Locations, Bodies and Sets

J. Jeremy Meyers

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Institute for Logic, Language and Computation
Universiteit van Amsterdam
Science Park 107
1098 XG Amsterdam
phone: +31-20-525 6051
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homepage: http://www.illc.uva.nl/

Locations, Bodies and Sets

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Co-promotors:	Prof. John Perry, Prof. Grigori Mints, Prof. Tom Ryckman

"*In terms of part-whole theory itself, the use or non-use of sets makes no substantial difference to expressive power. The sum axiom may be expressed as an axiom using sets, while with predicates a schema must be used if second-order quantification is to be avoided...*

Many of the writings of mereology are concerned with bringing logical refinement to the existing system, by suggesting new primitives, shorter axioms, and the like. While they are interesting in themselves, these contributions take the classical theory for granted and do not question its presuppositions. For someone who questions the cogency of some of these presuppositions it is remarkable how little work has been done on systems of mereology weaker than the classical ones of Leśniewski and Leonard and Goodman."

Peter Simons *Parts* pages 54 and 65

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Without comparison my greatest debts are to my parents, my son Jason, and my sister Felicite. They were by far my biggest source of support over the years, and I could not have completed this dissertation without their patient devotion.

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Chapter 1

Introduction

1.1 Background and Problems

Mereology (stemming from the Greek $\mu\epsilon\rho o\varsigma$ or 'part') is the theory of the parthood relation. Sophisticated thinking about part-to-whole relations begins at least as early as the Presocratics and continues in the writings of the famous ancient Greeks. In the *Parmenides* and the *Thaetetus*, Plato gives rather substantial treatment to mereological notions. Aristotle does as well notably in the *Metaphysics*, but also in the *Physics*, the *Topics*, and *De partibus animalium.* Mereology plays also a significant role in the writings of medieval ontologists and scholastic philosophers such as Garland the Computist, Peter Abelard, Thomas Aquinas, Raymond Lull, Walter Burley, and Albert of Saxony, as well as in Jungius's *Logica Hamburgensis* of 1638, Leibniz's *Dissertatio de arte combinatoria* of 1666 and *Monadology* of 1714, and Kant's early writings like the *Gedanken* of 1747 and the *Monadologia physica* of 1756.

Formal theories of parthood emerge in more contemporary thought mainly through the work of Franz Brentano and his pupils. For example, notable treatment is found in Husserl's third *Logical Investigation* of 1901. But it is not until Leśniewski's *Foundations of a General Theory of Manifolds* of 1916 that a pure theory of the part-relation was given an exact (albeit highly unique) formulation. As Leśniewski's work was published in Polish, it was never widely known. Consequently, it was not until the publication of Leonard and Goodman's *The Calculus of Individuals and its Uses* in

1940 that mereology became a more popular area of research. And since then formal mereology has become a theory of crucial significance in the fields of ontology and metaphysics. The formal turn would primarily have two causes. Firstly, many mathematicians and philosophers became frustrated with the inexactnesses and foibles of natural language as a medium for working. Secondly and most importantly, intrigue in formalization was spurred by a desire to elucidate the contrasts between concrete structures and sets.

Both are true of Stanisław Leśniewski. His obsession with rigor and his disavowal of the conception of distributive classes espoused by Russell and Whitehead led him to formalize a highly sophisticated theory of parts. Ultimately, his goal was to provide a formal mereology with the power and scope to supplant set theory and thus serve as a foundation for mathematics free of the antimonies of naïve set theory.

At least since the time of Leśniewski's seminal work, the theory of nominalism has maintained close theoretical ties with mereology. *Nominalism* is the view that (1) abstract objects do not exist, and (2) spatiotemporal (so-called *concrete*) particulars exist and are the only existents. The combined vision of nominalism and formal mereology will then be one of identifying a nominalistically acceptable formal theory of parthood. But how far-reaching should this vision be conceived?

Leśniewski espoused a nominalism of the most radical form. He rejected so-called "*general objects*" and all forms of platonic universals. His distaste for abstract objects was no doubt partly due to his aversion to set theory. But in general in his writings, he explicitly rejects anything which is not spatiotemporal and individual. Fascinatingly, he thought of his own logical systems in this way. For Leśniewski, formal languages are collections of concrete tokens. Some of these are printed on paper in books and articles. Others appear as utterances, chalk marks on blackboards, or pixelated patches upon computer screens. The radical nature of his view can be seen by comparing it with our modern conception of formal languages. If a logical system is a concrete complex of signs, then it cannot be infinite. Formal languages are not classes containing all syntactical formula types permitted by the formation rules. They change over time. Ideally, they grow as new theorems are proved. Although Leśniewski's understanding of his formal language was rigorously enforced at the level of practice, philosophical grounds for this view were never systematically argued. Nonetheless, Quine describes in *The Time of My Life* that he and Leśniewski spent many long evenings disputing whether the use of higher-order variables and quantification committed Leśniewski to abstract entities. Perhaps unsurprisingly Quine thought that it did, and Leśniewski thought otherwise.

More modern conceptions of mereological nominalism, especially in the field of philosophical ontology, are more nuanced and less far-reaching. Quine argued for a relaxed version of nominalism (see in particular [77] and [80]) with an important exception: existent objects are either concrete, physical entities or *abstract sets*—no other exceptions being permitted. Sets, he argued, are required to account for the presence of mathematical terms figuring in our best theories of physical science. But he rejected platonic universals and multiply located properties and relations. Still, although he proposed a view of the nature of the physical world based on objects postulated by the best theories of science, Quine thought part-to-whole structure figured centrally in the ground conception of reality.

Nominalistic theories espousing the existence of sets are nowadays known as those of *class nominalism*. Like Quine, David Lewis also held a type of class nominalism. One important difference between their versions is that, for Lewis, sets containing only concrete members (so called *impure sets*) are concrete ([57] 83 and [62] 59). This interpretation of impure sets has one obvious advantage in the context of nominalistic formal mereology. For, if impure sets are concrete, quantification over subsets of concrete entities will not involve one in ontological commitments to abstract sets. And thus without violating nominalism, highly complicated part-to-whole structures are captured by single sentences. But this boon hinges solely on the claim that impure sets are concrete. And on any view that is sensitive to their role in mathematical practice, sets are abstract. In the context of research in fields of pure mathematics, for example set theory, number theory and so on, mathematicians ascribe no space-time locations to the objects they study. And given a traditional understanding of sets, even impure classes contain objects—like the null set—which seem to defy an interpretation sympathetic to nominalism. Moreover, the number of impure sets is greater than that of the number of concrete particulars.

Thus there is an interesting connection between nominalistic ontology and formal mereology concerning the issue of capturing mereological *structure*. To capture the entire part-to-whole structure of physical objects will require that the adopted language have strong enough expressive resources. However if using certain formal devices is ontologically committing, they may outstrip the nominalist's repertoire of accepted entities.

Quine held that any set of material objects unrestrictedly combine [77]. Any selection of objects no matter how scattered or random combines to form a particular thing. Interestingly, Quine's regimented language of first-order logic will, for technical reasons, fail to capture certain highly complex infinite structures. Also intriguing is his rejection of properties, but his allowance of first-order predicates of any arity. He feels free to employ predicates without giving them equal representation in the ontology.

Concerning the structure of reality, other philosophers have argued rather extreme views to the contrary. For example, in his book *Material Beings* [99], Peter van Inwagen offers a mereological ontology which has the following unique characteristic: no material objects combine to form complex objects except living beings. Any other object is a mere "arrangement" of atoms. Here again, we have an ontological position which concerns the structure of reality. According to van Inwagen, although certain objects we might specify do not fuse or combine, others like humans, cats, and plants do. They persist through changes, move, and bear a multitude of mereological relations to other objects at various times. Thus van Inwagen's position implies that the primordial structure of reality involves not merely quarks, atoms, and spacetime locations, but also organisms. Other inanimate objects are seemingly free-floating particles without underlying cohesion.

The parthood relation also figures in a number of more contemporary versions of nominalistic ontology. For example, recent theories of four-dimensionalism are based on a mereological analysis of material objects. One notable recent theory of this type is due to Ted Sider [88]. According to this view, single objects do not wholly exist at various times. Strictly speaking, I cannot literally leave one spatial position and occupy another. Objects like you an I have various temporal parts. Biological species are four-dimensional “worms”: not only do we have bodies which are parts of us with hearts and lungs, but also parts that are temporally extended. You and I have one year-old parts, for example, in which our bodies extend from our birth to the age of one year. Sider’s way of understanding reality is thus in one way more austere than van Inwagen’s. Objects do not wholly exist and endure changes. Reality is static.

Let us wrap up this background with some general remarks. The identification of a theory of parthood requires one to address the following:

An Ontology. Identifying the mereological structure of objects requires an ontological theory of some sort. We must determine which objects are susceptible to mereological analysis. And most importantly, the selected ontology must correspond to our metaphysical doctrine. Only after identifying an acceptable ontology will we be prepared to answer any question about how objects are theoretically articulated and whether they unrestrictedly fuse.

The Ability to Capture Mereological Structure. Secondly, the formal system must be capable of “carving reality at reality’s joints,” to use a phrase from Plato. Not only must we provide a selection of primitives and logical devices to represent valid inferences over the domain, but we must also select a language with enough expressive power to capture the mereological structure implied by the proposed ontology.

Mereological Reasoning. Finally, I wish to analyze the ontology from the standpoint of logical reasoning. My approach will turn centrally on language selection. Which formal language should we select to represent the accepted mereo-ontology? My answer: to carve reality out in a way that corresponds to our ontology, it is better to use a scalpel than a kitchen knife! We should select a language which represents the inner structure of reality. But our selected language must presuppose no ontological commitments to entities outside the domain. The best way of controlling the conceptions at play is to eliminate any logical device or primitive which may denote an object which we cannot unequivocally countenance. In a word: *no logical distinctions without corresponding ontological ones.* Only after such a language has been erected, will we stand in a position to test precisely what details it can detect.

1.2 My Argument

This dissertation is a form of experimental metaphysics which begins with the following hypothesis. Suppose both that the theory of nominalism is true and that existent objects are mereologically analyzable. We shall answer two questions:

- *Can there be any such thing as a nominalistically acceptable formal mereology*?
- *And how much of the structure of reality can any remotely acceptable system capture*?

Ultimately, I claim that the answer to the first question must be in the negative. The relation of parthood will not be a particular thing of any kind. It is a relation wholly presented in various locations amid its relata. In contrast to Leśniewski's view, I claim formal languages are abstract. They cannot be particular things appearing in nature. They must be infinitely large and comprise every well-formed formula.

We must provide a maximally nominalistic ontology and corresponding logical framework to answer the second question. According to nominalism, the accepted ontology consists of mereological states of affairs involving individuals related part-to-whole. Meeting the demands of nominalism will affect our ability to make certain logical distinctions about the structure of spacetime. And it will be these very distinctions will allow us to capture the structural features of an infinite arrangement of objects.

1.3 Dissertation Abstract

The dissertation is divided into two parts. The first consists of an ontological analysis. And the second consists of a logical investigation.

Ontology. In the first four chapters I argue that by assuming nominalism is true, we must accept a maximally nominalistic ontology. We must reject the existence of *sets* in favor of extensional fusions. A formal mereology will have as its domain of discourse the entire range of concrete entities and each will be the fusion of its parts. In the class of existent objects are so-called *locations* which are fusions of either material or material-free objects. Locations are extensional in the mereological sense: two are identical if and only if their proper parts are. Locations, although geometrically variant, are identified solely in terms of their constituents and fuse unrestrictedly.

The first failure of the nominalistic hypothesis concerns the nature of the parthood relation. I claim the relation must be a multiply presented object, repeated wholly amid its relata, and not a particular thing of any kind. As the parthood relation is non-particular, it is not an individual *part* of reality but rather a *feature* of it. We next encounter a representation problem. The interconnectedness of actual locations requires recourse to sequences of intrinsically connected locations in topological contact. Formalizing such notions requires terms for sets, sequences, and higher-order quantification. By incorporating these notions directly, the interconnectedness of reality will lead to commitments to abstract objects.

More-or-less the nominalist must hold the view that reality contains living organisms. This violates the principle of mereological extensionality. Movements on the part of various objects and organisms imply that subparts of reality have less dimensions than that of the entire system. Organisms figure in a multitude of *localized* mereological arrangements. They are capable of persisting through changes in their proper parts. Although it might be thought that conceiving of reality in this way supersedes a purely nominalistic account or falls outside the pales of formal metaphysics, I claim, based on features of our relation to our bodies, that some such account must be adopted.

Perhaps ironically, a view of the physical world as a comprised of situated living beings provides a way to obviate explicit commitment to topological properties and sets required to represent the interconnectedness of physical universes. Organisms have intrinsically interconnected locations within a single spatiotemporally closed universe. Hence we arrive at a view that the objects postulated by nominalism are those connected via locations to our bodies.

The sum total of the requirements above will entail that our nominalist accept some notion of localized state of affairs involving the parthood relation. The ontology will therefore consist of concrete individuals and mereological arrangements involving them. Some states of affairs are localized and obtain at sub-locations of reality, but others will hold regardless of one's immediate location. The distinction between localized and non-localized situations helps to explain issues related to time, simultaneity at a distance, and tense.

Mereologic. Having provided a maximally nominalistic ontology, we can then turn our attention to modeling *reasoning* over the selected domain. Our pilot system will be a modal logic of mereology tailored precisely to the ontology. We employ a so-called hybrid language. Hybrid languages are extensions of standard modal languages in which reference can be made to individual objects by so-called "nominals" which are terms that function much like constants in the first-order language. Specifically, I adopt an extension $\mathcal{H}_{\mathsf{m}}$ of Arthur Prior's nominal tense language with additional operators for various part and extension relations. Although expressively weak in comparison to first-order mereologies, it is shown formally that $\mathcal{H}_{\mathsf{m}}$ is capable of denoting only acceptable states of affairs. Given the modal flavor of hybrid languages, both localized and non-localized types of situation are representable. Formulas are evaluated relative to a particular location. There are also formulas of $\mathcal{H}_{\mathsf{m}}$ that are capable of denoting objects irrespective of an occupied location.

Our first formal task is to show that each formula represents an acceptable state of affairs devoid of relations besides that of parthood. Complex and arithmetical relations and principles definable in first-order logic are undefinable in the selected language. An appropriate restriction then emerges: in $\mathcal{H}_{\mathsf{m}}$, one cannot count. The existence of various mereologics for classes of extensional mereological structures is then demonstrated. By a novel Henkin construction, completeness with respect to the classes of atomless and atomic structures is given. And I show that there is a sound and complete proof system for infinite atomic Boolean algebras with any infinitely large set of atoms.

In essence, any nominalistically acceptable mereological distinction is encapsulated in the notion of $\mathcal{H}_{\mathsf{m}}$-*bisimilarity*. Structural simulations are well-known in the fields of modal logic and formal ontology. These relations encode what structural details the language can distinguish. In a certain sense, I argue that nominalistic mereology just is the "mereo-bisimilar" fragment of the first-order language.

But now to the second question: can $\mathcal{H}_{\mathsf{m}}$ detect the subtle differences between distinct spatiotemporal locations? The best way to go about answering our second question is first to identify suitable models representing the structural features we wish to preserve. Then we proceed to test how much structure $\mathcal{H}_{\mathsf{m}}$ can "see" of them. Two mathematical models are indistinguishable by $\mathcal{H}_{\mathsf{m}}$-formulas if there is a mereo-bisimulation between them. Thus if $\mathcal{H}_{\mathsf{m}}$ detects no differences between two models—one which has the structural features of locations and another which clearly does not—then *a fortiori* $\mathcal{H}_{\mathsf{m}}$ will *not* be able to capture the corresponding structural details in reality.

Well-known, adequately proved results in the theory of Boolean algebras show that certain mathematical structures called complete Boolean algebras have the requisite features of the structure of unrestrictedly fused locations. Taking some results proven by Tarski [93] and MacNeille [61] in the thirties for granted, I show that any infinite n-dimensional atomless or atomic Boolean algebra with a finite valuation is mereo-bisimilar to its Boolean completion.

So, on the one hand, if reality contains infinitely many locations, we lack the ability to discriminate between uncountably many of them. Indeed if there are infinitely many locations and these decompose to a floor of atoms, then single formulas of $\mathcal{H}_{\mathsf{m}}$ will conflate reality with a finite structure. Again, if there are infinite locations and some of these contain no atoms or if all are completely atomless, then up to mereo-bisimulation, portions of reality will be conflated with "pixelated" or geometrically extended, unanalyzable objects. Capturing the structure of an infinite dimensional system will then be a major stumbling block for a nominalistic mereology. So, again, if reality contains infinitely many individuals, a formal language should be selected with terms for sets and set-quantifiers. I show the existence of $\mathcal{H}_{\mathsf{m}}$-logics of classical atomic and extended atomless spaces and demonstrate that there is a sound and complete proof system for the class of regular open sets of $\mathbb{R}^n$ for any $n \in \mathbb{N}$. And I show the same result holds for the class of powerset algebras of the real numbers.

The constructive and positive contribution of the dissertation is one to formal logic. Presently, all investigated modal mereologies are set-theoretic in the sense that propositional variables $\{p, q, r...\}$ are capable of being true at many distinct elements of Kripke models. Each proposition symbol represents either a subset of an implied domain of items or a "multiply located" entity. In Vakarelov [97], "A modal logic of set relations" the author investigates a modal logic whose atomic formulae range over subsets of the domain. Goranko and Vakarelov [46] also introduce a language with a set-theoretic semantics interpreted over powerset algebras whose modalities represent set operations based on membership relations implicit in the Boolean set-operations. In addition, Balbiani et al. [8] provide a modal logic based on membership modalities with a topological interpretation. But none of the above modal frameworks will allow us to discriminate localized situations from non-local. And all presuppose set-theoretic frameworks which should be excluded on nominalistic grounds. Our logical systems are all proper fragments of first-order logic with unique expressive properties which shed light on a number of contrasts between first-order mereologies and extended hybrid logics. And the formal results can be seen as independent contributions to the theory of space and modern spatial logic more generally.

1.4 The Philosophical Assumptions I Make

Before we move to the content of the dissertation, we should explain some of the important assumptions made and terms employed. Nominalism, it will be assumed, is not merely a rejection of abstract objects or, in particular, a rejection of universals. It is also the thesis that concrete particular objects exist. The theory has one of two positive forms. The first type is what I call *naturalistic*. There is a *single* universe which is presumed to be spatiotemporal. Parts of it are pairwise interconnected. According to this nominalist, the theory of naturalism is true in David Armstrong's ontological sense of the term. This is not a methodological or epistemological sense of 'naturalism'. And this form does not necessarily imply some notion of a final physics. Although I will argue that our relationship to our bodies involves us in postulating entities in which we are physically situated, the form of nominalism itself does not turn on empirical methodology. A presumption of naturalistic nominalism also does not imply that we cannot make sense of the conceptual possibility that human knowledge surpasses the contents of a universe of concrete entities. Indeed, this form of naturalism will not block the metaphysical *possibility* of abstracta. The form in mind will be just an espousal of the closure of the spatiotemporal world and its contents: that any instances of the kinds of object that actually exist are concrete and could not have existed except but within a single mutually interconnected dimensional system. The term 'dimensional system' is to refer to whatever it may be that in fact underlies the general systematic treatment of spatial and spatiotemporal regions, distances, and the geo-topological relations amid parts of the physical world.

No serious commitment to one treatment of the dimensional system over another will be made. Indeed, it will be fully within the spirit of my argument that the universe may have many or even fewer dimensions than four. It may be the case that the dimensional system is at least as fundamental as its contents (substantivalism). Moreover the dimensional system may be in some sense identical to the universe (supersubstantivalism). Or it may be less fundamental than its constituents (relationalism). If the last of these alternatives is true, then nominalism will be rejected. For, in this case, all existing objects will not be individuals, since there will be spatial relations. What I do claim in Chapter II is that in order for universes not to cleave or fragment into non-physical objects there must exist *some* so-called *closing substances* which I admit I suggestively call *locations*. Individual locations may be either material or material-free, and I do not propose a view one way or the other. I make only the disjunctive claim that any part of the physical universe is, perhaps among other things or in other modes or configurations, either an individual portion of matter, a region, or a fusion of both.

It is also possible that the dimensional system is in some sense discontinuous or quantized. I am inclined to the view that the physical world may be finitely or discretely articulated into extended geometrical entities. Indeed, it seems possible that, even if dimensional regions are quantized, they might not have been, and that despite the system's supposed non-Euclidean structure, it could have turned out otherwise.

What is most important to my account is that *universes* are natural objects. Whatever their geometrical structure, it crucial that locations within them unrestrictedly combine and decompose extensionally.

The second form nominalism might take is one which views the spatiotemporal parts of reality as possibly disconnected. One strong version is David Lewis' theory of modal realism according to which, roughly, every way a universe could be is a way that some spatiotemporal universe is. Thus there will not be a single universe, but rather a multitude of them. All objects are in some sense spatiotemporal, but some are disconnected from others and in different possible worlds. I will neither agree nor disagree with this position. I think it is possible that there are dimensional systems or objects which are spatiotemporally disconnected from our universe or, in other words, that some objects may be spatiotemporally "isolated", to use Lewis' jargon. Again, even on this weaker view, I will take no firm stand.

I assume according to tradition that concrete objects are spatiotemporal and causally efficacious, as defined earlier, but moreover, that they exist independently of minds. The existence of concrete objects and their constituents is not dependent upon subjective experiences and perspectives. The assumption will furthermore entail that universes are physical entities that do not reduce to mere logical structures. I plan in the future to commence study to address this view directly. But here I simply note these as assumptions.

Finally, I will presume that the parthood relation is a type of partial order. This is the most common way of understanding the relation. And I find it *prima facie* obvious. In short, I will in no way contest this standard way of understanding the parthood relation. However, I will provide an argument that the relation is a stronger type of supplementary partial order over the class of concrete entities.

1.5 Chapter Overview

1.5.1 Ontology

The first part of the thesis consists of three chapters and is devoted to clarifying what a maximally nominalistic mereo-ontology must be like.

Chapter 2: Mereological Structure and Physical Reality

I claim that if we accept the nominalist's position, we must reject the existence of any notion of set in favor of extensional fusions. Reality will consist *inter alia* of spacetime locations which can be either material or material-free. Locations will be extensional objects in the mereological sense—that is, two locations are identical if and only if their proper parts are. Locations are fusions of matter or regions. Although the latter are geometrically variant, it will be argued that they and their constituents are extensional in the mereological sense. In addition, I argue that locations fuse unrestrictedly.

Chapter 3: Mereological Situations and Locality

Existing in space and time will be just one non-particular, the parthood relation—a repeatable, multiply located uniformity existing amid individuals. According to this view, reality is comprised ultimately of situations involving these individuals and the parthood relation. A so-called nominalistically acceptable mereological state of affairs or situation is one devoid both of sets, various other properties and relations (besides that of parthood), and abstracta. The existence of localized mereological situations provides a way in which to explain the persistence of objects like organisms undergoing mereological changes. Non-local or closed mereological situations will also be deemed to exist.

Chapter 4: Inscriptional Nominalistic Mereologies

I argue that the theory of inscriptionalism, according to which formal logics are concrete particular "marks", will lead to an over-generation problem. The argument proceeds by assuming, toward reductio, that formal languages are physical aggregates of marks. On this view, by introducing a formal language for investigation, a physical "prototype" is produced: an example of a concrete vocabulary. Syntactic "protocols", the type-theoretic definiens of physical tokens or "marks", are then given by way of equiformity relations to ordered objects in the prototype. But it is argued that any protocol (as a complex system of types) must be applicable to the entire domain of concrete entities. As they are types, the must exist at least as early as the earliest point in time the could have existed. This leads to the absurdity that unintended structures appearing in nature are objects of formal languages. Thus it is claimed that they must be abstract objects of some kind.

1.5.2 Nominalistic Mereologic

In the final three chapters we propose a formal language and investigate its properties formally. We introduce various general types of logics for mereological reasoning. And then we make some claims about their philosophical significance.

Chapter 5: A Formal Language for Mereology

I argue that the referential success of first-order quantificational statements implies the existence of parametrizations—situations in which individuals are "abstracted away"—and arithmetical constructions appearing in the concrete world. Parametrizations, however, were rejected in chapter III. We briefly enquire into the possibility of infinitary languages and reject them on nominalistic grounds. A more suitable language $\mathcal{H}_{\mathsf{m}}$ is introduced all of whose formulas denote finitely complex mereological situations. We briefly introduce the $\mathcal{H}_{\mathsf{m}}$-invariance morphism and the general $\mathcal{H}_{\mathsf{m}}$-logic of models underlining its significance to the ontological investigation.

Chapter 6: Hybrid Mereological Languages $\mathcal{H}_{\mathsf{m}}$ and $\mathcal{H}_{\mathsf{o}}$

We investigate the formal properties of the languages of hybrid mereology. The expressive powers of both are analyzed in detail. Sound and complete proof systems for varieties of Boolean algebras and zero-deleted Boolean algebras are introduced.

We show that the atomless and atomic classes of Boolean algebras are definable and show sound and complete axiom systems for them. It is shown that in $\mathcal{H}_\mathsf{m}$ counting sentences and therefore generalized arithmetical principles cannot be expressed as required. Finally, by an ultraproduct construction analogous to van Benthem's original argument [10], it is proved that $\mathcal{H}_\mathsf{m}$ is the proper fragment of first-order logic closed under mereobisimulation.

Chapter 7: Capturing the Structure of Locations

Can $\mathcal{H}_\mathsf{m}$ detect the subtle differences between distinct locations? It is argued that the best way to go about answering this question will be to identify suitable mathematical models to represent the features of objects we wish to preserve (extensionality and unrestricted fusion). Well-known, adequately proved algebraic results suggest that certain mathematical structures have these properties. We demonstrate the existence of $\mathcal{H}_\mathsf{m}$-logics of classical atomic and extended atomless spaces. In particular, we show that there is a sound and complete proof system for the class of regular open sets of $\mathbb{R}^n$ for any $n \in \mathbb{N}$. And we show the same for the class of powerset algebras of the real numbers. This shows that if locations are dense and atomless, then nominalistic mereologies are highly insufficient. They cannot serve to represent the variable structures of regions and matter. Likewise if existent objects are infinite in number and atomic, $\mathcal{H}_\mathsf{m}$-formulas will be insufficient to distinguish the entire space from a finite model.

Chapter 8. Conclusion

I claim that any formal mereology without terms for sets and set-quantifiers will furnish a rather meagre source of intuitions, categories, and inferences. And I argue that any way of extending a formal mereology with conceptions sufficient to represent the dimensional, topological, and geometric properties of locations will require set-theoretic resources.

1.6 List of Items Published

- Material in Chapter 6 was substantially revised and submitted to the *Journal of Philosophical Logic* and accepted under the title "What is Nominalistic Mereology?" DOI 10.1007/s10992-012-9252-4

- Materials in both Chapters 6 and 7 were presented in various professional conferences from 2009 to 2010. It found substantial treatment at the University of Tampere, Finland in two separate short courses on Mereology and Modal Logic in 2010 and 2011, funded by the University of Tampere and the Finnish Academy of Sciences. Material in these was also accepted as a short presentation entitled "Hybrid Formal Ontology" in *Methods for Modalities 6* in Copenhagen, Denmark in 2009. And various proofs in chapter VII were presented at the University of Groningen in 2009 in the conference *Logic in Rationality.* Technical results in both chapters were also presented at Stanford University in the *Conference on Spatial Relations* in 2009.

- Other formal results relevant to this dissertation include a paper "Undecidable First-order Theories of Affine Geometry" submitted to the *Conference on Computer Science Logic 2012* by myself, Antti Kuusisto, and Jonni Virtema of the University of Tampere in Finland. A short version was accepted as a presentation at *IEEE Symposium on Logic in Computer Science (LICS) 2012.*

Chapter 2

Mereological Structure and Physical Reality

2.1 Nominalism and Mereology

Nominalism is the view that (1) abstract objects do not exist, and (2) spatiotemporal (so-called *concrete*) particulars exist and are the only existents.[1] Mereology, as a field of theoretical research, proceeds from the assumption that existent objects have parts and are susceptible to mereological analysis. Thus the combined vision of nominalistic formal mereology will then be to identify a nominalistically acceptable formal theory of parthood. Let us suppose that nominalism is true and that concrete objects exist independently of minds. Furthermore assume that concrete objects are susceptible to mereological analysis.

Based on the assumption of nominalism, I claim that all types of set should be rejected. The parthood relation itself will be understood as an object presented *in toto* amid its relata. In other words, it is wholly repeated amid the separate objects it relates. I will argue that the conception of a spatiotemporal universe is at the heart of the assumption of nominalism. A *universe* is a fusion of maximally

[1] There is also another version of nominalism according to which all objects are particular things. On such a view, abstract entities may exist. We will consider this view in the latter chapters of the dissertation and in some detached contexts.

pairwise interconnected objects. Universes are comprised of extensional fusions of interconnected *locations*. Locations are either material or material-free and are closed under unrestricted mereological fusion.

2.1.1 Reality and Abstract Objects

Although there are various conceptions of abstract objects, the most prominent is of causally inert and non-spatiotemporal entities.[2] Abstract objects are neither parts of the landscape of space-time nor capable of exerting physical forces. The reason for this characterization stems from the nature of the nominalist's qualms. She sees something dubious with the very notion of a non-spatiotemporal, causally inert entity. And typically, her worry is prompted by a naturalist or empiricist epistemology. In particular, she wonders how one can form reliable beliefs about objects that cannot, even theoretically, give rise to sensorily detectable effects.

In theory, nominalism is to be supported by serious empirical science. It will be maintained that, not only is the spatiotemporality of the universe commonsensical and intuitive to hold, but it is indeed true and confirmed. Needless to say, scientific advances in the nineteenth and twentieth century have made such a view more difficult to maintain. In recent theoretical physics, the motivation to preserve the character of observable objects has given way to that of coherent and comprehensive mathematical explanation.[3] But we hardly need to delve into the specifics of theoretical physics to sense that the spatiotemporality criterion may be too strong. On the one

[2]There are other proposals as to how to characterise abstract objects. One approach defines them as those the understanding of whose names involves a recognition that the named object is in the range of a certain functional expression (Dummett [24] p 485). It has also been thought that an abstract object is one that fails to exemplify existence (Zalta [104] p 12), or an object that could exist but does not (Zalta [104] pgs 60, 96). On another conception of abstract objects these are objects that cannot exist separately from other entities (Lowe [59] 514). For a discussion of the various ways of characterising the abstract/concrete distinction see Burgess and Rosen [12] 13-25)

[3]Presently, string theory is the most promising candidate for bridging the gap between quantum field theory (QFT) and the general theory of relativity, thus supplying a unified theory of all natural forces, including gravitation. Its basic idea is not to take zero-dimensional entities as fundamental but rather very tiny one-dimensional objects. Importantly, the newly accepted entities interact at extended distances and not merely at points. This ontological distinction between string theory and standard QFT is the reason why the former also encompasses the gravitational force which cannot be treated in the framework of QFT.

hand, even if the part of the *observable* world is spatiotemporal, it may be arranged into a more complex medium with objects of higher dimensions. Consequently, the nominalist may very well wish to consider these objects acceptable. On the other, as Kant thought, the three-dimensional character of objects, order of time, direction of causality, and four-dimensionality of events may be mere byproducts of the manner in which we are constituted to experience reality.

Still, a certain driving motivation for the nominalist will be to safeguard the perceived character of physical objects. So we shall assume that concrete objects are spatiotemporal and causally efficacious. If an object is *spatiotemporal*, we will take this to mean that it is located in three spatial dimensions, one temporal, and thus may be extended in four dimensions or less. Indeed I see no other way to faithfully interpret the theory of nominalism. If objects are spatiotemporal, then this implies that they reside in a dimensional system. Nonetheless we will periodically generalize the account to one for arbitrary dimensions. And if an object has a certain number of dimensions κ, we will say that it is κ-dimensional.

Even if all objects are concrete, this requirement will be insufficient for one sort of nominalist. Two additional properties will be demanded. Firstly, there will be a *single* reality: a composition (or *fusion* defined below) of all spatiotemporal objects. If objects are not a part of this one universe, they do not exist. Secondly, objects will be in some sense interconnected or contiguously arranged. These two requirements will be referred to as *the closure thesis* (CT). The reason for the interconnectedness requirement can be explained by considering the idea of a spatiotemporally "isolated" object. By accepting the notion of reality as simply a composition of arbitrarily many spatiotemporal objects, this will imply that even those entities having no space-time connection to one another comprise a single universe. And importantly, the objects spatiotemporally related to our bodies may not exhaust reality.

Another sort of nominalist will reject that all objects are spatiotemporally interconnected. For example, according to David Lewis' theory of modal realism, for every way a world could be, there exists one. One is our world—the actual world. But there are various other possible worlds that are as real as ours is to us. Although a metaphysical theory of modality, his theory has certain mereotopological consequences. He rejects interconnectivity tenet of CT. There is a single *reality*, but there are many isolated universes comprising worlds of their own. The actual world is the one spatiotemporally related or interconnected to our bodies. Others are disconnected from us. For Lewis, possible worlds are maximal sums of spatiotemporally related entities. A sum of spatiotemporally related objects is maximal if and only if nothing that is not part of the sum is spatiotemporally related to any part of it. And as sums of spatiotemporally related objects are sums of concrete objects, and sums of concrete objects are concrete objects, Lewisian possible worlds are concrete objects. So although there will be a fusion of merely existent objects, those in pairwise distinct worlds will be disconnected.

Although Lewis is a nominalist, his conception of the extent of reality is radically different from one whose motivation follows from naturalism or empiricism. And it appears to be either naturalism or empiricism which centrally informs the nominalist's tendency to reject abstract objects. She rejects them because either they have no capability of affecting her, being observed by her, or residing in the same universe as her body. And her rejection is not merely due their failing to have a specific structure or being causally inert. According to modal realism, non-actual possibilities are to be identified with existent disconnected universes. For a nominalist espousing CT, this cannot be. Although any fact that ϕ about a spatiotemporally isolated entity implies that possibly $\ulcorner\phi\urcorner$, it is not true that possibly $\ulcorner\phi\urcorner$ implies that there is a disconnected entity at which ϕ. The existence of spatiotemporally isolated entities will be merely conceivable or perhaps possible. And she will reject the notion of any isolated entity as worth her ontological commitment.

2.1.2 Fragmented and Tensed Versions of Reality

Our nominalist will reject certain views out of hand. On one theory, that of *presentism*, reality has a tensed configuration. The present is all that exists (Prior [73][76][75]). The past and future do not exist, but there are tensed *facts* constituting reality accounting for facts of the past and future. For example, consider the proposition that Elvis is dead. On this view, it is true that, previously, Elvis exists, even though, presently and henceforth, this is not the case. Thus there will be a fact of the matter that he was alive. Let ϕ be any proposition such that $\ulcorner\phi\urcorner$ is given in the present tense. In general, if $\ulcorner\phi\urcorner$ is false but, $\ulcorner$previously, $\phi\urcorner$ is true, then the tense realist will account for this fact by postulating the existence of a past tensed fact: *previously* $\ulcorner\phi\urcorner$. Likewise if ϕ *will be* true, there will be a future fact of the matter, and so on. Although various tensed facts constitute reality, objects which do not currently or presently exist will simply not exist at all. On the tensed view, a continuously "flowing" present time is postulated. Existent objects will be extended parts of a three-dimensional entity existing at the present time.

The traditional tensed theory of reality will have substantial problems that have been given substantial treatment elsewhere.[4] These problems can be described briefly as follows. As the facts constituting what is presently the case are in constant flux, complex or layered tensed facts will change as the present changes. Thus ultimately, reality will be constituted by inconsistent facts. Natural refinements of the tensed view include those which take other tenses to be the existent times. For example it might be claimed that only the past and present are real or the present and future. And these views, too, will yield analogous inconsistencies in facts as the present time changes.

On another tensed theory, fragments of reality are again indexed by times. But reality will be now configured as an ordered sequence of tensed fragments, each of which corresponding to a unique form the present might take but also containing uniquely configured past and future times relative to the present.[5] For various reasons which will not be fully explained here, this view does not entangle one in the logical inconsistencies of the former system. For each fragment will correspond to a present time and give rise to its own independent array of tensed facts which hold just at that time. As time "passes" and new present times are realized, reality itself passes into another configuration. So the problem will now not be one of inconsistency, but rather physicality. Like the traditional tensed account, whatever is physical and

[4] The earliest criticism is McTaggart [65]. For an especially precise criticism see Fine [35]. Finally for a so-called B-theoretical, four-dimensionalist account see Sider [88]). Also see Fine [37], [36], Evans [31], Horwitch [49], Mellor [66], [67].

[5] For an example of a view along these lines see Fine [35].

existent will be in some sense a present configuration. But for each present, there will be distinct non-physical modes within each fragment corresponding to the past and future.

According to nominalism, both tensed views of reality should be rejected. On the tensed view, material bodies, cars, and people, for example, will be either three-dimensional parts of space or, perhaps, brief four-dimensional intervals. Either way, the past and future will have a non-physical character. They will be physically and dimensionally disconnected from the present. And thus they will have an abstract modal character.

On yet another—the theory of *idealism*—reality is fragmented experientially. It will be comprised of various sequences of first-personal experiences, indexed by every first-personal standpoint. Obviously, if experiences are conceived as abstractions and reality is configured in this manner, any such theory will be nominalistically unacceptable. The same will hold for views of reality in which there are entities which are 'set apart'. Abstract or platonistic conceptions of meanings, propositions, or forms may be postulated to exist in different realms entirely. And for each such type of entity, there will be unelidable questions concerning how we know of them, how our brains can connect up to them, and how any physical connections to them can be obtained whatsoever. There are a multitude of metaphysical theories postulating the existence of entities that by definition cannot be construed as physically connected to concrete objects. None will be consistent with nominalism as proposed.

Connectivity will be understood to obtain amid all actual concrete objects. And if this is the case, various connections in the dimensional system will be postulated amid concrete entities. Thus it comes as no surprise that questions concerning the status of reality will concern those of the ontology of space and the existence of regions. A *region* is an object of arbitrary dimensions that a material object may occupy. Regions may be conceived as distinct from matter. Material portions will be thought to be objects occupiable by regions. And thus we must know what mereological relationships (if any) they stand in. On the other hand, it is possible that material objects simply *are* the regions they occupy. On this view—that of *supersubstantivalism*—any conceived structure of reality will be greatly simplified. Still it might be held either that regions or matter do not exist. If the notion of one or the other implies an inconsistency, then one shall be rejected. We return to a discussion of regions in the present Chapter.

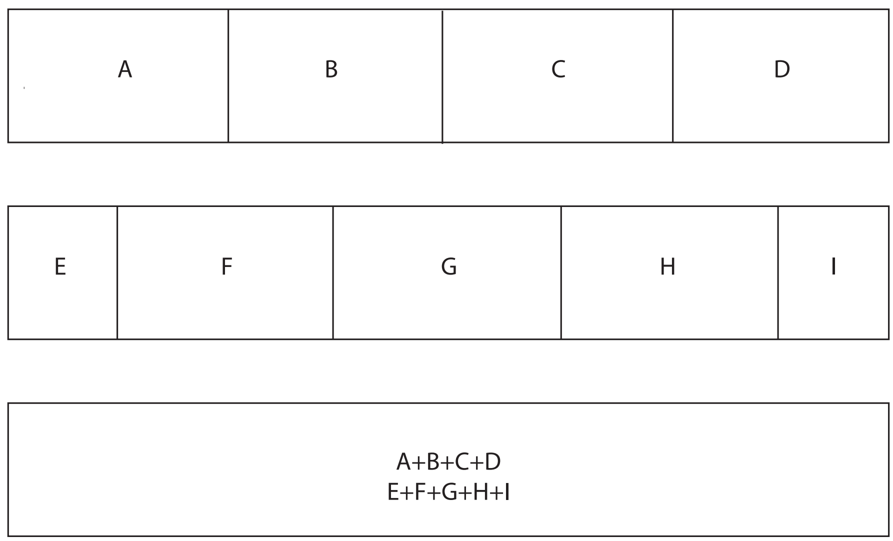
2.1.3 Mathematical Entities and Sets

Paradigmatically and in practice, mathematical objects are sets. And controversial axioms of set theory aside, sets have some rather irrefutable abstract features. For example, on any standard theory, a set is identified solely as a collection of objects which has its members necessarily. So if sets are concrete, there are concrete objects that have their parts necessarily. But theoretically, any concrete object could have parts at least slightly different than those it has.

Another well-known, justification for the abstractness of sets concerns the proper interpretation of mathematical practice. It is a plausible *prima facie* constraint on any metaphysical interpretation of successful research that one refrain from interpreting its investigative methods in a way which would render them misleading or inadequate. But observe that if mathematical entities like sets and numbers have space-time locations, the practice of mathematics is certainly misleading. For mathematicians do not ascribe physical locations to the objects they study; and, to suggest that they must, would render the nature of their research and their results inexplicable or inaccurate.

But it has been suggested that not all sets are abstract. There may be so-called *impure* sets that are concrete. An *impure set* is one containing a concrete member. And a *pure set* is one with no concrete members. Consider the set of all humans and the set of all planets. These are *completely impure*, since their members are concrete. Hence contrary to the common characterization of sets, it might be thought that any completely impure set is spatiotemporal (Lewis [57] p83 and Maddy [62], p59). But on any traditional understanding of sets, it would seem even completely impure sets are not concrete.

Firstly, all sets have the null set as a subset. And it is this feature that would most clearly indicate their abstractness. Only on a most mysterious theory of concrete entities would it follow that each has a single "empty" object as a necessary ingredient. Even a set of concrete *Urelements*, for example of either matter or regions, will contain the null set. And therefore any Urelement and the set containing it will be distinct.



Secondly, there are strictly more impure sets than concrete objects. The *extensional fusion* (henceforth *fusion*) $o_1 + ... + o_\kappa$ of a set $\{o_1 ... o_\kappa\}$ of objects is exactly the object whose parts are $o_1 ... o_\kappa$ and whose every part overlaps at least one of $\{o_1 ... o_\kappa\}$. A *division of an object o* is a selection of parts of o whose extensional fusion is o. Now observe that every division of o gives rise to a class whose members are its parts. But the nominalist will object that every *way* of dividing an object gives rise to a unique one. By in fact dividing an object one may obtain a new one; but no mere way of dividing an object shall be identified as a concrete entity. For example suppose we have a particular object $A + B + C + D$ divided in two ways as in the diagram. The top two objects represent ways the bottom one can be divided. The fusion $A + B + C + D$ is exactly $E + F + G + H + I$. However, construing the objects as *sets*, we have $\{A, B, C, D\} \neq \{E, F, G, H, I\}$, and neither $\{A, B, C, D\}$ nor $\{E, F, G, H, I\}$ are identical to the set $\{A + B + C + D\}$.

So each *manner* of dividing a single concrete entity gives rise to a distinct set. But any *manner* of dividing an object is not a concrete object. From the nominalistic standpoint, theoretical divisions of objects are not concrete. The concrete object is simply the fusion. Indeed, the nominalistic alternative to the notion of set is simply that of fusion which is by now so well-known in theories of extensional formal mereology. In these theories there is no analogous distinction between the relations of membership and subset. And there exists no "null" object which is a part of everything.

A theory of non-well-founded sets would furnish an alternative conception. In non-well-founded theories, sets are allowed to contain themselves and otherwise violate the axiom of well-foundedness according to which no set contains an infinitely descending membership chain. But in general it is not the well-foundedness *per se* of a set which nominalism objects to but rather the two features just mentioned: (1) the rather bizarre idea that there is a *single* "empty" concrete object that is a part of everything, and (2) that each theoretical division of a concrete entity is concrete. And the employment of non-well-founded sets will not alone eliminate these possibilities.

We will investigate the notion of set from the experimental domain, but will refrain from attributing commitment to them on the part of the nominalist. And we will yield to a somewhat standard interpretation of them. All will have *subsets* which necessarily include the null set. And for any set S, there will be a powerset $Pow(S)$ of S containing the set of all subsets of S. As is standard, for the size of a set we will understand its cardinality. Of course, if sets exist, they can have infinitely many members, and so on. The only axioms of the theory of sets which I shall not assume are that of choice and foundation.

Although terminological exceptions exist, we will therefore hold that sets—pure, impure, well-founded, and non-well-founded—are abstract entities. And in particular, when any conception of set, class, selection, or collection is conscientiously invoked, we will assume that the notion of fusion is not the one intended. Conversely, if an extensional fusion of objects is mentioned the intended meaning will not be that of class just described.

We rely heavily on the conception of fusion just defined. It is for good reason that its definition is formulated in the language of first-order logic and not in a higher-order language. Sets are, theoretically, as fusible as individuals. And various sets shall be definable in higher-order languages. But even mereological fusions of sets will be unacceptable. So we shall maintain the view that acceptable fusions must be concrete individuals only and defined according to the first-order formulation (given above).

2.1.4 Universals and Properties

Universals, too, are typically claimed to be abstract. Recall that a *universal* is a property conceived as capable of being *borne by* multiple, distinct things or having multiple, distinct (henceforth *various*) instances. Otherwise it is a *particular*. In philosophical parlance, a universal is said to be *instantiatible by* various objects. For example, if redness is a universal, then every red thing is an instance of it. But the things that are red, e.g. individual apples, cannot have various instances.

If universals are to be nominalistically acceptable, then by definition they must be spatiotemporal. Accordingly, it is more plausible that unary properties are located *in* and relations *amid* their instances than at other places. This view is the so-called *in re* theory of realism about universals. And it is to be contrasted with *ante rem* realism, according to which universals are located outside their instances.[6] The truth of *in re* realism will imply that universals are *capable of being multiply located* (abbreviation: *multiply presentable*), in the sense of being *wholly* located at the objects that instantiate them. For example, plausibly, there are various apples of precisely the same shade of red. Thus according to the *in re* theory of universals, the same property of *redness* is located at each red apple. Note that this has the strange consequence that a single universal can be at some distance from itself—for redness is wholly located at each red object of the same shade. But it is the *concrete particular* that the nominalist privileges (e.g. particular houses, people, and cars). And these cannot be simultaneously at spatial distances from themselves.

The notion of a concrete universal may also lead to a mereological perplexity

[6]Another, common way of putting this is that *ante rem* realists hold that properties exist independently of being true of anything; *in re* realists require that they be true of something.

concerning the structure of reality. Let u be wholly present at each pairwise non-overlapping location $l_1...l_n$. There are two cases. On the one hand, if being wholly present at a location implies being a part of that location, u is a part of each $l_1...l_n$. And we have an immediate contradiction. For each of the latter was to be non-overlapping. So assume, on the other hand, that being wholly present at a location does *not* imply being a part of that location. Then apparently, being present at a location does not imply being a part of *any* location; for, if a concrete object is to be deemed a part of some location, it would be absurd to suggest one other than that at which it is presented. And since universals will be concrete, the fusion of *all* locations will not include as a part (proper or otherwise) the fusion of all concrete items. But again we have a problem. The very point of adopting *in re* realism was to place all concrete objects *within* a single spatiotemporal universe. We shall call this the *locatedness problem for universals.*

So the nominalist's way of handling the location problem will be to eschew as many universals as she can. But if she is a formal mereologist, she must address the nature of the parthood relation itself. Suppose it does not have multiple instances and is not multiply presentable. If it does have a single instance, it is a particular with just one location. Then it is not borne by arbitrarily many pairs of individuals. Indeed, in this case the relation of parthood will be a rela*ton* relating at most two. However, in conception, the project of formal mereology presumes that all objects are capable of having parts. So, if, according to commonsense and nominalism, there are a great many objects and they are all residents of a single spatiotemporal whole, the view of parthood as a relaton is clearly incorrect.

Another alternative will be to view the relation as a set. And the most plausible candidate will be the extension of all pairs of concrete objects standing in the relation. However, as we have seen, sets are abstract. If parthood is a set of ordered pairs, it is completely pure, since no ordered pair is concrete. Observe that the decompositional structure of the members of the proposed set will follow from any specification of all concrete objects—some will have parts and some will be parts of others. For example, perhaps matter, regions, or some combined conception of substance will be adopted as a type of Urelement (see below). If two Urelements exist, they will stand in the relation of parthood given how they are constituted. And by providing the entire selection of Urelements, the proper parts of all entities will be included. So the set-theoretic construction of ordered pairs will be a logical construction posterior to parthood.

2.1.5 Is the Parthood Relation a Trope?

Another alternative will be to claim that the parthood relation is a relational trope. Recall that a *trope* is traditionally defined as an abstract particular. They are non-spatiotemporal, non-universals (see Campbell [14], Daly [21]). Campbell, Daly, and other philosophers have claimed that tropes can play the theoretical role of properties and relations. As these are abstract objects, the nominalist will reject them out of hand. But, as particulars, they will, again, be rela*tons*, and the same problem noted above will be inherited. However suppose that tropes are to be understood a bit differently. Perhaps both relations and properties are concrete individuals, but they are neither at some distance from themselves nor multiply presentable. It may be plausible that they can nonetheless be applicable to various objects.

But this conception of them, too, will give rise to a certain unelidable problem. To see this, suppose that the parthood relation is a particular but not wholly present amid various objects and therefore not at various distances from itself. As normal, it relates all or a multitude of objects. For example, let $o_1, o_2, o_3, o_4, o_5, o_6, \ldots$ be a selection of concrete objects. Suppose that o_1 is a part of o_2, o_3 is a part of o_4, and so on. Now observe that if the parthood relation is a trope, there are not *various* instances which are repeated as in

$$o_1 \leq o_2, \quad o_3 \leq o_4, \quad o_5 \leq o_6, \ldots$$

where $\leq$ is repeated and wholly present amid each pair. Rather, $\leq$ relates objects in some sense as

$$o_1, \quad o_3, \quad o_5, \ldots \quad \leq \quad o_2, \quad o_4, \quad o_6, \ldots$$

In the latter case $\leq$ will have a single location but this location will be "spread throughout" reality. At least a multitude of objects will be thought to have parts. So $\leq$ will in some sense "spreads across" them all. But how are we possibly to understand this proposal? If the relation is not to be a set, it will be exactly the fusion of its instances. But then what precisely will $\leq$ relate? It cannot relate individuals separately or singly, since this will require there to be multiple, repeated instances. Thus it must relate them as sets. That is as

$$\{o_1, \quad o_3, \quad o_5, ...\} \quad \leq \quad \{o_2, \quad o_4, \quad o_6, ...\}.$$

Indeed, if parthood is reflexive and every object has itself as a set, it must relate duplicate copies of the set of all concrete individuals. That is to say,

$$\{o_1, \quad o_2, \quad o_3, \quad o_4, \quad o_5, \quad o_6, ...\} \quad \leq \quad \{o_1, \quad o_2, \quad o_3, \quad o_4, \quad o_5, \quad o_6, ...\}.$$

I see no other way clearly to assess this situation than to hold that the notion of parthood as a trope implies the existence of impure sets. Moreover the parthood-trope must be viewed as an entity that relates a set S of concrete individuals to another T if and only if $S \subseteq T$. Thus it appears the trope is nothing more or less than that of the subset relation restricted to the domain of concrete impure sets.

Admittedly, from a formal perspective we may make sense of parthood in this way. By a famous result in the theory of Boolean algebras by M. H. Stone from 1936 [92], up-to-isomorphism, the parthood relation simply is the subset relation. And this may be one reason why the notion of a parthood trope seems to relate sets of objects in a rather intuitive way. However, it will obviously not be in keeping with nominalism. So this interpretation of the relation will be rejected.

So henceforth, we shall view the parthood relation as a multiply presented entity and necessarily wholly presented amid its relata. That is to say, the relation will be understood to be presented at each existent object. The justification for this will be that each has at least itself as a part. Consequently, if all multiply presented entities are universals, we shall have to accept at least one. Parthood instances will not be deemed parts of the locations at which they are presented, since they are not particular things. However their existence will be dependent upon the existence of the locations where they are presented. Therefore, the locatedness problem of universals will be avoided in this case. The parthood relation will be understood simply as *presented*. Presentedness will be unanalyzed and obviously deserves a thorough treatment in a separate investigation.[7]

2.1.6 Extensional and Intensional Objects

A nominalist favors particulars over universals on the basis of how the former are thought to be constituted. Adopting nominalism would *ceteris paribus* imply a preference for so-called *extensional* entities. In general, an object o is *extensional* if, for any x, $o = x$ if and only if the instances of x are exactly those of o. Depending upon an object's type, its extension may come in various forms. For example, the extension of a physical object is its total physical manifestation or *physical aggregate*; the extension of a property is the selection of objects which are instances of that property; the extension of a concept is the selection of objects to which that concept applies; the extension of a proposition is the total "state of affairs" or "world" making that proposition true; and so on.

Intensional objects, on the other hand, are those which are not identified solely by their extension. These are such things as concepts and properties. The concept of being a (well-formed) creature with a kidney and the concept of being a (well-formed) creature with a heart can be said to apply to exactly the same range of entities, despite being different concepts. Propositions, too, are typically considered intensional. Although the proposition that creatures with kidneys have kidneys and the proposition that creatures with hearts have kidneys are both true statements (of the total state of affairs), these propositions are not identical.

Importantly, intensional entities are also to be contrasted with extensional ones like sets or classes, which do satisfy the principle of extensionality. For example, the set of creatures with kidneys and the set of creatures with hearts are equivalent insofar

[7] In what follows, I will assume that parthood is presented amid relata. And this may very well figure as some sort of primitive relation in a conceived formal ontology. And I would even submit that if we are forced to accept the notion of arbitrary properties, then the relation of presentation might be required for a full metaphysical account of the implications of mereology. But such an examination would involve notions outside that of formal mereology.

as they have the same members and, accordingly, are identical. And consequently, extensional entities may be either abstract or concrete.

By the standard definition of extensional objects above, each of the following philosophically important types of entity is intensional: qualities, attributes, properties, relations, conditions, states, concepts, ideas, notions, and thoughts. All or most of these have been classified at one time or another as kinds of universals. Standard traditional views about the ontological status of universals naturally apply to intensional entities. Like universals, all intensional entities would be abstract. And nominalists would hold that they do not exist, and realists would hold that they do.[8]

2.2 "Mereologicality"

Traditionally, mereology proceeds from a two-part assumption: (i) existent objects are individuals, and (ii) each pair of individuals is capable of standing in the single relation of parthood.[9] We claimed that the parthood relation is multiply presented, hence not a particular, and therefore not an individual. Accordingly, the traditional assumptions above will be strengthened. Mereology will now rest on what we will call *the principal assumption of mereology* (AM) or the view that reality has a mereological structure R consisting of individuals related according the single multiply presented relation of parthood. In contrast to adopting an ontology of contacts between objects or sets, parts of objects will preferred. For at least some of an object's parts will be conceived as physical and extended, whereas the relation of contact will imply immediately the existence of topological and spatial configurations in which objects meet at zero-dimensional points or material atoms.

To understand AM, it is necessary to move from an intuitive understanding of parthood to a more formal one. We write '$x \leq y$' to indicate that x is a part of y or, in other words, y is an extension of x. In the standard way, we shall distinguish the

[8]Conceptualists would accept their existence but deem them to be mind-dependent. Platonists would hold them to be mind-independent.

[9]The traditional view has recently been reconsidered. For example, Kit Fine has defined a pluralistic view of mereology with various parthood modes for different object types. See his "Towards a Theory of Part" to appear in *Journal of Philosophy* on his website.

notions of *part* and *proper part*. On almost any theory, parthood $\leq$ is a partial order (reflexive, transitive, and antisymmetric), and we will not dispute this natural way of understanding the relation. Not only is parthood a type of pre-order, i.e. reflexive $\forall x(x \leq x)$ and transitive $\forall x \forall y \forall z(x \leq y \wedge y \leq z \rightarrow x \leq z)$, it is also anti-symmetric; i.e. $\forall x \forall y(x \leq y \wedge y \leq x \rightarrow x = y)$. That is to say, a part of an object o cannot also be an extension unless it is identical to o. Moreover, the proper part relation $<$ is a strict partial order, i.e. irreflexive $\forall x(x \not< x)$, transitive, and asymmetric $\forall x \forall y(x < y \rightarrow y \not< x)$. An object cannot be identified with one of its proper parts. And it also cannot be identified with any of its proper extensions. Challenging these intuitions would be tantamount to a failure to understand what is meant by the predicate.

2.2.1 Mereological Extensionality

There is a well-known mereological specification of the general notion of extensionality. An object is extensional in the mereological sense if and only if it is uniquely constituted *by its proper parts*. The nominalist's preference for mereologically extensional entities will follow from the idea that real objects are specifiable solely in terms of physical quantities or portions. She holds all objects *build up* and *decompose* in a uniform way, and no mere way of considering them eventuates in an outright deletion of or supplementation to their constituents. In short, their constitution will be impervious to consideration.[10] And if they are extended in time and space, then

[10] It might be thought that certain interpretations of the theory quantum mechanics may make this view difficult to maintain. For example, according to the relational interpretation of quantum mechanics, first proposed by Carlo Rovelli [86], observations such as those in the double-slit experiment result specifically from the interaction between the observer (measuring device) and the object being observed (physically interacted with), not any absolute property possessed by the object. In the case of an electron, if it is initially "observed" at a particular slit, then the observer-particle (photon-electron) interaction "includes" information about the electron's position. This partially constrains the particle's eventual location at the screen. If it is "observed" (measured with a photon) not at a particular slit but rather at the screen, then there is no "which path" information given as feature of the interaction. In this case the electron's "observed" position on the screen is determined strictly by its probability function. This makes the resulting pattern on the screen the same as if each individual electron had passed through both slits. It has also been suggested in a quasi-Leibnizian way, that space and distance themselves are relational, and that an electron can appear to be in "two places at once"— for example, at both slits—because its spatial relations to particular points

they will also decompose, if at all, in a uniform way.

Specifically o is *mereologically extensional* (henceforth abbreviated *extensional*) if and only if any object identical to o has exactly the same proper parts as o. The general principle obtained by this property—the so-called *principle of mereological extensionality* (ME)—is that *all* objects are extensional, or equivalently, (a) no two objects with the same proper parts are distinct and (b) no distinct objects have the same proper parts; i.e. $\forall x \forall y(x = y \leftrightarrow \forall z(z < x \leftrightarrow z < y))$. So a common theme in philosophical discussions of mereology is the question whether this principle is true in general. Some have argued that ME is not true. It will be claimed both that (i) there is a single individual with different proper parts, and (ii) there are distinct concrete, spatially extended individuals with the same proper parts. If two objects overlap (notation xOy), then they have a part in common, i.e. $\exists z(z \leq x \wedge z \leq y)$. It is well known that ME logically implies the following properties. Firstly, an extended object cannot be a proper part of another without a non-overlapping remainder, i.e. $\forall x \forall y(x < y \rightarrow \exists z(z \leq y \wedge \neg zOx))$ or *the principle of supplementation* (PS). And secondly, an object cannot fail to be a part of another without the latter having a non-overlapping remainder, i.e. $\forall x \forall y(x \nleq y \rightarrow \exists z(z \leq x \wedge \neg zOy))$ or *the principle of strong supplementation* (PSS). Moreover, it is well known that the latter implies extensionality, if parthood is assumed a partial order.

The Necessity Condition

Let us consider the question of the necessity of mereological extensionality (NME), i.e. $\forall x \forall y(x = y \rightarrow \forall z(z < x \leftrightarrow z < y))$.[11] If one intends to show that the necessity of mereological extensionality is not true in general, this will require a counterexample: a single individual which has distinct proper parts.

For a famous attempt, consider *Tibbles* the cat. *Tibbles* is involved in an accident,

on the screen remain identical from both slit locations (see Fink [34]). Still on each such view, it is the action of the waves and particles that is effected, not their constituents.

[11] Observe that this is just an instance of $\forall x \forall y(x = y \rightarrow \forall \phi(\phi(x) \leftrightarrow \phi(y)))$ where $\phi(x)$ may be assumed to be any unary property capable of having a non-empty extension. If the property ϕ is to be elementarily definable (i.e. a 1-type), then this condition will be an elementary version of Leibniz's Law. And otherwise it might be a second order condition, and thus perhaps an assumption on which Tarski's conception of a *solid* relied [94].

tragically loses his tail $Tail$ at time t, but is able to scurry off and survive. Thus according to commonsense, before t, $Tibbles = Tibbles^- + Tail$, and at t, $Tibbles = Tibbles^-$. Before t, $Tibbles$ is thought simply to have more feline tissue—the part $Tail$—than after. Now we have two consequences. Firstly, if $Tibbles$ is identical to both $Tibbles^-$ and $Tibbles^- + Tail$, then $Tibbles^- = Tibbles^- + Tail$ and there is a single individual with different proper parts, contradicting NME. But if $Tibbles$ is not identical to both, then presumably he is identical to one and not the other. So if he is not identical to both and NME is true, then either he fails to survive and $Tibbles^-$ is not $Tibbles$ or he, for some strange reason, springs into existence as $Tibbles^-$, not having been $Tibbles^- + Tail$.

An important observation is that the truth of NME does not follow from AM. To explain, suppose we believe the individual o to exist. According to AM, o will be susceptible to mereological analysis and identical to an individual in R. But if R is assumed to have an extensional structure and o is not extensional, we may reject o's existence entirely. In the case of $Tibbles$, if $Tibbles^-$ and $Tibbles^- + Tail$ are extensional we may hold them to exist and reject the existence of $Tibbles$. Likewise, we may be inclined on independent ontological grounds to accept the existence of $Tibbles$ but neither $Tibbles^-$ nor $Tibbles^- + Tail$. And if nominalism is to be rejected and $Tibbles$ exists, perhaps $Tibbles^-$ and $Tibbles^- + Tail$ are abstract forms or modes he takes at different times (see chapter 3 section 3.2.3).

In general, if AM is true, for any selection of purportedly existent objects, each object in the selection must be identified as one in R. As more objects are identified as real, this will affect the admissibility of further identifications. Nonetheless, we may reject NME and still maintain AM. On this view, some or even all individuals will be presumed to exist that do not have the same proper parts from time to time. Reality will then be configured non-extensionally but in some other mereological sense. Either way, a two-part ontology will be presumed. One will be an ontology of purportedly existent objects, and another will concern the structure of reality itself.

Going back as far as the ancients and continuing to the present, philosophers, mathematicians, logicians and scientists have either claimed or suggested that one way to solve these problems will be to suppose that all objects have a particular dimensional structure (Casati and Varzi [13], Goodman and Quine [44], Lewis [57], Quine [77][80][79][83], Sider [88], Tarski [94], Whitehead [100]). Tenses do not to figure in any proper explanation of the structure of reality. And at any rate this would seem to be implied by the spatiotemporality criterion, since four-dimensional locations will be deemed to exist; and some—like the past or the future—will be deemed not to exist on a tense-theoretic account of reality. Tenses and tensed facts will be abandoned entirely. The requisite criteria by which to identify objects will be thought to concern just their dimensional characteristics. For example, if $Tibbles$ exists, then, plausibly, he has three or more dimensions. Assume the former possibility. Suppose that *Tibbles endures* his accident. That is, suppose that *he* exists *through the change*, and therefore before and after it. He is wholly present at each moment through his tragedy. Then NME will be rejected. For, if $Tibbles$ exists at time t, presumably $Tibbles^- + Tail = Tibbles$. And he will have a different selection of proper parts than $Tibbles^-$. But if $Tibbles = Tibbles^-$, then he will have different proper parts than $Tibbles^- + Tail$. So supposing $Tibbles$ endures and is three-dimensional doesn't help us in securing NME.

So a natural solution will be to claim that existent objects are not three, but rather four-dimensional. On one way of understanding this proposal, the universe and its objects will all be four-dimensional but will be presumed to "grow". Future objects and facts do not exist; but present and past objects and facts do. Every non-future, four-dimensional segment of $Tibbles$ is a part of him, but as time progresses, he obtains more proper parts. Thus on this view $Tibbles$ has various temporal parts extending into the past, but only those up to the present. Observe that if NME is to be sustained, this solution must be rejected. For each new moment he continues to exist, *Tibbles*' proper parts change. And note that there will be a similar problem if we make the analogous, but rather unintuitive proposal that past objects and facts "disappear" out of existence as objects and facts including and after the present emerge.

On another theory, reality is also configured four-dimensionally. The universe is given in its entirety as a four-dimensional "block". *Tibbles* will now be a four-dimensional object, and his *proper* parts will include various four-dimensional sub-objects. There will again be a multitude of temporal parts. One of these will be a four-dimensional interval of temporally extended feline tissue before t and another after. In short, on this view *every* four-dimensional sub-part will be a part of *Tibbles*, beginning at the time of his birth until his eventual death. And every four-dimensional proper part of this "worm" will also be a proper part of him. Observe that NME on this view would seem to hold. That is, if *Tibbles* is identical to this four-dimensional object, any removal of a proper part of it would seem to preclude a most complete "version" or "profile" of him. This may make it seem that a not-too-exorbitant price is paid by assuming existent objects are four-dimensional. By employing the notion of dimensional structure, we must make numerical distinctions, but the benefit is that the survivability of spatiotemporal objects and their extensionality is assured. Although we should not confuse this theory of objects with the one of them as "growing" through time (as in the example above), we will call this view *four-dimensionalism* (Sider [88]).

A Failure to Account for Motion and Movement

The four-dimensionalist's way of dealing with the problem will also not be completely successful. On this view, *Tibbles* cannot change locations. Suppose l is his four-dimensional location and he, the four-dimensional worm w, is located precisely at l. A location in this sense (defined more precisely below) will not necessarily be a region. For now we observe the following. For all locations l and any object x, if an object x is located at l, then it is not necessarily the case that $x = l$. But if $x = l$, then x is located at l. *Tibbles* may be identical to his locations at various times, or he may be in some sense merely occupy them, and this will be immaterial to the following argument.

According to four-dimensionalism, $Tibbles'$ movements are, in some sense, to be reinterpreted as his *proper* parts. *He* is distinct from each of these. Thus if $p_1...p_\lambda$ are his proper subparts, $\forall i \in \{1...\lambda\}$, $p_i \neq w$. His proper parts $p_1...p_\lambda$, too, will have exact four-dimensional locations, say $l_1...l_\lambda$, respectively. And $\forall i \in \{1...\lambda\}$, l_i is a *proper* part of l and $l = l_1 + ... + l_\lambda$. And therefore we cannot say that *he* had one set of parts and now another, since he, w, does not *undergo* deletions of his proper parts. He is once and for all w at l. And therefore, as reality is four-dimensional, he will not be capable of having multiple distinct locations. That is to say, for any object x, if $x = Tibbles$, then $x = w$ and therefore located exactly at l. Indeed he is not located at any l_i for $i \in \{1...\lambda\}$, but only his proper parts $p_1...p_\lambda$ are, respectively.

Now witness that he can also not move. For suppose he can. He is w. And at each time, *he* will be one and the same four-dimensional worm w. And he will therefore have just one location l. To see this in another way, suppose toward reductio that he does indeed move and he is four-dimensional. Then he has a four-dimensional location l_i and after a time a four-dimensional location l_k where $l_i \neq l_k$. But, if he moves, another dimension along which he traverses over and above the four of l_i will be required. And thus the dimensional system must contain at least five dimensions, contrary to the theory of four-dimensionalism.

We can also not say that his proper parts move. Each of his proper parts p_i has a single precise location l_i, for $i \in \{1...\lambda\}$. That is if l' is any location such that $l_i \neq l'$, then p_i is not located at l'. Another way to see this is as follows. Supposing a proper part p_i of $Tibbles$ moves, then p_i will change four-dimensional locations, say from l_i to l_k. Thus if p_i occupies some dimensions $d_1 \times ... \times d_4$, in the movement from l_i to l_k, there must be at least a single dimension $d' \neq d_i$ for all $i \in \{1...4\}$ along which that change occurs. So as l_i is four-dimensional, the dimensional system must contain at least five dimensions, which is a contradiction.

The Sufficiency Condition

Now consider the question of the sufficiency of mereological extensionality (SME), i.e. $\forall x \forall y(\forall z(z < x \leftrightarrow z < y) \rightarrow x = y)$. What is peculiar about the case of $Tibbles$ will become clearer if we consider different types of object and potential counterexamples. Consider the following two examples. Possibly: (i) two words can be made up of the same letters, as with 'fallout' and 'outfall' (Hempel [47] 110; Rescher [84] 10); and (ii) the same flowers can compose a nice bunch or a scattered bundle, depending on the arrangements of the individual flowers (Eberle [29]: section 2.10).

A diehard nominalist may have less difficulty in the case of (i). She will attempt to interpret all words as spatiotemporal tokens and therefore 'fallout' and 'outfall' will be two words. They are now to be understood as tokens—individuals—of sequences (with the same symbol types). One can claim nonetheless that they are distinct individuals, since one is a word-token appearing in one location, and the other appears elsewhere. They are also different in various respects, since they contain different sequences (reading left to right). And for the nominalist, if there is one thing they comprise, this will be just their fusion.[12]

In (ii) flowers composing a nice bunch will have different spatiotemporal locations and therefore be different individuals in the first case than in the second. That is, they will be located one place when they are a nice bunch but located another place which they are scattered.

Thus what is interesting about the case of *Tibbles*, as has been pointed out sufficiently elsewhere by Simons [90] and Varzi [13], is that there appears no non-question-begging reason to assume that it is *impossible* that there are two, distinct selections of parts combining to form the same individual. And this seems rather clear at least in the case of organisms. Even if they lose certain parts, they survive as living beings and maintain their identity as an organism capable representing the world. In general, the puzzle will stand if we can point to certain commonsense cases or organisms, like *Tibbles*, which survive deletions, augment their parts (by for example in cases of transplants etc.), or move from location to location.

Matter, Regions, and ME

Although *Tibbles* provides a worthy counterexample of mereological extensionality, both conditions may hold for individuals of certain types of nominalistically acceptable objects. Suppose that we are committed to the notion of matter. In particular suppose individual portions of matter exist. They will be either extended or zero-dimensional. According to AM, they will be susceptible to mereological analysis. The parts of a portion of matter are further bits of matter. Ultimately, material portions will decompose into atoms, atomless matter or "gunk" (cf. Zimmerman [105]), or

[12] The token theory of linguistic entities will, however, be rejected in Chapter 3.

perhaps a combination of both. An *atom* is an object whose only part is itself, and an object that is *atomless* contains no atoms. Any object lacking an atomless part will be called *atomic*. If an object is atomic, then conceivably its atoms are either zero-dimensional or extended indivisible simples.

