Evidence in Epistemic Logic

–

A Topological Perspective

Aybüke Özgün

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A Topological Perspective

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Aybüke Özgün

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Anneme ve Babama

Contents

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Chapter 1

Introduction

This dissertation brings together epistemic logic and topology. It studies formal representations of the notion of evidence and its link to justification, justified belief, knowledge, and evidence-based information dynamics, by using tools from topology and (dynamic) epistemic logic.

Epistemic logic is an umbrella term for a species of modal logics whose main objects of study are knowledge and belief. As a field of study, epistemic logic uses modal logic and mathematical tools to formalize, clarify and solve the questions that drive (formal) epistemology, and its applications extend not only to philosophy, but also to theoretical computer science, artificial intelligence and economics (for a survey, see van Ditmarsch et al., 2015a). Hintikka (1962) is considered the founding father of modern epistemic logic. In his book *Knowledge and Belief: An Introduction to the Logic of the Two Notions* (1962)—inspired by insights in (von Wright, 1951)—Hintikka formalizes knowledge and belief as basic modal operators, denoted by K and B, respectively, and interprets them using standard possible worlds semantics based on (relational) Kripke structures. Ever since—as Kripke semantics provides a natural and relatively easy way of modelling epistemic logics—it has been one of the prominent and most commonly used semantic structures in epistemic logic, and research in this area has widely advanced based on the formal ground of Kripke semantics.

However, standard Kripke semantics possesses some features that make the notions of knowledge and belief it implements too strong—leading to the problem of logical omniscience—and is lacking the ingredients that make it possible to talk about the nature and grounds of acquired knowledge and belief. What triggered the work presented in this dissertation is the latter issue: we not only seek an easy way to model knowledge and belief, but also study the emergence, usage, and transformation of *evidence* as an inseparable component of a *rational* and *idealized* agent's justified belief and knowledge.

For this purpose, topological spaces are proven to be natural mathematical objects to formalize the aforementioned epistemic notions, and, in turn, evidence-based information dynamics: while providing a deeper insight into the evidence-based interpretation of knowledge and belief, topological semantics also generalizes the standard relational semantics of epistemic logic. Roughly speaking, topological notions like open, closed, dense and nowhere dense sets qualitatively and naturally encode notions such as *measurement/observation, closeness, smallness, largeness and consistency*, all of which will recur with an epistemic interpretation in this dissertation. Moreover, topological spaces are equipped with well-studied basic operators such as the interior and closure operators which—alone or in combination with each other—succinctly interpret different epistemic modalities, giving a better understanding of their axiomatic properties. To that end, we see topological spaces as information structures equipped with an elegant and strong mathematical theory that help to shed some light on the philosophical debates surrounding justified belief and knowledge, and to gain more insights into learning via evidence-acquisition.

The epistemic use of topological spaces as information structures can be traced back to the 1930s and 1940s, where topological spaces served as models for intuitionistic languages, and open sets are considered to be 'pieces of evidence', 'observable properties' concerning the actual state (see, e.g., Troelstra and van Dalen, 1988). This interpretation assigned to open sets constitutes the basic epistemic motivation behind our use of topological models, and will return often at various places (in modified forms) in the main body of this dissertation. Variations of this idea can also be found in domain theory in computer science (Abramsky, 1987, 1991; Vickers, 1989), and guide the research program of "topological" formal learning theory initiated by Kelly and others (Kelly, 1996; Schulte and Juhl, 1996; Kelly et al., 1995; Kelly and Lin, 2011; Baltag et al., 2015c) in formal epistemology.

The literature connecting (modal) epistemic logic and topology is developed based on two separate, yet strongly related topological settings. Our work in this dissertation justly benefits from both approaches. The first direction stems from the interior-based topological semantics of McKinsey (1941) and McKinsey and Tarski (1944) for the language of basic modal logic (some of the ideas could already be found in Tarski, 1938 and Tsao-Chen, 1938). In this semantics the modal operator $\Box$ is interpreted on topological spaces as the interior operator. These investigations took place in an abstract, mathematical context, independent from epistemic/doxastic considerations. McKinsey and Tarski (1944) not only proved that the modal system $\mathsf{S4}$ is the logic of all topological spaces (under the above-mentioned interpretation), but also showed that it is the logic of any dense-in-itself separable metric space, such as the rational line $\mathbb{Q}$, the real line $\mathbb{R}$, and the Cantor space, among others. This approach paved the way for a whole new area of spatial logics, establishing a long standing connection between modal logic and topology (see, e.g., Aiello et al., 2007 for a survey on this topic, in particular, see van Benthem and Bezhanishvili, 2007). Moreover, the completeness results concerning the epistemic system $\mathsf{S4}$ have naturally attracted epistemic logicians, and led to an *epistemic* re-evaluation of the interior semantics, seeing topologies as models for information. One branch of the epistemic logic-topology connection has thus been built on the interior-based topological semantics, where the central epistemic notion studied is knowledge (see, e.g.,van Benthem and Sarenac, 2004). What we add to this body of work, in Part I of this dissertation, are the missing epistemic components *evidence* and *belief*, as well as the *dynamics of learning new evidence*, strengthening the connection between epistemic logic and topology. We do so by reanalyzing the neighbourhood-based evidence models of van Benthem and Pacuit (2011) from a topological perspective. The way we represent evidence and how it connects to justified belief are inspired by the approach in (van Benthem and Pacuit, 2011), and the evidence transforming actions considered are adapted from the aforementioned influential work.

The second topological approach to epistemic logic was initiated by Moss and Parikh (1992). They introduced the so-called *topologic*, a bimodal framework to formalize reasoning about sets and points in a single modal system. Their topological investigations have a strong motivation from epistemic logic, suggesting that "simple aspects of topological reasoning are also connected with special-purpose logics of *knowledge*" (Moss and Parikh, 1992, p. 95). The key element Moss and Parikh (1992) introduced to the paradigm of epistemic logic is the abstract notion of epistemic *effort*. Effort can, roughly speaking, be described as any type of evidence-gathering—via, e.g., measurement, computation, approximation, experiment or announcement—that can lead to an increase in knowledge. The formalism of topologic therefore combines the static notion of knowledge with the dynamic notion of effort, thus, it is strongly related to *dynamic epistemic logic* (Baltag et al., 1998; van Ditmarsch et al., 2007; van Benthem, 2011; Baltag and Renne, 2016). In Part II of this thesis, we build a bridge between the two formalisms, which results in both conceptual and technical advantages. While dynamic epistemic logic expands the array of dynamic attitudes it studies, the topologic setting obtains epistemically more intuitive axiomatizations, clarifying the meaning of *effort* by linking it to well-understood instances such as *public* and *arbitrary announcements*.

***

The contributions of this thesis are presented in two parts. Below, we give a brief overview of each chapter. Every chapter starts with a brief introduction further elaborating its content and links to the relevant literature.

Chapter 2 provides the technical preliminaries that are essential for both parts of the dissertation. This includes, in the first half, a very brief introduction to the standard Kripke semantics for the basic modal logic. We recall the commonly studied static systems for epistemic/doxastic logics and the corresponding relational properties that render these logics sound and complete. In the second part, we introduce the elementary topological notions that will be used throughout this dissertation.

PART I: From Interior Semantics to Evidence Models

Part I is concerned with evidence-based interpretations of justified belief and knowledge. Starting with a by-now-standard topological interpretation of knowledge as the interior operator, we develop, in a gradual manner, a topological framework that (1) can talk about evidence not only semantically, but also at the syntactic level, thereby making the notion of evidence more explicit; (2) takes evidence as the most primitive notion, and defines belief and knowledge purely based on it, thereby linking these two crucial notions of epistemology at a deeper, more basic level. These investigations have considerable philosophical consequences as they allow us to discern, isolate, and study various aspects of the notion of evidence, and its relation to justification, knowledge and belief.

Chapter 3 introduces the interior-based topological semantics of McKinsey and Tarski (1944) as a way to model knowledge, points out its link to the standard relational semantics, and motivates the interpretation of knowledge as the topological interior operator. It then discusses an existing topological semantics for belief based on the derived set operator, and argues that it does not constitute a satisfactory semantics for belief, especially when considered in tandem with knowledge as the interior.

Chapter 4 shifts our focus from the topological interpretation of knowledge to the topological interpretation of belief, and presents the first step toward developing a topological theory of belief that works well in combination with knowledge as the interior operator. More precisely, the first part of this chapter presents a review of the topological belief semantics of (Özgün, 2013; Baltag et al., 2013), addressing the following questions:

- *Given the interior-based topological semantics for knowledge, how can we construct a topological semantics for belief that can also address the problem of understanding the relation between knowledge and belief? To what extent do topological notions capture the intuitive meaning of the intended notion of belief?*

The proposed semantics for belief is derived from Stalnaker's logical framework in which belief is realized as a weakened form of knowledge (Stalnaker, 2006), which leads to a belief logic of *extremally disconnected spaces*. While this static setting provides a satisfactory answer for the above questions, the dynamic extension with public announcement modalities runs into problems due to the structural properties of extremally disconnected spaces. This leads to the search for a public announcement friendly logic of knowledge and belief. The second part of this chapter (based on Section 4.2 of Baltag et al., 2015a) is devoted to solving this issue, and the proposed solution consists in interpreting knowledge and belief on *hereditarily extremally disconnected spaces*.

While this semantics for belief works well for Stalnaker's strong notion of belief as *subjective certainty*, from a more general perspective, it can be seen somewhat restrictive for two reasons. It is based on rather exotic classes of topological spaces, and the corresponding logics do not comprise evidence in a real sense as there is no syntactic representation of it. This constitutes part of the motivation for the next chapter, leading to more general and fundamental questions addressed there.

Chapter 5 contains the main contribution of Part I. Resting on the assumption that an agent's rational belief is based on the available evidence, we try to unveil the concrete relationship between an agent's evidence, beliefs and knowledge, and study the evidence dynamics that the designed static account supports. This project is motivated by both philosophical and technical questions, as well as the aforementioned drawbacks of our own work in Chapter 4. To be more precise, we focus on the following questions, among others:

- *How does a rational agent who is in possession of some possibly false, possibly mutually contradictory pieces of evidence put her evidence together in a consistent way, and form consistent beliefs?*
- *What are the necessary and sufficient conditions for a piece of evidence to constitute justification for one's beliefs? What properties should a piece of justification possess to entail (defeasible) knowledge?*
- *How does our formalization of the aforementioned notions help in understanding the discussions in formal epistemology regarding the link between justified belief and knowledge?*
- *What are the complete axiomatizations of the associated logics of justified belief, knowledge and evidence? Do they have the finite model property? Are they decidable?*

The above questions also drive the approach of van Benthem and Pacuit (2011); van Benthem et al. (2012, 2014), which inspired our work considerably. Addressing the first question requires defining a "smart" way of aggregating the available evidence, based on *finite* and *consistent* subcollections of it. Topologically, this leads to a move from a topological subbasis to a basis. This generates a topological evidence structure that allows us to work with many epistemic modalities capturing different notions of evidence, belief, and knowledge interpreted using topological notions. The explicit use of topologies is one of the features of our setting which separates it from that of van Benthem and Pacuit (2011). Once the evidence aggregation method is set, we take a *coherentist* and *holistic* view on justification, and, roughly speaking, define it as a piece of evidence that is consistent with every available evidence. Moreover, in our setting, defeasible knowledge requires a true justification. We then use our topological setting to formalize stability and defeasibility theories of knowledge (Lehrer and Paxson, 1969; Lehrer, 1990; Klein, 1971, 1981), as well as relevant notions such as (non-)misleading evidence, clarifying some of the philosophical debates surrounding them. Our main technical results concern completeness, decidability and the finite model property for the associated logics.

PART II: From Public Announcements to Effort

In Part II of this dissertation, we no longer discuss belief, but rather focus on notions of knowledge as well as various types of information dynamics comprising learning new evidence. This part takes the subset space setting of Moss and Parikh (1992) as a starting point, and is centered around the notions of *absolutely certain knowledge* and *knowability* as "*potential knowledge*", as well as the connections between the abstract notion of epistemic *effort* encompassing any method of evidence acquisition and the well-studied dynamic attitudes such as *public* and *arbitrary public announcements*.

Chapter 6 provides the background for Part II and motivates the paradigm shift between the two parts of this thesis. In particular, it introduces the subset space semantics of Moss and Parikh (1992) and the topological public announcement logic of Bjorndahl (2016).

Chapter 7 investigates extensions of the topological public announcement logic of Bjorndahl (2016) with the *effort* modality of Moss and Parikh (1992), as well as with a topological version of the arbitrary announcement modality of Balbiani et al. (2008). This work is of both conceptual and technical interest, aiming at clarifying the intuitively obvious, yet formally elusive connection between the dynamic notions *effort* and its seemingly special instances: public and arbitrary announcements. In particular, we address the following questions, and answer them positive:

- *Can we clarify the meaning of the effort modality by linking it to the aforementioned dynamic modalities?*
- *Does treating the effort modality together with public announcements in a topological setting provide any technical advantages regarding the complete axiomatization of its associated logic, decidability and the finite model property?*

We give a complete axiomatization for the *dynamic topologic of effort and public announcements*, which is epistemically more intuitive and, in a sense, simpler than the standard axioms of topologic (Georgatos, 1993, 1994; Dabrowski et al., 1996).

Our completeness proof is also more direct, making use of a standard canonical model construction. Moreover, we study the relations between this extension and other known logical formalisms, showing in particular that it is co-expressive with the simpler and older logic of interior and global modality (Goranko and Passy, 1992; Bennett, 1996; Shehtman, 1999; Aiello, 2002), which immediately provides an easy decidability proof both for the original topologic and for our extension.

Chapter 8 is concerned with the multi-agent generalization of the setting presented in the previous chapter. Modelling multi-agent epistemic systems in the style of subset space semantics is not a trivial task. We start the chapter by laying out some problems one encounters while working with multi-agent extensions of subset space logics. Our proposal for a multi-agent logic of knowledge and knowability and its further extensions with public and arbitrary announcements does not run into these problems and constitutes a novel semantics for the aforementioned notions. In addition, the multi-agent setting presented in this chapter is general enough not only to model fully introspective, i.e. S5-type knowledge, but also to interpret S4, S4.2 and S4.3-types of knowledge. This contrasts with and enriches the existing approaches to subset space semantics for knowledge, since the other approaches, to the best of our knowledge, can only work with S5 knowledge.

Origin of the material

- Chapter 4 is based on:

  Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. (2015a). The topological theory of belief. *Under review.* Available online at *http://www.illc.uva.nl/Research/Publications/Reports/PP-2015-18.text.pdf.*

  Part I of Chapter 4 (Sections 4.1-4.2.1) provides a review of (Özgün, 2013; Baltag et al., 2013), whereas the remainder of the chapter contains material not covered in (Özgün, 2013; Baltag et al., 2013) but presented in (Baltag et al., 2015a).

- Chapter 5 is based on two papers, where the latter is an extended version of the former:

  Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. (2016a). Justified belief and the topology of evidence. In *Proceedings of 23rd Workshop on Logic, Language, Information and Computation* (*WoLLIC 2016*), pp. 83-103.

  Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. (2016b). Justified belief and the topology of evidence–Extended version. Available online at *http://www.illc.uva.nl/Research/Publications/Reports/PP-2016-21.text.pdf.*

- Chapter 7 is based on:

  van Ditmarsch, H., Knight, S., and Özgün, A. (2014). Arbitrary announcements on topological subset spaces. In *Proceedings of the 12th European Conference on Multi-Agent Systems* (*EUMAS 2014*), pp. 252-266.

  Baltag, A., Özgün, A., and Vargas-Sandoval, A. L. (2017). Topo-Logic as dynamic epistemic logic. In *Proceedings of the 6th International Workshop on Logic, Rationality and Interaction* (*LORI 2017*). To appear.

- Chapter 8 is based on:

  van Ditmarsch, H., Knight, S., and Özgün, A. (2015b). Announcements as effort on topological spaces. In *Proceedings of the 15th Conference on Theoretical Aspects of Rationality and Knowledge* (*TARK 2015*), pp. 283-297.

  van Ditmarsch, H., Knight, S., and Özgün, A. (2015c). Announcements as effort on topological spaces–Extended version. Accepted for publication in *Synthese*.

Moreover, although the main results of the following papers are not included in this dissertation, the discussion concerning their conceptual content contributes to the present work to a great extent.

van Ditmarsch, H., Knight, S., and Özgün, A. (2017). Private announcements on topological spaces. *Studia Logica*. Forthcoming.

Bjorndahl, A., and Özgün, A. (2017). Logic and Topology for Knowledge, Knowability, and Belief. In *Proceedings of the 16th Conference on Theoretical Aspects of Rationality and Knowledge* (*TARK 2017*), pp. 88-101.

Chapter 2

Technical Preliminaries

In this chapter, we provide the technical preliminaries essential for the main body of the thesis. The original work presented in Parts I and II is based on two different, yet related topological frameworks. However, we occasionally resort to their connection with the relational semantics and the well-developed completeness results therein in order to obtain similar conclusions for the topological counterpart. We therefore primarily use three different formal settings in developing our original contribution: the standard relational semantics for the basic modal logic, the interior-based topological semantics à la McKinsey and Tarski (1944), and the subset space semantics introduced by Moss and Parikh (1992). While the relational setting serves only as a technical tool utilized in Parts I and II, the latter two topological settings have inspired the developments presented in these parts. We leave the background details of these topological settings for later chapters, and present here only the formal tools that are commonly used in both parts.

Outline

Our presentation in this chapter is two-fold. Section 2.1 briefly discusses the standard relational semantics for the language of basic modal logic, and the unimodal epistemic and doxastic systems that will be studied in later chapters. Section 2.2 introduces the purely topological preliminaries that will be used throughout the thesis. Additionally, this chapter also serves the purpose of fixing our notation for the main body of this dissertation. Readers who are familiar with the aforementioned topics should feel free to skip this chapter.

2.1 Relational Semantics for Modal Logics (of Knowledge and Belief)

Starting from the pioneering work of Hintikka (1962), if not earlier, modal logic and its relational semantics—also known as Kripke semantics—have been the main tools utilized in the formalization of knowledge and belief. Hintikka (1962) interpreted knowledge and belief as normal modal operators, K and B, respectively, on Kripke models. This enables us to formulate the properties of various notions of knowledge and belief (of different strength and type) by using modal formulas of a given epistemic/doxastic language.

In this section, we briefly present the standard relational semantics for the basic modal language and define some well-known epistemic and doxastic logics. This is in no way an exhaustive presentation of relational semantics for modal epistemic and doxastic logics: we here aim to fix notation in order to ease the presentation in the main chapters and summarize the results we later refer to. The presentation in this section is based on the basic modal language since we make use of the technical aspects of the relational setting to prove results almost exclusively regarding unimodal epistemic/doxastic systems.

2.1.1. Definition. [Syntax of $\mathcal{L}_\Box$] The language of *basic modal logic* $\mathcal{L}_\Box$ is defined recursively as

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi,$$

where $p \in \textsc{prop}$, a countable set of *propositional variables*.

Abbreviations for the Boolean connectives $\vee, \rightarrow$ and $\leftrightarrow$ are standard, and $\bot$ is defined as $p \wedge \neg p$. We employ $\Diamond\varphi$ as an abbreviation for $\neg\Box\neg\varphi$.

Since we, in general, work with the above defined modal language in an epistemic/doxastic setting, the particular languages we consider in this work typically include, instead of $\Box$, modalities such as K and B for knowledge and belief, respectively. Accordingly, $\mathcal{L}_K$ denotes the *basic epistemic language* and $\mathcal{L}_B$ the *basic doxastic language* defined as in Definition 2.1.1.

We are particularly interested in the modal systems that are commonly used in the formal epistemology literature to represent notions of knowledge and belief. Some of the interesting and widely used axioms and an inference rule formalizing properties of these notions are listed in Table 2.1.

We again use a similar notational convention as we did in case of the languages. For example, the axiom of Consistency for *belief* is denoted by (D_B) $B\varphi \rightarrow \neg B\neg\varphi$, Positive Introspection for *knowledge* is written as (4_K) $K\varphi \rightarrow KK\varphi$, etc.

Let CPL denote all instances of classical propositional tautologies (see, e.g., Chagrov and Zakharyaschev, 1997, Section 1.3 for an axiomatization of classical propositional logic). Throughout this thesis, we use Hilbert-style axiom systems in

($K_\Box$)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	Normality
($D_\Box$)	$\Box\varphi \to \neg\Box\neg\varphi$	Consistency
($T_\Box$)	$\Box\varphi \to \varphi$	Factivity
($4_\Box$)	$\Box\varphi \to \Box\Box\varphi$	Positive Introspection
($.2_\Box$)	$\neg\Box\neg\Box\varphi \to \Box\neg\Box\neg\varphi$	Directedness
($.3_\Box$)	$\Box(\Box\varphi \to \psi) \vee \Box(\Box\psi \to \varphi)$	Connectedness
($5_\Box$)	$\neg\Box\varphi \to \Box\neg\Box\varphi$	Negative Introspection
($Nec_\Box$)	from φ, infer $\Box\varphi$	Necessitation
(MP)	from $\varphi \to \psi$ and φ, infer $\vdash \psi$	Modus Ponens

Table 2.1: Some unimodal axiom schemes and a rule of inference for $\Box$

order to provide the syntactic definitions of the modal logics we work with. Recall that, the weakest/smallest *normal* modal logic, denoted by $\mathsf{K}_\Box$, is defined as the least subset of $\mathcal{L}_\Box$ containing all instances of propositional tautologies (CPL) and ($K_\Box$), and closed under the inference rules (MP) and ($Nec_\Box$). Then, following standard naming conventions, we define the following normal modal logics that are used to represent knowledge and belief of agents with different reasoning power, where $\mathsf{L}+(\varphi)$ denotes the smallest modal logic containing L and φ. In other words, $\mathsf{L}+(\varphi)$ is the smallest set of formulas (in the corresponding language) that contains L and φ, and is closed under the inference rules of L. For example:

$$\begin{array}{rcl} \mathsf{KT}_\Box & = & \mathsf{K}_\Box + (T_\Box) \\ \mathsf{S4}_\Box & = & \mathsf{KT}_\Box + (4_\Box) \\ \mathsf{S4.2}_\Box & = & \mathsf{S4}_\Box + (.2_\Box) \\ \mathsf{S4.3}_\Box & = & \mathsf{S4}_\Box + (.3_\Box) \\ \mathsf{S5}_\Box & = & \mathsf{S4}_\Box + (5_\Box) \\ \mathsf{KD45}_\Box & = & \mathsf{K}_\Box + (D_\Box) + (4_\Box) + (5_\Box) \end{array}$$

Table 2.2: Some normal (epistemic/doxastic) modal logics

While the systems $\mathsf{S4}_K, \mathsf{S4.2}_K, \mathsf{S4.3}_K$ and $\mathsf{S5}_K$ are considered to be logics for knowledge of different strength, much work on the formal representation of belief takes the logical principles of $\mathsf{KD45}_B$ for granted (see, e.g., Baltag et al. (2008); van Ditmarsch et al. (2007); Baltag and Smets (2008)). Hintikka (1962) considered $\mathsf{S4}_K$ to be the logic of knowledge, $\mathsf{S4.2}_K$ is defended by Lenzen (1978) and Stalnaker (2006). Van der Hoek (1993); Baltag and Smets (2008) studied $\mathsf{S4.3}_K$ as epistemic logics for agents of stronger reasoning power. While the system $\mathsf{S5}_K$ is used in applications of logic in computer science (Fagin et al., 1995; Meyer and van der Hoek, 1995; van Ditmarsch et al., 2007), it is, as a logic of knowledge, often deemed to be too strong and rejected by philosophers (see, e.g., Hintikka, 1962; Voorbraak, 1993, for arguments against $\mathsf{S5}_K$). In this thesis, we examine each of the above systems in different topological frameworks. In the following, we first present their standard relational semantics.

Before moving on to the standard relational semantics for the basic modal logic, we briefly recall the following standard terminology for Hilbert-style axiom systems, and set some notation. Given a logic L defined by a (finitary)[1] Hilbert-style axiom system, an L-*derivation/proof* is a finite sequence of formulas such that each element of the sequence is either an axiom of L, or obtained from the previous formulas in the sequence by one of the inference rules. A formula φ is called L-*provable*, or, equivalently, a *theorem* of L, if it is the last formula of some L-proof. In this case, we write $\vdash_{\mathsf{L}} \varphi$ (or, equivalently, $\varphi \in \mathsf{L}$). For any set of formulas Γ and any formula φ, we write $\Gamma \vdash_{\mathsf{L}} \varphi$ if there exist finitely many formulas $\varphi_1, \dots, \varphi_n \in \Gamma$ such that $\vdash_{\mathsf{L}} \varphi_1 \wedge \dots \wedge \varphi_n \to \varphi$. We say that Γ is L-*consistent* if $\Gamma \nvdash_{\mathsf{L}} \bot$, and L-*inconsistent* otherwise. A formula φ is *consistent with* Γ if $\Gamma \cup \{\varphi\}$ is L-consistent (or, equivalently, if $\Gamma \nvdash_{\mathsf{L}} \neg\varphi$). Finally, a set of formulas Γ is *maximally consistent* if it is L-consistent and any set of formulas properly containing Γ is L-inconsistent, i.e. Γ cannot be extended to another L-consistent set. We drop mention of the logic L when it is clear from the context.

2.1.2. Definition. [Relational Frame/Model] A *relational frame* $\mathcal{F} = (X, R)$ is a pair where X is a nonempty set and $R \subseteq X \times X$. A *relational model* $\mathcal{M} = (X, R, V)$ is a tuple where (X, R) is a relational frame and $V : \textsc{prop} \to \mathcal{P}(X)$ is a valuation map.

Relational frames/models are also called *Kripke frames/models*. Throughout this thesis, we use these names interchangeably. We say $\mathcal{M} = (X, R, V)$ is a relational model *based on the frame* $\mathcal{F} = (X, R)$. While elements of X are called *states* or *possible worlds*, one of which represents the actual state of affairs, called the *actual* or *real* state, R is known as the *accessibility* or *indistinguishability relation*. We let $R(x) = \{y \in X \mid xRy\}$. The set $R(x)$ represents the set of states that the agent considers possible at x. This way, roughly speaking, a relational structure models the agent's *uncertainty* about the actual situation via the truth conditions given in the following definition.

2.1.3. Definition. [Relational Semantics for $\mathcal{L}_\Box$] Given a relational model $\mathcal{M} = (X, R, V)$ and a state $x \in X$, *truth* of a formula in the language $\mathcal{L}_\Box$ is defined recursively as follows:

$$\begin{array}{lll} \mathcal{M}, x \models p & \text{iff} & x \in V(p), \text{ where } p \in \textsc{prop} \\ \mathcal{M}, x \models \neg\varphi & \text{iff} & \text{not } \mathcal{M}, x \models \varphi \\ \mathcal{M}, x \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, x \models \varphi \text{ and } \mathcal{M}, x \models \psi \\ \mathcal{M}, x \models \Box\varphi & \text{iff} & \text{for all } y \in X, \text{ if } xRy \text{ then } \mathcal{M}, y \models \varphi. \end{array}$$

[1] In Chapter 8, we work with a proof system with an infinitary inference rule. The notion of derivation for this infinitary logic, and other relevant notions, will be explained in Chapter 8.

It follows from the above definition that

$$\mathcal{M}, x \models \Diamond\varphi \quad \text{iff} \quad \text{there is } y \in X \text{ such that } xRy \text{ and } \mathcal{M}, y \models \varphi.$$

We adopt the standard notational conventions and abbreviations (see e.g., Blackburn et al., 2001, Chapter 1.3). If $\mathcal{M}$ does not make φ true at x, we write $\mathcal{M}, x \not\models \varphi$. In this case, we say that φ is *false* at x in $\mathcal{M}$. When the corresponding model is clear from the context, we write $x \models \varphi$ for $\mathcal{M}, x \models \varphi$.

We call a formula φ *valid in a relational model* $\mathcal{M} = (X, R, V)$, denoted by $\mathcal{M} \models \varphi$, if $\mathcal{M}, x \models \varphi$ for all $x \in X$, and it is *valid in a relational frame* $\mathcal{F} = (X, R)$, denoted by $\mathcal{F} \models \varphi$, if $\mathcal{M} \models \varphi$ for every relational model based on $\mathcal{F}$. Moreover, we say φ is *valid in a class* $\mathcal{K}$ *of relational frames*, denoted by $\mathcal{K} \models \varphi$, if $\mathcal{F} \models \varphi$ for every member of this class, and it is *valid*, denoted by $\models \varphi$, if it is valid in the class of all frames. These definitions can easily be extended to sets of formulas in the following way: a set $\Gamma \subseteq \mathcal{L}_\Box$ is *valid in a relational frame* $\mathcal{F}$ iff $\mathcal{F} \models \varphi$ for all $\varphi \in \Gamma$. We define $\|\varphi\|^{\mathcal{M}} = \{x \in X \mid \mathcal{M}, x \models \varphi\}$ and call $\|\varphi\|^{\mathcal{M}}$ the *truth set*, or equivalently, *extension* of φ in $\mathcal{M}$. In particular, we write $x \in \|\varphi\|^{\mathcal{M}}$ for $\mathcal{M}, x \models \varphi$. We omit the superscript $\mathcal{M}$ when the model is clear from the context. The crucial concepts of *soundness* and *completeness* that link the syntax and the semantics are defined standardly (see, e.g., Blackburn et al., 2001, Chapter 4.1).

We conclude the section by listing the relational soundness and completeness results for the important epistemic and doxastic logics defined in Table 2.2. To do so, we first list in Table 2.3 some important frame conditions, and then define some useful order theoretic notions that will also be used in later chapters.

Reflexivity	$(\forall x)(xRx)$
Transitivity	$(\forall x, y, z)(xRy \land yRz \to xRz)$
Symmetry	$(\forall x, y)(xRy \to yRx)$
Antisymmetry	$(\forall x, y)(xRy \land yRx \to x = y)$
Seriality	$(\forall x)(\exists y)(xRy)$
Euclideanness	$(\forall x, y, z)(xRy \land xRz \to yRz)$
Directedness	$(\forall x, y, z)((xRy \land xRz) \to (\exists w)(yRw \land zRw))$
No right branching	$(\forall x, y, z)((xRy \land xRz) \to (yRz \lor zRy \lor y = z))$
Total (Connected)	$(\forall x, y)(xRy \lor yRx)$
Preorder	reflexive and transitive
Partial order	reflexive, transitive and antisymmetric
Equivalence relation	reflexive, transitive and symmetric

Table 2.3: Relevant Frame Conditions

Following the traditional conventions in order theory, we also call a reflexive and transitive relational frame (X, R) a *preordered set*; and a reflexive, transitive and antisymmetric frame a *partially ordered set*, or, in short, a *poset*. The following order theoretic notions will be useful in later chapters.

2.1.4. DEFINITION. [Up/Down-set,Upward/Downward-closure] Given a preordered set (X, R) and a subset $A \subseteq X$,

- A is called an *upward-closed set* (or, in short, an *up-set*) of (X, R) if for each $x, y \in X$, xRy and $x \in A$ imply $y \in A$;
- A is called a *downward-closed set* (or, in short, a *down-set*) of (X, R) if for each $x, y \in X$, yRx and $x \in A$ imply $y \in A$;
- the *upward-closure* of A, denoted by $\uparrow A$, is the smallest up-set of (X, R) that includes A. In other words, $\uparrow A = \{y \in X \mid \exists x \in A \text{ with } xRy\}$;
- the *downward-closure* of A, denoted by $\downarrow A$, is the smallest down-set of (X, R) that includes A. In other words, $\downarrow A = \{x \in X \mid \exists y \in A \text{ with } xRy\}$.

For every element $x \in X$, we simply write $\uparrow x$ and $\downarrow x$ for the upward and downward-closure of the singleton $\{x\}$, respectively.

We can now state some of the well-known relational soundness and completeness results. For a more detailed discussion, we refer to (Chagrov and Zakharyaschev, 1997; Blackburn et al., 2001).

2.1.5. THEOREM (RELATIONAL (KRIPKE) COMPLETENESS).

- $\mathsf{S4}_\Box$ *is sound and complete with respect to the class of preordered sets;*
- $\mathsf{S4.2}_\Box$ *is sound and complete with respect to the class of directed preordered sets;*
- $\mathsf{S4.3}_\Box$ *is sound and complete with respect to the class of total preordered sets;*
- $\mathsf{S5}_\Box$ *is sound and complete with respect to the class of frames with equivalence relations;*
- $\mathsf{KD45}_\Box$ *is sound and complete with respect to the class of serial, transitive and Euclidean frames.*

Following Theorem 2.1.5, we sometimes refer to a class of relational frames/ models by the name of its corresponding logic. For example, a preordered set is also called an $\mathsf{S4}$-frame. Similarly, a relational model based on a serial, transitive and Euclidean frame is also called a $\mathsf{KD45}$-model, etc.

2.2 Background on Topology

In this section, we introduce the topological concepts that will be used throughout this thesis. We refer to (Dugundji, 1965; Engelking, 1989) for a thorough introduction to topology.

2.2.1. Definition. [Topological Space] A *topological space* is a pair (X, τ), where X is a nonempty set and τ is a family of subsets of X such that

- $X, \emptyset \in \tau$, and
- τ is closed under finite intersections and arbitrary unions.

The set X is a *space*; the family τ is called a *topology* on X. The elements of τ are called *open sets* (or *opens*) in the space. If for some $x \in X$ and an open $U \subseteq X$ we have $x \in U$, we say that U is an *open neighborhood* of x. A set $C \subseteq X$ is called a *closed set* if it is the complement of an open set, i.e., it is of the form $X \backslash U$ for some $U \in \tau$. We let $\bar{\tau} = \{X \backslash U \mid U \in \tau\}$ denote the family of all closed sets of (X, τ). Moreover a set $A \subseteq X$ is called *clopen* if it is both closed and open.

A point x is called an *interior point* of a set $A \subseteq X$ if there is an open neighbourhood U of x such that $U \subseteq A$. The set of all interior points of A is called the *interior* of A and is denoted by $Int(A)$. Then, for any $A \subseteq X$, $Int(A)$ is an open set and is indeed the largest open subset of A, that is

$$Int(A) = \bigcup \{U \in \tau \mid U \subseteq A\}.$$

Dually, for any $x \in X$, x belongs to the *closure* of A, denoted by $Cl(A)$, if and only if $U \cap A \neq \emptyset$ for each open neighborhood U of x. It is not hard to see that $Cl(A)$ is the smallest closed set containing A, that is

$$Cl(A) = \bigcap \{C \in \bar{\tau} \mid A \subseteq C\},$$

and that $Cl(A) = X \backslash Int(X \backslash A)$ for all $A \subseteq X$. It is well known that the interior Int and the closure Cl operators of a topological space (X, τ) satisfy the following properties (the so-called Kuratowski axioms) for any $A, B \subseteq X$ (see, e.g., Engelking, 1989, pp. 14-15)[2]:

(I1) $Int(X) = X$	(C1) $Cl(\emptyset) = \emptyset$
(I2) $Int(A) \subseteq A$	(C2) $A \subseteq Cl(A)$
(I3) $Int(A \cap B) = Int(A) \cap Int(B)$	(C3) $Cl(A \cup B) = Cl(A) \cup Cl(B)$
(I4) $Int(Int(A)) = Int(A)$	(C4) $Cl(Cl(A)) = Cl(A)$

[2]The properties (I1) – (I4) (and, dually, (C1) – (C4)) are what render the knowledge modality interpreted as the topological interior operator an $\mathsf{S4}$-type modality. We will elaborate on this in Chapter 3.

A set $A \subseteq X$ is called *dense* in X if $Cl(A) = X$ and it is called *nowhere dense* if $Int(Cl(A)) = \emptyset$. Moreover, the *boundary* of a set $A \subseteq X$, denoted by $Bd(A)$, is defined as $Bd(A) = Cl(A) \backslash Int(A)$.

A point $x \in X$ is called a *limit point* (or *accumulation point*) of a set $A \subseteq X$ if for each open neighborhood U of x, we have $A \cap (U \backslash \{x\}) \neq \emptyset$. The set of all limit points of A is called the *derived set* of A and is denoted by $d(A)$. For any $A \subseteq X$, we also let $t(A) = X \backslash d(X \backslash A)$. We call $t(A)$ the *co-derived set* of A. Moreover, a set $A \subseteq X$ is called *dense-in-itself* if $A \subseteq d(A)$. A space X is called *dense-in-itself* if $X = d(X)$.

2.2.2. Definition. [Topological Basis] A family $\mathcal{B} \subseteq \tau$ is called a *basis* for a topological space (X, τ) if every non-empty open subset of X can be written as a union of elements of $\mathcal{B}$.

We call the elements of $\mathcal{B}$ *basic opens*. We can give an equivalent definition of an interior point by referring only to a basis $\mathcal{B}$ for a topological space (X, τ): for any $A \subseteq X$, $x \in Int(A)$ if and only if there is an open set $U \in \mathcal{B}$ such that $x \in U$ and $U \subseteq A$.

Given any family $\Sigma = \{A_\alpha \mid \alpha \in I\}$ of subsets of X, there exists a unique, smallest topology $\tau(\Sigma)$ with $\Sigma \subseteq \tau(\Sigma)$ (Dugundji, 1965, Theorem 3.1, page 65). The family $\tau(\Sigma)$ consists of $\emptyset$, X, all finite intersections of the A_α, and all arbitrary unions of these finite intersections. Σ is called a *subbasis* for $\tau(\Sigma)$, and $\tau(\Sigma)$ is said to be *generated* by Σ. The set of finite intersections of members of Σ forms a basis for $\tau(\Sigma)$.

2.2.3. Definition. [Subspace] Given a topological space (X, τ) and a nonempty subset $P \subseteq X$, the topological space (P, τ_P) is called a *subspace* of (X, τ) (*induced by* P) where $\tau_P = \{U \cap P \mid U \in \tau\}$.

The closure Cl_P, the interior Int_P and the derived set d_P operators of the subspace (P, τ_P) can be defined in terms of the closure and interior operators of (X, τ) as, for all $A \subseteq P$,

$$
\begin{aligned}
Cl_P(A) &= Cl(A) \cap P \\
Int_P(A) &= Int((X \backslash P) \cup A) \cap P \\
d_P(A) &= d(A) \cap P.
\end{aligned}
$$

2.2.4. Definition. [Hereditary Property] A property of a topological space is called *hereditary* if each subspace of the space possesses this property.

2.2.5. Lemma. *For any two topological space (X, τ) and (X, τ'), if $\tau \subseteq \tau'$ then $Int_\tau(A) \subseteq Int_{\tau'}(A)$ for all $A \subseteq X$.*

We here end the presentation of the background material for this dissertation. In the next chapter, we introduce the interior-based topological semantics for the basic modal language and motivate the use of topological models in epistemic logic.

Part I

From Interior Semantics to Evidence Models

Chapter 3

The Interior Semantics

In this chapter, we provide the formal background for the interior-based topological semantics for the basic modal logic that originates from the work of McKinsey (1941), and McKinsey and Tarski (1944). In this semantics the modal operator $\Box$ is interpreted on topological spaces as the interior operator. As briefly discussed in Chapter 1, among other reasons, the fact that the epistemic system $\mathsf{S4}$ is the logic of all topological spaces, and the interpretation of open sets as 'observable properties' or 'pieces of evidence' put the interior-based topological semantics on the radar of epistemic logicians.

In the following, we briefly introduce the so-called topological interior semantics, focusing particularly on its epistemic insights, and explain how and why it constitutes a satisfactory interpretation for (evidence-based) knowledge, and, consequently, why—in certain contexts—it forms a richer semantics than the relational semantics. Our contribution in Part I is inspired by and developed on the basis of this setting. In later chapters, we extend and enrich the interior semantics in order to formalize different notions of (evidence-based) knowledge and justified belief, as well as various notions of evidence possession.

Outline

Section 3.1 is a technical section introducing the interior semantics together with its connection to the relational semantics (Section 3.1.2). In Section 3.1.3, we list the general topological soundness and completeness results for the systems $\mathsf{S4}, \mathsf{S4.2}$ and $\mathsf{S4.3}$ that will be used in later chapters. Section 3.2 then explains the motivation behind the use of the interior operator as a knowledge modality, where the main focus will be on the underlying evidence-based interpretation.

3.1 Background on the Interior Semantics

This section gives an overview of the essential technical preliminaries of the interior semantics. The presentation of this section follows (van Benthem and Bezhanishvili, 2007, Section 2). The reader who is familiar with the source and topic should feel free to continue with Section 3.2.

3.1.1 Syntax and Semantics

We work with the basic epistemic language $\mathcal{L}_K$ as given in Definition 2.1.1. Since we examine the interior semantics in an epistemic context, we prefer to use the modality $K\varphi$ (instead of $\Box\varphi$) that is read as "the agent knows φ (is true)". The dual modality $\hat{K}$ for *epistemic possibility* is defined as $\hat{K}\varphi := \neg K \neg \varphi$.

3.1.1. Definition. [Topological Model] A *topological model* (or, in short, a *topo-model*) $\mathcal{X} = (X, \tau, V)$ is a triple, where (X, τ) is a topological space and $V : \text{PROP} \to \mathcal{P}(X)$ is a valuation function.

3.1.2. Definition. [Interior Semantics for $\mathcal{L}_K$] Given a topo-model $\mathcal{X}=(X, \tau, V)$ and a state $x \in X$, *truth* of a formula in the langauge $\mathcal{L}_K$ is defined recursively as follows:

$$\begin{array}{lll}
\mathcal{X}, x \models p & \text{iff} & x \in V(p) \\
\mathcal{X}, x \models \neg\varphi & \text{iff} & \text{not } \mathcal{X}, x \models \varphi \\
\mathcal{X}, x \models \varphi \wedge \psi & \text{iff} & \mathcal{X}, x \models \varphi \text{ and } \mathcal{X}, x \models \psi \\
\mathcal{X}, x \models K\varphi & \text{iff} & (\exists U \in \tau)(x \in U \text{ and } \forall y \in U,\ \mathcal{X}, y \models \varphi)
\end{array}$$

It is useful to note the derived semantics for $\hat{K}\varphi$:

$$\mathcal{X}, x \models \hat{K}\varphi \quad \text{iff} \quad (\forall U \in \tau)(x \in U \text{ implies } \exists y \in U,\ \mathcal{M}, y \models \varphi)$$

Truth and *validity* of a formula φ of $\mathcal{L}_K$ are defined in the same way as for the relational semantics. We here apply similar notational conventions as we have set in Section 2.1. We let $[\![\varphi]\!]^{\mathcal{X}} = \{x \in X \mid \mathcal{X}, x \models \varphi\}$ denote the *truth set*, or equivalently, *extension* of a formula φ in topo-model $\mathcal{X}$. We emphasize the difference between $||\varphi||^{\mathcal{M}}$ and $[\![\varphi]\!]^{\mathcal{X}}$: while the former refers to the truth set in a relational model under the standard relational semantics (Definition 2.1.3), the latter is defined with respect to topo-models and the interior semantics (Definition 3.1.2). We again omit the superscript for the model when it is clear from the context.

The semantic clauses for K and $\hat{K}$ give us exactly the interior and the closure operators of the corresponding model. In other words, according to the interior semantics, we have

$$\begin{array}{lll}
[\![K\varphi]\!] & = & \mathit{Int}([\![\varphi]\!]) \\
[\![\hat{K}\varphi]\!] & = & \mathit{Cl}([\![\varphi]\!]).
\end{array}$$

3.1.2 Connection between relational and topological models

As is well known, there is a tight link between the relational semantics and the interior semantics at the level of reflexive and transitive frames: every reflexive and transitive Kripke frame corresponds to an *Alexandroff space*. The class of reflexive and transitive frames therefore forms a subclass of all topological spaces. This connection does not only help us to see how the interior semantics and the relational semantics relate to each other and how the former extends the latter, but it also provides a method to prove topological completeness results by using the already established results for the relational counterpart.

3.1.3. Definition. [Alexandroff space] A topological space (X, τ) is an *Alexandroff space* if τ is closed under arbitrary intersections, i.e., $\bigcap \mathcal{A} \in \tau$ for any $\mathcal{A} \subseteq \tau$.

A topo-model $\mathcal{X} = (X, \tau, V)$ is called an *Alexandroff model* if (X, τ) is an Alexandroff space. A very important feature of an Alexandroff space (X, τ) is that every point $x \in X$ has a smallest open neighbourhood. Given a reflexive and transitive Kripke frame (X, R), we can construct an Alexandroff space (X, τ_R) by defining τ_R to be the set of all up-sets of (X, R). The up-set $R(x) = {\uparrow}x = \{y \in X \mid xRy\}$ forms the smallest open neighborhood containing the point x. It is then not hard to see that the set of all down-sets of (X, R) coincides with the set of all closed sets in (X, τ_R), and that for any $A \subseteq X$, we have $Cl_{\tau_R}(A) = {\downarrow}A$, where Cl_{τ_R} denotes the closure operator of (X, τ_R). Conversely, for every topological space (X, τ), we define a *specialization preorder* $\sqsubseteq_\tau$ on X by

$$x \sqsubseteq_\tau y \text{ iff } x \in Cl(\{y\}) \text{ iff } (\forall U \in \tau)(x \in U \text{ implies } y \in U).$$

$(X, \sqsubseteq_\tau)$ is therefore a reflexive and transitive Kripke frame, i.e., a preordered set. Moreover, we have that $R = \sqsubseteq_{\tau_R}$, and that $\tau = \tau_{\sqsubseteq_\tau}$ if and only if (X, τ) is Alexandroff (see, e.g., van Benthem and Bezhanishvili, 2007). Hence, there is a natural one-to-one correspondence between reflexive and transitive Kripke models and Alexandroff models. In particular, for any reflexive and transitive Kripke model $\mathcal{M} = (X, R, V)$, we set $B(\mathcal{M}) = (X, \tau_R, V)$, and for any Alexandroff model $\mathcal{X} = (X, \tau, V)$, we can form a reflexive and transitive Kripke model $A(\mathcal{X}) = (X, \sqsubseteq_\tau, V)$. Moreover, any two models that correspond to each other in the above mentioned way make the same formulas of $\mathcal{L}_K$ true at the same states, as shown in Proposition 3.1.4.

3.1.4. Proposition. *For all $\varphi \in \mathcal{L}_K$,*

1. *for any reflexive and transitive Kripke model $\mathcal{M} = (X, R, V)$ and $x \in X$,*

$$\mathcal{M}, x \models \varphi \textit{ iff } B(\mathcal{M}), x \models \varphi;$$

2. for any Alexandroff model $\mathcal{X} = (X, \tau, V)$ and $x \in X$,

$$\mathcal{X}, x \models \varphi \text{ iff } A(\mathcal{X}), x \models \varphi.$$

Therefore, reflexive and transitive Kripke models and Alexandroff models are just different representations of each other with respect to the language $\mathcal{L}_K$. In particular, the modal equivalence stated in Proposition 3.1.4-(1) constitutes the key step that allows us to use the relational completeness results to prove completeness with respect to the interior semantics.

3.1.3 Soundness and Completeness for $\mathsf{S4}_K$, $\mathsf{S4.2}_K$ and $\mathsf{S4.3}_K$

Having explained the connection between reflexive-transitive Kripke models and Alexandroff models, we can now state the topological completeness results for $\mathsf{S4}_K$ and its two normal extensions $\mathsf{S4.2}_K$ and $\mathsf{S4.3}_K$ that are of interests in later chapters. In fact, Proposition 3.1.4-(1) entails the following more general result regarding all Kripke complete normal extensions of $\mathsf{S4}_K$.

3.1.5. Proposition (van Benthem and Bezhanishvili, 2007). *Every normal extension of $\mathsf{S4}_K$ (over the language $\mathcal{L}_K$) that is complete with respect to the standard relational semantics is also complete with respect to the interior semantics.*

Proof:
Let L_K be a normal extension of $\mathsf{S4}_K$ that is complete with respect to the relational semantics and $\varphi \in \mathcal{L}_K$ such that $\varphi \notin \mathsf{L}_K$. Then, by relational completeness of L_K, there exists a relational model $\mathcal{M} = (X, R, V)$ and $x \in X$ such that $\mathcal{M}, x \not\models \varphi$. Since L_K extends the system $\mathsf{S4}_K$, which is complete with respect to reflexive and transitive Kripke models, R can be assumed to be at least reflexive and transitive. Then, by Proposition 3.1.4-(1), we obtain $B(\mathcal{M}), x \not\models \varphi$. □

We can therefore prove completeness of the Kripke complete extensions of $\mathsf{S4}_K$ with respect to the interior semantics via their relational completeness. What makes the interior semantics more general than Kripke semantics is tied to soundness. For example, $\mathsf{S4}_K$ is not only sound with respect to Alexandroff spaces, but also with respect to all topological spaces.

3.1.6. Theorem (McKinsey and Tarski, 1944). *$\mathsf{S4}_K$ is sound and complete with respect to the class of all topological spaces under the interior semantics.*

Similar results have also been proven for $\mathsf{S4.2}_K$ and $\mathsf{S4.3}_K$ for the following restricted classes of topological spaces.

3.1.7. Definition. [Extremally Disconnected Space] A topological space (X, τ) is called *extremally disconnected* if the closure of each open subset of X is open.

For example, Alexandroff spaces constructed from directed preorders, i.e., from $\mathsf{S4.2}_K$-frames, are extremally disconnected. To elaborate, it is routine to verify that, given a directed preordered set (X, R) and an up-set U of (X, R), the downward-closure $\downarrow U$ of the set U is still an up-set. Recall that $Cl_{\tau_R}(U) = \downarrow U$, where (X, τ_R) is the corresponding Alexandroff space and Cl_{τ_R} is its closure operator. Therefore, since the set of all up-sets of (X, R) forms the corresponding Alexandroff topology τ_R, we conclude that (X, τ_R) is extremally disconnected. This, in fact, establishes the topological completeness result for $\mathsf{S4.2}_K$ via Proposition 3.1.5. It is also well known that topological spaces that are Stone-dual to complete Boolean algebras, e.g., the Stone-Čech compactification $\beta(\mathbb{N})$ of the set of natural numbers with a discrete topology, are extremally disconnected (Sikorski, 1964).

