The Logic of Kant's Temporal Continuum

Riccardo Pinosio

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Institute for Logic, Language and Computation
Universiteit van Amsterdam
Science Park 107
1098 XG Amsterdam
phone: +31-20-525 6051
e-mail: illc@uva.nl
homepage: http://www.illc.uva.nl/

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Faculteit der Geesteswetenschappen

to my family

to whom I owe everything

Contents

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Amsterdam
November, 2016.

Riccardo Pinosio

Chapter 1

Introduction

> “Tu sei studente, no?” mi disse. Io dissi di sì e lei volle sapere se ero alle tèniche. Le dissi che ero all'università.
> “Mària-vèrgola” disse la Gina.
> “Non s'impara niente” dissi.
> “Allora si vede che non studi.”
> “Per studiare studio” dissi. “Ma non imparo niente.”
> “Allora si vede che sei uno zuccone” disse la Gina. Poi mi domandò se studiavo da vocato. Io feci segno di no, e lei disse: “Da cosa studi tu, allora?”.
> “Filosofia” dissi. Lei mi domandò cosa si fa quando si è studiato da filosofia, e io le dissi che si prende la laura. Lei voleva sapere che mestiere si fa, e io dissi che volendo si può insegnare filosofia agli altri, ma di solito quelli che la sanno non la insegnano, mentre quelli che la insegnano non la sanno.
> “E cosa fanno allora quelli che la sanno?”
> “Se la tengono in mente” dissi.
> “E poi?”
> “E poi pensano, e tutto quello che pensano è filosofia”
> “E poi?”
> “E poi muoiono.”
> Poi lei ci salutò, e ripartì verso le fratture a oriente che saltano in Valsugana, per tornar giù in valle. Noi restammo lì senza far niente, alcune ore, e a un certo punto mi accorsi che si preparava un temporale.
>
> Luigi Meneghello, *I Piccoli Maestri*

In the present thesis I expound, with as much concision and precision as I can, the philosophical foundations of Kant's thoughts on space and time, and exhibit the underlying mathematical structure of the Kantian temporal *continuum*. I deem this work to be of interest not only for those who are concerned with Kant's thought on these matters, but also for those who are more generally interested in the development of mathematically rigorous phenomenological foundations for the concept of the (spatio-temporal) continuum. The first aim of this work is exegetical in nature: I wish to provide an analysis of the Kantian temporal continuum that is both formal and exegetically accurate. The second aim is speculative, as by setting Kant's theory of the continuum on a firm mathematical basis I wish to show its importance for contemporary debates in the philosophy of mathematics, of physics and of cognitive science, as well as its close relation to the Russellian-Whiteheadian or Aristotelian continuum. These aims are closely related to the more general enterprise, initiated in Achourioti and van Lambalgen (2011), of providing a mathematical formalization of Kant's theoretical philosophy.

I leave to my four readers the task of determining whether this thesis succeeds in achieving these aims. Over the past years I have invested so much time and effort in this work that I cannot detach myself from it and, as it were, consider it with the impartial gaze of the critic; I cannot step from the turmoil of the storm onto the shore and evaluate my efforts objectively. Thus, by way of an introduction, I will rather explain what motivated me to write this thesis and what specific topics are herein treated.

The original source of my interest in Kant's theory of time and its mathematical structure is of a religious nature, but the reader will not find any hint of this in the exposition that follows, and this is for two reasons. The first reason is that contemporary highly regimented academic standards do not take kindly to this and are not suited to the discussion of these matters, which are better treated, if anything, in literary form. The second reason is that I am not nearly a good enough writer, in Italian or English, to be able to convey a glimpse of these elusive thoughts in a beautifully written form, and one should write on such matters beautifully or not at all. Still, the religious motivation was of such importance to me that it must be remarked upon. The introduction seemed the most appropriate place for such musings.

Time is one of the most fundamental concepts by means of which we understand the world in as different fields as physics, cognitive science, philosophy and computer science. Nevertheless, the analysis of the content of the concept of time seems to differ rather wildly across these disciplines, to the extent that one might legitimately ask whether there is any concept of "time" at all, or whether "time" is just a sloppy shorthand for a host of superficially similar but radically different concepts. To be fruitful, however, any sort of skepticism must be positively formulated in the form of a challenge: is it possible to identify a set of features that are common to the seemingly different concepts of time used across the sciences, and that somehow capture the "essence" of time?

This question can be specialized further in relation to the apparent "gulf" between the subjective world of phenomenology and the objective world of physics. As an introspective being endowed with senses and consciousness I can directly observe not only the world "outside of myself", the representations of objects different from me, but most importantly the world "inside of myself", the world of my inner experience in relation to which, borrowing Kant's beautiful expression, I am as much a spectator as I am an originator. This world of inner experience takes the peculiar form that in our every day existence we call "time", and which can be described in non-temporal terms only metaphorically; it is the keen awareness of the unceasing flickering of existence, which always leaves behind a fading sign of its presence. This form of our inner experience is one of the fundamental subjective *data* at the dawning of our self-consciousness; the other, intimately tied with it, is of course fear, as the Bṛhadāraṇyaka Upaniṣad makes clear.

However, one cannot sit all day and ponder on this peculiar fact of consciousness, since the world must take its toll; from the dynamics of survival spring the technology of measurement and the slow but steady construction of scientific objectivity. At its height, this provides us with conceptual structures that are not only far removed from the world of immediate experience, but often flatly contradict some of the most entrenched intuitions we have regarding our own existence; through these objective constructions we not only harness the power of nature, and control that which is not ourselves, but also rebel against our insignificant place in the universe by understanding it.

A gulf between our immediate grasp of existence as unceasing temporal flux, in which primitive experiences succeed one another, and the structured explanations of scientific theories then opens wide. Should we take our inner existence as merely an epiphenomenon of the physical processes that we analyze in purely objective, non-experiential terms, or are there some properties of inner consciousness that are irreducible to any conceptual analysis, which can be grasped only by unmediated subjective awareness?

The present study is, in its essence, only the beginning of an attempt to construct a rigorous relationship between the experience of inner consciousness and the objective physical world by subsuming both in a unified ontology, which must ultimately be neither "mental" nor "physical", since these descriptors already embody a dualism that - if a relationship between inner consciousness and the objective world is to be established - must be superseded. Ultimately, I would like to be able to provide the fundamental concepts by means of which we understand the world, and in particular the concepts of time and becoming, with an abstract meaning that eschews the categories of the "mental" and the "physical".

The reader is most likely wondering at this point why anyone would take on such a task, and what does Kant have to do with it. I shall address these two points in turn.

I find this enterprise important because I believe the central task of philosophy to be that of struggling to formulate systems in which the multiplicity of being is unified in a totality without at the same time annihilating this multiplicity, or, to put it in a different but equivalent way, to combat the dispersion of the particulars - whether objective or subjective - through the creation of a system in which each particular finds its own place and acquires meaning in relation to every other particular. Thus, at the very beginning of the *Asclepius* we find:

> Alterum enim alterius consentaneum esse dinoscitur, omnia unius esse aut unum esse omnia; ita enim sibi est utrumque conexum, ut separari

> alterum ab utro non possit.[1]

This passage expresses with beauty and simplicity something which, in my mind, is the core of any good philosophical investigation: the struggle not to lose sight of the forest by being too engrossed in the examination of the trees, even if this means that the forest has to be created by an act of will, so to say, *ex nihilo*. Indeed, it is in this act of creation - the act of unifying and organizing scattered particulars under the single heading of a system - that I experience the fundamentally mystic, or religious, emotion that Einstein has so aptly described:

> The finest emotion of which we are capable is the mystic emotion. Herein lies the germ of all art and all true science. Anyone to whom this feeling is alien, who is no longer capable of wonderment and lives in a state of fear is a dead man. To know that what is impenetrable for us really exists and manifests itself as the highest wisdom and the most radiant beauty, whose gross forms alone are intelligible to our poor faculties — this knowledge, this feeling [...] that is the core of the true religious sentiment. In this sense, and in this sense alone, I rank myself among profoundly religious men. (Albert Einstein, letter to Hoffman and Dukas, 1946; see Einstein (2013)).

It is this sort of religious feeling that has been the source of this modest book.

As far as the second question is concerned, Kant's theoretical philosophy, and in particular his analysis of the spatial and temporal continuum, are of great interest for the general aim outlined above because Kant is perhaps the philosopher who thought the most deeply about time, and in particular about the relation between our intuitive conception of time and the objective, physical notion of time. Indeed, we find him struggling with this relationship throughout the *Metaphysical Foundations of Natural Science*, where he attempted to derive as much of physics *a priori* as possible, and in many beautiful pages of the *Opus Postumum*, where we see Kant grappling with the very same problem of building a philosophical system where the intuitive time of inner consciousness and the objective time of physics would be put in correspondence and unified under a general abstract heading - the "conditions for the possibility of experience". Thus, Kant's work provides us with unique insights into the nature of the temporal continuum, whose vibrations, as we shall see, resound in the works of later thinkers - James, Hermann Weyl and Whitehead - and of distant ancestors - Aristotle, above all. Furthermore, Kant's systematic philosophy, as we shall see, lends itself perfectly to mathematical formalization, this aspect being of paramount importance. Indeed, the reader should

[1] In Copenhaver (1995, p. 67) the passage is translated as follows:

> Admittedly, the one is consistent with the other: all are of one or all are one, for they are linked so that one cannot be separated from the other.

not infer from the discussion above that I somehow advocate a non-systematic approach to philosophical work. On the contrary, the present work is firmly grounded in the tradition of systematic philosophy that in the past century has been criticized, on different grounds, by the logical positivists, the existentialists and the postmodernists; in this regard, I conceive of mathematical formalization merely as the natural outcome of the construction of a system of philosophy as a science, following Husserl's remark in his third logical investigation:

> The progress from vaguely formed, to mathematically exact, concepts and theories is, here as everywhere [...] an inescapable demand of science. (Husserl, 2012, p. 41).

In this respect, I am keenly aware that philosophically inclined readers would not be pleased if they had to plough through a series of technical results in logic and topology to extract their philosophical meaning, while mathematically inclined ones would not be pleased if they had to sit through a careful examination of Kant's argument in the Transcendental Deduction to get to the mathematics. Thus, to please both audiences, I have attempted to organize the material of the present thesis in ascending order of mathematical sophistication, although in some places there are lapses from this grand scheme of things. More precisely, the thesis is structured as follows.

In chapter 2 I provide a justification for the application of mathematical methods in the exegesis of systematic philosophy. Indeed, the attempt to elucidate Kant's thought on space and time by means of contemporary mathematics might raise the eyebrows of many a philosophical reader. After all, the techniques of contemporary logic of which we make ample use were hardly available in Kant's times; it might then seem anachronistic, or even a downright falsification, to apply this modern technology to understand Kant's thought. Some readers, on the other hand, might object to this enterprise on a different ground, and doubt the fruitfulness of applying formal methods to Kant's exegesis; they might be worried that the crucial points of Kant's philosophy defy formalization, and that a formal approach to understanding Kant cannot but be superficial. Chapter 2 shall address these and related worries on a matter of principle, although I am of the opinion that the best justification for a methodology are the fruits that it bears.

In chapter 3 I provide a philosophical analysis of the distinction between the form of intuition and the formal intuition appearing in the footnote at B161n of the *Critique*, which lies at the core of my interpretation of Kant's theory of space and time. In particular, a formal theory of Kant's temporal continuum must begin with an exegetical analysis on which to ground *formal correlates* to Kant's informal notions. We shall see in chapter 2 that the development of this interpretative analysis is not neutral to the aim of formalization; rather, it is driven by it. It is also the case, however, that this formalization drives the interpretation itself, since the conclusions I shall infer from the formal approach will prompt us to dig deeper into Kant's own thought. Thus, there is here a clear instance of a circularity, which, as I argue in chapter 2, has to be thoroughly embraced.

In chapter 4 I begin to present the formal theory and the main insights that can be gleaned from it in relation to Kant's theory of time. This chapter is semi-formal in the sense that there are no proofs; the focus is on illustrating the philosophical foundations of the formal theory and how the latter illuminates Kant's conception of the continuum. Thus, another purpose of this work is to introduce the reader versed in Kant's scholarship, but with only a passing acquaintance of mathematical logic, to interpretative efforts that make essential use of logical formalization as an exegetical tool in its own right.

In chapter 5 the formal theory is developed to its full mathematical extent and the main theorems are proven. Although I provide some background on the mathematical tools employed in section 5.2, the reader will benefit from being acquainted with the basics of point-set topology, order theory, and inverse limits.

In chapter 6 I discuss the relation of the formal theory to the attempts by Russell and Walker to construct time from extended events, showing how the framework of this thesis subsumes these attempts as special cases, and provide a general topological view of the distinction between Russell's and Walker's constructions. Moreover, I briefly discuss the relation of the framework to formal topology, a constructive and predicative approach to point-free topology, and its relevance not only to understand the Kantian continuum but also to revive Russell's and Walker's project of constructing relativistic spacetimes from events.

I conclude the thesis in chapter 7 with an outline of the results that have been achieved and with an overview of some interesting open problems regarding Kant's transcendental logic, Kant's theory of space, and the relation of our framework to the Walker-Whitehead-Russell project of constructing relativistic spacetimes from events and constructive topology.

Chapter 2

The exegetical role of logic

This work is largely concerned with the application of tools from formal logic to the exegesis of Kant's distinction between the form of intuition and the formal intuition of time. As we remarked in chapter 1, this enterprise might be met with skepticism both from Kant scholars and logicians, albeit for different reasons. What is then the justification for the application of logical methods to Kant's exegesis, and, most importantly, why should they be useful in any way?

The first point that we wish to make very clear is that mathematical formalization cannot displace traditional exegetical methods. We are then not under the illusion, whose influence on scores of formal philosophers has been so aptly denounced in Rota (1991), that philosophical problems can be resolved by merely writing down some axioms in a formal language, supposedly characterizing the basic informal philosophical notions at hand, and then proceed to uncritically take as the real content of the philosophical theory the consequences that can be computed from these axioms. A naive approach of this sort does not work because no real effort is made to provide a *systematic* interpretation of the basic entities and of the consequences of the mathematical formalism with respect to the content of the informal philosophical concepts to be characterized. This means that in the naive approach the relationship between the mathematical framework and the informal philosophical theory is tenuous at best; it often reduces to an argument to the effect that the basic notions of the formal theory are "inspired", in some superficial, vague or ill-defined way, by the philosophical theory, after which the author sets aside worries of interpretation and trails off in the distance to prove his results. Such an approach is even more disastrous when one aims at formalizing Kant's theory of space and time, because in this case the network of interrelated concepts is so complex that one is bound to be led astray if one does not heed the need for a systematic interpretation of the formalism, both in relation to Kant's texts and scholarly interpretations of Kant's ideas.

We thus outright reject the idea that a naive formalization of Kant's philosophy, or, for that matter, any sort of mathematical formalization, could ever hope to replace traditional exegetical methods. We do believe, however, that mathematical formalization has an important role to play as an exegetical tool in its own right, which must remain closely tied to traditional exegetical methods but which has also been, unfortunately, largely ignored.

The exegetical role of mathematics begins by noting that one of the central aims of any system of philosophy is to obtain an understanding of the meaning of the concepts that constitute its fundamental focus, by relating these concepts logically as members of an organic whole, in such a way that their meaning is thereby clarified. Thus, it is appropriate to say that any philosopher who develops a philosophical system is always striving to achieve clarity with respect to those very concepts that are at the foundation of his system, which are such that either their meaning is unclear at the outset or that their received explication is judged defective. The clarification is then pursued by building a system in which these core concepts are analyzed and coherently related to each other, so that their meaning emerges from their role within the system as a whole.

It was already very clear to Kant himself that this is the fundamental way in which systematic philosophy proceeds. He discusses a similar point explicitly at A731/B759 in the CPR,[1] where we find:

> That in philosophy one must not imitate mathematics in putting the definitions first, unless perhaps as a mere experiment. For since they are analyses of given concepts, these concepts, though perhaps only still confused, come first, and the incomplete exposition precedes the complete one, so that we can often infer much from some marks that we have drawn from an as yet uncompleted analysis before we have arrived at a complete exposition, i.e., at a definition; in a word, it follows that in philosophy the definition, as distinctness is made precise, must conclude rather than begin the work.[2] (CPR A731/B759)

The point that Kant makes here regarding the nature of systematic philosophy is echoed by Rota throughout (Rota, 1991), where he claims that the main concepts of philosophy, such as the concepts of "reality", "mind" and so forth,

[1] We adopt the following system of abbreviations for Kant's works: CPR stands for The Critique of Pure Reason, CPJ stands for The Critique of the Power of Judgment, MFNS stands for the Metaphysical Foundations of Natural Science, OP stands for the Opus Postumum, and R stands for the reflections. Passages from the CPR are cited with their usual numbering from the A and the B edition, while all other passages are cited according to their volume and page number from the *Akademieausgabe* edition of the collected works of Immanuel Kant (Kant, Holger, Gerresheim, Heidemann, & Martin, 1908). The English translation of Kant's passages follows the Cambridge edition of the works of Immanuel Kant.

[2] Kant continues thus in a footnote at the same place: "[...] If one would not know what to do with a concept until one had defined it, then all philosophizing would be in a bad way. But since, however far the elements (of the analysis) reach, a good and secure use can always be made of them, even imperfect definitions, i.e., propositions that are not really definitions but are true and thus approximations of them, can be used with great advantage."

do have meaning despite being among the least precise concepts; indeed, forcing these concepts to be precise leads to misunderstanding them (Rota, 1991, p. 170). Later, in Section 7, he criticizes the definitions of "mathematizing philosophers", who wish to import the axiomatic method in philosophy uncritically, as follows:

> Whereas mathematics *starts* with a definition, philosophy *ends* with a definition. A clear statement of what it is we are talking about is not only missing in philosophy, such a statement would be the end of all philosophy. If we could define our terms, then we would dispense with philosophical argument. Actually, the "define your terms" imperative is deeply flawed in more than one way. While reading a formal mathematical argument, we are given to believe that the "undefined terms", or the "basic definitions" have been whimsically chosen out of a variety of possibilities [...] in actual fact, no mathematical definition is arbitrary. The theorems of mathematics motivate the definitions as much as the definitions motivate the theorems [...] there is, thus, a hidden circularity in formal mathematical exposition. The theorems are proved starting with definitions, but the definitions themselves are motivated by the theorems that we have previously decided ought to be right. (Rota, 1991, p. 172)

The insight offered by the passages above is that, in systematic philosophy, clear definitions are always an achievement of the process of analysis or exposition of the concepts, and never its beginning.[3] If this is true, however, it would seem to preclude the applicability of mathematical formalization to philosophical analysis or to the exegesis of any system of philosophy: after all, the development of a formalization does require mathematical precision, which, as we have seen, is misplaced when it is forced upon philosophy like a straitjacket.

It would be much too hasty, however, to conclude that there can be no place for mathematical formalization in (the exegesis of) systematic philosophy. Indeed, the considerations above show that it is only the naive approach that is dangerous, due to the false supposition that precise insights can be achieved while disregarding the exposition of the conceptual structure of an organic philosophical system. We believe that Rota's point was exactly that *this* specific use of mathematics by "mathematizing philosophers", with its "dominant" role of axiomatics with respect to philosophical argument, bestows illusory certainty and is not suited to

[3]In this respect, it is interesting to note in passing that this view is consistent with the idea that the analysis of the notions of "space" and "time" in the CPR first begins with a coarser characterization in the TA, and then unfolds into a better and more precise one in the TD and in the footnote at B161n. It then makes good sense to think with Longuenesse that Kant's conception of space and time is perfected in the TD, i.e., that to a certain extent he "rereads" the TA in the TD, where "rereading" means that Kant provides a deeper and more precise exposition of these notions and of how they relate to the other basic concepts of his system. Of course, one can dispute how much clearer Kant's characterization of "space" and "time" is at B161n, but that is a moot point.

the aims of philosophy. A different role for mathematics in systematic philosophy, however, is not excluded, namely, a role in which the relation between the formal tools and the informal analysis of the basic concepts at hand is tighter and more organic. Indeed, Rota's appreciation for Church's philosophical use of type theory in Kac, Rota, and Schwartz (1986, p. 170) and his remarks regarding the relation between mathematics and phenomenology suggest that he himself considered such a role for mathematics not only possible but advantageous.

How does such a principled use of mathematics in systematic philosophy look like? The best way to answer this question is to provide a rough description of a principled *exegetical* use of mathematical formalization, as we believe this to be the first and most fundamental application of mathematics to philosophy. Indeed, a fruitful use of mathematics as an exegetical tool ought to be a good starting point to be able to discriminate the good use of mathematics in philosophy from the bad, and is, in any case, all we need to justify the methodology of this work.

The most important point in this respect is that a principled use of mathematics in the exegesis of a philosophical system, such as Kant's critical philosophy, must face a circularity that is similar, even though not identical, to that mentioned by Rota in the passage above.

The first part of the circularity consists in the fact that every act of formalization of an informal philosophical text is, first and foremost, an act of interpretation, and is then dependent on a preliminary exegesis of the concepts involved. Thus, the choice of mathematical tools to formalize, say, Kant's theory of time is not at all neutral. Different choices are possible, depending on which concepts are taken as basic, on the interpretation of their role within the system as a whole, and on the background reading of their philosophical history. The choice of the mathematical formalism, then, already embodies an interpretational stance; it implies that certain aspects of the system, which are considered as fundamental, will be captured in the formalization, while others, which are considered as secondary, will be abstracted away.

Indeed, this "abstracting away" is crucial. A formalization of an informal philosophical system relates to the system being formalized in a way that is analogous, though certainly not identical, to the way in which the mathematical apparatus of a scientific theory relates to the phenomenon to be explained. In this latter case, only those aspects of the phenomenon that are considered crucial for understanding it and for developing predictions have a counterpart in the mathematical theory; other aspects deemed irrelevant are excluded. Analogously, a formalization of a system of philosophy privileges certain aspects of the system over others and is then dependent on a specific interpretation of the system itself. Of course, for a scientific theory the identification of the aspects of the system that can be safely abstracted away is a much less controversial matter than for a philosophical system, but the two cases are in principle not as different as they might at first seem, since all we have to go by to develop a formal interpretation of a philosophical system is the textual data as our "phenomenon".

The second part of the circularity is that the act of formalization, even though it is dependent on a preliminary interpretative stance, opens up new exegetical possibilities that might not be immediately apparent without the use of formal tools. This is because the goal of developing "formal correlates" to the central concepts of a system of philosophy influences the interpretation of their content, so that the aim of formalization drives the interpretation; whenever the formalization of certain concepts throws a specific light on the content of other concepts it prompts a revision of the interpretative stance towards extending and consolidating the formal theory.

The circularity intrinsic in formalizing a philosophical system is not to be regarded as a flaw but as an advantage since it denotes, so to speak, a virtuous "hermeneutic circle" between the philosophical theory and its purported formal counterpart. While the formalization is never final and its accuracy can always be criticized purely in light of the exegesis of the philosophical theory, it is also the case that the difference between competing interpretations is made far more precise by relating them to the formal theory, as criticism of the latter on interpretative grounds must mean the rejection of some of its assumptions or constructions. Thus, a principled exegetical use of mathematics is one that provides formal correlates to the philosophical concepts at hand while always questioning their accuracy in relation to the philosophical system, so that no formalization is ever final, but rather only a stage in a process of approximation to the informal theory. This sort of use of mathematics in philosophy is not only admissible but also advantageous, as it increases the precision of the exegesis of a system of philosophy without being prey to the dogmatism decried by Rota.

A further advantage offered by the formalization of traditional philosophical systems is that it makes their relevance for contemporary debates in science and philosophy more evident, since, as Husserl aptly observes in his third logical investigation, mathematization is the natural outcome of the development of a system (see chapter 1). For instance, our formal analysis of Kant's theory of time shows that there exists a surprising relation between Kant's theory of the temporal continuum and the basic philosophy underlying the causal set theory approach to quantum gravity (Reid (1999); see chapter 7), and the formalization of Kant's theory of the external representation of time in self-affection through motion is in remarkable agreement with contemporary results in cognitive science (see section 3.9). Hence, the usefulness of mathematical formalization in philosophy is not only limited to the resolution of exegetical disputes; it also helps bring new life to philosophical systems themselves by bringing them to bear on contemporary problems.

We hope that these brief remarks suffice to convince the skeptical reader that the enterprise of this thesis is at least worthy of consideration. Of course, as is always the case, the best justification for a methodology is not a matter of principle but rather a matter of fruitfulness, and the latter can only be proven in actual practice, as we shall soon attempt.

Chapter 3

Form of intuition and formal intuition

3.1 Introduction

The aim of the present chapter is that of clarifying the distinction between the form of intuition and the formal intuition[1] in Kant's critical philosophy, since our understanding of Kant's continuum and of its mathematical form is largely based upon this distinction. Kant introduces the distinction in a footnote at B161n in the CPR, at the height of the Transcendental Deduction[2] of the categories; the obscurity of the footnote has given rise to multiple competing interpretations, with some commentators (Falkenstein, 2004, p. 91) even claiming that the footnote is indeed just a flat contradiction. A reader not versed in the meanders of Kantian exegesis might be rightfully perplexed at the amount of ink that has been spent on trying to interpret such a seemingly cursory footnote. The reason for its importance is that it seems to paint space and time in a very different light than that used in the Transcendental Aesthetic;[3] it is here described as acquired through an active process of synthesis,[4] rather than being given *a priori* before any synthetic combination by the understanding. The fundamental role of space and time in the architecture of the CPR, as the forms which every intuition of objects must conform to, is then a sufficient justification to devote the utmost effort to understanding the footnote at B161n.

Nevertheless, there are further reasons to develop a clear understanding of the distinction that go beyond pure exegesis. Until his very last days, Kant struggled to understand not only what the ultimate properties of space and time are, but how the acquisition of space and time as representations is related to an attentive

[1]In the sequel, we speak simply of "the distinction" to refer to the distinction between the form of intuition and the formal intuition.

[2]Henceforth TD.

[3]Henceforth TA

[4]Indeed, Béatrice Longuenesse has gone so far as arguing (Longuenesse, 1998, p. 299) that the transcendental deduction of the categories is essentially a re-reading of the TA.

self-conscious subject that is situated in the world. Thus, he reflected deeply on the relation between space and time on the one hand, and motion, attention, and self-consciousness on the other. The distinction between the form of intuition and the formal intuition is at the core of Kant's reflections on these matters, and we believe that a thorough understanding of it provides insights that are relevant not only for contemporary metaphysical debates but also for the cognitive science of space and time; indeed, we shall return to the contemporary relevance of Kant's theory of time for cognitive science in section 3.9. In any case, the reader not interested in Kant's scholarship in itself should not be put off by the Kantian jargon in the following sections. While the use of Kant's sometimes baroque terminology is unavoidable the insights that we shall achieve will be well worth the hassle.

Let us then direct our attention to the passage at B161n, so that we might see why it has been a source of puzzlement for generations of scholars:

> Space, represented as an object (as is really required in geometry), contains more than the mere form of intuition, namely the comprehension of the manifold given in accordance with the form of sensibility in an intuitive representation, so that the form of intuition merely gives the manifold, but the formal intuition gives unity to the representation. In the Aesthetic I ascribed this unity merely to sensibility, only in order to note that it precedes all concepts, though to be sure it presupposes a synthesis, which does not belong to the senses but through which all concepts of space and time first become possible. For since through it (as the understanding determines the sensibility) space or time are first given as intuitions, the unity of this a priori intuition belongs to space and time, and not to the concepts of the understanding (CPR, B161n).

Before we examine the issues raised by the passage, note that while the distinction between the "form of intuition" and the "formal intuition" is in the first instance only applied to space, it is later applied to time as well; thus, we shall generally assume in what follows that the footnote applies to time as much as it does to space. This assumption seems uncontroversial: although the properties that Kant ascribes to space are certainly different from those he ascribes to time, the synthesis mentioned in the passage above, which is described as the source of the distinction, is applied to space and time equally, along with this mysterious "unity" that it produces and the "concepts of space and time" that it makes possible. Thus, in the sequel we focus mostly on time rather than on space for the simple reason that the formal elucidation of the footnote in the case of time is considerably simpler. The interpretation proposed in this chapter applies in the same way to space, however; we shall return to this point in chapter 4, once our formal interpretation is worked out.

Now, the above passage is puzzling for the following reasons:

(1) At the beginning of the passage it is said that space, as required in geometry, is "represented as an object"; it is also implied that space "represented as an object" is space as "formal intuition", which is opposed to space as "form of intuition". As noted in Onof and Schulting (2015, p. 4), conceiving of space as an "object" would seem to run contrary to Kant's objections to the Newtonian substantivalist view of space. Moreover, Kant refers to space as an "empty intuition without an object" (B348/A292), i.e., an *ens imaginarium*, and in (R4673 17:638-39) it is said that "space is not an object of intuitions (an object or its determination), but the intuition itself, which precedes all objects and [*crossed out* through which] in which if the latter are posited, the appearance of them is possible". While these claims are compatible with the claim at the end of the footnote that "space or time are first given *as intuitions*", they hardly seem to square with space as formal intuition being space "represented as an object". Of course, in light of what we said above, these remarks apply to time as well.

(2) Space and time as formal intuitions are described as bestowing a "unity" on the manifold that is given in agreement with the forms of intuition. This "unity" is then equated with a certain unity that is ascribed to sensibility in the TA, but Kant goes on to note that he ascribed it to sensibility only to make it explicit that it "precedes all concepts". He now reveals that it actually presupposes a synthesis, which "does not belong to the senses" but is such that "the understanding determines the sensibility"; by means of it "space or time are first given as intuitions". These remarks are puzzling for two reasons:

   (a) Kant seems to switch freely between two notions of unity. One is the unity of the manifold when it is comprehended in an intuitive representation: this unity pertains to the manifold itself, and it is brought about by its relation to the formal intuition. The second notion of unity is that of the unity of the formal intuition itself, i.e., of space and time represented as objects – space and time as intuitions themselves. This must be the unity that Kant means when he refers back to the TA since there the emphasis is exactly on investigating various properties of space and time, among which is their unity – or rather, as we shall later see, their *unicity*. Moreover, in the last sentence of the footnote Kant speaks of the unity of the *a priori* intuitions of space and time themselves. What is, then, the relation between these two notions of unity?

   (b) If the synthesis that produces space or time as formal intuitions precedes all concepts, and in particular, according to the last sentence of the passage, it precedes the concepts of the understanding (the categories), then there seems to be a conflict with the fact that, for Kant, any synthetic activity is carried out by the understanding (B130), which is the faculty of concepts or of rules. Indeed, Kant seems to explicitly say that the synthesis pertains to the understanding, since through it "the understanding determines the sensibility". Thus, the question arises: how can there be a synthesis of the understanding that does not involve the recognition of a manifold under a concept - not even a pure concept of the understanding, a category?

(3) Kant says that the synthesis which gives unity to the manifold is such that it "does not belong to the senses", but through it "all concepts of space and time first become possible". What are these "concepts of space and time"? Certainly, they cannot be space and time as forms of intuition, since this would run contrary to the main tenet of the TA that space and time are not concepts. It is also difficult to hold that these concepts are space and time as formal intuitions since the footnote seems to exclude this possibility: space and time are first given *as intuitions* through the synthesis, that is, they are given as formal intuitions, not as concepts. Hence, these "concepts of space and time" must be something else: what are they, and in what sense does the synthesis make them possible?

The reader must have realized by now why this footnote is so important and why, alas, it is so ill-understood: it extends into the deepest waters of Kant's theoretical philosophy, since the relation between space and time as the *a priori* forms of sensibility and the spontaneity of the understanding is at the core of Kant's attempt in the TD of providing a justification to the categories. The aim of the present chapter is then to shed light on Kant's notion of formal intuition, while the aim of the following chapters is that of supporting the interpretative efforts with a formalization, in the language of contemporary mathematical logic, of the main notions occurring at B161n.

The structure of the chapter is as follows. In section 3.2 we provide a brief outline of the interpretation of the distinction that we develop in this chapter. In section 3.3 we examine Kant's definition of the form of intuition. In section 3.4 we consider Kant's peculiar view of the temporal continuum in relation to the form of intuition of time, and in section 3.5 we consider the synthesis of apprehension, through which an empirical manifold is synthesized. Section 3.6 and section 3.7 then examine Kant's notion of the figurative synthesis, the *a priori* ground to the synthesis of apprehension. In section 3.8 we finally provide our reading of the distinction between the form of intuition and the formal intuition, with particular emphasis on the temporal case. In section 3.9 we point out various suggestive connections between Kant's treatment of space and time and contemporary research in cognitive science. We conclude the chapter in section 3.10 by summarizing the main points of our interpretation. We shall, however, return to a more careful analysis of how our interpretation bears on the interpretative problems presented in this section in the next chapter at section 4.7, when we have enough of the formal theory in place to discuss them in a more precise fashion.

3.2 A brief outline of the interpretation

It is expedient, before we delve into the detailed analysis of Kant's notion of form and formal intuition, to outline briefly the main features of the interpretation that we develop, so that the reader will be better equipped to follow the discussion in the upcoming sections.

The main point of our interpretation is that the formal intuition is produced by what Kant terms the *figurative synthesis* or *synthesis speciosa* , a process through which the subject affects its inner and outer sense and which, we claim, is governed by the categories and the unity of apperception. We also hold that while the form of intuition in the TA is nothing else than the formal intuition in the TD, there still exists a genuine distinction between the form of intuition and the formal intuition that resists all attempts to collapse the two. The distinction is, however, not clear-cut, but is a matter of degree: one can identify degrees of increasing "formality", so to say, of the form of outer and inner sense, which are intermediate between a purely passive form of intuition and the full-fledged notion of "space and time as objects", or the formal intuition that supports geometrical constructions and the analysis of motion.

This hierarchy of degrees of formality depends on the gradually increasing involvement of the *synthesis of the unity of apperception*, by means of which the understanding *determines* the sensibility (B161 and B154-155) by imposing additional *a priori* structure on the passive temporal manifold that the sensibility affords. Thus, what counts as the *temporal form* of a *possible experience* is gradually delimited.

Mathematically, this amounts to considering a hierarchy of first and second order axioms on structured sets of *events*, whose Kantian interpretation is as events of self-affection through *a priori* motion. The stronger the constraints on these structured sets, the more the manifold is brought to the unity of consciousness. Most importantly, the *categories* - in particular the categories of relation - play a role in the constitution of the formal intuition, as general rules of *combination* by means of which any manifold can be brought to the unity of apperception.

Thus, the "graded conceptualist" view supported by our formalization involves a "pre-discursive" role of the understanding and of the categories (M. Friedman, 2012; Longuenesse, 1998, p. 241ff), and is at odds with so-called "nonconceptualist" views of the distinction between the form and the formal intuition (Onof & Schulting, 2015; Melnick, 1973; Allison, 2000, 2004).

The final outcome of our interpretation is that the constraints imposed by the unity of apperception via the categories on the temporal form of a possible experience imply the existence of a special structure of events, time "as an object", which satisfies mathematical correlates of the properties that Kant ascribes to the form of time in the TA and the formal intuition of time in the TD. In particular, we shall see that:

1. Time is *unique* and every possible temporal form of an experience is a part of it, but it is not constituted by these parts; rather, every part is only possible through the whole

2. Time is *actually infinite* in the specific Kantian sense of representing the "unboundedness of the progress of intuition"

3. Time supports all the *concepts of time* that are required in geometry and physics

4. It admits of an *outward representation* in the form of a line, which is the *description of a space* through *a priori* motion and contains infinitesimals, in agreement with Kant's discussion in the MFNS.

Most of this chapter is devoted to clarifying the concepts appearing in italics above, while the next two chapters are devoted to make them mathematically precise, so as to obtain a coherent formal theory in which they all fit. The way in which we shall derive the formalization, by careful exegetical analysis of the textual evidence in light of extant scholarship will be, along with the solutions it provides to the problems in the introduction and its elegance, its main claim to accuracy.

3.3 The form of intuition in the TA

A good place to start our endeavors are the "General remarks on the Transcendental Aesthetic" in the CPR, where we find a definition of the form of intuition:

(1) [...] everything in our cognition that belongs to intuition [...] contains nothing but mere relations [Verhältnis], of places in one intuition (extension), alteration of places (motion), and laws in accordance with which this alteration is determined (moving forces) [...] it is not merely that the representations of **outer sense** make up the proper material with which we occupy our mind, but also the time in which we place these representations, which itself precedes the consciousness of them in experience and grounds the way in which we place them in mind as a *formal condition*, already contains relations of succession, of simultaneity, and of that which is simultaneous with succession (of that which persists). Now that which, as *representation*, can precede any act of thinking something is intuition and, if it contains nothing but relations, it is the form of intuition, which, since it does not represent anything except insofar as something is

> posited in the mind, can be nothing other than *the way in which the mind is affected by its own activity* [. . .] (CPR B68, our emphasis).

The form of the intuition of time consists only in relations of succession, of simultaneity and of persistence, according to which all our representations must be ordered,[5] but which itself precedes the consciousness of these representations in experience *a priori*. Similarly, in the TA Kant characterizes the form of intuition as "that which so determines the manifold of appearance that it allows of being ordered" (A20/B34). Now, three substantial problems arise for the logician in understanding this passage.

First, what are these relations of succession? Various possible temporal relations spring immediately to mind: temporal precedence, temporal overlap, simultaneity, and so forth. Kant's definition does not seem to specify whether all these relations, or only some of them, are salient.

Second, what exactly are the *relata* in question? Are they point-like instants of some sort, are they events or durations having some breadth or extension? Do they represent empirical events, such as the collision of two bodies in space, or are they something more "fundamental", i.e. *a priori*?

Third, how is one to construe the claim that the form of intuition is nothing other than "the way in which the mind is affected by its own activity"? This seems *prima facie* related to what Kant speaks of at B161n, but it is indeed rather obscure.

In order to obtain precise answers to these questions, and illuminate the meaning of the passage above, we must dig deeper, relying on the whole system of Kant's philosophy to illuminate the meaning of its parts. We focus first on the second and third questions, and return to the first question in the next chapter at section 4.2.

3.4 The Kantian continuum

We can begin to understand what sort of entities are related by the form of the intuition of time by considering the problem from the perspective of the philosophy of mathematics, since time has always been the foremost example of a continuum.

Historically,[6] one can identify two radically opposed main traditions regarding the problem of continuum.

[5]Kant speaks here only of ordering the representations of outer sense in time; however at A34 he remarks that *all* representations are ordered in time, not only those of outer sense, but also those of inner sense.

[6]To be sure, we cannot provide here an accurate history of the thought on the concept of the continuum. What we aim at is to simply provide a rough outline that serves our general purpose. The reader interested in the general problem of the continuum should consult Feferman (2009), van Dalen (2009).

On the one hand there is the Cantorean conception of the continuum, in which one conceives of the latter as built "bottom-up" from an actually infinite set of dimensionless points or monads, which are the only fundamental constituents or "parts" of it. The "extendedness" of the Cantorean continuum is then an emergent property: while all of its infinitely many constituent parts have the property of being dimensionless, the whole has the property of being extended. The commonly accepted theory of the continuum in contemporary mathematics is Cantorean in this sense. This theory can be presented either geometrically, as a set of points on which various axioms of order are imposed along the lines of Pieri (Pieri, 1899), Pasch (Pasch, 1882) and Hilbert (Hilbert, 1971), or algebraically, as the ordered field of points that can be obtained as the completion of the ordered field of the rationals, along the lines of Dedekind (Dedekind, 1963) and Cauchy. The essence of both constructions, however, is that of a Cantorean theory of the continuum that is the direct descendant of Grosseteste's medieval theory of the *compositio ex punctis*.[7] In particular, both constructions rely for their rigorous formulation on the full power of contemporary first order logic and of the theory of sets, two technologies that were only invented in the second half of the XIX century and that first allowed for the rigorous manipulation of actual infinities. The codification of the axiomatic method in algebra, geometry and the foundations of analysis, then, was essential to put the Cantorean theory on rigorous foundations and make it the standard view of the continuum in contemporary mathematics. Still, the notion of "rigor" employed here is radically modern, since it derives from Hilbert's and Pieri's requirement that "spatio-temporal intuition" be eliminated from the concept of (geometrical) proof.[8]

On the other hand there is what one might term the "continuist" (van Dalen, 2009) or "phenomenological" (Feferman, 2009) conception of the continuum, which rivaled the Cantorean conception before the advent of modern axiomatics tipped the scale in favor of the latter. Proponents of a continuist theory of the contin-

[7] See Maier (1966), and the discussion in Grunbaum (1977).

[8] Indeed, Pieri, *contra* Veronese and Enriques, wrote:

> If you maintain that the postulates of geometry are nothing but rigorous formulations of the intuitive concept of physical space (which merely impress stability and a seal of rationality on the facts of spatial intuition), you ascribe, in my opinion, too much importance to an objective representation, which you treat as a *conditio sine qua non* of the very existence of geometry, whereas the latter can, in fact, very well subsist without it. Today, geometry can exist independently of any particular interpretation of its primitive concepts, just like arithmetic. (Torretti, 2012, p. 224)

It then follows that one cannot appeal to spatial intuition in geometrical proofs, so that, e.g., continuity must be enforced by a second-order axiom and cannot be inferred from the diagram itself. In the last years, however, there have been various efforts towards making geometrical intuition mathematically respectable; see in particular Avigad, Dean, and Mumma (2009), Mumma (2010, 2012).

uum include, among others, Aristotle, C. S. Peirce, Russell, Whitehead, Brouwer, Weyl, and, most importantly for this work, Kant. While there are important differences between these authors' view on the continuum, they all share certain fundamental concerns which allow one to categorize them all broadly as continuists.

In particular, the continuist conception rejects the emergence of the extended continuum from an actually infinite collection of dimensionless points in favor of a "top-down" approach according to which the continuum is given as an extended whole before its parts, which are then introduced by a process of mathematical division. Hence, points are not constituents of the continuum, since nothing in the continuum is simple; parts of the continuum are themselves subcontinua and can be analyzed *ad infinitum*, while points supervene on these parts. For instance, both Aristotle and Kant conceive of points merely as boundaries delimiting the subcontinua, while Whitehead conceives of them as "nested families" of parts. Aristotle in particular formulates this attitude towards points most clearly in the case of points (instants) of time when he claims that "time is not composed of indivisible nows any more than any other magnitude is composed of indivisibles" (Physics, 239b 5) and that:

(2) The now is a link [συνέχεια] of time [...] for it links together past and future time, and in its general character of "limit" it is at once the beginning of time to come and the end of time past. But in the case of the "now" this is not so obvious as in that of the stationary point; for [...] it potentially divides time. And in this potentiality one "now" differs from another, but in its actual holding of time continuously together it always remains the same, as in the parallel case of *mathematical lines traced by moving points*, in which case the point too, if arrested as a divider, is not conceived as retaining its identity with the tracing point or another arrested point; for if we are dividing the line, the point differs at every division, but if we regard the line as a single undivided one, the point that traces it is the same all along.[9] (Physics, 222a 10-20, our emphasis)

Some continuists also hold that boundaries in the continuum are not really point-like but are extended, and that they do not have determinate locations or positions; they are "approximations" of points, or "thick boundaries", which are joined by an inhexaustible "in-between". In the words of Weyl (Weyl, 1994, p. 92):

(3) 1. An individual point in [a continuum] is non-independent, i.e., is pure nothingness when taken by itself, and exists only as a 'point of transition' (which, of course, can in no way be understood mathematically);
2. It is due to the essence of time (and not to contingent imperfections in our medium) that a fixed temporal point cannot be exhibited in any

[9]See also Aquinas' commentary on Aristotle's Physics, Bk. 6.861.

> way, that always only an approximate, never an exact determination is possible.

Similar themes are echoed by Brouwer:

(4) [The ur-intuition is] the substratum of all perception of change, which is divested of all quality, a unity of continuous and discrete, a possibility of the thinking together of several units, connected by a "between", which never exhausts itself by the interpolation of new units.[10]

Note that Aristotle's conception of the instant of time departs from the above descriptions, since Aristotle, and after him Augustine, held that instants of time are dimensionless and indivisible, even though they do not make up the temporal continuum.

The Cantorean and continuist conceptions also differ with respect to the possibility of first or last instants of time, that is, instants with empty past or empty future. On the Cantorean conception, since instants are primitive, the existence of first or last instants of time is not an issue. On the continuist conception, however, such instants are suspect: since boundaries supervene on parts of time, assuming the existence of a boundary without any time in the past or future seems, at the very least, unwarranted. Aristotle, in particular, excludes this possibility:

(5) Now since [...] the moment [is] a kind of middle-point, uniting as it does in itself [...] a beginning of future time and an end of past time, it follows that [...] there must always be time on both sides of it (Physics, 251b 13-17)

Finally, a common theme among continuists is the emphasis on *potential*, rather than actual, notions of infinity, which capture the "inhexaustibility" of the continuum. Since the continuum cannot be reduced to an actual infinity of simple parts, from which it can be built bottom-up, it must be the case that every division of the continuum into parts, which introduces new boundaries, can be further refined. This is in particular Aristotle's conception of infinity in the Physics, and is implicit in Brouwer's quote above.

This brief outline is all that we need for our present purposes, since it is now clear that Kant's theory of time is an instance of the continuist theory of the continuum. Indeed, Kant's views on the matter are very close to Aristotle's:

(6) The property of magnitudes on account of which no part of them is the smallest (no part is simple) is called their continuity. Space and time are quanta continua because no part of them can be given except as enclosed between boundaries (points and instants), thus only in such a way

[10]For the source of this quote and an illuminating discussion of Brouwer's continuum see van Dalen (2009, p. 3ff).

> that this part is again a space or a time. Space therefore consists only of spaces, time of times. Points and instants are only boundaries, i.e., mere places of their limitation; but places always presuppose those intuitions that limit or determine them, and from mere places, as components that could be given prior to space or time, neither space nor time can be composed. Magnitudes of this sort can also be called flowing, since the synthesis (*of the productive imagination*) in their generation is a progress in time, the continuity of which is customarily designated by the expression "flowing" ("elapsing"). All appearances whatsoever are accordingly continuous magnitudes [. . .] (A169-70/B211-2, our emphasis)

The passage above, which has clear Aristotelian undertones, implies that the entities related by time as the form of intuition cannot be point-like instants, since these only supervene as boundaries limiting parts of time, and are not fundamental components from which time itself might be built "bottom up". Indeed, it is clear from the TA that even the extended parts of time do not constitute time as a whole, but are only "specified" or "isolated" within it, by individuating their boundaries.

Kant differs from Aristotle and is closer to Weyl, however, in considering the possibility that instants are extended rather than point-like:

(7) are two different states separated by a time that is not filled through a continuous series of alterations[?] The instant in time can be filled, but in such a way that no time-series is indicated. All parts of time are in turn times. The instant. Continuity. (R4756, 17:700)

Note the occurrence of the modal *can*, which is important since it is akin to the modal description of the "I think" as that which "must *be able* to accompany all my representations", and is intimately connected to Kant's notion of infinity, which is closer to the continuist's potential infinity than to the Cantorean actual infinity. To be sure, on this point certain passages from the CPR (A25, B48) seem to cast the shadow of a doubt, as Kant says that space and time are "infinitely given" magnitudes. It would be a mistake, however, to construe these claims as supporting a Cantorean notion of actual infinity, exactly because of the modal characterization that Kant gives of the infinity of space and time. We return on these points in section 3.8.5, since they can be best addressed once we have examined the figurative synthesis.

We can then reliably infer that the *relata* of time as the form of intuition are parts of time, having some breadth or extension. It is these parts of time that are related according to relations of succession, simultaneity and persistence. But what are these parts of time exactly, i.e., what do they represent? In order to answer this question we must examine Kant's account of how our temporal experience is constructed.

3.5 The synthesis of apprehension

We have established that the form of the intuition of time in the TA consists in relations of succession, simultaneity and persistence whose *relata* are extended parts of time. We can obtain more insight into what these are by considering two closely related notions: the synthesis of apprehension of appearances and its *a priori* counterpart, the figurative synthesis. These two notions are at the core of our interpretation and on their basis the formalization will be developed.

The synthesis of apprehension appears in the B edition at §26, where the TD reaches its climax, and is glossed as an empirical synthesis that combines the manifold provided by sensibility into the perception of an appearance.[11] The elucidation of this synthesis at B162 makes it clear that it contains a reproductive aspect, according to which, in the terminology of the three-fold synthesis of the A edition, the manifold of intuition is not only "run through" but also "held together" in the present, so that objective, and not merely subjective, temporal relations can be perceived. The apprehension of two states of water, liquid and solid, "as ones standing in a relation of time to each other" does not only imply that the two states are "run through", but also that the former state is reproduced when the second state is intuited, so as to bind them according to a relation of objective temporal succession. The reproductive aspect is needed because we cannot apprehend the succession as a unit, since "apprehension, merely by means of sensation, fills only an instant" (A167/B209) and "as contained in one moment no representation can ever be anything other than absolute unity" (A99). Without reproduction no perception of the appearance, no "image" of it (A121) could ever arise:

(8) Now it is obvious that if I draw a line in thought, or think of the time from one noon to the next, or even want to represent a certain number to myself, I must necessarily first grasp one of these manifold representations after another in my thoughts. But if I were always to lose the preceding representations (the first parts of the line, the preceding parts of time, or the successively represented units) from my thoughts and not reproduce them when I proceed to the following ones, then no whole representation [...] *not even the purest and most fundamental representations of space and time, could ever arise.* (A102, our emphasis)

The synthesis of apprehension in the B edition then unifies into a single synthetic process the two empirical syntheses of apprehension and of reproduction in

[11] Thus Kant:

> First of all I remark that by the **synthesis of apprehension** I understand the composition of the manifold in an empirical intuition, through which perception, i.e., empirical consciousness of it (as appearance), becomes possible. (B160)

the imagination,[12] which in the A edition are distinguished as logically separate aspects of empirical synthesis.[13]

Moreover, the synthesis of apprehension is objective exactly because of the "holding together" of the manifold, which brings the latter to the intuition of an object and of the objective temporal relations among its parts. Thus, it is not to be confused with the merely subjective apprehension that appears in the analogies of experience, which does not contain this essential moment of *comprehension*. The subjective apprehension of the manifold of a house (B235/A190), for instance, is always successive, as my attentional focus shifts from the bolted main door to an ornate window at the second floor, but this succession which I apprehend is not objective, that is, it does not hold for every possible experience, but is merely - paraphrasing Kant - a "play of my own imagination", as the movement of my attentional focus determines the order in which the features are taken up into consciousness. The moment of comprehension, which contains the moment of reproduction and consists in apprehending the manifold in different orders and in the production of a unitary representation, is then necessary for the objectivity of apprehension. Most importantly, it is the same moment that is mentioned in the footnote at B161n and that is responsible for the cognition of simultaneity (CPJ 5:259). As we shall see in section 3.8.4, the *a priori* counterpart of empirical comprehension will be the key to understand how, according to Kant, the representation of space and time are generated in the first place.

3.5.1 The categories and the synthesis of apprehension

The synthesis of apprehension, considered on its own, produces an objective temporal order among appearances. Now, what mathematical properties does this temporal order *necessarily* satisfy? Unfortunately, since the synthesis of apprehension is merely empirical, not very many: even the most basic properties of time in the TA, such as linearity or one-dimensionality, fail.

Indeed, consider any temporal relation that might hold among appearances, say the relation $a \, \mathrm{R}_+ b$ meaning something along the lines of "appearance a begins after appearance b or simultaneously with it". This is a "relation of succession"; in Kant's example of the freezing of water at B162, for instance, the appearance of water in its solid state begins after the appearance of water in its liquid state. We remarked above that such a temporal relation between two appearances a, b can be apprehended only if b is reproduced in the fleeting present during which a is apprehended. Thus, transitivity of the relation R_+ implies that the reproduction of appearances that were apprehended in presents arbitrarily distant in the past is

[12] The reader should at this point draw a parallel between these Kantian notions and Husserl's notion of *retention*.

[13] In the A edition Kant remarks that "the synthesis of apprehension is therefore inseparably combined with the synthesis of reproduction" (A102); thus, both "happen at once", so to speak, and are such that no logical priority can be found between them.

always possible. Since the synthesis of apprehension is merely empirical, however, there is nothing that guarantees that this must be the case *for every possible experience*, i.e., according to Kant's strong reading of the notion of necessity. Transitivity can then fail unless there is some ground for it which applies to any possible experience and guarantees the possibility of reproduction of the past - i.e., its being taken up into consciousness - in the present, but this ground can then only be *a priori*.

It is of interest, *en passant*, that the situation just outlined is known in cognitive science[14] as one of "islands in time". A developmental dissociation between the ability to remember past events and the ability to arrange them in a linear order implies that individual memories might start off as "unconnected":

> There is no evidence that events are automatically coded by the times of their occurrence or that memory is temporally organized [...] many older events are difficult to discriminate by their ages [...] but are still presumably episodic memories; and it seems likely that we are poor at remembering the internal order of some episodic memories [...] what appear to be genuine episodic memories are more like the "islands in time" than memories one reaches by mentally travelling through some temporally organized representation (W. J. Friedman, 2007)

For a similar reason as the failure of transitivity, there is no guarantee that the temporal order of representations that is generated by the empirical synthesis of apprehension alone satisfies any sort of linearity, in the form $a\,\mathrm{R}_{+}b \vee b\,\mathrm{R}_{+}a$, which enforces the linearity of time.

We then conclude that the empirical synthesis of apprehension alone does not ensure that appearances must be given to us in the temporal form which Kant describes in the TA. An *a priori* ground is then needed that guarantees the possibility of the objective determination of empirical appearances according to temporal relations satisfying the properties ascribed to time in the TA for every possible experience.

Now, in Kant's framework *a priori* objective determination of spatiotemporal relations on a sensible manifold can be achieved only through the action of the categories, which, by subsuming all possible appearances under general rules, determine them with respect to their temporal relations for every possible experience. Only by means of rules for the "time-determination" of appearances, i.e. for their determination in objective temporal relations, can we really experience an "happening";[15] this is, indeed, the main point of Kant's example of the freezing of water at B162.

[14] We return to the relation between Kant's thoughts on time and contemporary cognitive psychology in section 3.9.

[15] Thus Kant:

> [...] as soon as I perceive or anticipate that there is in this sequence a relation

Prima facie it would then seem reasonable to conclude that the *relata* of time as the form of intuition are empirical appearances, and that the application of the categories to these appearances ensures that their temporal relations can be always determined objectively so that they satisfy the properties of time in the TA. One might then attempt to develop a formalization of Kant's form of time starting with a set of extended parts of time, representing the temporal component of empirical appearances and their alterations, ordered according to some temporal relations yet to be determined. These relations would satisfy various axioms of temporal order, whose justification would appeal to the action of the categories and the unity of apperception. One would then obtain a formal argument showing that the properties of time in the TA (e.g. one-dimensionality, unicity, divisibility, and so forth) hold of the temporal form of any possible experience only if the action of the categories and the unity of apperception are taken into account.

We believe that this argument carries a certain strength, and our analysis will ultimately boil down to providing an argument similar to this. There is a clear drawback to this specific argument, however: we are here considering merely the synthesis of apprehension of an empirically given manifold of intuition, which is of course not given *a priori*. This muddies the waters, since according to Kant time is supposed to be given *a priori*, whether as the form or formal intuition. One might then argue that time is given, along with all its phenomenal properties, *a priori* and prior to any role of the understanding; appearances would then be only situated within this already given time. There would then be no need of the categories or the unity of apperception to justify the necessary properties of time; the role of the categories would be only that of constituting objective temporal relationships among appearances, which are however already arranged in a temporal form that is given *a priori* and already conforms to that described in the TA.

In order to evaluate and, ultimately, reject this objection we must look beyond the synthesis of apprehension and - with Kant - "raise our object to the transcendental" by considering its *a priori* correlate: the figurative synthesis, or transcendental synthesis of the imagination. A careful examination of the role of this synthesis will show that the categories, as the unity of apperception, must play a role in the constitution of time.

> to the preceding state, from which the representation follows in accordance with a rule, I represent something as an occurrence, or as something that happens [...] it is also an indispensable **law of the empirical representation** of the temporal series that the appearances of the past time determine every existence in the following time, and that these, as occurrences, do not take place except insofar as the former determine their existence in time, i.e., establish it in accordance with a rule. (B243/A198 - B244/A199)

3.6 The figurative synthesis in the first step of the TD

3.6.1 The dual nature of the figurative synthesis

A fundamental feature of Kant's theoretical philosophy is that since the concept of an empirical synthesis does not by itself contain any notion of necessary validity every empirical synthesis that plays a role in the constitution of experience as a system of connected perceptions must be grounded on another *a priori* synthesis, which bestows on the empirical synthesis its applicability to every possible experience.[16] In Kant's system, then, syntheses always come in pairs; better, every synthesis is, so to speak, "two-sided", one side being empirical, the other side being *a priori*. The synthesis of apprehension is the single most important example of this dual aspect of Kant's concept of synthesis, since the TD relies on it crucially. In the TD Kant calls the *a priori* correlate to the synthesis of apprehension with a variety of names, such as the "transcendental synthesis of the imagination", the "pure productive synthesis of the imagination", or, in the B deduction, the "figurative synthesis" or *synthesis speciosa*, and describes it mysteriously as a synthesis through which the understanding "affects" or "determines" the sensibility (B153) in agreement with the categories. We shall denote it with the term "figurative synthesis" for the sake of brevity.

The dual nature of the synthesis of apprehension appears clearly in the threefold synthesis of the TD A, where Kant states that it must have an *a priori* ground, without which "we could never have *a priori* neither the representations of space nor of time" (A100). Shortly afterwards he claims that without a further *a priori* ground, an experience might be given such that the synthesis of reproduction cannot put forth any stable association or recollection of appearances:

(9) If cinnabar were now red, now black, now light, now heavy [. . .] then my empirical imagination would never even get the opportunity to think of heavy cinnabar on the occasion of the representation of the color red; or if a certain word were attributed now to this thing, now to that [. . .] without the governance of a certain rule to which the appearances *are already subjected in themselves*, then no empirical synthesis of reproduction could take place. There must therefore be something that itself makes possible this reproduction of the appearances by being the *a priori* ground of a necessary synthetic unity of them. (A101, our emphasis)

[16]This fundamental aspect of the architecture of Kant's philosophy is stated, beyond doubt, at the very beginning of the TD A:

> **Sense**, **imagination** and **apperception**; each of these can be considered empirically, namely in application to given appearances, but they are also elements or foundations *a priori* that make this empirical use itself possible (A115)

This *a priori* ground is identified at A102 with a transcendental synthesis of the imagination, which grounds the possibility of all experience and whose description we quoted in the previous section at Passage (8). Interestingly, Kant remarks again there that without this *a priori* ground, "not even the purest and most fundamental representations of space and time could ever arise".

Since the syntheses of apprehension and of reproduction are so inseparably combined in the three-fold synthesis that they are essentially fused into one in the B edition, it is very reasonable to conclude that their respective *a priori* grounds are, in the B edition, one and the same *a priori* synthesis of the imagination, the figurative synthesis. This conclusion is strengthened by the fact that, as we shall soon see, the figurative synthesis in the B edition is described in the exact same way as in (8): it "produces" the representations of space and time by the "drawing of a line in thought". Hence, the *a priori* ground at Passage (8) cannot be anything other than the figurative synthesis: in the transition from the A edition to the B edition the empirical syntheses of apprehension and of reproduction were combined into the single empirical synthesis of apprehension appearing at §26 of the TD B, and their respective *a priori* correlates, which had been considered as logically separate aspects of synthesis in the A edition, were combined into the single figurative synthesis appearing at §24 of the TD B.

The reader might wish at this point to be provided with a concrete, straightforward definition of the figurative synthesis and of the role it plays in our interpretation. However, this is not possible before we have examined more carefully the role played by the figurative synthesis in the transcendental deduction, and what relations it bears to other fundamental concepts of the CPR. Needless to say, one should always keep in mind Kant's dictum that philosophy does not begin with definitions, but ends with definitions.

3.6.2 The first step of the TD B: the intellectual synthesis and the unity of apperception

We start our investigation of the figurative synthesis from the B deduction, which, as is generally accepted by most commentators, has the following two-step structure. The first step, from §15 to §23, is devoted to showing the necessary applicability of the categories to any manifold of sensible intuition in general, regardless of the properties of *our* particular forms of intuition, space and time. In particular, the argument of this step relies only on a synthesis or combination (*verbindung*, *combinatio*) which is presented as necessary to ensure that any possible intuitive manifold, regardless of its form, stands under the most fundamental principle of Kant's theoretical philosophy: the *synthetic unity of apperception*, or synthetic unity of consciousness, which appears at the beginning of the TD B (B134). The second step, beginning at §24, is instead concerned with justifying the applicability of the categories to any manifold that is ordered according to *our* forms of intuition, that is, spatiotemporally. Each step is centered around a synthesis on which the overall argument relies; the first step concerns the *synthesis intellectualis* or intellectual synthesis, while the second step is where the figurative synthesis, and its empirical correlate, the synthesis of apprehension, appear.

We shall in this section consider the first step of the TD B, since an understanding of the figurative synthesis, and hence of the second step of the TD B, is impossible without a solid understanding of the unity of apperception.

In his lectures and in the passage known as the "Stufenleiter" (A320/B377) Kant distinguishes between "representations" and "perceptions", where the latter are representations that are accompanied by consciousness.[17] One then observes that the representation of the identity of the "instances of consciousness" that accompany different representations, that is, of the fact that these are instances of consciousness of a lasting, self-identical "I", is not at all a trivial matter, since it can be conceived as a *second-order representation*, the representation of a relation between representations. Kant terms this second-order representation the *analytic unity of apperception* (B134), which is the necessary feature of a subject since without it I could not say that "all my representations belong to me" (B134), because there would be no "me" to speak of. I would then have "as multicolored, as diverse a self as I have representations of which I am conscious" (B134), a "scattered" or "fragmented" consciousness, consisting of a chaotic plurality of different instances of consciousness each accompanying a particular feature of the manifold.

Now, for the analytic unity of consciousness to hold for every possible experience, of which it is a constitutive element, there must be an *a priori* principle that guarantees that it holds for any possible manifold of representations that might be given to me as a subject. This is the principle of the synthetic unity of apperception, which grounds the former analytic unity (B134), and consists in the necessity of the possibility[18] of the synthesis, or combination, of these different instances of consciousness in one single consciousness encompassing them.[19]

[17]See for instance (R5661, 18:318) and (24:752), where it is said that "representation may be combined also with apperception - the consciousness of the representation". We would argue that this distinction is why Kant, in the B edition, says that the "I think" must *be able* to accompany all my representations. The modal form of the verb here is significant, since Kant insists exactly on this point in one of his letters to Beck (11:395); the modal character of many of Kant's notions, in particular that of a "possible experience", will be important in what follows.

[18]Note the use of the locution "necessity of the possibility", since what is necessary is that my representations are "in accord with the condition under which alone they **can** stand together in a universal self-consciousness" (B133); we find here again the modal formulation, common to all Kant's principles in the CPR, that will be important in what follows.

[19]Thus Kant at B131:

> The consciousness of [one representation] ... is still always to be distinguished from the consciousness of [the other representation], and it is only the synthesis of this possible consciousness that is at issue here.

And, in his letter to Herz (11:50):

Semi-formally, let us denote two distinct representations with A, B, and with I_A, I_B the instances of consciousness accompanying them. Then $id(I_A, I_B)$ is the representation of the identity of the two instances of consciousness, and $I_{id(I_A,I_B)}$ is the consciousness of this latter representation, which is however only possible if A, B are synthesized or combined as parts of a single representation C, so that I_C, the consciousness of C and of its synthesis (B133), implies $I_{id(I_A,I_B)}$. Of course, C could now be combined with a representation D, yielding a representation E, so that I_E implies $I_{id(I_C,I_D)}$, and so forth. Thus, the representation of the identity of two instances of consciousness depends on their being synthesized as "parts" of a single consciousness, which is possible only though the synthesis of their respective representations into a whole representation accompanied by a single consciousness.