Intuitively, any atomless portion of matter is extended. This would be *prima facie* clear, since otherwise, an atomless object would be located in a non-extended "space" or point. And thus the corresponding matter located at or identical to that point would be an atom. If material particulars exist and atomless material portions are extended, we can show that any individual bit of matter is extensional in the mereological sense. Note that assuming parthood is a partial order, it is well known that PSS implies ME (cf. [90]).

Assume both that (1) individual bits of matter exist, and (2) atomless objects are extended. We can show that parthood is closed under PSS. *Demonstration.* Toward contradiction assume (1) and (2), but that PSS does not hold. There are p, q such that $p \nleq q$, but there is no $r \leq p$ which does not overlap q. As $p \nleq q$, there is a $s \leq p$ such that $s \nleq q$. *Subclaim 1.* Suppose s is atomic. Then as every part of s overlaps q, each atom of s overlaps q. But no object can partially overlap an atom, and thus each such atom must be a part of q. Hence $s \leq q$, which is a contradiction. *Subclaim 2.* Assume s has an atomless part w. By the reductio assumption, every part of s overlaps q. Thus every part of w overlaps q. As w is atomless, it is extended. So every extended part of w overlaps q. By assumption, both $w - q$ and $q - w$ are non-extended. Thus q and w have the same extension. And therefore either $w \leq q$ or w contains an atom, and we have reached a contradiction. *End of demonstration.*

If two individual bits of matter are identical, then they have the same proper parts, and vice versa. On this view, the cat *Tibbles* may not be numerically identical to the fusion of his material parts, as the original example suggested. But, assuming he does exist, the combined matter constituting his body will be identical to the fusion of the individual material objects *Tail* and $Tibbles^{-}$. And in general, if o is an individual material object, o is extensional in the mereological sense.

If we are committed to the notion of a *region*, the case may be similar. We assume that regions are not relations, but types of individuals and furthermore that they have arbitrarily many dimensions. On a Newtonian or classical view, we may presume them to be identified by their position within space and time, dimension, and extension. Thus, by definition, the necessity condition would hold immediately. And if there are two regions r_1 and r_2 with precisely the same proper-subregions, these will then have the same dimension, extension, and position. Therefore r_1 and r_2 will have the same dimension, extension, and position. So $r_1 = r_2$.

If this view is to be one of space-time regions, it will not be reconcilable with standardly held views in physics since the confirmation of the special theory of relativity. According to the Minkowski spacetime model and Lorentz covariance postulate, there is no absolute and well-defined state of rest—i.e no privileged frame of reference. A decrease in length can be detected by an observer of objects that travel at a non-zero velocity relative to that observer. And an actual difference of elapsed time between two events as measured by observers moving relative to one another or differently positioned relative to gravitational masses can be observed.

Still one way to save the extensional character of space may still be available. Assume the geometrical properties of spatiotemporal regions vary but that their number of dimensions and constituents are constant as is standard in theories of non-Euclidean topological spaces. That is, assume that one and the same region is (1) capable of stretching and contracting while maintaining their number of dimensions, and (2) comprising the same basic constituents. A region in this sense may be interpreted as a set (if, contra nominalism, abstract entities are deemed acceptable) or a fusion of points or extended regions. On this view points can be moved arbitrarily closer together along any dimensional axis so long as the object is still extended in that dimension (i.e. it does not reduce to zero). Supposing space not to decompose into points, a region may then be interpreted as a union of atomless sets which infinitely decompose (again, if abstract objects are countenanced) or a fusion of extended, atomless subregions. Thus if space is comprised of a combination of atomic and atomless portions, analogously we can assume that one and the same region can contract and extend. A region in this sense would also be extensional and identified solely in terms of its dimensionally invariant constituents. So, on any of these construals, from different frames of reference, one and the same region can contract or dilate.

Indeed the same problem with regions carries over to material bodies. Geometrical properties of matter will also be required to vary. In physics, a rigid body is an idealization of a single solid body of finite size. Their most important property is as follows. One and the same rigid body cannot undergo geometrical deformation. The distance between any two given points of a rigid body remains constant in time regardless of external forces exerted on it. In short, such an object cannot physically exist due to the special theory of relativity. Objects are normally perceived to be perfectly rigid. And this will hold if they are not moving near the speed of light. In classical mechanics a rigid body is usually considered a continuous mass distribution, while in quantum mechanics a rigid body is usually thought of as a collection of point masses. For instance, in quantum mechanics molecules consisting of the point masses—electrons and nuclei—are often seen as rigid bodies or rigid "rotors".

So on any analysis, the notion of a physical object with invariant geometrical properties must be rejected. Nonetheless, on the view suggested by nominalism and AM we must conceive of reality, if it is closed and extended, as comprised of constituents whose existence is not eliminated altogether by our position or relative velocity to other objects. We will therefore settle on a tentative view that one and the same non-zero-dimensional concrete object is capable of undergoing geometrical deformation in virtue of maintaining the same basic constituents. Constituents of regions will be regions. And constituents of matter will be material (described more fully below). As spatiotemporal objects are assumed, there may be objects of zero-to-four dimensions and all will be presumed to maintain their number. I find that this view is in keeping with a theoretically important idea related to nominalism and physicalism—namely that the *constituents* of concrete objects are recalcitrant to our position, velocity, or consideration despite their geometrical properties.

2.2.2 Unrestricted Fusion and the Closure Problem

In addition to the question of identity criteria for objects and the question of extensionality, there is a related important one concerning the circumstances under which objects fuse. Do all objects in reality unrestrictedly fuse? Or stated differently, is it the case that every *set* of concrete objects fuses? Again, if the principle is not true in general, does there exist a subset of concrete objects whose every subset fuses? Obviously, each of these questions presupposes that we have sets handy, and we have assumed otherwise. Therefore it may be thought that the question is really misleading in a certain respect and should be expressed in a different way. If we are not prepared to accept the existence of impure sets, there will be no clear way to understand the question. As Simons [90] has pointed out, a commitment to unrestricted fusions may also commit us to the view that every way of *specifying* an object gives rise to a new one. Consider the question of the fusion of my toe and Alpha Centauri, for example. There may be an individual portion of matter comprised of that of Alpha Centauri and my toe, but there is also the intuition that this purported "object" is simply hodgepodge. And such worries will arise if we see something more curious or special about persons, artifacts, or, in general, objects figuring prominently in our day-to-day lives. So it might be claimed that not all objects fuse unrestrictedly.

A *specification of individuals* is an instance of expressing the existence of individuals. By asserting that certain objects exist or by proposing some for consideration, we will specify them pluralistically. And, for the purpose of this argument, by specifying them we will assume that this does not at the outset imply the existence of a set comprising them.

We say that *a type T of concrete objects unrestrictedly fuses* if, for any specification S of objects type T, there is a fusion of S. To point out a problem, assume no type of concrete object fuses unrestrictedly. Some portions of matter will then be divided or cleaved. But cleaved into what? According to both modal realism and CT, there will be some interconnectivity amid the various parts of a single *universe*.

One unsuccessful attempt to preserve closure without appealing to unrestricted fusions will be to assume that, although there are some concrete objects which are cleaved, they may nonetheless occupy a *region* in the universe which is the fusion of the exact spatiotemporal regions of the objects specified. Thus on this attempt, the region of an object will be presumed distinct from the objects occupying it. And an *exact* region r of an object o will not include any subregion that is not occupied by part of o. Observe that on this view regions are concrete objects since they will be presumed to be parts of the universe. But then, by the reductio assumption, the exact region of an object will not unrestrictedly combine. So it now appears we require another medium or object within which to "close" these objects. But observe that each newly suggested medium will again be conceived as concrete and by assumption fragment or cleave. Reality itself will therefore be fragmented. And this was precisely what the nominalist intended to rule out by the interconnectivity of co-worldly objects.

Consider the relation of *being located at*. Naturally, we shall not assume that, necessarily, for all objects x and y, if x is located at y, then $x = y$. Nonetheless, if an object is located at another it may be the case that they are identical. The obvious intuition here is that if regions exist, they may be identical to their locations. However, possibly, material objects may not be identical to the regions they occupy.

The notion of location suggests more precisely what is meant by the property of cleavedness. A single object o is *cleaved* if and only if it is comprised of at least two concrete objects o_1 and o_2 such that the following hold: (i) $o_1 + o_2$ does not exist; and (ii) there are no spatiotemporal objects l_1 and l_2 such that o_1 is located at l_1, o_2 is located at l_2, and $l_1 + l_2$ exists.

Thus a cleaved object o will be comprised of concrete objects, say o_1 and o_2, for which there is no spatiotemporal fusion. Now we ask: what is it that we specify when we specify o_1 and o_2? Do we specify a concrete individual? A concrete individual is a spatiotemporal one. And any concrete individual will have a single spatiotemporal location. Thus o_1 and o_2 will have locations l_1 and l_2. However, by the definition of cleaved objects $l_1 + l_2$ does also not exist. And therefore there will be no single location for o_1 and o_2. I shall call the determination of the location of cleaved concrete objects *the closure problem.*

One way one might attempt to go about solving the closure problem without resorting to unrestricted fusions will be to claim that there is a single, spatiotemporal object of which *all* objects are a part or at which all objects are located although not exactly. Perhaps this object will be a single universal region occupied by all objects which is a fusion of all regions. Or perhaps it will be a single fusion of concrete objects. Yet again, it may be a fusion of a mixture of both regions and matter. The proposal will be that it is unnecessary that every *proper* specification of objects has a fusion, since there will be a single, maximal one of which all entities are a part. But note that even on this proposal, there will be "inner" cleaves despite outer closure. Although there will be a single universal medium, there will be inner unfusing objects. Suppose there is a specification of two individuals like this i_1 and i_2. On this view, they will be parts of the fusion of all objects $\mathcal{U}$; i.e. $i_1 \leq \mathcal{U}$ and $i_2 \leq \mathcal{U}$. However, their *exact* location which is presumably a part of $\mathcal{U}$ will be unfused. That is $i_1, i_2 \leq \mathcal{U}$ but there will be no $i_1 + i_2 \leq \mathcal{U}$. I see no other way to understand the proposal but to view this as a situation in which the set $\{i_1, i_2\}$ is a part of $\mathcal{U}$.

The purported solution that a maximal region will account in a satisfactory way for the cohesion of concrete objects would also run counter to the observed character of physical objects. Assume that we can denote or specify *proper* sub-objects—cars, buildings, and people—in a universe. Then if a fusion of them does not exist, on this proposal these sub-objects will have as an exact location the single universe $\mathcal{U}$. But this will imply that either sub-objects or sub-regions of reality are not specifiable. If the first is true, we have a direct contradiction. And if sublocations of the whole are not specifiable, then it is equally mysterious how we should be able to specify any sub-objects whatsoever. For presumably, since the maximal fusion $\mathcal{U}$ exists, each object will have a spatial "profile" or "shadow" that it projects onto the whole. Objects will have boundaries and there will be a location within those boundary. And this is simply what was originally conceived as the exact location of the object (or the closure of that location). But this location, as it is spatiotemporal, will now not exist. And therefore the object will not have a location with that boundary.

And I see no other way to explain this than to view cleaved objects as impure sets. Cleaves are therefore not merely theoretical divisions. Since cleaved objects will appear in the universe, they will be actually, intrinsically divided. Recall our earlier discussion of theoretical divisions and impure sets. We rejected them as abstract, since it appeared that there were more sets than concrete objects. Various divisions of a single concrete object will not be distinct. Thus if we assume that cleaves exist, we will *a fortiori* hold that reality is comprised of sets.

I would even suggest that the desire to reject a view of reality as cleaved is the real basis for the modal realist's demand for the spatiotemporal interconnectedness of worlds. Even if our nominalist does not hold CT, there will be various worlds or interconnected entities. Within each, for any specifiable entities, he will demand a fusion of their locations. For example, although Lewisian reality is disconnected, on this view worlds are spatiotemporally connected. And thus if there are distinct, specifiable objects in them, these will be identical to material or regional fusions.

2.2.3 What are closing substances?

We will therefore hold the view that concrete possible worlds and universes will have substances that are fusions of either regions or matter. And we shall not here subscribe to a spatial substantivalism. We shall call these fusions *closing substances* and, suggestively, *locations*. Locations are fusions of matter or regions. They are either material—i.e. containing matter—or material-free locations. So 'location' and 'closing substance' will be synonyms, but we shall use the word 'region' to denote only a material-free location. And matter will be thought of as a location as well, all of whose parts are material.

If regions do not exist independently of matter, a bit of matter will be a closing substance of itself and have as parts only further bits of matter. Similarly, regions, if they exist, will be closing substances as well, and have only regions as parts. However, if regions have existence independently of matter, then for any nominalistically acceptable specification of existent individuals in actuality, there will be a fusion of the regions those individuals exactly occupy.

If it is the case that regions and matter indeed independently exist, I should be inclined to the view that regions are the primitive closing entities, and matter will not be. Regions will be viewed as the "occupiees" or hosts enclosing material portions. For, possibly, material portions will not occupy every region. Consequently on this bifurcated view, occupation relations will be required to hold amid matter and the regions they occupy. Again, if regions are deemed not to exist, then, plausibly, matter will play the role of closing substance. And if both exist, but only in certain places, then one or the other at various parts of universes. Likewise if only regions exist, then clearly regions will play the role of closing substance.

This tentative dualistic view will arise from the idea that, possibly, a single location may be material-free. Consider for example the surface of a three-dimensional object. Although surfaces lie in space, they do not take up any [13]. Zero-dimensional points are also a case in point. If points are distinct from so-called zero-dimensional particles, they are not distinct on the basis of their filling a region, since both points and zero-dimensional particles do not take up or "fill" spaces. Importantly, note that for external contact to take place between concrete disjoint entities, material-free points and surfaces will be required. Entities in contact need not overlap and may meet at points along surfaces.

All existent individuals—people, cars, buildings, and universes—will have some relation to closing substances. That is to say, all will *have a location*. But what is the nature of this relation? We shall have more to say about the relation of locations and individuals in the next chapter. In short, they will either be correlated with locations and exist independently or they shall be identical to them. That is, it may be the case that various distinct individuals are co-located. And this will follow on account of asymmetries in how they may survive mereological changes (e.g. in their proper parts or extensions, described more carefully in the next chapter). What is rather clear is that if we do not allow classes, modes of locations, or multiply presented unary properties to appear in the ontology, then specifying purportedly *distinct* co-located individuals will be impossible.

The decompositional and compositional structure of locations will be maximally articulated. A location is maximally articulated if the mereological features of its parts—whether they be material or material-free—carry over to the location itself. Thus if a location is material and that material bit is an atom, then that location will be an atom. And if a location is a region and that region is a point, then that location will be a zero-dimensional point. If a location is material and atomless, then the location will be atomless. And if a location is a region and atomless (if such exists), then that location is atomless. Again, on the basis of our earlier observations concerning the status of matter and regions, not only the unrestricted fusion but also the mereo-extensionality of closing substances will be required. And if a location is observed in various frames of reference it may have different geometrical properties. However, its number of dimensions and its constituents will be invariant with respect to the occupied frame of reference.

By the unrestricted fusion of locations, if the universe is spatiotemporal, then there will be n-dimensional locations for every n-dimensional object existing within it. If a universe has a decompositional structure such that every object has an atom as a part—that is, if the universe is *atomic* and has a decompositional structure each of whose atoms have dimensions greater than or equal to $n \not\leq 4$—then there will be locations of n to 4-dimensions. Any atoms will fuse unrestrictedly. If, however, the universe has a decompositional non-atomic structure and therefore contains an atomless object, then there may be a number of dimensions k where $0 < k \leq 4$ such that all objects in the universe are minimally k-dimensional. Thus importantly, in either the atomless or atomic case, there will be 4-dimensional worms. That is if there are three-dimensional objects which undergo changes, as I will claim, then by upward closure under unrestricted fusion, there will also be four-dimensional objects. The upshot is therefore that locations are closed upward, but the minimal number of dimensions will be deemed to vary according to the decompositional structure of reality.

The latter characteristic is particularly important. Although I claim that three-dimensional organisms exist, there will be four-dimensional objects as well. Thus the four-dimensionalist's assertion that all objects are four-dimensional will be a proper restriction of the ontology proposed here.

2.2.4 Interconnectedness

The dialectic pursued thus far went as follows. The question of the truth of ME turned out to require an account of which types of objects exist and of the structure of reality. The notion of reality as tensed or fragmented into modes or experiences could not be accommodated in a nominalistically acceptable way. So in order to retrieve the viability of our intuitions about the extensionality of objects and their survivability through changes, we considered the possibility that reality and all existent objects have a four-dimensional structure. Although initially plausible, the view involved us in a trouble with explaining the possibility of change and movement. The question of unrestricted fusions as well gave rise to questions concerning the total mereological structure of reality arising from the criterion of interconnectedness. And it was claimed that some closing substance must be countenanced. However, the articulation of closing substance must meet requirements which are implied by the interconnectivity of objects within single universes.

If objects exist solely in virtue of having a particular dimensional structure, this will imply that an object and its parts may be fragmented into spatiotemporally isolated entities. Reality may then be cleaved. And we have seen that this will be unacceptable for the nominalist. The nominalist who holds CT must hold there to be cohesion within universes. And the nominalist about possible worlds will hold that there is a corresponding cohesion within worlds. To see this, let P be any property (dimensional or otherwise) which an object o may have such that having it does not imply that o be a connected part of the fusion of all spatiotemporal objects. By definition, having P will not be sufficient according to nominalism to guarantee the existence of o. For suppose that an object exists if and only if it has property P. Then trivially, it is possible that there exists an object with P that is not connected to the fusion of all concrete objects.

We cannot simply assume that all four-dimensional objects are actual. Possibly, there are four-dimensional objects that are completely disconnected from those in the actual universe or from those connected to our bodies. For example, there may be a four-dimensional region completely disconnected from all other regions. Or there may be an isolated bit of matter disconnected from any object in spatiotemporal relation to our bodies. And thus the problem we had identifying *Tibbles* with an object in reality is transfered again to the more specific category of four-dimensional objects. And although the modal realist thinks certain objects are disconnected or isolated, he will not think that all are. He will define actual objects as interconnected or non-isolated. So, if reality and/or actuality is closed and susceptible to mereological analysis according to AM, what does it mean for objects to be connected? And can the envisaged criterion be cashed out in a nominalistically acceptable way?

Let's suppose we have two objects o_1 and o_2 that are connected. Firstly, it is clearly not necessary that they be in contact, since they may be connected via another object. Two objects are *in contact* if and only if they overlap or there is exactly a zero-dimensional point between some parts of them. For example if o_1 and o_2 are material bodies, then there may be some positive distance between them. As a starting point, it might be claimed that two objects are connected if there is a sequence of objects $p_1...p_\kappa$ such that $\forall i \in \{1...\kappa\}$, r_i and its successor are in contact, o_1 is in contact with p_1, o_2 is in contact with p_κ, and the fusion of $o_1, o_2, p_1...p_\kappa$ exists. But if each p_i has an arbitrary structure, this will be insufficient. For possibly, each may be itself the fusion of a disconnected entity. So we must demand that each object in the connecting sequence be *intrinsically connected*, in the sense that they themselves have interconnected parts. This notion is analogous to a well known one in topology—that of a *connected* subset of a topological space. A *connected set* is one that cannot be represented as the union of two disjoint non-empty open subsets. As an example of a space that is not connected, one can delete an infinite line from the real Euclidean plane. Or consider the case of two disjoint disks in the two-dimensional Euclidean space. Thus an intrinsically connected physical object is one that cannot be represented as the fusion of two physical objects not in contact.

Suppose, according to classical Euclidean geometry, that worldly spaces decompose into zero-dimensional points. Consider the notion of *path-connectedness*. A *point* is a zero-dimensional region. Points by definition can occupy only a single dimensional medium—they are theoretically uncleavable—and will therefore be connected (i.e. the closure of a point is the point itself). A *path* is a *continuous* linear arrangement of points possibly traversing several dimensions. In topology, a path in a topological space X is a continuous map f from the unit interval $I = [0, 1]$ to X, $f : I \to X$. The initial point of the path is $f(0)$ and the terminal point is $f(1)$. And a set of regions R is *path-connected* if any two regions of the set is connected by a path. On this conception, an entity will be interconnected if a path exists between any two objects such that the fusion of the path and the objects exist. As each point in a path is intrisically connected and each complement of the point with respect to the path will be thought to "connect" or infinitely approach the point, the whole path will traversable by extended, co-worldly or co-real entities.

On the other hand, space may decompose ultimately into extended regions as Russell and Whitehead argued. They demanded that the objects of fundamentally primitive terms be *extended* or non-zero-dimensional entities. For example, a *ball*, seen as a region, is simply the space inside a sphere and therefore extended. It may be a *closed ball* which includes the boundary or surface points or an *open ball* which excludes them. Two regions o_1 and o_2 are *connected in the extended sense* if there is a sequence of balls $(b_i)_{1\leq i\leq \tau}$, such that b_1 overlaps r_1, b_τ overlaps r_2, and for each $j \in \{1, ..., \tau\}$, both the immediate predecessor and successor of b_j overlap b_j. On this view, reality will be interconnected if there is such a sequence between any two objects and there is a fusion of the objects and those in the sequence. In conception, any shape of extended objects many be selected if we suppose that the object take up a single dimensional space and is intrinsically dense. And if we choose a closed or open ball, both will be intrinsically dense and intra-traversable.

Conceivably, we may reinterpret each of these types of connection in a material way. On the first, we will accept the notion of a zero-dimensional particle and their thorough ubiquity throughout reality or the actual world. But if this is unacceptable, then we may interpret the connections as consisting of material spheres or extended, intrinsically connected objects of some sort.

2.2.5 Tentative Conclusions

Considerations of the previous section show quite clearly the limitations of mereology to capture the mereotopological properties of physical universes. The interconnectivity requirement will imply that we accept the existence of geometrically articulated entities in nature. And the specification of interconnectedness will involve commitment to sets and sequences of mereotopologically analyzable entities which we already deemed as nominalistically unacceptable.

An initial attempt at eschewing such demands would be to claim that the selected intrinsically connected item be deemed primitive and unanalyzed. But witness that our selection of them in the present context was deliberate. We sought to identify entities based on an analysis of their geometrical and topological structure. On the one hand, in either its material or spatial sense, a path was conceived as *prima facie* the sort of entity via which objects can be connected. On the other hand, the shape of a sphere was presumed to take up a single portion of reality whose parts were mutually interconnected. Hence the analysis was antecedent to the selection of the objects. And so the pure geometrical structures, sets, and sequences would be taken for granted.

If two objects are connected, there is nonetheless a sense in which they may be *inaccessible*—or *incapable of coming in contact* given their physical properties. For instance, material bodies located within one interconnected fusion of concrete objects may have characteristics which make them inaccessible. For example, suppose that reality consists of two disconnected four-dimensional objects f_1 and f_2 which are connected by a single one-dimensional path p. In other words, suppose $\mathcal{U} = f_1 + f_2 + p$. Given their dimensional characteristics, any part of f_1 of greater than 1 dimension is incapable of accessing any part of f_2.

Still, an object, say o, may be accessible to *some* objects. All of them will be connected to $\mathcal{U}$, but given an arbitrary situation or arrangement of objects, they may not be in contact with o. There are clear commonsense cases of this. One drives a car to a different place. Or I walk to a certain location. Or a particle a traverses a space and after a duration of time comes into contact with another say b. Thus movements of one object to another will imply the interaccessibility of each. And if o changes its location, it will change its extensions and be mereologically distinguished after the change. The potential of bodily and intentional movement suggests that there may be a solution to our worries by appealing to the existence of animated bodies or organisms. According to nominalism, organisms are presumed to exist within a single real and actual universe. The nominalist will understand the locations of organisms as intrinsically connected; and therefore the substance equal to the location of the organisms will also be intrinsically connected. So now we will turn to our attention to the body for a new understanding of mereo-reality from inside a space of substances.

Chapter 3

Mereological Situations and Locality

3.1 Interconnectedness and the Body

Building into an ontology machinery to guarantee the interconnectedness of objects involves one in commitments to sequences, sets, and primitive contact and connectedness relations. Sequences will either be continuous or consist of intrinsically dense extended locations. However, instead of incorporating abstractions such as these into the ontology, one may be able to get along by appealing simply to the nominalist's assumptions. According to her, persons are concrete parts of reality. She will have no serious desire to entertain a Cartesian skepticism or dualism. Persons are organisms with intrinsically connected locations.

For the nominalist who holds the thesis of closure, all *real* objects are connected to one's body. And for the nominalist about possible worlds, *actual* objects will be assumed to be connected to one's body. So by championing the nominalist's conception of the organisms, we will, at least initially, obviate the need to resort to abstractions. Still, this way of proceeding will give rise to ontological commitments peculiar to animate beings located within and connected to their environment.

There are benefits for adopting this strategy. We saw that yielding to the principle of mereological extensionality without exception leads to absurd consequences for the case of living organisms like $Tibbles$. The persistence of self or other-representative acts and intentional states and acts on the part of animated objects undergoing changes indicates strongly that they endure deletions of their parts. Given their subjective potential, organisms give rise to *localized* mereological states of affairs or situations relative to the position of their bodies. It is argued that the persistence of subjective potential implies survivability through changes. Organisms can move, exist in multiple spatial as well as temporal locations, and undergo changes in their proper parts.

There are also *closed* or *non-local* situations that obtain despite one's location. The mereological structures of various construals of the same inanimate object will be accounted for by the articulation of reality suggested by our notion of closing substance. Recall that closing substances were primordial locations with maximal articulation closed under unrestricted fusion. Global specifications of inanimate objects persisting through changes are then construable as either temporally arranged closing substances or four-dimensional objects.

It is suggested that if there are distinctions between inanimate objects and closing substances or if various distinct individuals can be co-located—like for example a location and a boat—this will require some commitment to either multiply presented properties or equivalence classes. Although the nominalist will reject these if it turns out that they are irreducibly abstract, I suggest various ways one might integrate properties and various modal constructions into the ontology of situations.

The combined view presented here is then as follows. If nominalism holds and reality is mereologically structured according to AM, then this will imply that an object o exists if and only if either (a) o is a mereological situation, (b) a concrete individual involved in one, or (c) the multiply presentable relation of parthood. We call this the localized theory of mereological reality (LMR).

3.2 First Personal Objects

A commitment to both concrete individuals and a conception of parthood as a multiply presented relation gives rise to a conception of mereological state of affairs. A *mereological situation* (synonym: *arrangement*) is a state of affairs involving individuals and the parthood relation. For example, if individuals a, b, and c exist and a is a part of b and a is an extension of c, then a state of affairs $\lceil a : \leq (b), \geq (c) \rceil$ obtains. This is the situation that *at* a, b is an extension, and c is a part. It will also not be excessive to allow the negations $\nleq$ and $\ngeq$ of these relations. Thus if it obtains that d is a non-part and e is a non-extension of a we shall also have $\lceil a : \ngeq (d), \nleq (e) \rceil$. If state of affairs exists, we also say that it *holds*.

An object given to itself "in person" or *first-personally* (abbreviation: *a first-personal object*) is one that exhibits either a capability of self-representation or other-representation which we shall call *subjective potential*. More specifically, an entity exhibits self-representational acts if it is capable of producing representations, notions, ideas or sensory images of itself, controlling or moving its body, or directing attention to its body; and it will exhibit other-representational acts if it can produce analogous representations about existing objects distinct from itself or intentional acts directed to other existing things. Obviously, various grades of these capabilities will be exhibited by different organisms. Living beings that are capable of reaction to external stimuli, sensory perception, or intentional mental acts or processes are first-personal. But in each case these abilities will be due to a certain potential of the body, its constituents and parts, and its structure. Subjective potential will be dependent *inter alia* on the status of the being's bodily parts. Roughly, a first-personal object endures deletions of or additions to its proper parts, if it keeps those parts in virtue of which it maintains the same subjective potential.

A multitude of animals and biological species will be first-personal objects: viruses, insects, horses, and humans. But clearly it is not necessary that *every* object be given to itself "in person", since possibly some are incapable of both self and other-representation. Allowing perhaps a bit of imprecision, we will also call a first personal object *animated*, and employ the term *organism*. An object is *inanimate* if has no subjective potential and animate otherwise.

Situations of the form $\lceil \sigma \rceil$ containing brackets '$\lceil, \rceil$' will be called *closed*, *global* or *non-local*. Henceforth, the expressions without the brackets '$\lceil, \rceil$' will be understood to denote a *localized* state of affairs obtaining *at* an organism. By CT there is a fusion $\mathcal{U}$ of all existent objects. Let '$\mathcal{E}$' designate the fusion of all Earthly objects. For example, although $\lceil o :\leq (\mathcal{E}) \rceil$ will obtain at any location if o is a part of the Earth,

$\leq(\mathcal{E})$ will obtain only at an Earthly organism. I am an Earthly object. So at myself and any Earthly organism, $\leq(\mathcal{E})$ obtains. And if o is an Earthly object, $\ulcorner o : \leq(\mathcal{E})\urcorner$ will obtain at any object whatsoever. Still it will not follow in general that $(o, \leq(\mathcal{E}))$ holds unless the location of that situation is a first-personal object o and o is Earthly. Thus localized situations will obtain under more or less specific circumstances. For example, the localized state of affairs (o) will obtain at a location l if and only if $o = l$. However the situation $\leq(\mathcal{E})$ containing no distinguished individual will hold for many first-personal objects. It will have various animated locations on earth at which it obtains.

A *first-personal situation* is one that holds *at* and *for* a first-personal object with subjective potential. That is, in a first-personal situation, the organism's subjective potential is exercised. A localized state of affairs may hold but nonetheless be unleveraged or unrepresented. For example, this may be the case if a being is at rest or incognizant of itself or its surroundings. Still the location of a first-personal object is one at which there exists subjective potential regardless of it being leveraged. Therefore if σ is a localized situation and obtains at location l, then l will be a first-personal object yet σ may not be a first-personal situation.

Subjective potential is sustained *inter alia* in virtue of the parts of an organism being in tact and bearing functional relations to others. For example, chemical and mereotopological properties must hold amid the various parts of the body. Animated proper parts of the body will themselves be involved in mereological states of affairs given the subjective potential of the entire body.

3.2.1 The Reality of Localized Situations

We have thus far merely introduced constructions for representing closed and local situations. But a certain nominalist, who we call the *globalist*, may wish to reject local ones on the grounds that closed situations can equally well account for all local ones. On the one hand, she may argue that local features are just closed ones in disguise. Or she may hold that although local situations may in some sense figure in a proper account of existent objects, they need not figure in any serious metaphysical account of reality. For, from a nominalistic point of view, it might be thought that local situations can be incorporated into third-personal explanations of psychological phenomena.

But if psychological phenomena are not to figure in the serious nominalistic account of reality, the nominalist has a serious problem. For by the lights of her very program, psychological states of affairs and features are thoroughly explainable in terms of physical processes and objects. Thus there will be no delineation of domains. The psychologist's domain is the nominalist's.

My Body and Its Movements

We will be unable to account for first-personal situations if we do not discriminate between closed and local ones. Let o be an organism. There will be a closed situation in which o is involved $\lceil o :\leq (\mathcal{U})\rceil$ (read: 'at $o, \mathcal{U}$ is an extension'), since each existent object figures in a closed situation. All closed states of affairs will hold at any object. At any location l, the holding of a localized situation σ at l implies that at any location l' we have $\lceil\sigma\rceil$ (even when $l \neq l'$). However the converse will not follow at arbitrary locations. For example, it will not necessarily be the case that at l, $\lceil\sigma\rceil$ implies that at l, σ; and in general it will not be the case that at l, $\lceil\sigma\rceil$ implies that at any l', σ.

According to nominalism, my existence follows only if I am a concrete particular. The globalist will claim that there exists a closed situation the obtaining of which implies that I exist and have a body b. Apparently this situation is $\lceil\mathcal{U} :\geq (b)\rceil$ or $\lceil b :\leq (\mathcal{U})\rceil$; that is, there exists a closed situation and an object involved in it and this object is my body and a part of the fusion of all objects. But the obtaining of this situation will not be sufficient to guarantee that (b)—i.e. the situation I am b or perhaps more specifically $(b :\leq (\mathcal{U}))$. In this situation I am b and b is a part of the fusion of all existent objects. That is, without the existence of a localized object or situation, we will have no way to explain what it is about the situation to hold at the location of my body. And this will be required, since according to nominalism, I am my body. And importantly, it is nonetheless conceivable, although contra nominalism, that I am not my body.

Observe that that I am my body is a situation that holds for me, this one is distinct from $\ulcorner b :\leq (\mathcal{U})\urcorner$. For the latter can hold at any location. That is $\ulcorner b :\leq (\mathcal{U})\urcorner$ will hold at any location, even an inanimate one, and thus not necessarily just at me. It is a global situation. However, for the identification to be one from me to the physical aggregate, the global fact will not suffice. It must obtain exactly if $(b, \leq (\mathcal{U}))$ is a situation obtaining at the location of my body. Specifically the situation $(b, \leq (\mathcal{U}))$ will obtain only at the location of my body and the latter situation will be localized for me when it is leveraged or recognized.[1]

There is an additional insufficiency of global facts, as we mentioned in the previous chapter, which concerns possessive parts of objects. First-personal objects have various proper parts. I have an arm a for example. For me to have an arm, a situation $\ulcorner a :\leq (\mathcal{U})\urcorner$ involving it must hold. Since this situation is global, it will obtain at any location. But for my arm to be *mine*, the state of affairs $\geq (a)$ must also hold at me. The latter situation is one obtaining at a location of which an individual has a part referred to by the possessive description 'my arm'.

In addition, for this state of affairs to be recognized by me, I must leverage it in a first-personal situation or be capable of representing this to myself. So one of the salutary features of LMR will be that there are states of affairs accounting for the possession of parts by individuals. And we noted this was a major deficiency with the four-dimensionalist's account of eternal, unchanging and immobile objects. No closed state of affairs will be sufficient to account for the local one in this case either. For example, apparently the state of affairs $\ulcorner$Jeremy Meyers $:\geq (a)\urcorner$ will be sufficient to account for the fact that my arm is a part of me. However, this fact will obtain for many other sentient persons. Indeed, by nominalism, it will obtain in virtue of reality being the way it is and independent of what may or may not be represented by any first-personal object.

For an organism to exercise its subjective potential, it must possess itself as one of its parts and be located exactly at and identical to its body. That is, the local state of affairs $(o, \geq (p_1), \geq (p_2), ...)$ must obtain *at* o whose parts are $p_1, p_2, ...$ Thus a

[1] In essence the insights I provide here are due to an idea of the body analogous to John Perry's view of indexical expressions containing "I"—i.e. those containing the "essential indexical".

first-personal situation relies on the existence of located mereological states of affairs. I can leverage my local situation and reposition or move *my* arm only if I am located exactly at and identical to my body and my body has an arm. In general, you cannot move my arm as its possessor, partly because you are not at my location and identical to me. And you cannot do this in the way that I can because you do not harness subject potential over my body. Roughly, in moving my arm, a self-representative act from my location is given. If my movement is a reaction to some external circumstance like an attempt at averting danger or if I intend to change an existing situation or my position, I am situated exactly in the proximity of my body. And I can glean certain proximal information regarding the situation in my extensions. And thus a representative act on the mereological states of affairs at my location makes available information about my body and my proximate environment.

Moreover subjectively intended movements will be assumed to take place on the part of organisms, first-personal situations will be presumed. They will be explained by a combination of facts traceable to facets of physiology, biochemistry, and neuroscience. It therefore follows that organisms are three-dimensional (or at any rate of lower dimensions than that of the universe). For otherwise, they will not be capable of the movement attributed to them.

Reality and the Body

By CT and AM, all *concrete* objects that are mereologically related to my body are real and parts of the fusion of all objects $\mathcal{U}$. In virtue of mereological relations adhering amid first-personal objects and the rest of reality, at each existent organism, there will be a single total localized situation involving all objects. I.e at any animate object $\leq((\mathcal{U}), \geq(r))$ obtains for each object r. Therefore there is a *maximal* localized situation involving all existent objects r, r', r''...: $\leq((\mathcal{U}), \geq(r), \geq(r'), \geq(r''), ...)$ at any location of a first-personal object. Take note of the scope of the subexpression $\leq((\mathcal{U}), ...)$ above. The situation is localized for an object that is a part of the fusion of all objects $\mathcal{U}$ such that at $\mathcal{U}$ there are various individual parts $r, r', r'', ...$ Each of the objects in $\mathcal{U}$, on this view, will be interconnected.

Reality will have a similar mereological structure relative to organisms according to the theory of nominalism about possible worlds. Observe that, for me to exist, I will again be a spatial body. At me, $\leq ((\mathcal{U}), \geq (r), \geq (r'), \geq (r''), ...))$ as before. The difference between the modal realist's account and the last will concern the topological nature of closing substances. That is, reality will consist of mutually disconnected entities. My body is both a part of one and connected to one, but there are others to which it is not connected. Nonetheless, there will be a world $\mathcal{A}$—the actual world—whose parts are mutually interconnected. And one of its parts will be my body. Hence there will be a closed situation $\ulcorner b :\leq (\mathcal{U})\urcorner$ as before. But now the maximal actual localized situation of all concrete objects will be one in which $((b), \leq ((\mathcal{A}), \geq (r), \geq (r'), \geq (r''), ...))$ obtains at me.

Essentially the same will hold if there are disconnected concrete parts of reality but reality is assumed to be *modally incomplete* or *partial*. In other words the same conception of reality will be valid if for every way a world could be, there is not a concrete world but only some disconnected ones. Organisms will give rise to total localized mereological situations in this case as well.

Although all degrees of representing oneself and one's environment will be to an extent physical, not all will be of the same kind. For example, although one may be involved in local situations, this will not mean that one completely grasps or attains knowledge of them. A case in point will be the total situations described above. Although these will obtain *at* objects and perhaps be denoted by them, obviously they will not be thoroughly considered or grasped. And this will mean that all situations cannot be leveraged and therefore first-personal.

3.3 Mereological Situations

Here we will discuss more formally the suggested notion. Again, a mereological situation will obtain *at* locations or closing substances. Theoretically any location can be localized. So we will present the various types of situation that might obtain. And on the basis of our argument for the existence of localized situations, various localized situations obtain at first-personal objects. If o is a first-personal object, then (o) is a so-called *distinguished local situation (DLS) at o*. Any DLS of o is a *mereological* *situation* of o (abbreviation: a *situation* or *arrangement* of o). Mereological situations will not be given recursively. But their structures will be one of the following forms. Now let r and s be either an animate or inanimate object and q be an organism. Suppose $\sigma_1 ... \sigma_\kappa$ are any κ situations. Then for any $\lambda \in \{1...\kappa\}$:

$$\begin{array}{rlcl}
(o) & \text{obtains at } q & \Longleftrightarrow & q = o, \\
(\sigma_\lambda)^- & \text{obtains at } q & \Longleftrightarrow & \sigma_\lambda \text{ does not obtain at } q, \\
(\sigma_1, ..., \sigma_\kappa) & \text{obtains at } q & \Longleftrightarrow & \sigma_1 \text{ and ... and } \sigma_\kappa \text{ obtain at } q, \\
\leq(\sigma_\lambda) & \text{obtains at } q & \Longleftrightarrow & \sigma_\lambda \text{ obtains at an extension of } q, \\
\geq(\sigma_\lambda) & \text{obtains at } q & \Longleftrightarrow & \sigma_\lambda \text{ obtains at a part of } q, \\
\nleq(\sigma_\lambda) & \text{obtains at } q & \Longleftrightarrow & \sigma_\lambda \text{ obtains at a non-extension of } q, \\
\ngeq(\sigma_\lambda) & \text{obtains at } q & \Longleftrightarrow & \sigma_\lambda \text{ obtains at a non-part of } q, \\
\lceil r : \sigma_\lambda \rceil & \text{obtains at } s & \Longleftrightarrow & \sigma_\lambda \text{ obtains at } r.
\end{array}$$

In the context of more complex situations the outer parentheses of a DLS may be removed as we have been doing. And in general, if R is any relation $\leq, \geq, \nleq, \ngeq$, for $R((o), ...)$ we can write just $R(o, ...)$ if no confusion results.

Although we have used the notion of set to define their structures, situations are mereological arrangements of actual objects. For any local situation of the form $(\sigma_1, ..., \sigma_\kappa)$ (at an location), no sub-situation σ_n where $n \in \{1...\kappa\}$ is the null set or "null" situation. And by definition, any two distinct situations are not mere divisions of some, more primordial entity (see section back to chapter 2 section 2.1.3). Suppose j exists. The local situation for j $(j, \sigma, \sigma', \sigma'', ...)$ involving every situation $\sigma, \sigma', \sigma'', ...$ at j we call *the total local situation of* j. The complex closed situation $\lceil \mathcal{U} : \tau, \tau', \tau'', ... \rceil$ involving all situations τ, τ', τ'' at $\mathcal{U}$ we call *mereo-reality.* If infinitely complex situations exist, for example where κ is some infinite number, it will not be assumed that the sub-situations $\sigma_1, ..., \sigma_\kappa$ are well-orderable.

Situations will *have* locations as well and their locations may be distinct from the location *at which* they obtain. For example, if a situation contains a negated sub-situation—i.e. one of the form $(\sigma)^-$, then the location of the situation will be the fusion of the locations at which it obtains. However, if a situation does not contain a sub-situation of negated form, then it will be called *positive.* Positive locations will have as their location the fusion of all the locations of the individuals involved in them.

By nominalism, each first-personal object o exists. Thus at each we have (o). And for every existent object p, at o we have either $(o, \leq (p))$, $(o, \geq (p))$, $(o, \nleq (p))$, or $(o, \ngeq (p))$. So we can show that there is a total local situation $(o, \xi, \xi'...)$ at o involving all objects in mereo-reality. That is, each $\xi, \xi', ...$ is a sub-situation of the total situation relative to o. And for each such sub-situation of the form $Rn...$ where R is any relation $\leq, \geq, \nleq, \ngeq$ and n is an individual, there exists a closed situation $\lceil n : ...\rceil$. Thus by CT, one such situation is mereo-reality $\lceil \mathcal{U} : \tau, \tau', \tau'', ...\rceil$.

Situations involving mereological fusions, products, and complements will also hold. For example, let i and j be names for existent objects. The global situation that f is the fusion of i and j is

$$\lceil f : (\geq (i, j), (\geq (((\geq (\leq i))^-, (\geq (\leq j))^-)^-)^-)^-)\rceil$$

; i.e. the one in which f has parts i and j and any part of f overlaps i or j. And we can abbreviate the expression of the situation to $\lceil f : +[i, j]\rceil$. Analogously, we can define the fusion situation in a localized way like $(f, +[i, j])$ and even $+[i, j]$ in which f is not identified in the situation. And more generally, for any individuals $i_1...i_\kappa$, $+[i_1, ..., i_\kappa]$ is the local situation of the fusion of those individuals. We can also generalize the expression to one for situations $\sigma_1, ..., \sigma_\kappa$: $((\geq (\sigma_1, ..., \sigma_\kappa), (\geq (((\geq (\leq \sigma_1))^-, ..., (\geq (\leq \sigma_\kappa))^-)^-)^-)^-))$ and indicate this by the expression $+[\sigma_1, ..., \sigma_\kappa]$. Likewise the global situation for products can be obtained in a similar fashion. For example if i and j overlap, then

$$\lceil p : ((\geq ((\leq (i, j))^-))^-, (\ngeq ((\leq (i, j))^-))^-)\rceil$$

is the situation in which p is exactly the individual whose parts are those both of i and j. And we can abbreviate the localized expression of the situation to $\times[i, j]$, and in general

$$((\geq ((\leq \sigma_1, ..., \leq \sigma_\kappa)^-))^-, (\ngeq ((\leq \sigma_1, ..., \leq \sigma_\kappa)^-))^-)$$

is the product and the expression of which can be abbreviated to $\times[\sigma_1, ..., \sigma_\kappa]$. Finally, the global situation $\lceil c : (\geq (((\leq (\geq (j))))^-)^-))^-, (\ngeq (((\leq (\geq ((j))))^-)^-))^- \rceil$ is one in which c is exactly the complement of i, which we again abbreviate to $\lceil c : \mathsf{C}[i] \rceil$ and thus the corresponding localized ones to $(c, \mathsf{C}[i])$ and $\mathsf{C}[i]$.

The existence of closing substances, their extensionality, and their unrestricted fusion will imply that unrestricted products, and complements also exist in the form of closing substances for any entities. As is well known since Tarski [95] the underlying mereological structure of our ontology will therefore have the structure of models of general extensional mereology. Tarski showed that models of general extensional mereology are essentially Boolean algebraic. And given our understanding of the fusion of locations of the last chapter, even if they are infinite, they will be unrestrictedly fused. So nominalistic considerations aside, the $\leq$-structure implicit in real situations will be isomorphic to the inclusion relation restricted to the set of all non-empty subsets of a given set, which is to say a *complete* Boolean algebra with the zero element removed. Complete Boolean algebras are akin to unrestrictedly fused models for general extensional mereology, in the sense that, in the former, there are arbitrary suprema for every subset of the domain. Mathematically, this will not imply any particular geometrical shape of mereo-reality. Locations will be understood to be geometrically variant. However it will imply that the part-wise structure of universes include *more* locations. These features are explained more fully in chapter 7.

3.3.1 The Question of Parametrized Situations

The most important feature about situations is that all objects have locations. And this holds for the parthood relation itself. We committed ourselves to just one multiply presented relation: the parthood relation. Although the parthood relation is not a part of any location, it is presented at each. We rejected all other relations. We rejected all ante rem universals and all other multiply presentable objects. We therefore turn our attention to the acceptability of so-called *parametrizations* of situations.

A parametrization of a situation is a reduction of a situation in which some or all the individuals involved in it are removed or "abstracted away". In order to represent their possibility, we will use variables $x, y, z, ...$ But variables will not *stand for* or *designate* entities. They will be understood to represent the lack of an individual in "arguments" appearing in situations.

To understand why parametrizations are unacceptable, consider a parameter like $(\leq (\mathcal{U}), \geq x)$ and suppose this is a situation. If so, its ontological status is entirely unclear. The parthood relation is presented at situations amid individuals and is a relation appearing in nature. And thus it is presented at various locations. But in contrast, as they do not figure amid individuals in situations, it is unclear whether parametrizations are multiply presentable. If it is a particular, it is not located amid any individuals, since this role has been set aside exclusively for the parthood *relation.* So presumably the parametrization will multiply presented. But it is also unclear if it is a spatiotemporal. Assume that it is nominalistically acceptable and therefore spatiotemporal. Then its existence will be superfluous. For either it will be involved in an existing situation (e.g. like the one containing the Eiffel Tower) or otherwise it will be a located "dangling feature" and overlapping an already instantiated uniformity. And if it is not a feature of an already existing situation, it will be preferable *ceteris paribus* to eliminate it for the instantiated version.

If a parametrized situation involves no existent individuals then it is yet more dubious. Consider $(\leq y, \geq x)$. It is difficult to say whether this relation can even be said to dangle in any position at all. Shall it be present in an existing situation? Is it a universal existing in a totally mysterious location or outside $\mathcal{U}$ altogether? We have at least the same puzzling possibilities as the last case. So by assuming nominalism, any parameterization of a situation is overkill or straightforwardly unacceptable.

There is final important consideration against parametrizations. Situations correspond to standard mereological relationships amid concrete entities. Apparently, the parametrization $\ulcorner z : +[x, y]\urcorner$ of any closed fusion situation $\ulcorner i : +[j, k]\urcorner$ would denote an arithmetical relation of the subsituations involved in the arrangement. And thus more complex pure arithmetical *principles* will be given in space-time as parameterizations. For example, at *any* location l

$$\ulcorner l : ((+[x, y], (+[y, x])^-)^-, (+[y, x], (+[x, y])^-))^-\urcorner$$

will obtain. That is, if one is at the fusion of x and y, then one is at the fusion of y and x and vice versa. And this is nothing other than the principle of commutativity of addition. For another example, we have the principle

$$\ulcorner l : ((+[x, +[y, w]], (+[+[y, w], x])^{-})^{-}, (+[+[y, w], x], (+[x, +[y, w]])^{-})^{-})\urcorner$$

of associativity also appearing as a situation. Likewise for products as well:

$$\ulcorner l : ((\times[x, y], (\times[y, x])^{-})^{-}, (\times[y, x], (\times[x, y])^{-}))^{-}\urcorner$$

$$\ulcorner l : ((\times[x, \times[y, w]], (\times[\times[y, w], x])^{-})^{-}, (\times[\times[y, w], x], (\times[x, \times[y, w]])^{-})^{-})\urcorner$$

Thus by admitting abstract concatenations of parametrized situations, patently arithmetical structures emerge implying a conception of reality as *constituted by* arithmetical properties, relations, and principles. It is therefore at odds with the nominalist's concrete conception of the world.

3.3.2 One Organism in Multiple Locations

In the last chapter, we argued that the exact constitution of an individual may involve matter, regions, or potentially both. And there is still the question whether individuals are simply fusions of matter and/or regions. Although the question will not be answered for the entire range of objects in the present dissertation, we will address one type of entity now. We have argued that organisms are *inter alia* animated locations or locations with subjective potential. A single potential animates one and only one location at any given time.

For an object to be capable of change, there must be a dimension along which the change occurs. Here we will assume this is a single temporal dimension. But this is not necessary, since there may be multiple dimensions along which changes occur. Nonetheless, it will be a natural position to adopt for the nominalist holding the spatiotemporality criterion. Organisms will be presumed to be three-dimensional. But strictly speaking this is also not necessary. What matters most is that organisms (whatever their dimensions) are understood to be wholly present at each location they animate. And thus whatever their exact dimensions are, they must be less than that of the entire system. Obviously, the reason they must have less dimensions is that they must be capable of changing and nonetheless being parts of one and the same dimensional system. If an n-dimensional organism o exists within a dimensional system s, then $o \leq s$. But for this part to change will require that it become relocated. And thus for o to become relocated and remain n-dimensional, s must be at least $n+1$-dimensional.

On this view times and spatial entities are also locations.[2] To explain, in a given situation *at* a given three-dimensional location l, the time t of l is the fusion of all spatial locations that are connected to l within the three-dimensions of l. The entire landscape of concrete objects will exist in a single closed dimensional mereo-universe as suggested by the conception lately introduced. Specifically, the structure of physical reality will consist of a sequence of three-dimensional objects or a four-dimensional product space such that all locations in the space are interconnected. This way of understanding time in terms of physical "chunks" is to my knowledge due to van Benthem [9]. Multiple locations can be animated by the same subjective potential but not distinct ones at one time. Consider the following example.

During the course of his life, *Tibbles* exists wholly at each of his spatial bodily locations. Let T be the location equal to the mereological fusion of each of these. For example, at time t' directly before his accident, let $p_{t'}$ be the individual involved in $\lceil p_{t'} : \times[T, t']\rceil$. $p_{t'}$ is the mereological product of the location of T and t'. The time t' is the part of reality equal to the fusion of all spatial objects existing within the dimensions of *Tibbles*'s body at the location of his accident. At time t of the accident, let p_t be the individual such that $\lceil p_t : \times[T, t]\rceil$. p_t is the mereological product of T and t. As *Tail* will be severed from *Tibbles* and organically deteriorate somewhere,

[2]Augmenting the mereo-ontology with temporal relations may or may not be acceptable. Temporal relations may be rejected on the idea that they suggest an overt dimensional or geometrical mereo-ontology and, for example, not one of mere matter. However, even on this view, the localized structure of situations will allow various located situations of objects. Scaling or scaffolding the present ontology may be acceptable in other ways. For example, a many dimensional ontology might be adopted on independent grounds for example if one wishes to incorporate the distinctions between space and time, or metric from mereological features, or topological properties of some sort. And thus a corresponding view of closing substance will be articulated on more granulated grounds in which first personal situations are located at more finely structured substances.

it will fail to be a part of *Tibbles* at some time. Thus the location of $Tail$ will not be included in p_t. If not immediately then soon thereafter, the tail will be rid of the subjective potential located at $Tibbles^{-}$'s body. $Tail$ and $Tibbles^{-}$ will share the same potential and not organically interact in any way.

Thus we have the following. $Tibbles^{-}$ and $Tibbles^{-} + Tail$ have different proper parts. At t' *Tibbles* is $Tibbles^{-} + Tail$ and located at $p_{t'}$. At t' he animates $p_{t'}$. And at t he animates p_t and is $Tibbles^{-}$. But at t *Tibbles* is not $Tibbles^{-} + Tail$. And at t' he is not $Tibbles^{-}$. In short, he is a single object in flux existing *within* reality, and he does not in any way exist as a fusion or a set of temporal parts. Rather he is a body enduring *through time*.

Still, from a third-personal perspective, we can view or represent *his life* as bodies at various times involved in a sequence of closed situations. If *Tibbles*' life spans the times $t_1...t_\tau$ we have: $\lceil p_1 : \times[T, t_1]\rceil ... \lceil p_\tau : \times[T, t_\tau]\rceil$. And from a first-personal perspective, we can represent *Tibbles* life as a sequence of localized situations at him: $\times[T, t_1] ... \times[T, t_\tau]$. There will also be other localized situations $\sigma ... \sigma_\xi$ involving objects in his outer environment. For example, at him at various times $\times[\leq (\sigma_1), t_1], ..., \times[\nleq (\sigma_\xi), t_\tau]$.

Localized situations will nonetheless not be him but merely indicate the existence of his subjective potential. At i, he *is* the animated body located exactly at p_i. And he will continue to persist until such time as his body fails to maintain his subjective potential. For example if he undergoes part deletions to the extent that both inner and outer localized situations are non-existent, he fails at such time to exist.

Importantly, neither the sequence nor the fusion of every dimensional body will be *him*. On the spatiotemporal view, he will persist through a sequence of changes. After each change he will be a three-dimensional object. By the unrestricted fusion of closing substances, there is four-dimensional fusion of each of his bodies at different times. But he will be distinct from this object.[3] What makes him an enduring individual will be that *Tibbles* is a first-personal object with a single potential to be

[3] Let us generalize this account to any organism of n dimensions. If he is n-dimensional, he will persist through a sequence of changes. After each change he will be a n-dimensional object. By the unrestricted fusion of closing substances, there is $n+1$ dimensional fusion of each of his specifications at different times. But he will be distinct from this $n+1$-dimensional object.

either aware of his three-dimensional body or aware of other objects. But this very potential will, according to nominalism, be attributed to his body at each stage.

Movements are explained similarly. Consider the case of a man whose has moved to Europe from the USA at some time t. Let $\mathcal{A}$ be the fusion of all objects on the continent of North America at time t and assume $\mathcal{E}$ to be the fusion of all on the continent of Europe at time t. Excluding the time elapsed traveling from one location to the other, before t his body will have location b_1 and be a part of $\mathcal{A}$ and at t the location b_2 of his body will be a part of $\mathcal{E}$. That is, the global situation $\lceil b_1 : ((\leq \mathcal{E})^-, \leq (\mathcal{A}))\rceil$ obtains before t, and after t, $\lceil b_2 : (\leq (\mathcal{E}), (\leq (\mathcal{A}))^-)\rceil$. But, similar to the case of part deletion, what makes these movements of *him* will be that at each stage he maintains the single subjective potential due to the status of his bodily parts to consider his environment. And this will mean that he be capable of producing representations of localized situations at $\times[t', b_1, \mathcal{A}]$ at any time t' before t and at $\times[t, b_2, (\mathcal{A})^-, \mathcal{E}]$ after.

3.3.3 Multiple Individuals in the Same Location?

Suppose feline tissue is a type of enduring substance. Let $Tissue$ be the feline substance before *Tibbles*' accident. And $Tissue'$ be the substance after. Directly after *Tibbles* accident, $Tissue = Tissue'$. And thus $Tissue$ will persist indefinitely after the severing of $Tail$ and $Tibbles^-$, and thus after t. Now presumably *Tibbles* and $Tissue$ once were exactly co-located. But at t, *Tibbles* survives as $Tibbles^-$ and at location $Tissue^-$. In contrast the feline tissue will survive as a spatially disconnected object $Tissue$ in the situation $\lceil Tissue : \times[+[T, Tail], t]\rceil$ of $Tissue^-$ and $Tail$, and any later scattering of those feline substances through time and space. So now we can point out the perceived asymmetry: as they survive as different substances, there is now the suggestion that *even before* t *Tibbles* $\neq Tissue$. *Tibbles* survives or decomposes in a way $Tissue$ possibly cannot. And thus there is a modal property the one has that the other does not. If feline tissue does not exist and $Tissue$ is a mere unmodified location, then the location $Tissue$ will be distinct from p_t (defined in the section above) and any three-dimensional object after t.

Given the asymmetry noted above, it may seem that there are two *individuals* at one location: $Tibbles$ and $Tissue$. Given our ontology of situations, we will be capable of distinguishing $Tibbles$ from his location without appealing to the idea of there being two individuals. All individuals have locations, but only some locations are animated, and the latter are animated at certain times. The only qualitative difference between locations will be that some are animated and some not. Any further way of distinguishing them, for example as substance *qua Tibbles* from substance *qua* feline tissue or the former and the latter from substance *qua* location, will require additional modal or property notions. And this may be required, if we wish to distinguish these further. But for the nominalist, she will wish to remain conservative until it is clear that another type must be admitted.

Intuitively, the conception of closing substance or location is therefore more akin to the notion of region. Recall it was only because we required the notions of matter and region but did not feel the need to unnecessarily sanction a spatial substantivalism that we settled on the uniform, unbiased notion of location. Closing substances will systematically decompose and compose into closing substances. And thus they will play the theoretical role of floor nominalistic substance. Their existence will be characterized, if at all, as animated; all other characterizations will be abandoned.

Thus I find the best solution will be to accommodate change in organisms without resorting to co-located individuals. We adopt the notion of *being identical to an animated location.* In other words, a body having subjective potential will have a localized relation to its body and environment. $Tibbles$ will persist not merely as a location but as one which gives rise to a localized state of affairs from his location. He will now be thought to survive in a different way than $Tissue$ whether the latter be feline tissue or merely a location. At the location of $Tissue$ we have the localized or animated situation $\times[Tibbles, t']$ at t'. And after the severing of $Tail$ the localized or animated situation moves to $\times[Tibbles^{-}, t]$ whereas the location of $Tissue$ persists in the unanimated closed situation $\lceil Tissue : \times[+[T, Tail], t]\rceil$. Therefore the surviving of organisms through changes can be accounted for by appealing to their localized situations.

On this view there are not two co-located individuals. There is one individual, and there is one subjective potential. One and the same potential is accounted for by the localized nature of the situation. And one and the same potential is capable of re-locating to different although non-distinct individuals. Thus the solution is to interpret potential as simply that of localized features of situations given rise to by organisms.

Is there then a mode or feature adopted? It would seem that our understanding of organism involves the notion of mode. An animated location is a location in a certain mode or configuration. Otherwise, it may require another multiply presentable feature. But this is only if we feel we have to explain why a localized situation emerges. Localized situations exist on account of the body. But the localized nature or character of a situation itself is something altogether different than a property. It is a situation as first-personally given and seemingly much different than a property of objects. Nevertheless, that a body have subjective potential is at least partly due to the nature of its parts. Therefore although a mode or property will be required, this would seem acceptable, since the requirement will follow from an explanation of the physical nature of organisms.

3.3.4 Tense and Simultaneity

For each organism, there is a time-location corresponding to the present-location relative to its animated location. In this sense the present is a location relative to an animated location—i.e. the fusion of all three-dimensional objects connected to the organism along the three dimensions in which it is located. Thus the present is a time as defined in the previous section.

The present time gives rise to corresponding future and present *locations* (although not times) relative to the present time. The future situation relative to the animated location will be the fusion of *all* three-dimensional objects which are after the present along the temporal dimension. Let t be the present time relative to an animated location l. And let f be the fusion of all three-dimensional objects after t but not including t. Then at l we have $(\nleq f, \ngeq f)$ and moreover we have the closed situation $\lceil l : (\nleq f, \ngeq f) \rceil$. The reason for the $\nleq$ and $\ngeq$ operations is that the future location relative to the occupied one will be a non-part, non-extension of the present location. And analogously, the past is the fusion of all three dimensional objects before the present along the temporal dimension. Therefore at l we have $(\nleq p, \ngeq p)$. But then the same mereological states of affairs for both past locations and futures locations will hold. There will only be mereological distinctions one can make on this way of conceiving the past and the future and they will therefore be indistinguishable. Hence tenses do not really figure in the mereological conception of time *per se* (although they may be provided in a tensed ontological extension of nominalistic mereology). They are rather "markers" relative an animated location. And every new present will obviously give rise to situations involving times which are parts of the locations past and future. Still, there will indeed be something objective about the past and the future at any given three dimensional location. And this will concern relations to the animated location (although not to time itself).

Observe now that this is a rather good feature and one which accords with features of time pertaining to the consequences the special theory of relativity. We can look at sequences of localized situations as reference frames. Each organism is capable of being animated at various locations through reality at different velocities. For example, one subjective potential may be traversing locations, moving to another location at a certain rate of speed while another is moving in a different direction at another rate of speed. And this is just to say that these subjective potentials animate various locations at different speeds. Given the structure of the dimensional system and the directions of their motions and their relative rates of speed, their localized situations will give rise to non-simultaneous present times. And theoretically, simultaneity at a distance will not be guaranteed, as required.

Still, although there will be total localized situations reflecting the relation of organisms within reality and therefore a quasi-indexing of reality according to the movement of animated substance, there will also be closed situations. Closed situations will reflect what is third-personal in reality. There will be situations involving all bodies and thus facts of the matter pertaining to their mereological situations. And these will be invariant with respect to localized situations. Obviously much more work needs to be done to see if the localized approach to mereology is fully consistent with a mereological view of reality and the special theory. And we note this as an area of further research.

3.3.5 Inanimate Objects and the *Ship of Theseus*

Like organisms, it is plausible that artifactual, inanimate objects survive various changes in their parts. However, as these items are inanimate, to explain in a non-question begging way how they are capable of enduring is comparatively more difficult. Consider the famous example of the Ship of Theseus. The Ship is originally constructed at time t. Over a period of time, the Ship's parts are removed. And it comes to pass that, at a certain point s, the original parts of the Ship have all been replaced by others. The parts that are replaced are not discarded, but subsequently used to create another ship whose construction is completed at time u. So which ship is the Ship of Theseus?

This question may not have an exact answer. If we privilege, for some reason, the idea of objects as constituted solely by their proper parts according to ME, then presumably we should say that the first time after t at which the "original" ship's first part was replaced, it ceased then to be the Ship of Theseus. And then perhaps strangely at time u the Ship re-enters existence. However, if the conception of a ship as a "working entity" is privileged—that is, one to which sailors return on a daily basis and one of which travelers have memories, etc.—then one may conceive of the Ship as the continually refurbished object. And thus even the one whose parts are completely replaced at time s is the Ship. Again, one may reject the notion entirely, as van Inwagen [99] has done, and suggest that there is no ship but rather atoms arranged "shipwise".

To provide an answer to the question which object is the Ship will be out of the scope of the present investigation, since to answer it will require an analysis of what being an object of a particular *kind* or *type* entails. Questions of meaning and nature of object-types aside, what we can ask is whether the ontology is strong enough to delineate between the various ways the Ship might be specified according to mereological structure. Thus we must ask if we have the proper number of distinct locations and whether we can make the appropriate number of structural distinctions. And it appears that we can, since at each time t, and each location l there will be an individual at that location. In the case of the Ship these will appear in closed situations. Thus the continually refurbished ship will have a three-dimensional location, the growing lot of parts of the refurbished ship will have a location, and the eventually re-constructed three-dimensional entity will also be present in an altogether different location. Likewise if the Ship is incapable of change and is four-dimensional, that specification, too, designates an existent location.

If there is any clear criteria by which to identify inanimate objects as things over and above locations, then this will involve us in making further distinctions. And these will not concern just the relations adhering amid individuals and their parts, but also concern either the different roles they play, the various multiply instantiatable properties they have, or the modes or configurations they may take. Although making them will lie outside of the purview of nominalistic mereological reasoning and the corresponding ontology, this may be accomplished most easily by adding to the present ontology various multiply presentable objects which are presented at locations. There are various ways these might be incorporated. On a rather natural way, we simply incorporate them into existing situations. And therefore involved in both localized and closed situations we will also have various unary properties or modal markers. We will return to this in chapter 4 when we present the logic of situations.

3.3.6 Unrestricted Fusions of Locations

Closing substances will appear in the total situation relative to an object. Locations at different parts of reality will be connected to all others via locations in sequential contact. The types of dense closing objects appearing in the sequence will not figure in the mereo-ontology. But all real, actual objects are indeed connected to the immediate location of the body. And the total mereological situation relative to the body will involve all objects in reality. If reality is the closed entity suggested by the principle of closure, then mereo-reality will expand out to all interconnected regions. And if reality is, indeed, disconnected, then mereo-reality will expand out from our bodies to a fusion of all existent objects.

In the preceding chapter it was argued that closing substance unrestrictedly combines and the decompositional structure of closing substance is extensional in the mereological sense. Now this can be seen more clearly. Consider the issue once again in the context of mereological situations. In particular consider the notion of a positive situation. A *positive situation* is one containing no negative situations. A *negative situation* is one of the form $(\tau)^-$ or one containing a situation of the form $(\sigma)^-$. A *positive localized (closed) situation* is a localized (closed) situation that is neither negative nor contains a negative sub-situation.

Let $(\leq (i_1, ..., i_\kappa), \geq (j_1, ..., j_\tau), \nleq (k_1, ..., k_\lambda), \ngeq (l_1, ..., l_\xi))$ be *any* localized situation comprised of existent individuals $i_1, ..., i_\kappa, j_1, ..., j_\tau, k_1, ..., k_\lambda, l_1, ..., l_\xi$ relative to one's present location. We now will attempt to show, intuitively, that there must be a mereological situation in which just those objects $i_1, ..., i_\kappa, j_1, ..., j_\tau, k_1, ..., k_\lambda, l_1, ..., l_\xi$ are involved. For at one's very location l—i.e. the location of one's body—there is a connected sequence of objects beginning at l and connecting to the other objects. Clearly this situation itself must have a location or closing substance. Thus the locations of those objects must be involved in a single closed location *not* involving my body and those connecting sequences from my body to each. And therefore their must be a fusion of those objects which is the fusion of the locations of $i_1, ..., i_\kappa, j_1, ..., j_\tau, k_1, ..., k_\lambda, l_1, ..., l_\xi$.

This will follow as well on the conception of reality implied by the theory of concrete possible worlds. Existent objects will be located in situations. Some of these locations will have interconnections with the body and some not. Those interconnected objects will be actual. However, in either case there will be existent localized mereological situations relative to one's body and one requiring a fusion location at which that situation obtains.

3.3.7 Idealism?

It may seem that LMR will commit us to a idealistic view of reality. There are two worries. For one, it might be thought that the existence of a person must itself must be demanded to imply the existence of reality. That is, it might be claimed that if one holds LMR, an object or reality itself will exist *only if* there is an object that is aware of it. But this is not so.

Perhaps ironically, the reason for the 'inside-out' approach was to obviate the need to resort to sets and instead rely on situations. Nominalism also demands the interconnectedness of actual objects. *If nominalism is not true*, CT may be rejected. Sets and sequences of geometrical objects appearing in nature may then be granted. And even actual spatiotemporal objects which are not interconnected may be countenanced. As we saw in the previous chapter, if nominalism is not maintained, it might be held that reality is disconnected and contains isolated fragments or cleaves. There will be parts of it that are spatiotemporally isolated from the rest. Or perhaps there will be various abstract objects like propositions, senses, meanings, and so on.

Still, the spirit of nominalism and in particular one held by proponents of CT will maintain that individuals live *within* worlds and their environment is susceptible to observation and theoretical and scientific re-evaluation from the inside. For example, if the nominalist has naturalist or empiricist leanings, these will only be borne out by assuming that our bodies reside in the same reality we observe and study. Even our commonsense understanding of reality would be consistent with this theory. Thus the notion of the body as living organism and requiring the status as real is not merely an expedient, as we earlier suggested. It is a necessity for the nominalist. And thus the view of reality as first-personally given must figure in the general conception of it and the proper conception of mereo-ontology. And this in no way implies that reality be perceived or observed to be real.

Secondly it might be thought that reality will then be fragmented into abstract first-personal experiences. However, this would only beg the question against the existence of closed states of affairs. Recall that closed states of affairs exist as well as localized ones. If our experiences are somehow detached from reality, then there will be no link between localized states of affairs an closed. But this will be diametrically opposed to the theory of nominalism.

Although our tack has been experimental and we have made the assumption that nominalism is true and therefore that the body is material, it may indeed be true, contra nominalism, that located situations are not mere experiences but physical goings-on or brain states of some sort. Nonetheless it may turn out that reality does indeed fragment into experiences. Reality will nonetheless be comprised of localized situations; and these will simply *be* the experiences. However, this will be a direct contradiction of nominalism and our hypothesis.

Chapter 4

Inscriptional Nominalistic Mereologies

Nominalism was defined as the rejection of abstract objects and the view that concrete individuals are the only existent entities. A most extreme version of formal mereology, essentially due to Leśniewski, thus presents itself. Let us call a *inscriptional nominalistic mereology* (INM) a formal theory of the parthood relation whose language is considered a concrete entity. *Inscriptionalism* is the theory that any logical system is a sum of concrete objects or "marks." The standard understanding of inscriptionalism is based on a view that languages are syntactical tokens. Therefore if inscriptionalism is true, formal languages are spatiotemporal particulars. In this chapter I argue that this view should be rejected. Whether we view marks as printed, written, or uttered tokens, these cannot be the elements of formal languages. Toward a reductio ad absurdum, we shall show that if this is the case, the assumption will involve us in an over-generation problem. Formulas of a language will exist long before logicians defined them; and this will contradict even a nominalistic view of them. Many objects which are clearly not parts of a formal language will then not be distinguishable from those that are. Our argument in this chapter will pave the way for the introduction to the mereo-language appearing in the next.