3.1.8. Definition. [Hereditarily Extremally Disconnected Space] A topological space (X, τ) is called *hereditarily extremally disconnected* (*h.e.d.*) if every subspace of (X, τ) is extremally disconnected.

Alexandroff spaces corresponding to total preorders, i.e., corresponding to $\mathsf{S4.3}_K$-frames, are hereditarily extremally disconnected. To see this, observe that for every nonempty $Y \subseteq X$, the subspace $(Y, (\tau_R)_Y)$ of (X, τ_R) is in fact the Alexandroff space constructed from the subframe $(Y, R \cap (Y \times Y))$ of (X, R). Moreover, every subframe of a total preorder (X, R) is still a total preorder, thus, is also a directed preorder. Therefore, the correspondence between total preorders and h.e.d spaces follows from the fact that Alexandroff spaces constructed from directed preorders are extremally disconnected. Another interesting and non-Alexandroff example of an hereditarily extremally disconnected space is the topological space $(\mathbb{N}, \tau)$ where $\mathbb{N}$ is the set of natural numbers and $\tau = \{\emptyset, \text{ all cofinite subsets of } \mathbb{N}\}$. In this space, the set of all finite subsets of $\mathbb{N}$ together with $\emptyset$ and X completely describes the set of closed subsets with respect to $(\mathbb{N}, \tau)$. It is not hard to see that for any $U \in \tau$, $Cl(U) = \mathbb{N}$ and $Int(C) = \emptyset$ for any closed C with $C \neq X$. Moreover, every countable Hausdorff extremally disconnected space is hereditarily extremally disconnected (Blaszczyk et al., 1993). For more examples of hereditarily extremally disconnected spaces, we refer to (Blaszczyk et al., 1993).

3.1.9. Theorem (Gabelaia, 2001). *$\mathsf{S4.2}_K$ is sound and complete with respect to the class of extremally disconnected topological spaces under the interior semantics.*

3.1.10. Theorem (Bezhanishvili et al., 2015). *$\mathsf{S4.3}_K$ is sound and complete with respect to the class of hereditarily extremally disconnected topological spaces under the interior semantics.*

3.2 The Motivation behind *Knowledge as Interior*

Having presented the interior semantics, we can now elaborate on its epistemic significance that has inspired our work in this dissertation, in particular, the content of Chapter 4 and Chapter 5.

We would first like to note that the conception of knowledge as interior is not the only type of knowledge we study in this thesis. We even question whether *knowledge as interior* is the "only" type of knowledge that a topological semantics can account for and answer in the negative (see Chapters 5-7). However, the aforementioned semantics can be considered as the most primitive, in a sense as the most direct way of interpreting an epistemic modality in this setting. We therefore argue that, even in this very basic form, the interior semantics works at least as well as the standard relational semantics for knowledge, and, additionally, it extends the relational semantics while admitting an evidential interpretation of knowledge.

The interior semantics is naturally epistemic and extends the relational semantics. The initial reason as to why the topological interior operator can be considered as knowledge is inherent to the properties of this operator. As noted in Section 2.2, the Kuratowski axioms (I1)-(I4) correspond exactly to the axioms of the system $\mathsf{S4}_K$, when K is interpreted as the interior modality (see Table 3.1 for the one-to-one correspondence). Therefore, elementary topological

	$\mathsf{S4}_K$ axioms	Kuratowski axioms
(K_K)	$K(\varphi \wedge \psi) \leftrightarrow (K\varphi \wedge K\psi)$	$Int(A \cap B) = Int(A) \cap Int(B)$
(T_K)	$K\varphi \rightarrow \varphi$	$Int(A) \subseteq A$
(4_K)	$K\varphi \rightarrow KK\varphi$	$Int(A) \subseteq Int(Int(A))$
(Nec_K)	from φ, infer $K\varphi$	$Int(X) = X$

Table 3.1: $\mathsf{S4}_K$ vs. Kuratowski axioms

operators such as the interior operator, or, dually, the closure operator produces the epistemic logic $\mathsf{S4}_K$ with no need for additional constraints (also see Theorem 3.1.6). In other words, in its most general form, topologically modelled knowledge is *Factive* and *Positively Introspective*, however, it does not necessarily possess stronger properties. On the other hand, this in no way limits the usage of interior semantics for stronger epistemic systems. In accordance with the case for the relational semantics, we can restrict the class of spaces we work with and interpret stronger epistemic logics such as $\mathsf{S4.2}_K, \mathsf{S4.3}_K$ (see Theorems 3.1.9 and 3.1.10) and $\mathsf{S5}_K$ in a similar manner (see, e.g., van Benthem and Bezhanishvili, 2007,

p. 253). To that end, topological spaces provide *sufficiently flexible* structures to study knowledge of different strength. They are moreover *naturally epistemic* since the most general class of spaces, namely the class of all topological spaces, constitutes the class of models of arguably the weakest, yet philosophically the most accepted normal system $\mathsf{S4}_K$. Moreover, as explained in Section 3.1.2, the relational models for the logic $\mathsf{S4}_K$, and for its normal extensions, correspond to the subclass of Alexandroff models (see Proposition 3.1.4). The interior semantics therefore generalizes the standard relational semantics for knowledge.

One may however argue that the above reasons are more of a technical nature showing that the interior semantics works as well as the relational semantics, therefore motivate "why we *could* use topological spaces" rather than "why we *should* use topological spaces" to interpret knowledge as opposed to using relational semantics. Certainly the most important argument in favour of the conception of *knowledge as the interior operator* is of a more 'semantic' nature: the interior semantics provides a deeper insight into the evidence-based interpretation of knowledge.

Evidence as open sets. The idea of treating 'open sets as pieces of evidence' is adopted from the topological semantics for intuitionistic logic, dating back to the 1930s (see, e.g., Troelstra and van Dalen, 1988). In a topological-epistemological framework, typically, the elements of a given open basis are interpreted as observable evidence, whereas the open sets of the topology are interpreted as properties that can be verified based on the observable evidence. In fact, the connection between evidence and open sets comes to exist at the most elementary level, namely at the level of a subbasis. We can think of a subbasis as a collection of observable evidence that is *directly* obtained by an agent via, e.g., testimony, measurement, approximation, computation or experiment. The family of directly observable pieces of evidence therefore naturally forms an open topological basis: closure under finite intersection captures an agent's ability to put finitely many pieces into a single piece, i.e., her ability to derive more refined evidence from direct ones by combining finitely many of them together. Therefore, a topological space does not only account for the plain conception of *evidence as open sets*, but it is rich enough to differentiate various notions of evidence possession. The above-mentioned correspondence between evidence and open sets constitutes the main motivation behind the topological frameworks developed in this dissertation and we will elaborate on different views and interpretations of *topological evidence* in later chapters, starting with Chapter 5.

On the other hand, the basic epistemic language $\mathcal{L}_K$ interpreted by the interior semantics is clearly not expressive enough to distinguish different types of open sets, e.g., it cannot distinguish a basic open from an arbitrary open, simply because the only topological modality K is interpreted as an existential claim of an open neighbourhood of the actual state that entails the known proposition:

$$x \in KP \text{ iff } x \in Int(P) \tag{3.1}$$
$$\text{iff } (\exists U \in \tau)(x \in U \text{ and } U \subseteq P) \tag{3.2}$$
$$\text{iff } (\exists U \in \mathcal{B}_\tau)(x \in U \text{ and } U \subseteq P) \tag{3.3}$$

where $\mathcal{B}_\tau$ is a basis for τ. Therefore, in its current form, the interior semantics does not form a sufficiently strong setting to account for (various type of) evidence possession alone. However, even based on this basic shape, the notion of knowledge as the interior operator yields an evidential interpretation at a purely semantic level. More precisely, from an extensional point of view[1], a proposition P is true at world x if $x \in P$. If an open U is included in a set P, then we can say that proposition P is *entailed/supported* by evidence U. Open neighbourhoods U of the actual world x play the role of sound (correct, truthful) evidence. Therefore, as basic open sets are the pieces of observable evidence, (3.3) means that the actual world x is in the interior of P iff there exists a sound piece of evidence U that supports P. That is, according to the interior semantics, the agent *knows* P at x iff she has a sound/correct piece of evidence supporting P. Moreover, open sets will then correspond to properties that are in principle verifiable by the agent: *whenever they are true, they are supported by a sound piece of evidence, therefore, can be known.* Dually, we have

$$x \notin Cl(P) \text{ iff } (\exists U \in \tau)(x \in U \text{ and } U \subseteq X \backslash P) \tag{3.4}$$

meaning that closed sets correspond to falsifiable properties: *whenever they are false, they are falsified by a sound piece of evidence.* These ideas have also been used and developed in (Vickers, 1989; Kelly, 1996) with connections to epistemology, logic and learning theory.

The interior-based semantics for knowledge has been extended to multiple agents (van Benthem et al., 2005), to common knowledge (Barwise, 1988; van Benthem and Sarenac, 2004) to logics of learning and observational *effort* (Moss and Parikh, 1992; Dabrowski et al., 1996; Georgatos, 1993, 1994), to topological versions of dynamic-epistemic logic (Zvesper, 2010) (see Aiello et al., 2007, for a comprehensive overview on the field). Belief on topological spaces, rather surprisingly, has not been investigated and developed as much as knowledge, especially in connection with topological knowledge.

3.3 Belief on Topological Spaces?

As explained in Section 3.2, as far as an evidential interpretation of knowledge is concerned, the interior semantics improves the standard relational semantics,

[1] *Extensional* here means any semantic formalism that assigns the same meaning to sentences having the same extension.

most importantly, for the reason that *evidential justification* for knowing something is embedded in the semantics. It then seems natural to ask whether a topological semantics can also account for notions of (*evidentially*) *justified belief*. Answering this question constitutes one of the main goals of Part I of this dissertation.

One of the crucial properties that distinguishes knowledge from belief is its veracity (formalised by the axiom (T_K)). However, no matter how idealized and rational the agent is, it must be possible for her to believe false propositions, yet she is expected to hold consistent beliefs (formalized by the axiom (D_B)). To the best of our knowledge, the first worked out topological semantics for belief is proposed by Steinsvold (2006) in terms of the co-derived set operator. According to the co-derived set interpretation of belief,

$$x \in BP \text{ iff } (\exists U \in \tau)(x \in U \text{ and } U \backslash \{x\} \subseteq P), \tag{3.5}$$

i.e., $x \in BP$ iff $x \in t(P)$. We here note that this topological semantics interpreting the modal operator $\Box$ as the co-derived set operator, or dually, $\Diamond$ as the derived set operator was also pioneered by McKinsey and Tarski (1944), and later extensively developed by the Georgian logic school led and inspired by Esakia, and their collaborators (see, e.g., Esakia, 2001, 2004; Bezhanishvili et al., 2005, 2009; Beklemishev and Gabelaia, 2014; Kudinov and Shehtman, 2014). Steinsvold (2006) was the first to propose to use this semantics to interpret belief, and proved soundness and completeness for the standard belief system $\mathsf{KD45}_B$. This account still requires having a *truthful* piece of evidence for the believed proposition, however, the proposition itself does not have to be true. Therefore, it is guaranteed that the agent may hold false beliefs. However, as also discussed in (Baltag et al., 2013; Özgün, 2013), and briefly recapped here, this semantics further guarantees that *in any topo-model and any state in this model, there is at least one false belief*, that is, the agent always believes the false proposition $X \backslash \{x\}$ at the actual state x. This is the case because for any topological space (X, τ) and $x \in X$, we have $x \notin d(\{x\})$, i.e., $x \in t(X \backslash \{x\})$, therefore, the clause (3.5) entails that $x \in B(X \backslash \{x\})$ always holds. This is an undesirable and disadvantageous property, especially if we also want to study dynamics such as belief revision, updates or learning. Always believing $X \backslash \{x\}$ prevents the agent to ever learning the actual state unless she believes everything. Formally speaking, $x \in B(\{x\})$ iff the singleton $\{x\}$ is an open, and in this case, the agent believes everything at x. In order to avoid these downsides and obtain $\mathsf{KD45}_B$, we have to work with the so-call DSO-spaces, as shown by Steinsvold (2006). A *DSO-space* is defined to be a dense-in-itself space (i.e., a space with no singleton opens) in which every derived set $d(A)$ is open.

Moreover, in a setting where knowledge as the interior and belief as the co-derived set operator are studied together, we obtain the equality

$$KP = P \cap BP,$$

stating that *knowledge is true belief*. Therefore, this semantics yields a formalization of knowledge and belief that is subject to well-known Gettier counterexamples (Gettier, 1963).[2]

In the next chapter, we present another topological semantics proposed by Baltag et al. (2013) for belief, in particular, for Stalnaker's notion of belief as *subjective certainty* (Stalnaker, 2006), in terms of the *closure of the interior operator* on extremally disconnected spaces. Baltag et al. (2013) have argued that this semantics is better behaved, especially when considered together with the notion of knowledge as the interior operator. They moreover provided a soundness and completeness result for the belief system $\mathsf{KD45}_B$ with respect to the class of extremally disconnected spaces, which extends the class of DSO-spaces. However, this setting still encounters problems when extended for *public announcements*. We then propose a solution consisting in interpreting belief in a similar way based on hereditarily extremally disconnected spaces, and axiomatize the belief logic of hereditarily extremally disconnected spaces.

[2]This connection has also been observed in (Steinsvold, 2006, Section 1.11), and an alternative topological semantics for knowledge in terms of clopen sets is suggested without providing any further technical results. Steinsvold (2006) does not elaborate on to what extend his proposed semantics for knowledge could give new insight into the Gettier problem and leaves this point open for discussion.

Chapter 4

A topological theory of "justified" belief: *an initial attempt*

Understanding the relation between knowledge and belief is an issue of central importance in epistemology. Especially after Gettier (1963) shattered the traditional account of knowledge as *justified true belief*, many epistemologists have attempted to strengthen the latter to attain a satisfactory notion of the former. According to this approach, one starts with a weak notion of belief (which is at least justified and true) and tries to reach knowledge by making the chosen notion of belief stronger in such a way that the defined notion of knowledge would no longer be subject to Gettier-type counterexamples (Gettier, 1963).[1] More recently, there has also been some interest in reversing this project—deriving belief from knowledge—or, at least, putting "knowledge first" (Williamson, 2000). In this spirit, Stalnaker (2006) has proposed a formal framework in which belief is realized as a weakened form of knowledge. More precisely, beginning with a logical system in which both belief and knowledge are represented as primitives, Stalnaker formalizes some natural-seeming relationships between the two, and proves on the basis of these relationships that belief can be *defined* from knowledge. To this end, Stalnaker's syntactic formalization seems to be analogous to the aforementioned *status quo* of the interior semantics for knowledge and of a topological interpretation for belief, where the interpretation of knowledge is given and a good semantics for belief is to be unveiled.

Baltag et al. (2013) and Özgün (2013), starting from Stalnaker's formalism, proposed to interpret belief, in particular Stalnaker's belief, as *subjective certainty*, in terms of *the closure of the interior operator* on extremally disconnected spaces (Section 4.2 explains the reason for restriction to extremally disconnected

[1]Among this category, we can mention the *defeasibility analysis of knowledge* (Lehrer and Paxson, 1969; Lehrer, 1990; Klein, 1971, 1981), "*no false lemma*" account (Clark, 1963), the *sensitivity account* (Nozick, 1981), the *contextualist account* (DeRose, 2009) and the *safety account* (Sosa, 1999). For an overview of responses to the Gettier challenge and a detailed discussion, we refer the reader to (Ichikawa and Steup, 2013; Rott, 2004).

spaces). This static setting, developed based on extremally disconnected spaces, however could not be extended with updates for *public announcements* due to some structural properties of the extremally disconnected spaces (see Section 4.2.2). One way of dealing with this problem based on *all topological spaces*, leading to weakening of the underlying knowledge and belief logics, has been presented in (Baltag et al., 2015b). In this chapter, we present a solution approaching the issue from the opposite direction, namely, we propose to restrict the class of spaces we work with to the class of *hereditarily extremally disconnected spaces.*

Outline

Section 4.1 presents Stalnaker's combined system of knowledge and belief, and lists the important aspects of his work that inspired (Özgün, 2013; Baltag et al., 2013). In Section 4.2, we review the topological belief semantics of (Özgün, 2013; Baltag et al., 2013), and, Section 4.2.2 recalls why updates do not work on extremally disconnected spaces. In Section 4.3, we introduce the material that goes beyond (Özgün, 2013; Baltag et al., 2013), and model belief, conditional beliefs and public announcements on hereditarily extremally disconnected spaces and present several completeness results regarding $\mathsf{KD45}_B$ and its extensions with conditional beliefs and public announcements.

This chapter is based on (Baltag et al., 2015a).

4.1 Belief as *subjective certainty*

Stalnaker (2006) focuses on the properties of knowledge, belief and the relation between the two. He approaches the problem of understanding the precise connection between knowledge and belief from an unusual perspective by following a "knowledge-first" approach. That is, unlike most proposals in the formal epistemology literature, he starts with a chosen notion of knowledge and weakens it to obtain belief. He bases his analysis on a strong conception of belief as "subjective certainty": from the point of the agent in question, her belief is *subjectively indistinguishable from her knowledge.*

Stalnaker (2006) works with the *bimodal language* $\mathcal{L}_{KB}$ given by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid B\varphi,$$

augmenting the logic $\mathsf{S4}_K$ with the additional axioms schemes presented in Table 4.1.

Let Stal denote this combined logic.[2] Most of the above axioms, such as $\mathsf{S4}_K$,

[2] What justifies the properties of knowledge and belief stated in Stal may be debatable, though not in the scope of this dissertation. We refer to (Bjorndahl and Özgün, 2017) for a topological-based reformulation of Stalnaker's system.

(D_B)	$B\varphi \to \neg B \neg \varphi$	Consistency of belief
(sPI)	$B\varphi \to KB\varphi$	Strong positive introspection
(sNI)	$\neg B\varphi \to K\neg B\varphi$	Strong negative introspection
(KB)	$K\varphi \to B\varphi$	Knowledge implies belief
(FB)	$B\varphi \to BK\varphi$	Full belief

Table 4.1: Stalnaker's additional axiom schemes

(D_B), (KB), are widely taken for granted by many formal epistemologists (see Section 2.1 for some sources). The properties (sPI) and (sNI) state that Stalnaker's agent has full introspective access to her beliefs. Finally, (FB) constitutes the key property of *belief as subjective certainty*, the notion of belief Stalnaker seeks to capture. In his setting, *the agent fully believes φ iff she believes that she knows it.*[3] He therefore studies a strong notion of belief that is very close to knowledge.

From these first principles formalizing the interplay between knowledge and belief, Stalnaker (2006) extracts the properties regarding the unimodal fragments for knowledge and for belief, as well as a definition of belief in terms of knowledge. More precisely, he shows that

- Stal derives $\mathsf{S4.2}_K$ as the pure logic of knowledge (although only $\mathsf{S4}_K$ was initially assumed);
- Stal derives $\mathsf{KD45}_B$ as the pure logic of belief; and
- it proves the equivalence $B\varphi \leftrightarrow \hat{K}K\varphi$.

He therefore argues—based on the first principles of the system Stal—that the "true" logic of knowledge is $\mathsf{S4.2}_K$, that the "true" logic of belief is $\mathsf{KD45}_B$, and that belief is definable in terms of knowledge as the *epistemic possibility of knowledge*. As a conclusion of the last item, Stal constitutes a formalization of knowledge and belief admitting conceptual priority of belief over knowledge. Moreover, given the interior semantics for knowledge, the equation $B\varphi \leftrightarrow \hat{K}K\varphi$ yields a natural topological semantics for full belief (Baltag et al., 2013; Özgün, 2013).

4.2 The Topological Semantics for Full Belief

The topological semantics for Stal, and in particular for full belief, was first studied in (Baltag et al., 2013; Özgün, 2013). They propose to extend the interior semantics for knowledge by a semantic clause for belief, and model belief as the *closure of the interior operator* on extremally disconnected spaces. The restriction

[3]The converse direction of (FB) is easily derivable in Stal.

to the class of extremally disconnected spaces is imposed by the axioms of Stal, that is, e.g., the axiom (D_B) as well as the derived principles such as (K_B) and ($.2_K$) define extremally disconnectness when K is interpreted as the interior operators and B is interpreted as the closure of the interior operator (see Gabelaia, 2001, Theorem 1.3.3 for ($.2_K$), and Özgün, 2013, Propositions 11 and 12 for (D_B) and (K_B)). Baltag et al. (2013) provide several topological soundness and completeness results for both bimodal and unimodal cases, in particular for Stal and $\mathsf{KD45}_B$, with respect to extremally disconnected spaces. In this section we give an overview of their proposal and list some of the results. The proofs can be found in (Özgün, 2013; Baltag et al., 2015a).

4.2.1. Definition. [Closure-interior semantics for $\mathcal{L}_{KB}$] Given a topo-model $\mathcal{X} = (X, \tau, V)$, the semantics for the formulas in $\mathcal{L}_{KB}$ is defined for Boolean cases and $K\varphi$ in the same way as in Definition 3.1.2. The semantics for $B\varphi$ is given by

$$[\![B\varphi]\!] = Cl(Int([\![\varphi]\!])).$$

Truth and validity of a formula as well as soundness and completeness of logics are defined in the same way as for the interior semantics.

4.2.2. Theorem (Baltag et al., 2013). Stal *is the sound and complete logic of knowledge and belief on extremally disconnected spaces under the closure-interior semantics.*

Moreover, Stalnaker's combined logic of knowledge and belief yields the systems $\mathsf{S4.2}_K$ and $\mathsf{KD45}_B$. It has already been proven that $\mathsf{S4.2}_K$ is sound and complete with respect to the class of extremally disconnected spaces under the interior semantics (see Theorem 3.1.9). This raises the question of topological soundness and completeness for $\mathsf{KD45}_B$ under the proposed semantics for belief in terms of the *closure and the interior operator*.

4.2.3. Theorem (Baltag et al., 2013). $\mathsf{KD45}_B$ *is sound and complete with respect to the class of extremally disconnected spaces under the closure-interior semantics.*

Theorem 4.2.3 therefore shows that the logic of extremally disconnected spaces is $\mathsf{KD45}_B$ when B is interpreted as the closure of the interior operator. Besides these technical results, the closure-interior semantics of belief comes with an intrinsic philosophical and intuitive value, and certain advantages compared to the co-derived set semantics as elaborated in the next section.

4.2.1 What motivates *topological* full belief

The closure-interior semantics provides an intuitive interpretation of Stalnaker's conception of (full) belief as *subjective certainty*. It does so through the definitions of the interior and closure operators and the concepts they represent, namely, the notions of *evidence* and *closeness*. We have discussed the role of open sets as pieces of evidence, and of open neighbourhoods of the actual state as pieces of truthful evidence in Section 3.2. Moreover, it is well known that the closure operator captures a topological, qualitative notion of *closeness*: x is said to be *close to* a set $A \subseteq X$ iff $x \in Cl(A)$. Recalling the proposed topological semantics for full belief, given a topological space (X, τ) and $P \subseteq X$, we have

$$x \in BP \text{ iff } x \in Cl(Int(P)) \tag{4.1}$$

$$\text{iff } x \in Cl(K(P)) \tag{4.2}$$

$$\text{iff } (\forall U \in \tau)(x \in U \text{ implies } U \cap KP \neq \emptyset) \tag{4.3}$$

Therefore, following (4.2), *topologically*, the set of states in which the agent believes P is *very close* to the set of states in which the agent knows P. Taking open sets as evidence pieces, (4.3) moreover states that an agent (fully) believes P at a state x iff every sound piece of evidence she has at x is consistent with her knowing P, i.e., she does not have any truthful evidence that distinguishes the states in which she has belief of P from the states in which she has knowledge of P. Belief, under this semantics, therefore becomes subjectively indistinguishable from knowledge. Hence, the closure-interior semantics naturally captures the conception of belief as "subjective certainty".

Moreover, the closure-interior belief semantics improves on the co-derived set semantics for the following reasons: (1) belief as the closure of the interior operator does not face the Gettier problem, at least not in the easy way in which the co-derived set semantics does, when considered together with the conception of knowledge as interior. More precisely, knowledge as interior cannot be defined as (justified) true full belief since, in general, $Int(P) \neq Cl(Int(P)) \cap P$, i.e., $KP \neq BP \wedge P$; (2) the class of DSO-spaces with respect to which $\mathsf{KD45}_B$ is sound and complete under the co-derived set semantics is a proper subclass of the class of extremally disconnected spaces (see Özgün, 2013, Proposition 13). Therefore, the closure-interior semantics for $\mathsf{KD45}_B$ is defined on a larger class of spaces.

Additionally, Özgün (2013) and Baltag et al. (2013) have studied a topological analogue of static conditioning—capturing static belief revision—by providing a topological semantics for conditional beliefs based on extremally disconnected spaces. However, this framework encounters problems when extended to a dynamic setting by adding update modalities for public announcements, formalized as model restriction by means of subspaces.

4.2.2 Problems with updates for public announcements

The topological semantics and associated logics we have studied so far were *static*, representing the epistemic state of an agent as isolated from receiving further information. Following the methodology of Dynamic Epistemic Logic (DEL), we can also represent knowledge and belief change brought about by a piece of new information by extending the static language with *dynamic modalities*, and designing *an update mechanism* that transforms the initial model into an "updated" structure. The resulting updated model expresses what is known/believed after the chosen epistemic action has been performed (see, e.g., van Ditmarsch et al., 2007; van Benthem, 2011; Baltag and Renne, 2016, for a detailed presentation of DEL).

The first, and maybe the most well-known, epistemic action studied in the literature of DEL is the so-called *public announcements* introduced by Plaza (1989) and Gerbrandy and Groeneveld (1997). Public announcements are concerned with learning "hard" information, i.e. information that comes with an inherent warranty of veracity, e.g. because of originating from an infallibly truthful source.[4] In DEL, in a qualitative setting based on relational semantics or a plausibility order, public announcements are standardly modelled by restricting the initial model to the truth set of the new information (see, e.g., Plaza, 1989, 2007; Gerbrandy and Groeneveld, 1997; van Ditmarsch et al., 2007). Its natural topological analogue, as recognized by Zvesper (2010); Baskent (2011, 2012) (among others), is a topological update operator using the restriction of the original topology to the subspace induced by a nonempty subset P. The described update mechanism for public announcements is sometimes called *update for hard information*, or *hard update* (van Benthem, 2011). In what follows, we simply refer to it as *update*.

In order for this interpretation to be successfully implemented, the subspace induced by the new information P should possess the same structural properties as the initial topology that renders the axioms of the underlying static knowledge/belief system sound. More precisely, we demand that the subspace induced by the new information P be in the class of structures with respect to which the (static) knowledge/belief logics in question are sound and complete. However, since extremally disconnectedness is not a hereditary property, the above mentioned topological interpretation of conditioning with true, hard information cannot be implemented on extremally disconnected spaces. This is obviously analogous to the problem of implementing updates on relational models based on directed preorders (see, e.g., Balbiani et al., 2012, for a more general explanation regarding preserving frame conditions in public announcement logic). Baltag et al. (2015b) present a solution for this problem by changing the semantics for belief as the *interior of the closure of the interiors operator*, and modelling public announcements on all topological spaces. In Section 4.3 though, we confine the

[4]The "public" aspect of an announcements is relevant only in a multi-agent settings, encoding the fact that all agents receive the same information conveyed by the announcement.

topo-models to the largest subclass that preserves extremally disconnectedness under taking arbitrary subspaces, namely to the class of *hereditarily extremally disconnected* (*h.e.d.*) *spaces*. This also requires a re-evaluation of the underlying static knowledge and belief systems. Before presenting the modified setting based on h.e.d spaces, we explain the problem regarding updates on extremally disconnected spaces in a more precise manner.

Topological updates for public announcements. We now consider the language $\mathcal{L}_{KB}^{!}$ obtained by adding to the language $\mathcal{L}_{KB}$ (existential) dynamic public announcement modalities $\langle !\varphi\rangle\psi$, reading "*$\varphi$ is true and after the public announcement of φ, ψ becomes true*". The dual operator $[!\varphi]$ is defined as usual as $\neg\langle !\varphi\rangle\neg$, and $[!\varphi]\psi$ reads as *"after the public announcement of φ, ψ becomes true*".

4.2.4. Definition. [Restricted Model] Given a topo-model $\mathcal{X} = (X, \tau, V)$ and $\varphi \in \mathcal{L}_{KB}^{!}$, the topo-model $\mathcal{X}^{\varphi} = (\llbracket\varphi\rrbracket, \tau^{\varphi}, V^{\varphi})$ is called the *restricted model*, where

- $\llbracket\varphi\rrbracket = \llbracket\varphi\rrbracket^{\mathcal{X}}$,
- $\tau^{\varphi} = \{U \cap \llbracket\varphi\rrbracket \mid U \in \tau\}$, and
- $V^{\varphi}(p) = V(p) \cap \llbracket\varphi\rrbracket$, for any $p \in$ PROP.

In other words, $(\llbracket\varphi\rrbracket, \tau^{\varphi})$ is the subspace of (X, τ) induced by $\llbracket\varphi\rrbracket$. The semantics for the dynamic modalities $\langle !\varphi\rangle\psi$ is then given as

$$\llbracket\langle !\varphi\rangle\psi\rrbracket^{\mathcal{X}} = \llbracket\psi\rrbracket^{\mathcal{X}^{\varphi}}.$$

Updates in general are expected to cause changes in an agent's knowledge and belief in some propositions, however, the way she reasons about her epistemic/doxastic state, in a sense the defining properties of the type of agent we consider, should remain unaffected. This amounts to saying that any restricted model should as well make the underlying *static* knowledge and belief logics sound. In particular, as we work with rational, highly idealized normal agents that hold consistent beliefs, we demand them not to lose these properties after an update with true information. With respect to the closure-interior semantics, these requirements are satisfied if and only if the resulting structure is extremally disconnected: under the topological belief semantics, both the axiom of *Normality*

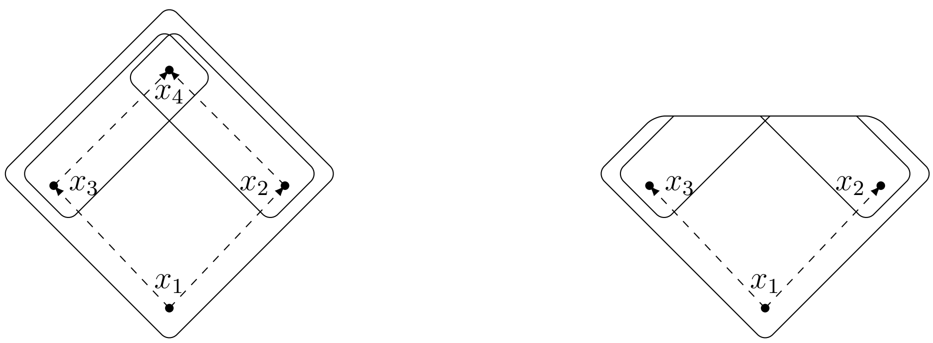
$$B(\varphi \wedge \psi) \leftrightarrow (B\varphi \wedge B\psi) \tag{K_B}$$

and the axiom of *Consistency of Belief*

$$B\varphi \rightarrow \neg B\neg\varphi \tag{D_B}$$

characterize extremally disconnected spaces (Özgün, 2013, Propositions 11 and 12). Therefore, if the restricted model is not extremally disconnected, the agent comes to have inconsistent beliefs after an update with hard true information. In order to avoid possible confusions, we note that $B\bot$ is never true with respect to the closure-interior semantics since $[\![B\bot]\!] = Cl(Int(\emptyset)) = \emptyset$. By an agent having inconsistent beliefs, we mean that she believes mutually contradictory propositions such as φ and $\neg\varphi$ at the same time, without in fact believing $B\bot$, as also illustrated by the following example.

4.2.5. Example. Consider the Alexandroff topo-model $\mathcal{X} = (X, \tau, V)$ where $X = \{x_1, x_2, x_3, x_4\}$, $\tau = \{X, \emptyset, \{x_4\}, \{x_2, x_4\}, \{x_3, x_4\}, \{x_2, x_3, x_4\}\}$ and $V(p) = \{x_1, x_2, x_3\}$ and $V(q) = \{x_2, x_4\}$ for some $p, q \in$ PROP. It is easy to see that $\mathcal{X}$ corresponds to a directed reflexive transitive relational frame as depicted in Figure 4.1a, where the reflexive and transitive arrows are omitted. It is easy to check that (X, τ) is an extremally disconnected space and $Bq \to \neg B\neg q$ is valid in $\mathcal{X}$. We stipulate that x_1 is the actual state and p is truthfully announced. The updated (i.e., restricted) model is then $\mathcal{X}^p = ([\![p]\!], \tau^p, V^p)$ where $[\![p]\!] = \{x_1, x_2, x_3\}$, $\tau^p = \{[\![p]\!], \emptyset, \{x_2\}, \{x_3\}, \{x_2, x_3\}\}$, $V^p(p) = \{x_1, x_2, x_3\}$ and $V^p(q) = \{x_2\}$. Here, $([\![p]\!], \tau^p)$ is not an extremally disconnected space (similarly, the underlying Kripke frame is not directed) since $\{x_3\}$ is an open subset of $([\![p]\!], \tau^p)$ but $Cl_p(\{x_3\}) = \{x_1, x_3\}$ is not open in $([\![p]\!], \tau^p)$. Moreover, as $x_1 \in [\![Bq]\!]^{\mathcal{X}^p} = Cl_p(Int_p(\{x_2\})) = \{x_1, x_2\}$ and $x_1 \in [\![B\neg q]\!]^{\mathcal{X}_p} = Cl_p(Int_p(\{x_1, x_3\})) = \{x_1, x_3\}$, the agent comes to believe both q and $\neg q$, implying that the restricted model falsifies (D_B) at x_1. Consequently, it also falsifies (K_B) since $[\![B(q \wedge \neg q)]\!]^{\mathcal{X}^p} = \emptyset$.



(a) (X, τ, V) (b) $([\![p]\!], \tau^p, V^p)$

Figure 4.1: Update of (X, τ, V) by p.

One possible solution for this problem is extending the class of spaces we work with: we can focus on all topological spaces instead of working with only extremally disconnected spaces and provide semantics for belief in such a way that the aforementioned problematic axioms become valid on all topological spaces. This way, we do not need to worry about any additional topological property that is supposed to be inherited by subspaces. This solution, unsurprisingly, leads to a weakening of the underlying static logic of knowledge and belief. It is well known that the knowledge logic of all topological spaces under the interior semantics is $\mathsf{S4}_K$ (Theorem 3.1.6), and the (weak) belief logic of all topological spaces is studied in (Baltag et al., 2015b). In the next section, we work out another solution which approaches the issue from the opposite direction: we further restrict our attention to *hereditarily extremally disconnected spaces*, thereby guaranteeing that no model restriction leads to inconsistent beliefs. As the logic of hereditarily extremally disconnected spaces under the interior semantics is $\mathsf{S4.3}_K$ (Theorem 3.1.10), the underlying static logic, in this case, would consist in $\mathsf{S4.3}_K$ as the logic of knowledge but again $\mathsf{KD45}_B$ as the logic of belief as shown in the next section.[5]

4.3 The Belief Logic of H.E.D Spaces

In this section, we present the underlying static logic of belief for the closure-interior semantics, and then extend this setting based on h.e.d. spaces for conditional beliefs and public announcements.

Even though we work with a more restricted class, the belief logic of h.e.d. spaces is still $\mathsf{KD45}_B$. While the soundness of this system follows from Theorem 4.2.3 since every h.e.d. space is extremally disconnected, its topological completeness will be shown by using its Kripke completeness. To this end, we first need to build a connection between $\mathsf{KD45}$-frames and h.e.d. spaces that is similar to the one presented in Section 3.1.2, and prove their modal equivalence for the language $\mathcal{L}_B$ analogous to Proposition 3.1.4-(1).

4.3.1 Connection between $\mathsf{KD45}$-frames and h.e.d. spaces

Recall that $\mathsf{KD45}$-frames are serial, transitive and Euclidean Kripke frames. Since truth of modal formulas with respect to the standard relational semantics is preserved under taking generated submodels (see, e.g., Blackburn et al., 2001, Proposition 2.6), we can use the following simplified relational structures as Kripke frames of $\mathsf{KD45}_B$.

4.3.1. Definition. [Brush/Pin]

- A relational frame (X, R) is called a *brush* if there exists a nonempty subset $\mathcal{C} \subseteq X$ such that $R = X \times \mathcal{C}$;
- A brush is called a *pin* if $|X \backslash \mathcal{C}| = 1$.

[5] The logical counterpart of the fact that extremally disconnected spaces ($\mathsf{S4.2}$-spaces) are not closed under subspaces is that $\mathsf{S4.2}$ is not a subframe logic (see Chagrov and Zakharyaschev, 1997, Section 9.4). The logical counterpart of the fact that hereditarily extremally disconnected spaces ($\mathsf{S4.3}$-spaces) are extremally disconnected spaces closed under subspaces is that the subframe closure of $\mathsf{S4.2}$ is $\mathsf{S4.3}$, (see Wolter, 1993, Section 4.7).

(a) Brush (b) Pin

Figure 4.2: An example of a brush and of a pin, where the top ellipses illustrate the final clusters and an arrow relates the state it started from to every element in the cluster.

Clearly, if such a $\mathcal{C}$ exists, it is unique; call it the *final cluster* of the brush. It is easy to see that every brush is serial, transitive and Euclidean (see Figure 4.2). For the proof of the following lemma see, e.g., (Chagrov and Zakharyaschev, 1997, Chapter 5) and (Blackburn et al., 2001, Chapters 2, 4).

4.3.2. Lemma. $\mathsf{KD45}_B$ *is a sound and complete with respect to the class of brushes, and with respect to the class of pins. In fact,* $\mathsf{KD45}_B$ *is sound and complete with respect to the class of finite pins.*

Similar to the construction in Section 3.1.2, we can build an Alexandroff h.e.d. space from a given pin. The only extra step consists in taking the reflexive closure of the initial pin. More precisely, for any frame (X, R), let R^+ denote the *reflexive closure* of R, defined as

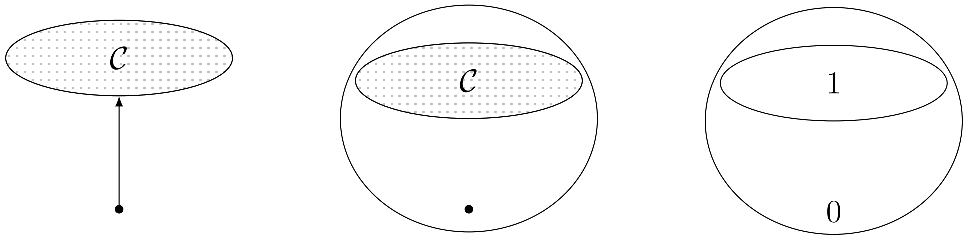
$$R^+ = R \cup \{(x, x) \mid x \in X\}.$$

Given a pin (X, R), the set $\tau_{R^+} = \{R^+(x) \mid x \in X\}$ constitutes a topology on X. In fact, in this special case of pins, we have $\tau_{R^+} = \{X, \mathcal{C}, \emptyset\}$ where $\mathcal{C}$ is the final unique cluster of (X, R). Therefore, it is easy to see that (X, τ_{R^+}) is an Alexandroff h.e.d. space. In fact, (X, τ_{R^+}) is a generalized Sierpiński space where $\mathcal{C}$ does not have to be a singleton (see Figure 4.3).

This construction leads to a natural correspondence between pins and Alexandroff h.e.d. spaces. In particular, for any Kripke model $\mathcal{M} = (X, R, V)$ based on a pin, we set $I(\mathcal{M}) = (X, \tau_{R^+}, V)$. Moreover, any two such models $\mathcal{M}$ and $I(\mathcal{M})$ make the same formulas of $\mathcal{L}_B$ true at the same states, as shown in Proposition 4.3.4.

4.3.3. Lemma. *Let* (X, R) *be a pin and* $\mathcal{C}$ *denote the final cluster of* (X, R), *and let Int and Cl denote the interior and closure operators, respectively, in the topological space* (X, τ_{R^+}). *Then for all* $x \in X$ *and every* $A \subseteq X$:

1. $R(x) = \mathcal{C} \in \tau_{R^+}$;



(a) (X, R) (b) (X, τ_{R^+}) (c) Sierpiński space

Figure 4.3: From pins to Alexandroff h.e.d. spaces

2. *$Int(A) \cap \mathcal{C} \neq \emptyset$ if and only if $A \supseteq \mathcal{C}$;*
3. *$Cl(A) = X$ if and only if $A \cap \mathcal{C} \neq \emptyset$;*
4. *if $Cl(Int(A)) \neq \emptyset$ then $Cl(Int(A)) = X$.*

Proof:
(1) follows from the fact that $R = X \times \mathcal{C}$ (Definition 4.3.1). (2) and (3) are direct consequences of the construction of τ_{R^+}, that is, $\tau_{R^+} = \{X, \mathcal{C}, \emptyset\}$. And, (4) follows from (2) and (3), since $Cl(Int(A)) \neq \emptyset$ implies that $Int(A) \neq \emptyset$. □

4.3.4. Proposition. *For all $\varphi \in \mathcal{L}_B$, any Kripke model $\mathcal{M} = (X, R, V)$ based on a pin and $x \in X$,*

$$\mathcal{M}, x \models \varphi \text{ iff } I(\mathcal{M}), x \models \varphi.$$

Proof:
The proof follows by subformula induction on φ; cases for the propositional variables and the Boolean connectives are elementary. So assume inductively that the result holds for ψ; we must show that it holds also for $\varphi := B\psi$.

$$\begin{aligned}
\mathcal{M}, x \models B\psi \text{ iff } & R(x) \subseteq ||\psi||^{\mathcal{M}} && \text{(the relational semantics of } B\text{)} \\
\text{iff } & \mathcal{C} \subseteq ||\psi||^{\mathcal{M}} && \text{(Lemma 4.3.3-1)} \\
\text{iff } & \mathcal{C} \subseteq [\![\psi]\!]^{I(\mathcal{M})} && \text{(induction hypothesis)} \\
\text{iff } & Int([\![\psi]\!]^{I(\mathcal{M})}) \cap \mathcal{C} \neq \emptyset && \text{(Lemma 4.3.3-2)} \\
\text{iff } & Cl(Int([\![\psi]\!]^{I(\mathcal{M})})) = X && \text{(Lemma 4.3.3-3)} \\
\text{iff } & x \in Cl(Int([\![\psi]\!]^{I(\mathcal{M})})) && \text{(Lemma 4.3.3-4)} \\
\text{iff } & I(\mathcal{M}), x \models B\psi && \text{(the closure-interior semantics of } B\text{)}
\end{aligned}$$

□

4.3.5. THEOREM. $\mathsf{KD45}_B$ *is sound and complete with respect to the class of hereditarily extremally disconnected spaces under the closure-interior semantics.*

Proof:
Soundness follows from Theorem 4.2.3 and the fact that every hereditarily extremally disconnected space is extremally disconnected. For completeness, let $\varphi \in \mathcal{L}_B$ such that $\varphi \notin \mathsf{KD45}_B$. Then, by Lemma 4.3.2, there exists a relational model $\mathcal{M} = (X, R, V)$, where (X, R) is a pin, and $x \in X$ such that $\mathcal{M}, x \not\models \varphi$. Therefore, by Proposition 4.3.4, we obtain $I(\mathcal{M}), x \not\models \varphi$. Since $I(\mathcal{M})$ is hereditarily extremally disconnceted, we obtain the desired result. $\square$

Theorem 4.3.5 therefore shows that the (belief) logic of h.e.d. spaces is also $\mathsf{KD45}_B$. The class of h.e.d. spaces of course restricts the class of extremally disconnected spaces, however, it is still a larger class than the class of DSO-spaces.

4.3.6. PROPOSITION. *Every DSO-space is hereditarily extremally disconnected, however, not every h.e.d. space is a DSO-space.*

Proof:
Recall that a DSO-space is a dense-in-itself topological space (i.e., a space with no singleton opens) in which every derived set $d(A)$ is open. Let (X, τ) be a DSO-space and (P, τ_P) its subspace induced by the nonempty set $P \subseteq X$. Observe that, for all $A \subseteq P$, we have $d_P(A) \in \tau_P$ since $d(A) \in \tau$ and $d_P(A) = d(A) \cap P$. Now suppose $U \in \tau_P$ and consider $Cl_P(U)$. Since $Cl_P(U) = d_P(U) \cup U$ and $d_P(U) \in \tau_P$, we immediately obtain that $Cl_P(U) \in \tau_P$. Therefore (P, τ_P) is extremally disconnected. Hence, every subspace of (X, τ) (including in particular (X, τ) itself) is extremally disconnected. As an example of an h.e.d. space that is not DSO, consider the Sierpiński space given in Figure 4.3c: the Sierpiński space has a singleton open, therefore, it is not dense-in-itself. $\square$

We can further generalize the belief semantics on h.e.d. spaces for static conditioning.

4.3.2 Static conditioning: *conditional beliefs*

Static conditioning captures the agent's revised beliefs about how the world was before learning new information. This is in general implemented by conditional belief operators $B^{\varphi}\psi$ read as *"if the agent would learn φ, then she would come to believe that ψ was the case before the learning"* (Baltag and Smets, 2008, p. 12). Conditional beliefs therefore are static and hypothetical by nature, hinting at possible future belief changes of the agent. In the DEL literature, the semantics for conditional beliefs is generally given in terms of sphere models (Grove, 1988), or equivalently, in terms of plausibility models (van Benthem, 2007; Baltag and Smets, 2008; van Benthem and Pacuit, 2011).

In this section, we provide a topological semantics for conditional beliefs based on h.e.d. spaces. This topological semantics has been studied in (Özgün, 2013; Baltag et al., 2013) based on extremally disconnected spaces, where the dynamic extension encountered the problem explained in Section 4.2.2.

We can obtain the semantics for a conditional belief modality $B^{\varphi}\psi$ in a natural and standard way by relativizing the semantics for the simple belief modality to the extension of the learned formula φ. By relativization we mean a local change that only affects one occurrence of the belief modality $B\varphi$, and that does not cause a real change in the model. Similar to the case in (Özgün, 2013; Baltag et al., 2013) for extremally disconnected spaces, we can relativize the belief semantics in two different ways. To recap, given a topo-model $\mathcal{X} = (X, \tau, V)$ based on an extremally disconnected topology τ, we can describe the extension of a belief formula in the following equivalent ways

$$[\![B\varphi]\!] \stackrel{(1)}{=} Cl(Int([\![\varphi]\!])) \stackrel{(2)}{=} Int(Cl(Int([\![\varphi]\!]))).$$

While the relativization of (1) leads to

$$[\![B^{\varphi}\psi]\!] = Cl([\![\varphi]\!] \cap Int([\![\varphi]\!] \to [\![\psi]\!])), \tag{4.4}$$

the relativization of (2) results in

$$[\![B^{\varphi}\psi]\!] = Int([\![\varphi]\!] \to Cl([\![\varphi]\!] \cap Int([\![\varphi]\!]^{\mathcal{M}} \to [\![\psi]\!]))), \tag{4.5}$$

where $[\![\varphi]\!] \to [\![\psi]\!]$ is used as an abbreviation for $(X \backslash [\![\varphi]\!]) \cup [\![\psi]\!]$.

However, as elaborated in (Özgün, 2013), the first semantics (4.4) does not work well as a generalization of belief on extremally disconnected spaces, and the same arguments still hold on h.e.d. spaces. For example, it validates the equivalences

$$K\varphi \leftrightarrow \neg B^{\neg\varphi}\top \leftrightarrow \neg B^{\neg\varphi}\neg\varphi$$

which give a rather unusual definition of knowledge in terms of conditional beliefs. The first of these equivalences also shows that the *conditional belief operator is not a normal modality* (as the Necessitation rule for conditional beliefs stated in Theorem 4.3.7 does not preserve validitiy). Moreover, this semantics validates only a few of the AGM postulates stated in terms of conditional beliefs as in Theorem 4.3.7 (see Alchourrón et al., 1985, for the classical AGM theory). On the other hand, the second relativization does not possess any of the above flaws, and moreover validates all the AGM postulates formulated in terms of conditional beliefs as shown below (see Baltag and Smets, 2008, 2006, for the treatment of AGM theory in terms of conditional beliefs as a theory of static belief revision). We refer to (Baltag et al., 2015a) for the proofs of the results stated in the remaining of this chapter.

4.3.7. THEOREM. *The following formulas are valid in h.e.d. spaces with respect to the topological semantics for conditional beliefs and knowledge given in (4.5)*

Normality:	$B^\theta(\varphi \to \psi) \to (B^\theta\varphi \to B^\theta\psi)$
Factivity:	$K\varphi \to \varphi$
Persistence of Knowledge:	$K\varphi \to B^\theta\varphi$
Strong Positive Introspection:	$B^\theta\varphi \to KB^\theta\varphi$
Success of Belief Revision:	$B^\varphi\varphi$
Consistency of Revision:	$\neg K\neg\varphi \to \neg B^\varphi\bot$
Inclusion:	$B^{\varphi\wedge\psi}\theta \to B^\varphi(\psi \to \theta)$
Rational Monotonicity:	$B^\varphi(\psi \to \theta) \wedge \neg B^\varphi\neg\psi \to B^{\varphi\wedge\psi}\theta$

Moreover, the Necessitation rule for conditional beliefs

$$\textit{from } \varphi, \ \textit{infer } B^\psi\varphi$$

preserves validity.

Given the semantics in (4.5), we also obtain the following validities defining conditional beliefs in terms of knowledge, and simple belief in terms of conditional belief, respectively:

- $B^\varphi\psi \leftrightarrow K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \psi)))$,
- $B\varphi \leftrightarrow B^\top\varphi$.

Adding these two equivalences to a complete axiomatization of $\mathsf{S4.3}_K$ therefore yields a complete logic of knowledge and conditional beliefs with respect to h.e.d. spaces.