The B deduction terms the synthesis or combination that brings any manifold under the unity of apperception the *synthesis intellectualis* (B151), or intellectual synthesis, in order to distinguish it from the figurative synthesis, which comes into play only at §24. The intellectual synthesis is nothing other than the synthesis of the manifold by means of the categories, which, as logical functions of judgments (B143), or, which is the same, as *a priori* rules for the connection of appearances (B201), make the comprehension of the manifold of representations in one consciousness possible:

(10) Synthetic unity of apperception *a priori* is the synthesis of the manifold in accordance with an *a priori* rule. The logical function is the action of unifying the same consciousness with many representations, i.e., of thinking a rule in general. The unity of intuition *a priori* is only possible through the combination of the manifold in one apperception, which must therefore take place *a priori* [18:282]

Thus, the pure concepts of the understanding, as *a priori* rules for the combination of representations, that is, as logical functions, guarantee the unity of the manifold of intuition, since they ensure that this manifold can be given in one consciousness, whatever the form of this manifold might be.[20] The unity of the

> I ascribe to the understanding the synthetic unity of apperception, through which alone the manifold of intuition (of whose *every feature* I may nevertheless be *particularly* conscious), in a unified consciousness, is brought to the representation of an object in general (whose concept is then determined by means of that manifold)

[20]Note that there can be no doubt that the notion of "combination", or *verbindung* that appears in the first step of the deduction B denotes a synthesis by means of the categories, since Kant says so explicitly at B151. Moreover, Kant gives a definition of "combination" in terms of *compositio* and *nexus* at B201, where the first denotes synthesis according to the categories of quantity and quality, and the latter according to the categories of relation and modality. The connection between the categories, the concept of combination and the unity of apperception is also made explicit in a letter to Tieftrunk, which echoes the argumentative line of the first step of the B deduction:

manifold that is achieved by means of combination according to the categories in the intellectual synthesis is, of course, objective rather than subjective (B140), in agreement with our treatment in section 3.5.1, and is represented discursively by means of judgments, whose logical functions are these same categories (A128 and B143). Since the analytic unity of apperception is a constitutive component of an experience, and the former is grounded on the synthetic unity of apperception and ultimately on the categories, it then follows that any experience must stand under the categories.

In particular, the culmination of the argument of the first step of the B deduction at §20 makes it very clear that its structure is that of a transcendental argument; in order to show the necessary applicability of the categories to any manifold of intuition one first assumes the principle of the unity of apperception, and then argues that the unity of apperception is guaranteed for any possible experience only if a discursive reflection of any manifold by means of judgments is possible, and thus in turn only if the categories, as the logical functions of these judgments, are necessarily applicable to any possible manifold.

Of course, the perennial interpretative question at this point is: if the first step of the B deduction already proves the necessary applicability of the categories to the manifold of any possible intuition, why is a second part at all needed, in which this argument is related specifically to *our* forms of intuition via the figurative synthesis? Doesn't the first part already suffice ? While the aim of this work is not to provide an answer to this question, the problem cannot be side-stepped, since the division of the TD B is closely tied to the distinction between the intellectual and the figurative synthesis. If we wish to understand the figurative synthesis, we must then have a theory about why Kant thought it necessary to present the argument in the B deduction as he did.

3.6.3 The imagination in the two steps of the TD A

What is the point of dividing the deduction of the categories into the two steps of the B edition, if the second step seems redundant? Before we tackle this question directly, a closer look to the structure of the A deduction might provide us with more data to provide a satisfactory answer. We shall then return to the above

> The concept of the *composed* [zusammensetzen] in general is not the concept of a particular category. Rather, it is included in every category (as *synthetic* unity of apperception). For that which is composed cannot as such be *intuited*; rather, the concept of consciousness of composing (a function that, as synthetic unity of apperception, is the foundation of all the categories) must be presupposed in order to think the manifold of intuition [...] as unified in one consciousness. (12:223)

Kant continues this passage by distinguishing the categories according to whether they express a *mathematical* or a *dynamical* function, a synthesis of what is homogeneous and not homogenous respectively; the former are the categories of quantity and quality, the latter those of relation and modality, a classification that is in agreement with B201 in the CPR.

question in the following section.

Certainly, in the TD B deduction Kant did not aim at replacing the argumentative line of the TD A with something radically different, since in a famous footnote in the MFNS he states that the deficiencies of the TD A concern only "the manner of presentation, and not the ground of explanation" [MFNS 476]. In this respect, while most commentators have focused on the two-step structure of the TD B, we believe that the TD A already contains a division in two steps, even though only *in nuce*.

Indeed, consider the third section of the TD A, beginning at A115. At this point, Kant has expounded his theory of the three-fold synthesis (see section 3.6.1), which is a synopsis of all the various elements that concur to provide a deduction of the categories; he now sets out to present "as unified and in connection" all that was expounded before "separately" (A115). If one carefully compares what follows with the text of the B deduction, one sees that the argumentative structure is analogous in the two cases. The paragraphs from A115 to A119 follow the same argumentative line as the first part of the B deduction (§15 to §23), and focus on the relation of the manifold of intuition to the unity of apperception via the synthesis according to the categories, without regard for our specific forms of intuition, space and time.

The paragraphs from A120 onwards cover the same ground as the second part of the B deduction (starting at §24), and exhibit "the necessary connection of the understanding with the appearances by means of the categories [. . .] by beginning from beneath, namely with what is empirical" (A120). We find here all the essential elements of the second step of the B deduction: the empirical synthesis of apprehension (which appears at B160), the transcendental or productive synthesis of the imagination (which appears at B151) and, crucially, the explanation of the possibility of *prescribing* the laws of nature to objects of *our* sensible intuition (A126 and B160), which is the culmination of the second step of the B deduction:

(11) Now the possibility of cognizing *a priori* **through categories** whatever objects **may come before our senses,** [. . .] as far as the laws of their combination are concerned, thus the possibility of as it were prescribing the law to nature and even making the latter possible, is to be explained. For if the categories did not serve in this way, it would not become clear why everything that may ever come before our senses must stand under the laws that arise *a priori* from the understanding alone. (B160)

This division of the argument of the TD A shows that a two-step structure is already present, albeit only *in nuce*, in the A edition. One might then formulate the hypothesis that in the B edition Kant only wished to make this division more transparent by neatly tearing apart the intellectual use of the imagination (A115-A119) from its sensible application, starting at A120. In the B deduction, the former will become the intellectual synthesis and the latter the figurative synthesis.

This reading of the transition from the A deduction to the B deduction is confirmed by the numerous ambiguities regarding the role of the faculty of the imagination in the two arguments.

In particular, in the first step of the TD A the synthesis that guarantees the relation of any manifold of representations to one apperception is attributed to the imagination, as a transcendental synthesis of this faculty. Still, Kant states there that the "unity of apperception in relation to [...] the transcendental synthesis of the imagination is the pure understanding" (A119), i.e., the categories. Thus, the role of the categories seems here that of constraining the synthesis of the imagination, so that its action brings the manifold of representations under the unity of apperception; indeed, at A124 Kant remarks that it is the pure apperception that "must be added to the synthesis of the imagination to make its function intellectual. For in itself the synthesis of the imagination [...] is nevertheless always sensible". This pure apperception, however, can be added to the synthesis of the imagination only through the categories, which then constitute the unity of its synthesis. Most importantly, at A130 Kant states that the sensible manifold belongs to consciousness with a certain "intellectual form", to which it is brought by the synthesis of the imagination:

(12) [...] the way in which the manifold of sensible representation (intuition) belongs to a consciousness precedes all cognition of the object, as its intellectual form, and itself constitutes an *a priori* formal cognition of all objects in general, insofar as they are thought (categories). Their synthesis through the pure imagination, the unity of all representations in relation to original apperception, precede all empirical cognition [...] (A130)

This "intellectual form", imposed *a priori* by the categories on the manifold so as to ensure its unity under one apperception, guarantees that they can be cognized and so "be something for us". Thus, in the A deduction the synthesis that brings the manifold of representations to the unity of apperception is, so to speak, an "intellectual" synthesis of the imagination.

In the first step of the TD B, on the other hand, the general concept of synthesis or combination (*combination*, *verbindung*) is not attributed to the imagination but directly to the understanding, as Kant states at B152 that the figurative synthesis "is distinct from the intellectual synthesis without any imagination merely through the understanding". However, later Kant seems to revert back to the account of the A edition, stating that "that which connects the manifold of sensible intuition is imagination, which depends on the understanding for the unity of its intellectual synthesis and on sensibility for the manifoldness of apprehension" (B165). This lapse, along with the fact that in the TD B the term "transcendental synthesis of the imagination" applies not to the purely intellectual synthesis of the categories but only to the figurative synthesis, seems to imply that in the TD B Kant attributed the role played by the transcendental synthesis of the imagination in the TD A to the understanding alone, in the form of an intellectual synthesis, so that in the TD B the term "transcendental synthesis of the imagination" comes to designate a different aspect of synthesis, for which the relation to both sensibility and the understanding is crucial. This shift in the role of the imagination is confirmed by a series of reflections that have been dated to around the time of the publication of the first critique (23:18 lbl b12). In these notes, the distinction between a pure sensible synthesis of the imagination, which as the figurative synthesis of the B edition "produces nothing but shapes [Gestalten]", and a transcendental synthesis of the same faculty, which is essentially that appearing at A118-119, is made much more explicit than in the TD A.[21]

It then seems that the two-step structure of the B deduction is actually the product of a revision of the synthetic processes that are attributed to the imagination and the understanding, respectively, in the TD A; but what is the deeper meaning of this revision? We shall address this question after we have take up again the question that was left unanswered in the previous section: why did Kant believe that the deduction of the categories would be incomplete without a second step relating them to *our* forms of intuition, space and time?

3.7 The figurative synthesis in the second step of the TD

3.7.1 The role of the second step of the transcendental deduction

Various commentators (Longuenesse, 1998, p. 211-33; Keller, 2001, p. 88-94; Dickerson, 2003, p. 196-201; Pollok, 2008) have argued that while in the first step of the transcendental deduction Kant shows *that* the categories must apply to the manifold provided by any sensibility, no matter its form, in the second step he

[21] Thus Kant:

> The transcendental synthesis of the imagination lies at the basis of all the concepts of our understanding [...] the productive imagination is I. empirical in apprehension 2. pure but sensible with regard to an object of pure sensible intuition, 3. transcendental with regard to an object in general. The first presupposes the second, and the second presupposes the third [...] the pure synthesis of the imagination is the ground of the possibility of the empirical synthesis in apprehension [...] it is possible *a priori* and produces nothing but shapes [Gestalten]. The transcendental synthesis of the imagination pertains solely to the unity of apperception in the synthesis of the *manifold in general* through the imagination [...] (23:18 lbl b 12, my emphasis)

aims to show *how* it is that the categories apply to a manifold given according to *our* forms of intuition, space and time.[22] But why is this second step at all needed? Is it just a mere elucidation of the first step?

In this respect, note that the emphasis in the second step of both deductions is, as expressed in Passage (11) above, on "prescribing the laws to nature". Thus, Kant's concerns in relating the categories specifically to our forms of intuition are due to providing an explanation of the lawfulness of nature, where the laws in question are the propositions of physics, whose formulation essentially depends on the properties of space and time. Indeed, the necessary unity of nature under physical laws does not follow from the first step of the TD. There, Kant has shown merely the necessity of the possibility of the unity of any manifold of representations, not necessarily sensible, under one apperception by means of the categories; this unity, however, is a logical one, hence purely intellectual, and so completely independent from space and time as conditions of sensibility. The possibility remains open that the unity bestowed on sensible manifolds in virtue of being given in the space and time of the TA is not that which they derive from their necessary logical relation to the unity of apperception. Indeed, Kant acknowledges this explicitly:

> In the sequel (§26) it will be shown from the way in which the empirical intuition is given in sensibility that its unity can be none other than the one the category prescribes to the manifold of a given intuition in general according to the preceding §20; thus by the explanation of its *a priori* validity in regard to all objects of our senses the aim of the deduction will first be fully attained (B145)

What is missing is a proof that sensible manifolds must stand under the categories from the mere fact that they can only be given to us in space and time, as this would ensure that the spatiotemporal unity of sensible manifolds harmonizes with the unity of apperception, and hence that particular physical laws are grounded on a more general *a priori* "lawfulness of appearances in space and time" that is itself grounded on the logical unity of the sensible manifolds under the categories (B165). This argument at B165 is echoed at (R4676, 17:656):

(13) If something is apprehended, it is taken up in the function of apperception. I am, I think, thoughts are in me [...] the I constitutes the substratum for a rule in general, and apprehension relates every appearance to it.

[22] Indeed, at the beginning of what we termed the second step of the A deduction Kant says that he will now "set the necessary connection of the understanding with the appearances by means of the categories [...] beginning from beneath, namely with what is empirical" (A120); this is also the form of the argument at §26 of the B deduction, which starts with the examination of the empirical synthesis of apprehension (B162). Thus, the second step of both deductions proceeds from empirical appearances, given according to *our* forms of intuition, to the *a priori* conditions of their unity, in order to show how the categories apply to sensible manifolds.

But how can apprehension by itself relate appearances to the unity of apperception? Only if by the mere act of being apprehended, appearances already stand under the categories. This, however, can only hold if the spatiotemporal form of apprehension is in agreement with the intellectual form mentioned in Passage (12). We shall return to the relation between apprehension and the intellectual form in section 3.8.

3.7.2 The homogeneity problem

A fundamental problem arises at this point of the argument, which, following Kant, can be termed the "homogeneity problem". Briefly stated, the problem is that it is unclear how one could ever prove that appearances must stand under the categories from the mere fact of being given in space and time, given that the former belong to a very different realm than the latter. Indeed, the categories are pure concepts, and as such belong to the understanding as logical functions of judgments; space and time are forms of inner and outer sense, and as such belong to sensibility. There seems to be here an unbridgeable gulf between the intellectual and sensible forms that prevents the proof we seek.

Kant was well aware of the homogeneity problem, since he devotes to it the chapter on the schematism of the understanding, and continued grappling with it well after the publication of the second edition of the CPR. In a note dating back to 1797 he explains that the categories are not homogeneous with the *form* of intuition; rather, "the application of the categories to the appearances is [...] made possible through the transcendental determination of time (because it is homogeneous with both the appearances and the categories) [...]" (R6359, 18:686), where transcendental time-determination is described as "a product of apperception in relation to the form of intuition".

Thus, the schematism chapter and the above note would seem to suggest that Kant tackled the homogeneity problem by arguing that while the categories are not homogeneous with time and space as the forms of intuition, they are homogeneous with the transcendental determination of these forms, which is somehow brought about by the unity of apperception. To make sense of this, one must understand what transcendental time-determination is and in what sense it makes appearances homogeneous with the categories. We must then finally appeal to the figurative synthesis, which in the TD B is described as follows:

(14) the understanding [...] can determine the manifold of given representations in accord with the synthetic unity of apperception, and thus think *a priori* synthetic unity of the apperception of the manifold of **sensible intuition**, as the condition under which all objects of our (human) intuition must necessarily stand, through which then the categories [...] acquire [...] application to objects that can be given to us in intuition.[23]

[23] Kant then continues by saying:

(B151)

This description identifies the figurative synthesis as the process responsible for the *a priori* determination of the form of sense in relation to the unity of apperception. Hence, the figurative synthesis is responsible for the time-determination mentioned in the aforementioned note; through it, the categories become homogeneous with the sensible manifold that *may* be given in intuition, and acquire “application to objects”. This, of course, meshes nicely with the schematism chapter, where time-determination is also attributed to the figurative synthesis.[24] In particular, Kant gives there the following description of the schema of a category:

(15) The schema of a pure concept of the understanding is something that can never be brought to an image at all, but is rather only the pure synthesis [...] in accord with a rule of unity [...] which the category expresses, and is a transcendental product of the imagination, which concerns the determination of the inner sense in general, in accordance with *conditions of its form (time)* in regard to all representations [...] (A142/B181, my emphasis)

In light of what we said above, we interpret this passage as stating that the figurative synthesis determines the inner sense in such a way that the categories become applicable to any manifold of representations that may be given according to this *determined* form of inner sense. In particular, the determination of inner sense must proceed in accordance with the *form of intuition*; Kant had already made that clear at B152 by stating that the figurative synthesis belongs to sensibility in virtue of the subjective conditions “under which it can give a corresponding intuition to the concept of understanding” (see Passage (14) above).

We infer from these observations that Kant attempted to solve the homogeneity problem by attributing to the imagination an *a priori* synthesis, the figurative synthesis, through which the sensibility is determined *a priori* for any possible manifold in relation to the categories and hence to the unity of apperception.

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the imagination, on account of the subjective condition under which alone it can give a corresponding intuition to the concepts of understanding, belongs to **sensibility**; but insofar as its synthesis is still an exercise of spontaneity [...] and can thus determine the form of sense *a priori* in accordance with the unity of apperception, [it is] to this extent a faculty for determining sensibility *a priori*, and its synthesis of intuitions, **in accordance with the categories**, must be the transcendental synthesis of the **imagination**, which is an effect of the understanding on sensibility [...] that which determines the inner sense is the understanding, and its original faculty of combining the manifold of intuition, i.e., of bringing it under an apperception. (B152 - B153)

[24]In Kant’s copy of the A edition, the very term “schematism” is glossed as follows: “the synthesis of the understanding is called thus if it determines the inner sense in accordance with the unity of apperception” (E LVII, 23:27). It is obvious from the above that Kant here is talking about the figurative synthesis.

Appearances are then homogeneous with the categories by the mere fact of being given according to this *determined* form of sensibility, which is then the sensible form that is isomorphic to the intellectual form that Kant speaks of at Passage (12).

Now, if space and time as forms of intuition are not homogeneous with the categories, but become homogeneous with the categories only via the action of the figurative synthesis, this implies that space and time as determined by the figurative synthesis are not merely space and time as forms of intuition.

3.8 The formal intuition

3.8.1 Form of intuition versus formal intuition

We then claim that Kant's solution to the homogeneity problem consists in distinguishing the form of intuition, which as the mere form of the receptivity of sensibility is purely passive and merely gives the manifold without any relation to the categories or the unity of apperception, from the formal intuition, which denotes space and time as *conscious* representations along with consciousness of their phenomenal properties, such as unity, infinity and so forth. The latter is produced by the spontaneity of the understanding that affects the sensibility through the figurative synthesis, as the synthesis of apprehension, only exercised *a priori* in agreement with both the form of inner and outer sense and the categories. Indeed, since the synthesis at B161n is described as the understanding affecting the sensibility, which is exactly how the figurative synthesis itself is defined at §24 of the TD B, there is little doubt that the synthesis at B161n must be the figurative synthesis. Since appearances are given with the formal intuition, as Kant claims at §26 of TD B, their formal content is then already determined in relation to the categories, in the sense of the chapter on the schematism of the understanding.

In this respect, in his response to Eberhard Kant writes:

(16) The *Critique* admits absolutely no implanted or innate *representations* [...] the ground of the possibility of sensory intuition [...] is the mere *receptivity* peculiar to the mind, when it is affected by something (in sensation) to receive a representation in accordance to its subjective constitution. Only this first formal ground [...] is innate, not the spatial representation itself. For impressions would always be required in order to determine the cognitive faculty to the representation of an object (which is always a specific act) in the first place. Thus arises the *formal* intuition called space, as an originally acquired representation (the form of outer objects in general), the ground of which (as mere receptivity) is nevertheless innate, and whose acquisition long precedes the determinate concepts of things that are in accordance with this form [...] (8:223)

Kant distinguishes here a "first formal ground of intuition", the subjective constitution of the receptivity of the mind, and space and time as representations of an object, which are originally acquired through sensory impressions. In light of the footnote at B161n, and following Longuenesse (Longuenesse, 1998, p. 252), it seems sensible to interpret this "first formal ground" as the mere form of intuition of the footnote at B161n, the subjective and passive constitution of sensibility, and this process of "original acquisition" of the formal intuition of space, which requires the spontaneity of the understanding, as the figurative synthesis of the B deduction.

Indeed, this reading meshes nicely with Kant's description of the form of intuition at §24 of the TD B, right after the figurative synthesis has been introduced:

(17) [the synthesis of the understanding] is nothing other than the unity of the action [...] through which it is capable of itself determining sensibility internally with regard to the manifold that *may* be given to it in accordance with the forms of its intuition [...] inner sense contains the mere **form** of intuition, but without combination of the manifold in it, and thus it does not yet contain any **determinate** intuition at all, which is possible only through the consciousness of the determination of the manifold through the [...] figurative synthesis (B154)

Note Kant's boldface on the term "form", which is contrasted here with the notion of a "determinate intuition". From this passage we understand that the form of intuition is purely passive, as it does not contain any combination of the manifold, and so does not yet contain any determinate intuition. Hence, it is merely the subjective constitution of the sensibility, the "first formal ground" mentioned above; it follows that the form of intuition cannot contain even the intuitions of space and time as unitary representations, which can only be acquired through the understanding "affecting the sensibility" and first providing it with *a priori* sensory impressions. This explains why Kant felt the need to distinguish the form of intuition from the formal intuition at B161n, and why he says that space and time are first "given as intuitions" through the figurative synthesis. Furthermore, it also throws light on the otherwise puzzling remarks at A97-98 and A99-100, where Kant says that the sensibility in its "original receptivity", i.e. in its pure passivity, only provides a manifold by distinguishing a multitude of representations, but that the unity of this manifold in one representation, as in the representation of space, requires the *a priori* correlate to the synthesis of apprehension, the figurative synthesis, which contains the essential moment of comprehension.

Thus, the crucial aspect of the formal intuition is that it is a unitary representation that is accompanied by consciousness, which is produced by the figurative synthesis, and this sets it apart from the merely passive form of intuition:

(18) With space and time one can only take two paths: 1. that they are

> concepts, 2. that they are mere intuitions. In the first case they are a. empirical or b. a priori concepts. In the second case they are 1. intuitions of things in themselves through observation and yet necessary, 2. formal intuition a priori, i.e., consciousness of the way in which objects of the senses are represented to us.[25] (R5649, 18:298)

Space and time as "formal intuition *a priori*" are the "consciousness of the way in which objects of the senses are given to us", which certainly cannot be obtained from the mere unity of apperception alone, the "I think", because Kant is adamant that from the "I think" alone "nothing manifold is given" (B136). Similarly, it cannot be provided *a priori* by the mere sensibility, since this is merely passive receptivity. But then what is the formal intuition exactly, and how does it ensure homogeneity with respect to the categories?

3.8.2 The formal intuition and the homogeneity problem

The formal intuition is nothing more than the consciousness of the necessary form of *any* act of apprehension of a sensible manifold, which being objective already contains a relation of the latter to the categories, that is, it is only possible in agreement with the categories as concepts of objective unity. While the synthesis of apprehension itself is always empirical, we can nevertheless become conscious of the form of *any* such act of apprehension through its *a priori* correlate, the figurative synthesis.

Indeed, note that in Passage (17) above the intellectual synthesis is nothing other than the "unity" of the action of the figurative synthesis. This action determines the sensibility with respect to the manifold that *may* be given according to the form of intuition; the modal *may* is, as always, important, as it implies that the determination of sensibility through the figurative synthesis is *a priori* and hence valid of all appearances. This *a priori* determination through the figurative synthesis in agreement with the categories represents to the subject the formal content of any act of apprehension of an object in general, and consciousness of this synthesis implies consciousness of the formal content of any act of

[25] Similar passages from the OP are:

> Consciousness of itself (*apperceptio*) is an act through which the subject makes itself in general into an object. It is not yet a *perception* [...] it is, rather, pure intuition, which, under the designations of space and time, contains merely the formal element of the composition (*coordination et subordinatio*) of the manifold of intuition [...] (22:413)

> Space is not an object of intuitions (an object or its determination), but the intuition itself [...] it is a pure intuition *a priori*. But how is such an intuition possible[?] it is nothing other than the consciousness of one's own receptivity for sensing representations (impression) of things in accordance with certain relations among them (17:639)

apprehension, the formal intuitions of space and time.

This, of course, explains why the footnote at B161n appears at §26, where Kant is busy with proving that the categories do indeed apply to appearances which are apprehended according to *our* forms of intuition, as the examples at B162 make clear. Indeed, in the footnote at B162 Kant claims to have shown that the synthesis of apprehension agrees with the synthesis of apperception "contained in the category" because "it is one and the same spontaneity that, there under the name of the imagination and here under the name of the understanding, brings combination under the manifold of intuition". The synthesis of apprehension proceeds in agreement with the categories, and already contains the relation of what is apprehended to the latter, as Passage (13) in the previous section make clear; but then space and time as the *forms of apprehension*, and not merely as the forms of inner sense, guarantee homogeneity with the categories. This is why Kant states that

(19) The unity of apprehension is necessarily combined with the unity of the intuition [of] space and time, for without this the latter would yield no real representation. (R4678, 17:660)

In other words, the unity of space and time as representations depends on the unity of the act of apprehension, which, if exercised *a priori*, is just the act of the figurative synthesis, the consciousness of whose form first gives space and time as intuitions:

(20) That we can affect ourselves [...] is possible only through our apprehending the representations of things that affect us [...] for thereby do we affect ourselves, and time **is properly the form of the apprehension** of representations which are related to something outside us. (R6310, 18:623)

Note that these passages square well with Kant's claims, at A99-100 and A102 (but see also Passage (8)), that space and time arise *as representations* only in virtue of the pure syntheses of apprehension and of reproduction. For in the second edition these pure syntheses have been combined into one, as we have seen, as the figurative synthesis.

Thus, space and time as formal intuitions ensure that appearances are homogeneous with the categories because the formal intuition, being produced by the figurative synthesis in agreement with the categories, is necessarily "isomorphic" to the purely logical "intellectual form" of Passage (12). Paraphrasing Kant, while the categories as logical functions of judgments are the "*a priori* formal cognition of all objects in general, insofar as they are thought" (A130), space and time as formal intuitions are the "*a priori* formal cognition of all objects in general, insofar as they are sensed" (see Passage (12)); it is the homogeneity between these two formal aspects of cognition that ensures the necessary applicability of the categories to *our* experience.

We can now also make sense of the change in the role of the imagination from the A edition to the B edition, which we discussed in section 3.6.3. In the second edition Kant attempted to provide a better explanation for why anything that is empirically apprehended, and is then in space and time, must stand under the categories; for this purpose he isolated the figurative synthesis from the intellectual synthesis, emphasizing the role of the former in ensuring the applicability of the categories to spatiotemporal manifolds by naming it the "transcendental synthesis of the imagination", a term which, in the A edition and in the notes at (23:18 lbl b 12), had been reserved for the intellectual synthesis. He then marked more strongly the distinction between space and time as purely passive, i.e., as the forms of intuition, and space and time as intuition themselves, which, being produced by the figurative synthesis in agreement with the categories, would relate the synthesis of apprehension to the latter *a priori*.

3.8.3 The formal intuition and self-consciousness

What is the exact nature of the affection of the sensibility by the understanding through the figurative synthesis?

Note that Kant often remarks that space and time are radically different from appearances because while the latter can be perceived, the former are "merely formal" intuitions, which "cannot be perceived in themselves" (B207).[26] But if space and time cannot be perceived, one might ask, in what way are they originally intuited? The discussion in the previous sections suggests that this is possible only through the apprehension of impressions and the consciousness of the necessary form of this act. Since space and time are given *a priori*, however, these impressions cannot be empirical; the question then remains whence they come from, i.e., what object they are impressions of.

Kant's answer to this question is that the manifold of these impressions is provided *a priori* by the subject, which makes itself the original object of apprehension and thereby structures itself spatiotemporally. In the OP we find:

(21) Our sensible intuition is, initially, not perception (empirical representation with consciousness), for a principle of positing oneself and of becoming conscious of this position precedes it; and the form[s] of this positing of the manifold, as thoroughly combined, are the pure intuitions, which are called space and time [...][27] (22:420)

[26]Thus, Kant says at (R5384, 18:158) that "space endures; but space itself can be perceived only by means of things in it"

[27]Other related passages from the OP are:

- The first act of the faculty of representation, through which the subject posits the manifold of its intuition and makes itself an object of the senses, is a synthetic *a priori* cognition of the *given* (*dabile*): space and time as the formal element of intuition [...] (OP 22:452)

Thus, the affection of the sensibility by the understanding through the figurative synthesis consists in the apprehension of a manifold of representations which I produce myself, as I apprehend myself as the original sensible object, and structure myself in space and time. Thus, Kant says that the subject does not merely perceive sensations in itself, but it "must arouse them and connect them synthetically, hence affect itself" (R6349, 18:674), and is thus not a thinking but an intuiting of itself. Consciousness of the form of this act of apprehension of myself as the original object means in turn consciousness of the necessary form of an act of apprehension of an object in general, i.e., space and time as formal intuitions.

In order for this sort of self-affection to be possible, however, there must be a distinction between the self as subject and the self as object, i.e., the self that apprehends and the self that is apprehended. The former, as the understanding, determines the latter, as sensibility, so that we are both "originators" and "spectators" (OP 22:421):

(22) That I am conscious of myself is a thought that already contains a twofold self, the self as subject and the self as object [...] that I, who think, can be an object (of intuition) to myself [...] is an undoubted fact [...] the self in the second sense [...] is capable of being known in many ways, among which time, the form of inner intuition, is that which underlies a priori all perceptions and their combination whose apprehension (apprehensio) conforms to the manner in which the subject is thereby affected, i.e., to the condition of time, in that *the sensory self is determined by the intellectual to take up this condition into consciousness.* [...] (20:270)

We find in the passage above a clear statement of the distinction between the sensible self and the intellectual self, where the former is determined by the latter; recall that this distinction is central at §24 in the TD B and nicely mirrors that between the sensible and the intellectual form. Most importantly, Kant states clearly that the condition of time is taken up into consciousness through the determination, or affection, of the sensory self by the intellectual self. It is through this act of affection that what Kant calls *empirical consciousness* of oneself is

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- Space and time are products (but primitive products) of our own imagination, hence self-created intuitions, inasmuch as the subject affects itself and is thereby appearance [...] (OP 22:37)
- (space and time) are the subjective element of the subject's self-affection (formally) (OP 22:33)
- The formal element of pure (not empirical) intuition is in representation *a priori* (in appearance); that is, represents the self-determination, how the subject affects itself (OP 22:480)

Note that Kant says that it is the faculty of the imagination, i.e., the figurative synthesis, that is responsible for the production of space and time through self-affection, i.e., the self-determination of the subject.

achieved, i.e., the consciousness of oneself as the original object of intuition that is structured in a spatiotemporal form. Indeed, empirical consciousness in this context just means consciousness of oneself as an object apprehended in a spatiotemporal form, as opposed to the transcendental consciousness represented by the "I" through which "no manifold is given"; it is then not to be confused with consciousness of an empirical manifold, which is *a posteriori* (see, in particular, R5661, 18:319).

The correctness of this reading is supported by examining Kant's arguments for the refutation of idealism, which are intimately bound up with the notion of self-affection in the figurative synthesis. The following passages from Kant's notes expose Kant's strategy quite clearly:

(23) We need space in order to construct time, and thus determine the latter by means of the former [...] in the representation of space we must be conscious of ourselves as being affected by outer things. We do not cognize this by means of an inference, rather it lies in the way in which we affect ourselves in order to construct time as the mere form of the representation of our inner state, for which something other, not belonging to this inner state, must still always be given (i.e., something outer, the construction of which at the same time contains the intuition of time and lies at its ground). (R6311, 18:613)

The representation of space consists in the consciousness of the form according to which we are related to outer objects, and in particular to ourselves as the original object that is apprehended in space and time. Note Kant's strong phrasing: we *construct* time as the form of the representation of our inner state, in the same way as we "institute" space as the form of the representation of our outer state.[28]

We now have an abstract account of the figurative synthesis and of its role in producing space and time as the formal intuitions. The last element required to complete the picture is an analysis of the role of motion as the concrete act of the figurative synthesis, to which we now turn.

3.8.4 The formal intuition, motion and comprehension

In the previous sections we established that space and time as formal intuitions are generated in the apprehension of oneself as the original object, since impressions

[28]Indeed, Kant says at (18:308) that we are conscious of the form of the relation of an outer object to ourself, and that

> If we were merely affected by our self yet without noticing this spontaneity, then only the form of time would be found in our intuition: and we would not be able to represent any space (existence outside of us) [...] That we ourselves must always simultaneously institute space and the determination of time [...] (R5653, 18:308)

are required to acquire the representations of space and time. How does one become an object of apprehension to oneself *a priori*, however?

Kant's answer is that one becomes an object to oneself through the *a priori* motion in which the subject describes a space and thereby affects its outer sense and its inner sense. This is the concrete content of the figurative synthesis:

(24) [...] we cannot think of a circle without **describing** it [...] we cannot even represent time without, in **drawing** a straight line (which is to be the external figurative representation of time), attending merely to the action of the synthesis of the manifold *through which we successively determine the inner sense*, and thereby attending to the succession of this determination in inner sense. Motion, as action of the subject [...] consequently the synthesis of the manifold in space, if we abstract from this manifold in space and attend solely to the action in accordance with which we determine the form of **inner sense**, first produces the concept of succession at all [...][29] (B154-B155)

The act of self-affection of the subject is then the intellectual self determining the sensible self through motion, so that a manifold is produced which is then apprehended in agreement with the conditions of unity for the understanding, the categories, and thereby space and time as unitary representations arise. In particular, note that this act of "apprehension" is objective in the sense of section 3.5, that is, the representations of outer sense which are produced through motion are not merely apprehended but also comprehended, i.e., apprehended in different sequences so that their simultaneity can first be cognized, and a unitary representation first arises (see R6314, 18:616). The moment of the comprehension of the manifold is then essential over and above the act of apprehension to cognize space and time as formal intuitions, only it is exercised *a priori*, as the footnote B161n makes clear.

Moreover, it is in describing spaces in outer sense that the subject first becomes conscious of oneself in the representations of space and time, the necessary forms of any act of apprehension of an object. After all, as we have seen, the representation of space and time requires sensory impressions of an object, but something, even *myself*, can be an object of the outer senses only through motion, because "only thereby can these senses be affected" (4:447). Pure *a priori* sensory impressions are given through the *a priori* motion of the subject itself

[29] As a footnote to this passage we find:

> Motion of an object in space does not belong in a pure science, thus also not in geometry; for that something is movable cannot be cognized *a priori* but only through experience. But motion, as **description** of a space, is a pure act of the successive synthesis of the manifold in outer intuition in general through productive imagination, and belongs not only to geometry but even to transcendental philosophy

which posits itself as the original object.[30]

Thus, we can have conscious representations of space, time and their properties only if we delineate *trajectories* in outer sense and focus on the form of how we are affected. Passage (23) is particularly explicit in this sense; the construction of a line "contains the intuition of time and lies at its ground", since if I produce a trajectory and focus on the successive determination of the inner sense I first produce the intuition of time. Note that it is important to speak of trajectories rather than merely geometrical constructions, since motion, as the original act of the description of a space, combines both inner and outer intuition, time and space[31]; thus, the description of a space is really a spatio-temporal act, and the subject can either focus on the outer manifold which is produced or focus on the determination of the inner sense in producing it. It thereby becomes conscious of the necessary properties of its forms of outer and inner sense, respectively.

The reader might at this point be puzzled by this characterization of the figurative synthesis. If the figurative synthesis consists in the *a priori* description of a space through motion, and in the comprehension of the manifold so produced, then one might say that it should only provide particular spaces, such as lines or circles, or particular times, namely the times during which these constructions are carried out. But Kant is clear, in the TA and elsewhere, that these particular spaces are only possible within the original intuitions of space and time, and that from the mere composition of particular spaces or particular times neither infinite space nor infinite time could ever arise.

However, we do not claim that the figurative synthesis produces space and time as formal intuitions by composing or synthesizing particular spaces or times. Rather, it is in the *act* of describing particular spaces of this kind that the subject becomes conscious of the properties of its forms of sensibility, and thereby first acquires the representations of space and time as the formal element of *any* sensible manifold. In this respect, recall that in the footnote at B161n Kant speaks of a "synthesis [. . .] through which all concepts of space and time become possible"; in the OP we find:

(25) Motion can be treated [. . .] mathematically, for it is nothing but concepts of space and time, which can be presented *a priori* in pure intuition; the understanding *makes* them [OP 22:516]

[30] At this point the reader might object that in Kant's comments on Kästner's treatises, at [417], Kant states that space and time are not generated through the sensible representation of an actual object of the senses. However, Kant is there talking specifically of empirical impressions on the senses, i.e., *a posteriori*, as his reference to the process of "observation, measurement and weighing up" makes clear; if space and time were produced in this fashion, then the propositions of mathematics would be merely empirical. The process of producing sensory impression *a priori* is a different matter, however, and this is what is at stake here.

[31] Kant remarks this explicitly in the phoronomy section of the MFNS ([489]) and at 22:440 of the OP, where he says that motion combines both the outer intuition (space) and the inner intuition (time) in one.

Motion, as the figurative synthesis, is indeed what first makes possible particular pure sensible concepts, such as the concept of a line, or the concept of the time during which this line is constructed. Space and time as formal intuitions, however, are not these pure sensible concepts; rather, in the act of describing these particular spaces we determine the form of intuition with respect to the categories and we become conscious of this determination for any possible manifold. It is the consciousness of this latter determination that constitutes space and time as formal intuitions.

3.8.5 The infinity of the formal intuition and the transcendental ideal

In the previous sections we have presented our exegesis of Kant's distinction between the form of intuition and the formal intuition. We shall now examine more closely the property of the infinity of space and time, as this aspect of Kant's theory has been hotly debated by commentators (Onof & Schulting, 2015, 2014).

We have seen that at A25 and B48 Kant claims that space and time are represented as infinitely given magnitudes. Now, Kant's definition of this sort of infinity is generally formulated in mereological terms: a magnitude is infinite if it is such that every magnitude of the same type is only a part of it (KT 20:419). In his comments on Kästner's treatises, Kant terms this infinity an "infinite in actuality" (20:421), and attributes it to what he calls "metaphysical space", which is original and unitary. Metaphysical space is then contrasted with the many derived geometric spaces that are constructed as the schemata of pure sensible concepts, whose infinity is merely potential, i.e., it is an "infinitely progressing construction". The infinity in actuality of metaphysical space is then the ground or the "foundation" of the potential infinity of geometric spaces, since the progression in the construction of, e.g., a line, always requires an "environment" space such that any line that has been drawn is only a part of it. It would then seem that Kant's discussion of infinity in Kant's comments on Kästner's treatises at (20:410-23) contradicts the interpretation presented above, as we have explicitly maintained that space and time as formal intuitions are generated by the subject in the description of geometrical spaces.

This is not the case, however, since as always in Kant one must be very careful to distinguish genetic relations from relations of logical dependence. Indeed, in the previous sections we did not claim that space and time are constructed by composing particular geometrical spaces, but that in the consciousness of the synthesis of particular spaces one *acquires* the consciousness of the necessary form of all our acts of apprehension whatsover. In particular, one acquires the consciousness that there are no bounds to what can be apprehended or synthesized, the unboundedness of space and time, which gives the principle of their infinity.

Indeed, Kant says that metaphysical space is infinite because "it consists in the pure form of the sensible mode of representation of the subject, as *a priori* intuition". At A25 Kant glosses infinity as the "boundlessness of the progress in intuition", and in the OP we find:

(26) [the quality of space and time as pure intuitions] consists therein that the subject posits itself as given (*dabile*); their quantity, however, in that the act of composition (as infinite in progression (*cogitabile*)) contains the intuition of an infinite whole [...] what is thought *in indefinitum* is here represented as given *in infinitum*. Space and time are infinite quanta. (OP 22:11)[32]

When I affect myself in the describing of a space I am conscious of the unboundedness of *any* possible act of the figurative synthesis, and in this case the distinction between a progress *in indefinitum* and a progress *in infinitum* essentially collapses: what is thought in a pure sensible concept as *in indefinitum* is represented intuitively as *in infinitum*, and thereby I acquire the intuition of the infinity of space and time (compare also A511/B539). The crucial point here is that space and time are closely related to the transcendental ideal (A579/B607), in that for space and time the distinction between what is possible and what is actual collapses (see R4515, 17:579 and R6290, 18:559): consciousness *that I can* always extend a line indefinitely means consciousness *of the representation* of an infinite line, of which all lines are but parts.

Now, this account is perfectly consistent with the claim that the construction of indefinitely extensible geometrical spaces presupposes metaphysical space, since it is an account of how the representation of the latter is acquired, while Kant's notion of "ground" or "foundation" at (KT 20:420) pertains to transcendental logic and merely means that the condition for the possibility of the construction of geometric spaces is the original infinite metaphysical space. Again, the comparison with the discussion of the transcendental ideal is instructive. The latter is defined as the concept of an object that instantiates all predicates that we might call *transcendentally positive*, i.e., such that they signify a "something", a "reality" which their negation merely removes[33]. The transcendental ideal is the condition

[32]Later, at 22:99, Kant states that

> What is formal in this intuition is the One and All, coordinated; [it] is the representation of space and time, which represents an infinity (unlimited magnitude) not analytically through concepts, but synthetically through the construction of concepts (OP 22:99)

In this passage the dependence between the infinity of space and time as formal intuitions and the act of the figurative synthesis, which constructs pure sensible concepts such as that of a line, is clear.

[33]See A575/B603. What Kant is doing there is essentially introducing an asymmetry among antonyms in transcendental logic. From the perspective of pure logic the two antonyms "bright" and "dark" are on a par, as the negation of one is the other. Transcendentally, however, Kant

for the possibility of the concept of a thoroughgoingly determined thing, or object in general, since the latter is the concept of an object to which exactly one of each pair of opposed predicates applies, and transcendentally is then just a limitation of the ideal. This is entirely analogous to the case of space (and of time) if we replace "space" for the transcendental ideal, "point" for the concept of a thoroughgoingly determined object, and "region of space" for a transcendentally positive predicate: space is the condition for the possibility of a point, since the latter can only be determined by deciding, for every possible region of space, whether the point is in it or not[34]. Thus Kant:

(27) Thus all the possibility of things [...] is regarded as derivative, and only that which includes all reality in it is regarded as original [...] all manifoldness of things is only so many different ways of limiting the concept of the highest reality [...] just as all figures are possible only as different ways of limiting infinite space [...] (A578/B606)

The terminology used by Kant here is exactly that used in his comments on Kästner's treatises; while the transcendental ideal is the condition for the possibility of the concept of a thoroughgoingly determined thing, metaphysical space is the condition for the possibility of geometric spaces, since the latter are merely limitations of the former. This relation of ground to conditioned, however, is merely transcendental; it does not explain how the representations of the transcendental ideal or of metaphysical space are acquired. For metaphysical space, we argue, Kant explained this in his footnote at B161n. Thus, there is no tension between Kant's comments on Kästner's essays and the conceptualist interpretation of the footnote that we presented.

3.9 Kant's theory of time and cognitive science

In the previous section we developed an interpretation of Kant's theory of time and space that sheds light on the distinction between the form of intuition and the formal intuition. Before we move on to the presentation of the formal theory, we wish to exhibit some relevant connections between our interpretation and contemporary results in cognitive science, in particular with respect to time perception. Of course, these remarks do not constitute a complete theory of human temporal cognition. Nevertheless, they are highly suggestive of the relevance that Kant's

claims that the latter is derived from the former, as it merely signifies a "taking away" of a reality, the absence of light, so that one acquainted only with bright objects could form the concept of a darkness merely negatively, but one acquainted with only dark objects could not form the concept of brightness.

[34]The reader acquainted with mathematical logic will be immediately aware that this is exactly how points are constructed in pointfree topology. This lends additional exegetical support to our formalization, in which the techniques of formal topology are central.

thought on these matters can have for contemporary cognitive science, not only as a high-level theory by means of which empirical results can be analyzed and interpreted, but, most importantly, as a source for novel empirical questions. We then hope that these brief remarks will be of use for psychologists and empirically minded philosophers of mind, and we ourselves plan to expand on them in more depth in the future.

The main point of interest here is the relationship between spatial representation and time cognition. It might seem from the discussion in the previous section that Kant thought of space and time symmetrically, since, as we argue, they are both produced by the figurative synthesis. This would be wrong, however, as for Kant there is an important sense in which time is dependent on space, and not viceversa: the determination of a unit of time by means of which one can measure the duration of events can only occur by representing time spatially, i.e., through the synthesis of a line, the "external representation" of time (B156). Thus, even though space and time as formal intuitions are originally acquired "simultaneously", there is a logical dependence of the latter on the former, at least as far as duration is concerned.

This claim about the determination of the duration of events might at first not seem very surprising, since it is well-known that temporal notions are often expressed linguistically by means of spatial metaphors, and that children become proficient in the use of spatial vocabulary before they become proficient in the use of temporal vocabulary (Clark, 1973). Indeed, theories of metaphorical mental representation (Lakoff & Johnson, 1999) postulate that abstract domains, in particular time and number, depend asymmetrically on other domains which we are directly acquainted to by means of sensorimotor experience, such as space, force and motion; thus, humans would *think* of time as abstracting from spatial representations. Still, one of the most influential theories of the relationship between space and time, Walsh's A Theory of Magnitude (ATOM) (Walsh, 2003), postulates overlapping brain regions for the processing of space, time and number (see Basso, Nichelli, Frassinetti, and di Pellegrino (1996) in particular), predicting a symmetrical relationship between time and space. Thus, the relationship between time and space is not at all obvious, and the debate is ongoing.

In particular, recent empirical results seem to support the metaphorical representation analysis. In Carelli (2011) it is shown that the estimation of the duration of observed complex events is made substantially easier if subjects are allowed to display events on a time-line as a retrieval support, and in Carelli and Forman (2012) it is shown that the use of a time-line increases accuracy for both children and adults alike, so much so that for short duration stimuli the difference in accuracy between age groups essentially disappears.

Even more in agreement with Kant's ideas are the psychophysical tasks studied in Casasanto and Boroditsky (2008), with nonlinguistic stimuli and responses. Subjects were shown dots or lines on a computer screen and were asked to estimate either their duration or their spatial displacement, marking off with the mouse the beginning and end of each spatial or temporal interval. The experiment showed an asymmetric dependence of duration estimation on spatial representation consistent with Kant's views, since spatial extent acted as a distractor for the estimation of temporal extent but not the other way around; in particular, the longer a line, the longer the duration of the stimulus according to subjects' estimation. In Casasanto, Fotakopoulou, and Boroditsky (2010), the results of analogous experiments on children are reported, which show that "Kindergarden and elementary school-aged children can ignore irrelevant temporal information when making judgments about space, but they have difficulty ignoring spatial information when making judgments about time"; the authors then conclude that space and time are asymmetrically related in children's minds. Note that since these studies do not rely on linguistic stimuli or responses they support the claim that humans' *thinking* about time, and not merely their *talking* about time, is inherently spatial.

The framework of metaphoric mental representation is in remarkable agreement not only with Kant's claims on the dependence of the metric of time on its spatial representation as a line, but also with Kant's account of number concepts. Indeed, recall that according to Kant we conceptualize a number by first producing a multiplicity of units successively (the mere apprehensive act) and then synthesizing this multiplicity into a whole representation (the comprehensive act); the latter act act, however, happens by placing these units in space, as the comprehension of a multiplicity of homogeneous units in a unitary representation occurs only if these are represented as simultaneous (R6314, 18:616; see section 3.8.4).

From a cognitive perspective, however, it is more interesting to investigate how these claims fit within Kant's overall theory of the original acquisition of the representations of space and time, which we expounded in the previous section. Indeed, the insight that time is closely tied to its spatial representation was in itself not new; Descartes and Barrow held similar views (Futch, 2008, p. 27), among others. Kant, however, additionally attempted to provide an account of the mental functions that are needed to construct the representation of time itself, along with the concept of duration. To bring this attempt to bear on current cognitive science one must provide possible cognitive correlates to such functions, which might be investigated empirically and which are to correspond to the formal correlates that we shall present in the following section.

Fortunately, Kant himself gives various hints on the empirical content of his theory of the *a priori* acquisition of space and time, as he brings in the picture the role of *attention* and *(self)-consciousness*. Empirically, that the subject affects itself through motion means that it affects itself originally through the motion of its limbs and body, so that it is for itself the first spatiotemporally structured object to be apprehended empirically. Thus, Kant says that we originally represent space as an object of experience by means of tactile awareness of our own body, or by drawing lines by moving one's hands and limiting those lines with points (OP 21:590). This process, however, is intimately tied to *acts* of attention, without which it would be impossible to bind the representations that have been so produced into a whole representation of space. Thus, Kant says that

> We are required to affect the inner sense [. . .] by means of attention [. . .] in order to have first of all in the intuition of ourself a knowledge of what inner sense is presenting to us; which then merely makes us aware of ourself as we appear to ourselves [. . .] (20:270)

Note that this role of acts of attention appears in the CPR at B156-157. Now, Kant's emphasis on the role of attention, self-affection through motion, and self-consciousness in order to produce the representation of time is quite suggestive in light of known empirical research.

Treisman and Gelade's feature integration theory of attention (FITA) (A. M. Treisman & Gelade, 1980), in particular, shows that there exists a close relation between space, time and visual attention, and can be given a Kantian reading on the basis of the interpretation presented in the previous section. Recall that FITA postulates two distinct stages for object perception. First, the visual scene is analyzed in parallel along different dimensions such as color, brightness, and so forth, but these features are "scattered" and not combined into the representation of an object. In Kant's parlance, parallel processing merely "provides the manifold" without its combination, which is then ordered merely according to the *form* of intuition. The second stage involves the serial processing of features by means of focal attention, so that the central attentive "fixation point" binds features that are found together into a whole. In Treisman and Gelade's words, "focal attention provides the "glue" which integrates the initially separable features into unitary objects". In particular, it is assumed that while the perceptual system can identify the presence in the visual scene of simple features such as colour or orientation via parallel processing without attentional focus, the identification of conjunctive features (e.g., a colored shape) requires serial scanning of the scene and therefore attention. In Kant's framework this second stage can be identified with the description of a space by the subject through motion, as the inner sense is thereby affected and the representation of space and time as formal intuition is acquired. In this sense, experiments VIII and IX in A. M. Treisman and Gelade (1980) are most interesting, showing that if attentional focus is prevented even simple features cannot be reliably located in space; they are, so to speak, "free floating", and the determination of their spatial location requires an additional attentional act (see also A. Treisman (1998)). Additional evidence supporting the role of attentional focus in time perception, not only with respect to duration but also with respect to temporal order and simultaneity, can be found, e.g., in Stelmach and Herdman (1991), Weiß and Scharlau (2011), Zakay (1992).

The relation between Kant's theory of time and FITA is also interesting in light of the current debate on "sensed change" versus "seen change" in psychology.

Recall that recent empirical results (Rensink, 2004; Busch, Fründ, & Herrmann, 2010) show that our awareness of change can be decomposed into "sensed change", which is merely the awareness that something has changed, and "seen change", the consciousness of what it is that has changed; the former can occur without the latter, thus giving rise to the experience of "pure change" (Arstila, 2016). Since according to Kant's theory of time the perception of the changes in the attributes of an object requires focal attention, one is led to formulate the hypothesis that "seen change" is dependent on the serial processing of the visual field, while "sensed change" can be detected by mere parallel processing. Moreover, only the former case should involve the perception of the passage of an interval of time of a determined duration, since as we have seen this also requires focal attention in Kant's theory.

These remarks are of interest not merely because they provide empirical support for some of Kant's insights about time, but, most importantly, because they show that it might be fruitful to conceive of Kant's theory of space and time as a high-level theory of human spatial and temporal cognition that is amenable to empirical investigation.[35] Indeed, from an empirical standpoint Kant's claims, as we construe them, are radical. Space and time are *constructed* as conscious representations by the subject, when the latter moves its body and focuses its attention on the serial processing of these movements, thereby affecting itself and ordering its inner experience in time and its outer experience in space. Most importantly, the *self-consciousness* of oneself as a thinking being arises in this process, and therefore depends on the act of structuring experiences spatiotemporally. What we suggest, then, is that on the basis of this thesis one can attempt to provide a unified theory of time perception that takes into account the role of both attention and consciousness. Indeed, while research in time perception varies widely in terms of focus and approach, it has been conducted in relative isolation from other psychological domains, and the individuation of unifying principles and models of time perception has proven to be a challenge. Still, there has recently been a rise of interest in the problem of identifying common principles for time perception, attention and memory (Matthews & Meck, 2016), and the role of consciousness with respect to time perception has also been the subject of investigation (Yin, Terhune, Smythies, & Meck, 2016). In light of these remarks we then believe that Kant's theory of time can be brought to bear on these issues, as a high-level theory of the relation between time perception, attention and consciousness; we then plan to pursue the task of developing a full-fledged cognitive theory on this basis in future work.

[35] To be sure, Kant's theory of cognition has already been of inspiration for empirical research; see, for instance, Palmer and Lynch (2010), Palmer (2008), Northoff (2012). His theory of space and time, however, has yet to be fully exploited in this sense.

3.10 Summary of the interpretation

We can now answer fully the second and third questions we posed regarding passage (1) in section 3.3.

Recall that Kant says there that time and space as the formal conditions of experience already contain relations of succession, simultaneity, relations of place, and so forth, and that these formal conditions are themselves representations. Most importantly, the form of intuition is described as "nothing other than the way in which the mind is affected by its own activity". Thus the *relata* of the form of intuition are just extended "events" of self-affection by the subject in describing spaces, which are spatiotemporally structured since they represent the synthesis of a certain space in a certain time. The mention of the way "in which the mind is affected by its own activity", then, clearly refers to the figurative synthesis in agreement with the categories, through which these pure events are generated and synthesized through apprehension and comprehension, so that a unitary representation of space and time arises. Hence, the form of intuition in the TA is nothing else than the formal intuition in the TD; still, there is a fundamental distinction, mentioned at B161n and A100, between a purely passive notion of form of intuition, that does not contain any determinate intuition at all but merely gives the manifold, and the conscious formal intuition produced by the figurative synthesis.

The figurative and intellectual syntheses are then two faces of the same coin, and the categories have a pre-discursive role as rules that constrain how the figurative synthesis proceeds in generating the formal intuition as the consciousness of the combination of the manifold generated through motion. The applicability of the categories to sensible manifolds as logical functions of judgments is thereby ensured, and so are the properties of time in the TA.

The exposition of the main points of our interpretation is now complete, with the exception of the issue of the degrees of formality, which is best addressed once the formal theory is in place.

Chapter 4

Philosophical foundations of the formal theory

4.1 Introduction

The purpose of this chapter is to outline a formal theory that clarifies the informal interpretation we proposed in the previous chapter, and which shall be treated from a more rigorous mathematical perspective in the following chapter; in particular, we discuss here how the formal theory bears on the interpretation of the footnote at B161n. The only prerequisite to understand this chapter is a basic acquaintance with first-order logic.

The chapter is structured as follows. In section 4.2 we answer the first question regarding Passage (1) of section 3.3, that is, what relations of succession of events of self affections one ought to consider; we also attempt to clarify further the notion of an event of self affection through motion. In section 4.3 we discuss the axioms of temporal order on events and, most importantly, we provide their justification in light of the interpretation presented in the previous chapter. In section 4.4 we provide an intuitive outline of how a construction of instants of time that is faithful to Kant's discussion of boundaries at Passage (6) in section 3.4 would go; a more rigorous treatment is provided in the next chapter from section 5.5 onwards. In section 5.10 we provide an intuitive outline of how Kant's notion of the potential or "modal" infinite divisibility of time can be modelled, and discuss more formally Kant's notion of infinity of time; a mathematically more rigorous treatment of these topics is provided in the following chapter from section 5.8 onwards. In section 4.6 we discuss the peculiar notion of "now" that emerges from our formalization of the Kantian continuum, and compare it with other attempts at formalizing the Aristotelian continuum. In section 4.7 we finally return to the problems that were discussed in section 3.1, and provide a solution to them in light of the formal theory.

4.2 A formal take on time-determination

4.2.1 Relations of succession

In the previous section we showed that the form of the intuition of time consists of "relations of succession" that hold among "pure events". These pure events, the *relata* of the form of intuition, are not point-like but extended; they are events representing the self-affection of inner and outer sense, generated by descriptions of trajectories in outer sense, and are therefore spatiotemporally structured. We must now address the question that was left unanswered in the previous section, namely, what sort of "relations of succession" among these events one ought to consider. Since Kant is never really explicit about which spatial or temporal relations are primitive, we must attempt to infer this from the consideration of his whole system.

Let us first examine those relations that are related to coexistence in time. A good candidate in this respect is the binary relation O with intuitive meaning: aOb if event a "temporally overlaps" with event b. Evidence for the salience of the overlap relation can be found in the third analogy at B257, where Kant glosses simultaneity in terms of temporal overlap. If I first observe A and then observe B and then observe A again I determine my representations of A and B in a relation of temporal overlap, so that they are cognized as being simultaneous in the sense that since they overlap there is a time in which they both exist, which is Kant's definition of simultaneity.

Along with relations encoding coexistence in time we must have relations encoding temporal order. In light of the work in philosophical logic on the construction of time from events (Russell, 1936; Thomason, 1984, 1989; Van Benthem, 2013) one might recur to the binary relation P of complete precedence, with intuitive meaning: aPb if event a ends before event b begins, i.e., a "completely precedes" b. The salience of this relation seems at first supported by Kant's emphasis on relations of "succession". However, this cannot be quite right in view of Kant's treatment of causality. Indeed, Kant holds that a cause does not have to completely precede its effect, but that, on the contrary, it can be simultaneous with its effect (A202/B248). Hence, complete precedence is too strong. We then take as primitive the binary relations R_{-}, R_{+}, with intuitive meaning: $a\,\mathrm{R}_{-}b$ if event a ends before or simultaneously with event b (a is in the past of b or b does not end before a), $a\,\mathrm{R}_{+}b$ if event a begins after event b or simultaneously with it (a is in the future of b or b does not begin after a). These relations are salient because the discussion at A202/B248 makes clear that to treat Kant's notion of causality formally one must be able to compare events with respect to their beginning and their end.

Finally, we define a binary relation $\preceq$ of "covering" as $a \preceq b$ iff $a\,\mathrm{R}_{-}b \wedge a\,\mathrm{R}_{+}b \wedge O(a,b)$. It is useful to think of the covering relation among events as a relation of temporal encompassment: given two events a, b of description of spaces, a covers

b if the temporal extent of a "encompasses" the temporal extent of b.[1]

4.2.2 The general form of an act of description

We now have salient temporal relations on events. A relation like $a\,\mathrm{R}_{+}b$, however, is already the representation of a conscious time-determination of inner sense, a relation of succession that is *objective* and not merely subjective. Indeed, recall that the apprehension of myself as the originally given object is an objective synthesis (see section 3.8.4), as it contains the essential moment of the *comprehension* of the manifold into a unitary representation, through which objective temporal relations are determined. Thus, $a\,\mathrm{R}_{+}b$ represents the consciousness of a succession, and not merely a succession of consciousness, i.e., it is already an objective *combination* that cannot be afforded by passive receptivity alone. Of course, there are in mere inner sense successions of representations; only, the conscious determination of a succession requires the action of the figurative synthesis in apprehending, comprehending and connecting according to rules the manifold given *a priori*, and is thus already spontaneity - formal intuition - and not mere receptivity.

We can gain more insight on this matter by examining what the general form of an act of self-affection in the description of a space is. Consider simplest case: a point that moves periodically between to locations in space, thereby describing a line segment. This act of description can be represented by the diagram in Figure 4.1.

The idea of the diagram is the following. The vertical axis represents the inner sense, the horizontal axis represents the outer sense. The portions of the curve labelled A to E represent acts of description of spaces in outer sense, through which certain spaces are described in certain times and the inner sense is thereby affected successively.

In particular, different acts are individuated by changes to the state of motion of the point, so that, e.g., A, C, E are states of approximate rest of the moving point, and B, D are states of approximate constant velocity. Of course, the word "approximate" is essential here, as the diagram in Figure 4.1 represents only a first approximation in the analysis of the act of description. The moving point is really only at rest at a moment of infinitesimal duration, and it undergoes a constant acceleration before reaching the constant velocity. However, this is a central point in Kant's analysis of motion, which is closely tied to his theory of the spatial and temporal continuum (see section 3.4): a motion is first given as a coarse whole, which can then be analyzed further by distinguishing finer and finer submotions identified by changes of state of the movable point determined more and more precisely, yielding a sequence of refinements. We shall provide a

[1]Nicod (Nicod, 2014) emphasizes this relation of "encompassment", which he takes as a primitive relation. Kant himself explicitly recognizes this relation as salient at (R4756, 17:701).

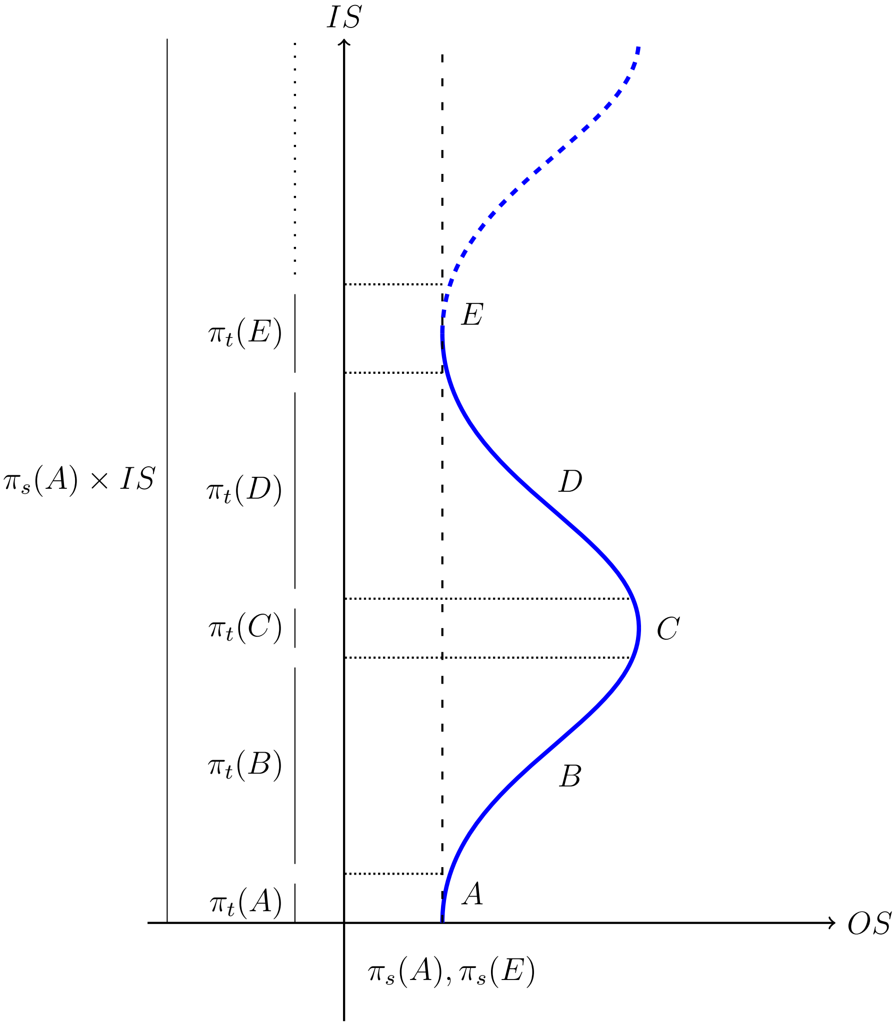


Figure 4.1: An act of description

way to formalize this process in section 4.5.

The dashed part of the curve indicates that the motion under consideration is periodic, as the point keeps moving back and forth between the two locations; this sort of motion is of importance as Kant states that the perception of simultaneity occurs whenever I become aware that I can apprehend the manifold in different orders "as many times as I want".

Now, for a given act of description, say A, we denote with $\pi_t(A)$ the time in which the space is described, which is really a representation with consciousness in inner sense of an object of outer sense, a space which we might denote with $\pi_s(A)$; everything which we encounter in our consciousness is a representation in inner sense of a manifold in outer sense.

Now, if the spaces I synthesize in outer sense did not *persist* through time, but disappeared as soon as the figurative synthesis moved to the synthesis of other spaces, then I would not be able to cognize, e.g., that $\pi_s(A)$ coexists in time with $\pi_s(C)$. For then $\pi_s(E)$ would be represented as a space wholly different than $\pi_s(A)$, a space synthesized anew, and I would be left with a series of synthesis of always different spaces, so that overlap, and hence simultaneity, could never be perceived. Moreover, according to Kant (R5348, 18:158), without the persistence of the spaces that I synthesize in outer sense I could not cognize the succession of the events of self-affection themselves, since to be conscious of a succession in inner sense one must be conscious that one has perceived the *same* thing at *different* times. Thus without persistence of spaces there is no cognition of identity through time, and I could not cognize even the difference between, say, $\pi_t(A)$ and $\pi_t(E)$, let alone that they stand in an objective succession. Thus, the representation of the persistence of the spaces synthesized in outer sense is necessary for the cognition of simultaneity and succession, so that the construction of time depends on space, as explained in section 3.8. In the diagram, the dashed vertical line and the vertical line labelled $\pi_s(A) \times IS$ denote the representation of the persistence of $\pi_s(A)$, against which the succession of, e.g., $\pi_t(A)$ and $\pi_t(E)$ is cognized. Still, it is only because I cognize simultaneity through the construction of geometrical spaces that I can consciously represent space as persistent; hence the representation of space itself depends on time. In other words, in the construction of geometrical spaces through motion I "simultaneously" institute space and time as intuitions.