4.0.8 Leśniewski's Conception of Formal Languages

Leśniewski's view of formal languages was inscriptional. So to put the present investigation in perspective, a short review of his conception will be helpful. Leśniewski viewed the language of *Mereology* (and indeed all languages) as equivalent exactly to its physical aggregate. His official ontological stance was that all objects are concrete particulars. Hence his conception of formal language in general is one of a finite sum of physical marks which grows over time.

Leśniewski defined his formal language by appealing to a notion which explains how formulas are graphically similar. He claimed that various tokens are the same formula if they are all *equiform*. Thus for example if a logical system is published and there are various copies, then there are as many such logics as there are copies. Assuming the rather unlikely case in which each is typographically identical, they are, in Leśniewski's terminology, all equiform. Obviously, in practical applications equiformity is never quite exact. But in the context of formula-recognition, minute variations are rather insignificant. For example, we will recognize handwritten manuscripts and other variants to be equiform with systems which are typographically rather different.

Most logicians view formal languages as Platonistic constellations susceptible to critique "from the outside", a view Leśniewski attributed to Whitehead and Russell with harsh criticism. He claimed that formal languages are not sets of any kind. And therefore the entire gamut of syntactically well-formed phenomena of a language cannot be given a standardized semantics. Truth-conditions for the entire range of well-formed "objects" cannot be given in advance, since no unwritten formulas exist. Nominalistic proclivities of this strict sort and those of inscriptionalism in general thus make it impossible to provide a systematic meta-logical semantics.

In nearly all modern logical systems, semantic definitions are said to be *confined to the metalanguage only*, and not to appear in the object language. This view is diametrically opposed to that of Leśniewski. He contended that the symbol '$=_{def}$' used by most logicians actually smuggles in an unrecognized primitive. Consequently, his understanding of definitions prompted him to formulate *Protothetic*—his propositional logic—based on equivalence alone. In so doing, the same connective is primitive that is used for definitions. Instead, Leśniewski employs an exceedingly complex system of definitions, which he calls 'terminological explanations', and inference rules, which he calls 'directives' (see [91]). On his view, definitions are object-language equivalences and should be recognized as such.

From the start, Leśniewski considered his systems to be comprised of constant expressions which are primitive, defined, and with an intended, invariant meaning. Given his understanding of definitions, these could be made clear only by examples, models, and intuition given by elucidations. For example, he demanded each terminological definition be independent of the others, a property he and his students demonstrated rigorously by employing various models. This highlights an important (and nowadays rather trivial) fact—namely, that any formal system is always interpreted and explained by appealing to other intuitions (like models, examples, and so on). And moreover, formal languages are significantly different due to the fact that they are *introduced*. The graphical features of the syntax and the conceptions formalized will be chosen explicitly. These issues suggest various general problems with the notion of an INM which we now cover.

4.0.9 Interpretability and Equiformity

If an INM is to be an inferential system based on validities or axioms, it must contain truth-apt entities. It was claimed in the first chapter that propositions or sentence meanings are typically taken to be intensional. Normally it is thought that two propositions can have the same extension without being the same. Obviously the truth of this commonly held view will depend upon what one takes to the be an extension of a proposition. For example if the extension of a proposition is thought to be its truth value, then propositions would be clearly be intensional. But there are also theories of propositions which take them to be extensional. Arguing that they are structured entities of some sort, like structured *Russelian* propositions, is one way one might formulate an extensional account of propositions. On this view propositions are structured concrete objects rather than, for example, Fregean senses. Still on other views propositions are functions from utterances or token inscriptions to worlds or states of affairs, etc.

However, from the standpoint of formal logic, the question of the existence of propositions or sentence meanings will be a an ancillary one (an obvious exception is the famous attempt by Church [17]). The very trade of the logician will commit him to a notion of sentence or formula, and he will therefore require an explanation of what they are. Although an intended interpretation will be envisaged, it will be divorced from the syntactical object. The syntax will be understood, in some sense, as devised, selected, or even created. Various syntactical manipulations will be intended to stand for semantical inferences. Less syntactical machinery will be desired, but at least as much to represent the required inferences and objects. Hence formal formulas will be assumed to be selected on the basis of what inferences are to be modeled. Formulas will require truth conditions, and, implicitly or explicitly, these will be "grafted on" posterior to a selection of their syntactical form. Therefore, it is natural to view syntactic objects as graphical "marks" and, if one is a nominalist, to view these as the formulas of a language.

According to tradition, given a sequence of symbols $\overline{s}$, there is a distinction between various tokens '$\overline{s}, \overline{s}, \overline{s}, ...$' of $\overline{s}$ and $\overline{s}$; or, stated in mathematical jargon, a single sequence of symbols can *have multiple occurrences.* Here we may take the occurrences '$\overline{s}, \overline{s}, \overline{s}, ...$' to be either inscriptions, utterances, lit patches of pixels upon a screen, or the like. Accordingly, as formulas will be claimed to be equivalent to physical marks, any distinction between the occurrences of formula and the formula *type* will be lost.

So the inscriptionalist must attempt somehow to define a formula by means of some features had by each physical occurrence of the formula. Like Leśniewski, I see no other way to accomplish this, than to employ the notion of equiformity. For some relations must adhere amid the various physical tokens of a single formula. Indeed if formulas translate into various physical media, there must be cross-media relations between the various tokens in these forms. For example, if a single formula is written and also appears on a screen, there will be some relation between the two guaranteeing that they are the same formula. In particular, it will be claimed that two formulae are the same if they are *intermedially equiform.*

If the strict version of nominalism presented in chapter 1 is correct, a formula ϕ cannot be *the class* of equiform tokens of ϕ, since classes are abstract and claimed not to exist. Moreover, a formula cannot be a so-called Lockean "abstraction" obtained by extracting the essential features of equiform instances; for, according to inscriptionalism, any formula is exactly its *total*, physical manifestation and not any selection of features. Moreover, our motivation has been that the only feature we will assume exists is the parthood relation.

Nevertheless, if two syntactical objects o_1 and o_2 are equiform, then there is a similarity relation—namely, the binary relation of equiformity—o_1 bears to o_2. This relation is both reflexive and symmetric on the set of token formulae. One and the same token formula is equiform to itself. And o_1 is equiform to o_2 if and only if o_2 is equiform to o_1. Although similarity relations are not in general equivalences, transitivity will also hold, since if o_1 and o_2 are equiform and o_2 and o_3 are equiform, then o_3 will be thought to be equiform in the precise respect as o_2 is to o_1—that is, equiformity is a relation which captures the relevant syntactical features amid objects which are the same formula. And if the relation were not transitive in this way, the absolute similarity would be lost.

Were the relation of equiformity to be a concrete particular, the inscriptionalist may succeed in using equiformity to define his language in a nominalistically acceptable way. Suppose this is correct. Then the relation of equiformity is spatiotemporal and not multiply presentable. Hence equiformity must be a so-called "rela*ton*" r, existing within time and space. But this proposal must fail for reasoning entirely analogous to that of section 2.1.5. By intent, the notion of equiformity was to relate *arbitrarily many* finite concrete particular tokens of the same formula, and any relaton is by definition not multiply presentable and can relate at most two. And thus if equiformity is a concrete particular it must relate *sets* of tokens. In short, *if the extreme nominalistic project is to be metaphysically sustainable, the notion of equiformity itself must be nominalistically acceptable, which it is not.*

So taken in the strictest sense, the inscriptional nominalistic project must again fail. In order to define any suitable language, our nominalist must appeal to a notion which is not nominalistically acceptable. However, in order to provide a most charitable account of inscriptionalism, let us assume as we did for the case of parthood that the relation is a uniformity existing in nature and therefore multiply presentable. It will therefore operate at the level of interpretation as a recognized existent relation amid syntactical tokens.

4.1 Formal Languages as Physical Objects

We have seen that the relation of equiformity functions in the recognition of sequences of symbols and their place within a language based on their graphical form. But by merely having such a notion, one will not be able to interpret the symbols of a language. In other words, in order to explain how a language is to be interpreted, equiformity will be insufficient. We shall now take pains to sketch this in some detail.

When a formal language $\mathcal{L}$ is proposed for study, $\mathcal{L}$ will be given an interpretation $\mathcal{I}$ (whether implicit or explicit), and a so-called prototype P of $\mathcal{L}$ will be introduced. The interpretation $\mathcal{I}$ will assign various objects to the non-logical symbols of the language; and (again either implicitly or explicitly) a selection of truth-conditions will be given for arbitrary formulas. The *prototype* of the vocabulary of the language is the initial physical token of the vocabulary, for example appearing in a book or article, which provides a way to consider the intended categories and symbol types used of $\mathcal{L}$ and their interpretation. Let σ be the vocabulary of $\mathcal{L}$. Thus for example if one considers for the first time the first-order language, then σ will be given in abbreviation and a list of variables, constants, Boolean connectives, and quantifiers will be provided by physical example. If any of the above categories are meant to be those of an infinite number of objects ellipses (...) are typically used to indicate that infinitely many are meant. If σ is intended to be infinite, P is obviously a token abbreviation of $\mathcal{L}$. And if σ is assumed to be finite and the syntactical formation rules are given by unrestricted inductive definitions as is normal, there will be infinitely many syntactically distinct formulas. Thus from the syntactical perspective, prototypes of formal languages are most often abbreviations.

The inscriptionalist will hold that the prototype P is the temporal incipience of the vocabulary of the INM $\mathcal{L}$ and a part of a potentially growing sum of marks which are equiform to P. If marks or inscriptions equiform to those in P are produced, these are also to be deemed parts of $\mathcal{L}$. Note that P is finite. On the one hand, assume that σ is presumed to be infinite. As new formulas are physically manifested (P grows), there will be elements of σ for which there is no equiform object already displayed in P; for example, every variable of σ will not be given in P. Thus non-equiform additions to P proposed to be in $\mathcal{L}$ will be introduced as objects of a particular "syntactic type" (e.g. as variables, constants, function symbols, propositional variables, etc.). So in order physically to expand an INM and maintain a pre-established set of syntactic categories (e.g. lexical from non-lexical items or logical from non-logical symbols), the inscriptionalist will have to invoke these notions explicitly when each new symbol is physically introduced. At each occasion in which a new symbol is added, then there may be a donning of that symbol. For example, if one wishes to use the symbol w as a variable and this item has not been used before, one will say "*let the symbol w be a variable*". On the other hand, if σ is assumed to be finite, then as tokens of an INM "grow", there will be new formulas introduced that are not equiform to any sequence appearing in P. Assume that ϕ is a formula of $\mathcal{L}$ just introduced which is not equiform to any formula appearing in P. Similar to the previous case, ϕ must be introduced as an item in the syntactic category *formula*.

We therefore conclude that the inscriptionalist must posit the existence of various *syntactic categories* of symbols in the vocabulary and symbol sequences (described more formally below). For the first, these correspond semantically either to types of object or non-logical symbol types (e.g. for either sets or individuals) or roles played by the logical constants. For the second, these correspond to atomic formulae, formulae, sentences, and sequences thereof.

Although the following is formally trivial, it is significant for our purposes. Suppose S is a set of syntactic categories. Let V and V' be pairwise non-equiform vocabularies. Suppose both are partitionable into S such that for each category C of S, the number of symbols of V in a category C is exactly the number of those in V' in C, and vice versa. Then there is a bijection $f : V \longrightarrow V'$ and x is of category C if and only if $f(x)$ is in category C. And any inductive formation rule R for formulas over V can be extended to one for V', and thus two pairwise non-equiform languages $\mathcal{L}$ $\mathcal{L}'$ can be obtained. That is, base clauses will be used to define the simplest formulas of $\mathcal{L}$ as the atomic formulae $\overline{s}$ in V, where $f[\overline{s}]$ is also to be a formula of $\mathcal{L}'$ in V';

and for any formulas (sequences of symbols) $\overline{s}, \overline{t}$ if R implies $\overline{u}(\overline{s}, \overline{t})$ is a formula in $\mathcal{L}$, then $f[\overline{u}](f[\overline{s}], f[\overline{t}])$ is a formula in $\mathcal{L}'$. Thus there are two pairwise non-equiform languages which can be given interpretations such that, for any formula ϕ of $\mathcal{L}$, there is a formula ϕ' of $\mathcal{L}'$ where ϕ is true if and only if ϕ' is true, and vice versa. That is, $\mathcal{L}$ and $\mathcal{L}'$ are mere *syntactic variants*.

Conversely, suppose we have two (isomorphic) structural copies of a single language. We may view these languages as concrete or set-theoretic, it makes no difference. Trivially, these may be given different interpretations. In particular, different semantic categories may be used to interpret the non-logical symbols of one language and not the other. For example, suppose we have two duplicate, equiform copies of a selection of symbols. We interpret the items of both as terms. But one is interpreted as a selection of sets and another as individuals. Thus two equiform symbolisms may give rise to completely different interpretations. We therefore conclude that the equiformity of symbols is independent of the interpretation of a language. And yet again we will fail to provide a good definition of language in terms of mere equiformity of symbols.

The argument suggests very strongly that all formal languages have two features which henceforth we shall assume them to have. Firstly, each *has* a set of syntactic categories. On the one hand we have, according to tradition, *symbols*. These are either logical or nonlogical in the vocabulary. And furthermore we have types of valid sequences of symbols or *formulas*. Secondly, equiform symbols will be intentionally linked to a symbol type; and formulas will be determined by holding fixed the type and the corresponding symbol types and then defining longer more complicated sequences of symbols. Again, let us strengthen our assumptions. We will now hold that there are these categories and they are exhibited concretely as multiply presentable categories appearing in the physical universe. And we will assume that this is acceptable to our nominalist. We now provide an example for how the inscriptionalist will view his language.

Along with the prototype, a *protocol* is given which establishes which objects are to be parts of the language based on the satisfaction of equiformity relations amid the symbols within a formula. Let us consider an example in the first-order language.

We first begin with the prototype. It will consist of a selection of symbols. The only non-logical symbols will be $\leq$ and various constants. On one way of maximally conserving their number, we have the following symbol types. (i) We have the *terms* (objects of type T) which are either variables $v, v', v'', ...$ (objects of type V) or constants $c, c', c'', ...$ (objects of type C); (ii) we have the Boolean connectives $\neg$ and $\vee$; (iii) we have a quantifier $\exists$; and the prime symbol $'$. This will conclude the prototype.

Let σ be any sequence of marks. We use the symbol 'e' as a symbol figuring in a unary predicate ranging over concrete particulars. Each such predicate is one of the following:

$$
\begin{array}{lll}
\mathsf{e}^T x & \Longleftrightarrow & x \text{ is equiform to one of type } T \\
\mathsf{e}^V x & \Longleftrightarrow & x \text{ is equiform to one of type } V \\
\mathsf{e}^C x & \Longleftrightarrow & x \text{ is equiform to one of type } C \\
\mathsf{e}^{prime} x & \Longleftrightarrow & x \text{ is equiform to } ' \\
\mathsf{e}^{\leq} x & \Longleftrightarrow & x \text{ is equiform to } \leq \\
\mathsf{e}^{\vee} x & \Longleftrightarrow & x \text{ is equiform to } \vee \\
\mathsf{e}^{\neg} x & \Longleftrightarrow & x \text{ is equiform to } \neg \\
\mathsf{e}^{\exists} x & \Longleftrightarrow & x \text{ is equiform to } \exists \\
\mathsf{e}^{)} x & \Longleftrightarrow & x \text{ is equiform to }) \\
\mathsf{e}^{(} x & \Longleftrightarrow & x \text{ is equiform to } (\\
\mathsf{e}^{\overline{s}} \overline{x} & \Longleftrightarrow & \overline{x} \text{ is equiform to } \overline{s}
\end{array}
$$

The type *finite sequence* will be used in the protocol. We denote an n-length finite sequence of objects by use of a predicate $Seq(P^a x_1, ..., P^b x_n)$ restricted to concrete token marks. This predicate is interpreted in the following way '$Seq(P^a x_1, ..., P^b x_n)$' is true if and only if '$\ll x_1 ... x_n \gg$' is an n-length sequence of physical marks such that x_1 has unary property P^a and ... and x_2 has unary property P^b'. We let $F^{\mathcal{L}} x$ be the predicate which is true if and only if x is a formula of $\mathcal{L}$. And if a token sequence $\overline{s}$ of marks is given and $\overline{s}$ is a formula, we write $F^{\mathcal{L}} \overline{s}$. The protocol will be given as follows:

- If $\overline{s}$ is any finite k-length sequence satisfying

$$Seq(\mathsf{e}^V x_1, \mathsf{e}^{prime} x_2, ..., \mathsf{e}^{prime} x_k),$$

then $\overline{s}$ is of type V;

- If $\overline{s}$ is any finite k-length sequence satisfying

$$Seq(\mathsf{e}^{C}x_1, \mathsf{e}^{prime}x_2, ..., \mathsf{e}^{prime}x_k),$$

then $\overline{s}$ is of type C;

- If $\overline{s}$ is a sequence satisfying $Seq(\mathsf{e}^{(}x_1, \mathsf{e}^{T}x_2, \mathsf{e}^{\leq}x_3, \mathsf{e}^{T}x_4, \mathsf{e}^{)}x_5)$, then $F^{\mathcal{L}^{\leq}}\overline{s}$;

- If $F^{\mathcal{L}^{\leq}}\overline{s}$ and $\overline{s}$ is a k-length sequence and $F^{\mathcal{L}^{\leq}}\overline{u}$ for some l-length sequence $\overline{u}$ and $\overline{t}$ is a $k+l+3$-length sequence satisfying

$$Seq(\mathsf{e}^{(}x_1, \mathsf{e}^{\overline{s}}\overline{x}, \mathsf{e}^{\vee}x_{k+2}, \mathsf{e}^{\overline{u}}\overline{y}, \mathsf{e}^{)}x_{k+l+3}),$$

then we have that $F^{\mathcal{L}^{\leq}}\overline{t}$;

- If $F^{\mathcal{L}^{\leq}}\overline{s}$ and $\overline{s}$ is a k-length sequence and $\overline{t}$ is a $k+3$-length sequence satisfying

$$Seq(\mathsf{e}^{(}x_1, \mathsf{e}^{\neg}x_2, \mathsf{e}^{\overline{s}}\overline{x}, \mathsf{e}^{)}x_{k+3}),$$

then we have that $F^{\mathcal{L}^{\leq}}\overline{t}$;

- If $F^{\mathcal{L}^{\leq}}\overline{s}$ and $\overline{s}$ is a k-length sequence and $\overline{t}$ is a $k+2$-length sequence satisfying

$$Seq(\mathsf{e}^{\exists}x_1, \mathsf{e}^{V}x_2, \mathsf{e}^{\overline{s}}\overline{x}),$$

then we have that $F^{\mathcal{L}^{\leq}}\overline{t}$;

And this will conclude the protocol. Observe that the protocol is not just the definition of the language. It is an identification of the vocabulary and the language. The language will then be defined as the physical aggregate of any token sequence satisfying the criteria for an object in $F^{\mathcal{L}^{\leq}}x$. Note as well that we have identified the protocol for the language whose vocabulary contains just one binary symbol $\leq$. If we wished to expand the relational vocabulary, then, clearly, an expansion of the protocol will be required. Obviously, we may use a potentially infinite number of predicates of any arity. Thus various other symbols may be required to discriminate various arities and unique relations. And this can be done in obvious ways by using not merely by using sequences of prime symbols $'''$... but also another like $***$....

What should be fairly clear, however, is that by seizing on the notion of syntactic category we make an anti-nominalist trade of concrete tokens for syntactic types, for the latter are nothing more or less than multiply presentable properties of inscriptions. And we can reduce the number of syntactic types by using various concatenations of the same symbol to discriminate them. Otherwise we will be forced to admit infinitely many types.

4.1.1 Formal Languages are Abstract

Intuitively, there are formulas of any formal language which do not exist in physical form. For example, syntactic formation rules permit an infinite number of (distinct) formulas to be written, but formal languages have only existed for a finite time. Hence, against the theory of the inscriptionalist, most logicians hold that inductively defined formal languages with infinite vocabularies are infinite. But if formal languages are INMs, we are left with the view that no grammatically well-formed objects that are nowhere written, typed, or uttered exist. Each time one wishes to expand the vocabulary, there will be an expansion of the protocol. The protocol will then give rise to new grammatically well-formed formulas.

So suppose formal languages are physical entities and grow in the way suggested by Leśniewski. It is only plausible to presume that formal languages are located exactly where the symbols or formulas have physical form. And there may be a multitude of types of physical form. These may be written, typed, or they may figure in brain states, etc.

Let us also assume that $\mathcal{L}$ is a formal language and C is the protocol or rules determining the acceptable extensions to its prototype P. C provides the syntactical criteria for token-symbols to be members of $\mathcal{L}$ based solely on their physical form—e.g. as terms, truth-functional connectives, atomic formulas, arbitrary formulas, etc.;

and the equiformity criterion imposes a constraint on the degree of physical variability which might be permitted of the symbols. We may view $\mathcal{L}$ as the language of first-order logic, a typical modal language, a finite fragment of first-order logic, or even a higher-order language.

The temporal incipience of C can be shown not to exist as follows. Observe that C is a type (in the metaphysical sense)—a particular selection of features or types capable of having various tokens. In other words, C is just a massive complex type. For example in the construction above we had the type Seq, the various equiformity predicates e, the type $F^{\mathcal{L}^{\leq}}$, and the vocabulary and formula conditions were couched inductively. In general, for a token o to be of a type T means simply that o have every feature in the selection T.

Now sensibly, any object-type T begins to exist (if at all) at least as early as the earliest point at which its tokens *could* exist. But for any time s, if tokens of T begin to exist at s, then, possibly, they exist at some time s' earlier than s. Consider the class S of each such point earlier than s at which tokens could begin to appear. If the inscriptionalist is correct, there must be some point s' at which the points of S converge or are bounded—and this will be precisely when the protocol is given. But clearly, a token of T could have appeared earlier than s', and so on. Thus C was never brought into existence. C will be multiply presented amid any marks or inscriptions throughout time.

An obvious objection to this argument would be that S does indeed converge or is bounded at a time point, say u, and that no token o of type T could exist earlier than u. Presumably, there is some necessary condition that must hold for o to appear, which cannot obtain earlier than u. In the case of formal languages, these may include various facts about the environment, the existence of intelligent life, humans' abilities, background conditions, and so forth. However, note that it is perfectly possible that these conditions too obtain at an earlier time than u. Thus this is not a worthy objection to view that protocols have no temporal incipience.

Suppose $\mathcal{L}$ is any formal language. Suppose the temporal beginning of $\mathcal{L}$ is t. It is possible that there are various token sequences $\overline{s}, \overline{t}, ...$ equiform to some in P appearing (for whatever reason) at some time t' earlier than t. The objects $\overline{s}, \overline{t}, ...$ may intentionally have been created or they may have been unintentional, non-artifactual objects. Random marks or structures appearing in nature my apply. Importantly, any objects whatsoever may be of the syntactic categories envisioned for $\mathcal{L}$, those for another language, or no language at all. Since the syntactic categories $T_1, T_2, ...$ of C imposed on the sequences of $\mathcal{L}$ have no temporal beginning, they will also be imposed on those in $\overline{s}, \overline{t}, ...$ Hence they meet the identity criteria for a part of $\mathcal{L}$ and are satisfactory syntactical extensions of C and parts of $\mathcal{L}$. But this is clearly absurd. If only these structural restrictions are conditions for membership in formal languages, such a position would lead to the view that unintended random marks, physical indentations, and structures occurring any time in the past are also parts of $\mathcal{L}$.

Hence it natural to at this point to claim that the domain needs to be restricted in some sense to a time or location. Thus this must appear explicitly in the protocol defining objects of the language. The protocol will then contain an additional property: *all formulas must exist after or at a certain spatiotemporal location and in a particular form etc.*. On this view, perhaps we may think of these as chalk boards, books, journals, brains, brains in intentional states, times after a particular time, etc. Indeed it is entirely conceivable that the elements of a formal language should be restricted to some spatial locations or to times after the prototype and protocol are given.

But this way of restricting the applicability of types appearing in a protocol will also imply an absurdity which we now present. In short, if the types are conceived as restricted, this is tantamount to the view that a location-restricted protocol is logically distinct from one obtained merely by raising the location-restriction of the first. To explain, assume that P_1 is not a location-restricted protocol but that P_2 is a protocol identical to P_1 in every way except that P_2 is location-restricted. Suppose as well that P_1 is donned earlier than a protocol P_2. Then the the non-restricted language will be distinct from the restricted one. But this is diametrically opposed to any commonsensical way of thinking of formal languages. Formal logical languages are clearly the sort of objects that are *not* temporally restricted. For example, if I propose a language and think I am the first to do so, but I find out that another has introduced it before, then I shall not say that the one I introduced is different solely on the basis that it was introduced at a different time.

Obviously, to incorporate a location restriction in the protocol itself was just a "fix". Perhaps it will then be claimed that languages are simply the *type* of object which has some sort of restricted manifestation. On this view languages do not exist but as restricted to certain times, locations and forms. But again, this will be tantamount to the view which we were originally against—namely, that types have some sort of restricted bounds. It is not necessary that a protocol, say C, begin at any specific time or place. Of course any token of it simply *could* have been given before any point at which it actually is.

Observe that the same argument goes through if we view P and C as "initial" brain states or a mental "donning" of a language. In this case, we may view E as the acceptable syntactic extensions of the protocol which are either brain-states: cognitive processes in which the formula is considered. But here again, the type C and its atemporal status is of crucial importance. Types of formal languages are not merely mental tokens, since their token formulas are of the sort of that could have been produced earlier than they are.

Thus we have the following: if $\mathcal{L}$ is a concrete particular, either (a) mysteriously, $\mathcal{L}$ exists in some spatiotemporal location apart from the location of the satisfactory extensions of C or (b) $\mathcal{L}$ exists in these locations, but every object in the universe equiform to the range of syntactically acceptable objects of $\mathcal{L}$ is in $\mathcal{L}$. Although both are bizarre, it seems (b) is more plausible than (a). Still implausibly, (b) implies that unintended, unconsidered objects throughout time and space can be elements of language. Thus it is preferable to reject both and claim that formal linguistic objects are abstract.

4.1.2 The Uses of Logic and Mereologic

I therefore claim that at least some sets must exist. Formal languages are sets and therefore abstract. We select sets primarily on the basis that they are extensional. And we have seen how unwieldy it is to attempt to define the language according to types of multiply instantiatable properties.

This way of viewing formal linguistic entities also has commonsense on its side. Formal languages can then be deemed infinite, as is typical. All grammatically acceptable formulas exist. In particular, we can envision the elements of formal languages as comprised of exemplars which are distinct from those items appearing in the physical incipience or prototype P. P will now be understood as a selection of graphically distinct "introductory" representations of a logical system. And then the significance of equiformity can be restricted to the psychology of formula-recognition. Moreover, we can also provide a systematic semantics for the language. And the denotations of the formulas and terms can be given in as physical or concrete of terms as possible.

Furthermore, by submitting to the abstractness of linguistic items one can nonetheless maintain the view that formal languages are *codified*, *introduced for consideration* or *defined* by the introduction of a prototype. And it would even be consistent with this view to claim that formula-tokens and formula-sequence-tokens (arguments/proofs) begin to exist at some point and then scatter ubiquitously thereafter.

The understanding of formal languages as sets accounts for all this information quite well. If languages are classes, they contain one exemplar of each equiform syntactical item. Accepting sets over, for example, clusters of abstract features is standard in class-nominalistic analyses. And if abstractions must be accepted, sets shall be preferred based on their being extensional.

Leśniewski developed his logical systems with the goal of providing a foundation for mathematics without the antimonies of existing systems but with a scope and power comparable to those of Frege or Whitehead and Russell. As we have seen this culminated in a radical, nominalistic approach. But if the relationship between nominalism and mereology was already complicated enough by problems concerning how formal languages are to be defined, then, coupled with the pursuit of a foundation for mathematics, their marriage was destined for divorce. Mathematical objects are abstractions *par excellence*. So if mereology is to be used to provide a foundation for mathematics, then a fully unrestricted nominalism must be rejected. On the other hand, if nominalism is to be rejected, then, on pain of principle, it would seem that we could adopt set theory from the outset.

But it is well-known that Leśniewski's gradual move from a prose-like presentation intermingled with variables to his more mature, formalized systems prevented him from considering his logic as uninterpreted (see [91] for an authoritative interpretation of Leśniewski's development). He considered all his axioms and theorems to be absolutely true. In this he was following Frege and Russell, who likewise did not envisage independently existing structures as conveying meaning upon expressions. But ironically, the development of logical semantics at the hands of Leśniewski's former student Alfred Tarski, quickly began to overtake the old approach. The watershed came with the publication of Tarski's seminal essay on the concept of truth in formalized languages. The paper was produced in preliminary form in 1929-30, revised when Gödel's famous incompleteness theorems became known in 1931, and subsequently published in Polish in 1933.[1] And in the case of mathematical structures, the Tarskian approach is very useful. Given an intuitive understanding of the structural features of an entity, a class of set-theoretic structures is to be selected bearing them. A language will then be selected with the logical and non-logical features desired. And truth conditions of formal languages will be provided relativized to the selected class of models.

Our arguments would thus far only suggest that formal linguistic entities and sets are abstract. The status of physical objects is still an open question. We should prefer a conception of physical object as concrete and particular and susceptible to mereological critique. Material bodies and spatial regions, for example, should be describable in terms of a logical analysis involving concrete particulars as capable of standing in the parthood relation.

What this means will become clearer as we press on. But tentatively, it suggests that to explain our conception of physical or concrete object not require recourse

[1]Leśniewski was known to be opposed to the Tarskian approach. As Simons notes, there are probably two reasons for this. One is that in his metalogical apparatus Tarski avails himself of set theory. Even though his use is not extensive, to Leśniewski it is an unwanted intrusion of set theory into the metatheory of logic. The other reason is probably that, whereas in earlier parts of the monograph Tarski adheres faithfully to a conception of finite types akin to Leśniewski's theory of semantic categories, in later amendments he distances himself from this as an unnecessary restriction and accepts transfinite types.

to arithmetical or set-theoretic constructions. In addition, given extensional motivations, the analysis should not necessarily involve intensional ones. For example, it is possible that our conception of physical object may involve notions that are irreducibly syncategorical or conceptual. Syncategorical features of logical languages would correspond to intensional or mental features. But then, these features will not correspond to the parthood relation or extensional concrete particulars in any way.

Arguably, the most useful aspect of formal languages is that there are well-developed tools of mathematical logic (model theory and proof theory, in particular) which allow us to demonstrate many important things about them. In general, this concerns the type of objects they can be said to *describe*. For instance, with the tools of the mathematical logician we can determine the sorts of objects and relations which can be defined by their formulas. And we can assess whether there are sound and complete axiom systems in languages for a class of structures. We also have devices which can help us determine when theories only have one model (up to isomorphism), and so on.

Although mathematical structures and sets are abstract according to the nominalist, their intended, non-trivial features may be somehow *intelligible*; and this is already suggested by our view of formal logics as abstract. Their intended structural features are represented as a set of axioms and may accurately represent the structural features of nominalistically acceptable entities. Such a view is what we would expect from most nominalists. For, it would be absurd to claim that the range of mathematical research is unintelligible. And we turn to these issues explicitly in the next chapter.

Chapter 5

A Formal Language for Mereology

5.1 The Question of Mereological Languages

We have seen that formal languages and their elements are not concrete. On the basis of a preference for extensional objects, we settled on a view of them as sets of well-formed of formulas. Sets are neither spatial nor temporal and are therefore not parts of natural worlds understood nominalistically. They have no spatiotemporal location whatsoever. Token inscriptions of formulas are physical and distinct from the formulas of formal languages. Accordingly we will provide a systematic semantics for formulas. And the latter will be understood themselves not to carry any "intrinsic" meaning. The semantics of the selected formal language will be bestowed or "grafted" on to the constellation. Various semantic relations between them and worldly situations will be postulated. But now we must ask what type of language we should select.

Naturally a formal mereology will contain the parthood predicate '$\leq$' as its only non-logical symbol. But what about the logical symbols? Purely logical features of formal languages follow from their selection of logical constants. There is therefore a question about what sort of logical constants should be admitted. A formal mereology of the sort we now wish to consider should be capable of making only nominalistically acceptable distinctions, or at least as far as possible. And, for reasons just discussed, this will imply that they not contain expressions which refer to non-situations or imply the existence of parametrizations of situations. Moreover situations are conceived as actual. So no so-called *empty names* will be admitted, and therefore no free logic of any kind will be acceptable (see appendix). Still in other contexts, we may wish to view the language as an entity of formal study. And there will therefore be two contexts: *a formal context and a real context.*

Although formal languages are sets, the purpose of nominalistic formal mereology will now be seen differently. Namely, it will be to model nominalistically acceptable inferences over mereologically arranged natural objects. And thus the question originally posed in the introduction will be restricted. We now ask whether there is a formal apparatus which can represent reasoning over concrete objects without construing them in any way as abstractions (e.g. universals or multiply presented concrete entities) or constituted fundamentally by abstractions.

All higher-order quantifiers are unacceptable, since sets have been rejected. We therefore start by considering the first-order language (FOL). Conceivably, as Quine argued, we can make nominalistically acceptable distinctions in FOL. Physical reality will contain concrete locations and organisms. And although first-order (FO) formulas will be abstract themselves, they will be thought to comprise a class of linguistic entities which permit of descriptions of mereological situations in which concrete objects figure.

However, it will be claimed that adopting FOL implies an espousal of parametrized situations. We saw that identifying any location for a parametrized situation seemed a will-o'-the-wisp. And adopting them as stand-alone entities seemed superfluous. They are instantiated or "embedded" already in the fabric of concrete universes. Indeed it was shown that commitment to them leads to a view of universes as constituted by highly complex parametrized mereological situations which are nothing more-or-less than arithmetical relations.

A rather lucky prospect is that there are already well-investigated classes of modal, hybrid languages similar to those developed by Arthur Prior in [73], [75], [74] which are expressively sufficient to allow reference to just types of mereological situation.

5.1.1 FO Formulas and Quantification

Assume that $\mathcal{L}$ has a signature containing the parthood predicate $\leq$ and a single constant naming each existent concrete object and no others. So if there are κ objects, then we shall understand there to be κ constants in $\mathcal{L}$.[1] And then the corresponding FO formulas will be defined inductively in a suitable way. So that if there are countably many objects in reality, then there shall be countably many constants. And if there are uncountably many then the formulas will be defined by transfinite induction. We can show that each closed, quantifier-free formula in $\mathcal{L}$ is true if and only if there exists a situation obtaining at any location satisfying the formula. Each closed atomic formula is of the form $c_i \leq c_j$ for constants c_i, c_j. We use the following satisfaction relation for each closed atomic formula $c_i \leq c_j$ in $\mathcal{L}$:

$$\ulcorner c_i \leq c_j \urcorner \text{ is true } \iff \lceil c_i :\leq c_j \rceil.$$

In particular, we will show that for any $\mathcal{L}$ closed atomic formula ϕ, we can extend this correspondence to a function σ on the closed atomic formulas of $\mathcal{L}$ such that $\ulcorner \phi \urcorner$ is true if and only if σ^ϕ is a situation obtaining at any location.

Proposition 5.1.1 (Situation and FO Atomic Sentence Correspondence)**.** Let $\phi \in \mathcal{L}$ be a FO atomic sentence. Then, $\ulcorner \phi \urcorner$ is true $\iff$ there is a situation σ^ϕ such that σ^ϕ obtains at any location.

Proof. By induction on the complexity of ϕ. *Base Case.* Let $\ulcorner c_i \leq c_j \urcorner$ be any closed atomic $\mathcal{L}$-formula. $\ulcorner c_i \leq c_j \urcorner$ is true $\iff \lceil c_1 :\leq c_2 \rceil$ is a closed situation and therefore true at any location. So in this case let $\sigma^{c_1 \leq c_2} = \lceil c_1 :\leq c_2 \rceil$. *Inductive Step.* We do cases for conjunction and negation. ($\wedge$) Now $\ulcorner \phi \wedge \psi \urcorner$ is true $\iff \ulcorner \phi \urcorner$ is true and $\ulcorner \psi \urcorner$ is true $\iff \sigma^\phi$ and σ^ψ hold at any location [by the Inductive Hypothesis] $\iff (\sigma^\phi, \sigma^\psi)$ obtains at any location. ($\neg$) Now $\ulcorner \neg\phi \urcorner$ is true $\iff \ulcorner \phi \urcorner$ is not true $\iff \sigma^\phi$ obtains at no location [by the Inductive Hypothesis] $\iff (\sigma^\phi)^-$ obtains at any location. □

[1] Theoretically, there should be no problem with the assumption concerning constants. For example, it seems clear that if there are a certain number of concrete objects, there will also be enough abstract structures to represent them and therefore enough constants.

Do FO Open Formulas represent Sets?

In $\mathcal{L}$, constants represent individuals and atomic, closed FO-formulas represent situations. But do open FO-formulas represent any real things? Given all that has been said thus far, if open formulas represent parametrized situations, they will be rejected as types of formulas in the envisaged language. But is it the case that they do? They do not clearly represent single closed situations. For closed situations involve particular objects and these do not appear in open formulas. Thus we consider the following other alternatives.

It might be thought that they represent sets. On this view an n-ary open formula will be thought to represent the set of all n-tuples of objects satisfying the formula. Suppose open formulas represent sets. Their members will be tuples of objects. And in our case, they will be tuples of concrete individuals. This conception of open formulas will be one of them as abstract impure sets (see section 2.1.3 pgs. 18-21). Hence not only will there be abstract sets which are formal languages, but now there will also be abstract sets for every property definable by a free FO formula. However, there are two problems with this proposal.

Firstly, the nominalist will reject the notion that concrete objects exist both in the world and as members of sets. Although formal linguistic systems are abstract, it is not necessary that the entire gamut of objects be contained in sets. An as we have seen, formal logics will contain formulas which are abstract. And although there are token inscriptions of formulas, these will be entirely distinct from the formulas themselves. In contrast, for the case of arbitrary concrete entities, the nominalist's position will be that they are individual parts of spatiotemporal universes. And it will be entirely antithetical to the nominalist's position to double-up their existence as elements of abstract impure sets. For the nominalist will claim that they are not *both* parts of the spatiotemporal world *and* elements of impure sets. Indeed it was this very consideration which called for the rejection of impure sets. By admitting them we exponentiate the number of concrete things. Not only are there fusions of naturally occurring objects but, if impure sets containing them exist, there are also objects for each way a fusion can be divided. If there were indeed sets of concrete objects as well as physical or natural fusions of them, then there would be no principled reason to prefer fusions over sets. And reality would simply bifurcate into a physical "side" and "set side". That is to say, concrete objects would be parts of two fundamentally distinct types of entity. Thus, despite our espousal of the existence of sets of formulas, at least tentatively it would be more frugal in the case of concrete objects to reject them as Urelements of sets. And obviously, for the same reasoning, higher type-theoretic objects will also be rejected as well.

Secondly, recall that the notion of a set of pairs of objects standing in the parthood relation is posterior to the relation appearing in the natural world amid objects. My arm a is attached to my body b and is a part of me. It is a part of me—i.e. *at* my location $(b, \geq a)$—not on account of there being a pure ordered set $\langle a, b\rangle \in S$. It is rather because my arm is connected to my body, that it bears certain functional relations to other parts of it, and in particular that there is a localized situation involving my arm *at me* and that I can leveraged this situation in various ways. And if we espouse the notion of persisting inanimate objects, the story is similar. If a door d is a part of a car c $\lceil d :\leq c\rceil$, then this is not due to their being a set S and a pair $\langle d, c\rangle \in S$. Rather it is because the door is connected to the car and bears certain functional relations to the whole. For example, to drive it, I will enter through the door; the door will be hinged to the car; if it remains in tact and connected to the car, then wherever I drive it the door will follow; it was manufactured with the door, for the purpose of safety, comfort, and so on.

Open Formulas in Other Guises

Is it possible that open formulas do represent an existent entity, but both not a set and not a parametrized situation? I argue that, if there are such things, they must be either so-called situated features of situations or *ante rem* universals. In the first case, they are already present in mereological situations, and therefore closed formulas will suffice to represent them. In the second case, they should be flatly rejected.

Open Formulas as Concrete. Let us suppose that FO open formulas represent objects which are concrete. We first note that an open FO formula cannot represent a particular entity of any kind. To see this we must generalize an earlier observation. The proposal is to be rejected by completely analogous reasoning to that found in chapter II section 2.1.5 concerning tropes. Let $\phi(x_1...x_n)$ be an open FO formula. If ϕ represents a particular thing, then what it represents can relate at most n things. However, we saw that assuming its applicability to more than n objects involved us viewing relations as entities relating sets which was unacceptable. The same is true in this case. If '$\phi(x_1...x_n)$' denotes a particular object $d(\phi)$ the predicate is true of many tuples of objects, then $d(\phi)$ must relate sets $X_1...X_n$. I.e. we have $d(\phi)(X_1, ..., X_n)$.

So FO open formulas must represent objects that are non-particulars. I see no other way to understand this proposal than to say they must be some "situated feature" of a mereological arrangement. A *situated feature* of a mereological situation is a series of parthood instances appearing amid particular objects in a given situation. For example, consider the situation

$$\ulcorner \text{wheel} :\leq (\text{car}, \geq (\text{door}, \nleq (\text{engine})))\urcorner.$$

The particular *organization* of parthood instances the situation can be represented in the following by using dashes $-$: $\ulcorner \leq (-, \geq (-, \nleq (-)))\urcorner$. In general, each situated series of parthood instances within a situation will have a particular organization. But this organization will be present *in* the situation. Two properties are to be considered crucial to situated features. Firstly, the existence of situated features themselves will be considered dependent upon their figuring as instances within situations. That is, they will not have existence independent of the situation. Secondly, they will be distinct from their organization. They are the instances of the relation of parthood in a situation and are features of no other. Therefore they are not parametrized situations since they do not exist independently of the situation in which they figure. But observe that since situated features do not exist apart from them, we shall have no need for FO open formulas. For what is expressed by atomic sentences of $\mathcal{L}$ will be sufficient to capture the content of situated features of situations.

Open formulas represent abstractions?. Let us suppose that open FO formulas represent some existent entity. Assume they are abstract. We have seen earlier that the nominalist will reject that they represent sets. By definition of abstractness, each object they represent will be non-spatiotemporal. They have neither temporal nor spatial location in any universe. Each such object, therefore, will not exist amid individuals in any situation and will not be a particular organization of parthood instances. And this will immediately imply that they not be *situated features* within a mereological situations. For if so, they would exist as spatiotemporally located features. And thus it will not be an *in re* feature of any sort. I see no other way to view this matter than to view them as *ante rem* objects and most likely universals of some kind. And therefore they shall be immediately rejected.

FO Open Formulas and FO Quantifiers

Another way of viewing the matter will be to claim that open formulas represent no thing whatsoever. If this is to be maintained, then we should eliminate them in the pursuit of a more parsimonious language. The only objects in a FO mereology will then be sentences. However, if this is the case, we will have a certain trouble with explaining how FO quantificational statements succeed in referring to objects and situations.

Consider an example of a universally quantified statement $\forall x_1 ... \forall x_n \phi(x_1 ... x_n)$. As $\mathcal{L}$ is presumed to contain sufficiently many constants, there is a closed atomic FO-formula '$\phi(c_1 ... c_n)$' expressing that ϕ holds of each tuple of objects $o_1 ... o_n$. As each is an atomic sentence, each denotes a situation true at any location by proposition 5.1.1. Suppose that $\ulcorner \forall x_1 ... \forall x_n \phi(x_1 ... x_n) \urcorner$ is true. As sets will not be acceptable, the nominalist will reject the idea that the formula denotes the set of situations

$$\{\sigma^{\phi(\bar{c})} \ : \ \bar{c} \text{ is a tuple of constants in the vocabulary of } \mathcal{L}\}.$$

Admittedly, the formula is true if and only if the single situation $(\sigma^{\phi(\bar{c}_1)}), ..., \sigma^{\phi(\bar{c}_\lambda)}))$ involving each $\sigma^{\bar{c}_i}$ for all $i \in \{1 ... \lambda\}$ (where λ is the number of all n-tuples of constants) obtains at any location. However, observe that *reference to* a possibly infinitely complex situation by a finite formula is obtained by ascribing to each situation a certain *feature* ϕ and obviously not by asserting the conjunction of every atomic sentence. More specifically, as an infinitely complex situation may be implied by the truth of a quantified formula and since in FOL reference to these is made by finitely long formulas, denoting the corresponding situations succeeds in virtue of their being a single finitely complex *aspect* of each situation. And in general for each arbitrary quantified sentence $Q_1x_1...Q_nx_n\psi(x_1...x_n)$ each such aspect ψ will be either a property or relation presented in each situation instance σ^{ψ}. And therefore in general reference to situations is achieved in the case of FO quantified formulas by committing to the existence of finitely complex multiply presentable properties and relations. And I see no other way to assess their status except as parametrizations of situations. That is to say, infinitely large situations will be denotable by FO-formulas by committing to features which are thought to exist in virtue of their source being existent situations.

A case may be made that existential theories could be acceptable. An existential FO theory would be one whose every quantified statement contained only existential quantifiers. However on this view, we should, yet again, prefer closed instances of situations. That is $\exists x_1...\exists x_n(\phi(x_1...x_n)$ is true if and only if there are objects $o_1...o_n$ such that $\phi(o_1...o_n)$. And as $\mathcal{L}$ contains sufficiently many constants, for any objects $o_1...o_n$ satisfying the formula, there are constants $c_1...c_n$, and therefore a sentence in $\mathcal{L}$ such that $\phi(c_1...c_n)$. Thus closed situations will be deemed sufficient.

5.1.2 Arithmeticality in First-Order Logic

In a sense, the constant-free fragment of FO logic implies a view of objects as mere "indicatables" or "existables". In interpretation, these correspond to "moments" from syntactic forms to arbitrary objects and not one from names to distinguished particulars. And therefore it comes as no surprise that in the case of FO formal mereologies, the quantifiers can be used to express rather complex general arithmetical relations and counting statements. For example, we have seen that the predicate $x + y = z$ (i.e. the fusion of x and y is identical to z) will be expressible in the first order language with relational vocabulary $\{\leq\}$ by:

$$y \leq z \wedge x \leq z \rightarrow \forall w(w \leq z \rightarrow \exists v(v \leq w \wedge v \leq x) \vee \exists u(u \leq w \wedge u \leq y)).$$

In this vocabulary the identity relation will be definable as $x \leq y \wedge y \leq x$. And for example, we will be able to express commutativity and associativity of fusion

$$\forall x \forall y \forall z (x + y = y + x)$$

$$\forall x \forall y \forall z ((x + y) + z = x + (y + z)).$$

These properties would ensure that situations have a sort of arithmetical "organization". This alone will of course be entirely acceptable for the nominalist. However, what will be unsatisfactory, as was claimed in the last section, is that commitment to the parameter itself will now be implied. The two formulas above express that all objects o_1, o_2 have feature $(x + y = y + x)$ and for any objects o_1, o_2, o_3, they have feature $((x + y) + z = x + (y + z))$. And this will imply that at any location of reality and for any objects o_1 and o_2, there is parameter

$$((+[x, y], (+[y, x])^-)^-, (+[y, x], (+[x, y])^-))^-$$

in which objects o_1 and o_2 can be substituted. That is, if one is at the fusion of x and y, then one is at the fusion of y and x and vice versa. And again the valid substitutability of the parameter by arbitrary individuals is nothing other than the principle of commutativity of addition. And for the other example, the same will hold of the parameter

$$((+[x, +[y, w]], (+[+[y, w], x])^-)^-, (+[+[y, w], x], (+[x, +[y, w]])^-)^-).$$

It is hardly surprising that arithmetical features arise in this way. It is well known that the relation $x = \bigoplus[y : \phi(y)]$, i.e. that x is a fusion of objects of any elementary set defined by $\phi(y)$ is expressible in the FO language as

$$\forall z(\phi(z) \rightarrow z \leq x) \wedge \forall w(w \leq x \rightarrow \exists v(v \leq w \wedge \phi(v))).$$

And the unrestricted fusion principle that every elementary property with a non-empty extension has a fusion is also expressible: $\exists x \phi(x) \rightarrow \exists x(x = \bigoplus[y : \phi(y)])$. It is well known that Tarski showed that the models of so called *extensional mereology* satisfying ME and the property of unrestricted fusions are essentially Boolean algebras (BAs)—the only exception being the existence of the bottom 0 (see [90]). Moreover all FO mereologies will contain counting sentences (e.g. "There are *exactly* n/*at least* n/*at most* n such that ϕ"). So for example, one can express the general arithmetical proposition that for any x, y and z, if x is the fusion of two disjoint objects, say y and z, where y is the fusion of exactly two atoms and z is the fusion of four, then x contains exactly six.

Thus reference to concrete objects by first-order quantified formulas will imply that the nominalist commit to the existence of parameterizations. And these will not be situated features, as defined earlier. They will imply the nominalistic acceptability of pure numerical distinctions. Although it may be viewed as a boon that the existence of parameterizations will imply the nominalistic legitimacy of asserting pure arithmetical principles, making the distinctions in this way will be, if not straightforwardly unacceptable, at least out of the purview of nominalistic mereology.

5.1.3 Upshot

The conclusions we must draw from the previous sections of this chapter are as follows. If FOL is adopted, then FO-open formulas must be rejected. Either they involve us in commitments to *ante rem* universals or they are superfluous and we should prefer closed formulas denoting various closed mereological situations. But even if we restrict our attention to theories of FO sentences, FO-quantified statements imply commitments to parametrized statements which we should reject. For they imply that relations besides that of parthood be independently existing. Indeed the FO language will deliver a great amount of arithmetical power. Even without arbitrary predicates, FO pure mereologies will allow us a great many arithmetical distinctions. It therefore stands to reason that the first-order language is nominalistically unacceptable.

5.2 The Question of Modal Mereological Languages

In many early formal mereologies of the nineteenth and twentieth centuries, selected languages were highly expressive. And expression of arithmetical and higher order relations and functions in a mereology was highly desirable. Indeed Leśniewski's ultimate goal was to supplant theories of sets with a mereological foundation for mathematics. On this view, there is no line whatsoever to be drawn between mathematical and mereological enquiry. Consequently, there will be no fundamental demarcation between the respective domain of entities postulated. This view basically flowed from his understanding of formal languages. For Leśniewski, languages were conceived as independently meaningful and physical. We will deal with this proposal in due course. However, our observations have indicated that logics, even the first-order language and its corresponding mereologic, will contain many abstract features.

Quine suggested, for example, that Leśniewski's use of higher-order variables committed him to platonic objects. On this score we have sided with Quine. Any formalize theory of physical science containing irreducible second-order quantifiers will import into its corresponding ontology a commitment to sets. Let us call any mereology with terms for sets or higher-order quantifiers a *set theoretic calculus of individuals* (SCI). So all SCIs modeling inferences over the domain of concrete entities are nominalistically unacceptable on the basis that higher-order quantifiers would range over sets of Urelements and terms for sets would denote impure sets.

However our stance is yet more conservative than Quine's. So naturally, we should be inclined to look into less expressive systems. In particular, since a compromise between nominalistic restrictions and mereological necessities must be met, the most natural way to proceed is to search for a nominalistically acceptable, proper *fragment* of the FO language. The formal methods of modal logic therefore present themselves as a viable means to acquire a suitable language and logic.

A natural alternative to standard modal languages which import a conception of property or proposition are hybrid logics. A principal motivation for hybrid logic is to add further expressive power to ordinary modal logic with the aim of being able to formalize statements containing reference to particular times and objects. This is achieved by adding to ordinary modal logic a second sort of propositional symbol called the *nominal*. Given a modified Kripke semantics each nominal is true relative to exactly one point. A natural language statement of the second kind like

it is 10 PM 1 March 2005

is then formalized using a nominal, not an ordinary propositional symbol which would be used to formalize a simple characterization like

it is raining.

The fact that a nominal is true at exactly one location implies that a nominal can be considered a term. For example, if i is a nominal that stands for "it is 10 PM 1 March 2005", then the nominal can be considered a term referring to the time 10 PM 1 March 2005.

Formal mereology obviously will require a language whose terms designate individuals. And therefore any modal-style mereology must contain just such a device, since we require a language with terms for individuals. So we shall select an improper fragment of nominal tense logic which contains nominals but without propositional variables.

On the view I propose, formal mereology is a branch of modern modal logic with a separate meta-theoretical semantics for situations and thus not an inscriptional theory of any kind. Mereological operators will be understood as representing parthood relations relative to organisms located within reality, but also as relations in closed situations relating non-local individuals. As required, the parthood operations will represent the parthood relation itself. And we will admit no other relations besides $\leq$.

Presently, all investigated modal mereologies are quasi-set-theoretic mereologies or *set-theoretic calculi of individuals* with the typical propositional variables $\{p, q, r...\}$ of standard modal languages which, over Kripke models, are capable of being true at various states or elements of models. Each represents either a subset of an implied domain of items or a multiply located entity of some sort. Consider Vakarelov [97], "A modal logic of set relations." As the title notes, the author investigates a modal logic whose atomic formulae range over subsets of the domain. For another example, Goranko and Vakarelov [46] introduce a language with a set-theoretic semantics interpreted over powerset algebras whose modalities represent set operations based on membership relations implicit in the Boolean operative notions. And Balbiani et al. [8] provide a modal logic based on membership modalities with a topological interpretation.[2]

Thus the difficulty using any of the currently proposed systems for nominalistic mereology is that they imply ontological commitments to sets or arbitrary unary properties. By selecting a standard modal system, standard modal valuation functions will map proposition symbols to *sets* of objects. So on the one hand, the SCIs currently studied will all import a conception of set into the corresponding ontology. On the other hand, even if we do not countenance a quasi-Kripke-model interpretation over situations, standard modal logics have syntactic elements that are capable of being true at multiple objects. In the present context of mereological situations, these modal mereologics would express that individuals in situations be capable of bearing multi-instantiatable properties or making true various propositions.

Other approaches extend the vocabulary of classical mereologics with relations that are not FO definable in terms of parthood. For instance, in the spirit of Whitehead's [100] original logic which defines parthood in terms of the contact relation, Nenov and Vakarelov [69] introduce a modal *mereotopology* in a language with both parthood and contact modalities. Similarly in Kontchakov et al. [51] the authors investigate spatial constraint languages with equality, contact and connectedness predicates, as well as Boolean operations on regions, interpreted over low-dimensional Euclidean spaces. However, the question of nominalism motivates an investigation into axiomatizations without any background topological and set-theoretic notions. Contact relations bring about rather immediately questions peculiar to space and topology.

[2] Admittedly, the Stone representation theorem will imply that any logic sound and complete with respect to arbitrary BAs will also be so with respect to its Boolean set algebra isomorph. Hence in the context of set-theoretic characterization theorems, extensional mereological models cannot be mathematically divorced from their corresponding Boolean set algebras.

5.2.1 A Hybrid Language for Mereology

Now we will introduce the selected language. On this approach, the ontology justifies the necessity and acceptability of logical devices. We require a way to express situations from a given one. But we must eliminate reference to all abstractions. And all devices standing for sets and multiply presented objects are to be eliminated.

General Syntax and Semantics

Here we will give the syntax and semantics of the language informally. Let Ω be a countable set $\{\mathcal{U}, i, j, k...\}$ of atomic formulae which we call *nominals*. $\mathcal{U}$ will be the nominal naming the fusion of all concrete particulars. And let d be the denotation function sending names to locations. Consider the following inductively defined language which we call $\mathcal{H}$:

$$\phi \ ::= \ \top \mid i \mid \neg\phi \mid \phi \wedge \psi \mid \langle\leq\rangle\phi \mid \langle\geq\rangle\phi \mid \langle\curlywedge\rangle\phi \mid \langle\curlyvee\rangle\phi \mid @_i\phi$$

Objects of the language will be evaluated at locations. $\top$ is to be interpreted simply as an arbitrary tautology. For the case of nominals

$$\ulcorner i \urcorner \text{ is true at location } l \Longleftrightarrow l = d(i).$$

The Boolean connectives are read in the standard way but also at locations. The operators $\langle\leq\rangle\phi$ and $\langle\geq\rangle\phi$ are straightforward parthood operators.

$$\ulcorner\langle\leq\rangle\phi\urcorner \text{ is true at } l \Longleftrightarrow \text{ there is a location } l' \quad l \leq l' \text{ and at } l', \ulcorner\phi\urcorner \text{ is true.}$$

$$\ulcorner\langle\geq\rangle\phi\urcorner \text{ is true at } l \Longleftrightarrow \text{ there is a location } l' \quad l \geq l' \text{ and at } l', \ulcorner\phi\urcorner \text{ is true.}$$

We uncover the technical reasons why in the next chapter, but for now note that the other two operators are defined in a subtle way.

$$\ulcorner\langle\nleq\rangle\phi\urcorner \text{ is true at } l \Longleftrightarrow \text{ there is a location } l' \quad l \nleq l' \text{ and at } l', \ulcorner\phi\urcorner \text{ is false.}$$

Relations	Expression	Abbreviation
Part	$x \leq y$	$x \leq y$
Proper Part	$x \leq y \wedge x \neq y$	$x < y$
Proper Extension	$y \leq x \wedge x \neq y$	$x > y$
Overlap	$\exists w(w \leq x \wedge w \leq y)$	xOy
Proper Overlap	$xOy \wedge \neg(x \leq y \vee x \geq y)$	$xPOy$
Disjoint	$\neg xOy$	Dy
Tarski Fusion	$x \leq z \wedge y \leq z \wedge \forall w(w \leq z \rightarrow wOx \vee wOy)$	$x + y = z$
Product	$\forall w(w \leq z \leftrightarrow w \leq x \wedge w \leq y)$	$x \times y = z$
Complement	$\forall w(w \leq x \leftrightarrow wDy)$	$x =\sim y$
Part	$\langle\leq\rangle i$	$\langle\leq\rangle i$
Proper Part	$\langle\leq\rangle i \wedge \neg i$	$\langle<\rangle i$
Proper Extension	$\langle\geq\rangle i \wedge \neg i$	$\langle>\rangle i$
Overlap	$\langle\geq\rangle\langle\leq\rangle i$	$\langle O\rangle i$
Proper Overlap	$\langle O\rangle i \wedge \neg(\langle\leq\rangle i \vee \langle\geq\rangle i)$	$\langle PO\rangle i$
Disjoint	$\neg\langle O\rangle i$	$\langle DR\rangle i$
Tarski Fusion	$\langle\geq\rangle i \wedge \langle\geq\rangle j \wedge [\geq](\langle O\rangle i \vee \langle O\rangle j)$	$\langle+\rangle(i, j)$
Product	$[\geq](\langle\leq\rangle i \wedge \langle\leq\rangle j) \wedge [\curlywedge](\langle\leq\rangle i \wedge \langle\leq\rangle j)$	$\langle\times\rangle(i, j)$
Complement	$[\geq]\langle DR\rangle i \wedge [\curlywedge]\langle DR\rangle i$	$\langle\sim\rangle i$

Table 5.1: Definitions of mereological relations and operators.

$$\ulcorner\langle\ngeq\rangle\phi\urcorner \text{ is true at } l \iff \text{ there is a location } l' \quad l \ngeq l' \text{ and at } l', \ulcorner\phi\urcorner \text{ is false.}$$

The @-operator will allow us to assess non-local situations:

$$\ulcorner @_i\phi\urcorner \text{ is true at } l \iff \text{ at } d(i) \quad \ulcorner\phi\urcorner \text{ is true.}$$

The ability to refer to individuals with a localized semantics permits the expression of possessive, mereological features. For example, if 'j' is the name for my arm and I am i, then the sentence '$i \wedge \langle\geq\rangle j$' is the expression that one of my parts is my arm.

The possessive term 'my arm' in this context is conceived as a name phrase. But if we wish to characterize the individual j as *an* arm or an object with the property of having an arm, then we will have to extend the language with various property variables over and above the nominals. And therefore the we must adopt two sorts of atomic symbols. One sort will be Ω and another will be the set Φ of properties $p, q, r, ...$. Thus if p is a property symbol in Φ, its truth conditions will be given by:

$$\ulcorner p \urcorner \text{ is true at } o \Longleftrightarrow \quad o \text{ has property } P$$

and will be understood as one in which the object at that location has property P. For example, to express that j is a part of me and moreover an object with the property a of being an arm would be '$\langle \geq \rangle (j \wedge a)$'. A language $\mathcal{H}^+$ obtained by adding property symbols to $\mathcal{H}$ is equivalent to a proper extension of the language of Arthur Prior's original hybrid tense logic [7] containing in addition to the latter just the final two converse modalities $\langle \curlyvee \rangle$, $\langle \curlywedge \rangle$.

Let R be any of the following relation symbols $\leq, \geq, \nleq, \ngeq$. If ϕ is a formula, we will abbreviate expressions of the form $\neg\langle R \rangle \neg \phi$ by use of the box as in $[R]\phi$. Thus for example, '$[\leq]\neg i$' will express that it is not the case that the given object is part of i—in other words, every extension of the occupied location is not one to which i is identical.

The Boolean connective expressions are straightforward, so consider the operator expressions. The first two state that the given object has a part or extension at which σ^ϕ, respectively. Finally, the other two are *converse operators* expressing that the given object has a non-part and a non-extension from the given one, respectively. For the final two operator types, the inner situation is negative, thus allowing us, for technical reasons, to make claims with fewer symbols. Converse operators will be required, since not all objects are accessible as extensions or parts from a given location. For some will only be accessible if there are various "lateral moves" to them. It can be easily checked that if $\ulcorner [\curlyvee]\phi \urcorner$ is true at location l then any situation such that σ^ϕ obtains at an extension of l. Likewise if $\ulcorner [\curlywedge]\phi \urcorner$ is true at location l, then any situation such that σ^ϕ obtains at a part of l. It is a rather lucky state of affairs that the language provides us a way to express all the important relationships in the theory of extensional mereology as operator expressions whose arguments refer to distinguished, existent individuals. These are provided in the table.

Situation Correspondence

We can define a rather tight correspondence between $\mathcal{H}$-formulas and mereological situations. Each atomic formula of $\mathcal{H}$ will be of the form i for some nominal i. We use the following satisfaction relation for each nominal i in $\mathcal{L}$:

$$\ulcorner i \urcorner \text{ is true at } o \Longleftrightarrow (d(i)) \text{ obtains at } o$$

For any $\mathcal{H}$-formula ϕ, this correspondence can be extended to a function σ sending every $\mathcal{H}$-formula to a mereological situation such that

$$\ulcorner \phi \urcorner \text{ is true at } o \text{ if and only if } \sigma^{\phi} \text{ obtains at } o.$$

Proposition 5.2.1 ($\mathcal{H}$/Situation Correspondence)**.** Let $\phi \in \mathcal{H}$. Then, $\ulcorner \phi \urcorner$ is true $\Longleftrightarrow$ there is a situation σ^{ϕ} such that σ^{ϕ} obtains at any location.

Proof. By induction on the complexity of ϕ. *Base Case.* Let i be any nominal. For any location l, $\ulcorner i \urcorner$ is true at $l \Longleftrightarrow (d(o))$ is true at l. So in this case let $\sigma^{i} = (d(o))$. *Inductive Step.* First, we do cases for conjunction and negation. ($\wedge$) Now $\ulcorner \phi \wedge \psi \urcorner$ is true $\Longleftrightarrow \ulcorner \phi \urcorner$ is true and $\ulcorner \psi \urcorner$ is true $\Longleftrightarrow \sigma^{\phi}$ and σ^{ψ} hold at any location [by the Inductive Hypothesis] $\Longleftrightarrow (\sigma^{\phi}, \sigma^{\psi})$ obtains at any location. ($\neg$) Now $\ulcorner \neg\phi \urcorner$ is true $\Longleftrightarrow \ulcorner \phi \urcorner$ is not true $\Longleftrightarrow \sigma^{\phi}$ obtains at no location [by the Inductive Hypothesis] $\Longleftrightarrow (\sigma^{\phi})^{-}$ obtains at any location. The operator cases are all analogous. So we do just one case. ($\langle\leq\rangle$) $\ulcorner \langle\leq\rangle\varphi \urcorner$ is true at $o \Longleftrightarrow o$ is a part of a location o' at which $\ulcorner \varphi \urcorner$ is true $\Longleftrightarrow o$ is a part of a location o' at which σ^{φ} obtains [by the Inductive Hypothesis] $\Longleftrightarrow\ \leq(\sigma^{\varphi})$ obtains at o. □

Let o be any location and let p be any animated or inanimate location. Given the correspondence above, the satisfaction relation for $\mathcal{H}$-formulas can be given as follows.

$$\begin{array}{lclcl}
\ulcorner i \urcorner & \text{is true at } o & \iff & (d(i)) & \text{obtains at } o \\
\ulcorner \neg\varphi \urcorner & \text{is true at } o & \iff & (\sigma^{\varphi})^{-} & \text{obtains at } o \\
\ulcorner \varphi \wedge \psi \urcorner & \text{is true at } o & \iff & (\sigma^{\varphi}, \sigma^{\psi}) & \text{obtains at } o \\
\ulcorner \langle \leq \rangle \varphi \urcorner & \text{is true at } o & \iff & \leq (\sigma^{\varphi}) & \text{obtains at } o \\
\ulcorner \langle \geq \rangle \varphi \urcorner & \text{is true at } o & \iff & \geq (\sigma^{\varphi}) & \text{obtains at } o \\
\ulcorner \langle \curlyvee \rangle \varphi \urcorner & \text{is true at } o & \iff & \nleq ((\sigma^{\varphi})^{-}) & \text{obtains at } o \\
\ulcorner \langle \curlywedge \rangle \varphi \urcorner & \text{is true at } o & \iff & \ngeq ((\sigma^{\varphi})^{-}) & \text{obtains at } o \\
\ulcorner @_i \varphi \urcorner & \text{is true at } p & \iff & \lceil d(i) : \sigma^{\varphi} \rceil & \text{obtains at } p
\end{array}$$

For any name 'i', $\ulcorner i \urcorner$ will be true at o if and only if $(d(i))$ obtains at i if and only if, by earlier definitions, $d(i) = o$. And this is a clear case of diachronic identity. For example, since localized situations obtain at first-personal objects only, in $\mathcal{H}$, one can claim "I am i". $\top$ will be definable as, for example, as $((i, (i)^{-}))^{-}$.

5.2.2 The Question of Infinitary Nominalistic Mereology

We have argued that the preferred language should be comprised of just finitely long formulas. But is this a mistake and should we allow them to be infinitely long? To my knowledge no infinitary mereology has ever been seriously proposed for its philosophical significance. And rarely is consideration ever given to infinitary mereologies.

Nonetheless, recall that we defined mereological situations in an open-ended way. If it turns out that there are infinitely many concrete objects, then we will be committed to infinitely large situations involving them, not least of which mereo-reality itself. Thus it is at least questionable whether, against tradition, we should admit them.

Any notion of a formula type with uncountably many symbols will certainly involve us in rather controversial set-theoretic hypotheses concerning their well-orderability. And this is one rather good reason for rejecting them. But problems with sequences of uncountably many symbols not only concern questions endemic to the theory of sets. They concern combined questions in theoretical geometry and set theory. For although we have subscribed to an abstract conception of symbols and formulas, both have geometrical structures. The former have particular shapes and the latter are linear sequences of the former. Obviously no *particular* distances will be involved in their internal structures since they are abstract. But symbols will nonetheless be closed under certain geometrical transformations variant with respect to size; and portions of symbols, for example, will bear sequential relations to others so that the entire structure of the symbol is preserved. Thus the question of their well-orderability will not be directly analogous to that of numbers or points.

Sentences which are of countably infinite length may be less worrisome. And although the first-order language was claimed to be nominalistically unacceptable, it might be claimed that the vocabulary of $\mathcal{H}$ can be used to provide an acceptable infinitary language. Consider the language $\mathcal{H}(\omega_1, \omega_1)$ which consists of all formulas in the vocabulary of $\mathcal{H}$ allowing infinite conjunctions, infinite disjunctions and infinitely many operators of the type defined in the preceding section.

$\mathcal{H}(\omega_1, \omega_1)$ will have some nominalistically acceptable features. For example, as in $\mathcal{H}$ it will not involve commitments to parameterizations of situations. And all references will be made by name and not by way of variables or properties. However $\mathcal{H}(\omega_1, \omega_1)$ will also have very strong arithmetical properties. For example, the standard model of arithmetic $\mathbb{N} = (N, +, \times, s, 0)$ will not be characterizable. However we can characterize a distinguished, or in other words, fully named restriction $\mathbb{N}^- = (N, +, -, 0, 1, 2, ...)$ of $\mathbb{N}$

$$\begin{array}{l}
\bigwedge_{m\in\omega} \left(\bigwedge_{n\in\omega} \left(@_{i_{m+n}}\langle+\rangle(i_m, i_n)\right)\right) \\
\bigwedge_{m\in\omega} \left(\bigwedge_{n\in\omega} \left(@_{i_{m-n}}\langle-\rangle(i_m, i_n)\right)\right) \\
\bigwedge_{m\in\omega} \left(\bigwedge_{n\in\omega-\{m\}} \neg @_{i_m} i_n\right) \\
\bigvee_{m\in\omega} i_m
\end{array}$$

where we let $n = i_n^{\mathbb{N}^-}$. And therefore, despite reality's being finite or infinite, if $\mathcal{H}(\omega_1, \omega_1)$ is adopted, even within reality we will be able to express that there is a sequence of its objects which is isomorphic to this arithmetical structure. Adopting $\mathcal{H}(\omega_1, \omega_1)$ may as well incite worries with regard to accepting any infinite vocabulary. However, if reality is finite, then such expressivity would seem superfluous.