4.3.8. THEOREM. *The sound and complete logic* KCB *of knowledge and conditional beliefs with respect to the class of h.e.d. spaces is obtained by adding the following equivalences to any complete axiomatization of* $\mathsf{S4.3}_K$:

1. $B^\varphi\psi \leftrightarrow K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \psi)))$
2. $B\varphi \leftrightarrow B^\top\varphi$

Against this static background, we can further axiomatize the logic of public announcements, knowledge and conditional beliefs, following the standard DEL–technique: This is done by adding to KCB a set of reduction axioms that give us a recursive rewriting algorithm to step-by-step translate every formula containing public announcement modalities to a provably equivalent formula in the static language. The completeness of the dynamic system then follows from the soundness of the reduction axioms and the completeness of the underlying static logic (see, e.g., Section 7.4 of van Ditmarsch et al., 2007 for a detailed presentation of completeness by reduction, and see Wang and Cao, 2013 for an elaborate discussion of axiomatizations of public announcement logics).

4.3.9. Theorem. *The sound and complete dynamic logic* !KCB *of knowledge, conditional beliefs and public announcements with respect to the class of h.e.d. spaces is obtained by adding the following reduction axioms to any complete axiomatization of the logic* KCB:

1. $\langle !\varphi \rangle p \leftrightarrow (\varphi \wedge p)$
2. $\langle !\varphi \rangle \neg \psi \leftrightarrow (\varphi \wedge \neg \langle !\varphi \rangle \psi)$
3. $\langle !\varphi \rangle (\psi \wedge \theta) \leftrightarrow (\langle !\varphi \rangle \psi \wedge \langle !\varphi \rangle \theta)$
4. $\langle !\varphi \rangle K\psi \leftrightarrow (\varphi \wedge K(\varphi \to \langle !\varphi \rangle \psi))$
5. $\langle !\varphi \rangle B^{\theta}\psi \leftrightarrow (\varphi \wedge B^{\langle !\varphi \rangle \theta} \langle !\varphi \rangle \psi)$
6. $\langle !\varphi \rangle \langle !\psi \rangle \chi \leftrightarrow \langle ! \langle !\varphi \rangle \psi \rangle \chi$

4.4 Conclusions and Continuation

In this chapter, we presented our very first attempt to formalize a notion of evidence-based "justified" belief by using topological semantics based on extremally disconnected spaces, first proposed in (Özgün, 2013; Baltag et al., 2013). The belief semantics based on hereditarily extremally disconnected spaces was later investigated in (Baltag et al., 2015a).

To summarize, starting with the conception of knowledge as the interior operator, and building on Stalnaker's principles regarding the relation between knowledge and belief (Table 4.1), we proposed a topological semantics of belief as *subjective certainty* in terms of the closure of the interior operator. While the proposed topological semantics provides an intuitive and natural interpretation for the conception of belief as subjective certainty (see Section 4.2.1), it also yields the standard logic of belief $\mathsf{KD45}_B$ both on extremally and hereditarily extremally disconnected spaces (Theorems 4.2.3 and 4.3.5, respectively). The transition from extremally disconnected spaces to hereditarily extremally disconnected spaces is motivated by the fact that the topological semantics based on extremally disconnected spaces falls short of dealing with public announcements as shown in Section 4.2.2. However, even this restricted class of h.e.d. spaces generalizes the topological belief semantics based on the co-derived set operator since $\mathsf{KD45}_B$ is the logic of DSO-spaces when belief is interpreted as the co-derived set operator, and the class of DSO-spaces is a proper subclass of the class of h.e.d. spaces (Proposition 4.3.6). Moreover, when studied in tandem with the notion of knowledge as the interior, the belief semantics in terms of the closure of the interior operator does not yield a definition of knowledge as true belief (unlike belief as the co-derived set operator, see Section 3.3).

At a high level, this chapter takes a further small step toward developing a satisfactory epistemic/doxastic formal framework in which we can talk about evidential grounds of knowledge and belief. It does so by extending the interior-based topological semantics for knowledge by a semantic clause for belief, which arguably works better than the aforementioned proposal based on the co-derived set operator. However, within the current setting, everything we can say about evidence has to be said at a purely semantic level (see Section 3.2 and Section 4.2.1 to recall the topological, evidence-related readings of knowledge and belief, respectively). As we have not yet introduced any "evidence modalities", the modal language cannot really say anything concerning the link between evidence and belief, or evidence and knowledge, let alone represent different notions of evidence possession.

This provides motivation for the framework we develop in the next chapter. Chapter 5, improving on the evidence logic of van Benthem and Pacuit (2011) based on neighbourhood semantics, introduces a new topological semantics for various notions of evidence, evidence-based justified belief, knowledge and learning, where the studied notions of evidence are made explicit in the corresponding syntax via matching modalities.

Chapter 5

Justified Belief, Knowledge and the Topology of Evidence

In this chapter, we propose a topological semantics for various notions of *evidence*, *evidence-based justification, belief*, and *knowledge*, and explore the connection between these epistemic notions. The work presented in this chapter is to a great extent based on taking a new, *topological* perspective to the models for evidence, belief and evidence-management proposed by van Benthem and Pacuit (2011), and developed further by van Benthem et al. (2012, 2014). The framework developed in this chapter moreover generalizes and improves on our own work on a topological semantics for Stalnaker's doxastic-epistemic logic presented in Chapter 4.

The influential approach, initiated by van Benthem and Pacuit (2011); van Benthem et al. (2012, 2014), represents evidence semantically—roughly speaking, as sets of possible worlds—based on neighbourhood structures as well as syntactically by introducing evidence modalities. Their setting goes beyond and generalizes the formal treatment of the aforementioned epistemic notions in terms of relational structures, such as Kripke and plausibility models, and non-relational models, such as Grove sphere models. We here take a further step toward improving the formal, modal theoretical treatment of evidence, justified belief and knowledge by revealing the hidden topological structure of the evidence models of van Benthem and Pacuit (2011). The topological perspective enables more fine-grained and refined mathematical representations of various notions of evidence, such as *basic evidence, combined evidence, factive evidence* and (*non-*)*misleading evidence*, as well as relevant epistemic notions such as *argument* and *justification* (based on evidence), and, in turn, *justified belief* and (*in-*)*defeasible knowledge*. Consequently, we obtain a semantically and syntactically rich setting that provides a more in-depth logical analysis regarding the role of evidence in reaching an agent's epistemic/doxastic state. We also examine several types of evidence dynamics introduced in (van Benthem and Pacuit, 2011) and apply this setting to analyze and address key issues in epistemology such as "no false lemma" Gettier examples, misleading defeaters, and undefeated justification versus undefeated belief. Our main technical results are concerned with completeness, decidability and finite model property for the associated logics.

Outline

Section 5.1 serves as a semi-formal introduction and summary of the chapter, emphazising the important features of its content. In Section 5.2, we introduce the evidence models of van Benthem and Pacuit (2011) as well as our *topological* evidence models, and provide semantics for the notions of basic, combined and factive evidence. We moreover provide topological definitions for argument and justification. In Section 5.3, we propose a topological semantics for a notion of justified belief while comparing our setting to that of van Benthem and Pacuit (2011). We then generalize our semantics of (simple) belief for conditional beliefs. Section 5.4 defines the model transformations induced by evidence-based information dynamics such as public announcements, evidence addition, evidence upgrade and feasible evidence combination. In Section 5.5, we propose a topological interpretation for a notion of fallible knowledge and connect our formalism to some important discussions emerged in the post-Gettier epistemology literature, such as stability/defeasibility theories of knowledge, misleading vs. genuine defeaters etc. Finally, Section 5.6 presents all our technical results. The reader who is interested in the technical aspect only can jump to Section 5.6 directly.

This chapter is based on (Baltag et al., 2016a,b)

5.1 Introduction

One of our main goals in this chapter, that we also share with van Benthem and Pacuit (2011); van Benthem et al. (2012, 2014), is to study notions of *belief and knowledge for a rational agent who is in possession of some* (*possibly false, possibly mutually contradictory*) *pieces of evidence*. A central underlying assumption is that an agent's rational belief and knowledge is based on the available evidence, namely, the evidence she has acquired via, e.g., direct observation, measurements, testimony from others etc. We therefore do not take belief or knowledge as the primitive notions, they are represented as "derived" notions purely based on evidence. Toward designing a formal setting that can capture these ideas (among others), we use the uniform evidence models of van Benthem and Pacuit (2011), with a special focus on the topology generated by the evidence. In the following, we provide a detailed overview of the epistemic notions studied in this chapter, introduce the modalities we consider, and explain where our work stands in the relevant literature.

A crucial reason as to why the approach presented in this chapter improves on the settings of Chapters 3 and 4 is that we here introduce evidence modalities in order to also provide syntactic representations of notions of evidence, and eventually to build *evidence logics*. In particular, we study the operator of "having (a piece of) evidence for a proposition P" proposed by van Benthem and Pacuit (2011), but we also investigate other interesting variants of this concept: "having (combined) evidence for P", "having a (piece of) *factive* evidence for P" and "having (combined) factive evidence for P". Table 5.1 below lists the corresponding evidence modalities together with their intended readings.

$E_0\varphi$	the agent has a basic (piece of) evidence for φ
$E\varphi$	the agent has a (combined) evidence for φ
$\Box_0\varphi$	the agent has a factive basic (piece of) evidence for φ
$\Box\varphi$	the agent has factive (combined) evidence for φ

Table 5.1: Evidence modalities and their intended readings

The *basic pieces of evidence* possessed by an agent are modelled as nonempty sets of possible worlds. A *combined evidence* (or just "evidence", for short) is any nonempty intersection of finitely many pieces of evidence. This notion of evidence is not necessarily factive[1], since the pieces of evidence are possibly false (and possibly inconsistent with each other). The family of (combined) evidence sets forms a topological basis, that generates what we call the *evidential topology*. This is the smallest topology in which all the basic pieces of evidence are open, and it will play an important role in our setting. In fact, the modality $\Box\varphi$ capturing the concept of "having factive evidence for φ" coincides with the *interior operator* in the evidential topology (see Section 5.2.2). We therefore use the interior semantics of McKinsey and Tarski (1944) to interpret a notion of *factive* evidence (this is unlike the case in Chapter 4, where the interior operator was treated as knowledge). We also show that the two factive variants of evidence-possession operators ($\Box_0$ and $\Box$) are more expressive than the non-factive ones (E_0 and E): when interacting with the global modality, the two factive evidence modalities $\Box_0\varphi$ and $\Box\varphi$ can define the non-factive variants $E_0\varphi$ and $E\varphi$, respectively, as well as many other doxastic/epistemic operators.

The notion of *justified belief* we study in this chapter will be defined purely by means of the notions of evidence mentioned above. We propose a *"coherentist" semantics for justification* and *justified belief*, that is obtained by extending, generalizing, and (to an extent), streamlining the evidence-model framework for

[1]Factive evidence is true in the actual world. In epistemology it is common to reserve the term "evidence" for factive evidence. But we follow here the more liberal usage of this term in (van Benthem and Pacuit, 2011), which agrees with the common understanding in day to day life, e.g. when talking about "uncertain evidence", "fake evidence", "misleading evidence" etc.

beliefs introduced by van Benthem and Pacuit (2011). The main idea behind the belief definition of van Benthem and Pacuit (2011) seems to be that the rational agent tries to form consistent beliefs, by looking at all *strongest finitely-consistent collections of evidence*, and she believes whatever is entailed by all of them.[2] Their belief definition therefore crucially depends on a notion of "strongest" evidence, and it works well in the finite case (whenever the agent has finitely many pieces of basic evidence) as well as in *some* infinite cases. But, as already noted in (van Benthem et al., 2014), this setting has the shortcoming that it can produce *inconsistent beliefs* in the general infinite case. A more technical defect of this setting is that the corresponding doxastic logic does not have the finite model property (see van Benthem et al., 2012, Corollary 2.7 or van Benthem et al., 2014, Corollary 1). In this chapter, we propose an "improved" semantics for evidence-based belief, obtained by, in a sense, weakening the definition from (van Benthem and Pacuit, 2011). According to us, a proposition P is believed if P is entailed by *sufficiently strong finitely-consistent collections of evidence.* This definition coincides with the one of van Benthem and Pacuit (2011) for the models carrying finite evidence collections, but involves a different generalization of their notion in the infinite case. In fact, our semantics *always ensures consistency of belief*, even when the available pieces of evidence are mutually inconsistent. We also provide a formalization of *argument* and a *"coherentist" view on justifications.* An *argument* essentially consists of one or more evidence sets supporting the same proposition (thus providing multiple evidential paths towards a common conclusion); a *justification* is an argument that is not contradicted by any other available evidence. Our definition of belief is equivalent to requiring that P *is believed iff there is some* (*evidence-based*) *justification for* P, therefore, accurately captures the concept of "justified belief". Our proposal is also very natural from a topological perspective; it is equivalent to saying that P *is believed iff it is true in "almost all" epistemically possible states*, where "almost all" is interpreted topologically as "all except for a nowhere-dense set". We moreover generalize this belief semantics for conditional beliefs. Table 5.2 below lists the belief modalities we study in this chapter.

$B\varphi$	the agent has justified belief in φ
$B^{\varphi}\psi$	the agent believes ψ conditionally on φ

Table 5.2: Belief modalities and their intended readings

[2]To be sure, this is still vague since we have not yet specied what a "strongest finitely-consistent collections of evidence" means (we return to formalize these notions in Section 5.2.1), however, this much precision should be sufficient to explain the rough idea behind the belief definition of van Benthem and Pacuit (2011), and our notion of justified belief studied in this chapter.

Moving on to *knowledge*, there are a number of different notions one may consider. First, there is "absolutely certain" or "infallible" knowledge, akin to Aumann's concept of partitional knowledge (Aumann, 1999) or van Benthem's concept of hard information (van Benthem, 2007). In our single-agent setting, this can be simply defined as the global modality (quantifying universally over all epistemically possible states). There are very few propositions that can be known in this infallible way (e.g., the ones known by introspection or by logical proof). Most facts in science or real life are unknown in this sense. It is therefore more interesting to look at notions of knowledge that are less-than-absolutely-certain, namely, the so-called *defeasible knowledge*. In our framework, we consider both absolutely certain knowledge and defeasible knowledge, but our main focus will be on the latter notion. See Table 5.3 below for the corresponding knowledge modalities and their readings.

$[\forall]\varphi$	the agent infallibly knows φ
$K\varphi$	the agent fallibly (or defeasibly) knows φ

Table 5.3: Knowledge modalities and their intended readings

The famous Gettier counterexamples (Gettier, 1963) show that simply adding "factivity" to belief will not give us a "good" notion of defeasible knowledge: true (justified) belief is extremely fragile (i.e., it can be too easily lost), and it is consistent with having only wrong justifications for an accidentally true conclusion. We here formalize a notion of defeasible knowledge saying that "*P is (fallibly) known if there is a factive justification for P*". We therefore study a notion of knowledge defined as *correctly justified belief*. As elaborated in Section 5.5.1, this less-than-absolutely-certain notion of knowledge finds its place in the post-Gettier literature as being stronger than the one charaterized by the "no false lemma" of Clark (1963) and weaker than the conception of knowledge described by the defeasibility theory of knowledge championed by Lehrer and Paxson (1969); Lehrer (1990); Klein (1971, 1981).

Yet another path leading to our setting in this chapter goes via our previous work (Baltag et al., 2013, 2015a), presented in Chapter 4, on a topological semantics for the doxastic-epistemic axioms of Stalnaker (2006). Recall that Stalnaker's system Stal (see Table 4.1) is meant to capture a notion of fallible knowledge, in close interaction with a notion of "strong belief" defined as *subjective certainty*. The main principle specific to this system was that "believing implies believing that you know" captured by the axiom of Full Belief ($B\varphi \to BK\varphi$). The topological semantics that we proposed for these concepts in (Özgün, 2013; Baltag et al., 2013, 2015a) was overly restrictive (being limited to the rather unfamiliar class of extremally disconnected and hereditarily extremally disconnected topologies). In this chapter, we show that these notions can be interpreted on arbitrary topological spaces, without changing their logic. Indeed, our definitions of belief and knowledge can be seen as the natural generalizations to arbitrary topologies of the notions in (Özgün, 2013; Baltag et al., 2013, 2015a).

We completely axiomatize the various resulting logics of evidence, knowledge, and belief, and prove decidability and finite model property results. We moreover study a few dynamic extensions, encoding different types of evidential dynamics. Our technically most challenging result is the completeness of the richest logic containing the two factive evidence modalities $\Box_0\varphi$ and $\Box\varphi$, as well as the global modality $[\forall]\varphi$. This logic can define all the modal operators mentioned above. While the other proofs are more or less routine, the proof of this result involves a nontrivial combination of known methods.

5.2 Evidence, Argument and Justification

In this section, we introduce the (uniform) evidence models of van Benthem and Pacuit (2011) as well as our *topological* version, and provide the formal semantics of the evidence modalities given in Table 5.1. More precisely, we focus on the operator "having a basic (piece of) evidence for a proposition P" (from van Benthem and Pacuit, 2011), as well as the variants capturing "having (combined) evidence for P", "having a basic (piece of) *factive* evidence for P" and "having (combined) factive evidence for P". We explain how a rational agent can put her basic evidence pieces together in a "finitely consistent" way toward forming *combined evidence*, *strongest* and *strong enough evidence*, and eventually, her beliefs. We moreover provide topological definitions for *argument* and *justification* purely based on evidence.

5.2.1 Evidence à la van Benthem and Pacuit

5.2.1. Definition. [Evidence Models (van Benthem and Pacuit, 2011)] An *evidence model* is a tuple $\mathfrak{M} = (X, \mathcal{E}_0, V)$, where

- X is a nonempty set of *possible world* (or *states*),
- $\mathcal{E}_0 \subseteq \mathcal{P}(X)$ is a family of sets called *basic evidence sets* (or *pieces of evidence*), satisfying $X \in \mathcal{E}_0$ and $\emptyset \notin \mathcal{E}_0$, and
- $V : \text{PROP} \to \mathcal{P}(X)$ is a valuation function.

The evidence models presented in (van Benthem and Pacuit, 2011; van Benthem et al., 2014) are more general, covering cases in which evidence depends on the actual world, i.e., in which each state may be assigned different set of neighbourhoods. In this chapter, however, we stick with what they call "uniform" models (given in Definition 5.2.1), which corresponds to working with agents who are "evidence-introspective"[3].

Note that evidence models are not necessarily based on topological spaces, i.e., $\mathcal{E}_0$ is not defined to be a topology (it may not even constitute a topological basis). However, topo-models given in Definition 3.1.1 constitute a special case of evidence models.[4] We would like to elaborate more on the structural properties of evidence models and explain which epistemic concepts they intend to represent.

The family $\mathcal{E}_0$ is almost an arbitrary nonempty collection of subsets of a given domain, carefully designed to capture certain aspects of the type of evidence that is intended to be formalized. First of all, the subset $\mathcal{E}_0$ represents the set of evidence the agent has acquired about the actual situation[5] directly via, e.g., testimony, measurement, approximation, computation or experiment. It is the collection of evidence the agent gathered so far, and it is all our rational, idealized agent has to form her beliefs and knowledge. The collection of evidence the agent possesses is uniform across the states, i.e., the set of evidence the agent has does not depend on the actual state. This corresponds to working with an "evidence-introspective" agent, that is, the agent is absolutely sure about what evidence she has and what it entails.

The two properties of $\mathcal{E}_0$, namely, $X \in \mathcal{E}_0$ and $\emptyset \notin \mathcal{E}_0$ impose the following constraints, respectively:

- Tautologies are always evidence, and
- Contradictions never constitute direct evidence.

Unlike the common practice in epistemology, where the term "evidence" is generally reserved for factive evidence, van Benthem and Pacuit (2011) and van Benthem et al. (2012, 2014) follow a more liberal, in a sense, more realistic view on evidence which agrees with the common usage in day to day life, e.g. when talking about "uncertain evidence", "fake evidence", "misleading evidence". They not only consider evidence gathered from absolutely reliable and truthful sources, but also take into account *fallible* information coming from a possibly unreliable source: *a piece of evidence in $\mathcal{E}_0$ does not have to contain the actual state.* Moreover, the evidence gathered from different sources (or even from a single source) may be mutually inconsistent: *the intersection of evidence pieces may be empty.* Therefore, the evidence models of van Benthem and Pacuit (2011) (as

[3]Since we never consider the more general case and focus only on the topological extension of their uniform evidence models, we use the term "evidence model" exclusively for the uniform evidence models of van Benthem and Pacuit (2011); van Benthem et al. (2014), given above in Definition 5.2.1.

[4]As an even more special case, we can also think of Grove/Lewis Sphere spaces. These are topological spaces in which the open sets are "nested", i.e. for every $U, U' \in \tau$, we have either $U \subseteq U'$ or $U' \subseteq U$ (see, e.g., Example 5.3.1).

[5]Standardly, as in the relational semantics and the interior semantics, the actual situation is represented by a state x of X called the *actual state* or the *real world*.

well as our topological evidence models) take into account that the agent might be collecting evidence from different sources that may or may not be reliable, however, it is assumed that all her current sources are equally reliable (or equally unreliable) as no special order or quantitative measure is defined on the elements of $\mathcal{E}_0$. Under these assumptions, what is expected from a rational agent toward forming consistent beliefs based on the collection of evidence pieces she has, is to evaluate every piece of evidence she possesses in a coherent and holistic way, and put them together in a finite and consistent manner. This leads to the notions of (*finite*) *bodies of evidence* and *combined evidence*, conceptions with crucial roles in formation of consistent beliefs based on fallible evidence, and of the evidential topology. In what follows, we provide technical definitions of the evidence-related auxiliary notions that are adopted from van Benthem and Pacuit (2011), and will be used throughout this chapter.

Bodies of evidence, Evidential Support and Strength

We call a collection of evidence pieces $F \subseteq \mathcal{E}_0$ *consistent* if $\bigcap F \neq \emptyset$, and *inconsistent* otherwise. In order to ease the notation, we let $A \subseteq_{fin} B$ to be read as *A is a finite subset of B.*

5.2.2. Definition. [(Finite) Body of Evidence] Given an evidence model $\mathfrak{M} = (X, \mathcal{E}_0, V)$, a *body of evidence* is a nonempty family $F \subseteq \mathcal{E}_0$ of evidence pieces such that every nonempty finite subfamily is consistent. More formally, a nonempty family $F \subseteq \mathcal{E}_0$ is a body of evidence if

$$(\forall F' \subseteq_{fin} F)(F' \neq \emptyset \text{ implies } \bigcap F' \neq \emptyset).$$

A *finite body of evidence* $F \subseteq_{fin} \mathcal{E}_0$ is therefore simply a finite set of mutually consistent pieces of evidence, that is, $F \subseteq_{fin} \mathcal{E}_0$ such that $\bigcap F \neq \emptyset$.

Therefore, a body of evidence is simply a collection of evidence pieces that has the finite intersection property, and that represents the agent's ability of putting evidence pieces together in a *finitely consistent* way.

Given an evidence model $\mathfrak{M} = (X, \mathcal{E}_0, V)$, we denote by

$$\mathcal{F} := \{F \subseteq \mathcal{E}_0 \mid (\forall F' \subseteq_{fin} F)(F' \neq \emptyset \text{ implies } \bigcap F' \neq \emptyset)\}$$

the *family of all bodies of evidence* over $\mathfrak{M}$, and by

$$\mathcal{F}^{fin} := \{F \subseteq_{fin} \mathcal{E}_0 \mid \bigcap F \neq \emptyset\}$$

the *family of all finite bodies of evidence*. Both the interpretation of evidence-based belief of van Benthem and Pacuit (2011) and our proposal for justified belief, as well as the notion of defeasible knowledge we study in this chapter crucially rely on the notion of body of evidence. But, in order to be able to talk about these *evidence-based* informational attitudes, we first need to specify what it means for a proposition to be *supported* by a body of evidence.

5.2.3. Remark. Throughout Sections 5.2-5.5, we use the following conventions to ease the presentation. Given an evidence model $\mathfrak{M} = (X, \mathcal{E}_0, V)$ (or, a topo-e-model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ defined later), we call any subset $P \subseteq X$ a *proposition*. We say a proposition $P \subseteq X$ is true at x if $x \in P$. The Boolean connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, on propositions are defined standardly as set operations: for any $P, Q \subseteq X$, we set $\neg P := X \backslash P$, $P \wedge Q := P \cap Q$, $P \vee Q := P \cup Q$ and $P \rightarrow Q := (X \backslash P) \cup Q$. Moreover, the Boolean constants $\top$ and $\bot$ are given as $\top := X$ and $\bot := \emptyset$. Following this convention, we define the semantics of the aforementioned modal operators for evidence, belief and knowledge introduced in Tables 5.1-5.3 as set operators from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ (and for the binary modality of conditional belief, from $\mathcal{P}(X) \times \mathcal{P}(X)$ to $\mathcal{P}(X)$). These set operators give rise to the interpretations of the corresponding modalities of the full language $\mathcal{L}$ (given in Section 5.6) in a standard way.

5.2.4. Definition. [Evidential Support] Given an evidence model $\mathfrak{M}$=$(X, \mathcal{E}_0, V)$ and a proposition $P \subseteq X$, a body of evidence F *supports* P if P is true in every state satisfying all the evidence in F, i.e., if $\bigcap F \subseteq P$.

It is easy to see that a body of evidence F is inconsistent iff it supports every proposition (since $\emptyset \subseteq P$, for all P). The *strength order* between bodies of evidence is given by inclusion: $F \subseteq F'$ means that F' is at least as strong as F. Note that stronger bodies of evidence support more propositions: if $F \subseteq F'$ then every proposition supported by F is also supported by F'. A body of evidence is *maximal* ("strongest") if it is a maximal element of the poset $(\mathcal{F}, \subseteq)$, i.e., if it is not a proper subset of any other such body. We denote by

$$Max_{\subseteq}\mathcal{F} := \{F \in \mathcal{F} \mid (\forall F' \in \mathcal{F})(F \subseteq F' \Rightarrow F = F')\}$$

the *family of all maximal bodies of evidence* of a given evidence model. By Zorn's Lemma, *every body of evidence can be strengthened to a maximal body of evidence*, i.e.,

$$\forall F \in \mathcal{F}\, \exists F' \in Max_{\subseteq}\mathcal{F}(F \subseteq F').$$

Therefore, in particular, every evidence model has at least one maximal body of evidence, that is, $Max_{\subseteq}\mathcal{F} \neq \emptyset$.

In fact, for *finite* bodies of evidence, the notions of evidential support and strength can be represented in a more concise way via the notion of combined evidence, which, to anticipate further, is represented by basic open sets of the evidential topology generated from $\mathcal{E}_0$ (see Section 5.2.2).

Combined Evidence and Evidential Basis

5.2.5. Definition. [Combined Evidence]
Given an evidence model $\mathfrak{M} = (X, \mathcal{E}_0, V)$, a *combined evidence* (or, *evidence*, for short) is any nonempty intersection of finitely many basic evidence pieces. In

other words, a nonempty subset $e \subseteq X$ is a combined evidence if $e = \bigcap F$, for some $F \in \mathcal{F}^{fin}$.

A combined evidence therefore is just a repackaging of a finite body of evidence in terms of its intersection. We denote by

$$\mathcal{E} := \{\bigcap F \mid F \in \mathcal{F}^{fin}\}$$

the *family of all* (*combined*) *evidence*, which in fact constitutes a topological basis over X. We will return to the topological versions of evidence models in Section 5.2.2.

The definitions *evidential support* and *strength* are adapted for the elements of $\mathcal{E}$ in an obvious way. A (combined) evidence $e \in \mathcal{E}$ *supports* a proposition $P \subseteq X$ if $e \subseteq P$. In this case, we also say that e is *evidence for* P. The natural strength order between combined evidence sets therefore is given by the reverse inclusion: $e \supseteq e'$ means that e' is at least as strong as e. This is both to fit with the strength order on bodies of evidence (since $F \subseteq F'$ implies $\bigcap F \supseteq \bigcap F'$), and to ensure that stronger evidence supports more propositions (since, if $e \supseteq e'$, then every proposition supported by e is supported by e').

Recall that $\mathcal{E}_0$ represents the collection of evidence pieces that are directly observed by the agent. The elements of the derived set $\mathcal{E}$ therefore serve as indirect evidence which is obtained by combining finitely many pieces of direct evidence together in a consistent way. This does not mean that all of this evidence is necessarily true. We say that some (basic or combined) evidence $e \in \mathcal{E}$ is *factive evidence* at state $x \in X$ whenever it is true at x, i.e., if $x \in e$. Similarly, a body of evidence F is factive if all the pieces of evidence in F are factive, i.e., if $x \in \bigcap F$.

Having presented the primary semantic concepts used in the representation of (basic and combined) evidence, we proceed with our topological setting.

5.2.2 Evidence on *Topological* Evidence Models

For any nonempty set X and any family Σ of subsets of X, we can construct a topology on this domain by simply closing Σ under finite intersections and arbitrary unions (see Section 2.2). Therefore, every evidence model $\mathfrak{M} = (X, \mathcal{E}_0, V)$ can be associated with an *evidential topology* that is generated by the set of basic evidence pieces $\mathcal{E}_0$, or equivalently, by the family of all combined evidence $\mathcal{E}$. In this section, we introduce the topological evidence models, generated from evidence models of van Benthem and Pacuit (2011) in the above described way, and provide topological formalizations of a notion of *argument* and a "coherentist" form of *justification* (in the spirit of Lehrer (1990)) based on the topological models. We moreover give the precise interpretations of the modalities $E_0\varphi$ and $E\varphi$ for basic and combined evidence possession, respectively, as well as their factive versions $\Box_0\varphi$ and $\Box\varphi$.

5.2.6. Definition. [Topological Evidence Model] A *topological evidence model* (or, in short, a *topo-e-model*) is a tuple $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$, where $(X, \mathcal{E}_0, V)$ is an evidence model and $\tau = \tau_{\mathcal{E}}$ is the topology generated by the family of combined evidence $\mathcal{E}$ (or equivalently, by the family of basic evidence sets $\mathcal{E}_0$), which is called the *evidential topology*.

The families $\mathcal{E}_0$ and $\mathcal{E}$ obviously generate the same topology: $\mathcal{E}$ is the closure of $\mathcal{E}_0$ under nonempty finite intersections. We denote the evidential topology by $\tau_{\mathcal{E}}$ only because the family $\mathcal{E}$ of combined evidence forms a *basis* of this topology. Since any family $\mathcal{E}_0 \subseteq \mathcal{P}(X)$ generates a topology over X, topo-e-models are just another presentation of evidence models described in Definition 5.2.1. We use this special terminology to stress our focus on the topology, and to avoid ambiguities, since our definition of belief in topo-e-models will be different from the definition of belief in evidence models of van Benthem and Pacuit (2011).

Argument and Justification. Given a topo-e-model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ and a proposition $P \subseteq X$, we say

- an *argument for* P is a union $U = \bigcup \mathcal{E}'$ of some nonempty family of (combined) evidence $\mathcal{E}' \subseteq \mathcal{E}$, each separately supporting P (i.e., $e \subseteq P$ for all $e \in \mathcal{E}'$, or equivalently, $U \subseteq P$).

Epistemologically, an argument for P provides *multiple evidential paths* $e \in \mathcal{E}'$ to support the common conclusion P. Topologically, an argument for P is the same as a *nonempty open subset of* P: a set of states U is an argument for P iff $U \in \tau$ and $U \subseteq P$. Therefore, the open $Int(P)$ forms the *weakest (most general) argument for* P, since it is the largest open subset of P.

- A *justification for* P is an argument U for P that is consistent with every (combined) evidence (i.e., $U \cap e \neq \emptyset$ for all $e \in \mathcal{E}$, that is, $U \cap U' \neq \emptyset$ for all $U' \in \tau \backslash \{\emptyset\}$).

Justifications are thus defined to be arguments that are undefeated (i.e., whose negations are not supported) by any available evidence or any other argument based on this evidence. Topologically, a justification for P is just a *dense open subset* of P: a set of states U is a justification for P iff $U \in \tau$ such that $U \subseteq P$ and $Cl(U) = X$. As for evidence, an argument or a justification U for P is said to be *factive* (or "correct") if it is true in the actual world, i.e., if $x \in U$.

The fact that arguments are open in the generated topology encodes the principle that *any argument should be evidence-based*: whenever an argument is correct, then it is supported by some factive evidence. To anticipate further: in our setting, justifications will form the basis of *belief*, while correct justifications will form the basis of *fallible (defeasible) knowledge*. But before moving to justified belief and fallible knowledge, we introduce a stronger, irrevocable form of knowledge that is captured by the global modality.

Infallible Knowledge: possessing hard information. We use $[\forall]$ for the so-called *global* modality, which associates to every proposition $P \subseteq X$, some other proposition $[\forall]P$, given by putting:

$$[\forall]P := \begin{cases} X & \text{if } P = X \\ \emptyset & \text{otherwise.} \end{cases}$$

In other words, $[\forall]P$ holds (at any state) iff P holds at all states. In this setting, $[\forall]P$ is interpreted as "absolutely certain, *infallible knowledge*", defined as truth in all the worlds that are consistent with the agent's information[6]. This is a limit notion capturing a very strong form of knowledge encompassing all epistemic possibilities. It is *irrevocable*, i.e., it cannot be lost or weakened by any information gathered later. In this respect, $[\forall]P$ could be best described as *possession of hard information.* Its dual $[\exists]P := \neg[\forall]\neg P$ expresses the fact that P is consistent with (all) the agent's hard information.

We would like to note here that infallible knowledge $[\forall]\varphi$ is not the most interesting notion of knowledge we study in this chapter, and it is harshly criticized by many epistemologists (see, e.g., Hintikka, 1962). However, having this strong modality in our framework is useful for both conceptual and technical reasons: while it helps us to see the difference between infallible and fallible knowledge, the global modality, in general, adds to the expressive power of modal languages. In particular, it will allow us to express all the other modalities we work with in terms of only the modalities $\Box_0\varphi$ and $\Box\varphi$ when interacting with the global modality $[\forall]\varphi$ (see Proposition 5.6.2).

Having *Basic* Evidence for a Proposition. Van Benthem and Pacuit (2011) define, for every proposition $P \subseteq X$, another proposition E_0P by [7]:

$$E_0P := \begin{cases} X & \text{if } \exists e \in \mathcal{E}_0\,(e \subseteq P) \\ \emptyset & \text{otherwise.} \end{cases}$$

The modal sentence E_0P therefore intends to capture possession of basic (direct) evidence for the proposition P, thus reads as "the agent *has basic evidence for P*". In other words, E_0P states that P *is supported by some basic piece of evidence.* Additionally, we introduce a *factive* version of this proposition, $\Box_0P$, that is read as "the agent has *factive basic evidence for P*", and is given by

$$\Box_0P := \{x \in X \mid \exists e \in \mathcal{E}_0\,(x \in e \subseteq P)\}.$$

[6]In a multi-agent model, some worlds might be consistent with one agent's information, while being ruled out by another agent's information. Therefore, in a multi-agent setting, $[\forall_i]$ will only quantify over all the states in agent i's current information cell (according to a partition Π_i of the state space reflecting agent i's hard information). We will present a multi-agent epistemic system in Chapter 8.

[7]Van Benthem and Pacuit (2011) denote this by $\Box P$, and it is denoted by $[E]P$ in (van Benthem et al., 2014). We use E_0P for this notion, since we reserve the notation EP for having *combined* evidence for P, and $\Box P$ for having *combined factive* evidence for P.

Having (Combined) Evidence for a Proposition. The above notions of evidence possession based on having basic evidence for a propositions can be generalized to having (combined) evidence for a proposition. This way, we obtain two other evidence operators: EP, meaning that "the agent *has* (*combined*) *evidence for* P", and $\Box P$, meaning that "the agent *has factive* (*combined*) evidence for P". More precisely, EP and $\Box P$ are given as follows:

$$EP := \begin{cases} X & \text{if } \exists e \in \mathcal{E}\,(e \subseteq P) \\ \emptyset & \text{otherwise} \end{cases}$$

$$\Box P := \{x \in X \mid \exists e \in \mathcal{E}\,(x \in e \subseteq P)\}.$$

Since $\mathcal{E}$ is a basis of the evidential topology $\tau_{\mathcal{E}}$, we have that *the agent has evidence for a proposition P iff she has an argument for P.* So EP can also be interpreted as "having an argument for P". Similarly, $\Box P$ can be interpreted as "having a *correct* argument for P". Moreover, $\Box$ operator for having combined factive evidence coincides with the topological interior operator (see equations (3.1)-(3.3) in Section 3.2), thus, it coincides with the knowledge operator under the interior semantics presented in Chapter 3. This observation therefore points to a major difference between the framework introduced in this chapter and the approach based on the interior semantics presented in Chapters 3 and 4: while in the interior semantics the interior operator represents "knowledge of" something, in our interpretation the interior represents only "having true evidence for" something. The difference arises from the fact that an agent may be in possession of some evidence that happens to be true, without the agent necessarily knowing, or even believing, that this evidence is true. To better understand the difference, we need a topological understanding of *belief*.

5.3 Justified Belief

In this section, we propose a topological semantics for a notion of evidence-based justified belief. We do this by modifying, and in a sense, eliminating the "bugs" in the belief definition proposed by van Benthem and Pacuit (2011) based on evidence models. While our proposal coincides with that of van Benthem and Pacuit (2011) on evidence models carrying a finite set of basic evidence pieces $\mathcal{E}_0$ and in some infinite cases, in general ours is "better" behaved. To name a few reasons, among others, our proposal leads to a notion of belief that is *topologically natural*, *always consistent*, and in fact, it *satisfies the axioms of the standard doxastic logic* KD45 *on all topo-e-models*. To better explain the origins and inspiration of our proposal, we first recapitulate the belief definition of van Benthem and Pacuit (2011). We then introduce our definition of justified belief, and show how and when the two proposals coincide. We also provide several equivalent characterizations of our proposed notion of justified belief, and generalize this setting for conditional beliefs.

5.3.1 Belief à la van Benthem and Pacuit

In their work, van Benthem and Pacuit (2011) present an evidence-based notion of belief defined on the evidence models. According to their definition,

P is believed iff every maximal (i.e., strongest) body of evidence supports P.

We denote this notion by Bel. More formally, given an evidence model $\mathfrak{M} = (X, \mathcal{E}_0, V)$ and a proposition $P \subseteq X$,

$$BelP \text{ holds (at any state) iff } (\forall F \in Max_{\subseteq}\mathcal{F})(\bigcap F \subseteq P).^{8}$$

However, as can be seen directly from the above definition, Bel is inconsistent on evidence models whose every maximal body of evidence is inconsistent.

5.3.1. Example. Consider the evidence model $\mathfrak{M} = (\mathbb{N}, \mathcal{E}_0, V)$, where the state space is the set $\mathbb{N}$ of natural numbers, $V(p) = \emptyset$, and the basic evidence family is $\mathcal{E}_0 = \{[n, \infty) \mid n \in \mathbb{N}\}$ (see Figure 5.1). The only maximal body of evidence in $\mathcal{E}_0$ is $\mathcal{E}_0$ itself. However, $\bigcap \mathcal{E}_0 = \emptyset$. So $Bel\bot$ holds in $\mathfrak{M}$.

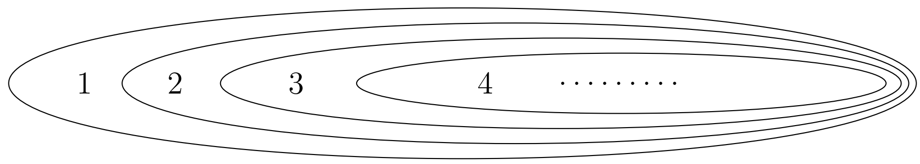


Figure 5.1: $\mathfrak{M} = (\mathbb{N}, \mathcal{E}_0, V)$

This phenomenon only happens in (some cases of) *infinite* models, so it is *not* due to the inherent mutual inconsistency of the available evidence. At a high level, the source of the problem seems to be the tension between the way the agent combines her evidence pieces and the way she forms her beliefs based her evidence: while she puts her evidence pieces together in a *finitely* consistent way, having consistent beliefs requires possibly infinite collections to have nonempty

[8] As already noticed in (van Benthem and Pacuit, 2011; van Benthem et al., 2014), in many but not all cases, this is equivalent to treating plausibility models as a special case of evidence models where the plausibility relation is given by the *evidential* plausibility order $\sqsubseteq_{\mathcal{E}}$ defined as

$$x \sqsubseteq_{\mathcal{E}} y \text{ iff } (\forall e \in \mathcal{E}_0)(x \in e \text{ implies } y \in e) \text{ iff } (\forall e \in \mathcal{E})(x \in e \text{ implies } y \in e),$$

and applying the standard semantics of belief on plausibility models as "truth in all the most plausible states". The relation between evidence models and plausibility models, as well as the connection between the notions of belief defined on these structures are subtle. We skip the details on this issue here, and refer to (van Benthem and Pacuit, 2011, Section 5) and (van Benthem et al., 2014, Section 3) for details.

intersections. More precisely, even though it is guaranteed by definition that every finite subfamily of a maximal body of evidence is consistent, the whole maximal body of evidence may actually be inconsistent. Therefore, in order to avoid this problem, we could instead focus on *maximal finite bodies* of evidence as blocks of evidence forming beliefs: these are, by definition, guaranteed to be always consistent. However, this solution inevitably restricts the class of evidence models we can work with, simply because an infinite evidence model might not bear any maximal finite body of evidence. To illustrate this, we can think of the evidence model presented in Example 5.3.1: the set of basic evidence $\mathcal{E}_0$ is the only maximal body of evidence in $(\mathbb{N}, \mathcal{E}_0, V)$, and it is infinite. Therefore, in order to eventually be able to provide a belief logic of all evidence models that formalizes a notion of consistent belief, further adjustments in the definition of Bel are warranted. To this end, we propose to "weaken" the belief definition of van Benthem and Pacuit (2011) in the sense that we focus on *all finite bodies of evidence that are "strong enough"* instead of focusing on all the "*strongest*" such bodies.

5.3.2 Our Justified Belief

It seems to us that the intended goal (only partially fulfilled) in (van Benthem and Pacuit, 2011) was to ensure that the agents are able to form consistent beliefs based on the (possibly false and possibly mutually contradictory) available evidence. We think this to be a natural requirement for *idealized rational* agents, and so we consider doxastic inconsistency to be "a bug, not a feature", of the van Benthem-Pacuit framework. Hence, we now propose a notion that produces in a natural way—with no need for further restrictions—only consistent beliefs, and also that agrees with the one in (van Benthem and Pacuit, 2011) in many cases specified below.

The intuition behind our proposal is that a proposition P is believed iff *it is supported by all "sufficiently strong" evidence*. We therefore say that P *is believed*, and write BP, iff *every finite body of evidence can be strengthened to some finite body of evidence which supports* P. More formally, given an evidence model $\mathfrak{M} = (X, \mathcal{E}_0, V)$ and a proposition $P \subseteq X$,

$$BP \text{ holds (at any state)} \quad \text{iff} \quad \forall F \in \mathcal{F}^{fin} \exists F' \in \mathcal{F}^{fin} (F \subseteq F' \text{ and } \bigcap F' \subseteq P).$$

The notion of belief B (like Bel) is a "global" notion, which depends only on the agent's evidence, not on the actual world, so it is either true in all possible worlds, or false in all possible worlds. We therefore have

$$BP := \begin{cases} X & \text{if } \forall F \in \mathcal{F}^{fin} \exists F' \in \mathcal{F}^{fin} (F \subseteq F' \text{ and } \bigcap F' \subseteq P) \\ \emptyset & \text{otherwise.} \end{cases}$$

This reflects the assumption that beliefs are internal (and fully transparent) to the agent (Baltag et al., 2008).

It is easy to see that, unlike Bel, our notion of belief B is *always consistent* (i.e., $B\bot = B\emptyset = \emptyset$), since no finite body of evidence has an empty intersection. Moreover, it satisfies the axioms of the standard doxastic logic KD45 (see Section 5.6.2). As shown in Example 5.3.2, our notion of belief B and Bel are in general incompatible (even in cases when Bel is consistent). On the other hand, these two notions coincide on a restricted class of evidence models (see Proposition 5.3.3).

5.3.2. Example. We now present two models showing that B and Bel are not comparable in general. More precisely, the first example below illustrates that BP does not imply $BelP$, and the second model shows that $BelP$ does not imply BP even when Bel is consistent.

Consider the evidence model $\mathfrak{M} = (\mathbb{N} \cup \{\spadesuit\}, \mathcal{E}_0, V)$, where $\mathbb{N}$ is the set of natural numbers, $V(p) = \emptyset$, and the set of basic evidence is $\mathcal{E}_0 = \{e_i \mid i \in \mathbb{N}\} \cup \{\{n\} \mid n \in \mathbb{N}\}$ where $e_i = [i, \infty) \cup \{\spadesuit\}$ (see Figure 5.2).

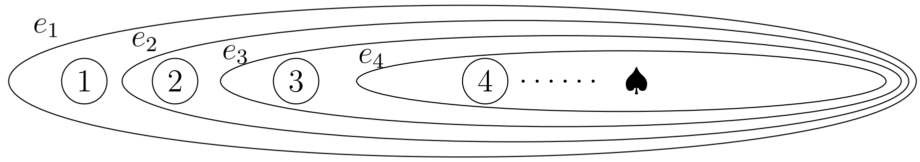


Figure 5.2: $\mathfrak{M} = (\mathbb{N} \cup \{\spadesuit\}, \mathcal{E}_0, V)$

We then have that

$$Max_{\subseteq}(\mathcal{F}) = \{\{e_i \mid i \in \mathbb{N}\}\} \cup \{\{e_i \mid i \leq n\} \cup \{\{m\}\} \mid n, m \in \mathbb{N} \text{ with } m \geq n\}.$$

Therefore, for any $F \in Max_{\subseteq}(\mathcal{F})$, we have

$$\bigcap F = \begin{cases} \{\spadesuit\} & \text{if } F = \{e_i \mid i \in \mathbb{N}\}, \\ \{m\} & \text{if } F = \{e_i \mid i \leq n\} \cup \{\{m\}\} \text{ with } m \geq n. \end{cases}$$

We thus obtain that $\bigcup_{F \in Max_{\subseteq}(\mathcal{F})} \bigcap F = \mathbb{N} \cup \{\spadesuit\}$. This means that $Bel(\mathbb{N} \cup \{\spadesuit\}) = Bel\top$ holds in $\mathfrak{M}$, and moreover, $\mathbb{N} \cup \{\spadesuit\}$ is the only proposition that is believed according to the belief definition of van Benthem and Pacuit (2011). Thus, in particular, $Bel(\mathbb{N}) = \emptyset$, hence, $Bel(\mathbb{N})$ does not hold in $\mathfrak{M}$ (i.e., no state in $\mathbb{N} \cup \{\spadesuit\}$ makes $Bel(\mathbb{N})$ true). On the other hand, we have $F \in \mathcal{F}^{fin}$ iff $F = \{e_i \mid i \in I\}$, or $F = \{e_i \mid i \in I\} \cup \{\{m\}\}$ for some $I \subseteq_{fin} \mathbb{N}$ and $m \geq max(I)$, where $max(I)$ is the greatest natural number in I. Therefore, for every $F \in \mathcal{F}^{fin}$, we have

$$\bigcap F = \begin{cases} [max(I), \infty) \cup \{\spadesuit\} & \text{if } F = \{e_i \mid i \in I\}, \\ \{m\} & \text{if } F = \{e_i \mid i \in I\} \cup \{\{m\}\} \text{ for } m \geq max(I). \end{cases}$$

This implies that, any finite body F of the form $\{e_i \mid i \in I\} \cup \{\{m\}\}$ already supports $\mathbb{N}$. Moreover, if $F = \{e_i \mid i \in I\}$, there exists a stronger finite body F'

of the form $F' = \{e_i \mid i \in I\} \cup \{\{m\}\}$ for some $m \geq max(I)$ that supports $\mathbb{N}$. We therefore have that $B(\mathbb{N})$ holds in $\mathfrak{M}$. Hence, in general, BP does not imply $BelP$.

Now consider the evidence model $\mathfrak{M}' = (\mathbb{N} \cup \{\spadesuit\}, \mathcal{E}'_0, V)$ based on the same domain as $\mathfrak{M}$, and where $V(p) = \emptyset$ and the basic evidence family $\mathcal{E}'_0 = \{[n, \infty) \cup \{\spadesuit\} \mid n \in \mathbb{N}\}$ (see Figure 5.3). The only maximal body of evidence in $\mathcal{E}'_0$ is $\mathcal{E}'_0$

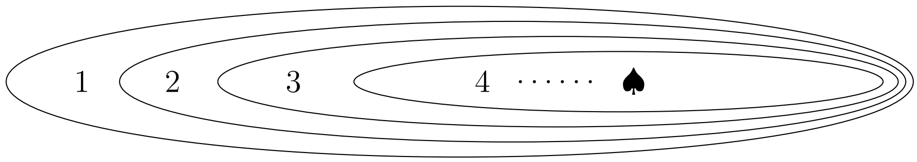


Figure 5.3: $\mathfrak{M}' = (\mathbb{N} \cup \{\spadesuit\}, \mathcal{E}'_0, V)$

itself, and $\bigcap \mathcal{E}'_0 = \{\spadesuit\}$. Therefore, we have $\neg Bel\bot$ in $\mathfrak{M}'$, i.e., Bel is consistent in $\mathfrak{M}'$. Moreover, in particular, $Bel\{\spadesuit\}$ holds in $\mathfrak{M}$. On the other hand, for all finite bodies $F \in \mathcal{F}^{fin}$, we have $\{\spadesuit\} \subsetneq \bigcap F$, implying that $\neg B\{\spadesuit\}$ in $\mathfrak{M}'$. Therefore, even when Bel is consistent, $BelP$ does not imply BP.