The Diagram in Figure 4.1 also highlights the role that infinitesimals play with respect to the notion of the description of a space, since according to Kant a point is at rest at a location, say at $\pi_s(E)$, if it has infinitesimal velocity, i.e. a velocity that is smaller than any given velocity, and not null velocity. Thus, while discussing the motion of an object projected upward from a point A, and reversing direction of motion at point B under the influence of gravity, Kant states that:

> [...] the speed at point B is not completely diminished, but only to a degree that is smaller than any given speed. With this speed,

> therefore, the body would, if it were to be viewed always as still rising [...] uniformly traverse with a mere moment of speed (the resistance of gravity here being set aside) a space smaller that any given space in any given time no matter how large. (4:486)

Hence, the events of self-affection labelled $\pi_t(A), \pi_t(C), \pi_t(E)$ represent the point at rest, i.e., moving with infinitesimal velocity, and are to be conceived of as representing clusters of infinitesimals. We shall return on the formalization of these infinitesimals in section 5.11.

Note finally that the act of description, since it is produced by the figurative synthesis, must be constrained by the categories as rules for the objective determination of inner and outer sense. Thus the representation of the permanence of $\pi_s(A)$, the event $\pi_s(A) \times IS$, depends on the category of substance, while the representation of the objective succession of $\pi_t(A)$ and $\pi_t(E)$ depends on the category of cause, which determines the imagination so that the latter event succeeds the former. We shall have more to say on the role of the categories in the figurative synthesis in the following section.

4.3 An axiom system for Kant's intuition of time

4.3.1 Axioms as representations of the influence of the understanding

In the previous sections we have seen that the "pure events" that are the *relata* of the form of intuition in the TA are events of self affection in the description of spaces, and that the temporal relations holding among such events are objective and already represent time as formal. What is then the formal correlate of the merely passive form of the intuition of time appearing at b161n? In the present setting this is just a set of events without any temporal relation whatsoever, that is, so that the relations are interpreted as the empty set; this corresponds to the "merely given", purely passive manifold mentioned at passage (17). Now, through the action of the figurative synthesis in the description of a space that we outlined above this manifold originally provided by sensibility is structured in spatiotemporal relations. There must be certain axioms, however, ensuring that the relations produced by the figurative synthesis satisfy the properties of time in the TA; these axioms, as pure *a priori* rules for the combination of all possible appearances in space and time, must be grounded on the categories, or better on their schemata.

For any axiom on temporal relations, whatever its logical form, is already a *determination* of inner sense for any possible experience in relation to a possible understanding, although not necessarily our own. To represent the determination of inner sense in relation to *our* understanding, one must consider axioms which are grounded in the categories, so that this determination relates anything that might be apprehended in sensibility to the unity of apperception in a judgment. In particular, without axioms grounded in the categories the figurative synthesis would be unruly, determining the relations of time in a way that is unlike that described by Kant in the TA. Not even the simplest properties of time - transitivity and linearity, say - would be assured, for similar reasons as those presented in section 3.5.1. The axioms on sets of events then represent the spontaneity of the understanding that "drives" or "constrains" the figurative synthesis, which, as we have seen, must proceed "in agreement with the categories". The stronger the axiom system on sets of events, the more the understanding constrains the action of the figurative synthesis that determines our temporal form.

To capture these considerations formally we start with finite sets of events of self-affection which are ordered according to the temporal relations presented above. Every finite set of events represents a manifold that may be produced by the figurative synthesis through finitely many acts of description, such as acts A to E in Figure 4.1. We consider finite sets because, as we have seen in section 3.8.5, the infinity of time consists in the consciousness of the unboundedness of any act of apprehension; starting with actually infinite sets of events would run contrary to Kant's dictum that an actual infinity of distinct units cannot be intuited, and so cannot be a possible experience. Now, on these sets of events we shall impose axioms of temporal order that can be derived from the analysis in section 4.2.2, and justify them in terms of the categories.

4.3.2 The axioms of temporal order

We consider the following axioms of temporal order:

1. Explicit definition of $\preceq$ ("covering relation")
   (a) $a \preceq b \leftrightarrow a \, \mathrm{R}_{+} b, a \, \mathrm{R}_{-} b$
2. Reflexivity, symmetry of overlap
   (a) aOa
   (b) $aOb \rightarrow bOa$
3. Conditions for overlap
   (a) $cOb \wedge c \, \mathrm{R}_{+} a \wedge b \, \mathrm{R}_{-} a \rightarrow aOb$
4. Transitivity
   (a) $a \, \mathrm{R}_{+} b \wedge b \, \mathrm{R}_{+} c \rightarrow a \, \mathrm{R}_{+} c$
5. Conditional transitivity for O:

(a) $aOc \wedge cOb \wedge c\,\mathrm{R}_{+}b \wedge c\,\mathrm{R}_{+}a \to aOb$

6. Linearity

   (a) $b\,\mathrm{R}_{+}a \vee a\,\mathrm{R}_{+}b$

7. Covering axiom

   (a) $\exists c(a \preceq c \wedge b \preceq c)$

8. Substitution principle

   (a) Any sentence ϕ obtained from the above axioms by replacing R_{-} for R_{+} and R_{+} for R_{-}.

An *event structure* can now be defined as a tuple $\mathcal{W} = (W, \mathrm{R}_{+}, \mathrm{R}_{-}, O, \preceq)$ satisfying the axioms above; this definition shall soon be modified, however, by the addition of two partial operations. All the intuitive properties of O, R_{+}, R_{-} follow from the axioms above, as we shall see in section 5.3. As we said in the previous section the axioms must be given a Kantian justification in terms of the categories and of the figurative synthesis. In particular, the problematic axioms from a Kantian perspective are the transitivity axioms, the linearity axioms, the covering axiom and the substitution axiom.

4.3.3 The transitivity axioms

We already hinted in section 3.5.1 that transitivity is a strong principle in our context. In particular, let a, b, c be events of self affection in the description of a space. If $a\,\mathrm{R}_{-}b, b\,\mathrm{R}_{-}c$ then in the fleeting present during which the figurative synthesis produces event b event a is reproduced, and in the present during which event c is produced event b is reproduced. In order to be able to conclude that $a\,\mathrm{R}_{-}c$, the two acts of reproduction must be "composed" so as to be able to reproduce event a when event c is produced. Thus, for transitivity to hold such acts of reproduction must *necessarily* be able to be composed, since temporal relations between two events can only be established when they are both present before me in one consciousness. The necessity of the possibility of composing such reproductions, however, relies on an objective ground that guarantees it and that extends to all appearances. This objective ground is the category of cause, whose influence on the sensibility determines the objective successions expressed via the relations R_{-}, R_{+} in the first place; since causality is itself transitive, the latter temporal relations are also transitive.

4.3.4 The linearity axioms

In section 3.5.1 we saw that the merely empirical synthesis of apprehension does not ensure that our temporal experience satisfies the linearity axioms. To see how this is achieved in the figurative synthesis, consider a set E of events. In general, the events in E represent acts of self-affection in the description of different spaces, such as lines or circles, which might be constructed "simultaneously"; there is then no guarantee that they arise from the description of a single geometrical space, as in Figure 4.1. Thus, if we represent descriptions of distinct geometrical spaces as subsets of E we obtain a family $\alpha_0, \alpha_1, \ldots$ of subsets whose union is E. Each set α_i contains events that pertain to the description of the same spatial manifold in outer sense, e.g. a line or a circle. We also assume that any event belongs to exactly one subset, so that the family of subsets is really a partition of E. This is a simplification, since events can in principle belong to various acts of description; an event corresponding to the synthesis of part of the side of a triangle, for instance, belongs to the act of description of the side itself and of the whole triangle. Still, this simplification is harmless for the purpose at hand, and we shall assume it. If we now index every event according to the set it belongs to we obtain pairs of the form (α_i, e) with $\alpha_i \subseteq E$, $e \in \alpha_i$.

Now, it is clear that when restricted to any given α_i the linearity axioms hold, because if $e, e' \in \alpha_i$ then $\pi_s(e), \pi_s(e')$ are two spaces that are in each other's vicinity and are produced in the same act of description, and hence the events of self-affection are causally related. A similar situation holds in the empirical case: a set of events representing attributes of the same empirical appearance satisfies the linearity axioms because the beginning and end of these events can be simultaneously compared due to their relative closeness.[2] However, this does not ensure that the linearity axioms hold in general for events that belong to different acts of description that may be very far apart in space and time. If the cognizing subject produces two line segments in different directions starting at different places it is not obvious that events of self-affection belonging to the two line segments must satisfy the excluded middle; they could just be encoded as temporally incomparable, and as lying in different time-lines.

The justification for the unrestricted use of the linearity axioms in the axiom system above comes is then grounded on the restriction imposed by the category of community on the figurative synthesis, to the effect that all substances must be in "thoroughgoing interaction" and causal determination. This means, in turn, that between any two events pertaining to different acts of description, or, in

[2]Isaac Barrow seems to make a similar point when he states that "time, abstractly speaking, is the continuance of each thing in its own being" (Barrow (1976), *Lectio* I.), since Arthur (1995) observes that "since some things continue to exist longer than others, these times are durations with respect to the beings in question, and thus are relative measures"; hence, the order of succession of events is indexed by the substances or beings they are modifications or attributes of.

the empirical case, between any two attributes pertaining to different substances, there must be causal influence - a sort of action at distance which ensures that events are always comparable with respect to their beginning and end, so that they belong to a single time-line. The one-dimensionality of time then requires the action of the category of community in the form of the linearity axioms.

4.3.5 The covering axiom

The justification of the covering axiom introduces a further important theme, the different philosophical status of universal axioms and existential axioms. The logical form of the axioms considered above was universal; thus, these axioms restrict the class of possible temporal forms of experiences by constraining the possible relations of temporal order among events. The covering axiom, instead, *posits* additional events which the understanding produces of its own and *a priori*. In the empirical case, this means that events which need not have been perceived are produced by the understanding itself. In the *a priori* case of self-affection, which grounds the empirical case, this means that events which need not have been actually described by the subject are produced by the understanding itself. We shall term events that are introduced by existential axioms *transcendental*, as they are postulated by the understanding *a priori*.

The justification for the covering axiom stems from the category of substance as the "persistence of the real in time", since as we saw in section 4.2.2 the representation of the persistence of the spaces synthesized in outer sense depends on the influence of the category of substance on the sensibility, which alone enables the intuition of something as persisting in the first place.

Note that the covering axiom turns the preorder $\preceq$ into a directed preorder.[3] In the case of a finite set of events E, iterated application of the covering axiom implies the existence of an event w with the property that $a \preceq w$ for any $a \in E$. We term w a *universal cover*, which can be thought of as a representation at the level of a single set of events of the infinite, unbounded time of which all times are part, thus capturing Kant's dictum that "different times are only part of one and the same time" (A31-2/B47). Moreover, it was argued in Achourioti and van Lambalgen (2011), and it will be of importance in section 4.5, that directedness is closely related to the synthesis of the unity of apperception, as it implies that any two representations are related as parts of an encompassing whole representation. As we saw in section 4.2.2, that any two events e, e' must be part of a larger encompassing event e'' is a precondition for e, e' to be in an objective temporal relation. At the same time, the covering axiom ensures that the instances of consciousness accompanying event e and that accompanying event e' are related to each other, and so it is a precondition for the thought of their numerical identity. In the presence of a universal cover w the numerical identity

[3]Recall that a preorder $(P, \leq)$ is directed if for any $x, y \in P$ there exists $z \in P$ with $x, y \leq z$.

of all the I's accompanying the events, which is ensured by their being parts of w, implies the existence of a unified spatio-temporal world, in which all events can be temporally related.

4.3.6 The substitution axiom and the arrow of time

The substitution axiom states a duality between R_-, R_+ that is itself the expression of a perfect symmetry between the "past" and the "future". In particular, given an event structure $\mathcal{W}$, one can obtain a "dual" event structure $op(\mathcal{W})$ by letting $op(W) = W$ and letting $\mathrm{R}_+^{op(\mathcal{W})} = \mathrm{R}_-^{\mathcal{W}}$ (similarly for $\mathbb{R}_-$), $\mathrm{R}_-^{op(\mathcal{W})} = \mathrm{R}_+^{\mathcal{W}}$ (similarly for $\mathbb{R}_+$), and $O^{op(\mathcal{W})} = O^{\mathcal{W}}$ (similarly for $\mathbb{O}$). The dualization operation $op(\cdot)$, which as we shall see is actually an endofunctor in the category of event structures and event maps, effectively exchanges the orientation of the past and the future; therefore, our axioms do not provide a way to choose one orientation over the other. From a Kantian perspective, however, this is not as problematic as it may seem, since the orientation of the timeline must depend on the subsumption of events under concepts, so that a cause and effect relation between *types* of events also determines what can precede a given even and what cannot. In other words, from a Kantian perspective the direction of time is determined causally - that is, on the basis of causal laws. The formulation of a formal setting in which all this can be analyzed would require the combination of the formal theory of Kant's concept of time presented in this work with the formalization of Kant's transcendental logic developed in Achourioti and van Lambalgen (2011), but this is still ongoing work, and hence it will not feature in this thesis (see chapter 7).

4.3.7 The temporal operations, their axioms and justification

The axioms in section 4.3.2 concern the relations of temporal order that we introduced in section 4.2. We shall now enrich the language by introducing two partial binary operations $\oplus, \ominus$ on events. That $\oplus, \ominus$ are partial binary operations just means that they are ternary relations $\oplus, \ominus \subseteq W^3$ on the set W of events satisfying the constraint of functionality below. We write $a \oplus b = c$ rather than $\oplus(a, b, c)$ and denote with $a \oplus b$ the unique c such that $a \oplus b = c$ if it exists; similarly for $\ominus$. Before we comment on the meaning of these operations we give their axioms:

9. Partial binary operations $\oplus, \ominus$ on events.

   (a) $a \oplus b = y \wedge a \oplus b = z \to y = z$ (functionality)

   (b) $a\,\mathrm{R}_+ b \vee aOb \leftrightarrow \exists(a \oplus b)$ (explicit domain of definition)

   (c) $\exists(a \oplus b) \to a \oplus b\,\mathrm{R}_+ b$

(d) $\exists(a \oplus b) \rightarrow a \oplus b\, \mathrm{R}_{+} a$

(e) $\exists(a \oplus b) \rightarrow a\, \mathrm{R}_{-} a \oplus b \wedge a \oplus b\, \mathrm{R}_{-} a$

(f) $\exists(a \oplus b) \wedge a\, \mathrm{R}_{+} b \rightarrow a\, \mathrm{R}_{+} a \oplus b$

(g) $\exists(a \oplus b) \wedge b\, \mathrm{R}_{+} a \rightarrow b\, \mathrm{R}_{+} a \oplus b$

(h) $a \oplus a = a$

(i) $(a \oplus b) \oplus c = (a \oplus c) \oplus b$

(j) $(a \oplus b) \oplus b = a \oplus b$

(k) $(a \oplus b) \ominus c = (a \ominus c) \oplus b$

Where $\exists P$ is shorthand for $\exists y(P = y)$ for P an atomic formula in which only $\oplus, \ominus$ occur.

On the basis of Kantian philosophy two interpretations can be given of the operations, one in terms of the synthesis of apprehension and one in terms of the category of causality.

In relation to the synthesis of apprehension we interpret the events introduced by $\oplus, \ominus$ as "potential" binding of other events: given a collection $A = \{a_0, \ldots, a_n\}$ of events and an event e in the range of $(\oplus, \ominus)$, e binds A if $A = \{a \in E \mid a \preceq e\}$; i.e., e "encompasses" all and only the events in A, and represents their potential binding into a unity. In the *a priori* case of the description of a line these transcendental events represent actual bindings, since in this case all events are produced *a priori* by the subject and enjoy only the property of spatio-temporal extension; hence, there are here no constraints on binding sets of events into unities. In the empirical case, in which events may be tenure events of *a posteriori* qualities by means of which they fall under empirical concepts, there are more constraints on what counts as an objective unity; thus, only some of the potential bindings of events are realized - what Kant calls "the comprehension of the manifold given in accordance with the form of sensibility in an intuitive representation".

The second interpretation of the operations relies on the category of causality. We have seen in section 4.3.4 that the category of community is essential to ensure that the linearity axioms hold, and with them the linearity of the determinations of inner sense produced by the figurative synthesis - the linearity of time. We can then justify the operations $\oplus, \ominus$ in terms of the action of the category of causality as follows: given two events a, b we interpret $a \oplus b$ as "the part of a that can be causally influenced by b", and $a \ominus b$ as "the part of a that can causally influence b". The existence of the events in the range of $\oplus, \ominus$ is then justified in terms of the transcendental action of the category of causality.

4.4 Instants of time

We now wish to provide a construction of boundaries in terms of events that is faithful to the textual evidence, in particular to Passage (6) of section 3.4. The fundamental idea of the construction is that there are two sorts of instants in the Kantian continuum. The first sort consists of time instants, which we term "boundaries", are constructed as "limitations" between events, in agreement with Passage (6). The second sort, which we term "infinitesimal intervals", consists of maximal overlapping classes of events that arise in the "clefts" between boundaries of the first sort; under certain assumptions on the $\preceq$ ordering they are generated by events that are minimal in this ordering, which we call $\preceq$-minimal events. The interpretation of the latter sort of boundaries as "infinitesimal" is justified by the fact that they exhibit a close relation to infinitesimal quantities, as shown in section 5.11. In Figure 4.1, for instance, the points of the curve at which the movable point is at rest will be represented by infinitesimal intervals. What is more, one can understand infinitesimal intervals as representing the fleeting time that lies "in-between" two adjacent boundaries, in agreement with Brouwer's quote (4) in section 3.4, which closely describes the core of the continuist conception of the continuum.

Since our focus here is on how our formalization explains the distinction between the form and the formal intuition we shall not provide an in-depth discussion of the technical details of the construction of instants from events, and shall content ourselves with an intuitive elucidation of the construction. The reader interested in the details can consult chapter 5 starting from section 5.5.

First, note that the axioms of temporal order allow us to represent a given (finite) event structure $\mathcal{W}$ by laying off its events along a timeline, along the lines of Thomason (Thomason, 1984, 1989); see Figure 4.2, where events are labelled with letters from a to f. Of course, only the relative position of the endpoints of the events is relevant, while their length is not, as the language of the theory of event structures does not have metric primitives.[4] Note that f a universal cover, whose existence follows from the covering axiom, and that for the sake of clarity we have not drawn the events in the range of the operations $\oplus, \ominus$. Hence, Figure 4.2 displays only what we might term the set of *generators* of the event structure that is obtained by closing under $\oplus, \ominus$, but this is irrelevant for the purpose of our illustration here.

In Figure 4.2 boundaries and infinitesimal intervals supervene on the events and are represented by labelled line segments perpendicular to the events themselves. In particular, $-\infty, x, y, +\infty$ are boundaries, since they can be conceived as "limitations" or "separations", in agreement with passage (6) of section 3.4. These boundaries are defined as tuples (P, C, F) where $P, C, F \subseteq W$ represent

[4]This marks a substantial difference with other approaches to the formalization of the Aristotelian continuum, in particular Hellman and Shapiro (2013); but see section 4.6.

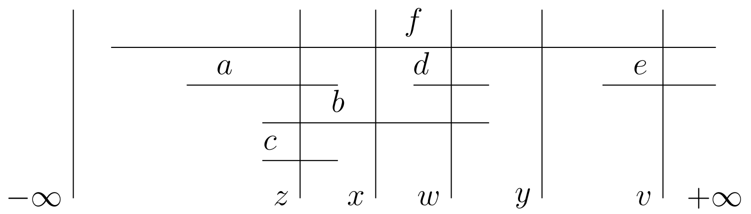


Figure 4.2: An event structure and its boundaries

the past, the present ("current") and the future of the boundary respectively. The past of a boundary is given by all events lying strictly to the left of the vertical line, its future by all events lying strictly to the right of the vertical line, and its present by all events intersecting the vertical line. Boundary x, for instance, is defined as the triple $(\{a, c\}, \{b, f\}, \{d, e\})$. Note that for any boundary, any event in its present overlaps with an event its past and an event in its future, if these are not empty, and moreover any two events in the present overlap. This neatly captures Aristotle's intuition of an instant as "the link of time" that connects past and future; see passage (2) in section 3.4.

We term x, y in particular "two-sided boundaries", since they comply with Aristotle's requirement on instants expressed in passage (5) of section 3.4: both their past and future are not empty. Boundaries $-\infty, +\infty$ have a special interpretation, since they are purely "formal" boundaries that provide a correlate within a single event structure to the infinity or "unboundedness" of time: they are instants that lie in the infinite past or infinite future.

The vertical lines labelled with letters z, w, v do not indicate static boundaries but "infinitesimal displacements", the fleeting times lying between two adjacent instants mentioned by Brouwer. Under suitable conditions on the $\preceq$ ordering an "infinitesimal displacement" or "infinitesimal interval" is defined as a class of events which is maximal with respect to the property that any two events in the class overlap, i.e., a maximal class of pairwise overlapping events, which is generated by the upset with respect to $\preceq$ of a $\preceq$-minimal event; see section 5.6 and section 5.7.

One of the main results of chapter 5 is that the set of boundaries and infinitesimal intervals on an event structure can be naturally endowed with a total order and a topology that turn it into a connected totally ordered topological space satisfying various nice properties enjoyed by the unit interval $[0, 1]$. Indeed, the reader will undoubtedly have noticed that in Figure 4.2 boundaries and infinitesimal intervals can be totally ordered. Given an event structure $\mathcal{W}$, we shall denote this totally ordered topological space with $K(\mathcal{W})$; the description of its construction is given in section 5.6. Of course, the intuitive descriptions given above must be made mathematically precise in order to study the properties of $K(\mathcal{W})$. In particular, the formal definition of boundaries as triples (P, C, F) is a variation on Walker's construction of instants from events (Walker, 1947), while infinitesimal intervals are closely related to Russell's construction of instants from events (Russell, 1936). Thus, in order to capture Kantian temporal continua we use both a Walker-type and a Russell-type construction of instants. This is of interest because there is a long-standing debate in the literature regarding the relation between Walker's and Russell's constructions and their relative merits, but we provide a general construction of boundaries from event structures that subsumes both and shows them to be essentially complementary; this construction is provided starting from section 5.5, and is related to Russell's and Walker's original constructions in chapter 6. Our approach sheds also new light on the relation between Russell's construction of instants and point-free topology (Johnstone, 1983), which has been recently investigate in Mormann (2009). In particular, we exhibit Walker and Russell instants as neighborhood filters of respectively closed and open points belonging to connected well-formed topological spaces; these are studied in the context of *digital topology* (Kong & Rosenfeld, 1989). I am also currently investigating this connection to digital topology in relation to the foundations of relativity; but see chapter 6 for more detail on such and related matters.

This informal elucidation will suffice for now. We turn to providing a formal analysis of Kant's notion of infinity, and with it of the distinction between the form and the formal intuition of time.

4.5 The formal intuition of time as an inverse limit

In order to provide a formal analysis of Kant's concept of formal intuition and the infinity of time, as we have seen, we are mostly concerned with the class of finite event structures. Finite event structures represent the possible "temporal forms" of the experiences of a being that can only process and store finite amounts of information - as a Kantian subject must. Any finite event structure can be conceived as the representation of the temporal form, or formal temporal content, of a possible experience and is produced *a priori* by the figurative synthesis. Thus, the axioms constrain the class of possible experiences, since they constrain the temporal forms that the figurative synthesis can produce.

We saw in section 3.8, however, that the action of the figurative synthesis in the description of a space comprises the consciousness of the fact that the source of this action lies in the subject and not in the object. We argued that this consciousness grounds Kant's insistence on a modal reading of infinity, infinite divisibility, and the refinement of boundaries. The action of the figurative synthesis, since it is accompanied by the consciousness that all possible manifolds produced by it *can* always be divided, extended, and unified as parts of a whole, yields time as the formal intuition *at once*, since for space and time everything that is possible is also actual.

We can model the relation between the figurative synthesis and time as the formal intuition by considering maps between event structures. Let $\mathcal{W}, \mathcal{W}'$ be event structures; a function $f : \mathcal{W} \to \mathcal{W}'$ is a *map* if it preserves the signature, that is, the relations and partial operations R_+, R_-, $O; \oplus, \ominus$. A map $f : \mathcal{W} \to \mathcal{W}'$ is a *retraction map* (or simply a *retraction*) if $\mathcal{W}'$ is a submodel of $\mathcal{W}$ and $f(a) = a$ for any $a \in W$.[5].

We can use retraction maps between finite event structures to model the Kantian potential, or modal, infinity and infinite divisibility of time. Figure 4.3 illustrates how this is done.

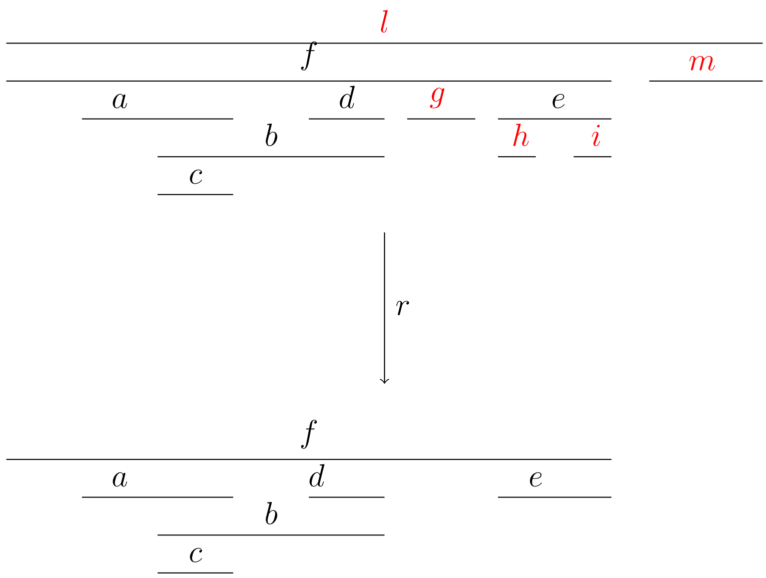


Figure 4.3: A retraction between two event structures

Figure 4.3 displays again only the set of *generators* $\mathcal{W}_g, \mathcal{W}'_g$ of two event structures $\mathcal{W}, \mathcal{W}'$, where $\mathcal{W}'$ is a submodel of $\mathcal{W}$ and $W'_g \subseteq \mathcal{W}_g$. A retraction $r : \mathcal{W} \to \mathcal{W}'$ is represented by the downward arrow in the figure, and can be explicitly constructed as follows. Define a map $r_0 : \mathcal{W}_g \to \mathcal{W}'$ by letting, for all $a \in \mathcal{W}_g$ such that $a \in \mathcal{W}'_g$, $r_0(a) = a$; let moreover $r_0(m) = r_0(h) = r_0(i) = e, r_0(l) = f, r_0(g) = f \oplus d$. Then from r_0 a retraction map $r : \mathcal{W} \to \mathcal{W}'$ can be defined recursively according to the following:

$$\begin{aligned} &r(g) = r_0(g) && \text{for all } g \in \mathcal{W}_g \\ &r(a \oplus b) = r(a) \oplus r(b) \\ &r(a \ominus b) = r(a) \ominus r(b) \end{aligned}$$

It is a straightforward matter to check that the map r is well-defined and that it is a retraction. Philosophically, the map r encodes that the figurative

[5]Hence, f is surjective, i.e. for any $y \in Y$ there is $x \in X$ with $f(x) = y$.

synthesis *can* always refine what is given at a fixed stage of approximation. For instance, event g in $\mathcal{W}$, which is interpolated between d and e, implies that the instant between d and e (see section 4.4) can be extended, that is, that a series in the instant "can be indicated". The relation between $\mathcal{W}$ and $\mathcal{W}'$ is then of a logical, and not temporal, nature; its ground is the figurative synthesis, which is constrained by the categories and the unity of apperception, and which can extend and refine the event structure *ad infinitum*.

Abstracting from this concrete example, it should be the case that for any two finite event structures $\mathcal{W}, \mathcal{W}'$ there exists an event structure $\mathcal{W}''$ which retracts to both $\mathcal{W}$ and $\mathcal{W}'$. If this were not the case, then the figurative synthesis could produce two wholly incompatible *a priori* temporal experiences, which could not be unified as parts of the same whole. As we have seen before, this would in turn imply that the identity of the "I" accompanying the two distinct temporal experiences could not be thought, since the temporal experiences could not be "taken up" into the same consciousness, and we would have a violation of the principle of the unity of apperception.

Fortunately the example above generalizes nicely. Indeed, one can prove (see section 5.9) that the class of finite event structures forms an *inverse system*, or a directed diagram, under retraction maps. This means that for any two finite event structures, there is an event structure that retracts to both, in such a way that this family of retractions satisfies a condition of global consistency. The situation can be pictured with a diagram of the following form:

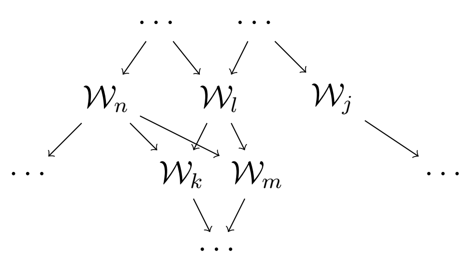


Figure 4.4: Inverse system of finite event structures

The arrows in Figure 4.4 represent retraction maps. Furthermore, if we indicate with $r_{\mathcal{W}\mathcal{W}'}$ the retraction map from $\mathcal{W}$ to $\mathcal{W}'$, the following global consistency condition holds: for any $\mathcal{W}'', \mathcal{W}', \mathcal{W}$ such that $\mathcal{W}''$ retracts to $\mathcal{W}'$ and $\mathcal{W}'$ retracts to $\mathcal{W}$, then $\mathcal{W}''$ retracts to $\mathcal{W}$, and $r_{\mathcal{W}'\mathcal{W}''} \circ r_{\mathcal{W}\mathcal{W}'} = r_{\mathcal{W}\mathcal{W}''}$.

The diagram represents the action of the figurative synthesis, along with its potential nature, and its relation to the unity of apperception, embodied in the directedness constraint.[6] Any two temporal experiences that the figurative synthesis can produce *a priori* must be able to be unified as parts of a temporal

[6] The relation between the unity of apperception and directedness of an inverse system was originally pointed out in Achourioti and van Lambalgen (2011).

experience subsuming both, and must thereby be able to be related to the stable "I" or apperception. The infinite divisibility of parts of time, which *can* be divided; or of instants, in which a series *can* be indicated; or the infinity of time, are all reduced to the action of the figurative synthesis and its relation to the unity of apperception.

We are now finally able to provide a formal correlate to time as the formal intuition, or time "as an object". In section 3.8 we said that time as the formal intuition is nothing else than the consciousness of the necessary form of our self-affection through the figurative synthesis. However, the form of our self-affection is represented mathematically by the inverse system of finite event structures pictured in Figure 4.4. Time as the formal intuition, then, must be a structure that somehow encodes the information of this diagram.

The structure in question is the *inverse limit* of the inverse system of finite event structures, which we denote with $\mathcal{V}$. Abstractly, $\mathcal{V}$ can be defined as the event structure, up to isomorphism, satisfying the following properties:

1. There exists a family of retractions $\{r_{\mathcal{V}\mathcal{W}} : \mathcal{V} \to \mathcal{W} \mid \mathcal{W} \text{ finite}\}$, from $\mathcal{V}$ to every finite event structure $\mathcal{W}$, such that $r_{\mathcal{V}\mathcal{W}} \circ r_{\mathcal{W}\mathcal{W}'} = r_{\mathcal{V}\mathcal{W}'}$ for any finite $\mathcal{W}, \mathcal{W}'$;

2. for any event structure $\mathcal{V}'$ satisfying the above property there exists a unique map $u : \mathcal{V}' \to \mathcal{V}$ such that $r_{\mathcal{V}'\mathcal{W}} = r_{\mathcal{V}\mathcal{W}} \circ u$ for any $\mathcal{W}$

The fundamental intuition underlying the definition above, which might seem obscure but is merely a particular instance of a well-known construction,[7] is that $\mathcal{V}$ is the "limit structure" of the structures in the diagram of Figure 4.4. The first condition requires that the limit retracts to any structure in the diagram in a consistent manner, while the second condition ensures that the limit is the "smallest possible structure" that satisfies this property. It is possible to provide a concrete construction of $\mathcal{V}$, but since it is a technical matter we shall not do so here; the reader can find the details in chapter 5.

The inverse limit $\mathcal{V}$ is, in our setting, the formal correlate of time as the formal intuition. It represents the intuition that is produced in the consciousness of the synthetic activity of the figurative synthesis as constrained by the categories and the unity of apperception, which is represented by the diagram in Figure 4.4, and which reveals to the subject the necessary form of any possible temporal experience.

Since $\mathcal{V}$ is an event structure, one might wonder whether the space of boundaries $K(\mathcal{V})$ on the inverse limit can be given a Kantian interpretation. This question can be answered affirmatively by showing that $K(\mathcal{V})$ is such that the

[7] The reader acquainted with mathematical logic will note that our definition is merely a particular instance of the general definition of the limit of a directed diagram in category theory.

real line arises as a quotient of it.[8] $K(\mathcal{V})$ can accordingly be interpreted as the "outward representation" of time through motion that Kant mentions in the CPR; but more on this will be said in chapter 5.

4.5.1 The infinity of time and the transcendental justification of the axioms

We argued in chapter 3 that time as the form of intuition occurring in the TA is nothing more than time as the formal intuition of the footnote at B161. It then follows that if $\mathcal{V}$ is the formal correlate of time as the formal intuition, it should satisfy formal correlates of all those properties that Kant ascribes to time in the TA, in particular infinity and unicity. We focus here on the property of infinity, since the property of unicity will be treated in section 4.7.

As far as infinity is concerned, then, recall that in the TA Kant provides a mereological definition of the infinity of time: the intuition of time is infinite since every determined magnitude of time is possible only as a part of it (see section 3.8.5). Within our formal setting this can be modelled in two ways, which capture different aspects of this notion of infinity.

The first way consists in the fact that since $\mathcal{V}$ retracts to any finite event structure then there exists an embedding of any finite event structure into $\mathcal{V}$, that is, there is an injective map from any finite event structure into $\mathcal{V}$. Hence, any finite event structure is in this sense a "part" of $\mathcal{V}$, which is then mereologically infinite in the Kantian sense.

It is important to note that this does not imply that time as a whole is made up by adding together parts of time, i.e., by adding together finite event structures, since this would run contrary to Kant's dictum that time can be composed out of its parts. Instead, it is the axioms that we imposed on event structures that ensure, as a matter of logical necessity, the existence of $\mathcal{V}$. The existence of an inverse limit relies on the fact that the class of finite event structures is directed under retraction maps, and this in turn relies on the above axioms, which can be justified only by appealing to the categories. In the absence of any of those axioms we would not have the directedness property that is crucial to prove the existence of $\mathcal{V}$, and it would not be possible to model the process pictured in Figure 4.3.

These observations also provide us with a *transcendental justification* of our axioms, in the form of the following transcendental argument. First, a necessary condition for the unity of consciousness is that any two temporal experiences must be able to be unified as parts of the same temporal whole, since only in this

[8] This property can actually be strengthened, since one can show that any compact connected separable total order is a quotient of $K(\mathcal{V})$, and more generally that any compact separable total order arises as the subspace of closed points of a quotient of $K(\mathcal{V})$. If compact separable total orders are conceived as orders of instants of time, this result strengthens the canonicity of $\mathcal{V}$, since any order of instants of time can be generated from it.

way the identity of the consciousness accompanying each of them can be thought; this is represented formally by the directedness condition. Second, the axioms presented above are necessary for the class of finite event structures to satisfy the directedness condition. It follows that the axioms presented above are necessary for the unity of consciousness in experience; indeed, their justification relies on the action of the categories as concepts bringing any possible manifold to the unity of apperception.

The second way in which the infinity of time can be modelled relies on the inverse system of Figure 4.4. Let $\mathcal{W}, \mathcal{W}'$ be finite event structures such that $\mathcal{W}'$ is a submodel of $\mathcal{W}$. We say that $\mathcal{W}'$ is *future bounded in* $\mathcal{W}$ if there exists $a \in \mathcal{W}$ such that $b\,\mathrm{R}_{-}a \wedge a \mathbb{O} b$ for any $b \in \mathcal{W}'$. The notion of *past bounded in* $\mathcal{W}$ is defined dually, and if $\mathcal{W}'$ is both past and future bounded in $\mathcal{W}$ we simply say it is *bounded*. It is now easy to see from Figure 4.3 that for any finite event structure $\mathcal{W}'$ there exists an event structure $\mathcal{W}$ retracting to $\mathcal{W}'$ such that $\mathcal{W}'$ is bounded in $\mathcal{W}$. Intuitively, this means that any event structure can be extended by adding events to the left and the right of its universal covers, thereby introducing a larger universal cover, as pictured in Figure 4.3. Thus, the inverse system captures formally the dynamic notion of the potential infinity of time; this was informally analyzed, in very similar terms but for the case of space, in M. Friedman (2012). In section 5.10, we shall see that this sort of infinity of time can be actually modelled by a directed system of embeddings that is induced by the inverse system.

4.6 The structure of the present

In this section we provide some brief remarks comparing our formalization of Kantian instants, presented in section 4.4, to James' and Aristotle's conceptions of the present, so as to clarify further its peculiarities. The reader should recall the discussion in section 3.4, since what follows relates to it closely.

In the *Principles of psychology* (James, 2013), William James discusses the notion of "now" in the following terms:

> [T]he practically cognized present [i.e. the specious present] is no knife-edge, but a saddle-back, with a certain breadth of its own on which we sit perched, and from which we look in two directions of time. The unit of composition of our perception of time is a *duration*, with a bow and a stern, as it were–a rearward- and a forward-looking end. It is only as parts of this *duration-block* that the relation of *succession* of one end to the other is perceived. We do not first feel one end and then feel the other after it, and from the perception of the succession infer an interval of time between, but we seem to feel the interval of time as a whole, with its two ends embedded in it. The experience is from the outset a synthetic datum, not a simple one; and to sensible perception its elements are inseparable,

> although attention looking back may easily decompose the experience, and distinguish its beginning from its end. (James, 2013, p. 574-5)

The doctrine of the *specious* or extended present championed by James in this passage is in agreement with the formalization of Kant's notion of instant that was presented in section 4.4. In particular, our formalization is in agreement with the *extensionalist* or *retentionalist* model of the specious present (Arstila, 2016), which holds that while the contents of a temporal experience are extended durations, the temporal experiences themselves objectively occur during near-momentary instants or "snapshots". Indeed, even though an instant is a class of pairwise overlapping extended events, the consciousness accompanying this class only lasts a moment or unity, which is why it is possible to represent instants in a linear order. Evidence of the relevance of the specious present for contemporary neuroscience can be found in Varela (1999), Pockett (2003).

It is also noteworthy that James' extended present, while given phenomenologically as a synthetic whole, can be further decomposed into parts whenever the attentional focus is directed back to it. This interpretation provides us with a cognitive justification for the continuous maps between event structures which we introduced in section 4.5. Indeed, the axiom of excluded middle forces not only the past, but also the future to be determined, in the sense that events are in the future with respect to a temporal boundary, and not with respect to the agent. Thus, within a given event structure an event can be in the future of an instant despite it being already past from the perspective of the subject's "now". The perspectival *coming to be* of the future can instead be represented by an inverse system of event structures connected by maps, where new instants that are maximal in the linear order are introduced by splitting or dividing events, in particular the universal cover as pictured in Figure 4.3, and thus "come to be". Similarly, events can be divided in order to introduce new instants which are not maximal in the linear order of boundaries, which can be interpreted as arising from a process of analysis of past events by means of attention, along the lines outlined by James.

As we already remarked in section 3.4, the Aristotelian theory of time is closely related to Kant's, in particular with respect to the notion of the present. We do not have here the space to provide an in depth comparison of the two theories, but will only sketch some salient points, and we shall compare our theory with some recent attempts at formalizing the Aristotelian continuum (Roeper, 2006; Hellman & Shapiro, 2013).

The most fundamental understanding of continuity in Aristotle is that of a binary relation which holds between substances if these are such that they constitute a "unity":

> The continuous [τό δὲ συνεχὴς] is a subdivision of the contiguous

> [ἐχόμενος]:[9] things are continuous when the touching limits of each become one and the same and are, as the word implies, contained in each other: continuity is impossible if these extremities are two. This definition makes it plain that continuity belongs to things that naturally in virtue of their mutual contact form a unity [εἷς]. (227a 10-15)

Thus, two substances are continuous if, when they are brought together, their respective boundaries "fuse" into each other and are ultimately absorbed into the whole; imagine, for instance, two bodies of water, which are continuous, versus two coins, which when brought together are merely contiguous.[10]

A substance is then said to be continuous if any partition of the substance gives rise to two continuous substances by "actualizing" the boundary separating them, a boundary which is first only potential; contrary to merely contiguous substances, which can be divided without the creation of anything new, the division of a continuous substance brings forth the "shared" boundary as a new entity. Thus a continuous magnitude enjoys a certain "viscosity" or "unity", which makes it impossible for it to arise by adding together indivisible points, since these cannot be continuous because they can be in contact only "as whole with whole".[11] The temporal continuum then, as we have already seen, cannot be made up from instants, or "nows". The "now" in time is but the division or the "link" of time, that which connects past and future (see passage (2)). This basic description of the properties of "now" fits quite well with the formal treatment above, since the boundaries are indeed links between the past and the future: any event in the present of "two-sided" boundaries, which in any case are the only boundaries that Aristotle admits (see passage (5)), overlaps with an event in the past and an event in the future. We depart from Aristotle, however, in taking boundaries to be extended, and therefore divisible; that this was Kant's own stance on the matter, in agreement with James, can be inferred by passage (7).

Aristotle's notion of boundary gives his continuum a weak form of indecomposability: splitting any continuum into two parts will make actual the division between them, thus creating a new entity, the boundary itself. Hellman and Shapiro (Hellman & Shapiro, 2013) note that both classical (Dedekind-Cantor)

[9]For Aristotle, two things are contiguous if their boundaries are in contact, i.e., when their extremities "are together"; see the *Physics*, (226b21).

[10]St Thomas Aquinas' commentary on Aristotle's Physics elaborates on the same point:

> Ex hoc autem ulterius concludit, quod continuatio esse non potest, nisi in illis ex quibus natum est unum fieri secundum contactum. Ex eadem enim ratione aliquod totum est secundum se unum et continuum, ex qua ex multis fit unum continuum, vel per aliquam conclavationem, vel per aliquam incollationem, vel per quemcumque modum contingendi, ita quod fiat unus terminus utriusque [...]. (Aquinas' commentary on Aristotle's physics, Bk. V, *Lectio* 5)

[11]See the illuminating introduction in Hellman and Shapiro (2013).

and intuitionistic theories of the continuum satisfy stronger forms of indecomposability. Moreover, they also note that their theory does not capture Aristotle's peculiar notion of indecomposability, as they define points as a superstructure in terms of Cauchy sequences of nested regions defined by the operation of bisection (Hellman & Shapiro, 2013, p. 498). Our formalization does capture this aspect of Aristotle's theory more closely, since splitting an event into two parts gives rise to a new event structure in which there exists a boundary separating these parts, albeit an extended one. In section 5.4 we shall see that our topological approach also provides us with much stronger forms of indecomposability, indeed, much stronger than those satisfied by the classical continuum. Another important element of the Aristotelian and Kantian view of the continuum, as we saw in section 3.4, is the rejection of actual infinities in favour of potential infinities: points come into existence only via the iterated process of division of parts of time, which are at every stage of division always finite in number.[12] Both formalizations of the Aristotelian continuum under consideration (Hellman & Shapiro, 2013; Roeper, 2006) have actual infinities, i.e., every model of the axioms will contain infinitely many regions. For instance, the axioms of Hellman and Shapiro (2013) imply that the set of regions is atomless, and one can show that every interval is equal to the fusion of two non-overlapping congruent parts. Hellman and Shapiro thus interpret Aristotle's "breaking in two" or "splitting" only as metaphorical; they note, however, that this poses a challenge in interpreting both Aristotle's notion of boundaries passing from potential to actual on breaking, and his views of potential infinity. An approach by means of inverse systems of event structures as that outlined in section 4.5 provides, in our opinion, a closer match to Aristotle's notions of the potentiality of boundaries and the potential infinite divisibility of time.

Note, however, that Hellman and Shapiro's axiomatization is more expressive than ours, as they include various geometric notions, such as congruence, which allow them to give a translation axiom and ultimately to prove the Archimedean property for their "gunky" line. We can achieve a similar result only via the defined points, by embedding event structures into their linear order of boundaries.

The last point which we turn our attention to in this brief discussion is St. Augustine's argument against the extended present. We find it in B. XI, chapter XV of the *Confessions*:

> If any fraction of time be conceived that cannot now be divided even into the most minute momentary point, this alone is what we may call time present. But this flies so rapidly from future to past that it cannot be extended by any delay. For if it is extended, it is then divided into past and future. But the present has no extension whatever.

The issue here seems to be that an extended present would have successive

[12] See Bk. III, part VI of the Physics.

parts, some of which would be in the past, while others would be in the future; but then the present could be split into smaller parts, and it would be difficult to see how it would still be a present. This objection can be neutralized by noticing that it only holds if the domain of primitive objects are points, as in the classical continuum; then one could certainly split an extended present, represented by an interval, into a succession of two intervals, and conclude that the original interval was not a present after all. In such a setting, a present can only be an indivisible point. However in our formalization the present is extended but does not have parts succeeding one another, since all the events constituting it pairwise overlap; thus, it cannot be "split" within a given event structure.

We have now concluded our outline of the formalization of Kant's theory of time. We can now finally apply these insights to the interpretative problems of the passage at B161n, to which we now turn.

4.7 Solution to the interpretative problems

Now that we have outlined the basics of the formalization of Kant's theory of the time continuum, which we shall investigate in more detail in the next chapter, we can go back to the exegetical issues presented in section 3.1, and tackle them in light of what has been achieved.

4.7.1 Solution to problem (1): space and time as objects

The first point that we must address is problem (1): what does it mean that space (and time) as formal intuitions are "represented as an object"? The problem is quite pressing because Kant seems to blatantly contradict himself on this point at various places, claiming at the same time that "space and time are not objects of intuition, but pure intuition itself" and that "space and time are objects of intuition".[13] How can this conundrum be resolved?

In light of the analysis developed in chapter 3, what Kant seems to be saying at B161 and other related passages such as (18), albeit often in a sloppy way, is that there is a distinction between being the object of an intuition, which space and time cannot be because "they cannot be perceived" in themselves (B207), and being an intuition that is produced *a priori* by the self-affection of the subject.

[13] For instance:

> Space and time are not objects of intuition but pure intuition itself [...] (OP 22:439)

But slightly later:

> Space and time, as objects of intuition, regarded as unity - the one of outer intuition, the other of inner - are given *a priori* (OP 22:449)

While empirical objects belong to the former class, space and time belong to the latter, and in this sense they are the "original sense-object", "pure intuition itself" that is produced by the "consciousness of the way in which objects of the senses are represented to us". This reading is in agreement with the formal theory presented in the previous section. The inverse limit $\mathcal{V}$, which we hold to be the correlate of the formal intuition, is produced by the consciousness of the action of the figurative synthesis, whose correlate is the inverse system of finite event structures. We furthermore remarked that this inverse system is possible because of the unity of apperception, which through the categories constrains the action of the figurative synthesis. Hence, time as the formal intuition is an immediate consequence of the application of the synthesis of the unity of apperception on the *a priori* manifold that the subject produces in its self-affection, and is, then, not the intuition "of something", but a primitive intuition that does not refer to any "object" other than itself. Thus, when Kant speaks of space and time represented as "objects" he really means space and time represented as the original intuition of oneself, which is produced by the subject affecting itself through motion.

4.7.2 Solution to problem (2)a: the two notions of unity

We now address the problems concerning the concept of unity that appears at B161.

Problem (2)a can be easily solved by distinguishing the two notions of unity, namely, the unity of the manifold given in space and time and the unity of space and time themselves. The former means that there is only one "experience" proper, and that the talk of "experiences" is just a shorthand for the talk of "parts of the one experience". Similarly, the unity of space and time means that there exists one space and one time, and that "spaces" or "times" are only parts of these all-encompassing wholes. Now, the former unity must be grounded in the latter. Appearances form one "experience" only if they are subject to the categories in such a way that they must constitute a system of perceptions connected by laws. To be homogeneous with the categories, however, appearances must be given in the formal intuition, as we have argued in section 3.7.2. Thus, the unity of space and time as formal intuitions is the ground for the law-like connection of appearances that constitutes experience.

The formal theory presented in the previous section models the unity of time as formal intuition, on which the unity of experience is grounded, by means of the strong logical and categorical properties of $\mathcal{V}$.

First, $\mathcal{V}$ is unique in the sense that any finite event structure embeds in it, and thus any "part of time" is just a part of it[14]. Hence, there cannot be parts of time that are, so to speak, wholly distinct from $\mathcal{V}$. This form of unity can be

[14] Although one should not conclude that $\mathcal{V}$ is the sum of its parts, which would run contrary to Kant; see section 5.4 on this point.

used to elucidate the sense in which the unity of time is the ground of the unity of the manifold of experience. If $\mathcal{W}, \mathcal{W}'$ are two event structures representing the temporal form of two different "parts of experience" A, B, then an embedding of $\mathcal{W}, \mathcal{W}'$ into $\mathcal{V}$ amounts to the determination of their reciprocal temporal relations by taking them to be parts of a temporal whole. Thus, the temporal laws to which A abides must be the same as those to which B abides; which amounts to saying that there really is only *one* system of perceptions connected by laws, i.e., one experience. A further elucidation of this point, in connection with the notion of "thoroughgoing determination" of appearances in time, is given in the next chapter at section 5.9.

Second, it is straightforward to prove using the second property in the definition of $\mathcal{V}$ that any two inverse limits of the inverse system of finite event structures are isomorphic. This is quite a strong unicity property, as it means that $\mathcal{V}$ is for all practical purposes completely determined by our axiom system.

4.7.3 Solution to problem (2)b: the synthesis and the formal intuition

We now turn to problem (2)b, which is one of the most controversial points of the footnote and presents the greatest challenge to our interpretation.

Recall that at B161n it is said that there must be a synthesis that yields "space and time as intuitions" by giving unity to the form of intuition, and that this unity "precedes all concepts" and "belongs to space and time, and not to the concepts of the understanding". First, let us reiterate that it seems beyond doubt that the synthesis mentioned at B161n is the figurative synthesis. We are led to this conclusion not only by our interpretation in the previous sections but also by the brute fact that the synthesis at B161n is such that through it "the understanding determines the sensibility", which is exactly how Kant describes the figurative synthesis in the section immediately preceding. Moreover, as we shall soon see, the fact that Kant claims that through the synthesis at B161n "all concepts of space and time first become possible" unambiguously identifies it as the figurative synthesis. Hence, the claim that it is the figurative synthesis that yields the formal intuition, i.e., that brings unity to the form of intuition, seems uncontroversial.

The problem at hand, then, can be decomposed into the following subproblems:

(1) does the figurative synthesis involve an active role of the categories?

(2) does the formal intuition amount to the space and time of the TA, that is, is the "unity" produced by the synthesis that of the TA?

A negative answer to the first part of subproblem (1) immediately leads to an empasse. If we say that the figurative synthesis does not involve an active role of the categories then we are not only running contrary to the textual evidence, as Kant says that the figurative synthesis proceeds "in agreement with the categories", but we are effectively introducing a synthesis that does not involve the understanding but is at the same time supposed to produce a unity, and this is a difficult proposition to accept on the grounds of Kant's definition of synthesis. We must then conclude that the figurative synthesis involves the categories, as our interpretation in the previous sections established.

But then we run into the empasse mentioned at problem (2)b: how can a synthesis yield a unity that precedes all concepts, even the categories? This question is particularly pressing, because the justification of the axioms of the formal theory relies on the categories, which, in our reading, constrain the action of the figurative synthesis. Before we tackle this problem, however, it is expedient to first examine subproblem (2), as it provides additional useful context to answer this question.

In the interpretation we provided in the previous sections we maintained that the formal intuition at B161n is nothing other space and time in the TA, even though there exists, in the TD in particular, a distinction between a purely passive form of intuition and the formal intuition. Hence, we answer the question of subproblem (2) positively. Most nonconceptualists, however, answer the question negatively and hold that the formal intuition provides a different sort of unity than that of space and time in the TA, as they maintain that these are given with all their properties originally and without the need of any sort of synthesis or construction.

Onof and Schulting, for instance, argue that the unity provided by the formal intuition is a unity from the perspective of the understanding (Onof & Schulting, 2015, p. 27ff), i.e., that what is at stake in the footnote is the grasp by the understanding of the unicity of space; the latter, however, is not generated by the understanding but belongs to the intuitions of space and time originally. Thus, it is the *taking as a unity* of the unicity of space and time that requires a synthesis, and not this unicity or space and time themselves as given infinite magnitudes (Onof & Schulting, 2015, p. 29-33). Other nonconceptualists, such as Fichant (Fichant, 1997), have instead argued that space and time as formal intuitions are not the space and time of the TA but *particular* spaces and times, that is, geometrical spaces, whose unity is synthesized but is also grounded on the original unity of space and time of the TA.

We find these takes on the matter unsatisfactory, however, on both philosophical and exegetical grounds.

First, such nonconceptualist approaches amount to renouncing the possibility of providing an explanation for why the properties of space and time are what they are, since such explanations would inevitably involve exhibiting the properties of space and time as constructed from non spatiotemporal entities and processes. More specifically, how is one to justify, say, the one-dimensionality that is attributed to time *a priori* in the TA without recurring to the categories?

Rejecting a role for the categories would seem to undermine the very possibility to provide a justification for any axiom of temporal order and to provide as precise a *transcendental justification* as possible, on the basis of the unity of apperception, for the properties of space and time; a justification that we attempted to provide in the previous sections by means of inverse systems of event structures. One would then have to merely accept the existence of two mysterious intuitions, called "space" and "time", that just spring into existence with a set of peculiar properties more or less by *fiat*. This sort of explanation is not only philosophically unsatisfactory, but also runs contrary to the spirit of Kant's enterprise, considering the emphasis that Kant puts on the "construction" or "self-production" of space and time, which we examined in section 3.8.3.

Second, it should be noted that space and time in the TA of the B edition are described in a way that exactly relates to the discussion of the self-affection of the subject in the description of a space, the notion that we put as the cornerstone of our interpretation. Indeed, in Passage (1) of section 3.3 the form of intuition is explicitly described as "the way in which the mind is affected by its own activity", which, in our reading, is just the affection of the sensibility by the understanding: the self-affection of the subject through the figurative synthesis in the description of a space.

Third, in relation to our discussion of self-affection the notion of "unity" occurring at B161n acquires much clearer contours. Kant says that this unity of the formal intuition, which is produced by a synthesis, amounts to the *comprehension* (*Zusammenfassung*) of the manifold of the form of intuition in an intuitive representation. In the CPJ, as we already noted, Kant remarks that this act of comprehension is necessary for the cognition of simultaneity of space, as we tried to make clear in section 4.2.2. But this essential moment of comprehension, as we argued in section 3.5, is just the moment of "holding" or "taking together" appearing in the synthesis of apprehension; indeed, Kant says that this moment is essential for the "unity of intuition [...] as [...] in the representation of space" (A99), without which we would have no representations of space nor of time (see also Passage (8) in section 3.5). Thus, the *taking as a unity* on which (Onof & Schulting, 2015) insists is actually the act of apprehension and comprehension, so that the nonconceptualist interpretation claiming that the synthesis at B161n is the figurative synthesis responsible for the production of geometrical spaces is correct, in that this is exactly what apprehension and comprehension *a priori* amount to. But we also claim that it is exactly in the construction of these particular spaces *a priori*, through the self-affection of the figurative synthesis, that the original intuitions of space and time of the TA are first acquired as unities with all their properties. This in particular explains the pervading terminological ambiguity noted by Fichant (Fichant, 1997) between the unity of particular spaces and times (plural) and the unity of space and time (singular), as a representation of the latter is acquired in the representation of the former. As we remarked in sections 3.8.4 and 3.8.5 this does not contradict the fact that geometrical spaces presuppose, in a transcendental logical sense, the one original space and time of which they are but parts.

Hence, if we are correct, in the act of describing spaces in agreement with the categories the figurative synthesis first yields the intuitions of space and time with the properties ascribed to them in the TA. We must, however, address the problem we left hanging above, namely, how can the unity so produced be described as "preceding all concepts" and as belonging to space and time rather than to the categories?

The key to solve this problem lies in providing a correct interpretation of what it means that the figurative synthesis proceeds "in agreement with" the categories. In particular, in our formalization the role played by the categories in constraining the figurative synthesis is that of *a priori* rules that are expressed in first-order logic as geometric formulas involving only primitives of temporal order. In other words, the categories are employed merely as *schematic rules* of the productive imagination. This use of the categories is akin to that in the schematism chapter of the CPR and, even though it originates from the understanding, does not involve the subsumption of the manifold under concepts. It merely means that the "blind and indispensable function of the soul" that, as synthesis in general (A78/B103), is independent of the understanding must be constrained by the influence of the categories so as to be able to produce a unity. In other words, the imagination bridges the gap between the sensibility and the understanding since it imposes categorial constraints on the manifolds that can be afforded by the sensibility, but this does not amount to and is only a first step towards full subsumption under concepts. After all, Kant would not say that the unity of a particular geometrical space, say a line, belongs to the categories, but that it is a spatiotemporal unity, even though - and this is quite uncontroversial given how Kant describes the figurative synthesis - the synthesis of the particular geometrical space must proceed "in agreement with" the categories.

We conclude this section by noting that on this point our interpretation also differs from the most prominent conceptualist takes on the problem. Friedman (M. Friedman, 2012), in particular, attempts to solve the problem by identifying the figurative synthesis or transcendental synthesis of the imagination with a synthesis pertaining directly to the unity of apperception, which precedes both the construction of geometrical concepts and the schematized categories. We believe, however, that the textual evidence in support of such a move is not strong. First, Kant always related the figurative synthesis directly to the categories and to the description of geometrical spaces, not only at §25 of the CPR but also in his notes, as section 3.6.3 made clear. Second, at the beginning of the TD B he claims that a manifold must always be brought to the unity of apperception through an act of combination (B135), but combination (*combinatio*, *Verbindung*) is always used by Kant in relation to the categories or the construction of geometrical concepts, for instance in the footnote at B202. Finally, as we noted in section 3.6.2, in a letter to Tieftrunk (12:233) Kant remarks that the concept of the "composed" is included in every category, and that "the concept of consciousness of composing (a function that, as synthetic unity of apperception, is the foundations of all the categories) must be presupposed in order to think the manifold of intuition [...] as unified in a consciousness". Hence there seems to be no substantial textual support for assuming a synthesis that "bypasses" the categories. But, as we noted above, there is no need to do so to solve the problem at hand. Friedman's worries about the fact that geometrical spaces presuppose metaphysical space can be defused by noting that this is a logical, and not genetic, presupposition, as we did in section 3.8.5, and the worries about the synthesis at B161n "preceding all concepts" can be dealt with by noting that indeed, logically speaking, the unity of space and time precedes all concepts, although it still presupposes the categories as pure spatiotemporal schemata constraining the figurative synthesis, as explained above and captured in the formal model.

4.7.4 Solution to problem (3): concepts of space and time

It then remains to address problem (3), that is: what are the "concepts of space and time" which are made possible by the synthesis at B161n? Our interpretation provides us with a straightforward answer to this question, which is expressed in a nutshell in passage (25) in 3.8.4. The "concepts of space and time" mentioned here are the pure sensible concepts that are constructed by the action of the figurative synthesis in the "describing" of a space. Examples of these concepts are the geometrical concepts, such as "triangle" and "circle" but also "distance" and "direction", and the physical concepts such as "duration", "speed", "momentum", and so forth. This reading has very substantial textual support; see, among many others, passages (25), (24), and A165/B206 in the CPR.

To be sure, the formal interpretation we outlined does not provide a fully worked out theory of the relation between time as the formal intuition and such pure sensible concepts, mainly because a Kantian formalization of space as the formal intuition, along similar lines as those presented for the case of time in this thesis, is still lacking, although we shall say more on the external representation of time as a line in the next chapter. The technical challenge involved in developing a Kantian formalization of space as the formal intuition, in any case, is that space has more degrees of freedom than time, and for this reason developing a Kantian theory of geometrical constructions that is to be point-free is not straightforward. Nevertheless, Friedman has proposed in M. Friedman (2012) a group-theoretic interpretation of Kant's theory of space that is closely related to the approach presented in this work, and about which we shall say more in chapter 7.

Chapter 5

A formal theory of the Kantian time continuum

5.1 Introduction

In this chapter we flesh out the mathematical details of the formalization of Kant's theory of time that was only sketched in the previous chapter in relation to its philosophical application. The material in this chapter grew out of a manuscript written jointly with my supervisor, Michiel van Lambalgen (Pinosio & van Lambalgen, 2016). The present chapter differs from the manuscript in important respects; in particular, the formal treatment has been substantially improved, allowing better proofs of the main theorems and a more incisive philosophical discussion.

The chapter is organized as follows. In section 5.2 we provide the logical, topological and categorical notions that are needed for the present chapter but also for chapter 6. In section 5.3 we look more closely at the axiom system already introduced in section 4.3. In section 5.4 we provide a formal correlate for Kant's dictum that time is not composed out of its parts by means of the topological notion of ultra-connectedness. In section 5.5 we provide the construction of Kantian boundaries, while in section 5.6 we examine the construction of the infinitesimal intervals which, together with the Kantian boundaries, give rise to the general topological form of Kantian continuum, and introduce the notion of maps between event structures. In section 5.7 we consider the role of the partial operations $\oplus, \ominus$ already introduced in section 4.3 with respect to the construction of boundaries. In section 5.8 we discuss infinite divisibility and retraction maps, while in section 5.9 we begin the discussion of inverse systems of finite event structures as the formal correlate to the action of the figurative synthesis and of the properties of their limits. Finally, in section 5.10, we discuss the inverse limit on the inverse system of all finite models of the axiom system and retraction maps, which was already sketched in section 4.5, and provide the construction of the Kantian continuum as the space of instants on this limit.

5.2 Mathematical preliminaries

In this section we shall introduce some basic mathematical notions, from logic, order theory, topology and category theory that will be used in this and the following chapter. The reader might skip this section, however, and return to it later when a reference is needed.

5.2.1 Orders

Let X be a set. A *preorder* on X is a binary relation $\leq\ \subseteq X \times X$ which is (i) reflexive, meaning that $x \leq x$ for any $x \in X$, and (ii) transitive, meaning that $x \leq y, y \leq z$ implies $x \leq z$ for any $x, y, z \in X$. We often abuse our notation and write simply X for the preorder $(X, \leq)$. A *partial order* or poset is a preorder X which satisfies antisymmetry: $x \leq y, y \leq x$ implies $x = y$ for any x, y. Given a preorder X and a set $S \subseteq X$, the *downset* generated by S is defined as: $\downarrow S = \{x \in X \mid x \leq y \text{ for some } y \in S\}$. The *upset* generated by S is defined similarly. We shall abuse our notation and denote the downset of a singleton set $\{x\}$ as $\downarrow x$, and similarly for the upset of a singleton set. A subset $D \subseteq X$ is a *downset*, or a *downward closed* subset of X, if $\downarrow D = D$. The notion of an *upset* or *upward closed* subset is defined similarly. If we wish to make explicit the order with respect to which downsets and upsets are taken, we simply write $\downarrow_{\leq} S, \uparrow_{\leq} S$.

Given a preorder X and a point $x \in X$ the *order open initial segment* generated by x is the set $(x, \leftarrow) = \{y \in X \mid y \leq x, y \neq x\}$, while the *order open final segment* generated by x is the set $(x, \rightarrow) = \{y \in X \mid y \geq x, y \neq x\}$. We call the set of order open initial and final segments of a preorder its set of *rays*. *Order closed* final and initial segments are defined similarly.

Given a preorder X and a subset $U \subseteq X$, we say U is *order convex*, or simply convex if the context is clear, if whenever $x, y \in U$ with $x \leq z \leq y$ then $z \in U$.

A preorder X is a *total preorder* if $x \leq y \vee y \leq x$ for any $x, y \in X$; if the preorder is a partial order then X is a *linear order*. A linear order is *complete* if for any $Z \subseteq X$ and $U := \{b \in X \mid \forall x \in Z(x \leq b\}$ non-empty it holds that U has a $\leq$-minimal element.

Let X be a preorder. A subset $\mathcal{F} \subseteq X$ is said to be a *filter* if it satisfies the following conditions:

1. $\mathcal{F}$ is not empty
2. $\mathcal{F}$ is an upset: $\uparrow\mathcal{F} = \mathcal{F}$
3. $\mathcal{F}$ is down-directed: for any $x, y \in \mathcal{F}$ there exists $z \in \mathcal{F}$ with $z \leq x, z \leq y$

A filter $\mathcal{F}$ is *maximal* if it cannot be properly extended: there is no filter $\mathcal{F}'$ such that $\mathcal{F} \subset \mathcal{F}'$.