In addition, $\mathcal{H}(\omega_1, \omega_1)$ contains a sort of weak second-order quantifier. For example consider the generalized quantifier language $\mathcal{L}(Q_0)$ obtained from $\mathcal{L}$ by adding a new quantifier symbol Q_0 and interpreting $Q_0 x \phi(x)$ as *there exist infinitely many x such that $\phi(x)$*. Now let $Q_0\phi(\overline{c})$ be a version of this quantifier expressing that there exist infinitely many named objects $\overline{c}$. To express this:

$$\bigwedge_{m\in\omega}\left(\bigwedge_{n\in\omega-\{m\}}\neg @_{i_m} i_n\right) \wedge \neg \bigvee_{n\in\omega}(\phi \to i_0 \vee ... \vee i_n)$$

Thus if second-order quantifiers of this sort indicate to our nominalist a capability that exceeds that of a mereological system in the envisaged sense, then infinitary languages will be unacceptable. However there is yet another reason why we might not want to resort to an infinitary language.

Arguably, there will be at least a preference for finite specifications. Firstly, it is *prima facie* true that there are finitely many objects in reality—or at least this will be the case if nominalism is true. However, it is entirely unclear that there are infinitely many. So in the spirit of conserving the number of assumptions and unnecessary hypotheses, we should prefer a finitary language.

Moreover, there will be a certain respect the nominalist has for modeling finite portions of reality. She, as a formal mereologist no doubt, assumes that she can model reality formally. However for the same reason that she does not wish to postulate supernatural entities, she will also prefer to restrict what she models to what she has observed. And this will clearly be the case for the nominalist espousing naturalistic nominalism and, in particular, the closure thesis.

Still, from a purely formal perspective it may be quite interesting to investigate formally infinitary versions of the hybrid mereological language. But this is obviously for another occasion. We will henceforth note this as an additional avenue of research, but one which displays comparatively more infelicities than finitary approaches.

5.2.3 Mereobisimulation and Nominalistic Mereology

What is nominalistic mereology? And what are its limitations? To answer this question, we entertain first the following example. Suppose that there are finitely

many real objects. Then theoretically these can be named. And then there is what we might call a *finite diagram* of reality consisting of the set of $\mathcal{H}$-formulas of the form $@_x\langle\leq\rangle y$ where $x, y \in \Omega$: $\{@_i\langle\leq\rangle j, @_k\langle\leq\rangle l...\}$. However it stands to reason that there may be individuals which are unnamed. For example, from a practical perspective, there are sure to be objects of which no one is aware. Or for another example, if there are uncountably many objects, yet only countably many names, uncountably objects will be unnamed. Consider the two finite situations σ_1 and σ_2 shown above (reflexive and transitive loops have been omitted). Some objects in both are unnamed.

σ_1 on the left has fewer objects than σ_2 on the right. And there are more unnamed objects in σ_2 than there are in σ_1. Can we furnish a $\mathcal{H}$-formula which distinguishes these situations? Or in other words, is there an $\mathcal{H}_\mathsf{m}$ formula which is true in one situation and false in the other? Actually, it can be shown that we cannot; and specifically why this is the case will be described in the following chapters. Indeed, the set of $\mathcal{H}$-formulas true of σ_1 is exactly the set true on σ_2. That is, there will be a fair amount of structure that $\mathcal{H}$ "misses" or "fails to detect". And we shall see in the final technical chapter that over infinitely complex situations $\mathcal{H}$ will fail radically in this regard. However in the FO language $\mathcal{L}$ it is easy to show that there are various sentences that are true of σ_1 but untrue of σ_2 and vice versa.

However, what is particularly interesting and important about this is that by showing the indistinguishability, we essentially show the limitations of nominalistic mereology. To explain why, let us back up and describe our tack up to this point. Our goal in the early chapters of the dissertation was to provide a maximally nominalistic ontology. And we settled on a rather conservative selection of primitive entities which were both nominalistically acceptable and capable of capturing mereological distinctions. And essentially, $\mathcal{H}$ was selected as a maximally acceptable way to restrict our ontological commitments to just mereological situations and the individuals involved in them.

The manner of proving that two situations above are $\mathcal{H}$-indistinguishable relies on showing that there exists a certain type of relation R between the objects in them. For example, j in σ_1 will be mapped just to j in σ_2 and vice versa. Likewise for $\mathcal{U}$ and k. Intuitively, if we wish to show two models are structurally similar in every way, the other points must also be linked. And in order to show structural identity, we must show that there are one-to-one links which map all objects to others preserving any of the various relations in the models. But in general, varying degrees of structural similarity will be required for various maps.

These relations—so-called *invariance morphisms*—are of fundamental importance in all types of mathematics. The notion of homomorphism is one such relation. A homomorphism is a relational preserving function amid models. Homomorphisms preserve varying degrees of structure and can be shown also to preserve the truth of certain formulas of various formal languages amid models. Yet stronger than homomorphism is isomorphism which preserves all relational structure. Indeed isomorphism preserves the truth of formulas of all higher-logical languages. For example, if there is an isomorphism amid two structures, then every formula of first and second order logic that is true of one is true of the other, and vice versa.

One rather interesting fact with regard to FOL is that, despite its high degree of expressivity, it cannot capture all models up to isomorphism. Now a set of sentences in FOL can be used diagram *finite* models up to isomorphism. But one and the same FO-theory with at least one infinite model may have infinitely many structurally distinct infinite models. For example, if we formalize general extensional mereology in FOL, then it can be shown formally (and will be in the following chapters) that FOL does not capture up to isomorphism unrestrictedly fused infinite structures. The reason is as follows: although there does not exist a countably infinite unrestrictedly fused model for general extensional mereology, any first-order theory will have countable models (Koppelberg (2006)[52]) by the downward Löwenheim Skolem theorem. Thus theories of first-order logic never capture uncountable structures up-to-isomorphism.

Now suppose our models accurately represent the organization of parthood instances in physical reality. We will return to this assumption in chapter VII, however now note the following important observation. In a certain sense, if we are committed to capturing all fine structure of locations and there are infinite locations, we will *a fortiori* be committed to the idea that even FOL is expressively deficient to represent the mereological structure of reality.

I therefore believe using invariance morphisms to test the strength of formal languages provides a good way of assessing their limitations. Given all that has been said about our reasons for selecting $\mathcal{H}$, we should view the $\mathcal{H}$-invariance morphism or so-called *mereobisimulation* as encapsulating the nominalistic, mereological criteria by which situations can be distinguished. But as we shall see mereo-bisimulation is, in a certain respect, weaker than homomorphisms and of course far weaker than isomorphism.

The notion of mereobismulation will also be used to determine the fragment of FOL to which $\mathcal{H}$ is equivalent. It is rather clear that FOL and even the FO language $\mathcal{L}$ (defined earlier in the chapter) are far stronger than $\mathcal{H}$. As the mereological operators and the @-operator are based on rather clear first-order quantifications, it will be easy to show that $\mathcal{H}$ can be translated into $\mathcal{L}$. And in the latter half of the next chapter, we show that if two mathematical models are related by a mereobisimulation, then they are indistinguishable. Conversely, we show that if they are indistinguishable by $\mathcal{H}$-formulas, then there is a mereo-bismulation. Putting the two results together will imply that we can specify precisely the nominalistically acceptable mereological fragment of first-order logic.

5.2.4 Empty Names, Quantifiers, and Variables

A nominalist will typically try to eschew referents of empty terms, for example, by interpreting them as mere syntactic marks or improper descriptions. The less nominalistically inclined may be content with construing these as abstractions of various sorts, like sets or Fregean senses. Research into the *meaning* of empty names is a highly developed area in the philosophy of language and metaphysics. Hence the ontological status of what (if anything) we mean and refer to with empty names is left for a different discussion. Formally, however, we must have an explanation for how to interpret the expressions in our own system, and we turn now to this issue.

According to tradition, there are two ways to understand the first-order expression "$\exists xFx$". To understand the quantifier *substitutionally* is to understand it as true if and only if there is some name n which can be substituted for the "x" in "Fx" such that the resulting sentence Fn is true. This contrasts with the typical *objectual* understanding, according to which it is true if and only if there is some *object* which is F. Therefore, if the only objects which are F lack names, then "$\exists xFx$" will be true via the objectual reading and false on the substitutional.

Recall Leśniewski's construal of quantification. His interpretation of the quantifier is non-referential and therefore neither substitutional nor objectual. The first-order expression "$\exists xFx$" is to be read "something Fs" or "there is (exists) an Fer", and "$\forall xFx$" is to be read roughly as "all (existent) things are Fers". We may refer to plurals in the same manner: "$\exists pFp$" which would then mean "some thing is an Fer or some things are Fers". Both types of quantification imply the existence of the quantified item(s). Hence if empty terms are admitted in the language, inferences of the type $Fq \vdash \exists pFp$ are unsound. However, according to *Mereology* such inferences are sound. His style of quantification is ontologically uncommitting in the sense that quantifying over variables of any type does not commit one in anyway to a corresponding, actually existent referent. So for *Mereology* the question concerning empty names does not appear. There is no antecedent demand for reference.

There are various problems and deficiencies with both understandings of the quantifier. Firstly, the advocate of the substitutional quantification has the trouble of determining the referents and meanings of free variables and for explaining how existence demands terms standing for individuals. Secondly, Leśniewski's understanding of quantification is simply too difficult to comprehend, since his existential quantifier—if we can even call it that—makes no claim on the world. He wishes to use the existential quantifier but in some sense also to deflate its meaning. Finally, the objectual reading requires that we view our eventual system or calculus of arbitrary objects as ontologically committing, even though, against what we claim, we may not wish to understand the formal language in this strict way.

Which interpretation is to be preferred for our formal mereology? It seems, given this perplexity, our intuitions would prefer the objectual reading. Still, this theory of quantifiers is also not suitable for hypothetical exploration in metaphysics. There's no mystery here. Hypothetical contexts like these abound. For example, mathematicians often experiment with different axiomatic systems of set-theory to determine how the truth of the same statement in each might differ. Likewise, physicists use different models with various, underlying axiomatic bases and languages.

Our view will be that there are two separate contexts within which to view the language. The first is a hypothetical or experimental one in which we may test the strength of various theories of $\mathcal{H}$ with a model-theoretic interpretation of the formulas. However, the language will also be understood as first-personally employable within a physical reality.

Chapter 6

$\mathcal{H}_{\mathsf{m}}$ and $\mathcal{H}_{\mathsf{o}}$

We now begin a formal analysis of two languages $\mathcal{H}_{\mathsf{m}}$ and $\mathcal{H}_{\mathsf{o}}$. The first will contain only nominals and the second will also contain propositional symbols denoting properties of individuals. They differ from $\mathcal{H}$ and $\mathcal{H}^{+}$ described in the previous chapter in only one respect. Both contain, in addition to the items introduced previously, an atom constant α. The atom constant is, as its name suggests, a constant atomic formula true at location if and only if that location is an atom. The notions of both material atom and an atomic region are of obvious importance to mereology since the time of Democritus. And as we shall see in the next chapter, the property is of fundamental importance in the context of the ontology of space. With α we shall wish to distinguish a point—a type of atom which is a zero-dimensional region—from an extended region. Its addition is therefore formally motivated. With the atom constant, atomic models as well as atomless ones shall be definable. And we shall be able to say much more about the limitations of mereology with the α than without.

The property of being an atom is definable in the first-order language with relational signature $\{\leq\}$ and in many others. However, the reason why it is explicitly added here is that in $\mathcal{H}$ the property is not definable in general over unnamed objects. Only on a case-by-case basis and *per* distinguished individual can it be expressed in $\mathcal{H}$: $@_i[\geq]i$. Although we shall tackle all formal results for languages with α, those applicable to $\mathcal{H}$ and $\mathcal{H}^{+}$ (and there shall be many) will be simple restrictions of those we demonstrate. Thus if there is a worry with regard to the symbol implying anti-nominalistic ontological commitments, in the results below it can be ignored.

In $\mathcal{H}_{\mathsf{m}}$ we demonstrate also that the important Boolean algebraic operators for suprema, infima, and complements are expressible. The reason to investigate Boolean algebras (BAs) and Boolean algebraic notions concerns the mereological structure of locations. As indicated previously, models for general extensional mereology and Boolean algebras differ only due to the existence of a bottom element 0. However, as Tarski showed, all other structural features are preserved. BAs and FO theories of BAs have been well-investigated over the last century. Thus in order to demonstrate the expressive power of our own system, it suffices to consider just BAs. Still we provide axiomatizations and expressivity results for both BAs and GEMS structures.

Boolean algebras are typically considered the most appropriate model-type to investigate the decompositional structure of space. From the classical perspective, regions are suprema of *sets* of points in types set BAs. Even traditionally, the class of regions is typically seen as extensional, despite whether points or extended entities are deemed primitive. And we have seen that a region, if they exist independently of matter, must figure as a type of closing substance. They are therefore extensional. Any two regions with the same parts are identical. As BAs are extensional in this respect, this is another reason why we shall study them.

We show that the classes for BAs and GEMSs are definable in both $\mathcal{H}_{\mathsf{m}}$ and $\mathcal{H}_{\mathsf{o}}$, as are the atomic and atomless frames. In $\mathcal{H}_{\mathsf{o}}$ even the class of Boolean complete and unrestrictedly fused structures is definable. This marks a difference between the FOL and $\mathcal{H}_{\mathsf{o}}$ that we note: there is no first-order formula defining this class of models. We show that by extending the standard naming axioms of hybrid logic with various others that we can obtain complete axiom systems for varieties of BAs. Showing axiomatic completeness is not "business as usual." A different Henkin model construction is required to build *fully named* models in our selected languages.

In the final sections, we investigate the two invariance notions for $\mathcal{H}_{\mathsf{m}}$ and $\mathcal{H}_{\mathsf{o}}$. We enquire especially into mereo-bisimulations. By demonstrating the existence of mereo-bisimulations amid models, we show that in various proper fragments of $\mathcal{H}_{\mathsf{m}}$ standard mereological relationships are inexpressible. On the other hand, it is proved that over BAs and GEMSs, properties requiring counting are inexpressible. This is rather important, since we one of our main motivations was to eliminate arithmetical principles and relations. And therefore by removing the ability to count, arithmetical properties are avoided. Finally, we provide a characterization of the mereological fragment of first-order logic; that is, we show which formulas are equivalent to the first-order translation of $\mathcal{H}_{\mathsf{m}}$ and $\mathcal{H}_{\mathsf{o}}$-formulas.

6.1 Syntax and Semantics

Let us begin a formal explanation of our simplest mereological language $\mathcal{H}_{\mathsf{m}}$. Let $\Omega = \{\mathbf{0}, \mathbf{1}, i, j, k, ...\}$ be a countably infinite set of atomic formulas. The members of Ω are called *nominals*. In $\mathcal{H}_{\mathsf{m}}$, Ω is our only set of atomic formulae. We define $\mathcal{H}_{\mathsf{m}}$ by the recursive definition:

$$\phi \ ::= \ \top \mid i \mid \alpha \mid \neg\phi \mid \phi \wedge \psi \mid [\leq]\phi \mid [\geq]\phi \mid [\curlywedge]\phi \mid [\curlyvee]\phi \mid @_i\phi.$$

1 and **0** are two important nominals. Over the models which we shall study, they will be given denotations appearing the in first-order signature. We shall call the symbol α the *atom symbol* or *atom constant*.

$\mathcal{H}_{\mathsf{o}}$ has only one additional symbol-type. Let $\Phi = \{p, q, r, ...\}$ be a countably infinite set of atomic formulas. The members of Φ are called *propositional variables* or *property symbols*. In $\mathcal{H}_{\mathsf{o}}$ the atomic formulas are either nominals, the atom constant, or proposition symbols. We define $\mathcal{H}_{\mathsf{o}}$ by the recursive definition:

$$\phi \ ::= \ \top \mid i \mid p \mid \alpha \mid \neg\phi \mid \phi \wedge \psi \mid [\leq]\phi \mid [\geq]\phi \mid [\curlywedge]\phi \mid [\curlyvee]\phi \mid @_i\phi.$$

We call $[\leq]$ the extension operator, and $[\geq]$ the part operator. $[\curlyvee]$ and $[\curlywedge]$ are inverses of $[\leq]$ and $[\geq]$, respectively. If a formula does not contain proposition symbols, we call it *pure*. Thus note that each $\mathcal{H}_{\mathsf{m}}$-formula is pure. Both languages are interpreted over models whose frames are of the Boolean type $(W, \leq, 1, 0)$. And they can be simplified to ones interpreted over models whose frames are of GEMS type $(W, \leq, 1)$.

Definition 6.1.1 (Models of Mereological Type)**.** We call a model $\mathcal{M} = (\mathcal{F}, V)$ *of* *mereological type* if satisfies the following two conditions: (1) $\mathcal{F}$ is either of type $(W, \leq, 1, 0)$ or type $(W, \leq, 1)$ where W is a nonempty set, $\leq$ is a binary relation and (2) V is a either hybrid or pure hybrid valuation. V is a *pure hybrid valuation* if it is a function with domain Ω such that for all $i \in \Omega$, $V(i)$ is a singleton subset of the domain. V is a hybrid valuation if it is a function with domain $\Omega \cup \Phi$ such that for all $i \in \Omega$, $V(i)$ is a singleton subset of the domain and for all $p \in \Phi$, $V(p)$ is a subset of the domain.

Definition 6.1.2 (BA and BO Models)**.** A *hybrid Boolean algebra model* (or *BA-model* for short) is a 5-tuple $(W, \leq, 1, 0, V)$ where $(W, \leq, 1, 0)$ is a BA and V is a pure hybrid valuation. A *hybrid Boolean ontological model* (or *BO-model* for short) is also a 5-tuple $(W, \leq, 1, 0, V)$ where $(W, \leq, 1, 0)$ is a BA and V is a hybrid valuation.

Definition 6.1.3 (GEM and GEO Models)**.** A *GEMS* is a 0 deleted BA $(W, \leq, 1)$. A *hybrid General Extensional Mereological model* (or *GEM*-model for short) is a 4-tuple $(W, \leq, 1, V)$ where $(W, \leq, 1)$ is a GEMS and V is a pure hybrid valuation. A *hybrid General extensional ontological model* (or *GEO-model* for short) is also a 4-tuple $(W, \leq, 1, V)$ where $(W, \leq, 1)$ is a GEMS and V is a hybrid valuation.

For the FO formula $w \neq 0 \wedge \forall v((v \leq w \wedge v \neq 0) \rightarrow v = w)$, which over BA-models is true if and only if w is an atom, we write $At(w)$. Over GEM-models, a slightly different formulation is required: $\forall v(v \leq w \rightarrow v = w)$.

6.1.1 Truth Conditions over Models

Definition 6.1.4 (Truth in $\mathcal{H}_{\mathsf{m}}$)**.** Suppose $\mathcal{M} = (\mathcal{F}, V)$ is model of the Boolean type, V is a pure hybrid valuation, and $w \in W$. Then the satisfaction relation for

$\mathcal{H}_{\mathsf{m}}$ formulas is defined as follows:

$$
\begin{array}{lcl}
\mathcal{M}, w \models \top & \Longleftrightarrow & w = w \\
\mathcal{M}, w \models \mathbf{1} & \Longleftrightarrow & w = 1 \\
\mathcal{M}, w \models \mathbf{0} & \Longleftrightarrow & w = 0 \\
\mathcal{M}, w \models i & \Longleftrightarrow & \{w\} = V(i) \text{ where } i \in \Omega \\
\mathcal{M}, w \models \alpha & \Longleftrightarrow & At(w) \\
\mathcal{M}, w \models \neg\varphi & \Longleftrightarrow & \mathcal{M}, w \not\models \varphi \\
\mathcal{M}, w \models \varphi \wedge \psi & \Longleftrightarrow & \mathcal{M}, w \models \varphi \wedge \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models [\leq]\varphi & \Longleftrightarrow & \forall v \in W \ w \leq v \Rightarrow \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models [\geq]\varphi & \Longleftrightarrow & \forall v \in W \ v \leq w \Rightarrow \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models [\curlyvee]\varphi & \Longleftrightarrow & \forall v \in W \ \mathcal{M}, v \models \varphi \Rightarrow w \leq v \\
\mathcal{M}, w \models [\curlywedge]\varphi & \Longleftrightarrow & \forall v \in W \ \mathcal{M}, v \models \varphi \Rightarrow v \leq w \\
\mathcal{M}, w \models @_{\mathbf{1}}\varphi & \Longleftrightarrow & \exists v \in W \ v = 1 \wedge \ \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models @_{\mathbf{0}}\varphi & \Longleftrightarrow & \exists v \in W \ v = 0 \wedge \ \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models @_{i}\varphi & \Longleftrightarrow & \exists v \in W \ \mathcal{M}, v \models \varphi \wedge \{v\} = V(i)
\end{array}
$$

A semantics for models whose frames are of the signature $(W, \leq, 1)$ (like GEM and GEO-models) can be simplified, since semantic rules for $\mathbf{0}$ and $@_{\mathbf{0}}\phi$ are not required. Note also that over models whose frames are of the signature $(W, \leq, 1)$, the atom constant will be given the appropriate formulation:

$$\mathcal{M}, w \models \alpha \Longleftrightarrow \forall v (v \leq w \rightarrow v = w).$$

The box operator expressions are the "simplest" dominance quantifications over models of the mereological type. This is best seen by a reflection on the dual, diamond formulation of the operators:

$$
\begin{array}{lcl}
\mathcal{M}, w \models \langle\leq\rangle\varphi & \Longleftrightarrow & \exists v \in W \ w \leq v \wedge \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models \langle\geq\rangle\varphi & \Longleftrightarrow & \exists v \in W \ v \leq w \wedge \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models \langle\curlyvee\rangle\varphi & \Longleftrightarrow & \exists v \in W \ w \not\leq v \wedge \mathcal{M}, v \not\models \varphi \\
\mathcal{M}, w \models \langle\curlywedge\rangle\varphi & \Longleftrightarrow & \exists v \in W \ v \not\leq w \wedge \mathcal{M}, v \not\models \varphi
\end{array}
$$

We choose these additional modalities in part on the basis of their simplicity. But we also select operators like them out of necessity. In the latter sections of the paper, it will be proved that the inverse operators are not definable by the others.

Definition 6.1.5 (Truth in $\mathcal{H}_{\mathsf{o}}$)**.** Suppose $\mathcal{M} = (\mathcal{F}, V)$ is a model of the mereological type, V is a hybrid valuation, and $w \in W$. Then the satisfaction relation of $\mathcal{H}_{\mathsf{o}}$-formulas is defined as in Definition 4, but with just the additional clause:

$$\mathcal{M}, w \models p \Longleftrightarrow w \in V(p) \text{ where } p \in \Phi.$$

6.1.2 First-order Translation

Given a particular interpretation, we can translate $\mathcal{H}_{\mathsf{o}}$-formulas into classical ones for the purpose of formal evaluation. We conveniently identify the nominals of hybrid logic with the constants of the first-order correspondence language.

Definition 6.1.6 (FO Translation)**.** The first-order translation of $\mathcal{H}_{\mathsf{m}}/\mathcal{H}_{\mathsf{o}}$-formulas is given in the following, where $i \in \Omega$ and $p \in \Phi$:

$$\begin{array}{lcl}
ST_x(\top) & = & x = x \\
ST_x(i) & = & i = x \\
ST_x(p) & = & Px \\
ST_x(\alpha) & = & x \neq 0 \wedge \forall y((y \leq x \wedge y \neq 0) \rightarrow y = x) \\
ST_x(\neg\varphi) & = & \neg ST_x(\varphi) \\
ST_x(\varphi \wedge \psi) & = & ST_x(\varphi) \wedge ST_x(\psi) \\
ST_x([\leq]\varphi) & = & \forall y(x \leq y \rightarrow ST_y(\varphi)) \\
ST_x([\geq]\varphi) & = & \forall y(y \leq x \rightarrow ST_y(\varphi)) \\
ST_x([\curlyvee]\varphi) & = & \forall y(ST_y(\varphi) \rightarrow x \leq y) \\
ST_x([\curlywedge]\varphi) & = & \forall y(ST_y(\varphi) \rightarrow y \leq x) \\
ST_x(@_i\varphi) & = & \exists y(y = i \wedge ST_y(\varphi))
\end{array}$$

where y is a variable that has not been used so far in the translation.

The translation serves us formally given a model-theoretic interpretation. But it is not the final analysis of our hybrid languages. It merely gives us a means of comparing the strength of our languages against the language of first order logic. The translation above is suitable for BA-models. However, for GEM-models the translation clause for α would be

$$ST_x(\alpha) = \forall y(y \leq x \rightarrow y = x)$$

We say φ is valid on $\mathcal{M}$ and write $\mathcal{M} \models \varphi$ if and only if $\mathcal{M}, w \models \varphi$ for every state w of $\mathcal{M}$.

Proposition 6.1.7 (Local and Global Correspondence on Models)**.** For all formulas φ of $\mathcal{H}_{\mathsf{m}}/\mathcal{H}_{\mathsf{o}}$, hybrid models $\mathcal{M}$, states $w \in M$

$$\begin{array}{ll} (i) & \mathcal{M}, w \models \varphi \Longleftrightarrow \mathcal{M} \models ST_x(\varphi)[w] \\ (ii) & \mathcal{M} \models \varphi \Longleftrightarrow \mathcal{M} \models \forall x ST_x(\varphi). \end{array}$$

Proof. (i) An easy induction on the complexity of ϕ. (ii) An easy consequence of (i) □

6.1.3 Logical Expressions

Consider the well-known existential operator E of extended modal languages whose truth conditions are given by $\mathcal{M}, w \models E\phi \Longleftrightarrow \exists v \in |\mathcal{M}|\ \mathcal{M}, v \models \phi$. The dual A of E has truth conditions $\mathcal{M}, w \models A\phi \Longleftrightarrow \forall v \in |\mathcal{M}|\ \mathcal{M}, v \models \phi$.

Proposition 6.1.8 (Existential Operator Definability)**.** The existence modality is definable in $\mathcal{H}_{\mathsf{m}}$.

Proof. The formula $\langle \leq \rangle \phi \vee \langle \Upsilon \rangle \neg \phi$ defines the existence modality. □

A model is *named* if every state in the model is the denotation of some nominal (i.e. $\forall w \in W$, $\exists i \in \Omega$ where $V(i) = \{w\}$).

Proposition 6.1.9 (Diagram)**.** Finite named models whose frames are in the FO signature $(\leq, c_1 ... c_n)$ (i.e. with a single binary relation and finitely many constants) are diagrammable in $\mathcal{H}_{\mathsf{m}}$ by a single sentence.

Proof. Let $\mathcal{M} = (\mathcal{F}, V)$ be any finite named pure hybrid model such that $\mathcal{F}$ is a frame in the signature $(\leq, c_1...c_n)$. $\mathcal{M}$ has say $m \geq n$ elements which are denotations of nominals $i_1, ..., i_k$ where $k \geq m$. Without loss of generality, let $\{V(i_1), ..., V(i_m)\} = |\mathcal{M}|$. Formulas of the form

$$A\left(\bigvee_{x=1}^{m} i_x\right) \wedge \left(\bigwedge_{1\leq l \neq h \leq m} \neg @_{i_l} i_h\right) \wedge (@_{i_v} j_1 \wedge ... \wedge @_{i_w} j_{k-m})$$

express that there are distinct named objects $i_1...i_m$ such that there are maximally $k-m$ of them $i_v...i_w$ with multiple names. The relation $\leq^{\mathcal{F}}$ can be diagrammed in an obvious way as a conjunction of sentences of the form $@_{i_x}\langle\leq\rangle i_y \iff V(i_x) \leq V(i_y)$ and $@_{i_x}\langle\not\leq\rangle i_y \iff V(i_x) \leq V(i_y)$. □

6.1.4 Mereological and Boolean Relationships in $\mathcal{H}_{\mathsf{m}}$

Boolean Algebraic Relations

Below are the canonical, first-order formulations of the important expressions of mereological relationships over Boolean algebras in the dominance signature.

Overlap	$\exists w(w \neq 0 \wedge w \leq x \wedge w \leq y)$	xOy
Proper Overlap	$xOy \wedge \neg(x \leq y \vee x \geq y)$	$xPOy$
Disjoint	$\neg xOy$	$xDRy$
Supremum	$x \leq z \wedge y \leq z \wedge \forall w(x \leq w \wedge y \leq w \rightarrow z \leq w)$	$x \vee y = z$
Infimum	$z \leq x \wedge z \leq y \wedge \forall w(w \leq x \wedge w \leq y \rightarrow w \leq z)$	$x \wedge y = z$
Complement	$\forall z(x \vee y = z \rightarrow z = 1) \wedge \forall z(x \wedge y = z \rightarrow z = 0)$	$x =\sim y$

Note that the traditional rendering of the overlap relation $\exists z(z \leq x \wedge z \leq y)$ will not capture the intended meaning over BAs. Interpreting the relationship in the traditional way would imply $\forall x \forall y(xOy)$, as every object dominates 0. It would also appear natural (especially in the context of physical objects) that 0 would have no representation in space, despite how centrally it figures in spatial *reasoning*. Accordingly, we can meta-theoretically analyze the notions of part, proper part, and proper extension in BAs:

Part	$x \leq y \wedge x \neq 0$	xPy
Proper Part	$x \leq y \wedge x \neq y \wedge x \neq 0$	$xPPy$
Proper Extension	$y \leq x \wedge x \neq y \wedge y \neq 0$	$xPEy$

Over BAs, this motivates a meta-analysis of the parthood relation as a subset of the dominance relation. Let $(W, \leq, 1, 0)$ be a BA. Parthood would then be a subset of the dominance relation: $\leq -\{(0, v) \mid v \in W\}$.

Proposition 6.1.10 (Definability of BA Operators)**.** The Boolean algebraic interpretations of the relationships of overlap, proper overlap, disjointness, supremum, infimum, and complements can be given modal operator interpretations in $\mathcal{H}_{\mathsf{m}}$ and $\mathcal{H}_{\mathsf{o}}$ over nominals.

Proof. We do the case for the overlap, least upper bound, and complement operators only. The other definitions are found in Table 6.1.4. Let $\mathcal{M}$ be any hybrid BA-model and w a state in $\mathcal{M}$. We first show that $\langle\geq\rangle(\neg\mathbf{0} \wedge \langle\leq\rangle i)$ defines the overlap operator.

$$\mathcal{M}, w \models \langle\geq\rangle(\neg\mathbf{0} \wedge \langle\leq\rangle i) \iff \exists v \in W\ (v \leq w \wedge w \neq 0 \wedge v \leq i).$$

We next show that $\langle\geq\rangle i \wedge \langle\geq\rangle j \wedge [\curlyvee](\langle\geq\rangle i \wedge \langle\geq\rangle j)$ defines a supremum operator with two arguments for nominals i and j.

$$\begin{aligned}\mathcal{M}, w \models \langle\geq\rangle i \wedge \langle\geq\rangle j \wedge [\curlyvee](\langle\geq\rangle i \wedge \langle\geq\rangle j) \iff\ & \mathcal{M}, w \models \langle\geq\rangle i \wedge \langle\geq\rangle j \\ & \wedge \mathcal{M}, w \models [\curlyvee](\langle\geq\rangle i \wedge \langle\geq\rangle j)\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}, w \models \langle\geq\rangle i \wedge \langle\geq\rangle j &\iff j \leq w \wedge i \leq w \\ \mathcal{M}, w \models [\curlyvee](\langle\geq\rangle i \wedge \langle\geq\rangle j) &\iff \forall v \in W(\mathcal{M}, v \models (\langle\geq\rangle i \wedge \langle\geq\rangle j) \Longrightarrow w \leq v) \\ &\iff \forall v \in W(i \leq v \wedge j \leq v \Longrightarrow w \leq v)\end{aligned}$$

as required. The complement operator is defined by

$$(\neg(\mathbf{0} \vee \mathbf{1}) \rightarrow ([\geq]\langle DR\rangle j \wedge [\curlywedge]\langle DR\rangle j)) \wedge (\mathbf{0} \leftrightarrow @_j \mathbf{1}) \wedge (\mathbf{1} \leftrightarrow @_j \mathbf{0}).$$

Observe that the three conjuncts represent the three important cases. **1** and **0** are complements of each other; thus the final two conjuncts are sufficient. The first clause expresses that $\mathcal{M}, w \not\models \mathbf{0} \wedge \mathbf{1}$ implies

$$\mathcal{M}, w \models [\geq]\langle DR\rangle j \wedge [\curlywedge]\langle DR\rangle j \iff \forall v \in W(v \leq w \iff vDRj)).$$

For this case, note that the condition above is sufficient, since if $w \neq 0$ and $w \neq 1$, the complement of w dominates exactly those objects which are disjoint from w. □

In $\mathcal{H}_{\mathsf{m}}$ one can also express that the present state is the supremum of a finite number n of named states for any $n \in \mathbb{N}$. We simply iterate the required number of dominance claims as in the following:

$$\langle\vee\rangle(i_1, ..., i_n) \;=_{def}\; \langle\geq\rangle i_1 \wedge ... \wedge \langle\geq\rangle i_n \wedge [\curlyvee](\langle\geq\rangle i_1 \wedge ... \wedge \langle\geq\rangle i_n).$$

The infimum operator is defined as: $\langle\leq\rangle i \wedge \langle\leq\rangle j \wedge [\curlywedge](\langle\leq\rangle i \wedge \langle\leq\rangle j)$. Analogously, we can express that the present state is the infimum of a finite number n of named states.

$$\langle\wedge\rangle(i_1, ..., i_n) \;=_{def}\; \langle\leq\rangle i_1 \wedge ... \wedge \langle\leq\rangle i_n \wedge [\curlywedge](\langle\leq\rangle i_1 \wedge ... \wedge \langle\leq\rangle i_n).$$

Relations	**Expression**	**Abbreviation**
	BA formulations:	
Part	$x \leq y \wedge x \neq 0$	xPy
Proper Part	$x \leq y \wedge x \neq y \wedge x \neq 0$	$xPPy$
Proper Extension	$y \leq x \wedge x \neq y \wedge y \neq 0$	$xPEy$
Overlap	$\exists w(w \neq 0 \wedge w \leq x \wedge w \leq y)$	xOy
Proper Overlap	$xOy \wedge \neg(x \leq y \vee x \geq y)$	$xPOy$
Disjoint	$\neg xOy$	$xDRy$
Supremum	$x \leq z \wedge y \leq z \wedge \forall w(x \leq w \wedge y \leq w \rightarrow z \leq w)$	$x \vee y = z$
Infimum	$z \leq x \wedge z \leq y \wedge \forall w(w \leq x \wedge w \leq y \rightarrow w \leq z)$	$x \wedge y = z$
Complement	$\forall z(x \vee y = z \rightarrow z = 1) \wedge \forall z(x \wedge y = z \rightarrow z = 0)$	$x =\sim y$
Part	$\langle\leq\rangle i \wedge \neg\mathbf{0}$	$\langle P\rangle i$
Proper Part	$\langle\leq\rangle i \wedge \neg(i \vee \mathbf{0})$	$\langle PP\rangle i$
Proper Extension	$\langle\geq\rangle i \wedge \neg i \wedge \neg\mathbf{0}$	$\langle PE\rangle i$
Overlap	$\langle\geq\rangle(\neg\mathbf{0} \wedge \langle\leq\rangle i)$	$\langle O\rangle i$
Proper Overlap	$\langle O\rangle i \wedge \neg(\langle\leq\rangle i \vee \langle\geq\rangle i)$	$\langle PO\rangle i$
Disjoint	$\neg\langle O\rangle i$	$\langle DR\rangle i$
Supremum	$\langle\geq\rangle i \wedge \langle\geq\rangle j \wedge [\curlyvee](\langle\geq\rangle i \wedge \langle\geq\rangle j)$	$\langle\vee\rangle(i,j)$
Infimum	$\langle\leq\rangle i \wedge \langle\leq\rangle j \wedge [\curlywedge](\langle\leq\rangle i \wedge \langle\leq\rangle j)$	$\langle\wedge\rangle(i,j)$
Complement	$\neg(\mathbf{0} \vee \mathbf{1}) \rightarrow ([\geq]\langle DR\rangle i \wedge [\curlywedge]\langle DR\rangle i)$	
	$\wedge(\mathbf{0} \leftrightarrow @_i\mathbf{1}) \wedge (\mathbf{1} \leftrightarrow @_i\mathbf{0})$	$\langle\sim\rangle i$
	GEMS formulation:	
Part	$x \leq y$	$x \leq y$
Proper Part	$x \leq y \wedge x \neq y$	$x < y$
Proper Extension	$y \leq x \wedge x \neq y$	$x > y$
Overlap	$\exists w(w \leq x \wedge w \leq y)$	xOy
Proper Overlap	$xOy \wedge \neg(x \leq y \vee x \geq y)$	$xPOy$
Disjoint	$\neg xOy$	$xDRy$
Tarski Fusion	$x \leq z \wedge y \leq z \wedge \forall w(w \leq z \rightarrow wOx \vee wOy)$	$x + y = z$
Product	$\forall w(w \leq z \leftrightarrow w \leq x \wedge w \leq y)$	$x \times y = z$
Complement	$\forall w(w \leq x \leftrightarrow wDy)$	$x =\sim y$
Part	$\langle\leq\rangle i$	$\langle\leq\rangle i$
Proper Part	$\langle\leq\rangle i \wedge \neg i$	$\langle<\rangle i$
Proper Extension	$\langle\geq\rangle i \wedge \neg i$	$\langle>\rangle i$
Overlap	$\langle\geq\rangle\langle\leq\rangle i$	$\langle O\rangle i$
Proper Overlap	$\langle O\rangle i \wedge \neg(\langle\leq\rangle i \vee \langle\geq\rangle i)$	$\langle PO\rangle i$
Disjoint	$\neg\langle O\rangle i$	$\langle DR\rangle i$
Tarski Fusion	$\langle\geq\rangle i \wedge \langle\geq\rangle j \wedge [\geq](\langle O\rangle i \vee \langle O\rangle j)$	$\langle+\rangle(i,j)$
Product	$[\geq](\langle\leq\rangle i \wedge \langle\leq\rangle j) \wedge [\curlywedge](\langle\leq\rangle i \wedge \langle\leq\rangle j)$	$\langle\times\rangle(i,j)$
Complement	$[\geq]\langle DR\rangle i \wedge [\curlywedge]\langle DR\rangle i$	$\langle\sim\rangle i$

Extensional Mereological Expressions

Displayed below are some important mereological relationships which are interpretations suitable for GEMSs.

Proper Part	$x \leq y \wedge x \neq y$	$x < y$
Proper Extension	$y \leq x \wedge x \neq y$	$x > y$
Overlap	$\exists w(w \leq x \wedge w \leq y)$	xOy
Proper Overlap	$xOy \wedge \neg(x \leq y \vee x \geq y)$	$xPOy$
Disjoint	$\neg xOy$	Dy
Tarski Fusion	$x \leq z \wedge y \leq z \wedge \forall w(w \leq z \rightarrow wOx \vee wOy)$	$x + y = z$
Product	$\forall w(w \leq z \leftrightarrow w \leq x \wedge w \leq y)$	$x \times y = z$
Complement	$\forall w(w \leq x \leftrightarrow wDy)$	$x =\sim y$

The formulation of the relationships of overlap, proper overlap, and disjointness are standard. The binary fusion equation given above labelled "Tarski fusion" is a first-order formulation of the fusion relation that appears in [94]. Intuitively, it says that the fusion z of two elements y and z dominates them such that any part of z overlaps either x or y. Thus, z is composed of x and y and no other objects.

The product formulation above is non-standard. Like Tarski fusion, it is an extensional formulation: the product z of two objects x and y is exactly the object whose parts are both parts of x and y. Finally, the complement relationship is a simplification of more typical complement versions. It is more simple also than the complement relationship given for BAs. A complement y of x is the object whose parts are exactly those disjoint from x.

Again, these relationships are definable in both $\mathcal{H}_{\mathsf{m}}$ and $\mathcal{H}_{\mathsf{o}}$ by the following:

Proper Part	$\langle\leq\rangle i \wedge \neg i$	$\langle<\rangle i$
Proper Extension	$\langle\geq\rangle i \wedge \neg i$	$\langle>\rangle i$
Overlap	$\langle\geq\rangle\langle\leq\rangle i$	$\langle O\rangle i$
Proper Overlap	$\langle O\rangle i \wedge \neg(\langle\leq\rangle i \vee \langle\geq\rangle i)$	$\langle PO\rangle i$
Disjoint	$\neg\langle O\rangle i$	$\langle DR\rangle i$
Tarski Fusion	$\langle\geq\rangle i \wedge \langle\geq\rangle j \wedge [\geq](\langle O\rangle i \vee \langle O\rangle j)$	$\langle+\rangle(i,j)$
Product	$[\geq](\langle\leq\rangle i \wedge \langle\leq\rangle j) \wedge [\curlywedge](\langle\leq\rangle i \wedge \langle\leq\rangle j)$	$\langle\times\rangle(i,j)$
Complement	$[\geq]\langle DR\rangle i \wedge [\curlywedge]\langle DR\rangle i$	$\langle\sim\rangle i$

Again, a fusion operator with a finite number n of objects is definable:

$$\langle\geq\rangle i_1 \wedge ... \wedge \langle\geq\rangle i_n \wedge [\geq](\langle O\rangle i_1 \vee ... \vee \langle O\rangle i_n)$$

And the product of a finite number n is definable by:

$$[\geq](\langle\leq\rangle i_1 \wedge ... \wedge \langle\leq\rangle i_n) \wedge [\curlywedge](\langle\leq\rangle i_1 \wedge ... \wedge \langle\leq\rangle i_n)$$

6.1.5 $\mathcal{H}_{\mathsf{o}}$ Expressivity

Propositional variables introduce the notion of *property* and, in the context of the model-theoretic interpretation, the notion of *class*. Extensions of the proposition symbols can be related in more or less complex ways to single objects in the model. This is one way in which some of the algebraic meta-theory is expressible in $\mathcal{H}_{\mathsf{o}}$.

Filters and Ideals. By Stone's famous representation theorem, every BA is isomorphic to a set-algebra. Thus up to mathematical equivalence, the objects of a BO model are also subsets of the domain, and the denotations of the propositional variables are therefore families of sets.

The following proposition shows that some important mathematical properties are expressible in $\mathcal{H}_{\mathsf{o}}$.

Proposition 6.1.11. In $\mathcal{H}_{\mathsf{o}}$ the following expressions are definable over BO-models

(1) i is the supremum of p

(2) i is the infimum of p

(3) p is the principal filter generated by i

(4) p is the principal ideal generated by i

Proof. It is easily checked that each of the following below expresses the corresponding existence claim above:

(1) $A(p \rightarrow \langle\leq\rangle i) \wedge @_i[\geq](\neg\mathbf{0} \rightarrow \langle O\rangle p)$

(2) $A(p \rightarrow \langle\geq\rangle i) \wedge @_i[\leq](\neg\mathbf{0} \rightarrow \langle O\rangle p)$

(3) $@_i[\leq]p \wedge A(p \rightarrow \langle\geq\rangle i)$

(4) $@_i[\geq]p \wedge A(p \rightarrow \langle\leq\rangle i)$

□

Mereological changes. Propositional variables introduce the notion of *property* and, in the context of the model-theoretic interpretation, the notion of *class*. In the context of a language for egocentric logic, they can be understood as corresponding to the presence of an organism:

$$@_{t'}(p_{Tibbles} \wedge \langle\geq\rangle i_{Tail}) \wedge @_t(p_{Tibbles} \wedge \neg\langle\geq\rangle i_{Tail})$$

Here we may understand the propositional variable as denoting the existence of a subjective potential peculiar to *Tibbles* being had by the location of his body at various times. Sequences of formulae correspond to the passage of localized situations as in. Let $\ll l_1 ... l_\lambda \gg$ be a sequences of λ locations of Tibbles life before and after his accident.

$$\ll (p_{Tibbles} \wedge l_1 \wedge \langle\geq\rangle i_{Tail}) ... (p_{Tibbles} \wedge l_\lambda \wedge \neg\langle\geq\rangle i_{Tail}) \gg$$

Temporally-ordered phases of Tibbles' life are represented by *temporal* sequences of localized situations at which $p_{Tibbles}$ is true. Obviously more work needs to be accomplished to *linearize* $Tibbles'$ presence in a model. That is (at least from a commonsense perspective) $Tibbles$ will be required to exist at one and only sublocation at each time. And it would seem that this would require additional temporal notions. We note this a future avenue of research.

6.2 Some Definable Frame Classes

6.2.1 Frame Definability and Validity over Fully Named Models

We say that ϕ is valid on a model $\mathcal{M}$, $\mathcal{M} \models \phi$, if for all $w \in \mathcal{M}$, $\mathcal{M}, w \models \phi$. As normal, we say φ is valid on a frame $\mathcal{F}$ and write $\mathcal{F} \models \varphi$ if and only if φ is valid on $\mathcal{M} = (\mathcal{F}, V)$ for any pure hybrid valuation V. We write $\mathcal{F} \models \Sigma$ if and only if, for all $\phi \in \Sigma$, $\mathcal{F} \models \phi$. Let K be a class of frames. We say $\Sigma \subseteq \mathcal{H}_{\mathsf{m}}$ *defines (or characterizes)* K if, for all frames $\mathcal{F}$, $\mathcal{F} \in \mathsf{K} \iff \mathcal{F} \models \Sigma$. If $\Sigma = \{\phi\}$, for some single $\phi \in \mathcal{H}_{\mathsf{m}}$, we say that ϕ defines K.

Lemma 6.2.1 (Frame Definability via Pure Formulas)**.** Each formula of $\mathcal{H}_{\mathsf{m}}$ defines an elementary frame condition.

Proof. Assume that $\mathcal{F} \models \phi$ where $\phi \in \mathcal{H}_{\mathsf{m}}$. Observe that ϕ contains some finite number n of nominals $i_1, ..., i_n$. To indicate this, we write $\phi(i_1, ..., i_n)$.

$$\begin{aligned}
\mathcal{F} \models \phi \iff & (\mathcal{F}, V) \models \phi(i_1, ..., i_n) \text{ for any hybrid valuation } V \\
\iff & (\mathcal{F}, V) \models \forall x ST_x(\phi(i_1, ..., i_n)) \text{ for any hybrid valuation } V \\
& \text{[by proposition 6.1.7]} \\
\iff & (\mathcal{F}, V) \models \forall x ST_x(\phi(i_1/x_1, ..., i_n/x_n))[s(x_1), .., , s(x_n)] \\
& \text{for any first-order variable assignment } s \\
& \text{and where } x_1, ..., x_n \text{ are a fresh stock of variables} \\
& \text{not occurring in } \forall x ST_x(\phi(i_1, ..., i_n)). \\
\iff & \mathcal{F} \models \forall x_1 ... \forall x_n (\forall x ST_x(\phi(i_1, ..., i_n))[i_1/x_1, ..., i_n/x_n])
\end{aligned}$$

Since $ST_x(\phi(i_1, ..., i_n))$ is FO, $\forall x_1 ... \forall x_n(\forall x ST_x(\phi(i_1, ..., i_n))[i_1/x_1, ..., i_n/x_n])$ is FO. And we are done. □

We say ψ is a *pure instance* of ϕ if ψ is obtained from ϕ by uniformly substituting nominals for nominals.

If $\varphi \in \mathcal{H}_{\mathsf{m}}$ and ϕ is used as an axiom, then any pure instance of φ is a theorem. Consider the formula $ST_x(\varphi[i_1/x_1, ..., i_k/x_k])$ obtained by uniformly substituting variables $x_1, ..., x_k$ for constants $i_1, ..., i_k$ in $ST_x(\varphi)$. Similar to lemma 6.2.1, what the next lemma shows is that when φ is used as axiom over fully named models, these are equivalent to conditions of the form $\forall x \forall x_1, ..., \forall x_k(ST_x(\varphi[i_1/x_1, ..., i_k/x_k]))$.

Lemma 6.2.2 ([6] p. 437)**.** Let $\mathcal{M} = (\mathcal{F}, V)$ be a named model and ϕ a pure formula. Suppose that for all pure instances ψ of ϕ, $\mathcal{M} \models \psi$. Then $\mathcal{F} \models \phi$.

Proof. Assume $\phi \in \mathcal{H}_{\mathsf{m}}$ and $\mathcal{M}$ is a named model. Suppose for all pure instances ψ of ϕ, $\mathcal{M} \models \psi$. Let $\mathcal{M} \models \phi(i_1 ... i_n)$ where $i_1 ... i_n$ are the nominals in ϕ. By proposition 6.1.7 we have $\mathcal{M} \models \forall x ST_x(\phi(i_1 ... i_n))$. $\mathcal{M} \models (\forall x ST_x(\phi(i_1 ... i_n)))[i_1/j_1 ... i_n/j_n]$ by assumption for any $j_1 ... j_n \in \Omega$. As $\mathcal{M}$ is fully named, $\mathcal{M} \models \forall x_1 ... \forall x_n \forall x ST_x(\phi(x_1 ... x_n))$. And as the latter is a nominal-free closed formula, $\mathcal{F} \models \forall x_1 ... \forall x_n \forall x ST_x(\phi(x_1 ... x_n))$. Thus for any valuation V, $\mathcal{F}, V \models \phi(i_1 ... i_n)$. □

6.2.2 The Definability of the BA and GEM Conditions

Definability of the Class of BAs

We shall show that there is a method for building a named BA-model satisfying any consistent set of $\mathcal{H}_{\mathsf{o}}$ or $\mathcal{H}_{\mathsf{m}}$-sentences. And since $\mathcal{H}_{\mathsf{m}}$ contains no proposition symbols, for any condition defined by a $\mathcal{H}_{\mathsf{m}}$-formula, completeness with respect to that class is immediate.

Along with any complete axiom system for first order logic, the sentences below axiomatize the first-order theory of Boolean algebras.

(BA1) $\forall x(x \wedge 1 = x)$
(BA2) $\forall x(x \vee 0 = x)$
(BA3) $\forall x(\sim x \wedge x = 0)$
(BA4) $\forall x(\sim x \vee x = 1)$
(BA5) $\forall x \forall y \forall z((x \wedge (y \vee z)) = ((x \wedge y) \vee (y \wedge z)))$
(BA6) $\forall x \forall y \forall z((x \vee (y \wedge z)) = ((x \vee y) \wedge (y \vee z)))$
(BA7) $\forall x \forall y(x \vee y = y \vee x)$
(BA8) $\forall x \forall y(x \wedge y = y \wedge x)$

By the standard translation, 6.1.7, and 6.1.10 it is easy to show that each formula below defines the corresponding condition above.

(HBA1) $@_i \langle \wedge \rangle (i, \mathbf{1})$
(HBA2) $@_i \langle \vee \rangle (i, \mathbf{0})$
(HBA3) $@_{\mathbf{0}} \langle \wedge \rangle (\langle \sim \rangle i, i)$
(HBA4) $@_{\mathbf{1}} \langle \vee \rangle (\langle \sim \rangle i, i)$
(HBA5) $\langle \vee \rangle (i, \langle \wedge \rangle (j, k)) \leftrightarrow \langle \vee \rangle (\langle \wedge \rangle (i, j), \langle \wedge \rangle (i, k))$
(HBA6) $\langle \wedge \rangle (i, \langle \vee \rangle (j.k)) \leftrightarrow \langle \wedge \rangle (\langle \vee \rangle (i, j), \langle \vee \rangle (i, k))$
(HBA7) $\langle \wedge \rangle (i, j) \leftrightarrow \langle \wedge \rangle (j, i)$
(HBA8) $\langle \vee \rangle (i, j) \leftrightarrow \langle \vee \rangle (j, i)$

We call the set containing the eight hybrid formulas above BA.

Proposition 6.2.3. Let $\mathcal{M} = (\mathcal{F}, V)$ be a named model. Suppose that for all pure instances ψ of each formula ϕ in BA, $\mathcal{M} \models \psi$. Then, $\mathcal{F}$ is a BA.

Proof. As each formula in BA is pure and defines the required property, then by lemma 6.2.2 the desired result is immediate. $\square$

Definability of Properties of GEMS structures

To axiomatize the class of GEM-models in first-order logic, a first-order version of Tarski's 1929 system suffices [94]. This axiom system is that of the partial order with the following additional formulae. We give the axioms the following labels:

(TEM1) $\forall x(x \leq x)$
(TEM2) $\forall x \forall y(x \leq y \wedge y \leq x \rightarrow x = y)$
(TEM3) $\forall x \forall y \forall z(x \leq y \wedge y \leq z \rightarrow x \leq z)$
(TEM4) $\exists x \phi(x) \rightarrow \exists z(\forall w(\phi(w) \rightarrow w \leq z) \wedge \forall w(w \leq z \rightarrow \exists v(vOw \wedge \phi(v))))$
for each formula $\phi \in FO$
(TEM5) $\forall x(x \leq 1)$

Simons [90] shows that (TEM4) implies $\leq$-extensionality:

$$\forall x \forall y((\exists z\ z < x \vee \exists z\ z < y) \rightarrow (x = y \leftrightarrow \forall z(z < x \leftrightarrow z < y)))$$

And (TEM4) implies that there is a unit: take the instance of that axiom obtained by simply substituting $x = x$ for ϕ. Finally to ensure that this object is the denotation of '1' we include (TEM5). Thus its models are GEMSs. Again, by the standard translation, 6.1.7, and 6.1.10 it is easy to show that each formula below defines the corresponding condition above.

(HTEM1) $@_i\langle\leq\rangle i$
(HTEM2) $@_i\langle\leq\rangle j \wedge @_j\langle\leq\rangle i \rightarrow @_i j$
(HTEM3) $@_i\langle\leq\rangle j \wedge @_j\langle\leq\rangle k \rightarrow @_i\langle\leq\rangle k$
(HTEM4) $E\phi \rightarrow E([\curlywedge]\phi \wedge [\geq]\langle O\rangle\phi)$ for each formula $\phi \in \mathcal{H}_{\mathsf{m}}$
(HTEM5) $A[\leq]\mathbf{1}$

We call the infinite set formulas (HTEM1)-(HTEM5) M.

Proposition 6.2.4. Let $\mathcal{M} = (\mathcal{F}, V)$ be a named model. Suppose that for all pure instances ψ of each formula ϕ in M, $\mathcal{M} \models \psi$. Then, $\mathcal{F}$ is a GEMS.

Proof. As each formula in M is pure and defines the required property, by lemma

6.2.2 the desired result is immediate. □

6.2.3 Atomic and Atomless BA Classes

If $\mathfrak{A}$ is a BA and $x \in \mathfrak{A}$, we say that x is *atomic* if for every non-zero element $z < x$, there is an atom $y \leq z$. This can be expressed in $\mathcal{H}_{\mathsf{m}}$ if the object x is named by a nominal, say i, as follows: $i \wedge [\geq]((\neg\mathbf{0} \wedge \neg i) \rightarrow \langle\geq\rangle\alpha)$. In addition x is called *atomless* if it has no atoms below it. This notion can be expressed as an operator: $[\geq](\neg\mathbf{0} \rightarrow \neg\alpha)$. Moreover, a BA $\mathfrak{A}$ is *atomic* if for all $x \in \mathfrak{A}$, there is an atom y such that $y \leq x$. $\mathfrak{A}$ is *atomless* if $\mathfrak{A}$ contains no atoms.

Proposition 6.2.5. In $\mathcal{H}_{\mathsf{m}}$, the atomic and atomless classes frames are definable.

Proof. $A\langle\geq\rangle\alpha$ expresses $\forall x \exists y(y \leq x \wedge At(y))$. $A\neg\alpha$ expresses that there exists no atom. □

Proposition 6.2.6. $\mathcal{H}_{\mathsf{m}}$ lacks the finite frame and model properties.

Proof. Consider the closed formula $A(\langle\geq\rangle\top \wedge \neg\mathbf{0} \wedge \neg\alpha)$. □

6.2.4 Definability in $\mathcal{H}_{\mathsf{o}}$

The unrestricted composition condition is definable in $\mathcal{H}_{\mathsf{o}}$ over GEO models. We simply use the axiom schema (HTEM4) as a $\mathcal{H}_{\mathsf{o}}$-formula:

$$(\mathsf{COMP}) \;\; Ep \rightarrow E([\curlywedge]p \wedge [\geq]\langle O\rangle p)$$

which defines the following second-order property:

$$\forall P(\exists x Px \rightarrow \exists z(\forall w(Pw \rightarrow w \leq z) \wedge \forall w(w \leq z \rightarrow \exists v(vOw \wedge Pv)))).$$

The sentence above expresses that for any set of objects, there is a Tarski extensional fusion. There is also a Boolean formulation of this principle which defines the condition of Boolean completeness, i.e the property that every set of objects in a BA has a supremum.

$$(\mathsf{BC})\quad Ep \to E([\curlywedge]p \wedge [\geq](\neg \mathbf{0} \to \langle O \rangle p)).$$

The antecedent existence condition is necessary unless we wish to countenance the existence of any object describable in the language. Used as an axiom, it is also equivalent to a second-order condition. Is there a first-order set of formulas expressing the existence of suprema or fusions of arbitrary classes of the domain? We answer this question now. We need to understand some rather important mathematical facts regarding complete BAs.

Definition 6.2.7. The *cofinality* $cf(A)$ of a BA A is the least limit ordinal κ such that A is the union of an increasing chain of length κ of proper subalgebras of A, provided such a chain exists.

Proposition 6.2.8 (Koppelberg (2006) [52])**.** Every infinite complete BA has cofinality $\aleph_1$.

Clearly the cofinality of an infinite Boolean algebra B is an infinite regular cardinal bounded by the size of B. If C is an infinite quotient of B the $cf(B) \leq cf(C)$. Koppelberg showed that $Pow(\omega)$, and in fact every infinite complete Boolean algebra, has cofinality $\aleph_1$. Moreover, since every infinite Boolean algebra has an infinite quotient of size $\leq 2^{\aleph_0}$, there is no Boolean algebra whose cofinality exceeds $2^{\aleph_0}$. Let A be an infinite complete BA. By the previous proposition there is a strictly increasing chain $(B_\alpha)_{\alpha<\aleph_1}$ of subalgebras of A such that $A = \bigcup_{\alpha<\aleph_1} B_\alpha$. Thus, since the construction is strictly increasing, A has at least $\aleph_1$ members. Thus we have.

Corollary 6.2.9. There does not exist a countably infinite complete BA.

Theorem 6.2.10. Boolean completeness is not first-order definable and, in particular, is not EC_Δ.

Proof. Suppose there were a set of first order sentences Σ defining Boolean completeness. Let $\mathcal{BA}$ be the FO theory of Boolean algebra. By the downward Löwenheim Skolem theorem, $\Sigma \cup \mathcal{BA}$ has a countable model. But, by the previous corollary, such a model does not exist. □

6.3 Logics

We noted that when pure formulas are used as axioms they are immediately complete with respect to the class of frames they define. This hinged on satisfying a formula which was closed under uniform substitution on a fully named model. Here we show how one builds named models.

6.3.1 $\mathbf{K}_{\mathsf{hm}}$

We introduce the logic $\mathbf{K}_{\mathsf{hm}}$ of the class of structures with unit in the language of $\mathcal{H}_{\mathsf{m}}$. $\mathbf{K}_{\mathsf{hm}}$ is therefore also the logic of rooted models. Specifically, we show completeness of $\mathbf{K}_{\mathsf{hm}}$ with respect to models whose frames have root 1 and which are of the Boolean type $(W, \leq, 1, 0)$. After presenting this logic, we discuss how to adapt the proof to one for type $(W, \leq, 1)$.

Definition 6.3.1. $\mathbf{K}_{\mathsf{hm}}$ is the set of $\mathcal{H}_{\mathsf{m}}$-formulas containing the tautologies, (**K**) $[\leq](\phi \to \psi) \to ([\leq]\phi \to [\leq]\psi)$ and dual $\langle\geq\rangle\phi \leftrightarrow \neg[\geq]\neg\phi$, the axioms listed below closed under the following rules of proof: $[\leq]$-generalization, modus ponens, sorted substitution (if $\phi \in \mathbf{K}_{\mathsf{hm}}$ and θ results from ϕ by uniformly replacing proposition letters by arbitrary formulas and nominals by nominals), and the rules below closed under uniform substitution of formulas ξ, and θ:

$$
\begin{array}{ll}
(\langle\leq\rangle\text{-}\langle\geq\rangle) & @_i\langle\leq\rangle j \leftrightarrow @_j\langle\geq\rangle i \\
(\langle\curlyvee\rangle\text{-}\langle\geq\rangle) & @_i\langle\curlyvee\rangle\phi \leftrightarrow E(\neg\langle\geq\rangle i \wedge \neg\phi) \\
(\langle\curlywedge\rangle\text{-}\langle\leq\rangle) & @_i\langle\curlywedge\rangle\phi \leftrightarrow E(\neg\langle\leq\rangle i \wedge \neg\phi) \\
(\mathbf{K}_{@}) & @_i(\phi \to \psi) \to (@_i\phi \to @_i\psi), \\
(\mathsf{Self\ Dual}) & @_i\phi \leftrightarrow \neg @_i \neg\phi, \\
(\mathsf{Intro}) & i \wedge \phi \to @_i\phi. \\
(\mathsf{ref}) & @_i i, \\
(\mathsf{sym}) & @_i j \leftrightarrow @_j i, \\
(\mathsf{nom}) & @_i j \wedge @_j\phi \to @_i\phi, \\
(\mathsf{agree}) & @_j @_i\phi \leftrightarrow @_i\phi, \\
(\mathsf{back}) & \langle\leq\rangle @_i\phi \to @_i\phi, \\
(\mathsf{E}) & \phi \to E\phi, \\
(\alpha\text{-}\langle\leq\rangle) & @_i\alpha \leftrightarrow @_i(\neg\mathbf{0} \wedge \neg\langle\geq\rangle(\neg\mathbf{0} \wedge \neg i)) \\
(\mathsf{Clip}) & (@_i\alpha \wedge @_j(\langle\leq\rangle i \wedge \neg\mathbf{0})) \to @_i j \\
(\mathsf{Gen}@) & \text{If } \vdash \xi, \text{ then } \vdash @_i\xi \text{ for any } i \in \Omega \\
(\mathsf{NAME}) & \text{If } \vdash j \to \theta, \text{ then } \vdash \theta \\
(\mathsf{PASTE}) & \text{If } \vdash @_i\langle\leq\rangle j \wedge @_j\xi \to \theta, \text{ then } \vdash @_i\langle\leq\rangle\xi \to \theta \\
(\mathsf{SPLIT}) & \text{If } \vdash @_i\langle\geq\rangle j \wedge @_j\xi \to \theta, \text{ then } \vdash @_i\langle\geq\rangle\xi \to \theta \\
(\mathsf{INV1}) & \text{If } \vdash @_j(\neg\langle\geq\rangle i \wedge \neg\xi) \to \theta, \text{ then } \vdash @_i\langle\curlyvee\rangle\xi \to \theta \\
(\mathsf{INV2}) & \text{If } \vdash @_j(\neg\langle\leq\rangle i \wedge \neg\xi) \to \theta, \text{ then } \vdash @_i\langle\curlywedge\rangle\xi \to \theta
\end{array}
$$

In the final five, j is a nominal distinct from i that does not occur in ξ or θ.

Where $\mathbf{1}$ and $\mathbf{0}$ appear, no other nominals will be substituted for them. And it is also a relative of hybrid tense logic of [7]. It contains many axioms of Prior's original system [73]. The axioms $(\langle\curlyvee\rangle\text{-}\langle\geq\rangle)$ $(\langle\curlywedge\rangle\text{-}\langle\leq\rangle)$, and $(\alpha\text{-}\langle\leq\rangle)$ are $\langle\leq\rangle$-interaction principles. The axioms $\mathbf{K}_{@}$ through agree are naming validities which are well known hybrid logical axioms. Finally the last five rules allow us to expand any set of formulas to a maximally consistent set of formulas with the required number of *named* witnesses.

It is well known result in hybrid logic that if we replace ϕ by $\neg\phi$ in the Intro axiom, contrapose, and make use of Self Dual, we obtain $(i \wedge @_i\phi) \to \phi$; we call this the Elim formula. Observe that the transitivity of naming follows from nom; for example, by substituting the nominal k for ϕ yields $@_ij \wedge @_jk \to @_ik$. expresses the interaction between @ and $\langle\leq\rangle$: Back $\langle\leq\rangle @_i\phi \to @_i\phi$.

It is well known that with Back we can derive $\langle\leq\rangle i \wedge @_i\phi \to \langle\leq\rangle\phi$, called Bridge, which is a valid @-$\langle\leq\rangle$ interaction principle.

Lemma 6.3.2. Bridge is provable in $\mathbf{K}_{\mathsf{hm}}$.

Proof. We do a sketch. By Elim we have $(i \wedge @_i\phi) \to \phi$ or tautologously $(@_i\phi \wedge i) \to \phi$. As $\mathbf{K}_{\mathsf{hm}}$ is a normal modal logic, we can prove $\langle\leq\rangle(@_i\phi \wedge i) \to \langle\leq\rangle\phi$. As in all normal modal logics, in $\mathbf{K}_{\mathsf{hm}}$, $([\leq]\varphi \wedge \langle\leq\rangle\psi) \to \langle\leq\rangle(\varphi \wedge \psi)$ is theorem where φ and ψ are any $\mathcal{H}_{\mathsf{m}}$-formulas. As an instance we have $([\leq]@_i\phi \wedge \langle\leq\rangle i) \to \langle\leq\rangle(@_i\phi \wedge i)$. By tautologous reasoning $([\leq]@_i\phi \wedge \langle\leq\rangle i) \to \langle\leq\rangle\phi$ and $[\leq]@_i\phi \to (\langle\leq\rangle i \to \langle\leq\rangle\phi)$. By substituting $\neg\phi$ for (back) we can prove that $@_i\phi \to [\leq]@_i\phi$. Thus again by tautologous reasoning we have $@_i\phi \to (\langle\leq\rangle i \to \langle\leq\rangle\phi)$ which implies $\langle\leq\rangle i \wedge @_i\phi \to \langle\leq\rangle\phi$ as desired. □

It is clear that are axioms are sound. But what about completeness? Let us say that a $\mathbf{K}_{\mathsf{hm}}$-maximally consistent set (henceforth $\mathbf{K}_{\mathsf{hm}}$-MCS) is *named* if and only if it contains a nominal, and call any nominal belonging to a $\mathbf{K}_{\mathsf{hm}}$ a name for that MCS. Now, $\mathbf{K}_{\mathsf{hm}}$ is strong enough to prove a lemma which is fundamental to proving the Existence Lemma and Truth Lemma for our main result. Inside any $\mathbf{K}_{\mathsf{hm}}$-MCS are a collection of named MCSs with a number of salutary properties:

If Γ is a $\mathbf{K}_{\mathsf{hm}}$-MCS and i is a nominal, then we will call $\{\phi \mid @_i\phi \in \Gamma\}$ *the set named i yielded by* Γ and denote this set by Δ_i.

Lemma 6.3.3. Let Γ be a $\mathbf{K}_{\mathsf{hm}}$-MCS. Then:
(i) For every nominal i, Δ_i is a $\mathbf{K}_{\mathsf{hm}}$-MCS that contains i.
(ii) For all nominals i and j, if $i \in \Delta_j$, then $\Delta_j = \Delta_i$.
(iii) For all nominals i and j, $@_i\phi \in \Delta_j$ iff $@_i\phi \in \Gamma$ [Agreement Property].
(iv) If k is a name for Γ, then $\Gamma = \Delta_k$.

Proof. See [6] pg 439 Lemma 7.24. □

A $\mathbf{K}_{\mathsf{hm}}$-MCS is *pasted* if $@_i\langle\leq\rangle\phi \in \Gamma$ implies that for some nominal j, $@_i\langle\leq\rangle j \wedge @_j\phi \in \Gamma$. A $\mathbf{K}_{\mathsf{hm}}$-MCS is *split* if $@_i\langle\geq\rangle\phi \in \Gamma$ implies that for some nominal j, $@_i\langle\geq\rangle j \wedge @_j\phi \in \Gamma$. Also a $\mathbf{K}_{\mathsf{hm}}$-MCS is *1-shifted* if $@_i\langle\curlyvee\rangle\phi$ implies that for some nominal j $@_i\neg\langle\geq\rangle j \wedge @_j\neg\phi \in \Gamma$. Andt a $\mathbf{K}_{\mathsf{hm}}$-MCS is *2-shifted* if $@_i\langle\curlywedge\rangle\phi$ implies that for some nominal j $@_i\neg\langle\leq\rangle j \wedge @_j\neg\phi \in \Gamma$. If a $\mathbf{K}_{\mathsf{hm}}$-MCS is named, pasted, split, 1 and 2-shifted, we say that it is *decomposed*.

Lemma 6.3.4 (Lindenbaum Lemma)**.** Let Ω' be a countably infinite set of nominals disjoint from Ω. Suppose $\mathcal{L}'$ is the language obtained by adding all these new nominals to $\mathcal{H}_{\mathsf{m}}$. Then every $\mathbf{K}_{\mathsf{hm}}$-consistent set of formulas in language $\mathcal{H}_{\mathsf{m}}$ can be extended to a decomposed $\mathbf{K}_{\mathsf{hm}}$-MCS in language $\mathcal{L}'$.

Proof. Enumerate Ω'. Given a consistent set of $\mathcal{H}_{\mathsf{m}}$-formulas Σ, define Σ_k to be $\Sigma \cup \{k\}$, where k is the first new nominal in Ω'. Toward contradiction suppose that Σ_k is inconsistent. Then for some conjunction of formulas θ from Σ, $\vdash k \rightarrow \neg\theta$. But as k is a new nominal, it does not occur in θ; hence, by the NAME rule, $\vdash \neg\theta$. But this contradicts the consistency of Σ, so Σ_k must be consistent.

We now decompose. Enumerate all the formulas of $\mathcal{L}'$, define Σ^0 to be Σ_k, and suppose we have defined Σ^m, where $m \geq 0$. let ϕ_{m+1} be the $(m+1)$-th formula in the enumeration of $\mathcal{L}'$. We define Σ^{m+1} as follows. If $\Sigma^{m+1} \cup \{\phi_{m+1}\}$ is inconsistent, let $\Sigma^{m+1} = \Sigma^m$. Otherwise let:

(i) $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\}$ if ϕ_{m+1} is in none of the following forms $@_i\langle\leq\rangle\phi$, $@_i\langle\geq\rangle\phi$, $@_i\langle\curlyvee\rangle\phi$, or $@_i\langle\curlywedge\rangle\phi$. (Here i can be any nominal.)