There are some special cases where Bel and B do coincide. First of all, our notion of belief B coincides with Bel on the evidence models with finite basic evidence sets $\mathcal{E}_0$. More generally, Bel and B coincide on all *maximally compact* evidence models: the ones in which every body of evidence is equivalent to a finite body of evidence. More formally, an evidence model $\mathfrak{M} = (X, \mathcal{E}_0, V)$ is called *maximally compact* if it satisfies the property

$$\forall F \in \mathcal{F} \exists F' \in \mathcal{F}^{fin} (\bigcap F = \bigcap F') \tag{MC}$$

5.3.3. Proposition. *For all maximally compact evidence models* $\mathfrak{M}{=}(X, \mathcal{E}_0, V)$ *and* $P \subseteq X$, *we have* $BelP = BP$.

Proof:
Let $\mathfrak{M} = (X, \mathcal{E}_0, V)$ be a maximally compact evidence model and $P \subseteq X$.

($\subseteq$) Suppose $BelP$ holds in $\mathfrak{M}$, i.e., suppose that for all $F \in Max_{\subseteq}\mathcal{F}$, we have $\bigcap F \subseteq P$. Now let $F' \in \mathcal{F}^{fin}$. By Zorn's Lemma, F' can be extended to a maximal body of evidence $F'' \in \mathcal{F}$. Note that, since F'' extends F', i.e., $F' \subseteq F''$, we have $\bigcap F'' \subseteq \bigcap F'$. Since $\mathfrak{M}$ is maximally compact, there is $F_0 \in \mathcal{F}^{fin}$ such that $\bigcap F'' = \bigcap F_0$. Now consider the family of evidence $F_0 \cup F'$. Since $\bigcap F_0 = \bigcap F'' \subseteq \bigcap F'$, we have $\bigcap(F_0 \cup F') = \bigcap F_0 \cap \bigcap F' = \bigcap F_0 \neq \emptyset$. Therefore, the family of evidence $F_0 \cup F'$ is a finite body of evidence, i.e., $F_0 \cup F' \in \mathcal{F}^{fin}$. Obviously, $F_0 \cup F'$ extends F', i.e., $F' \subseteq F_0 \cup F'$. Moreover, since $BelP$ holds in $\mathfrak{M}$, we have that $\bigcap F'' \subseteq P$. We then obtain $\bigcap(F_0 \cup F') = \bigcap F_0 = \bigcap F'' \subseteq P$. We have therefore proven that the finite body of evidence $F_0 \cup F'$ extends F' and it entails P. As F' has been chosen arbitrarily from $\mathcal{F}^{fin}$, we conclude that BP holds in $\mathfrak{M}$.

($\supseteq$) Suppose BP holds in $\mathfrak{M}$, i.e., suppose that for all $F \in \mathcal{F}^{fin}$, there exists $F' \in \mathcal{F}^{fin}$ such that $F \subseteq F'$ and $\bigcap F' \subseteq P$. Let $F'' \in Max_{\subseteq}\mathcal{F}$. Then, since $\mathfrak{M}$ is maximally compact, there exists $F_0 \in \mathcal{F}^{fin}$ such that $\bigcap F'' = \bigcap F_0$. Moreover, since BP holds in $\mathfrak{M}$, there exists $F_1 \in \mathcal{F}^{fin}$ such that $F_0 \subseteq F_1$ and $\bigcap F_1 \subseteq P$. Besides, since $\bigcap F_1 \subseteq \bigcap F_0 = \bigcap F''$ and F'' is maximal, we in fact have $F_1 \subseteq F''$ (otherwise, there exists $e \in \mathcal{E}_0$ such that $e \in F_1$ but $e \notin F''$. Therefore, as $\bigcap F_1 \subseteq \bigcap F''$, we would have $\bigcap F_1 \subseteq \bigcap(F'' \cup \{e\})$, and thus $\bigcap(F'' \cup \{e\}) \neq \emptyset$, contradicting maximality of F''.) Therefore, $\bigcap F'' \subseteq \bigcap F_1$, and thus, $\bigcap F_1 = \bigcap F''$. Then, together with $\bigcap F_1 \subseteq P$, we obtain $\bigcap F'' \subseteq P$. As F'' has been chosen arbitrarily from $Max_{\subseteq}\mathcal{F}$, we conclude that $BelP$ holds in $\mathfrak{M}$. $\square$

Another important feature of our belief definition is that B is a *purely topological notion*, as stated in the following proposition which, in turn, constitutes a justification for our use of topo-e-models rather than working with only evidence models.

5.3.4. PROPOSITION. *In every topo-e-model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$, the following are equivalent, for any proposition $P \subseteq X$:*

1. *BP holds (at any state)*
   (i.e., $\forall F \in \mathcal{F}^{fin} \exists F' \in \mathcal{F}^{fin}(F \subseteq F'$ and $\bigcap F' \subseteq P)$);

2. *every evidence can be strengthened to some evidence supporting P*
   (i.e., $\forall e \in \mathcal{E}\, \exists e' \in \mathcal{E}(e' \subseteq e \cap P)$);

3. *every argument (for anything) can be strengthened to an argument for P*
   (i.e., $\forall U \in \tau \backslash \{\emptyset\}\, \exists U' \in \tau \backslash \{\emptyset\}(U' \subseteq U \cap P)$);

4. *there is a justification for P, i.e., there is some argument for P which is consistent with any available evidence*
   (i.e., $\exists U \in \tau(U \subseteq P$ and $\forall e \in \mathcal{E}(U \cap e \neq \emptyset)))$;

5. *P includes some dense open set*
   (i.e., $\exists U \in \tau(U \subseteq P$ and $Cl(U) = X)$);

6. *$Int(P)$ is* dense *in τ (i.e., $Cl(Int(P)) = X$), or equivalently, $X \backslash P$* is nowhere dense (i.e., *$Int(Cl(X \backslash P)) = \emptyset$);*

7. *$[\forall]\Diamond\Box P$ holds (at any state) (i.e., $[\forall]\Diamond\Box P = X$), or equivalently, $[\forall]\Diamond\Box P \neq \emptyset$.*

Proof:
The equivalence between (1), (2) and (3) is easy, and follows directly from definitions of combined evidence and argument. The equivalence of (5) and (6) is

also straightforward (recall that $Int(P)$ is the largest open contained in P). The equivalence between (4) and (5) simply follows from the definitions of arguments and dense sets. For the equivalence of (6) and (7), recall that $[\forall]$ is the global modality, $\Box$ is interior and $\Diamond$ is closure. For the equivalence of (3) and (4):

(3)⇒(4): Suppose (3) holds and consider the open set $Int(P)$. We will show that $Int(P)$ is a justification for P, i.e., $Int(P) \cap e \neq \emptyset$ for all $e \in \mathcal{E}$. Let $e \in \mathcal{E}$. By (3), since $e \in \mathcal{E} \subseteq \tau \backslash \{\emptyset\}$, there exists $U_0 \in \tau \backslash \{\emptyset\}$ such that $U_0 \subseteq e \cap P$. We then have $Int(U_0) \subseteq Int(e \cap P) = Int(e) \cap Int(P)$. Therefore, since U_0 and e are open sets, we obtain $U_0 \subseteq e \cap Int(P)$. As $U_0 \neq \emptyset$, we conclude that $e \cap Int(P) \neq \emptyset$.

(4)⇒(3): Suppose (4) holds, i.e., suppose that there exists $U_0 \in \tau$ such that (a) $U_0 \subseteq P$ and (b) $U_0 \cap e \neq \emptyset$ for all $e \in \mathcal{E}$. Let $U \in \tau$ with $U \neq \emptyset$. Now consider the open set $U \cap U_0$. Since $\mathcal{E}$ is a basis of τ, there exists $e \in \mathcal{E}$ such that $e \subseteq U$. Therefore, by (b), the intersection $U \cap U_0 \neq \emptyset$, thus, $U \cap U_0 \in \tau \backslash \{\emptyset\}$. By (a), we also have $U \cap U_0 \subseteq U \cap P$. $\Box$

Proposition 5.3.4 deserves a closer look as it describes the topological properties of our notion of belief, as well as states that our belief is the same as *justified belief* that is coherent with every available evidence. The equivalence between (1), (2) and (3) shows that we can define BP in equivalent ways by using only basic evidence pieces (i.e., the elements of $\mathcal{E}_0$), or by using only combined evidence (i.e., the elements of $\mathcal{E}$), or by using only the open sets of the generated evidential topology $\tau_{\mathcal{E}}$. Proposition 5.3.4-(4) proves that our definition of belief indeed gives us a conception of *evidentially justified belief*. The requirement that any justification of a believed proposition must be *open* in the evidential topology simply means that the justification is ultimately based on the available evidence; while the requirement that the justification is *dense* (in the same topology) means that all the agent's beliefs must be coherent with all her evidence. Therefore, believed propositions, according to our definition, are those for which there is some evidential justification that is consistent with all available (basic or combined) evidence. Moreover, whenever a proposition P is believed, there exists a *weakest (most general) justification* for P, namely the open set $Int(P)$. Proposition 5.3.4-(5-7) provide topological reformulations of the above items. In particular, Proposition 5.3.4-(6) shows that our proposal is very natural from a topological perspective: it is equivalent to saying that *P is believed iff the complement of P is nowhere dense*. Since nowhere dense sets are one of the topological concepts of "small" or "negligible" sets, this amounts to believing propositions iff they are true in *almost all* epistemically-possible worlds, where "almost all" spelled out topologically as "everywhere but a nowhere dense part of the model". Finally, Proposition 5.3.4-(7) tells us that belief is definable in terms of the operators $[\forall]$ and $\Box$.

We will provide further technical results such as the soundness and completeness of the belief logic with respect to the topo-e-models in Section 5.6.2. We now proceed with formalizing a notion of conditional beliefs on topo-e-models.

5.3.3 Conditional Belief on Topo-e-models

The belief semantics given in Section 5.3.2 can be generalized to *conditional beliefs* $B^Q P$ by relativizing the simple belief definition BP to the given condition Q, in a way similar to how we obtained conditional belief semantics in Section 4.3.2. However, this current setting requires a somewhat more careful treatment (as recognized already in van Benthem and Pacuit, 2011) since some of the agent's evidence might be inconsistent with the condition Q. While evaluating beliefs under the assumption that the given condition Q is true, one should focus only on the evidence that is consistent with Q by neglecting the evidence pieces that are disjoint with Q. Therefore, in order to define conditional beliefs, we need a "relativized" version of the notion of consistent (bodies of) evidence.

Given an evidence model $\mathfrak{M} = (X, \mathcal{E}_0, V)$, for any subsets $Q, A \subseteq X$, we say that A is *Q-consistent* iff $Q \cap A \neq \emptyset$. Moreover, a body of evidence F is called Q-consistent iff $\bigcap F \cap Q \neq \emptyset$. We can then define conditional beliefs based on these notions of "conditional consistency". We say that P *is believed given* Q, and write $B^Q P$, iff *every finite Q-consistent body of evidence can be strengthened to some finite Q-consistent body of evidence supporting the proposition $Q \to P$* (i.e. $\neg Q \cup P$)).

An analogue of Proposition 5.3.4 providing different characterizations can also be proven for conditional belief:

5.3.5. PROPOSITION. *In every topo-e-model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$, the following are equivalent, for any two propositions $P, Q \subseteq X$ with $Q \neq \emptyset$:*

1. *$B^Q P$ holds (at any state);*

2. *every Q-consistent evidence can be strengthened to some Q-consistent evidence supporting $Q \to P$*
(i.e., $\forall e \in \mathcal{E}(e \cap Q \neq \emptyset \Rightarrow \exists e' \in \mathcal{E}(e' \cap Q \neq \emptyset$ and $e' \subseteq e \cap (Q \to P))))$;

3. *every Q-consistent argument can be strengthened to a Q-consistent argument for $Q \to P$*
(i.e., $\forall U \in \tau(U \cap Q \neq \emptyset \Rightarrow \exists U' \in \tau(U' \cap Q \neq \emptyset$ and $U' \subseteq U \cap (Q \to P))))$;

4. *there is some Q-consistent argument for $Q \to P$ whose intersection with any Q-consistent evidence is Q-consistent*
(i.e., $\exists U \in \tau(U \cap Q \neq \emptyset$ and $U \subseteq Q \to P$ and $\forall e \in \mathcal{E}(e \cap Q \neq \emptyset \Rightarrow (U \cap e) \cap Q \neq \emptyset)))$;

5. *$Q \to P$ includes some Q-consistent open set which is dense in Q*
(i.e., $\exists U \in \tau(U \cap Q \neq \emptyset$ and $U \subseteq Q \to P$ and $Q \subseteq Cl(U \cap Q)))$;

6. *$Int(Q \to P)$ is* dense *in Q*
(i.e., $Q \subseteq Cl(Q \cap Int(Q \to P)))$;

7. *$\forall(Q \to \Diamond(Q \wedge \Box(Q \to P)))$ holds (at any state) (i.e., $\forall(Q \to \Diamond(Q \wedge \Box(Q \to P))) = X$), or equivalently, $\forall(Q \to \Diamond(Q \wedge \Box(Q \to P))) \neq \emptyset$.*

Proof:
The equivalence between (1), (2), (3) is easy and directly follows from the semantics of $B^Q P$, and the definitions of Q-consistent evidence and Q-consistent argument. For the equivalence between (5) and (6), consider the weakest argument $Int(Q \to P)$ for $Q \to P$. And, for the equivalence of (6) and (7), recall that $[\forall]$ is the universal quantifier, $\Box$ is interior and $\Diamond$ is closure. We here show only the equivalence between (3) and (4), and between (4) and (5) in details.

(3)⇒(4): Suppose (3) holds and consider the weakest argument $Int(Q \to P)$ for $Q \to P$. Since $X \in \mathcal{E}$ and X is Q-consistent, by (3), there exists a stronger $U \in \tau$ such that $U \cap Q \neq \emptyset$ and $U \subseteq Q \to P$. Since $Int(Q \to P)$ is the largest open with $Int(Q \to P) \subseteq Q \to P$, we obtain $U \subseteq Int(Q \to P) \subseteq Q \to P$ for any such U, therefore, $Int(Q \to P)$ is also Q-consistent. Let $e \in \mathcal{E}$ be such that $e \cap Q \neq \emptyset$. Therefore, since $\mathcal{E} \subseteq \tau$, by (3), there exists $U' \in \tau$ such that $U' \cap Q \neq \emptyset$ and $U' \subseteq e \cap (Q \to P)$. By the previous argument, we know that $U' \subseteq Int(Q \to P)$, thus, $U' \subseteq e \cap Int(Q \to P) \neq \emptyset$. And, since U' is Q-consistent, the result follows.

(4)⇒(3): Suppose (4) holds, i.e., suppose that there is $U_0 \in \tau$ such that (a) $U_0 \cap Q \neq \emptyset$, (b) $U_0 \subseteq Q \to P$ and (c) for all $e \in \mathcal{E}$ with $e \cap Q \neq \emptyset$, we have $(U_0 \cap e) \cap Q \neq \emptyset$. Let $U \in \tau$ be such that $U \cap Q \neq \emptyset$ and consider the open set $U \cap U_0$. Since $U \cap Q \neq \emptyset$ and $\mathcal{E}$ is a basis for τ, there exists $e_0 \in \mathcal{E}$ such that $e_0 \subseteq U$ and $e_0 \cap Q \neq \emptyset$. Therefore, by (c), we have that $(U_0 \cap e_0) \cap Q \neq \emptyset$, thus, the open set $U_0 \cap e_0$ is Q-consistent. Moreover, since $U_0 \subseteq Q \to P$ and $e_0 \subseteq U$, we obtain $U_0 \cap e_0 \subseteq U \cap (Q \to P)$.

(4)⇔(5): For the left-to-right direction, suppose (4) holds as in the above case, and toward showing $Q \subseteq Cl(U_0 \cap Q)$, let $x \in Q$ and $e \in \mathcal{E}$ such that $x \in e$. Therefore, e is Q-consistent, i.e., $e \cap Q \neq \emptyset$. Then, by (4), we obtain $(U_0 \cap e) \cap Q \neq \emptyset$, implying that $x \in Cl(U_0 \cap Q)$. For the right-to-left direction, suppose (5) holds with U_0 the witness and let $e \in \mathcal{E}$ be such that $e \cap Q \neq \emptyset$. This means that there is $y \in e \cap Q$, thus, $y \in Q$. Then, by (5), $y \in Cl(U_0 \cap Q)$. Therefore, as $y \in e \in \mathcal{E}$, we conclude $(U_0 \cap Q) \cap e \neq \emptyset$. □

5.4 Evidence Dynamics

What we have presented so far focuses on how an agent forms beliefs based on a *fixed* collection of evidence pieces she *has gathered so far*. However, collecting and evaluating evidence is not a one-time process: the agent might receive further information or re-evaluate her current evidence set, thus, she might need to revise her beliefs and knowledge accordingly. There are different ways one can incorporate new information into the initial evidence structure depending on, e.g., the information source and how the agent regards the new information. Van Benthem and Pacuit (2011) presents a wide range of evidence dynamics as model transformations, and in this section, we study their dynamic operators such as public announcements, evidence addition, evidence upgrade and (a feasible version of) evidence combination implemented on topo-e-models. While the only domain changing operator is the so-called updates for public announcemets; evidence addition, upgrade and combination only affect the agent's initial basic evidence set $\mathcal{E}_0$, and thus the combined evidence set $\mathcal{E}$ and the generated topology $\tau_{\mathcal{E}}$. We here only describe the corresponding model changes and leave the presentation of the corresponding dynamic logics for Section 5.6.6. Throughout this section, we are given a topo-e-model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ and some proposition $P \subseteq X$, with $P \neq \emptyset$.

Public Annoucements. Public announcements involve learning a new fact P with absolute certainty. The announced proposition P is taken as "hard information", that is, a true information coming from an infallible source. The standard way of interpreting this—as also mentioned in Section 4.2.2—is via model restrictions, both on relational and neighbourhood structures (see, e.g., Definition 4.2.4). For evidence models, this means keeping only the worlds in P and only the P-consistent evidence pieces. Topologically, this is a move from the original space (X, τ) to the *subspace* (P, τ_P) induced by P.

5.4.1. Definition. [Public Announcements]
The model $\mathfrak{M}^{!P} = (X^{!P}, \mathcal{E}_0^{!P}, \tau^{!P}, V^{!P})$ is defined as follows: $X^{!P} = P$, $\mathcal{E}_0^{!P} = \{e \cap P \mid e \in \mathcal{E}_0 \text{ with } e \cap P \neq \emptyset\}$, $\tau^{!P} = \{U \cap P \mid U \in \tau\}$, and $V^{!P}(p) = V(p) \cap P$ for each $p \in$ PROP.

It is easy to check that $\mathfrak{M}^{!P}$ is a topo-e-model with the set of combined evidence

$$\mathcal{E}^{!P} = \{e \cap P \mid e \in \mathcal{E} \text{ with } e \cap P \neq \emptyset\}.$$

Evidence addition. An agent can also regard and admit the new information on par with her old evidence without assuming it is hard information. In this case, the natural thing to do is to add the new piece of evidence to the initial basic evidence set and generate the evidential topology from the new evidence collection. This action simply describes the most straightforward way an agent collects individually consistent evidence pieces.

5.4.2. Definition. [Evidence Addition]
The model $\mathfrak{M}^{+P} = (X^{+P}, \mathcal{E}_0^{+P}, \tau^{+P}, V^{+P})$ is defined as follows: $X^{+P} = X$, $\mathcal{E}_0^{+P} = \mathcal{E}_0 \cup \{P\}$, τ^{+P} is the topology generated by $\mathcal{E}_0^{+P}$, and $V^{+P} = V$.

Again, $\mathfrak{M}^{+P}$ is a topo-e-model, since $\emptyset \notin \mathcal{E}_0^{+P}$ and $X^{+P} = X \in \mathcal{E}_0^{+P}$, and τ^{+P} is the evidential topology generated by $\mathcal{E}_0^{+P}$. Moreover, the set of combined evidence $\mathcal{E}^{+P}$ of $\mathfrak{M}^{+P}$ can be described as

$$\mathcal{E}^{+P} = \mathcal{E} \cup \{e \cap P \mid e \in \mathcal{E} \text{ with } e \cap P \neq \emptyset\},$$

which clearly constitutes a basis for τ^{+P}.

Evidence upgrade. The operator of *evidence upgrade* $\Uparrow P$ incorporates P into all other pieces of evidence, thus making P the most important available evidence.

5.4.3. Definition. [Evidence Upgrade]
The model $\mathfrak{M}^{\Uparrow P} = (X^{\Uparrow P}, \mathcal{E}_0^{\Uparrow P}, \tau^{\Uparrow P}, V^{\Uparrow P})$ is defined as follows: $X^{\Uparrow P} = X$, $\mathcal{E}_0^{\Uparrow P} = \{e \cup P \mid e \in \mathcal{E}_0\} \cup \{P\}$, $\tau^{\Uparrow P}$ is the topology generated by $\mathcal{E}_0^{\Uparrow P}$, and $V^{\Uparrow P} = V$.

$\mathfrak{M}^{\Uparrow P}$ is obviously a topo-e-model for the same reasons given above, and the set of combined evidence $\mathcal{E}^{\Uparrow P}$ of $\mathfrak{M}^{\Uparrow P}$ can be described as

$$\mathcal{E}^{\Uparrow P} = \{e \cup P \mid e \in \mathcal{E}\} \cup \{P\}.$$

The following observation proves that evidence upgrade with P in fact makes the proposition P the most important evidence piece in the sense that the believed propositions in $\mathfrak{M}^{\Uparrow P}$ are exactly those entailed by P.

5.4.4. Proposition. *Given a topo-e-model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ and propositions $P, Q \subseteq X$ with $P, Q \neq \emptyset$,*

$$P \subseteq Q \textit{ iff } BQ \textit{ holds in } \mathfrak{M}^{\Uparrow P}.$$

Proof:
Suppose $P \not\subseteq Q$. This means, by definition of $\mathcal{E}^{\Uparrow P}$, that there is no argument in $\mathfrak{M}^{\Uparrow P}$ that supports Q (since every element e of $\mathcal{E}^{\Uparrow P}$ includes P). Therefore, by Proposition 5.3.4-(4), we obtain that BQ does not hold in $\mathfrak{M}^{\Uparrow P}$. For the other direction, suppose $P \subseteq Q$ and let $e \in \mathcal{E}^{\Uparrow P}$. By the definition of $\mathcal{E}^{\Uparrow P}$, either $e = P$ or there is $e' \in \mathcal{E}$ such that $e = e' \cup P$. If $e = P$, then obviously $e \cap Q = P \cap Q = P \neq \emptyset$ (where we used the assumption $P \subseteq Q$). If $e = e' \cup P$, then $e \cap Q = (e' \cup P) \cap Q = (e' \cap Q) \cup (P \cap Q) = (e' \cap Q) \cup P \supseteq P \neq \emptyset$ (where we again used the assumption $P \subseteq Q$). Therefore, by Proposition 5.3.4-(4), we obtain that BQ holds in $\mathfrak{M}^{\Uparrow P}$. $\square$

Feasible evidence combination. Another dynamic operation considered in (van Benthem and Pacuit, 2011) is *evidence combination*. We here adapt it to our topological setting, which assumes that agents can combine only finitely many pieces of evidence at a given time. This is what we call *feasible evidence combination*, in contrast to the infinitary combinations allowed in (van Benthem and Pacuit, 2011). The dynamic operation of evidence combination is concerned with *internal* re-evaluation of the evidence pieces the agent possesses, it does not involve any new external information. Feasible evidence combination, intuitively speaking, produces a model in which every evidence previously regarded as combined evidence becomes a basic piece of evidence.

5.4.5. Definition. [Feasible Evidence Combination]
The model $\mathfrak{M}^{\#} = (X^{\#}, \mathcal{E}_0^{\#}, \tau^{\#}, V^{\#})$ is defined as follows: $X^{\#} = X$, $\mathcal{E}_0^{\#}$ is the smallest set closed under nonempty, finite intersections and containing $\mathcal{E}_0$, and $\tau^{\#}$ is the topology generated by $\mathcal{E}_0^{\#}$, and $V^{\#} = V$.

$\mathfrak{M}^{\#}$ is clearly a topo-e-model. In fact, since $\mathcal{E}_0^{\#}$ is obtained by closing $\mathcal{E}_0$ under finite and nonempty intersections, we have $\mathcal{E}_0^{\#} = \mathcal{E}^{\#} = \mathcal{E}$, and therefore, the topology stays the same, i.e., $\tau = \tau^{\#}$.

The precise syntax capturing the above evidence dynamics, and the complete axiomatizations of the corresponding logics will be provided in Section 5.6. We now continue with our proposal for a defeasible type of knowledge based on topo-e-models.

5.5 Knowledge

The only notion of knowledge we have considered so far in this chapter was the so-called *infallible knowledge*—represented by the global modality $[\forall]$—that conveys absolute certainty (Section 5.2.2). However, there are very few things we could know in this strong sense, maybe, say, only logical-mathematical tautologies. We now define a "softer" (weaker) notion of knowledge that approximates better the common usage of the word than infallible knowledge. In particular, in this section, we study a notion of (fallible) knowledge based on *factive justification*. Formally, given a topo-e-model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$, we set

$$KP := \{x \in X \mid \exists U \in \tau \, (x \in U \subseteq P \text{ and } Cl(U) = X)\},$$

stating that KP *holds at* x iff P *includes a dense open neighborhood of* x. Similarly to the cases for belief and conditional beliefs (see Propositions 5.3.4 and 5.3.5), we can provide several equivalent definitions of KP on topo-e-models as follow.

5.5.1. Proposition. *Let $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ be a topo-e-model, and assume $x \in X$ is the actual world. The following are equivalent for all $P \subseteq X$:*

1. *KP holds at x in $\mathfrak{M}$*
   (i.e., $\exists U \in \tau\,(x \in U \subseteq P$ and $Cl(U) = X)$);

2. *there is some factive justification for P at x, i.e., there is some factive argument for P at x which is consistent with any available evidence*
   (i.e., $\exists U \in \tau(x \in U \subseteq P$ and $\forall e \in \mathcal{E}(U \cap e \neq \emptyset))$);

3. *$Int(P)$ contains the actual state and is* dense *in τ*
   (i.e., $x \in Int(P)$ and $Cl(Int(P)) = X$);

4. *$\Box P \wedge BP$ holds at x.*

Proof:
The proof is similar to the proof of Proposition 5.3.4. For the equivalence between (1) and (2), recall that $\mathcal{E}$ constitutes a basis for τ. The equivalence of (2) and (3) is also straightforward (recall that $Int(P)$ is the largest open set contained in P). For the equivalence of (3) and (4), see Proposition 5.3.4-(6) and recall that $\Box$ is interpreted as the interior operator. $\Box$

Therefore, as the equivalence between Proposition 5.5.1-(1) and (2) shows, we propose to define knowledge as *correctly justified belief.* In other words, we here study a notion of knowledge that is characterized as *belief based on true justification.* We would like to emphasize that the above-defined notion of knowledge does *not* boil down to "justified true belief". This would clearly be vulnerable to Gettier-type counterexamples (Gettier, 1963). To explain better, we illustrate the semantics we propose for justified belief and knowledge, as well as the connection between the two notions in the example below.

5.5.2. Example. Consider the topo-e-model $\mathfrak{M} = ([0,1], \mathcal{E}_0, \tau, V)$, where $\mathcal{E}_0 = \{(a,b) \cap [0,1] \mid a, b \in \mathbb{R},\ a < b\}$ and $V(p) = \emptyset$. The generated topology τ is the standard topology on $[0,1]$. Let $P = [0,1] \backslash \{\frac{1}{n} \mid n \in \mathbb{N}\}$ be the proposition stating that "the actual state is *not* of the form $\frac{1}{n}$, for any $n \in \mathbb{N}$" (see Figure 5.4). Since the complement $\neg P = [0,1] \backslash P = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is nowhere dense (i.e., $Int(Cl(\neg P)) = Int(\neg P) = \emptyset$), the agent *believes* P, and e.g. $U = \bigcup_{n \geq 1}(\frac{1}{n+1}, \frac{1}{n})$ is a justification for P, that is, U is a dense open subset of P. This belief is *true at world* $0 \in P$. But this *true belief is not knowledge* at 0: no justification for P is true at 0, since P does not include any open neighborhood of 0, so $0 \notin Int(P)$ and hence $0 \notin KP$. This shows that $KP \neq P \wedge BP$. Moreover, P is known in all the *other* states $x \in P \backslash \{0\}$, since

$$\forall x \in P \backslash \{0\}\, \exists \epsilon > 0 (x \in (x - \epsilon, x + \epsilon) \subseteq P),$$

therefore $x \in Int(P)$.

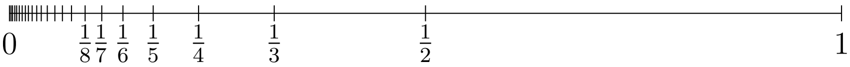


Figure 5.4: $([0,1],\tau)$

Going back to Stalnaker's epistemic-doxastic system Stal, it is easy to see that K together with justified belief B satisfies Stalnaker's Full Belief principle $BP = BKP$ (see Table 4.1). These operators in fact satisfy all the axioms and rules of the system Stal on all topo-e-models, thus, on *all topological spaces*, not only on the restricted class of extremally disconnected spaces. We prove the soundness and completeness of Stalnaker's system Stal with respect to all topo-e-models in Section 5.6.4.

One interesting property of this weaker type of knowledge is it being *defeasible* in the light of new information, even when the new information is true. In contrast, the usual assumption in epistemic logic is that *knowledge acquisition is monotonic*. As a result, logicians typically assume that knowledge is "irrevocable": once acquired, it cannot be defeated by any further evidence gathered later. In our setting, the only irrevocable knowledge is the absolutely certain one (true in all epistemically-possible worlds), captured by the operator $[\forall]$. Clearly, K is *not* irrevocable.

5.5.1 Knowledge is *defeasible*

Gettier (1963)—with his famous counterexamples against the account of knowledge as justified true belief—triggered an extensive discussion in epistemology that is concerned with understanding what knowledge is, and in particular, with identifying the exact properties and conditions that render a piece of justified true belief knowledge. Epistemologists have made various proposals such as, among others, the *no false lemma* (Clark, 1963), the *defeasibility analysis of knowledge* (Lehrer and Paxson, 1969; Lehrer, 1990; Klein, 1971, 1981), the *sensitivity account* (Nozick, 1981), the *safety account* (Sosa, 1999), and the *contextualist account* (DeRose, 2009)[9]. While there is still very little agreement about these questions, the extent of the post-Gettier literature at the very least shows that the relation between justified belief and knowledge is very delicate, and it is not an easy task, if possible, to identify a unique notion of knowledge that can deal with all kinds of intuitive counterexamples. However, as Rott (2004) states, one can accept that all these proposals "capture important intuitions that can in some way or other be regarded as relevant to the question whether or not a given belief constitutes a piece of knowledge" (Rott, 2004, p. 469). Providing an extensive philosophical analysis regarding the aforementioned theories of knowledge is way

[9] For an overview of responses to the Gettier challenge and a detailed discussion, we refer the reader to (Rott, 2004; Ichikawa and Steup, 2013).

beyond the scope of this dissertation. However, in this section, we argue that our conception of knowledge captured by the modality K is stronger than Clark's "no false lemma" (Clark, 1963), and very close to (though subtly different from) the so-called defeasibility theory of knowledge held by Lehrer and Paxson (1969); Lehrer (1990); Klein (1971, 1981).

Clark's influential "no false lemma" proposal is to require a *correct "justification"*—one that doesn't use any falsehood—for a piece of belief to constitute knowledge (Clark, 1963). As similar as this sounds to our knowledge K, our proposal imposes a stronger requirement than Clark's, since our concept of justification requires consistency with all the available (combined) evidence. In our terminology, Clark only requires a factive *argument* for P. So Clark's approach is 'local', assessing a knowledge claim based only on the truth of the evidence pieces (and the correctness of the inferences) that are used to justify it. Our proposal is coherentist, and thus 'holistic', assessing knowledge claims by their coherence with all of the agent's acceptance system: justifications need to be checked against all the other arguments that can be constructed from the agent's current evidence.

On the other hand, the defeasibility theory of knowledge, roughly speaking, defends that knowledge can be defined as *justified belief that cannot be defeated by any* factive *evidence gathered later* (though it may be defeated by false evidence). Therefore, knowledge is equated with *undefeated justified belief*. In its simplest version, as formalized by Stalnaker (2006), *the agent knows P if and only if*

1. *P is true*
2. *she believes that P, and*
3. *her belief in P cannot be defeated by new factive information.*

In other words, given a true proposition P, the agent knows P iff she does not give up her belief in P after receiving any true information, i.e., her belief in P is *stable* for true information. As Rott (2004) pointed out, this is a simple version of defeasibility theory of knowledge as it requires only the belief in P itself to be stable. For this reason, Rott (2004) calls this *stable belief theory* or *stability theory of knowledge*. The above version has been challenged for being too weak to form knowledge. The full-fledged version of the defeasibility theory, as held by Lehrer and others, insists that, in order to know P, not only the belief in P has to stay stable, but also its justification (i.e. what we call here "an argument for P") should be undefeated. More precisely, according to this strong version of defeasibility theory, *the agent knows P if and only if*

1. P is true
2. she believes that P,

3. her belief in P cannot be defeated by new factive information, and

4. *her justification is undefeated by new factive information.*

In other words, for the agent to know P, there must exist an argument for P that is believed conditional on every true evidence. Clearly, this implies that the belief in P is stable, however, it is not at all obvious whether having stable belief in P would imply its justification being undefeated. Indeed, Lehrer claims that this is not the case. The problem is that, when confronted with various new pieces of evidence, the agent might keep switching between different justifications (for believing P); thus, she may keep believing in P conditional on any such new true evidence, without actually having any good, robust justification (i.e., one that remains itself undefeated by all true evidence) (see Example 5.5.4). To have knowledge, we thus need a *stable justification.*[10]

However, the above interpretation (of both the stability and the defeasibility theory) was also attacked as being *too strong*: if we allow as potential defeaters *all* factive propositions (i.e. all sets of worlds P containing the actual world), then there are intuitive examples showing that knowledge KP *can* be defeated (Klein, 1980, 1981). Here is such an example discussed by Klein (1981), a leading proponent of the defeasibility theory. Loretta filled in her federal taxes, following very carefully all the required procedures on the forms, doing all the calculations and double checking everything. Based on this evidence, she correctly believes that she owes \$500, and she seems perfectly justified to believe this. So it seems obvious that she *knows* this. But suppose now that, being aware of her own fallibility, she asks her accountant to check her return. The accountant finds no errors (when there are in fact some errors in her calculation, yet not affecting the correct result that she owes \$500), and so he sends her his reply reading "Your return contains *no* errors"; but he inadvertently leaves out the word "no". If Loretta would learn the true fact that the accountant's letter actually reads "Your return contains errors", she would lose her true belief that she owed \$500! So it seems that there exist defeaters that are true but "misleading". We can formalize this counterexample as follows, and show that our knowledge K is neither stable nor indefeasible:

5.5.3. Example. Consider the model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$, where $X = \{x_1, x_2, x_3\}$, $V(p) = \emptyset$, $\mathcal{E}_0 = \{X, O_1, O_2\}$, $O_1 = \{x_1, x_2\}$, $O_2 = \{x_2, x_3\}$ (see Figure 5.5). The resulting set of combined evidence is $\mathcal{E} = \{X, O_1, O_2, \{x_2\}\}$. Assume the actual world is x_1. Then O_1 is *known*, since $x_1 \in \mathit{Int}(O_1) = O_1$ and $\mathit{Cl}(O_1) = X$. Now

[10] Lehrer uses the metaphor of an *Ultra-Justification Game* (Lehrer, 1990), according to which 'knowledge' is based on arguments that survive a game between the Believer and an omniscient truth-telling Critic, who tries to defeat the argument by using both the Believer's current "justification system" and *any new true evidence* (see Fiutek, 2013, Section 5.2 for a formalization of Lehrer's ultra-justification game).

consider the model $\mathfrak{M}^{+O_3} = (X, \mathcal{E}_0^{+O_3}, \tau^{+O_3}, V)$ obtained by adding the new evidence $O_3 = \{x_1, x_3\}$ (as in Definition 5.4.2). We have $\mathcal{E}_0^{+O_3} = \{X, O_1, O_2, O_3\}$, so $\mathcal{E}^{+O_3} = \{X, O_1, O_2, O_3, \{x_1\}, \{x_2\}, \{x_3\}\}$. Note that the new evidence is *true* ($x_1 \in O_3$). However, O_1 is *not even believed* in $\mathfrak{M}^{+O_3}$ anymore, since $O_1 \cap \{x_3\} = \emptyset$, so O_1 is no longer dense in τ^{+O_3}. Therefore, O_1 is *no longer known* after the true evidence O_3 was added!

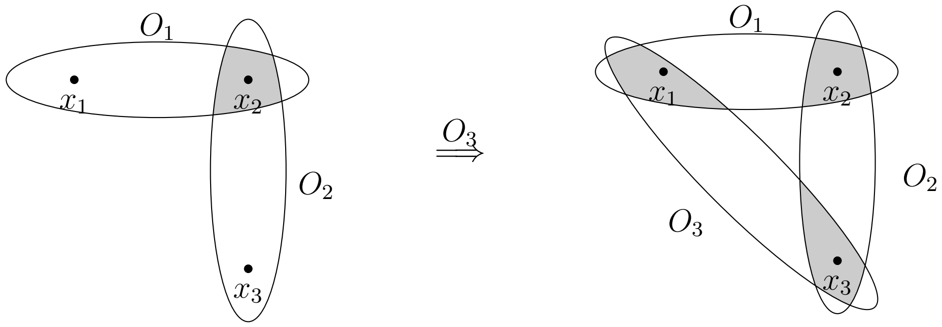


Figure 5.5: From $\mathfrak{M}$ to $\mathfrak{M}^{+O_3}$

Klein's story corresponds to taking O_1 to represent Loretta's direct evidence (based on careful calculations) that she owes \$500, O_2 to represent her prior evidence (based on past experience) that the accountant doesn't make mistakes in his replies to her, and O_3 the potential new evidence provided by the letter. In conclusion, our notion of knowledge is incompatible with the above-mentioned strong interpretations of both stability and defeasibility theory, thus confirming the objections raised against them.

Klein's solution is that one should exclude such *misleading* defeaters, which may "unfairly" defeat a good justification. But how can we distinguish them from genuine defeaters? Klein's diagnosis, in Foley's more succinct formulation, is that "a defeater is misleading if it justifies a falsehood in the process of defeating the justification for the target belief" (Foley, 2012, p. 96). In the example, the falsehood is that the accountant had discovered errors in Loretta's tax return. It seems that the new evidence O_3 (the existence of the letter as actually written) supports this falsehood, but how? According to us, it is the combination $O_2 \cap O_3$ of the new (true) evidence O_3 with the old (false) evidence O_2 that supports the new falsehood: the true fact (about the letter saying what it says) entails a falsehood *only* if it is taken in conjunction with Loretta's prior evidence (or blind trust) that the accountant cannot make mistakes. So intuitively, *misleading defeaters are the ones which may lead to new false conclusions when combined with some of the old evidence.*

Misleading evidence and weakly indefeasible knowledge. We proceed now to formalize the distinction between misleading and genuine (i.e., nonmisleading) defeaters. Given a topo-e-model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$, a state $x \in X$ and a proposition $Q \subseteq X$,

- Q is *misleading at* $x \in X$ *with respect to* $\mathcal{E}$ if evidence-addition with Q produces some false new evidence;

equivalently, and more formally, if there is some $e \in \mathcal{E}^{+Q} \backslash \mathcal{E}$ such that $x \notin e$, i.e., if there is some $e \in \mathcal{E}$ such that $x \notin (e \cap Q)$ and $(e \cap Q) \notin \mathcal{E} \cup \{\emptyset\}$. A proposition $Q \subseteq X$ is called *nonmisleading* if Q is not misleading. It is easy to see that *old evidence* $e \in \mathcal{E}$ is by definition nonmisleading with respect to $\mathcal{E}$ (i.e., each $e \in \mathcal{E}$ is nonmisleading with respect to $\mathcal{E}$), and *new nonmisleading* evidence must be true (i.e., if $Q \subseteq X$ is nonmisleading at x and $Q \notin \mathcal{E}$, then $x \in Q$).

We are now in the position to formulate precisely the "weakened" versions of both stability and defeasibility theories that we are looking for. The weak stability theory will stipulate that *the agent knows* P *if and only if*

1. P is true
2. she believes that P,
3. her belief in P cannot be defeated by any *nonmisleading* evidence,

On the other hand, the weak defeasibility theory requires that there exists some justification (argument) for P that is undefeated by every nonmisleading proposition. More precisely, the weak defeasibility theory strengthens the above described weak stability theory by the following "stable justification" clause:

4. her belief in its justification is undefeated by any *nonmisleading* evidence.

Finally, we also provide a third formulation, which one might call *epistemic coherence theory*, saying that P *is known iff there exists some justification (argument) for* P *which is consistent with every nonmisleading proposition.* While our proposed notion of knowledge is stronger than the one described by the weak stability theory, as illustrated by Example 5.5.4, it coincides with the ones defined by the weak defeasibility and epistemic coherence theories (see Proposition 5.5.5). In particular, the following counterexample shows that weak stability is (only a necessary, but) not a sufficient condition for knowledge K:

5.5.4. Example. Consider the model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$, where $X = \{x_0, x_1, x_2\}$, $V(p) = \emptyset$, $\mathcal{E}_0 = \{X, O_1, O_2\}$ with $O_1 = \{x_1\}$, $O_2 = \{x_1, x_2\}$ (see Figure 5.6). The resulting set of combined evidence is $\mathcal{E} = \mathcal{E}_0$. Assume the actual world is x_0, and let $P = \{x_0, x_1\}$. Then, P is *believed* in $\mathfrak{M}$ (since its interior $Int(P) = \{x_1\}$ is dense in τ) but it is *not known* (since $x_0 \notin Int(P) = \{x_1\}$). However, we can show that P *is believed in* $\mathfrak{M}^{+Q}$ *for any nonmisleading* Q *at* x_0. For this, note that the family of nonmisleading propositions (at x_0) is $\mathcal{E} \cup \{P, \{x_0\}\} = \{X, O_1, O_2, P, \{x_0\}\}$. It is easy to see that for each set Q in this family, BP holds in $\mathfrak{M}^{+Q}$.

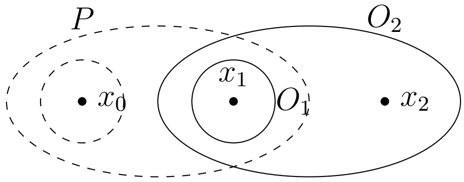


Figure 5.6: $\mathfrak{M} = (X, \mathcal{E}_0, V)$: The continuous ellipses represent the currently available pieces of evidence, while the dashed ones represent the other nonmisleading propositions.

One should stress that our counterexample agrees with the position taken by most proponents of the defeasibility theory: stability of (justified) belief is not enough for knowledge. Intuitively, what happens in the above example is that, although the agent continues to believe P given any nonmisleading evidence, her justification keeps changing. For example, while the only justification for believing P in $\mathfrak{M}$ is O_1, the evidence O_1 is no longer dense in model $\mathfrak{M}^{+\{x_0\}}$, therefore, cannot constitute a justification for P in $\mathfrak{M}^{+\{x_0\}}$. On the other hand, another argument in $\mathfrak{M}^{+\{x_0\}}$, namely $\{x_0, x_1\}$ forms a justification for P in $\mathfrak{M}^{+\{x_0\}}$, thus P is still believed in $\mathfrak{M}^{+\{x_0\}}$, but, based on a different justification. Therefore, there is *no* uniform justification for P that works for every nonmisleading evidence Q.

The next result shows that *our notion of knowledge exactly matches the weakened version of defeasibility theory*, as well as the *epistemic coherence formulation*:

5.5.5. PROPOSITION. *Let $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ be a topo-e-model, and $x \in X$ is the actual world. The following are equivalent for all $P \subseteq X$:*

1. *KP holds at x in $\mathfrak{M}$.*

2. *There is an argument (justification) for P that cannot be defeated by any nonmisleading proposition; i.e. $\exists U \in \tau \setminus \{\emptyset\}$ such that $U \subseteq P$ and BU holds in $\mathfrak{M}^{+Q}$ for all nonmisleading $Q \subseteq X$ (at x with respect to $\mathcal{E}$).*

3. *There is an argument (justification) for P that is consistent with every nonmisleading proposition; i.e. $\exists U \in \tau \setminus \{\emptyset\}$ such that $U \subseteq P$ and $U \cap Q \neq \emptyset$ for all nonmisleading $Q \subseteq X$ (at x with respect to $\mathcal{E}$).*

Proof:

(1) $\Rightarrow$ (2): Suppose $x \in KP$. This means, by Proposition 5.5.1-(3), that $x \in Int(P)$ and $Cl(Int(P)) = X$. Now consider the argument $Int(P)$. Obviously $Int(P) \in \tau \setminus \{\emptyset\}$ and $Int(P) \subseteq P$. Let Q be a nonmisleading proposition at x with respect to $\mathcal{E}$, and Cl^{+Q} and Int^{+Q} denote the closure and the interior operators of τ^{+Q}, respectively. We only need to show that $Int^{+Q}(Int(P))$ is dense in (X, τ^{+Q}),

i.e., that for all $e \in \mathcal{E}^{+Q}$, we have $e \cap \mathit{Int}^{+Q}(\mathit{Int}(P)) \neq \emptyset$. Let $e \in \mathcal{E}^{+Q}$. Then, by the definition of $\mathcal{E}^{+Q}$, we have two cases: (1) $e \in \mathcal{E}$, or (2) $e \notin \mathcal{E}$ but $e = e' \cap Q$ for some $e' \in \mathcal{E}$. Since Q is nonmisleading, the latter case entails that $x \in e$. If $e \in \mathcal{E}$, we have $e \cap \mathit{Int}^{+Q}(\mathit{Int}(P)) \neq \emptyset$ since $\mathit{Int}(P) \subseteq \mathit{Int}^{+Q}(\mathit{Int}(P))$ (by Lemma 2.2.5) and $\mathit{Int}(P)$ is dense in (X, τ). If $e \notin \mathcal{E}$ and $e = e' \cap Q$ for some $e' \in \mathcal{E}$ with $x \in e$, we obtain $x \in e \cap \mathit{Int}^{+Q}(\mathit{Int}(P))$ since $x \in \mathit{Int}(P) \subseteq \mathit{Int}^{+Q}(\mathit{Int}(P))$, thus, $e \cap \mathit{Int}^{+Q}(\mathit{Int}(P)) \neq \emptyset$. Therefore, $\mathit{Int}^{+Q}(\mathit{Int}(P))$ is dense in (X, τ^{+Q}), i.e., $B(\mathit{Int}(P))$ holds in $\mathfrak{M}^{+Q}$.

(2) $\Rightarrow$ (3): Suppose (2) holds, i.e., there is a $U \in \tau \backslash \{\emptyset\}$ such that $U \subseteq P$ and $\mathit{Cl}^{+Q}(\mathit{Int}^{+Q}(U)) = X$ for all nonmisleading $Q \subseteq X$ (at x with respect to $\mathcal{E}$). Let Q be nonmisleading at x with respect to $\mathcal{E}$. Since $\mathit{Cl}^{+Q}(\mathit{Int}^{+Q}(U)) = X$, we have that $e \cap \mathit{Int}^{+Q}(U) \neq \emptyset$ for all $e \in \mathcal{E}^{+Q}$. As Q is nonmisleading at x, we in particular have $\emptyset \neq Q = Q \cap X \in \mathcal{E}^{+Q}$ (by the definition of $\mathcal{E}^{+Q}$ and the fact that $X \in \mathcal{E}$). Hence, it follows from (2) that $Q \cap \mathit{Int}^{+Q}(U) \neq \emptyset$. Since $\mathit{Int}^{+Q}(U) \subseteq U$, we obtain $U \cap Q \neq \emptyset$.

(3) $\Rightarrow$ (1): Assume that $U \in \tau \backslash \{\emptyset\}$ is such that $U \subseteq P$ and $U \cap Q \neq \emptyset$ holds for all nonmisleading Q (at x with respect to $\mathcal{E}$). Clearly, this implies that U is consistent with all $e \in \mathcal{E}$, i.e., that $e \cap U \neq \emptyset$ (since available evidence is by definition nonmisleading), so U is a justification for P (i.e., $X = \mathit{Cl}(U) = \mathit{Cl}(\mathit{Int}(P))$). So, to show that KP holds at x, it is enough to show that $x \in \mathit{Int}(P)$. For this, take the proposition $Q = \{x\}$, which obviously is nonmisleading at x, hence by (3) we must have $U \cap \{x\} \neq \emptyset$, i.e. $x \in U$. Then, $x \in U \in \tau$ and $U \subseteq P$ give us $x \in \mathit{Int}(P)$, as desired. □

5.6 Logics for evidence, justified belief, knowledge, and evidence dynamics

This section constitutes the technical heart of this chapter and is devoted to our results concerning soundness, completeness, decidability and finite model property for several logics of evidence, belief and knowledge (Sections 5.6.2-5.6.5). We then continue with introducing the formal syntax and the semantics for the aforementioned dynamic evidence modalities for public announcements, evidence addition, evidence upgrade and feasible evidence combination, and provide sound and complete axiomatizations for the associated logics (Section 5.6.6). In order to keep this section self-contained and fix some notation, we first recapitulate, in a concise way, the formal syntax and the semantics capturing the static notions we have presented in the previous sections (Section 5.6.1).

5.6.1 Logics for evidence, justified belief and knowledge

Syntax. The full (static) language $\mathcal{L}$ of evidence, belief, and knowledge we consider is defined recursively by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid E_0\varphi \mid E\varphi \mid \Box_0\varphi \mid \Box\varphi \mid B\varphi \mid B^{\varphi}\varphi \mid K\varphi \mid [\forall]\varphi$$

where $p \in \textsc{prop}$. We employ the usual abbreviations for propositional connectives $\top$, $\bot$, $\vee$, $\rightarrow$, $\leftrightarrow$, and for the dual modalities $\hat{B}$, $\hat{K}$, $\hat{E}$ etc. except that some of them have special abbreviations: $[\exists]\varphi := \neg[\forall]\neg\varphi$ and $\Diamond\varphi := \neg\Box\neg\varphi$. Several fragments of the language $\mathcal{L}$ is of particular interest: $\mathcal{L}_B$ the fragment having the belief modality B as the only modality; $\mathcal{L}_K$ having only the knowledge modality K; and some bimodal fragments such as $\mathcal{L}_{KB}$ having only operators K and B; $\mathcal{L}_{[\forall]K}$ having only operators $[\forall]$ and K; and the trimodal fragment $\mathcal{L}_{[\forall]\Box_0\Box}$ having only the modalities $[\forall]$, $\Box_0$ and $\Box$.