Let X, X' be preorders. A *map* from X to X' is a function $f : X \to X'$ which satisfies:

$$x \leq y \text{ implies } f(x) \leq' f(y) \text{ for any } x, y \in X \text{ (order preservation).}$$

A map between preorders is an *isomorphism* if it is injective and surjective, that is, if $x \neq y$ then $f(x) \neq f(y)$ and for any $x' \in X'$ there exists $x \in X$ with $f(x) = x'$.

5.2.2 Topology

In this section I will briefly introduce some basic notions from topology; the reader not acquainted with topology is however encouraged to consult the first chapters of any textbook on general topology.

Let X be a set. A *family of sets* is simply a collection of subsets of X, i.e., a subset of the powerset $\mathcal{P}X$ of X. We denote a set X equipped with a family of sets $\mathcal{F} \subseteq \mathcal{P}X$ as a tuple $(X, \mathcal{F})$.

A family of sets $(X, \mathcal{F})$ is said to be *closed under finite intersections* if for any $U, V \in \mathcal{F}$, $U \cap V \in \mathcal{F}$. A family of sets $(X, \mathcal{F})$ is said to be *closed under arbitrary intersections* if for any $\mathcal{G} \subseteq \mathcal{F}$, $\bigcap\{U \mid U \in \mathcal{G}\}$ belongs to $\mathcal{F}$. The definitions for a family of sets to be *closed under finite unions* and *closed under arbitrary unions* are analogous.

Let X be a set. A *topology* τ on X is simply a family of sets $\tau \subseteq \mathcal{P}X$ which is (i) closed under finite intersections, (ii) closed under arbitrary unions, and (iii) the whole set X and the empty set $\emptyset$ belong to τ. A set X equipped with a topology is called a *topological space* and is denoted as (X, τ). We always use small greek letters τ, θ etc. to refer to topologies on sets. The sets which are elements of the family of sets τ are called the *open sets* of the topological space, while the sets which are complements of open sets are the *closed sets* of the topological space. If, given a point $x \in X$, $\{x\}$ is either open or closed, then we say x is *decided*; otherwise it is *undecided*.

As in the case of preorders we shall often abuse our notation and denote a topological space merely with X.

A topological space X is a T_0 *topological space* if for any two distinct points $x, y \in X$ there exists an open set U which contains *exactly* one of the two points x, y. It is T_1 if all points of the space are closed. It is T_2 or Hausdorff if for any two points x, y there exists open sets U, V with $x \in U, y \in V$ and $U \cap V = \emptyset$ (any two points can be separated by open sets).

Given a topological space X, the *specialization ordering* $\sqsubseteq \subseteq X \times X$ of X is defined by letting $x \sqsubseteq y$ if $U \in \tau, x \in U$ implies $y \in U$. The specialization ordering is always a preorder; it is a poset if the topological space is T_0, and it is the trivial ordering if the topological space is T_1.

Let X be a linear order. The *order topology* on X is generated by the subbasis of all rays.

Let X, X' be two topological spaces. A *map* $f : X \to X'$ is a function satisfying:

$$\text{For any } U \subseteq X, U \in \tau' \text{ it holds that } f^{-1}[U] \in \tau^{1}$$

Here $f^{-1}[U]$ denotes the *inverse image* of U under the function f, i.e., $f^{-1}[U] = \{x \in X \mid f(x) \in U\}$. Maps of this sort are called *continuous*. A map $f : X \to X'$ between topological spaces is said to be an *isomorphism* if it satisfies the following additional conditions:

1 f is a bijection

2 For any $U \in \tau$, $f[U] \in \tau'$ must hold[2]

Topological spaces and continuous maps form the category **Top** of topological spaces.

In the sequel we shall also consider sets equipped with two distinct topologies, which are called *bitopological spaces*. The reader should consult Kelly (1963) for relevant background information on bitopological spaces. We shall in particular need the following notion: let (X, τ, τ') be a bitopological space, then the *join topology* on X, denoted as $\tau \vee \tau'$, is the topology having $\tau \cup \tau'$ as a subbase. In the sequel we shall also need some notions from the theory of ordered topological spaces (Nachbin, 1965). An *ordered topological space* is simply a tuple $(X, \leq, \tau)$ where X is a set, τ is a topology on X, and $\leq$ is a partial order on X. For instance, a linear order equipped with the order topology defined above is a LOTS or *linearly ordered topological space*. Given an ordered topological space X we can consider the *lower topology* $\mathcal{L}_X$ and the *upper topology* $\mathcal{U}_X$ on X defined by letting:

$$\mathcal{U}_X = \{D \subseteq X \mid D \in \tau, \uparrow D = D\}$$
$$\mathcal{L}_X = \{D \subseteq X \mid D \in \tau, \downarrow D = D\}$$

Clearly, $(X, \mathcal{L}_X, \mathcal{U}_X)$ is a bitopological space. Finally, we say an ordered topological space is *convex* if it has a subbase of open upsets and downsets.

5.2.3 Alexandroff topological spaces and the Alexandroff correspondence

In the sequel we shall make ample use of a simple correspondence between preorders and a specific class of topological spaces, called *Alexandroff topological spaces*, which are closed not merely under finite intersections but under arbitrary

[1]This means that preimages of open sets are open.
[2]This means that images of open sets are open.

intersections. The correspondence between preorders and Alexandroff topological spaces was first noticed by Alexandroff in Alexandroff (1937) and works as follows.

Let X be a preorder; the corresponding Alexandroff topology $\mathcal{A}_{\leq} \subseteq \mathcal{P}X$ is defined as the set of all upsets of the preorder. In this topology, the downsets are the closed sets. In the other direction, for an Alexandroff topology $\tau \subseteq \mathcal{P}X$ on X one simply takes the specialization preorder $\leq_\tau = \sqsubseteq^\tau$. It is a classic result by Alexandroff that these constructions establish a bijective correspondence between preorders and Alexandroff spaces on a set X, that is:

5.2.1. THEOREM. *For all preorders $\leq$ on a set X it holds that $\leq_{\mathcal{A}_{\leq}} = \leq$ and for all Alexandroff topologies τ it holds that $\mathcal{A}_{\leq_\tau} = \tau$.*

This correspondence can be easily extended to order preserving functions and continuous maps so as to yield a categorical equivalence.

5.2.4 Model theoretic notions

I will recall here very briefly some notions from model theory that will be needed in the sequel; a good reference for the basics of model theory is (Hodges, 1997). Some of these notions are mostly recalled for the purpose of fixing notation, while other are slightly more specific to the work that lies ahead.

A *signature* L is specified by giving a set of constants, a set of n-ary relation symbols, and a set of n-ary function symbols.

A *structure* $\mathcal{A}$ in the signature L is specified by providing (i) a domain $dom(\mathcal{A})$ of objects, (ii) an element $a \in dom(\mathcal{A})$ for any constant c of L, (iii) a set of n-tuples of objects of $\mathcal{A}$ for any n-ary relation symbol R of L, (iv) an n-ary operation of type $dom(\mathcal{A})^n \to dom(\mathcal{A})$ for any function symbol F of L.

If $\mathcal{A}$ is a L structure and R is a relation symbol of L, we denote the relation named by R in $\mathcal{A}$ as $R^{\mathcal{A}}$, and similarly for function and constant symbols. With these basic notion one can proceed to define the notions of *terms*, *formulas*, *satisfaction in a model*, and so forth, as usual. In particular, a *positive primitive* formula of L is a formula $\psi(\bar{x})$ which contains only occurrences of $\wedge, \vee, \bot$ and $\exists$. A *geometric implication* of L is a formula of the form $\forall \bar{x}(\theta(\bar{x}, \bar{y}) \to \psi(\bar{x}, \bar{y}))$ where θ, ψ are positive primitive. Given a structure $\mathcal{A}$, a substructure $\mathcal{B}$ of $\mathcal{A}$ is an L structure such that $dom(\mathcal{B}) \subseteq dom(\mathcal{A})$, $R^{\mathcal{B}} = R^{\mathcal{A}} \cap B^n$ for any n-ary relation symbol R, and $F^{\mathcal{B}} = F^{\mathcal{A}}|B^n$ for any n-ary operation F.

Given a theory T in a signature L, a *model* of the theory is an L structure that satisfies the axioms of T. We always distinguish the notion of a substructure of a model from that of a *submodel*, where the latter is a substructure that is also a model for T.

A *homomorphism* from an L structure $\mathcal{A}$ to an L structure $\mathcal{B}$ is a map $f : dom(\mathcal{A}) \to dom(\mathcal{B})$ satisfying:

- For each constant c of L, $f(c^{\mathcal{A}}) = c^{\mathcal{B}}$
- For each n-ary symbol R of L and n-tuple $\bar{a}$ of elements of $dom(\mathcal{A})$, if $\bar{a} \in R^{\mathcal{A}}$ then $f(\bar{a}) \in R^{\mathcal{B}}$
- For each n-ary function symbol F of L and n-tuple $\bar{a}$ of elements of $dom(\mathcal{A})$, it holds that $f(F^{\mathcal{A}}(\bar{a})) = F^{\mathcal{B}}(f(\bar{a}))$

Let $\mathcal{A}, \mathcal{B}$ be L structure such that $\mathcal{B}$ is a substructure of $\mathcal{A}$. A homomorphism $f : \mathcal{A} \to \mathcal{B}$ is a *retraction* if $f|\mathcal{B} : \mathcal{B} \to \mathcal{B}$ is the identity. Then $\mathcal{B}$ is said to be a *retract* of $\mathcal{A}$. Given any two L structures $\mathcal{A}, \mathcal{B}$ and a homomorphism $f : \mathcal{A} \to \mathcal{B}$, we say that a formula $\phi(\bar{x})$ is *preserved by* f if for any sequence of objects $\bar{a}$ from $\mathcal{A}$, $\mathcal{A} \models \phi(\bar{a})$ implies $\mathcal{B} \models \phi(f\bar{a})$.

The following results will be of importance for what follows:

5.2.2. LEMMA. *Let $\phi(\bar{x})$ be a positive primitive formula and let $f : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Then f preserves $\phi(\bar{x})$.*

For geometric *sentences* we obtain a similar result if the maps under consideration are retractions:

5.2.3. LEMMA. *Let ϕ be the geometric sentence $\forall \bar{x}(\psi(\bar{x}, \bar{y}) \to \chi(\bar{x}, \bar{y}))$ and let $f : \mathcal{A} \to \mathcal{B}$ be a retraction map. Then f preserves ϕ.*

Proof:
Assume that $\mathcal{A} \models \phi$. We need to show that $\mathcal{B} \models \phi$. Assume then that $\mathcal{B} \models \psi(\bar{a})$ for some $\bar{a}$ in $\mathcal{B}$. Since the map f is a retraction this means that $\mathcal{B}$ is a submodel of $\mathcal{A}$, hence there is an embedding of $\mathcal{B}$ into $\mathcal{A}$. Thus $\mathcal{A} \models \psi(\bar{a})$ and hence $\mathcal{A} \models \chi(\bar{a})$ since $\mathcal{A} \models \phi$. Since f is a homomorphism from Lemma 5.2.2 we have $\mathcal{B} \models \chi(f\bar{a})$ and thus $\mathcal{B} \models \chi(\bar{a})$ since $f\bar{a} = \bar{a}$. Hence $\mathcal{B} \models \phi$. □

5.2.5 Inverse systems and inverse limits

Let $P = (X, \leq)$ be a partially ordered set. We say that P is *up-directed*, or simply *directed*, if for any $s, t \in X$ there is $u \in X$ with $s \leq u, t \leq u$. Let now L be a first order signature and let P be a directed poset. An *inverse system*, or *projective system* of L-structures indexed by P consists of a set $\{\mathcal{A}_s \mid s \in P\}$ of L-structures indexed by P, along with a set of homomorphisms $\{f_{ts} : \mathcal{A}_t \to \mathcal{A}_s\}$ for any t, s with $s \leq t$, satisfying the following:

- $f_{ss} : A_s \to A_s$ is the identity map
- For any $s, t, u \in P$ with $s \leq t \leq u$ and element a of $\mathcal{A}_u$ we have $f_{ts} \circ f_{ut} = f_{us}$.

An inverse system of L-structures is said to be an *inverse sequence* if the index set P is a linear order. Given an inverse system, we can define its *inverse limit* $\mathcal{L}$: an L-structure that is a mathematical formalization of the "limit" at infinity to which all the L-structures in the system "converge". It is defined as follows. The domain $dom(\mathcal{L})$ is the set consisting of all the elements $\xi \in \Pi_{s \in P} dom(\mathcal{A}_s)$ such that:

$$\text{if } s \leq t \text{ then } f_{ts}(\xi(t)) = \xi(s)$$

Here $\Pi_{s \in P} dom(A_s)$ denotes the cartesian product of all the $\mathcal{A}_s$; the elements of this product are basically functions which choose, for any $s \in P$, one element from $\mathcal{A}_s$. Hence, $\xi(t)$ denotes the component of the element $\xi \in \Pi_{s \in P} \mathcal{A}_s$ which is chosen from $\mathcal{A}_t$. The restriction above ensures that we keep only those elements of the product which corresponds to "threads" in the inverse system, i.e., those elements which behave well with respect to the maps. Now that we have the domain of the limit $\mathcal{L}$ we need to turn this into an L-structure; we can do this as follows:

- For any n-ary relation symbol R, let $(\xi, \xi', \cdots) \in R^{\mathcal{L}}$ iff $(\xi_s, \xi'_s, \cdots) \in R^{\mathcal{A}_s}$ for any $s \in P$
- For any constant symbol c we let $c^{\mathcal{L}}$ be the thread ξ such that $\xi(s) = c^{\mathcal{A}_s}$ for all s; note that this is a thread because the maps f_{ts} are homomorphisms.

Note that in the case of a countable inverse sequence of L-structures the elements of the limit have a very simple form: they are (possibly infinite) sequences of the form $(a_0, a_1, \cdots)$, where $f_{10}(f_{21}(a_2)) = f_{20}(a_2) = a_0$ and so forth. In general, given an arbitrary inverse system (or even an inverse sequence) we are not ensured that the inverse limit is not empty. However, if all models are finite we do have the following:

5.2.4. Theorem. *Let $\{P, \{A_s \mid s \in P\}, \{f_{st}\}\}$ be an inverse system of L-structures such that each $\mathcal{A}_s$ is finite. Then the inverse limit $\mathcal{L}$ is not empty.*

Proof:
Equip the finite L-structures with the (auxiliary) discrete topology, and use the result that the inverse limit of an inverse system of compact Hausdorff spaces is non-empty. □

5.2.5. Lemma. *Let $\{P, \{A_s \mid s \in P\}, \{f_{st}\}\}$ be an inverse system of L-structures such that the inverse limit $\mathcal{L}$ is not empty. Then the* projection map $\pi_s : \mathcal{L} \to A_s$ *defined by $\pi_s(\xi) = \xi(s)$ is a homomorphism which in addition satisfies:*

$$f_{ts}(\pi_t(\xi)) = \pi_s(\xi)$$

For all s, t with $s \leq t$. Moreover, if the homomorphisms f_{st} are retractions, so are the projections.

The following result states the standard categorical universal property of inverse limits:

5.2.6. THEOREM. *[Universality]* $\{P, \{A_s \mid s \in P\}, \{f_{st}\}, \mathcal{V}\}$ *be an inverse system of L-structures, where* $\mathcal{V}$ *is the inverse limit. If* $(\mathcal{N}, \rho_s)$ *is an L-structure that satisfies the same diagrams as* $(\mathcal{V}, \pi_s)$*, then there exists a unique homomorphism* $\iota : \mathcal{N} \longrightarrow \mathcal{V}$ *satisfying* $\rho_s = \iota \circ \pi_s$.

Given an inverse system of L structure, a natural question is what classes of formulas are *preserved to inverse limits*, i.e., what formulas are such that if they are satisfied at every L structure $\mathcal{A}_s$ for $s \in T$, then they are satisfied at the limit $\mathcal{V}$. We shall not address this question here, however, as we prefer to come back to it when we consider the more specific inverse systems of finite event structures in section 5.9. Most of the results presented there, however, hold in general for L structures.

5.3 The axiom system for objective time

We start by recalling the axiomatic definition of an event structure from chapter 4. An intuitive semantics for the axioms can be obtained by constructing diagrammatic representations along the lines of those presented in chapter 4, for which recall the following intuitive interpretation of the primitive relations and partial operations:

- $a\,\mathrm{R}_+ b$ means "a begins simultaneously with or after b", i.e. "a is in the future of b"
- $a\,\mathrm{R}_- b$ means "a ends simultaneously with or before b", i.e. "a is in the past of b"
- aOb means "a and b overlap"
- $a \oplus b$ denotes the "future cut of a by b", the maximal event which is covered by a and is in the (causal) future of b
- $a \ominus b$ denotes the "past cut of a by b", the maximal event which is covered by a and is in the (causal) past of b

A more rigorous semantics will be provided later in this section and even more generally in chapter 6 in terms of ordered topological spaces, but for the moment, and for readers mainly interested in the philosophical import of the formal theory, the intuitive interpretation will suffice.

5.3.1. DEFINITION. An event structure is a first-order structure in the signature $(\mathrm{R}_+, \mathrm{R}_-, O; \preceq; \oplus, \ominus)$ that is a model of the following axioms:

(1) $a \preceq b \leftrightarrow a\,\mathrm{R}_+ b \wedge a\,\mathrm{R}_- b$ (Explicit definition of $\preceq$)

(2) $aOb \rightarrow bOa$ (symmetry of O)

(3) aOa (reflexivity of O)

(4) $cOb \wedge c\,\mathrm{R}_+ a \wedge b\,\mathrm{R}_- a \rightarrow aOb$ (condition for overlap)

(5) $a\,\mathrm{R}_+ b \wedge b\,\mathrm{R}_+ c \rightarrow a\,\mathrm{R}_+ c$ (Transitivity of order)

(6) $aOc \wedge cOb \wedge c\,\mathrm{R}_+ b \wedge c\,\mathrm{R}_+ a \rightarrow aOb$ (conditional transitivity of O)

(7) $b\,\mathrm{R}_+ a \vee a\,\mathrm{R}_+ b$ (linearity)

(8) $\exists c(a \preceq c \wedge b \preceq c)$ (covering axiom)

(9) Partial binary operations $(\oplus, \ominus)$ on events (we write $a \oplus b = y$ for $\oplus(y, a, b)$).

   (a) $a \oplus b = y \wedge a \oplus b = z \rightarrow y = z$ (functionality)
   (b) $a\,\mathrm{R}_+ b \vee aOb \leftrightarrow \exists y(y = a \oplus b)$ (explicit domain of definition)
   (c) $a \oplus b\,\mathrm{R}_+ b$
   (d) $a \oplus b\,\mathrm{R}_+ a$
   (e) $a\,\mathrm{R}_- a \oplus b$
   (f) $a \oplus b\,\mathrm{R}_- a$
   (g) $a\,\mathrm{R}_+ b \rightarrow a\,\mathrm{R}_+ a \oplus b$
   (h) $b\,\mathrm{R}_+ a \rightarrow b\,\mathrm{R}_+ a \oplus b$
   (i) $a \oplus a = a$
   (j) $(a \oplus b) \oplus c = (a \oplus c) \oplus b$
   (k) $(a \oplus b) \oplus b = a \oplus b$
   (l) $(a \oplus b) \ominus c = (a \ominus c) \oplus b$

(10) Any sentence ϕ obtained from the above axioms by replacing R_- for R_+, R_+ for R_-, $\ominus$ for $\oplus$ and $\oplus$ for $\ominus$ (substitution principle)

Note that free variables are understood as universally quantified, and moreover that the axioms of group (9) are actually geometric formulas, as they are of the form

$$\exists y(a \oplus b = y) \rightarrow \Phi$$

We provide some important comments on the axiom system in section 5.3.1. Moreover, in what follows we shall sometimes use symmetry and reflexivity of O without mention.

5.3.2. DEFINITION. Let $\mathcal{W}$ be an event structure. For convenience we define the following abbreviations:

- $a \equiv_- b$ if $a\,\mathrm{R}_- b, b\,\mathrm{R}_- a$
- $a \equiv_+ b$ if $a\,\mathrm{R}_+ b, b\,\mathrm{R}_+ a$
- $a \equiv b$ if $a \preceq b, b \preceq a$
- $a \mathbb{O} b$ if $\neg(aOb)$
- $a\,\mathbb{R}_- b$ if $\neg(a\,\mathrm{R}_- b)$
- $a\,\mathbb{R}_+ b$ if $\neg(a\,\mathrm{R}_+ b)$

The abbreviation $\mathbb{O}$, in particular, will be of importance in the discussion of boundaries in section 5.5, where it will be interpreted as an operation of type $\mathcal{P}W \to \mathcal{P}W$ on the powerset of W through which boundaries can be constructed.

We have the following elementary consequences from the axioms:

5.3.3. LEMMA. *The following are elementary consequences from the axiom system of Definition 5.3.1:*

(1) Transitivity of precedence (see section 5.3.1)

  (a) $aOb \to aOc \vee cOb \vee b\,\mathrm{R}_+ c \vee a\,\mathrm{R}_- c$

(2) Reflexivity

  (a) $a\,\mathrm{R}_- a$ *(from linearity)*

  (b) $a\,\mathrm{R}_+ a$ *(from linearity)*

(3) Interaction between overlap and covering

  (a) $a \preceq b \to aOb$ *(from axiom (4) and symmetry of O)*

  (b) $c \preceq a \wedge cOb \to aOb$ *(from axioms (4), (10), and linearity)*

  (c) $a\,\mathbb{R}_- b \wedge a\,\mathbb{R}_+ b \to aOb$

  (d) $a\mathbb{O}b \to a\,\mathbb{R}_+ b \vee b\,\mathbb{R}_+ a$

(4) $a\,\mathrm{R}_+ b \wedge b\,\mathrm{R}_+ a \to aOb$ *(from linearity, reflexivity of O, and axiom (4))*

(5) $a\mathbb{O}b \to b\mathbb{O}a$ *(from symmetry of O)*

(6) Transitivity for $\mathbb{R}_+$, $\mathbb{R}_-$

(a) $a\,\mathbb{R}_{-}b \wedge b\,\mathbb{R}_{-}c \rightarrow a\,\mathbb{R}_{-}c$ *(from linearity, excluded middle and transitivity of* R_{-}*)*

(b) $a\,\mathbb{R}_{+}b \wedge b\,\mathbb{R}_{+}c \rightarrow a\,\mathbb{R}_{+}c$ *(from linearity, excluded middle and transitivity of* R_{-}*)*

(7) *Exchange*

(a) $a\,\mathbb{R}_{-}b \rightarrow b\,\mathrm{R}_{-}a$ *(from linearity)*

(b) $a\,\mathbb{R}_{+}b \rightarrow b\,\mathrm{R}_{+}a$ *(from linearity)*

Proof:
We only provide a proof of (3)b and (1)a by way of illustration, as the proof for the other claims are similar and routine. For (3)b, then, assume that $c \preceq a, cOb$. Then aOc by (3)a and $c\,\mathrm{R}_{+}a, c\,\mathrm{R}_{-}a$ by definition of $\preceq$. Now if either $b\,\mathrm{R}_{-}a$ or $b\,\mathrm{R}_{+}a$ then aOb follows respectively from either axiom (4) directly or axiom (4) plus axiom (10) of Definition 5.3.1. Otherwise, because of excluded middle we must have $b\,\mathbb{R}_{-}a, b\,\mathbb{R}_{+}a$ and by linearity and substitution we have $a\,\mathrm{R}_{-}b, a\,\mathrm{R}_{+}b$ which implies $a \preceq b$ and so aOb again by (3)a.

For (1)a, assume $aOb, a\mathbb{O}c, c\mathbb{O}b, b\,\mathbb{R}_{+}c$; we show that $a\,\mathrm{R}_{-}c$. To prove this we show that $c\,\mathrm{R}_{-}a$ leads to a contradiction, whence by linearity $a\,\mathrm{R}_{-}c$. So assume $c\,\mathrm{R}_{-}a$. First, note that we must have $b\,\mathrm{R}_{-}c$. Indeed, since $b\,\mathbb{R}_{+}c$ implies $c\,\mathrm{R}_{+}b$ by linearity then if $c\,\mathrm{R}_{-}b$ then $c \preceq b$, but then cOb by (3)a, contradiction; hence, $b\,\mathrm{R}_{-}c$ by linearity. Similarly, we must have $a\,\mathrm{R}_{+}c$ since from $c\,\mathrm{R}_{+}a$ and the assumption that $c\,\mathrm{R}_{-}a$ we obtain $c \preceq a$ which leads to a contradiction, so by linearity $a\,\mathrm{R}_{+}c$. Then from axiom (4) by replacing a for c and viceversa we obtain $aOb \wedge a\,\mathrm{R}_{+}c \wedge b\,\mathrm{R}_{-}c \rightarrow cOb$ and hence cOb, which gives a contradiction with the assumption that $c\mathbb{O}b$, and we are done. □

As far as the operations $\oplus, \ominus$ are concerned, we have the following:

5.3.4. LEMMA. *The following are consequences from the axiom system of Definition 5.3.1:*

(1) $a\,\mathrm{R}_{+}b \wedge a \preceq c \rightarrow a \preceq c \oplus b$ *(maximality)*

(2) $(a \oplus b) \ominus b = (a \ominus b) \oplus b$ *(follows from axiom (9)l)*

(3) $(a \oplus b) \ominus b \preceq b$

(4) $(a \oplus b) \ominus b \preceq (b \oplus a) \ominus a$

(5) $aOb \wedge cOa \wedge cOb \rightarrow cO(a \oplus b) \ominus b$

(6) $c \preceq a \wedge c \preceq b \rightarrow c \preceq (a \oplus b) \ominus b$ *(follows from maximality)*

(7) $a \preceq b \to a \oplus c \preceq b \oplus c$ *(follows from maximality)*

(8) $a \preceq b \leftrightarrow (a \oplus b) \ominus b \equiv a$ *(follows from maximality)*

Proof:
We only show (1) as the proof for the other cases is similar. Let then $\mathcal{W}$ be an event structure and $a, b, c \in W$ with $c\,\mathrm{R}_+ b, c \preceq a \to c \preceq a \oplus b$. We have that $a \oplus b \equiv_- a$ by axioms (9)e and (9)f and hence, since $c \preceq a$, we get $c\,\mathrm{R}_- a \oplus b$. To see that $c\,\mathrm{R}_+ a \oplus b$ we use the linearity axioms. Then either $a\,\mathrm{R}_+ b$ or $b\,\mathrm{R}_+ a$; in the former case by axiom (9)g then $a\,\mathrm{R}_+ a \oplus b$ and thus $c\,\mathrm{R}_+ a \oplus b$ and we are done. In the latter case by axiom (9)h then $b\,\mathrm{R}_+ a \oplus b$, hence $c\,\mathrm{R}_+ a \oplus b$ and we are done. □

The following lemma introduces the notion of an exact cover of a finite set of events:

5.3.5. Lemma. *Let $\mathcal{W}$ be an event structure and let $A \subseteq W \times W$ be a finite multiset of events with enumeration $a_1, \ldots, a_n$ such that for all k, $a_k\,\mathrm{R}_+ a_1$ and $a_k\,\mathrm{R}_- a_n$. Then there exists c such that for all k, $a_k \preceq c$, $c\,\mathrm{R}_+ a_1$ and $c\,\mathrm{R}_- a_n$. Such c is called an* exact cover *of A*

Proof:
Consider a multiset of events A with cardinality $|A| \leq n$ for $n \in \omega$. A number n of applications of axiom (8) of Definition 5.3.1 yields an event d such that $a_k \preceq d$ for all $k \leq n$. Set then $c = (d \ominus a_n) \oplus a_0$. It is tedious but straightforward to check that the axioms for $\oplus, \ominus$ ensure that c satisfies the required properties. □

The following proposition states that given an event structure W and an event $a \in W$, the downset of a in the preorder $\preceq$ is also a model of the axioms in Definition 5.3.1:

5.3.6. Proposition. *Let $\mathcal{W}$ be an event structure and let $a \in \mathcal{W}$. Then $\downarrow_{\preceq} a$, the principal ideal generated by a with respect to $\preceq$, is an event structure.*

We also define the notion of a "universal cover", namely, an event which covers all other events in the event structure. The "universal cover", which as we saw in chapter 4 represents the unboundedness of time, will be of great importance to model potential infinite divisibility in section 5.8.

5.3.7. Definition. Let $\mathcal{W}$ be an event structure and $c \in W$. Then c is a *universal cover* if for any $a \in W$ it holds that $a \preceq c$.

Finally, note that while the $\preceq$ relation is binary it is possible in our vocabulary to extend it to a relation $\preceq \subseteq W \times \mathcal{P}W$ between elements of W and subsets of W, as follows:

5.3.8. Definition. Let $\mathcal{W}$ be an event structure. We say that a set $C \subseteq W$ *finitely covers*, or simply *covers*, an event $a \in W$, denoted $a \preceq C$, if there exists a finite subset $C' \subseteq C$ and enumeration $(a_1, \cdots, a_n)$ of C' such that:

$$a\,\mathrm{R}_+ a_1 \wedge a\,\mathrm{R}_- a_n \bigwedge_{1 \leq i \leq n-1} a_i O a_{n+1}$$

It is a straightforward matter to show that the above definition faithfully captures the relation of finite covering between a bounded interval and a set of bounded intervals on the real line under the interpretation of the language that we shall give in the following section. Interestingly, however, this only holds for the linear case. As soon as we give up the linearity axioms and move to more general models in which R_-, R_+ can give rise to non-total preorders the vocabulary of the axiom system of Definition 5.3.1 does not suffice to capture the finite covering relation, which must then be taken as primitive and its properties be axiomatized. We shall say more about this in chapter 6, where the finitary covering relation above will play an important role in relating our framework to constructive topology.

5.3.1 Remarks on the axioms

We have already provided ample philosophical commentary on the justification of the axiom system in chapter 4; here we limit ourselves to mostly technical and terminological considerations that will be useful later.

Terminology

The full set of axioms in Definition 5.3.1 will be referred to as GT, where "GT" stands for geometry of time, a reference to Kant's insistence on the necessity of an "outer" (geometric) representation of time (B154). When we use the term "event structure", then, we shall in general mean a model of GT, unless specified further. In what follows we shall also work in the subsystem GT_0 consisting of GT minus the axioms of group (9) and axiom (8), in which the exact covering Lemma 5.3.5 does not hold. We also make use of an intermediate axiom system $\mathrm{GT}_1 = \mathrm{GT}_0 +$ (8), which will be important in the treatment of boundaries in section 5.5. The philosophical meaning of these different axiom systems has been already treated in chapter 4.

The axiom system and geometry

The reader will have undoubtedly drawn diagrams along the lines of those in chapter 4 to clarify the meaning of the axioms and of Lemma 5.3.5. Now, if we think of the universal cover c of an event structure, if it exists, as the empty form of time, then the geometric content of the axioms is to construct orthogonal projections from events to c, in a way analogous to that illustrated in section 4.2.2. Euclid's Bk.I, Proposition 12 shows how to construct orthogonal projections, and the theory of parallels (Bk. I, Propositions 27 – 32) shows that projection preserves the primitive relations and operations, as well as quantitative relationships. Although these constructions and proofs are simple, they require Euclid's five postulates (as well as his principles for comparison of magnitudes). In other words, there is a close connection between geometric principles and time as an object, that is, as formal intuition (B161n).

A closer relation to Euclidean geometry can also be obtained by considering the following definitions, which specialize to our setting the notions of positive primitive formulas and geometric implications introduced in section 5.2.4:

5.3.9. Definition. A formula in the signature $(\mathrm{R}_+, \mathrm{R}_-, O; \preceq; \oplus, \ominus)$ is *positive primitive* if it is constructed using only $\vee, \bigvee, \wedge, \exists, \bot$.

5.3.10. Definition. A formula is *geometric* or a *geometric implication* if it is of the form

$$\forall \bar{x}(\theta(\bar{x}, \bar{y}) \rightarrow \psi(\bar{x}, \bar{y}))$$

where θ and ψ are positive primitive.

Note that the logical form of Euclid's problems and theorems in the Elements is that of geometric sentences in which disjunctions in the consequence occur very rarely. Now, our axiom system only consists of geometric sentences, and, if we exclude the linearity axioms, only of geometric sentences whose consequent does not contain disjunctions, which highlights how their logical form is close to Euclid's spirit.

The abbreviation $\mathbb{O}$ will also be useful in discussing boundaries in section 5.5.

Total precedence

The reader who is acquainted with the logical literature on time, events and "periods" (Van Benthem, 2013) might be surprised that we do not have, like (Russell, 1936; Walker, 1947; Thomason, 1984, 1989), a primitive relation P encoding complete precedence. This choice was philosophically justified in section 4.2, but the technical justification for using primitives like R_+, R_- instead of precedence is, as we shall see, that they have a natural topological interpretation.

Nevertheless, we can define total precedence by letting $aPb := a\mathbb{O}b \wedge a\,\mathrm{R}_- b$. We can then understand consequence (1)a as enforcing transitivity of P. Indeed:

5.3.11. Lemma. *Let P be defined as aPb if $a\mathbb{O}b \wedge a\,\mathrm{R}_- b$. Then the axioms of Definition 5.3.1 imply that P is transitive.*

Proof:
Assume that bPc, cPa. Then clearly $c\mathbb{O}b, b\,\mathrm{R}_-c$, $a\mathbb{O}c, c\,\mathrm{R}_-a$. If $b\,\mathrm{R}_+c$ then $b \preceq c$ which implies that bOc by Lemma 5.3.3 (3)a, contradiction; hence $c\,\mathrm{R}_+b$ by linearity. An argument along these lines shows that $c\,\mathrm{R}_-a$, and if we reformulate consequence (1)a as follows:

$$a\mathbb{O}c \wedge c\mathbb{O}b \wedge c\,\mathrm{R}_+b \wedge c\,\mathrm{R}_-a \rightarrow a\mathbb{O}b$$

we obtain $a\mathbb{O}b$. Transitivity of R_- then implies that $b\,\mathrm{R}_-a$, whence bPa follows. □

It is straightforward to check that our axioms provide us with all the properties of P beyond transitivity that are required for the construction of instants from events proposed by Russell and Walker, but see chapter 6 for more details on this matter.

5.3.2 Finite model property

The axioms collected in GT_0 are universal; as a consequence, GT_0 is in a sense complete with respect to finite models. The same result, however, holds for GT. We can formulate the result precisely if we consider the class of geometric formulas, which were argued to be the formal analogue of Kant's judgments in Achourioti and van Lambalgen (2011).

5.3.12. Theorem. *Let φ be a geometric implication in the signature of GT. Then $GT \models \varphi$ iff φ holds on all finite models of GT.*

Proof:
The direction from left to right is trivial. For the direction from right to left, we prove the contrapositive. Assume $GT \not\models \varphi(\bar{x}, \bar{y})$, where $\varphi(\bar{x}, \bar{y})$ is of the form $\forall \bar{y}(\theta(\bar{x}, \bar{y}) \rightarrow \psi(\bar{x}, \bar{y}))$ for $\theta(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})$ positive primitive formulas. Then for some countable structure $\mathcal{M}$,

$$\mathcal{M} \models GT + \exists \bar{y}(\theta(\bar{x}, \bar{y}) \wedge \neg\psi(\bar{x}, \bar{y})).$$

Thus there must be a tuple $\bar{a}$ of objects of $\mathcal{M}$ such that $\mathcal{M} \models \theta(\bar{a}), \neg\psi(\bar{a})$. Since the tuple $\bar{a}$ is finite, Lemma 5.3.5 provides us with an object c which covers every object of $\bar{a}$. We can now define a submodel $\mathcal{M}'$ of $\mathcal{M}$ having as objects of the domain the objects in $\bar{a}$, the covering event c, and all events which can be obtained from these by closing under the operations $\oplus, \ominus$. It is straightforward to check that $\mathcal{M}'$ is a submodel of $\mathcal{M}$ and that the equational theory of $\oplus, \ominus$ ensures that $dom(\mathcal{M}')$ is finite. Hence $\mathcal{M}'$ is the desired finite model. □

Note that the formula expressing the existence of a universal cover

$$\exists x \forall y (y \preceq x)$$

is true on all finite models of GT but not on arbitrary models, showing that Theorem 5.3.12 above cannot be extended beyond geometric formulas.

5.3.3 Standard models

In the context of GT the relation O has a strong, constructive, interpretation as *overlap*, which is enforced by the axioms of group (9); GT_1 (and GT_0), on the other hand, allow for a weaker interpretation of O as *proximity*, the relation of being "infinitesimally close". We can highlight the difference in the allowed interpretations of O in GT and GT_1 more precisely by considering set-based models for the axioms. Consider in particular the set of all nonempty open intervals of the unit interval $\mathbb{I} = [0, 1]$, equipped with its natural order. These intervals are connected and order-convex. We then have the following:

5.3.13. LEMMA. *Define the structure $E(\mathbb{I})$ with domain*

$$\{a \subseteq \mathbb{I} \mid a \textit{ open, nonempty and order-convex}\}$$

by letting for any a, b:

(1) $a \mathrm{R}_+ b$ if $a \subseteq \uparrow b$

(2) $a \mathrm{R}_- b$ if $a \subseteq \downarrow b$

(3) aOb if $cl(a) \cap cl(b) \neq \emptyset$

(4) $a \oplus b = \uparrow b \cap a$ if $\uparrow b \cap a \neq \emptyset$, otherwise undefined

(5) $a \ominus b = \downarrow b \cap a$ if $\downarrow b \cap a \neq \emptyset$, otherwise undefined

Then $E(\mathbb{I}) \models GT_1$ but $E(\mathbb{I}) \not\models GT$

Proof:
The first part is easily verified by checking that all the axioms not involving $\oplus, \otimes$ are valid according to the given interpretation. The second part follows from the failure of the axioms for $\oplus, \ominus$. Indeed, we have $[0, x)O(x, 1]$, but $[0, x) \oplus (x, 1]$ is undefined. □

If in Lemma 5.3.13 we interpreted O differently, however, by letting aOb if $a \cap b \neq \emptyset$, we would have obtained a model of GT. Most importantly, GT has a concrete interpretation in the structure of rational open intervals:

5.3.14. LEMMA. *Let $E_\omega(\mathbb{I})$ be the structure having as domain all the nonempty open order-convex subsets of $\mathbb{I} \cap \mathbb{Q}$ and as relations those of Lemma 5.3.13, but let aOb if $a \cap b \neq \emptyset$. Then $E_\omega(\mathbb{I}) \models GT$.*

The proof of the following result is then tedious but straightforward:

5.3.15. PROPOSITION. *Given a finite event structure $\mathcal{W}$, there is a finite submodel $\mathcal{V}$ of $E_\omega(\mathbb{I})$ which is isomorphic to $\mathcal{W}$.*

Where the notion of isomorphism employed here relies on event maps, which shall be defined in section 5.6.2

5.3.4 Definability, operations and extensionality

The reader has perhaps wondered, when inspecting the axioms in Definition 5.3.1, whether we cannot dispense with the primitive relations O, R_+, R_- by defining them in terms of the operations $\oplus, \ominus$. In this respect, consider the following:

5.3.16. LEMMA. *Let $\mathcal{W}$ be a model of GT, and let $a, b \in W$ such that aOb. Then $c = (a \oplus b) \ominus b$ is such that $c \preceq a, c \preceq b$ and c is maximal with this property in the $\preceq$ ordering.*

Proof:
A routine argument using the group axioms (9). □

Lemma 5.3.16 above provides what we might call a "partial pseudo-meet" operation, defined as $a \sqcap b = (a \oplus b) \ominus b$. This operation is only a partial pseudo-meet and does not turn $\preceq$ into a semilattice, however, since it is only partial and $\preceq$ is merely a preorder and not a partial order. Indeed, the various properties of $\oplus, \ominus$ proven in Lemma 5.3.4 show that $\sqcap$ satisfies the properties of a meet (i) when it is defined and (ii) only up to equivalence $\equiv$ under covering. For instance, $a \sqcap b \equiv b \sqcap a$ using 5.3.4 (4), but $a \sqcap b, b \sqcap a$ are not required to be equal. Thus, since $\sqcap$ is only a partial operation we cannot define the O relation in terms of $\sqcap$. Indeed, the O relation is used in GT to determine the domain over which the partial operations $\oplus, \ominus$, and hence the partial operation $\sqcap$, is defined (axiom (9)b).

Things begin to change, however, if we introduce additional strength in the form of the *extensionality axiom*, forcing $\preceq$ to be a partial order:

$$a \preceq b \wedge b \preceq a \rightarrow a = b \tag{5.1}$$

The extensionality axiom turns $a \sqcap b = (a \oplus b) \ominus b$ into a partial meet, and we could do without the equational theory for $\oplus, \ominus$ (axioms (9)i-(9)l), since those axioms would follow straightforwardly from extensionality. The covering relation $\preceq$ would then become definable directly in terms of $\oplus, \ominus$, since

$$a \preceq b \leftrightarrow a \sqcap b = a$$

would become provable.

In order to turn $\sqcap$ into a proper meet operation one could, moreover, turn $\oplus, \ominus$ into total operations, by postulating the existence of a distinguished event 0, the "empty event", satisfying the following constraints for any events a, b:

(1) $0 \preceq a$

(2) $a\,\mathrm{R}_{+}0 \vee a\,\mathrm{R}_{-}0 \rightarrow a \preceq 0$

(3) $aOb \rightarrow a \neq \bot$

(4) $a \oplus 0 = 0 \oplus a = 0$

(5) $aOb \rightarrow a \oplus b \neq \bot$

The intuitive import of these constraints is clear. Interpreting events as intervals of $\mathbb{I}$, as we did in section 5.3.3, then 0 represents the empty set $\emptyset$, and the above constraints encode the properties of the empty set under that interpretation of the signature. In particular, O should only hold between non-empty events, as aOb is interpreted as $a \cap b \neq \emptyset$ for a, b intervals. From this fact we see that we could not merely add the above axioms to GT because this would yield an inconsistency, since in the axiomatization of O of Definition 5.3.1 it is assumed that all events are not-empty (for instance, O is axiomatized as a reflexive relation, which yields immediately an inconsistency with (3) above). To remedy the situation one could modify some of the axioms specifying that some of the events involved must be non empty. For example, axiom (3) becomes

$$a \neq 0 \rightarrow aOa$$

while (4) becomes:

$$cOb \wedge c\,\mathrm{R}_{+}a \wedge b\,\mathrm{R}_{-}a \wedge a \neq 0 \rightarrow aOb$$

and similarly for axiom (4), (9)h and (9)e; the latter in particular becomes

$$a \oplus b \neq 0 \rightarrow a\,\mathrm{R}_{-}a \oplus b$$

Most importantly, all these axioms can be easily put in geometric form using disjunctions, e.g., the axiom above is equivalent to:

$$a \oplus b \rightarrow a\,\mathrm{R}_{-}a \oplus b \vee a \oplus b = 0$$

Having done all this, one can turn $\oplus, \ominus$ into total operations by simply dropping axiom (9)b, as one easily checks that then:

$$a\,\mathrm{R}_{+}b \wedge a\mathbb{O}b \to b \oplus a = 0$$

Finally, in the presence of the extensionality axiom it follows in general that the relations R_{+}, R_{-} become explicitly definable in terms of $\oplus, \ominus$ since

$$a\,\mathrm{R}_{+}b \leftrightarrow a \oplus b = a$$

becomes provable from the axioms, so one could in principle reformulate the axiom system without using R_{+}, R_{-}. Since the meet operation is now total, we can also explicitly define O since

$$aOb \leftrightarrow a \sqcap b \neq 0$$

becomes provable. Hence, the extensionality axiom and the empty event would in principle allow one to reformulate the axioms only in terms of $\oplus, \ominus$ and 0.

One must, however, consider whether the extensionality axiom and the empty event are acceptable from a Kantian perspective, since, because of what we discussed in chapter 2, we do not want to be carried away by the formalism: every axiom must be inspected for transcendental content in agreement with Kant's philosophy for its addition to be justified. As far as the empty event is concerned, for instance, there are good Kantian reason to reject it, but we prefer to postpone the matter to section 6.6, where we shall see that the Kantian way to make $\oplus, \ominus$ into total operations is not to add an empty event but to provide O with a stronger interpretation: for a, b *possibly empty* intervals, aOb if and only if $a \cap b \neq \emptyset$. This interpretation relates O with Kant's discussion of the transcendental ideal, as it represents a positive way, in the tradition of constructive mathematics, to deal with non empty sets; but more on this in section 6.6.

Furthermore, from a mathematical standpoint, it is interesting not to explicitly define R_{+}, R_{-}, O in terms of $\oplus, \ominus$ because the weaker axiom systems GT_0 and GT_1 are interesting in their own right, since it is possible to provide a philosophically interesting construction of instants of time that does not rely on $\oplus, \ominus$ (see section 5.5). What is more, the maps between event structures that are of interest for the purpose of this work are not maps preserving $\sqcap$, but rather overlap-preserving maps, i.e., they preserve only *non-empty* meets. Thus, to define these maps we would have to define overlap in terms of $\sqcap$ anyway, so that it is more principled to have it as a primitive from the start.

A few words more should be said with respect to the extensionality axiom. Philosophically, the problem with this axiom is that it implies the identity of events existing at exactly the same times. This is in general unacceptable, since such events might for instance occur at different places; indeed, Thomason (Thomason, 1989), Russell (Russell, 1936) and Walker (Walker, 1947) do not have such an axiom. Moreover, the axiom would seem to prevent us from expressing the simultaneity of events, since in its presence there cannot be distinct events that are simultaneous. Note that this fact does relate to Kant's views on the matter, since for Kant simultaneity is not a merely temporal notion, but, as we highlighted in section 4.2.2, it is rather a spatio-temporal notion, so that in inner sense itself there is not simultaneity ("different times are not simultaneous but successive").

On the other hand, there is a case to be made for the extensionality axiom if we assume that the elements of an event structures are merely the "periods" during which events exist, and not the events themselves, following the distinction in Van Benthem (2013). We have seen in the previous chapters that Kant conceives of an event as the act of drawing a certain space in a certain time, and in this sense an event can be decomposed into a time-component and a space-component; if the elements of an event structure are taken to be the time-components of events, then the extensionality axiom is justified. Given an event structure $\mathcal{W}$, spatiotemporal, as opposed to purely temporal, events might then be represented by a set E and a map $e : E \to W$ that assigns every event to its time-period. This would allow one to discuss simultaneity of events while still assuming $\preceq$ to be a partial order.

In any case, from a mathematical point of view, for the treatment that follows the presence or absence of the extensionality axiom does not make much difference, since in its absence we could always work with *setoids* rather than sets. Recall that a setoid is simply a set A equipped with an equivalence relation $\equiv$, a "defined" notion of equality. In our setting, the relevant equivalence relation is $\equiv$, so that given an event structure $\mathcal{W}$ it is of no mathematical import whether we work with $\mathcal{W}/\equiv$, the quotient of $\mathcal{W}$ under $\equiv$ enforcing the extensionality axiom, or whether we treat $\equiv$ as our relevant notion of equality. We shall then for the moment work without the extensionality axiom, but shall return on this matter in section 5.8.

5.4 Topologies on event structures and connectedness

In this section we introduce some notions that will be of importance in the next sections on the construction of the Kantian continuum, and we also discuss the connectedness properties of event structures in relation to Kant's *dictum* that "parts of time are times" but that "time is not composed out of its parts".

5.4.1 Operations on sets of events

The relations $\mathrm{R}_+, \mathrm{R}_-$ are reflexive and transitive, that is, preorders. Hence, they lend themselves to the construction of Alexandroff topological spaces from preorders discussed in section 5.2.3. In particular, we shall denote with $\mathcal{A}_+, \mathcal{A}_-$ the Alexandroff topologies generated on an event structure $\mathcal{W}$ by $\mathrm{R}_+, \mathrm{R}_-$ respectively. For example, the open sets of $\mathcal{A}_+$ are those of the form $\{U \subseteq W \mid a \in$

$U, b\,\mathrm{R}_+ a$ implies $b \in U\}$, and similarly for $\mathcal{A}_-$. For simplicity, we also refer to $\mathcal{A}_+$ as the *future topology* and to $\mathcal{A}_-$ as the *past topology*, and speak of *past-open* and *future-open* sets.

Equipping an event structure $\mathcal{W}$ with the past and future topology turns it into a bitopological space (see section 5.2.2). Furthermore, on the basis of GT_0 one can prove, using the linearity axioms, that past-open and future-open sets can be totally ordered:

5.4.1. LEMMA. *For any two future-open sets A, B: $A \subseteq B$ or $B \subseteq A$, and similarly for past-open sets*

The relations $O, \mathbb{O}$ also have topological meaning. Let $\mathcal{W}$ be an event structure; we define two relations $O, \mathbb{O} \subseteq \mathcal{P}W \times \mathcal{P}W$ as follows:

(1) AOB if $\exists a \in A \exists b \in B\; aOb$

(2) $A\mathbb{O}B$ if $\forall a \in A \forall b \in B\; a\mathbb{O}b$

for $A, B \subseteq W$. The two relations $O, \mathbb{O}$ on sets so defined are a sort of *proximity relation* and *apartness relation* on $\mathcal{P}W$, respectively. In particular, although $\mathbb{O}$ is a symmetric relation, it gives rise to two asymmetric operations on open sets, where the asymmetry derives from the existence of the past and future topologies:

5.4.2. DEFINITION. Let $\mathcal{W}$ be an event structure. The unary operations $(\cdot)\mathbb{O} : \mathcal{A}_+ \to \mathcal{P}W$ and $\mathbb{O}(\cdot) : \mathcal{A}_- \to \mathcal{P}W$ are defined by letting:

$$A\mathbb{O} = \mathbb{O}A = \{a \in W \mid \forall b \in A\; a\mathbb{O}b\}$$

for $A \in \mathcal{A}_+$ future-open or $A \in \mathcal{A}_-$ past-open.

5.4.3. LEMMA. *Let $\mathcal{W}$ be an event structure and let U, V be past-open and future-open respectively. The following hold:*

1. *$U\mathbb{O}$ is future-open.*

2. *$\mathbb{O}V$ is past-open.*

Proof:
To see that $U\mathbb{O}$ is future-open let $b \in U\mathbb{O}$ and $c\,\mathrm{R}_+ b$. Choose $a \in U$; we must show $a\mathbb{O}c$. We have $a\mathbb{O}b$, which implies $a\,\mathbb{R}_- b$ or $a\,\mathbb{R}_+ b$. The first possibility implies $b \in U$, which is impossible. But $a\,\mathbb{R}_+ b$ together with $a\mathbb{O}b$ and $c\,\mathrm{R}_+ b$ implies $a\mathbb{O}c$. That $\mathbb{O}V$ is past-open is proven similarly. □

Thus, the relation $\mathbb{O}$ can be used to define two unary operations, one from the set $\mathcal{A}_-$ of past-open sets of an event structure $\mathcal{W}$ to the set $\mathcal{A}_+$ of future-open sets, and the other from the set of future-open sets to the set of past-open sets. Note however that, even though $\mathcal{A}_+, \mathcal{A}_-$ are complete distributive lattices (complete Heyting algebras), the two operations are not lattice maps, as they do not preserve meets. These operations will prove useful to provide a definition of Kantian boundary in the next section.

Finally, recall that the defined covering relation $\preceq$ is transitive and reflexive, hence a preorder; it then also gives rise to an Alexandroff topological space via the correspondence in section 5.2.3. In the sequel we shall make use of the self-duality of the Alexandroff topologies generated by $\preceq$, and use both the topology whose closed sets are the upsets and the topology whose closed sets are the downsets. It will always be clear from the context which topology we are using.

5.4.2 Connectedness of event structures

Continuity and connectedness as synthetic *a priori* principles for time occur under various guises in Kant's works. As far as connectedness of time is concerned, in particular, one finds two common characterizations of this notion. The first is mereological in nature, stating that "parts of time" are themselves times that cannot be "detached" from the encompassing whole of time, i.e., parts of time "can be distinguished, but not separated" (R.4425, 17:541), so that *divisio non est realis, sed logica*. The second characterization has to do with the instants in time, and states that there are no "jumps" or "leaps" from one state of a substance to another, without intermediate transitions in between.[3] In this section we analyze briefly the first notion of connectedness in relation to event structures, while in the following sections on boundaries in time we shall mostly focus on the second characterization.

In the context of point-set topology, connectedness is defined as an indecomposability condition: a topological space is connected if there are no disjoint non-empty open sets whose union is the whole space. Kant's notion of indecomposability is much stronger, but before we can elucidate it we must define what it means for a space to be connected in a bitopological setting. Since the future-open sets are linearly ordered by inclusion (Lemma 5.4.1), any event structure is trivially connected in the R_+ topology, and likewise for the R_- topology. To be able to say something more interesting we therefore need both topologies:

5.4.4. Definition. The event structure $\mathcal{W}$ is *biconnected* if there are no non-empty U, V such that U is past-open, V is fut-open, $U \cap V = \emptyset$ and $U \cup V = W$.

As a consequence of the covering axiom, we obtain the following:

[3] Thus Kant:

> There is nothing simple in appearance, hence no immediate transition from one determinate state (not of its boundary) into another [...] a hiatus, a cleft, is a lack of interconnection among appearances, where their transition is missing. (R.4756, 17:699)

5.4.5. LEMMA. *Let $\mathcal{W}$ be an event structure. Then $\mathcal{W}$ is biconnected.*

The covering axiom implies a still stronger form of connectedness, however, which can be better seen by considering the join topology on an event structure $\mathcal{W}$, that is, the topology $\mathcal{A}_+ \vee \mathcal{A}_-$ on $\mathcal{W}$ (See section 5.2 for the notion of a join topology). It is straightforward to see that in this topology open sets are downsets with respect to $\preceq$; we could then say that in this topology open sets are closed under the relation "is a logical part of". This in turn implies that closed sets are upsets with respect to $\preceq$, as the complement of a downset is an upset. This immediately yields:

5.4.6. LEMMA. *Let $\mathcal{W}$ be an event structure and let $A, B \subseteq W$ be non-empty sets closed in $\mathcal{A}_{\mathrm{R}_+} \vee \mathcal{A}_{\mathrm{R}_-}$. Then $A \cap B$ is not empty*

Proof:
Choose $a \in A, b \in B$. If $a = b$, we are done. Otherwise, by the covering axiom choose c with $a, b \preceq c$. Then since both A, B are upsets with respect to $\preceq$ we have $c \in A \cap B$. □

The result above suggests that the following stronger notion of connectedness is more appropriate for event structures:

5.4.7. DEFINITION. A topological space is *ultra-connected* if any two non-empty closed sets have non-empty intersection.

Note that in our bitopological setting, this concept is non-trivial only for sets closed in $\mathcal{A}_{\mathrm{R}_+} \vee \mathcal{A}_{\mathrm{R}_-}$, hence in the preceding definition "closed" will be taken in this sense. We then have:

5.4.8. LEMMA. *Event structures with the $\mathcal{A}_{\mathrm{R}_+} \vee \mathcal{A}_{\mathrm{R}_-}$ topology are ultra-connected.*

In the context of finite event structures, which shall be our main focus in the approximation results starting from section 5.8, the above lemma reduces to the statement that there exists a universal cover. We can also use the ultra-connectedness of event structures to model Kant's *dictum* that time cannot be "made up" from its parts. Indeed, note that given an event structure $\mathcal{W}$ and an event $a \in W$, the principal ideal $\downarrow_{\preceq} a$ of a under $\preceq$ is open in $\mathcal{A}_{\mathrm{R}_+} \vee \mathcal{A}_{\mathrm{R}_-}$, and recall that Proposition 5.3.6 states that $\downarrow_{\preceq} a$ is itself an event structure. We might then take these ideals to denote "parts of time":

5.4.9. DEFINITION. Let $\mathcal{W}$ be an event structure and let $a \in W$. Then $\downarrow_a \preceq$, the principal ideal generated by a in the $\preceq$ ordering, is a *part of time* in $\mathcal{W}$. A part of time $\mathcal{W}'$ is said to be *proper* if $\mathcal{W}' \neq \mathcal{W}$.

Now, it is straightforward to see that ultra-connectedness can be formulated for open sets of $\mathcal{A}_{R_+} \vee \mathcal{A}_{R_-}$ as follows: for any open sets U, V, $W = U \cup V$ implies either $W = U$ or $\mathcal{W} = V$. This immediately yields:

5.4.10. PROPOSITION. *Let $\mathcal{W}$ be an event structure. Then W cannot be written as the union of proper parts of time.*

The above proposition goes a long way to capture Kant's idea that time "as a whole" cannot be composed or constructed from its parts. It now remains to investigate the definition of connectedness as absence of "clefts" in time, for which we must turn to the discussion of instants or boundaries of time.

5.5 Boundaries as limitations

In this section we provide a construction of boundaries in time that is faithful to the Kantian principles discussed in chapter 4 and that relies on the topological interpretation of the primitive relations of Definition 5.3.1 presented in section 5.4. Throughout this section, unless differently specified, we only rely on axiom system GT_1 and not on the stronger GT.

The main aim of this section is then to show that the set of events can be given the structure of a one dimensional continuum, which may have some instants that arise as boundaries. We argue as follows. Kant conceives of boundaries in time as "limitations", as we saw in chapter 4; that is, for Kant time is prior to its parts, which have the same structure as time itself – in particular there are no smallest parts allowing the construction of time as a set. Recall Passage (6) in section 3.4:

> Space therefore consists only of spaces, time of times. Points and instants are only boundaries, i.e., mere places of their limitation; but places always presuppose those intuitions that limit or determine them, and from mere places, as components that could be given prior to space or time, neither space nor time can be composed. (CPR A170/B212)

This notion of boundary, however, is a topological concept, not an order-theoretic one. Informally, a temporal boundary in an event structure $\mathcal{W}$ determines a set of events $P \subseteq W$ in the past of that boundary, and likewise a set of events $F \subseteq W$ in the future of the boundary, so that it "limits" but simultaneously "links" the past and the future, as not only Kant, but also Aristotle, held (see Passage (2) in section 3.4). Furthermore, P and F ought to be $\mathbb{O}$-apart; this implies that they are set-theoretically disjoint, and the complement of $P \cup F$ can be viewed as a representation of the temporal boundary between P and F, which we might rightly call the present C. The topologies defined in the previous section will then have a temporal meaning, as the open sets of $\mathcal{A}_+$ can be used to represent the future of a boundary, the open sets of $\mathcal{A}_-$ its past and the open sets of $\mathcal{A}_{\preceq}$ its present. Note moreover that the division into past and future of a boundary is relative to the domain W of $\mathcal{W}$, and all $e \in W$ may be situated in the past from the standpoint of the *now*; that is, if $a \in F$ it does not mean that a is still to come. Temporal progression in the sense of *coming to be* is represented not by a single event structure, but by a system of event structures linked by continuous maps, as we already mentioned in chapter 4.

5.5.1 A first attempt at defining boundaries

In light of the previous observations, one might attempt to define boundaries as follows:

5.5.1. DEFINITION. Given an event structure $\mathcal{W}$, a boundary in $\mathcal{W}$ is a triple (P, C, F) of subsets of W such that the following hold:

(1) $P \cup C \cup F = W$

(2) P is past-open and F is future-open

(3) $P \mathbb{O} F$, that is, P, F are $\mathbb{O}$-apart

(4) $C = (P \cup F)^c$

5.5.2. LEMMA. *C is open in $\mathcal{A}_{\preceq}$, i.e., $d \in C$ and $d \preceq c$ implies $c \in C$.*

This simple minded definition does not fully work, however, because we cannot on its basis prove that boundaries are linearly ordered. Indeed, one might then attempt to define a linear order on boundaries by letting

5.5.3. DEFINITION. $(P, C, F) \leq (P', C', F')$ if $P \subseteq P'$.

For two boundaries $(P, C, F), (P', C', F')$. This suggestion does not yield the expected result, however, since given P only, F can be chosen independently subject only to the constraint that P and F are $\mathbb{O}$-apart; it then follows that the event structure is, in a loose sense, two-dimensional. This problem can be avoided if F is somehow completely determined by P. As we shall see, this issue is connected to the nature of the boundary between P and F, i.e. C. For instance, the present should not contain any pair of events a, b such that $a \mathbb{O} b$; if there were such a pair of events, the present could be split into two parts, one containing a but not b and the other containing b but not a, and hence it would not really constitute a boundary. In order to improve on the above definition we shall approach the problem using the operations $(\cdot)\mathbb{O}, \mathbb{O}(\cdot)$ introduced in section 5.4.

5.5.2 Boundaries from closure operators

We shall provide a better definition of Kantian boundaries by means of a construction starting from an even structure $\mathcal{W}$ and the set of pairs of the form (U, V) with $U \subseteq W$ past-open and $V \subseteq U$ future-open. This construction really amounts to defining two closure operators, one on $\mathcal{A}_+$ and the other on $\mathcal{A}_-$. A boundary will then be represented as a pair (P, F) where P, F are closed with respect to the former and the latter closure operator, respectively. Note, most importantly, that these closure operators are distinct from the topological closure operators of the past and future topologies themselves, so that the reader should not confuse the two.

5.5.4. Definition. Let $\mathcal{W}$ be an event structure and define endomaps $L(\cdot), R(\cdot)$ on the set of past-open and future-open sets, respectively, by letting

$$L(\cdot) : U \mapsto \mathbb{O}(U\mathbb{O}) \tag{5.2}$$
$$R(\cdot) : V \mapsto (\mathbb{O}V)\mathbb{O} \tag{5.3}$$

Where $U, V \subseteq W$ are respectively past-open and future-open.

We then have:

5.5.5. Theorem. *Let $\mathcal{W}$ be an event structure and let $L(\cdot), R(\cdot)$ be as in Definition 5.5.4. Then the following hold:*

(1) $L(\cdot)$ is monotone: $U \subseteq U'$ entails $\mathbb{O}(U\mathbb{O}) \subseteq \mathbb{O}(U'\mathbb{O})$

(2) $L(\cdot)$ is extensive: $U \subseteq \mathbb{O}(U\mathbb{O})$

(3) $R(\cdot)$ is monotone and extensive

(4) for any U we have $L(U) = LL(U)$, and similarly $R(V) = RR(V)$

(5) if U is $L(\cdot)$-closed, then $V = U\mathbb{O}$ is $R(\cdot)$-closed and we have $\mathbb{O}V = U$. Analogously if V is $R(\cdot)$-closed, then $U = \mathbb{O}V$ is $L(\cdot)$-closed and $U\mathbb{O} = V$

(6) the set of $L(\cdot)$-closed sets (resp. $R(\cdot)$-closed sets) is a complete linear order under inclusion

(7) $W, \emptyset$ are closed for both operators.

Proof:
For claim (1) one checks that $U \subseteq U'$ entails $U'\mathbb{O} \subseteq U\mathbb{O}$, which in turn entails $\mathbb{O}(U\mathbb{O}) \subseteq \mathbb{O}(U'\mathbb{O})$; these latter sets are past-open by Lemma 5.4.3.

For claim (2) we note that Lemma 5.4.3 implies $\mathbb{O}(U\mathbb{O})$ is past-open. Choose $a \in U, b \in U\mathbb{O}$, then by definition $a\mathbb{O}b$, hence $a \in \mathbb{O}(U\mathbb{O})$. Claim (3) is proven similarly.

For claim (4), observe that by claim (1) we have

$$\mathbb{O}(U\mathbb{O}) \subseteq \mathbb{O}((\mathbb{O}(U\mathbb{O}))\mathbb{O})$$

and by claim (3), setting $V = U\mathbb{O}$, we obtain the converse inclusion, whence

$$\mathbb{O}(U\mathbb{O}) = \mathbb{O}((\mathbb{O}(U\mathbb{O}))\mathbb{O})$$

To prove claim (5) we note that $U = \mathbb{O}(U\mathbb{O})$ implies $(\mathbb{O}V)\mathbb{O} = (\mathbb{O}(U\mathbb{O}))\mathbb{O} = U\mathbb{O} = V$.

To prove claim (6), let $U = \bigcup_i U_i$ be a union of $L(\cdot)$-closed sets. Then U need not be $L(\cdot)$-closed, but $\mathbb{O}(\bigcup_i U_i\mathbb{O})$ is the least closed set larger than the U_i. Furthermore, given a pair (U, V) we can construct two increasing sequences, where $\subseteq$ is interpreted coordinate-wise:

$$(U, V) \subseteq (U, U\mathbb{O}) \subseteq (\mathbb{O}(U\mathbb{O}), U\mathbb{O})$$

and

$$(U, V) \subseteq (\mathbb{O}V, V) \subseteq (\mathbb{O}V, (\mathbb{O}V)\mathbb{O})$$

Since the past-opens are linearly ordered by $\subseteq$ (Lemma 5.4.1), we may fuse the two sequences by ordering them linearly according to the first coordinate

$$(U, V) \leq (\mathbb{O}(U\mathbb{O}), V) \leq ((\mathbb{O}V, V) \leq (\mathbb{O}V, (\mathbb{O}V)\mathbb{O}),$$

which gives, by Theorem 5.5.5 (5), for any past-open U least and greatest extensions that are fixpoints.