(ii) $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\} \cup \{@_i\langle\leq\rangle j \wedge @_j\phi\}$, if ϕ_{m+1} is of the form $@_i\langle\leq\rangle\phi$. (Here j is the next nominal in the nominal enumeration of Ω' that does not occur in Σ^m or $@_i\langle\leq\rangle\phi$.)

(iii) $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\} \cup \{@_i\langle\geq\rangle j \wedge @_j\phi\}$, if ϕ_{m+1} is of the form $@_i\langle\geq\rangle\phi$. (Here j is the next nominal in the nominal enumeration of Ω' that does not occur in Σ^m or $@_i\langle\geq\rangle\phi$.)

(iv) $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\} \cup \{@_i\neg\langle\geq\rangle j \wedge @_j\neg\phi\}$, if ϕ_{m+1} is of the form $@_i\langle\curlyvee\rangle\phi$. (Here j is the next nominal in the nominal enumeration of Ω' that does no occur in Σ^m or $@_i\langle\curlyvee\rangle\phi$.)

(v) $\Sigma^{m+1} = \Sigma^m \cup \{\phi_{m+1}\} \cup \{@_i\neg\langle\leq\rangle j \wedge @_j\neg\phi\}$, if ϕ_{m+1} is of the form $@_i\langle\curlywedge\rangle\phi$. (Here j is the next nominal in the nominal enumeration of Ω' that does no occur in Σ^m or $@_i\langle\curlywedge\rangle\phi$.)

Let $\Sigma^+ = \bigcup_{n\geq 0}\Sigma^n$. Clearly this set is decomposed. Furthermore, it is consistent. The consistency of the second, third, fourth, and fifth steps are precisely what the PASTE, SPLIT, INV1, and INV2 rules guarantee. □

Definition 6.3.5. Let Γ be a decomposed $\mathbf{K}_{\mathsf{hm}}$-MCS. The *named model yielded by* Γ, is $\mathcal{M}^\Gamma = (W^\Gamma, \leq^\Gamma, 1^\Gamma, 0^\Gamma, V^\Gamma)$, where $W^\Gamma = \{\Delta_i \mid i \in \Omega$, $\leq$ is the restriction to W^Γ such that:

- $u \leq^\Gamma v \Longleftrightarrow$ for all formulas $\phi, \phi \in v$ implies $\langle\leq\rangle\phi \in u$.
- $u \leq^\Gamma v \Longleftrightarrow$ for all formulas $\phi, \phi \in u$ implies $\langle\geq\rangle\phi \in v$.
- $1^\Gamma = \Delta_{\mathbf{1}}$
- $0^\Gamma = \Delta_{\mathbf{0}}$
- V^Γ is the usual canonical valuation, i.e. for each $i \in \Omega$, $V^\Gamma(i) = \{\Delta_i\}$.

Lemma 6.3.6 (Existence Lemma)**.** Let Γ be a decomposed $\mathbf{K}_{\mathsf{hm}}$-MCS, and let $\mathcal{M} = (W, \leq, 1, 0, V)$ be the named model yielded by Γ. (i) Suppose $u \in W$ and $\langle\leq\rangle\phi \in u$. Then there is a $v \in W$ such that $u \leq v$ and $\phi \in v$. (ii) Suppose $u \in W$ and $\langle\geq\rangle\phi \in u$. Then there is a $v \in W$ such that $v \leq u$ ad $\phi \in v$. (iii) Suppose $u \in W$ and $\langle\curlyvee\rangle\phi \in u$. Then there is a $v \in W$ such that $\neg v \leq u$ where $\neg\phi \in v$. (iv) Suppose $u \in W$ and $\langle\curlywedge\rangle\phi \in u$. Then there is a $v \in W$ such that $\neg v \geq u$ where $\neg\phi \in v$.

Proof. We do cases (i) and (iii) only. (ii) is proven similar to (i), and (iv) is proven similar to (iii).

(i) As $u \in W$, for some nominal i we have that $u = \Delta_i$. Hence as $\langle\leq\rangle\phi \in u$, $@_i\langle\leq\rangle\phi \in \Gamma$. But Γ is pasted; so for some nominal j, $@_i\langle\leq\rangle j \wedge @_j\phi \in \Gamma$, and so $\langle\leq\rangle j \in \Delta_i$ and $\phi \in \Delta_j$. If we could show that $\Delta_i \leq \Delta_j$, then Δ_j would be a suitable choice of v. So suppose $\psi \in \Delta_j$. This means that $@_j\psi \in \Delta_i$. But $\langle\leq\rangle j \in \Delta_i$. Hence by Bridge, $\langle\leq\rangle\psi \in \Delta_i$ as required.

(iii) Since $u \in W$, for some nominal i we have that $u = \Delta_i$. Hence as $\langle\curlyvee\rangle\phi \in u$, $@_i\langle\curlyvee\rangle\phi \in \Gamma$. But Γ is 1-shifted; so for some nominal j, $@_i\neg\langle\geq\rangle j \wedge @_j\neg\phi \in \Gamma$, and so $\neg\langle\geq\rangle j \in \Delta_i$ and $\neg\phi \in \Delta_j$. If we could show that $\neg\Delta_j \leq \Delta_i$, then Δ_j would be a suitable choice of v. What we must do is then show that there is a formula ψ such that $\psi \in \Delta_j$ but $\langle\geq\rangle\psi \notin \Delta_i$. This is easy, since $j \in \Delta_j$ and we already have that $\langle\geq\rangle j \notin \Delta_i$. □

Lemma 6.3.7 (Atom Lemma)**.** Let Γ be a decomposed $\mathbf{K}_{\mathsf{hm}}$-MCS, and let $\mathcal{M} = (W, \leq, 1, 0, V)$ be the named model yielded by Γ. $\alpha \in u \in W \iff \mathcal{M}, u \models \alpha$.

Proof. ($\Rightarrow$) Let $\alpha \in u \in W$. For some nominal i we have that $u = \Delta_i$. Hence as $\alpha \in u$, $@_i\alpha \in \Gamma$. Let $v \leq \Delta_i$ and $v \neq 0$. Hence for some nominal, say j, we have $v = \Delta_j$. If we could show that $\Delta_i = \Delta_j$ we would confirm that $At(v)$ as required. It suffices to show that $i \in \Delta_j$, by lemma 6.3.3. Since $\Delta_j \leq \Delta_i$, by definition of $\leq$, we have that for all formulas ϕ, $\phi \in v$ implies $\langle\leq\rangle\phi \in u$. Thus since $i \in \Delta_i$ we have $\langle\leq\rangle i \in \Delta_j$; moreover, since $v \neq 0$, $\neg\mathbf{0} \in \Delta_j$; and thus $@_j(\langle\leq\rangle i \wedge \neg\mathbf{0}) \in \Gamma$. Therefore we have $@_i\alpha \wedge @_j(\langle\leq\rangle i \wedge \neg\mathbf{0}) \in \Gamma$ by consistency and maximality. And by the (CLIP) axiom we have $@_j i \in \Gamma$. Hence finally $i \in \Delta_j$ as required. Thus $At(v)$ and $\mathcal{M}, u \models \alpha$.
($\Leftarrow$) Let $\mathcal{M}, u \models \alpha$. We have $u \neq 0$ and $\forall y \in W(y \leq u \Rightarrow y \neq 0 \Rightarrow y = u)$. Let $v \in W$ be any object where $v \neq 0$ and $u \neq v$. As $\mathcal{M}$ is named, there is some $i \in \Omega$ such that $V(i) = \{u\}$. Hence, by definition of V and $\mathcal{M}$, $(\neg\mathbf{0} \wedge \neg i) \in v$. As $At(u)$, we have $v \nleq u$. Thus by definition of $\leq$, we have $\neg\langle\geq\rangle(\neg\mathbf{0} \wedge \neg i) \in u$. By consistency and maximality, $i \wedge \neg\mathbf{0} \wedge \neg\langle\geq\rangle(\neg\mathbf{0} \wedge \neg i) \in u$. By the Intro axiom $@_i(\neg\mathbf{0} \wedge \neg\langle\geq\rangle(\neg\mathbf{0} \wedge \neg i)) \in u$. By ($\alpha$-$\langle\geq\rangle$), $@_i\alpha \in u$. Then $\alpha \in u$. □

Lemma 6.3.8 (Truth Lemma)**.** Let $\mathcal{M} = (W, \leq, 1, 0, V)$ be the named model yielded by a decomposed $\mathbf{K}_{\mathsf{hm}}$-MCS Γ, and let $u \in W$. Then for all formulas ϕ, $\phi \in u \iff \mathcal{M}, u \models \phi$.

Proof. Induction on ϕ. *Base Case:* The case for α follows by the Atom Lemma. The case for nominals is well-known. *Inductive Step:* We do the case for $\langle\curlyvee\rangle$. The case for $\langle \curlywedge \rangle$ is analogous to the latter. The others are straightforward.

$$\begin{aligned}
\mathcal{M}, w \models \langle \curlyvee \rangle \phi &\iff \exists v \in W\ w \nleq v \text{ and } \mathcal{M}, v \models \neg\phi \\
&\iff \exists v \in W\ w \nleq v \text{ where } \neg\phi \in v \text{ [by IH]} \\
&\implies \exists v \in W \exists i \in \Omega\ i \in w \text{ where } \neg\langle \geq \rangle i \wedge \neg\phi \in v \\
&\qquad \text{[def. of } \leq \text{ and maximality]} \\
&\implies E(\neg\langle \geq \rangle i \wedge \neg\phi) \in v \quad \text{[by (}\mathsf{E}\text{)]} \\
&\implies @_i \langle \curlyvee \rangle \phi \in v \text{ [by } (\langle \curlyvee \rangle\text{-}\langle \leq \rangle)\text{]} \\
&\implies \langle \curlyvee \rangle \phi \in w.
\end{aligned}$$

The converse implication $\langle \curlyvee \rangle \phi \in x \implies \exists v \in W\ w \nleq v$ where $\neg\phi \in v$ follows from the Existence lemma. □

Theorem 6.3.9 (General Completeness)**.** Every $\mathbf{K}_{\mathsf{hm}}$-consistent set of formulas is satisfied on a named countable model.

Proof. Given a $\mathbf{K}_{\mathsf{hm}}$-consistent set of formulas Σ, use the Lindenbaum Lemma to expand it to a decomposed set Σ^+ in a countable language $\mathcal{L}'$. Let $\mathcal{M} = (W, \leq, 1, 0, V)$ be the named model yielded by Σ^+. By item (iv) of lemma 6.3.3, because Σ^+ is named, $\Sigma^+ \in W$. By the truth lemma, $\mathcal{M}, \Sigma^+ \models \Sigma$. The model is countable because each state is named by some $\mathcal{L}'$ nominal, and there are only countably many of these. □

The $\mathcal{H}_{\mathsf{m}}$-logic of models whose frames are of type $(W, \leq, 1)$ is slightly different. The (CLIP) and (α-$\langle \leq \rangle$) would be replaced by $@_i\alpha \leftrightarrow @_i[\geq]i$ and $@_i\alpha \wedge @_j\langle \leq \rangle i \to @_i j$, respectively. As in the completeness proof just shown, we first expand Γ to a MCS as normal. However, the definition of the named model yielded by Γ would not contain the distinguished element Δ_{0}. Besides these differences the proof would be entirely analogous to the one for models of Boolean type.

6.3.2 $\mathbf{K}_{\mathsf{hm}} + \mathsf{BA}$ and $\mathbf{K}_{\mathsf{hm}} + \mathsf{M}$

In this section we investigate logics for BAs and GEMSs.

Proposition 6.3.10 (Soundness)**.** $\mathbf{K}_{\mathsf{hm}} + \mathsf{BA}$ is sound with respect to the class of BAs.

Proof. Suppose that $\mathcal{F}$ is any Boolean algebra. We first show soundness of **K**. Let $\mathcal{F}$ be any frame, V be any pure hybrid valuation, and $w \in \mathcal{F}$ such that $(\mathcal{F}, V), w \models [\leq](\phi \to \psi)$. Suppose $(\mathcal{F}, V), w \models [\leq]\phi$. It suffices to show $(\mathcal{F}, V), w \models [\leq]\psi$. $\forall v \in (\mathcal{F}, V)$, $w \leq v$ implies $(\mathcal{F}, V), v \models \phi \to \psi$ and $(\mathcal{F}, V), v \models \phi$. By modus ponens $(\mathcal{F}, V), v \models \psi$. So $(\mathcal{F}, V), w \models [\leq]\psi$. We next show the soundness of dual. Again suppose that $\mathcal{F}$ is any frame, V is any pure hybrid valuation V, and $w \in \mathcal{F}$ such that $(\mathcal{F}, V), w \models \langle\geq\rangle\phi$. Thus $\forall w \in (\mathcal{F}, V)$, $(\mathcal{F}, V), w \models \langle\geq\rangle\phi$. Then $\exists v \in (\mathcal{F}, V)$ such that $w \leq v$ and $(\mathcal{F}, V), v \models \phi$. Thus it is not the case that $\forall v \in (\mathcal{F}, V)$ $w \leq v$ implies $(\mathcal{F}, V), v \models \neg\phi$. Hence $(\mathcal{F}, V), w \models \neg[\leq]\neg\phi$ as required; and similarly for the other direction. That the interaction axioms ($\langle\leq\rangle$-$\langle\geq\rangle$), ($\langle\curlyvee\rangle$-$\langle\geq\rangle$), ($\langle\curlywedge\rangle$-$\langle\leq\rangle$) are all sound is trivial since they are semantic equivalences which are valid with respect to the dominance relation $\leq$. And it is equally clear that the (NAME), (PASTE), (SPLIT), (INV1), and (INV2) rules are all sound. The final nine modal naming rules are trivially sound. It therefore remains to check that the BA axioms are correct. Observe that by 6.1.10 the definitions of the operators is sufficient. Thus each represents an instance of one of the BA axioms and is therefore sound.

□

Corollary 6.3.11. (i)$\mathbf{K}_{\mathsf{hm}}$+BA is sound and complete with respect BAs. (ii) $\mathbf{K}_{\mathsf{hm}}$+M is sound and complete with respect GEMSs.

Proof. Let Λ be the $\mathcal{H}_{\mathsf{m}}$-logic obtained by $\mathbf{K}_{\mathsf{hm}}$ and the axiom set BA. By theorem 6.3.9, every Λ consistent set of formulas is satisfied on a named model $\mathcal{M} = (\mathcal{F}, V)$. By proposition 6.2.3, $\mathcal{F}$ is a BA. By the previous proposition we are done. (ii) Analogous to (i), we use the M axioms. □

Henceforth we use the function $\mathsf{n} : \mathcal{H}_{\mathsf{m}} \to Pow(\Omega)$ which assigns to each $\mathcal{H}_{\mathsf{m}}$-formula ϕ the set of nominals appearing in ϕ. If V is a hybrid valuation, define V^{ϕ} to be the restriction of V such that $dom(V^{\phi}) = dom(V) \cap \mathsf{n}(\phi)$. A *proper nominal expansion* $(V^{\phi})^{+}$ of V^{ϕ} with respect to ϕ is any hybrid valuation such that $V^{\phi} \subseteq (V^{\phi})^{+}$ and for each $(i, \{w\}) \in (V^{\phi})^{+} - V^{\phi}$, we have $i \notin \mathsf{n}(\phi)$.

In order to prove the existence of a $\mathcal{H}_{\mathsf{m}}$-logic for the class of atomic BAs with arbitrarily many atoms we require another rule which we now introduce. If ϕ is a

$\mathcal{H}_{\mathsf{m}}$-formula of the form $@_i(\neg j \wedge \alpha)$ where i and j are different nominals we call ϕ an *atomic witness*. Consider the following rule: (ATOM) If $\vdash \delta \rightarrow \theta$, then $\vdash \theta$, where δ is any $\mathcal{H}_{\mathsf{m}}$-conjunction of a consistent set of atomic witnesses such that $\mathsf{n}(\delta) \cap \mathsf{n}(\theta) = \varnothing$.

Proposition 6.3.12. (i) $\mathcal{F}, V, w \models \phi \Longleftrightarrow \mathcal{F}, V^{\phi}, w \models \phi$. (ii) For any proper nominal expansion $(V^{\phi})^+$ of V^{ϕ}, $\mathcal{F}, V, w \models \phi \Longleftrightarrow \mathcal{F}, (V^{\phi})^+, w \models \phi$.

Proof. (i) and (ii) are straightforward proofs by induction on the complexity of ϕ. □

Proposition 6.3.13. ATOM is sound w.r.t. the class of infinite atomic BAs.

Proof. It suffices to show that ATOM is valid on the class of infinite atomic BAs. So suppose that $\delta \rightarrow \theta$ is valid on the class of infinite atomic frames. Suppose toward contradiction that there is infinite atomic BA $\mathcal{F}$, valuation V, and $w \in \mathcal{F}$ such that $\mathcal{F}, V, w \models \neg\theta$. By the previous proposition, as $\mathsf{n}(\delta) \cap \mathsf{n}(\theta) = \varnothing$, we have $\mathcal{F}, V^{\phi}, w \models \neg\theta$. Since $\mathcal{F}$ is an infinite atomic BA, by the previous proposition $\mathsf{n}(\delta) \cap \mathsf{n}(\theta) = \varnothing$ there is a valuation V^+ extending V such that $\mathcal{F}, V^+ \models \delta$. Thus $\mathcal{F}, V^+, w \models \delta \rightarrow \neg\theta$, a contradiction. □

Corollary 6.3.14. Let $\kappa \geq \aleph_0$ be a cardinal number. $\mathbf{K}_{\mathsf{hm}} + \mathsf{BA} + A\langle\geq\rangle\alpha + \mathsf{ATOM}$ (abbrev: Λ) is sound and complete w.r.t. the class of atomic BAs with κ atoms.

Proof. Let Ω_1 be a set of κ nominals disjoint from Ω. We define two sets of formulae. First let $\Delta_i = \{@_i(\neg j \wedge \alpha) : i, j \in \Omega_1 \text{ and } j \neq i\}$. And we set $\Delta_{\Omega_1} = \bigcup\{\Delta_i : i \in \Omega_1\}$. Next let Σ be any Λ-consistent set. *Claim*: $\Sigma \cup \Delta_{\Omega_1}$ is Λ-consistent. Toward contradiction suppose otherwise. Then for some conjunction θ of formulas of Σ and some conjunction δ of formulas of Δ_{Ω_1}, $\vdash \delta \rightarrow \neg\theta$. But as the nominals of Ω_1 are disjoint from those in Ω, by the ATOM rule, $\vdash \neg\theta$. But this contradicts the consistency of Σ, so $\Sigma \cup \Delta_{\Omega_1}$ must be Λ-consistent. Next, by a transfinite induction analogous to that of the Lindenbaum Lemma, expand $\Sigma \cup \Delta_{\Omega_1}$ to a decomposed set Σ^+ with, again, a fresh set of κ nominals disjoint from those in Ω and Ω_1. At each limit ordinal $\lambda < \kappa$ use Zorn's lemma. Again, there is a named model $\mathcal{M} = (W, \leq, 1, 0, V)$ yielded by Σ^+. Analogous to the general case (theorem 6.3.9) we have $\mathcal{M}, \Sigma^+ \models \Sigma$. Note that $\mathcal{M}$ is κ large because each state is named by some nominal and there are only κ of these. As Δ_{Ω_1} implies that there are κ distinct atoms, $\mathcal{M}$ has exactly κ atoms. As in corollary 6.3.11, the other axioms imply that $\mathcal{M}$ is a BA. Thus we are done.

□

6.3.3 Remarks about a "$\mathbf{K}_{\mathsf{ho}}$"

What about the corresponding logics of $\mathcal{H}_{\mathsf{o}}$? If we use the identical axiom systems of BA and GEM-models over a general logic $\mathbf{K}_{\mathsf{ho}}$ in $\mathcal{H}_{\mathsf{o}}$, we get immediate completeness with respect to GEO and BO-models. As for the general logic, the guise of the axioms appearing in definition 6.3.1 would be different: for each formula symbol ϕ and ψ in the list we would replace this with p and q, respectively. The other axioms are pure. And thus we will have no issue.

However, if we use the impure axiom (COMP) or (BC) it is unclear whether any consistent set of sentences in those logics can be satisfied on a corresponding complete or unrestrictedly fused model. The reason stems from the fact that there may be certain properties amid the extensions of the proposition symbols which may be definable in $\mathcal{H}_{\mathsf{o}}$, but which may be unsatisfiable on any completed or unrestrictedly fused model. These properties are unknown. And thus we note investigations into them as worthy of further research.

6.4 Invariance and Characterization

We show that $\mathcal{H}_{\mathsf{m}}$ is sufficiently expressive but not excessively so. We introduce the invariance notions. In several fragments of $\mathcal{H}_{\mathsf{m}}$, we demonstrate that the inverse operators are undefinable. The additional expressive capability obtained by the inverse operators is required to define the Boolean operators. We next demonstrate that $\mathcal{H}_{\mathsf{m}}$ is not excessively strong. Recall that it was claimed that a strict nominalistic mereology would be incapable of proving generalized arithmetical facts. We show that all extensions of $\mathbf{K}_{\mathsf{hm}}$ lack general arithmetical principles by showing that non-equinumerous models are mereobisimilar.

6.4.1 Mereobisimulations

Definition 6.4.1 (Mereobisimulation)**.** Let $\mathcal{M} = (W, \leq, V)$ and $\mathcal{M}' = (W', \leq', V')$ be two models. A nonempty binary relation $Z \subseteq W \times W'$ is called a *mereobisimulation* between $\mathcal{M}$ and $\mathcal{M}'$ (notation: $Z : \mathcal{M} \triangleq \mathcal{M}'$) if the following conditions are satisfied:

1. $wZw' \Longrightarrow (V^{-1}(\{w\}) = V'^{-1}(\{w'\}))$.
2. $(V(i) = \{w\}$ and $V'(i) = \{w'\}$ for some $i \in \Omega) \Longrightarrow wZw'$.
3. $wZw' \Longrightarrow (At(w) \Leftrightarrow At(w'))$.
4. $(wZw'$ and $w \leq v) \Longrightarrow \exists v' \in W'(vZv'$ and $w' \leq' v')$ (*Back*).
5. $(wZw'$ and $w' \leq' v') \Longrightarrow \exists v \in W(vZv'$ and $w \leq v)$ (*Forth*).
6. $(wZw'$ and $w \geq v) \Longrightarrow \exists v' \in W'(vZv'$ and $w' \geq' v')$ (*Back*).
7. $(wZw'$ and $w' \geq' v') \Longrightarrow \exists v \in W(vZv'$ and $w \geq v)$ (*Forth*).
8. $(wZw'$ and $w \nleq v) \Longrightarrow \exists v' \in W'(vZv'$ and $w' \nleq' v')$ (*Back*).
9. $(wZw'$ and $w' \nleq' v') \Longrightarrow \exists v \in W(vZv'$ and $w \nleq v)$ (*Forth*).
10. $(wZw'$ and $w \ngeq v) \Longrightarrow \exists v' \in W'(vZv'$ and $w' \ngeq' v')$ (*Back*).
11. $(wZw'$ and $w' \ngeq' v') \Longrightarrow \exists v \in W(vZv'$ and $w \ngeq v)$ (*Forth*).

If, in addition to the conditions above, $Z \subseteq W \times W'$ satisfies the following condition:

12. If wZw', then w and w' satisfy the same proposition letters.

then we say that Z is a *ontobisimulation* between $\mathcal{M}$ and $\mathcal{M}'$ (notation: $Z : \mathcal{M} \doteq \mathcal{M}'$). We write $w \triangleq w'$ if those states are mereobisimilar and $w \doteq w'$ if they are ontobisimilar. And we let $w \leftrightsquigarrow_{\mathcal{L}} w'$ denote that those states are indistinguishable by $\mathcal{L}$-formulas, for some language $\mathcal{L}$.

Theorem 6.4.2. Let $\mathcal{M}, \mathcal{M}'$ be two models. (a) Then for every $w \in W$ and $w' \in W'$, $w \triangleq w'$ implies that $w \leftrightsquigarrow_{\mathcal{H}_{\mathsf{m}}} w'$. (b) $w \in W$ and $w' \in W'$, $w \doteq w'$ implies that $w \leftrightsquigarrow_{\mathcal{H}_{\mathsf{o}}} w'$.

The converses of the above are easy to prove for a restricted case. We say that $\mathcal{M}$ is *image finite* if for each state $u \in \mathcal{M}$, the set $\{(w, v) \mid w \leq v\}$ is finite. We name this the Mereo-Hennessy Milner Theorem given its similarity to that seminal result. The proof is entirely analogous to the original.

Theorem 6.4.3 (Mereo-Hennessy-Milner Theorem)**.** Let $\mathcal{M}$ and $\mathcal{M}'$ be two image-finite models. Then, for every $w \in W$ and $w' \in W'$, $w \mathrel{\mathring{=}} w'$ iff $w \leftrightsquigarrow_{\mathcal{H}_{\mathsf{o}}} w'$.

6.4.2 Expressivity in $\mathcal{H}_{\mathsf{m}}$

By $\mathcal{H}(O_1, ..., O_n, \alpha)$ we denote the recursively defined hybrid language obtained by a countable set of nominals Ω, the atom constant α, closed under the Boolean operations, and operators $O_1, ..., O_n$. It follows from 6.4.2 that if $\{O_1, ..., O_n\}$ is a subset of the operators in $\mathcal{H}_{\mathsf{o}}$, then a corresponding invariance result for $\mathcal{H}(O_1, ..., O_n, \alpha)$ follows. We now show that that our language is well motivated in terms of its selection of operators. That is, by properly restricting the set of operators, we shall not be able to define important Boolean and mereological operators. We will show in each case that the operator is not definable by demonstrating the existence of two models such that at an object in one the relevant property holds, but in the other it does not.

Proposition 6.4.4. The inverse modalities $[\curlywedge]\phi$ and $[\curlyvee]\phi$ are inexpressible in the fragment $\mathcal{H}([\leq], [\geq], \alpha)$ of $\mathcal{H}_{\mathsf{m}}$ (i.e. in the standard hybrid tense language with the atom operator).

Proof. Let $\mathcal{M}$ be a single, unnamed reflexive loop. Let $\mathcal{M}'$ contain two unnamed reflexive loops. These two models are $\mathcal{H}([\leq], [\geq], \alpha)$-bisimilar; yet, in the second model at either point, say x, we have $\mathcal{M}, x \models \langle \curlywedge \rangle \bot$, which is not true at the single point in $\mathcal{M}$. And similarly for the $[\curlyvee]\phi$ operator. □

However, we require at least the $[\curlywedge]$ modality to express the complement modality:

Proposition 6.4.5. Over BA and GEM-models, no formula in $\mathcal{H}([\leq], [\geq], \alpha) \subseteq \mathcal{H}_{\mathsf{m}}$ defines a complement operator with respect to an arbitrary nominal i.

Proof. Consider the two BA-models in figure 6.1. $\mathcal{M}$ is the atomic BA-model with exactly 4 atoms. $\mathcal{M}'$ is the atomic BA-model with 3 atoms. In both models there is a single point that is named i. The complement of the point named i in $\mathcal{M}'$ is w'. However w is not the complement of the point named i in $\mathcal{M}'$ but $w \leftrightsquigarrow w'$. The restriction of the models and the bisimulation to those points properly above demonstrates the corresponding result over GEM-models. □

6.4.3 The lack of counting in $\mathcal{H}_{\mathsf{m}}$

Proposition 6.4.6. There exists no formula in $\mathcal{H}_{\mathsf{m}}$ expressing that there are exactly n individuals.

Proof. Take two completely unnamed models $\mathcal{M} = \{W\}$, $\mathcal{M}' = \{W'\}$, one with n objects and another with k where $k \neq n$. Assume that the objects stand in no relation to one another. Every object in both is an atom. Let Z be the cross product $W \times W'$. It is easily seen that the two models are completely mereobisimilar. □

We can express that there are three objects of a sort: $\alpha \wedge \langle\leq\rangle(\neg\alpha \wedge \langle\curlywedge\rangle\bot)$—namely there is an atom, say x, with a proper extension, say y, where there is an object z such that $z \not\leq y$. Many of the arithmetical properties of Boolean algebras ultimately boil down to arithmetical statements about atoms. Thus it is natural to ask whether there is an expression "there are at least n-atoms" in $\mathcal{H}_{\mathsf{m}}$. We can express

$$\bigwedge_{1\leq k\leq n} @_{i_k}\alpha \wedge \bigwedge_{1\leq k\neq l\leq n} @_{i_k}\neg i_l$$

which implies that there are n *named* atomic states $i_1, ..., i_n$. But in general there is no such formula expressing the existence of n distinct atoms. Even a stronger claim holds.

Proposition 6.4.7. Over BA and GEM models, there exists no formula in $\mathcal{H}_{\mathsf{m}}$ expressing that there are at least n atoms for $n > 2$.

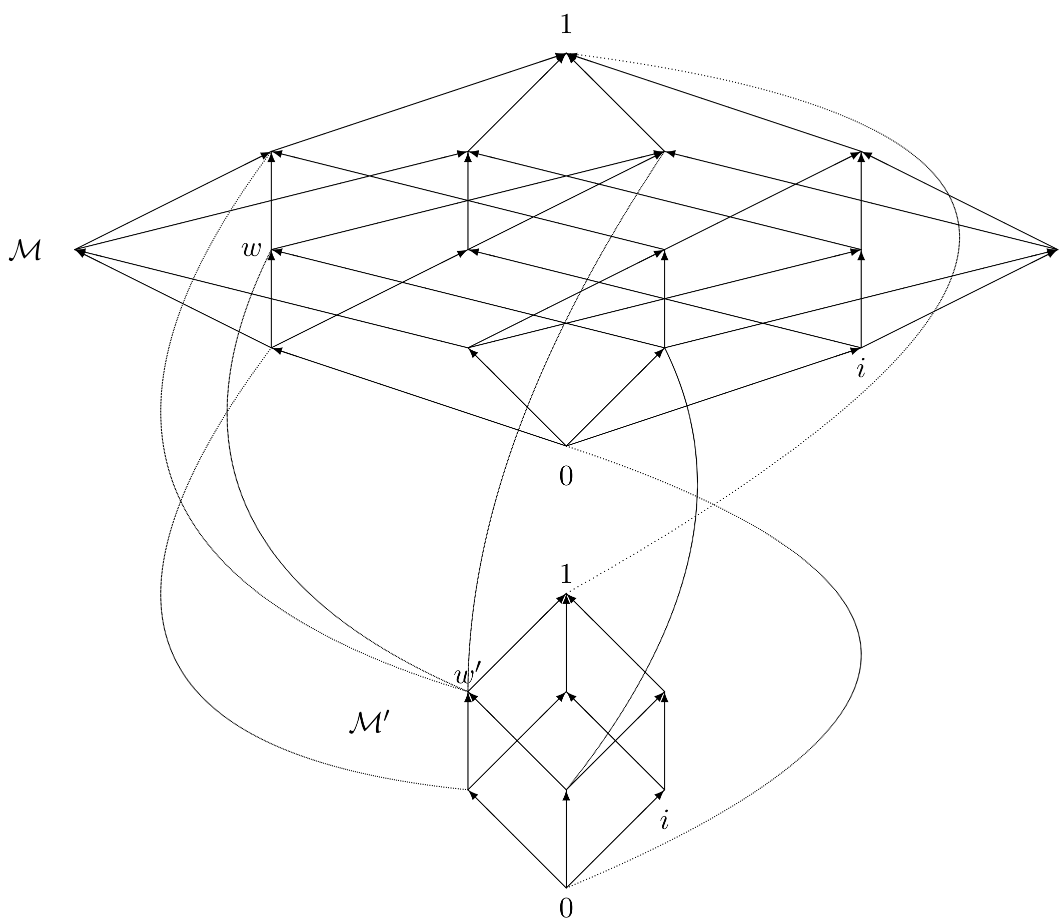


Figure 6.1: Two BA-models that have $\mathcal{H}([\leq], [\geq], \alpha)$-bisimilar points. In $\mathcal{M}'$, w' is the complement of the denotation of i. In $\mathcal{M}$, w is not the complement of the denotation of i. However, $w \leftrightsquigarrow_{\mathcal{H}([\leq],[\geq],\alpha)} w'$. Transitive edges and reflexive loops are omitted.

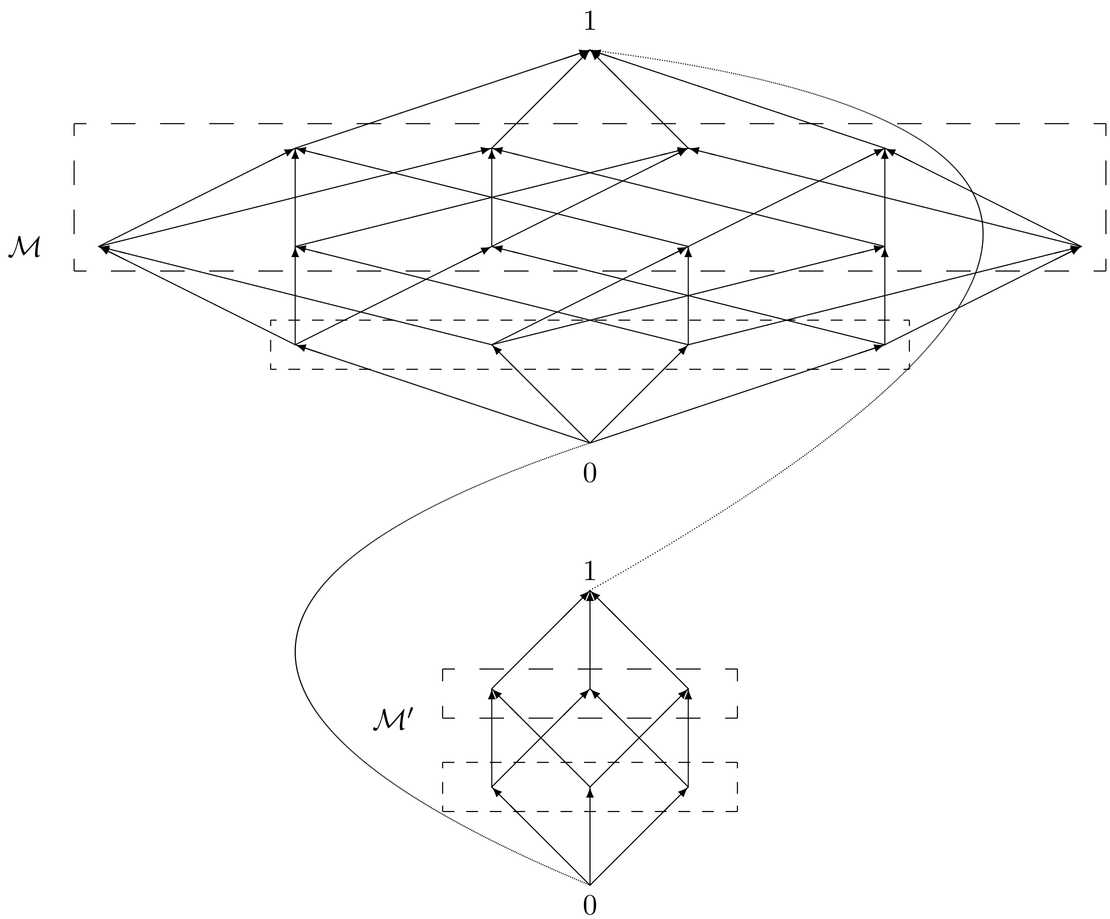


Figure 6.2: Two BA-models that are totally mereobisimilar. Transitive edges and reflexive loops are omitted.

Proof. Consider the BA-models in figure 6.3 which both are unnamed except for 1 and 0. One is the BA with 3 atoms and the other is the BA with 4. Let Z be the following relation:

$$Z = \{(x, x') \in (\mathcal{M} - \{0,1\}) \times (\mathcal{M}' - \{0,1\}) \mid At(x) \Leftrightarrow At(x')\} \cup \{(1,1),(0,0)\}.$$

□

6.4.4 Characterization of $\mathcal{H}_{\mathsf{m}}$ and $\mathcal{H}_{\mathsf{o}}$

In this section, we show which formulas are equivalent to the standard translation of an $\mathcal{H}_{\mathsf{m}}/\mathcal{H}_{\mathsf{o}}$-formula. This is done in a fashion totally analogous to Johan van Benthem's original characterization of modal logic [10]. We show the case for $\mathcal{H}_{\mathsf{o}}$-formulas. The $\mathcal{H}_{\mathsf{m}}$ case is an immediate consequence.

Modal Saturation

In order to prove the characterization theorem, we first require a notion of $\mathcal{H}_{\mathsf{o}}$-saturation.

Definition 6.4.8 (Hennessy-Milner Classes)**.** We say a class K of models of mereological type have the Hennessy-Milner Property if for every two models $\mathcal{M}, \mathcal{M}' \in \mathsf{K}$ and any two states w, w' of $\mathcal{M}$ and $\mathcal{M}'$, respectively, $w \leftrightsquigarrow_{\mathcal{H}_{\mathsf{o}}} w$ implies $\mathcal{M}, w \mathrel{\mathring{=}} \mathcal{M}', w'$.

We now introduce a notion of modal completeness. To explain informally, suppose that we are working over a model $\mathcal{M}$ of mereological type with unit and $w \in \mathcal{M}$ where w has successors $v_0, v_1, v_2, ...$ and, respectively, $\phi_0, \phi_0 \wedge \phi_1, \phi_0 \wedge \phi_1 \wedge \phi_2, ...$ hold. If there is no successor v of w where all formulas from Σ hold *at the same time*, then the model is in some sense *modally incomplete*. To formalize the corresponding notion of completeness observe the following definition.

Definition 6.4.9 (Modal Saturation)**.** Let $\mathcal{M}$ be a model of mereological type, $X \subseteq W$, and Σ a set of $\mathcal{H}_{\mathsf{o}}$-formulas. Σ is *satisfiable* in the set X if there is a state $x \in X$ such that $\mathcal{M}, x \models \phi$ for all ϕ in Σ; Σ is *finitely satisfiable* in X if every finite subset of Σ is satisfiable in X. The model $\mathcal{M}$ is called *modally saturated* or *m-saturated*, for short, if it satisfies the following conditions for every state $w \in W$ and every set Σ of $\mathcal{H}_{\mathsf{o}}$-formulas:

- *If* Σ is finitely satisfiable in the set of $\leq$-successors of w, *then* Σ is satisfiable in the set of $\leq$-successors of w.
- *If* Σ is finitely satisfiable in the set of $\geq$-successors of w, *then* Σ is satisfiable in the set of $\geq$-successors of w.
- *If* Σ is finitely satisfiable in the set of $\nleq$-successors of w, *then* Σ is satisfiable in the set of $\nleq$-successors of w.
- *If* Σ is finitely satisfiable in the set of $\ngeq$-successors of w, *then* Σ is satisfiable in the set of $\ngeq$-successors of w.

Proposition 6.4.10. Let K be the class of models of Boolean type $(W, \leq, 1, 0, V)$. Then the class $\mathsf{K}' \subseteq \mathsf{K}$ of m-saturated models of K has the Hennessy-Milner Property.

Proof. It suffices to prove that the relation $\leftrightsquigarrow_{\mathcal{H}_{\mathsf{o}}}$ between states in $\mathcal{M}$ and states in $\mathcal{M}'$ (where $\mathcal{M}, \mathcal{M}'$ are any members of K') is an ontobisimulation. The conditions concerning the nominals, proposition symbols are trivially satisfied, as is the case for the atom constant. The forth and back conditions are analogously proved and are virtually immediate by the definition of m-saturation. We do just the forward case for $\leq$.

Let $\mathcal{M} = (W, \leq, 1, 0, V)$ and $\mathcal{M}' = (W', \leq', 1', 0', V')$ be models of Boolean type. Assume that $w, v \in W$ and $w' \in W'$ are such that $w \leq v$ and $w \leftrightsquigarrow_{\mathcal{H}_{\mathsf{o}}} w'$. Let Σ be the set of formulas true at v. It is clear that for every finite subset Δ of Σ we have $\mathcal{M}, v \models \bigwedge \Delta$. Hence $\mathcal{M}, w \models \langle \leq \rangle \bigwedge \Delta$. As $w \leftrightsquigarrow_{\mathcal{H}_{\mathsf{o}}} w'$, it follows that $\mathcal{M}', w' \models \langle \leq \rangle \bigwedge \Delta$, so w' has an $\leq'$-successor v_Δ such that $\mathcal{M}', v_\Delta \models \bigwedge \Delta$. In other words, Σ is finitely satisfiable in the set of successors of w'; but then by m-saturation, Σ itself is satisfiable in a successor v' of w'. Thus $v \leftrightsquigarrow_{\mathcal{H}_{\mathsf{o}}} v'$.

□

The Ultraproduct Construction

As with the classical result, we need to use the notion of an ultrapower construction, which we now define.

Definition 6.4.11 (Filters and Ultrafilters)**.** Let W be a non-empty set. A *filter* F over W is a set $F \subseteq \mathcal{P}(W)$ such that (i) $W \in F$, (ii) If $X, Y \in F$, then $X \cap Y \in F$, and (iii) $X \in F$ and $X \subseteq Z \subseteq W$ implies $Z \in F$. A filter is called *proper* if it is distinct from $\mathcal{P}(W)$. An *ultrafilter over* W is a proper filter U such that for all $X \in \mathcal{P}(W)$, $X \in U$ if and only if $(W - X) \notin U$.

Suppose that $I \neq \varnothing$, U is an ultrafilter over I, and for each $x \in I$, W_x is a non-empty set. Let $C = \Pi_{x \in I} W_x$ be the cartesian product of those sets. That is: C is the set of all functions f with domain I such that for each $x \in I$, $f(x) \in W_x$. For two functions $f, g \in C$ we say that f and g are U-*equivalent* (notation $f \sim_U g$) if $\{x \in I \mid f(x) = g(x)\} \in U$. It is easy to check that $\sim_U$ is an equivalence relation on C.

Definition 6.4.12 (Ultraproducts of Sets)**.** Let f_U be the equivalence class of f modulo $\sim_U$, that is $f_U = \{g \in C \mid g \sim_U f\}$. The *ultraproduct of* W_x *modulo* U, denoted as $\Pi_U W_x$, is the set of all equivalence classes of $\sim_U$. So

$$\Pi_U W_x = \{f_U \mid f \in \Pi_{x \in I} W_x\}.$$

If every W_x is identical (i.e. if $W_x = W$ for all $x \in I$), the ultraproduct is called the *ultrapower of* W *modulo* U, and written $\Pi_U W$.

Analogous to the general definition of the ultraproduct of first-order models, we can define an ultraproduct of models of mereological type.

Definition 6.4.13 (Ultraproduct of Hybrid Models of Mereological Type)**.** Let $\mathcal{M}_x (x \in I)$ be a set of models of mereological type. The *ultraproduct* $\Pi_U \mathcal{M}_x$ *of* $\mathcal{M}_x$ *modulo* U is the model described as follows:

(i) The universe W_U of $\Pi_U \mathcal{M}_x$ is the set $\Pi_U W_x$, where W_x is the universe of $\mathcal{M}_x$.

(ii) Let V_x be the hybrid valuation of $\mathcal{M}_x$. Then the hybrid valuation V_U and distinguished elements 1_U and 0_U of $\Pi_U \mathcal{M}_x$ are defined by

$$\begin{array}{lcll} f_U \in V_U(p) & \iff & \{x \in I \mid f(x) \in V_x(p)\} \in U & \text{for } p \in \Phi \\ \{f_U\} = V_U(i) & \iff & \{x \in I \mid \{f(x)\} = V_x(i)\} \in U & \text{for } i \in \Omega \\ 1_U & = & \{(x, 1_x) \mid x \in I\}_U & \\ 0_U & = & \{(x, 0_x) \mid x \in I\}_U & \end{array}$$

(iii) Let $\leq_x$ be the dominance relation in the model $\mathcal{M}_x$. The relation $\leq_U$ in $\Pi_U \mathcal{M}_x$ is given by

$$f_U \leq_U g_U \iff \{x \in I \mid f(x) \leq_x g(x)\} \in U.$$

(iv) And let $At(f_U) \iff \{x \in I \mid At_x(f(x))\} \in U$, where $At_x(y)$ indicates that in model $\mathcal{M}_x$ we have $At(y)$.

Proposition 6.4.14. Let $\Pi_U \mathcal{M}$ be an ultrapower of $\mathcal{M}$ where $\mathcal{M}$ is a model of mereological type $(W, \leq, 1, 0)$. Then, for all $\mathcal{H}_{\mathsf{o}}$-formulas we have $\mathcal{M}, w \models \phi \iff \Pi_U \mathcal{M}, (f_w)_U \models \phi$, where f_w is the constant function such that $f_w(x) = w$, for all $x \in I$.

Proof. By induction on ϕ. We do the unique cases. *Base Case.*

$$\begin{array}{lcl} \Pi_U \mathcal{M}, (f_w)_U \models i & \iff & \{(f_w)_U\} = V_U(i) \\ & \iff & \{x \in I \mid \{f_w(x)\} = V_x(i)\} \in U \\ & \iff & \{x \in I \mid \{w\} = V_x(i)\} \in U \\ & \iff & \{w\} = V(i) \; [\Leftarrow \text{ as } U \text{ is a filter } I \in U] \\ & \iff & \mathcal{M}, w \models i \end{array}$$

$$\begin{array}{lcl} \Pi_U \mathcal{M}, (f_w)_U \models \alpha & \iff & At((f_w)_U) \\ & \iff & \{x \in I \mid At_x(f_w(x))\} \in U \\ & \iff & \{x \in I \mid At_x(w)\} \in U \\ & \iff & At(w) \; [\Leftarrow \text{ as } U \text{ is a filter } I \in U] \\ & \iff & \mathcal{M}, w \models \alpha \end{array}$$

Inductive Step. The Boolean cases are well-known. To prove closure under negation requires that U is an ultrafilter and in particular $X \in U$ if and only if $(W - X) \notin U$. We do only the case for the inverse operator $\langle \Upsilon \rangle \phi$.

$$\begin{aligned}
\Pi_U \mathcal{M}, (f_w)_U \models \langle \Upsilon \rangle \phi &\iff \exists g_U \in \Pi_U \mathcal{M}\ (f_w)_U \not\leq g_U\ \ \Pi_U \mathcal{M}, g_U \not\models \phi \\
&\iff \{x \in I \mid f_w(x) \leq_x g(x)\} \\
&\qquad \cup \{x \in I \mid \mathcal{M}_x, g(x) \models \phi\} \notin U \text{ [by IH]} \\
&\iff \{x \in I \mid f_w(x) \leq_x g(x) \text{ or } \mathcal{M}_x, g(x) \models \phi\} \notin U \\
&\iff \{x \in I \mid f_w(x) \not\leq_x g(x) \text{ and } \mathcal{M}_x, g(x) \not\models \phi\} \in U \\
&\qquad \text{[by definition of } U] \\
&\iff \exists g(x) \in W, w \leq g(x)\ \mathcal{M}, g(x) \not\models \phi \\
&\iff \mathcal{M}, w \models \langle \Upsilon \rangle \phi
\end{aligned}$$

□

Let $\Gamma(x)$ be a set of first-order formulas in which a single individual variable x may occur free. We call $\Gamma(x)$ a *type*. We say that a first-order model $\mathcal{M}$ *realizes* a type $\Gamma(x)$ if there is an element $w \in \mathcal{M}$ such that for all $\gamma \in \Gamma(x)$, $\mathcal{M} \models \gamma[w]$.

Assume that $\mathcal{M}$ is a model for a given first-order language $\mathcal{L}^1$ with domain W. For a subset $A \subseteq W$, $\mathcal{L}^1[A]$ is the language obtained by extending $\mathcal{L}^1$ with new constants $\underline{a}$ for all elements $a \in A$. $\mathcal{M}_A$ is the expansion of $\mathcal{M}$ to a structure for $\mathcal{L}^1[A]$ in which each $\underline{a}$ is interpreted as a. We now recall the notion of κ-saturated models.

Definition 6.4.15 (κ-saturated Models)**.** Let κ be a natural number or ω. A model $\mathcal{M}$ is *κ-saturated* if for every subset $A \subseteq W$ of size less than κ, the expansion $\mathcal{M}_A$ realizes every set $\Gamma(x)$ of $\mathcal{L}^1[A]$-formulas (with only x occurring free) that is consistent with the first-order theory of $\mathcal{M}_A$. An ω-saturated model is called *countably saturated.*

Lemma 6.4.16 (Hennessy-Milner property)**.** Let $\mathcal{M}$ be an model of mereological type with unit. If $\mathcal{M}$ is countably saturated, then it is m-saturated. It follows that the class of countably saturated models of mereological type has the Hennessy-Milner property.

Proof. Assume that $\mathcal{M}$ is of mereological type and, viewed as a first-order model, is countably saturated. We do only the case for the $\leq$-relation. The others are similar. Let a be a state in W, and consider a set of Σ of $\mathcal{H}_{\mathsf{o}}$-formulas which is finitely satisfiable in the $\leq$-successor set. Define Σ' to be $\Sigma' = \{\underline{a} \leq x\} \cup ST_x(\Sigma)$, where $ST_x(\Sigma)$ is the set $\{ST_x(\phi) \mid \phi \in \Sigma\}$ of standard translations of formulas in Σ. Clearly, Σ' is consistent with the first-order theory of $\mathcal{M}_a$: $\mathcal{M}_a$ realizes every finite subset of Σ', namely in some successor of a. So, by the countable saturation of $\mathcal{M}$, Σ' is realized in some state b. By $\mathcal{M}_a \models \underline{a} \leq x[b]$ it follows that b is a successor of a. By proposition 6.1.7 and $\mathcal{M}_a \models ST_x(\phi)[b]$ for all $\phi \in \Sigma$, it follows that $\mathcal{M}, b \models \Sigma$. So Σ is satisfiable in a successor of $\underline{a}$. □

To build countably saturated models of mereological type, we use ultraproducts based on a special type of ultrafilter. An ultrafilter is *countably incomplete* if it is not closed under countable intersections (but it will be closed under finite intersections). For an example, consider an ultrafilter over $\mathbb{N}$ which does not contain any singletons $\{n\}$. Then, for any n, $(\mathbb{N} - \{n\}) \notin U$. But $\varnothing = \bigcap_{n \in \mathbb{N}}(\mathbb{N} - \{n\}) \notin \mathbb{N}$. Thus U is countably incomplete.

Lemma 6.4.17. Let $\mathcal{L}$ be a countable first-order language, U a countably incomplete ultrafilter over a non-empty set I, and $\mathcal{M}$ an $\mathcal{L}$-model. The ultrapower $\Pi_U \mathcal{M}$ is countably saturated.

Proof. A standard result. See Chang [16] Theorem 6.1.1. □

We are now ready to prove the crux of the characterization theorem, the so-called Detour Lemma.

Lemma 6.4.18 (Detour Lemma)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be models of mereological type and w and v states in $\mathcal{M}$ and $\mathcal{N}$, respectively. Then the following are equivalent:

(i) For all $\mathcal{H}_{\mathsf{o}}$-formulas ϕ: $\mathcal{M}, w \models \phi \iff \mathcal{N}, v \models \phi$.

(ii) There exist ultrapowers $\Pi_U \mathcal{M}$ and $\Pi_U \mathcal{N}$ and a bismulation Z: $\Pi_U \mathcal{M}, (f_w)_U \mathrel{\mathring{=}} \Pi_U \mathcal{N}, (f_v)_U$ linking $(f_w)_U$ and $(f_v)_U$, where $f_w(f_v)$ is the constant function mapping every index to $w(v)$.

Proof. (ii) $\Rightarrow$ (i). By proposition 6.4.14 $\mathcal{M}, w \models \phi$ iff $\Pi_U\mathcal{M}, (f_w)_U \models \phi$. By assumption this is equivalent to $\Pi_U\mathcal{N}, (f_v)_U \models \phi$ and the latter is equivalent to $\mathcal{N}, v \models \phi$. (i)$\Rightarrow$ (ii). Assume that for all $\mathcal{H}_{\mathsf{o}}$-formulas ϕ we have $\mathcal{M}, w \models \phi$ iff $\mathcal{N}, v \models \phi$. We need to create bisimilar ultrapowers. Take the set of natural numbers $\mathbb{N}$ as the index set and let U be a countably incomplete ultrafilter (as in the example above). By lemma 6.4.17, the ultrapowers $\Pi_U\mathcal{M}, (f_w)_U$ and $\Pi_U\mathcal{N}, (f_v)_U$ are countably saturated. Now $(f_w)_U$ and $(f_v)_U$ are $\mathcal{H}_{\mathsf{o}}$-equivalent: for all $\mathcal{H}_{\mathsf{o}}$-formulas ϕ we have $\Pi_U\mathcal{M}, (f_w)_U \models \phi$ iff $\Pi_U\mathcal{N}, (f_v)_U \models \phi$. This follows from the assumption that w and v are $\mathcal{H}_{\mathsf{o}}$-equivalent together with proposition 6.4.14. Next use 6.4.16: as $(f_w)_U$ and $(f_v)_U$ are $\mathcal{H}_{\mathsf{o}}$-equivalent and $\Pi_U\mathcal{M}$ and $\Pi_U\mathcal{N}$ are countably saturated, there is the required ontobisimulation Z. □

Definition 6.4.19. A first-order formula $\phi(x)$ in $\mathcal{L}^1$ in the signature of mereological type is *invariant for ontobisimulations* if for all models $\mathcal{M}$ and $\mathcal{N}$ and all states w in $\mathcal{M}$, $v \in \mathcal{N}$, and all ontobisimulations Z between $\mathcal{M}$ and $\mathcal{N}$ such that wZv, we have $\mathcal{M} \models \phi(x)[w]$ iff $\mathcal{N} \models \phi(x)[v]$. We are now ready to prove the principle result of this section.

Theorem 6.4.20 (Characterization Theorem)**.** Let $\phi(x)$ be a first-order formula in $\mathcal{L}^1$, where the latter is in the signature of mereological types, Then $\phi(x)$ is invariant for ontobisimulations iff it is equivalent to the standard translation of a $\mathcal{H}_{\mathsf{o}}$-formula.

Proof. ($\Leftarrow$) follows from theorem 6.4.2. ($\Rightarrow$) With the detour lemma, this direction is proven analogously to van Benthem's original argument [10]; or perhaps see [6] pg. 103. □

6.5 Conclusion

Our hybrid languages have allowed us a way take seriously the demands of nominalism and simultaneously arrive at *bona fide* mereologies. What we showed is as follows (the major results are listed with an asterisk ($*$)).

- Any mereological language must be sufficiently expressive. In section 6.2.2, it was shown that all important mereological relationships and operations over nominals are expressible in $\mathcal{H}_\mathsf{m}$.

- And we contrasted the formal mereological language $\mathcal{H}_\mathsf{m}$ with that of $\mathcal{H}_\mathsf{o}$. It was shown in section 6.2.3 that special types of relationships linking nominals and properties were expressible in $\mathcal{H}_\mathsf{o}$: filter, ideal, supremum and infimum relationships.

- We demonstrated both the class of GEMSs and BAs were definable in $\mathcal{H}_\mathsf{m}$ in section 6.3.2. And in 6.3.4 we showed that we could express the atomlessness and atomicity of states and define the atomless and atomic classes of frames. In $\mathcal{H}_\mathsf{o}$ it was also shown that we could define a second order class of frames (i.e. the Boolean complete class of frames) which are not definable in FOL.

- ($*$) We proved soundness and completeness results for various general classes of structures with unit in section 6.4. And in 6.4.2 we showed axiomatizations for the classes of atomic and atomless BAs. And we explained how the analogous results can be obtained for the class of GEMS structures. In particular we showed that there is a logic complete with respect to countable atomless Boolean algebras and another complete with respect to infinite atomic Boolean algebras with a set of κ atoms for every infinite cardinal κ.

- There was the possibility that our language might be too excessive. So in section 6.5.2, we showed that by restricting the language to $\mathcal{H}([\leq],[\geq],\alpha)$ we could not define the complement operator, which is an obvious expressive necessity for extensional mereology. And in $\mathcal{H}([\leq],[\geq],[\curlywedge],\alpha)$ even the supremum operator was not definable. The latter result implies trivially that Tarski's fusion operator of [94] is also not definable in $\mathcal{H}([\leq],[\geq],[\curlywedge],\alpha)$.

- In the previous chapter, it was argued that counting principles exceeded the threshold of a nominalistic mereology. In section 6.5.3, we proved that in $\mathcal{H}_\mathsf{m}$ one cannot count, as required.

- (∗) Finally in section 6.5.4, we characterized the languages of nominalistic mereology as proper fragments of the first-order language.

Thus it is in the mereobisimilar-fragment of first-order language where we glean the power of a streamlined nominalistic mereology. In comparison to the standard modal or hybrid languages, $\mathcal{H}_{\mathsf{m}}$ is comparatively strong. However, in contrast to traditional mereological languages, $\mathcal{H}_{\mathsf{m}}$ is much weaker. Consequently, it is clear that abstract features of formal languages—feature instantiation via variable binding, sets and set quantifiers, and so on—greatly enrich our ability to represent concrete entities. Indeed, they may even be indispensable. Hints of this are already present the indefinability result of theorem 6.2.10 where it was shown that the Boolean complete frame class is not elementary. And this presents itself as the next question concerning our logics. To wit: how much subtle structure of space can one capture in $\mathcal{H}_{\mathsf{m}}$? We visit this in the next chapter.

Chapter 7

Capturing the Structure of Locations

$\mathcal{H}_{\mathsf{m}}$ represents an almost purely nominalistic picture of reality. For any finite mereological arrangement of objects, say $\ulcorner a :\leq b, \geq c, \nleq d, ...\urcorner$, there is a corresponding formula $@_a(\langle\leq\rangle b \wedge \langle\geq\rangle c \wedge \langle\nleq\rangle d \wedge ...)$ expressing that that situation obtains. The same is true of localized situations. For example, let o be a first-personal object. Then if $(\leq b, \geq c, \nleq d, ...)$ is a finite situation holding at o, then there is a corresponding formula $\langle\leq\rangle b \wedge \langle\geq\rangle c \wedge \langle\nleq\rangle d \wedge ...$ expressing that that situation holds at o.

Basically, what we demonstrated in the last section is that there are logics in $\mathcal{H}_{\mathsf{m}}$ which, over structures extensional in the mereological sense, allow to derive all the logical consequences that follow from a given set of assumptions. That is, if there is a set Σ of $\mathcal{H}_{\mathsf{m}}$-formulas which are all true of a extensional structure $\mathcal{S}$—and by these formulas, we envision a corresponding set of situations—then any further fact expressible in $\mathcal{H}_{\mathsf{m}}$ which is a logical consequence of Σ of $\mathcal{S}$ is demonstratable by use of the axioms and rules of the logic. But not all extensional structures are complete or unrestrictedly fused. In the example above $\mathcal{S}$ is not necessarily complete. Thus we must show that the logics we introduced in the last chapter allow us to derive all the truths over completed or unrestrictedly fused structures.

We can achieve this in a variety of ways. The most natural way to do this is to show that any infinite structure $\mathcal{S}$ which is a GEMS or BA, is mereobisimilar to one with a finite valuation, say $\mathcal{S}'$ that is a completed version of $\mathcal{S}$. This is the best way to proceed in our case. What this will show is that for any location l in the uncompleted version $\mathcal{S}$, there is a location l' in the completed version $\mathcal{S}'$ at which every formula true at l is also true at l' and vice versa. Now imagine this in the case of situations. If they are infinite, then many of the objects in $\mathcal{S}'$ will be indistinguishable. They will be, up-to-mereobisimulation, the same as those in $\mathcal{S}'$.

We will show that *if there are infinitely many locations, $\mathcal{H}_{\mathsf{m}}$ will not be strong enough to capture their mereological structure*. In particular, if there are infinitely many locations and the universe has an atomic structure, then any single formula describing such a structure will be true also of finite situation involving finitely many locations. Likewise if there are infinitely many locations but some are atomless, then each atomless location will be indistinguishable from a countable one which is not unrestrictedly fused. Moreover, in the atomless case, the structure of atomless locations will be indistinguishable from ones whose objects have a "pixelated" character.

7.1 Complete Extensional Models

We have seen that $\mathcal{H}$-formulas can be interpreted as situation descriptions by the correspondence result of proposition 5.2.1. And we have also observed in chapter 3 that each situation has a location. As nominalism privileges physical objects, the accurate describability of locations would be a reasonable demand. But how should we go about testing whether they are sufficient? Although the nominalist may object to the method, we will rely on mathematical models of a particular sort. We will address her qualms in the next section. But first, let us introduce them.

The notion of unrestricted fusion is appropriate for GEMS structures. But the corresponding property in the theory of BAs is that of *completeness*: A BA $\mathfrak{A}$ is *complete* if for any subset S of $\mathfrak{A}$, there is a supremum of S. Admittedly, a consequence of 6.2.10 is that even FOL will fail to distinguish a great many distinct objects in a complete BA. But we are interested especially in $\mathcal{H}_{\mathsf{m}}$ from a philosophical perspective. For, as we have seen, generalized arithmetical properties that require counting sentences are not expressible in $\mathcal{H}_{\mathsf{m}}$ (see chapter 6 section 6.5.3). In addition, unlike in FOL, arbitrary properties definable by open formulas in FO are not definable in $\mathcal{H}_{\mathsf{m}}$. Hence it is significant that we obtain a formal gauge on the language's strength. Thus it is worthwhile to know how precisely the structure of locations can be described without reference to sets, arbitrary properties definable by $\leq$, and arithmetical notions.

An investigation into the expressive strengths of formal *mereologies* also contributes to the standing philosophical dialogue concerning mereology's fundamentality and wide applicability. Over the last century, many rather influential philosophers have claimed or suggested that the physical world can be ultimately described in mereological terms (see e.g.[77],[88],[42],[43],[44]). Many of these accounts present mereology as a more basic and general way to understand the structure of the world without resorting to set theory. And it is often suggested that the validities of mereological reasoning are independent of the content of scientific theories.

Locations are much like regions. The only difference is that a location may be either material-free or material whereas a region is material-free. But in the context of mathematical accounts of space, the question of the physical contents of regions is irrelevant. Traditional as well as modern approaches to spatial logic centrally concern the notion of *structure*. And thus formal investigations into spatial logics for complete Boolean algebras will be entirely relevant to questions concerning the structure of locations.

There are two typical mathematical models of the mereological breakdown of space. On the one hand, we have the regular open algebra of the real numbers which is an *atomless* BA traditionally used to model space according to the extended approach in for example Tarski [94], Vakarelov et al [98], and Whitehead [100]. On the other hand, we have the infinite atomic Boolean algebra with $2^{\aleph_0}$ atoms. This model is isomorphic to the powerset algebra of the real numbers—the mereological set structure implicit in the classical Euclidean spaces.

First, we will see that the $\mathcal{H}_{\mathsf{m}}$-logic of the BA obtained by the regular open sets $RO(\mathbb{R}^n)$ of $\mathbb{R}^n$ for any $n \in \mathbb{N}$ is just $\mathbf{K}_{\mathsf{hm}} + \mathsf{BA} + A\neg\alpha$. As Tarski [93] and MacNeille [61] showed, in order to arrange $RO(\mathbb{R}^n)$ Boolean algebraically, the topological properties of the space necessitate a "regularized" interpretation of the Boolean operations. We introduce these notions first. Next, we identify a countable atomless Boolean subalgebra $\mathfrak{CH}$ of the BA $\mathfrak{RO}$ obtained from $RO(\mathbb{R}^n)$ whose members are regular unions of n-dimensional open boxes. $\mathfrak{CH}$ is a "pixellated" breakdown of $\mathbb{R}^n$. We demonstrate that for any finite pure hybrid valuation V on the subalgebra, $(\mathfrak{RO}, V)$ is mereobisimilar to $(\mathfrak{CH}, V)$. By a well-known fact in the theory BAs, any countable $\mathbf{K}_{\mathsf{hm}}+\mathsf{BA}+A\neg\alpha$ canonical frame is isomorphic to $\mathfrak{CH}$. And completeness with respect to $RO(\mathbb{R}^n)$ is a consequence.

Locations may decompose also atomically. So to give a non-biased argument, we also inquire into infinite atomic BAs. We demonstrate that the $\mathcal{H}_{\mathsf{m}}$-logic of complete atomic BAs with κ many atoms for any cardinal κ is the logic obtained by $\mathbf{K}_{\mathsf{hm}}+\mathsf{BA}+A\langle\geq\rangle\alpha$ closed under the rule (ATOM). By a standard result in the theory of BAs, there is only one powerset algebra up to isomorphism. In this context, the relevant Boolean set algebras are the Boolean Cartesian n-product spaces of the real numbers. Only slightly more difficult than the atomless case, this atomic completeness result is again carried out by demonstrating a mereobisimulation between a finitely named canonical BA-model and its completion. Indeed, the employed mereobisimulation implies that any infinite atomic BA-model is mereobisimilar to a finite one, and we therefore obtain the finite model property over this class of structures as a consequence.

A two part philosophical conclusion is drawn from our technical results. Firstly, the logics studied can now be said to be the $\mathcal{H}_{\mathsf{m}}$-mereologics of traditional models of locations and space, independent of whether the extended or atomic approach is correct. Secondly, since these logics conflate natural models of space with those that are far more coarse, it appears that nominalistic mereological reasoning is highly insufficient over infinite spaces. In particular, it is far weaker than elementary mereologics and detects much less subtle spatial structures. Obviously, a classical formalism or a set-theoretic calculus of individuals would more accurately represent spatial reasoning. But as such languages import in arithmetical and set-theoretic principles, resorting to first-order languages requires jettisoning the strict nominalistic program.

7.2 Is the Employment of Mathematical Models Nominalistically Acceptable?

Obviously, one immediate challenge is to convince the nominalist that we can use models to test the strength of formal languages. For, if she does not believe they exist, then she may doubt the content or intelligibility of the results. And consequently she may be entitled to believe, despite all we will say in this chapter, that the expressivity of $\mathcal{H}_{\mathsf{m}}$ is indeed sufficient to describe the mereological structure of infinitely many locations.

Let us suppose, as I have claimed, that locations are unrestrictedly fused. Assume that the nominalist is prepared to admit this. She views the world as mereologically analyzable and therefore the target entities of the description will be locations related according to the parthood relation. Furthermore, assume the proofs in the argument that follows are mathematically sound according to the commonly held standards of proof demonstration. If she fails to find them convincing, then conceivably she will do so on grounds that either (i) the models are unintelligible or do not exist; or (ii) the models do not accurately represent the structure of unrestrictedly fused objects.

If (i) is true and she rejects the existence or intelligibility of models, she cannot do so on the basis that they do not correspond to objects standardly used in scientific practice. For the vast majority of scientific theories are based on models. Nearly all scientists hold them to be both intelligible and sufficient to represent a vast array of objects. Thus given the nominalist's likely naturalist or physicalist leanings, she will typically not make this move. And if she is unrelenting in this regard, it is hard not view her own intuitions of physical structure as mysterious.

It is also quite unreasonable that there should be any general problem with modeling physical objects as individuals related according to parthood. Any view of physical objects will imply a conceived specification of entities. And this means nothing more or less than that one must espouse a view of the world as comprising objects of some sort. And the parthood relation is arguably the most intuitive of conceptions available to us. So since one's very subjective notion or specification of real objects will be tantamount to a model, she must accept some notion that parts of reality are modelable.

But suppose, for whatever reason she still holds (i). There must be some problem with the nature of the models. It is likely that her worry will be with the status of them as abstract set-theoretic constructions. But I suggest that her worries may be assuaged if these models are finite. Theoretically, we can represent finite models in a number of acceptable ways. This can be done linguistically, graphically, or digitally. Even if a finite model is exceedingly large, there will be, theoretically, some computational means by which to assess its features. And at any rate, any problem identified with representing a model will be a mere practical one. Thus I do not see a legitimate qualm with finite set-theoretic models, for in theory, these can be given nominalistically acceptable surrogates.

Then apparently her real worry is with the status of infinite set-theoretic constructions. Assuming she holds that mathematicians do engage in substantive research, then she envisions some nominalistically acceptable way to understand infinite sets. There will then be some *bona fide* nominalistically acceptable mathematical subject matter. Thus if this is the case, I see no other way to understand her qualm with infinite structures than to attribute it to them being infinite or ungraspable. But note that, if this is the case, she must also discount any *subjective notion* of the physical world as infinite. For clearly, any such notion will be a subjective model or representation and equally as dubious or under-grasped as any infinite set-theoretic construction. And importantly, this will imply either that she is skeptical of any notion of the world as infinite or is outright committed to a finite account of its structure. In these cases, I admit that she may be entitled to discount the results in what follows. Still, if this is the case, I will argue in the conclusion of the dissertation that she may have a problem understanding the *geometric structure* of atomic extended locations. So we shall postpone any further argument concerning this view until the next chapter. Nonetheless, if she views infinite notions about reality as indeed intelligible, I see no reason why she cannot understand the results that follow. She will be able to intuit both the requisite properties of mereological extensionality and unrestricted fusion and thereby be capable of interpreting set-theoretic constructions in a suitable nominalistically acceptable way.

If she does not hold that mathematicians engage in substantive research, I can do nothing other than simply disagree. And unless mathematicians are wrong about the structures of complete BAs, and I doubt very seriously that they are, I do not see how she can hold (ii) either. For she will likely defer to experts and hold a view of the division of research and theoretical labor in the sciences. Accordingly, mathematicians will be the source experts in their field of study. And the Boolean algebraist will be the most informed of BAs. However, barring the unlikely possibility that mathematical research in the theory of Boolean algebras is incorrect, I must simply disagree with her view of mathematical research.

7.2.1 Regions and Spatial Logic

It was mentioned in chapter 2 that closing substances may very well be regions. The results presented in this chapter concern mereological structure locations, whether they be regions or matter. And therefore if some locations are regions, the formal demonstrations will apply equally well to the structure of regions. Consequently the following work will be a case of theoretical spatial logic. So before we entertain the formal results, we will now consider briefly some traditional issues concerning spatial logic and the ontology of regions.

Our modern conception of spatial structure is conditioned by mereological representations which follow from the geometrical foundations laid by both Euclid and Descartes. That is to say, although their analyses of space are geometrical, they imply a mereological understanding which is sedimented in our ordinary views.