Semantics. We interpret the language $\mathcal{L}$ on topo-e-models in an obvious way, following the definitions of the corresponding operators provided in previous sections.

5.6.1. Definition. [Topo-e-Semantics for $\mathcal{L}$] Given a topo-e-model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$, we extend the valuation map V to an interpretation map $[\![.]\!] : \mathcal{L} \rightarrow \mathcal{P}(X)$ recursively as follows:

$$\begin{array}{lcl}
[\![p]\!] & = & V(p) \\
[\![\neg\varphi]\!] & = & X \backslash [\![\varphi]\!] \\
[\![\varphi \wedge \psi]\!] & = & [\![\varphi]\!] \cap [\![\psi]\!] \\
[\![E_0\varphi]\!] & = & \{x \in X \mid \exists e \in \mathcal{E}_0 (e \subseteq [\![\varphi]\!])\} \\
[\![E\varphi]\!] & = & \{x \in X \mid \exists e \in \mathcal{E}\,(e \subseteq [\![\varphi]\!])\} \\
[\![\Box_0\varphi]\!] & = & \{x \in X \mid \exists e \in \mathcal{E}_0\,(x \in e \subseteq [\![\varphi]\!])\} \\
[\![\Box\varphi]\!] & = & \{x \in X \mid \exists U \in \tau\,(x \in U \subseteq [\![\varphi]\!])\} \\
[\![B\varphi]\!] & = & \{x \in X \mid \exists U \in \tau\,(U \subseteq [\![\varphi]\!] \text{ and } Cl(U) = X)\} \\
[\![B^{\theta}\varphi]\!] & = & \{x \in X \mid \exists U \in \tau\,(\emptyset \neq U \cap [\![\theta]\!] \subseteq [\![\varphi]\!] \text{ and } Cl(U \cap [\![\theta]\!]) \supseteq [\![\theta]\!])\} \\
[\![K\varphi]\!] & = & \{x \in X \mid \exists U \in \tau\,(x \in U \subseteq [\![\varphi]\!] \text{ and } Cl(U) = X)\} \\
[\![[\forall]\varphi]\!] & = & \{x \in X \mid [\![\varphi]\!] = X\}
\end{array}$$

It is not hard to see that the above defined semantics for the modalities of $\mathcal{L}$ corresponds exactly to the semantic operators given in Sections 5.2-5.5: e.g. $[\![[\forall]\varphi]\!] = [\forall][\![\varphi]\!]$, $[\![\Box\varphi]\!] = \Box[\![\varphi]\!] = Int([\![\varphi]\!])$, etc. Moreover, while all modalities except for E_0 and $\Box_0$ capture topological properties of topo-e-models, i.e., they can be interpreted directly in (X, τ), the expressivity of the full language goes beyond the purely topological properties: the meaning of E_0 and $\Box_0$ does not only depend on the evidential topology, but also depends on the basic evidence set $\mathcal{E}_0$. From the point of expressivity, the most important fragment of $\mathcal{L}$ is the trimodal language $\mathcal{L}_{[\forall]\Box_0\Box}$ since it is equally expressive as the full language $\mathcal{L}$ with respect to the topo-e-models:

5.6.2. Proposition. *The following equivalences are valid in all topo-e-models:*

1. $B\varphi \leftrightarrow [\forall]\Diamond\Box\varphi$
2. $E\varphi \leftrightarrow [\exists]\Box\varphi$
3. $E_0\varphi \leftrightarrow [\exists]\Box_0\varphi$
4. $K\varphi \leftrightarrow \Box\varphi \wedge [\forall]\Diamond\Box\varphi$
5. $B^\theta\varphi \leftrightarrow [\forall](\theta \to \Diamond(\theta \wedge \Box(\theta \to \varphi)))$

Proof:
The proof follows easily from the semantics clauses of the modalities given in Definition 3.1.2. □

Therefore, all the other modalities of $\mathcal{L}$ can be defined in $\mathcal{L}_{[\forall]\Box_0\Box}$. In fact, all our dynamic modalities can also be expressed in $\mathcal{L}_{[\forall]\Box_0\Box}$ (see Section 5.6.6). For this reason, instead of focusing on the full language $\mathcal{L}$, we present soundness, completeness and decidability results for the *factive evidence fragment* $\mathcal{L}_{[\forall]\Box_0\Box}$: its importance comes from its expressive power. We moreover provide sound and complete axiomatizations for the pure doxatic fragment $\mathcal{L}_B$, the pure epistemic fragments $\mathcal{L}_K$ and $\mathcal{L}_{[\forall]K}$, and finally for the epistemic-doxastic fragment $\mathcal{L}_{KB}$. As the semantics of $[\forall]$, B and K can be defined only based on the evidential topology (without referring to $\mathcal{E}_0$), we will state the corresponding soundness and completeness results simply with respect to topo-models. For $\mathcal{L}_{[\forall]\Box_0\Box}$, we need the complete structure of the topo-e-models as the semantics of $\Box_0$ depends on the basic evidence set $\mathcal{E}_0$, and cannot be recovered purely topologically.

5.6.2 The belief fragment $\mathcal{L}_B$: $\mathsf{KD45}_B$

In this section, we prove that the logic of belief on all topo-models is the standard belief system $\mathsf{KD45}_B$, and it moreover has the finite model property with respect to the class of topo-models.

Soundness of $\mathsf{KD45}_B$:

5.6.3. Lemma. *Given a topological space (X, τ) and any two subsets $U_1, U_2 \subseteq X$, if U_1 is open dense and U_2 is dense, then $U_1 \cap U_2$ is dense.*

Proof:
Let (X, τ) be a topological space and $U_1, U_2 \subseteq X$. Suppose U_1 is an open dense and U_2 is a dense set in (X, τ). Since U_1 is open and dense we have that $W \cap U_1$ is open and non-empty for any non-empty open set W. Thus, since U_2 is dense, we also have that $(W \cap U_1) \cap U_2 \neq \emptyset$. Therefore, $W \cap (U_1 \cap U_2) \neq \emptyset$ for any nonempty $W \in \tau$, i.e., $U_1 \cap U_2$ is dense as well. □

5.6.4. Proposition. $\mathsf{KD45}_B$ *is sound with respect to the class of all topo-models.*

Proof:
The soundness, as usual, is shown by proving that all axioms are validities and that all derivation rules preserve validities. The cases for the axioms (4_B) and (5_B) and the inference rules are elementary, whereas the validity of (K_B) in the class of *all* topological spaces follows from Lemma 5.6.3 as follows. Let $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ and $\varphi, \psi \in \mathcal{L}_B$. We need to show that $[\![B(\varphi \wedge \psi) \leftrightarrow B\varphi \wedge B\psi]\!] = X$, i.e., that $[\![B(\varphi \wedge \psi)]\!] = [\![B\varphi \wedge B\psi]\!]$. Let $x \in B(\varphi \wedge \psi)$. This implies, by the semantics of B that $[\![B(\varphi \wedge \psi)]\!] = X$, i.e., $Cl(Int([\![\varphi \wedge \psi]\!])) = X$. We therefore obtain, $X = Cl(Int([\![\varphi \wedge \psi]\!])) = Cl(Int([\![\varphi]\!]) \cap Int([\![\psi]\!])) \subseteq Cl(Int([\![\varphi]\!])) \cap Cl(Int([\![\psi]\!])) = [\![B\varphi \wedge B\psi]\!]$. For the other direction, suppose $x \in [\![B\varphi \wedge B\psi]\!]$. We therefore have $x \in [\![B\varphi]\!]$ and $x \in [\![B\psi]\!]$. Then, by the semantics of B, we obtain $Cl(Int([\![\varphi]\!])) = X$ and $Cl(Int([\![\psi]\!])) = X$. This means that both $Int([\![\varphi]\!])$ and $Int([\![\psi]\!])$ are dense in (X, τ). Hence, by Lemma 5.6.3, we obtain $Cl(Int([\![\varphi]\!]) \cap Int([\![\psi]\!])) = X$. Similarly to the argument above, we then have $X = Cl(Int([\![\varphi]\!]) \cap Int([\![\psi]\!])) = Cl(Int([\![\varphi \wedge \psi]\!])) = [\![B(\varphi \wedge \psi)]\!]$. $\square$

Completeness of $\mathsf{KD45}_B$:

For completeness, we use the connection between the $\mathsf{KD45}$-Kripke frames and topological spaces presented in Section 4.3.1. We only need to show that the two semantics—the relational semantics and the proposed semantics on topo-e-models—are equivalent for the language $\mathcal{L}_B$. To recall the definition of relational frame called a pin, see Definition 4.3.1, page 37.

5.6.5. PROPOSITION. *For all $\varphi \in \mathcal{L}_B$ and any Kripke model $\mathcal{M} = (X, R, V)$ based on a pin,*

$$\|\varphi\|^{\mathcal{M}} = [\![\varphi]\!]^{I(\mathcal{M})}.$$

Proof:
The proof follows by subformula induction on φ; cases for the propositional variables and the Boolean connectives are elementary. So assume inductively that the result holds for ψ; we must show that it holds also for $\varphi := B\psi$. Observe that, given a Kripke model $\mathcal{M} = (X, R, V)$ based on a pin (X, R) and $\varphi \in \mathcal{L}_B$, we have

$$\|B\varphi\|^{\mathcal{M}} = \begin{cases} X & \text{if } \|\varphi\|^{\mathcal{M}} \supseteq \mathcal{C} \\ \emptyset & \text{otherwise} \end{cases} \quad \text{and, } [\![B\varphi]\!]^{I(\mathcal{M})} = \begin{cases} X & \text{if } [\![\varphi]\!]^{I(\mathcal{M})} \supseteq \mathcal{C} \\ \emptyset & \text{otherwise} \end{cases}$$

where $\mathcal{C}$ is the final cluster of (X, R). By induction hyposthesis, we have $[\![\varphi]\!]^{I(\mathcal{M})} = \|\varphi\|^{\mathcal{M}}$, therefore, $[\![B\varphi]\!]^{I(\mathcal{M})} = \|B\varphi\|^{\mathcal{M}}$. $\square$

5.6.6. THEOREM. $\mathsf{KD45}_B$ *is sound and complete with respect to the class of all topo-e-models. Moreover,* $\mathsf{KD45}_B$ *has the finite model property.*

Proof:
Soundness is given in Proposition 5.6.4. For completeness, let $\varphi \in \mathcal{L}_B$ such that $\varphi \notin \mathsf{KD45}_B$. Then, by Lemma 4.3.2, there exists a finite pin $\mathcal{M} = (X, R, V)$ with $\|\varphi\|^{\mathcal{M}} \neq X$. Thus, by Propositition 5.6.5, we have that $[\![\varphi]\!]^{I(\mathcal{M})} \neq X$, where $I(\mathcal{M}) = (X, \tau_{R^+}, V)$ is the corresponding topological model. Since $I(\mathcal{M}) = (X, \tau_{R^+}, V)$ is finite, we have also shown that $\mathsf{KD45}_B$ has the finite model property. $\square$

5.6.3 The knowledge fragments $\mathcal{L}_K$ and $\mathcal{L}_{[\forall]K}$: $\mathsf{S4.2}_K$ and $\mathsf{Know}_{[\forall]K}$

In this section, we focus on the two knowledge fragments $\mathcal{L}_K$ and $\mathcal{L}_{[\forall]K}$, and provide sound and complete axiomatizations for the associated logics. While the fragment having only the modality K leads to the familiar system $\mathsf{S4.2}_K$, the full knowledge fragment having both K and $[\forall]$ gives us the axiomatization $\mathsf{Know}_{[\forall]K}$ presented below.

Soundness and Completeness of $\mathsf{S4.2}_K$

The proof of soundness is again a standard validity check. The relatively harder case of the normality axiom (K_K) for the knowledge modality K follows from Lemma 5.6.3 and the fact that the interior operator commutes with finite intersections (see, e.g., Table 3.1). For completeness, we follow a similar strategy as in the proof of Theorem 5.6.6.

Let (X, R) be a *transitive* Kripke frame. A nonempty subset $\mathcal{C} \subseteq X$ is called *cluster* if (1) for each $x, y \in \mathcal{C}$ we have xRy, and (2) there is no $D \subseteq X$ such that $\mathcal{C} \subsetneq D$ and D satisfies (1). A point $x \in X$ is called a *maximal point* if there is no $y \in X$ such that xRy and $\neg(yRx)$. We call a cluster a *final cluster* if all its points are maximal. It is not hard to see that for any final cluster $\mathcal{C}$ of (X, R) and any $x \in \mathcal{C}$, we have $R(x) = \mathcal{C}$. A transitive Kripke frame (X, R) is called *cofinal* if it has a unique final cluster $\mathcal{C}$ such that for each $x \in X$ and $y \in \mathcal{C}$ we have xRy.

5.6.7. Lemma. $\mathsf{S4.2}_K$ *is sound and complete with respect to the class of reflexive and transitive cofinal frames.*

Proof:
See, e.g., (Chagrov and Zakharyaschev, 1997, Chapter 5). $\square$

Recall that, given a reflexive and transitive Kripke frame (X, R), we can construct an Alexandroff space (X, τ_R) by defining τ_R to be the set of all upsets of (X, R) (see Section 3.1.2).

5.6.8. Lemma. *For every reflexive transitive cofinal frame (X, R) and nonempty $U \in \tau_R$, we have $Cl(U) = X$ in (X, τ_R).*

Proof:
Let (X, R) be a reflexive and transitive cofinal frame and let $\mathcal{C} \subseteq X$ denote its final cluster. By construction, $\mathcal{C} \in \tau_R$ and moreover $\mathcal{C} \subseteq U$, for all nonempty $U \in \tau_R$. Therefore, for every nonempty $U, V \in \tau_R$, we have $V \cap U \supseteq \mathcal{C} \neq \emptyset$. Hence, $Cl(U) = X$ for any nonempty $U \in \tau_R$. □

5.6.9. Proposition. *For every reflexive and transitive cofinal Kripke model $\mathcal{M} = (X, R, V)$ and all $\varphi \in \mathcal{L}_{[\forall]K}$,*

$$\|\varphi\|^{\mathcal{M}} = [\![\varphi]\!]^{B(\mathcal{M})},$$

where $B(\mathcal{M}) = (X, \tau_R, V)$.

Proof:
The proof follows by subformula induction on φ; cases for the propositional variables, the Boolean connectives and the modality $[\forall]$ are elementary. So assume inductively that the result holds for ψ; we must show that it holds also for $\varphi := K\psi$. Let $\mathcal{M} = (X, R, V)$ be a reflexive and transitive cofinal Kripke model, $x \in X$ and $\varphi \in \mathcal{L}_K$.

($\subseteq$) Suppose $x \in \|K\psi\|^{\mathcal{M}}$. This implies that $x \in R(x) \subseteq \|\psi\|^{\mathcal{M}}$. By induction hypothesis, we obtain $R(x) \subseteq [\![\psi]\!]^{B(\mathcal{M})}$. Since $x \in R(x) \in \tau_R$, we have $x \in Int([\![\psi]\!]^{B(\mathcal{M})})$. Then, by Lemma 5.6.8, $Cl(Int([\![\psi]\!]^{B(\mathcal{M})})) = X$. Therefore, $x \in [\![K\psi]\!]^{B(\mathcal{M})}$.

($\supseteq$) Suppose $x \in [\![K\psi]\!]^{B(\mathcal{M})}$. This means, by the topological semantics of K, that $x \in Int([\![\psi]\!]^{B(\mathcal{M})})$ and that $Cl(Int([\![\psi]\!]^{B(\mathcal{M})})) = X$. Then, by induction hypothesis, $x \in Int(\|\psi\|^{\mathcal{M}})$ and $Cl(Int(\|\psi\|^{\mathcal{M}})) = X$. The former implies that there is an open set $U \in \tau_R$ such that $x \in U \subseteq \|\psi\|^{\mathcal{M}}$. In particular, since $R(x)$ is the smallest open neighbourhood of x, we obtain $R(x) \subseteq \|\psi\|^{\mathcal{M}}$. Therefore, $x \in \|K\psi\|^{\mathcal{M}}$. □

5.6.10. Theorem. $\mathsf{S4.2}_K$ *is sound and complete with respect to the class of all topo-models.*

Proof:
For completeness, let $\varphi \in \mathcal{L}_K$ such that $\varphi \notin \mathsf{S4.2}_K$. Then, by Lemma 5.6.7, there exists a Kripke model $\mathcal{M} = (X, R, V)$ based on the reflexive and transitive cofinal frame (X, R) such that $\|\varphi\|^{\mathcal{M}} \neq X$. Thus, by Propositition 5.6.9, we have $[\![\varphi]\!]^{B(\mathcal{M})} \neq X$, where $B(\mathcal{M}) = (X, \tau_R, V)$ is the corresponding topological model.
□

Soundness and Completeness of $\mathsf{Know}_{[\forall]K}$:

The full knowledge fragment $\mathcal{L}_{[\forall]K}$ having both K and $[\forall]$ yields the axiomatic system $\mathsf{Know}_{[\forall]K}$ given in Table 5.4 below.

(CPL)	all classical propositional tautologies and (MP)
($\mathsf{S5}_{[\forall]}$)	all $\mathsf{S5}$ axioms and rules for the modality $[\forall]$
($\mathsf{S4}_K$)	all $\mathsf{S4}$ axioms and rules for the modality K
(Ax-1)	$[\forall]\varphi \to K\varphi$
(Ax-2)	$[\exists]K\varphi \to [\forall]\hat{K}\varphi$

Table 5.4: The axiomatization of $\mathsf{Know}_{[\forall]K}$

5.6.11. THEOREM. $\mathsf{Know}_{[\forall]K}$ *is sound and complete with respect to the class of all topo-models.*

Proof:
Soundness is easy to see, we here only prove that the axiom ($[\exists]K\varphi \to [\forall]\hat{K}\varphi$) is valid on all topo-models. Let $\mathfrak{M} = (X, \tau, V)$ be a topo-model, $\varphi \in \mathcal{L}_{[\forall]K}$, and $x \in X$ such that $x \in [\![[\exists]K\varphi]\!]$. This means that there exist $y \in X$ such that $y \in \mathit{Int}([\![\varphi]\!])$ and $\mathit{Cl}(\mathit{Int}([\![\varphi]\!])) = X$. Note that for any $z \in X$,

$$z \in [\![\hat{K}\varphi]\!] \text{ iff } z \notin \mathit{Int}([\![\neg\varphi]\!]) \text{ or } \mathit{Cl}(\mathit{Int}([\![\neg\varphi]\!])) \neq X,$$

(see Proposition 5.5.1-(3)). Therefore, in order to show $[\![\hat{K}\varphi]\!] = X$, it suffices to show that $\mathit{Cl}(\mathit{Int}([\![\neg\varphi]\!])) \neq X$. Since $y \in \mathit{Int}([\![\varphi]\!])$, we know that $\mathit{Int}(\mathit{Cl}([\![\varphi]\!])) \neq \emptyset$ (as $\mathit{Int}([\![\varphi]\!]) \subseteq \mathit{Int}(\mathit{Cl}([\![\varphi]\!]))$). Hence, $\mathit{Cl}(\mathit{Int}([\![\neg\varphi]\!])) \neq X$. We therefore obtain $[\![\hat{K}\varphi]\!] = X$, hence, $[\forall]\hat{K}$ holds everywhere in $\mathfrak{M}$.

For completeness, we use a well-known Kripke completeness result for the logic obtained by extending $\mathsf{S4.2}_K$ with the universal modality $[\forall]$. More precisely, it has been shown in (Goranko and Passy, 1992) that the modal system $\mathsf{Know^0}_{[\forall]K} := \mathsf{S5}_{[\forall]} + \mathsf{S4.2}_K + ([\forall]\varphi \to K\varphi)$, simply obtained by replacing (Ax-2) in Table 5.4 by the axiom $(.2_K)$:=$\hat{K}K\varphi \to K\hat{K}\varphi$, is complete with respect to the class of reflexive and transitive cofinal Kriple frames when K is interpreted as the standard Kripke modality and $[\forall]$ as the global modality. It is not hard to see that the axiom $(.2_K)$ is derivable in $\mathsf{Know}_{[\forall]K}$ (by using (Ax-1) and (Ax-2) in Table 5.4), hence, $\mathsf{Know}_{[\forall]K}$ is stronger than $\mathsf{Know^0}_{[\forall]K}$, i.e., that $\mathsf{Know^0}_{[\forall]K} \subseteq \mathsf{Know}_{[\forall]K}$. Let $\varphi \in \mathcal{L}_{[\forall]K}$ such that $\varphi \notin \mathsf{Know}_{[\forall]K}$. Thus, $\varphi \notin \mathsf{Know^0}_{[\forall]K}$. Then, by the relational completeness of $\mathsf{Know^0}_{[\forall]K}$, there exists a reflexive and transitive cofinal Kripke model $\mathcal{M} = (X, R, V)$ such that $\|\varphi\|^{\mathcal{M}} \neq X$. Then, by Proposition 5.6.9, we obtain $[\![\varphi]\!]^{B(\mathcal{M})} \neq X$, where $B(\mathcal{M}) = (X, \tau_R, V)$. $\square$

5.6.4 The knowledge-belief fragment $\mathcal{L}_{KB}$: Stal revisited

In this section, we show that Stalnaker's system Stal of knowledge and belief (see Table 4.1) is sound and complete with respect to the class of all topo-models under the semantics of knowledge and belief presented in this chapter. Recall that, in Chapter 4, we provided a topological completeness result for this system for the restricted class of extremally disconnected spaces. Therefore, we here show that the topological semantics presented in this chapter generalizes the one provided in Chapter 4 for Stalnaker's combined system Stal.

5.6.12. Theorem. Stal *is sound and complete with respect to the class of all topo-models.*

Proof:
For soundness, we here only show the validity of the axiom (FB): the validity proofs of the other axioms are either trivial or follow from the previous results. Let $\mathfrak{M} = (X, \tau, V)$ be a topo-model, $\varphi \in \mathcal{L}_{KB}$ and $x \in X$. Suppose $x \in [\![B\varphi]\!]$. Hence, $[\![B\varphi]\!] \neq \emptyset$. This implies, by the semantics of B, that $[\![B\varphi]\!] = Cl(Int([\![\varphi]\!])) = X$. Recall that $x \in [\![K\varphi]\!]$ iff $x \in Int([\![\varphi]\!])$ and $Cl(Int([\![\varphi]\!])) = X$. By the assumption, we already know that $Cl(Int([\![\varphi]\!])) = X$. Thus, in this particular case, $[\![K\varphi]\!] = Int([\![\varphi]\!])$. Therefore, $X = Cl(Int([\![\varphi]\!])) = Cl(Int(Int([\![\varphi]\!]))) = Cl(Int([\![K\varphi]\!]))$ implying that $BK\varphi$ holds everywhere in $\mathfrak{M}$.

For completeness, we follow a similar method as in the proof of Theorem 5.6.11. Let $\varphi \in \mathcal{L}_{KB}$ such that $\varphi \notin \mathsf{Stal}$. Then, since $\vdash_{\mathsf{Stal}} B\varphi \leftrightarrow \hat{K}K\varphi$, there exists a $\psi \in \mathcal{L}_K$ such that $\vdash_{\mathsf{Stal}} \varphi \leftrightarrow \psi$ (this is obtained by replacing every occurrence of B in φ by $\hat{K}K$). Therefore, $\psi \notin \mathsf{Stal}$. Moreover, since $\mathsf{S4.2}_K \subseteq \mathsf{Stal}$ (see Section 4.1), we obtain $\psi \notin \mathsf{S4.2}_K$. Then, by Theorem 5.6.10, there exists a topo-model $\mathfrak{M} = (X, \tau, V)$ such that $[\![\psi]\!] \neq X$. Since Stal is sound with respect to all topo-models and $\vdash_{\mathsf{Stal}} \varphi \leftrightarrow \psi$, we conclude $[\![\varphi]\!] \neq X$. $\square$

5.6.5 The factive evidence fragment $\mathcal{L}_{[\forall]\Box_0\Box}$: $\mathsf{Log}_{\forall\Box\Box_0}$

The logic $\mathsf{Log}_{\forall\Box\Box_0}$ of factive evidence is given by the axiom schemas and inference rules in Table 5.5 over the language $\mathcal{L}_{[\forall]\Box_0\Box}$.

This section presents the proof of the following theorem. Strong completeness and strong finite model property are defined standardly (see, e.g., Blackburn et al., 2001, Definition 4.10-Proposition 4.12 and Definition 6.6, respectively).

5.6.13. Theorem. *The logic* $\mathsf{Log}_{\forall\Box\Box_0}$ *of factive evidence is sound and strongly complete with respect to the class of all topo-models. Moreover, it has the strong finite model property, therefore, it is decidable.*

(CPL)	all classical propositional tautologies and (MP)
($\mathsf{S5}_{[\forall]}$)	all $\mathsf{S5}$ axioms and rules for the modality $[\forall]$
($\mathsf{S4}_{\Box}$)	all $\mathsf{S4}$ axioms and rules for the modality $\Box$
($4_{\Box_0}$)	$\Box_0\varphi \to \Box_0\Box_0\varphi$
Universality (U)	$[\forall]\varphi \to \Box_0\varphi$
Factive Evidence (FE)	$\Box_0\varphi \to \Box\varphi$
Pullout[11]	$(\Box_0\varphi \wedge [\forall]\psi) \to \Box_0(\varphi \wedge [\forall]\psi)$
Monotonicity rule for $\Box_0$	from $\varphi \to \psi$, infer $\Box_0\varphi \to \Box_0\psi$

Table 5.5: The axiomatization of $\mathsf{Log}_{\forall\Box\Box_0}$

The proof of Theorem 5.6.13 is technically the most challenging result of this chapter. The key difficulty consists in guaranteeing that the natural topology for which $\Box$ acts as interior operator is exactly the topology generated by the neighborhood family associated to $\Box_0$. Though the main steps of the proof may look familiar, involving known methods (a canonical quasi-model construction, a filtration argument, and then making multiple copies of the worlds to yield a finite model with the right properties), addressing the above-mentioned difficulty requires a non-standard application of these methods, as well as a number of additional notions and results, and a careful treatment of each of the steps. The plan of the proof is as follows. Since the soundness proof is straightforward, we here focus on completeness and the finite model property (then decidability follows immediately). We first prove strong completeness of $\mathsf{Log}_{\forall\Box\Box_0}$ with respect to a *canonical quasi-model*. We then continue with proving the strong finite quasi-model property for $\mathsf{Log}_{\forall\Box\Box_0}$ via a filtration argument. In the last step, we prove that every finite quasi-model is equivalent to a finite Alexandroff quasi-model by making multiple copies of the worlds in order to put the model in the right shape. As Alexandroff quasi-models are modally equivalent to Alexandroff topo-e-models (Proposition 5.6.14), the result follows.

Quasi-model Construction

A *quasi-model* is a tuple $\mathcal{M} = (X, \mathcal{E}_0, \leq, V)$, where $(X, \mathcal{E}_0, V)$ is an evidence model and $\leq$ is a preorder such that every $e \in \mathcal{E}_0$ is an up-set of $(X, \leq)$ (see Definition 2.1.4, page 14 to recall the definition of an up-set). Given a preordered set $(X, \leq)$, the set $Up_{\leq}(X)$ denotes the *set of all up-sets of* $(X, \leq)$. We use the same notations as for topo-e-models, for example, $\mathcal{E}$ for the closure of $\mathcal{E}_0$ under nonempty finite intersections, and $\tau_{\mathcal{E}}$ for the topology generated by $\mathcal{E}$.

The *semantics* for the language $\mathcal{L}_{[\forall]\Box_0\Box}$ on quasi-models is defined the same way as on topo-e-models (see Definition 5.6.1), *except* that for $\Box$ we (do *not* use the topology, but instead we) use the standard Kripke semantics based on the relation $\leq$. More precisely, the semantics for the modalities $[\forall]$, $\Box_0$ and $\Box$ are given by the following clauses:

$$\begin{array}{lcl} \|[\forall]\varphi\|^{\mathcal{M}} & = & \{x \in X \mid \|\varphi\|^{\mathcal{M}} = X\} \\ \|\Box_0\varphi\|^{\mathcal{M}} & = & \{x \in X \mid \exists e \in \mathcal{E}_0\, (x \in e \subseteq \|\varphi\|^{\mathcal{M}})\} \\ \|\Box\varphi\|^{\mathcal{M}} & = & \{x \in X \,|\, \forall y \in X(x \leq y \text{ implies } y \in \|\varphi\|^{\mathcal{M}})\} \end{array}$$

We again omit the superscripts for the model when it is clear from the context.

A quasi-model $\mathcal{M} = (X, \mathcal{E}_0, \leq, V)$ is called *Alexandroff* if the topology $\tau_{\mathcal{E}}$ is Alexandroff and $\leq\, = \,\sqsubseteq_{\mathcal{E}}$ is the specialization preorder. There is a natural one-to-one correspondence between Alexandroff quasi-models and Alexandroff topo-e-models, given by putting, for any Alexandroff quasi-model $\mathcal{M} = (X, \mathcal{E}_0, \leq, V)$, $B(\mathcal{M}) = (X, \mathcal{E}_0, \tau_{\mathcal{E}}, V)$. Moreover, *$\mathcal{M}$ and $B(\mathcal{M})$ satisfy the same formulas of $\mathcal{L}_{[\forall]\Box_0\Box}$ at the same points*, as shown in Proposition 5.6.14 below.

5.6.14. PROPOSITION. *For all $\varphi \in \mathcal{L}_{[\forall]\Box_0\Box}$ and every Alexandroff quasi-model $\mathcal{M} = (X, \mathcal{E}_0, \leq, V)$, we have*

$$\|\varphi\|^{\mathcal{M}} = [\![\varphi]\!]^{B(\mathcal{M})}.$$

Proof:
The proof follows by subformula induction on φ; cases for the propositional variables, the Boolean connectives and the modalities $[\forall]$ and $\Box_0$ are trivial as the semantics for these cases are defined exactly the same way in both structures. For the modality $\Box$, recall that it is interpreted as the interior operator of the topology $\tau_{\mathcal{E}}$, thus, this case is analogous to Proposition 3.1.4-(1). $\Box$

Therefore, as stated by Proposition 5.6.14, Alexandroff quasi-models provide just another presentation of Alexandroff topo-e-models with respect to the language $\mathcal{L}_{[\forall]\Box_0\Box}$.

5.6.15. PROPOSITION. *For every quasi-model $\mathcal{M} = (X, \mathcal{E}_0, \leq, V)$ the following are equivalent:*

1. *$\mathcal{M}$ is Alexandroff (hence, equivalent to an Alexandroff topo-e-model);*
2. *$\tau_{\mathcal{E}} = Up_{\leq}(X)$;*
3. *for every $x \in X$, $\uparrow x$ is in $\tau_{\mathcal{E}}$.*

Proof:
(1)$\Rightarrow$(3): Suppose $\mathcal{M}$ is Alexandroff, i.e., $\tau_{\mathcal{E}}$ is Alexandroff and $\leq\, = \,\sqsubseteq_{\mathcal{E}}$. Let $x \in X$. Then we have: $\uparrow x = \{y \in X \mid x \leq y\} = \{y \in X \mid x \sqsubseteq_{\mathcal{E}} y\} = \{y \in X \mid \forall U \in \tau_{\mathcal{E}}(x \in U \Rightarrow y \in U)\} = \bigcap\{U \in \tau_{\mathcal{E}} \mid x \in U\}$. Since $\tau_{\mathcal{E}}$ is an Alexandroff space, we have $\bigcap\{U \in \tau_{\mathcal{E}} \mid x \in U\} \in \tau_{\mathcal{E}}$, and hence $\uparrow x = \bigcap\{U \in \tau_{\mathcal{E}} \mid x \in U\} \in \tau_{\mathcal{E}}$.

(3)⇒(2): It is easy to see that $\tau_{\mathcal{E}} \subseteq Up_{\leq}(X)$ (since $\tau_{\mathcal{E}}$ is generated by $\mathcal{E}_0$ and every element of $\mathcal{E}_0$ is upward-closed with respect to $\leq$). Now let $A \in Up_{\leq}(X)$. Since A is upward-closed, we have $A = \bigcup\{\uparrow x \mid x \in A\}$. Then, by (3) (and $\tau_{\mathcal{E}}$ being closed under arbitrary unions), we obtain $A \in \tau_{\mathcal{E}}$.

(2)⇒(1): Suppose $\tau_{\mathcal{E}} = Up_{\leq}(X)$ and let $\mathcal{A} \subseteq \tau_{\mathcal{E}}$. By (2), every $U \in \mathcal{A}$ is upward-closed, hence, $\bigcap \mathcal{A}$ is upward-closed. Therefore, by (2), $\bigcap \mathcal{A} \in \tau_{\mathcal{E}}$. This proves that $\tau_{\mathcal{E}}$ is Alexandroff. (2) also implies that $\uparrow x$ is the least open neighbourhood of x in $\tau_{\mathcal{E}}$, i.e., $\uparrow x \subseteq U$, for all U such that $x \in U \in \tau_{\mathcal{E}}$. Therefore, $\leq$ is included in $\sqsubseteq_{\mathcal{E}}$. For the other direction, suppose $x \sqsubseteq_{\mathcal{E}} y$. This implies, in particular, that $y \in \uparrow x$ (since $x \in \uparrow x \in \tau_{\mathcal{E}}$), i.e., $x \leq y$. □

Having introduced the auxiliary notions and facts, we are ready to prove Theorem 5.6.13. This proof goes through *three steps*:

1. strong completeness for quasi-models;
2. strong finite quasi-model property; and
3. every finite quasi-model is modally equivalent to a finite Alexandroff quasi-model (hence, to a topo-e-model).

Step 1: Strong Completeness for Quasi-Models. The proof follows via a canonical quasi-model construction.

5.6.16. Lemma (Lindenbaum's Lemma). *Every* $\mathsf{Log}_{\forall\Box\Box_0}$*-consistent set can be extended to a maximally consistent one.*

Let us now fix a consistent set of sentence Φ_0. Our goal is to construct a quasi-model for Φ_0. By Lemma 5.6.16, there exists a maximally consistent set T_0 such that $\Phi_0 \subseteq T_0$. For any two maximally consistent sets T and S, we put:

$$T \sim S \text{ iff for all } \varphi \in \mathcal{L}_{[\forall]\Box_0\Box} : ([\forall]\varphi \in T \Rightarrow \varphi \in S)\,,$$
$$T \leq S \text{ iff for all } \varphi \in \mathcal{L}_{[\forall]\Box_0\Box} : (\Box\varphi \in T \Rightarrow \varphi \in S)\,.$$

Since $[\forall]$ is an $\mathsf{S5}$ modality, $\sim$ is an equivalence relation. Similarly, as $\Box$ is an $\mathsf{S4}$ modality, $\leq$ is a preorder. Moreover, since $\vdash_{\mathsf{Log}_{\forall\Box\Box_0}} [\forall]\varphi \to \Box\varphi$ (by axioms (U) and (FE) in Table 5.5), we obtain that $\leq$ is included in $\sim$.

5.6.17. Definition. [Canonical Quasi-Model for T_0] The canonical quasi model for T_0 is defined as $\mathcal{M} = (X, \mathcal{E}_0, \leq, V)$, where

- $X := \{T \subseteq \mathcal{L}_{[\forall]\Box_0\Box} \mid T \text{ is a maximally consistent set with } T \sim T_0\}$;
- $\mathcal{E}_0 := \{\widehat{\Box_0\varphi} \mid \varphi \in \mathcal{L}_{[\forall]\Box_0\Box} \text{ with } [\exists]\Box_0\varphi \in T_0\}$, where $\widehat{\theta} := \{T \in X \mid \theta \in T\}$ for any $\theta \in \mathcal{L}_{[\forall]\Box_0\Box}$;

- $\leq$ is the restriction of the above preorder $\leq$ to X; and
- $V(p) := \hat{p}$.

In the following, variables $T, S, \ldots$ range over X.

5.6.18. Lemma. *$\mathcal{M} = (X, \mathcal{E}_0, \leq, V)$ is a quasi-model.*

Proof:
In order to show that $\mathcal{M}$ is a quasi model, we need to show that (1) $X \in \mathcal{E}_0$ and $\emptyset \notin \mathcal{E}_0$, (2) $\leq$ is a preorder, and (3) every element of $\mathcal{E}_0$ is upward-closed with respect to $\leq$. Note that (2) follows from the fact that $\Box$ is an $\mathsf{S4}$ modality.

(1): Since $\vdash_{\mathsf{Log}_{\forall\Box\Box_0}} \Box_0\top$ (by Nec$_{[\forall]}$ and axiom (U) in Table 5.5), we have $\widehat{\Box_0\top} = X$. Moreoever, by axiom (T$_{[\forall]}$), we obtain $[\exists]\Box_0\top \in T_0$, hence, $\widehat{\Box_0\top} = X \in \mathcal{E}_0$. And, obviously, $\emptyset \notin \mathcal{E}_0$.

(3): Let $e \in \mathcal{E}_0$. By the definition of $\mathcal{E}_0$, we have $e = \widehat{\Box_0\varphi}$ for some $\varphi \in \mathcal{L}_{[\forall]\Box_0\Box}$ such that $[\exists]\Box_0\varphi \in T_0$. Now suppose $T, S \in X$ with $T \in \widehat{\Box_0\varphi}$ (i.e., $\Box_0\varphi \in T$) and $T \leq S$. Note that $\vdash_{\mathsf{Log}_{\forall\Box\Box_0}} \Box_0\varphi \to \Box\Box_0\varphi$ (by axioms ($4_{\Box_0}$) and (FE)). Therefore, $\Box\Box_0\varphi \in T$. Since $T \leq S$, we then obtain $\Box_0\varphi \in S$, i.e., $S \in \widehat{\Box_0\varphi}$. Thus, as S has been chosen arbitrarily, we conclude that e is upward-closed with respect to $\leq$.

$\Box$

5.6.19. Lemma (Existence Lemma for $[\forall]$). *For every $\varphi \in \mathcal{L}_{[\forall]\Box_0\Box}$,*

$$\widehat{[\exists]\varphi} \neq \emptyset \text{ iff } \widehat{\varphi} \neq \emptyset.$$

Proof:
($\Rightarrow$) Suppose $\widehat{[\exists]\varphi} \neq \emptyset$, i.e., there is $T \in X$ such that $T \in \widehat{[\exists]\varphi}$. This means $[\exists]\varphi \in T$. This implies that the set $\Gamma := \{[\forall]\psi \mid [\forall]\psi \in T\} \cup \{\varphi\}$ is consistent. Otherwise, there exist finitely many sentences $[\forall]\psi_1, \ldots, [\forall]\psi_n \in T$ such that $[\forall]\psi_1 \wedge \ldots \wedge [\forall]\psi_n \to \neg\varphi$ is a theorem of $\mathsf{Log}_{\forall\Box\Box_0}$. But then, since $[\forall]$ is an $\mathsf{S5}$-modality, we obtain that $[\forall]\psi_1 \wedge \ldots \wedge [\forall]\psi_n \to [\forall]\neg\varphi$ is also a theorem. Hence, as $[\forall]\psi_1 \wedge \ldots \wedge [\forall]\psi_n \in T$, we get $[\forall]\neg\varphi \in T$, which combined with $[\exists]\varphi \in T$, implies that T is inconsistent, contradicting T being consistent. Therefore, given that Γ is consistent, by Lindenbaum's Lemma, there exists some maximally consistent set S such that $\Gamma \subseteq S$. It is easy to see that this implies $\varphi \in S$ and $S \sim T \sim T_0$ (i.e., $S \in X$). Therefore, $S \in \widehat{\varphi}$ implying that $\widehat{\varphi} \neq \emptyset$.

($\Leftarrow$) Suppose $\widehat{\varphi} \neq \emptyset$, i.e., there is $T \in X$ such that $T \in \widehat{\varphi}$. Then, since $\varphi \to [\exists]\varphi \in T$ (by axiom (T$_{[\forall]}$)), we obtain $[\exists]\varphi \in T$, implying that $\widehat{[\exists]\varphi} \neq \emptyset$. $\Box$

5.6.20. Lemma (Existence Lemma for $\Box$). *For every $\varphi \in \mathcal{L}_{[\forall]\Box_0\Box}$ and $T \in X$, $T \in \widehat{\Diamond\varphi}$ iff there is $S \in \widehat{\varphi}$ such that $T \leq S$.*

Proof:

($\Rightarrow$) Assume $T \in \widehat{\Diamond\varphi}$, that is, $\Diamond\varphi \in T$. This implies that the set $\Gamma := \{\Box\psi \mid \Box\psi \in T\} \cup \{\varphi\}$ is consistent. Otherwise there exist finitely many sentences $\Box\psi_1, \ldots, \Box\psi_n \in T$ such that $(\Box\psi_1 \wedge \ldots \wedge \Box\psi_n) \to \neg\varphi$ is a theorem. But then, since $\Box$ is an $\mathsf{S4}$-modality, we obtain that $\Box\psi_1 \wedge \ldots \wedge \Box\psi_n \to \Box\neg\varphi$ is also a theorem. Hence, as $\Box\psi_1 \wedge \ldots \wedge \Box\psi_n \in T$, we get $\Box\neg\varphi \in T$, which combined with $\Diamond\varphi \in T$, implies that T is inconsistent, contradicting T being consistent. Therefore, given that Γ is consistent, by Lindenbaum's Lemma, there exists some maximally consistent set S such that $\Gamma \subseteq S$. It is easy to see that this implies $\varphi \in S$ and $T \leq S$. Since $\leq$ is included in $\sim$, we also obtain $S \sim T \sim T_0$, i.e., $S \in X$. Therefore, $S \in \widehat{\varphi}$.

($\Leftarrow$) Suppose there is $S \in \widehat{\varphi}$ such that $T \leq S$. Then, by definition of $\leq$, $\Diamond\varphi \in T$, i.e., $T \in \widehat{\Diamond\varphi}$. $\Box$

5.6.21. Lemma (Existence Lemma for $\Box_0$). *For every $\varphi \in \mathcal{L}_{[\forall]\Box_0\Box}$ and $T \in X$, $T \in \widehat{\Box_0\varphi}$ iff there exist $e \in \mathcal{E}_0$ such that $T \in e \subseteq \widehat{\varphi}$.*

Proof:

($\Rightarrow$) Suppose $T \in \widehat{\Box_0\varphi}$, i.e. $\Box_0\varphi \in T$. Since $T \sim T_0$, we get $[\exists]\Box_0\varphi \in T_0$. This means $\widehat{\Box_0\varphi} \in \mathcal{E}_0$. Taking $e := \widehat{\Box_0\varphi}$, we get $e \in \mathcal{E}_0$ and $T \in e$. Moreover, since $\vdash_{\mathsf{Log}_{\forall\Box\Box_0}} \Box_0\varphi \to \varphi$, we obtain $e = \widehat{\Box_0\varphi} \subseteq \widehat{\varphi}$.

($\Leftarrow$) Suppose there is $e \in \mathcal{E}_0$ such that $T \in e \subseteq \widehat{\varphi}$. Then, by the definition of $\mathcal{E}_0$, we obtain that $e = \widehat{\Box_0\theta}$ for some θ such that $[\exists]\Box_0\theta \in T_0$. Therefore, $T \in e = \widehat{\Box_0\theta} \subseteq \widehat{\varphi}$. This implies that the set $\Gamma := \{\Box_0\theta\} \cup \{\forall\psi : \forall\psi \in T\} \cup \{\neg\varphi\}$ is inconsistent. Otherwise, by Lindenbaum's Lemma, there exists a $S \in X$ such that $\Box_0 \in S$ and $\neg\varphi \in S$. The former means that $S \in \widehat{\Box_0\theta}$ and the latter means (since S is maximal) that $S \notin \widehat{\varphi}$. Thus, $S \in \widehat{\Box_0\theta} \backslash \widehat{\varphi}$, contradicting the assumption $\widehat{\Box_0\theta} \subseteq \widehat{\varphi}$. Therefore, given that Γ is inconsistent, there exists a *finite* set $\{[\forall]\psi_1, \ldots, [\forall]\psi_n\} \subseteq \Gamma$ such that $\vdash \bigwedge_{i\leq n}[\forall]\psi_i \to (\Box_0\theta \to \varphi)$. Since $[\forall]$ is a normal modality and T is maximal, $\bigwedge_{i\leq n}[\forall]\psi_i = [\forall]\gamma$ for some $[\forall]\gamma \in T$. We then have

1. $\vdash [\forall]\gamma \to (\Box_0\theta \to \varphi)$
2. $\vdash ([\forall]\gamma \wedge \Box_0\theta) \to \varphi$
3. $\vdash \Box_0([\forall]\gamma \wedge \Box_0\theta) \to \Box_0\varphi$ (Monotonicity of $\Box_0$)
4. $\vdash \Box_0\Box_0([\forall]\gamma \wedge \theta) \to \Box_0\varphi$ (Pullout axiom)
5. $\vdash \Box_0([\forall]\gamma \wedge \theta) \to \Box_0\varphi$ (since $\vdash_{\mathsf{Log}_{\forall\Box\Box_0}} \Box_0\varphi \leftrightarrow \Box_0\Box_0\varphi$)
6. $\vdash ([\forall]\gamma \wedge \Box_0\theta) \to \Box_0\varphi$ (Pullout axiom)

Therefore, since $[\forall]\gamma, \Box_0\theta \in T$ and T is maximal, we obtain $\Box_0\varphi \in T$, i.e., $T \in \widehat{\Box_0\varphi}$. $\Box$

5.6.22. Lemma (Truth Lemma). *For every formula $\varphi \in \mathcal{L}_{[\forall]\Box_0\Box}$, we have*

$$\|\varphi\|^{\mathcal{M}} = \widehat{\varphi}.$$

Proof:
The proof follows standardly by subformula induction on φ, where the inductive step for each modality uses the corresponding Existence Lemma, as usual. □

5.6.23. Proposition. $\mathsf{Log}_{\forall\Box\Box_0}$ *is sound and strongly complete for quasi-models.*

Proof:
Let Φ_0 be a $\mathsf{Log}_{\forall\Box\Box_0}$-consistent set of formulas. Then, by Lindenbaum's Lemma (Lemma 5.6.16), Φ_0 can be extended to a maximally consistent set T_0. We can then construct a canonical quasi-model $\mathcal{M} = (X, \mathcal{E}_0, \leq, V)$ for T_0 as in Definition 5.6.17, and by Lemma 5.6.22 obtain that $\mathcal{M}, T_0 \models \varphi$ for all $\varphi \in \Phi_0$. □

Step 2: Strong Finite Quasi-Model Property. In this section, we prove that the logic $\mathsf{Log}_{\forall\Box\Box_0}$ has the strong finite quasi-model property. We do so via a filtration argument using the canonical model described in Definition 5.6.17.

Let φ_0 be a $\mathsf{Log}_{\forall\Box\Box_0}$-consistent formula. By Lemma 5.6.16, there exist a maximally consistent set T_0 such that $\varphi_0 \in T_0$. Consider the canonical quasi-model $\mathcal{M} = (X, \mathcal{E}_0, \leq, V)$ for T_0 (as given in Definition 5.6.17). We will use two facts about this model:

1. $\|\varphi\|^{\mathcal{M}} = \widehat{\varphi}$, for all $\varphi \in \mathcal{L}_{[\forall]\Box_0\Box}$; and
2. $\mathcal{E}_0 = \{\widehat{\Box_0\varphi} \mid [\exists]\Box_0\varphi \in T_0\} = \{\|\Box_0\varphi\|^{\mathcal{M}} \mid [\exists]\Box_0\varphi \in T_0\}$.

Closure conditions for Σ: Let Σ be a *finite* set such that: (1) $\varphi_0 \in \Sigma$; (2) Σ is closed under subformulas; (3) if $\Box_0\varphi \in \Sigma$ then $\Box\Box_0\varphi \in \Sigma$; (4) Σ is closed under single negations; (5) $\Box_0\top \in \Sigma$. For $x, y \in X$, put

$$x \equiv_\Sigma y \text{ iff for all } \psi \in \Sigma\, (x \in \|\psi\|^{\mathcal{M}} \Longleftrightarrow y \in \|\psi\|^{\mathcal{M}}),$$

and denote by $|x| := \{y \in X \mid x \equiv_\Sigma y\}$ the equivalence class of x modulo $\equiv_\Sigma$. Also, put $X^f := \{|x| \mid x \in X\}$, and more generally put $e^f := \{|x| \mid x \in e\}$ for every $e \in \mathcal{E}_0$. We now define a "*filtrated model*" $\mathcal{M}^f = (X^f, \mathcal{E}_0^f, \leq^f, V^f)$, where

- $X^f := \{|x| \mid x \in X\}$;
- $|x| \leq^f |y|$ iff for all $\Box\psi \in \Sigma$ $\big(x \in \|\Box\psi\|^{\mathcal{M}} \Rightarrow y \in \|\Box\psi\|^{\mathcal{M}}\big)$;
- $\mathcal{E}_0^f := \{e^f \mid e = \widehat{\Box_0\psi} = \|\Box_0\psi\|^{\mathcal{M}} \in \mathcal{E}_0$ for some ψ such that $\Box_0\psi \in \Sigma\}$;

- $V^f(p) := \{|x| : x \in V(p)\}$.