The proof of claim (7) is straightforward. □

The theorem above shows that $L(\cdot), R(\cdot)$ are *closure operators* on the lattices of past-open and future-open sets of an event structure, respectively. We can now define boundaries of an event structure in terms of these closure operators by considering $L(\cdot), R(\cdot)$ as a single operation $(L(\cdot), R(\cdot))$ on the product lattice $\mathcal{A}_- \times \mathcal{A}_+$ of past-open and future-open sets, as follows:

5.5.6. Definition. Let $\mathcal{W}$ be an event structure. A *boundary* is a tuple (P, C, F) of subsets of W such that the following hold:

1. P, F are past-open and future-open respectively
2. (P, F) is closed for the closure operator $(L(\cdot), R(\cdot))$
3. $F = P\mathbb{O}$ (and hence $P = \mathbb{O}F$ because of Theorem 5.5.5 (5))

4. $C = (P \cup F)^c$

5.5.7. LEMMA. *Let $\mathcal{W}$ be an event structure and let $a, b \in W$ be such that $a \mathbb{O} b, a\,\mathrm{R}\_b$, i.e., a completely precedes b. Then there exists a boundary (P, C, F) with $a \in P$, $b \in F$.*

Proof:
Let $U_a = \{c \mid c\,\mathrm{R}\_a\}$. Since $a\mathbb{O}b$, $b \in U_a\mathbb{O}$, $b \notin \mathbb{O}(U_a\mathbb{O})$. Thus, we let $P = \mathbb{O}(U_a\mathbb{O})$, $F = P\mathbb{O}$, $C = (P \cup F)^c$. Clearly $a \in P$ because of Theorem 5.5.5, and since $a\mathbb{O}b$, $b \in F$. □

The boundaries from Definition 5.5.6 satisfy various properties, which allow us to provide an explicit definition of when a triple of sets is such a boundary. In particular:

5.5.8. PROPOSITION. *Let $\mathcal{W}$ be an event structure and let (P, C, F) be a boundary according to Definition 5.5.6. Then the following hold:*

(1) P and F are $\mathbb{O}$-apart

(2) $P \cup C \cup F = W$ and P, C, F are all disjoint.

(3) C is empty if and only if either P or F are empty

(4) $C \neq W$

(5) For any $a \in W$, if a overlaps with an event in P and an event in F then it belongs to C

(6) For any $a, b \in C$ aOb

(7) For any $a \in C$ if P is not empty then there is $b \in P$ with aOb

(8) For any $a \in C$ if F is not empty then there is $b \in F$ with aOb

Proof:
Property (1) follows from the definition of the closure operators.

Property (2) is obvious from the fact that $C = (P \cup F)^c$ and property (1).

For property (3) consider first the left-to-right direction; we show the contrapositive, i.e., that if P, F are both not empty then C is not empty. Assume then that P, F are not empty, and pick $a \in P, b \in F$. Axiom (8) of Definition 5.3.1 (the covering axiom) implies that there exists an event $c \in W$ with $a, b \preceq c$. But then aOc, bOc, hence by property (1) $c \notin P$ and $c \notin F$, but then $c \in C$.

For the right-to-left direction it suffices to note that if P is empty then $F = P\mathbb{O} = W$, hence $C = (P \cup F)^c = W^c = \emptyset$, and similarly if F is empty. Property (4) then follows straightforwardly.

Property (5) follows from properties (1) and property (2).

For property (6), assume that $a, b \in C$ but $a\mathbb{O}b$. Then by linearity one of a, b precedes the other; assume w.l.o.g. that a precedes b. It is easy to show from the axioms of Definition 5.3.1 that if cOb for $c \in P$ then $a\,\mathrm{R}_{-}c$ so that since P is past-open we would have $a \in P$, which is impossible because of property (2). Hence $\{b\}$ and P are $\mathbb{O}$-apart, so that $b \in F$, which yields a contradiction.

For property (7) assume P is not empty and pick $a \in C$. If $\{a\}$ and P are $\mathbb{O}$-apart then $a \in F$ which yields a contradiction with property (2). Hence there exists $b \in P$ with aOb. □

In light of Proposition 5.5.8 we then have the following characterization of boundaries:

5.5.9. PROPOSITION. *Let $\mathcal{W}$ be an event structure. A tuple (P, C, F) of subsets of W is a boundary if and only if it satisfies:*

(1) $P \cup C \cup F = W$

(2) If one of P or F are empty then so is C

(3) P is past-open and F is future-open

(4) P and F are $\mathbb{O}$-apart

(5) For any $a \in C$ there is $b \in P$ with aOb

(6) For any $a \in C$ there is $b \in F$ with aOb

Proof:
For the left-to-right direction, let (P, C, F) be a boundary; then Proposition 5.5.8 implies that (P, C, F) satisfies all conditions of Proposition 5.5.9. For the right-to-left direction, let (P, C, F) be a tuple of subsets of W that satisfies the conditions of Proposition 5.5.9; we show that it satisfies the properties of Definition 5.5.6.

Obviously P, F are past-open and future-open respectively because of (3) above.

To see that (P, F) are closed for the closure operator $(L(\cdot), R(\cdot))$, consider first $L(P) = \mathbb{O}(P\mathbb{O})$. Clearly $P \subseteq \mathbb{O}(P\mathbb{O})$ since $L(\cdot)$ is extensive, so we only need to show that $\mathbb{O}(P\mathbb{O}) \subseteq P$. Now, if P is empty then clearly so is $\mathbb{O}(P\mathbb{O})$, and we are done. Otherwise, let $a \in W$; we show that if $a \notin P$ then $a \notin \mathbb{O}(P\mathbb{O})$. Let then $a \notin P$; then either $a \in C$ or $a \in F$ because of (1) above.

Assume $a \in F$. Then by property (4) we have that $F \subseteq P\mathbb{O}$ and so $a \in P\mathbb{O}$ which implies $a \notin \mathbb{O}(P\mathbb{O})$.

Assume $a \in C$. Then by property (2) above also F is not empty, hence by property (6) above there is $b \in F$ with aOb. By property (4) we have that $F \subseteq P\mathbb{O}$ so $b \in P\mathbb{O}$, but then $a \notin \mathbb{O}(P\mathbb{O})$ and we are done.

The proof of the claim that $R(F) = F$ is similar.

To see that $F = P\mathbb{O}$ we reason as follows. First, $F \subseteq P\mathbb{O}$ because of property (4) above. We then need to show that $P\mathbb{O} \subseteq F$. If P is empty then so is C by property (2), hence by property (1) $F = W$ and we are done. Otherwise, let $a \notin F$; we show that $a \notin P\mathbb{O}$. By property (1) either $a \in P$ or $a \in C$. In the former case clearly $a \notin P\mathbb{O}$ by reflexivity of O. In the latter case then since since P is not empty this implies by property (5) above that a overlaps with an event in P, but then also $a \notin P\mathbb{O}$.

Finally, we show that $C = (P \cup F)^c$. To show this because of property (1) it suffices to show that C is disjoint from both P, F. If C is empty we are done. Otherwise consider any $a \in C$; by property (2) it follows that F, P are not empty, hence by properties (6) and (5) a overlaps with some event in F and some event in P, and so it cannot be in P nor F because of property (4). □

We can now show that the set of boundaries defined from an event structure $\mathcal{W}$ can be linearly ordered, as follows:

5.5.10. PROPOSITION. *Let $\mathcal{W}$ be an event structure and $\leq$ be a binary relation on the set of boundaries of $\mathcal{W}$ defined by letting $(P, C, F) \leq (P', C', F')$ if $P \subseteq P'$. Then $\leq$ is a complete linear order.*

Proof:
The inclusion order is linear because pasts of boundaries are past-open sets and Lemma 5.4.1. The linear order is complete because of Theorem 5.5.5 (6). □

5.5.11. COROLLARY. *The set of boundaries of an event structure equipped with the order topology is Hausdorff and compact.*

Proof:
The first observation is standard. Compactness follows from the fact that in a linear order that is a complete lattice, each closed interval is compact, combined with Proposition 5.5.10. □

5.5.12. DEFINITION. Let $\mathcal{W}$ be an event structure. The space of boundaries $B(\mathcal{W})$ of $\mathcal{W}$ is the set of boundaries according to Definition 5.5.6 ordered under inclusion of pasts as in Proposition 5.5.10.

In what follows, we shall often write $x \in B(\mathcal{W})$ for a given boundary of $\mathcal{W}$, and denote the past, present and future of x as P_i, C_i, F_i, respectively. Note that $B(\mathcal{W})$ can be considered as a LOTS by taking the order topology.

The following result, which we shall need in the sequel, only holds if the event structure is a model of GT:

5.5.13. LEMMA. *Let $\mathcal{W}$ be an event structure that satisfies the full set of axioms GT and let $x, y \in B(\mathcal{W})$ be boundaries such that $x < y$. Then there exists an event $a \in \mathcal{W}$ such that $a \in P_y \cap F_x$.*

Proof:
Since $x < y$ we must have that $P_x \subset P_y$ hence there exists $a \in P_y, a \notin P_x$. If $a \in F_x$ we are done. Otherwise $a \notin F_x, a \in C_x$. By Proposition 5.5.9 since C_x is not empty then F_x is also not empty and there exists $c \in F_x$ with aOc. We let $b = a \oplus c$ and we are done. □

5.5.3 Boundaries and the infinity of time

The boundaries introduced in the previous section capture Kant's conception of boundaries in time as "limitations" between parts of time quite well. There is an interesting philosophical distinction to be made, however, between what we shall term "two-sided boundaries" and "formal boundaries".

5.5.14. DEFINITION. Let $\mathcal{W}$ be an event structure and $x \in B(\mathcal{W})$ a boundary. Then x is a *two-sided boundary* if C_x, and consequently P_x, F_x are not empty, while it is a *formal boundary* if C_x is empty.

There are only two formal boundaries, those of the form $(\emptyset, \emptyset, W), (W, \emptyset, \emptyset)$, which are respectively the minimum and maximum of the complete lattice of boundaries. Recall Passage (5) in section 3.4, in which Aristotle argues that instants, since they are limitations of parts of time, cannot have empty past or future, but are always preceded by a time; two-sided boundaries capture exactly this intuition formally. Formal boundaries, on the other hand, are different, since they lie strictly "beyond" any event, and can be taken as a formal representation at the level of boundaries of Kant's construal of the infinity of time.

More precisely, let us recall Kant's notion of infinity of time, which we already examined in section 3.8.5 and section 4.5.1:

> The infinitude of time signifies nothing more than that every determinate magnitude of time is only possible through limitations of a single time grounding it. The original representation of time must there be given as unlimited. But where the parts themselves [...] can be determinately represented only through limitation, there the entire representation cannot be given through concepts [...] but immediate intuition must ground them. (A31-2/B47-8)

We interpret "magnitude of time" as a function defined on intervals that are determined by boundaries. Given Kant's concept of number, a "determinate magnitude" is a function that takes only rational values; since magnitudes must be continuous, and the values of the "determinate magnitude" are closed points, the domain of the magnitude must consist of closed intervals. In an event structure $\mathcal{W}$, the time elapsed until *now* corresponds to a closed interval, with *now* represented as the maximum of the set of two-sided boundaries; that is, $N = \bigvee\{x \in B(\mathcal{W}) \mid x \text{ is two-sided.}\}$. Obviously time does not stop *now*; indeed, the events contained in F_N, which are contained in the interval between N and the formal boundary $(W, \emptyset, \emptyset)$ and, as it is easy to see, are pairwise overlapping, represent the *potentiality* for the coming to be of the future. In Husserlian terms, they are a formal representation of the *protension* of the subject towards the future. Thus, the interval between the maximum and minimum of the set of two-sided boundaries - the *now*, as we have seen, and what we might call the *origin* - is the "bounded" time; while the whole space $\mathcal{W}$, which is contained between the two formal boundaries, is the "infinite" or "unbounded" time.

5.5.4 Closure operators and geometric formulas

GT_1 is a first-order theory of events, but we have here been mostly concerned with higher order constructions: pasts are sets of events, boundaries are triples of such sets, and the linear order is a set of pairs of these triples. Since our focus is the linear continuum derived from event structures, the logical and set theoretic principles involved in the construction have to be scrutinized for their Kantian content (or lack thereof). Here it is essential to recall that events are only virtual parts of time; they cannot be detached and collected into a set by means of a comprehension axiom. To have a set of events is then to have a rule (or rules) for marking these events on the timeline. Since there is an intimate connection between constructive rules and geometric formulas, we have to investigate whether the sets of interest – e.g. the closed sets of the closure operator $U \mapsto \mathbb{O}(U\mathbb{O})$ – are somehow definable geometrically. Using the excluded middle, we may write $c \in \mathbb{O}(U\mathbb{O})$ as

$$\forall b(bOc \to \exists a \in U \ aOb).$$

Suppose we start with a past-open U_0 which is given by a geometric formula. Applying the operation, we obtain the past-open U_1 defined by

$$U_1 = \{c \mid \forall b(bOc \to \exists a \in U_0 \ aOb)\},$$

from which it follows that U_1 is also given by a geometric formula. Moreover U_1 is a closed set for the $L(\cdot)$ closure operator, hence the rule is determined in two steps. As we shall see later, geometric formulas have preservation properties that are important in dealing with boundaries on infinite event structures.

5.6 Infinitesimal intervals and the general form of Kantian continua

The construction of boundaries presented in the previous section is quite satisfactory, not only mathematically but also philosophically, as it captures closely Kant's (and Aristotle's) claims about boundaries in time. It is, however, wanting from one respect: the space of boundaries $B(\mathcal{W})$ of an event structure $\mathcal{W}$ is compact but it is not necessarily connected, and hence it can hardly be said to be a *continuum*. The problem lies in the fact that a total order is connected only if it does not have any *jumps*, i.e., no pair of points (x, y) with $x < y$ and $\neg\exists z\ x < z, z < y$. However, we cannot certainly impose an axiom on event structures that ensures density of boundaries, as (Russell, 1936) or (Walker, 1947) would have it, because this would rule out finite models for our axioms, and we are especially interested in these finite models as a correlate to the figurative synthesis.

We shall then pursue a different strategy and fill the jumps in the set of boundaries with a new sort of "instants" that we shall rather call "infinitesimal intervals", since they represent the fleeting time between two boundaries that Brouwer mentions at Passage (4) and that act as a sort of "glue" between boundaries. Moreover, they are closely related to Kant's notion of infinitesimals generated by a flowing magnitude, which will be considered in 5.11. Hence, we adopt a terminology that is slightly more specialized than Kant's: the term *instant* will refer to either boundaries or infinitesimal intervals in time. In this section we again assume only GT_0, unless specified further.

5.6.1 Infinitesimal intervals

We begin with the important definition of a maximal overlapping class of events, already employed in Russell (1936):

5.6.1. Definition. Let $\mathcal{W}$ be an event structure and $A \subseteq W$. We say that A is an *overlapping class* of events if any two events in the class overlap. It is a *maximal overlapping class* if it is an overlapping class such that there is no overlapping class B with $A \subset B$.

We can now relax Definition 5.5.6 to the definition of an *instant*:

5.6.2. Definition. Let $\mathcal{W}$ be an event structure. An *instant* of $\mathcal{W}$ is a tuple (P, C, F) of subsets of W such that the following hold:

(1) P, F are past-open and future-open respectively

(2) (P, F) is closed for the closure operator $(L(\cdot), R(\cdot))$

(3) $F \subseteq P\mathbb{O}$ (hence $P \subseteq \mathbb{O}F$ because of Theorem 5.5.5)

(4) $C = (P \cup F)^c$ is an overlapping class

The point of the definition of an instant is to relax Definition 5.5.6 so that the pair P and F need not necessarily be "adjacent" or "touching" but can be apart, as long as the cleft between them is "small enough" that no determination of succession, in the form of complete precedence, is possible within it. Indeed, note that condition (3) is now just equivalent to requiring P, F to be $\mathbb{O}$-apart. Thus, while in Definition 5.5.6 P and F are always "touching", here we aim to allow them to be "almost touching". In this latter case the cleft between them, i.e. the present $(P \cup F)^c$, will be interpreted as an "enduring" present, by means of which we shall elucidate further Kant's notion of the description of a space (see section 5.11). We then posit the following:

5.6.3. Definition. Let $\mathcal{W}$ be an event structure and let (P, C, F) be an instant. Then i is an *infinitesimal interval* if $F \subset P\mathbb{O}$, i.e., F is a strict subset of $P\mathbb{O}$.

Note that if an instant is not an infinitesimal interval then it is simply a boundary in agreement with Definition 5.5.6, since condition (4) of Definition 5.6.2 follows from Definition 5.5.6 (see Proposition 5.5.8 (6)). Hence the properties that we proved in the previous section still hold conditionally for these boundaries. For the infinitesimal intervals we instead have:

5.6.4. Proposition. *Let $\mathcal{W}$ be an event structure and let i be an infinitesimal interval according to Definition 5.6.2. Then the following hold:*

(1) Properties (1), (2), (5) and (6) of Proposition 5.5.8 hold of i

(2) C_i is a maximal overlapping class

Proof:
We only show that C_i is a maximal overlapping class as the proof for the other properties is merely a variation on the proof of Proposition 5.5.8. First note that if i is an infinitesimal interval then $F_i \subset P_i\mathbb{O}$ and $P_i \subset \mathbb{O}F_i$. Let then $a \in W$ be such that aOb for any $b \in C_i$. We show that $b \in C_i$, i.e. $b \notin P_i, b \notin F_i$. Assume towards a contradiction that $b \in P_i$. Since $F_i \subset P_i\mathbb{O}$ there exists $c \in W$ with $c \in P_i\mathbb{O}$ and $c \notin F_i$; it then follows from property (2) of Proposition 5.5.8 that $c \in C_i$, and by assumption that cOb, which yields a contradiction. The proof for $b \notin F_b$ is analogous, and we are done. □

Note that it is not the case that if C_i is a maximal overlapping class then i is an infinitesimal interval, unless one imposes additional conditions on event structures, as the following example shows:

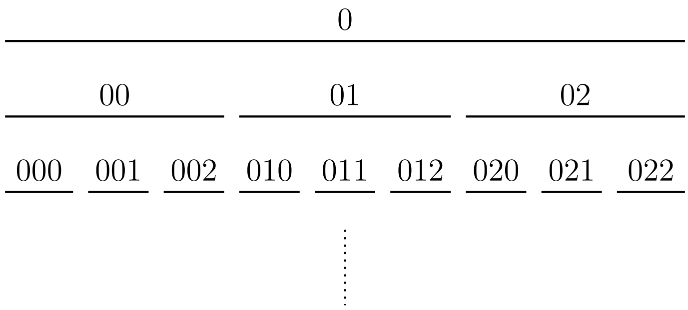
5.6.5. Example. Let $T = \bigcup_{n\in\omega}\{0,1,2\}^n$ be the set of all the finite ternary sequences. Denote with $s\star$ for $\star \in \{0,1,2\}$ the sequence obtained by extending s with $\star$ in the last position. Define relations on T by letting, for any $s, s' \in T$:

(1) $s\,\mathrm{R}_{-}s'$, $s\mathbb{O}s'$ if $s(n) < s'(n)$ for some n

(2) $s \preceq s'$ if s' is an initial segment of s

(3) $s0 \equiv_{+} s$, $s2 \equiv_{-} s$.

Then $\mathcal{T}$ defined in this way is an event structure (a model of GT_1) which looks as follows:



It is straightforward to check that the sets of events $P = \bigcup_{n\in\omega}\{a \mid a\,\mathrm{R}_{-}01^n0\}$ and $F = \bigcup_{n\in\omega}\{a \mid a\,\mathrm{R}_{+}02^n0\}$ are the past and future of a boundary $x \in B(\mathcal{T})$, i.e., $P\mathbb{O} = F$, and moreover that $C = \bigcup_{n\in\omega}\{01^n\}$ is a maximal overlapping class.

As we already did in section 5.5, of course, we can explicitly characterize when a tuple of the form (P, C, F) is an instant:

5.6.6. Proposition. *Let $\mathcal{W}$ be an event structure. A tuple (P, C, F) of subsets of W is an instant if and only if it satisfies:*

(1) $P \cup C \cup F = W$

(2) P, F are disjoint from C

(3) P is past-open and F is future-open

(4) P and F are $\mathbb{O}$-apart

(5) C is an overlapping class

(6) For any $a \in C$ there is b such that aOb and $b\mathbb{O}c$ for any $c \in P$

(7) For any $a \in C$ there is b such that aOb and $b\mathbb{O}c$ for any $c \in F$

Proof:
A straightforward variation on the proof of Proposition 5.5.9. □

Of course, the question now arises whether we can endow the set of instants with a total order and a topology that improves on what was achieved in the previous section. To this effect, recall that a jump of a linear order $(L, \leq)$ is a pair of elements $x, y \in L$ such that $x < y$ and there exists no $z \in L$ with $x < z < y$. The following result shows that if we construct the linear order of boundaries $B(\mathcal{W})$ on an event structure, the jumps of the linear order are in one to one correspondence with the infinitesimal intervals as introduced above:

5.6.7. LEMMA. *Let $\mathcal{W}$ be an event structure and let $B(\mathcal{W})$ be its set of boundaries. Then there exists a bijection j from the set of jumps of $B(\mathcal{W})$ to the set of infinitesimal intervals on $\mathcal{W}$.*

Proof:
Let $\mathcal{W}$ be an event structure. We construct a bijection between the set of jumps of $B(\mathcal{W})$ and the set of infinitesimal intervals of $\mathcal{W}$ as follows. Let $x, y \in B(\mathcal{W})$ such that (x, y) is a jump, and consider the tuple (P_x, C, F_y), where $C = (P_x \cup F_y)^c$; we claim that this tuple is an infinitesimal interval. Conditions (1), (2), (3) of Definition 5.6.2 are trivially verified; we show that C is an overlapping class. Indeed, choose $a, b \in (P_x \cup F_y)^c$ and assume towards a contradiction that $a \mathbb{O} b$. Then because of the linearity axioms one of a, b precedes the other, say a precedes b. But then by Lemma 5.5.7 there exists a boundary z with $a \in P_z, b \in F_z$, and it is a straightforward matter to check that $P_x \subset P_z \subset P_y$ so that $x < z < y$; this leads to a contradiction since (x, y) was assumed to be a jump, and we are done. To show that (P_x, C, F_y) is an infinitesimal interval it remains to show that $P_x \mathbb{O} \neq F_y$, but this follows because otherwise $x = y$, but (x, y) was supposed to be a jump.

We can thus define a map j from the set of jumps of $B(\mathcal{W})$ to the set of infinitesimal intervals by associating to any jump (x, y) the infinitesimal interval (P_x, C, F_y) defined as above. Clearly, if $(x, y), (z, w)$ are two distinct jumps then $j(x, y) \neq j(z, w)$ since they will have distinct pasts, hence the map is injective. To see that j is surjective let i be an infinitesimal interval and consider $x = (P_i, (P_i \cup P_i \mathbb{O})^c, P_i \mathbb{O})$ and $y = (\mathbb{O} F_i, (\mathbb{O} F_i \cup F_i)^c, F_i)$; clearly $x, y \in B(\mathcal{W})$ with $x < y$ since $P_i \subset \mathbb{O} F_i$ as i is an infinitesimal interval. We now claim that (x, y) is a jump. Indeed, assume towards a contradiction that there exists $z \in B(\mathcal{W})$ with $x < z < y$. Hence $P_i \subset P_z \subset \mathbb{O} F_i$ which implies $F_z \subset F_i$. Hence there are $a \in P_z, a \notin P_i$ and $b \in F_z, b \notin F_i$, hence $a, b \in C_i$ and $a \mathbb{O} b$, but C_i is an overlapping class, contradiction. □

Note that the results above allow us to linearly order the set of infinitesimal intervals on an event structure $\mathcal{W}$, where this linear order is simply that induced by the linear order of the jumps of $B(\mathcal{W})$. Hence, we obtain:

5.6.8. Lemma. *Let $\mathcal{W}$ be an event structure and let i be an infinitesimal interval, then there are $x, y \in B(\mathcal{W})$ with $x = i^-, y = i^+$.*

5.6.9. Proposition. *Let $\mathcal{W}$ be an event structure. The set of infinitesimal intervals of $\mathcal{W}$, denoted $I(\mathcal{W})$, is linearly ordered by letting $i \leq i'$ if $P_i \subseteq P_{i'}$ for any $i, i' \in I(\mathcal{W})$.*

Most importantly the whole set of instants on $\mathcal{W}$ according to Definition 5.6.2 - the boundaries and infinitesimal intervals - can be linearly ordered. We must, however, define the order by requiring not only inclusion of pasts but also inclusion of futures, because in general a boundary and infinitesimal interval can have the same past or the same future, but not both. In particular, the past and future of an infinitesimal interval $i \in I(\mathcal{W})$ are determined by the past and future of the boundaries $x, y \in B(\mathcal{W})$ such that $j(i) = (x, y)$, where j is the bijection of Lemma 5.6.7. We then have:

5.6.10. Proposition. *Let $\mathcal{W}$ be an event structure. The set of instants of $\mathcal{W}$ according to Definition 5.6.2, denoted with $K(\mathcal{W})$, is totally ordered by letting $x \leq y$ if $P_x \subseteq P_y$ and $F_y \subseteq F_x$ for any $x, y \in K(\mathcal{W})$, and we have:*

(1) $\leq$ is a complete linearly ordered lattice

(2) $B(\mathcal{W})$ is dense in $I(\mathcal{W})$, that is for any $i, i' \in I(\mathcal{W})$ with $i < i'$ there exists $x \in B(\mathcal{W})$ with $i < x < i'$

(3) Two boundaries $x, y \in B(\mathcal{W})$ define a jump (x, y) if and only if there exists exactly one infinitesimal interval $i \in I(\mathcal{W})$ with $x < i < y$.

We must now address the question of which topology to impose on $K(\mathcal{W})$, which is essential to discuss Kant's properties of time that have to do with continuity and connectedness. We have seen that the order topology of $B(\mathcal{W})$ can be totally disconnected, for instance when $B(\mathcal{W})$ is finite. In order to remedy this situation, we introduced the infinitesimal intervals; Lemma 5.6.7 shows that $I(\mathcal{W})$ acts as a kind of "filling glue" for jumps in $B(\mathcal{W})$. We cannot, however, impose the order topology on $K(\mathcal{W})$, because this would just reproduce the previous situation and yield in general a disconnected topological space, since an infinitesimal interval and one of the adjacent boundaries would induce a partition of the space into disjoint open sets. We shall then impose on $K(\mathcal{W})$ the order topology of $B(\mathcal{W})$ rather than the order topology of $K(\mathcal{W})$ itself; in other words, the topology on $K(\mathcal{W})$ will be generated only by the subbasis of order-open rays of $B(\mathcal{W})$, as follows:

5.6.11. Theorem. *Let $\mathcal{W}$ be an event structure. Let $\tau \subseteq \mathcal{P}K(\mathcal{W})$ be the topology on $K(\mathcal{W})$ generated by the subbase of sets of the form $\{(x, \leftarrow) \mid x \in B(\mathcal{W})\} \cup \{(x, \rightarrow) \mid x \in B(\mathcal{W})\}$. Then $(K(\mathcal{W}), \tau)$ is a compact connected T_0 ordered topological space such that every boundary is closed and every infinitesimal interval is open.*

Proof:
The T_0 property follows straightforwardly using Proposition 5.6.10.

To see compactness, consider first any infinitesimal interval $i \in I(\mathcal{W})$ and boundary $x \in B(\mathcal{W})$ such that $x = i^-$ or $x = i^+$, that is $P_x = P_i$ or $F_x = F_y$; it is straightforward to show that for any subbasic open set U:

$$\text{If } x \in U \text{ then } i \in U \tag{5.4}$$

Hence $x \sqsubseteq i$ for the specialization ordering $\sqsubseteq$. Now consider any open cover $\mathcal{C}$ of $K(\mathcal{W})$. Then $\mathcal{C}$ is also an open cover of $B(\mathcal{W})$, but by compactness and the fact that τ is the order-topology when restricted to $B(\mathcal{W})$ we have that there exists a finite subcover $\mathcal{C}' \subseteq \mathcal{C}$. However this subcover must cover the whole space $K(\mathcal{W})$; indeed take any $i \in I(\mathcal{W})$ and let $x = (P_i, (P_i \cup P_i\mathbb{O})^c, P_i\mathbb{O})$, then there exists $U \in \mathcal{C}'$ with $x \in U$, but by 5.4 it follows that $i \in U$ and we are done.

To see connectedness just consider that $B(\mathcal{W})$ with its order topology could be disconnected only because any jump (x, y) induces a separation $S = \{(y, \leftarrow), (x, \rightarrow)\}$ of $B(\mathcal{W})$. However, in $K(\mathcal{W})$ any such S is not a separation, because the infinitesimal interval between x, y belongs to both $(y, \leftarrow)$ and $(x, \rightarrow)$.

Finally, clearly any $i \in I(\mathcal{W})$ is open in τ since $\{i\} = (i^+, \leftarrow) \cap (i^-, \rightarrow)$, and any $x \in B(\mathcal{W})$ is closed in τ since $\{x\} = ((x, \leftarrow) \cup (x, \rightarrow))^c$. □

We shall abuse our notation and indicate with $K(\mathcal{W})$ the ordered topological space of Theorem 5.6.11. From the perspective of ordered topological spaces, Theorem 5.6.11 shows that $K(\mathcal{W})$ is a COTS, that is, a *connected ordered space* (Khalimsky, Kopperman, & Meyer, 1990). This fact highlights that there is a close relation between the theory of well-formed ordered spaces (of which COTS are a special case), as studied in Kopperman, Kronheimer, and Wilson (1998), and event structures as studied in this paper. We shall return on this topic in chapter 6.

We now have a way to construct a continuum from events which closely captures not only Kant's insights on the matter, but also the general continuist insight expressed by Brouwer in Passage (4) that between any two instants of time there is an inexhaustible "in-between": the infinitesimal intervals capture exactly the latter, and their role in this sense will become clearer once we consider the potential infinite divisibility of time in section 5.8.

5.6.2 Representation of events as intervals

A fundamental question in the logical literature on the construction of time from events is: given an event structure, how can an event be represented as an interval of its space of instants so that the relevant temporal relations are preserved? In our setting, this amounts to representing an event in an event structure $\mathcal{W}$ as an interval of $K(\mathcal{W})$ so that some topologically meaningful relations among those of

Defintion 5.3.1 are preserved. Of course, an answer to this problem depends on deciding what counts as a *map* between event structures, i.e., we must provide a category of event structures and event maps. In Thomason (1989), Thomason considers as maps those functions that preserve complete precedence. We, on the contrary, shall take as maps those functions that preserve R_+, R_-, O and also the operations $\oplus, \ominus$ up to equivalence $\equiv$. The philosophical reason for this choice is that these maps preserve just enough structure to be able to model the potential infinite divisibility of time, as we outlined in section 4.5. The mathematical reason is that these maps correspond to the "right" maps on the space of instants on event structures, that is, order-preserving continuous maps. In particular, such maps would allow us to obtain an equivalence of categories that is closely related to that in point-free topology between topological spaces and locales (Johnstone, 1986) or, even closer to the present setting, between topological spaces and formal topologies (Sambin, 2003) (see chapter 6). For the moment, the following suffices:

5.6.12. DEFINITION. Let $\mathcal{W}, \mathcal{W}'$ be event structures. A function $f : \mathcal{W} \to \mathcal{W}'$ is an *event map*, or simply a *map*, if it preserves O, R_+, R_-, and, in GT, it preserves the operations $\oplus, \ominus$ up to $\equiv$, that is:

$$f(a \oplus b) \equiv f(a) \oplus f(b)$$

and similarly for $\ominus$.

Note that a map must preserve the covering relation $\preceq$, and that because of the Alexandroff correspondence (see section 5.2.3) it can be seen in two complementary ways, that is, either as a homomorphism of models of a first-order theory or as a bicontinuous map between bitopological spaces preserving the "tolerance" or "proximity" O. We shall often switch between these perspectives in the sequel, but note that these two ways of seeing maps between event structures are not in general equivalent when one takes inverse limits of inverse systems of event structures; see in particular section 5.9. Furthermore, note that we require preservation of $\oplus, \ominus$ up to the defined equality $\equiv$; we shall discuss this choice in section 5.8 in relation to the axiom of extensionality.

We can now construct a map f from an event structure $\mathcal{W}$ to a canonical event structure induced by its space of instants $K(\mathcal{W})$ as follows:

5.6.13. DEFINITION. Let $\mathcal{W}$ be an event structure and let $S \subseteq W$ be a subset of W. We define the left and right endpoints of S, denoted as $l(S), r(S)$ for $l(S), r(S) \in K(\mathcal{W})$, by letting:

(1) $l(S) = \bigvee \{x \in K(\mathcal{W}) \mid S \subseteq F_x\}$

(2) $r(S) = \bigwedge \{x \in K(\mathcal{W}) \mid S \subseteq P_x\}$

If $S = \{a\}$ for some $a \in W$, then we abuse our notation and denote $l(\{a\})$ simply as $l(a)$, and similarly for $r(\cdot)$.

The instants $l(S), r(S)$ of Definition 5.6.13 are well defined since by Proposition 5.6.10 $K(\mathcal{W})$ is a complete lattice. Moreover we have:

5.6.14. Lemma. *Let $\mathcal{W}$ be an event structure and $S \subseteq W$. Then $l(S), r(S) \in B(\mathcal{W})$ and moreover $S \subseteq F_{l(S)}$, $S \subseteq P_{r(S)}$.*

Proof:
Consider an event structure $\mathcal{W}$ and fix $S \subseteq W$; we show both claims for $l(S)$ since for $r(S)$ the reasoning is symmetric.

For the first claim, assume towards a contradiction that $l(S) \in I(\mathcal{W})$, i.e., the smallest upper bound of the set $A = \{x \in K(\mathcal{W}) \mid S \subseteq F_x\}$ is an infinitesimal interval. Then clearly $S \not\subseteq F_{l(S)^+}$, otherwise $l(S)$ would not be an upper bound of A; but since $F_{l(S)^+} = F_{l(S)}$ because $l(S) \in I(\mathcal{W})$, this implies that $S \not\subseteq F_{l(S)}$. However, since $l(S)^-$ is not an upper bound of A there must be an instant $x > l(S)^-$ with $x \in A$, which yields a contradiction since necessarily $F_x \subseteq F_{l(S)}$, and we are done.

For the second claim we can then assume that $l(S) \in B(\mathcal{W})$. Now if $l(S) = \bigvee\{x \in K(\mathcal{W}) \mid S \subseteq F_x\}$ is a formal boundary then the claim follows trivially. Otherwise let $L = \{P_x \mid x \in B(\mathcal{W}), S \subseteq F_x\}$. Then $\bigvee\{x \in B(\mathcal{W}) \mid S \subseteq F_x\} = \mathbb{O}(\bigcup L\mathbb{O})$ by Theorem 5.5.5. Since $S \subseteq F_x$ for all x, we have $a\mathbb{O}b$ for any $a \in S$, $b \in \bigcup L$ and hence $S \subseteq \bigcup L\mathbb{O}$, hence $a\mathbb{O}c$ for any $a \in S$, $c \in \mathbb{O}(\bigcup L\mathbb{O})$, thus no element a of S can be in the present of $\bigvee\{x \in B(\mathcal{W}) \mid a \in F_x\}$ because of Proposition 5.5.9 (5), and this implies that any $a \in S$ must be in the future of $\bigvee\{x \in B(\mathcal{W}) \mid S \subseteq F_x\} = l(S)$ so $S \subseteq F_{l(S)}$ and we are done. □

We can then define a canonical event structure induced by the space of instants on an event structure $\mathcal{W}$ as follows:

5.6.15. Definition. Let $\mathcal{W}$ be an event structure and let $K(\mathcal{W})$ be its space of instants. The event structure generated by $K(\mathcal{W})$, denoted as $E(K(\mathcal{W}))$, is defined by letting W be the set

$$\{U \subseteq K(\mathcal{W}) \mid U \text{ is open, nonempty and order-convex}\}$$

And by defining the event structure relations and operations as in Lemma 5.3.13.

It is a routine matter to verify that Definition 5.6.15 does indeed yield an event structure satisfying the axioms of GT. Furthermore, the above definition is clearly a generalization of the event structure induced by the unit interval $E(\mathbb{I})$ considered in section 5.3.1 to the more general class of linearly ordered topological spaces that can be constructed as spaces of instants of an event structure. We can now formulate, however:

5.6.16. PROPOSITION. *Let $\mathcal{W}$ be an event structure and let $f : \mathcal{W} \to E(K(\mathcal{W}))$ be defined by letting:*

$$f(a) = (l(a), r(a))$$

For any $a \in W$. Then f is an event map, that is, it preserves R_+, $\mathrm{R}_-, O, \oplus, \ominus$.

The proposition above answers the question we posed as it allows us to see the events in an event structure $\mathcal{W}$ as intervals of the linear order of instants $K(\mathcal{W})$.

This suggests that one ought to be able to set up a pair of functors between the category of event structures and event maps on one side, and a suitable category of totally ordered topological spaces on the other side. This would improve on the approach in Thomason (1989), which provides functors between a category of event structures and homomorphisms, and a category of linear orders and monotone multi-valued maps. In this sense, it is helpful to note that our event maps can be seen as an instance of the "approximate maps" which have been introduced in Banaschewski and Pultr (2010) as a representation of localic morphisms.

5.7 Instants in the context of GT

The constructions in the previous sections only relied on the axiom system GT_1, and to a large extent they could have been carried out even for the weaker GT_0. We shall now consider the behaviour of instants in the context of the stronger system GT, and in the presence of further assumptions on the covering relation $\preceq$. In this section, unless specified further, we shall only consider models of GT.

5.7.1 Minimal events and overlapping classes

In this section we show that the axioms of GT imply that maximal overlapping classes of events are maximal filters in the $\preceq$ preorder, and introduce the concept of a $\preceq$-minimal event, an event that is minimal in the $\preceq$ preorder. This latter notion is of particular interest since under certain assumptions on event structures maximal overlapping classes of events have a canonical presentation as the upset of a $\preceq$-minimal event under the covering ordering $\preceq$, which in turn means that maximal overlapping classes are principal maximal filters. The reader should be aware, however, that we are not extending the ontology of instants, which remains composed only of the notions of boundary and infinitesimal interval.

5.7.1. PROPOSITION. *Let $\mathcal{W}$ be an event structure. Then $A \subseteq W$ is a maximal overlapping class if and only if it is a maximal filter under $\preceq$.*

Proof:
Let $\mathcal{W}$ be an event structure and let $A \subseteq W$ be a maximal overlapping class. Then A is an upset under $\preceq$, since if $a \preceq b$ for $a \in A, b \in W$ then aOc for all $c \in A$, but this implies bOc for all $c \in A$; hence $b \in A$ since A is maximal overlapping. We now show that A is down-directed under $\preceq$. Let then $a, b \in A$; since aOb then $a \sqcap b$, i.e., $(a \oplus b) \ominus b$ is defined. We show that $cOa \sqcap b$ for any $c \in A$, which implies $a \sqcap b \in A$ since A is maximal overlapping, and in turn implies A is down-directed under $\preceq$. Choose then any $c \in A$; then cOa, cOb and aOb, hence $cOa \sqcap b$ by Lemma 5.3.4 (5), and we are done. Hence A is a filter. Suppose now that $A \subset B$ for a filter B. Then there exists $b \in B, b \notin A$; but since B is down-directed under $\preceq$ then for any $a \in A$ there exists $c \in B$ with $c \preceq a, b$, but this implies that bOa for any $a \in A$, hence $b \in A$ since A is maximal overlapping, contradiction. Hence A is a maximal filter.

For the other direction we show the contrapositive: assume A is not a maximal overlapping class, we show that A is not a maximal filter. Now, if A is not an overlapping class then it is not down-directed under $\preceq$ and hence it is also not a filter, so we can assume that A is an overlapping class. Since A is not maximal then there exists A' maximal overlapping class with $A \subset A'$. By the proof of the left-to-right direction then A' is a maximal filter, but then clearly A cannot be a maximal filter, and we are done. □

Hence, in the context of GT maximal overlapping classes are nothing more than maximal filters under the covering ordering $\preceq$. We now introduce the concept of a $\preceq$-minimal event:

5.7.2. Definition. Let $\mathcal{W}$ be an event structure and let $\mu \in \mathcal{W}$. We say that μ is $\preceq$-minimal in $\mathcal{W}$ if

$$\mathcal{W} \models \forall x(x \preceq \mu \rightarrow \mu \preceq x)$$

5.7.3. Lemma. *Let $\mathcal{W}$ be an event structure, $\mu \in \mathcal{W}$ be a $\preceq$-minimal event, and $x \in B(\mathcal{W})$. Then either $\mu \in P_x$ or $\mu \in F_x$.*

Proof:
Let $\mathcal{W}$, μ, x be as in the statement of the lemma. We show that μ cannot be in C_x, which then means that it is either in P_x or F_x. Assume then towards a contradiction that $\mu \in C_x$; then both P_x, F_x are not empty because of Proposition 5.5.9. Then there must be c, d such that $c \in P_x, d \in F_x$ with $\mu Oc, \mu Od$. But then $\mu \ominus c$, $\mu \oplus d$ are defined and we have $(\mu \ominus c)\mathbb{O}(\mu \oplus d), \mu \ominus c \preceq \mu, \mu \oplus d \preceq \mu$, hence $\mu \ominus c, \mu \oplus d$ are strictly covered by μ, which contradicts the $\preceq$-minimality of μ. □

5.7.4. Lemma. *Let $\mathcal{W}$ be an event structure, $\mu \in \mathcal{W}$ be a $\preceq$-minimal event, and $a \in W$. Then $aO\mu$ implies $\mu \preceq a$.*

Proof:
If $aO\mu$ but a does not cover μ then either $\mu\,\mathbb{R}_+\mu\oplus a, \mu\oplus a \preceq \mu$ or $\mu\,\mathbb{R}_-\mu\ominus a, \mu\ominus a \preceq \mu$, but then μ is not $\preceq$-minimal. □

We are now interested in imposing some conditions on event structures so that maximal overlapping classes are always generated as upsets of $\preceq$-minimal events. These conditions are of interest because, as we shall see starting from section 5.9, inverse limits of inverse systems of finite event structures are such that maximal overlapping classes are always generated by $\preceq$-minimal events.

Under the assumption of well-foundedness of $\preceq$, maximal overlapping classes are canonically generated by $\preceq$-minimal events. We first need the following concepts:

5.7.5. Definition. Let P be a preorder. We say that P is *classically well-founded* if for any non-empty subset $S \subseteq P$ there exists $a \in P$ such that for any $b \in P$, if $b \preceq a$ then $a \preceq b$. Such a is then a *minimal element* of S.

Clearly, Definition 5.7.2 implies that $\preceq$-minimal events are minimal in the $\preceq$ preorder in the sense of Definition 5.7.5. We then posit:

5.7.6. Definition. Let $\mathcal{W}$ be an event structure. Then $\mathcal{W}$ is *well-founded* if the preorder $\preceq$ is well-founded according to Definition 5.7.5.

We then have:

5.7.7. Proposition. *Let $\mathcal{W}$ be a well-founded event structure. Then $A \subseteq W$ is a maximal overlapping class if and only if $A = \uparrow_{\preceq}\mu$ for a $\preceq$-minimal $\mu \in W$.*

Proof:
For the left-to-right direction assume $A \subseteq W$ is a maximal overlapping class. Choose any $a \in S$, and using choice construct a sequence $C = (\cdots \preceq a_2 \preceq a_1 \preceq a_0)$ of elements of A by letting $a_0 = a$ and $a_i \in A \setminus \{a_0, \cdots, a_{i-1}\}$ be such that $a_i \preceq a_{i-1}$ for all $i > 0$. Then C is inextensible, and by well-foundedness there exists an event $\mu \in C \subseteq A$ which is minimal in C under $\preceq$, that is, $\mu \preceq c$ for all $c \in C$. We show that $A = \uparrow_{\preceq}\mu$ and that μ is $\preceq$-minimal. First note that since $\mu \in A$ then $\mu O a$ for all $a \in A$, and that clearly μ is minimal in A under $\preceq$. Now if $\mu O a$ but a does not cover μ, then since $\mathcal{W}$ is a model of GT either $\mu\,\mathbb{R}_+\mu \oplus a, \mu \oplus a \preceq \mu$ or $\mu\,\mathbb{R}_-\mu \ominus a, \mu \ominus a \preceq \mu$; in either case μ is not minimal w.r.t. $\preceq$ in A, which yields a contradiction. Hence (i) $\mu \preceq a$ for any $a \in A$, but then clearly $A = \uparrow_{\preceq}\mu$, where the inclusion from right to left follows since A is a maximal overlapping class. To see that μ is $\preceq$-minimal just note that if $b \preceq \mu$ for any $b \in W$ then bOa for all $a \in A$ because of (i), but then $b \in A$ since A is maximal overlapping, hence $\mu \preceq b$, and we are done. □

A similar result holds if one assumes that the topology $\mathcal{A}_{\preceq}$ of the event structure is compact:

5.7.8. PROPOSITION. *Let $\mathcal{W}$ be an event structure such that $\mathcal{A}_{\preceq}$ is compact; then $A \subseteq W$ is a maximal overlapping class if and only if $A = \uparrow_{\preceq}\mu$ for a $\preceq$-minimal $\mu \in W$.*

Proof:
For the right-to-left direction, note that if $\mu \in W$ is a $\preceq$-minimal event and $a \in W$ is such that aOb for any $b \in \uparrow_{\preceq}\mu$, then $aO\mu$; but this means that $\mu \preceq a$ by Lemma 5.7.4, hence $a \in \uparrow_{\preceq}\mu$, hence $\uparrow_{\preceq}\mu$ is a maximal overlapping class.

For the left-to-right direction let $A \subseteq W$ be a maximal overlapping class. Since $\mathcal{W}$ is a model of GT this implies that the family $\mathcal{F} = \{\downarrow_{\preceq}a \mid a \in A\}$ has the finite intersection property: for any two $a, b \in A$ we have aOb but then $a \sqcap b$ is defined and belongs to $\downarrow_{\preceq}a \cap \downarrow_{\preceq}b$. Moreover, the sets in this family are closed.

By compactness, then, $\bigcap \mathcal{F} \neq \emptyset$, which means that there must be an event $\mu \in \bigcap \mathcal{F}$. We claim that μ is a $\preceq$-minimal event. Indeed, let $\mu' \in W$ be such that $\mu' \preceq \mu$. Then $\mu' \preceq a$ for any $a \in A$ by transitivity of $\preceq$, hence $\mu' O a$ for any $a \in A$, hence $\mu' \in A$ since A is a maximal overlapping class; but then $\mu \preceq \mu'$ by construction. Moreover, clearly $\mu \in A$ and as any $a \in A$ is such that $\mu \preceq a$ then $\uparrow_{\preceq}\mu = A$. □

Hence, if an event structure is either well-founded or such that $\mathcal{A}_{\preceq}$ is compact, then all its maximal filters are principal:

5.7.9. PROPOSITION. *Let $\mathcal{W}$ be an event structure that is well-founded or such that $\mathcal{A}_{\preceq}$ is compact. Then every maximal filter of $\mathcal{W}$ is principal.*

Proof:
If A is a maximal filter of $\mathcal{W}$ then it is a maximal overlapping class by Proposition 5.7.1, and by Proposition 5.7.7 or Proposition 5.7.8 then $A = \uparrow_{\preceq}\mu$ for some $\preceq$-minimal $\mu \in W$, that is, A is principal. □

We now turn to the consideration of infinitesimal intervals in the context of GT; what we shall say on the topic will be of importance also in chapter 6.

5.7.2 Infinitesimal intervals in GT

The following is the main result of this section:

5.7.10. PROPOSITION. *Let $\mathcal{W}$ be an event structure such that $\mathcal{A}_{\preceq}$ is compact or $\preceq$ is well-founded, and let $A \subseteq W$. Then the following hold:*

(1) A is a maximal overlapping class if and only if $A = C_i$ for $i \in I(\mathcal{W})$

(2) Every infinitesimal interval $i \in I(\mathcal{W})$ is of the form $(P_{l(\mu)}, \uparrow_{\preceq}\mu, F_{r(\mu)})$ for $\mu \in W$ a $\preceq$-minimal event.

(3) *The set of infinitesimal intervals and the set of maximal overlapping classes are in bijective correspondence*

Proof:
We only show the claims under the assumption that $\mathcal{A}_{\preceq}$ is compact, as if $\preceq$ is well-founded the proofs are analogous.

The right-to-left direction of claim (1) follows from Proposition 5.6.4. For the left-to-right direction assume A is a maximal overlapping class. Then by Proposition 5.7.8 $A = \uparrow_{\preceq}\mu$ for a $\preceq$-minimal $\mu \in W$. Let then i be the boundary $(P_{l(\mu)}, \uparrow_{\preceq}\mu, F_{r(\mu)})$. To see that i is an infinitesimal interval it suffices to note that $\mu \in P_{l(\mu)}\mathbb{O}$ but $\mu \notin F_{r(\mu)}$.

Claim (2) follows from Proposition 5.6.4 and 5.7.8 and the left-to-right direction of (1).

For claim (3), let then $j : I(\mathcal{W}) \to \mathcal{P}W$ be the map defined by letting $j(i) = C_i$. Clearly the map is injective, and the left-to-right direction of the claim above shows that the map is surjective. □

An important consequence of the previous proposition is that the converse of property (2) of Proposition 5.6.4 holds:

5.7.11. COROLLARY. *Let $\mathcal{W}$ be an event structure such that $\mathcal{A}_{\preceq}$ is compact or $\preceq$ is well-founded, and let $i \in K(\mathcal{W})$ be such that C_i is a maximal overlapping class. Then $i \in I(\mathcal{W})$ is an infinitesimal interval.*

The results in the previous sections are not only useful for the rest of this work but they are interesting in their own right, as they shed light upon the debate on the most appropriate method to construct a linear order of instants from events. As we already mentioned in the previous sections, we shall return on this problem in chapter 6.

5.8 Retraction maps and infinite divisibility

In this section we begin to consider inverse systems of finite event structures and how they can be used to model Kant's notion of potential infinite divisibility and the synthesis of the unity of apperception as applied to the temporal form of any possible experience. In particular, we shall employ retraction maps to show how a specific sort of inverse sequences of event structures and retraction maps can be used to model Kant's potential infinite divisibility of time.

5.8.1 Retractions

Recall that for Kant time is divisible to infinity potentially. At A524/B552, in particular, Kant states that the division of something which is given as a whole in intuition must proceed to infinity, even though the division can never reduce the whole to simple parts. Since time, as stated in the TA, is given as a whole and is not composed from its parts, we can infer that time is divisible to infinity, i.e., ever smaller subdivision of times can be introduced. Kant's conception of potential infinite divisibility is also closely tied to his characterization of continuity:

> The property of magnitudes on account of which no part of them is the smallest (no part is simple) is called their continuity. (A169/B211)

There are two main aspects to this notion of continuity "in the small", which are both grounded on Kant's notion of parthood. The first aspect is that "time does not consist of smallest parts", which means that any event in an event structure can be further divided into subevents. The second aspect is that "instants can be filled" (see passage (7) in section 3.4), which implies that instants in time *can* be only "approximations" of points, and thus *can* be further refined. The use of modal expressions such as "can be divided" or "can be refined" is important here and betrays Kant's modal conception of the continuity and divisibility of time: every event structure is finite, and determines a finite linear order of instants, but it can be further analyzed *ad infinitum*. See chapter 3 for a more detailed discussion of these matters.

The starting point for a formalization of all these notions is then Kant's conception of the *divisio logica* of a part of time. This can be understood as an operation of the subject, through which the original whole representation of a part of time is subdivided into subparts, in such a way, however, that the whole representation is preserved by the act of division. To be able to formalize this act of division we shall need the full set of axioms GT and the flexibility of the event maps introduced in section 5.6.2. In particular, we model the fact that the act of division preserves the representation of the whole by means of a specific type of event map, a retraction map, which specializes to our present context the general definition of a retraction in section 5.2.4:

5.8.1. Definition. Let $\mathcal{W}$ be an event structure and let $\mathcal{W}'$ be a submodel of $\mathcal{W}$. An event map $r : \mathcal{W} \to \mathcal{W}'$ is a *retraction*, and $\mathcal{W}'$ is a *retract* of $\mathcal{W}$, if the restriction of r to $\mathcal{W}'$ is the identity.

Retraction maps can be used to model the "extension" of an event structure $\mathcal{W}'$ to a "larger" event structure $\mathcal{W}$. Since the *divisio logica* implies that no part of time is lost in the division process, new events are added to a part of time while preserving the old, i.e. $\mathcal{W}'$ is a submodel of $\mathcal{W}$; the new events are conceived as added through "virtual splittings" of either events or boundaries. The map itself represents the position of events on the timeline at coarser stages of logical division, and thus must be equal to the identity on $\mathcal{W}'$.

We now address the question whether there exist some conditions ensuring that an event structure retracts to some of its substructures, so that we can use retractions to model this process of logical division. We start with a preliminary notion:

5.8.2. PROPOSITION. *Let $\mathcal{W}$ be a model of GT_0 and let W' be a substructure of $\mathcal{W}$. Then $W' \models GT_0$*

Proof:
Obvious, since GT_0 is a universal theory. □

The following theorem answers our question:

5.8.3. THEOREM. *Let $\mathcal{W}$ be an event structure and let $\mathcal{W}'$ be a finite submodel of $\mathcal{W}$. Then $\mathcal{W}'$ is a retract of $\mathcal{W}$.*

Proof:
Let $\mathcal{W}$ be an event structure and let $\mathcal{W}'$ be a finite submodel of $\mathcal{W}$. We can easily construct a retraction map $r : \mathcal{W} \to \mathcal{W}'$ as follows. Fix a universal cover $c \in \mathcal{W}'$; for any $a \in \mathcal{W}$ let $m_p(a) \in \mathcal{W}'$ be an event with the property that $a\,\mathrm{R}_- m_p(a)$ and there exists no $b \in \mathcal{W}'$ with $a\,\mathrm{R}_- b, b\,\mathrm{R}_- m_p(a), m_p(a)\,\mathbb{R}_- b$, if such an event exists. Otherwise let $m_f(a) = c$. Define m_f dually by replacing R_+ for R_- and $\mathbb{R}_+$ for $\mathbb{R}_-$. Let then the map $r : \mathcal{W} \to \mathcal{W}'$ be defined as follows:

$$\begin{array}{ll} r(a) = a & \text{if } a \in \mathcal{W}' \\ r(a) = (c \oplus m_f(a)) \ominus m_p(a) & \text{otherwise} \end{array}$$

We show that r so defined is a retraction map. First note that it preserves R_+, R_-, O. For instance, if $a\,\mathrm{R}_+ b$ then clearly $m_f(a)\,\mathrm{R}_+ m_f(b)$, but then the axioms for $\oplus$ imply that $c \oplus m_f(a)\,\mathrm{R}_+ c \oplus m_f(b)$ and hence that $(c \oplus m_f(a)) \ominus m_p(a)\,\mathrm{R}_+ c \oplus m_f(b) \ominus m_p(b)$ and we are done. Furthermore, r preserves $\oplus, \ominus$ up to the defined equality $\equiv$. Indeed, let $a, b \in \mathcal{W}$ be such that $a \oplus b$ is defined. We show that $r(a \oplus b) \equiv r(a) \oplus r(b)$, as the case for $\ominus$ is analogous. Now, by linearity either $a\,\mathrm{R}_+ b$ or $b\,\mathrm{R}_+ a$. We consider the two cases separately.

If $a\,\mathrm{R}_+ b$ then it follows from the axioms for $\oplus$ that $a \equiv a \oplus b$, hence $r(a \oplus b) \equiv r(a) \oplus r(b)$, and we are done.

If $b\,\mathrm{R}_+ a$ then $a \oplus b \equiv_- a$ and also $b \equiv_+ a \oplus b$. Then $r(a \oplus b) \equiv_- r(a)$, and $r(b) \equiv_+ r(a \oplus b)$. But then clearly since $b\,\mathrm{R}_+ a$ then $r(b)\,\mathrm{R}_+ r(a)$, hence by the same reasoning $r(a) \oplus r(b) \equiv_+ r(b)$, $r(a) \oplus r(b) \equiv_- r(a)$, and $r(a) \oplus r(b) \equiv r(a \oplus b)$ follows by transitivity of $\equiv_+, \equiv_-$.

Thus r is an event map. Moreover, it is clearly the identity on $\mathcal{W}'$ by definition, hence it is a retraction. □

One might now wonder whether the above result can be weakened by requiring that $\mathcal{W}'$ be only a model of GT_1, but this is readily seen not to be the case, as the following example shows.

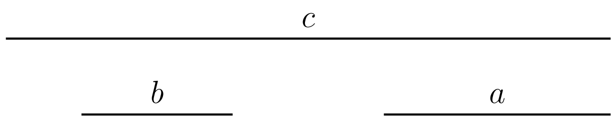


Figure 5.1: GT_1 does not ensure the existence of retraction maps

5.8.4. EXAMPLE. Consider the model of GT_1 defined by letting $W = \{a, b, c\}$, $a \equiv_- c$, $a \mathbb{O} b$, $b \, \mathrm{R}_- a$, $c \, \mathbb{R}_+ b$; the model is displayed in Figure 5.1.

Let then $\mathcal{W}$ be the model of GT obtained by closing this model of GT_1 under $\oplus, \ominus$, and let $\mathcal{W}'$ be the substructure of $\mathcal{W}$ induced on the subset $W' = \{b, c\}$. Then $\mathcal{W}'$ is a model of GT_1 but not of GT, and there exists no retraction map $r : \mathcal{W} \to \mathcal{W}'$. Indeed, if $r(e) = e$ for any $e \in \{b, c\}$ then if $r(a) = c$ we have $r(a) \, \mathbb{R}_+ r(b)$ which implies R_+ is not preserved; if $r(a) = b$ we have $r(c) \, \mathbb{R}_- r(a)$ which implies R_- is not preserved.

Example 5.8.4 shows that the "cutting" operations $\oplus, \ominus$ guarantee that a map preserving $\mathrm{R}_+, \mathrm{R}_-, O$ can always be defined from an event structure $\mathcal{W}$ to a finite submodel $\mathcal{W}'$, and that is their essential mathematical role.

Indeed, it is not difficult to show that if $\mathcal{W}'$ is only a model of GT_1, then there always exists a map $f : \mathcal{W} \to \mathcal{W}'$ that is the identity on $\mathcal{W}'$ and preserves O, R_+, which we might call a *f-retraction*, and that there always exists a map $g : \mathcal{W} \to \mathcal{W}'$ that is the identity on $\mathcal{W}'$ and preserves O, R_-, which we might call a *p-retraction*. Only, to ensure that *both* R_+, R_- are preserved, so that we obtain a full retraction map, it is necessary that $\mathcal{W}'$ is a submodel of $\mathcal{W}$, i.e., that is satisfies the axioms of GT. Philosophically, this means that transcendental events, in particular those introduced by $\oplus, \ominus$, are crucial to model infinite potential divisibility.

5.8.2 The extensionality axiom and setoids

In the treatment up to this point we have not assumed the extensionality axiom stating that:

$$a \equiv b \to a = b$$

Nevertheless, since the following discussion of inverse systems and limits of event structures will be simpler if this axiom is assumed, we shall assume it. In its presence, then, an event map preserves $\oplus, \ominus$ up to identity $=$ rather than equality $\equiv$, so that we can use the standard definitions of an inverse system and inverse limit of first-order models provided in section 5.2.

The reader who has philosophical qualms about this axiom in view of section 5.3.4 should note, however, that the whole discussion from this point onwards could be reformulated in the absence of this axiom, at the expense of a slightly more involved treatment, in terms of setoids. In particular, the definition of inverse systems and inverse limits would be modified accordingly, so that, for instance, a *thread* of the inverse limit would satisfy the coherence conditions of section 5.2.5 up to the defined equivalence $\equiv$ rather than equality $=$, and the results of the following sections would be carried over entirely, so that the presence of the extensionality axiom is immaterial and only useful in that it allows for a simpler discussion.

5.8.3 Infinite divisibility

Let us return to the original problem of modelling the act of *divisio logica* of time. We can model this act effectively as follows:

5.8.5. DEFINITION. Let $\mathcal{W}, \mathcal{W}'$ be event structures such that $\mathcal{W}'$ is a submodel of $\mathcal{W}$, let $r : \mathcal{W} \to \mathcal{W}'$ be a retraction map, and let $x \in K(\mathcal{W}')$. We say that $\mathcal{W}$ *splits* x if there exists $a, b \in W$ with $a \mathbb{O} b$ such that

(1) $\{a, b\}$ and $r^{-1}P_x \cup r^{-1}F_x$ are $\mathbb{O}$-apart

(2) $a, b \preceq c$ for any $c \in C_x$

While the above definition formulates the concept of "splitting" by mere reference to instants, it is important to note that it captures two distinct philosophical concepts. The first is the concept of "dividing" an event, which occurs if $\mathcal{W}$ is finite and $x \in I(\mathcal{W}')$. In this case $\mathcal{W}$ introduces two non-overlapping events that are covered by the $\preceq$-minimal event μ that generates the infinitesimal instant in $I(\mathcal{W}')$; this $\preceq$-minimal event must exist since $\mathcal{W}'$ is finite and thus $\preceq$ is compact (See Proposition 5.7.10). Hence, $\mathcal{W}$ effectively *splits* this minimal event, thereby splitting any event that covers μ, and thus it splits the infinitesimal interval that μ generates. The second concept is that of "splitting" or "refining" a boundary, which correspond to Kant's claim that "a boundary can be filled, but in such a way that the series is not indicated": this occurs when $x \in B(\mathcal{W}')$. Now, we remarked above that Kant talks about divisions of parts of time and of boundaries in modal terms. This is to be interpreted as reflecting a notion of potentiality: parts of time *can* always be subdivided to infinity, and boundaries *can* always be split, since there are no simple parts in time. We can then express this modal take on infinite potential divisibility as follows:

5.8.6. DEFINITION. A countable (inverse) sequence $\mathcal{S}$ of event structures

$$\dots \mathcal{W}_3 \longrightarrow_{r_{32}} \mathcal{W}_2 \longrightarrow_{r_{21}} \mathcal{W}_1 \longrightarrow_{r_{10}} \mathcal{W}_0$$

is said to be an *infinite divisibility sequence* if it satisfies the following conditions:

1. each $\mathcal{W}_i$ is finite and the bonding maps r_{ij} are retraction maps

2. for any instant $x \in K(\mathcal{W}_i)$ there exists $j \geq i$ such that $\mathcal{W}_j$ splits x

Of course, the above definition can be easily generalized to inverse systems of finite event structures and retraction maps. In the following section we shall be mostly concerned with such inverse systems; in particular, we shall be interested in the class of *all* finite event structures, which, as we shall show, can be endowed with the structure of an infinite divisibility inverse system. This inverse system is the fundamental formal correlate of Kant's figurative synthesis, as it captures its fundamentally modal nature, and we shall study carefully its inverse limit and space of instants as correlates to Kant's formal intuition of time.

5.9 Unity, universality and limits

We now begin to investigate more thoroughly how Kant's figurative synthesis can be modelled by means of inverse systems of finite event structures. We aim in particular to provide a formal correlate for Kant's notion of time as formal intuition, i.e., time as a unique and fully determined object through which "all concepts of time first become possible" (B161n). We take as our point of departure the collection of finite models of GT, which we conceive as a formal correlate for Kant's notion of possible "temporal forms" of experience.

In particular, recall that according to Kant if we consider any possible experience of succession of perceptions, we are immediately aware that the judgment regarding the temporal order of such perceptions is merely subjective, unless it is subsumed under a universal rule which makes this succession objective - and, ultimately, able to be communicated. An objective succession then requires subsumption of perceptions under the category of causality. This subsumption, however, requires itself a manifold on which it can be applied, and in particular it requires a temporal intuition which can encompass any possible succession of perceptions - not just actual experiences - and which supports the formulation of judgments of objective temporal succession. The consequences of this objective temporal determination are then fully determined by the properties of this all-encompassing temporal intuition. We provide a formal correlate to the thoroughgoing determination of time as formal intuition with respect to judgments of temporal order by means of Theorem 5.9.15 in this Section.