Euclid described the plane in terms of its types of geometrical part, among these *points*, *lines*, *surfaces*, and *angles*. His achievement was a formal study of these objects within an axiomatic system involving them as primitives. Descartes provided a numerical interpretation of all geometrical entities by analyzing zero-dimensional points in an n-dimensional space into n-tuples of numerical values. Two distinct points specify a line, and polygons are representable as a sequence of the vertex points of various lines. Hence according to his view, regions of space are theoretically and fundamentally part-wise decomposable into collections of points.

The structure of three-dimensional Euclidean space can be axiomatized in terms of the quaternary metrical relation of equidistance Tarski [96] and Tarski and Givant [41] over points. Equidistance is analyzable into other conceptions in a sufficiently rich language. One may also axiomatize the space in terms of ternary betweeness. Combinations of other geometrical relationships like orthogonality and relative closeness might also be selected as primitives. Still the assumption here again is that points, lines, and various polygons are parts of space.

Although the point-based analysis has been the traditional approach to spatial representation, there are plausible grounds for looking for alternative views. For one might also deploy arguments according to which all spatial entities are *extended*, non-atomic regions. Two early advocates of the region-based approach were Alfred North Whitehead and Bertrand Russell. They held that objects of perception can be the only referents of fundamentally primitive terms. Whitehead [101] argued explicitly that extended regions are more fundamental than points, as the former are in some way sensorily detectable, and points, being zero-dimensional, are not. And the view of regions as extended is also natural if one views them as closing substances or "hosts" of matter.

The earliest rigorous and fully formal theory of space in which extended regions appear as the basic entity appears in Tarski's *Geometry of Solids* in [94]. His axiomatization was one of an atomless GEMS. Tarski's selected language is a set-theoretic calculus of individuals with terms for sets and second-order universal quantifiers. He demonstrates the categoricity result that every model of size $2^{\aleph_0}$ of the investigated theory is isomorphic to the set of all regular open sets of the Euclidean space. Defining points via extended regions requires class abstraction of spheres from the space.

Subsequently, a number of other calculi and formalizations have been proposed. The spatial theories of extended regions appearing in Clark [18] and Clark [19] are based on Whitehead's contact relation interpreted over regular open (closed) regions. Contact in this sense is either the relation of sharing a point in the case of closed regions or, in the case of regular open sets, that of the closure of two regions sharing a point. Current work in region-based theories has attracted much attention from AI researchers. By now, one famous system is the *Region Connection Calculus* (Randell et al. [87]), a first-order theory whose only primitive is the connection relation. Most AI researchers are interested in investigating mereotopological relationships that may function in belief states by intelligent systems. And it is typically claimed that the region-based approach is more suitable for any intelligent system outfitted with a human-like perceptual function.

Nonetheless, several recent results show that the region and point-based approaches are actually not altogether distinct depending upon the level of expressivity of the languages employed. Even Whitehead observed that by considering classes of regions, points can be defined as infinite sets of nested regions which converge to a point. One very important recent result in this context is due to Pratt and Lemon [72] and Pratt-Hartmann [71]. The authors formally show that any sufficiently strong axiomatization of the polygonal regions of a plane can be interpreted in terms of the classical point model of the Euclidean plane. Thus, given sufficient logical resources, the region-based theory is not more simple in its ontological commitments. We can interpret extended regions as fundamental in some sense, but zero-dimensional regions will necessarily crop up at the meta-level. And this is also why I take pains to show both an atomless and an atomic result in what follows. For, from an ontological perspective, it is unclear whether logically definable points correspond to any object in physical reality.

7.3 Mereobisimulations between Models of Locations

In this section, we describe in general and informally the technical methods we employ. We begin with explaining the method of proving the atomless result and then move to the atomic, comparing the two approaches.

We have already shown that $\mathbf{K}_{\mathsf{hm}} + \mathsf{BA} + A\neg\alpha$ is complete with respect to the class of atomless BAs. For any $\phi \in \mathcal{H}_{\mathsf{m}}$ such that $\nvdash_{\mathbf{K}_{\mathsf{hm}}+\mathsf{BA}+A\neg\alpha} \phi$, there is a countable atomless BA $\mathfrak{B}$ and valuation V such that $(\mathfrak{B}, V) \not\models \phi$. This implies $(\mathfrak{B}, V^{\phi}) \not\models \phi$, where V^{ϕ} is the restriction of V whose domain is the set of nominals appearing in ϕ. What we will show is that there is a total mereobisimulation between $\mathcal{B} = (\mathfrak{B}, V^{\phi})$ and $\mathcal{RO} = (\mathfrak{RO}, V^{\phi})$ where $\mathfrak{RO} = (RO(\mathbb{R}^n), \subseteq, \mathbb{R}^n, \varnothing)$ is the regular open algebra of

$\mathbb{R}^n$. By $\mathcal{H}_{\mathsf{m}}$-invariance, the desired completeness result follows.

The required mereobisimulation Z between $\mathcal{B}$ and $\mathcal{RO}$ must contain a bijection of objects uniquely true of some $\mathcal{H}_{\mathsf{m}}$-formula. The remainder of the objects which are Z-related must play some similar role in relation to these distinguished states. One way this can be done is as follows. Since $ran(V^{\phi})$ is finite, it generates a finite Boolean subalgebra $\mathfrak{A}$ of $\mathfrak{B}$. Hence there is a Boolean monomorphism $h : \mathfrak{A} \longrightarrow \mathfrak{B}$. Moreover, by some straightforward, well-known results, there is a Boolean monomorphism $g : \mathfrak{B} \longrightarrow \mathfrak{RO}$. Thus $g \circ h : \mathfrak{A} \longrightarrow \mathfrak{RO}$ is an injective Boolean homomorphism. As $\mathfrak{A}$ is finite, it is atomic. Consider the atoms $Atom(|\mathfrak{A}|)$ of $\mathfrak{A}$. For each $x \in Atom(|\mathfrak{A}|)$, $g(h(x))$ is an atomless regular open set. Let $\mathsf{C} = g \circ h[Atom(|\mathfrak{A}|)]$. As C is (under $(g \circ h)^{-1}$) the set of atoms of the subalgebra $\mathfrak{A}$, two nice properties hold. Firstly, the elements of C are mutually disjoint, and, secondly, the supremum of C is the top $\mathbb{R}^n$. Thus each nonempty object of $\mathcal{RO}$ overlaps at least one member of C. That is to say, each regular open set has a covering of a regular supremum of members of C.

For any two sets, X, Y if there is a set Z such that $(Z \neq \varnothing \wedge Z \subseteq X \wedge Z \subseteq Y)$ we say that X *and* Y *overlap* and write $O(X,Y)$. Moreover if $O(X,Y)$ and $Y \nsubseteq X$, we say that X *partially overlaps* Y. We can now define the desired mereobisimulation Z. Let $R \in \mathcal{RO}$ be any regular open set. We let $b\mathsf{Z}R$ where $(b,R) \in \mathcal{B} \times \mathcal{RO}$ if and only if (a) $g(b)$ and R *dominate exactly the same members of* C and (b) $g(b)$ and R *partially overlap exactly same members of* C. We can identify an object in $b \in \mathcal{B}$ where $b\mathsf{Z}R$ which meets the two conditions above. Define three sets:

- $\mathsf{C}_1 = \{Y \in \mathsf{C} \mid Y \subseteq R\} = \{C^1_1, ..., C^1_k\}$
- $\mathsf{C}_2 = \{Y \in \mathsf{C} \mid Y \nsubseteq R \wedge O(Y,R)\} = \{C^2_1, ..., C^2_l\}$
- $\mathsf{C}_3 = \{Y \in \mathsf{C} \mid O(Y,R)\} = \{C^3_1, ..., C^3_{k+l}\}$

witnessing that $\mathsf{C}_1 \cap \mathsf{C}_2 = \varnothing$, $\mathsf{C}_1 \cup \mathsf{C}_2 = \mathsf{C}_3$, and moreover if $|\mathsf{C}| = j$ then $k + l \leq j$. For any $C^2_i \in \mathsf{C}_2$, there exists a $g^{-1}(C^2_i) \in \mathfrak{B}$. As $\mathfrak{B}$ is atomless, there is a $y_i \in \mathfrak{B}$ such that $y_i <^{\mathfrak{B}} h^{-1}(C^2_i)$. Since the members of C are pairwise disjoint, they can be dealt with separately. So to define b, we combine each $g^{-1}(C_i^1)$ and y_i:

$$b = \left(\bigvee_{i=1}^{k} g^{-1}\left(C_i^1\right) \right) \vee \left(\bigvee_{i=1}^{l} y_i \right).$$

To show that Z admits of the back and forth conditions is not difficult.

The point of using the atoms of the subalgebra generated by $ran(V^{\phi})$ is so that both the suprema and infima of *named* elements are uniquely Z linked. Thus there will be very few regions of the entire space which are distinguishable. The number of them will concern the size of the formula ϕ. Observe that if there were atoms in the model, further distinctions could be made. Obviously, in the case of atomless BAs, the atom constant α is of no avail in discriminating one region from another. For any object $C \in \mathsf{C}$, any proper part of C is $\mathcal{H}_{\mathsf{m}}$-indistinguishable. However, over models with atoms, the situation is more interesting.

Let Λ_{ATOM} be the logic obtained by the axiom system $\mathbf{K}_{\mathsf{hm}} + \mathsf{BA} + A\langle\geq\rangle\alpha$ closed under the rule (ATOM). For the atomic result, we again rely on the fact of general completeness: any Λ_{ATOM}-non-theorem ϕ is falsified on an infinite atomic BA. It is easy to use a transfinite Henkin construction to show that ϕ is falsified on a structure $\mathfrak{B}$ with continuum many atoms. The completion $\mathfrak{C}$ of $\mathfrak{B}$ (the precise conception of which to be explained below) will be related by appealing again to the set of atoms of the finite Boolean subalgebra $\mathfrak{A}$ generated by the denotations of the nominals appearing in ϕ. As before there will be the injective Boolean homomorphism $h : \mathfrak{A} \longrightarrow \mathfrak{B}$. However, there is a *special* Boolean embedding $g : \mathfrak{B} \longrightarrow \mathfrak{C}$ called a *complete* Boolean embedding. These mappings preserve suprema that happen already to exist in $\mathfrak{B}$. Let again $\mathsf{C} = g \circ h[Atom(|\mathfrak{A}|)]$. Suppose we used the earlier mereobisimulation. If an element is named i in $\mathfrak{B}$, it appears also (under the embedding) in $\mathfrak{C}$. Thus the as *named* atoms will be homomorphically embedded in $\mathfrak{C}$ to atoms in $\mathfrak{C}$. But by using the mereobisimulation above, atom-to-atom mereobisimilar links will not be ensured for those objects properly below those in C. For any objects x, y, if $y \not\leq x$ and xOy, we say that x *partially overlaps* y.

The mereobisimulation in for the atomic case is defined by strengthening the one for the atomless case with (c). We let $b\mathsf{Z}c$ if and only if (a) $g(b)$ and c *dominate* *exactly the same members of* C and (b) $g(b)$ and c *partially overlap the exactly same members of* C, and (c) for any corridor z partially overlapped by x, we have $x \wedge z$ *is an atom if and only if* $y \wedge z$ *is an atom.*

7.4 A Pixellated Geometry of Solids

Seen as a Boolean algebra of sets, there are various constructions of the countable atomless BA isomorph. For example, we can interpret its domain as either a set of individuals, a family of sets whose members are individuals, or one whose members are tuples of numerical values. Similar to Tarski's *Geometry of Solids*, interest in $\mathcal{RO}$ is motivated by how clearly the structure represents the breakdown of space into combinations of open *regions*—so-called *solids*—within the Euclidean space. Sets of tuples of multi-dimensional models are able to represent some geometrical properties of regions. Therefore, we will define the subalgebra $\mathcal{B}$ (noted earlier) as a geometrical substructure of the larger, more intricate constellation $\mathcal{RO}$. This provides us a clearer way to compare the strengths of an SCI, like Tarski's logic of solids, which is ω_1-categorical and thus captures $\mathfrak{RO}$ up to isomorphism.

7.4.1 Regular Open Sets

Recall that an open set in a topological space X is *regular open* if it coincides with the interior of its closure. The following theorem due MacNeille and Tarski asserts that the regular open sets constitute a complete BA of sets, *the regular open algebra of* X. If S is a set, we denote the closure of S by S^-. And by $S^\perp$, we denote the complement of the closure $\sim (S^-)$ of S.

Theorem 7.4.1 ([61],[93]). The class of all regular open sets of a topological space X, abbreviated $RO(X)$ is a complete BA with respect to the distinguished Boolean elements and operations defined by:

$$
\begin{array}{llcl}
(1) & 0 & = & \varnothing, \\
(2) & 1 & = & X, \\
(3) & P \wedge Q & = & P \cap Q, \\
(4) & P \vee Q & = & (P \cup Q)^{\perp\perp}, \\
(5) & \sim P & = & P^{\perp},
\end{array}
$$

The infimum and the supremum of a family $\{P_i\}$ of regular open sets are, respectively, $(\bigcap_i P_i)^{\perp\perp}$ and $(\bigcup_i P_i)^{\perp\perp}$.

We now describe some of the background intuitions underlying this seminal result. Note that a finite intersection of regular open sets yields a regular open set. But the union of two regular open sets is not necessarily regular open. For example, let P and Q be disjoint open half-planes in $\mathbb{R}^2$ separated by a line (a nowhere dense set). Say, P consists of the set of points to the right of the y axis, and Q consists of the points to the left. Then, $P \cup Q$ is open, but not regular, since $(P \cup Q)^{\perp\perp} = \mathbb{R}^2$. The point of construing the supremum of finite unions in this way is to fill in "cracks"—nowhere dense sets—that may result from unions. These nowhere dense sets are subsets of the boundaries of their respective objects:

Proposition 7.4.2. For any cardinal κ,

$$
\left(\left(\bigcup_{i=1}^{\kappa} P_i\right)^{\perp\perp} - \left(\bigcup_{i=1}^{\kappa} P_i\right)\right) \subseteq \left(\left(\bigcup_{i=1}^{\kappa} P_i\right)^{-} - \left(\bigcup_{i=1}^{\kappa} P_i\right)\right).
$$

Proof. Let κ be any cardinal. We have

$$
\begin{array}{rcl}
x \in (\bigcup_{i=1}^{\kappa} P_i)^{\perp\perp} - (\bigcup_{i=1}^{\kappa} P_i) & \Longrightarrow & x \notin \left(\sim \left((\bigcup_{i=1}^{\kappa} P_i)^{-}\right)\right)^{-} \cup \bigcup_{i=1}^{\kappa} P_i \\
& \Longrightarrow & x \notin \sim \left((\bigcup_{i=1}^{\kappa} P_i)^{-}\right) \cup (\bigcup_{i=1}^{\kappa} P_i) \\
& \Longrightarrow & x \in (\bigcup_{i=1}^{\kappa} P_i)^{-} \cap \sim (\bigcup_{i=1}^{\kappa} P_i). \\
& \Longrightarrow & x \in (\bigcup_{i=1}^{\kappa} P_i)^{-} - (\bigcup_{i=1}^{\kappa} P_i).
\end{array}
$$

□

Finally, note that $\perp$ is required instead of $\sim$ so that the complement of a regular open set is, again, regular open and not closed.

As a consequence of theorem 7.4.1, standard Boolean equivalences like distributivity and De Morgan's law follow under the regular open interpretation. For example, for any regular open sets P, Q, and R we have

Identity	$(P \cup \varnothing)^{\perp\perp}$	$=$	P
	$(P \cap X)$	$=$	P
Commutivity	$(P \cup Q)^{\perp\perp}$	$=$	$(Q \cup P)^{\perp\perp}$
	$(P \cap Q)$	$=$	$(Q \cap P)$
Distributivity	$(P \cup Q)^{\perp\perp} \cap R$	$=$	$((P \cap R) \cup (Q \cap R))^{\perp\perp}$
	$((P \cap Q) \cup R)^{\perp\perp}$	$=$	$(P \cup R)^{\perp\perp} \cap (Q \cup R)^{\perp\perp}$
De Morgan's Law	$((P \cup Q)^{\perp\perp})^{\perp}$	$=$	$(P^{\perp} \cap Q^{\perp})$
	$(P \cap Q)^{\perp}$	$=$	$(P^{\perp} \cup Q^{\perp})^{\perp\perp}$
Double Negation	$P^{\perp\perp}$	$=$	P

Definition 7.4.3. If X is a set such that $X = (\bigcup_{i=1}^{\kappa} X_i)^{\perp\perp}$, where each X_i is regular open, we say that X is *a regular union of κ regular open sets.* If X is a set such that $X = Y^{\perp}$, we say that X is *the regular complement of Y*.

We shall not revisit the entire proof of theorem 7.4.1, but a demonstration of some facts required to prove it will help us.

Lemma 7.4.4. (i) If $P \subseteq Q$, then $Q^{\perp} \subseteq P^{\perp}$. (ii) If P is open, then $P \subseteq P^{\perp\perp}$. (iii) If P is open, then $P^{\perp} = P^{\perp\perp\perp}$.

Proof. (i) It suffices to observe that closure preserves set-inclusions and complementation reverses them. (ii) Since $P \subseteq P^{-}$, we have, by complementation, that $P^{\perp} \subseteq \sim P$. Now apply closure: since $\sim P$ is closed, it follows that $P^{\perp -} \subseteq \sim P$, and this is the complemented version of what is desired. (iii) Applying part (i) to the conclusion of part (ii) we have $P^{\perp\perp\perp} \subseteq P^{\perp}$. Apply part (ii) to the open set $P^{\perp}$ (by substituting $P^{\perp}$ for P) to get the reverse inclusion. □

7.4.2 Regular Serial and Chequered Open Sets

We now define the classes of countable Boolean set algebras of interest to us. Recall that an *interval of rational (real) numbers* is a set I with the property that whenever x and y are in I, then so is every rational (real) number between x and y. And a rational (real) interval is said to be *open* if it is one of the following five forms: $\varnothing, (-\infty, \infty), (x, -\infty), (-\infty, y), (x, y)$ where $x < y$ and x and y are rational (real) numbers. Note that any open interval is then regular.

Definition 7.4.5 (Regular Serial Sets)**.** A set S is *regular serial* if it is equal to a finite regular union $(I_1 \cup ... \cup I_n)^{\perp\perp}$ of open intervals $I_1, ..., I_n$. We denote the set of all regular serial sets of a set X as $RS(X)$.

It is well known that both $(RS(\mathbb{Q}), \subseteq, \mathbb{Q}, \varnothing)$ and $(RS(\mathbb{R}), \subseteq, \mathbb{R}, \varnothing)$ whose domains are the regular serial open subsets of $\mathbb{Q}$ and $\mathbb{R}$, respectively, form BAs. Now $(RS(\mathbb{Q}), \subseteq, \mathbb{Q}, \varnothing)$ is countable and $(RS(\mathbb{R}), \subseteq, \mathbb{R}, \varnothing)$ uncountable. They are not regular open algebras, as they are not complete. But we will see that they are Boolean subalgebras of regular open algebras with respect to the regular interpretation of the Boolean operations.

Definition 7.4.6 (Open Boxes and Regular Chequered Open Sets)**.** For $n \geq 2$ we call X an *n-dimensional open box* (or just an nD *open box* for short), if $X = I_1 \times ... \times I_n$, where all the I_i's are open intervals. And X is *regular n-chequered-open* (or just $nRCH$ *open*) if it is a regular union of a finite set of nD open boxes. If X^n is a set of n-tuples, then by $RCH(X^n)$, we denote the set of all nRCH open sets of X^n.

Proposition 7.4.7. Both $(RS(\mathbb{Q}), \subseteq, \mathbb{Q}, \varnothing)$ and $(RCH(\mathbb{Q}^n), \subseteq, \mathbb{Q}^n, \varnothing)$ form BAs with the regular open interpretation of the Boolean operations.

Proof. We do the case for $(RCH(\mathbb{Q}^n), \subseteq, \mathbb{Q}^n, \varnothing)$. The one for $(RS(\mathbb{Q}), \subseteq, \mathbb{Q}, \varnothing)$ is a subcase. It's obvious that a regular union of finitely many nRCH open sets is nRCH open. It is easy to check that an intersection of finitely many nRCH open sets is again nRCH open. It suffices then to check the closure of nRCH open sets under $\perp$-complementation. Let $X \in RCH(\mathbb{Q}^n)$. Then $X = (B_1 \cup ... \cup B_k)^{\perp\perp}$ for some k open boxes $B_1, ..., B_k$. For all $i \in \{1, ..., k\}$, $B_i = I_1 \times ... \times I_n$, and for each $j \in \{1, ..., n\}$, I_j is an open interval. By part (iii) lemma 7.4.4 and De Morgan's law, $X^\perp = B_1^\perp \cap ... \cap B_k^\perp$ and each $B_i^\perp = I_1^\perp \times .... \times I_n^\perp$ where $1 \leq i \leq k$. In particular, for each j where $1 \leq j \leq n$ we have

$$I_j^\perp = \begin{cases} \varnothing & \text{if } I_j = (-\infty, \infty), \\ (-\infty, \infty) & \text{if } I_j = \varnothing, \\ (-\infty, x) & \text{if } I_j = (x, \infty) \text{ and } x \neq -\infty, \\ (x, \infty) & \text{if } I_j = (-\infty, x) \text{ and } x \neq \infty, \\ ((-\infty, x) \cup (y, \infty))^{\perp\perp} & \text{if } I_j = (x, y) \text{ where } x \neq -\infty \text{ and } y \neq \infty. \end{cases}$$

In consideration of the final clause of $I_j^\perp$ above, each $B_i^\perp$ is equivalent to the regular union of, *maximally*, 2^n nD open boxes. I.e., for any j where $1 \leq j \leq n$,

$$\begin{aligned} &(I_1^\perp \times ... \times ((-\infty, x) \cup (y, \infty))^{\perp\perp}) \times ... \times I_n^\perp) \\ &= (I_1^\perp \times ... \times (-\infty, x) \cup (y, \infty) \times ... \times I_n^\perp) \\ &= ((I_1^\perp \times ... \times (-\infty, x) \times ... \times I_n^\perp) \cup ... \cup (I_1^\perp \times ... \times (y, \infty) \times ... \times I_n^\perp) \cup ...) \\ &= ((I_1^\perp \times ... \times (-\infty, x) \times ... \times I_n^\perp) \cup ... \cup (I_1^\perp \times ... \times (y, \infty) \times ... \times I_n^\perp) \cup ...)^{\perp\perp} \end{aligned}$$

So, by the case for regular unions, each $B_i^\perp$ is nRCH open. Thus $X^\perp$ is equivalent to an intersection of finitely many nRCH open sets. By the intersections case, $X^\perp$ is nRCH open. □

Countable Boolean subalgebras of $(RS(\mathbb{R}), \subseteq, \mathbb{R}, \varnothing)$ and $(RCH(\mathbb{R}^n), \subseteq, \mathbb{R}^n, \varnothing)$ are definable in the following way. If X is a set of regular serial sets, then by X^* we denote the set of all regular serial sets of X which are regular unions of intervals *with rational endpoints*. If X is a set of nRCH open sets of a topological space, then by X^* we denote the set of all nRCH open sets of X which are regular unions of nD open boxes obtained by a finite Cartesian product of open intervals *with rational endpoints*. Thus $(RS(\mathbb{R})^*, \subseteq, \mathbb{R}, \varnothing)$ is isomorphic to $(RS(\mathbb{Q}), \subseteq, \mathbb{Q}, \varnothing)$, countable, and is a Boolean subalgebra of $(RS(\mathbb{R}), \subseteq, \mathbb{R}, \varnothing)$. Analogous to the 1D case, $RCH(\mathbb{R}^n)^*$ is countable and isomorphic to $RCH(\mathbb{Q}^n)$. And by the preceding proposition $RCH(\mathbb{R}^n)^*$ forms a BA with the regular open interpretation of the Boolean operations.

Theorem 7.4.8. $\mathbf{K}_{\mathsf{hm}} + \mathsf{BA} + A\neg\alpha$ is the $\mathcal{H}_{\mathsf{m}}$-logic of both $(RS(\mathbb{R})^*, \subseteq, \mathbb{R}, \varnothing)$ and also $(RCH(\mathbb{R}^n)^*, \subseteq, \mathbb{R}^n, \varnothing)$ for any $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$ and Γ be any $\mathbf{K}_{\mathsf{hm}} + \mathsf{BA} + A\neg\alpha$-consistent set of formulae in $\mathcal{H}_{\mathsf{m}}$. Let $\mathcal{M} = (\mathcal{F}, V)$ be the named model generated by Γ. $\mathcal{M}$ is countable and atomless. Every countably infinite atomless BA is isomorphic. $\mathcal{F}$ is isomorphic to $(RS(\mathbb{R})^*, \subseteq, \mathbb{R}, \varnothing)$ and $(RCH(\mathbb{R}^n)^*, \subseteq, \mathbb{R}^n, \varnothing)$. It is easy to use the isomorphism to generate pure hybrid valuations for these structures. And as isomorphism implies total mereobisimulation, Γ is satisfiable on both. $\square$

7.4.3 The Atomless Mereobisimulation

We now explain the first mereobisimulation in detail. The case of regular serial sets of $\mathbb{R}$ is the one-dimensional subcase of that of nRCH open algebras. So we show only the result for the latter. Let $\mathfrak{CH} = (RCH(\mathbb{R}^n)^*, \subseteq, \mathbb{R}^n, \varnothing)$ and define $\mathfrak{RO} = (RO(\mathbb{R}^n), \subseteq, \mathbb{R}^n, \varnothing)$ for some fixed $n \in \mathbb{N}$. Moreover, for any BA $\mathfrak{B}$, let $At^{\mathfrak{B}}(X)$ denote the set-theoretic version of the atom property (modulo $\mathfrak{B}$): $X \in |\mathfrak{B}| \wedge X \neq \varnothing \wedge \forall Y(Y \neq \varnothing \rightarrow (Y \subseteq X \rightarrow Y = X))$.

Now suppose $\mathcal{RO} = (\mathfrak{RO}, V)$ and $\mathcal{CH} = (\mathfrak{CH}, V)$ where V is a *finite* pure hybrid valuation in $\mathfrak{CH}$. Since $\mathfrak{CH}$ is a substructure of $\mathfrak{RO}$, V is also a well-defined valuation on $RO(\mathbb{R}^n)$.

Let $\mathfrak{A}$ be the Boolean subalgebra of $\mathfrak{CH}$ finitely generated by $\bigcup ran(V)$. $\mathfrak{A}$ is a Boolean set algebra of $\mathbb{R}^n$ of nRCH open sets whose domain is a finite topology on $\mathbb{R}^n$. As $\mathfrak{A}$ is a subalgebra of $\mathfrak{CH}$, it is also one of $\mathfrak{RO}$. There is therefore a single Boolean monomorphism h from $\mathfrak{A}$ into both $\mathfrak{CH}$ and $\mathfrak{RO}$, namely, the identity function on $|\mathfrak{A}|$. So, henceforth we view $\mathfrak{A}$ as a Boolean subalgebra of elements of $\mathfrak{CH}$ and therefore of $\mathfrak{RO}$. It is well known that any finitely generated Boolean subalgebra is finite. So as $\bigcup ran(V)$ is assumed finite, $\mathfrak{A}$ is finite. Set

$$\mathsf{C} = \{X \in |\mathfrak{A}| \;:\; At^{\mathfrak{A}}(X)\}.$$

That is, C is the set of atoms of the subalgebra generated by $\bigcup ran(V)$. Although the objects in C are atoms in $\mathfrak{A}$, they are not in $\mathcal{RO}$. So we call them *corridors*. Note as $(\bigcup \mathsf{C})^{\perp\perp} = \mathbb{R}^n$, we have that $\forall X \in \mathcal{RO}$, $\exists \mathsf{F} \subseteq \mathsf{C}$ such that $X \subseteq (\bigcup \mathsf{F})^{\perp\perp}$. I.e. every regular open set has a covering which is a finite regular union of corridors. More specifically, for each $C \in \mathsf{C}$, if $\mathsf{S}^C = \{X \in \mathfrak{RO} \mid X \subseteq C \in \mathsf{C}\}$, then $\mathsf{S} = \{\mathsf{S}^C \mid C \in \mathsf{C}\}$ is a partitioning of the set of objects of $\mathfrak{CH}$ dominated by $C \in \mathsf{C}$. Since each element of C is an element of $\mathcal{RO}$, and $\mathcal{CH}$ is a subalgebra of $\mathcal{RO}$, S is also a partitioning of the set of objects of $\mathfrak{RO}$ under some corridor.

Definition 7.4.9 (Atomless Corridor Functions)**.** π, o, κ: $|\mathcal{RO}| \rightarrow Pow(\mathsf{C})$ are defined as follows:

- $\pi(X) = \{C \in \mathsf{C} \mid C \subseteq X\}$,
- $o(X) = \{C \in \mathsf{C} \mid C \nsubseteq X \wedge O(C, X)\}$
- $\kappa(X) = \{C \in \mathsf{C} \mid O(C, X)\}$,

We call $\kappa(X)$ the *footprint* of X in C. By definition of π and o, note that $\kappa(X) = \pi(X) \cup o(X)$. Since C is the set of atoms in $\mathfrak{A}$, corridors are *disjoint*:

$$\forall x \in \mathbb{R}^n \, \forall C \in \mathsf{C} \, (x \in C \in \mathsf{C} \Longrightarrow \forall B \in \mathsf{C} \, (B \neq C \Longrightarrow x \notin B)) \, .$$

We now introduce the mereobisimulation between $\mathcal{RO}$ and $\mathcal{CH}$.

Definition 7.4.10 (Atomless Corridor Configuration)**.** $\forall X, Y \in |\mathcal{RO}|$, if $\pi(X) = \pi(Y)$ and $o(X) = o(Y)$, we say that X and Y have the same *atomless corridor configuration* and write $X \boxplus Y$. Set $\mathsf{Z} = \{(X, X') \in |\mathfrak{RO}| \times |\mathfrak{CH}| \mid X \boxplus X'\}$.

For each $X \in |\mathcal{RO}|$, we can find a $X' \in |\mathcal{CH}|$ such that $X \boxplus X'$. Assume $X \in |\mathcal{RO}|$. As corridors are pairwise disjoint, we deal with each separately and then recombine the result. For each corridor $C_i \in \kappa(X) = \{C_1, ..., C_k\}$ we select a $X' \in |\mathcal{CH}|$ such that $C_i \cap X \boxplus C_i \cap X'$. This will then imply $X \boxplus (\bigcup_{i=1}^{k} (C_i \cap X'))^{\perp\perp}$. Specifically, select an arbitrary function $f : |\mathfrak{RO}| \rightarrow |\mathfrak{CH}|$ where

$$f(X) = Y \text{ where } Y \text{ is an open nD box } \textit{proper} \text{ subset of } X.$$

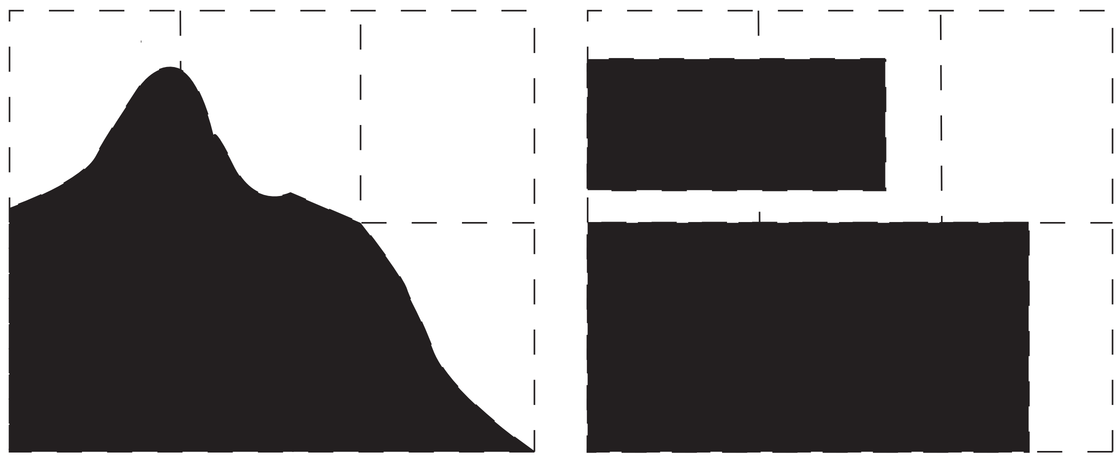


Figure 7.1: Two objects that have the same corridor configuration.

Since (1) every regular open set in $\mathcal{RO}$ is a regular union of a (possibly infinite) set of nD boxes and (2) $\mathcal{CH}$ is atomless, f exists. We employ f to select proper subsets of regular open sets which are nRCH open. Let $\Delta^{\mathcal{RO}} = |\mathcal{RO}| - |\mathcal{CH}|$ and $x \in \Delta^{\mathcal{RO}}$. Assume $\{C_1, ..., C_k\} = o(X)$. Then by definition of f, clearly $f(C_1) \in |\mathcal{CH}|$,..., $f(C_k) \in |\mathcal{CH}|$. Hence

$$\left(\left(\bigcup \pi(X)\right)^{\perp\perp} \cup \left(\bigcup_{i=1}^{k} f(C_i)\right)^{\perp\perp}\right)^{\perp\perp} \in |\mathcal{CH}|.$$

As we shall see, this has the consequence that like-named states in both models are uniquely linked. For an example of two objects with the same corridor configuration, see the figure.

7.4.4 Corridors, Coverings, and Monotonicity

In this section we prove a number of lemmas which will be used heavily in the atomless mereobisimulation argument.

Lemma 7.4.11 (Pixellated Covering)**.** Suppose V is a finite pure hybrid valuation on $\mathfrak{CH}$. Assume $\mathcal{RO} = (\mathfrak{RO}, V)$, $\mathcal{CH} = (\mathfrak{CH}, V)$, and $\mathfrak{A}$ is the finite subalgebra of $\mathfrak{RO}$ generated by $\bigcup ran(V)$. Suppose $\mathsf{C} = \{X \in |\mathfrak{A}| : At^{\mathfrak{A}}(X)\}$. For each $X \in \mathcal{RO}$, X has a covering $(\bigcup \kappa(X))^{\perp\perp} \in |\mathfrak{CH}|$ where $\kappa(X) \subseteq \mathsf{C}$ is defined by $\kappa(X) = \{C \in \mathsf{C} \mid O(C, X)\}$.

Proof. As $\mathfrak{A}$ is a Boolean subalgebra of $\mathcal{RO}$, there is a Boolean embedding $h : \mathfrak{A} \longrightarrow \mathfrak{RO}$. Thus $h(\mathbb{R}^n) = \mathbb{R}^n$. Since $\forall X \in \mathcal{RO}$, $X \subseteq \mathbb{R}^n$ and $\mathbb{R}^n$ is nRCH open, $\mathbb{R}^n$ covers X. Moreover, as h is an embedding, $(\bigcup \mathsf{C})^{\perp\perp} = \mathbb{R}^n$. Thus as C is a finite set of nRCH open sets, any X has a covering that is regular union of finitely many nRCH open sets. For all $X \in \mathcal{RO}$, if $\kappa(X) = \{C \in \mathsf{C} \mid O(C, X)\}$, then $(\bigcup \kappa(X))^{\perp\perp}$ is clearly the smallest nRCH open covering of X which is a regular union of members of C. □

To visualize how C embeds into $\mathcal{RO}$, note that corridors are nRCH open sets. So for each $C \in \mathsf{C}$, if $\varnothing \neq C \neq \mathbb{R}^n$, then the complement of $(C^{\perp} \cup C)$ represents a set of "cracks" between the regular open sets C and $C^{\perp}$: i.e. the boundary of both C and $C^{\perp}$. Indeed, as $(\bigcup \mathsf{C})^{\perp\perp} = \mathbb{R}^n$, the (normal) union consisting of each such boundary is also a nowhere dense set in the corresponding topology. For an example in the real plane, consider the four named nRCH open sets where, for a pure hybrid valuation V we have $\bigcup ran(V) = \{A, B, C, D\}$. Then the corridors generated by $\bigcup ran(V)$ are given in the following list. See the picture 7.2.

$$
\begin{aligned}
1 &= \mathbb{R}^n \cap ((A \cup B \cup C \cup D)^{\perp\perp})^{\perp}, \\
2 &= (\mathbb{R}^n \cap A) \cap ((B \cup C \cup D)^{\perp\perp})^{\perp}, \\
3 &= (\mathbb{R}^n \cap B) \cap ((A \cup C \cup D)^{\perp\perp})^{\perp}, \\
4 &= (\mathbb{R}^n \cap C) \cap ((A \cup B \cup D)^{\perp\perp})^{\perp}, \\
5 &= (\mathbb{R}^n \cap D) \cap ((A \cup B \cup C)^{\perp\perp})^{\perp}, \\
6 &= (\mathbb{R}^n \cap A \cap B) \cap ((C \cup D)^{\perp\perp})^{\perp}, \\
7 &= (\mathbb{R}^n \cap A \cap C) \cap ((B \cup D)^{\perp\perp})^{\perp}, \\
8 &= (\mathbb{R}^n \cap B \cap C) \cap ((A \cup D)^{\perp\perp})^{\perp}, \\
9 &= (\mathbb{R}^n \cap A \cap B \cap C) \cap (D^{\perp\perp})^{\perp}.
\end{aligned}
$$

Thus given the way A, B, C, and D overlap, $\bigcup ran(V)$ gives rise to 8 corridors.

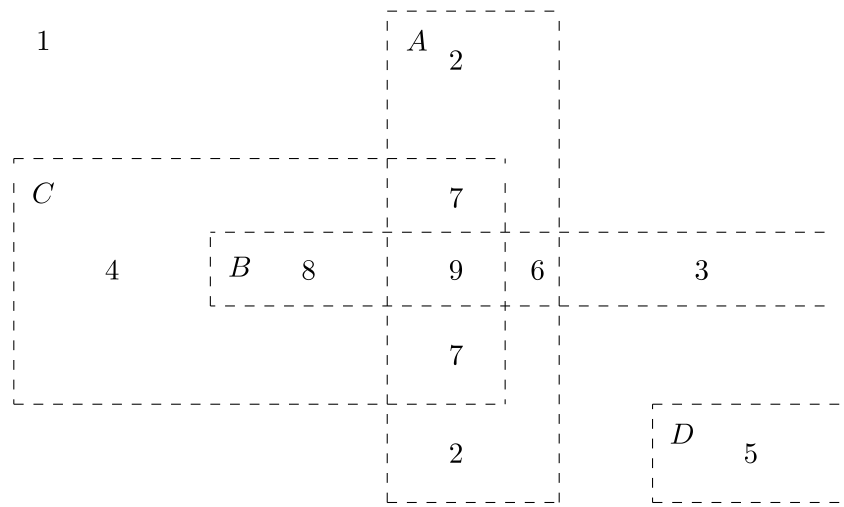


Figure 7.2: 8 corridors generated by 4 named nRCH open subsets of $\mathbb{R}^2$.

Another subset of nRCH opens of the same size could give rise to more corridors. According to a well-known result, a *finite* subset E of a BA generates a Boolean subalgebra no larger than $2^{2^{|E|}}$. Thus, overlapped differently, a generating set of size four could potentially give rise to 16 corridors.

Proposition 7.4.12. (π/κ-*Monotonicity*) Let $X, X', Y, Y' \in \mathcal{RO}$. (i) $X \subseteq Y \Rightarrow \pi(X) \subseteq \pi(Y)$. (ii) $X \subseteq Y \Rightarrow \kappa(X) \subseteq \kappa(Y)$. (*Regular Union Inclusion*) (iii) If $X \subseteq X'$ and $Y \subseteq Y'$, then $(X \cup Y)^{\perp\perp} \subseteq (X' \cup Y')^{\perp\perp}$.

Proof. (i) Let $X \subseteq Y$ and $C \in \pi(X)$. Thus $C \subseteq X$. By transitivity, $C \subseteq Y$. By definition of π, $C \in \pi(Y)$. (ii) Let $X \subseteq Y$ and $C \in \kappa(X)$. Thus $O(C, X)$. Hence since $X \subseteq Y$, $O(C, Y)$. By definition of κ, $C \in \kappa(Y)$. (iii) As $X \subseteq X'$ and $Y \subseteq Y'$ we have $(X \cup X') \subseteq (Y \cup Y')$. By two applications of lemma 3.4 part (i), we have the desired result. $\square$

Lemma 7.4.13. Let V be a finite pure hybrid valuation on $\mathfrak{CH}$. Assume $\mathcal{CH} = (\mathfrak{CH}, V)$ and $\mathcal{RO} = (\mathfrak{RO}, V)$. Suppose A^V is the finite subalgebra of $\mathfrak{RO}$ generated by $\bigcup ran(V)$. Let $\mathsf{C} = \{X \in |A^V| : At^{A^V}(X)\}$. $\forall X, Y \in \mathcal{RO}$:

1. $\bigcap \mathsf{C} = \varnothing$ and $(\bigcup \mathsf{C})^{\perp\perp} = \mathbb{R}^n$,

2. $(\bigcup \mathsf{C})^{-} = \mathbb{R}^n$,

3. $\pi(X) \cap o(X) = \varnothing$,

4. $X = \left((\bigcup \pi(X))^{\perp\perp} \cup \left((\bigcup o(X))^{\perp\perp} \cap X\right)\right)^{\perp\perp}$,

5. $\exists i \in \Omega \ (V(i) = \{X\}) \Longrightarrow$
$$\exists \mathsf{F} \subseteq \mathsf{C} \ \left(\left(\bigcup \mathsf{F}\right)^{\perp\perp} = X \text{ where } \pi(X) = \mathsf{F} \text{ and } o(X) = \varnothing\right),$$

6. $\forall x \in \mathbb{R}^n \ \forall C \in \mathsf{C} \ \left(x \in C \Longrightarrow \forall \mathsf{F} \subseteq \mathsf{C} \ \left(x \notin \left((\bigcup \mathsf{F})^{-} - \bigcup \mathsf{F}\right)\right)\right)$,

7. $\forall C \in \mathsf{C} \left(C \subseteq (\bigcup \pi(X))^{\perp\perp} \Longrightarrow C \in \pi(X)\right)$,

8. $\pi(X) = \pi\left((\bigcup \pi(X))^{\perp\perp}\right)$,

9. $\pi(X) \subseteq \pi(Y) \Longleftrightarrow (\bigcup \pi(X))^{\perp\perp} \subseteq (\bigcup \pi(Y))^{\perp\perp}$.

Proof. 1. As A^V is a Boolean subalgebra of sets of $\mathcal{CH}$ and $\mathcal{RO}$, there is an *Boolean* embedding h where $h : A^V \to \mathcal{CH}$ and $h : A^V \to \mathcal{RO}$. So $h((\bigcup \mathsf{C})^{\perp\perp}) = \mathbb{R}^n$ and $h(\bigcap \mathsf{C}) = \varnothing$.

2. Let $x \in \mathbb{R}^n$. Thus we have $x \in (\bigcup \mathsf{C})^{\perp\perp} = \mathbb{R}^n \Longleftrightarrow x \notin (\bigcup \mathsf{C})^{\perp} = \varnothing \Longleftrightarrow x \notin\sim ((\bigcup \mathsf{C})^{-}) = \varnothing \Longleftrightarrow x \in (\bigcup \mathsf{C})^{-} = \mathbb{R}^n$.

3. $\forall X \in |\mathcal{RO}| \ \forall C \in \mathsf{C}$ we have $C \in \pi(X) \Longleftrightarrow (C \in \kappa(X) \wedge C \subseteq X) \Longleftrightarrow (C \in$

$\kappa(X) \wedge C \notin o(X)) \Longleftrightarrow C \notin o(X)$.

$$
\begin{array}{llll}
4. \ X & = & (\bigcup \kappa(X))^{\perp\perp} \cap X & \text{[Lemma 7.4.11]} \\
& = & (\bigcup (\pi(X) \cup o(X)))^{\perp\perp} \cap X & \text{[Substitution]} \\
& = & (\bigcup \pi(X) \cup \bigcup o(X))^{\perp\perp} \cap X & \text{[Set equivalence]} \\
& = & ((\bigcup \pi(X))^{\perp} \cap (\bigcup o(X))^{\perp})^{\perp} \cap X & \text{[De Morgan's]} \\
& = & ((\bigcup \pi(X))^{\perp\perp} \cup (\bigcup o(X))^{\perp\perp})^{\perp\perp} \cap X & \text{[De Morgan's]} \\
& = & (((\bigcup \pi(X))^{\perp\perp} \cap X) \cup ((\bigcup o(X))^{\perp\perp} \cap X))^{\perp\perp} & \text{[Distributivity]} \\
& = & ((\bigcup \pi(X))^{\perp\perp} \cup ((\bigcup o(X))^{\perp\perp} \cap X))^{\perp\perp} & \text{[Definition of } \pi]
\end{array}
$$

5. Suppose $\exists i \in \Omega$ where $V(i) = \{X\}$. Then, $X \in \bigcup ran(V)$ and by the existence of the embedding $h : A^V \to |\mathcal{RO}|$, there is a $h^{-1}(X) \in |A^V|$. We set $\mathsf{F} = \{C \in A^V \mid C \subseteq^{A^V} h^{-1}(X) \text{ and } At(C)\}$. By definition of C, $\mathsf{F} \subseteq \mathsf{C}$. So $\pi(X) = \mathsf{F}$ and $(\bigcup \mathsf{F})^{\perp\perp} = X$. By part 3, $o(X) = \varnothing$.

6. Let $x \in \mathbb{R}^n$ and $x \in C$ for some $C \in \mathsf{C}$. Then, we have $x \in \bigcup \mathsf{C}$ and $x \notin (\bigcup \mathsf{C})^{-} - (\bigcup \mathsf{C})$. By part 2 we have $x \notin \mathbb{R}^n - \bigcup \mathsf{C}$. Let $\mathsf{F} \subseteq \mathsf{C}$. Clearly $((\bigcup \mathsf{F})^{-} - \bigcup \mathsf{F})) \subseteq (\mathbb{R}^n - \bigcup \mathsf{C})$. Therefore we have $x \notin ((\bigcup \mathsf{F})^{-} - \bigcup \mathsf{F}))$.

7. Let $C \in \mathsf{C}$ and $C \subseteq (\bigcup \pi(X))^{\perp\perp}$. Let $x \in C$. Then $x \in (\bigcup \pi(X))^{\perp\perp}$ and $x \notin (\bigcup \pi(X))^{\perp -}$. Thus as $(\bigcup \pi(X))^{\perp} \subseteq (\bigcup \pi(X))^{\perp -}$, $x \notin (\bigcup \pi(X))^{\perp}$. Hence $x \in (\bigcup \pi(X))^{-}$. By proposition 7.4.2, we have

$$\left(\left(\bigcup \pi(X)\right)^{\perp\perp} - \bigcup \pi(X)\right) \subseteq \left(\left(\bigcup \pi(X)\right)^{-} - \bigcup \pi(X)\right).$$

By part 6, $x \notin ((\bigcup \pi(X))^{-} - \bigcup \pi(X)))$. Thus $x \notin ((\bigcup \pi(X))^{\perp\perp} - \bigcup \pi(X))$. Therefore $x \in (\bigcup X)$. As C is the set of atoms of A^V, if $x \in C$, then $\forall B \in \mathsf{C}$ if $B \neq C$, we have $x \notin B$. So $C \in \pi(X)$ as required.

8. ($\Rightarrow$) Suppose $C \in \pi(X)$. Hence $C \subseteq \bigcup \pi(X)$. By lemma 7.4.4 part (ii) $C \subseteq (\bigcup \pi(X))^{\perp\perp}$. Then by definition of π we have $C \in \pi((\bigcup \pi(X))^{\perp\perp})$. ($\Leftarrow$) Suppose $C \in \pi((\bigcup \pi(X))^{\perp\perp})$. By definition of π we have $C \subseteq (\bigcup \pi(X))^{\perp\perp}$. As $C \in \mathsf{C}$, by 7 we have $C \in \pi(X)$.

9. ($\Rightarrow$) Via two applications of lemma 7.4.4 part (i). ($\Leftarrow$) Suppose $(\bigcup \pi(X))^{\perp\perp} \subseteq$

$(\bigcup \pi(Y))^{\perp\perp}$ and let $C \in \pi(X)$. Now

$$\begin{aligned}(\bigcup \pi(X))^{\perp\perp} \subseteq (\bigcup \pi(Y))^{\perp\perp} &\implies \pi((\bigcup \pi(X))^{\perp\perp}) \subseteq \pi((\bigcup \pi(Y))^{\perp\perp}) \\ & \text{[by } \pi\text{-monotonicity]} \\ &\implies \pi(X) \subseteq \pi(Y) \text{ [by part 8].}\end{aligned}$$

□

7.4.5 The Mereobisimulation between $\mathcal{CH}$ and $\mathcal{RO}$

Theorem 7.4.14. Let V be a finite hybrid valuation on $\mathfrak{CH}$. Suppose $\mathcal{RO} = (\mathfrak{RO}, V)$ and $\mathcal{CH} = (\mathfrak{CH}, V)$. Then $\mathcal{RO} \triangleq \mathcal{CH}$.

Proof. We show that Z is a mereobisimulation between $\mathcal{RO}$ and $\mathcal{CH}$. Let C be the corridors generated from $\bigcup ran(V)$. Assume $X \in |\mathcal{RO}|$ and $X' \in |\mathcal{CH}|$.

Condition 1. Suppose $X\mathsf{Z}X'$ and X is named by some nominal i. Thus $V(i) = \{X\}$. By part 5 of lemma 7.4.13, there is a $\mathsf{F} \subseteq \mathsf{C}$ such that $\pi(X) = \mathsf{F}$, $(\bigcup \mathsf{F})^{\perp\perp} = X$, and $o(X) = \varnothing$. Since $X\mathsf{Z}X'$ $o(X) = o(X') = \varnothing$ and $\pi(X) = \pi(X') = \mathsf{F}$. Hence $(\bigcup \mathsf{F})^{\perp\perp} = X'$, as required. Supposing X' is named the desired result follows analogously.

Condition 2. As $\mathcal{RO}$ and $\mathcal{CH}$ have the same valuation, this case is immediate.

Condition 3. We do only the back cases. The forth cases are entirely analogous. Fix an arbitrary function $f : \mathfrak{RO} \to \mathfrak{CH}$ such that $f(U) = W$ if an only if W is an open box proper subset of U.

Back-$\leq$ Case. Suppose $X \subseteq Y$ and $X\mathsf{Z}X'$. We must show that $\exists Y' \in |\mathcal{CH}|$ such that $Y\mathsf{Z}Y'$ and $X' \subseteq Y'$. We first define Y'. Note that the footprint $\kappa(Y)$ of Y is $\pi(Y) \cup (o(X) \cap o(Y)) \cup (o(Y) - o(X))$. Let $A = (\bigcup \pi(Y))^{\perp\perp}$. Assume $o(X) \cap o(Y) = \{B_1, ..., B_k\}$ and set

$$B = \left(\bigcup_{i=1}^{k} \left(B_i \cap \left(f\left(B_i \cap (X')^{\perp} \right) \right)^{\perp} \right) \right)^{\perp\perp}.$$

That is, for each $B_i \in o(X) \cap o(Y)$, we identify an open box $f(B_i \cap (X')^{\perp})$ proper

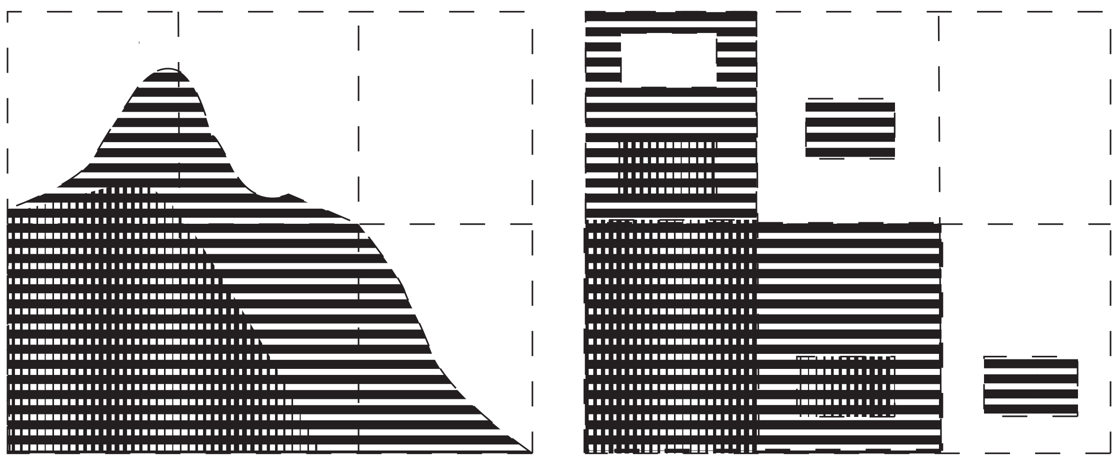


Figure 7.3: Example in the real plane for the Back-$\leq$ case. The same 6 corridors are represented on both sides. On the left we have two objects $X, Y \in |\mathcal{RO}|$. Y is the horizontally striped region whose footprint consists of 5 corridors, and X is the vertically striped region whose footprint consists of 3. On the right we have two nRCH sets. Y' is the horizontally striped region, and X' is the vertically striped. Corridors subsets are selected so that $X \mathsf{Z} X'$ and $Y \mathsf{Z} Y'$.

subset of $B_i \cap (X')^{\perp}$, and take the regular complement of the box with respect to B_i: $B_i \cap (f(B_i \cap (X')^{\perp}))^{\perp}$. Hence B is equivalent to the regular union of each region obtained, for all i. Next assume $o(Y) - o(X) = \{C_1, ..., C_l\}$ and set $C = (\bigcup_{i=1}^{l} f(C_i))^{\perp\perp}$. Let $Y' = (A \cup B \cup C)^{\perp\perp}$. Since $\kappa(Y')$ is finite and f selects a single open box, we have $Y' \in |\mathcal{CH}|$, as required.

Claim 1. $X' \subseteq Y'$. As $X \mathsf{Z} X'$, we have $\pi(X) = \pi(X')$ and therefore $(\bigcup \pi(X))^{\perp\perp} = (\bigcup \pi(X'))^{\perp\perp}$. As $X \subseteq Y$, we have $\pi(X) \subseteq \pi(Y)$ by monotonicity. Therefore we have $(\bigcup \pi(X))^{\perp\perp} \subseteq (\bigcup \pi(Y))^{\perp\perp}$ by part 9 lemma 7.4.13. Hence $(\bigcup \pi(X))^{\perp\perp} \subseteq A$ by definition of A. Thus $(\bigcup \pi(X'))^{\perp\perp} \subseteq A$. We next show that $((\bigcup o(X'))^{\perp\perp} \cap X') \subseteq B$. First we prove that for each $B_i \in \{B_1, ..., B_k\} = o(X')$, $(B_i \cap X') \subseteq (B_i \cap (f(B_i \cap (X')^{\perp}))^{\perp})$. Let $x \in (B_i \cap X')$. As $B_i \in o(X')$, we have $(B_i \cap X') \subset B_i$, and therefore $x \notin B_i \cap (B_i \cap (X')^{\perp})$. And as $f(B_i \cap (X')^{\perp})$ is an open box proper subset of $B_i \cap (X')^{\perp}$, we have $x \notin B_i \cap f(B_i \cap (X')^{\perp})$. So we have $x \in B_i \cap (f(B_i \cap (X')^{\perp}))^{\perp}$. Thus by regular union inclusion, $(\bigcup_{i=1}^{k}(B_i \cap X'))^{\perp\perp} \subseteq (\bigcup_{i=1}^{k}(B_i \cap f(B_i \cap (X')^{\perp})))^{\perp\perp}$. By distributivity, $((\bigcup_{i=1}^{k} B_i)^{\perp\perp} \cap X') \subseteq (\bigcup_{i=1}^{k}(B_i \cap f(B_i \cap (X')^{\perp})))^{\perp\perp}$. By definition of $o(X')$ and B, $((\bigcup o(X'))^{\perp\perp} \cap X') \subseteq B$. Finally, as we have shown $(\bigcup \pi(X'))^{\perp\perp} \subseteq A$ and $((\bigcup o(X'))^{\perp\perp} \cap X') \subseteq B$, by regular union inclusion we have $((\bigcup \pi(X'))^{\perp\perp} \cup ((\bigcup o(X'))^{\perp\perp} \cap X'))^{\perp\perp} \subseteq (A \cup B)^{\perp\perp}$. By lemma 7.4.13 part 4, we have $X' \subseteq (A \cup B)^{\perp\perp}$. Since $(A \cup B)^{\perp\perp} \subseteq (A \cup B \cup C)^{\perp\perp}$, by transitivity and definition of Y' we have $X' \subseteq Y'$.

Claim 2. $Y \mathsf{Z} Y'$. By lemma 7.4.13 part 3, we have $\kappa(Y') = \pi(Y') \cup o(Y')$ and $\varnothing = \pi(Y') \cap o(Y')$. No two distinct corridors overlap, hence by definition of B and C, $o(Y') = (o(X) \cap o(Y)) \cup (o(Y) - o(X)) = o(Y)$. And moreover, by definition of A, we have $\pi(Y) = \pi(Y')$.

Back-$\geq$ Case. Suppose $Y \subseteq X$ and $X \mathsf{Z} X'$. We must show $\exists Y' \in |\mathcal{CH}|$ such that $Y \mathsf{Z} Y'$ and $Y' \subseteq X'$. Observe first that as $Y \subseteq X$ and $X \mathsf{Z} X'$, by κ-monotonicity, we have $\kappa(Y) \subseteq \kappa(X)$. Thus, by part 3 of lemma 7.4.13, $\pi(Y) \cup o(Y) \subseteq \pi(X) \cup o(X)$. And as $X \mathsf{Z} X'$, we have $\pi(Y) \cup o(Y) \subseteq \pi(X') \cup o(X')$. By π-monotonicity, we have $\pi(Y) \subseteq \pi(X')$. And $o(Y) = (o(X') \cap o(Y) \cup (\pi(X') \cap o(Y))$. Hence, the footprint $\kappa(Y)$ of Y is $\pi(Y) \cup (o(X') \cap o(Y)) \cup (\pi(X') \cap o(Y))$. To define Y', we select piecemeal the proper corridors and corridor-fragments. Let $A = (\bigcup \pi(Y))^{\perp\perp}$. Let $B = (\bigcup (o(X') \cap o(Y)))^{\perp\perp} \cap X'$. Assume that $\pi(X') \cap o(Y) = \{C_1, ..., C_k\}$ and set $C = (\bigcup_{i=1}^{k} f(C_i))^{\perp\perp}$.

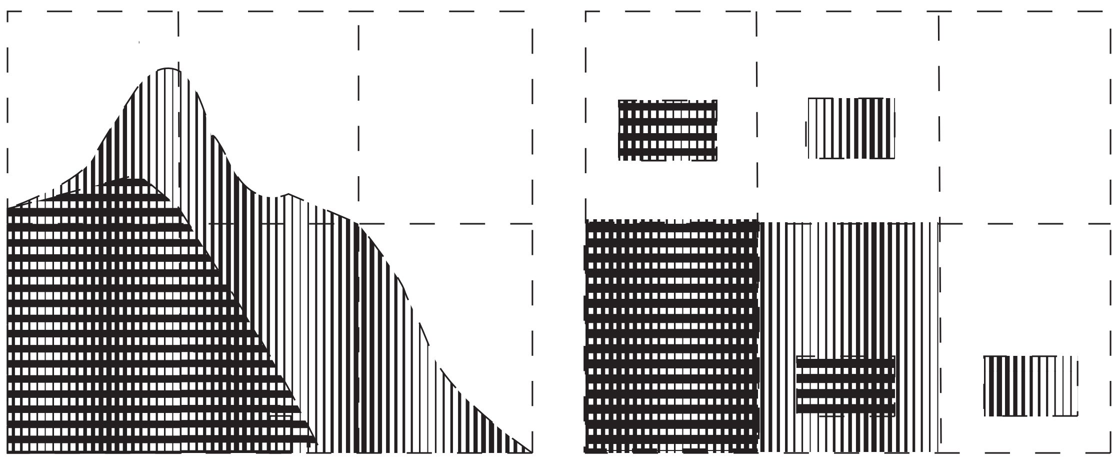


Figure 7.4: Example in the real plane for the Back-$\geq$ case. The same 6 corridors are represented on both sides. On the left we have two objects $X, Y \in |\mathcal{RO}|$. Y is the horizontally striped region, and X is the vertically striped region. On the right we have two nRCH sets. Y' is the horizontally striped region, and X' is the vertically striped. Corridors subsets are selected so that $X \mathsf{Z} X'$ and $Y \mathsf{Z} Y'$.

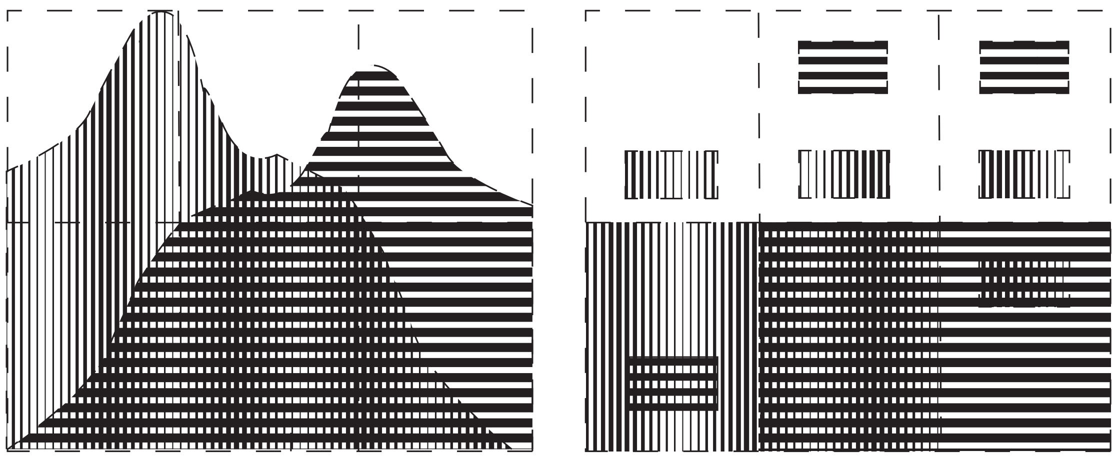


Figure 7.5: Example in the real plane for the Back-$\nleq$ case. The Back-$\ngeq$ is entirely analogous. The right side represents six corridors in $\mathcal{CH}$. The left side represents the same six corridors in $\mathcal{RO}$. On the left, X is the vertically striped region and Y is the horizontally striped region. On the right, X' is the vertically striped nRCH region and Y' is the vertically striped.

Now set $Y' = (A \cup B \cup C)^{\perp\perp}$.

Claim 1. $Y' \subseteq X'$. As $X\mathsf{Z}X'$, we have $\pi(X) = \pi(X')$ and therefore $(\bigcup \pi(X))^{\perp\perp} = (\bigcup \pi(X'))^{\perp\perp}$. As $Y \subseteq X$, we have $\pi(Y) \subseteq \pi(X)$ by monotonicity. Therefore we have $(\bigcup \pi(Y))^{\perp\perp} \subseteq (\bigcup \pi(X))^{\perp\perp}$ by part 9 lemma 7.4.13. Hence by definition of A, $A \subseteq (\bigcup \pi(X))^{\perp\perp}$ and $A \subseteq (\bigcup \pi(X'))^{\perp\perp}$. Clearly by definition of C, $C \subseteq (\bigcup \pi(X'))^{\perp\perp}$, and thus $(A \cup C) \subseteq (\bigcup \pi(X'))^{\perp\perp}$. Next observe we have $B \subseteq (\bigcup(o(X') \cap o(Y)))^{\perp\perp} \cap X'$; and as $(\bigcup(o(X') \cap o(Y)))^{\perp\perp} \cap X' \subseteq (\bigcup(o(X'))^{\perp\perp} \cap X')$, by transitivity we have $B \subseteq (\bigcup(o(X'))^{\perp\perp} \cap X')$. By regular union inclusion we have $(A \cup B \cup C)^{\perp\perp} \subseteq ((\bigcup \pi(X))^{\perp\perp} \cup (\bigcup(o(X'))^{\perp\perp} \cap X'))^{\perp\perp}$. By definition of Y' and part 4 lemma 7.4.13 we have $Y' \subseteq X'$.

Claim 2. $Y\mathsf{Z}Y'$. No two distinct corridors overlap. Therefore, we have $o(Y') = (o(X') \cap o(Y)) \cup (\pi(X') \cap o(Y)) = o(Y)$ and $\pi(Y) = \pi(Y')$.

Back-$\nleq$ Case. Suppose $X \nsubseteq Y$ and $X\mathsf{Z}X'$. We must show that $\exists Y' \in |\mathcal{CH}|$

such that $Y\mathsf{Z}Y'$ and $X' \nsubseteq Y'$. The footprint $\kappa(Y)$ of Y is $\pi(Y) \cup (\pi(X) \cap o(Y)) \cup (o(X) \cap o(Y)) \cup (o(X) - o(Y))$. Let $A = (\bigcup \pi(Y))^{\perp\perp}$. Assume that $\pi(X) \cap o(Y) = \{B_1, ..., B_k\}$ and set $B = (\bigcup_{i=1}^{k} f(B_i))^{\perp\perp}$. Let $o(X) \cap o(Y) = \{C_1, ..., C_l\}$ and set $C = (\bigcup_{i=1}^{l} f(C_i \cap (X')^{\perp}))^{\perp\perp}$. Assume that $o(X) - o(Y) = \{D_1, ..., D_m\}$ and set $D = (\bigcup_{i=1}^{m} f(D_i))^{\perp\perp}$. And let $Y' = (A \cup B \cup C \cup D)^{\perp\perp}$.

Claim 1. $X' \nsubseteq Y'$. There are two cases. First, assume that $\pi(X) \subseteq \pi(Y)$. Then $\exists Z \in |\mathcal{RO}|$ and $\exists E \in o(X)$ such that $Z = X \cap E$ and $Z \nsubseteq Y$. Hence $E \in (o(X) \cap o(Y)) \cup (o(X) - o(Y))$. If $E \in o(Y) \cap o(X)$, then by definition of C, $f(E \cap (X')^{\perp})) - (X' \cap E) \neq \varnothing$ and therefore $X' \nsubseteq Y'$ as required. If $E \in o(X) - o(Y)$, then $D \neq \varnothing$ and hence $X' \nsubseteq Y'$, and we are done. Finally, supposing that $\pi(X) \nsubseteq \pi(Y)$, then trivially $X' \nsubseteq Y'$.

Claim 2. By definition of Y' and the disjointness of corridors, $Y\mathsf{Z}Y'$.

Back-$\not\geq$ Case. Suppose $Y \nsubseteq X$ and $X\mathsf{Z}X'$. We must show that $\exists Y' \in |\mathcal{CH}|$ such that $Y\mathsf{Z}Y'$ and $Y' \nsubseteq X'$. The footprint $\kappa(Y)$ of Y is $\pi(Y) \cup (\pi(Y) \cap o(X)) \cup (o(Y) \cap o(X)) \cup (o(Y) - o(X))$. Let $A = (\bigcup \pi(Y))^{\perp\perp}$. Assume that $\pi(Y) \cap o(X) = \{B_1, ..., B_k\}$ and set $B = (\bigcup_{i=1}^{k} f(B_i))^{\perp\perp}$. Set $o(Y) \cap o(X) = \{C_1, ..., C_l\}$ and let $C = (\bigcup_{i=1}^{l} f(C_i \cap (X')^{\perp}))^{\perp\perp}$. Suppose $o(Y) - o(X) = \{D_1, ..., D_m\}$ and set $D = (\bigcup_{i=1}^{m} f(D_i))^{\perp\perp}$. And let $Y' = (A \cup B \cup C \cup D)^{\perp\perp}$.

Claim 1. $Y' \nsubseteq X'$. There are two cases. First assume that $\pi(Y) \subseteq \pi(X)$. Then $\exists Z \in |\mathcal{RO}|$ and $\exists E \in o(Y)$ such that $Z = Y \cap E$ and $Z \nsubseteq X$. Hence $E \in (o(Y) \cap o(X)) \cup (o(Y) - \kappa(X)))$. If $E \in o(Y) \cap o(X)$, then by definition of C, $f(E \cap (X')^{\perp})) - (X' \cap E) \neq \varnothing$ and therefore $X' \nsubseteq Y'$ as required. If $E \in o(Y) - \kappa(X)$, then $D \neq \varnothing$ and hence $X' \nsubseteq Y'$, and we are done. Finally, supposing that $\pi(Y) \nsubseteq \pi(X)$, then trivially $Y' \nsubseteq X'$.

Claim 2. Again, by definition of Y' and corridor disjointness, $Y\mathsf{Z}Y'$. □

Theorem 7.4.15. For any $n \in \mathbb{N}$, $\mathbf{K}_{\mathsf{hm}} + \mathsf{BA} + A\neg\alpha$ is the logic of $\mathfrak{RO} = (RO(\mathbb{R}^n), \subseteq, \mathbb{R}^n, \varnothing)$.