5.6.24. Lemma. *$\mathcal{M}^f$ is a finite quasi-model (of size bounded by a computable function of φ_0).*

Proof:
Since Σ is finite, there are only finitely many equivalence classes modulo $\equiv_\Sigma$. Therefore, X^f is finite. In fact, X^f has at most $2^{|\Sigma|}$ states. It is obvious that $\leq^f$ is a preorder. Moreover, since $X = \|\Box_0\top\|^{\mathcal{M}}$ and $\Box_0\top \in \Sigma$, we have $X^f \in \mathcal{E}_0^f$. Also, since $e \neq \emptyset$ for all $e \in \mathcal{E}_0$, we have each $e^f \in \mathcal{E}_0^f$ nonempty. So we only have to prove that the evidence sets e^f are upward–closed. For this, let $e^f \in \mathcal{E}_0^f$, $|x|, |y| \in X^f$ such that $|x| \in e^f$ and $|x| \leq^f |y|$. We need to show that $|y| \in e^f$. By the definition of $\mathcal{E}_0^f$, we know that $e = \widehat{\Box_0\psi} = \|\Box_0\psi\|^{\mathcal{M}}$ for some $\Box_0\psi \in \Sigma$. From $|x| \in e^f$, it follows that there is some $x' \equiv_\Sigma x$ such that $x' \in e = \|\Box_0\psi\|^{\mathcal{M}}$, and since $\Box_0\psi \in \Sigma$, we have $x \in \|\Box_0\psi\|^{\mathcal{M}}$. Therefore, since $\vdash_{\mathsf{Log}_{\forall\Box\Box_0}} \Box_0\psi \to \Box\Box_0\psi$ (this is easy to see from axioms ($4_{\Box_0}$) and (FE) stated in Table 5.5), we have $x \in \|\Box\Box_0\psi\|^{\mathcal{M}}$. But $\Box\Box_0\psi \in \Sigma$ (by the closure assumptions on Σ), so $|x| \leq^f |y|$ gives us $y \in \|\Box\Box_0\psi|\|^{\mathcal{M}}$. By the axiom ($\mathrm{T}_\Box$), we obtain $y \in \|\Box_0\psi\|^{\mathcal{M}} = \widehat{\Box_0\psi} = e$, hence $|y| \in e^f$. $\Box$

5.6.25. Lemma (Filtration Lemma). *For every formula $\varphi \in \Sigma$, we have*

$$\|\varphi\|^{\mathcal{M}^f} = \{|x| \mid x \in \|\varphi\|^{\mathcal{M}}\}.$$

Proof:
The proof follows by subformula induction induction on $\varphi \in \Sigma$; cases for the propositional variables, the Boolean connectives and the modalities $[\forall]\varphi$ and $\Box\varphi$ are treated as usual (in the last case using the filtration property of $\leq^f$ that: if $x \leq y$ than $|x| \leq^f |y|$). We only prove here the inductive case for $\varphi := \Box_0\psi$:

($\Rightarrow$) Let $|x| \in \|\Box_0\psi\|^{\mathcal{M}^f}$. This means that there exists some $e^f \in \mathcal{E}_0^f$ s.t. $|x| \in e^f \subseteq \|\psi\|^{\mathcal{M}^f}$. By the definition of $\mathcal{E}_0^f$, there exists some χ such that $\Box_0\chi \in \Sigma$ and $e = \widehat{\Box_0\chi} = \|\Box_0\chi\|^{\mathcal{M}} \in \mathcal{E}_0$. From $|x| \in e^f$, it follows that there is some $x' \equiv_\Sigma x$ such that $x' \in e = \|\Box_0\chi\|^{\mathcal{M}}$, and since $\Box_0\chi \in \Sigma$, we have $x \in \|\Box_0\chi\|^{\mathcal{M}} = e$. Now let $y \in e$ be *any* element of e. Then, by the definition of e^f and the assumption that $e^f \subseteq \|\psi\|^{\mathcal{M}^f}$, we obtain $|y| \in e^f \subseteq \|\psi\|^{\mathcal{M}^f}$. So, $|y| \in \|\psi\|^{\mathcal{M}^f}$. Therefore, by the induction hypothesis, $y \in \|\psi\|_{\mathcal{M}}$, hence, $e \subseteq \|\psi\|^{\mathcal{M}}$. Thus, we have found an evidence set $e \in \mathcal{E}_0$ such that $x \in e \subseteq \|\psi\|^{\mathcal{M}}$, i.e., shown that $x \in \|\Box_0\psi\|^{\mathcal{M}}$.

($\Leftarrow$) Let $x \in \|\Box_0\psi\|^{\mathcal{M}}$. It is easy to see that $[\exists]\Box_0\psi \in x$ (since $\vdash_{\mathsf{Log}_{\forall\Box\Box_0}} \Box_0\psi \to [\exists]\Box_0\psi$), and so also $[\exists]\Box_0\psi \in T_0$ (since $x \in X$, thus, $x \sim T_0$). This means that the set $e := \widehat{\Box_0\psi} = \|\Box_0\psi\|^{\mathcal{M}} \in \mathcal{E}_0$ is an evidence set in the canonical model (see Definition 5.6.17), and since $\Box_0\psi \in \Sigma$, we conclude that $e^f \in \mathcal{E}_0^f$. We obviously have $x \in e$, and so $|x| \in e^f$. Since $\vdash_{\mathsf{Log}_{\forall\Box\Box_0}} \Box_0\psi \to \psi$, we have

$e = \|\Box_0\psi\|^{\mathcal{M}} \subseteq \|\psi\|^{\mathcal{M}}$, and hence $e^f \subseteq \{|y| \mid y \in \|\psi\|^{\mathcal{M}}\} = \|\psi\|^{\mathcal{M}^f}$ (by the induction hypothesis). Thus, we have found $e^f \in \mathcal{E}_0^f$ such that $|x| \in e^f \subseteq \|\psi\|^{\mathcal{M}^f}$, i.e., shown that $|x| \in \|\Box_0\psi\|^{\mathcal{M}^f}$. □

5.6.26. THEOREM. $\mathsf{Log}_{\forall\Box\Box_0}$ *has strong finite quasi-model property.*

Proof:
Let φ_0 be a $\mathsf{Log}_{\forall\Box\Box_0}$-consistent formula. Then, by Lindenbaum's Lemma (Lemma 5.6.16), φ_0 can be extended to a maximally consistent set T_0 such that $\varphi_0 \in T_0$. We can then construct a canonical quasi-model $\mathcal{M} = (X, \mathcal{E}_0, \leq, V)$ for T_0 as in Definition 5.6.17, and by Lemma 5.6.22 obtain that $\mathcal{M}, T_0 \models \varphi_0$. Then, by Lemma 5.6.25, we have $\mathcal{M}^f, |T_0| \models \varphi_0$, where $\mathcal{M}^f$ is the filtrated model of $\mathcal{M}$ through the finite set Σ that is obtained by closing $\{\varphi_0\}$ under the closure conditions (1)-(5). By Lemma 5.6.24, we know that $\mathcal{M}^f$ is a finite model whose size is bounded by $2^{|\Sigma|}$, therefore we conclude that $\mathsf{Log}_{\forall\Box\Box_0}$ has the strong finite quasi-model property. □

Step 3: Equivalence of Finite Quasi-Models and Finite Alexandroff Quasi-Models. In this section, we prove that every finite quasi-model is modally equivalent to a finite Alexandroff quasi-model, and therefore, to a topo-e-model with respect to the language $\mathcal{L}_{[\forall]\Box_0\Box}$.

Let $\mathcal{M} = (X, \mathcal{E}_0, \leq, V)$ be a finite quasi-model. We form a new structure $\tilde{\mathcal{M}} = (\tilde{X}, \tilde{\mathcal{E}}_0, \tilde{\leq}, \tilde{V})$, by putting:

- $\tilde{X} := X \times \{0, 1\}$;
- $\tilde{V}(p) := V(p) \times \{0, 1\}$;
- $(x, i)\tilde{\leq}(y, j)$ iff $x \leq y$ and $i = j$;
- $\tilde{\mathcal{E}}_0 := \{e_i \mid e \in \mathcal{E}_0, i \in \{0, 1\}\} \cup \{e_i^y \mid y \in e \in \mathcal{E}_0, i \in \{0, 1\}\} \cup \{\tilde{X}\}$, where we used notations
  - $e_i := e \times \{i\} = \{(x, i) \mid x \in e\}$, and
  - $e_i^y := \uparrow y \times \{i\} \cup e \times \{1 - i\} = \{(x, i) \mid y \leq x\} \cup e_{1-i}$.

5.6.27. LEMMA. $\tilde{\mathcal{M}}$ *is a finite quasi-model.*

Proof:
It is easy to see that $\tilde{\mathcal{M}}$ is finite, in fact, it is of size $2{\cdot}|X|$. It is guaranteed by definition that $\tilde{X} \in \tilde{\mathcal{E}}_0$ and $\emptyset \notin \tilde{\mathcal{E}}_0$. To show that every element of $\tilde{\mathcal{E}}_0$ is upward-closed with respect to $\tilde{\leq}$, let $\tilde{e} \in \tilde{\mathcal{E}}_0$ and $(x, i), (y, j) \in \tilde{X}$ such that $(x, i) \in \tilde{e}$ and $(x, i)\tilde{\leq}(y, j)$. Then, by the definition of $\tilde{\leq}$, we know that $x \leq y$ and $i = j$. We

have two cases: if $\tilde{e} = e \times \{i\}$ for some $e \in \mathcal{E}_0$, then $y \in e$ (since e is upward closed with respect to $\leq$, $x \in e$ and $x \leq y$), therefore, $(y,i) \in e \times \{i\} = \tilde{e}$. If $\tilde{e} = e^z_k$ for some $z \in X$ and $k \in \{0,1\}$, we again have two cases. If $k = 1-i$, then the result follows as in the first case. If $k = i$, then $\uparrow z \times i \subseteq \tilde{e}$. Since $(x,i) \in \tilde{e}$, we obtain that $z \leq x$, and thus, $z \leq y$ (since $\leq$ is transitive). We therefore conclude that $(y,i) \in \uparrow z \times i \subseteq \tilde{e}$. □

Notation: For any set $\tilde{Y} \subseteq \tilde{X}$, put $\tilde{Y}_X := \{y \in X \mid (y,i) \in \tilde{Y} \text{ for some } i \in \{0,1\}\}$ for the set consisting of first components of all members of $\tilde{Y}$. It is easy to see that we have $(\tilde{Y} \cup \tilde{Z})_X = \tilde{Y}_X \cup \tilde{Z}_X$, and $\tilde{X}_X = X$.

5.6.28. Lemma. *If $y \in e \in \mathcal{E}_0$, $i \in \{0,1\}$ and $\tilde{e} \in \{e_i, e^y_i\}$, then we have:*

1. $\tilde{e}_X = e$;
2. $e^y_i \cap e_i = \uparrow(y,i)$, *where* $\uparrow(y,i) = \{\tilde{x} \in \tilde{X} \mid (y,i) \tilde{\leq} \tilde{x}\} = \{(x,i) \mid y \leq x\}$.

Proof:

(1): If $\tilde{e} = e_i$, then $\tilde{e}_X = (e \times \{i\})_X = e$. If $\tilde{e} = e^y_i$, then $\tilde{e}_X = (\uparrow y \times \{i\})_X \cup (e \times \{1-i\})_X = \uparrow y \cup e = e$ (since e is upward-closed and $y \in e$, so $\uparrow y \subseteq e$).

(2): $e^y_i \cap e_i = (\uparrow y \times \{i\} \cup e \times \{1-i\}) \cap (e \times \{i\}) = (\uparrow y \cap e) \times \{i\} = \uparrow y \times \{i\} = \uparrow(y,i)$ (since $\uparrow y \subseteq e$). □

5.6.29. Lemma. *$\tilde{\mathcal{M}}$ is an Alexandroff quasi-model (and thus also a topo-e-model).*

Proof:

By Proposition 5.6.15, it is enough to show that, for every $(y,i) \in \tilde{X}$, the upward-closed set $\uparrow(y,i)$ is open in the topology $\tau_{\tilde{\mathcal{E}}}$ generated by $\tilde{\mathcal{E}}_0$: this follows directly from Lemma 5.6.28-(2). □

5.6.30. Lemma (Modal-Equivalence Lemma). *For all $\varphi \in \mathcal{L}_{[\forall]\Box_0\Box}$,*

$$\|\varphi\|^{\tilde{\mathcal{M}}} = \|\varphi\|^{\mathcal{M}} \times \{0,1\}.$$

Proof:

The proof follows by subformula induction on φ; cases for the propositional variables, the Boolean connectives and the modalities $[\forall]\varphi$ and $\Box\varphi$ are straightforward. We only prove here the inductive case for $\varphi := \Box_0\psi$.

($\Rightarrow$) Suppose that $(x,i) \in \|\Box_0\psi\|^{\tilde{\mathcal{M}}}$. Then there exists some $\tilde{e} \in \tilde{\mathcal{E}}_0$ such that $(x,i) \in \tilde{e} \subseteq \|\psi\|^{\tilde{\mathcal{M}}} = \|\psi\|^{\mathcal{M}} \times \{0,1\}$ (where we used the induction hypothesis for ψ at the last step). From this, we obtain that $x \in \tilde{e}_X \subseteq (\|\psi\|^{\mathcal{M}} \times \{0,1\})_X = \|\psi\|^{\mathcal{M}}$. But by the construction of $\tilde{\mathcal{E}}_0$, $\tilde{e} \in \tilde{\mathcal{E}}_0$ means that either $\tilde{e} = \tilde{X}$ or there exist

$e \in \mathcal{E}_0$, $y \in e$ and $j \in \{0,1\}$ such that $\tilde{e} \in \{e_j, e_j^y\}$. If the former is the case, we have $x \in \tilde{e}_X = X \subseteq \|\psi\|^{\mathcal{M}}$. Since $X \in \mathcal{E}_0$, by the semantics of $\Box_0$, we obtain $x \in \|\Box_0\psi\|_{\mathcal{M}}$. If the latter is the case, by Lemma 5.6.28-(1), we have $\tilde{e}_X = e$, so we conclude that $x \in \tilde{e}_X = e \subseteq \|\varphi\|^{\mathcal{M}}$. Therefore, again by the semantics of $\Box_0$, we have $x \in \|\Box_0\psi\|^{\mathcal{M}}$.

($\Leftarrow$) Suppose that $x \in \|\Box_0\psi\|^{\mathcal{M}}$. Then, there exists some $e \in \mathcal{E}_0$ such that $x \in e \subseteq \|\psi\|^{\mathcal{M}}$. Take now the set $e_i = e \times \{i\} \in \tilde{\mathcal{E}_0}$. Clearly, we have $(x,i) \in e_i \subseteq \|\psi\|^{\mathcal{M}} \times \{i\} \subseteq \|\psi\|^{\mathcal{M}} \times \{0,1\} = \|\psi\|^{\tilde{\mathcal{M}}}$ (where we used the induction hypothesis for ψ at the last step), i.e., we have $(x,i) \in \|\Box_0\psi\|^{\tilde{\mathcal{M}}}$. $\Box$

5.6.31. THEOREM. *Every finite quasi-model is modally equivalent to a finite Alexandroff quasi-model, therefore, to a topo-e-model with respect to the language* $\mathcal{L}_{[\forall]\Box_0\Box}$.

Proof:
The proof immediately follows from Lemma 5.6.30: the same formulas are satisfied at x in $\mathcal{M}$ as at (x,i) in $\tilde{\mathcal{M}}$. $\Box$

Proof of Theorem 5.6.13: Theorem 5.6.13 (completeness and finite model property for topo-e-models) is thus obtained as an immediate corollary of Proposition 5.6.23, Theorems 5.6.26 and 5.6.31.

5.6.6 Dynamics Extensions of $\mathcal{L}_{[\forall]\Box_0\Box}$

Moving on to *dynamic extensions*, we consider PDL-style languages $\mathcal{L}^{!}_{\forall\Box\Box_0}$, $\mathcal{L}^{+}_{\forall\Box\Box_0}$, $\mathcal{L}^{\Uparrow}_{\forall\Box\Box_0}$, and $\mathcal{L}^{\#}_{\forall\Box\Box_0}$ obtained by adding to $\mathcal{L}_{[\forall]\Box_0\Box}$ dynamic modalities $[!\varphi]\psi$ for public announcements, respectively $[+\varphi]\psi$ for evidence addition, $[\Uparrow\varphi]\psi$ for evidence upgrade and $[\#]\psi$ for feasible evidence combination with the following intended readings:

$[!\varphi]\psi := \psi$ becomes true after the public announcement of φ

$[+\varphi]\psi := \psi$ becomes true after φ is accepted as an admissible piece of evidence

$[\Uparrow\varphi]\psi := \psi$ becomes true after φ is accepted as the most important evidence

$[\#]\psi := \psi$ becomes true after the basic evidence is feasibly combined

The *semantics* for dynamic operators uses the corresponding model change presented in Section 5.4 (as standard in Dynamic Epistemic Logic). More precisely, given a topo-e-model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ and $x \in X$, the semantics for the above mentioned dynamic operators are defined as

$$\begin{array}{lll} x \in [\![[!\varphi]\psi]\!] & \text{iff} & x \in [\![\varphi]\!] \text{ implies } x \in [\![\psi]\!]^{\mathfrak{M}^{![\![\varphi]\!]}} \\ x \in [\![[+\varphi]\psi]\!] & \text{iff} & x \in [\![[\exists]\varphi]\!] \text{ implies } x \in [\![\psi]\!]^{\mathfrak{M}^{+[\![\varphi]\!]}} \\ x \in [\![[\Uparrow\varphi]\psi]\!] & \text{iff} & x \in [\![[\exists]\varphi]\!] \text{ implies } x \in [\![\psi]\!]^{\mathfrak{M}^{\Uparrow[\![\varphi]\!]}} \\ x \in [\![[\#]\varphi]\!] & \text{iff} & x \in [\![\varphi]\!]^{\mathfrak{M}^{\#}} \end{array}$$

where we denote by $[\![\psi]\!]^{\mathfrak{M}^{![\![\varphi]\!]}}$ the extension of ψ in the updated model $\mathfrak{M}^{![\![\varphi]\!]}$, etc. The precondition $x \in [\![\varphi]\!]$ in the above clause for public announcements encodes the fact that public announcements are factive: so one can only update with *true* sentences φ. The preconditions $x \in [\![[\exists]\varphi]\!]$ in the clauses for evidence addition and upgrade encodes the fact that, in order to qualify as (new) evidence, φ has to be *consistent* (i.e. $[\![\varphi]\!] \neq \emptyset$). In the following, we present the sound and complete axiomatizations for the corresponding dynamic systems. These will be obtain by adding a set of reduction axioms for each dynamic modality to the axiomatization $\mathsf{Log}_{\forall\Box\Box_0}$, as standard in Dynamic Epistemic Logic (Baltag et al., 1998; van Ditmarsch et al., 2007; van Benthem, 2011). We only prove the validity of the reduction axiom for the modality $\Box_0$ in each case and leave the other cases for the reader since they follow either trivially or similar to the case for $\Box_0$.

5.6.32. THEOREM. *The sound and complete logic $\mathsf{Log}^{!}_{\forall\Box\Box_0}$ of evidence and public announcements with respect to the class of all topo-e-models is obtained by adding the following reduction axioms to the system $\mathsf{Log}_{\forall\Box\Box_0}$:*

1. $[!\varphi]p \leftrightarrow (\varphi \to p)$
2. $[!\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[!\varphi]\psi)$
3. $[!\varphi](\psi \wedge \chi) \leftrightarrow ([!\varphi]\psi \wedge [!\varphi]\chi)$
4. $[!\varphi]\Box_0\psi \leftrightarrow (\varphi \to \Box_0[!\varphi]\psi)$
5. $[!\varphi]\Box\psi \leftrightarrow (\varphi \to \Box[!\varphi]\psi)$
6. $[!\varphi][\forall]\psi \leftrightarrow (\varphi \to [\forall][!\varphi]\psi)$
7. $[!\varphi][!\psi]\chi \leftrightarrow [!\langle\varphi\rangle\psi]\chi$

Proof:
Let $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ be a topo-e-model, $x \in X$ and $\varphi, \psi \in \mathcal{L}^{!}_{\forall\Box\Box_0}$.

Axiom-4:

$$\begin{aligned}
x \in [\![[!\varphi]\Box_0\psi]\!] &\text{ iff } x \in [\![\varphi]\!] \text{ implies } x \in [\![\Box_0\psi]\!]^{\mathfrak{M}^{![\![\varphi]\!]}} \\
&\text{ iff } x \in [\![\varphi]\!] \text{ implies } \exists e \in \mathcal{E}_0^{![\![\varphi]\!]}(x \in e \subseteq [\![\psi]\!]^{\mathfrak{M}^{![\![\varphi]\!]}}) \\
&\text{ iff } x \in [\![\varphi]\!] \text{ implies } \exists e' \in \mathcal{E}_0(x \in e' \cap [\![\varphi]\!] = e \subseteq [\![\psi]\!]^{\mathfrak{M}^{![\![\varphi]\!]}}) \\
&\hspace{6cm}(\text{by defn. of } \mathcal{E}_0^{![\![\varphi]\!]}) \\
&\text{ iff } x \in [\![\varphi]\!] \text{ implies } \exists e' \in \mathcal{E}_0(x \in e' \subseteq [\![[!\varphi]\psi]\!]) \\
&\text{ iff } x \in [\![\varphi]\!] \text{ implies } x \in [\![\Box_0[!\varphi]\psi]\!] \\
&\text{ iff } x \in [\![\varphi \to \Box_0[!\varphi]\psi]\!]
\end{aligned}$$

□

5.6.33. THEOREM. *The sound and complete logic $\mathsf{Log}^{+}_{\forall\Box\Box_0}$ of evidence and evidence addition with respect to the class of all topo-e-models is obtained by adding the axiom K_+ and the Necessitation rule (Nec_+) for the evidence addition modalities as well as the following reduction axioms to $\mathsf{Log}_{\forall\Box\Box_0}$:*

1. $[+\varphi]p \leftrightarrow ([\exists]\varphi \to p)$

2. $[+\varphi]\neg\psi \leftrightarrow ([\exists]\varphi \to \neg[+\varphi]\psi)$

3. $[+\varphi](\psi \wedge \chi) \leftrightarrow ([+\varphi]\psi \wedge [+\varphi]\chi)$

4. $[+\varphi]\Box_0\psi \leftrightarrow ([\exists]\varphi \to (\Box_0[+\varphi]\psi \vee (\varphi \wedge [\forall](\varphi \to [+\varphi]\psi))))$

5. $[+\varphi]\Box\psi \leftrightarrow ([\exists]\varphi \to (\Box[+\varphi]\psi \vee (\varphi \wedge \Box(\varphi \to [+\varphi]\psi))))$

6. $[+\varphi][\forall]\psi \leftrightarrow ([\exists]\varphi \to [\forall][+\varphi]\psi)$

Proof:
Let $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ be a topo-e-model, $x \in X$ and $\varphi, \psi \in \mathcal{L}^+_{\forall\Box\Box_0}$. Observe that

$$x \in [\![[\exists]\varphi]\!] \text{ implies } [\![\psi]\!]^{\mathfrak{M}^{+[\![\varphi]\!]}} = [\![[+\varphi]\psi]\!] \tag{5.1}$$

Axiom-4:

$x \in [\![[+\varphi]\Box_0\psi]\!]$
iff $x \in [\![[\exists]\varphi]\!]$ implies $x \in [\![\Box_0\psi]\!]^{\mathfrak{M}^{+[\![\varphi]\!]}}$
iff $x \in [\![[\exists]\varphi]\!]$ implies $\exists e \in \mathcal{E}_0^{+[\![\varphi]\!]}(x \in e \subseteq [\![\psi]\!]^{\mathfrak{M}^{+[\![\varphi]\!]}})$
iff $x \in [\![[\exists]\varphi]\!]$ implies $(\exists e' \in \mathcal{E}_0(x \in e' \subseteq [\![\psi]\!]^{\mathfrak{M}^{+[\![\varphi]\!]}})$ or $(x \in [\![\varphi]\!] \subseteq [\![\psi]\!]^{\mathfrak{M}^{+[\![\varphi]\!]}}))$
(by defn. of $\mathcal{E}_0^{+[\![\varphi]\!]}$)

iff $x \in [\![[\exists]\varphi]\!]$ implies $(\exists e' \in \mathcal{E}_0(x \in e' \subseteq [\![[+\varphi]\psi]\!])$ or $x \in [\![\varphi]\!] \subseteq [\![[+\varphi]\psi]\!])$
(by (5.1))
iff $x \in [\![[\exists]\varphi]\!]$ implies $((x \in [\![\Box_0[+\varphi]\psi]\!])$ or $(x \in [\![\varphi]\!]$ and $x \in [\![[\forall](\varphi \to [+\varphi]\psi]\!]))$
iff $x \in [\![[\exists]\varphi]\!]$ implies $(x \in [\![\Box_0[+\varphi]\psi]\!]$ or $x \in [\![\varphi \wedge [\forall](\varphi \to [+\varphi]\psi]\!])$
iff $x \in [\![[\exists]\varphi \to (\Box_0[+\varphi]\psi \vee (\varphi \wedge [\forall](\varphi \to [+\varphi]\psi)))]\!]$

The proof for the modality $\Box$ follows in a similar way with minor differences because of the fact that for every $e \in \mathcal{E}^{+[\![\varphi]\!]}$ there is some combined evidence $e' \in \mathcal{E}$ such that either $e = e'$ or $e = e' \cap [\![\varphi]\!]$. Therefore, we have

Axiom-5:

$x \in [\![[+\varphi]\Box\psi]\!]$
iff $x \in [\![[\exists]\varphi]\!]$ implies $\exists e \in \mathcal{E}^{+[\![\varphi]\!]}(x \in e \subseteq [\![\psi]\!]^{\mathfrak{M}^{+[\![\varphi]\!]}})$
iff $x \in [\![[\exists]\varphi]\!]$ implies $\exists e' \in \mathcal{E}(x \in e' \subseteq [\![\psi]\!]^{\mathfrak{M}^{+[\![\varphi]\!]}}$ or $x \in e' \cap [\![\varphi]\!] \subseteq [\![\psi]\!]^{\mathfrak{M}^{+[\![\varphi]\!]}})$
iff $x \in [\![[\exists]\varphi]\!]$ implies $\exists e' \in \mathcal{E}((x \in e' \subseteq [\![[+\varphi]\psi]\!])$
or $(x \in [\![\varphi]\!]$ and $x \in e' \subseteq [\![\varphi \to [+\varphi]\psi]\!]))$
iff $x \in [\![[\exists]\varphi]\!]$ implies $(x \in [\![\Box[+\varphi]\psi]\!]$ or $(x \in [\![\varphi]\!]$ and $x \in [\![\Box(\varphi \to [+\varphi]\psi]\!]))$
iff $x \in [\![[\exists]\varphi \to (\Box[+\varphi]\psi \vee (\varphi \wedge \Box(\varphi \to [+\varphi]\psi)))]\!]$

$\square$

5.6.34. THEOREM. *The sound and complete logic $\mathsf{Log}^{\Uparrow}_{\forall\Box\Box_0}$ of evidence and evidence upgrade with respect to the class of all topo-e-models is obtained by adding the axiom $K_{\Uparrow}$ and the Necessitation rule ($Nec_{\Uparrow}$) for the evidence addition modalities as well as the following reduction axioms to $\mathsf{Log}_{\forall\Box\Box_0}$:*

1. $[\Uparrow\varphi]p \leftrightarrow ([\exists]\varphi \to p)$
2. $[\Uparrow\varphi]\neg\psi \leftrightarrow ([\exists]\varphi \to \neg[\Uparrow\varphi]\psi)$
3. $[\Uparrow\varphi](\psi \wedge \chi) \leftrightarrow ([\Uparrow\varphi]\psi \wedge [\Uparrow\varphi]\chi)$
4. $[\Uparrow\varphi]\Box_0\psi \leftrightarrow ([\exists]\varphi \to ((\Box_0[\Uparrow\varphi]\psi \vee \varphi) \wedge [\forall](\varphi \to [\Uparrow\varphi]\psi)))$
5. $[\Uparrow\varphi]\Box\psi \leftrightarrow ([\exists]\varphi \to ((\Box[\Uparrow\varphi]\psi \vee \varphi) \wedge [\forall](\varphi \to [\Uparrow\varphi]\psi)))$
6. $[\Uparrow\varphi][\forall]\psi \leftrightarrow ([\exists]\varphi \to [\forall][\Uparrow\varphi]\psi)$

Proof:
Let $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ be a topo-e-model, $x \in X$ and $\varphi, \psi \in \mathcal{L}^{\Uparrow}_{\forall\Box\Box_0}$. Similar to the above case, we have

$$x \in [\![[\exists]\varphi]\!] \text{ implies } [\![\psi]\!]^{\mathfrak{M}^{\Uparrow[\![\varphi]\!]}} = [\![[\Uparrow\varphi]\psi]\!] \tag{5.2}$$

Axiom-4:

$x \in [\![[\Uparrow\varphi]\Box_0\psi]\!]$
iff $x \in [\![[\exists]\varphi]\!]$ implies $\exists e \in \mathcal{E}_0^{\Uparrow[\![\varphi]\!]}(x \in e \subseteq [\![\psi]\!]^{\mathfrak{M}^{\Uparrow[\![\varphi]\!]}})$
iff $x \in [\![[\exists]\varphi]\!]$ implies $(\exists e' \in \mathcal{E}_0(x \in e' \cup [\![\varphi]\!] \subseteq [\![\psi]\!]^{\mathfrak{M}^{\Uparrow[\![\varphi]\!]}})$ or $(x \in [\![\varphi]\!] \subseteq [\![\psi]\!]^{\mathfrak{M}^{\Uparrow[\![\varphi]\!]}}))$ (by defn. of $\mathcal{E}_0^{\Uparrow[\![\varphi]\!]}$)
iff $x \in [\![[\exists]\varphi]\!]$ implies $(\exists e' \in \mathcal{E}_0(x \in e' \cup [\![\varphi]\!] \subseteq [\![[\Uparrow\varphi]\psi]\!])$ or $(x \in [\![\varphi]\!] \subseteq [\![[\Uparrow\varphi]\psi]\!]))$ (by (5.2))
iff $x \in [\![[\exists]\varphi]\!]$ implies $(\exists e' \in \mathcal{E}_0(x \in e' \subseteq [\![[\Uparrow\varphi]\psi]\!]$ and $[\![\varphi]\!] \subseteq [\![[\Uparrow\varphi]\psi]\!])$ or $(x \in [\![\varphi]\!] \subseteq [\![[\Uparrow\varphi]\psi]\!]))$
iff $x \in [\![[\exists]\varphi]\!]$ implies $(x \in [\![\Box_0[\Uparrow\varphi]\psi]\!]$ and $x \in [\![[\forall](\varphi \to [\Uparrow\varphi]\psi)]\!])$ or $(x \in [\![\varphi \wedge [\forall](\varphi \to [\Uparrow\varphi]\psi)]\!])$
iff $x \in [\![[\exists]\varphi \to ((\Box_0[\Uparrow\varphi]\psi \vee \varphi) \wedge [\forall](\varphi \to [\Uparrow\varphi]\psi))]\!]$

The validity of the axiom 5 follows similarly where we replace the basic evidence set $\mathcal{E}_0$ by the corresponding combined evidence set $\mathcal{E}$. $\square$

5.6.35. Theorem. *The sound and complete logic* $\mathsf{Log}^{\#}_{\forall\Box\Box_0}$ *of evidence and feasible evidence combination with respect to the class of all topo-e-models is obtained by adding the axiom* $K_{\#}$ *and the Necessitation rule* $(Nec_{\#})$ *for the evidence addition modalities as well as the following reduction axioms to* $\mathsf{Log}_{\forall\Box\Box_0}$:

1. $[\#]p \leftrightarrow p$
2. $[\#]\neg\varphi \leftrightarrow \neg[\#]\varphi$
3. $[\#](\varphi \wedge \psi) \leftrightarrow ([\#]\varphi \wedge [\#]\psi)$
4. $[\#]\Box\varphi \leftrightarrow \Box[\#]\varphi$
5. $[\#]\Box_0\varphi \leftrightarrow \Box[\#]\varphi$
6. $[\#][\forall]\varphi \leftrightarrow [\forall][\#]\varphi$

Proof:
Let $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ be a topo-e-model, $x \in X$ and $\varphi \in \mathcal{L}^{\#}_{\forall\Box\Box_0}$.

Axiom-5:

$$\begin{aligned}
x \in [\![[\#]\Box_0\varphi]\!] &\text{ iff } x \in [\![\Box_0\varphi]\!]^{\mathfrak{M}^{\#}} \\
&\text{ iff } \exists e^{\#} \in \mathcal{E}_0^{\#}(x \in e^{\#} \subseteq [\![\varphi]\!]^{\mathfrak{M}^{\#}}) \\
&\text{ iff } \exists e^{\#} \in \mathcal{E}_0^{\#}(x \in e^{\#} \subseteq [\![[\#]\varphi]\!]) \\
&\text{ iff } \exists e \in \mathcal{E}(x \in e \subseteq [\![[\#]\varphi]\!]) \qquad (\text{since } \mathcal{E}_0^{\#} = \mathcal{E}^{\#} = \mathcal{E}) \\
&\text{ iff } x \in [\![\Box[\#]\varphi]\!]
\end{aligned}$$

The validity of the axiom 5 follows similarly since $\mathcal{E} = \mathcal{E}^{\#}$. □

5.7 Conclusions and Further Directions

In this chapter, we studied a topological semantics for various notions of *evidence*, *evidence-based justification, argument,* (*conditional*) *belief*, and *knowledge*. We did so by using topological structures based on the (uniform) evidence models of van Benthem and Pacuit (2011). Several soundness, completeness, finite model property and decidability results concerning the logics of belief, knowledge and evidence on *all* topological (evidence) models have been shown. We also discussed some dynamic evidence modalities such as public announcements, evidence addition, evidence upgrade and feasible evidence combination, and provided sound and complete axiomatizations for the associated logics by means of a set of reduction axioms for each dynamic modality.

Our topological approach contributes to the evidence setting of van Benthem and Pacuit (2011); van Benthem et al. (2012, 2014) in many ways. First of all, this topological approach, we believe, gives mathematically more natural meanings to the epistemic/doxastic modalities we considered by providing a precise match between epistemic and topological notions. The list of the epistemic notions studied together with their topological counterparts is given in Table 5.6 below.

Epistemology	Topology
Basic Evidence	Subbasis of a topology ($\mathcal{E}_0$)
(Combined) Evidence	Basis of a topology ($\mathcal{E}$)
Arguments	Open Sets ($\tau_{\mathcal{E}}$)
Justifications	Dense Open Sets
Belief	Dense interior (nowhere dense complement)
Knowledge (of P)	$x \in Int(P)$ and $Int(P)$ is dense

Table 5.6: Matching epistemic and topological notions

Besides, concerning the belief interpretation, our proposal yields a notion of belief that coincides with the one of van Benthem and Pacuit (2011) in "good" cases, and that behaves better in general. More precisely, our justified belief is always consistent, in fact, it satisfies the axioms and rules of the standard belief system $\mathsf{KD45}_B$ on *all* topological spaces (Section 5.6.2). It moreover admits a natural topological reading in terms of dense-open sets (or equivalently, in terms of nowhere dense sets) as "*truth in most states of the model*", where "most" refers to "everywhere but a nowhere dense part". We have also shown that the logic of evidence models under our proposed semantics has the finite model property, whereas this was not the case in (van Benthem and Pacuit, 2011; van Benthem et al., 2012, 2014).

The formalism developed in this chapter improves also on our own work on another topological semantics for Stalnaker's epistemic-doxastic system, presented in Chapters 4. While in Chapters 3 and 4 we could talk about evidential grounds of knowledge and belief only on a semantic level, the current setting provides syntactic representations of evidence, therefore, makes the notion of evidence a part of the logic. Moreover, we showed that knowledge and belief can be interpreted on arbitrary topological spaces (rather than on extremally disconnected or h.e.d. spaces), without changing their logic. To this end, the semantics of knowledge and belief proposed in this chapter generalizes the setting of Chapter 4.

In the rest of this section, we name a few directions for future research:

Connection to "topological" formal learning theory. One line of inquiry involves adding to the semantic structure a larger set $\mathcal{E}_0^{\diamond} \supseteq \mathcal{E}_0$ of *potential evidence*, meant to encompass all the evidence that might be learnt in the future. This would connect well with the topological program in Inductive Epistemology started by Kelly and others (Kelly, 1996; Schulte and Juhl, 1996; Kelly et al., 1995; Kelly and Lin, 2011; Baltag et al., 2015c), in which a topological version of Formal Learning Theory is used to investigate convergence of beliefs to the truth in the limit, when the agent observes a stream of incoming evidence. A formal setting that involves both actual evidence $\mathcal{E}_0$ and potential evidence $\mathcal{E}_0^{\diamond} \supseteq \mathcal{E}_0$

would combine coherentist justification with predictive learning. A logical syntax appropriate for this setting could be obtained by extending our language with operators borrowed from topo-logic (Moss and Parikh, 1992), such as an operator $\Diamond\varphi$, expressing the fact that φ can become true after more evidence is learnt. Inductive *learnability* of φ is then captured by the formula $\Diamond K\varphi$, where K is our defeasible knowledge (rather than the absolutely certain knowledge operator of topo-logic).

Multi-agent extensions. Another line of research involves extending our framework to a *multi-agent* setting. It is straightforward to generalize our semantics to multiple agents, though obtaining a completeness result might not be that easy. However, the real interesting challenge comes when we look at notions of *group knowledge*, for some group G of agents. For *common knowledge*, there are at least two different natural options: (1) the standard Lewis-Aumann concept of the infinite conjunctions of "everybody knows that everybody knows etc." (Lewis, 1969; Aumann, 1976), and (2) a stronger concept, based on *shared evidence* (the intersection $\bigcap_{a\in G}\mathcal{E}_0^a$ of the evidence families $\mathcal{E}_0^a$ of all agents $a \in G$). The two concepts differ in general, and this is related to Barwise's older observation on the distinction of concepts of common knowledge in a topological framework (Barwise, 1988), in contrast to Kripke models, where all the different versions collapse to the same notion (see also van Benthem and Sarenac, 2004 and Bezhanishvili and van der Hoek, 2014, Section 12.4.2.5 for a discussion on the different formalizations of common knowledge on topological spaces). Similarly, in this evidence-based setting, the standard notion of *distributed knowledge* does not seem appropriate to capture a group's *epistemic potential*. Standardly, a group of agents G is said to have distributed (implicit) knowledge of φ if φ is implied by the knowledge of all individuals in G pooled together (see, e.g., Fagin et al., 1995, Chapter 2 for a standard treatment of distributed knowledge based on relation models). In our setting though, a natural way to think about a group's epistemic potential is to let the agents share all their evidence, and compute their knowledge based on the evidence family obtained by taking the union $\mathcal{E}_0^G = \bigcup_{a\in G}\mathcal{E}_0^a$ of all the evidence families $\mathcal{E}_0^a$ of all agents a in G. This corresponds to moving to the smallest topology that includes all agents' evidential topologies τ^a, which also gives us a natural way to define a consistent notion of (potential) *group belief*. However, this setting has some apparent 'defects', that is, some facts known by one individual in the group might be defeated by another member's false or misleading evidence, therefore, the individual knowledge of these facts will be lost after the group members share all their evidence. This is in contrast with the standard notion of distributed knowledge that is group monotonic: the distributed knowledge of a larger group always includes the distributed knowledge of any of its subgroups, and so in particular it includes everything known by any member of the group. One option is to simply give up the dogma that groups are always wiser than their members, and retain the evidence-based model of group knowledge as providing a better representation of the epistemic potential of a group. Learning from others might not always be epistemically beneficial: it all depends on the quality of the others' evidence. There are also ways to avoid this conclusion, pursued by Ramirez (2015), via natural modifications of our models and by defining knowledge to be undefeated by any potential evidence that the agent may learn. This way Ramirez (2015) re-establishes group monotonicity, but showing completeness for the resulting logic possess technical challenges (see Ramirez, 2015, for details).

Part II

From Public Announcements to Effort

Chapter 6

Topological Subset Space Semantics

In this chapter, we present the two topological frameworks, on the basis of which the work presented in the second part of this dissertation was developed. The first is the so-called subset space semantics of Moss and Parikh (1992), and its topological version developed by Georgatos (1993, 1994) and Dabrowski et al. (1996). The second is the topological public announcement formalism introduced by Bjorndahl (2016). We also point out the connections and differences between the epistemic use of topological spaces in Parts I and II of this thesis, especially regarding the types of evidence represented and the notion of knowledge studied.

Outline

In Section 6.1, we present the subset space framework, providing its syntax and semantics as well as the complete axiomatizations of the associated logics with respect to subset spaces and topological spaces. Section 6.2 introduces the topological public announcement logic of Bjorndahl (2016), and provides several expressivity results concerning the languages studied in the aforementioned settings.

6.1 The Subset Space Semantics and TopoLogic

The formalism of "topologic", introduced by Moss and Parikh (1992), and investigated further by Dabrowski et al. (1996), Georgatos (1993, 1994), Weiss and Parikh (2002) and others, represents a *single-agent* subset space logic (SSL) for the notions of knowledge and *effort*. One of the crucial aspects of this framework is that it is concerned not only with the representation of knowledge, but also aimed at giving an account of information gain or knowledge increase in terms of *observational effort*.[1] It is the latter feature of this work that makes the use

[1] Moss and Parikh (1992) is partly inspired by Vickers' work on reconstruction of topology via a logic of finite observation (Vickers, 1989).

of subset spaces significant. While the knowledge modality $K\varphi$ has the standard reading "the agent knows φ (is true)", in the subset space setting, the effort modality $\Box\varphi$ captures a notion of effort as any action that results in an increase in knowledge and is read as "φ stays true no matter what further evidence-gathering efforts are made". The modality $\Box$ therefore captures a notion of *stability* under evidence-gathering. Effort can be in the form of measurement, computation, approximation, or even announcement, depending on the context and the information source. To illustrate the underlying intuition of the subset space semantics, and the notions of knowledge, effort, and evidence it represents, suppose for instance, that you have measured your height and obtained a reading of 5 feet and 10 inches ± 3 inches. The measuring devices we use to calculate such quantities always come with a certain error range, therefore giving us an approximation rather than the precise value. With this measurement in hand, you cannot be said to know whether you are less than 6 feet tall, as your measurement, i.e., the current evidence you have, does not rule out that you are taller or shorter. However, if you are able to spend more resources and take a more precise measurement, e.g., by using a more accurate meter with $\pm$ 1 error range, you come to know that you are less than 6 feet tall (Bjorndahl and Özgün, 2017). Subset space logics are designed to represent such situations, and therefore involve two modalities: one for knowledge K, and the other one for effort $\Box$.

The formulas in the bimodal language are interpreted on subset spaces $(X, \mathcal{O})$, where X is a nonempty domain and $\mathcal{O}$ is an *arbitrary* nonempty collection of subsets of X. The elements of $\mathcal{O}$ represent *possible observations*, and more effort corresponds to a more refined truthful observation, thus, a possible increase in knowledge. A subset space is not necessarily a topological space, however, topological spaces do constitute a particular case of subset spaces and topological reasoning provides the intuition behind this semantics, as we will elaborate below.[2] While presenting the most general case of subset spaces in this section, our main results in later sections will still be based on purely topological models.

In this section, we provide the formal background for the subset space semantics of Moss and Parikh (1992), explaining how these "topological" structures constitute models that are well-equipped to give an account for evidence-based knowledge and its dynamics. We also point out the differences and the connection between the two topological approaches developed in Chapter 5 and Part II,

[2] The subset space setting also comes with an independent technical motivation. Many of the aforementioned sources are concerned with axiomatizing the logics of smaller classes of subset spaces meeting particular closure conditions on the set of subsets $\mathcal{O}$. For example, while Moss and Parikh (1992) axiomatized the logic of subset spaces, Georgatos (1993, 1994) and Dabrowski et al. (1996) provided an axiomatization of the logic of topological spaces, and complete lattice spaces. Moreover, Georgatos (1997) axiomatized the logic of treelike spaces, and Weiss and Parikh (2002) presented an axiomatization for the class of directed spaces. These results are quite interesting from a modal theoretical perspective, however, in this dissertation, we are primarily interested in the applications of topological ideas in epistemic logic. We therefore focus on the epistemic motivation behind the topologic formalism.

respectively. In particular, we compare the evidence representation on evidence models of van Benthem and Pacuit (2011) with the one on subset models of Moss and Parikh (1992), and in turn, the type of evidence-based knowledge studied on these structures.

6.1.1 Syntax and Semantics

In their influential work, Moss and Parikh (1992) consider the bimodal language $\mathcal{L}_{K\Box}$ given by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \Box\varphi,$$

and interpret it on subset spaces, a class of models generalizing topological spaces.

6.1.1. Definition. [Subset Space/Model] A *subset space* is a pair $(X, \mathcal{O})$, where X is a nonempty set of states and $\mathcal{O}$ is a collection of subsets of X. A *subset model* is a tuple $\mathcal{X} = (X, \mathcal{O}, V)$, where $(X, \mathcal{O})$ is a subset space and $V : \text{PROP} \to \mathcal{P}(X)$ a valuation function.

It is not hard to see that subset spaces are just like the evidence models of van Benthem and Pacuit (2011) (given in Definition 5.2.1), but with no constraints on the set of subsets $\mathcal{O}$.[3] However, the way the truth of a formula is defined on subset models leads to a crucial difference between the two settings, especially concerning the type of evidence represented by the elements of $\mathcal{O}$, and the characterization of the notion of knowledge interpreted based on evidence. This point will become clear once we present the formal semantics below.

Subset space semantics interprets formulas not at worlds x but at *epistemic scenarios* of the form (x, U), where $x \in U \in \mathcal{O}$. Let $ES(\mathcal{X})$ denote the collection of all such pairs in $\mathcal{X}$. Given an epistemic scenario $(x, U) \in ES(\mathcal{X})$, the set U is called its *epistemic range*; intuitively, it represents the agent's current information as determined, for example, by the measurements she has taken. The language $\mathcal{L}_{K\Box}$ is interpreted on subset spaces as follows:

6.1.2. Definition. [Subset Space Semantics for $\mathcal{L}_{K\Box}$] Given a subset space model $\mathcal{X} = (X, \mathcal{O}, V)$ and an epistemic scenario $(x, U) \in ES(\mathcal{X})$, *truth* of a formula in the language $\mathcal{L}_{K\Box}$ is defined recursively as follows:

$$\begin{array}{lll}
\mathcal{X}, (x, U) \models p & \text{iff} & x \in V(p), \text{ where } p \in \text{PROP} \\
\mathcal{X}, (x, U) \models \neg\varphi & \text{iff} & \text{not } \mathcal{X}, (x, U) \models \varphi \\
\mathcal{X}, (x, U) \models \varphi \wedge \psi & \text{iff} & \mathcal{X}, (x, U) \models \varphi \text{ and } \mathcal{X}, (x, U) \models \psi \\
\mathcal{X}, (x, U) \models K\varphi & \text{iff} & (\forall y \in U)(\mathcal{X}, (y, U) \models \varphi) \\
\mathcal{X}, (x, U) \models \Box\varphi & \text{iff} & (\forall O \in \mathcal{O})(x \in O \subseteq U \text{ implies } \mathcal{X}, (x, O) \models \varphi)
\end{array}$$

[3] We could in fact define the subset spaces exactly the same way as evidence models by putting the constraints $X \in \mathcal{O}$ and $\emptyset \notin \mathcal{O}$. This would technically make no difference, however, we here prefer to present the most general case.

We say that a formula φ is *valid in a model* $\mathcal{X}$, and write $\mathcal{X} \models \varphi$, if $\mathcal{X}, (x, U) \models \varphi$ for all scenarios $(x, U) \in ES(\mathcal{X})$. We say φ is *valid*, and write $\models \varphi$, if $\mathcal{X} \models \varphi$ for all $\mathcal{X}$. We let $[\![\varphi]\!]^U_{\mathcal{X}} = \{x \in U \mid \mathcal{X}, (x, U) \models \varphi\}$ denote the *truth set*, or equivalently, *extension of* φ *under* U *in the model* $\mathcal{X}$. We again omit the notation for the model, writing simply $(x, U) \models \varphi$ and $[\![\varphi]\!]^U$, whenever $\mathcal{X}$ is fixed.

Epistemic readings of subset space semantics: current vs potential evidence

In subset space semantics, the points of the space represent "possible worlds" (or, *states* of the world). However, having the units of evaluation as pairs of the form (x, U)—rather than a single state x—allows us to distinguish the evidence that the agent currently has in hand from the potential evidence she can *in principle* obtain. More precisely, elements of $\mathcal{O}$ can be thought of as potential pieces of evidence meant to encompass all the evidence that might be learnt in the future, while the epistemic range U of an epistemic scenario (x, U) corresponds to the current evidence, i.e., "evidence-in-hand" by means of which the agent's knowledge is evaluated.[4] This is made precise in the semantic clause for $K\varphi$, which stipulates that the agent knows φ just in case φ is entailed by her *factive*[5] evidence-in-hand. The knowledge modality K therefore behaves like the global modality within the given epistemic range U. For this reason, in various places, we will often refer to K as the global modality. Thus, the type of knowledge captured by the modality K in this setting is absolutely certain, infallible knowledge based on the agent's current truthful evidence. These points already underline the substantial differences between the two evidence-based epistemic frameworks studied in this thesis: while $\mathcal{E}_0$ of an evidence model $(X, \mathcal{E}_0, V)$ represents the set of evidence pieces the agent *has already acquired about the actual situation*, the set $\mathcal{O}$ of a subset model $(X, \mathcal{O}, V)$ represents the set of *potential evidence the agent can in principle discover, even if she does not happen to personally have it in hand at the moment*. A subset model is therefore intended to carry all pieces of evidence the agent currently has and can potentially gather later, hence, supports model-internal means to interpret evidence-based information dynamics, as displayed, e.g., by the effort modality.[6] In this framework, more effort means acquiring more evidence for the actual state of affairs, therefore, a better approximation of the real state. The effort modality $\Box\varphi$ is thus interpreted in terms of *neighbourhood-shrinking* and read as "φ is stably true under evidence-acquisition", i.e., φ is true, and will stay true no matter what further factive evidence is obtained.