5.9.1 Inverse systems of finite event structures

The guiding intuitions of the constructions that follow are that "parts of time are times" (A169/B211) and that "different times are only parts of one and the same time." (A31-2/B47). We interpret the former as meaning that there exist finite families of "parts of time", which obey the same axioms as time itself ("are times"). These parts of time contain both empirical and transcendental events, and hence are interpreted in our framework as models of GT. The second quote we understand to mean that there exists a unified time ("one and the same time") which is in some sense universal. In section 5.4 and section 5.8 we introduced ways of creating parts of time, parts of parts of time, and so forth; we then argued that the resulting structures are related by retractions so as to model the potential divisibility of time. In light of these considerations we posit:

5.9.1. Definition. Let T be a directed partial order. An *inverse system of finite event structures* indexed by T is a family $\{\mathcal{W}_t \mid t \in T\}$ of finite models of GT together with a family of retractions $\mathcal{F} = \{r_{ts} : \mathcal{W}_t \to \mathcal{W}_s \mid s, t \in T, t \geq s\}$ satisfying:

(1) $r_{ss} \in \mathcal{F}$ is the identity for any $s \in T$

(2) $r_{ts} \circ r_{vt}(a) = r_{vs}(a)$ for any $v, t, s \in T$ with $v \geq t \geq s$

Of course, we are here considering inverse system of finite event structures and retraction maps - not merely homomorphisms - according to the definition in section 5.2. We shall be particularly interested in the inverse limits of such inverse systems, so as to obtain a formal correlate to time as the formal intuition; the requirement of directedness is then crucial, as it ensures that the inverse limit is well behaved, and it provides us with a formal correlate for the unity of apperception, since it implies that for any two finite event structures $\mathcal{W}_s, \mathcal{W}_t$ in the system there must be an event structure $\mathcal{W}_u$ which retracts to both.

The question now arises whether the class of *all* finite event structures, which is a formal correlate to the class of all possible temporal forms of experience that can be produced by the figurative synthesis, can be given the structure of an inverse system according to Definition 5.9.1. Note in this sense that the directedness of the index set of an inverse system implies that all indexed models must satisfy the same geometric theory; e.g., the sentence

$$\exists a_1 ... \exists a_n (a_1 \mathbb{O} a_2 \wedge a_1 \mathbb{R}_+ a_2 \wedge \ldots \wedge a_{n-1} \mathbb{O} a_n \wedge a_{n-1} \mathbb{R}_+ a_n)$$

expressing the existence of an antichain of length n, is not in this theory. We then have:

5.9.2. Lemma. *Let $\mathcal{W}, \mathcal{W}'$ be finite event structures that satisfy the same geometric extension of GT. Then there exists a finite event structure $\mathcal{W}''$ and retraction maps $g : \mathcal{W}'' \to \mathcal{W}, r : \mathcal{W}'' \to \mathcal{W}'$.*

Proof:
Let $\mathcal{W}, \mathcal{W}'$ be such that they satisfy the same geometric extension $\mathcal{G}$ of GT. By means of a "dynamic proof" (Coquand, 2002), we can construct a finite model $\mathcal{W}''$ of $\mathcal{G}$ such that $\mathcal{W}, \mathcal{W}'$ are submodels of $\mathcal{W}''$. By Theorem 5.8.3, there are

retraction maps $r : \mathcal{W}'' \to \mathcal{W}'$ and $g : \mathcal{W}'' \to \mathcal{W}$. □

Thus, the class of finite event structures can be endowed with the structure of an inverse system by simply indexing the finite event structures by a set T and letting $s \leq t$ if there exists a retraction map $r : \mathcal{W}_t \to \mathcal{W}_s$; the set T is then directed because of Lemma 5.9.2. This presentation of the inverse system of all finite event structures is particularly pleasant since it makes clear that the formal correlate of the process of *divisio logica* of time by the figurative synthesis is not the index set of an inverse system of finite event structures but the finite event structures and retraction maps among them. The inverse system of *all* finite event structures will be called the *finitary spectrum* of GT.

We shall in particular be interested in the inverse systems of finite event structures that satisfy the conditions of Definition 5.8.6:

5.9.3. DEFINITION. Let $(T, \mathcal{W}_t, r_{vs})$ be an inverse system of finite event structures. We say that it *satisfies infinite divisibility*, or that is is an *infinite divisibility inverse system*, if it satisfies the conditions given in Definition 5.8.6 of section 5.8.

Clearly, the finitary spectrum of GT is an infinite divisibility inverse system.

5.9.2 Limits of inverse systems and preservation of formulas

Now that we have introduced inverse systems of finite event structures as formal correlates of parts of time related via potential divisibility we focus on the limits of such inverse systems. Limits will be used to provide a formal correlate to Kant's notion of time as a *formal intuition*, time as an object of which all times are but parts. Note that Theorem 5.2.4 implies that the inverse limit of an inverse system of finite event structures is non empty; indeed, this is obvious since our maps are not mere homomorphisms but retractions, and hence they are surjective, so that an element of the inverse limit can be constructed by simple induction. Moreover, recall that Lemma 5.2.5 specializes to our setting with retraction maps so that the projections from the inverse limit to the elements of the inverse system are also retractions.

A natural question at this point is whether the inverse limit of an inverse system of finite event structures can itself be regarded as an event structure. To answer this question we need to go beyond the results presented in section 5.2, and investigate which formulas are preserved to the inverse limit.

5.9.4. LEMMA. *Let $(T, \mathcal{W}_s, r_{ts}, \mathcal{V})$ be an inverse system of finite event structures, where $\mathcal{V}$ is the inverse limit of the system, and let $\phi(\bar{x})$ be a positive primitive formula. Then for any tuple $\bar{a}$ of objects of $\mathcal{V}$, $\mathcal{V} \models \phi(\bar{a})$ if and only if $\mathcal{W}_s \models \phi(\pi_s(\bar{a}))$ for every $s \in T$.*

Proof:
The proof, by induction on the complexity of $\phi(\bar{x})$, can be found in Achourioti and van Lambalgen (2011). $\square$

Since Lemma 5.2.3 implies that only geometric *sentences* are preserved by retraction, we cannot expect to obtain a result as Lemma 5.9.4 for geometric formulas. We obtain, however, the following, weaker result:

5.9.5. LEMMA. *Let $(T, \mathcal{W}_s, r_{ts}, \mathcal{V})$ be an inverse system of finite models of GT, and let $\phi(\bar{x}, \bar{y})$ be a geometric formula. Then for any tuple $\bar{a}$ of objects of $\mathcal{V}$, $\mathcal{V} \models \phi(\bar{a})$ if and only if $\mathcal{W}_s \models \phi(\pi_s(\bar{a}))$ for all $s \in S$, where $S \subseteq T$ a cofinal subset of T.*

Proof:
Let $\forall \bar{x}(\psi(\bar{x}, \bar{y}) \rightarrow \chi(\bar{x}, \bar{y}))$ be a geometric formula in the distinguished vocabulary, and let $\bar{a}$ be a tuple of objects from $\mathcal{V}$ such that $\mathcal{V} \models \psi(\bar{a}) \rightarrow \chi(\bar{a})$. Then $\mathcal{V} \models \neg\psi(\bar{a}) \vee \chi(\bar{a})$, hence either $\mathcal{V} \models \neg\psi(\bar{a})$ or $\mathcal{V} \models \chi(\bar{a})$. In the former case because of Lemma 5.9.4 there must be an index $i \in T$ such that $\mathcal{W}_i \models \neg\psi(\pi_t\bar{a})$; since negations of positive primitive formulas in the distinguished vocabulary are preserved upwards, we then have $\mathcal{W}_j \models \neg\psi(\pi_s\bar{a})$, and hence $\mathcal{W}_j \models \psi(\pi_s\bar{a}) \rightarrow \chi(\pi_s\bar{a})$ for all $j \geq i$, which is a cofinal set of models. If $\mathcal{V} \models \chi(\bar{a})$ then by Lemma 5.9.4 we obtain $\mathcal{W}_t \models \chi(\pi_t\bar{a})$ for all $t \in T$, hence $\mathcal{W}_t \models \psi(\pi_t\bar{a}) \rightarrow \chi(\pi_t\bar{a})$ for all $t \in T$, which is obviously cofinal. For the direction from right to left, let $\bar{a}$ be a tuple of objects from $\mathcal{V}$ such that $\mathcal{W}_s \models \neg\psi(\pi_s\bar{a}) \vee \chi(\pi_s\bar{a})$ for all $s \in S$, $S \subseteq T$ a cofinal set of indices. Clearly, if there exists $s' \in S$ with $\mathcal{W}_{s'} \models \neg\psi(\pi'_s\bar{a})$ then $\mathcal{V} \models \neg\psi(\bar{a})$ since $\mathcal{V}$ retract to $\mathcal{W}_{s'}$, and we are done. Otherwise $\mathcal{W}_s \models \chi(\pi_s\bar{a})$ for all $s \in S$, hence $\mathcal{W}_t \models \chi(\pi_t\bar{a})$ for any $t \in T$ since S is cofinal in T, hence by Lemma 5.9.4 we obtain $\mathcal{V} \models \chi(\bar{a})$ and the result follows. $\square$

For geometric sentences, we have:

5.9.6. LEMMA. *Let $(T, \leq, \{\mathcal{W}_s \mid s \in T\}, r_{ts}, \mathcal{V})$ be an inverse system of finite event structures and let ϕ be a geometric sentence. Then $\mathcal{V} \models \phi$ iff $\mathcal{W}_s \models \phi$ for all $s \in T$*

Proof:
Since the projection maps π_s are retractions, the direction from left to right follows straightforwardly from Lemma 5.2.3. For the direction from right to left follows straightforwardly from Lemma 5.9.5. $\square$

Note that Lemma 5.9.4 and Lemma 5.9.5 are general model-theoretic results about inverse systems of first-order models, and would go through even if the maps r_{st} where mere homomorphism.

We then obtain that the inverse limit $\mathcal{V}$ of an inverse system of finite event structures is itself a model of GT, since GT is a geometric theory, that is, its axioms are geometric formulas. Moreover, to show this fact only the right-to-left direction of Lemma 5.9.6 is needed; the latter does not rely on Lemma 5.2.3, which assumes that the r_{ts} are retractions, but only on Lemma 5.9.5, which considers general homomorphisms. Hence, in category theoretic terms, this means that the category of event structures and event maps - not necessarily retractions - has limits.

Note moreover that although in general only geometric sentences are preserved by retractions (Lemma 5.2.3), and not geometric formulas, in our particular context certain geometric formulas are preserved. For instance, we have:

5.9.7. Lemma. *Let $\mathcal{W}, \mathcal{W}'$ be models of GT and let $r : \mathcal{W} \to \mathcal{W}'$ be a retraction map. Then r preserves the following geometric formula:*

$$\forall x(x \preceq y \to y\,\mathrm{R}_{-}x \vee y\,\mathrm{R}_{+}x)$$

Proof:
Assume that $\mathcal{W} \models \forall x(x \preceq b \to b\,\mathrm{R}_{-}x \vee b\,\mathrm{R}_{+}x)$ for some b in $\mathcal{W}$. We need to show that $\mathcal{W}' \models \forall x(x \preceq f(b) \to f(b)\,\mathrm{R}_{-}x \vee f(b)\,\mathrm{R}_{+}x)$. Let then a be an object in $\mathcal{W}'$ such that $\mathcal{W}' \models a \preceq f(b)$, and assume towards a contradiction that $\mathcal{W}' \models f(b)\,\mathbb{R}_{-}a, f(b)\,\mathbb{R}_{+}a$. Since $\mathbb{R}_{-}$, $\mathbb{R}_{+}$ are reflected by event maps, we must have that $\mathcal{W} \models b\,\mathbb{R}_{-}a, b\,\mathbb{R}_{+}a$, and thus $\mathcal{W} \models a \preceq b$ by excluded middle. Since we assumed that $\mathcal{W} \models \forall x(x \preceq b \to b\,\mathrm{R}_{-}x \vee b\,\mathrm{R}_{+}x)$, however, we obtain a contradiction. Thus either $\mathcal{W}' \models f(b)\,\mathrm{R}_{-}a$ or $\mathcal{W}' \models f(b)\,\mathrm{R}_{+}a$, which concludes the proof. □

The lemma above can be understood as follows. Define an auxiliary relation $a \ll b$ on events of an event structure, where $a \ll b$ is defined as $b\,\mathbb{R}_{+}a \wedge b\,\mathbb{R}_{-}a$. The relation $\ll$ is, in the set-based interpretation of our axioms of section 5.3.3, the "well inside" relation from topology; we recall that in topology an open set A is well inside another open set B if the closure of A is contained in B. If an event satisfies the formula of Lemma 5.9.7 then it is such that whenever another event is covered by it, this latter event is not well inside it, but it so to speak “sticks to an edge”. The lemma then shows that this formula is preserved downwards by event maps. Note, however, that the geometric formula:

$$\forall x(x \preceq y) \to (y \preceq x)$$

expressing $\preceq$-minimality is not preserved by event maps nor by retraction maps.

5.9.3 The topology on the limit of inverse systems

We have seen in the previous section that the limit $\mathcal{V}$ of an inverse system of finite event structures is itself an event structure. We can then investigate its topological properties. In particular, we have:

5.9.8. PROPOSITION. *Let $(T, \mathcal{W}_s, r_{ts}, \mathcal{V})$ be an inverse system of finite event structures. Then $\mathcal{V}$ has a universal cover.*

Proof:
First, note that it is a standard fact that the inverse limit of a cofinal subsystem of an inverse system is isomorphic to the inverse limit of the whole system, and that if the inverse system is countable, then the cofinal subsystem can be chosen to be a countable cofinal sequence. Since an inverse system of finite event structures is countable, we can then assume that $\mathcal{V}$ is the limit of an inverse sequence of the form $\mathcal{W}_0 \leftarrow \mathcal{W}_1 \leftarrow \cdots$. Let then $C(i) = \{c \in \mathcal{W}_i \mid c \text{ is a universal cover}\}$ for any $i \in \omega$. Since the formula $\forall x(x \preceq y)$ is preserved by homomorphisms then $r_{ij}|C(i) \subseteq C(j)$; hence we obtain an infinite finitely branching tree whose nodes at each level are the elements of $C(i)$. By König's lemma then there exists an infinite branch ξ which is also a thread of the inverse system and hence $\xi \in \mathcal{V}$. By Lemma 5.9.5 then ξ is a universal cover of $\mathcal{V}$. $\square$

5.9.9. COROLLARY. *Let $(T, \mathcal{W}_s, r_{ts}, \mathcal{V})$ be an inverse system of finite event structures. Then $op(\mathcal{A}_{\preceq})$ is compact.*

Proof:
$op(\mathcal{A}_{\preceq})$ is the Alexandroff topology whose open sets are downsets under $\preceq$; now every open open cover of $\mathcal{V}$ by downsets must contain a downset containing a universal cover, since universal covers exist in $\mathcal{V}$ by Proposition 5.9.8. But then this downset is already a cover of the whole space, and we are done. $\square$

One might wonder if the topology $\mathcal{A}_{\preceq}$ on an inverse limit $\mathcal{V}$ of an inverse system of finite event structures is also compact. We shall soon see that the answer is negative. The reader acquainted with inverse limits of topological spaces might be surprised at this fact, since the inverse limit of an inverse system of finite topological spaces is always compact. Indeed, we have:

5.9.10. PROPOSITION. *Let $(T, \mathcal{W}_s, r_{ts}, \mathcal{V})$ be an inverse system of finite event structures. Equip $\mathcal{V}$ with the topology τ having as a basis the family $\{\pi_s^{-1}(D) \mid s \in T, D \subseteq \mathcal{W}_s, D = \downarrow_{\preceq} D\}$ of preimages of downsets from $\mathcal{W}_s$ for any $s \in T$. Then $\mathcal{V}$ is a compact topological space.*

Proof:
Equip every $\mathcal{W}_s$ for $s \in T$ with the discrete topology; since the inverse limit of finite discrete topological spaces is always compact, and τ is a coarsening of this topology on the limit, it is also compact. $\square$

The reader should note, however, that we do not compute the limit of an inverse system of finite event structures in the category of topological or bitopological spaces, but in the category of event structures. Hence, the topology $\mathcal{A}_{\preceq}$ on the limit need not coincide with the topology τ induced on it by taking preimages of downsets under $\preceq$; indeed, in general τ is coarser than $\mathcal{A}_{\preceq}$ on the limit. The remarks in the following section will clarify this aspect further.

Now, since in general the topology $\mathcal{A}_{\preceq}$ on the limit $\mathcal{V}$ is not compact, the results in section 5.7 that rely on the compactness of $\mathcal{A}_{\preceq}$ do not apply to $\mathcal{V}$. In particular, Proposition 5.7.8 does not apply, which then means that also Proposition 5.7.10 and Corollary 5.7.11 do not apply. Still, the question remains whether for limits $\mathcal{V}$ of inverse systems of finite event structures, despite the failure of compactness of $\mathcal{A}_{\preceq}$, the correspondence between maximal overlapping classes and $\preceq$-minimal events of Proposition 5.7.8 still stands; this would in turn imply that all the other propositions in section 5.7 also stand. The following result answers this question in the affirmative:

5.9.11. PROPOSITION. *Let $(T, \mathcal{W}_s, r_{ts}, \mathcal{V})$ be an inverse system of finite event structures. Then $A \subseteq V$ is a maximal overlapping class if and only if $A = \uparrow_{\preceq}\mu$ for a $\preceq$-minimal $\mu \in V$.*

Proof:
For the right-to-left direction the proof of Proposition 5.7.8 applies. We then consider only the left-to-right direction.

For the left-to-right direction, equip $\mathcal{V}$ with the limit topology τ having as a basis the family $\{\pi_s^{-1}(D) \mid s \in T, D \subseteq \mathcal{W}_s, D = \downarrow_{\preceq} D\}$ of preimages of downsets from $\mathcal{W}_s$ for any $s \in T$. By Proposition 5.9.10 this topology is compact. Let then $A \subseteq V$ be a maximal overlapping class, and let $\mathcal{F} = \{\pi_s^{-1}(\downarrow\xi_s) \mid \xi \in A, s \in T\}$, where downsets are all taken with respect to $\preceq$. First, note that $\mathcal{F}$ has the finite intersection property. Indeed $\xi \in \pi_s^{-1}(\downarrow\xi_s), \rho \in \pi_t^{-1}(\downarrow\rho_t)$ for any two sets in $\mathcal{F}$, but $\xi O \rho$ since A is an overlapping class, but then because of the GT axioms there exists $\delta \in V$ with $\delta \preceq \xi, \rho$; then $\delta_s \preceq \xi_s$, $\delta_t \preceq \rho_t$, so $\delta \in \pi_s^{-1}(\downarrow\xi_s), \pi_t^{-1}(\downarrow\rho_t)$.

Second, note that all the sets in $\mathcal{F}$ are closed in τ. Since τ is compact then $\bigcap \mathcal{F}$ is not empty, and we choose $\mu \in \bigcap \mathcal{F}$. Then $\mu \in \pi_s^{-1}(\downarrow\xi_s)$ for all $\xi \in A, s \in T$, but then $\mu \in \pi_s(\pi_s^{-1}(\downarrow\xi_s)) = \downarrow\xi_s$ for all $\xi \in A, s \in T$, where the equality follows from surjectivity. But then $\mu_s \preceq \xi_s$ for any $\xi \in A, s \in T$, hence $\mu \preceq \xi$ for any $\xi \in A$.

Finally, note that μ is $\preceq$-minimal: if $\rho \preceq \mu$ then $\rho \preceq \xi$ for any $\xi \in A$ hence $\rho \in A$ since A is an overlapping class hence $\mu \preceq \rho$ by the above. The fact that $A = \uparrow_{\preceq}\mu$ is clear by Lemma 5.7.4. □

5.9.12. COROLLARY. *Let $(T, \mathcal{W}_s, r_{ts}, \mathcal{V})$ be an inverse system of finite event structures and let $a \in V$. Then there exists a $\preceq$-minimal event μ with $\mu \preceq a$*

Proof:
Any singleton set $\{a\} \subseteq V$ can be extended to a maximal overlapping class A using the axiom of choice; by Proposition 5.9.11 there must be a $\preceq$-minimal event $\mu \preceq a$. $\square$

The above proposition effectively states that the compactness of the topology induced on the limit by the basis of preimages of upsets under $\preceq$ is sufficient to ensure the correspondence between maximal overlapping classes and $\preceq$-minimal events, even in the absence of compactness for the topology $\mathcal{A}_{\preceq}$ on $\mathcal{V}$; we can then use all of the results of section 5.7 for $\mathcal{V}$. In the following section the mismatch between the inverse limit topology and the topology generated by the primitive relations on the limit is further investigated.

5.9.4 Expanding the language

The O, R_{-}, R_{+} vocabulary is not sufficiently rich to serve as a language with which to describe the inverse limit. We add distinguished monadic predicates $U_0, U_1, U_2 \ldots; V_0, V_1, V_2 \ldots; C_0, C_1, C_2 \ldots$ to the signature of GT satisfying the following axiom schemes:

(1) $\forall y \forall x (U_i(x) \wedge y \,\mathrm{R}_{-} x \rightarrow U_i(y))$

(2) $\forall y \forall x (V_i(x) \wedge y \,\mathrm{R}_{+} x \rightarrow V_i(y))$

(3) $\forall y \forall x (C_i(x) \wedge x \,\mathrm{R}_{+} y \wedge x \,\mathrm{R}_{-} y \rightarrow C_i(y))$

We assume that for each $\mathcal{W}_t$, all basic open sets of $\mathcal{W}_t$ of both the past and future topology are represented by a predicate. The interpretation of U_i on the inverse limit is thus a past-open set, and likewise for the other predicates. Note that the textbook way of inducing a topology on the inverse limit is to define an induced basis consisting of sets $\pi_t^{-1}(U)$ for open $U \subseteq \mathcal{W}_t$; if the $\mathcal{W}_t$ are finite, this topology is always second-countable. We however employ a finer topology that is generated by the interpretations of the distinguished predicates on the limit. The new open sets are then countable intersections of open sets from the induced basis, and this topology, though first-countable, is no longer second-countable.

To allow GT to express facts about the topology on the limit generated by the basis defined by the U_i, V_j, we enlarge its logical vocabulary with the infinitary operations $\bigwedge, \bigvee$ and define infinitary geometric formulas by allowing $\bigwedge, \bigvee$ in positive primitive formulas. The expanded language allows us to express that for $I \subseteq \mathbb{N}$,

$$\bigwedge_{i \in I} U_i, \bigvee_{i \in I} U_i$$

are past-open. If $\varphi(x)$ is a (possibly infinitary) formula defining a past-open set, then

$$\forall u (\forall v (\varphi(v) \rightarrow u \mathbb{O} v) \rightarrow u \mathbb{O} x)$$

defines the past generated by φ; as observed previously, if φ is positive primitive, the formula defining the past is geometric

$$\forall u(uOx \to \exists v(\varphi(v) \wedge uOv)).$$

We will abbreviate this formula as

$$\mathbb{O}(\varphi\mathbb{O})(x).$$

The pasts determine a subtopology of the topology on the limit; the following lemma ensures that the subtopology of the pasts is still Alexandroff:

5.9.13. LEMMA. *Define $R(x,y)$ iff $x \in \mathbb{O}(\{y\}\mathbb{O})$. Then*
(i) R is transitive and reflexive
(ii) if U is a fixpoint, $y \in U$ and $R(x,y)$ then $x \in U$.

Proof:
Since $x \in \mathbb{O}(\{x\}\mathbb{O})$, we have $R(x,x)$. Assume $R(x,y) \wedge R(y,z)$, i.e.

$$x \in \mathbb{O}(\{y\}\mathbb{O}) \wedge y \in \mathbb{O}(\{z\}\mathbb{O}).$$

Since $\mathbb{O}(\{z\}\mathbb{O})$ is a fixpoint, we have

$$x \in \mathbb{O}(\{y\}\mathbb{O}) \subseteq \mathbb{O}(\mathbb{O}(\{z\}\mathbb{O})\mathbb{O}) = \mathbb{O}(\{z\}\mathbb{O}).$$

It follows that $R(x,y)$.
Assume $y \in U = \mathbb{O}(U\mathbb{O})$, then $x \in \mathbb{O}(\{y\}\mathbb{O}) \subseteq \mathbb{O}(U\mathbb{O}) = U$. □

Hence the subtopology of the pasts is also Alexandroff, and not second-countable in the inverse limit considered.

We may now define, for instance, a boundary as a formula $\beta(x) = \varphi(x) \vee \nu(x) \vee \psi(x)$ such that φ defines a past, ψ the corresponding future and $\nu(x) \leftrightarrow \neg(\varphi(x) \vee \psi(x))$ the present.

5.9.14. DEFINITION. We define a linear order $<$ on β-formulas by putting $\beta < \beta'$ if $\forall x(\varphi(x) \to \varphi'(x))$.

Since the inverse limit $\mathcal{V}$ of an inverse system of finite event structures is itself a model of GT, one can construct the space of boundaries $B(\mathcal{W})$ and $K(\mathcal{W})$ on $\mathcal{V}$. The construction of this space of boundaries can now be expressed wholly within the language of GT, using the above results; in particular, the linear order on the boundaries induced by the inclusion of pasts can now be expressed in terms of β formulas as outlined above.

5.9.5 Universality of GT

In the discussion above we have considered arbitrary inverse systems of finite event structures. In order to achieve the required universality result for time as formal intuition, however, we need to consider the finitary spectrum of GT, i.e., the inverse system of *all* finite models of GT. The Kantian justification for considering this rather special inverse system, as we explain before, particularly in chapter 4, lies in Kant's notion of the temporal form of a "possible experience", which can be produced by the action of the figurative synthesis. In formal terms, the temporal form of a possible experience is nothing else that a finite model of GT. Since time as formal intuition is the all-encompassing time in which all possible experiences must be able to be determined, this means that the inverse limit of the inverse system of *all* finite models of GT is the best model to understand the universality of time as formal intuition, as produced by the action of the figurative synthesis.

Let then $(T, \mathcal{W}_s, r_{vs}, \mathcal{V})$ be a countable geometrically complete inverse system of finite event structures, that is, each geometric sentence not derivable from GT is represented by a countermodel in the system by the finite model property of GT; note that the finitary spectrum of GT is clearly such a system. We have:

5.9.15. THEOREM. *The following hold:*

(i) $\mathcal{V} \models GT$

(ii) for any geometric sentence φ, $\mathcal{V} \models \varphi$ *iff* $GT \vdash \varphi$

Hence, the inverse limit of all finite models of GT is a universal model for the theory of GT, and it is then "thoroughgoingly determined" with respect to the schemata of possible temporal judgments, as time as formal intuition should be, in a similar way in which Euclidean geometry is "thoroughgoingly determined" since it can be axiomatized by a complete theory having a universal model (Tarski, 1959). This thoroughgoing determination grounds the necessary thoroughgoing determination in time of empirical appearances. Note, moreover, that the inverse limit is universal for $(T, \mathcal{W}_S, r_{vs})$ in the sense of Theorem 5.2.6, which implies that it is "unique", in the specific categorical sense.

5.10 The time continuum as the limit on the finitary spectrum

In this section we investigate the space of instants $K(\mathcal{V})$ on inverse limits of inverse systems of finite event structures and, in particular, of inverse systems satisfying infinite divisibility, such as the finitary spectrum of GT. We shall see that the continuum emerging from this analysis is not quite the real continuum, although it bears to it an interesting relation that will be exploited also in the following chapter. Recall that given any inverse system of finite event structures, not necessarily satisfying infinite divisibility, we immediately obtain that the space of boundaries $B(\mathcal{V})$ on the limit $\mathcal{V}$ is compact Hausdorff because of Corollary 5.5.11 in section 5.5. The full space of instants $K(\mathcal{V})$ on $\mathcal{W}$ is not Hausdorff but only T_0 in virtue of the infinitesimal intervals that are inserted in the jumps of $B(\mathcal{V})$; it however still compact and is connected (see Theorem 5.6.11). Moreover, note that the subspace of boundaries of $K(\mathcal{V})$ is just $B(\mathcal{V})$. We now consider the direct limit construction on inverse systems of finite event structures.

5.10.1 Direct limits

Let $(T, \mathcal{W}_s, r_{vs}, \mathcal{V})$ be an inverse system of finite event structures. Since any map $r_{vs} : \mathcal{W}_v \to \mathcal{W}_s$ is a retraction map this implies that $\mathcal{W}_s$ embeds in $\mathcal{W}_v$, i.e., there exists a map $e_{sv} : \mathcal{W}_s \to \mathcal{W}_v$ which is an embedding; in our case, it is just the identity map. We are therefore entitled to consider the *direct system* $(T, \mathcal{W}_t, \sqsubseteq, e_{sv})$ where $\sqsubseteq = op(\leq)$ is the opposite order of $\leq$, i.e., $t \sqsubseteq s$ iff $s \leq t$ for any $s, t \in T$, and e_{sv} is the identity embedding. For this direct system it is possible to define a direct limit $\mathcal{D}$ in the usual way (see Hodges (1997)). We then have:

5.10.1. LEMMA. *Let $(T, \mathcal{W}_s, r_{vs}, \mathcal{V})$ be an inverse system of finite event structures. The direct limit $\mathcal{D}$ of the direct system $(T, \mathcal{W}_s, \sqsubseteq, e_{sv})$ is a countable model of GT, it is isomorphic to a submodel of $\mathcal{V}$, and hence it is a retract of $\mathcal{V}$.*

Proof:
The fact that $\mathcal{D}$ is a model of GT follows because $\mathcal{D}$ is the direct limit of a direct system of models of a geometric theory, and geometric formulas (actually, all Π_2 formulas) are preserved by direct systems (Hodges (1997), Theorem 2.4.6). Countability follows because all of the $\mathcal{W}_t$ are finite. To see that $\mathcal{D}$ is isomorphic to a substructure of $\mathcal{V}$ it suffices to consider the map $e : \mathcal{D} \to \mathcal{V}$ defined by mapping any element $a^{\sim}$ in the domain of $\mathcal{D}$ for $a \in \mathcal{W}_t$ to the thread $\xi \in \mathcal{V}$ defined by letting $\xi_s = r_{ts}(a)$ if $s \leq t$, and $\xi_s = a$ otherwise; this is easily checked to be an isomorphism onto the image. Since $\mathcal{D} \models$ GT, then, this substructure is also a model of GT. Finally, because of Theorem 5.8.3, $\mathcal{D}$ is a retract of $\mathcal{V}$. □

The direct limit $\mathcal{D}$ is isomorphic to the submodel of $\mathcal{V}$ given by all the threads $\xi \in \mathcal{V}$ which become eventually constant, i.e., those threads such that there exists $t' \in T$ with $\xi(t) = a$ for all $t \geq t'$. We shall abuse our notation and call this submodel of $\mathcal{V}$ also $\mathcal{D}$.

5.10.2 Complete event structures

We introduce here the notion of a complete event structure, which is of interest since it allows us to present the finitary spectrum of GT in a very canonical simple form.

5.10.2. DEFINITION. Let $\mathcal{W}$ be a finite model of GT. $\mathcal{W}$ is said to be *complete* if for any event $a \in W$ we have $a \equiv e$ for $e \in W$ the exact cover of two $\preceq$-minimal events $\mu, \mu' \in W$

5.10.3. PROPOSITION. *Let $\mathcal{W}$ be a finite event structure. Then there exists a complete event structure $H(\mathcal{W})$ and a retraction map $r : H(\mathcal{W}) \to \mathcal{W}$*

Proof:
Let $\mathcal{W}$ be a finite event structure; we construct $H(\mathcal{W})$ in two steps as follows. First, note that the auxiliary relations $\equiv_+, \equiv_-$ on an event structure are equivalence relations, and it is also clear that the R_+, R_- linearly order the equivalence classes under $\equiv_+, \equiv_-$ respectively. We extend $\mathcal{W}$ to an event structure $\mathcal{W}'$ as follows. For any equivalence class under $\equiv_+$ such that there exists no $\preceq$-minimal event in it, let a be any event which is minimal with respect to $\preceq$ in the equivalence class. Such event must exist because $\mathcal{W}$ is finite and $\preceq$ reduces to a total preorder on the equivalence class. We then extend $\mathcal{W}$ by adding an event μ satisfying: $\mu \equiv_+ a$ and $\mu \mathbb{O} c$ for any c with $c^{\equiv_+} > a^{\equiv_+}$. It is easily checked that μ is a $\preceq$-minimal event covered by a (and hence by any event in $a^{\equiv_+}$). A similar procedure can be performed for $\equiv_-$ equivalence classes. Closing the resulting structure under $\oplus, \ominus$ yields a finite model $\mathcal{W}'$ of GT such that for any event a in the model $a \equiv c$ where c is the exact cover of two $\preceq$-minimal events. Clearly, $\mathcal{W}$ is a submodel of $H(\mathcal{W})$, hence by Theorem 5.8.3 there must be a retraction map $r : H(\mathcal{W}) \to \mathcal{W}$. □

The following corollary of Proposition 5.10.3 makes it clear that the finitary spectrum can be replaced by the inverse system of all *complete* finite models of GT.

5.10.4. COROLLARY. *The set of all complete finite models of GT forms an inverse system of finite event structures which is cofinal with the finitary spectrum; the two inverse systems have then isomorphic limits.*

Proof:
Cofinality follows from Proposition 5.10.3. The fact that the limits are isomorphic is standard; see the proof of Proposition 5.9.8. □

5.10.3 The space of instants on the limit of inverse systems

We now come to the consideration of the space of instants on the limit of inverse systems of finite event structures, and in particular of inverse systems that satisfy infinite divisibility. The following result shows that the space of instants $K(\mathcal{V})$ on an inverse limit $\mathcal{V}$, not necessarily satisfying infinite divisibility, is separable:

5.10.5. Proposition. *Let $(T, \mathcal{W}_s, r_{ts}, \mathcal{V})$ be an inverse system of finite event structures. Then the following hold:*

(1) $B(\mathcal{V})$ is separable

(2) $K(\mathcal{V})$ is separable if and only if $B(\mathcal{V})$ has countably many jumps

(3) $K(\mathcal{V})$ is second countable if and only if $B(\mathcal{V})$ has countably many jumps

Proof:
Let $(T, \mathcal{W}_s, r_{ts})$ be an inverse system of finite event structures and let $\mathcal{V}$ be its inverse limit. First, note that since the set of all finite event structures is countable, then T is also countable. Hence, we can always consider a cofinal subsequence of T, whose limit is isomorphic to $\mathcal{V}$. We then for simplicity assume that the inverse system is an inverse sequence.

We begin by proving separability of $B(\mathcal{V})$, that is, we show that there exists a countable subset $Q \subseteq B(\mathcal{V})$ such that $U \cap Q \neq \emptyset$ for any non-empty open $U \subseteq B(\mathcal{V})$. Consider then the set $Q \subseteq B(\mathcal{V})$ defined by letting Q be the set of all points $q \in B(\mathcal{V})$ such that $P_q = L(\pi_s^{-1} P_x)$ for some $s \in T, x \in B(\mathcal{W}_s), P_x \neq \emptyset$. The set Q is countable, since $\bigcup_{s \in T} B(\mathcal{W}_s)$ is countable.

We first show the following claim: let $x, y \in B(\mathcal{V})$ be such that (x, y) is not a jump, then there exists $q \in Q$ with $x < q < y$.

To prove this claim, assume (x, y) is not a jump. Then there must be $z \in B(\mathcal{V})$ with $x < z < y$. We can then apply Lemma 5.5.13 and obtain threads ξ, ξ' such that $\xi \in P_z, \xi \in F_x$ and $\xi' \in P_y, \xi' \in F_z$. Since $\xi \in P_z, \xi' \in F_z$ we must have that $\xi \mathbb{O} \xi'$. Hence there must be a least $s \in T$ such that $\mathcal{W}_s \models \xi_s \mathbb{O} \xi'_s$, and thus the set $\{x \in B(\mathcal{W}_s) \mid \xi_s \in P_x, \xi'_s \in F_x\}$ is not empty and finite; let w be the minimal element of this set of boundaries. Clearly we have that $\xi \in \pi_s^{-1} P_w$. Moreover we have that $\zeta \mathbb{O} \xi'$ for any $\zeta \in \pi_s^{-1} P_w$; this follows because $\xi'_s \in F_w$ and hence $\xi' \in \pi_s^{-1}(\xi'_s) \subseteq \pi_s^{-1} F_w$, and P_w, F_w are $\mathbb{O}$-remote.

Since $\xi \in \pi_s^{-1} P_w$ and $\zeta \mathbb{O} \xi'$ for any $\zeta \in \pi_s^{-1} P_w$, it must be the case that $\xi \in L(\pi_s^{-1} P_w)$ and $\xi' \in L(\pi_s^{-1} P_w)\mathbb{O}$, so if we let q be the boundary whose past is $L(\pi_n^{-1} P_w)$ we must have $x < q < y$ and we are done.

We can now show separability of the order topology on $B(\mathcal{V})$ by showing that any nonempty basic open set in the order topology contains a boundary from Q. Consider a basic open set of the form $\{(x, y) \mid x, y \in B(\mathcal{V})\}$. If the pair (x, y) defines a jump, then this basic open set is actually empty, and we are done. Otherwise we just apply the result above to obtain a boundary $q \in Q$ which lies strictly between x and y, and we are done. Hence the space $B(\mathcal{V})$ is separable.

We now show (2). For the left-to-right direction, assume $B(\mathcal{V})$ has uncountable many jumps; we show $K(\mathcal{V})$ cannot be separable. Indeed, if $B(\mathcal{V})$ has uncountably many jumps, then by Lemma 5.6.7 $I(\mathcal{V})$ is uncountable, and each of these instants is open in $K(\mathcal{V})$ by Theorem 5.6.11. Since any subset dense in

$K(\mathcal{V})$ will have to contain $I(\mathcal{V})$, clearly $K(\mathcal{V})$ is not separable. For the right-to-left direction, just note that again by Lemma 5.6.7 $K(\mathcal{V})$ has countably many infinitesimal intervals; hence a countable dense subset of $K(\mathcal{V})$ is just $Q \cup I(\mathcal{V})$, with Q defined as above.

We now show (3). For the left-to-right direction, assume $B(\mathcal{V})$ had uncountably many jumps; we show $K(\mathcal{V})$ cannot be second countable. Indeed, if $B(\mathcal{V})$ has uncountably many jumps than $I(\mathcal{V})$ is uncountable by Lemma 5.6.7, and since every $x \in I(\mathcal{V})$ is open in $K(\mathcal{V})$ then $I(\mathcal{V})$ with the subspace topology is discrete and hence not second countable. But the subspace of a second countable space is second countable, so $K(\mathcal{V})$ is not second countable. For the right-to-left direction suppose that $B(\mathcal{V})$ has countably many jumps. Then the family of open sets $\{(q, \leftarrow) \mid q \in Q\}$, $\{(q, \rightarrow) \mid q \in Q\}$, along with the sets $(x, \rightarrow), (y, \leftarrow)$ for (x, y) a jump provide us with a countable subbasis for the topology on $K(\mathcal{V})$, and hence the space is second countable. □

The above proposition immediately yields the following corollary:

5.10.6. Corollary. *Let $(T, \mathcal{W}_s, r_{ts}, \mathcal{V})$ be an inverse system of finite event structures. Then $K(\mathcal{V})$ is separable if and only if it is second countable.*

We now focus our attention on the space of boundaries on limits of inverse systems of finite event structures that satisfy the requirement of infinite divisibility, since these are the culmination of our treatment in this chapter: they provide the Kantian continuum. We begin with the following result:

5.10.7. Lemma. *Let $(T, \mathcal{W}_s, r_{ts}, \mathcal{V})$ be an infinite divisibility inverse system. Then for any $\xi, \xi' \in \mathcal{V}$, if $\xi \mathbb{O} \xi'$ then there exists ξ'' between ξ and ξ', i.e., $\xi \mathbb{O} \xi'', \xi'' \mathbb{O} \xi'$ and $\xi'' \,\mathrm{R}_+ \xi, \xi'' \,\mathrm{R}_- \xi'$. Moreover, ξ'' can be taken to be a thread $\xi'' \in \mathcal{D}$.*

Proof:
Let $\xi, \xi' \in \mathcal{V}$ be such that $\xi \mathbb{O} \xi'$, and let s be the least index such that $\mathcal{W}_s \models \xi_s \mathbb{O} \xi'_s$. If there exists an event $a \in \mathcal{W}_s$ between ξ_s, ξ'_s we are done, since the eventually constant thread γ defined by a will be between ξ and ξ' in $\mathcal{V}$. Otherwise there is a boundary $x \in B(\mathcal{W}_s)$ with $\xi_s \in P_x, \xi'_s \in F_x$; by infinite divisibility then there must be $t \geq s$ and an event $a \in \mathcal{W}_t$ such that a is between ξ_t and ξ'_t in $\mathcal{W}_t$, and we can take the eventually constant thread defined by a as above. □

The result above states a sort of density for event structures, which the limit $\mathcal{V}$ of an infinite divisibility inverse system satisfies: given any two events which do not overlap, a third event can be found in between not overlapping both. We now have:

5.10.8. Lemma. *Let $(T, \mathcal{W}_s, r_{ts}, \mathcal{V})$ be an infinite divisibility inverse system. Then for any $\xi \in \mathcal{D}$, the set $\{\mu \in \mathcal{V} \mid \mu \preceq \xi, \mu \preceq -minimal\}$ is uncountable, and hence $I(\mathcal{V})$ is uncountable.*

Proof:
Let $(T, \mathcal{W}_s, r_{ts})$ be an infinite divisibility inverse system with limit $\mathcal{V}$. Again, we assume for simplicity that the inverse system is a countable inverse sequence, which can be assumed to be of the form $\mathcal{W}_0 \leftarrow \mathcal{W}_1 \leftarrow \cdots$. Choose any $\xi \in \mathcal{D}$. Then there must be a least index i such that $\xi_j = c$ for all $j \geq i$. Consider then the cofinal sequence obtained by taking all $\{\mathcal{W}_j \mid j \geq i\}$; the limit $\mathcal{V}'$ of this sequence is isomorphic to $\mathcal{V}$. We then construct uncountably many $\preceq$-minimal threads in $\mathcal{V}'$ as follows. Since $\mathcal{W}_i$ is finite we can choose any $\preceq$-minimal $\mu \in W_i$, and consider $\downarrow_{\preceq}\mu$; this is a cluster under $\preceq$ of $\preceq$-minimal events. By infinite divisibility let k be the least index $k > i$ such that $\mathcal{W}_k$ splits μ, and consider $r_{ki}^{-1}(\downarrow_{\preceq}\mu)$; this is a downset under $\preceq$ and by finiteness the set $M_k = \{\mu' \mid \mu' \in W_k, \mu' \in r_{ki}^{-1}(\downarrow_{\preceq}\mu), \mu' \preceq\text{-minimal}\}$ is not empty; moreover, since $\mathcal{W}_k$ splits μ, there are at least two $\mu_0, \mu_1 \in M_k$ such that $\mu_0 \mathbb{O} \mu_1$. Define then the initial segment of a thread by letting $\gamma_i = r_{ki}(\mu_0)$ for all $i < j$, and $\gamma_k = \mu_0$, and similarly for γ_1. Recursive iteration of this construction yields an uncountable set of threads $\{\gamma_\sigma \mid \sigma \in 2^\omega\}$ of $\mathcal{V}'$ such that for any γ_σ we have $\pi_s(\gamma_\sigma)$ is a $\preceq$-minimal event in $\mathcal{W}_j$ for all $j \in T, j \geq i$, i.e., $\mathcal{W}_j \models \forall x(x \preceq \pi_j(\gamma_\sigma) \rightarrow \pi_j(\gamma_\sigma) \preceq x)$ for all $j \in T, j \geq i$. We can now apply the right to left direction of Lemma 5.9.5 to conclude that γ_σ is a $\preceq$-minimal event in $\mathcal{V}'$ for any $\sigma \in 2^\omega$. It is then straightforward to see that any such σ is also a $\preceq$-minimal thread in $\mathcal{V}$ and that $\sigma \preceq \xi$ for all σ □

5.10.9. Corollary. *Let $(T, \mathcal{W}_s, r_{ts}, \mathcal{V})$ be an infinite divisibility inverse system. Then $\mathcal{A}_{\preceq}$ on $\mathcal{V}$ is not compact.*

Proof:
By Corollary 5.9.12 and Lemma 5.10.8 the family $\{\uparrow_{\preceq}\mu \mid \mu \in V \text{ is } \preceq\text{-minimal}\}$ is an uncountable open cover of $\mathcal{V}$ which does not have a finite subcover. □

5.10.4 The Kantian continuum as the Alexandroff COTS

We can now provide a general characterization of the space of instants $K(\mathcal{V})$ on the limit of infinite divisibility inverse systems, i.e., a general characterization of the Kantian continuum itself. Since the space of boundaries $B(\mathcal{V})$ on the limit of an inverse system of finite event structures is a compact and separable linear order, we can make use of the characterization of this type of orders which has been given by Ostaszewski in Ostaszewski (1974).

More specifically, for the total order $B(\mathcal{V})$ where $\mathcal{V}$ is the limit of an inverse system of finite event structures let us define two equivalence relations $\equiv, \sim \subseteq B(\mathcal{V}) \times B(\mathcal{V})$ as follows. We let $x \equiv y$ if the cardinality of the set of points between x, y is countable, where a point z is between x, y if $x \leq z \leq y$ or

$y \leq z \leq x$. Clearly this is an equivalence relation, and for simplicity we denote the equivalence class of a point $x \in B(\mathcal{V})$ under $\equiv$ as $\widehat{x}$. We then let $x \sim y$ if $x = y$ or if $\widehat{x} = \{x, y\}$. Note that for any two points $x, y \in B(\mathcal{V})$, if $x \sim y$ and $x \neq y$ then (x, y) defines a jump in $B(\mathcal{V})$, since given any $z \in L$ with $x < z < y$, $x \equiv y$ implies $x \equiv z$ and this implies that $z \in \widehat{x}$, which yields a contradiction. We denote with $\widetilde{x}$ the equivalence class under $\sim$ of any $x \in B(\mathcal{V})$. These equivalence relations can be used to characterize the total order of boundaries $B(\mathcal{V})$ on the inverse limit $\mathcal{V}$ of an infinite divisibility inverse system as follows:

5.10.10. THEOREM. *Let $(T, \mathcal{W}_s, r_{ts}, \mathcal{V})$ be an infinite divisibility inverse system. Then the following hold:*

(1) For any $x \in B(\mathcal{V})$, x has either an immediate predecessor x^-, or an immediate successor x^+, but not both.

(2) For any $x, y \in B(\mathcal{V})$, if y has no immediate predecessor and $x < y$, then there are uncountably many boundaries between x and y (and similarly if y has no immediate successor and $y < x$)

Proof:
Let $(T, \leq, \mathcal{W}_s, r_{ts})$ be an infinite divisibility inverse system, $\mathcal{V}$ be its inverse limit, and let $x \in B(\mathcal{V})$. We first show that x cannot have both an immediate predecessor x^- and an immediate successor x^+. Suppose otherwise. Then $(x^-, x), (x, x^+)$ define jumps, and hence (i) $\{y \in B(\mathcal{V}) \mid x^- < y < x^+ = \{x\}$. By Lemma 5.5.13 there are events $\xi, \xi' \in \mathcal{V}$ such that $\xi \in P_x \cap F_{x^-}$ and $\xi' \in P_{x^+} \cap F_x$, hence $\xi \mathbb{O} \xi'$. By Lemma 5.10.7 then there must be a third event ξ'' between ξ and ξ', and hence by Lemma 5.5.7 there are boundaries z, w with $x^- < z < w < x^+$, which is a contradiction with (i) above. Hence x cannot have both an immediate predecessor and an immediate successor.

We now show that x must have either an immediate predecessor or an immediate successor. Consider then C_x. Since it is a pairwise overlapping set of events, it can be extended to a maximal overlapping set of events A. Hence by Proposition 5.7.10, which holds for $\mathcal{V}$ because of Proposition 5.9.11, $A = C_i$ for an infinitesimal interval i of the form $(P_{l(\mu)}, \uparrow_{\preceq}\mu, F_{r(\mu)})$ for $\mu \in V$ a $\preceq$-minimal event. Now, either $i < x$ or $x > i$; without loss of generality assume $i < x$. Then i defines a jump (z, w) in $B(\mathcal{W})$ according to Lemma 5.6.7, and in particular $z = l(\mu) = \bigvee\{y \in K(\mathcal{V}) \mid \mu \in F_y\}$; we claim that $l(\mu)$ is an immediate predecessor of x, i.e., $x = w = r(\mu)$. Suppose towards a contradiction that $l(\mu) < r(\mu) < x$. Then by Lemma 5.5.13 there exists $a \in P_x \cap F_{r(\mu)}$, and hence $\mu \mathbb{O} a, \mu\, \mathrm{R}\_a$. It is then straightforward to check that for any $b \in C_i$, $a \preceq b$ because $\mu \preceq b, bOc$ for some $c \in C_x$. But then $a \preceq \mu$ which gives a contradiction, and we are done.

To show claim (2) let $x, y \in B(\mathcal{V})$ be such that $x < y$ and y has no immediate predecessor. We first show that there must be an eventually constant thread

$\gamma \in \mathcal{D}$ with $\gamma \in F_x \cap P_y$. Indeed, since (x, y) is not a jump there is z with $x < z < y$ and by Lemma 5.5.13 there are threads ξ, ξ' with $\xi \in P_z \cap F_x, \xi' \in P_y \cap F_z$. Hence by Lemma 5.10.7 there must be an eventually constant thread $\gamma \in \mathcal{D}$ between ξ, ξ', and hence $\gamma \in F_x \cap P_y$; by Lemma 5.10.8 there must then be uncountably many $\preceq$-minimal events covered by γ, which yield uncountably many boundaries between x and y by taking $\{l(\mu) \mid \mu \preceq \gamma\}$. □

5.10.11. COROLLARY. *Let $(T, \mathcal{W}_s, r_{ts}, \mathcal{V})$ be an infinite divisibility inverse system. Then every $x \in B(\mathcal{V})$ is adjacent to exactly one infinitesimal interval $i \in I(\mathcal{V})$, and $|\widehat{x}| = 2$ for any $x \in B(\mathcal{V})$.*

Given a limit $\mathcal{V}$ of an infinite divisibility inverse system of finite event structures, we can now endow the set of equivalence classes $\widetilde{B(\mathcal{V})} = \{\widetilde{x} \mid x \in B(\mathcal{V})\}$ with a linear order, by letting $\widetilde{x} \leq \widetilde{y}$ if $x \leq y$ or $\widetilde{x} = \widetilde{y}$. Note that this linear order is still a complete lattice; the join of a subset $S \subseteq \widetilde{B(\mathcal{V})}$, for instance, can be defined as $\widetilde{\bigvee_{\widetilde{x} \in S} x}$, and similarly for the meet. We then have:

5.10.12. THEOREM. *Let $(T, \mathcal{W}_s, r_{ts}, \mathcal{V})$ be an infinite divisibility inverse system. Then $\widetilde{B(\mathcal{V})}$ is order-isomorphic and hence homeomorphic to the unit interval $\mathbb{I}$.*

Proof:
The order topology of $\widetilde{B(\mathcal{V})}$ is compact because $\leq$ is a lattice. Moreover, we have that $\widetilde{B(\mathcal{V})}$ does not have jumps, i.e., it is dense. One can now check that the set $\{\widetilde{q} \mid q \in Q\}$, where Q is the countable set defined in the proof of Proposition 5.10.5, is a countable dense subset set which is also dense in $\widetilde{B(\mathcal{V})}$. Since any countable dense linear order without endpoints is order isomorphic to $(0, 1) \cap \mathbb{Q}$, we have that there is an isomorphism between $\widetilde{Q}$ and $(0, 1) \cap \mathbb{Q}$. Because of compactness of $\widetilde{B(\mathcal{V})}$ this isomorphism can be extended to an isomorphism between $\widetilde{B(\mathcal{V})}$ and $[0, 1]$. □

Again following (Ostaszewski, 1974), we obtain:

5.10.13. COROLLARY. *Let $(T, \mathcal{W}_s, f_{ts}, \mathcal{V})$ be an infinite divisibility inverse system. Then $B(\mathcal{V})$ is order-isomorphic and hence homeomorphic to $\mathbb{I} \times \{0, 1\}$ with the lexicographic ordering.*

Proof:
The claim follows from Theorem 5.10.12 and the main Theorem of (Ostaszewski, 1974). □

The results above state that the space of boundaries $B(\mathcal{V})$ on an infinite divisibility inverse system is order-isomorphic to the space obtained from the unit interval $\mathbb{I}$ by splitting every real $r \in \mathbb{I}$ into two points (r_0, r_1) that define a jump and taking the lexicographic ordering. Now, since there is a one-to-one correspondence between jumps and infinitesimal intervals (Lemma 5.6.7) and infinitesimal intervals are situated in the order of $K(\mathcal{V})$ between the two boundaries that define the corresponding jump (Proposition 5.6.10), we have that there is an order-isomorphism f from $K(\mathcal{V})$ to the space $\mathbb{I} \times \{0, 1/2, 1\}$ with the lexicographic ordering; note in particular that $f^{-1}(1/2) = I(\mathcal{V})$. However, the order-isomorphism is not a homeomorphism, because the topology on $K(\mathcal{V})$ is not the order topology, but is coarser (See section 5.6). To turn this order-isomorphism into a homeomorphism of topological spaces we must then consider a coarser topology on $\mathbb{I} \times \{0, 1/2, 1\}$. We then have:

5.10.14. Definition. The *Alexandroff COTS* is the ordered topological space whose underlying set is $\mathbb{I} \times \{0, 1/2, 1\}$, ordered lexicographically, and the topology is that induced by the subbasis of order-open rays of the form $\{((x, i), \leftarrow) \mid x \in \mathbb{I}, i \in \{0, 1\}\} \cup \{((x, i), \rightarrow) \mid x \in \mathbb{I}, i \in \{0, 1\}\}$. We denote this space by $\mathcal{A}$.

5.10.15. Theorem. *Let $(T, \mathcal{W}_s, f_{ts}, \mathcal{V})$ be an infinite divisibility inverse system. Then $K(\mathcal{V})$ is order-isomorphic and homeomorphic, that is, it is isomorphic in the category of ordered topological spaces, to the Alexandroff COTS $\mathcal{A}$.*

Proof:
Corollary 5.10.13, along with Proposition 5.6.10, give the order-isomorphism; continuity of the order-isomorphism and of its inverse is easily checked. □

We adopt the name "Alexandroff COTS" for the topological space $\mathcal{A}$ in Definition 5.10.14 since the topological space $\mathbb{I} \times \{0, 1\}$ with the lexicographic ordering is known as the "Alexandroff split interval", as it was originally introduced by Alexandroff to provide an example of a separable but not metrizable space; but $\mathcal{A}$ is also a connected ordered topological space in the sense of (Khalimsky et al., 1990), as we shall better see in chapter 6. Since the full space of instants $K(\mathcal{V})$ on the limit of an infinite divisibility inverse system is order-isomorphic and homeomorphic to $\mathcal{A}$, we then conclude that $\mathcal{A}$ provides the general topological structure of the Kantian continuum. Most importantly, $\mathcal{A}$ is (isomorphic to) the space of instants on the finitary spectrum of GT; hence the definition of an infinite divisibility inverse system is quite canonical in that all these inverse systems, and in particular the finitary spectrum, give rise to the same space of instants. Philosophically, the open points of $\mathcal{A}$, which correspond to the infinitesimal intervals in $I(\mathcal{V})$, represent the inexhaustible "in-between" that Brouwer mentions in Passage (4) of section 3.4. Indeed, the process of *divisio logica* could, in principle, proceed to transfinite heights by splitting the instants further. This is witness to the radical impossibility of exhausting the continuum, and one finds it unavoidable to meditate on the elusive nature of the Totality, which, even though it always seems at arm's length, escapes even the transfinite realms.

5.11 Infinitesimals

In this section we discuss, on the basis of the achievements in the previous sections, the flowing aspect of the Kantian continuum, through which time is represented as an object both in inner sense and in outer sense; these two aspects of time are represented by structures that are very different yet intimately related. An important reason why time must be represented as an object, i.e. as formal intuition, is the need to possess a substrate on which continuous ("flowing") magnitudes can be defined, as well as metrics which uniformly assign duration to events. Michael Friedman (M. Friedman, 1992) has rightly pointed out that the expression "flowing magnitude" ("*fliessende Grösse*") should be taken in the sense of Newton's *fluents*, independent variables which nonetheless vary continuously with time, and which therefore can be viewed as motions. Some of these are transcendental, e.g., drawing a line that is the external representation of time, and these motions are only required to be continuous. As a geometrical construction, drawing a line occurs in time as inner sense, as a function on the space of boundaries. These boundaries live on an inverse limit constructed under the constraint of the transcendental unity of apperception. This is time as inner sense, from which we somehow have to fashion the external representation of time in outer sense.

5.11.1 Duration

Of the three fundamental modes of time listed by Kant: succession, simultaneity and duration, the latter mode has received scant attention so far. We now investigate whether the Kantian continuum supports duration in the form of a metric:

5.11.1. Definition. A *metric* on a space X is a function $\delta(x,y) : X \times X \to [0,+\infty)$ satisying

1. $\delta(x,y) \in [0,\infty)$
2. $\delta(x,y) = \delta(y,x)$
3. $\delta(x,y) = 0$ iff $x = y$
4. $\delta(x,y) \leq \delta(x,z) + \delta(z,y)$

If the space X carries a topology τ, one says that (X,τ) is *metrizable* if there exists a metric δ on X such that the set of open balls $B(x,r) = \{y \mid \delta(x,y) < r\}$ generates τ. This can happen only if τ is Hausdorff. A function $\psi : X \times X \to [0,+\infty)$ satisfying conditions 1, 3, 4 only will be called a *pseudo-metric*. Pseudometrizability is defined as before; but the Hausdorff property is no longer implied.

5.11.2. LEMMA. *The linear order $K(\mathcal{W})$, with the topology as defined in Theorem 5.6.11, is neither metrizable nor pseudo-metrizable.*

Proof:
According to Theorem 5.6.11, $K(\mathcal{W})$ is a T_0 compact connected ordered topological space. In the presence of T_0, a pseudo-metric is a metric. However, if $K(\mathcal{W}), \tau$ were metrizable, it would have to be Hausdorff, but the material on the Alexandroff COTS in the previous section shows it is not. □

Hence if we equate the duration of an interval with distance assigned by a metric with values in $[0, \infty)$, not all intervals have a non-negative real-valued duration. We shall interpret this to mean that not all durations of intervals are *commensurable.* When Kant writes

> The instant in time can be filled, but in such a way that no time-series is indicated. (R4756, 17:700)

this suggests that an instant has a magnitude which is incommensurable with real-valued duration. Since one cannot extract a linear order of parts of an instant, this must mean that these parts are ordered as a complete graph, such that there is an edge between e and d if $e \leq d \wedge d \leq e$. An example will help.

Configurations of the type $x < i < y$ (x, y boundaries of a jump, i infinitesimal interval) are now interpreted as $x \leq i \leq y \leq x$, where $x \leq y \wedge y \leq x$ does not imply $x = y$. That is, what were previously three distinct instants now constitute a single "filled" instant $\{x, i, y\}$. As a consequence, $\leq$ is a preorder (a transitive and reflexive relation), not a linear order.

We will now proceed to define the magnitudes incommensurable with the reals.

5.11.2 Nilsquare infinitesimals

Consider the polynomial ring $\mathbb{R}[X]$ and take the quotient $\mathbb{R}[X]/(X^2)$ of $\mathbb{R}[X]$ by the ideal (X^2). In this structure, $X^2 = 0$ and each element can be written as $q + Xr$, for real q, r; this 2-dimensional vector space over $\mathbb{R}$ is the *ring of dual numbers.* Multiplication is given by

$$(q + Xr)(s + Xt) = qs + X(rs + qt).$$

All elements of the form Xr for $r > 0$ satisfy $(Xr)^2 = 0$; hence $1 - Xr$ is still invertible. If f is a smooth function, the Taylor expansion shows that for all $a \in \mathbb{R}$ and nilsquare ϵ

$$f(a + \epsilon) = f(a) + f'(0)\epsilon.$$

The ring $\mathcal{R} := \mathbb{R}[X]/(X^2)$ cannot be linearly ordered, but carries a preorder $\leq$ which is compatible with the ring operations and leaves the position of the nilsquares undecided.

(1) $x \leq y$ implies $x + z \leq y + z$

(2) $x \leq y$, $0 \leq r$ implies $xr \leq yr$

(3) $0 \leq 1$

(4) if e is nilsquare, $0 \leq e \wedge e \leq 0$

(5) if e is nilsquare, r a real, then er is nilsquare (incommensurability)

We then obtain:

5.11.3. LEMMA. *The following are true for* $(\mathbb{R}[X]/(X^2), \leq)$*:*

(1) $\leq$ *is reflexive, transitive, not anti-symmetric*

(2) if d, e *are nilsquare, then by transitivity* $e \leq d \wedge d \leq e$ *(nilsquares do not have a definite magnitude)*

(3) more generally if r *is a real and* d, e *are nilsquare then* $r + e \leq r + d \wedge r + d \leq r + e$

(4) $\leq$ *restricted to the reals is a linear order*

The relation $\leq$ defines an Alexandroff topology on $\mathcal{R}$ determined by $\leq$-downsets and $\geq$-upsets closed sets. E.g. if $1 - 2X$ is in the $\leq$-downwards closed set U, then so are 0 and $1 + 5X$. This topology is connected but not T_0 since $\leq$ is not a partial order, and no point is a cut point. To quote Weyl (Weyl, 1994, p. 92)

> An individual point in [a continuum] is non-independent, i.e., is pure nothingness when taken by itself, and exists only as a "point of transition" (which, of course, can in no way be understood mathematically).

The failure of T_0 allows us to show that $\mathcal{R}$ is a pseudo-metrizable space. We first need an auxiliary notion, the *semi-norm*:

5.11.4. DEFINITION. Let $x = a+d, y = c+e$ (d, e nilsquare), then the real-valued *semi-norm* $\parallel x - y \parallel$ is defined by

$$\sqrt{(a-c)^2 + (d-e)^2} = \sqrt{(a-c)^2};$$

here we use that in $\mathcal{R}$ any nilsquare can be represented as rX, r a real.

5.11.5. LEMMA. $\parallel x - y \parallel$ *satisfies*

(1) if r *is a real, then* $\parallel xr - yr \parallel = r \parallel x - y \parallel$ *(homogeneity)*

(2) if $z = a + e$, e nilsquare, $\| (x - z) - (y - z) \| = \| x - y \|$ (translation invariance)

(3) $x = 0$ implies $\| x \| = 0$, but not conversely (as one would require for a norm).

5.11.6. Definition. The semi-norm $\| x - y \|$ defines a homogeneous translation invariant pseudo-metric δ on $\mathcal{R}$ by putting $\delta(x, y) = \| x - y \|$.

5.11.7. Lemma. *δ generates the Alexandroff topology on $\mathcal{R}$ determined by $\leq$.*

The closed interval $[1, 2]$ contains $1 + rX$ and $2 + sX$ for all reals r, s, but $\delta(1 + rX, 2 + sX) = 1$ for all r, s: $\| 2 + sX - (1 + rX) \| = \| 1 + (s - r)X \| = 1$. The "instants" $1, 2$ can thus be viewed as filled ("fat") instants in the Kantian sense, which only give diffuse boundaries around an interval. More precisely, we need an assignment of instants of $K(\mathcal{W})$ to dual numbers given by a function $F : K(\mathcal{W}) \longrightarrow \mathcal{R}$ which is injective and satisfies $x \leq y \Leftrightarrow F(x) \leq F(y)$, i.e. F, F^{-1} are continuous w.r.t. the Alexandroff topologies associated to the preorders. As a consequence of this definition, if x, y differ only infinitesimally, then so do $F(x), F(y)$, and conversely. We may call such magnitudes "flowing", to be distinguished from "static" magnitudes, whose range does not include infinitesimals.[4] We take this to be an explication of Weyl's view that (Weyl, 1994, p. 92)

> It is due to the essence of time (and not to contingent imperfections in our medium) that a fixed temporal point cannot be exhibited in any way, that always only an approximate, never an exact determination is possible.

This is why the strict linear order on $K(\mathcal{W})$ is not suitable for representing duration and (infinitesimal) persistence. Note however that the two-dimensional numbers used to represent duration do not mean that time is somehow two-dimensional; precisely because the strict linear order $K(\mathcal{W})$ is what it is, one needs two mutually incommensurable types of numbers to quantify it.

Finally, we remark that we can now provide a formal correlate to Kant's notion of the "external representation of time" in the *drawing* of a line. Note the emphasis on "drawing": the external representation of time is not a line as a mere geometrical object, but it is the act of construction of this geometrical object by the imagination itself in time. Formally, we can capture this act by means of an order-preserving map $D : K(\mathcal{W}) \rightarrow \mathcal{R}$ from the Kantian continuum equipped with the preorder described above to the ring of dual numbers.

The next section discusses a surprising application of these notions to Kant's work on physics.

[4] In this sense the pseudo-metric (i.e. duration) is a static magnitude.