Proof. Let $\nvdash_{\mathbf{K}_{\mathsf{hm}}+\mathsf{BA}+A\neg\alpha} \phi$. By general completeness, there is a countable canonical atomless BA-model $\mathcal{M}$ such that $\mathcal{M} \not\models \phi$. Note that ϕ is falsified on the corresponding atomless, countable BA-frame $\mathcal{F}$ of $\mathcal{M}$ with valuation V^{ϕ} obtained by restricting the valuation V to the finite subvaluation whose domain is just those nominals occurring in ϕ. Let $n \in \mathbb{N}$ and $\mathfrak{CH} = (RCH(\mathbb{R}^n)^*, \subseteq, \mathbb{R}^n, \varnothing)$. Every countable atomless BA is isomorphic. So since the canonical frame $\mathcal{F}$ is a countable atomless BA there is an isomorphism $g : \mathcal{F} \to \mathfrak{CH}$. Generate a valuation V^g for $\mathfrak{CH}$ such that $dom(V^\phi) = dom(V^g)$ and $ran(V^g) = \{g(V(i)) \mid \text{ where } i \in dom(V^\phi)\}$. Define $\mathcal{CH} = (\mathfrak{CH}, V^g)$. Thus $\mathcal{CH} \not\models \phi$. Let $\mathcal{RO} = (\mathfrak{RO}, V^g)$. V^h is finite valuation naming elements all of which appear in $\mathcal{CH}$ and therefore in $\mathcal{RO}$. By the theorem directly above $\mathcal{CH} \triangleq \mathcal{RO}$. Hence by mereobisimilar invariance, $\mathcal{RO} \not\models \phi$. □

7.5 Complete Atomic BAs

A Contrast: the FO Theory of Infinite Atomic Complete BAs. We will show that any infinite atomic complete BA model $(\mathcal{F}, V)$ with a finite pure hybrid valuation V is mereobisimilar to a model whose frame is the completion of $\mathcal{F}$. What makes the result informative is that, in comparison to FO logic, it is not generally true that an infinite atomic BA model (with relational signature $\{\leq\}$) is elementarily equivalent to is completion. For instance, it is easily seen that the formula

$$(\mathsf{M})\ \ x < y < z \wedge \forall w(x < w < y \to \exists v(x < v < w)) \wedge \forall w(y < w < z \to \exists v(w < v < z))$$

is satisfied on the infinite complete atomic BA $\mathfrak{B}$ with countably many atoms. For let $y \in \mathfrak{B}$ be the supremum of a non-finite/non-cofinite set of atoms w.r.t. $|\mathfrak{B}|$, x be the supremum of a finite number of atoms such that $x < y$, and z be the supremum of a cofinite number of atoms such that $y < z$. However on the finite/cofinite BA of the natural numbers for example, (M) is clearly unsatisfiable.

A Bit of Background. A *completion* $\mathfrak{A}$ of a BA $\mathfrak{B}$ is a BA with the following properties: (1) $\mathfrak{B}$ is a Boolean subalgebra of $\mathfrak{A}$; (2) $\forall S \subseteq |\mathfrak{B}|$, $\bigvee h[S] \in |\mathfrak{A}|$; and (3) $\forall y \in |\mathfrak{A}|$, $y = \bigvee h[S]$ for some $S \subseteq |\mathfrak{B}|$. It is well-known that condition (3) in the definition above is equivalent to the proposition that $\mathfrak{B}$ is a *dense* subset of $\mathfrak{A}$, in the sense that every non-zero element in $\mathfrak{A}$ is above a non-zero element in $\mathfrak{B}$. Hence, some simple reasoning shows that a BA $\mathfrak{A}$ is a completion of a BA $\mathfrak{B}$ if and only if $\mathfrak{B}$ is a dense subalgebra of $\mathfrak{A}$ and every subset of $\mathfrak{B}$ has supremum in $\mathfrak{A}$. Thus every element in a completion of a BA is the supremum of the set of all elements that are below it.

Next consider some well-known concepts. A non-empty subset I of a partially ordered set $(P, \leq)$ is an *ideal*, if the following conditions hold: (1) $\forall x \in I$, $y \leq x \Longrightarrow y \in I$, and (2) $\forall x, y \in I$, $\exists z \in I$ such that $x \leq z$ and $y \leq z$. The smallest ideal that contains a given element x is a *principal ideal* and in this case x is said to be the generator of the ideal. MacNeille [61] and Tarski [93] proved that every BA $\mathfrak{C}$ has a completion which is the set of all (so-called) complete ideals of $\mathfrak{C}$. An ideal I of a BA $\mathfrak{A}$ is *complete* provided that whenever the supremum of a set of elements in I exists in $\mathfrak{A}$, that supremum is a member of I. It is well-known that the class of complete ideals of $\mathfrak{A}$ forms a complete BA. To fix intuitions we consider the following two well-known theorems and two well-known facts (see [40] for the details of these proofs):

Theorem 7.5.1. Every BA $\mathfrak{B}$ has a completion $\mathfrak{A}$ which is an isomorphic copy of the BA of complete ideals in $\mathfrak{B}$ closed under the Boolean set operations.

Theorem 7.5.2. Any two completions of a BA $\mathfrak{A}$ are isomorphic via a mapping that is the identity on $\mathfrak{A}$.

Recall that a *complete embedding* is a homomorphism preserving all suprema (and consequently all infima) that happen to exist. We can define a complete embedding f of any BA $\mathfrak{A}$ into its completion. To understand why such an embedding exists, suppose $\mathfrak{B}$ is a BA and $\mathfrak{A}$ is its completion. Witness that $\forall x \in \mathfrak{B}$, the principal ideal $\downarrow x$ generated by x is a complete ideal, and hence in $\mathfrak{A}$. So we can define a mapping $f : \mathfrak{B} \longrightarrow \mathfrak{A}$ by $f(x) = \downarrow x$. If $x, y \in \mathfrak{B}$, we have:

$$f(x \vee y) = \downarrow (x \vee y) = \downarrow x \vee \downarrow y = f(x) \vee f(y)$$

$$f(\sim x) = \downarrow (\sim x) = \sim (\downarrow x) = \sim f(x).$$

Thus, f is a homomorphism. And it is one-to-one: $f(x) = f(y) \Rightarrow \downarrow x = \downarrow y \Rightarrow x = y$, since the generator of a principal ideal is the largest element in the ideal. By the facts above, it follows that the monomorphism f above is complete, since $ran(f)$ is the set of all principal ideals in $\mathfrak{A}$. Moreover the following is a well-know result:

Lemma 7.5.3. Suppose $\mathfrak{A}$ is the completion of a BA $\mathfrak{B}$. Let g be the corresponding complete embedding $g : \mathfrak{A} \rightarrow \mathfrak{B}$ Then we have (i) ($x \in \mathfrak{A}$ and $At(x)$) $\Leftrightarrow$ ($g(x) \in \mathfrak{B}$ and $At(x)$) (ii) $\mathfrak{A}$ is atomic iff $\mathfrak{B}$ is atomic.

The significance of lemma 7.5.3 to our case can be understood as follows: there exist no atoms in the completion are that not atoms in the inverse of the corresponding Boolean embedding; and atoms in the original model are embedded to atoms in the completion.

The Corridor Proof Method. If V is a pure hybrid valuation on a structure with domain A and z is any function on A, we let $z[V] = \{(i, \{z(x)\} \mid (i, \{x\}) \in V\}$. Let $\mathfrak{E}$ be any BA. We write $\mathcal{C}(\mathfrak{E})$ to denote the completion of $\mathfrak{E}$. Suppose $z : \mathfrak{E} \longrightarrow \mathcal{C}(\mathfrak{E})$ is the corresponding complete embedding. If $\mathcal{E} = (\mathfrak{E}, V)$ is a BA-model, then we let $\mathcal{C}^z(\mathcal{E}) = (\mathcal{C}(\mathfrak{E}), z[V]))$ (or just $\mathcal{C}(\mathcal{E})$ when the context is clear). We denote the set of atoms of a set S by $Atom(S)$.

If a BA $\mathfrak{E}$ is a Boolean subalgebra of a BA $\mathfrak{F}$ we write $\mathfrak{E} \trianglelefteq \mathfrak{F}$. It is well known that if $\mathfrak{E} \trianglelefteq \mathfrak{F}$, then there is a Boolean monomorphism from $\mathfrak{E}$ into $\mathfrak{F}$. In particular, as we just noted, if $\mathfrak{F} = \mathcal{C}(\mathfrak{E})$, then there is a complete embedding of $\mathfrak{E}$ into $\mathfrak{F}$—namely the identity map on $\mathfrak{E}$ (see pp.74-104 [40]).

We now explain briefly the proof method. So let $\mathcal{B} = (\mathfrak{B}, V)$ be an infinite atomic BA-model where V is a *finite* pure hybrid valuation. There exists a complete embedding $g : \mathfrak{B} \longrightarrow \mathcal{C}(\mathfrak{B})$. As g is a complete embedding, g is also Boolean embedding from $\mathcal{B}$ into $\mathcal{C}^g(\mathcal{B})$ (abbrev: $\mathcal{C}(\mathcal{B})$). It suffices show $g[\mathcal{B}] \triangleq \mathcal{C}(\mathcal{B})$ where $g[\mathcal{B}]$ is the full BA-model image $(g[|\mathcal{C}(\mathcal{B})|], \{(g(x), g(y)) \mid x \leq^{\mathcal{B}} y\}, g(1), g(0), V^g)$ of $\mathcal{B}$ in $\mathcal{C}(\mathcal{B})$. Thus in order to show the mereobisimulation we must link objects of $|\mathcal{C}(\mathcal{B})| - |g[\mathcal{B}]|$ back to ones in $g[\mathcal{B}]$ based on some mereobisimilar role. Let $\mathfrak{A}$ be the *finite* subalgebra of $\mathfrak{B}$ generated by $\bigcup ran(V)$. There is a Boolean monomorphism $h : \mathfrak{A} \longrightarrow \mathfrak{B}$. Set $\mathsf{C} = g \circ h[Atom(|\mathfrak{A}|)]$, and let $\mathcal{A} = (\mathfrak{A}, h^{-1}[V])$. Elements of C

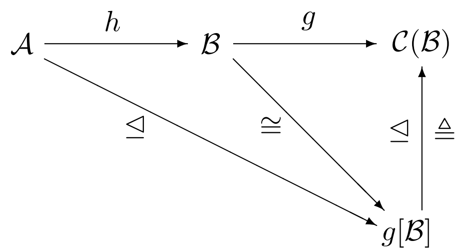


Figure 7.6: Models of the proof and their relations.

might not be atoms in either $\mathcal{B}$ or $\mathcal{C}(\mathcal{B})$, so we call them *corridors*. Since $g \circ h[\mathsf{C}] \subseteq g[\mathcal{B}] \subseteq |\mathcal{C}(\mathcal{B})|$, C gives rise to a partitioning of $Atom(\mathcal{C}(\mathcal{B}))$; i.e. $\forall x \in Atom(\mathcal{C}(\mathcal{B}))$, $\exists c \in \mathsf{C}$ such that $x \leq^{\mathcal{C}(\mathcal{B})} c$. Since $\mathfrak{A} \trianglelefteq \mathfrak{B} \trianglelefteq \mathcal{C}(\mathfrak{B})$, $g \circ h : \mathfrak{A} \longrightarrow \mathcal{C}(\mathfrak{B})$ is a Boolean monomorphism. Moreover clearly $g \circ h : \mathcal{A} \longrightarrow \mathcal{C}(\mathcal{B})$ is a Boolean monomorphism. If xOy and $y \nleq x$, we say that x *partially overlaps* y and write $xPVy$. We define the mereobisimulation as follows. Let $x\mathsf{Z}y$ if and only if

- $\forall z \in \mathsf{C}\ (z \leq x \Longleftrightarrow z \leq y)$,
- $\forall z \in \mathsf{C}\ (xPVz \Longleftrightarrow yPVz)$,
- $\forall z \in \mathsf{C}\ (xPVz \Longrightarrow (At(x \wedge z) \Leftrightarrow At(y \wedge z)))$.

Let $w \in \mathcal{C}(\mathcal{B})$. We can select a $w' \in g[\mathcal{B}]$ such that $w \triangleq w'$. Define three sets:

- $\mathsf{C}_1 = \{x \in \mathsf{C} \mid x \leq w\} = \{c_1^1, ..., c_k^1\}$,
- $\mathsf{C}_2 = \{x \in \mathsf{C} \mid xPVw\} = \{c_1^2, ..., c_l^2\}$,
- $\mathsf{C}_3 = \{x \in \mathsf{C} \mid xOw\} = \{c_1^3, ..., c_{k+l}^3\}$.

witnessing that $\mathsf{C}_1 \cap \mathsf{C}_2 = \varnothing$, $\mathsf{C}_1 \cup \mathsf{C}_2 = \mathsf{C}_3$, and moreover if $|\mathsf{C}| = j$ then $k+l \leq j$. Since the members of C are pairwise disjoint, they can be dealt with separately. Consider first C_2. Each member of C_2 is partially overlapped by w. By lemma 7.5.3, there is a function $f : \mathcal{C}(\mathcal{B}) \to g[\mathcal{B}]$ such that

$$f(x) = \begin{cases} x & \text{if} \quad At(x) \\ y & \quad\quad \text{otherwise} \end{cases}$$

where $y = a_1 \vee a_2$ for some a_1 and a_2 such that $At(a_1), At(a_2)$, and $(a_1 \vee a_2) \leq x$. We call any function meeting these conditions an *atomic matching function*. Finally let

$$w' = \left(\bigvee_{i=1}^{k} c_i^1\right) \vee \left(\bigvee_{i=1}^{l} f(c_i^2)\right).$$

Thus for any object $w \in \mathcal{C}(\mathcal{B})$, we can select an object $w' \in g[\mathcal{B}]$ with the same corridor configuration. We simply combine objects selected piecemeal from $\pi(w)$ and $o(w)$ with matching corridor configurations. We will show that the identification of such objects can be made satisfying the back and forth conditions of mereobisimulation.

Definition 7.5.4 (Corridor Functions)**.** Suppose V is a finite pure hybrid valuation on an atomic BA-model $\mathcal{B} = (\mathfrak{B}, V)$ where $g : \mathcal{B} \longrightarrow \mathcal{C}(\mathcal{B})$ is a complete embedding. Let $\mathfrak{A}$ be the finite subalgebra of $\mathcal{B}$ generated by $\bigcup ran(V)$ and $h : \mathcal{A} \longrightarrow \mathcal{B}$ the corresponding Boolean monomorphism from $\mathcal{A} = (\mathfrak{A}, h^{-1}[V])$ into $\mathcal{B}$. Suppose $\mathsf{C} = g \circ h[Atom(|\mathfrak{A}|)]$. Define functions κ, π, and o from $\mathcal{C}(\mathcal{B})$ into C such that

- $\pi(x) = \{y \in \mathsf{C} \mid y \leq x\}$,
- $o(x) = \{y \in \mathsf{C} \mid xPVy\}$,
- $\kappa(x) = \{y \in \mathsf{C} \mid xOy\}$.

For the remainder of the section, fix an infinite atomic BA-model $\mathcal{B} = (\mathfrak{B}, V)$ with a finite pure hybrid valuation V and a complete embedding $g : \mathcal{B} \rightarrow \mathcal{C}^g(\mathcal{B})$. And we abbreviate $\mathcal{C}^g(\mathcal{B})$ to $\mathcal{C}(\mathcal{B})$ as we have been doing. Finally by $\mathcal{A}$ and C we understand objects defined as in the previous definitions.

Definition 7.5.5 (Corridor Configuration)**.** For all $x, y \in \mathcal{C}(\mathcal{B})$, if

(1) $\pi(x) = \pi(y)$,

(2) $o(x) = o(y)$,

(3) $\forall c \in \mathsf{C}(c \in o(x) \Longrightarrow (At(x \wedge c) \Leftrightarrow At(y \wedge c)))$, we say that x and y *have the same corridor configuration* and write $x \boxdot y$. And set

$$\mathsf{Z} = \{(x, y) \in |g[\mathcal{B}]| \times |\mathcal{C}(\mathcal{B})| \mid x \boxdot y\}.$$

Moreover we call $\kappa(x)$ the *corridor covering of* x.

Lemma 7.5.6 (Fit Lemma)**.** Let $X \subseteq \mathsf{C}$ and Y be a finite subset of $Atom(\mathcal{C}(\mathcal{B}))$. Then $\bigvee X \vee \bigvee Y \in g[\mathcal{B}]$.

Proof. Each corridor is an element of $g[\mathcal{B}]$; thus as X is finite, $\bigvee X \in g[\mathcal{B}]$. Moreover, as $Atom(g[\mathcal{B}]) = Atom(\mathcal{C}(\mathcal{B}))$, Y is finite, and $g[\mathcal{B}]$ is a BA-model, $\bigvee Y \in g[\mathcal{B}]$. Again, as $g[\mathcal{B}]$ is a BA model, $\bigvee X \vee \bigvee Y \in g[\mathcal{B}]$. □

Monotonicity and Configurations. The most salutary property of the subalgebra $\mathfrak{A}$ is that it is finite. Let at be a characteristic function on $|\mathcal{A}|$ such that for all $at(x) = 1$ if $At(x)$ and $at(x) = 0$ if $\neg At(x)$. Let AT be a function on $Pow(|\mathcal{A}|)$ such that for each $X \in Pow(|\mathcal{A}|)$ $AT(X) = \{(x, at(x)) \mid x \in X\}$. Essentially, the principal result will show that for all $w, v \in \mathcal{C}(\mathcal{B})$, $w \triangleq v$ only if

$$\langle \pi(w), AT(\{w \wedge x \mid x \in o(w)\})\rangle = \langle \pi(v), AT(\{v \wedge x \mid x \in o(v)\})\rangle.$$

To show this hinges on the rather obvious fact that $\forall x \in \mathcal{C}(\mathcal{B})$: $x \leq^{\mathfrak{B}} \bigvee \kappa(x)$ and other ideas implicit in the following two propositions.

Proposition 7.5.7 (κ/π-Quasi-monotonicity)**.** Let $w, v \in \mathcal{C}(\mathcal{B})$. We have

(a) $w \leq v \Longrightarrow \kappa(w) \subseteq \kappa(v)$.

(b) $w \leq v \Longrightarrow \pi(w) \subseteq \pi(v)$.

Proof. Straightforward by definition of π and κ. □

Proposition 7.5.8. Let $w, v \in \mathcal{C}(\mathcal{B})$.

(a) $\bigvee \mathsf{C} = 1$ and $\bigwedge \mathsf{C} = 0$,

(b) $w \leq \bigvee \kappa(w)$,

(c) $(c, c' \in \kappa(w)$ and $c \neq c') \Longrightarrow \neg cOc'$,

(d) $\pi(w) \cap o(w) = \varnothing$,

(e) $w = \bigvee \pi(w) \vee (\bigvee o(w) \wedge w)$,

(f) If w is named, then $\exists \mathsf{F} \subseteq \mathsf{C}$ such that $\bigvee \mathsf{F} = w$,

(g) $\pi(w) \subseteq \pi(v) \Longleftrightarrow \bigvee \pi(w) \leq \bigvee \pi(v)$,

(h) If $At(w)$, then

(i) $\pi(w) \neq \varnothing$ implies $\pi(w) = \{w\}$ and $o(w) = \varnothing$, and
(ii) $\pi(w) = \varnothing$ implies $\exists!1 c \in \mathsf{C}$ where $\{c\} = o(w)$.

Proof. (a) $\mathfrak{A}$ is a finite Boolean subalgebra of $\mathcal{C}(\mathfrak{B})$. Thus $g \circ h[\bigvee Atom(\mathfrak{A})] = 1$ and $g \circ h[\bigwedge Atom(\mathfrak{A})] = 0$.

(b) By reflexivity $w \leq w$. So $w \leq (\bigvee\{y \in \mathsf{C} \mid y \leq w\} \vee \bigvee\{y \in \mathsf{C} \mid wOy$ and $w \not\leq y\})$. Thus $w \leq \bigvee(\pi(w) \cup o(w))$ and $w \leq \bigvee \kappa(w)$.

(c) By definition, $\mathsf{C} = g \circ h[Atom(\mathfrak{A})]$. Let $a, a' \in Atom(\mathfrak{A})$ and $a \neq a'$. Thus $a \wedge^{\mathfrak{A}} a' = 0$. As $h : \mathfrak{A} \to \mathfrak{B}$ is a Boolean embedding, we have $h(a) \wedge^{\mathfrak{B}} h(a') = 0$. So as g is a Boolean embedding, $g(h(a)) \wedge^{\mathcal{C}(\mathcal{B})} g(h(a')) = g(0) = 0$. Thus the members of C are pairwise disjoint. Thus $\kappa(w)$ is a subset of $\mathcal{C}(\mathcal{B})$ whose members are pairwise disjoint.

(d) Trivial by definition of π and o and (c).

(e) $\forall c \in \kappa(w)$, either $c \leq^{\mathcal{C}(\mathfrak{B})} w$ or $wPVc$. By part (b), we have $x \leq^{\mathcal{C}(\mathfrak{B})} \bigvee \kappa(x)$. By definition $x \leq^{\mathcal{C}(\mathfrak{B})} \bigvee(\pi(x) \cup o(x))$. Hence $x \leq^{\mathcal{C}(\mathfrak{B})} \bigvee \pi(x) \vee \bigvee o(x)$. By definition of π, $\bigvee \pi(x) \leq^{\mathcal{C}(\mathfrak{B})} x$. By part (d) and definition of o, we have $(x - \bigvee \pi(x)) \leq^{\mathcal{C}(\mathfrak{B})} o(x)$. So $(x - \bigvee \pi(x)) = o(x) \wedge x$. Thus $w = \bigvee \pi(w) \vee (\bigvee o(w) \wedge w)$ as desired.

(f) If w is named, $g^{-1}(w)$ is in the generating set for $\mathfrak{A}$. So $h^{-1}(g^{-1}(w)) \in \mathfrak{A}$. As $\mathfrak{A}$ is finite, it's atomic; and therefore there is a set of atoms A in $\mathcal{A}$ such that $h^{-1}(g^{-1}(w)) = \bigvee A$. As $g \circ h : \mathfrak{A} \to \mathcal{C}(\mathfrak{B})$ is a Boolean embedding, $w = g(h(\bigvee A)) = \bigvee g[h[A]]$. By definition $g[h[A]] \subseteq \mathsf{C}$.

(g) ($\Rightarrow$) is obvious. ($\Leftarrow$). Assume $\bigvee \pi(w) \leq \bigvee \pi(v)$ and let $x \in \pi(w)$. Thus $x \leq \bigvee \pi(w)$. By transitivity, $x \leq \bigvee \pi(v)$. By quasi-monotonicity, $\pi(x) \leq \pi(\bigvee \pi(v))$. As $\pi(\bigvee \pi(v)) = \pi(v)$ we have $x \in \pi(v)$.

(h) Let $At(w)$. (i) Assume $\pi(w) \neq \varnothing$. Note that if $x \in Atom(\mathcal{A})$, then as g and h are Boolean embeddings, $g(h(x)) \neq 0$. Thus as $At(w)$ and any $c \in \pi(w)$ is such that $c \leq w$, we have $\pi(w) = \{w\}$. Hence $w = \bigvee \pi(w)$ and, by (e), $\bigvee \pi(w) = \bigvee \pi(w) \vee (\bigvee o(w) \wedge w)$. So $o(w) = \varnothing$. (ii) Suppose $\pi(w) = \varnothing$. By (e), $w = \bigvee \pi(w) \vee (\bigvee o(w) \wedge w)$. So $w = \bigvee o(w) \wedge w$. Hence by part (b), $w \leq \bigvee o(w)$. And by disjointness of corridors, there is exactly one $c \in \mathsf{C}$ such that $w \leq c$. □

c-Configurations. Under corridors, there is a complexity which arises due to the ability to discriminate atoms from non-atoms. This concerns property (3) of definition 7.5.5: $\forall c \in \mathsf{C}(c \in o(w) \Longrightarrow (At(w \wedge c) \Longleftrightarrow At(w' \wedge c)))$. Suppose $w\mathsf{Z}w'$. Let $v' \leq w'$ and $o(v') \cap o(w') \neq \varnothing$. As $w\mathsf{Z}w'$, note $o(v') \cap o(w') = o(v') \cap o(w)$. We must show that $\exists v \in g[\mathcal{B}]\ \ v \leq w$ and $v\mathsf{Z}v'$ which implies we must demonstrate the existence of objects below each member of $o(v') \cap o(w)$ meeting condition (3) with a supremum, say c, that is such that $c \leq w$. In general, the property must hold for each relation $\leq, \geq, \nleq, \ngeq$.

Lemma 7.5.9 (c-configurations)**.** Let R be any of $\leq, \geq, \nleq, \ngeq$ uniformly substituted throughout the scope of either (i) or (ii) below.

(i) Let $w', v' \in \mathcal{C}(\mathcal{B})$, $w \in g[\mathcal{B}]$ and $w'Rv'$. Suppose that $w\mathsf{Z}w'$ and $o(w') \cap o(v') = \{c'_1, ..., c'_k\}$. Then $\exists c \in g[\mathcal{B}]$ such that:

(a) $o(c) \cap o(w) = o(v') \cap o(w')$;

(b) $\forall i \in \{1, ..., k\} : (c'_i \wedge w)R(c'_i \wedge c)$ and $c'_i \in o(v') \Longrightarrow (At(c'_i \wedge v') \Leftrightarrow At(c'_i \wedge c))$.

(ii) Let $w, v \in g[\mathcal{B}]$, $w' \in \mathcal{C}(\mathcal{B})$ and wRv. Suppose that $w\mathsf{Z}w'$ and $o(w) \cap o(v) = \{c_1, ..., c_k\}$. Then $\exists c' \in \mathcal{C}(\mathcal{B})$ such that:

(a) $o(c') \cap o(w') = o(v) \cap o(w)$;

(b) $\forall i \in \{1, ..., k\} : (c_i \wedge w')R(c_i \wedge c')$ and $c_i \in o(v) \Longrightarrow (At(c_i \wedge v) \Leftrightarrow At(c_i \wedge c'))$.

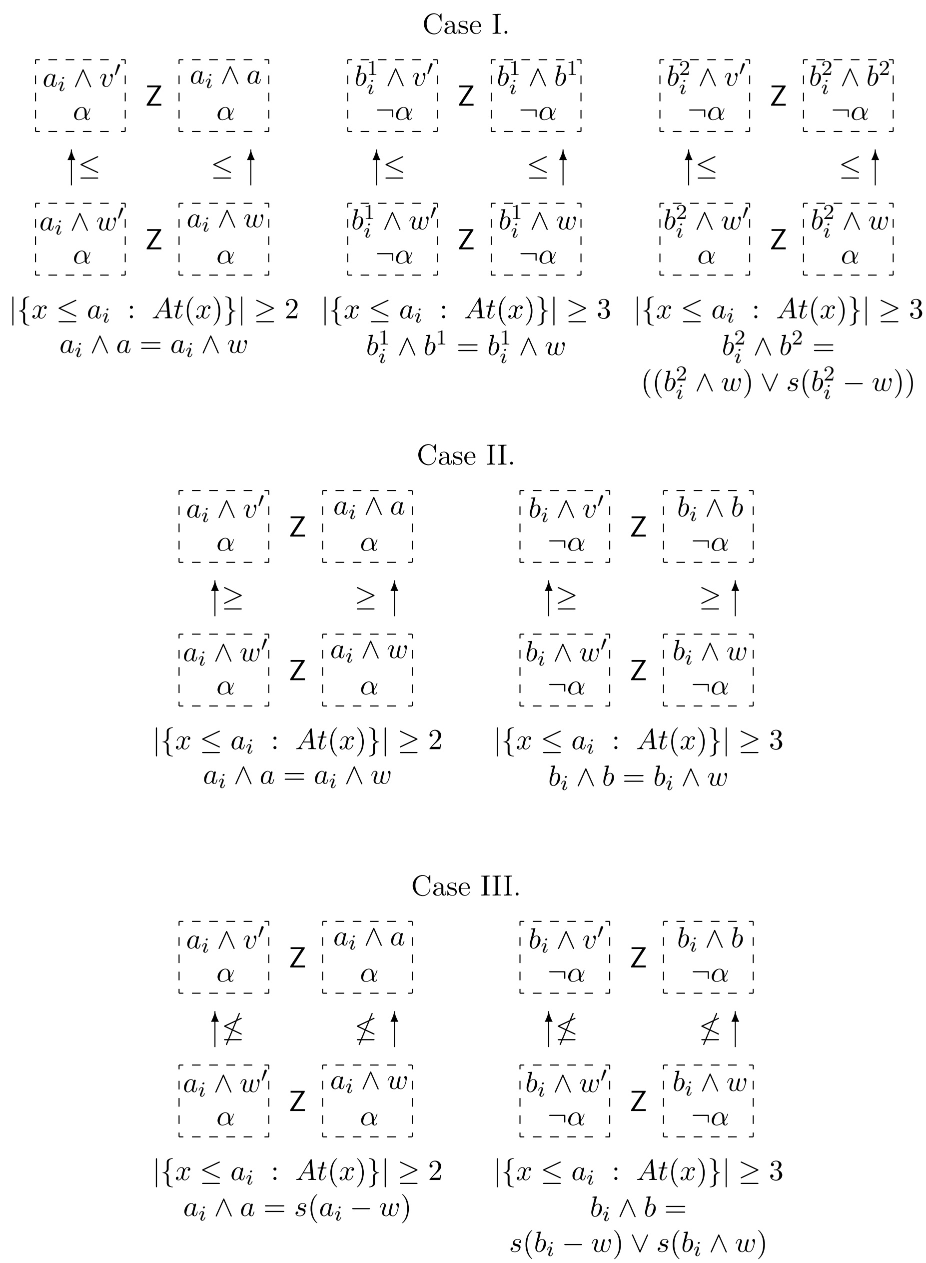


Figure 7.7: Pictorial representation of the proof of lemma 7.5.9. Squares represent objects $x \wedge y$, where x is a non-atomic corridor and y is an object such that $x \in o(y)$. α indicates $At(x \wedge y)$ and $\neg\alpha$ that $\neg At(x \wedge y)$. The selected function $s : (\mathcal{C}(\mathcal{B}) - \{0\}) \to Atom(\mathcal{C}(\mathcal{B}))$ sends its input z to an atom below z.

Proof. We prove the lemma in subcases for each of $\leq$, $\geq$, $\not\leq$, $\not\geq$. Moreover we do only the proofs for (i) in each case, as those for (ii) are proven analogously. Select a function $s : (\mathcal{C}(\mathcal{B}) - \{0\}) \to Atom(\mathcal{C}(\mathcal{B}))$ which assigns to each non-zero $x \in \mathcal{C}(\mathcal{B})$ a single atom below x.

I. $\leq$ *Case*. Let $B = C - A = \{b_1, ..., b_{k-l}\}$ where

$$\begin{array}{lclcl} C & = & o(w') \cap o(v') & = & \{c'_1, ..., c'_k\} \\ A & = & \{c'_i \in C \mid At(c'_i \wedge v')\} & = & \{a_1, ..., a_l\} \\ B^1 & = & \{b_i \in B \mid \neg At(b_i \wedge w)\} & = & \{b^1_1, ..., b^1_{k-l-n}\} \\ B^2 & = & B - B^1 & = & \{b^2_1, ..., b^2_n\}. \end{array}$$

First set $a = \bigvee_{i=1}^{l}(w \wedge a_i)$ and $b^1 = \bigvee_{i=1}^{k-l}(b^1_i \wedge w)$. Observe that for every $i \in \{1, ...n\}$, $b^2_i \in C - A$ implies $\neg At(b^2_i \wedge v')$; thus since $w \wedge b^2_i \in o(w)$, each element of B^2 must dominate at least 3 atoms. Set $b^2 = \bigvee_{i=1}^{n}((b^2_i \wedge w) \vee s(b^2_i - w))$. By definition of B^2, since for all $b^2_i \in B^2$, both v' and w partially overlap b^2_i, each $b^2 \wedge b^2_i$ partially overlaps each corridor b^2_i . We let $c = a \vee b^1 \vee b^2$. By the fit lemma, $c \in g[\mathcal{B}]$.

Claim (a): $o(c) \cap o(w) = o(v') \cap o(w')$. ($\Rightarrow$) Let $c'_i \in o(c) \cap o(w)$. By definition of c, $c'_i \in (A \cup B^1 \cup B^2) = C = o(v') \cap o(w')$. ($\Leftarrow$) Assume $c'_i \in o(v') \cap o(w')$. By proposition 7.5.8(c), the members of $o(v') \cap o(w')$ are pairwise disjoint. By definition of c, $c'_i \neq (c'_i \wedge c) \neq 0$.

Claim (b) part 1: $\forall i \in \{1, ..., k\} : (c'_i \wedge w) \leq (c'_i \wedge c)$. For all $c'_i \in C$ $c'_i \in A \cup B^1 \cup B^2$. For each $a_i \in A$, by definition of a, $(a_i \wedge w) \leq a \leq c$. For all $b^1_i \in B^1$, by definition of b^1, $(b^1_i \wedge w) \leq b^1 \leq c$. And for all $b^2_i \in B^2$, by definition of b^2, $(b^2_i \wedge w) \leq b^2 \leq c$. Thus $\forall i \in \{1, ..., k\} : (c'_i \wedge w) \leq (c'_i \wedge c)$.

Claim (b) part 2: $\forall i \in \{1, ..., k\} : c'_i \in o(v') \Longrightarrow (At(c'_i \wedge v') \Leftrightarrow At(c'_i \wedge c))$. Assume that $c'_i \in o(v')$ for some i. ($\Leftarrow$) Let $At(c'_i \wedge v')$. Then $c'_i = a_j$ for some $a_j \in A$ such that $At(w \wedge a_j)$ and $(w \wedge a_j) \leq c$. Hence $(c'_i \wedge c) \neq 0$. Now by proposition 7.5.8(c), $\forall x \in C$, if $x \neq c'_i$ then $x \wedge c'_i = 0$. Thus by definition of c, $a_j \wedge c = w \wedge a_j$. ($\Rightarrow$) If $\neg At(c'_i \wedge v')$, then $c'_i \in B$. Now if $\neg At(c'_i \wedge w)$, then by definition of b^1, $\neg At(c'_i \wedge c)$. On the other hand, if $At(c'_i \wedge w)$, then by definition of b^2, $\neg At(c'_i \wedge c)$, and, in particular, $c'_i \wedge c$ dominates exactly 2 atoms.

II. $\leq$ *Case*. As $w\mathsf{Z}w'$, we have $o(w') \cap o(v') = o(w) \cap o(v') = \{c'_1, ..., c'_k\}$. Hence for each $i \in \{1, ..., k\} : (c'_i \wedge w') \neq 0$ and $(c'_i \wedge w) \neq 0$. So it suffices to show $\forall i \in \{1, ..., k\} : \exists x_i \leq (c'_i \wedge w)$ and $At(x_i) \Leftrightarrow At(v' \wedge c'_i)$. So set $o(w) \cap o(v') = C$ and $A = \{a_1, ..., a_l\} = \{c'_i \in C \mid At(c'_i \wedge v')\}$. Now $\forall a_i \in A$, since $(a_i \wedge w) \neq 0$ and $g[\mathcal{B}]$ is atomic, $\exists y_i \leq (a_i \wedge w)$ such that $At(y_i)$. Let $a = \bigvee_{i=1}^{l} y_i$. Next let $B = C - A = \{b_1, ..., b_{k-l}\}$. Now $\forall b_i \in B$, we must select a $z_i \leq (w \wedge b_i)$ such that $\neg At(z_i)$. Set $b = \bigvee_{i=1}^{k-l}(w \wedge b_i)$. Since $v' \leq' w'$ and $o(w) = o(w')$, $\forall c'_i \in o(w) \cap o(v')$, $\neg At(v' \wedge c'_i) \Longrightarrow \neg At(w' \wedge c'_i) \Longrightarrow \neg At(w \wedge c'_i)$. So by definition of B, $\forall b_i \in B$ we have $\neg At(v' \wedge b_i)$ and thus $\neg At(w \wedge b_i)$. Finally let $c = a \vee b$. Clearly (a) and (b) hold, and by the fit lemma $c \in g[\mathcal{B}]$.

III. $\not\leq$ *Case*. Let $C = o(w') \cap o(v')$. Since $w\mathsf{Z}w'$, $C = o(w) \cap o(v')$. By definition, $c'_i \in C$ implies $\neg At(c'_i)$. Set $A = \{a_1, ..., a_l\} = \{c'_i \in C \mid At(c'_i \wedge v')\}$. For all $a_i \in A$, as $o(w) = o(w')$, w partially overlaps a_i. Thus $a_i - w$ dominates an atom. Set $a = \bigvee_{i=1}^{l} s(a_i - w)$. Now let $B = C - A = \{b_1, ..., b_{k-l}\}$. For any $i \in \{1, ..., k-l\}$, since v' partially overlaps b_i and $\neg At(b_i \wedge v')$, b_i is the supremum of at least three atoms. So set $c = \bigvee_{i=1}^{k-l}(s(b_i - w) \vee s(b_i \wedge w))$. Let $c = a \vee b$. By the fit lemma $c \in g[\mathcal{B}]$, and it is clear by definition of c that (a) and (b) hold.

IV. $\leq$ *Case*. Analogous to the previous case. □

The mereobisimulation. Recall we suppose that $\mathcal{B} = (\mathfrak{B}, V)$ is an infinite atomic BA-model and $g : \mathcal{B} \to \mathcal{C}(\mathcal{B})$ is the corresponding complete embedding where $\mathcal{C}(\mathcal{B}) = (\mathcal{C}(\mathfrak{B}), V^g)$. Let $g[\mathcal{B}] = (g[|\mathcal{C}(\mathcal{B})|], \{(g(x), g(y)) \mid x \leq^{\mathcal{B}} y\}, g(1), g(0), V^g)$. We will show $g[\mathcal{B}] \triangleq \mathcal{C}(\mathcal{B})$. As $g[\mathcal{B}] \cong \mathcal{B}$, it will be immediate that $\mathcal{B} \triangleq \mathcal{C}(\mathcal{B})$ as required.

Lemma 7.5.10. Suppose $w \in g[\mathcal{B}]$ and $w' \in \mathcal{C}(\mathcal{B})$.

(i) If $w\mathsf{Z}w'$ then $(V^g)^{-1}[\{w\}] = (V^g)^{-1}[\{w'\}]$.

(ii) If $V(i) = \{w\}$ and $V(i) = \{w'\}$ for some $i \in dom(V)$, then $w\mathsf{Z}w'$.

(iii) If $w\mathsf{Z}w'$, then $At^{g[\mathcal{B}]}(w) \Leftrightarrow At^{\mathcal{C}(\mathcal{B})}(w')$.

Proof. (i) Let $w\mathsf{Z}w'$. Supposed w is named. Then by proposition 7.5.8(f), $\exists \mathsf{F} \subseteq \mathsf{C}$ such that $\bigvee \mathsf{F} = w$. Trivially, $\bigvee \mathsf{F} \leq w$ and therefore $\mathsf{F} = \pi(w)$. Thus as $\pi(w) \cap o(w) = \varnothing$,

$o(w) = \varnothing$. As $\pi(w) = \pi(w')$ and $o(w) = o(w')$, $\bigvee \mathsf{F} = w'$ and therefore $w = w'$. Thus w and w' have the same name. (ii) Suppose that w and w' have the same name. Then $w = w'$ and trivially they have the same corridor configuration. (iii) Suppose that $w\mathsf{Z}w'$. ($\Rightarrow$) There are two cases. If $\pi(w) \neq \varnothing$, then by proposition 7.5.8(h), $\pi(w) = \{w\}$ and $o(w) = \varnothing$. As $w\mathsf{Z}w'$, $\pi(w') = \{w\}$, $o(w') = \varnothing$, and therefore $At(w')$. If $\pi(w) = \varnothing$ then there is a single $c \in \mathsf{C}$ where $\{c\} = o(w)$. Again, as $o(w) = o(w')$, $\{c\} = o(w')$. Thus by condition (3) of definition 7.5.5 $At(w')$. ($\Leftarrow$) Same as ($\Rightarrow$). □

We do the forth cases only. The back cases are entirely analogous.

Lemma 7.5.11 (Forth Cases)**.** Let $w\mathsf{Z}w'$ where $w', v' \in \mathcal{C}(\mathcal{B})$ and $w \in g[\mathcal{B}]$. (a) $w' \leq v'$ implies $\exists v \in g[\mathcal{B}]$ such that $w \leq v$ and $v\mathsf{Z}v'$. (b) $v' \leq w'$ implies $\exists v \in g[\mathcal{B}]$ such that $v \leq w$ and $v\mathsf{Z}v'$. (c) $w' \not\leq v'$ implies $\exists v \in g[\mathcal{B}]$ such that $w \not\leq v$ and $v\mathsf{Z}v'$. (d) $v' \not\leq w'$ implies $\exists v \in g[\mathcal{B}]$ such that $v \not\leq w$ and $v\mathsf{Z}v'$.

Proof. Select any atomic matching function f. f will be used in all cases.

(a) Let $w\mathsf{Z}w'$ and $w' \leq v'$. Observe that the corridor covering of v':

$$\kappa(v') = \pi(v') \cup (o(v') \cap o(w')) \cup (o(v') - o(w')).$$

We now define v as a supremum of objects selected piecemeal from those in $\pi(v')$, $o(v') \cap o(w')$, and $o(v') - o(w')$. Let $a = \bigvee \pi(v')$. Assume $o(v') - o(w') = \{b_1, ..., b_l\}$ and set $b = \bigvee_{i=1}^{l} f(b_i \wedge v')$. Let $o(v') \cap o(w') = \{c_1, ..., c_m\}$. By lemma 7.5.9.I, $\exists c \in g[\mathcal{B}]$ such that $o(w) \cap o(c) = o(w') \cap o(v')$ and $c \boxdot \bigvee(o(v') \wedge o(w')) \wedge v'$. Finally we set $v = a \vee b \vee c$. By selection of f and the fit lemma, $v \in g[\mathcal{B}]$.

Claim 1. $v\mathsf{Z}v'$. By proposition 7.5.8.d, $\forall x \in \mathcal{C}(\mathcal{B})$, $\pi(x) \cap o(x) = \varnothing$. So by definition of a, b, and c: $\pi(v) = \kappa(a) = \pi(v')$ and $o(v) = \kappa(b \vee c)$. Moreover:

$$\begin{aligned} x \in o(v) &\iff x \in \kappa(b \vee c) \\ &\iff x \in \kappa(\textstyle\bigvee_{i=1}^{l} f(b_i \wedge v')) \cup (o(c) \cap o(w)) \\ &\iff x \in (o(v') - o(w')) \cup (o(v') \cap o(w')) \text{ [by lemma 7.5.9.I]} \\ &\iff x \in \{b_1, ..., b_l\} \cup \{c_1, ..., c_m\} \\ &\iff x \in o(v') \text{ [by proposition 7.5.8(c)].} \end{aligned}$$

Hence we have shown $\pi(v) = \pi(v')$ and $o(v) = o(v')$. Finally we show the atomic condition is preserved. Let $x \in o(v)$. We must prove $At(x \wedge v) \Leftrightarrow At(x \wedge v')$. Observe that either $x = c_i \in \{c_1, ..., c_m\}$ or $x = b_i \in \{b_1, ..., b_l\}$. If the former, then the atomic condition holds in this case by lemma 7.5.9.I. If the latter, the atomic condition holds by the selection of f.

Claim 2. $w \leq v$. By quasi-monotonicity of π, $\pi(w') \subseteq \pi(v')$. As $w\mathsf{Z}w'$, $\pi(w) \subseteq \pi(v')$. By claim 1, $\pi(w) \subseteq \pi(v)$. By proposition 7.5.8(g), $\bigvee \pi(w) \leq \bigvee \pi(v)$. Thus $\bigvee \pi(w) \leq a \leq v$. By proposition 7.5.8(e), it remains to prove that $(w \wedge \bigvee o(w)) \leq v$. As $b \leq v$ clear by definition of b, it suffices to show: $(*)$ $(w \wedge \bigvee o(w)) \leq (c \vee \bigvee \pi(v))$. By assumption $w' \leq v'$. So $o(w') \subseteq \kappa(v')$ and, by claim 1, $o(w) \subseteq \kappa(v)$. By definition of κ, $o(w) \subseteq (\pi(v) \cup o(v))$. Let $x \in o(w)$. (A) If $x \in \pi(v)$ then $x \notin o(v)$ by proposition 7.5.8(d). Therefore we have $((x \wedge w) \wedge \bigvee o(w)) \leq \bigvee \pi(v)$ as required. (B) Assume $x \notin \pi(v)$. By proposition 7.5.8(d), $x \in o(v)$. Thus $x \in o(w) \cap o(v)$. By claim 1, $x \in o(w') \cap o(v')$. By lemma 7.5.9I, $(x \wedge w) \vee \bigvee o(w) \leq c$. So $(*)$ holds. Finally as $\bigvee \pi(w) \leq a$ we have $\bigvee \pi(w) \vee (w \wedge \bigvee o(w)) \leq (a \vee c)$. By proposition 7.5.8(e), $w \leq (a \vee c)$. So $w \leq v$, as required.

(b) Let $w\mathsf{Z}w'$ and $v' \leq w'$. As w' dominates v', the corridor covering of v':

$$\kappa(v') = \pi(v') \cup (o(v') \cap o(w')) \cup (o(v') \cap \pi(w')).$$

We define v based on the corridor covering. Let $a = \bigvee \pi(v')$. Assume $\pi(w) \cap o(v') = \{b_1, ..., b_k\}$ and set $b = \bigvee_{i=1}^{k} f(b_i \wedge v')$. Let $o(w) \cap o(v') = \{c_1, ..., c_l\}$. By lemma 7.5.9.II, $\exists c \in g[\mathcal{B}]$ such that $o(c) \cap o(w) = o(v') \cap o(w')$ and $c \boxdot \bigvee(o(v') \wedge o(w')) \wedge v'$.

We set $v = a \vee b \vee c$. By selection of f and the fit lemma, $v \in g[\mathcal{B}]$.

Claim 1. $v\mathsf{Z}v'$. By proposition 7.5.8(d), $\forall x \in \mathcal{C}(\mathcal{B})$, $\pi(x) \cap o(x) = \varnothing$. So by definition of a, b, and c: $\pi(v) = \kappa(a) = \pi(v')$ and $o(v) = \kappa(b \vee c)$. For the claim that $o(v) = o(v')$:

$$\begin{aligned} x \in o(v) &\iff x \in \kappa(b \vee c) \\ &\iff x \in \kappa(\textstyle\bigvee_{i=1}^{l} f(b_i \wedge v')) \cup (o(c) \cap o(w)) \\ &\iff x \in (\pi(w') \cap o(v')) \cup (o(v') \cap o(w')) \text{ [by lemma 7.5.9.II]} \\ &\iff x \in \{b_1, ..., b_l\} \cup \{c_1, ..., c_m\} \\ &\iff x \in o(v') \text{ [by proposition 7.5.8(c)].} \end{aligned}$$

Hence we have shown $\pi(v) = \pi(v')$ and $o(v) = o(v')$. We must yet show the atomic condition is preserved. Let $x \in o(v)$. We must prove $At(x \wedge v) \Leftrightarrow At(x \wedge v')$. Observe that either $x = c_i \in \{c_1, ..., c_m\}$ or $x = b_i \in \{b_1, ..., b_l\}$. If the former, then the atomic condition holds in this case by lemma 7.5.9.II. If the latter, the atomic condition holds by the selection of f.

Claim 2. $v \leq w$. As $a = \bigvee \pi(v') = \bigvee \pi(v) \leq \bigvee \pi(w)$ and both b and c are factors of w, we have $a \vee b \vee c = v \leq w$.

(c) Let $w\mathsf{Z}w'$ and $w' \not\leq' v'$. The corridor covering of v'

$$\kappa(v') = \pi(v') \cup (o(v') \cap \pi(w')) \cup (o(v') \cap o(w')) \cup (o(v') - \kappa(w')).$$

Let $a = \bigvee \pi(v')$. Let $o(v') \cap \pi(w') = \{b_1, ..., b_k\}$ and set $b = \bigvee_{i=1}^{k} f(b_i \wedge v')$. Next let $o(v') \cap o(w') = \{c_1, ..., c_l\}$. By lemma 7.5.9.III, $\exists c \in h[\mathcal{B}]$ such that $o(c) \cap o(w) = o(v') \cap o(w')$ and $c \boxdot \bigvee(o(v') \wedge o(w')) \wedge v'$. Suppose $o(v') - \kappa(w') = \{d_1, ..., d_m\}$ and set $d = \bigvee_{i=1}^{m} f(d_i \wedge v')$. We set $v = a \vee b \vee c \vee d$. By selection of v and f and according to the fit lemma, $v \in W$.

Claim 1. $v\mathsf{Z}v'$. Analogous to the previous cases, $\kappa(a) = \pi(v)$ and $\kappa(b \vee c \vee d) = o(v)$. For the claim that $o(v) = o(v')$ observe that:

$$\begin{aligned} x \in o(v) &\iff x \in \kappa(b \vee c \vee d) \\ &\iff x \in \kappa(\textstyle\bigvee_{i=1}^{k} f(b_i \wedge v')) \cup (o(c) \cap o(w)) \cup \kappa(\bigvee_{i=1}^{m} f(d_i \wedge v')) \\ &\iff x \in (o(v') \cap \pi(w')) \cup (o(v') \cap o(w')) \cup (o(v') - \kappa(w')) \\ &\qquad \text{[by lemma 7.5.9.III]} \\ &\iff x \in o(v') \end{aligned}$$

We have shown $\pi(v) = \pi(v')$ and $o(v) = o(v')$. We must yet show the atomic condition is preserved. Let $x \in o(v)$. We must prove $At(x \wedge v) \Leftrightarrow At(x \wedge v')$. Either $x = b_i \in \{b_1, ..., b_k\}$, $x = c_i \in \{c_1, ..., c_l\}$, or $x = d_i \in \{d_1, ..., d_m\}$. If the second holds, then the atomic condition holds in this case by lemma 7.5.9.III. If the first or the latter, then the atomic condition holds by the selection of f.

Claim 2. $w \not\leq v$. We consider the two exhaustive cases:

(1) $\pi(w) \subseteq \pi(v)$ and $o(w) \subseteq o(v)$ or (2) $\pi(w) \not\subseteq \pi(v)$ or $o(w) \not\subseteq o(v)$.

Assume (1). By proposition 7.5.8(e), $\bigvee \pi(w) \leq \bigvee \pi(v)$. By corridor disjointness, it therefore suffices to ensure that there exists a non-zero $z \in g[\mathcal{B}]$ such that

$$z \leq \left(\left(\bigvee o(w) \wedge w\right) - \left(\bigvee o(v) \wedge v\right)\right).$$

By (c)1. above, $o(w) \subseteq o(v)$ implies $o(w') \subseteq o(v')$. $o(w') \cap o(v') = o(w')$. By lemma 7.5.9.III, $o(c) \cap o(w) = o(v') \cap o(w')$ and for each $i \in \{1, ..., l\}$, $(c_i \wedge w) \not\leq (c_i \wedge c)$. Thus, $w \not\leq v$. Next assume (2). If both $\pi(w) \not\subseteq \pi(v)$ and $o(w) \not\subseteq o(v)$, then trivially $w \not\leq v$. And if $\pi(w) \not\subseteq \pi(v)$ and $o(w) \subseteq o(v)$, then $\exists z \in \pi(w)$ such that $z \notin \pi(v)$; thus clearly $z \not\leq v$ and thus $w \not\leq v$. So suppose that $o(w) \not\subseteq o(v)$ and $\pi(w) \subseteq \pi(v)$. By π quasi-monotonicity, we have $\bigvee \pi(w) \leq \bigvee \pi(v)$. Moreover, there is a $e \in o(w)$ such that $e \notin o(v)$. Hence $e \wedge w \neq 0$ and, by the disjointness of corridors, $(w \wedge e) \not\leq v$. Thus $w \not\leq v$.

(d) Let $w\mathsf{Z}w'$ and suppose $v' \not\leq w'$. Let $a = \bigvee \pi(v')$. Next let $o(v') \cap o(w') =$

$\{c_1, ..., c_l\}$. By lemma, $\exists c \in g[\mathcal{B}]$ such that $o(c) \cap o(w) = o(v') \cap o(w')$ and $c \boxdot \bigvee(o(v') \wedge o(w')) \wedge v'$. Let $o(v') \cap \pi(w') = \{b_1, ..., b_k\}$ and set $b = \bigvee_{i=1}^{k} f(b_i \wedge v')$. Finally, let $o(v') - \kappa(w') = \{d_1, ..., d_m\}$ and set $d = \bigvee_{i=1}^{m} f(d_i \wedge v')$. We set $v = a \vee b \vee c \vee d$. By definition of v and f and by the fit lemma, $v \in g[\mathcal{B}]$. Both $v\mathsf{Z}v'$ and $v \not\leq w$ are proven in a manner completely analogous to case (c). □

For each case in the lemma above, the method of defining the vs can be used analogously to define the v's required for the back cases. Thus we have:

Lemma 7.5.12 (Back Cases)**.** Let $w\mathsf{Z}w'$ where $w, v \in g[\mathcal{B}]$ and $w' \in \mathcal{C}(\mathcal{B})$. (a) $w \leq v$ implies $\exists v' \in \mathcal{C}(\mathcal{B})$ such that $w' \leq v'$ and $v\mathsf{Z}v'$. (b) $v \leq w$ implies $\exists v' \in g[\mathcal{B}]$ such that $v' \leq w'$ and $v\mathsf{Z}v'$. (c) $w \not\leq v$ implies $\exists v' \in g[\mathcal{B}]$ such that $w' \not\leq v'$ and $v\mathsf{Z}v'$. (d) $v \not\leq w$ implies $\exists v' \in g[\mathcal{B}]$ such that $v' \not\leq w'$ and $v\mathsf{Z}v'$.

Theorem 7.5.13. $g[\mathcal{B}] \triangleq \mathcal{C}(\mathcal{B})$.

Proof. Follows from lemmas 7.5.11, 7.5.10, and 7.5.12. □

Theorem 7.5.14. $\mathbf{K}_{\mathsf{hm}} + \mathsf{BA} + A\langle\geq\rangle\alpha + \mathsf{ATOM}$ (abbrev: Λ) is sound and complete w.r.t. the class of infinite complete atomic BAs with κ atoms for any $\kappa \geq \aleph_0$.

Proof. Let $\not\vdash_\Lambda \phi$. By corollary 6.3.14, ϕ is falsified on a infinite atomic BA-model $(\mathfrak{B}, V)$ with $\kappa \geq \aleph_0$ atoms. By proposition 6.3.12, there is a finite valuation V^ϕ where $dom(V^\phi) = dom(V) \cap \mathsf{n}(\phi)$ such that $(\mathfrak{B}, V^\phi) \not\models \phi$. By theorem 7.5.13, $(\mathfrak{B}, V^\phi) \triangleq \mathcal{C}(\mathfrak{B}, V^\phi)$ where $\mathcal{C}(\mathfrak{B}, V^\phi)$ is the BA-model completion of $(\mathfrak{B}, V^\phi)$. Thus ϕ is falsified on the completion of $(\mathfrak{B}, V^\phi)$, and we are done. □

Corollary 7.5.15. $\mathbf{K}_{\mathsf{hm}} + \mathsf{BA} + A\langle\geq\rangle\alpha + \mathsf{ATOM}$ is the logic of the powerset algebra of the real numbers.

Proof. Immediate by the previous theorem where the number of atoms $\kappa = 2^{\aleph_0}$. □

7.6 Finite Model Property

Let $\sharp : \mathcal{H}_{\mathsf{m}} \longrightarrow \mathbb{N}$ be defined by $\sharp(x) = |\mathsf{n}(x)|$. For each $\phi \in \mathcal{H}_{\mathsf{m}}$ if $\mathcal{M} = (\mathcal{F}, V)$ is a hybrid BA-model, and ϕ is satisfiable on $\mathcal{M}$, we let $\mathcal{M}^{\phi}$ denote the BA model $(\mathcal{F}^{\phi}, V^{\phi})$, where $\mathcal{F}^{\phi}$ is the BA subalgebra of $\mathcal{F}$ generated by $V^{\phi}[\mathsf{n}(\phi)]$. Define $\mathsf{C}^{\phi/\mathcal{F}} = Atom(\mathcal{F}^{\phi})$ and call $\mathsf{C}^{\phi/\mathcal{F}}$ the corridors generated by ϕ over $\mathcal{M}$. We say that a BA is $\mathfrak{A}$ *finitely generated* if it is a subalgebra of a BA, say $\mathfrak{B}$, generated from some finite set $E \subseteq \mathfrak{B}$.

Proposition 7.6.1 ([40] pp.75-83)**.** Let $\mathfrak{A}$ be a finitely generated BA. Then $|\mathfrak{B}| = 2^m$ where $m = Atom(\mathfrak{B})$. Moreover if $|E| = n$ where E is the generating set, then $\mathfrak{B}$ has *at most* 2^n atoms and thus has at most 2^{2^n} elements.

Proposition 7.6.2. Let $\mathcal{M} = (\mathcal{F}, V)$ be a BA-model and $w \in \mathcal{M}$. Assume $\phi \in \mathcal{H}_{\mathsf{m}}$ and $\mathcal{M}, w \models \phi$. If $\sharp(\phi) = n$, then $|\mathsf{C}^{\phi/\mathcal{F}}| \leq 2^n$.

Proof. Follows immediately from proposition 7.6.1. □

Recall that a logic Λ has the *finite model property* (fmp) if there is a class of models M of Λ such that any non-theorem ϕ of Λ is falsified by some finite model in M.

Definition 7.6.3. Let $\mathcal{B} = (\mathfrak{B}, V)$ be a BA-model and $\phi \in \mathcal{H}_{\mathsf{m}}$. Let $h : \mathcal{B}^{\phi} \longrightarrow \mathcal{B}$ be the corresponding Boolean monomorphism from the subalgebra of $\mathcal{B}$ generated from $\mathsf{n}(\phi)$ into $\mathcal{B}$. Define

$$\begin{aligned} M_1 &= \{x \in \mathsf{C}^{\phi/\mathcal{B}} \ : \ At^{\mathcal{B}}(h(x))\} \\ M_2 &= \{x \in \mathsf{C}^{\phi/\mathcal{B}} \ : \ |\{y \leq^{\mathcal{B}} x \ : \ At^{\mathcal{B}}(y)\}| = 2\} \\ M_3 &= \{x \in \mathsf{C}^{\phi/\mathcal{B}} \ : \ |\{y \leq^{\mathcal{B}} x \ : \ At^{\mathcal{B}}(y)\}| \geq 3\} \\ M_4 &= \{x \in Atom(\mathcal{B}) \ : \ \exists y \in M_2 \text{ where } x \leq h(y)\}. \end{aligned}$$

For each $c_i \in M_3$ we select a set of exactly 2 atoms $\{a_i^1, a_i^2\} \subseteq Atom(\mathcal{B})$ such that $(a_i^1 \vee a_i^2) \leq^{\mathcal{B}} h(c)$ and set $M_5 = \{a_i^1, a_i^2, h(c_i) - (a_i^1 \vee a_i^2) \mid c_i \in M_3\}$. For any such M_5, we call the subalgebra $\mathcal{F}$ of $\mathcal{B}$ generated from $M_6 = h[M_1] \cup M_4 \cup M_5$ a *finite frame reduction* (ffr) of $\mathcal{B}$ *modulo* ϕ. We let the valuation $V^{\mathcal{F}}$ of $\mathcal{F}$ be $h \circ V^{\phi}$ and call

$(\mathcal{F}, V^{\mathcal{F}})$ *the finite mereobisimilar reduction* (fmr) of $\mathcal{B}$ *modulo* ϕ. And we call M_6 the *m-generator* of $\mathcal{B}$.

Obviously, for any $\phi \in \mathcal{H}_{\mathsf{m}}$ that is satisfiable on an infinite atomic BA model $\mathcal{B}$, there is a fmr $\mathcal{M}$ of $\mathcal{B}$. We can therefore show the following.

Theorem 7.6.4 (Finite Model Property)**.** $\mathbf{K}_{\mathsf{hm}} + \mathsf{BA} + A\langle\geq\rangle\alpha + \mathsf{ATOM}$ has the finite model property and is therefore decidable.

Proof. We only do a sketch. Let ϕ be Λ-consistent. By completeness, ϕ is satisfied at an element $w \in \mathcal{B}$ where $\mathcal{B}$ an infinite atomic BA-model. There is some fmr $\mathcal{M}$ mod ϕ of $\mathcal{B}$. As $\mathcal{M}$ is finite, it suffices to show that ϕ is satisfied on $\mathcal{M}$. It therefore suffices to show $\mathcal{B} \triangleq \mathcal{M}$. Let h be the corresponding Boolean monomorphism from $\mathcal{M}$ into $\mathcal{B}$. As $\mathcal{M}$ is an fmr of $\mathcal{B}$, there is some set F which is an m-generator for $\mathcal{M}$. Define functions from $\mathcal{B}$ into $\mathcal{B}$:

- $\pi(x) = \{y \in \mathsf{C}^{\phi/\mathcal{B}} \mid y \leq x\}$,
- $o(x) = \{y \in \mathsf{C}^{\phi/\mathcal{B}} \mid xPVy\}$,
- $\kappa(x) = \{y \in \mathsf{C}^{\phi/\mathcal{B}} \mid xOy\}$.

We have $\kappa(x) = \pi(x) \cup o(x)$. As before the monotonicity properties of π and κ hold. And the covering property and proofs for the analogs of proposition 7.5.8 hold. For any $x, y \in \mathcal{B}$: if

(1) $\pi(x) = \pi(y)$,

(2) $o(x) = o(y)$, and

(3) $\forall c \in \mathsf{C}^{\phi/\mathcal{F}}\ (c \in o(x) \Longrightarrow (At(x \wedge c) \Leftrightarrow At(y \wedge c)))$,

then we say that x and y have the same *corridor configuration.* Set

$$\mathsf{Z} = \{(x, y) \in h[\mathcal{M}] \times \mathcal{B}\ :\ x \text{ has the same corridor configuration as } y\}.$$

It is easily checked that each distinct corridor configuration of $\mathcal{B}$ is represented in $h[\mathcal{M}]$. And thus to show that Z is a mereobisimulation between $h[\mathcal{M}]$ and $|\mathcal{B}|$ is virtually identical to the proofs of lemmas 7.5.9, 7.5.11, 7.5.12. Finally, observe that for any atomic BA-model, any fmr has maximally $3 \times 2^{\sharp(\mathsf{n}(\phi))}$ atoms. Thus by fact 7.6.1 $\mathcal{M}$ is finite, and as $\mathcal{M} \cong h[\mathcal{M}]$, the latter is also finite. □

Corollary 7.6.5. The $\mathcal{H}_{\mathsf{hm}}$-logic of the class of infinite complete atomic BAs has the fmp and is decidable. In particular, if $\phi \in \mathcal{H}_{\mathsf{m}}$, then the problem of deciding whether ϕ is satisfiable on an atomic Boolean algebra of any infinite cardinality is solvable within time $\sum_{i=0}^{3\times 2^{\sharp(\mathsf{n}(\phi))}} 2^{i^{\sharp(\mathsf{n}(\phi))}}$ and therefore in 3-EXPTIME.

Proof. By corollary 7.5.15 and the finite model property the logic in question is decidable. Let ϕ be satisfiable on an infinite atomic BA-model. By the previous theorem it is satisfiable on an fmr mod ϕ of some infinite atomic BA-model. By definition, any such fmr has maximally $3 \times 2^{\sharp(\mathsf{n}(\phi))}$ atoms. Finite atomic BAs are determined up to isomorphism by their number of atoms. Thus to find a satisfying BA-model it suffices to check $\forall k \in \{0, ..., 3 \times 2^{\sharp(\mathsf{n}(\phi))}\}$ every BA-model obtained by the finite BA-frame $\mathcal{F}_k$ with exactly k atoms. For each such BA-frame $\mathcal{F}_k$, $|\mathcal{F}_k| = 2^k$. So there are $2^{k^{\sharp(\mathsf{n}(\phi))}}$ pure hybrid valuations definable over $\mathcal{F}_k$. Therefore there are $\sum_{i=0}^{3\times 2^{\sharp(\mathsf{n}(\phi))}} 2^{i^{\sharp(\mathsf{n}(\phi))}}$ BA-models to check. So the satisfiability problem is solvable in 3-EXPTIME. □

7.7 Concluding Observations

The two major results of this section are as follows:

- $\mathbf{K}_{\mathsf{hm}} + \mathsf{BA} + A\neg\alpha$ is the logic of the regular open sets of the real numbers; and
- $\mathbf{K}_{\mathsf{hm}} + \mathsf{BA} + A\langle\geq\rangle\alpha$ is the logic of the powerset algebra of the reals.

We have a intuitive understanding of space as "carved out" in a continuous manner. But, we have seen that the mereologic for atomless structures $\mathbf{K}_{\mathsf{hm}} + \mathsf{BA} + A\neg\alpha$ deals in a rather heavy-handed way with extended regions, conflating chequered opens with the regular open hyperregions of the same number of dimensions. Our first task was to isolate a type of countable pixelated Boolean algebra. And for each finite dimension $n \in \mathbb{N}$ we found a so-called n-chequered open algebra $\mathcal{CH}$. It was immediate that the canonical model on the Henkin construction of the previous chapter was isomorphic to $\mathcal{CH}$. Then we showed that the completion of $\mathcal{CH}$ was mereobisimilar to $\mathcal{CH}$. This shows that any atomless location will be indistinguishable from its completion, and therefore how insufficient nominalistic mereologies are to represent reasoning over the geometrical structure of *arbitrary* subregions.

As for the atomic result, it was shown that nominalistic mereological reasoning in $\mathcal{H}_{\mathsf{m}}$ is incapable of treating spatial extents as a suprema of an arbitrary finite number of atoms. Although points from complex regions are distinguishable, the variety of complex extents are not. Over our selected models of space, formulas of the language represent a finite nominalistic ratiocination of the entire mereological landscape of spatial objects. We identified a selection of objects which were of particular importance—the corridors. But "under" these objects, the only aspects we can distinguish are atoms from non-atoms. This is most notable in the finite model property argument where it was shown that the "areas" under corridors can be reduced to a size of at most 3. Various subregions—indeed, uncountably many in the case of Euclidean spaces—are conflated. But according to commonsense, proper parts of corridors are distinguishable. In the case of atomic BAs, this suggests that our understanding of space involves unrestricted quantifications over regions—that is to say, not just distinguishing them by name, but by the *amount* of objects within them. Simply put, we need to count to understand space the way it appears to us.

Chapter 8

Conclusion

Let us return to the two questions posed in the introduction. Firstly we asked: *Can there be any such thing as a nominalistically acceptable formal mereology?* We saw that there does not exist a nominalistic formal mereology in the strict sense. The parthood relation itself is not a particular thing, but rather a multiply located object wholly presented amid its relata. And in chapter four we saw that Leśniewski's inscriptionalist vision could not be sustained. Obscurely, inscriptionalism will entail that parts of formal languages are exhibited *wherever* structural similarity relations agree with the initial prototype of the language.

Secondly we asked: *How much of the structure of reality can any remotely acceptable system capture?* In chapter V, we identified a language whose formulas represent finitely complex mereological situations. In chapter VI we identified general logics in this language which could be deployed over mereological structures. However in chapter VII, it was shown that any remotely acceptable one will be incapable of capturing the mereological structure implicit in the classical atomic models of space as well as those of extended regions investigated by Tarski and MacNeille. Nominalistic mereology will provide an unacceptable source of distinctions to capture infinite mereological structures.

8.1 A Summary of the Argument

Ontological Analysis. Firstly, we observed that there is an unavoidable problem with viewing the parthood relation as a concrete particular. And this was seen most clearly in our discussion of the possibility of the parthood relation being a relational trope (chapter II section 2.1.5). I argued that the relation could be neither a relational trope nor a particular. For, if it is not wholly present in various locations, it must be of higher logical type than an individual. That is, if it *is* a particular object, it will relate sets of concrete individuals. Consequently, by viewing the relation as a particular, we will violate the assumption of nominalism.

The conception of locations as interconnected posed another problem. By providing an explanation and formalization of the required notions, we determined that various abstract conceptions were required. So we made use of a likely nominalistic assumption about the status of living organisms. Accounting for situations in which organisms are related to their environment gave rise to the notion of localized situation. And to accommodate these conceptions, we argued for a two-sorted theory of mereological states of affairs involving individuals.

A second unavoidable problem with nominalistic mereology concerned the ontological status of formal languages. By nominalism, any formal language is not an abstract set of formulas, as is standard in modern modal logic. Instead formal languages comprise physical aggregates of formulas which are concrete particulars. Thus a formula's membership in a formal language will be determined by it being a token of a highly complex system of types comprising a so-called *syntactic protocol*. Protocols, as we showed, have no temporal beginning. And thus implausibly, all manner of non-artifactual objects meeting the structural criteria of the protocol will be objects of the language. Thus formal languages must be abstract. We made a compromise and settled on a view of them as sets of formulas, as is common in modern formal logic.

Logical Analysis. We showed that the hybrid language $\mathcal{H}_{\mathsf{m}}$ met the goal of a system of references implying no ontological distinctions beyond those of the acceptable ontology. Provided that there are an at-most countable number of individuals, we showed that the expressive power of $\mathcal{H}_{\mathsf{m}}$ was, theoretically, sufficient to denote each finite mereological situation. All traditionally important mereological relations were shown to be definable as operators in $\mathcal{H}_{\mathsf{m}}$. And we proved that the language had the expressive power to define the important classes of frames of extensional mereology. We introduced a new notion of mereobisimulation as a the invariance notion of $\mathcal{H}_{\mathsf{m}}$. And we used it to show that unacceptable expressions like counting statements and arithmetical properties are inexpressible in $\mathcal{H}_{\mathsf{m}}$.

The real blood, sweat, and tears was seen in the final chapter. Suppose $\mathcal{R}$ is the structure of real locations. In this structure all locations are represented according to the parthood relation. Our arguments of chapter II indicated that $\mathcal{R}$ is unrestrictedly fused. And I claimed that the part-to-whole structure of $\mathcal{R}$ is modeled by some mathematical structure, say $\mathcal{M}$. According to our results, there is a reduct $\mathcal{M}^-$ of $\mathcal{M}$ with uncountably fewer objects than $\mathcal{M}$ such that $\mathcal{M} \triangleq \mathcal{M}^-$. This means that for every object $w \in \mathcal{M}$, there is one $w' \in \mathcal{M}^-$ such that every formula true at w is true at w', and vice versa. And since $\mathcal{M}^-$ is a model of a *proper* fragment of $\mathcal{R}$, nominalistic reasoning must be highly insufficient to represent the structure of locations. In short, there simply aren't enough acceptable distinctions we can make to distinguish them! We demonstrated this result over two types of structure: atomless and infinite atomic Boolean algebras. In either case, if there are infinitely many concrete objects, then uncountably many of them will be $\mathcal{H}_{\mathsf{m}}$-indistinguishable. Thus nominalistic mereology will provide an unacceptable source of distinctions for infinite mereological structures.