[4]The term "evidence-in-hand" is borrowed from (Bjorndahl and Özgün, 2017), where the elements of $\mathcal{O}$ are described as "evidence-out-there".

[5]As in the previous chapters, $x \in U$ expresses the factivity of evidence.

[6]In later sections, we study other dynamic modalities, such as the so-called public and arbitrary announcement modalities, interpreted on topological spaces in the style of the effort modality, that is, without leading to any global change in the initial model.

As every topological space is a subset space, the above readings of the modalities also apply to the topological models. However, the additional structure that topological spaces possess helps us to formalize naturally some further aspects of evidence aggregation (similar to Part I). For example, when $\mathcal{O}$ is closed under finite intersections, we can consider the epistemic range U of a given epistemic scenario (x, U) as a finite stream of truthful information $(O_1, \dots, O_n)$ the agent has received and put together: $x \in U = \bigcap_{i \leq n} O_i \in \mathcal{O}$ (Baltag et al., 2015c). Moreover, as noted in (Moss and Parikh, 1992), we can express some topological concepts in the language $\mathcal{L}_{K\Box}$ that, in fact, lead to concise modal reformulations of *verifiable* and *falsifiable* propositions (as also noted in Georgatos, 1993). To be more precise, given a topo-model $\mathcal{X} = (X, \tau, V)$ and a propositional variable $p \in$ PROP, $V(p)$ is open in τ iff $p \to \Diamond Kp$ is valid in $\mathcal{X}$. Recall that the open sets of a topology are meant to represent potential evidence, i.e., properties of the actual state that are in principle *verifiable*: *whenever they are true, they are supported by a sound piece of evidence that the agent can in principle obtain, therefore, can be known* (Vickers, 1989; Kelly, 1996). Therefore, we can state that

- p is *verifiable* in $\mathcal{X}$ iff $p \to \Diamond Kp$ is valid in $\mathcal{X}$.

In contrast, $V(p)$ is closed in τ iff $\Box\hat{K}p \to p$ is valid in $\mathcal{X}$, and closed sets correspond to properties that are in principle *falsifiable*: *whenever they are false, their falsity can be known.* In a similar manner, this can be formalized in the language $\mathcal{L}_{K\Box}$ as

- p is *falsifiable* in $\mathcal{X}$ iff $\neg p \to \Diamond K\neg p$, or equivalently, $\Box\hat{K}p \to p$ is valid in $\mathcal{X}$.

As remarked in (Vickers, 1989; Kelly, 1996), the closure properties of a topology are satisfied in this interpretation. First, contradictions ($\emptyset$) and tautologies (X) are in principle verifiable (as well as falsifiable). The conjunction $p \wedge q$ of two verifiable facts is also verifiable: if $p \wedge q$ is true, then both p and q are true, and since both are assumed to be verifiable, they can both be known, and hence $p \wedge q$ can be known. Finally, if $\{p_i \mid i \in I\}$ is a (possibly infinite) family of verifiable facts, then their disjunction $\bigvee_{i \in I} p_i$ is verifiable: in order for the disjunction to be true, then there must exist some $i \in I$ such that p_i is true, and so p_i can be known (since it is verifiable), and as a result the disjunction $\bigvee_{i \in I} p_i$ can also be known (by inference from p_i).

6.1.2 Axiomatizations: SSL and TopoLogic

Moss and Parikh (1992) provided a sound and complete axiomatization of their logic of knowledge and effort with respect to the class of subset spaces. Its purely topological version was later studied by Georgatos (1993, 1994), and Dabrowski et al. (1996), who independently provided complete axiomatizations and proved decidability. In this section, we give the axiomatizations for the logic of subset spaces (SSL) and of topological spaces ($\mathsf{TopoLogic}$). We state the relevant completeness, decidability and finite model property results, and refer to the aforementioned sources for their proofs.

The axiomatization of the subset space logic, denoted by SSL, is obtained by augmenting the logic $\mathsf{S5}_K + \mathsf{S4}_\Box$ for the language $\mathcal{L}_{K\Box}$ with the additional axiom schemes (AP) and (CA) presented in Table 6.1.

(AP)	$(p \to \Box p) \land (\neg p \to \Box \neg p)$, for $p \in \textsc{prop}$	Atomic Permanence
(CA)	$K\Box\varphi \to \Box K\varphi$	Cross Axiom

Table 6.1: Additional axiom schemes of SSL

Therefore, the effort modality on subset spaces is $\mathsf{S4}$-like. The axiom (AP) states that the truth value of the propositional variables does not depend on the given epistemic range, but only depends on the actual state. In fact, this is the case for all Boolean formulas in $\mathcal{L}_{K\Box}$, and can be proven in the system SSL. The cross axiom is also interesting since it links the two modalities of this system.

6.1.3. Theorem (Moss and Parikh, 1992). SSL *is sound and complete with respect to the class of all subset spaces.*

It was shown in (Dabrowski et al., 1996) that the logic of subset spaces does not have the finite model property, however, its decidability was proven by using non-standard models called *cross axiom* models (see Dabrowski et al., 1996, Section 2.3).

Concerning the logic of topological spaces for $\mathcal{L}_{K\Box}$, i.e., the so-called $\mathsf{TopoLogic}$, it is axiomatized by adding the following axiom schemes to the axiomatization of SSL:

(WD)	$\Diamond\Box\varphi \to \Box\Diamond\varphi$	Weak Directedness
(UN)	$\Diamond\varphi \land \hat{K}\Diamond\psi \to \Diamond(\Diamond\varphi \land \hat{K}\Diamond\psi \land K\Diamond\hat{K}(\varphi \lor \psi))$	Union Axiom

Table 6.2: Additional axiom schemes of $\mathsf{TopoLogic}$

6.1.4. Theorem (Georgatos, 1993, 1994). $\mathsf{TopoLogic}$ *is sound and complete with respect to the class of all topological spaces. Moreover, it has the finite model property, therefore, it is decidable.*

The literature on subset space semantics goes far beyond the presentation of this section. However, we here confine ourselves to the material we will use in later sections, and refer the reader to (Parikh et al., 2007) for a survey of the further technical results, extensions, and variations of the topologic formalism. In this dissertation, we are particularly interested in revealing the connection between the effort modality, and the well-known dynamic epistemic modalities such as the public and arbitrary announcement modalities. To that end, we use the topological public announcements introduced by Bjorndahl (2016), presented in the next section.

6.2 Topological Public Announcements

The epistemic motivation behind the subset space semantics and the dynamic nature of the effort modality clearly suggests a link between the subset space setting and dynamic epistemic logic, in particular dynamics known as public announcements (Plaza, 1989, 2007; Gerbrandy and Groeneveld, 1997). The information intake represented by the effort modality intuitively encompasses any method of evidence acquisition, including public announcements, a precise and well-studied instance. This connection was also noted by Georgatos (2011), and further studied in (Baskent, 2011, 2012; Balbiani et al., 2013; Wáng and Ågotnes, 2013b; Bjorndahl, 2016), proposing different interpretations for the so-called public announcement modalities. For example, Baskent (2011, 2012) and Balbiani et al. (2013) propose modelling public announcements on subset spaces by deleting the states or the neighbourhoods falsifying the announcement, following the common approach in public announcement logics (see, e.g., van Ditmarsch et al., 2007). However, this method is obviously *not* in the spirit of the effort modality, in the sense that effort, as interpreted on subset spaces, does not lead to a global model change but manifests itself *locally* as a transition from one neighbourhood to a smaller one, i.e., as a neighbourhood shrinking operator. To the best of our knowledge, Wáng and Ågotnes (2013b) were the first to propose semantics for public announcements on subset spaces in terms of epistemic range refinement rather than model restriction. Bjorndahl (2016) then proposed a revised *topological semantics* (in the style of subset space semantics) for the syntax of public announcement logic (*without* the effort modality), that assumes as precondition of learning φ the sentence $\mathsf{int}(\varphi)$, saying, roughly speaking, that φ *is (potentially) knowable*. Topologically, this corresponds to the interior operator of McKinsey and Tarski (1944). Bjorndahl's formalism therefore brings three separate yet connected logical frameworks together: public announcement logic, the interior semantics of McKinsey and Tarski (1944), and the subset space semantics of Moss and Parikh (1992). It thus constitutes a rich enough background to study the connection between effort and the public announcements as well as their connection to so-called arbitrary announcements.

In this section, we present Bjorndahl's topological public announcement logic, and briefly explain the main intuition and motivation behind his formalism. The main body of the work presented in Part II crucially relies on Bjorndahl's setting, and explores its extensions with the aforementioned dynamic modalities both in single and multi-agent cases.

6.2.1 Syntax and Semantics

Bjorndahl (2016) considers the language $\mathcal{L}^{!}_{K\mathsf{int}}$ given by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \mathsf{int}(\varphi) \mid [\varphi]\varphi,$$

where $K\varphi$ is as in Section 6.1, $[\varphi]\psi$ is the public announcement operator, and int is called the "knowability" modality, which, in this setting, plays the role of a precondition of an announcement (Bjorndahl, 2016). The operator $[\varphi]\psi$ is often denoted by $[!\varphi]\psi$ in the public announcement logic literature (as well as in Part I); we skip the exclamation sign, but we will use the notation $[!]$ for this modality when we do not want to specify the announcement formula φ (so that ! functions as a placeholder for the content of the announcement). We prefer this notation here to emphasize the difference from the update operators studied in Part I (which were interpreted in a standard way via model restrictions, where the precondition of an announcement is only that the announced formula is true). The dual modalities for $K\varphi$ and $[\varphi]\psi$ are defined as usual, and we let $\mathsf{cl}(\varphi) := \neg\mathsf{int}(\neg\varphi)$.

Bjorndahl (2016) interprets the above language on topological spaces, in the style of subset space semantics, by extending the subset space semantics of the epistemic language $\mathcal{L}_K$ with semantic clauses for the additional modalities.

6.2.1. Definition. [Topological Semantics for $\mathcal{L}^{!}_{K\mathsf{int}}$] Given a topo-model $\mathcal{X} = (X, \tau, V)$ and an epistemic scenario $(x, U) \in ES(\mathcal{X})$, *truth* of formulas in $\mathcal{L}^{!}_{K\mathsf{int}}$ is defined for the propositional variables and the Boolean cases as in Definition 6.1.2, and the semantics for K, $\mathsf{int}(\varphi)$ and $[\varphi]\psi$ is given recursively as

$$\begin{array}{lll}
(x,U) \models K\varphi & \text{iff} & (\forall y \in U)((y,U) \models \varphi) \\
(x,U) \models \mathsf{int}(\varphi) & \text{iff} & x \in Int([\![\varphi]\!]^U) \\
(x,U) \models [\varphi]\psi & \text{iff} & (x,U) \models \mathsf{int}(\varphi) \text{ implies } (x, Int([\![\varphi]\!]^U)) \models \psi
\end{array}$$

where Int is the interior operator of (X, τ), and $[\![\varphi]\!]^U$ is as defined on p. 106.

To elaborate, the semantic clause for K is exactly the same as in Definition 6.1.2, and is repeated here: as is standard in subset space semantics, knowledge is entailed by the agent's current evidence U. On the other hand, the precondition of an announcement in Bjorndahl's setting is captured by the topological interior operator that refers to the existence of a piece of factive *potential* evidence entailing the announcement:

$$(x,U) \models \mathsf{int}(\varphi) \text{ iff } (\exists O \in \tau)(x \in O \subseteq [\![\varphi]\!]^U).$$

More precisely, $\mathsf{int}(\varphi)$ means that φ *is knowable at the actual state* (though not necessarily knowable in general, at other states) in the sense that there exists some potential evidence—an open set containing the actual state—that entails φ. Therefore, for the precondition of an announcement, Bjorndahl (2016) requires not only that the announced formula is true, but also that it is entailed by a piece of (factive) evidence the agent could possibly obtain. In this respect, a true proposition cannot be announced if it does not have any *open* subsets including the actual state. For example, on a topo-model with no singleton opens, the agents can never know the actual state, not every true proposition can come to be known (as in Georgatos, 1994, Example 1, p. 149). It is this evidence-based interpretation of public announcements that makes Bjorndahl-style updates different than standard update operators (interpreted via model restrictions). In a framework where knowledge is based on the agent's current evidence, and every piece of evidence the agent might acquire later is represented within the given model in terms of open sets of a topology, the operator int as the precondition for learning something seems to be the right notion to consider. It is a good fit with the intuition behind the subset space/topological semantics and the evidence-based learning we study in this part (see Bjorndahl, 2016, for some examples).

6.2.2. Remark. It is worth noting that the intuition behind reading $\mathsf{int}(\varphi)$ as "φ is knowable" can falter when φ is itself an epistemic formula. For instance, if φ is the Moore sentence $p \wedge \neg Kp$, then $K\varphi$ is not satisfiable in any subset model, in particular, $\Diamond K\varphi$ is never true. Therefore, in this sense, φ can never be known; nonetheless, $\mathsf{int}(\varphi)$ *is* satisfiable. This is because $\mathsf{int}(\varphi)$ abstracts away from the temporal and dynamic dimension of knowability, and is simply concerned with *potential* knowledge. On the other hand, $\Diamond K\varphi$ is a *dynamic* schema that states "the agent *comes to know* φ after having spent some effort, having acquired some further evidence". In this respect, $\mathsf{int}(\varphi)$ might be more accurately glossed as "one could come to know what φ *used to express* (before you came to know it)". Since primitive propositions do not change their truth value based on the agent's epistemic state, this subtlety is irrelevant for propositional knowledge and knowability (Bjorndahl and Özgün, 2017).[7]

Bjorndahl (2016) then proceeds with providing a sound and complete axiomatization for the associated dynamic logic $\mathsf{PAL}^+_{\mathsf{int}}$ (called *public announcement logic with* int), using natural analogues of the standard reduction axioms of public announcement logic, and shows that this formalism is co-expressive with the simpler (and older) logic of interior $\mathsf{int}(\varphi)$ and global modality $K\varphi$ (previously

[7]For a discussion of different notions of knowability and their link to Fitch's famous Paradox of Knowability (Fitch, 1963; Brogaard and Salerno, 2013), we refer the interested reader to (Fuhrmann, 2014; van Ditmarsch et al., 2012). In particular, Fuhrmann (2014) discusses a notion of knowability as potential knowledge in the spirit of ours, and van Ditmarsch et al. (2012) consider dynamic notions of knowability.

investigated by Goranko and Passy (1992); Bennett (1996); Shehtman (1999); Aiello (2002), extending the work of McKinsey and Tarski (1944) on interior semantics). The axiomatizations $\mathsf{EL}_{\mathsf{int}}$ and $\mathsf{PAL}^{+}_{\mathsf{int}}$ for the languages $\mathcal{L}_{K\mathsf{int}}$ and $\mathcal{L}^{!}_{K\mathsf{int}}$, respectively, are given in Table 6.3.[8]

	(I) Axioms of system $\mathsf{EL}_{\mathsf{int}}$:
(CPL)	all classical propositional tautologies and Modus Ponens
($S5_K$)	all $\mathsf{S5}$ axioms and rules for the knowledge modality K
($S4_{\mathsf{int}}$)	all $\mathsf{S4}$ axioms and rules for the interior modality int
(K-int)	*Knowledge implies knowability*: $K\varphi \to \mathsf{int}(\varphi)$
	(II) Additional reduction axioms of $\mathsf{PAL}^{+}_{\mathsf{int}}$:
(R_p)	$[\varphi]p \leftrightarrow (\mathsf{int}(\varphi) \to p)$
($R_\neg$)	$[\varphi]\neg\psi \leftrightarrow (\mathsf{int}(\varphi) \to \neg[\varphi]\psi)$
($R_\wedge$)	$[\varphi](\psi \wedge \chi) \leftrightarrow [\varphi]\psi \wedge [\varphi]\chi$
(R_K)	$[\varphi]K\psi \leftrightarrow (\mathsf{int}(\varphi) \to K[\varphi]\psi)$
(R_{int})	$[\varphi]\mathsf{int}(\psi) \leftrightarrow (\mathsf{int}(\varphi) \to \mathsf{int}([\varphi]\psi)$
($R_{[\mathsf{comp}]}$)	$[\varphi][\psi]\chi \leftrightarrow [\mathsf{int}(\varphi) \wedge [\varphi]\mathsf{int}(\psi)]\chi$

Table 6.3: The axiomatizations for $\mathsf{EL}_{\mathsf{int}}$ and $\mathsf{PAL}^{+}_{\mathsf{int}}$.

We conclude the section by stating the completeness results for $\mathsf{EL}_{\mathsf{int}}$ and $\mathsf{PAL}^{+}_{\mathsf{int}}$, and continue our presentation in the next section with a detailed discussion on the expressive power of $\mathcal{L}^{!}_{K\mathsf{int}}$ and its fragments, also in comparison to $\mathcal{L}_{K\Box}$, with respect to topo-models.

6.2.3. Theorem (Shehtman, 1999). $\mathsf{EL}_{\mathsf{int}}$ *is sound and complete with respect to the class of all topo-models.*

Bjorndahl (2016) also presents a canonical topo-model construction for $\mathsf{EL}_{\mathsf{int}}$ (see Bjorndahl, 2016, Theorem 1). He moreover proves the completeness and soundness of $\mathsf{PAL}^{+}_{\mathsf{int}}$:

6.2.4. Theorem (Bjorndahl, 2016). $\mathsf{PAL}^{+}_{\mathsf{int}}$ *is sound and complete with respect to the class of all topo-models.*

[8] In Table 6.3, we present Bjorndahl's original axiomatization as it appears in (Bjorndahl, 2016). In Chapter 7, we propose an alternative set of axioms for the public announcement modality from which Bjorndahl's axioms are derivable. For this reason, we denote his original system by $\mathsf{PAL}^{+}_{\mathsf{int}}$, and reserve the more standard notation $\mathsf{PAL}_{\mathsf{int}}$ for our version presented in Chapter 7.

6.2.2 Expressivity

This section provides several expressivity results concerning the above defined languages with respect to topo-models. We focus in particular on the expressive power of $\mathcal{L}^!_{K\mathsf{int}}$ and its fragments as provided in (Bjorndahl, 2016), as well as the connection between $\mathcal{L}_{\mathsf{int}}$ and $\mathcal{L}_{K\Box}$ (see, e.g., Parikh et al., 2007, Section 4.3). The reader who is familiar with the aforementioned sources can skip this section.

6.2.5. THEOREM (BJORNDAHL, 2016). *$\mathcal{L}^!_{K\mathsf{int}}$, $\mathcal{L}_{K\mathsf{int}}$ and $\mathcal{L}^!_K$ are equally expressive with respect to topo-models.*

Proof:
For the proof details of the co-expressivity between $\mathcal{L}^!_{K\mathsf{int}}$ and $\mathcal{L}_{K\mathsf{int}}$, we refer to (Bjorndahl, 2016, Proposition 5). $\mathcal{L}^!_{K\mathsf{int}}$ and its fragment $\mathcal{L}^!_K$ are equally expressive since the modality int can be defined in terms of the public announcement modalities. In particular, for all $\varphi \in \mathcal{L}^!_{K\mathsf{int}}$, we have $\mathsf{int}(\varphi) \leftrightarrow \langle\varphi\rangle\top$ valid in all topo-models. To prove this, let $\mathcal{X} = (X, \tau, V)$ be a topo-models and $(x, U) \in ES(\mathcal{X})$.

$$\begin{aligned}(x,U) \models \mathsf{int}(\varphi) &\text{ iff } x \in Int([\![\varphi]\!]^U) && \text{(by the semantics of int)}\\ &\text{ iff } x \in Int([\![\varphi]\!]^U) \text{ and } (x, Int([\![\varphi]\!]^U)) \models \top\\ &\text{ iff } (x,U) \models \langle\varphi\rangle\top && \text{(by the semantics of public announ. [!])}\end{aligned}$$

□

On the other hand, not surprisingly, the modality int increases the expressive power of the purely epistemic fragment $\mathcal{L}_K$. And, similarly, the global modality K increases the expressivity of $\mathcal{L}_{\mathsf{int}}$:

6.2.6. THEOREM. *$\mathcal{L}_{K\mathsf{int}}$ is strictly more expressive than $\mathcal{L}_K$, and than $\mathcal{L}_{\mathsf{int}}$. Moreover, $\mathcal{L}_K$ and $\mathcal{L}_{\mathsf{int}}$ are incomparable.*

Proof:
In order to show that $\mathcal{L}_{K\mathsf{int}}$ is strictly more expressive than $\mathcal{L}_K$, we use the example in (Bjorndahl, 2016, Proposition 3).[9] Consider the topo-models $\mathcal{X} = (\{x, y\}, 2^{\{x,y\}}, V)$ and $\mathcal{Y} = (\{x, y\}, \{\emptyset, \{y\}, \{x, y\}\}, V)$ such that $V(p) = \{x\}$ (see Figure 6.1). Let $Int_{\mathcal{X}}$ and $Int_{\mathcal{Y}}$ denote the interior operators of $\mathcal{X}$ and $\mathcal{Y}$, respectively. It is obvious that $\mathcal{X}$ and $\mathcal{Y}$ are modally equivalent with respect to $\mathcal{L}_K$. In other words, for all $\varphi \in \mathcal{L}_K$ and all $(z, U) \in ES(\mathcal{X}) \cap ES(\mathcal{Y})$, we have $\mathcal{X}, (z, U) \models \varphi$ iff $\mathcal{Y}, (z, U) \models \varphi$ (in Bjorndahl, 2016, this argument is given by a notion of bisimulation). However, while $\mathcal{X}, (x, \{x, y\}) \models \mathsf{int}(p)$ since

[9] The topo-models presented in this proof are in fact quite standard examples that are used in order to compare the expressivity of the global modality and an S4-type Kripke modality on relational structures. We here adopt these relational structures to our setting by presenting them as topo-models.

(a) $\mathcal{X}$ (b) $\mathcal{Y}$

Figure 6.1: Squares represent the open sets in the corresponding topologies.

$x \in \{x\} = Int_{\mathcal{X}}(\llbracket p \rrbracket^{\{x,y\}})$, we also have $x \notin \emptyset = Int_{\mathcal{Y}}(\llbracket p \rrbracket^{\{x,y\}})$. Therefore, $\mathsf{int}(p)$ can distinguish $\mathcal{X}, (x, \{x, y\})$ from $\mathcal{Y}, (x, \{x, y\})$, thus it cannot be equivalent to any formula in $\mathcal{L}_K$.

To show that $\mathcal{L}_{K\mathsf{int}}$ is strictly more expressive than $\mathcal{L}_{\mathsf{int}}$, consider again the model $\mathcal{X} = (\{x, y\}, 2^{\{x,y\}}, V)$, and the topo-model $\mathcal{X}' = (\{x, y\}, 2^{\{x,y\}}, V')$ such that $V'(p) = \emptyset$ (see Figure 6.2).

(a) $\mathcal{X}$ (b) $\mathcal{X}'$

Figure 6.2: Squares represent the open sets in the corresponding topologies.

Observe that, for all $\varphi \in \mathcal{L}_{\mathsf{int}}$, $\mathcal{X}, (y, \{y\}) \models \varphi$ iff $\mathcal{X}', (y, \{y\}) \models \varphi$ (this can be shown easily by a subformula induction on φ). On the other hand, $\mathcal{X}, (y, \{y\}) \models \hat{K}p$ whereas $\mathcal{X}', (y, \{y\}) \not\models \hat{K}p$. Therefore, $\hat{K}p$ can distinguish $\mathcal{X}, (y, \{y\})$ from $\mathcal{X}', (y, \{y\})$, thus it cannot be equivalent to any formula in $\mathcal{L}_{\mathsf{int}}$.

Moreover, the first example shows that $\mathsf{int}(p) \in \mathcal{L}_{\mathsf{int}}$ is not equivalent to any formula in $\mathcal{L}_K$, and the second example shows that $\hat{K}p \in \mathcal{L}_K$ is not equivalent to any formula in $\mathcal{L}_{\mathsf{int}}$, hence, $\mathcal{L}_{\mathsf{int}}$ and $\mathcal{L}_K$ are incomparable. $\square$

We also compare $\mathcal{L}_{\mathsf{int}}$ and $\mathcal{L}_{K\Box}$, and thereby, see the exact connection between the interior semantics and the subset space style topological semantics. We here follow the presentation in (Parikh et al., 2007, Section 4.3). We first show that $\mathcal{L}_{\mathsf{int}}$ is embedded in the language $\mathcal{L}_{K\Box}$ via the following translation:

6.2.7. Definition. [Translation $* : \mathcal{L}_{\mathsf{int}} \to \mathcal{L}_{K\Box}$] For each $\varphi \in \mathcal{L}_{\mathsf{int}}$, the translation $(\varphi)^*$ of φ into $\mathcal{L}_{K\Box}$ is defined recursively as follows:

$$\begin{array}{lcl} p^* & = & p, \text{ where } p \in \textsc{prop} \\ (\neg\varphi)^* & = & \neg(\varphi)^* \\ (\varphi \wedge \psi)^* & = & \varphi^* \wedge \psi^* \\ (\mathsf{int}(\varphi))^* & = & \Diamond K \varphi^* \end{array}$$

6.2.8. Definition. [Bi-persistent Formula of $\mathcal{L}_{K\Box}$ (on topo-models)] A formula $\varphi \in \mathcal{L}_{K\Box}$ is called *bi-persistent* if for all topo-models $\mathcal{X} = (X, \mathcal{O}, V)$, and all $(x, U), (x, O) \in ES(\mathcal{X})$ we have $(x, O) \models \varphi$ iff $(x, U) \models \varphi$.

6.2.9. Proposition. *For all $\varphi \in \mathcal{L}_{\mathsf{int}}$, the corresponding formula $\varphi^* \in \mathcal{L}_{K\Box}$ is bi-persistent on topo-models.*

Proof:
The proof follows by subformula induction on φ; cases for the propositional variables and the Boolean connectives are elementary. So assume inductively that the result holds for ψ; we must show that it holds also for $\varphi := \mathsf{int}(\psi)$. Let $\mathcal{X} = (X, \tau, V)$ be a topo-model and $(x, O), (x, U) \in ES(\mathcal{X})$. We then have

$$\begin{aligned}
(x,U) \models (\mathsf{int}(\psi))^* \text{ iff } & (x,U) \models \Diamond K\psi^* && \text{(by the definition of } *) \\
\text{iff } & (\exists U' \in \tau)(x \in U' \subseteq U \text{ and } (x,U') \models K\psi^*) && \\
& && \text{(by the semantics of } \Box) \\
\text{iff } & (\exists U' \in \tau)(x \in U' \subseteq U \text{ and } [\![\psi^*]\!]^{U'} = U') && \\
& && \text{(by the semantics of } K)
\end{aligned}$$

Now, consider the set $U' \cap O$. It is easy to see that $U' \cap O \in \tau$ (since τ is a topology), and that $x \in U' \cap O \subseteq O$. So, we only need to show that $(x, U' \cap O) \models K\psi^*$, i.e., that $U' \cap O = [\![\psi^*]\!]^{U' \cap O}$. But, since ψ^* is bi-persistent (by induction hypothesis), $U' \cap O \subseteq U'$ and $[\![\psi^*]\!]^{U'} = U'$, we have $[\![\psi^*]\!]^{U' \cap O} = [\![\psi^*]\!]^{U'} \cap O = U' \cap O$. Therefore, $(x, U' \cap O) \models K\psi^*$. Moreover, as $x \in U' \cap O \subseteq O$, we obtain $(x, O) \models \Diamond K\psi^*$. The other direction follows similarly. $\Box$

6.2.10. Proposition (Dabrowski et al., 1996, Proposition 3.5). *For all $\varphi \in \mathcal{L}_{\mathsf{int}}$, all topo-models $\mathcal{X} = (X, \tau, V)$ and all $(x, U) \in ES(\mathcal{X})$,*

$$(x, U) \models \varphi \textit{ iff } (x, U) \models \varphi^*.$$

Proof:
The proof follows by subformula induction on φ; cases for the propositional variables and the Boolean connectives are elementary. So assume inductively that the result holds for ψ; we must show that it holds also for $\varphi := \mathsf{int}(\psi)$. Let $\mathcal{X} = (X, \tau, V)$ be a topo-model and $(x, U) \in ES(\mathcal{X})$ such that $(x, U) \models \mathsf{int}(\psi)$, i.e., $x \in Int([\![\psi]\!]^U)$. This means that there is $O \in \tau$ such that $x \in O \subseteq [\![\psi]\!]^U$. Then, by induction hypothesis, we obtain $O \subseteq [\![\psi^*]\!]^U$, i.e., $(y, U) \models \psi^*$ for all $y \in O$. By Proposition 6.2.9, we know that ψ^* is bi-persistent on topo-models. Therefore, we infer that $(y, O) \models \psi^*$ for all $y \in O$. Hence, by the semantics of K, we obtain $(x, O) \models K\psi^*$. As $x \in O \subseteq U$, we conclude $(x, U) \models \Diamond K\psi^*$. The other direction follows similarly. $\Box$

Therefore, the language $\mathcal{L}_{K\Box}$ completely embeds the language $\mathcal{L}_{\mathsf{int}}$ as its fragment consisting of the propositional variables, and closed under the Boolean operators and the modalities $\Diamond K$. As shown in (Parikh et al., 2007, Proposition 6.8), the language $\mathcal{L}_{K\Box}$ is in fact strictly more expressive than $\mathcal{L}_{\mathsf{int}}$ on topo-models:

6.2.11. Proposition (Parikh et al., 2007, Proposition 6.8). *$\mathcal{L}_{K\Box}$ is strictly more expressive than $\mathcal{L}_{\mathsf{int}}$ with respect to topo-models.*

Proof:
It follows from Proposition 6.2.10 that for every $\varphi \in \mathcal{L}_{\mathsf{int}}$, there exists ψ, namely φ^*, such that φ and φ^* are true at the same epistemic scenarios of every topo-model. Moreover, the second example in the proof of Theorem 6.2.6 shows that $\hat{K}p$ is not equivalent on the class of topo-models to φ^* for any $\varphi \in \mathcal{L}_{\mathsf{int}}$ (see Parikh et al., 2007, Proposition 6.8 for a different example). $\Box$

6.3 Conclusions and Continuation

In this chapter, we presented the subset space semantics introduced by Moss and Parikh (1992), mainly focusing on its topological versions. While the standard TopoLogic formalism à la Georgatos (1993, 1994); Dabrowski et al. (1996) completely axiomatizes the logic of topological spaces for the language $\mathcal{L}_{K\Box}$ of knowledge and effort, Bjorndahl (2016) studies the variant $\mathcal{L}_{K\mathsf{int}}$ with the interior operator of McKinsey and Tarski (1944) and the knowledge modality K, and its extension $\mathcal{L}^{!}_{K\mathsf{int}}$ with a topological update operator. We therefore have different axiomatizations for the class of topological spaces, using subset space style semantics based on different languages. The expressivity results concerning the aforementioned languages and their fragments have been discussed in Section 6.2.2, and are summarized in Figure 6.3 below. As we see in Figure 6.3, the languages $\mathcal{L}_{K\Box}$ and $\mathcal{L}_{K\mathsf{int}}$ are also co-expressive with respect to topo-models. We leave the proof of this result for the next chapter (see Theorem 7.1.19).

At this stage we still do not have a logical formalism that analyzes the public announcement modality and the effort modality in one system, although Bjorndahl (2016) provides topological semantics for public announcements that matches the way effort is evaluated on topological spaces. This constitutes one of the topics of the next chapter: we extend the topologic framework with the Bjorndahl-style update modalities, or equivalently, study the extensions of $\mathcal{L}_{K\mathsf{int}}$ and $\mathcal{L}^{!}_{K\mathsf{int}}$ by the effort modality $\Box$, and develop a formal framework that elucidates the relation between effort and public announcements.

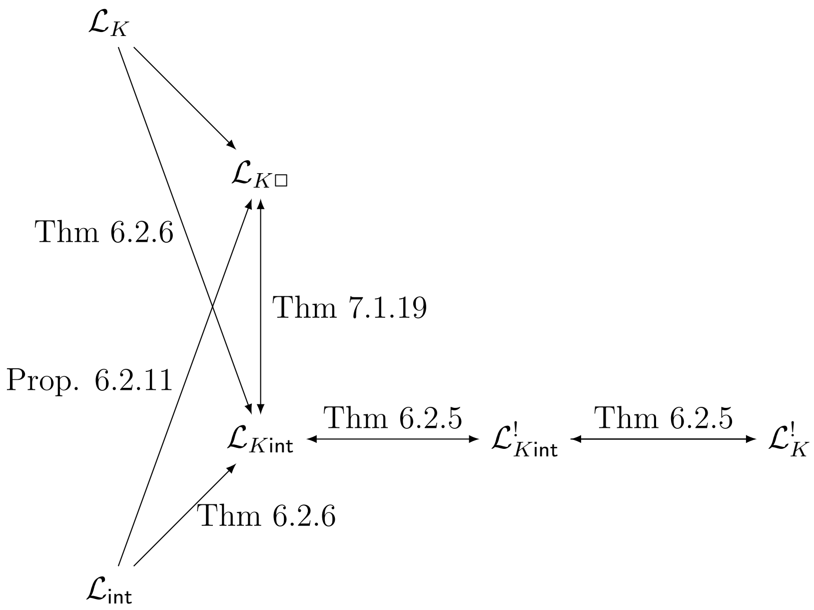


Figure 6.3: Expressivity diagram (Arrows point to the more expressive languages, and reflexive and transitive arrows are omitted. Arrows without tags can be obtained as easy consequences from the others.)

Chapter 7

TopoLogic as Dynamic Epistemic Logic

This chapter studies two different extensions of Bjorndahl's setting for topological public announcements: one with the *effort modality* of Moss and Parikh (1992), and the other with the so-called *arbitrary announcement modality* of Balbiani et al. (2008). We first explore the logic of topological spaces for the language $\mathcal{L}^{!}_{K\mathsf{int}\Box}$, obtained by extending Bjorndahl's language $\mathcal{L}^{!}_{K\mathsf{int}}$ with the effort modality introduced in the previous chapter. This way, we design a formal framework which reveals the link between effort and (topological) public announcements, resulting in both conceptual and technical advantages.

Yet another close relative of both the effort modality and the public announcement modalities is the so-called *arbitrary announcement modality* $\circledast$ that was introduced by Balbiani et al. (2008) and studied on Kripke models. Roughly speaking, the arbitrary announcement modality $\circledast\varphi$ is read as "φ stays true after every announcement". It therefore generalizes public announcements by quantifying over all such announcements. On the other hand, the effort modality *seems* stronger than the arbitrary announcement modality as the former quantifies over *all* open neighbourhoods of the actual state, not only over the epistemically definable ones. In this chapter, we also look at the connection between these three dynamic operators, by providing a topological semantics for $\circledast\varphi$ that quantifies universally over Bjorndahl-style public announcements (similar to the way standard arbitrary public announcement in (Balbiani et al., 2008) quantifies over standard public announcements).

Outline

Section 7.1 presents the Dynamic TopoLogic which combines the topologic formalism with Bjorndahl's public announcements presented in Chapter 6. While Section 7.1.2 provides several expressivity results, Section 7.1.3 focuses on the completeness proof of the proposed axiomatization for Dynamic TopoLogic. In Section 7.2, we study arbitrary announcements on topo-models and demonstrate that, in fact, the arbitrary announcement and the effort modality are equivalent in our single-agent framework.

This chapter is based on (van Ditmarsch et al., 2014; Baltag et al., 2017).

7.1 Dynamic TopoLogic

In this section, based on (Baltag et al., 2017), we investigate a natural extension of topologic, obtained by adding to it Bjorndahl's topological update operators. In other words, we revisit TopoLogic as a dynamic epistemic logic with public announcements. The resulting "Dynamic TopoLogic" forms a logic of evidence-based *knowledge* $K\varphi$, *knowability* $\mathsf{int}(\varphi)$, *learning* of new evidence $[\varphi]\psi$, and *stability* $\Box\varphi$ (of some truth φ) under any such evidence-acquisition.

To recall briefly, Moss and Parikh (1992) gave a sound and complete axiomatization with respect to the class of all subset spaces (Theorem 6.1.3). The axiomatization for topological spaces was later studied by Georgatos (1993, 1994) and Dabrowski et al. (1996), who independently provided completeness and decidability proofs for TopoLogic (Theorem 6.1.4). These existing completeness and decidability results involve technically interesting, yet rather complicated constructions. Moreover, one of the main axioms of the original TopoLogic, the so-called Union Axiom, capturing closure of the topology under binary unions (see Table 6.2), is very complex and looks rather unintuitive from an epistemic perspective. Against this background, our investigations in this chapter lead to results of conceptual and technical interest as the extended syntax explicates the notion of effort in terms of public announcements, and entertains an epistemically more intuitive and clear complete axiomatization.

In the following, we present several expressivity results concerning this extended language, denoted by $\mathcal{L}^{!}_{K\mathsf{int}\Box}$, and its fragments, and thus expand Figure 6.3. In particular, we show that this extension is co-expressive with Bjorndahl's language $\mathcal{L}^{!}_{K\mathsf{int}}$ of topological public announcements (Bjorndahl, 2016), and therefore with the simpler language $\mathcal{L}_{K\mathsf{int}}$. This elucidates the relationships between TopoLogic and other modal (and dynamic-epistemic) logics for topology. In particular, TopoLogic is directly interpretable in the simplest logic of topo-models for $\mathcal{L}_{K\mathsf{int}}$, which immediately provides an easy decidability proof both for TopoLogic and for our extension.

We also give a complete axiomatization for Dynamic TopoLogic, which is in a sense more transparent than the standard axioms of TopoLogic. Although we have more axioms, each of them is natural and easily readable, directly reflecting the intuitive meanings of the connectives. More precisely, our axiomatization consists of a slightly different version of Bjorndahl's axiomatization of $\mathsf{PAL}^{+}_{\mathsf{int}}$ (ours includes a few other standard axioms and rules of public announcement logic), together with *only two additional proof principles governing the behavior of the topologic "effort" modality* ($\Box\varphi$, what we call "stable truth"): an introduction rule and an elimination axiom. Everything to be said about the effort modality is therefore fully captured by these two simple principles, which together express the fact that this modality quantifies universally over all updates with any new evidence. In particular, the complicated Union Axiom of TopoLogic (see Table 6.2) is not needed in our system (though of course it can be proved from our axioms). Unlike the existing completeness proofs of TopoLogic (Georgatos, 1993; Dabrowski et al., 1996), ours makes direct use of a *standard canonical topo-model construction* (as, e.g., the canonical topo-model construction for S4 in Aiello et al., 2003, Section 3).[1] This simplicity shows the advantage of adding dynamic modalities: when considered as a fragment of a dynamic-epistemic logic, topologic becomes a more transparent and natural formalism, with intuitive axioms and canonical behavior.

7.1.1 Syntax, Semantics and Axiomatizations

The language $\mathcal{L}^{!}_{K\mathsf{int}\Box}$ of Dynamic TopoLogic is obtained by extending Bjorndahl's language $\mathcal{L}^{!}_{K\mathsf{int}}$ with the effort modality $\Box$ from the language of topologic $\mathcal{L}_{K\Box}$ (Moss and Parikh, 1992); or, equivalently, by extending the usual syntax of topologic with both the interior operator int of McKinsey and Tarski (1944) and with Bjorndahl's dynamic modalities for topological public announcements. As noted earlier, the interior operator is definable using topological public announcements (by putting $\mathsf{int}(\varphi) = \langle\varphi\rangle\top$). Therefore, keeping the modality int in the language as primitive is mainly a design decision, but it also simplifies our completeness proof. Therefore, our syntax is essentially given by adding the language $\mathcal{L}_{K\Box}$ of topologic *only* the dynamic public announcement modalities, hence, we use the name "Dynamic TopoLogic". We start our presentation by formally introducing the syntax and semantics for Dynamic TopoLogic.

Syntax and Semantics. The language $\mathcal{L}^{!}_{K\mathsf{int}\Box}$ of Dynamic TopoLogic is defined recursively by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \mathsf{int}(\varphi) \mid [\varphi]\varphi \mid \Box\varphi,$$

where $p \in \textsc{prop}$. Several fragments of the language $\mathcal{L}^{!}_{K\mathsf{int}\Box}$ are of both technical and conceptual interest. To recall, for the fragments of $\mathcal{L}^{!}_{K\mathsf{int}\Box}$, we use our standard notational convention listing all the modalities of the corresponding language as a subscript of $\mathcal{L}$ except that ! for public announcements appears as a superscript. For example, $\mathcal{L}_{\mathsf{int}}$ denotes the fragment of $\mathcal{L}^{!}_{K\mathsf{int}\Box}$ having only the modality int; $\mathcal{L}_{K\mathsf{int}}$ having only the modalities K and int; $\mathcal{L}_{K\Box}$ having only the modalities K and $\Box$; $\mathcal{L}^{!}_{K\mathsf{int}}$ having the modalities K, int and [!] etc.

[1] Dabrowski et al. (1996) also consider a canonical model, but their completeness proof of TopoLogic uses McKinsey-Tarski's theorem of the topological completeness of S4 (Theorem 3.1.6). In our setting, having the modality int that matches the topological interior operator in the language makes it easier to directly build a canonical model.

We interpret this language on topo-models in an obvious way by putting together the subset space semantics for $\mathcal{L}_{K\Box}$ (Definition 6.1.2) and Bjorndahl's semantics for the fragment $\mathcal{L}^{!}_{K\mathsf{int}}$ (Definition 6.2.1). This is recapitulated in the following definition.

7.1.1. Definition. [Topological Semantics for $\mathcal{L}^{!}_{K\mathsf{int}\Box}$] Given a topo-model $\mathcal{X} = (X, \tau, V)$ and an epistemic scenario $(x, U) \in ES(\mathcal{X})$, *truth* of formulas in $\mathcal{L}^{!}_{K\mathsf{int}\Box}$ is defined for the propositional variables and the Booleans as in Definition 6.1.2, and the semantics for the modalities is given recursively as follows:

$$\begin{array}{lll}
(x,U) \models K\varphi & \text{iff} & (\forall y \in U)((y,U) \models \varphi) \\
(x,U) \models \mathsf{int}(\varphi) & \text{iff} & x \in \mathit{Int}([\![\varphi]\!]^U) \\
(x,U) \models [\varphi]\psi & \text{iff} & (x,U) \models \mathsf{int}(\varphi) \text{ implies } (x, \mathit{Int}([\![\varphi]\!]^U)) \models \psi \\
(x,U) \models \Box\varphi & \text{iff} & (\forall O \in \tau)\,(x \in O \subseteq U \text{ implies } (x,O) \models \varphi)
\end{array}$$

Axiomatizations. Given a formula $\varphi \in \mathcal{L}^{!}_{K\mathsf{int}\Box}$, we denote by P_φ the set of all propositional variables occurring in φ (we will use the same notation for the necessity and possibility forms defined in Definition 7.1.22). The Dynamic TopoLogic, hereby denoted as $\mathsf{DTL_{int}}$, is the smallest subset of $\mathcal{L}^{!}_{K\mathsf{int}\Box}$ that contains the axioms, and is closed under the inference rules given in Table 7.1 below. The system $\mathsf{EL_{int}}$ is defined in a similar way over the language $\mathcal{L}_{K\mathsf{int}}$ by the axioms and inference rules in group (I) of Table 7.1 (as also given in Table 6.3), and $\mathsf{PAL_{int}}$ is defined over the language $\mathcal{L}^{!}_{K\mathsf{int}}$ by the axioms and inference rules in groups (I) and (II).

The first six items in Table 7.1 are standard. The Replacement of Equivalents rule ([!]RE) for [!] says that *updates are extensional*, that is, learning equivalent sentences gives rise to equivalent updates, while the reduction axiom (R[$\top$]) says that *updating with tautologies is redundant*. The reduction axioms (R_p), ($\mathrm{R}_\neg$) and (R_K) are exactly the same as in the axiomatization $\mathsf{PAL^{+}_{int}}$ of Bjorndahl (2016), and the reduction law ($\mathrm{R}_{[!]}$) for the iterative announcements is equivalent to ($\mathrm{R}_{[\mathsf{comp}]}$) but formulated in a simpler way (see Table 6.3 for $\mathsf{PAL^{+}_{int}}$). Bjorndahl's axiomatization also includes reduction laws for the connective $\wedge$ (denoted by ($\mathrm{R}_\wedge$)) and the modality int (denoted by ($\mathrm{R}_{\mathsf{int}}$)), however, as shown in Proposition 7.1.2, these can be derived in $\mathsf{PAL_{int}}$.The only key new components of our system are the last axiom and inference rule for $\Box$, i.e., the elimination axiom ([!]$\Box$-elim) and the introduction rule ([!]$\Box$-intro) for the effort modality. Taken together, they state that *θ is a stable truth after learning φ iff θ is true after learning every stronger evidence $\varphi \wedge \rho$*. The left-to-right implication in this statement is directly captured by ([!]$\Box$-elim), while the converse is captured by the rule ([!]$\Box$-intro). The "freshness" of the variable p in this rule ensures that it represents any "generic" further evidence. This is similar to the introduction rule for the universal quantifier. In essence, the effort axiom and rule express the fact that *the effort modality is a universal quantifier* (*over potential evidence*). One can compare the transparency and simple nature of our axioms with the complexity

	(I) Axioms and rules of system $\mathsf{EL}_{\mathsf{int}}$:
(CPL)	all classical propositional tautologies and Modus Ponens
($S5_K$)	all S5 axioms and rules for the knowledge modality K
($S4_{\mathsf{int}}$)	all S4 axioms and rules for the interior modality int
(K-int)	*Knowledge implies knowability*: $K\varphi \to \mathsf{int}(\varphi)$
	(II) Additional axioms and rules for $\mathsf{PAL}_{\mathsf{int}}$:
($K_!$)	$[\varphi](\psi \to \theta) \to ([\varphi]\psi \to [\varphi]\theta)$
($Nec_!$)	from θ, infer $[\varphi]\theta$
([!]RE)	*Replacement of Equivalents for* [!]: from $\varphi \leftrightarrow \psi$, infer $[\varphi]\theta \leftrightarrow [\psi]\theta$
	Reduction axioms:
(R[$\top$])	$[\top]\varphi \leftrightarrow \varphi$
(R_p)	$[\varphi]p \leftrightarrow (\mathsf{int}(\varphi) \to p)$
($R_\neg$)	$[\varphi]\neg\psi \leftrightarrow (\mathsf{int}(\varphi) \to \neg[\varphi]\psi)$
(R_K)	$[\varphi]K\psi \leftrightarrow (\mathsf{int}(\varphi) \to K[\varphi]\psi)$
($R_{[!]}$)	$[\varphi][\psi]\chi \leftrightarrow [\langle\varphi\rangle\psi]\chi$
	(III) Axioms and rules of the effort modality for $\mathsf{DTL}_{\mathsf{int}}$:
([!]$\Box$-elim)	$[\varphi]\Box\theta \to [\varphi \wedge \rho]\theta$ ($\rho \in \mathcal{L}^{!}_{K\mathsf{int}\Box}$ arbitrary formula)
([!]$\Box$-intro)	from $\psi \to [\varphi \wedge p]\theta$, infer $\psi \to [\varphi]\Box\theta$ ($p \notin \mathrm{P}_\psi \cup \mathrm{P}_\theta \cup \mathrm{P}_\varphi$)

Table 7.1: The axiomatizations of $\mathsf{DTL}_{\mathsf{int}}, \mathsf{PAL}_{\mathsf{int}}$ and $\mathsf{EL}_{\mathsf{int}}$

of the standard axiomatization of TopoLogic that contains, among others, the rather intricate Union Axiom (also given in Table 6.2):

$$\Diamond\varphi \wedge \hat{K}\Diamond\psi \to \Diamond(\Diamond\varphi \wedge \hat{K}\Diamond\psi \wedge K\Diamond\hat{K}(\varphi \vee \psi)) \tag{UN}$$

Proposition 7.1.2 states some important theorems and inference rules derivable in $\mathsf{DTL}_{\mathsf{int}}$, which will be used in our completeness proofs. While the denotations for the other items listed in the following proposition are obvious, (RE) is the full rule of *Replacement of Equivalents*, where $\varphi\{\psi/\chi\}$ denotes the formula obtained by replacing the occurrences of ψ in φ by χ.

7.1.2. Proposition. *The first seven schemas and the rule (RE) are provable both in* $\mathsf{PAL}_{\mathsf{int}}$ *and* $\mathsf{DTL}_{\mathsf{int}}$ *(for languages* $\mathcal{L}^{!}_{K\mathsf{int}}$ *and* $\mathcal{L}^{!}_{K\mathsf{int}\Box}$*, respectively). The ninth schema and the inference rule below can be derived in our full proof system* $\mathsf{DTL}_{\mathsf{int}}$:

1.	($\langle ! \rangle$)	$\langle \varphi \rangle \psi \leftrightarrow (\mathsf{int}(\varphi) \wedge [\varphi]\psi)$
2.	($R_\bot$)	$[\varphi]\bot \leftrightarrow \neg\mathsf{int}(\varphi)$
3.	($R_\wedge$)	$[\varphi](\psi \wedge \theta) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\theta)$
4.	(R_{int})	$[\varphi]\mathsf{int}(\psi) \leftrightarrow (\mathsf{int}(\varphi) \to \mathsf{int}([\varphi]\psi))$
5.	($R[\mathsf{int}]$)	$[\mathsf{int}(\varphi)]\psi \leftrightarrow [\varphi]\psi$
6.	($R_{[\mathsf{comp}]}$)	$[\varphi][\psi]\chi \leftrightarrow [\mathsf{int}(\varphi) \wedge [\varphi]\mathsf{int}(\psi)]\chi$
7.	($R_{[p]}$)	$[\varphi][p]\psi \leftrightarrow [\varphi \wedge p]\psi \quad (p \in \textsc{prop}$ arbitrary)
8.	(RE)	from $\psi \leftrightarrow \chi$, infer $\varphi \leftrightarrow \varphi\{\psi/\chi\}$
9.	($\Box$-elim)	$\Box\theta \to [\rho]\theta \quad (\rho \in \mathcal{L}^!_{K\mathsf{int}\Box}$ arbitrary formula)
10.	($\Box$-intro)	from $\psi \to [p]\theta$, infer $\psi \to \Box\theta \quad (p \notin \mathrm{P}_\psi \cup \mathrm{P}_\theta$ atom)

Proof:
We here present abridged derivations, some of the obvious steps are omitted. We start with the $\Box$-axioms and inference rules.