5.11.3 Infinitesimals in *Metaphysical foundations of natural science*

Infinitesimals provide an important bridge between the CPR and the *Metaphysical foundations of natural science* (MFNS). Recall from the discussion in section 3.8 that at B155n we find:

> Motion of an object in space does not belong in a pure science, thus also not in geometry; for that something is movable cannot be cognized a priori but only through experience. But motion, as description of a space, is a pure act of the successive synthesis of the manifold in outer intuition in general through the productive imagination, and belongs not only to geometry but even to transcendental philosophy.

This point is developed further in the first chapter ("Phoronomy") of the MFNS:

> In phoronomy, since I am acquainted with matter through no other property but its movability, and may thus consider it only as a point, motion can only be considered as the describing of a space - in such a way, however, that I attend not solely, as in geometry, to the space described, but also to the time in which, and thus to the speed with which, a point describes the space.

Surprisingly, speed, hence differentiation, is treated in the MFNS with a liberal sprinkling of infinitesimals. While discussing the change in velocity of an object projected upward from a point A, and reversing direction of motion at point B, Kant raises the question whether the object can be said to be at rest at point B, and answers affirmatively, with the following argument:

> The reason for this lies in the circumstance that the motion [of this object] is not thought of as uniform at a given speed but rather first as uniformly slowed down and thereafter as uniformly accelerated. Thus, the speed at point B is not completely diminished, but only to a degree that is smaller than any given speed. With this speed, therefore, the body would, if it were to be viewed always as still rising ... uniformly traverse with a mere moment of speed (the resistance of gravity here being set aside) a space smaller that any given space in any given time no matter how large. And hence it would absolutely not change its place (for any possible experience) in all eternity. It is therefore put into a state of enduring presence at the same place – i.e., of rest – even though this is immediately annulled because of the continual influence of gravity (i.e., the change of this state). (4:486)

Let q be a smooth function representing the height of the object as a function of time, so that q' represents the speed. We have generally for all smooth f, for all reals a, b, nilsquare ϵ,

$$f(a + b\epsilon) = f(a) + f'(a)b\epsilon$$

Kant claims that at the turning point $B = q(r)$ the speed is a non-zero infinitesimal representing the 'mere moment of speed'. Thus we cannot equate speed at B with $q'(r)b\epsilon$, since this quantity equals 0.[5] Instead we must expand the expression $q'(r + c\epsilon)$ to

$$q'(r + c\epsilon) = q'(r) + q''(r)c\epsilon = q''(r)c\epsilon,$$

which is a non-zero infinitesimal. This is possible because q' is also smooth. Now, even if c goes to infinity, the distance traversed between r and $r + c\epsilon$ is the integral of the infinitesimal-valued function $q''(r)c\epsilon$ over an infinitesimal interval, and this is indeed "a space smaller that any given space in any given time no matter how large.". If we equate "possible experience (of motion)" with with a process taking place in real (i.e. non-infinitesimal) time, then "it would absolutely not change its place (for any possible experience) in all eternity. It is therefore put into a state of enduring presence at the same place – i.e., of rest.". We have thus obtained a formalization of Kant's notion of rest, which we already discussed intuitively in section 4.2.2.

[5] At this point our analysis differs from Michael Friedman's, who in his magisterial (M. Friedman, 2013, p. 50) assumes the speed equals 0.

Chapter 6

Topology and the construction of time from experience

6.1 Introduction: Russell, Walker, and relativistic spacetimes

In the present chapter we relate the theory of event structures presented in chapter 5 to notions in digital topology (Kong & Rosenfeld, 1989) and formal topology (Sambin, 2003).

We show how in this context the definition of an event structure can be simplified and how the constructions of instants from events proposed independently by Russell (Russell, 1914, 1936) in philosophy and Walker (Walker, 1947) in physics are special cases of our approach to the Kantian continuum; this sheds light on their mutual relationship and on which of the two constructions is the most satisfactory (Thomason, 1989). Indeed, we shall see that the two constructions are complementary, as Walker and Russell identified the notion of the neighbourhood filter of a point for a special class of totally ordered topological spaces that arise in digital topology: only, they did so for two special, mutually disjoint classes of points of such spaces. We also show that every compact separable linear order of instants of time arises from a quotient of the Alexandroff COTS $\mathcal{A}$ introduced in Definition 5.10.14. As $\mathcal{A}$ is the formal correlate of the Kantian continuum, we interpret this result to mean that the continuum of time as the formal intuition contains all possible temporal orders of instants within itself, if one requires "temporal orders of instants" to be both compact and separable - very reasonable assumptions, as the temporal orders constructed by finite beings are certainly compact and separable.

This chapter also paves the way for the generalization of our approach to the non-linear case of relativistic spacetimes in view of reviving Russell's and Walker's original aim of constructing relativistic spacetimes from events. Indeed, while in philosophy the discussion on Russell's and Walker's constructions has overwhelmingly focused on the case of linear time, one should not forget that both Walker and Russell were mostly interested in constructing relativistic spacetimes, rather than linear time, from events.[1] While our focus here is in showing how our approach subsumes Russell's and Walker's in the case of linear time we shall do so by means of techniques in point-free topology that show how our approach can be extended to relativistic spacetimes, in agreement with their original project. We do, however, go beyond Russell's and Walker's concerns, as we are particularly interested in investigating the possibility of recovering spacetimes and the Kantian continuum in a finitary, point-free but also predicative and constructive fashion.

Of course, one might ask why the project of constructing relativistic spacetimes from events is still relevant and in particular why the Kantian continuum, which combines Russell's and Walker's constructions with inverse systems capturing infinite divisibility, should be applicable to the foundations of relativity.

With respect to metaphysical concerns, the answer to this question has already been given, to a certain extent, in the introduction to this thesis: we should like to know whether the structures occurring in fundamental physics, which are so far removed from experience, can be constructed from an ontology consisting only of primitives with an intuitive experiential interpretation, so that we might not only clarify the logical structure of the scientific theory but also fulfill that very central aim of philosophical investigation that is the "alterum enim alterius consentaneum esse dinoscitur, omnia unius esse aut unum esse omnia", the unification of dispersed knowledge into a coherent, all-encompassing system.

From the perspective of the foundations of physics and of quantum gravity, however, there is a further motivation to undertake this enterprise that has to do with investigating the role that the continuum plays in contemporary relativistic physics. We can better understand this point by examining again the difference between the Cantorean and the Aristotelian-Kantian continuum, and by then relating this fundamental distinction to the causal set approach to quantum gravity (Reid, 1999).

As we briefly outlined in section 3.4, the Cantorean view of the continuum became the most prominent view of the continuum in mathematics only with the rise of the axiomatic method in the XIX century, while before the continuist conception had still been a solid contender in the arena. In particular, the Cantorean continuum can be interpreted as arising from a process of infinitary idealization of the act of measurement, by means of which the connection between algebra and physical systems is abstracted away. In Feferman (2009, p. 9) Feferman has pointed out that perhaps the oldest technique to measure a line segment proceeds as follows. Given a line segment L, let us fix a unit line segment U with length $l(U)$ set arbitrarily to one, and let us first lay off U within L as many times

[1] As we shall see in the next section, Russell's motivations for this enterprise stem from the concerns of the logical positivist tradition, while Walker's motivations are more physical, as he aimed at providing foundations to relativity that would employ only strictly observable notions, in the tradition of Edward A. Milne's work on relativity Milne, 1935

as possible. This yields a first approximation to $l(L)$, the length of L, equal to $n_0 l(U)$. As there might still be a part of L that has not been covered in this process, we can divide U into k equal parts for some integer k, and lay off as many of these parts a possible within $L - n_0 U$, obtaining a better approximation of $l(L)$ equal to $n_0 U + n_1 U/k$. This process can of course be repeated over and over again, yielding better and better approximations of $l(L)$. Of course, each sequence of this kind is either a rational number or is the limit of an increasing sequence of rational numbers; taking then the space of all bounded non-decreasing sequences of rational numbers yields the Cantorean continuum constructed via Cauchy sequences. Still, note that constructing the continuum in this fashion requires one to abstract away from the physical limitations of measuring instruments and from computational resources; a sequence of rationals converging to an irrational number like π can be taken *as an object* only assuming an infinite power of discrimination through which such sequences can be taken as completed by *fiat*.

From a physical perspective, on the other hand, the procedure outlined above only provides one with a *rule* through which better and better approximations can be constructed, but one is certainly not thereby warranted in taking the sequence of rationals to be a completed object. In this respect, Kant's discussion of irrational magnitudes heeds this objection to the Cantorean construction and would seem to be closely related to the intuitionistic continuum, in that Kant conceives of, e.g., $\sqrt{2}$, essentially as denoting an algorithm to generate arbitrarily precise finite approximations to a number:

> The understanding, which arbitrarily makes for itself the concept of $\sqrt{2}$, cannot also bring forth the complete number-concept, namely, by means of its rational ratio to unity, but rather [...] it must give way in this determination to follow an infinite approximation to a number (11, 208:23-28)

And later:

> [...] The parts of the unit, which are to serve as the denominators of a series of fractions decreasing to infinity, are allowed to grow in accordance with a certain proportion [...] and this series, because it can never be completed, although it can be brought as near to completion as one wishes, expresses the root [...] (209:25-29)

Kant then goes as far as to claim that $\sqrt{2}$ is "itself no number, but only the rule for approximation to a number" (210:13-14), and that it is only "a determination of magnitude by means of a rule of enumeration [Zälen]" (14, 57:6-7); see in particular M. Friedman (1992, p. 110ff) for a masterful discussion of irrational magnitudes in Kant. From a Kantian and intuitionist perspective, then, the classical construction of the continuum with its actual infinites embodies an unacceptable idealization, which is far removed from the physical reality of finite powers of discrimination. Indeed, while mathematics was for a long time intertwined with physical concerns related to measurement - so much so that at the beginning of the XIX century students practiced a sort of "hybrid mathematics" with variables interpreted as physical quantities (Warwick, 2003) -, the mathematical advances of the XIX century where born by abstracting away from physical and computational limitations; this loftier and more powerful mathematics would then return to physics in the form of "applied mathematics", or "mathematical physics", in which terms both the theory of relativity and the theory of quantum mechanics were to be formulated. Thus, extending the approach of the previous chapters to the construction of relativistic spacetimes by relating it to formal topology would amount to grounding relativistic physics on a more restrictive conception of the continuum, for which computational and physical idealizations are central; indeed, formal topology has immediate computational content and would allow for the separation of the "infinitary" content of relativistic spacetimes from their "finitary" content (see section 7.3).

If the application of classical mathematics to physics has been so successful, however, why should one bother with investigating the role that different foundations of mathematics might play in physics, in particular with respect to such a basic concept as that of the continuum? For the physicist, I believe that the answer to this question is simply that a better understanding of the logical structure of relativity theory is of use for the task of providing a quantum theory of gravity that would recover relativistic spacetimes as its classical limit, and that great insights can be obtained by "going back to basics" - examine critically the foundational concepts of, in this specific case, relativity theory, and experiment with alternative observational interpretations for these concepts. For instance, Porter remarks in Porter, 2002 that

> Even from the start, the space-time "manifold" was considered "unphysical". It involved numerous powerful mathematical concepts that were inherently beyond observation, although providing apparently essential tools for developing the physical theory. More recently the "unphysicality" has stimulated attempts to use an observational approach to model the differential, dynamic aspects of space-time (and eventually to quantize it) using discrete, algebraic, or combinatorial models.

The remark above described most accurately the approach to a theory of quantum gravity known as causal set theory, which questions exactly the role played by infinity and the continuum in relativity and attempt to build a quantum theory of gravity on the basis of techniques from discrete mathematics (R. Sorkin, 1991).

In particular, the fundamental tenet of causal set theory is that spacetime is discrete at the Planck scale, that is, that there exists a smallest length scale and that time flows with in a series of "ticks" having a certain fixed duration. It is then assumed that at the Planck scale spacetime has the structure of a "causal set", which is just a partial order of fundamental elements that is discrete in the sense of being locally finite: given any two elements x, y of the order with $x \leq y$, the set $\{z \mid x \leq z \leq y\}$ is finite. Here the order $\leq$ is supposed to encode causal connectibility, in the relativistic sense. It is a classical result (Robb, 2014; Malament, 1977) that the causal structure determines almost all of the information of the metric tensor of a relativistic spacetime; more precisely, it determines the metric up to a multiplicative factor termed "local conformal factor", which is related to the volume of a local region of spacetime. Sorkin's inference (R. D. Sorkin, 2005, p. 3ff) from this fact is that in the continuum the causal order $\preceq$ does not suffice to specify the volume element, which then has to be given externally, but in a discrete structure such as a causal set the volume of a region can actually be determined by associating to each fundamental element a fundamental unity of volume and then by counting the number of fundamental elements in the region.

Now, most of the work in causal set theory has focused on finding ways to determine whether a causal set can be faithfully embedded into a manifold, preserving the causal order and moreover ensuring that the volume computed in the causal set for a region is proportional to the volume according to the metric. The problem has been approached by reverse engineering: given a manifold M, one randomly chooses points in the manifold proportionally to the volume by means of a Poisson random process so that local Lorenz invariance is preserved; this process, known as "sprinkling", yields a causal set that is faithful to the original manifold. Still, it has proven quite difficult to determine suitable conditions guaranteeing that an abstract causal set, not constructed through sprinkling from a manifold, can be faithfully embedded in a spacetime manifold.

Despite the fact that causal set theorists have mostly focused on investigating this process of sprinkling I am of the opinion that algebraic, topological and categorical approaches such as Christensen and Crane, 2005; Martin and Panangaden, 2006, 2010; Mallios and Raptis, 2001 offer a more principled way to tackle the problem of recovering the spacetime manifold from discrete structures because they allow for a greater understanding of the nature of causal sets and, most importantly, bring to the fore the foundational issues of the theory. These approaches have been pioneered by Sorkin himself, who in R. D. Sorkin (1991) shows how to approximate compact Hausdorff spaces by means of quotients of limits of inverse systems of finite T_0 spaces. The underlying idea is that the possible "finitary substratum" of the macroscopic continuum resembles the latter so that its structure is that of a finitary T_0 topological space, which via the Alexandroff correspondence corresponds to a finitary partial order encoding causal connectibility. The very same results in R. D. Sorkin (1991) were obtained independently in Kopperman and Wilson (1997), where it is shown that every compact Hausdorff space is the Hausdorff reflection of the inverse limit of its T_0 quotients, where the Hausdorff reflection is just its space of closed points. If the compact Hausdorff space in question is the unit interval $\mathbb{I}$ then the inverse limit of its T_0 quotients is known as the Smyth interval, which, as we shall see in section 6.5, is a special type of totally ordered topological space, and its Hausdorff reflection is just the unit interval. What is more, the n-square $\mathbb{I}^n$ is approximated in the same fashion, by considering products of T_0 quotients of each of the unit components - see (Webster, 2006, p. 443). Since $\mathbb{I}^n$ is isomorphic to a closed causal diamond in n-dimensional Minkowski spacetime in the category of ordered topological spaces one can approximate any closed diamond in n-dimensional Minkowski spacetime, and the causal order can be recovered purely topologically by regarding M^n as a bitopological space, as we shall see in section 6.5. In our point-free construction presented in chapter 5, on the other hand, we obtain the unit interval as the quotient of the Alexandroff COTS $\mathcal{A}$. Recall that the latter is not second-countable, and hence cannot be obtained as the limit on an inverse system of finite topological spaces and continuous maps. Thus, our category of event structures and event maps provides us with a different route to the recovery of $\mathbb{I}$, and we recover in a point-free setting not only the topology but also the order $\leq$ on the unit interval, namely, $\mathbb{I}$ is a quotient of $\mathcal{A}$ in the category of ordered topological spaces. Now, analogously to the above results in digital topology this approach is not limited to the unit interval; by generalizing the notion of an event structure and the construction of points in chapter 5, one should in principle be able to approximate by means of inverse limits any compact connected separable partially ordered topological space having a subbase of open upsets and downsets, that is, a compact connected separable ordered topological space. Thus, a suitable generalization of our approach would allow us to approximate, say, any closed diamond in a connected strongly causal spacetime M, since the latter, equipped with the causal order $\leq$ and the manifold topology M, is a connected non-compact Nachbin space (Nachbin, 1965).

However, before an investigation along these lines can be pursued so as to revive the Russell-Walker project it is useful to start by relating event structures as discussed in chapter 5 to digital and formal topology, which is the main content of the present chapter. This is important because in this context it becomes clear that the treatment in chapter 5 subsumes Russell's and Walker's constructions and that moreover it can be generalized to the case of partially ordered topological spaces. The use of formal topology also allows for the development of a constructive treatment of the Kantian continuum and, in turn, of the approximation of the topology and order of relativistic spacetimes, in keeping with the concerns mentioned above.

Of course, one might object that the greatest challenge in an approach of this sort is not that of approximating the topology or the causal order but rather the spacetime metric, and in this respect even the simplest case of two-dimensional Minkowski spacetime is highly non trivial. I thoroughly agree, but I am again of the opinion that point-free categorical approaches offer a more principled perspective on this problem than the sprinkling method favored by causal set theorists, so that a treatment of the approximation of the topology and order of relativistic spacetimes in a constructive point-free setting would pave the way for attacking the metric problem by importing tools from the field of "tolerance" or "fuzzy" geometry (Poston, 1971), which is akin in spirit to digital topology. Indeed, in (Christensen & Crane, 2005) techniques similar to those of tolerance geometry are employed to show (Theorem 3.3) how to approximate the length of a timelike geodesics in Minkowski spacetime by means of finite causal paths, which are akin to the notion of a path in a tolerance space. This result is analogous to that proven by Poston (Poston, 1971, p. 157) that the length of a bounded curve in $\mathbb{R}^n$ is faithfully approximated by imposing a unit tolerance on $\mathbb{R}^n$, so as to regard the latter as as a tolerance space, and then computing the "hop distance" between the two endpoints. In light of this fact I believe that the more general approach in (Poston, 1971) might be the right way to tackle the generalization of theorem 3.3 to curved spacetimes, which is stated in conjecture 3.5 of (Christensen & Crane, 2005). Moreover, in (Poston, 1971, p. 113) various tools for a tolerance version of differential geometry - a difference geometry - are developed, and the notion of a *matroid* - a combinatorial generalization of graphs and of independence in vector spaces - is used to provide a combinatorial version of the tangent bundle of a differential manifold. These techniques, and perhaps also the results in Webster (2006) where it is shown how to approximate measures on a continuum by means of measures on finite spaces, might offer a way to handle the problem, but I must leave an investigation of these matters to future work.

The structure of the chapter is as follows. In section 6.2 we discuss the original constructions of instants of time from extended events proposed by Russell and Walker and how they have been interpreted in the literature. In section 6.3 we discuss the relation between our definition of event structure and Russell's and Walker's definitions, while in section 6.4 we show how our approach allows us to subsume both Russell's and Walker's constructions. In section 6.5 we discuss some notions from digital topology and prove the representation theorem for compact separable linear orders mentioned above. In section 6.6 we move to the point-free setting and show how event structures as presented in chapter 5 are a special class of formal topologies, which allow for the development of the constructive Kantian continuum and for the generalization of our framework to the case of partially ordered topological spaces, in view of constructing relativistic spacetimes.

6.2 The construction of time in Russell and Walker

In this section we discuss Russell's and Walker's constructions of instants of time from events along with their philosophical background. Russell's construction of instants of time from events is presented in Russell (1914) and Russell (1936), and is carefully analyzed in Anderson (1989), Lück (2006), Bostock (2010), Mormann (2009); Walker's construction, on the other hand, is presented in Walker (1947) and discussed in Withrow (1961). There seems to be no evidence that either Russell or Walker were aware of each other's work; rather, they seem to have proceeded in their investigations independently.

6.2.1 Russell's construction and event structures

Russell was concerned with the logical construction of instants of time from extended events primarily because he aimed to provide a foundation for relativity theory in terms of concepts that were drawn from *experience*, i.e., that had a solid "phenomenological standing". In Russell (1914), for instance, Russell remarks that:

> It is to be observed that we cannot give what may be called *absolute* dates, but only dates determined by events. We cannot point to time itself, but only to some event occurring at that time. There is therefore no reason in experience to suppose that there are times as opposed to events: the events, ordered in the relations of simultaneity and succession, are all that *experience* provides. Hence, unless we are to introduce superfluous metaphysical entities, we must, in defining what mathematical physics can regard as an instant, proceed by means of some construction which assumes nothing beyond events and their temporal relations. (p. 117, our emphasis)

This passage makes it quite clear that Russell's concerns here were largely within the broad scope of logical positivism, as they betray the aim of providing an ontology for mathematical physics that would eschew all entities that cannot be justified on the basis of experience, and hence are not "metaphysically parsimonious". Russell's approach to the construction of time from events was greatly influenced by Whitehead's construction of points in space and spacetime by means of "enclosure-series", i.e., nested series of extended regions and four-dimensional spatiotemporal regions.[2] In Russell (1914, p. 122) however, Russell departs substantially from Whitehead's approach, since he defines instants as maximal classes of pairwise overlapping events (recall Definition 5.6.1), where the events are not required to form a nested series as Whitehead's region-based theory of space would have it.

We can better understand Russell's construction by reconsidering the notion of an event structure. In the most general terms, an *event structure* is just a first-order structure $\mathcal{W} = (W, \sigma, I)$ that is a model for a first order axiomatic theory A, where the elements of the carrier W are conceived of as "events" or "durations", the relation and function symbols in the first-order signature σ are

[2]Various studies of Whitehead's point-free geometry, from both a philosophical and mathematical standpoint, exist (Biacino & Gerla, 1996; Vakarelov, 2012, 2014). It is also important to note that a similar approach was investigated by De Laguna in De Laguna (1922).

given a temporal interpretation, and I is just the usual interpretation function. Of course, such a definition is not very useful unless the signature σ and the axiom system A are specified, as we did for our definition of an event structure in section 5.3. In the logical literature on Russell's and Walker's theories of time, however, there is no standard selection of temporal primitives and axioms. There is a general understanding that the members of σ should encode the most basic (functional) relations that hold between, say, the intervals of the real line as the standard representation of events on a linear temporal continuum; this constrains the possible range of primitives somewhat, however different choices are still possible, as the variety of Allen's fundamental temporal relations on intervals shows (Allen, 1983).

In Russell (1914) Russell takes only two binary relations on events as primitive: a relation which (Anderson, 1989) glosses as "the times at which a exists coincide (in part or whole) with the times at which b exists", and that we shall denote with aOb, and the relation glossed as "a temporally wholly precedes b, i.e., every time at which a exists is temporally precedent to any time at which b exists", which we shall denote with aPb. Note however that the primitive relations cannot be defined in terms of instants or time; rather, these are merely elucidations of the relations, but it is assumed that they are known by direct acquaintance in experience. Indeed, Russell himself simply describes O as "overlap" and P as "complete precedence". In Russell (1936), on the other hand, he takes only P as primitive, and lets

$$aOb \leftrightarrow \neg aPb \wedge \neg bPa \tag{D}$$

Of course, whether one takes O as a primitive relation or not one must provide some axioms that the primitive relations satisfy, i.e., a definition of what counts as an event structure in the above sense, and a method to construct instants of time from these event structures. Russell approaches both problems first in Russell (1914), with O as primitive, and returns to them some twenty years later in Russell (1936). An instant, says Russell, is a maximal class of overlapping events:

> Let us take a group of event of which any two overlap, so that there is some time, however short, when they all exist. If there is any other event which is simultaneous with all of these, let us add it to the group; let us go on until we have constructed a group such that no event outside the group is simultaneous with all of them, but all the events inside the group are simultaneous with each other. Let us define this whole group as an instant of time. (Russell, 1914, p. 126).

Note that the relation of "simultaneity" mentioned in this passage is nothing other than the relation O of overlap. The basic intuition of the construction is that a specific instant or "date" can be conceived as the class of all events that were occurring at that instant. Clearly, assuming that two events occurring at the same instant overlap it follows that an instant must be a class of pairwise overlapping events, and moreover assuming that if an event a does not occur at the instant then there must be a "witnessing event" b occurring at the instant which is not "simultaneous" with a, a highly non-trivial assumption, then the instant must be a maximal class of pairwise overlapping events.

Now, several commentators have taken Russell here to be striving towards a formulation of the notion of a maximal filter (Mormann, 2009, 2006; Bostock, 2010). This is correct, but one has to be quite careful here and note that with Russell's primitives this is not immediately apparent, since he does not have anything resembling a preordered relation of "covering" between events. Nevertheless, one can see how Russell's definition would become that of a maximal filter in a preorder by considering his axioms for event structures. Indeed, (Anderson, 1989) and before him (Wiener, 1914) note that the only axioms required to carry out Russell's construction in Russell (1936) are the following:

A1 $\neg aPb$ (irreflexivity of P)

A2 $aEb \wedge bPc \rightarrow aPc$

Where the relation aEb is defined by letting $aEb \leftrightarrow \exists c(aPc \wedge cOb)$, with intuitive reading "a ends before b", and O is defined as in (D). Now, clearly one can define, in an analogous fashion as E, the relation B with intuitive reading "a begins after b" by letting $aBb \leftrightarrow \exists c(cPa \wedge cOb)$, and then the relation $a \preceq b$, that is, "a is covered or encompassed by b", by letting

$$a \preceq b \leftrightarrow \neg(bEa) \wedge \neg(bBa)$$

It is a straightforward matter, using transitivity and irreflexivity of P, to check that the relation $\preceq$ is reflexive and transitive and therefore a preorder. Suppose we now strengthen Russellian event structures by adding the following axiom:

$$\forall a \forall b(aOb \rightarrow \exists c(c \preceq a \wedge c \preceq b \wedge \forall d(d \preceq a \wedge d \preceq b \rightarrow d \preceq c)) \qquad \text{(A3)}$$

Axiom (A3) requires that if two events a and b overlap, then there exists an event c covered by both which is maximal with this property; that is, c is a sort of pseudo-meet in a preorder analogous to that provided in section 5.3.4, but which is not required to begin simultaneously with one among a, b and end simultaneously with the other. An involved, but straightforward proof then yields:

6.2.1. PROPOSITION. *Let $\mathcal{W}$ be an event structure satisfying axioms 6.2.1, 6.2.1,* (A3)*. Then aOb, cOa, cOb implies cOd for some d with $d \preceq a, b$.*

Of course, the above proposition is the analogue of claim (5) in Lemma 5.3.4. We then immediately obtain:

6.2.2. Proposition. *Let $\mathcal{W}$ be an event structure satisfying axioms 6.2.1, 6.2.1,* (A3). *A subset of events $A \subseteq W$ is a maximal overlapping class if and only if it is a maximal filter.*

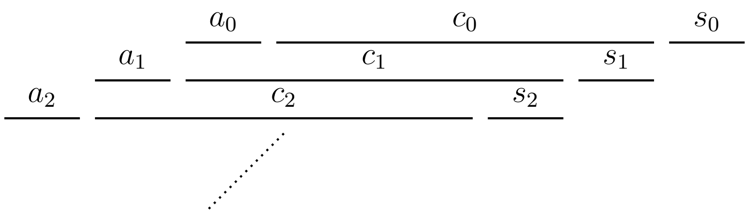
Proof:
A straightforward adaptation of the proof of Proposition 5.7.1 using Proposition 6.2.1. □

Note that a maximal filter $\mathcal{F}$ is also prime, as its complement is clearly an ideal; moreover, for the above proposition to hold it does not suffice to require a weaker version of axiom (A3) where the maximality condition has been removed, that is:

$$\forall a \forall b (aOb \rightarrow \exists c(c \preceq a \wedge c \preceq d)) \tag{A4}$$

Does not suffice, as the following example shows.

6.2.3. Example. Let $\mathcal{W}$ be the event structure with domain $\{\star_i \mid \star \in \{a, c, s\}, i \in \omega\}$ and interpretation such that $a_i P c_i, c_i P s_i, a_{i+1} P a_i, s_{i+1} P s_i$ for all $i \in \omega$. The following diagram represents such an event structure $\mathcal{W}$:



Then $\mathcal{W}$ is a model of 6.2.1, 6.2.1, (A4), and moreover $\{c_i \mid i \in \omega\}$ is a maximal overlapping class, but it is not down-directed under $\preceq$; indeed, it is an infinite antichain.

Thus, Russell's definition of an instant of time as a maximal overlapping class of events amounts to that of a maximal filter in the covering preorder if one strengthens his framework somewhat by adding axiom (A3). From an epistemological standpoint, however, such an axiom is not too problematic, since after all if two events overlap in time it seems plausible that one ought to be able to exhibit a period of time during which they coexist.

Having given his definition of an instant, Russell then attempted to provide additional axioms for his event structures that would ensure that the set of Russellian instants has the properties desired of a time series. Axioms 6.2.1, 6.2.1 already suffice to linearly order the instants, by letting, for any Russell instants i, i':

$$i \leq i' \text{ if } \exists a \in i \exists b \in i'(aPb) \tag{6.1}$$

It is straightforward to show that the relation $\leq$ on instants so defined is a linear order. Russell was also concerned, however, with providing axioms ensuring that the linear order of instants was dense and complete. We shall not examine Russell's attempts in this sense since, as we have seen, we are more interested in answering such questions via inverse systems, as these allow one to model a process of approximation of infinite structures by finite structures; thus, they are more suited to modelling an agent as a finite information processor, and in this sense they are more in agreement with experience. Indeed Russell's philosophical project, as much as Walker's, does not merely consist in developing a relationist theory of time and spacetime, but, most importantly, a phenomenological and operationalist theory. One can be a relationist because one believes that theories of space-time should ultimately be grounded on a set of primitive entities and relations that hold among them, such as the relation of "betweenness" on points (Goldblatt, 2012; Schutz, 1997), without worrying about the experiential or operationalist meaning of the primitives and relations. Russell's and Walker's requirements are more stringent: the relations and primitives must have a clear observational interpretation, and extensionless instants of time, or points in spacetime, must be conceived as idealized logical constructions in terms of bounded extended spacetime regions. Our emphasis on inverse systems can be then conceived as bringing Russell's and Walker's operationalist approach even further.

Finally, note that while Russell's and Whitehead's constructions are different, their ultimate aims are essentially the same: present the *continuum* of relativity theory as a logical construction in terms of primitive notions that are fundamentally grounded in experience. In particular, Russell's construction of a linear order of time instants was supposed to elucidate how the construction of relativistic spacetimes from events would proceed. In philosophy, Russell's approach has been mostly discussed in the context of logical constructions and "quasi-analysis" on the one hand (Leitgeb, 2007), and region-based theories of space and time on the other (Mormann, 2006, 2009; Bostock, 2010). Recently, Mormann (2009) has interpreted Russell's construction of points from events in terms of the construction of points of a topological space as maximal round filters of the corresponding lattice of open sets, firmly grounding the discussion of Russell's approach in the theory of frames and locales, which is the standard approach to a point-free, region-based account of topological spaces in mathematics (Johnstone, 1983).

6.2.2 Walker's construction

Walker's construction has not itself enjoyed as much popularity as Russell's, most likely because Walker was not a philosopher but a physicist. Indeed, his most important foundational work (Walker, 1948) has had mostly an impact on axiomatizations of relativity theory (Schutz, 1997), but is still based on a set $\mathcal{I}$ of primitive instants or spacetime points and a set of particles $\mathcal{R} \in \mathcal{PI}$, on whose basis various axioms involving a relation of temporal order $< \subseteq \mathcal{I} \times \mathcal{I}$ on instants and a relation of signal correspondence $\wedge \subseteq \mathcal{R} \times \mathcal{R}$ on particles are formulated. It is not assumed at the outset, but it is a consequence of the axioms, that the instants in a particle $R \in \mathcal{R}$ are linearly ordered, so that a particle is actually just a world-line in spacetime. Walker then recovers, as models of his axioms, Milne's Kinematical relativity, and shows that his axiom system is consistent with respect to spacetimes in GR of zero or negative curvature. At the very beginning of this work, however, Walker voices his discontent with an approach that has point-like instants as primitive ontological notions by saying:

> It may be argued that we are not in agreement with experience in taking our undefined element of time to be an instant, and that this element should be a *duration*, to be pictured as an interval. This is certainly true, and we hope later to replace the instants, temporal relations and the temporal axioms of the present paper by still more fundamental ideas in closer agreement with *experience*. These will give rise to instants as defined elements, and, except for signal correspondences which will then refer to durations, the remainder of the present paper will be valid. (Walker, 1948, p. 321)

Note how the passage above resonates perfectly with Russell's and Whitehead's concerns for a logical construction of relativistic spacetimes from notions that are grounded in experience; thus, Walker is engaged in exactly the same philosophical project as Russell's. Nevertheless, the construction presented in Walker (1947), which he developed to solve the problem mentioned in the passage above but did not have as much of an impact in the axiomatic foundations of relativity, differs radically from Russell's construction. This is all the more surprising since his primitive ontology is the same as Russell (1936), namely, the signature of an event structure comprises only one primitive binary relation P of precedence and the relation O of overlap is defined exactly as in (D) above. The axioms on events he considers are axiom 6.2.1 above, and moreover:

$$aPb, bOc, cPd \to aPd \tag{6.2}$$

It is a matter of simple substitution, however, to see that axiom (6.2) is equivalent to axiom 6.2.1 above, and hence that Russell's event structures and Walker's event structures amount to the same thing. Walker's axiom (6.2), however, is in a sense more pleasant than axiom 6.2.1 since its form is universal and not existential and it does not require the introduction of a further defined notion.

A Walker instant is then defined as a triple $(P, C, F) \in (\mathcal{PW})^3$ of subsets of the set W of events such that

(1) $P \cup F \cup C = W$

(2) P, F are not empty

(3) aPb for any $a \in P, b \in F$

(4) If $c \in C$ then there exists $a \in P, b \in F$ with cOa, cOb

Walker's definition captures the idea that the present of an instant of time is a "separation" but also a "link" between its past and its future, in agreement with Aristotle's and Kant's characterizations of the "now in time" (see section 3.4). In what follows, given a Walker instant x we shall denote its past, present and future by P_x, C_x and F_x respectively.

Since Walker's setting for event structures is the same as Russell's a natural question is whether, say, the present of a Walker instant can also be characterized as a filter in some way. Suppose then that we adopt the definitions of $E, B, \preceq$ as in Russell's construction. First note that we immediately have

6.2.4. PROPOSITION. *Let $\mathcal{W}$ be an event structure and (P, C, F) be a Walker instant. Then C is a pairwise overlapping class.*

Proof:
If (C) is not an overlapping class there are $a, b \in C$ such that, without loss of generality, aPb. Then let $P' = \{c \in W \mid a\mathbb{E}c\}$, $F' = \{c \in W \mid \neg cOa\}$, and $C' = P' \cup F'^c$. It is a straightforward matter to check that (P', C', F') is a Walker instant, and that $P \subset P' \subset P$, which yields a contradiction. Hence C is an overlapping class. □

Of course, the proof of Proposition 6.2.4 essentially relies on the fixpoint construction presented in section 5.5, which can also be carried out for Walker's original setting. Suppose then we also assume axiom (A3); Proposition 6.2.1 then holds, and we have

6.2.5. PROPOSITION. *Let $\mathcal{W}$ be an event structure satisfying axiom 6.2.1,* (6.2), (A3)*, and let (P, C, F) be a Walker instant. Then C is a prime filter.*

Proof:
It is straightforward to show that C is upward closed with respect to $\preceq$, since if $a \in C$ and $a \preceq b$ then aOc implies bOc for any $c \in W$, and since a must overlap with some event is the past and some event in the future, because of condition (4) of a Walker instant, then so does b. Hence b must be in C because of conditions (3) and (1) of a Walker instant.

To show that C is down-directed under $\preceq$ let $a, b \in C$; then by Proposition 6.2.5 aOb, hence by axiom (A3) there exists $d \preceq a, b$ maximal with this property. Now if aEb then there must be $c \in F$ with aOc, but this implies bOc, hence by Proposition 6.2.5 we have that cOd, so $d \notin P$; a similar reasoning applies if bEa or $\neg(aEb), \neg(bEa)$. The case to show that $c \notin F$ is analogous.

To see that C is prime we show that C^c is an ideal. Choose then $a \in C^c$ and $b \preceq a$. Now if $b \in C$ then □

The converse of Proposition 6.2.5 does not, of course, hold. For instance, it is easy to see that a maximal filter under $\preceq$ in a finite event structure cannot be the present of a Walker instant, since finiteness and directedness imply that the filter will be the upset under $\preceq$ of a minimal element in the $\preceq$-ordering, i.e., a $\preceq$-minimal event according to Definition 5.7.2; this event, however, cannot be in the present, because if it were it would have to overlap with an event in the past and an event in the future and, by axiom (A3), it would then not be minimal.

In Walker (1947) Walker then proves that the set of instants can be totally ordered by letting

$$x \leq y \text{ if } P_x \subseteq P_y \tag{6.3}$$

And he also proves that the order is complete in the sense that any bounded subset of it has a least upper bound (Theorem 17 in Walker (1947)). Note that maximal pairwise overlapping classes of events do not appear anywhere in Walker's construction.

Now that we have examined Russell's and Walker's constructions the question is what the relation between the two is, and in particular if they can be combined. Indeed, in the last half-century various works have evaluated the respective merits of Russell's and Walker's constructions (Thomason, 1989; Lück, 2006), mostly with respect to how well they can recover something resembling the real continuum. These works do not, however, attempt to relate both of them to well-established results in topology or point-free topology.[3] Furthermore, the importance of the theory for the foundations of relativity, which was the main concern of Russell, Whitehead and Walker alike, has been largely sidelined; none of the recent technical work seems to mention it aside from some cursory remarks of historical character, the focus being mostly on the case of the linear temporal continuum or on the spatial case. In the following sections we aim at resolving the discussion about the respective merits of Russell's and Walker's approaches by showing that they are special cases of a more general construction, which exhibits a specific class of totally ordered topological spaces studied in digital topology (Kong & Rosenfeld, 1989) as the space of prime filters on a specific class of semilattices that is obtained by strengthening the axioms on event structures. We show that an approach to the construction of the time continuum that combines order and topology in a point-free setting unifies Russell's and Walker's construc-

[3]There are of course results regarding the relation of Russell's construction to point-free topology (Mormann, 2009; Bostock, 2010), although we shall see that our approach helps to shed further light on the matter.

tions and shows that they are complementary. Furthermore, even though our focus here is also on the case of linear time, we share Russell's and Walker's ultimate interest in a reconstruction of relativistic spacetimes. We the aim at laying down the techniques for extending the theory of event structures to the relativistic case, and, as it were, to pave the way towards it; elucidating the relation of the theory to digital topology and point-free topology is essentially directed to this aim.

6.3 Walker's construction revisited

In section 6.2 we have examined the original constructions of instants by Russell and Walker and noticed that, in the presence of axiom (A3), they both give rise to filters under a defined covering relation $\preceq$. In order to proceed further and identify a common core to the two constructions, however, it is expedient to modify the presentation of an event structure so that the defined relation $\preceq$ becomes easier to handle, and the intrinsic assumption of linearity of Russell-Walker event structures is made explicit. We shall then modify the signature and axioms of event structures so as to take an event structure to be just a model of GT_1, i.e., of the axioms in Definition 5.3.1 excluding those pertaining to the operation $\oplus, \ominus$. The important point here, however, is that event structures as models of GT_1 are inter-interpretable with Russell-Walker event structures. In particular, it is straightforward to see that if we explicitly define

$$aPb \leftrightarrow a\,\mathrm{R}_{-}b \wedge \neg(aOb)$$

as we already did in section 5.3, then both axioms 6.2.1 and (6.2) are provable; hence, whatever follows from Russell-Walker event structures follows from GT_1 using the explicit definition above. On the other hand, for Russell-Walker event structures we can explicitly define:

$$a\,\mathrm{R}_{+}b \leftrightarrow \neg bBa \tag{6.4}$$

$$a\,\mathrm{R}_{-}b \leftrightarrow \neg bEa \tag{6.5}$$

Where E, B are themselves defined in terms of P as in section 6.2, and O is defined as in D. Note that $\neg bEa$ is equivalent to $aEb \vee (\neg(aEb) \wedge \neg(bEa))$, that is, "$a$ ends before b, or neither a ends before b nor b ends before a", and similarly for $\neg bBa$, which shows the linearity assumption implicit in Russell-Walker event structures: if b does not end before a then either a ends before b or they end simultaneously. It is now a tedious but straightforward exercise to check that the axioms of GT_1, with the single exception of axiom (8), follow from axioms 6.2.1 and (6.2) under the above explicit definitions. Hence, by reformulating Russell-Walker event structures as models of GT_1 the only additional strength is given by axiom (8).

Moreover, it is also clear that the boundaries of section 5.5 are essentially Walker instants. Indeed, it is straightforward to show that Walker instants as defined in section 6.2 comply with the constraints of Proposition 5.5.9; in particular, the present component of a Walker instant is now always not empty, thanks to the addition of axiom (8). On the other hand, a boundary such that its past and future components are not empty, that is, a "two-sided boundary", is a Walker instant. In other words, the only way in which the definition of boundaries given in section 5.5 generalizes the construction of Walker instants is by adding what we termed the formal boundaries, i.e., the boundaries $(\emptyset, \emptyset, W)$ and $(W, \emptyset, \emptyset)$, and this turns the complete linear order of Walker instants into a complete lattice. Henceforth, we shall consider the construction of Walker instants that includes the formal boundaries.

Now that we have reformulated the notion of an event structure so as to turn it into that used in chapter 5 we can proceed to show how Russell and Walker constructions can be combined.

6.4 Combining the two constructions

In this section we show how Walker's and Russell's constructions can be combined. In order to achieve this aim we must first uniform the presentation of Russell instants to that of Walker instants. To this effect, let $\mathcal{W}$ be an event structure; henceforth we shall only consider event structures as models of GT_1. We can identify every Russell instant, that is, every maximal overlapping class of events i, with a triple (P_i, C_i, F_i) where

$$C_i = i \tag{6.6}$$

$$P_i = \{a \in W \mid aPb \text{ for some } b \in C\} \tag{6.7}$$

$$F_i = \{a \in W \mid bPa \text{ for some } b \in C\} \tag{6.8}$$

Clearly, two distinct Russell instants give rise to two distinct triples of this kind, and any triple of this kind corresponds to a Russell instant via the forgetful map $u : \mathcal{P}W^3 \to \mathcal{P}W$ defined by $u : (P, C, F) \to C$. One easily verifies that $i \leq j$ if and only if $P_i \subseteq P_j$, for any Russell instants i, j. Hence, we might as well take Russell instants to be such triples, with the order defined by inclusion of pasts as in (6.3); we have then brought Russell instants in the form of Walker instants. Interestingly, note that a Walker instant is completely determined by its P, F components but not by its mere present component, as the event structure in Example 6.2.3 shows; on the other hand, a Russell instant is completely determined by its present component.

The next question is then whether one can define a linear order on Russell and Walker instants altogether. Given a Russell instant i, however, the tuples

$(P_i, (P_i \cup P_i\mathbb{O})^c, P_i\mathbb{O})$, $(\mathbb{O}F_i, (\mathbb{O}F_i \cup F_i)^c, F_i)$ are easily seen to be Walker instants, where $(\cdots)\mathbb{O}, \mathbb{O}(\cdots)$ are defined as in section 5.4; hence, neither inclusion of pasts nor inclusion of futures guarantee that the order is linear, as in general it might turn out to be a total preorder. One might then attempt to define an order on Russell and Walker instants by requiring inclusion of past *and* future as we did in Proposition 5.6.10, but this will also generally give rise to a total preorder and not a linear order; indeed, Example (5.6.5) in section 5.6 shows that the classes of Walker and Russell instants are not disjoint in general, as a Walker instant can be such that its present is a maximal overlapping class, in which case the triple counts both as a Walker and as a Russell instant.

The best way to solve the mismatch between Russell's and Walker's approaches and obtain a unified construction, then, would seem to consist in taking Walker's construction as fundamental, and let Russell instants be all triples whose past and future are fixpoints with respect to the L, R operations respectively (see section 5.5), whose present is a maximal overlapping class, *and* which are not already Walker instants. This is essentially the approach adopted in section 5.6, where the infinitesimal intervals are exactly instants of this sort, which are "inserted" in the jumps between Walker instants.

Consider now event structures satisfying additionally axiom (A3). Now, axiom (A3) is not a geometric implication, but this can be remedied by replacing it with the partial binary operations $\oplus, \ominus$ satisfying the axioms of Definition 5.3.1. In this setting, then, axiom (A3) follows from the existence of the pseudo-meet operation $\sqcap$. Of course, a more conservative solution would have been the addition of a mere pseudo-meet satisfying the property of maximality with respect to $\preceq$, as encoded in axiom (A3), rather than $\oplus, \ominus$. As we have seen in section 5.8, however, the operations are essential to ensure the existence of retractions from an event structure to any of its submodels, so that the inverse system construction relies on them essentially. If one aims at analyzing potential infinite divisibility by means of inverse systems so as to remain as close as possible to experience, then, one must adopt the operations $\oplus, \ominus$, and one is left with system GT.

In the context of GT, then, the present of both Walker and Russell instants is a filter; moreover, if a maximal filter $\mathcal{F} \subseteq \mathcal{P}W$ is principal, i.e. $\mathcal{F} = \uparrow_{\preceq}\mu$ for a $\preceq$-minimal $\mu \in W$, then $\mathcal{F}$ cannot be the present of a Walker instant because of Lemma 5.7.3. This in turn implies that in an event structure in which all maximal filters are principal the class of Walker and Russell instants as originally defined are actually disjoint, so that equipping their union with an order determined by inclusion of past and future as in Proposition 5.6.10 yields a linear order; but this linear order is just $K(\mathcal{W})$ as it was defined in section 5.6, since if all maximal filters are principal then Russell instants are just infinitesimal intervals, so that we have effected a unification of Walker's and Russell's constructions.

One might now wonder whether in an event structure every filter is either a Walker or a Russell instant. This is not the case, but we shall better understand why by considering in more detail the topological interpretation of Walker and

Russell instants, which will also shed further light on their interaction.

6.5 Digital topology

In this section we introduce well-formed spaces, selective spaces and COTS, and state various results that we shall need to provide a topological interpretation of Russell and Walker instants. Some of these results are simply recalled from the relevant literature (Kopperman et al., 1998; Kopperman & Wilson, 1999), while other results are proven here and are important in their philosophical application. In particular, we show that the Alexandroff COTS $\mathcal{A}$ of Definition 5.10.14, obtained as the space of instants on the inverse limit of infinite divisibility inverse systems in section 5.10, is such that every compact separable linear order is a subspace of a quotient of $\mathcal{A}$. Hence, $\mathcal{A}$ generates in this sense any possible linear order of instants of time.

6.5.1 Well-formed spaces, selective spaces and COTS

The most fundamental notion we shall employ is that of a well-formed space as found in Kopperman et al. (1998):

6.5.1. DEFINITION. A *well-formed space* is a convex linearly ordered topological space.

Recall that an ordered topological space is convex if its topology has a subbase of rays, i.e., of upsets and downsets in the order. Of course, a total order equipped with the order topology, that is, a LOTS, is a well-formed space. Well-formed spaces satisfy the following properties:

6.5.2. LEMMA. *Let X be a well-formed space. Then the following are true:*

(1) For any closed (resp. open) subset $K \subseteq X$, $\uparrow K, \downarrow K$ are closed (resp. open)

(2) For any $S \subseteq X$, x is a limit point of $\downarrow S$ if and only if either $x \in \downarrow S$ or x is a limit point of S (similarly for $\uparrow S$)

Proof:
The first claim is proven in Kopperman et al. (1998), Lemma 3.3. For the second claim, let $S \subseteq X$ and let x be a limit point of $\downarrow S$ such that $x \notin \downarrow S$. Then every neighborhood U of x has non empty intersection with $\downarrow S$, and since $x \notin S$ this means that $S \subseteq \downarrow x$ by linearity. It follows that U has non empty intersection with S and hence x is a limit point of S. The direction from right to left is trivial, and the argument for $\uparrow S$ is similar. □

We now consider subspaces and quotients of well-formed spaces. We always restrict our attention to quotients of ordered topological spaces where the equivalence classes are convex; in the totally ordered case, this amounts to the equivalence classes being intervals.

6.5.3. PROPOSITION. *Any subspace of a well-formed space X is well-formed, and any quotient of a well-formed space is well-formed.*

Proof:
The first claim is obvious. The second claim is proven as Theorem 4.1 in Kopperman et al. (1998) □

In what follows we shall also need various properties of *connected* well-formed spaces which are observed in Kopperman et al. (1998).

6.5.4. DEFINITION. Let X be a connected well-formed space. A point $x \in X$ is an ordered cut point of X if $(\leftarrow, x), (x, \rightarrow)$ are the maximal connected components of $X \setminus \{x\}$.

Using the notion of an ordered cut point it is possible to characterize precisely when a totally ordered topological space is well-formed, i.e., convex:

6.5.5. LEMMA. *A connected totally ordered space X which is locally connected and such that every point is either an order cut point or an end point is well-formed.*

Proof:
See Kopperman et al. (1998), Lemma 9.5. □

We then obtain the following properties for connected well-formed spaces, the proof of which can be found in Kopperman et al. (1998), Section 9.

6.5.6. LEMMA. *Let X be a connected well-formed space. The following are satisfied:*

(1) Any convex subspace of X is connected

(2) X is locally connected

(3) Every cut point of X is an ordered cut point and is decided, and every decided point is either an ordered cut point or an end point

It is clear that the notion of a well-formed space is very general, as the interaction axioms between the order and the topology are quite weak, and no separation axioms are required of the underlying topology; indeed, a well formed space need not even be sober. Before we put additional constraints on well-formed spaces we note that they admit of two other characterizations. We could have defined a well-formed space as a totally ordered topological space $(X, \tau, \leq)$ satisfying (i) τ has a basis of convex sets and (ii) X satisfies the condition that the upset and downset of an open set are open; clearly, this definition is equivalent to the one given above. More importantly, we remark that any well-formed space $(X, \leq, \tau)$ is a totally ordered bitopological space $(X, \leq, \mathcal{L}_\tau, \mathcal{U}_\tau)$, where $\mathcal{L}_\tau$, $\mathcal{U}_\tau$ are respectively the lower and upper topologies generated by τ, so that $\tau = \mathcal{L}_\tau \vee \mathcal{U}_\tau$ is the coarsest topology refining both $\mathcal{L}_\tau$ and $\mathcal{U}_\tau$.

Well-formed spaces have been studied from a bitopological perspective in Kopperman and Wilson (1999), a perspective which we shall also adopt in what follows because of its fruitfulness. Indeed, we immediately posit:

6.5.7. Definition. A bitopological space (X, τ, τ') is *weak pairwise* T_0 if for every pair of distinct points there exists a set that is either τ-open or τ'-open containing one point but not the other.

Note that the property of being weak pairwise T_0 is quite a weak separation property, as it does not imply that τ or τ' must be T_0; it does, however, imply that $\tau \vee \tau'$ is T_0. For well-formed spaces, however, we also have:

6.5.8. Proposition. *Let X be a well-formed space. Then X is T_0 if and only if $(X, \mathcal{L}, \mathcal{U})$ is weak pairwise T_0.*

Proof:
The right-to-left direction holds in general as we remark above. For the left-to-right direction, let X be a T_0 well-formed space and let $x, y \in X$ be distinct points. Then there exists a basic open set I such that, without loss of generality, $x \in I, y \notin I$. By linearity then either $x \leq y$ or $y \leq x$; assume without loss of generality $x \leq y$, then $x \in \downarrow i, y \notin \downarrow i$, but $\downarrow i \in \mathcal{L}$, and we are done. □

We can now show that the order $\leq$ of a T_0 well-formed space can be recovered purely topologically:

6.5.9. Proposition. *Let $(X, \leq, \tau)$ be a T_0 well formed space. Then $x \leq y$ iff any of the following two equivalent conditions holds:*

(1) $x \sqsubseteq_{\mathcal{U}\tau} y$ and $y \sqsubseteq_{\mathcal{L}\tau} x$

(2) $x \in \downarrow U, y \in \uparrow V$ for any $U \in \mathcal{N}(y), V \in \mathcal{N}(x)$.

Proof:
Assume X is a T_0 well-formed space and let $x, y \in X$. If $x \leq y$ then clearly whenever $y \in U$, $U \in \mathcal{L}\tau$ then $x \in U$ and whenever $x \in U$, $U \in \mathcal{U}\tau$ then $y \in U$. For the other direction, assume $x \sqsubseteq_{\mathcal{U}\tau} y$ and $y \sqsubseteq_{\mathcal{L}\tau} x$, and suppose towards a contradiction that $x \not\leq y$. Then by the T_0 separation property there exists $U \in \tau$ with either $x \in U, y \notin U$ or $y \in U, x \notin U$. In the former case we have $x \in \uparrow U, y \notin \uparrow U$ which yields a contradiction with $x \sqsubseteq_{\mathcal{U}\tau} y$. In the latter case we have $y \in \downarrow U, x \notin \downarrow U$ which yields a contradiction with $y \sqsubseteq_{\mathcal{L}\tau} x$, and we are done. The equivalence between the two conditions is straightforward. □

We shall now narrow down further the class of well-formed spaces by considering selective spaces and COTS (connected ordered topological spaces); the latter in particular will be crucial for our topological interpretation of Walker and Russell instants.

6.5.10. Definition. Let X be a connected totally ordered space. Then X is a COTS if any point of X is either an ordered cut point, or an end point.

6.5.11. Definition. Let X be a well-formed space. X is a *selective space* if it is T_0 and moreover whenever $x \sqsubseteq y$ then either $x = y$ or x, y are adjacent in the ordering.

Recall that given a subset F of a totally ordered set X, we denote with F_g the topology on X generated by the subbasis of order-open rays $\{(\leftarrow, x) \mid x \in F\} \cup \{(x, \rightarrow) \mid x \in F\}$. We then have the following properties for selective spaces (see Kopperman et al. (1998), Section 7 and theorem 9.13):

6.5.12. Lemma. *Let X be a selective space. Then the following hold:*

(1) X is sober

(2) Every ray of X is either open or closed

(3) if X is connected then its subspace of decided points is a well-formed COTS

(4) If X is connected then $\tau = F_g$, where $F \subseteq X$ is the set of closed points of X.

(5) If X is connected and $F \subseteq X$ is the subspace of closed points of X, then the order topology on F is equal to the subspace topology on F

Proof:
Claims (1) – (4) are proven in Kopperman et al. (1998). Claim (5) follows straightforwardly from (4). □

The notion of a COTS was originally introduced by Khalimsky, Kopperman and others (Khalimsky et al., 1990) in digital topology[4] as a connected topological space such that, of any three points, there is one that if removed leaves the other two points in disjoint connected components of the remainder. It is shown in Khalimsky et al. (1990) that COTS can be equivalently characterized as in Definition 6.5.10. Lemma 6.5.6 (1) along with Lemma 6.5.5 then imply that a COTS is well-formed if and only if it is locally connected. It then follows that if a COTS is locally connected and T_0,[5] then it is selective and Lemma 6.5.12 (2) along with Lemma 6.5.6 (3) imply that every point is decided. Viceversa, using again condition (3) of Lemma 6.5.6 it follows that whenever a connected well-formed space is such that every point is decided, then it is a COTS.

There are a number of other notions in the literature which are equivalent to those presented above; see Smyth and Webster (2002) for a treatment and a comparison of these equivalent notions. There it is also remarked (p. 7) that the fact that COTS were not originally required to be locally connected was perhaps an anomaly. Indeed, the discussion above makes clear that a COTS is well-formed only if it is locally connected, and hence it might be argued that the condition of being locally connected should be added in the definition of a COTS. In order to avoid confusion with the extant literature, however, it is useful to distinguish COTS and locally connected COTS. We shall therefore call the latter the *well-formed COTS*.

Well-formed COTS are of great interest because, as we shall see, they allow us to understand how Russell's and Walker's constructions of (space)-time from events complement each other, in a topological sense, and hence represent the general topological form of the Russell-Walker temporal continuum.

6.5.13. Example. The real line $\mathbb{R}$ and the unit interval $[0, 1]$ with their natural order and order topology are well-formed COTS. Another example of a well-formed COTS is the *Khalimsky line* $\mathbb{K}$ (Khalimsky et al., 1990), the set $\mathbb{Z}$ of integers with their natural order equipped with the topology generated by the following subbase:

$$\{\downarrow 2x \mid x \in \mathbb{Z}\} \cup \{\uparrow 2x \mid x \in \mathbb{Z}\}$$

It is straightforward to check that any subspace of $\mathbb{K}$ whose underlying set is convex is also a well-formed COTS. These spaces are termed Khalimsky spaces in Khalimsky et al. (1990); finite Khalimsky spaces can be obtained as quotients of the unit interval, as explained in Remark 6.5.16.

[4]Digital topology was primarily developed for the analysis of digital images with combinatorial topological methods; see Kong and Rosenfeld (1989) for a review of the field.

[5]The only COTS which is not T_0 is the 2-point set with the indiscrete topology.

6.5.14. Example. A further example of a well-formed COTS is of course the Alexandroff COTS of Definition 5.10.14. In particular, let the *Alexandroff split interval*, or simply *split interval*, be the totally ordered space $[0,1]\times\{0,1\}$ with the lexicographic ordering. This space equipped with the order topology is compact Hausdorff, separable, but not second countable and therefore not metrizable, and it is also not connected. Define then the *Alexandroff COTS*, denoted as $\mathcal{A}$, as follows: the underlying set is $A=[0,1]\times\{0,1/2,1\}$ ordered lexicographically, and the topology is that induced by the subbasis of the rays of the form $\{((x,i),\leftarrow) \mid x\in\mathbb{I}, i\in\{0,1\}\}\cup\{((x,i),\rightarrow) \mid x\in\mathbb{I}, i\in\{0,1\}\}$. It is straightforward to check that the space $\mathcal{A}$ so defined is a well-formed COTS which is not second countable nor separable, and that the projection $\pi:\mathcal{A}\to\mathbb{R}$ is continuous. Moreover, $\mathcal{A}$ is compact as its topology is coarser than the order topology, which is compact since $\mathcal{A}$ is a complete lattice. Furthermore, $\mathcal{A}$ has a basis of compact open sets. Indeed, consider any basic open $I={\uparrow}(x,1/2)\cap{\downarrow}(y,1/2)$ and let C be a cover of I. Then $C'=C\cup\{{\downarrow}(x,1/2),{\uparrow}(y,1/2)\}$ is a cover of $I^c={\uparrow}(x,0)\cap{\downarrow}(y,1)$ and the latter is compact since it is closed. Then there is a finite subcover C'' of C' that covers I^c, and $C''\setminus\{{\downarrow}(x,1/2),{\uparrow}(y,1/2)\}$ is a finite subcover of C covering I.

6.5.15. Example. An important example of a selective space which is not a COTS is the *Smyth line*. Let $\mathbb{D}=\{\frac{m}{2^n}\mid n\in\mathbb{N}\}$ be the set of dyadic rationals and define $S=\mathbb{D}\times -1\cup\mathbb{R}\times\{0\}\cup\mathbb{D}\times 1$. If we endow S with the lexicographic ordering we obtain an ordered extension of $\mathbb{R}$ in which every dyadic rational $(x,0)$ has been endowed with an immediate predecessor $x^-=(x,-1)$ and an immediate successor $x^+=(x,1)$. Enriching the total order with the topology generated by the subbase of rays of the form

$$\{(x,\rightarrow)\mid x\in\mathbb{D}\}\cup\{(\leftarrow,x)\mid x\in\mathbb{D}\}$$

Yields a a connected selective space which is however not a COTS, since $\{x^-,x^+\}$ are not decided and are not cut-points for any $x\in\mathbb{D}$.

As in the case of the Khalimsky line in Example 6.5.13, any subspace of S whose underlying set is convex is a connected selective space - a *Smyth space*. Smyth spaces can be constructed as limits of sequences of Khalimsky spaces, as illustrated in Kopperman et al. (1998); this also shows that the class of well-formed COTS is not closed under limits in the category of ordered topological spaces. Indeed, the notion of a selective space, along with other equivalent notions, was first introduced in order to relax the requirements on the COTS so as to obtain a class of spaces closed under limits in the category of ordered topological spaces and containing the (well-formed) COTS (Smyth & Webster, 2002, p. 207).

One can construct finite Khalimsky spaces as quotients of the unit interval:

6.5.16. REMARK. Finite Khalimsky spaces can be obtained as quotients of the unit interval $[0,1]$ as follows. Let $\mathcal{C}$ be a cover of $[0,1]$ by finitely many subbasic elements, and consider the quotient space $[0,1]/E$ induced by the equivalence relation defined by xEy iff $x \in U \Leftrightarrow y \in U$ for any $U \in \mathcal{C}$. Then the equivalence classes under E are convex, and the quotient space $[0,1]/E$ is a finite connected selective space; hence its subspace of decided points is a finite Khalimsky space (see Lemma 6.5.12 (3)). Alternatively, finite Khalimsky spaces can be generated directly by quotienting the unit interval according to a partition of it into finitely many open and closed intervals.

After having discussed in some detail the selective spaces and the COTS, we turn to a characterization of compact separable LOTS in terms of the Alexandroff COTS $\mathcal{A}$.

6.5.2 Characterization of compact separable LOTS

In Theorem 7.10 of Kopperman et al. (1998) selective spaces are characterized as those well-formed spaces which are quotients of a LOTS and are such that every point adjacent in the ordering to two other points is decided. In Ostaszewski (1974), as we have seen, a characterization of compact separable LOTS is provided in terms of the Alexandroff split interval. It is of interest that compact separable LOTS admit in turn a characterization in terms of the Alexandroff COTS $\mathcal{A}$ that is quite pleasing. To obtain this characterization we recall some notions from Ostaszewski (1974), some of which we have already recalled in chapter 5.

Let then X be a compact separable LOTS and define two equivalence relations $\equiv, \sim \subseteq X \times X$ on X as follows. First, let $x \equiv y$ for $x, y \in X$ if the set of points between x and y is countable. Clearly, this is an equivalence relation, and for simplicity we denote the equivalence class of a point x according to $\equiv$ as $\widehat{x}$. Secondly, we let $x \sim y$ if either $x = y$ or $\widehat{x} = \{x, y\}$. This is also an equivalence relation, and clearly if $x \neq y$ and $x \sim y$ hold then $\{x, y\}$ defines a jump in X and for any $z \in X, z \neq x, z \neq y$ the set of points between z and x or between z and y is uncountable. We denote with $\widetilde{x}$ the equivalence class under $\sim$ of any $x \in X$; clearly it holds that $|\widetilde{x}| \leq 2$. Moreover, the set $\widetilde{X}$ of equivalence classes under $\sim$ can be endowed with the induced total order from X.

We now obtain the following lemma, by means of which the main result in Ostaszewski (1974) is proven:

6.5.17. LEMMA. *Let X be a compact separable LOTS. Then $\widetilde{X}$ is order-isomorphic and homeomorphic to a closed subset of* $[0,1]$.

Let us now consider again the Alexandroff COTS $\mathcal{A}$. Of course, there is a continuous projection map $\pi : \mathcal{A} \to [0,1]$. We can combine this projection map and the existence of the homeomorphism of Lemma 6.5.17 to obtain the following:

6.5.18. THEOREM. *A LOTS is compact separable iff it is order-isomorphic and homeomorphic to the subspace of closed points of a compact COTS obtained as a quotient of the Alexandroff COTS $\mathcal{A}$.*

Proof:
Let $X = (X, \leq)$ be a compact separable LOTS, and denote with $\phi : \widetilde{X} \to Y$ the isomorphism of Lemma 6.5.17, where $Y \subseteq [0,1]$ is closed. Let moreover $\pi : \mathcal{A} \to [0,1]$ be the projection map from the Alexandroff COTS to $[0,1]$. We first define a map $\Phi : X \to \mathcal{A}$ by letting:

$$\Phi(x) = \begin{cases} inf(\pi^{-1}(\phi(\widetilde{x}))) & \text{if either } \widetilde{x} = \{x\} \text{ or } \widetilde{x} = \{x, y\} \text{ and } x \leq y, x \neq y \\ sup(\pi^{-1}(\phi(\widetilde{x}))) & \text{otherwise.} \end{cases}$$

It is straightforward to verify that Φ is an order-embedding of X into $\mathcal{A}$, i.e., Φ is both order-preserving and order-reflecting, and that the range $Y = \Phi[X]$ of Φ consists only of points closed in $\mathcal{A}$. Note moreover that in general, the order topology induced by the restriction of the ordering on Y is coarser than the subspace topology on Y, and so Φ need not be a topological embedding of X into $\mathcal{A}$.

We can, however, construct a COTS such that X with its order topology is order-isomorphic and homeomorphic to the subspace of the closed points of the COTS equipped with the induced ordering and the subspace topology. We shall construct this COTS as follows.