8.2 Mereologic and Beyond

Have our results been for nothing? Absolutely not! We have used more sensitive modeling tools which have enriched the discourse of formal ontology. Further studies in formal ontology can deploy the languages and mereologics I have introduced to model further physical types of situations and phenomena.

Interestingly, if there are only finitely many concrete objects, $\mathcal{H}_{\mathsf{m}}$ will be expressively sufficient. Or at least in one sense this will be true: if there are finitely many individuals, then the *mereological* structure of reality will be diagrammable by single formulas. In this case we will not need the entire expressive resources of the first-order language! Thus on these speculative grounds, $\mathcal{H}_{\mathsf{m}}$ supplies an interesting low-level mereology that is of independent philosophical interest. Moreover, it adds a fresh line to the current tradition of spatial logics for example found in recent trends like Aiello et al [1]. And research into $\mathcal{H}_{\mathsf{m}}$ fits rather nicely with newer trends in spatial logic like for example Tamar Lando's recent dissertation [53].

By combining geometrical predicates and operators to the language, new exciting questions now arise concerning the nature of spatially extended objects. If there are only finitely many concrete objects, then all will be spatially extended and have a finite decompositional structure. And thus there will be spatially extended atoms. But questions concerning the geometrical structure of atoms and their interconnections naturally arise. How must these atoms fit together to form a universe? In terms of computational complexity, questions like these may dwarf those of mereology and involve deep issues in real analysis and arithmetic. Atoms will now be required to have particular geometrical structures. Patterns in similarly structured extended objects will be required to admit of repetition and *cover* the entire dimensional system. These problems are dealt with in the subject area of finitistic geometry. Obviously, we shall have no opportunity to delve into this subject area here. But such questions do imply various avenues for future research.

We also have barely touched on questions concerning $\mathcal{H}_{\mathsf{o}}$ and its logics. Our base system of formal mereology $\mathbf{K}_{\mathsf{hm}}$ and its various extensions are incapable of providing a source of categories to represent objects of *qualitative diversity*. But this is not to say, of course, that by resorting to $\mathcal{H}_{\mathsf{o}}$ will solve the structural representability problem posed by infinite structures. For it was shown that even in first-order logic we lack the expressive resources to distinguish uncountably many token locations. Thus $\mathcal{H}_{\mathsf{o}}$, as a proper fragment of the first-order language, will not do much better than $\mathcal{H}_{\mathsf{m}}$. Nonetheless, since it will do somewhat better, another selection of formal research questions will deal with the expressivity and axiomatizability of spatially interesting structures in $\mathcal{H}_{\circ}$. How much more structure can $\mathcal{H}_{\circ}$ capture?

In sum, it would appear any formal mereology without terms for sets and set-quantifiers will furnish a rather meagre source of expressions and categories. Thus we might consider the question of extending $\mathcal{H}_{\circ}$ with *propositional quantifiers*: $\forall p \phi$. It is well known that if one extends the standard hybrid language with the propositional quantifier and the existence modality, the resultant language it is expressively equivalent to monadic second order logic (see ten Cate [15]). Tarski's logic in *Geometry of Solids* is expressible in just such a system. The results of this dissertation suggest that, in terms of expressive resources, his logic is unparalleled. And it appears that any way of extending a formal mereology with conceptions to represent the dimensional, topological, and geometric properties of locations will require set-theoretic resources.

8.3 Final Vision

I envision future research in mereology as a more variegated field in which mereological relations and predicates are incorporated with spatially interesting operators and relations. Logical extensions too, as we have seen, will give rise to interesting extensions of mereology.

I hope also that the "localized" approach will be taken more seriously in philosophically motivated research in mereology. Although many formal logicians and philosophical logicians are convinced of its importance, aversion to incorporating these features into philosophical and formal metaphysics is still rather strong. Nonetheless, we saw that the notion of a possessive part was easily modeled by adopting modal notions. And this suggested distinctions between being within a universe and discussing it from an "external" vantage point. For example in the case of *Tibbles*, it helped to show distinctions that were rather difficult to cash out in a non-localized semantics.

Moreover I hope to see philosophical approaches in ontology—both formal and philosophical—carried out with more control and subtlety. Questions of ontological commitment can be dealt with in a much more nuanced way by selecting piecemeal the acceptable conceptions and then closing them under certain logical operations. That is: *no logical distinctions without ontological ones*! We are not, as in the days of 'ol, restricted to canonical notation and the conceptions and distinctions they imply. We can sculpt a new ontologies, logics for them, and stand in a better place from which to judge their worthiness.

Appendix A

Boolean Algebras

A *partial order* is a relation that is *reflexive, transitive,* and *antisymmetric*. A *Boolean algebra* (BA) is a structure $(A, \wedge, \vee, \sim, 1, 0)$ such that $\wedge$ and $\vee$ are two binary operations on A, $\sim$ is a unary operation, and 1 and 0 are two distinguished elements, and satisfying the following axioms:

(1)	$\sim 0 = 1$	$\sim 1 = 0$
(2)	$p \wedge 0 = 0$	$p \vee 1 = 1$
(3)	$p \wedge 1 = p$	$p \vee 0 = p$
(4)	$p \wedge \sim p = 0$	$p \vee 0 = p$
(5)	$\sim (\sim p) = p$	
(6)	$p \wedge p = p$	$p \vee p = p$
(7)	$\sim (p \wedge q) = \sim p \vee \sim q$	$\sim (p \vee q) = \sim p \wedge \sim q$
(8)	$(p \wedge q) = (q \wedge p)$	$(p \vee q) = (q \vee p)$
(9)	$p \wedge (q \wedge r) = (p \wedge q) \wedge r$	$p \vee (q \vee r) = (p \vee q) \vee r$
(10)	$p \wedge (q \vee r) = (p \wedge q) \vee (q \wedge r)$	$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

It is well known that the axioms (3), (4), (8), (10) form a set of axioms for the theory of BAs. In other words, they imply the others. Any BA A is also equivalent to one of the form $(A, \leq, 1, 0)$ where $\leq = \{(x, y) \in A \times A \mid x \wedge y = x\}$. $\leq$ is called *the dominance relation* of A and is a partial order. We say that x *dominates* y if $x \leq y$. If E is a subset of a partially ordered set S, the *supremum* of E is the $x \in S$ such that x dominates every member of E and every object dominating each object in E dominates x.

Let X be an arbitrary set and let $Pow(X)$ be the class of all subsets of X. The structure $(Pow(X), \cup, \cap, \sim, X, \varnothing)$ whose domain is $Pow(X)$ together with the operations of union, intersection, and complementation, and the distinguished subsets of X is called the *powerset algebra* on X.

A.1 Topology and regular open algebras

A *topological space* is a set (X, τ) such that τ is a set of subsets of X such that $\varnothing, X \in \tau$ and τ is closed under finite intersections and arbitrary unions. The elements of X are called points and the elements of τ are called *open sets.* A set P of X is called *closed* if it is the complement of an open set with respect to X. Sets that are both closed and open are called *clopen.* The *interior* of a set P is the union of the open sets that are included in P. The *closure* of a set P is defined to be the intersection of all closed sets that include P. A set P is said to be *dense* if its closure is the entire space. A set P is said to be *dense in an open set* Q if the closure of P includes Q. A set is *nowhere dense* if it is not dense in any non-empty open set. A set is called *regular open* if it coincides with the interior of its closure.

The following theorem due MacNeille and Tarski asserts that the regular open sets constitute a complete BA of sets, *the regular open algebra of* X. If S is a set, we denote the closure of S by S^-. And by $S^\perp$, we denote the complement of the closure $\sim (S^-)$ of S.

Theorem A.1.1 ([61],[93])**.** The class of all regular open sets of a topological space X, abbreviated $RO(X)$ is a complete BA with respect to the distinguished Boolean elements and operations defined by:

$$
\begin{array}{llcl}
(1) & 0 & = & \varnothing, \\
(2) & 1 & = & X, \\
(3) & P \wedge Q & = & P \cap Q, \\
(4) & P \vee Q & = & (P \cup Q)^{\perp\perp}, \\
(5) & \sim P & = & P^{\perp},
\end{array}
$$

The infimum and the supremum of a family $\{P_i\}$ of regular open sets are, respectively, $(\bigcap_i P_i)^{\perp\perp}$ and $(\bigcup_i P_i)^{\perp\perp}$.

A.2 Boolean subalgebras

A *Boolean subalgebra* of a BA A is a substructure B of A such that B together with the distinguished elements and operations of A (restricted to the set B) is a BA. The algebra A is called a *Boolean extension* of B. If A is a subalgebra of B there is an embedding of A into B. If E is an arbitrary subset of a BA A, then the intersection B of all the subalgebras that happen to include E is a Boolean subalgebra of A. The subalgebra B is said to be *generated by* E, and E is called a set of *generators* of B. For each element i in a BA A and j in $2 = \{0, 1\}$, write $p(i, j) = i$ if $j = 1$, and $\sim i$ if $j = 0$. Finally, write 2^E for the set of 2-valued functions on E, that is to say, the set of functions from E to 2. Given such a function a, the value of $p(i, a(i))$, for each i in E, is either i or $\sim i$; denote the meet of these values by p_a, so that $p_a = \bigwedge_{i \in E} p(i, a(i))$. The following are well-known.

Theorem A.2.1. Let B be the subalgebra generated by a finite subset E of a BA. The atoms of B are the non-zero elements of the form $p_a = \bigwedge_{i \in E} p(i, a(i))$ and the elements of B are the joins of these atoms. Every element of B can be written in one and only one way as a join of atoms.

Corollary A.2.2. Every finitely generated Boolean algebra A if finite, and the number of its elements is 2^m, where m is the number of atoms in A. If a generating set of A has n elements, then A has at most 2^n atoms, and hence it has at most 2^{2^n} elements.

Theorem A.2.3. An element of a BA is in the subalgebra generated by a set E if and only if it can be written as a finite join of finite meets of elements and complements of elements from E.

A.3 Homomorphisms, Atoms, and Finite BAs

A *Boolean homomorphism* is a mapping f from a BA B, say, to a BA A such that

$$f(p \wedge q) = f(p) \wedge f(q)$$
$$f(p \vee q) = f(p) \vee f(q)$$
$$f(\sim p) = \sim (f(p))$$

A *Boolean monomorphism*, also called an *embedding*, is a Boolean homomorphism that is one-to-one: if $f(p) = f(q)$, then $p = q$. An *Boolean epimorphism* is a homomorphism that is onto: every element of A is equal to $f(p)$ for some $p \in B$. A Boolean homomorphism that is a bijection, that is to say, it is one-to-one and onto, is called an *isomorphism.* If there is an isomorphism from one BA onto another, the two algebras are said to be *isomorphic.*

An *atom* of a BA is an element that has no non-trivial proper subelements , i.e. $At(q) \iff q \neq 0$ and if there are only two elements p such that $p \leq q$, namely 0 an q. A BA is *atomic* if every non-zero element dominates at least one atom. A BA is *atomless* if it has no atoms. Every finite BA is atomic. And we have the following:

Proposition A.3.1. (i) Every finite BA A is isomorphic to the field $Pow(n)$ or, equivalently, to the BA 2^n, for some non-negative integer n. And the number of atoms in A is n. (ii) Any two finite BAs with the same number of atoms are isomorphic.

Proposition A.3.2. Any two countable, atomless BAs with more than one element are isomorphic.

A *completion* A of a BA $\mathfrak{B}$ is a BA with the following properties: (1) B is a Boolean subalgebra of A; (2) $\forall S \subseteq |B|$, $\bigvee h[S] \in |A|$; and (3) $\forall y \in |A|$, $y = \bigvee h[S]$

for some $S \subseteq |B|$. A Boolean *ideal* in a BA B is a subset M of B such that $0 \in M$, if $p \in M$ and $q \in M$, then $p \vee q \in M$, and if $p \in M$ and $q \in B$, then $p \wedge q \in M$. A *complete ideal* of a BA B is an ideal such that if $\{p_i\}$ is a family in M with a supremum of p in B, then $p \in M$; if $p \in M$ and $q \in B$, then $p \wedge q \in M$. A Boolean *filter* in a BA B is a subset N such that $1 \in N$, if $p \in N$ and $q \in N$, then $p \wedge q \in N$, and if $p \in N$ and $q \in B$, then $p \vee q \in N$.

Theorem A.3.3. Every BA A has a completion, namely (an isomorphic copy of) the BA of complete ideals in A.

Theorem A.3.4. Any two completions of a BA A are isomorphic via a mapping that is the identity relation on A.

Corollary A.3.5. (i) Two atomic BAs with the same number of atoms have isomorphic completions. (ii)The completion of a BA A is atomic if and only if A is atomic. (iii) Every complete BA is it own completion, and thus every finite BA is its own completion.

A.4 Formal theories of BAs

By far the most important result in logics for BAs is Tarski's theorem on the subject.

Theorem A.4.1 (Tarski (1951))**.** The elementary theory of BAs in the signature $\{\leq, 1, 0\}$ is decidable.

As far as stronger languages we have also:

Theorem A.4.2. The second-order theory of the complete BA with $\aleph_0$ atoms is $2^{\aleph_0}$-categorical and finitely axiomatizable.

Theorem A.4.3 (Rubin)**.** The elementary theory of BAs with a distinguished subalgebra—that is, the FO-theory of Boolean pairs—is undecidable.

Appendix B

Survey of Mereology

B.1 Types of mereologies

On the one hand, we have set-theoretic calculi of individuals (SCIs), or those within languages containing terms and quatification for sets. On the other, we have pure calculi of individuals (PCIs), or those containing only terms for individuals. The logical language used in traditional PCIs is that of first-order logic in the signature $\{\leq, =\}$ with the two place relation $x \leq y$ standing for the expression 'x is a part of y'.

The language typically employed in SCIs is expressively equivalent to a second-order language in the same signatire. Although the selection of a language is always a rather controversial matter, there are various similarities amid these theories, most often seen in the underlying axiomatization of the parthood relation.

B.2 Mereological Concepts

If x is a part of y, we write $x \leq y$. If x is a proper part of y, then we write $x < y$. Here x and y are *singular* terms, and the entities they denote are to be understood as *individuals*, that is, those of the lowest logical type. Traditionally, the relation of parthood is considered reflexive and transitive. Proper parthood is irreflexive, asymmetric, and transitive. These principles are so central to the meaning of these relations that seriously disputing them is tantamount to a failure to grasp them. Often the identity relation is defined in the normal way with $\leq$, i.e. $x = y \Leftrightarrow \forall z(z \leq x \leftrightarrow z \leq y)$, but this is not always the case.

Two individuals, x and y overlap, written xOy, if they have a part in common. It is easily seen that the relation is symmetric and reflexive. Two objects are disjoint xDy if and only if they do not overlap: $\neg xOy$.

Most mereological theories assume that at least some classes of individuals have suprema or mereological sums. Most often overlap is a prerequisite; however this is not always the case. It is, however, a central thesis of *General Extensional Mereology* that any two individuals linguistically specifiable have a mereological sum. For two individuals x and y we denote their sum by $x + y$. We shall have much more to say about the various types of mereological sum, but roughly, the sum is the individual which something overlaps if and only if it overlaps at least one of x and y.

If x and y overlap, by definition, they have a part in common. And if they do, they will have a mereological product. For the binary product of two objects we write $x \times y$. This object is the mereological analog for the intersection of two sets. The obvious crucial difference is that, in set theory, the intersection of sets with no common members exist: it is $\varnothing$. The same is true for BAs, for if two objects fail to overlap their infimum nonetheless exists. On primarily philosophical grounds, most mereologists are averse to admitting the null individual or any object that is a proper part of every individual. Supposing that no such individual exists, not any two individuals will have a mereological product and thus the operation gives rise to a partial function on the domain. Consequently, if x and y do not overlap, there is no $x \times y$, and thus the expression is equivalent to an improper description or empty name.

If x and y are two individuals, then their mereological difference, $x - y$ is the "largest" individual contained in x which has no part in common with y. There are two ways of understanding this operation, depending upon whether the null element is countenanced. If a null element, say 0 is admitted and $y \leq x$, then $x - y = 0$. If this is not the case, the expression is viewed as denoting no entity whatsoever.

Of course the existence of sums of *arbitrarily*-membered classes is not guaranteed by the existence of binary sums. But, if arbitrary binary sums are admitted, then clearly any sum of a finite set will also exist. Depending upon the further principles adopted, non-finite classes of individuals will also have sums. For example, if both unrestricted complementation is espoused along with the existence of a top or universe element over an infinite domain, then finite as well as cofinite sets of points will have sums.

Moreover, we might wish to consider the notion of a mereological sum of objects meeting a certain condition relative to the expressivity of the language. Let $\mathcal{L}$ be a language. To cover cases of sums of sets whose members meet a certain $\mathcal{L}$-specifiable condition, say $\phi(x) \in \mathcal{L}$, we write $\bigoplus x\phi(x)$. We shall encounter a variety of operators of varying strengths; thus, often we shall use a subscript, say i, to distinguish them as in $\bigoplus_i x\phi(x)$. Similarly, the mereological product of a set of objects specified by $\phi(x)$ is written $\bigotimes x\phi(x)$. Again, fusions for arbitrary subsets (like those that are neither finite nor cofinite) will still not, in general, exist, for the existence of arbitrary sums will be entirely dependent upon the expressivity of the language $\mathcal{L}$. If $\phi(x)$ is a rather complicated expression, then we shall use parentheses, as in $\bigoplus x(x \leq a \wedge xOb \wedge xDc)$.

If, however, arbitrary sums do exist, then there exists a sum of all objects. We denote this individual U for 'the universe'. In Boolean algebra, it is the top or 1. Traditionally, the existence of U is less controversial than the existence of arbitrary mereological sums.

Assuming that differences and U exist, then $U - x$ would denote an individual as well, named the *complement of* x. We abbreviate this individual $\sim x$. When mereological complements exist, these are obviously not the same as their Boolean counterparts, owing to the non-existence of the null individual in the former and the existence of the bottom in the latter. And as expected, most extensional mereologies would not espouse a complement for U.

Finally, we observe the predicate $At(x)$ which is to be read '*x is an atom*.' We may also formulate an expression in which we substitute a term for the variable x as in $At(\bigotimes x\phi(x))$. An atom is an individual that is indivisible. In the formal sense observed here, these are not to be confused with "atoms" in physics and chemistry, which may have proper subatomic parts. These are atoms in the strictest theoretical sense, and by definition have no proper parts whatsoever.

B.3 Symbolism for Formal Systems

We shall observe some conventions for portraying the systems in this appendix and, unless otherwise stated, will suspend them in the chapters that follow. Axioms are presented schematically and if not otherwise stated, metavariables which are unquantified are assumed to be universally quantified at the outer scope of the formula. We shall use the following variable types for the following purposes.

- Singular terms: $s, t, u, s_1, ...$
- Singular variables: $x, y, z, w, x_1, ...$
- Leśniewski-style nominal variables: $a, b, c, d, e, a_1, ...$
- Unary Predicates: $F, G, H, F_1, ...$
- Binary Predicates: $P, Q, R, P_1, ...$
- Class variables: $S, S_1, S_2, ...$
- Formula variables: $A, B, C, A_1, ...$

The use of truth functional connectives will accord with standard modern usage: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$. As normal, these connectives will bind their arguments in the order of strength displayed. The use of parentheses will be reduced. For example, $A \rightarrow B \rightarrow C$ is to be read $A \rightarrow (B \rightarrow C)$; otherwise we shall put in parentheses. The languages presented are assumed to be inductively defined in the normal way. We will use the quantifiers $\forall, \exists, \iota$ in their normal interpretation.

B.4 Mereological Principles

From the mathematical standpoint of the theory of relations, a mereology is a type of partial ordering. Mereological structures have attracted little attention from mathematicians. Perhaps the reason is that general mereological structures (GEMSs)—the models of *General Extensional Mereology* or GEM—are simply 0-deleted Boolean algebras. It has been argued wrongly in many places that models of general extensional mereologies are actually all complete BAs with zero deleted. But *infinite* models for extensional mereological structures without sets are not necessarily complete unless the language has the facility to quantify over arbitrary sets. For example, take the BA formed by the set of just the finite and cofinite sets of a countably infinite set. This model satisfies the axioms of general extensional mereological systems without quantification over arbitrary sets. The reason is that infinite non-cofinite sets are not definable in the standard first-order language in any of the standard mereological signatures. In order to get a grasp of the various axiomatizations, we shall cover the field in generality first before moving on to various, famous instances.

B.4.1 Parthood Axiomatizations and Concepts

The concept of part is often investigated alongside the concept of whole. In this extended sense, it is conceived as a theory of the relation of part to whole and the relation of part to part. However, when the concept of whole is added to the discussion, topological aspects of space are required—namely, continuity and contiguity. Thus, mereology is best understood as the pure theory of parthood.

Although there are dissenters with legitimate concerns, the parthood relation is understood minimally as a partial order. (The reader is directed to Simons (1986) for an overview of the various oppositions to a partial order axiomatization and a rather detailed list of sources and articles pertaining to the topic.) Thus, the first-order theory contains the following as axioms:

SA0 Any axiomatization for first-order logic with identity

SA1 $\forall x(x \leq x)$

SA2 $\forall x \forall y \forall z(x \leq y \land y \leq z \to x \leq z)$

SA3 $\forall x \forall y(x \leq y \land y \leq x \to x = y)$

We call the resulting system PO. One could just as well use the relation of proper part as primitive, since the following is implied by the axioms: $\forall x \forall y (x \leq y \leftrightarrow (x < y \vee x = y))$. In such case we would require the strict order axiomatization. But a drawback with choosing proper-parthood as primitive is that equality cannot be defined without introducing it also as a primitive. Hence equality is often admitted, when the strict order logic is employed.

It is well known that a partial order is too weak to model the mereological decomposition of an object. To see this consider a world with only two objects, one of which being a proper part of the other. Clearly such model is unrepresentative of a *mereological decomposition* in the desired sense. Hence, we must extend the logic to account for *supplementation principles* or those which represent the regularities of the accumulation of remainders in the course of a decomposition. There are several ways of varying strength and subtlety to express supplementation:

SF1 $\forall x \forall y (x < y \rightarrow \exists z (z < y \wedge z \neq x))$

SF2 $\forall x \forall y (x < y \rightarrow \exists z (z < y \wedge z \nleq x))$

SF3 $\forall x \forall y (x < y \rightarrow \exists z (z \leq y \wedge \neg z O x))$

SF4 $\forall x \forall y (x \nleq y \rightarrow \exists z (z \leq x \wedge \neg z O y))$

SF5 $\forall x \forall y ((\exists z\, z < x \vee \exists z\, z < y) \rightarrow (x = y \leftrightarrow \forall z (z < x \leftrightarrow z < y)))$

Consider SF1. This principle states that every proper part must be accompanied by another. The desired axiomatization is stronger than SF1, however. For, even though it rules out the two-element model lately discussed, it's easy to see that it doesn't rule out an infinite model that never fully splits, such as one in which each proper part taken completely overlaps the whole from which it was removed—for example, an infinite completely overlapping chain.

Now consider the other SF2. This principle is stronger than SF1 but fails to force the existence of a non-overlapping remainder. For example, it is satisfied on a model in which a whole is decomposed into proper parts *every one of which* overlap although not completely.

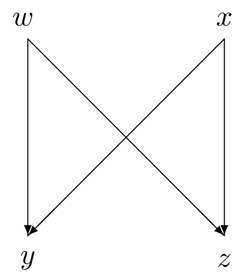


Figure B.1: A non-extensional SF3 frame.

According to the third principle SF3, for each proper part there is another with which it doesn't overlap. As far as supplementation goes, this principle seems sufficient. However, there might be a worry. For, suppose the domain consists of four distinct objects: w, x, y, and z. And assume in a decomposition y and z are both parts of w and x as in figure B.1. Such a model would be consistent with SF3. Some theorists have claimed that, although unintuitive, there might be exist distinct individuals capable of sharing all their proper parts. For example, the famous clay statue elicits the plausibility of SF3, since we might somehow argue that the clay is different from the statue, yet has all the same proper parts. We pass over the philosophical issues now, and return to this later.

Nonetheless, SF4 rules out non-extensionality, implies SF3, and is thus stronger. Indeed any PO logic containing SF4 defines the class of part-extensional frames—those satisfying SF5—from which it follows that no composite objects having the same proper parts are distinct. Given its importance, we shall name the logic satisfying axioms from PO and SF4 the label EM.

B.4.2 Composition Principles and Mereo-Boolean Variants

Decompositional principles are only part of the story. We also require those that take us from parts back to wholes. Indeed, the signature feature of mereology is the notion of mereological sum. And with the notion of sum follows also the Boolean analogs for products, differences, and so on. EM is, barring extensionality, still not as strong as classical systems. It is well known that for full GEM there is only one

n-element model for $n \in \mathbb{N}$ up to isomorphism. The reason for the difference between them turns on the existence of mereological sums. Thus, the most natural way to increase the strength of EM is to add to the axiomatization the existence of these various bounds.

There are myriad of conceivable conditions on composition. Consider the weakest:

$$\forall x \forall y (\psi xy \to \exists z (x \leq z \land y \leq z))$$

This principle asserts that any suitably related entities have an *upper bound*, where ψ is the hotly debated composition condition. Van Inwagen [99] has called the question of exactly what ψ should express the "Special Composition Question". One rather immediate intuition is to identify ψxy with mereological overlap. According to this interpretation, the principle is fairy weak. It holds in any domain with the unit. Thus, the existence of upper bounds does not imply that of sums or least upper bounds. And therefore conditions stronger than overlap are generally claimed to be necessary. One stronger conception would require any pair of suitably related entities to have a "smallest" underlapper. Here are three different versions of this:

SA6 $\forall x \forall y (\psi xy \to \exists z (x \leq z \land y \leq z \land \forall w (x \leq w \land y \leq w \to z \leq w)))$

SA7 $\forall x \forall y (\psi xy \to \exists z (x \leq z \land y \leq z \land \forall w (w \leq z \to wOx \lor wOy)))$

SA8 $\forall x \forall y (\psi xy \to \exists z \forall w (wOz \leftrightarrow wOx \lor wOy))$

The various principles correspond to three different operators which we can define in the following way:

SD6 $x +_a y = \iota z \forall w (x \leq w \land y \leq w \leftrightarrow z \leq w)$

SD7 $x +_b y = \iota z (x \leq z \land y \leq z \land \forall w (w \leq z \to wOx \lor wOy))$

SD8 $x +_c y = \iota z \forall w (wOz \leftrightarrow wOx \lor wOy)$

The first is the idea that the fusion of two objects is just their *least upper bound* relative to P (see e.g. Bostock [11] and van Benthem [9]). But this principle is

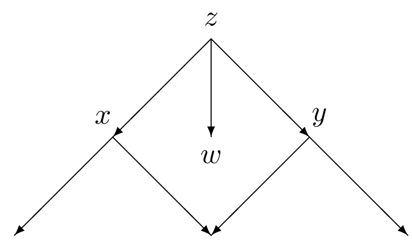


Figure B.2: Least upper bound counterexample.

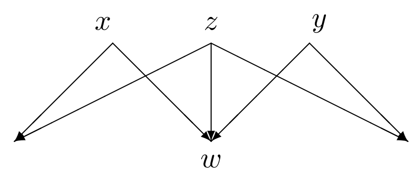


Figure B.3: A non-extensional model for fusion.

typically seen as too weak. It is satisfied on a model in which the least upper bound, z, of two overlapping objects, x and y, is one which has another dangling atom w disjoint from both x and y, like in figure B.2. It would clearly be a mistake to say that z is composed only of x and y. And for finite partial orders it is simple to show that SD6 is equivalent to mere upper bound existence, hence just as weak for that class.

Consider the second principle, corresponding to a notion found in Tarski [94]. For some metaphysical musings, like non-extensional decompositions, this principle seems too strong. It rules out the statue clay model lately noted but also rules out the situation in figure B.3. In contrast to the last example, this is a situation in which z may be legitimately understood as an object composed of x and y; yet it is an example of a non-extensional decomposition. Hence a position against extensionality like this would require a rejection of SF4 and therefore also SD7.

Along with a partial order axiomatization, the last principle SD8 strikes a nice bargain for metaphysicians interested in eschewing extensionality on complicated grounds. It is strong enough to rule out the first model but weak enough to allow the last. And it is therefore the most standard in treatments of mereology. We note in passing that it is well known that when SD8 used in the context of a strongly supplementary SD4 partial order axiomatization, SD6 is equivalent to SD7.

Here also an important relationship between finite 0-deleted Boolean algebra axiomatizations and extensional mereologies can be determined. For, it is well known that if we add to PO $\exists x \forall y (y \leq x)$ and $\forall x \forall y (\exists ! 1 z (z = x +_c y)$, it is easily checked that all the *finite* models of these structures are BAs.[1] The general case for zero-deleted BAs requires the existence of complements, which we define shortly.

We can strengthen the sum principles further by considering bounds and fusions for every expressible condition ϕ. For example, we might stipulate that any non-empty set of entities satisfying a suitable condition, say χ, has an upper bound. We can achieve the required degree of generality by relying on an axiom schema in which first-order specifiable classes are identified by arbitrary formulas. In the first order language, the following axiom schema will do, where '$\phi(w)$' is any formula in the language with one free variable:

$$(\exists w \phi(w) \wedge \forall w (\phi(w) \rightarrow \chi(w))) \rightarrow \exists z \forall w (\phi(w) \rightarrow w \leq z).$$

The other sum axioms can also be strengthened analogously. First define the following operators:

FDA $\bigoplus_a x \phi(x) = \iota z \forall w (z \leq w \leftrightarrow \forall v (\phi(v) \rightarrow v \leq w))$

FDB $\bigoplus_b x \phi(x) = \iota z \forall w (\phi(w) \rightarrow w \leq z \wedge \forall v (v \leq z \rightarrow \exists u (\phi(u) \wedge v O u)))$

FDC $\bigoplus_c x \phi(x) = \iota z \forall w (O w z \leftrightarrow \exists v (\phi(v) \wedge O w v)))$

Thus, existence principles for χ-restricted fusion for each of these can be given:

ARFA $(\exists w \phi(w) \wedge \forall w (\phi(w) \rightarrow \chi(w))) \rightarrow \exists y (y = \bigoplus_a x \phi(x))$

ARFB $(\exists w \phi(w) \wedge \forall w (\phi(w) \rightarrow \chi(w))) \rightarrow \exists y (y = \bigoplus_b x \phi(x))$

ARFC $(\exists w \phi(w) \wedge \forall w (\phi(w) \rightarrow \chi(w))) \rightarrow \exists y (y = \bigoplus_c x \phi(x))$

[1] One achieves the same models if the second formula is replaced with $\forall x \forall y (\exists ! 1 z (z = x +_b y)$.

Observe that the condition '$\exists w\phi(w)$' ensures that 'ϕ' selects a non-empty set, so there is no worry in asserting the unconditional existence of null entities. The condition of existence, however can be used alone without any additional conditions in the following *unrestricted* versions:

AUFA $\exists w\phi(w) \rightarrow \exists y(y = \bigoplus_a x\phi(x))$

AUFB $\exists w\phi(w) \rightarrow \exists y(y = \bigoplus_b x\phi(x))$

AUFC $\exists w\phi(w) \rightarrow \exists y(y = \bigoplus_c x\phi(x))$

The first-order logic obtained by adding to PO every instance of the axiom schema AUFC is known as *General Extensional Mereology* (or GEM). It's obvious that the existence of a universe element is guaranteed with the GEM axioms. Complementation is expressible in the language: $\sim x = \iota \bigoplus_c z(zDx)$. In each case, if x is not the universe, there are elements disjoint from it. Thus, the GEM is also the first-order logic of 0-deleted BAs.

B.5 PCIs

Let us recall our rubrics. By a 'CI' we understand a general extensional mereological first-order theory, possibly with terms for sets and quantification over sets, possibly without. A PCI is system devoid of sets and their machinery. And a SCI contains set-references and quantification. The first SCIs were laid down by Leonard and Goodman [55] and Tarski [94] independently. However, later in *The Structure of Appearance*, Goodman [45] rejected sets on nominalistic grounds. Accordingly, he switched to a classical PCI. Eberle [27] and Varzi [13] have also constructed systems with a non-set-theoretic mereological base.

B.5.1 GPCI

GEM is obviously a PCI. Goodman, however, uses a slightly different axiomatization. We begin our discussion of PCIs by introducing his here.

Goodman [45] selects overlap 'O' as primitive and defines identity and the remainder of the mereological connectives. Moreover, like GEM, the language he employs contains unary predicate symbols. Thus, his system differs from PCIs without the capability of multiply *characterizing* individuals. Hence, in the appropriate sense, his calculus is ontological, as the syntax for properties is countenanced. This does not necessarily imply any sort of platonism, since one might interpret these symbols as designating mereological extensions. Nonetheless, the language is strengthened considerably on account of this additional feature. For example, one can express that the extension of a predicate is exactly a filter or an ultrafilter, for example. And one can guarantee the existence of a unique mereological sum of individuals having color, say P, for some predicate letter. Here are his definitions:

GPCID1 $xDy \leftrightarrow \neg xOy$

GPCID2 $x \leq y \leftrightarrow \forall z(zOx \rightarrow zOy)$

GPCID3 $x = y \leftrightarrow \forall z(zOx \leftrightarrow zOy)$

GPCID4 $x < y \leftrightarrow x \leq y \wedge y \nleq x$

GPCID5 $\bigoplus_c xFx = \iota z \forall y(yOz \leftrightarrow \exists x(Fx \wedge yOx))$

GPCID6 $x + y = \bigoplus_c z(z \leq x \wedge z \leq y)$

GPCID7 $x \times y = \bigoplus_c z(z \leq x \wedge z \leq y)$

GPCID8 $U = \bigoplus_c x(x = x)$

GPCID9 $\sim x = \bigoplus_c z(zDx)$

GPCID10 $x - y = x \times \sim y$

GPCID11 $\bigotimes xFx = \bigoplus_c x \forall y(Fy \rightarrow x \leq y)$

Thus, he defines the binary operations in terms of the general sum operator introduced in the last section. And in addition all the standard terms and relationships are expressible.

Axioms of GPCI

GPCIA0 Any axiomatization of first-order logic without identity.

GPCIA1 $xOy \leftrightarrow \exists z\forall y(wOz \rightarrow wOx \wedge wOy)$

GPCIA2 $\exists xFx \rightarrow \exists x\forall y(yOx \leftrightarrow \exists z(Fz \wedge yOz))$

GPCIA3 $\forall z(zOx \leftrightarrow zOy) \rightarrow (\phi \rightarrow \phi[x/y])$

where $\phi[x/y]$ is any formula obtained from ϕ by replacing all free occurrences of x with y or a suitable alphabetic variant so that the introduced variable does not get inappropriately bound by a quantifier already appearing in the formula. And GPCIA3 is simply the Leibnizian axiom for identity according to which whenever two individuals are identical they have the same properties or are parts of precisely the same mereological extensions.

GPCIA1 says that two individuals overlap if and only if the binary product of the two exist. GPCIA2 guarantees the existence of a mereological sum of any non-empty extension of a unary predicate symbol. The axioms can be used to derive the following:

GPCIT1 $x = y \leftrightarrow \forall z(zDx \leftrightarrow zDy$

GPCIT2 $x = y \leftrightarrow \forall z(z \leq x \leftrightarrow z \leq y$

GPCIT3 $xOy \leftrightarrow \exists z(z \leq x \wedge z \leq y$

These theorems show that in Goodman's system disjointness and parthood can be also taken as primitive relations. Moreover, given the previous definitions, one can easily check that GEM and GPCI are actually equivalent. For the sake of completeness, we introduce one other PCI.

B.5.2 EPCI

In the late sixties and early seventies, Eberle [27] investigated several PCIs in a manner more formal than had been achieved before. He proved completeness of his mereologies with respect to certain algebraic structures with $\leq$ interpreted as a GEM-style partial order. And he investigated a number of stronger axiomatizations, to include atomistic ones. Eberle uses $\leq$ as primitive and defines the mereological relationships and $=$. We next consider some of the definitions explicitly.

EPCID1 $xOy \leftrightarrow \exists z(z \leq x \wedge z \leq y)$

EPCID2 $x = y \leftrightarrow x \leq y \wedge y \leq x$

To these definitions we add GPCID1, GPCID4, and GPCID5 from the last section.

Axioms of EPCI

EPCIA0 Any axiomatization of first-order logic without identity.

EPCIA1 $x \leq y \leftrightarrow \forall z(zOx \rightarrow zOy)$

EPCIA2 $\exists x Fx \rightarrow \exists x \forall y(yOx \leftrightarrow \exists z(Fz \wedge yOz))$

EPCIA3 $\forall z(zOx \leftrightarrow zOy) \rightarrow (\phi \rightarrow \phi[x/y])$

This logic can be shown to be equivalent to GEM and therefore GPCI owing to the fact that O and $\leq$ are interdefinable under the mereological interpretation.

B.6 SCIs

The first calculus of individuals employing explicit quantification over sets is due to Leonard and Goodman. Only a portion of the hierarchy of sets is required: sets of individuals. Sets of sets of individuals, and so on, are not required. The sum and product operators $\bigoplus_c$ and $\bigotimes$, which are devised for taking the mereological sum of objects meeting first-order specifiable conditions are replaced by the operators Su

and Pr to remind the reader that the latter operators take *sets* as their arguments. Specifically, we use the following syntax: 'x Su S' which is to be read 'x is the mereological sum of the set S of individuals'.

The system is constructed over the part of *Principia Mathematica* required to handle individuals, sets of individuals, predicates, and identity. The disjointness relation is primitive, and identity is presupposed given. Witness the following definitions:

LGSCID1 $x \leq y \leftrightarrow \forall z(zDy \rightarrow zDx)$

LGSCID2 $xOy \leftrightarrow \exists z(z \leq x \wedge z \leq y)$

LGSCID3 $x \ \mathsf{Su}\ S \leftrightarrow \forall y(yDx \leftrightarrow \forall z(z \in S \rightarrow yDz))$

LGSCID4 $x \ \mathsf{Pr}\ S \leftrightarrow \forall y(y \leq x \leftrightarrow \forall z(z \in S \rightarrow y \leq z))$

LGSCID5 as GPCID4

Axioms of LGSCI

Here, unlike earlier and later presentations, we add quantifiers for explicitness:

LGSCIA0 Any set-theory (like ZFC or PM) with first-order logic and identity

LGSCIA1 $\exists x(x \in S) \rightarrow \exists x(x \ \mathsf{Su}\ S)$ for any set S

LGSCIA2 $\forall x \forall y(x \leq y \wedge y \leq x \rightarrow x = y)$

LGSCIA3 $\forall x \forall y(xOy \leftrightarrow \neg xDy)$

Similarly, the following definition is also of interest:

LGSCID5 $\mathrm{U} = \iota z(z \ \mathsf{Su}\ \{x \mid x = x\})$

The binary operators $+$, $\times$, and $\sim$ are definable with set abstraction in the obvious ways.

Leonard and Goodman's approach was basically paved for by Tarski who devised two systems, one atomistic and another neutral on that score. We now take a look at his system.

B.6.1 TSCI

Tarski's system also takes a set-theoretic logic for granted. Parthood $\leq$ is taken as primitive and $=$ is assumed. Here are the relevant definitions.

TSCID1 $xDy \leftrightarrow \neg\exists z(z \leq x \wedge z \leq y)$

TSCID2 $x \mathsf{\ Su\ } S \leftrightarrow \forall y(y \in S \rightarrow y \leq x) \wedge \neg\exists y(y \leq x \wedge \forall z(z \in S \rightarrow zDy))$

Axioms of TSCI

TSCIA0 The same as LGSCIA0.

TSCIA1 $\forall x \forall y \forall z(x \leq y \wedge y \leq z \rightarrow x \leq z)$

TSCIA2 $\exists x(x \in S) \rightarrow \exists x(x \mathsf{\ Su\ } S \rightarrow x = y)$ for any set S

Appendix C

Leśniewski's Systems

From the perspective of language selection, there are two principal worries with sets. The first concerns the coherence of set-theoretical reasoning itself. For example, Leśniewski's system *Mereology* was to be a correction of set theory. He was motivated initially by Russell's paradox, and *Mereology* arose out of his struggles to pinpoint what he considered to be the mistake in the reasoning leading to the paradox in naive set theory. Thus, his intention was to use *Mereology* and its logical underlying framework to provide a new foundation for mathematics. For reasons which would lead us too far afield, this objective was not met, and Leśniewski's writings were destroyed in the Second World War.

We should understand worries about coherence as distinct from (although perhaps in various ways related to) those concerning the metaphysical nature of sets. Metaphysical worries generally concern the abstractness of sets.[1] Plainly, sets have an altogether different nature than material entities or regions. For example, material entities lie in regional wholes, not in sets. But, this modern conception of sets as abstract menageries is different than Leśniewski's conception.

Lesniewski's *Mereology* is a theory of *collective* classes, and is to be distinguished

[1] I shall not engage in a deliberate philosophical discussion of the distinction between abstract and concrete objects here. In Chapter 4 I will, however, discuss whether regions and regional wholes are abstractions and whether or not our understanding of regional wholes is distinct from our mathematical account of them. I shall engage in the discussion only in the context of a conceptual analysis of matter and its relation to regions.

from the more familiar theories of Russell, Zermelo, et al, which formalize the notion of a *distributive* class. Collective classes differ from distributive in several respects. In the collective sense of "class" there is no empty set, whereas the empty set does exist in the distributive. If we consider the class H of Holland, then in the distributive sense of "class" the provinces North Brabant and Limburg are elements of H, but Amsterdam is not. In the collective sense, however, not only are the several provinces elements of H but so are Rotterdam, Tilburg, and Arnhem. In the collective sense of "class" elementhood is transitive—that is, if x is an element of y, and y is an element of z, then x is an element of z. In the distributive sense, elementhood is not a transitive relation. In the collective sense of "class", but not in the distributive sense, if some class is a unit class, then it is the same object as its only element. Similarly, Frege (1895) distinguishes "concrete" from "logical" classes. His indication of this distinction was prompted by his criticism of Schröder, whom he accused of confusing these concepts. In this work, Frege claims that a calculus of what we have called *collective* classes is no more or less than a calculus of part-whole relationships.

In almost all of his writings, Tarski (1929) employs distributive sets. For example, in "Foundations for the Geometry of Solids", he allows himself reference to sets of n-dimensional open balls of an n-dimensional hyperspace. He therefore shows how equivalence classes of concentric balls can be isomorphically related to points in Euclidean spaces; we shall return to his categoricity result and a similar result in due course.

From the formal perspective, the differences between Leśniewski's system and the various CIs are the result of the underlying languages employed, not due to the specific axiomatizations of parthood. The languages adopted by Leśniewski and his students and followers are radically different from first-order and even higher-order formalisms. Indeed, Leśniewski's understanding of the metaphysical nature of his syntax was concrete. Elements of the language are viewed as physical objects which grow as they are employed and therefore not as platonic collections of entities with independent existence. I follow Peter Simons that the principal differences between *Ontology* and Leśniewski's underlying typed calculus, and languages of SCIs and PCIs are due to three factors: terms, quantification, and definitions.

C.0.2 Terms

In first-order logic, all constants and terms are essentially *singular*. These items denote individuals of the lowest logical type. In so-called first-order "free logics", there is a caveat: terms and definite descriptions might be improper. They may not refer to anything. Leśniewski's *Ontology* permits empty terms. Indeed, while in standard first-order logic constants refer to one and only one individual, in his languages, terms can refer to multiple individuals. Thus, Leśniewski's languages contain the ability of *plural reference*. In this respect, the language of *Ontology* contains important features of natural languages. Like the expressions, 'Bonnie and Clyde' and 'Americans', the language of *Ontology* has terms for objects multiply construed. This is a rather nice feature of the mereology and speaks against the rather pointless assumption that mereology must be a calculus of individuals per se.

Let us add to our current bag of symbols $\bigwedge$, which is Leśniewski's symbol designating the *empty name*. Then we add plural names $a, b, c, ...$ To fix intuitions consider a cake with four slices x, y, z, w. How many of these do we need to have a name for every plural combination of slices? There are $2^4 = 16$ ways of referring to a selection of these slices (including the null case). Therefore, we have 12 more combinations requiring plural names.

The term $\bigvee$ is a standard item for the universe in *Ontology*. Its dual is $\bigwedge$. It is crucial to observe the distinction between U and $\bigvee$. The first term is one which designates the mereological sum of all individuals. Thus, the former is a singular term. The second is a plural expression referring to all individuals multiply understood.

Ontology is the fundamental logic containing the propositional calculus and on which Leśniewski erected *Mereology*. Hence the addition of plural terms in the language of *Ontology* culminates in an extremely powerful mereological system. Both singular and plural variables may be quantified over, and complex expressions can be formulated with plural referents. To those schooled in the differences between first and higher-order logic, this is a definite indication of the overwhelming strength of the language. For example, relations only expressible in higher-order formalisms, like *outnumber* and *surround*, are expressible in *Mereology*. Nonetheless, from the set-theoretic standpoint, there is a limit to plural expressiveness, since the objects to be denoted must be specifiable within the formalism. Unless rather controversial axioms are adopted concerning infinity, higher cardinalities cannot be invoked. And it is in this connection which we observe the linguisitic and theoretical relativity of the concept of arbitrary quantities of individuals. We shall see that set-theoretic predilections are in disagreement with those of the mereological stamp. The issue turns fundamentally on whether the existence of objects must be linguistically specifiable. From the standpoint of mereology, the notion of subsets of arbitrary cardinality is just an axiomatic assumption of set theory. Ultimately this will be challenged in due course, for non-set-theoretic criteria might be plausibly leveraged to support the existence of arbitrary sums in some domains.

C.0.3 Quantification

Leśniewski's interpretation of the quantifier is non-referential. Let us compare this to the first-order quantifier. The first-order expression "$\exists x Qx$" is to be read "something Qs" or "there exists a Qer", and "$\forall x Qx$" is to be interpreted basically as "all existent things are Qers". We may refer to plurals in the same manner: "$\exists p Qp$" which would then mean "some thing is an Qer or some things are Qers". In keeping with Leśniewski's terminology, we call such singular and plural objects *manifolds*. According to the first-order understanding, quantification implies reference. Accordingly supposing the extension of Q to be empty, inferences such as $Qa \vdash \exists b Qb$ are invalid. However, since Leśniewski's construal of the quantifier is non-referential, such inferences are valid in *Mereology*. Thus the existential quantifier is not, properly speaking, an existential quantifier for Leśniewski.

Proper interpretation aside, Leśniewski's understanding of the quantifier is nonetheless problematic and reminiscent of the Meinongian problem of existence. Just as Meinong is seen as equivocating about existence with his distinction between subsisting and existing objects, Leśniewski seems also be equivocating here. In some sense, we must understand the quantificational expression as existential, for example in contexts of hypothetical reasoning like "if a Q'er exists, then ...". We at least wish hypothetically or modally to countenance the existence of an entity when we use the quantifier. This highlights a rather important feature of formal ontologies and signals its hypothetical essence.

Given the uniqueness of Leśniewski's quantifiers, we indicate this by using different symbols. We write Π for the universal quantifier and Σ for the existential.

C.0.4 Formal Definitions

For Leśniewski, definitions are not metalinguistic abbreviations, they are axioms. He regarded all definitions as object language equivalences. In order to preclude circularity, paradoxes, and other incoherent possibilities, he placed syntactic restrictions on definitions. Following his usage, we shall use the letter 'D' to signal that a definition is being given.

C.0.5 ***Ontology***

Ontology is a logical extension of *Protothetic*, Leśniewski's counterpart to the propositional calculus, which the latter contains as a proper fragment. Protothetic is nearly identical to the system of propositional types such as those of Church (1940) or Henkin (1963). *Ontology* extends the logic of propositional types with a strong quantificational ability and various other devices. It brings together a logic of Schröder with a quasi simple type theory of functions. Functions are eliminated on nominalistic grounds for functors or, in other words, the expressions denoting those functions.

On account of Leśniewski's rather free-wheeling view of quantification, he finds it unproblematic to quantify over variables of any type, as his understanding of the calculus only commits him to the existence of concrete marks and not to pre-existent abstract entities. Thus the reader might well think that the proper interpretation of Leśniewskian quantifier is as a device ranging over expressions and not over objects. For example, Quine (1969b) argues that Leśniewski's style of quantification is best understood as *substitutional*. But this cannot be right. For given his understanding of syntax as a growing expanse of physical marks, the proper conception of the quantifier cannot be a device which "ranges over" expressions. There are various scholarly interpretations of his quantifiers in the literature.[2]

Given Leśniewski's understanding of definitions, an indefinite number of predicates may be added to *Ontology*. Since within *Ontology* the possibility of defining new objects is a matter of axiomatizing, the notion of primitivity does not apply. However, the most important relational symbol is ε denoting *singular* inclusion. The second term in $a \varepsilon b$ is understood as possibly plural. The truth conditions of the formula $a \varepsilon b$ are as follows: it is true if and only if 'a' designates one and only one individual that is also one of the possibly many individuals indicated by b. Thus an easy way to read "$a \varepsilon b$" is "a is one of b" or just simply "a is a b" or "a is among b". This gives rise to a rather interesting set of expressions. In most typed theories of sets, expressions of the form $x \in x$ are either syntactically incoherent or simply false. But, in *Ontology* expressions of the form $a \varepsilon a$ are true if and only if the manifold referred to by x is an individual.

In short, in set-theoretic reasoning the distinction between individuals and pluralities is avoided by eliminating pluralities for the set-theoretic property of cardinality. Cardinalities are *features of* sets. And interestingly, sets remain singular individuals. A plurality, in contrast, "wears its cardinality on its sleeves" and therefore is not a feature of its contents.

In both *Ontology* and *Mereology*, definitions are of two kinds: *propositional* and *nominative*. Consider the following examples and not in complete generality, since we shall only present the axiomatization and not the entire linguistic landscape.

Propositive **ODS1** $\Pi a \Pi a_1 ... \Pi a_n (P(a_1, ..., a_n)) \leftrightarrow A)$

The example definition defines an n-place predicate P. The right-hand side contains only $a_1, ..., a_n$ free, and otherwise constants which have already been introduced.

Nominative **ODS2** $\Pi a \Pi a_1 ... \Pi a_n (a \varepsilon f(a_1, ..., a_n)) \leftrightarrow \Sigma b(a \varepsilon b \wedge B))$

The example definition above defines an n-place operator f, where the conjunct B freely contains at most those names in the definition and otherwise previously introduced constants. Of course, the nominative definition above is rather complex, and can be replaced with a 0-ary function or constant as in the following:

[2] See for example, Simons (1985), Küng and Canty (1970)

ODS3 $\Pi a \Pi a_1 ... \Pi a_n (a \ \varepsilon \ f(a_1, ..., a_n) \leftrightarrow \Sigma b(a \ \varepsilon \ b \wedge B))$

Thus, with ε, $\bigvee$, $\bigwedge$, and the multi-typed variable and quantificational apparatus, a multitude of new predicates and names may be given or "defined" within the system. Here are a few examples (all unquantified occurrences of terms are Π-quantified):

	Expression	**Concept**	**Truth conditions**
OD1	$Ea \leftrightarrow \Sigma b(b \ \varepsilon \ a)$	Existence	'a' designates at least one individual.
OD2	$!a \leftrightarrow \Pi b \Pi c(b \ \varepsilon \ a \wedge c \ \varepsilon \ a \rightarrow b \ \varepsilon \ c)$	Uniqueness	'a'designates at most one individual.
OD3	$E!a \leftrightarrow a \ \varepsilon \ a$	Singular existence	'a' designates exactly one individual.
OD4	$a \simeq b \leftrightarrow \Pi c(c \ \varepsilon \ a \leftrightarrow c \ \varepsilon \ b)$	Identity	'a' and 'b'designate the same individuals or are both empty
OD5	$a = b \leftrightarrow a \ \varepsilon \ b \wedge b \ \varepsilon \ a$	Singular identity	'a' and 'b'designate the same individual
OD6	$a \approx b \leftrightarrow !a \wedge a \simeq b$	Non-plural identity	'a' and 'b'designate the same individual or are both empty
OD7	$a \cong b \leftrightarrow Ea \wedge a \simeq b$	Existent identity	'a' and 'b'designate the same (existent) individual
OD8	$a \subseteq b \leftrightarrow \Pi c(c \ \varepsilon \ a \rightarrow c \ \varepsilon \ b)$	Inclusion	'b' designates whatever 'a' designates, if anything
OD9	$a \sqsubseteq b \leftrightarrow Ec \wedge a \subseteq b$	Nonempty Inclusion	'b' designates whatever 'a' designates (at least one item)
OD10	$a \ \varepsilon \ \bigvee \leftrightarrow a \ \varepsilon \ a$	Universal name	'$\bigvee'$ designates any individual
OD11	$a \ \varepsilon \ \bigwedge \leftrightarrow a \ \varepsilon \ a$	Empty name	'$\bigwedge'$ designates no individual
OD12	$a \ \varepsilon \ (b \cup c)$ $\leftrightarrow (a \ \varepsilon \ a \wedge (a \ \varepsilon \ b \vee a \ \varepsilon \ c))$	Nominal union	individual designated by '$a'$is among those designated by either 'b' or 'c'
OD13	$a \ \varepsilon \ (b \cap c) \leftrightarrow (a \ \varepsilon \ b \wedge a \ \varepsilon \ c)$	Nominal intersection	individual designated by '$a'$is among those designated by both 'b' and 'c'
OD14	$a \ \varepsilon \ \wr b \leftrightarrow a \ \varepsilon \ a \wedge \neg a \ \varepsilon \ b$	Nominal negation	individual designated by '$a'$is not among those designated by 'b'

Axiomatization of Ontology

OA0 Any axiomatization sufficient for propositional logic and rules for quantification of variables of any category.

OA1 $a \varepsilon b \leftrightarrow \Sigma c(c \varepsilon a) \wedge \Pi c(c \varepsilon a \to c \varepsilon b) \wedge \Pi c \Pi d(c \varepsilon a \wedge d \varepsilon a \to c \varepsilon d)$

The second axiom is rather important. The right side of the equivalence is equivalent to the proposition that there is at least one and most one a and whatever is a is also b. Observe that it's form is obscure, since ε appears on both sides of the equivalence. Any time one writes the simpler $a \varepsilon b$, one may write the longer version. This is a consequence of the missing meta-level in Leśniewski's system. Hence the definition is not explicit. Finally we must add the following axiom:

OA2 $\Pi f \Pi g \Pi a_1 ... \Pi a_n (\Pi a(a \varepsilon f(a_1 ... a_n) \leftrightarrow a \varepsilon g(a_1, ..., a_n)) \leftrightarrow \Pi F(F(f) \leftrightarrow F(g)))$

In the axiom above f and g are n-adic variables for operator names, and F is a monadic variable taking in the operator names for arguments. Both terms $f(a_1, ..., a_n)$ and $g(a_1, ..., a_n)$ contain only $a_1, ..., a_n$ free, and must be previously defined.

As the reader can perhaps tell, more work in proving theorems within this system are done by terms and term-forming functors. Below are some theorems of interest:

OT1 $E!a \leftrightarrow Ea \wedge !a$

OT2 $E!a \leftrightarrow a = a$

OT3 $E!a \leftrightarrow \Sigma b(a \varepsilon b)$

OT4 $Ea \leftrightarrow \neg a \simeq \bigwedge$

OT5 $Ea \leftrightarrow a \sqsubseteq \bigvee$

OT6 $a = b \leftrightarrow E!a \wedge a \simeq b$

OT7 $a \varepsilon b \leftrightarrow E!a \wedge a \subseteq b$

OT8 $a \varepsilon b \leftrightarrow !a \wedge a \sqsubseteq b$

OT9 $!a \leftrightarrow \Pi c \Pi d(c \varepsilon a \wedge d \varepsilon a \rightarrow c = d$

OT10 $a \varepsilon b \leftrightarrow \Sigma c(a \varepsilon c \wedge c \varepsilon b)$

OT11 $a \varepsilon b \wedge b \varepsilon c \rightarrow a \varepsilon c)$

OT12 $a \simeq b \leftrightarrow \Pi F(F(a) \leftrightarrow F(b))$

C.0.6 *Mereology*

Mereology is the most famous of Leśniewski's systems. According to the natural way of understanding *parthood, overlap, and disjointness*, these relations are of type (i, i). Likewise, any sum or product of individuals is, again, and individual. However, given the multitude of term types in *Ontology*, the mereological extension must be able to express the typical relationships under these more complex conditions. Consider the sentence 'a is a proper part of b'. In first-order logic, the expression is given a tripartite form in which the non-variable expressions are given over to one syntactic item $\leq$. Leśniewski, on the other hand, employs plural terms and singular inclusion together to obtain the same expression with four syntactic components. In particular, he splits the expression '( ) is a proper part of ( )' into a combination of the copulative '( ) is a ( )' and functor 'proper part of ( )' and combines them into '( ) is a (proper part( ))'. The same is true, as we shall see, for the other typical mereological relations and operators.

A multitude of primitives can be used to give rise to the same system. The goal here is merely to introduce the system, so we shall concern ourselves only with the system obtained based on the traditional functor *pt* which, in effect, is Leśniewski's analog for $\leq$.

System M.

The following definitions conform to the definitional requirements laid out in the last section for *Ontology*.

MD1 $a \varepsilon ppt(b) \leftrightarrow a \varepsilon pt(b) \wedge \neg a = b$

MD2 $a \varepsilon ov(b) \leftrightarrow a \varepsilon a \wedge b \varepsilon b \wedge \Sigma c(c \varepsilon pt(a) \wedge c \varepsilon pt(b))$

MD3 $a \varepsilon ex(b) \leftrightarrow a \varepsilon a \wedge b \varepsilon b \wedge \Pi c(c \varepsilon pt(a) \rightarrow c \varepsilon pt(b))$

MD4 $a \varepsilon Sm(b) \leftrightarrow a \varepsilon a \wedge \Pi c(c \varepsilon b \rightarrow c \varepsilon pt(a)) \wedge \Pi c(c \varepsilon pt(a) \rightarrow \Sigma d \Sigma e(d \varepsilon b \wedge e \varepsilon pt(c) \wedge e \varepsilon pt(d)))$

MD5 $a \varepsilon fu(b) \leftrightarrow a \varepsilon a \wedge \Pi c(c \varepsilon pt(a) \rightarrow \Sigma d \Sigma e(d \varepsilon b \wedge d \varepsilon pt(a) \wedge e \varepsilon pt(c) \wedge e \varepsilon pt(d)))$

To compare the symbolism of M with CIs, consider the following table:

Mereological Concept	**Calculus of Individuals**	***Mereology***
Part	$x \leq y$	$x \varepsilon pt(y)$
Proper Part	$x < y$	$x \varepsilon ppt(y)$
Overlap	xOy	$x \varepsilon ov(y)$
Disjointness	xDy	$x \varepsilon ex(y)$
Binary sum	$x + y$	$Bpr(x, y)$
Binary product	$x \times y$	$Bsm(x, y)$
Difference	$x - y$	$Cm(x, y)$
General sum	$\bigoplus x(Ax)$	$Sm(a)$
General product	$\bigotimes x(Ax)$	$Pr(a)$
Universe	U	U
Complement	$\sim x$	$Cpl(x)$
Atom	$At(x)$	$x \varepsilon atm$

Observe definitions MD4 and MD5. $a \varepsilon fu(b)$ is slightly different than $a \varepsilon Sm(b)$ and is to be read 'a is a fusion of bs". In the previous table the Sm operator is likened to $\bigoplus$, the CI-operator. This if for good reason. The crucial difference between the Sm

and the fu-operator is that, in contrast to the Sm operator, all the bs need not be "fused" so to speak. The fusion might be a plural expression and therefore there may exist many fusions of the bs. In the literature, a fusion is sometimes referred to as "a class" or "a collection". Obviously, in order to obviate confusion, we prefer to use a different term to avoid confusion.

MD6 $a \varepsilon U \leftrightarrow a \varepsilon Sm(\bigvee)$

MD7 $a \varepsilon nu(b) \leftrightarrow a \varepsilon a \wedge Eb \wedge \Pi c(c \varepsilon b \rightarrow a \varepsilon pt(c))$

We read the expression '$a \varepsilon nu(b)$' as 'a is a nucleus of bs'. Thus, the nucleus is not unique, but is rather any object which is part of all the bs. For the unique case, consider the following definition:

MD8 $a \varepsilon Pr(b) \leftrightarrow a \varepsilon Sm(nu(b))$

The product is just the complete or maximal fusion of bs, and is to be distinguished, therefore, from the conception of a nucleus. And it is just at this point that the major differences between a set and fusion and set and nucleus can be seen. The set can have inner patterns and characteristics or *depth*. However, the pluralities of *Mereology* have "horizontal" fusion relationships which give rise to various patterns.

Binary operators and the atom constant term can be defined in the following ways:

MD9 $a \varepsilon Bpr(b,c) \leftrightarrow b \varepsilon b \wedge c \varepsilon c \wedge Pr(b \cup c)$

MD10 $a \varepsilon Bsm(b,c) \leftrightarrow b \varepsilon b \wedge c \varepsilon c \wedge a \varepsilon Sm(b \cup c)$

MD11 $a \varepsilon Cm(b,c) \leftrightarrow b \varepsilon b \wedge c \varepsilon c \wedge a \varepsilon Sm(pt(b) \cap ex(c))$

MD12 $a \varepsilon Cpl(b) \leftrightarrow a \varepsilon Cm(U,b)$

MD13 $a \varepsilon atm \leftrightarrow a \varepsilon a \wedge \Pi b(b \varepsilon pt(a) \rightarrow b = a)$

The notion of discreteness is important in the axiomatization of *Mereology*. There are two types of discreteness, one stronger than the other.

MD14 $desc(a) \leftrightarrow \Pi b \Pi c(b \varepsilon a \wedge c \varepsilon a \rightarrow b = c \vee b \varepsilon ex(c))$

MD15 $wdesc(a) \leftrightarrow \Pi b \Pi c(b \varepsilon a \wedge c \subseteq a \wedge b \varepsilon pt(Sm(c)) \rightarrow b \varepsilon c)$

'wdesc(a)' is to be read 'as are weakly discrete' and is implied by discreteness, however the converse does not hold. To explain, the weak discreteness of as is the condition that any a which is part of the sum of one or more as is one of these as, which implies that no new a is obtained by putting several other as together and therefore that no a is ever a proper part of another a.

Axioms of *Mereology*

To arrive at the axiomatization of *Mereology*, we add to the axioms of *Ontology* the following:

MA1 $a \varepsilon pt(b) \rightarrow b \varepsilon b$

MA2 $a \varepsilon pt(b) \wedge b \varepsilon pt(c) \rightarrow a \varepsilon pt(c)$

MA3 $a \varepsilon Sm(c) \wedge b \varepsilon Sm(c) \rightarrow a = b$

MA4 $\Sigma c(c \varepsilon a) \rightarrow \Sigma c(c \varepsilon Sm(a))$

The first guarantees that if an object is a part, then it must exist or 'be an individual'. The second ensures the transitivity of parthood. The last two imply that if something is an a, then there is a unique sum of as. Below I list some theorems of *Mereology*.

MT1 $a \varepsilon a \rightarrow a \varepsilon pt(a)$

MT2 $a \varepsilon pt(b) \wedge b \varepsilon pt(a) \rightarrow a = b$

MT3 $a \varepsilon pt(b) \leftrightarrow a \varepsilon ppt(b) \vee a = b$

MT3 obviously implies that the *ppt* functor can also be selected as primitive.

MT4 $a \varepsilon ppt(b) \wedge b \varepsilon ppt(c) \rightarrow a \varepsilon ppt(c)$

MT5 $a \varepsilon ppt(b) \rightarrow b \varepsilon \wr (ppt(a))$

Leśniewski's first, 1916 axiomatization of *Mereology* consisted of MA3 and MA4 together with MD4. Other concepts can also be taken as primitive, as the following shows.

MT6 $a \, \varepsilon \, a \leftrightarrow a = Sm(a)$

MT7 $a \, \varepsilon \, a \leftrightarrow a = Sm(pt(a))$

MT8 $a \, \varepsilon \, pt(b) \leftrightarrow \Sigma c(a \, \varepsilon \, c \wedge b \, \varepsilon \, Sm(c))$

Thus, Sm may be used as primitive. Observe, however, that to achieve something comparable to the bound 'c' in a PCI would be impossible, for such would be tantamount to quantifying over unary predicate variables. Although we do not show this here, overlap and externality can also be used as primitives. The predicate *dcsr* is important, as it bears on the issue of cardinality.

MT9 $a \, \varepsilon \, ex(b) \leftrightarrow a \, \varepsilon \, a \wedge b \, \varepsilon \, b \wedge a \neq b \wedge dscr(a \cup b)$

The theorem above shows that *dscr* can be used also as primitive. We have noted that Leśniewski attempted to use *Mereology* as a foundation for mathematics. The predicate *dscr* played an important role in his attempts to produce such a foundation. To see why note the rather obvious fact that if a collection of objects which are discrete, the all its subsets are also discrete. That is,

MT10 $dscr(a) \wedge b \subseteq a \rightarrow dscr(b)$

Moreover, we have

MT11 $dscr(a) \wedge dscr(b) \rightarrow (a \simeq b \leftrightarrow Sm(a) \simeq Sm(b))$

which implies that mereological fusions of discrete objects can act under certain circumstances isomorphically to sets. Indeed, Leśniewski proved an analogous cardinality theorem akin to Cantor's original concerning the relation of the size of a set relative to its powerset—to whit, that if some as are pairwise discrete, then the number of fusions of these as is greater than the as.

Appendix D

Free Logics

So-called *free logics* are those logics free of existence assumptions with respect to its terms, singular and general, and whose quantifiers are treated exactly as in standard first-order logic. These logics do not presume that either singular or general terms have existential import. A singular term 's' has existential import if and only if s exists. And a general term say 'F' has existential import just in case F exist.

Examples of cases of import and non-import holding terms abound in colloquial English. For example, 'Washington Monument', 'New York', 'Queen Anne', '2/0', 'the number of galaxies in my hand', and 'the integer that is identical to $\sqrt{7}$' are all cases of singular terms. Obviously some of these hold existential import and some not. The same holds for cases of general terms. 'Gold' and 'is an artist' hold existential import and 'is divisible by 0' does not.

Free logic treats quantifiers as in the traditional first-order interpretation. This means roughly that the existential quantifier is to be read 'there is an existent object such that...' and the universal quantifier is to be read 'for all existent objects,...'.

The culprit is not seen as the meaning of the quantifiers, but in terms of the axiomatization employed. The distinctive property of free logics is a rejection of the principle of universal instantiation in first-order logic.

Returning to mereology, there are good reasons for countenancing empty terms. For example, according to the standard GEM interpretation, the term 'the mereological product of x and y' would be non-referrential if x and y do not overlap. Hence, mereological free calculi have been employed to overcome strained readings of propositions in other systems.

Here we provide an example of a free logic without sets due to Simons [90], based in a first-order formalism due to Lambert and van Fraassen [39]. $\leq$ is presumed primitive. We also introduce $\approx$, where $t \approx t'$ iff the terms 't' and 't' denote the same individual or are both empty. The syntax is built up inductively in the normal way with the first-order quantifiers $\forall$, $\exists$, and ι.

Definitions

LFFLD1 $\exists!1 \leftrightarrow \exists x(x \approx s)$

LFFLD2 $s = t \leftrightarrow s \approx t \wedge \exists!1s$

LFFLD3 $s < t \leftrightarrow s \leq t \wedge \neg t \leq s$

LFFLD4 $sOt \leftrightarrow \exists x(x \leq s \wedge x \leq t)$

LFFLD5 $sDt \leftrightarrow \neg sOt$

LFFLD6 $\bigoplus_c xFx \approx \iota z \forall y(yOz \leftrightarrow \exists x(Fx \wedge yOx))$

LFFLD7 $s \times t \leftrightarrow \bigoplus_c z(z \leq s \wedge z \leq t)$

LFFLD8 $s + t \approx \bigoplus_c z(z \leq s \vee z \leq t)$

LFFLD9 $\bigwedge \approx \iota x \neg(x \approx x)$

Rules

LFFLR1 Modus ponens

LFFLR2 If $\vdash A$, and $t_1, .., , t_n$occur in A,
then $\vdash \forall x_1, ..., \forall x_n(A[t_1, ..., t_n/x_1, ..., x_n])$
where $x_1, ..., x_n$ are distinct, and $t_1, ..., t_n$ are distinct

Axioms

LFFLA0	Tautologies of propositional logic
LFFLA1	$\forall x(A \to B) \to \forall x A \to \forall x B$
LFFLA2	$A \to \forall x A,$ where $x$is not free in A
LFFLA3	$\forall y(\forall x A \to a[x/y])$
LFFLA4	$s \approx t \to (A \to A[s/t])$ where t is not inappropriately bound
LFFLA5	$\iota x A \approx s \leftrightarrow \forall y(s \approx y \leftrightarrow A[x/y] \wedge \forall x(A \to y \approx x))$ where x and y are distinct.
LFFLA6	$s \leq t \to \exists!1s \wedge \exists!1t$
LFFLA7	$s \leq t \wedge t \leq u \to s \leq u$
LFFLA8	$s \leq t \wedge t \leq s \to s \approx t$
LFFLA9	$\forall z(zOx \to zOs) \to x \leq s$
LFFLA10	$\exists x Fx \to \exists x \forall y(yOx \leftrightarrow \exists z(Fz \wedge yOz))$

Note the first mereological axiom LFFLA6; it says that for things to stand in parthood relationships, they must exist. In this system $\forall x(x \nleq \bigwedge)$ is a theorem, as with $\forall x(\neg xO \bigwedge)$. If it weren't for the restriction of the first member in the consequent of LFFA9, we would be able to obtain $\forall x(\bigwedge \leq s)$, contradicting the first theorem. Since we are working in a free logic, we want to take advantage of the existence of non-referring terms like $\bigwedge$. Observe that although the term $\bigwedge$ exists in the language, it does not designate any individual. This reflects the extent to which the logic can become considerably more complicated by allowing non-referring terms.

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