Consider first the image $\phi[\widetilde{X}]$ of $\widetilde{X}$; since it is a closed subset of $[0,1]$ then $\phi[\widetilde{X}]^c = [0,1] \setminus \phi[\widetilde{X}]$ is an open set. Thus, $\phi[\widetilde{X}]^c$ can be partitioned into a (countable) family of maximal connected components that are open intervals of $[0,1]$, and hence, since the preimage under π of an open convex set is open convex, $K = \pi^{-1}[\phi[\widetilde{X}]^c]$ can be partitioned into a (countable) family $\{O_i\}_{i \leq j}$ of maximal connected components that are open intervals of $\mathcal{A}$. Define then a relation $E \subseteq \mathcal{A} \times \mathcal{A}$ by letting, for any $x, y \in \mathcal{A}$, xEy if one of the following conditions holds:

- $x = y$
- $\pi(x) = \pi(y) = r$ for some $r \in [0,1]$, $\phi^{-1}(r)$ is a singleton set
- x, y belong to the same connected component O_i for some $i \leq j$

Clearly, E is an equivalence relation on $\mathcal{A}$, and it is such that the equivalence classes under E are intervals of $\mathcal{A}$; we can consider then the quotient space $[\mathcal{A}]^E$, and let $q : \mathcal{A} \to [\mathcal{A}]^E$ be the quotient map. This space is compact connected since $\mathcal{A}$ is compact connected, it is well-formed, and it is straightforward to check that every point is decided; hence it is a compact COTS. Moreover, one can verify that (i) the map $q \circ \Phi$ is an order-isomorphism of X into the set of closed points of $[\mathcal{A}]^E$ with the induced ordering, and (ii) using Lemma 6.5.12 (iv), that $q \circ \Phi$ is a homeomorphism of X into the subspace of closed points of $[\mathcal{A}]^E$.

For the other direction, let X be a LOTS and let $[\mathcal{A}]^E$ be a compact COTS obtained as a quotient of $\mathcal{A}$ such that X is order-isomorphic and homeomorphic to the subspace F of closed points of $[\mathcal{A}]^E$. Let $q : \mathcal{A} \to [\mathcal{A}]^E$ be the quotient map. It suffices to show that F is compact separable. Since the set of closed points of $[\mathcal{A}]^E$ is closed and compactness is weakly hereditary, then F is compact.

To see separability we proceed as follows. Let $A \subseteq \mathcal{A}$ be the subspace of closed points of $\mathcal{A}$; this is isomorphic to the Alexandroff split interval. Let $D = \{x \in A \mid \pi(x) \in \mathbb{Q}\}$; it is easy to see that D is a countable dense subset of A. Let moreover $\widehat{q} : A \to [\mathcal{A}]^E$ be the continuous order-preserving restriction of q to A. Note that $F \subseteq \widehat{q}[A]$. Define a set $D' \subseteq F$ by letting:

$$D' = \{sup(\downarrow\widehat{q}(x) \cap F) \mid x \in D\} \cup \{inf(\uparrow\widehat{q}(x) \cap F) \mid x \in D\}$$

Note that infima and suprema of any subset of F must exist because F is compact. It is straightforward to show that the set D' so defined is countable and dense in F. □

The reader should note at this point that the proof of Theorem 6.5.18 also shows that any compact separable linear order embeds into the Alexandroff COTS $\mathcal{A}$. Hence, Theorem 6.5.18 provides a correlate to Kant's unity of the formal intuition of time, just on the side of the instants of time - as any compact separable linear order of instants is a "part" of the order of instants of the formal intuition of time.

6.6 Kant's continuum and constructive topology

In this section we show how the framework of event structures discussed in chapter 5 is related to formal topology (Sambin, 2003), which is a predicative and constructive approach to topology in the tradition of frames and locales. In particular, we show that assuming classical logic event structures are a special subclass of a class of structures that we term "formal bitopologies", since they are essentially the predicative[6] version of the biframes of (Banaschewski, Brümmer, & Hardie, 1983). This shows that event structures, as studied by Walker, Russell and Thomason (Walker, 1947; Russell, 1936; Thomason, 1989), can be (classically) seen as formal topologies [7]. If the background logic is intuitionistic,

[6]We note, at this point, that predicativity is an important issue for a mathematical theory of the phenomenological continuum, one that occupied Weyl's thoughts in Weyl (1994).

[7]While the relation between the Russell-Whitehead approach to the construction of (space)-time and point-free topology has been studied by various authors (Mormann, 2006, 2009; Vakarelov, 2012, 2014), we shall here specifically point out the relation that the notion of

however, event structures as defined in chapter 5 correspond to a more restricted class of formal bitopologies with very strong decidability assumptions. These facts raise the question of whether classical logic is suitable as the background logic for the construction of the Kantian continuum, since Kant does seem to emphasize constructive reasoning at certain places in the CPR, and of how a fully constructive development of the Kantian continuum would proceed.[8] While we can only touch upon such issues here, we can already point out that the move to a constructive logic will radically change the properties of the Kantian continuum.

We also discuss here the construction of the *spectrum* of points of a formal bitopology, which, in general, is a bitopological space X whose join topology is sober, and which can be equipped with an induced partial order that makes it into a convex bitopological space by simply taking $op(\sqsubseteq_\tau) \cap \sqsubseteq_{\tau'}$. Furthermore, the spectrum of points on an event structure as a formal bitopology is seen classically to be a connected well-formed space, and the Walker and Russell points can be put in correspondence to the closed and open points of the space, respectively.

Finally, the notion of a formal bitopology, as a predicative and constructive version of biframes, will be essential for developing a constructive approach to the approximation of the order and topology of relativistic spacetimes, which we aim to pursue in further work. The understanding of the next sections will be easier if the reader has a passing acquaintance with the theory of frames and locales. In particular, recall that given a topological space X its set of opens can be given the structure of a (complete) lattice, where the lattice operations $\bigvee, \wedge$ are defined to be arbitrary union and finite intersection respectively. This complete lattice satisfies the infinite distributivity law

$$(\bigvee U) \wedge a = \bigvee \{u \wedge a \mid u \in U\} \tag{6.9}$$

A complete lattice that satisfies (6.9) is a *frame*. Frames form a category under *frame homomorphism*, which are maps that preserve all joins, including the bottom 0, and all finite meets, including the top 1. It is straightforward to check that there exists a contravariant functor $\Omega : \mathbf{Top} \to \mathbf{Frm}$ from the category of topological spaces and continuous maps to the category of frames and frame homomorphisms; on the objects the functor just yields the frame of opens of the topological space, while given a continuous maps $f : X \to X'$ between two topological spaces X and X' a frame homomorphism $\Omega f : \Omega X' \to \Omega X$ is obtained by letting

$$\Omega f(U) = f^{-}(U) \text{ for any open } U \subseteq X'$$

event structure bears to that of a formal topology in the context of ordered topological spaces.

[8] From a neo-Kantian perspective formal topology has already been employed by Boniolo and Valentini in Boniolo and Valentini (2008, 2012), but we focus here on exploring a possible relation between the Kantian continuum and the constructive continuum of formal topology, such that presented in Negri and Soravia (1999).

The category of frames, and its dual, the category of locales, are the main object of study of point-free topology; a good textbook on frames and locales is Picado and Pultr (2011).

6.6.1 The constructive meaning of overlap

In order to see the relation between event structures as presented in chapter 5 and point-free topology we begin by strengthen our set of axioms somewhat. Recall that the extensionality axiom (5.1)

$$a \equiv b \to a = b$$

means that we can do without axioms (9)i-(9)l for $\oplus, \ominus$. Moreover, in its presence the $\sqcap$ operation in GT becomes a partial semilattice.

Next, we strengthen axiom (8), the covering axiom, by demanding that there always exists a universal cover 1, i.e., an event such that $a \preceq 1$ for all events a. From a Kantian standpoint this move is justifiable, since it is closely related to Kant's claims that time and space are given as "wholes" of which all particular times are parts; indeed, as we have seen in section 5.9, inverse limits of finite event structures always have a top element, a universal cover. In the presence of the extensionality axiom and a universal cover, any event a can then be written as $a = 1 \oplus a \sqcap 1 \ominus a$. Thus, if we add the extensionality axiom and the existence of the universal cover to the axioms of an event structure then any event structure $\mathcal{W}$ is a partial meet-semilattice under $\sqcap$, such that the set

$$G = \{1 \oplus a \mid a \in W\} \cup \{1 \ominus a \mid a \in W\}$$

generates $\mathcal{W}$: every event $a \in W$ can be written as a meet of finitely many events from G. Next, we should like to make the partial meet operation total, and to achieve this we should like to make $\oplus, \ominus$ total operations. One possibility to do so is to adjoin an "empty event" as described in section 5.3.4, but while the addition of the empty event in that fashion is mathematically unproblematic, is it acceptable from a philosophical, and in particular Kantian, standpoint? This is a thorny issue, since it is unclear what the transcendental meaning of the empty event is. To be sure, Kant does seem to imply the possibility of a transcendental role for the empty event in the anticipations of perception at B208, where he says

> [...] thus there is also possible a synthesis of the generation of the magnitude of a sensation from its beginning, the pure intuition = 0, to any arbitrary magnitude.

Kant is talking here in particular about intensive magnitudes, that is, magnitudes whose intensity is independent of the size of the spatial region they occupy and which have a "degree of influence on sense". However later, while discussing magnitudes at A168/B210 - A170/B212, he says that every appearance is continuous as either an extensive or intensive magnitude, and that "multiplicity can only be represented as approximation to negation = 0"; moreover, he claims (A172/B214) that

> [...] No perception [...] is possible that [...] would prove an entire absence of everything real in appearance, i.e., a proof of empty space or empty time can never be drawn from experience [...]

Thus, the evidence regarding whether or not an empty event is acceptable from a Kantian standpoint is contradictory.

Still, relating the issue of the background logic of the Kantian continuum to the discussion, at section 3.8.5, of the transcendental ideal might allow us to make a decision. Indeed, expressing that an event a is not empty by stating that $a \neq 0$ for 0 the empty event is unacceptable constructively; rather, one usually expresses this without negation by means of a positive predicate denoting that a is "inhabited". Now, we have seen in section 3.8.5 that Kant distinguishes properties according to whether they are "transcendentally primitive", i.e. they express a reality, or whether they are defined as the negation or limitation of a transcendentally primitive property. For instance, the transcendental ideal contains the property "brightness" but not the property "darkness", since the latter is defined as a limitation or negation of the former, and not the other way around: "brightness" cannot be defined as "not darkness", or at least it cannot be so defined by someone who has never experience "brightness" in the first place. Thus, it would seem that defining "extended" or "inhabited" as "not empty" runs contrary to Kant's discussion of the transcendental ideal, as the transcendentally fundamental property is that of being "inhabited". We cannot then include the empty event 0 as outline in section 5.3.4. Since we cannot recur to the empty event, then, how can we turn $\oplus, \ominus$ into total operations? The best way to do so, suggested by the discussion in section 5.3.4, is simply to strengthen the transcendental interpretation of the overlap relation O. In particular, in the axiomatization of Definition 5.3.1 we assumed that the events in W are always "inhabited" or "real"; that is why we imposed, for instance, the reflexivity axiom for O. However, we might take O to have a transcendentally stronger meaning, that is, aOb if a, b are "real" or of "non-void" extent and they overlap, thus allowing for events that satisfy $\neg(aOa)$, namely, events whose transcendental significance is that of representing the "void" or "unreal" as the negation of the "real". In this way, then, we also obtain a clear Kantian interpretation of the overlap relation in terms of the transcendental ideal. Mathematically, aOb, for a, b intervals as in section 5.3.3, means that $a \cap b$ is inhabited, where now however we also want to allow the possibility that a, b are empty.

Taking into account these considerations, then, we reformulate the definition of an event structure as follows:

6.6.1. DEFINITION. An *event structure* is a tuple $(W, \mathrm{R}_+, \mathrm{R}_-, O; \preceq; Pos; \oplus, \ominus, 1)$ where $\oplus, \ominus$ are binary operations, O, R_+, $\mathrm{R}_-, \preceq$ are binary relations, Pos is a unary predicate, and $1 \in W$, that is a model of the following axioms:

(1) $a \preceq b \leftrightarrow a\,\mathrm{R}_+ b \wedge\ \mathrm{R}_- b$ (explicit definition of $\preceq$)

(2) $Pos(a) \leftrightarrow aOa$ (explicit definition of positivity)

(3) $a \preceq b \wedge b \preceq a \rightarrow a = b$ (extensionality)

(4) $a \preceq 1$ (universal cover)

(5) $aOb \rightarrow bOa$ (symmetry of O)

(6) Positivity conditions for O:

- (a) $aOb \rightarrow Pos(a)$
- (b) $Pos(a) \vee a\,\mathrm{R}_+ c$

(7) $cOb \wedge c\,\mathrm{R}_+ a \wedge b\,\mathrm{R}_- a \rightarrow aOb$ (condition for overlap)

(8) $a\,\mathrm{R}_+ b \wedge b\,\mathrm{R}_+ c \rightarrow a\,\mathrm{R}_+ c$ (transitivity of order)

(9) $aOc \wedge cOb \wedge c\,\mathrm{R}_+ b \wedge c\,\mathrm{R}_+ a \rightarrow aOb$ (conditional transitivity of O)

(10) $b\,\mathrm{R}_+ a \vee a\,\mathrm{R}_+ b$ (linearity)

(11) binary operations $(\oplus, \ominus)$ on events

- (a) $Pos(a \oplus b) \rightarrow Pos(b) \wedge Pos(a)$ (positivity for $\oplus, \ominus$)
- (b) $aOb \rightarrow Pos(a \oplus b)$
- (c) $a \oplus b\,\mathrm{R}_+ b$
- (d) $a \oplus b\,\mathrm{R}_+ a$
- (e) $Pos(a \oplus b) \rightarrow a\,\mathrm{R}_- a \oplus b$
- (f) $a \oplus b\,\mathrm{R}_- a$
- (g) $a\,\mathrm{R}_+ b \rightarrow a\,\mathrm{R}_+ a \oplus b$
- (h) $Pos(a \oplus b) \wedge b\,\mathrm{R}_+ a \rightarrow b\,\mathrm{R}_+ a \oplus b$

(12) Any sentence ϕ obtained from the above axioms by replacing R_- for R_+, R_+ for R_-, $\ominus$ for $\oplus$ and $\oplus$ for $\ominus$ (substitution principle)

Where free variables are understood as universally quantified.

The only changes of the above Definition with respect to Definition 5.3.1 are that the covering axiom has been replaced by the top element 1 and that the interpretation of the overlap relation has changed so that now aOb if $a \cap b \neq$, where both a, b are allowed to be empty, so that for instance the operation $\oplus$, interpreted as $a \oplus b = a \cap \uparrow b$ for a, b open intervals of a well-formed space, can now be treated as a total operation. We remark that the modified axioms above can be obtained from the discussion in section 5.3.4 by expressing $a \neq 0$ positively as aOa. In particular, an event structure as in theorem 5.3.1 is clearly a model of the above axioms; viceversa, given a model of the above axioms we can obtain an event structure as in Definition 5.3.1 by defining:

$$a\boldsymbol{O}b \text{ iff } Pos(a) \wedge Pos(b) \rightarrow aOb$$

Finally, we note that the construction of boundaries from a model of the above axioms remains that examined in chapter 5, with the only additional requirement that any event a in the present of a boundary or infinitesimal interval is required to be positive, and that event maps remain those that we considered in chapter 5.

6.6.2 Event structures and formal bitopologies

The discussion in the previous section suggested that we can understand event structures as a semilattice generated by the sets of events $\{1 \oplus a \mid a \in W\}, \{1 \ominus a \mid a \in W\}$. This in turn suggests a correspondence with formal topologies which we now illustrate. In particular, we shall need the following definitions:

6.6.2. DEFINITION. A *bisemilattice* is a tuple (W, L, U) where W is a semilattice with top element and L, U are sublattices that generate W, that is for any $w \in W$:

$$w = l \cdot u \text{ for } l \in L, u \in U$$

We remark that the above definition of a bisemilattice is different from that in, e.g., Romanowska and Smith (1981), where a bisemilattice is defined as a set equipped with two semilattice operations. Furthermore, it is easy to see that $1 \in L, 1 \in U$, since $1 = l \cdot u$ for $l \in L, u \in U$ but then $1 \lhd l, 1 \lhd u$ and so $1 = l, 1 = u$ as 1 is the top element of the semilattice. We now posit:

6.6.3. DEFINITION. A *formal bitopology* is a tuple $(W, L, U, \lhd, Pos)$ where (W, L, U) is a bisemilattice, Pos is a unary predicate on W, and $\lhd \subseteq W \times \mathcal{P}W$ is a relation between elements and subsets of W satisfying the following conditions:

(1) $a \in U$ implies $a \lhd U$ (reflexivity)

(2) $a \lhd U$ and $u \lhd V$ for any $u \in U$ then $a \lhd V$ (transitivity)

(3) $a \lhd U$ implies $a \cdot b \lhd U$ ($\cdot$-left)

(4) $a \lhd U$ implies $a \cdot b \lhd U \cdot b$ ($\cdot$-right)

Where $U \cdot b = \{a \cdot b \mid a \in U\}$; and for Pos we have:

(1) $Pos(a)$ and $a \lhd U$ implies $Pos(b)$ for some $b \in U$ (monotonicity)

(2) $Pos(a) \rightarrow a \lhd U$ implies $a \lhd U$ (positivity)

We sometimes write $U \lhd U'$ for $a \lhd U'$ for all $a \in U$. The above definition is, essentially, a variation on the definition of a formal topology, developed by Sambin and his group (Sambin, 1987, 2003) to provide constructive and predicative foundations for topology; indeed, we termed it a "formal bitopology" because it is essentially a predicative version of the biframes studied in Banaschewski et al. (1983). Note in particular that the relation $\lhd$ is supposed to formalize the relation of covering between a basic element - in our specific case, an open interval of a well-formed space - and a (possibly infinite) set of basic elements, and that the unary predicate Pos denotes that an open set is "inhabited".

If the predicate Pos is decidable, moreover, positivity is intuitionistically equivalent to the following (Negri & Soravia, 1999):

$$\neg Pos(a) \text{ implies } a \lhd \emptyset$$

Finally, note that the binary trace of the relation $\lhd$ is just the ordering $\leq$ of the semilattice. The last notion from formal topology we shall need is that of a *Stone*, or *finitary*, formal topology (Negri, 1996). This is merely a formal topology such that $\lhd$ is a Stone cover, meaning that whenever $a \lhd U$ then there exists a finite subset K of U such that $a \lhd K$. Clearly, the same notion can be employed for a formal bitopology, so we shall speak of Stone formal bitopologies.

Recall, moreover, that in the theory of event structures as presented in chapter 5 we defined a finitary covering relation $\preceq\, \subseteq W \times \mathcal{P}W$ between events and subsets of events in terms of the binary covering relation $\preceq$, so that an interval a is "finitely covered" by a set of events U if there exists a finite sequence $a_1, \cdots, a_n$ of events in U such that $a\,\mathrm{R}_{+}a_1$, $a\,\mathrm{R}_{-}a_n$, and $a_1, \cdots, a_n$ are such that $a_i O a_{i+1}$ for all $0 \leq i < n$ (see Definition 5.3.8). Let us call such a set of events an *covering chain* for a. Note that in a second-order language that quantifies over finite subsets, whether a subset finitely covers an event a can be expressed by a second-order geometric formula. We then have:

6.6.4. Proposition. *Let $\mathcal{W}$ be an event structure. Then $\mathcal{W}$ is a Stone formal bitopology.*

Proof:
Let $\mathcal{W}$ be an event structure. We let $L = \{1 \oplus a \mid a \in W\}$, $U = \{1 \ominus a \mid a \in W\}$, and $a \lhd U$ if

$$aOa \Rightarrow \exists C \subseteq U, C \text{ a covering chain for } a$$

and, moreover, we let $Pos(a)$ if aOa and let $a \cdot b$ be $a \sqcap b$. The discussion in the previous section shows that (W, L, U) is a bisemilattice, and it is straightforward to verify that all the axioms of Definition 6.6.3 are satisfied. In particular, the implicational definition of covering above ensures that positivity is satisfied. To see monotonicity, if aOa and U finitely covers a then there is an overlapping covering chain $a_0, \cdot, a_n$ for a. Assume $a\,\mathrm{R}_+ a_i, a\,\mathrm{R}_- a_i$ for some i. Then since $a\,\mathrm{R}_+ b \wedge aOa \to bOb$ is a consequence of the axioms, and aOa, then a_iOa_i and we are done since then $a \lhd a_i$ and $Pos(a_i)$. Otherwise there must be a_i, a_{i+1} with a_iOa_{i+1} but then, for instance, $Pos(a_i)$ follows from the axioms. □

Hence, an event structure is a Stone formal bitopology. The converse does not of course hold unless we impose some further constraints on formal bitopologies. In particular:

6.6.5. DEFINITION. A *linear bitopology* is a formal bitopology $(W, L, U, \lhd, Pos)$ that satisfies the following additional conditions:

(1) $\neg(l \lhd l')$ implies $l' \lhd l$ and similarly for U (linearity)

(2) $Pos(l \cdot u) \wedge l \cdot u \lhd l' \cdot u'$ implies $l \lhd l', u \lhd u'$ (directedness)

(3) For any $l \in L, u \in U$, $Pos(l \sqcap u)$ implies $1 \lhd \{l, u\}$ (totality)

The fundamental intuition of the above definition is that of encoding, using the tools of formal topology, the relations between rays of a well-formed space. In particular, linearity encodes that given one of the two topologies of the bitopological space its open sets are linearly ordered under inclusion, although it does so without building in decidability of the covering relation $\lhd$, as we shall soon see. Directedness, on the other hand, encodes that the open sets of the two topologies are convex sets in a linear order. Finally, totality encodes that any two upset and downset whose meet is not empty must cover the whole space. Note also that from the above definition it follows that $l \cdot u \lhd u'$ implies $u \lhd u'$, and similarly for L. Indeed, if $l \cdot u \lhd u'$ then $l \cdot u \lhd 1 \cdot u'$, and since $1 \in L$ then $u \lhd u'$ by convexity. Moreover, if $a \in W$ and $a = l \cdot u = l' \cdot u'$ for $l, l' \in L, u, u' \in U$ then $u \lhd u', u' \lhd u$ and hence $u' = u \cdot u' = u$ so $u = u'$. Thus, we can write $a \in W$ as a meet of unique $l \in L$ and $u \in U$, which we shall denote as l_a and u_a respectively. Finally, we remark that:

$$Pos(l \cdot u) \wedge l \cdot u \lhd l' \text{ implies } l \lhd l' \text{ and similarly for U} \tag{6.10}$$

follows from the definition of a linear bitopology. Indeed, if $Pos(l \cdot u)$ and $l \cdot u \lhd l'$ then clearly $l \cdot u \lhd l' \cdot 1$ and hence by convexity it follows that $l \lhd l'$. We then obtain the following result:

6.6.6. PROPOSITION. *Assume classical logic. Then a Stone linear bitopology is an event structure.*

Proof:
Let $(W, L, U, \lhd, Pos)$ be a Stone linear bitopology, and define R_{+}, $\mathrm{R}_{-}, O, \oplus, \ominus$ on W by letting:

(1) $a \oplus b = a \cdot u_b$

(2) $a \ominus b = a \cdot l_b$

(3) aOb if $Pos(a \cdot b)$

(4) $a\,\mathrm{R}_{+}b$ if $a \lhd u_b$

(5) $a\,\mathrm{R}_{-}b$ if $a \lhd l_b$

Then one checks that $(W, \mathrm{R}_{+}, \mathrm{R}_{-}, O, \oplus, \ominus)$ is an event structure, i.e., all the axioms of Definition 6.6.1 hold. We only check some of the interesting axioms.

$aOb \to aOa$ holds because if $Pos(a \cdot b)$ then since $a \cdot b \lhd a$ by monotonicity then $Pos(a)$, hence aOa.

We now show that $aOa \vee a\,\mathrm{R}_{+}c$, that is, $Pos(a \cdot a) \vee a \lhd u_c$. We show in particular that $\neg Pos(a) \to a \lhd u_c$, from which using classical logic the claim follows. Indeed, if $\neg Pos(a)$ then $\neg Pos(a) \to a \lhd u_c$ follows intuitionistically, from which by positivity it follows that $a \lhd u_c$.

We now show that $cOb \wedge c\,\mathrm{R}_{+}a \wedge b\,\mathrm{R}_{-}a \to aOb$, that is, we assume that $Pos(c \cdot b), c \lhd u_a, b \lhd l_a$ and show $Pos(a \cdot b)$. Now since $b \lhd l_a$ then $l_b \lhd l_a$ and similarly $u_c \lhd u_a$ hence $l_b \cdot u_c \cdot u_b \lhd u_a \cdot l_a = a$. Since $Pos(c \cdot b)$ then $Pos(prayb \cdot u_c \cdot u_b)$ follows, and since $l_b \cdot u_c \cdot u_b \lhd b$ then $l_b \cdot u_c \cdot u_b \lhd a \cdot b$ and hence $Pos(a \cdot b)$ by monotonicity.

We now show linearity, that is, $a\,\mathrm{R}_{+}b \vee b\,\mathrm{R}_{+}a$, which is expanded as $a \lhd u_b \vee b \lhd u_a$. We show that $\neg(a \lhd u_b) \to b \lhd u_a$ follows intuitionistically, which classically implies the linearity axiom. Assume $\neg(a \lhd u_b)$, that is, $\neg(l_a \cdot u_a \lhd u_b)$. Then if $u_a \lhd u_b$ then $u_a \cdot l_a \lhd u_b$ which yields a contradiction, hence $\neg(u_a \lhd u_b)$. However this implies by the definition of a linear bitopology that $u_b \lhd u_a$ and hence $l_b \cdot u_b \lhd u_a$ and we are done.

Finally, we show that $(a \oplus b)O(a \oplus b) \to a\,\mathrm{R}_{-}a \oplus b$, that is, that $Pos(a \cdot u_b) \to a \lhd l_{a \cdot u_b}$, but to see this it suffices to consider that:

$$Pos(l_a \cdot (u_a \cdot u_b)) \wedge l_a \cdot (u_a \cdot u_b) \lhd l_{l_a \cdot (u_a \cdot u_b)}$$

follows, and by means of (6.10) above we obtain $l_a \lhd l_{a \cdot u_b}$, which immediately yields $a \lhd l_{a \cdot u_b}$. □

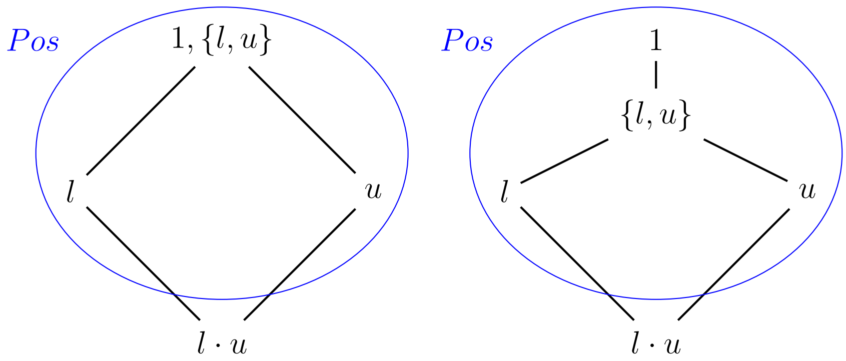
Now, there are various important observations to be made at this point. The first observation is that while the above proof showing that a Stone linear bitopology is an event structure assumes classical logic, this is only needed to be able to show that the disjunctive axioms in Definition 6.6.1 hold. If, however, we reformulated those axioms in Definition 6.6.1 that are of the form $\phi \vee \psi$ as $\neg\phi \rightarrow \psi$ then the above proof would be fully constructive. This just highlights that in a move from a classical to a constructive setting we are not warranted in general to assume decidability of positivity or, for that matter, of the predicates R_+, R_-, as done in Definition 6.6.1. This is also why we formulated the linearity constraint in the definition of a linear formal bitopology as $\neg(l \lhd l') \Rightarrow l' \lhd l$ rather than $l \lhd l' \vee l' \lhd l$, since in general we are not warranted in assuming that $\lhd$ is decidable.

A second point of reflection is the following: the above results show that event structures are (assuming classical logic) just a special class of formal bitopologies satisfying additional constraints of linearity, directedness and totality. Thus, one might ask, why bother with event structures at all, and not introduce formal topologies from the beginning? The answer to this question is important, and relates back to the discussion in chapter 2: in this thesis the formalism is at the service of the philosophy, and not the other way around. Had we started with a definition like Definition 6.6.5, it would have been impossible to provide a convincing transcendental justification for our axioms like that provided in chapter 4. In particular, we have seen in chapter 5 and at the beginning of this chapter that the weaker axiom systems GT_0 and GT_1 are interesting in their own right, and that many constructions - for instance the construction of instants - can be carried out in these weaker systems; in this sense, then, their strength is exactly their weakness, since it brings to the fore the transcendental presuppositions of Kantian constructions that would be thoroughly hidden in a definition like Definition 6.6.5.

We also remark that the formal topology of the reals considered in Negri and Soravia (1999) is a special case of a linear formal bitopology; indeed, the considerations above show that the covering relation of the Stone formal topology considered there in order to inductively define the formal topology of the reals (Negri & Soravia, 1999, p. 5) can be reconstructed wholly on the basis of its binary trace, i.e., the meet operation of the semilattice, as the discussion of the covering chains in Proposition 6.6.4 shows.

6.6.3 Connected formal bitopologies

A further point to be discussed is whether Propositions 6.6.4 and 6.6.6 provide us with as tight a correspondence as possible between event structures and linear formal bitopologies. In this respect, consider the linear formal bitopology $\mathcal{W}$ pictured in Figure 6.1 (a). This formal bitopology consists of four elements $W = \{1, l, u, l \cdot u\}$; we let $Pos^{\mathcal{W}} = \{1, l, u\}$, and moreover we let $\lhd$ be defined according to the Hasse diagram[9], so that the formal bitopology is essentially disconnected as $1 \lhd \{l, u\}$ and $l \cdot u$ is not positive.



(a) disconnected bitopology (b) connected bitopology

Figure 6.1: From a disconnected to a connected formal bitopology

Now, applying the construction of Proposition 6.6.6 to the formal topology of Figure 6.1 (a) yields an event structure such that $1\,\mathrm{R}_{+}l, 1\,\mathrm{R}_{-}u$ but $\neg(lOu)$, that is, $\neg(1 \preceq \{l, u\})$, and applying the construction of Proposition 6.6.4 we obtain a linear formal bitopology such that $\neg(1 \lhd \{l, u\})$, pictured in Figure 6.1 (b); and this latter formal bitopology is connected in the sense that $\neg(1 \lhd \{l, u\})$. In other words, we have effected a *connectification* of the linear formal bitopology. We can then present the situation as follows. First, we need (a variation of) the definition of a connected formal topology (see Vickers (2012), Proposition 14, and Negri and Valentini (1997)):

6.6.7. DEFINITION. A formal bitopology is *connected* iff $Pos(1)$ and whenever $1 \lhd U$ and $a, b \in U$ are both positive then there is a "connecting sequence" $c_k \in U$ $(0 \leq k \leq n)$ such that $c_0 = a, c_n = b$ and if $0 \leq k < n$ then $c_k \cdot c_{k+1}$ is positive.

It is a straightforward matter to check that the formal bitopology constructed from an event structure according to Proposition 6.6.4 is connected in the above sense, and that this yields a one-to-one correspondence between connected linear Stone formal bitopologies and event structures.

One might then wish to extend this correspondence, making a category out of formal bitopologies by generalizing maps between formal topologies, that is, the continuous or "approximable" relations of (Maietti & Valentini, 2004; Sambin, Valentini, & Virgili, 1996), to the setting of formal bitopologies. One should then be able formulate a pair of functors F, G between the category of event structures and the category of connected Stone linear formal bitopologies (as a subcategory

[9]This means: if a, b are nodes in the Hasse diagram and $a < b$ or a, b belong to the same node then $a \lhd b$.

of the category of formal bitopologies) that would yield a categorical equivalence. This should prove quite straightforward; indeed, the reader can check that given event structures $\mathcal{W}, \mathcal{W}'$, connected Stone linear formal bitopologies $F(\mathcal{W}), F(\mathcal{W}')$ obtained according to Proposition 6.6.4, and an event map $f : \mathcal{W} \to \mathcal{W}'$, if we let $F(f) \subseteq F(\mathcal{W}) \times F(\mathcal{W}')$ be defined by letting $aF(f)b$ if $b \in \uparrow_{\preceq} f(a)$ for all $a \in F(W)$ we obtain a relation satisfying all the axioms of continuous relations in the sense of (Sambin et al., 1996, p. 22). We shall not spell out the details here, however, since morphisms between formal bitopologies are mainly interesting for our present purposes in relation to obtaining a constructive rendition of the inverse limit construction in chapter 5, which we have to demand to future work (see chapter 7), so that these matters will also be best discussed then.

6.6.4 Points of a formal bitopology

We shall now discuss how to construct the points of a formal (linear) bitopology. The following definition is just the usual definition of the points of a formal topology (Sambin, 2003):

6.6.8. DEFINITION. Let $\mathcal{W} = (W, L, U, \lhd, Pos)$ be a formal bitopology. A *point* of $\mathcal{W}$ is a subset $\alpha \subseteq W$ such that:

(1) $1 \in \alpha$

(2) $a \in \alpha, b \in \alpha$ implies $a \cdot b \in \alpha$

(3) if $a \in \alpha$ and $\alpha \lhd U$ then $b \in \alpha$ for some $b \in U$

Of course, the above definition is the formal topological equivalent of the construction of points as completely prime filters from frames or locales; note that from condition (3) and positivity it follows that $Pos(a)$ for any $a \in \alpha$. We now define the *spectrum* of a formal bitopology $\mathcal{W}$, generalizing the usual notion of the spectrum of a formal topology, as follows. Define a topology $\mathcal{L}$ on $Pt(\mathcal{W})$, the set of formal points of $\mathcal{W}$, from the basis

$$\{\varphi(l) \mid l \in L\}$$

where

$$\varphi(a) = \{\alpha \in Pt(\mathcal{W}) \mid a \in \alpha\} \text{ for any } a \in W$$

note that the above set is clearly closed under intersection and so is a basis for a topology on $Pt(\mathcal{W})$. Define similarly a topology $\mathcal{U}$ on $Pt(\mathcal{W})$ by considering $\{\varphi(u) \mid u \in U\}$ as a basis. Then $Pt(\mathcal{W})$ is a bitopological space, and we can moreover define a relation $< \subseteq Pt(\mathcal{W}) \times Pt(\mathcal{W})$ on $Pt(\mathcal{W})$ by letting $\alpha < \beta$ if the following conditions hold:

(1) $u \in \alpha$ implies $u \in \beta$ for any $u \in U$

(2) $l \in \beta$ implies $l \in \alpha$ for any $l \in L$

(3) there exists $l \in L$ with $l \in \alpha, l \notin \beta$ or there exists $u \in U$ with $u \in \beta, u \notin \alpha$

And letting $\alpha \leq \beta$ if $\neg(\beta < \alpha)$. The tuple $(Pt(\mathcal{W}), \mathcal{L}, \mathcal{U}, <)$ is the spectrum of a formal bitopology $\mathcal{W}$, denoted as $\Sigma\mathcal{W}$.

6.6.9. Lemma. *Let $\mathcal{W}$ be a linear formal bitopology. Then $\alpha < \beta$ if and only if there exists $l \in L$ with $l \in \alpha, l \notin \beta$ or there exists $u \in U$ with $u \in \beta, u \notin \alpha$*

Proof:
Assume $\mathcal{W}$ is a linear formal bitopology and $\alpha < \beta$. The right-to-left direction is obvious by definition of $<$. For the left-to-right direction we show that conditions (1) and (2) of the definition of $<$ follow from each of the disjuncts of condition (3). We only consider the first disjunct $l \in \alpha, l \notin \beta$ for some $l \in L$ as the other case is analogous. We first show condition (2): $l' \in \beta$ implies $l' \in \alpha$ for every $l' \in L$. Let then $l' \in \beta$. If $l' \lhd l$ then $l \in \beta$ by the definition of a formal point, but this yields a contradiction. Hence $\neg(l' \lhd l)$ which implies that $l \lhd l'$ by the linearity condition of a linear formal bitopology, but then $l' \in \alpha$ since $l \in \alpha$, and we are done. To see condition (1), let $u \in \alpha$ for some $u \in U$. But then $l \cdot u \in \alpha$ and hence $Pos(l \cdot u)$, but then $1 \lhd \{l, u\}$ by the totality condition of event structures, which implies $u \in \beta$ by disjunctive syllogism and conditions (3) and (1) in the definition of a formal point. □

We then immediately obtain the following results:

6.6.10. Proposition. *Let $\mathcal{W}$ be a formal bitopology. Then the following hold:*

(1) $\Sigma\mathcal{W}$ is a bitopological space and $<$ is a strict partial order on $\Sigma\mathcal{W}$.

(2) V is an open downset for any $V \in \mathcal{L}$ and O is an open upset for any $O \in \mathcal{U}$

(3) Assuming classical logic, if $\mathcal{W}$ is a linear formal bitopology then $<$ is a strict linear order.

Proof:
For (1), that $\Sigma\mathcal{W}$ is a bitopological space follows from the above considerations. To see that $<$ is a partial order we show that it is irreflexive and transitive. Irreflexivity is clear. For transitivity, let $\alpha < \beta, \beta < \gamma$; we show $\alpha < \gamma$. Clearly for any $u \in U$ if $u \in \alpha$ then $u \in \beta$ since $\alpha < \beta$ and then $u \in \gamma$ since $\beta < \gamma$, and a similar reasoning shows that $l \in \gamma$ implies $l \in \alpha$ for any $l \in L$. Since $\alpha < \beta$ assume without loss of generality that $l \in \alpha, l \notin \beta$ for some $l \in L$. Then since $\beta < \gamma$ either there exists $l' \in L$ with $l' \in \beta, l' \notin \gamma$ or there exists $u \in U$

with $u \in \gamma, u \notin \beta$. In the first case then $l' \in \alpha$ and we are done. In the second case then clearly $u \notin \alpha$, since if $u \in \alpha$ then $u \in \beta$ as $\alpha < \beta$ and we have a contradiction. Hence $<$ is irreflexive and transitive, i.e., a strict partial order.

For (2) it suffices to note that $\varphi(l)$ is a downset for any $l \in L$: choose $\alpha \in \varphi(l)$ then $l \in \alpha$, but then if $\beta < \alpha$ then by definition $l \in \beta$ and hence $\beta \in \varphi(l)$. Since intersections and unions of downsets are downsets we are done.

Claim (3) is obvious. □

We remark that if we assume intuitionistic rather than classical logic, then the order $<$ of $\Sigma\mathcal{W}$ is in general only a partial order, even when the formal bitopology under consideration is linear in the sense of Definition 6.6.5. To be sure, in particular cases it is possible to prove that $<$ is a constructive linear order, as for instance in the case of the formal reals in Negri and Soravia (1999), where the decidability of the covering relation $\lhd$ on the rationals plays an essential role, but in general this does not hold. This, of course, shows that if we take the background logic of the Kantian continuum to be intuitionistic then there are dramatic effects on the properties of the Kantian continuum, even down to the order properties.

6.6.11. Lemma. *Let $\mathcal{W}$ be a connected formal bitopology. Then $\Sigma\mathcal{W}$ equipped with the join topology $\mathcal{L} \vee \mathcal{U}$ is convex, T_0, and connected.*

Proof:
$\Sigma\mathcal{W}$ is clearly well-formed since the open sets in $\mathcal{U}, \mathcal{L}$ are subbasic elements and are upsets and downsets respectively because of Proposition 6.6.10. To see T_0 note that a constructive version of the T_0 separation property can be formulated as follows:

$$\mathcal{N}(x) = \mathcal{N}(y) \Rightarrow x = y$$

where x, y are points and $\mathcal{N}(x)$ denotes the set of basic opens that contain x (Aczel & Fox, 2005, p. 6). This separation property is classically equivalent to the usual one and follows by construction. Connectedness follows from the connectedness condition for formal topologies (Definition 6.6.7). □

We can now relate the spectrum of points of a linear formal bitopology to the construction of Walker and Russell points, since this allows us to show that Walker and Russell points correspond respectively to closed and open points of a connected well-formed space that, moreover, satisfies the condition that every maximal proper filter is principal. In particular, let an *atom* in a formal topology be an element $a \in W$ such that

$$Pos(a) \wedge \forall b(b \lhd a \wedge Pos(b) \rightarrow a \lhd b)$$

Clearly, this is the formal topological equivalent of an atom in a semilattice with bottom element, or of our $\preceq$-minimal covering events in section 5.7. We then have:

6.6.12. PROPOSITION. *Assume classical logic. Let $\mathcal{W}$ be a connected linear formal bitopology such that every maximal proper filter $\mathcal{F}$ under $\lhd$ is principal, that is, $\mathcal{F} = \uparrow_{\lhd} a$ for some atom $a \in W$. Let $P \subseteq L$ be non-empty and downward-closed under $\lhd$, i.e., $l \in P, l' \lhd l$ implies $l \in P$, and let:*

$$F = \{u \in U \mid \neg Pos(l \cdot u) \text{ for some } l \in P\}$$

Moreover, let $C = \{a \in W \mid Pos(l \cdot a) \wedge Pos(a \cdot u) \text{ for some } l \in P, l \in F\}$. Then, we have:

(1) if F is not empty then C is a formal point and it is closed

(2) $\mathcal{F}$ is a formal point and it is open for every maximal proper filter $\mathcal{F}$

(3) The set of all triples (P, C, F) is a complete linear order under inclusion of P

(4) If α is open then it is a maximal proper filter, and if it is closed then it is a triple of the form (P, C, F) or an endpoint.

The proof of the above proposition is essentially a variation on the techniques presented in chapter 5 when discussing boundaries and infinitesimal intervals. Of course, the triples (P, C, F) are the two-sided boundaries of section 5.5 or Walker instants, while the maximal proper filters are the infinitesimal intervals or Russell instants; we need to assume that every maximal proper filter is principal because otherwise, as we pointed out in section 6.4, the classes of Russell and Walker instants are not necessarily disjoint. In particular, a clear picture of the relationship between Russell and Walker instants can be obtained if the linear formal bitopology is finite. In this case, all maximal proper filters are of course principal, and the spectrum of a finite linear formal bitopology is as in Figure 6.2

The lower part of the figure represents the linear bitopology by displaying only the elements of the generating semilattices L and U, in the same fashion as we did in chapter 4 to represent event structures. The vertical lines, labelled $a - h$, denote the formal points of the formal bitopology: given a vertical line, a formal point consists of all the elements of the semilattices L, U - the horizontal lines - intersecting that vertical line. The upper part of the figure, on the other hand, represents the linear order of formal points and whether a point is closed (filled dot), open (filled dot surrounded by a circle) or undecided (filled triangle). Note that the two-sided Walker instants are c, d which are both closed, and the Russell instants are e, f, h which are all open; the undecided points b, g are neither Walker nor Russell instants. The point a, while closed, is not a Walker instant, since it cannot be constructed as a fixpoint: indeed, in the construction presented in chapter 5 a point of this sort is replaced by the formal boundary at $-\infty$. Note finally that the space $\Sigma\mathcal{W}$ portrayed in the figure, while a connected well-formed space, is not selective; for instance, points d, h are such that $d \sqsubseteq h$ but they are

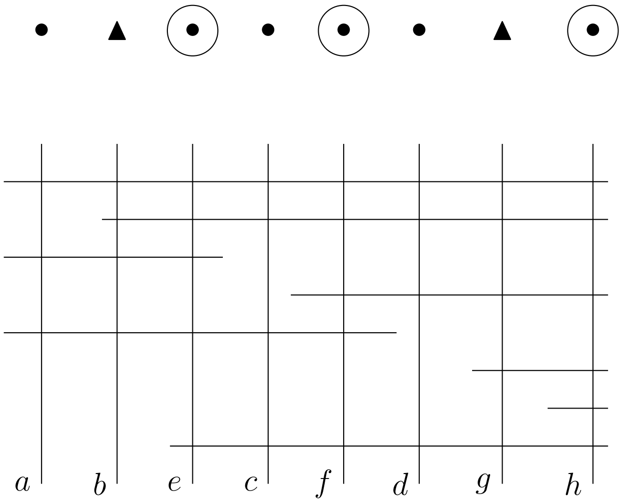


Figure 6.2: The spectrum of a finite linear connected bitopology

not adjacent in the linear ordering. The space does, however, quotient to a COTS by mapping the undecided points to the closest open point in the specialization ordering, e.g., collapsing b and e, and g and h yields a COTS, which is isomorphic to the subspace of decided points of the space.

Observe that the requirement of connectedness is essential in the proof of Proposition 6.6.12, since if the formal topology is not connected then the Walker instants are not necessarily formal points. For instance, the formal topology pictured in Figure 6.1 (a) is such that the triple (P, C, F) defined by letting $P = \{l\}, C = \{1\}, F = \{u\}$ is a Walker instant in the sense of Proposition 6.6.12, but is not a formal point because the third condition of the definition - that a formal point splits covers - fails. The connection between connectedness and Walker instants in well-formed spaces is relevant to our interpretation of them as Kantian or Aristotelian boundaries in time. Indeed, it expresses formally Aristotle's intuition that the now in time both "separates" and "connects" past and future, since even if the formal topology is disconnected the construction of instants presented in chapter 5 will interpolate a boundary that forces the space of points to be connected. Thus, using the tools of contemporary formal logic we obtained a rendition of instants of time or "nows" that, I feel, would please both Aristotle and Kant greatly.

6.6.5 Concluding remarks

We conclude by considering what has been achieved here from a philosophical standpoint.

First, we have seen that the construction of time from event structures developed by Russell and Walker is subsumed by our treatment of the Kantian continuum, and that the latter is, in turn, closely related to the treatment in formal topology of bitopological spaces. This raises the question of whether one can set the Kantian continuum on a predicative and constructive basis, by extracting suitable constructive content from the inverse limit procedure described in chapter 5. This will involve a deeper investigation of the category of formal bitopologies than we have provided here.

We have also provided a topological interpretation showing that Walker and Russell instants, under suitable conditions, are complementary, as they represent the closed and open points of a connected well-formed space. Thus, Walker and Russell were wrong and right at the same time: they just identified different subclasses of points of a special sort of connected well-formed spaces.

Finally, the notion of a formal bitopology introduced here paves the way for the construction of relativistic spacetimes from events in a point-free and constructive fashion, as we discussed in section 6.1. Of course, this aim is closely related to providing a constructive treatment of the Kantian continuum, since the construction of inverse limits must be scrutinized for constructive content.

Chapter 7

Conclusion

In this thesis I have provided an analysis of the conceptual foundations of Kant's theory of the temporal continuum and of its logical and topological structure. The mathematical theory I have developed provides formal correlates to Kant's temporal continuum (the so-called Alexandroff COTS) and to numerous related notions. Most importantly, it clarifies various elusive distinctions that Kant makes on this subject, in particular that between the form of intuition and the formal intuition. The work in this thesis is part of the more general project, initiated in Achourioti and van Lambalgen (2011), of setting Kant's critical philosophy on a firm mathematical basis; a project that does not aim to supersede traditional exegesis, but rather to complement it and relate it to current scientific developments (see chapter 2). In this respect, I can best present to the reader the achievements of this work in the table of Figure 7.1, where a correspondence is established between informal notions of Kant's theory of space and time in the left column and formal notions in the right column.

The analysis of the continuum presented here is also relevant, more generally, for the development of rigorous phenomenological foundations for the concept of the (spatiotemporal) continuum. After all, as we have seen in section 3.4, the Kantian continuum bears a close relationship to Aristotle's analysis of the concept and, more generally, to the "continuist" continuum.

Finally, I have also shown here that my approach to the Kantian continuum subsumes and extends Russell's and Walker's constructions of instants from events and that it is closely related to point-free topology in the predicative and constructive tradition of formal topology. This paves the way for reviving the Russell-Walker-Whitehead project of constructing relativistic spacetimes from events, which, as I argue, would shed further light on the foundations of relativity and on the viability of the causal set approach to quantum gravity.

I should like to conclude this thesis by discussing various open-ended possibilities for future work that may be relevant for the interested reader.

Informal notion	**Formal correlate**
Form of intuition	Mere multiplicity without constraints (a set), (section 4.3.1)
Part of time	Submodel of a model of GT (section 5.4)
Influence of the categories on sensibility	Graded notion: axiom systems GT_0, GT_1 and GT (see chapter 4)
Kantian-Aristotelian temporal boundaries	Two-sided boundaries (sections 4.4 and 5.5)
Infinity of time	Formal boundaries (section 5.5.3), directed system of event structures (section 4.5.1)
Formal intuition	Graded notion: axiom systems $\mathrm{GT}_0, \mathrm{GT}_1$ and GT (section 4.3.2)
Infinite divisibility of time	Infinite divisibility inverse systems (section 5.8)
Synthesis of the unity of apperception	Directedness of the class of finite event structures under retractions (sections 4.5 and 5.10)
Figurative synthesis	Inverse system of finite event structures (finitary spectrum of GT, sections 4.5 and 5.10)
Unity of time as an object	Inverse limit of the finitary spectrum (section 5.9.1) and its unicity up to isomorphism (Theorem 5.2.6, section 5.9.5), Universality of the Kantian continuum (Theorem 6.5.18)
Thoroughgoing determination of time as an object	Inverse limit of the finitary spectrum as a universal model (Theorem 5.9.15)
The Kantian continuum	The Alexandroff COTS $\mathcal{A}$ (Theorem 5.10.15)
Kantian infinitesimals and flowing magnitudes	Infinitesimal intervals (sections 5.6 and 5.11) and nilpotents in the ring of dual numbers (section 5.11)
External representation of time	Functions from $K(\mathcal{W})$ to $\mathcal{R}$ (see section 5.11)

Figure 7.1: Informal notions and formal correlates

7.1 A unified formal theory of Kant's transcendental philosophy

The immediate steps that should follow the investigation of the present thesis in relation to the analysis of the formal structure of Kant's transcendental philosophy are clear. In Achourioti and van Lambalgen (2011), Achourioti and van Lambalgen have proposed a formalization of Kant's transcendental logic which not only inspired the work in this thesis but lent to it some of the central mathematical techniques, in particular inverse systems and their limits. Nevertheless, it is still unclear how the two frameworks can be combined. For, they must be combined to obtain a full account of Kant's transcendental philosophy, since time (and, as we shall see in a moment, space) are of crucial importance to understand Kant's transcendental logic as opposed, in particular, to his general logic. On the other hand, however, it is also the case that an account of time that does not consider transcendental logic and the role of empirical or *a priori* concepts for temporal cognition is incomplete. In particular, our understanding of the relation between time and the category of *causality*, which in the present thesis has merely played the role of a pure schema constraining the action of the figurative synthesis (chapter 4), would certainly increase by enriching the formal framework of the thesis with the resources to formulate causal laws and concepts - for which the theory of judgments of transcendental logic is essential. Thus, I formulate the following open problem: how can the framework of this thesis be combined with that in Achourioti and van Lambalgen (2011) towards a full theory of Kant's transcendental philosophy?

7.2 The problem of space

The material of the present work focuses on Kant's temporal continuum and on the distinction between time as the form of intuition and as the formal intuition, but as I claimed in the first part of the thesis it also sheds light on Kant's conception of space as an object. One should still like, however, to have a formalization of Kant's theory of space along the same lines as that presented here for time. In this respect, note that a mere generalization of the present approach in which one takes, for instance, product spaces will not suffice. For instance, one could generalize our construction by considering the axiomatic theory obtained by taking products of event structures, and consider perhaps inverse systems of tuples $(\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2)$ where $\mathcal{W}_0, \mathcal{W}_1$ are linear event structures and $\mathcal{W}_2$ is their product. In light of results in digital topology regarding finitary approximations of the digital plane and approximation of function spaces (Webster, 2006) it seems quite clear that this approach yields a space of points on the inverse limit that is just the subspace of decided points of the product, in the category of ordered topological spaces, of the Alexandroff COTS $\mathcal{A}$ with itself. Now, while these investigations have certainly their interest, I am of the opinion that to obtain a satisfactory formalization of Kant's theory of space something else is needed, since such an approach does not address the fundamental technical obstacle I have always found in my way whenever I tried to address this issue, namely:

> How does one combine in a philosophically satisfactory way an axiomatic theory of geometrical constructions and an approximate and point-free approach to spatial topology?

As yet, I have not been able to find a satisfactory solution to this question. For instance, in his discussion of Kant's figurative synthesis in M. Friedman (2012, p. 242) Friedman defines a *perspectival space* to be the *a priori* correlate of the arrangement of empirical objects on the line of sight around a subject, in agreement with Kant's discussions of the intrinsic directionality of space in, for instance, his "On the first grounds of the distinction of directions in space" (2:375-383). In Friedman's account, the class of all perspectival spaces constitutes a correlate to Kant's metaphysical space (see the discussion at section 3.8.5 and M. Friedman (2012, p. 247)). Presumably, such *a priori* structure could be formalized along the following lines. Consider the plane $\mathbb{R}^2$ and fix a point $O \in \mathbb{R}^2$ as the origin, i.e., the "subject", and let C be the closed unit circle centered at O. Then any point p that lies outside C defines a point on C by taking the intersection between C and the line segment joining O with p, where the line segment represents the "line of sight" of the subject. One then obtains a *cyclic order* on such points (Huntington, 1916), but one can obtain a point-free structure by taking intervals of the cyclic order and axiomatizing the relations of order on their endpoints. One would then obtain a circular version of our event structures, which can be embedded in the plane, as a formal correlate to Friedman's perspectival space (see Figure 7.2).

Starting from the class of all finite perspectival spaces axiomatized in this fashion one would then provide an inverse limit construction along the lines of that presented in this work. Given our interpretation of the synthesis of the unity of the unity of apperception as the directedness of the inverse system, this approach would also have the merit of being closely related to Friedman's account of this synthesis in M. Friedman (2012), although, as I discussed in section 4.7, I do not quite agree with the notion of a precategorial synthesis. Next, however, Friedman seems to suggest that the class of all perspectival spaces, as metaphysical space, is united into a totality because every perspectival space can be transformed into any other perspectival space by a suitable sequence of translations and rotations. Here, however, the problem outlined above becomes more acute. In order to speak of translations and rotations it would seem that some sort of metric on perspectival spaces is needed, and whence this metric should come from is unclear. One cannot, I believe, start by assuming the real plane (or $\mathbb{R}^3$) with its metric and consider all possible coordinate systems and coordinate transformations among them as metaphysical space, since this clearly begs the question: whence does the

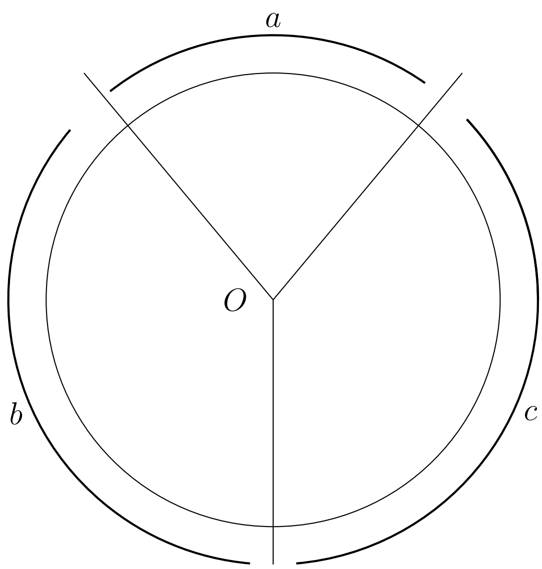


Figure 7.2: A perspectival space

metric come from, and how is this structure first obtained through a process of synthesis such as that which I have formalized in this work for the case of time? I am currently of the opinion that a solution to this problem should be algebraic in nature, and exploit, for instance, the relation between cyclic orders and cyclically ordered groups. Perhaps, moreover, Weyl's analogy between a subject and a barycentric coordinate system (Bell, 2000, p. 11) can be of help here, since it bears a close relation to Friedman's analysis of metaphysical space. Still, the problem of how to reconcile point-free topology, geometrical constructions and a process of synthesis formalized through inverse systems is open.

7.3 Relativistic spacetimes and the constructive Kantian continuum

This problem can be briefly stated. We have seen in section 6.6 that a close relation can be established between the theory of event structures and formal topologies (Sambin, 2003), to the effect that event structures form a special class of formal topologies. Now, can the construction of the Kantian continuum presented in chapter 5 be made fully constructive? This will involve investigating the constructive content of the inverse limit construction, a topic that I have not embarked upon in this thesis but which is interesting not only in relation to Kant's continuum but also with respect to developing, on a constructive basis, the recovery of relativistic spacetimes discussed in section 6.1.

Indeed, the discussion in section 6.1 and in section 6.6 shows that the notion of a formal bitopology can be of general use to obtain a predicative, pointfree and constructive approach to bitopological spaces whose bitopologies encode a partial order relation, and therefore to strongly causal spacetimes by considering open upsets and downsets under the light-like or time-like partial order $\preceq$. A treatment of the topology of relativistic spacetimes along the lines of formal topology would also allow one to apply type theory to relativity, as formal topology can be entirely formulated in this framework (Sambin, 2003). Furthermore, a suitable generalization of the continuum of formal topology to relativistic spacetimes offers the intriguing possibility of precisely distinguishing the finitary content of relativistic spacetimes from their infinitary content. This can be achieved by means of the inductive construction of coverings that formal topology makes possible (see Sambin (2003) section 1.3.5) and by generalizing the normal form theorem for the formal continuum (see section 1.3.6), and could open the way for the application of inductive methods in relativity theory. Finally, in relation to the above, it might be of technical interest to investigate better how a rendition of notions like biframes (Banaschewski et al., 1983) and d-frames (Jung & Moshier, 2006) would proceed in formal topology.

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Index

Samenvatting

De Logica van Kant's Tijdscontinuum

In dit proefschrift behandel ik de filosofische grondslagen alsmede de wiskundige structuur van Kant's tijdscontinuum. Ik richt me voornamelijk op het formaliseren van het tijdscontinuum zoals het wordt beschreven in de *Kritiek van de zuivere rede* en andere geschriften uit Kant's kritische periode. Echter, het grootste deel van mijn resultaten zijn evenzeer relevant voor het ontwikkelen van een wiskundige exacte grondslag voor een fenomenologisch begrip van het continuum. In het bijzonder betoog ik dat de topologische structuur van het Kantiaanse continuum gerepresenteerd kan worden door de lineaire ordening van Kantiaanse tijdspunten beschouwd op de inverse limiet van alle eindige modellen die de theorie $\mathcal{T}$ waarmaken, waarbij $\mathcal{T}$ staat voor de predicaat-logische theorie die Kant's begrip 'temporele vorm van de ervaring' formaliseert.

De wiskundige resultaten in dit proefschrift worden o.a. gebruikt om het nagenoeg ongrijpbare onderscheid – geintroduceerd in B161n van de eerste *Kritiek* – tussen tijd (en ruimte) als 'vorm van de aanschouwing' en als 'formele aanschouwing'. In het bijzonder betoog ik dat de formele aanschouwing wordt voortgebracht door de werking van wat Kant noemt de 'figuurlijke synthese' of *synthesis speciosa*, waarbij het subject innerlijk op zichzelf inwerkt door het traceren ('beschrijven' in de meetkundige zin) van ruimtes in de uiterlijke aanschouwing, geleid door de zuivere verstandsbegrippen ('categorieën'). Ik verdedig daarmee een conceptualistische lezing van B161n, waarin tijd (en ruimte) als formele aanschouwing worden voortgebracht door het beschrijven van tijden (en ruimtes) middels de figuurlijke synthese, hoewel de formele aanschouwing niet vereenzelvigd mag worden met individuele beschrijvingen van tijden (ruimtes). Ik betoog verder dat de begrippen 'formele aanschouwing' en 'vorm van de aanschouwing' weliswaar equivalent zijn in de context van de *Transcendentale esthetica*, maar dat er in de *Kritiek* als geheel wel degelijk een onderscheid is tussen formele aanschouwing enerzijds, en een volledig passief begrip van aanschouwingsvorm anderzijds. Dit onderscheid is echter niet absoluut, maar gradueel; vormen van de aanschouwing verschillen in de mate waarin ze formeel zijn. Deze interpretatie sluit naadloos aan bij de voorgestelde formalisering van Kant's tijdscontinuum.

Tot slot toon ik aan dat mijn reconstructie van Kant's continuum een generalisatie is van Russell's en Walker's constructies van tijdspunten uit 'gebeurtenissen', en dat de verkregen structuren nauw verwant zijn aan wat bekend staat als 'formal topology', een constructieve variant van verzamelingstheoretische topologie. Dit opent de mogelijkheid het Russell-Walker-Whitehead project – de constructie van relativistische ruimte-tijd uit gebeurtenissen – weer tot leven te wekken; bovendien werpt het een nieuw licht op de poging van 'quantum gravity' om quantum mechanica te verzoenen met relativiteitstheorie middels 'causal sets'.

Het proefschrift begint met enkele filosofische uiteenzettingen. In hoofdstuk 2 betoog ik dat het mogelijk is middels wiskundige technieken filosofische analyses transparanter te maken. Hoofdstuk 3 is gewijd aan mijn interpretatie van het onderscheid tussen vorm van de aanschouwing en formele aanschouwing. Hoofdstuk 4 bevat een eerste aanzet tot een formalisering van dit onderscheid, waarbij de nadruk ligt op de filosofische betekenis van de formalisering, meer dan op de achterliggende wiskundige theorie. Dat verandert in hoofdstuk 5, waar de wiskundige theorie van het Kantiaanse continuum wordt gepresenteerd. Hoofdstuk 6 bouwt hierop voort, en onderzoekt in hoeverre de in hoofdstuk 5 geconstrueerde temporele continua licht kunnen werpen op de unificatie van relativiteitstheorie en quantum mechanica.

Summary

The Logic of Kant's Temporal Continuum

In this thesis I provide an account of the philosophical foundations and mathematical structure of Kant's temporal continuum. I mainly focus on the development of a formalization of Kant's temporal continuum as it appears in the Critique of Pure Reason and in other works of Kant's critical period; most of my results, however, are generally relevant for the problem of developing mathematically rigorous foundations for a phenomenological concept of the continuum. In particular, I argue that the general topological form of the Kantian continuum is that of the Alexandroff COTS: a totally ordered topological space that is the space of Kantian instants on the limit of all finite models of a first-order theory that formalizes Kant's notion of the temporal form of an experience.

The formal apparatus of the thesis is also applied to the elucidation of the elusive distinction, at B161n of the Critique of Pure Reason, between space and time as "forms of intuition" and as "formal intuitions". In particular, I argue that the formal intuition is produced by the action of what Kant terms the figurative synthesis or *synthesis speciosa*, which consists in the self-affection of the subject in the description of spaces in outer sense in agreement with the categories. Thus, I propose a conceptualist reading of the distinction at B161n which holds that space and time as formal intuitions are produced in the act of description of particular spaces or times by the figurative synthesis, even though no particular such description is space or time as formal intuition. Moreover, while I argue that the formal intuition is nothing over and above the form of intuition of the TA, I also maintain that there exists in the Critique a distinction between the formal intuition and a purely passive notion of form of intuition that cannot be ignored. This distinction is, however, not sharp but graded, in that different levels of "formality" can be identified. This interpretation of the distinction between the form of intuition and the formal intuition is supported by the formalization.

Finally, I also show that my analysis of the Kantian continuum subsumes and extends Russell's and Walker's constructions of instants from events and that it is closely related to point-free topology in the predicative and constructive tradition of formal topology. This paves the way for a constructive and predicative treatment of bitopological spaces in formal topology and for reviving the Russell-Walker-Whitehead project of constructing relativistic spacetimes from events; I argue that this would shed further light on the foundations of relativity and on the viability of the causal set approach to quantum gravity.

The thesis is structured as a progression from informal philosophical analysis in the first part to technical mathematical treatment in the second part. In particular, the first two chapters are philosophical: chapter 2 provides a justification for the use of mathematical methods in the exegesis of systematic philosophy, while chapter 3 fleshes out my interpretation of the distinction between the form of intuition and the formal intuition. Chapter 4 is in part philosophical and in part technical, since I provide an overview of the basics of the formal theory, but without proofs and focusing on its philosophical import in relation to Kant's theory of the continuum and the distinction between form and formal intuition. Chapter 5 is technical and is where the main mathematical results of the thesis, and in particular the topological construction of the Kantian continuum, are given. Finally, chapter 6 relates the mathematical framework of the thesis to the constructions of time from events proposed by Russell and Walker and to point-free topology. I show that Russell's and Walker's constructions are special cases of the construction of Kantian instants, so that the two constructions can be unified and given a clear topological interpretation, which sheds light on the debate regarding the most satisfactory construction of instants of time from events. I also show that my account of Kant's continuum is closely related to point-free topology in the constructive and predicative tradition of formal topology, and argue that reviving the Russell-Walker-Whitehead project of constructing relativistic spacetimes on these grounds promises to deliver useful insights in the foundations of relativity.

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