($\Box$-elim):

1. $\vdash \Box\theta \leftrightarrow [\top]\Box\theta$ (R[$\top$])
2. $\vdash [\top]\Box\theta \to [\top \wedge \rho]\theta$, (for arbitrary $\rho \in \mathcal{L}^!_{K\mathsf{int}\Box}$) ([!]$\Box$-elim)
3. $\vdash [\top \wedge \rho]\Box\theta \to [\rho]\theta$, (for arbitrary $\rho \in \mathcal{L}^!_{K\mathsf{int}\Box}$) ($\vdash (\top \wedge \rho) \leftrightarrow \rho$ and ([!]RE))
4. $\vdash \Box\theta \to [\rho]\theta$, (for arbitrary $\rho \in \mathcal{L}^!_{K\mathsf{int}\Box}$) (1-3, CPL)

($\Box$-intro): proof follows analogously to the above case by using R[$\top$], and [!]$\Box$-intro with $\varphi := \top$.

(RE): The proof follows standardly by subformula induction on φ. Suppose $\vdash \psi \leftrightarrow \chi$. For the base case $\varphi := \psi$, we have $\varphi\{\psi/\chi\} = \chi$. Therefore, the equivalence $\vdash \varphi \leftrightarrow \varphi\{\psi/\chi\}$ boils down to $\vdash \psi \leftrightarrow \chi$, hence follows from the assumption. Now assume inductively that the statement holds for σ and θ. The cases for the Booleans, K and int are standard, where the latter two follows from the corresponding K-axioms and Necessitation rules. For [!], we use ($K_!$), ($Nec_!$), ([!]RE). For $\Box$, it is sufficient to show that we can derive the K-axiom ($K_\Box$) and the Necessitation rule ($Nec_\Box$) for $\Box$. The derivation of ($Nec_\Box$) easily follows from ($Nec_!$) and ($\Box$-intro). For ($K_\Box$), we have

1. $\vdash (\Box(\theta \to \gamma) \wedge \Box\theta) \to ([p](\theta \to \gamma) \wedge [p]\theta)$ ($p \notin \mathrm{P}_\theta \cup \mathrm{P}_\gamma$, ($\Box$-elim))
2. $\vdash ([p](\theta \to \gamma) \wedge [p]\theta) \to [p]\gamma$ ($K_!$)
3. $\vdash (\Box(\theta \to \gamma) \wedge \Box\theta) \to [p]\gamma$ (1, 2, CPL)
4. $\vdash (\Box(\theta \to \gamma) \wedge \Box\theta) \to \Box\gamma$ ($p \notin \mathrm{P}_\theta \cup \mathrm{P}_\gamma$, ($\Box$-intro))

($\langle ! \rangle$): follows from the definition $\langle \varphi \rangle \psi := \neg[\varphi]\neg\psi$ and the axiom ($R_\neg$).

($R_\wedge$): follows from ($K_!$) and ($Nec_!$).

($R_\bot$): is an easy consequence of ($R_\wedge$), (R_p) and ([!]RE)

(R_{int}):

1. $\vdash \mathsf{int}(\psi) \leftrightarrow \neg[\psi]\bot$ ($R_\bot$)
2. $\vdash [\varphi]\mathsf{int}(\psi) \leftrightarrow [\varphi]\neg[\psi]\bot$ (1, ([!]RE))
3. $\vdash [\varphi]\neg[\psi]\bot \leftrightarrow (\mathsf{int}(\varphi) \rightarrow \neg[\varphi][\psi]\bot)$ ($R_\neg$)
4. $\vdash (\mathsf{int}(\varphi) \rightarrow \neg[\varphi][\psi]\bot) \leftrightarrow (\mathsf{int}(\varphi) \rightarrow \mathsf{int}(\langle\varphi\rangle\psi))$ (($R_{[!]}$), ($R_\bot$))
5. $\vdash [\varphi]\mathsf{int}(\psi) \leftrightarrow (\mathsf{int}(\varphi) \rightarrow \mathsf{int}(\langle\varphi\rangle\psi))$ (2-4, CPL)
6. $\vdash (\mathsf{int}(\varphi) \rightarrow \mathsf{int}(\langle\varphi\rangle\psi)) \leftrightarrow (\mathsf{int}(\varphi) \rightarrow \mathsf{int}(\mathsf{int}(\varphi) \wedge [\varphi]\psi))$ (($\langle ! \rangle$), (RE))
7. $\vdash (\mathsf{int}(\varphi) \rightarrow \mathsf{int}(\mathsf{int}(\varphi) \wedge [\varphi]\psi)) \leftrightarrow (\mathsf{int}(\varphi) \rightarrow (\mathsf{int}(\varphi) \wedge \mathsf{int}([\varphi]\psi)))$ ($\mathsf{S4}_{\mathsf{int}}$, (RE))
8. $\vdash (\mathsf{int}(\varphi) \rightarrow (\mathsf{int}(\varphi) \wedge \mathsf{int}([\varphi]\psi))) \leftrightarrow (\mathsf{int}(\varphi) \rightarrow \mathsf{int}([\varphi]\psi))$ (CPL)
9. $\vdash [\varphi]\mathsf{int}(\psi) \leftrightarrow (\mathsf{int}(\varphi) \rightarrow \mathsf{int}([\varphi]\psi))$ (5-8, CPL)

(R[int]): follows by subformula induction on ψ by using the reduction axioms and the fact that int is an S4 modality. For case $\psi := \Box\chi$, we use ([!]$\Box$-elim) and ([!]$\Box$-intro).

Base case $\psi := p$

1. $\vdash [\mathsf{int}(\varphi)]p \leftrightarrow (\mathsf{int}(\mathsf{int}(\varphi)) \rightarrow p)$ (R_p)
2. $\vdash (\mathsf{int}(\mathsf{int}(\varphi)) \rightarrow p) \leftrightarrow (\mathsf{int}(\varphi) \rightarrow p)$ ($S4_{\mathsf{int}}$)
3. $\vdash (\mathsf{int}(\varphi) \rightarrow p) \leftrightarrow [\varphi]p$ (R_p)
4. $\vdash [\mathsf{int}(\varphi)]p \leftrightarrow [\varphi]p$ (1-3, CPL)

The cases for $\psi := \neg\chi$, $\psi := \chi \wedge \sigma$, $\psi := K\chi$ and $\psi := \mathsf{int}(\chi)$ follow in a similar way by using the corresponding reduction axioms.

Case $\psi := [\chi]\sigma$

1. $\vdash [\mathsf{int}(\varphi)][\chi]\alpha \leftrightarrow [\langle\mathsf{int}(\varphi)\rangle\chi]\alpha$ ($R_{[!]}$)
2. $\vdash [\langle\mathsf{int}(\varphi)\rangle\chi]\alpha \leftrightarrow [\mathsf{int}(\mathsf{int}(\varphi)) \wedge [\mathsf{int}(\varphi)]\chi]\alpha$ ($\langle ! \rangle$)
3. $\vdash [\mathsf{int}(\mathsf{int}(\varphi)) \wedge [\mathsf{int}(\varphi)]\chi]\alpha \leftrightarrow [\mathsf{int}(\varphi) \wedge [\mathsf{int}(\varphi)]\chi]\alpha$ ($S4_{\mathsf{int}}$, (RE))
4. $\vdash [\mathsf{int}(\varphi) \wedge [\mathsf{int}(\varphi)]\chi]\alpha \leftrightarrow [\mathsf{int}(\varphi) \wedge [\varphi]\chi]\alpha$ (IH on χ, (RE))
5. $\vdash [\mathsf{int}(\varphi) \wedge [\varphi]\chi]\alpha \leftrightarrow [\langle\varphi\rangle\chi]\alpha$ (($\langle ! \rangle$), (RE))
6. $\vdash [\langle\varphi\rangle\chi]\alpha \leftrightarrow [\varphi][\chi]\alpha$ ($R_{[!]}$)
7. $\vdash [\mathsf{int}(\varphi)][\chi]\alpha \leftrightarrow [\varphi][\chi]\alpha$ (1-6, CPL)

Case $\psi := \Box\chi$

We here only show the direction $\vdash [\mathsf{int}(\varphi)]\Box\chi \rightarrow [\varphi]\Box\chi$; the other direction follows similarly.

1. $\vdash [\mathsf{int}(\varphi)]\Box\chi \to [\mathsf{int}(\varphi) \wedge p]\chi$ $\quad (p \notin \mathrm{P}_{\varphi} \cup \mathrm{P}_{\chi}$, ([!]$\Box$-elim))
2. $\vdash [\mathsf{int}(\varphi) \wedge p]\chi \leftrightarrow [\mathsf{int}(\mathsf{int}(\varphi) \wedge p)]\chi$ $\quad$ (IH)
3. $\vdash [\mathsf{int}(\mathsf{int}(\varphi) \wedge p)]\chi \leftrightarrow [\mathsf{int}(\varphi \wedge p)]\chi$ $\quad$ ($\mathsf{S4}_{\mathsf{int}}$, (RE))
4. $\vdash [\mathsf{int}(\varphi \wedge p)]\chi \leftrightarrow [\varphi \wedge p]\chi$ $\quad$ (IH)
5. $\vdash [\mathsf{int}(\varphi)]\Box\chi \to [\varphi \wedge \ p]\chi$ $\quad$ (1-4, CPL)
6. $\vdash [\mathsf{int}(\varphi)]\Box\chi \to [\varphi]\Box\chi$ $\quad (p \notin \mathrm{P}_{\mathsf{int}(\varphi)} \cup \mathrm{P}_{\chi}$, ([!]$\Box$-intro))

($\mathrm{R}_{[\mathsf{comp}]}$):

1. $\vdash [\mathsf{int}(\varphi) \wedge [\varphi]\mathsf{int}(\psi)]\chi \leftrightarrow [\mathsf{int}(\varphi) \wedge \mathsf{int}[\varphi]\psi)]\chi$ $\quad$ (($\mathrm{R}_{\mathsf{int}}$), (RE))
2. $\vdash [\mathsf{int}(\varphi) \wedge \mathsf{int}[\varphi]\psi)]\chi \leftrightarrow [\mathsf{int}(\mathsf{int}(\varphi)) \wedge \mathsf{int}[\varphi]\psi)]\chi$ $\quad$ ($\mathsf{S4}_{\mathsf{int}}$, (RE))
3. $\vdash [\mathsf{int}(\mathsf{int}(\varphi)) \wedge \mathsf{int}[\varphi]\psi)]\chi \leftrightarrow [\mathsf{int}(\mathsf{int}(\varphi) \wedge [\varphi]\psi)]\chi$ $\quad$ ($\mathsf{S4}_{\mathsf{int}}$, (RE))
4. $\vdash [\mathsf{int}(\mathsf{int}(\varphi) \wedge [\varphi]\psi)]\chi \leftrightarrow [\mathsf{int}(\varphi) \wedge [\varphi]\psi]\chi$ $\quad$ (R[int])
5. $\vdash [\mathsf{int}(\varphi) \wedge [\varphi]\psi]\chi \leftrightarrow [\langle\varphi\rangle\chi]\alpha$ $\quad$ (($\langle ! \rangle$), (RE))
6. $\vdash [\langle\varphi\rangle\chi]\alpha \leftrightarrow [\varphi][\chi]\alpha$ $\quad$ ($\mathrm{R}_{[!]}$)
7. $\vdash [\mathsf{int}(\varphi) \wedge [\varphi]\mathsf{int}(\psi)]\chi \leftrightarrow [\varphi][\chi]\alpha$ $\quad$ (1-6, CPL)

($\mathrm{R}_{[p]}$):

1. $\vdash [\varphi][p]\psi \leftrightarrow [\langle\varphi\rangle p]\psi$ $\quad$ ($\mathrm{R}_{[!]}$)
2. $\vdash [\langle\varphi\rangle p]\psi \leftrightarrow [\mathsf{int}(\varphi) \wedge [\varphi]p]\psi$ $\quad$ (($\langle ! \rangle$), (RE))
3. $\vdash [\mathsf{int}(\varphi) \wedge [\varphi]p]\psi \leftrightarrow [\mathsf{int}(\varphi) \wedge p]\psi$ $\quad$ ((R_p), (RE))
4. $\vdash [\mathsf{int}(\varphi) \wedge p]\psi \leftrightarrow [\mathsf{int}(\mathsf{int}(\varphi) \wedge p)]\psi$ $\quad$ (R[int])
5. $\vdash [\mathsf{int}(\mathsf{int}(\varphi) \wedge p)]\psi \leftrightarrow [\mathsf{int}(\varphi \wedge p)]\psi$ $\quad$ ($\mathsf{S4}_{\mathsf{int}}$ and (RE))
6. $\vdash [\mathsf{int}(\varphi \wedge p)]\psi \leftrightarrow [\varphi \wedge p]\psi$ $\quad$ (R[int])
7. $\vdash [\varphi][p]\psi \leftrightarrow [\varphi \wedge p]\psi$ $\quad$ (1-6, CPL)

$\Box$

7.1.3. Corollary. $\mathsf{PAL}_{\mathsf{int}}$ *is sound and complete with respect to the class of all topo-models.*

Proof:
Soundness of $\mathsf{PAL}_{\mathsf{int}}$ is easy to see. The completeness proof follows from Theorem 6.2.4 and Proposition 7.1.2: since Bjorndahl's axiomatization $\mathsf{PAL}^{+}_{\mathsf{int}}$ is complete and our system $\mathsf{PAL}_{\mathsf{int}}$ can prove all his additional reduction rules ($\mathrm{R}_{\wedge}$), ($\mathrm{R}_{\mathsf{int}}$) and ($\mathrm{R}_{[\mathsf{comp}]}$), our system $\mathsf{PAL}_{\mathsf{int}}$ is complete as well. $\Box$

7.1.2 Soundness and Expressivity

In this section, we introduce a more general class of models for our full language $\mathcal{L}^!_{K\mathsf{int}\Box}$, called *pseudo-models.* These are a special case of the (even more general) subset models of Moss and Parikh (1992). Pseudo-models include all topo-models, as well as other subset models, but they have the nice property that the interior operator $\mathsf{int}(\varphi)$ can still be interpreted in the standard way. These structures, though interesting enough in themselves, are for us only an auxiliary notion, playing an important technical role in our completeness proof of $\mathsf{DTL}_{\mathsf{int}}$. For now though, we first prove the soundness of our full system $\mathsf{DTL}_{\mathsf{int}}$ with respect to pseudo-models (and thus also with respect to topo-models), and then provide several expressivity results concerning the above defined languages with respect to both topo and pseudo-models.

The definition of pseudo-models requires a few auxiliary notions, such as a more general class of models called *pre-models.*

7.1.4. Definition. [Lattice spaces and Pre-models] A subset space $(X, \mathcal{O})$ is called a *lattice space* if $\emptyset, X \in \mathcal{O}$, and $\mathcal{O}$ is closed under finite intersections and finite unions. A *pre-model* $(X, \mathcal{O}, V)$ is a triple where $(X, \mathcal{O})$ is a lattice space and $V : \mathrm{Prop} \to \mathcal{P}(X)$ is a valuation map.

Although a lattice space $(X, \mathcal{O})$ is not necessarily a topological space, the family $\mathcal{O}$ constitutes a topological basis over X. Therefore, every pre-model $\mathcal{X} = (X, \mathcal{O}, V)$ has an *associated topo-model* $\mathcal{X}_\tau = (X, \tau_{\mathcal{O}}, V)$, where $\tau_{\mathcal{O}}$ is the topology generated by $\mathcal{O}$ (i.e., the smallest topology on X such that $\mathcal{O} \subseteq \tau_{\mathcal{O}}$).

Given a pre-model $\mathcal{X} = (X, \mathcal{O}, V)$, we define the semantics for $\mathcal{L}^!_{K\mathsf{int}\Box}$ on pre-models for *all* pairs of the form (x, Y), where $Y \subseteq X$ is an arbitrary subset such that $x \in Y$. It is important to notice that, for a given evaluation pair (x, Y) on a pre-model, the set Y is *not necessarily* an element of $\mathcal{O}$. The reason for this adjustment will be explained in Remark 7.1.6, after we have defined the semantics for $\mathcal{L}^!_{K\mathsf{int}\Box}$ on pre-models.

7.1.5. Definition. [Pre-model Semantics for $\mathcal{L}^!_{K\mathsf{int}\Box}$] Given a pre-model and a pair of the form (x, Y) such that $x \in Y \subseteq X$, *truth* of formulas in $\mathcal{L}^!_{K\mathsf{int}\Box}$ is defined for the propositional variables and the Booleans as in Definition 6.1.2, and the semantics for the modalities is given recursively as follows:

$$
\begin{array}{lll}
(x, Y) \models K\varphi & \text{iff} & (\forall y \in Y)((y, Y) \models \varphi) \\
(x, Y) \models \mathsf{int}(\varphi) & \text{iff} & x \in \mathit{Int}([\![\varphi]\!]^Y) \\
(x, Y) \models [\varphi]\psi & \text{iff} & (x, Y) \models \mathsf{int}(\varphi) \text{ implies } (x, \mathit{Int}([\![\varphi]\!]^Y)) \models \psi \\
(x, Y) \models \Box\varphi & \text{iff} & (\forall O \in \mathcal{O})(x \in O \subseteq Y \text{ implies } (x, O) \models \varphi)
\end{array}
$$

where Int is the interior operator of $\tau_{\mathcal{O}}$.

7.1.6. Remark. Notice that the consequent of the semantic clause for $[\varphi]\psi$ requires $(x, Int(\llbracket\varphi\rrbracket^Y))$ to be a "well-defined" evaluation pair. If we were to restrict the evaluation pairs in a pre-model to the so-called epistemic scenarios of the form (x, U) with $x \in U \in \mathcal{O}$ (as in the case for topo-models), we could not have guaranteed that a pair of the form $(x, Int(\llbracket\varphi\rrbracket^U))$ would be well-defined: since pre-models are *not* necessarily based on topological spaces, the open set $Int(\llbracket\varphi\rrbracket^U)$ might not be an element of $\mathcal{O}$. Therefore, in order to render the above defined semantics well-defined for the public announcement modalities $[\varphi]\psi$, and thus, for the language $\mathcal{L}^!_{K\mathsf{int}\Box}$, we have generalized the satisfaction relation on pre-models to any pair (x, Y) with $x \in Y \subseteq X$.

Validity on pre-models on the other hand is defined by *restricting to epistemic scenarios* (x, U) such that $x \in U \in \mathcal{O}$, as in the case for the topo-models. More precisely, we say that a formula φ is *valid in a pre-model* $\mathcal{X}$, and write $\mathcal{X} \models \varphi$, if $\mathcal{X}, (x, U) \models \varphi$ for all *epistemic scenarios* $(x, U) \in ES(\mathcal{X})$. A formula φ is *valid*, denoted by $\models \varphi$, if $\mathcal{X} \models \varphi$ for all $\mathcal{X}$. We are now ready to define pseudo-models for the language $\mathcal{L}^!_{K\mathsf{int}\Box}$.

7.1.7. Definition. [Pseudo-models for $\mathcal{L}^!_{K\mathsf{int}\Box}$] A *pseudo-model* $\mathcal{X} = (X, \mathcal{O}, V)$ is a pre-model such that $\llbracket\mathsf{int}(\varphi)\rrbracket^U \in \mathcal{O}$, for all $\varphi \in \mathcal{L}^!_{K\mathsf{int}\Box}$ and $U \in \mathcal{O}$.

It is obvious that the class of pseudo-models includes all topo-models, and that all formulas of $\mathcal{L}^!_{K\mathsf{int}\Box}$ that are valid on pseudo-models are also valid on topo-models: this is because the *satisfaction relation for epistemic scenarios* in any pseudo-model that happens to be a topo-model agrees with the topo-model satisfaction relation.

Soundness of $\mathsf{DTL}_{\mathsf{int}}$

We now continue with the soundness proofs for $\mathsf{DTL}_{\mathsf{int}}$ with respect to topo and pseudo-models. Once we prove the soundness of $\mathsf{DTL}_{\mathsf{int}}$ for pseudo-models, its soundness for topo-models follows from the facts that every topo-model is a pseudo-model and that validity on both structures is defined with respect to epistemic scenarios. It is not hard to see that all the axiom schemas in group (I) and (II) in Table 7.1 are valid, and the inference rules ($\mathrm{Nec}_!$) and ([!]RE) preserve validity on pseudo-models. In the following, we focus on the axiom schema ([!]$\Box$-elim) and the inference rule ([!]$\Box$-intro).

7.1.8. Lemma. *Let $\mathcal{X} = (X, \mathcal{O}, V)$ and $\mathcal{X}' = (X, \mathcal{O}, V')$ be two pseudo-models and $\varphi \in \mathcal{L}^!_{K\mathsf{int}\Box}$ such that $\mathcal{X}$ and $\mathcal{X}'$ differ only in the valuation of some $p \notin \mathrm{P}_\varphi$. Then, for all $U \in \mathcal{O}$, we have $\llbracket\varphi\rrbracket^U_{\mathcal{X}} = \llbracket\varphi\rrbracket^U_{\mathcal{X}'}$.*

Proof:
The proof follows by subformula induction on φ. The base case $\varphi := q$ follows

from the fact that $V(q) = V'(q)$ for all $q \in \mathrm{P}_\varphi$. The cases for Booleans are straightforward, we here only prove the cases for the modalities.

Case $\varphi := K\psi$: Note that $\mathrm{P}_{K\psi} = \mathrm{P}_\psi$. Then, by induction hypothesis (IH), we have that $[\![\psi]\!]^U_{\mathcal{X}} = [\![\psi]\!]^U_{\mathcal{X}'}$. Due to the semantics of K, we have two cases (1) if $U = [\![\psi]\!]^U_{\mathcal{X}} = [\![\psi]\!]^U_{\mathcal{X}'}$, then $[\![K\psi]\!]^U_{\mathcal{X}} = [\![K\psi]\!]^U_{\mathcal{X}'} = U$, and (2) if $[\![\psi]\!]^U_{\mathcal{X}} = [\![\psi]\!]^U_{\mathcal{X}'} \neq U$, then we have $[\![K\psi]\!]^U_{\mathcal{X}} = [\![K\psi]\!]^U_{\mathcal{X}'} = \emptyset$.

Case $\varphi := \mathsf{int}(\psi)$: Note that $\mathrm{P}_{\mathsf{int}(\psi)} = \mathrm{P}_\psi$. By the semantics of int, we have $[\![\mathsf{int}(\psi)]\!]^U_{\mathcal{X}} = \mathit{Int}([\![\psi]\!]^U_{\mathcal{X}})$. Since $\mathcal{X}$ and $\mathcal{X}'$ generate the same topology $\tau_{\mathcal{O}}$ (they are based on the same lattice space), by IH, we obtain $\mathit{Int}([\![\psi]\!]^U_{\mathcal{X}}) = \mathit{Int}([\![\psi]\!]^U_{\mathcal{X}'})$, i.e., $[\![\mathsf{int}(\psi)]\!]^U_{\mathcal{X}} = [\![\mathsf{int}(\psi)]\!]^U_{\mathcal{X}'}$.

Case $\varphi := [\chi]\psi$: Note that $\mathrm{P}_{[\chi]\psi} = \mathrm{P}_\chi \cup \mathrm{P}_\psi$. Suppose $x \in [\![[\chi]\psi]\!]^U_{\mathcal{X}}$ and $x \in \mathit{Int}([\![\chi]\!]^U_{\mathcal{X}'})$. By IH, we have $[\![\chi]\!]^U_{\mathcal{X}'} = [\![\chi]\!]^U_{\mathcal{X}}$. Therefore, since $\mathcal{X}$ and $\mathcal{X}'$ generate the same topology $\tau_{\mathcal{O}}$, we obtain $\mathit{Int}([\![\chi]\!]^U_{\mathcal{X}'}) = \mathit{Int}([\![\chi]\!]^U_{\mathcal{X}})$. Hence, since $x \in [\![[\chi]\psi]\!]^U_{\mathcal{X}}$ and $x \in \mathit{Int}([\![\chi]\!]^U_{\mathcal{X}}) \subseteq U$, we have $\mathcal{X}, (x, \mathit{Int}([\![\chi]\!]^U_{\mathcal{X}})) \models \psi$, i.e., $x \in [\![\psi]\!]^{\mathit{Int}([\![\chi]\!]^U_{\mathcal{X}})}_{\mathcal{X}}$. Similarly, by IH, we then obtain $x \in [\![\psi]\!]^{\mathit{Int}([\![\chi]\!]^U_{\mathcal{X}'})}_{\mathcal{X}'}$ and therefore, $x \in [\![[\chi]\psi]\!]^U_{\mathcal{X}'}$. The other direction follows similarly.

Case $\varphi := \Box\psi$: Suppose $x \in [\![\Box\psi]\!]^U_{\mathcal{X}}$. This means, by the semantics of $\Box$, that for all $O \in \mathcal{O}$ with $x \in O \subseteq U$ we have that $\mathcal{X}, (x, O) \models \psi$, i.e., that $x \in [\![\psi]\!]^O_{\mathcal{X}}$. Therefore, by IH and the fact that $\mathrm{P}_{\Box\psi} = \mathrm{P}_\psi$, we obtain $x \in [\![\psi]\!]^O_{\mathcal{X}'}$. Since $\mathcal{X}$ and $\mathcal{X}'$ carry the same collection $\mathcal{O}$, we conclude that $x \in [\![\Box\psi]\!]^U_{\mathcal{X}'}$. The opposite direction follows similarly. $\Box$

7.1.9. Theorem. $\mathsf{DTL}_{\mathsf{int}}$ *is sound with respect to the class of all pseudo-models* (*and hence also with respect to the class of all topo-models*).

Proof:
The soundness proof follows by a simple validity check. We here only prove that ([!]$\Box$-elim) is valid and ([!]$\Box$-intro) preserves validity on pseudo-models.

([!]$\Box$-elim): Let $\mathcal{X} = (X, \mathcal{O}, V)$ be a pseudo-model and $(x, U) \in ES(\mathcal{X})$ such that $(x, U) \models [\theta]\Box\varphi$. This means that if $x \in \mathit{Int}([\![\theta]\!]^U)$ then for all $O \in \mathcal{O}$ with $x \in O \subseteq \mathit{Int}([\![\theta]\!]^U)$, we have $(x, O) \models \varphi$. Now let $\rho \in \mathcal{L}^!_{K\mathsf{int}\Box}$ and suppose $x \in \mathit{Int}([\![\theta \wedge \rho]\!]^U)$. Since $\mathit{Int}([\![\theta \wedge \rho]\!]^U) = \mathit{Int}([\![\theta]\!]^U) \cap \mathit{Int}([\![\rho]\!]^U) \subseteq \mathit{Int}([\![\theta]\!]^U)$, we obtain $x \in \mathit{Int}([\![\theta]\!]^U)$. Thus, by the first assumption that $(x, O) \models \varphi$ for all $O \in \mathcal{O}$ such that $x \in O \subseteq \mathit{Int}([\![\theta]\!]^U)$, we in particular obtain $(x, \mathit{Int}([\![\theta \wedge \rho]\!]^U)) \models \varphi$. Hence, $(x, U) \models [(\theta \wedge \rho)]\varphi$.

([!]$\Box$-intro): Suppose, toward a contradiction, that $\models \psi \to [\theta \wedge p]\varphi$ and $\not\models \psi \to [\theta]\Box\varphi$ where $p \notin \mathrm{P}_\psi \cup \mathrm{P}_\theta \cup \mathrm{P}_\varphi$. The latter means that there is a pseudo-model $\mathcal{X} = (X, \mathcal{O}, V)$ and an epistemic scenario $(x, U) \in ES(\mathcal{X})$ such that $\mathcal{X}, (x, U) \models \psi$ but $\mathcal{X}, (x, U) \not\models [\theta]\Box\varphi$, i.e., $\mathcal{X}, (x, U) \models \langle\theta\rangle\Diamond\neg\varphi$. Therefore, applying the semantics, we obtain $x \in \mathit{Int}([\![\theta]\!]^U_{\mathcal{X}})$ and there exists $U_0 \subseteq$

$Int(\llbracket\theta\rrbracket^U_{\mathcal{X}})$ such that $x \in U_0$ and $\mathcal{X}, (x, U_0) \models \neg\varphi$. Now, consider the model $\mathcal{X}' = (X, \mathcal{O}, V')$ such that $V'(p) = U_0$ and $V'(q) = V(q)$ for all $q \in$ PROP with $p \neq q$. Then, by Lemma 7.1.8, we have that $\llbracket\psi\rrbracket^U_{\mathcal{X}} = \llbracket\psi\rrbracket^U_{\mathcal{X}'}$, $\llbracket\theta\rrbracket^U_{\mathcal{X}} = \llbracket\theta\rrbracket^U_{\mathcal{X}'}$, and $\llbracket\neg\varphi\rrbracket^{U_0}_{\mathcal{X}} = \llbracket\neg\varphi\rrbracket^{U_0}_{\mathcal{X}'}$. Therefore, $\mathcal{X}', (x, U) \models \psi$ and $\mathcal{X}', (x, U_0) \models \neg\varphi$. It is easy to see that $Int(\llbracket\theta \wedge p\rrbracket^U_{\mathcal{X}'}) = Int(\llbracket\theta\rrbracket^U_{\mathcal{X}'}) \cap Int(\llbracket p\rrbracket^U_{\mathcal{X}'}) = V'(p) = U_0$ (since $Int(\llbracket p\rrbracket^U_{\mathcal{X}'}) = U_0 \subseteq Int(\llbracket\theta\rrbracket^U_{\mathcal{X}}) = Int(\llbracket\theta\rrbracket^U_{\mathcal{X}'})$). We therefore obtain (1) $x \in Int(\llbracket\theta \wedge p\rrbracket^U_{\mathcal{X}'})$ and (2) $\mathcal{X}', (x, Int(\llbracket\theta \wedge p\rrbracket^U_{\mathcal{X}'})) \models \neg\varphi$. Hence, by the semantics of [!], we have $\mathcal{X}', (x, U) \models \langle\theta \wedge p\rangle\neg\varphi$. Then, since $\mathcal{X}, (x, U) \models \psi$ and $\llbracket\psi\rrbracket^U_{\mathcal{X}} = \llbracket\psi\rrbracket^U_{\mathcal{X}'}$, we obtain $\mathcal{X}', (x, U) \models \psi \wedge \langle\theta \wedge p\rangle\neg\varphi$. Therefore, $\mathcal{X}', (x, U) \not\models \psi \to [\theta \wedge p]\varphi$, contradicting the validity of $\psi \to [\theta \wedge p]\varphi$. $\Box$

Expressivity on pseudo and topo-models

In this part, we establish several expressivity results with respect to both pseudo and topo-models, concerning our full language $\mathcal{L}^!_{K\mathsf{int}\Box}$ and its important fragments $\mathcal{L}^!_{K\mathsf{int}}, \mathcal{L}_{K\mathsf{int}}$ and $\mathcal{L}_{K\Box}$ studied in Chapter 6. The reason to consider the more general case of pseudo-models (not only topo-models) is that the co-expressivity of the languages $\mathcal{L}_{K\mathsf{int}}$ and $\mathcal{L}^!_{K\mathsf{int}\Box}$ for pseudo-models will be used in the completeness proof of $\mathsf{DTL}_{\mathsf{int}}$ (Corollary 7.1.37).

We first show the co-expressivity of $\mathcal{L}^!_{K\mathsf{int}}$ and $\mathcal{L}_{K\mathsf{int}}$ with respect to pseudo-models (Proposition 7.1.11). Its proof is standard, using the reduction laws to push dynamic modalities inside the formulas and then eliminating them. This requires an inductive proof on a non-standard complexity measure on formulas in $\mathcal{L}^!_{K\mathsf{int}}$ which induces a well-founded strict partial order on $\mathcal{L}^!_{K\mathsf{int}}$ satisfying the properties given in Lemma 7.1.10. Such a complexity measure is defined by Bjorndahl (2016) to prove the co-expressivity of $\mathcal{L}^!_{K\mathsf{int}}$ and $\mathcal{L}_{K\mathsf{int}}$ for topo-models (see Bjorndahl, 2016, Proposition 5), as well as for the completeness result of $\mathsf{PAL}^+_{\mathsf{int}}$ (see Bjorndahl, 2016, Corollary 1). Bjorndahl's simple complexity measure on $\mathcal{L}^!_{K\mathsf{int}}$ would in fact suffice for our expressivity result on pseudo-models for the languages $\mathcal{L}_{K\mathsf{int}}$ and $\mathcal{L}^!_{K\mathsf{int}}$. However, in order to prove the completeness of $\mathsf{DTL}_{\mathsf{int}}$ (in Section 7.1.3), we need a complexity measure on the formulas of the extended language $\mathcal{L}^!_{K\mathsf{int}\Box}$ taking into account the effort modality $\Box$ as well. A similar complexity measure will also be needed in Lemma 8.3.16 in Chapter 8. To this end, we define a more elaborate complexity measure on $\mathcal{L}^!_{K\mathsf{int}\Box}$ that we can use throughout this and the next chapter. The definition of this complexity measure is given in Appendix A.1.

7.1.10. LEMMA. *There exists a well-founded strict partial order $<^S$ on formulas of $\mathcal{L}^!_{K\mathsf{int}}$ such that*

1. $\varphi \in Sub(\psi)$ *implies* $\varphi <^S \psi$,
2. $\mathsf{int}(\varphi) \to p <^S [\varphi]p$,
3. $\mathsf{int}(\varphi) \to \neg[\varphi]\psi <^S [\varphi]\neg\psi$,
4. $[\varphi]\psi \wedge [\varphi]\chi <^S [\varphi](\psi \wedge \chi)$,

5. $\mathsf{int}(\varphi) \to \mathsf{int}([\varphi]\psi) <^S [\varphi]\mathsf{int}(\psi)$,

6. $\mathsf{int}(\varphi) \to K[\varphi]\psi <^S [\varphi]K\psi$,

7. $[\langle\varphi\rangle\psi]\chi <^S [\varphi][\psi]\chi$.

Proof:
See Lemmas A.1.4 and A.1.5. □

7.1.11. PROPOSITION. *$\mathcal{L}^!_{K\mathsf{int}}$ and $\mathcal{L}_{K\mathsf{int}}$ are co-expressive with respect to pseudo-models. In other words, for every formula $\varphi \in \mathcal{L}^!_{K\mathsf{int}}$ there exists a formula $\psi \in \mathcal{L}_{K\mathsf{int}}$ such that $\varphi \leftrightarrow \psi$ is valid in all pseudo-models.*

Proof:
The proof follows by $<^S$-induction on φ. The base case $\varphi := p$ follows from the fact that the languages $\mathcal{L}^!_{K\mathsf{int}}$ and $\mathcal{L}_{K\mathsf{int}}$ are defined based on the same set of propositional variables PROP. The cases for the Booleans $\varphi := \neg\psi$, $\varphi := \psi \wedge \chi$, and the cases for the modalities $\varphi := K\psi$ and $\varphi := \mathsf{int}(\psi)$ follow standardly using Lemma 7.1.10-(1). We here only prove the cases for $\varphi := K\psi$, and $\varphi := [\psi]\chi$:

Case $\varphi := K\psi$: Since $\psi <^S K\psi$ (Lemma 7.1.10-(1)), by induction hypothesis, there exists a $\psi' \in \mathcal{L}_{K\mathsf{int}}$ such that $\psi \leftrightarrow \psi'$ is valid in all pseudo-models. Then, by the soundness of (RE) (which follows from Proposition 7.1.2 and Theorem 7.1.9), we obtain $\models K\psi \leftrightarrow K\psi'$, where $K\psi' \in \mathcal{L}_{K\mathsf{int}}$.

Case $\varphi := [\psi]\chi$: Theorem 7.1.9 implies that the reduction laws given in Table 7.1 and Proposition 7.1.2 for the language $\mathcal{L}^!_{K\mathsf{int}}$ are valid in all pseudo-models. Therefore, applying the appropriate reduction (e.g., if $\chi := p$ apply (R_p), if $\chi := \neg\sigma$ apply ($\mathrm{R}_\neg$) etc.) we obtain a formula $\gamma \in \mathcal{L}^!_{K\mathsf{int}}$ such that $[\psi]\chi \leftrightarrow \gamma$ is valid in all pseudo-models. By Lemma 7.1.10.(2-7), we know that $\gamma <^S [\psi]\chi$. Hence, by induction hypothesis, there exists $\gamma' \in \mathcal{L}_{K\mathsf{int}}$ such that $\models \gamma \leftrightarrow \gamma'$. As γ is semantically equivalent to $[\psi]\chi$, we conclude that $\models [\psi]\chi \leftrightarrow \gamma'$, where $\gamma' \in \mathcal{L}_{K\mathsf{int}}$. □

Next, we prove that $\mathcal{L}^!_{K\mathsf{int}\Box}$ and $\mathcal{L}_{K\mathsf{int}}$ are equally expressive with respect to pseudo-models. This result will also be useful in the completeness proof of $\mathsf{DTL}_{\mathsf{int}}$ for topo-models (Corollary 7.1.38). In proving the co-expressivity of $\mathcal{L}^!_{K\mathsf{int}\Box}$ and $\mathcal{L}_{K\mathsf{int}}$, we follow a similar strategy as in (Balbiani et al., 2008; van Ditmarsch et al., 2014). Our proof follows the same steps as in the proof of co-expressivity between $\mathcal{L}_{K\mathsf{int}}$ and $\mathcal{L}^!_{K\mathsf{int}\circledast}$ for topo-models (see van Ditmarsch et al., 2014, Theorem 19), where $\mathcal{L}^!_{K\mathsf{int}\circledast}$ denotes the extension of $\mathcal{L}^!_{K\mathsf{int}}$ with the so-called arbitrary announcement modality $\circledast$. We will study the arbitrary announcement modality $\circledast\varphi$ and its connection to the effort modality $\Box\varphi$ in Section 7.2.

The proof of the co-expressivity result between $\mathcal{L}^!_{K\mathsf{int}\Box}$ and $\mathcal{L}_{K\mathsf{int}}$ (as well as the co-expressivity of $\mathcal{L}_{K\mathsf{int}}$ and $\mathcal{L}^!_{K\mathsf{int}\circledast}$) relies on the fact that for every formula φ in $\mathcal{L}_{K\mathsf{int}}$, there exists a $\psi \in \mathcal{L}_{K\mathsf{int}}$ in "normal form" such that φ and ψ are semantically equivalent in pseudo(topo)-models. Normal forms for formulas in the language $\mathcal{L}_{K\mathsf{int}}$ are defined similarly to the normal forms of the basic modal language in such a way that the modality int can occur in the scope of K (see Meyer and van der Hoek, 1995, for normal forms for the basic epistemic language).

7.1.12. Definition. [Normal form for the language $\mathcal{L}_{K\mathsf{int}}$] We say a formula $\psi \in \mathcal{L}_{K\mathsf{int}}$ is in *normal form* if it is a disjunction of conjunctions of the form

$$\delta := \alpha \wedge K\beta \wedge \hat{K}\gamma_1 \wedge \cdots \wedge \hat{K}\gamma_n$$

where $\alpha, \beta, \gamma_i \in \mathcal{L}_{\mathsf{int}}$ for all $1 \leq i \leq n$.

Our normal forms for the language $\mathcal{L}_{K\mathsf{int}}$ are similar to the so-called *disjunctive normal forms* introduced in (Georgatos, 1993, Definition 34) for the language $\mathcal{L}_{K\Box}$. More precisely, given a formula in $\mathcal{L}_{K\mathsf{int}}$ in normal form, we obtain a formula in $\mathcal{L}_{K\Box}$ in disjunctive normal form in the sense of Georgatos (1993) by replacing every occurrence of the modality int by $\Diamond K$.

7.1.13. Lemma (Normal From Lemma). *For every formula $\varphi \in \mathcal{L}_{K\mathsf{int}}$ there is a formula $\psi \in \mathcal{L}_{K\mathsf{int}}$ in normal form such that $\varphi \leftrightarrow \psi$ is valid in all pseudo-models, therefore, also valid in all topo-models.*

Proof:
The proof is given in Appendix A.2. $\square$

Having proven the Normal Form Lemma—the first crucial step toward the desired expressivity results—we now proceed with the proof of Theorem 7.1.17. For this, we need a few more validities in which *bi-persistent formulas on pseudo-models* in the language $\mathcal{L}^!_{K\mathsf{int}\Box}$ play an important role. Bi-persistent formulas in $\mathcal{L}^!_{K\mathsf{int}\Box}$ for pseudo-models are defined similarly as in Definition 6.2.8 with respect to *epistemic scenarios*. Informally speaking, these are the formulas of $\mathcal{L}^!_{K\mathsf{int}\Box}$ whose truth value on pseudo-models depends *only* on the actual state, *not* on the epistemic range.

7.1.14. Lemma. *Every formula of $\mathcal{L}_{\mathsf{int}}$ is bi-persistent on pseudo-models.*

Proof:
The proof is similar to the proof of Proposition 6.2.9, by subformula induction on φ: cases for the propositional variables and the Boolean connectives are elementary. So assume inductively that the result holds for ψ; we must show that it holds also for $\varphi := \mathsf{int}(\psi)$. Let $(X, \mathcal{O}, V)$ be a pseudo-model and $(x, O), (x, U) \in ES(\mathcal{X})$. We then have

$$\begin{aligned}(x, U) \models \mathsf{int}(\psi) &\text{ iff } x \in \mathit{Int}(\llbracket\psi\rrbracket^U)\\ &\text{ iff } (\exists U' \in \mathcal{O})(x \in U' \subseteq \llbracket\psi\rrbracket^U) \qquad \text{(since } \mathcal{O} \text{ is a basis for } \tau_{\mathcal{O}})\end{aligned}$$

Now, consider the open set $U' \cap O$. It is easy to see that $x \in U' \cap O$. So, we only need to show that $U' \cap O \subseteq [\![\psi]\!]^O$. Let $y \in U' \cap O$. Since $y \in U'$, we have that $(y, U) \models \psi$ (by $U' \subseteq [\![\psi]\!]^U$). Then, by induction hypothesis, we obtain $(y, O) \models \psi$, i.e., $y \in [\![\psi]\!]^O$. We therefore have that $x \in U' \cap O \subseteq [\![\psi]\!]^O$, i.e., that $x \in Int([\![\psi]\!]^O)$ (since $U' \cap O \in \mathcal{O} \subseteq \tau_{\mathcal{O}}$). Therefore, $(x, O) \models \mathsf{int}(\psi)$. The other direction follows similarly. □

7.1.15. Proposition. *For any $\varphi, \varphi_i \in \mathcal{L}_{\mathsf{int}}$, the following is valid in all pseudo-models:*

$$\Diamond(\varphi \wedge K\varphi_0 \wedge \bigwedge_{1\leq i\leq n} \hat{K}\varphi_i) \leftrightarrow (\varphi \wedge \mathsf{int}(\varphi_0) \wedge \bigwedge_{1\leq i\leq n} \hat{K}(\mathsf{int}(\varphi_0) \wedge \varphi_i)) \qquad (\mathrm{EL}_n^\Box)$$

Proof:
The proof follows similarly to the proof of (van Ditmarsch et al., 2014, Proposition 18). Let $\mathcal{X} = (X, \mathcal{O}, V)$ be a pseudo-model and $(x, U) \in ES(\mathcal{X})$. It is important to notice that every $\varphi, \varphi_i \in \mathcal{L}_{\mathsf{int}}$ is bi-persistent, we will use this fact several times.

We prove the statement only for $n = 1$.

($\Rightarrow$) Suppose $(x, U) \models \Diamond(\varphi \wedge K\varphi_0 \wedge \hat{K}\varphi_1)$. By the semantics, we have

$$(x, U) \models \Diamond(\varphi \wedge K\varphi_0 \wedge \hat{K}\varphi_1) \text{ iff } (\exists V \in \mathcal{O})(x \in V \subseteq U \text{ and } (x, V) \models \varphi \wedge K\varphi_0 \wedge \hat{K}\varphi_1)$$

We therefore have (1) $(x, V) \models \varphi$, (2) $(x, V) \models K\varphi_0$, and (3) $(x, V) \models \hat{K}\varphi_1$. We want to show that $(x, U) \models \varphi \wedge \mathsf{int}(\varphi_0) \wedge \hat{K}(\mathsf{int}(\varphi_0) \wedge \varphi_1)$. Now (1) and Lemma 7.1.14 imply $(x, U) \models \varphi$; and (2) implies that $(x, V) \models \mathsf{int}(\varphi_0)$. Then, by Lemma 7.1.14, we have $(x, U) \models \mathsf{int}(\varphi_0)$ (since $\mathsf{int}(\varphi_0)$ is bi-persistent).

In order to show $(x, U) \models \hat{K}(\mathsf{int}(\varphi_0) \wedge \varphi_1)$, we need to prove that there is a $y \in U$ such that $(y, U) \models \mathsf{int}(\varphi_0) \wedge \varphi_1$. Item (3) implies that there is a $z \in V$ such that $(z, V) \models \varphi_1$. Then, by Lemma 7.1.14, we have $(z, U) \models \varphi_1$. Moreover, (2) implies $(z, V) \models K\varphi_0$, and thus $(z, V) \models \mathsf{int}(\varphi_0)$. Then again by Lemma 7.1.14, $(z, U) \models \mathsf{int}(\varphi_0)$. So, $(z, U) \models \mathsf{int}(\varphi_0) \wedge \varphi_1$, and thus $(x, U) \models \hat{K}(\mathsf{int}(\varphi_0) \wedge \varphi_1)$.

($\Leftarrow$) Suppose $(x, U) \models \varphi \wedge \mathsf{int}(\varphi_0) \wedge \hat{K}(\mathsf{int}(\varphi_0) \wedge \varphi_1)$. We have:

$$\begin{array}{ll} & (x, U) \models \varphi \wedge \mathsf{int}(\varphi_0) \wedge \hat{K}(\mathsf{int}(\varphi_0) \wedge \varphi_1) \\ \text{iff} & (x, U) \models \varphi \text{ and } (x, U) \models \mathsf{int}(\varphi_0) \text{ and } \exists y \in U \text{ with } (y, U) \models \mathsf{int}(\varphi_0) \wedge \varphi_1 \\ \text{iff} & (x, U) \models \varphi \text{ and } (x, U) \models \mathsf{int}(\varphi_0) \text{ and } \exists y \in Int([\![\varphi_0]\!]^U) \text{ with } (y, U) \models \varphi_1 \end{array}$$

We want to show $(x, U) \models \Diamond(\varphi \wedge K\varphi_0 \wedge \hat{K}\varphi_1)$, i.e., we want to prove that there is a $V \in \mathcal{O}$ with $x \in V \subseteq U$ such that $(x, V) \models \varphi \wedge K\varphi_0 \wedge \hat{K}\varphi_1$.

We now claim that for $V := Int([\![\varphi_0]\!]^U)$, we obtain the desired result. It is easy to see that $x \in Int([\![\varphi_0]\!]^U) \subseteq U$ (since $(x, U) \models \mathsf{int}(\varphi_0)$). And, since $\mathcal{X}$ is a pseudo-model, it is guaranteed that $Int([\![\varphi_0]\!]^U) \in \mathcal{O}$. We want to show that $(x, Int([\![\varphi_0]\!]^U)) \models \varphi \wedge K\varphi_0 \wedge \hat{K}\varphi_1$. Since $(x, U) \models \varphi$, by Lemma 7.1.14, we obtain

$(x, Int(\llbracket\varphi_0\rrbracket^U)) \models \varphi$. Since $Int(\llbracket\varphi_0\rrbracket^U)) \subseteq \llbracket\varphi_0\rrbracket^U$, we have that $(z, U) \models \varphi_0$ for all $z \in Int(\llbracket\varphi_0\rrbracket^U)$. Therefore, as φ_0 is bi-persistent (Lemma 7.1.14), we obtain $(z, Int(\llbracket\varphi_0\rrbracket^U) \models \varphi_0$ for all $z \in Int(\llbracket\varphi_0\rrbracket^U)$, thus, $(x, Int(\llbracket\varphi_0\rrbracket^U)) \models K\varphi_0$. By the assumption, we have $\exists y \in Int(\llbracket\varphi_0\rrbracket^U)$ such that $(y, U) \models \varphi_1$, and thus, by Lemma 7.1.14, we obtain $(y, Int(\llbracket\varphi_0\rrbracket^U)) \models \varphi_1$. Therefore, by the semantics, we have $(x, Int(\llbracket\varphi_0\rrbracket^U)) \models \hat{K}\varphi_1$. □

The proof of the following lemma is straightforward, and follows directly from the semantics for $\Diamond$ and $\vee$.

7.1.16. Lemma. *For all $\varphi, \psi \in \mathcal{L}^{!}_{K\mathsf{int}\Box}$, the formula $\Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$ is valid in all pseudo-models.*

We now have sufficient machinery to show that $\mathcal{L}^{!}_{K\mathsf{int}\Box}$ and $\mathcal{L}_{K\mathsf{int}}$ are equally expressive with respect to pseudo-models.

7.1.17. Theorem. *$\mathcal{L}^{!}_{K\mathsf{int}\Box}$ and $\mathcal{L}_{K\mathsf{int}}$ are co-expressive with respect to pseudo-models.*

Proof:
We need to prove that for all $\varphi \in \mathcal{L}^{!}_{K\mathsf{int}\Box}$ there exists $\theta \in \mathcal{L}_{K\mathsf{int}}$ such that $\varphi \leftrightarrow \theta$ is valid in all pseudo-models. The proof follows by subformula induction on φ. The base case $\varphi := p$ follows from the fact the languages $\mathcal{L}_{K\mathsf{int}}$ and $\mathcal{L}^{!}_{K\mathsf{int}\Box}$ are defined based on the same set of propositional variables PROP. The cases for the Booleans $\varphi := \neg\psi$, $\varphi := \psi \wedge \chi$, and the cases for the modalities $\varphi := K\psi$ and $\varphi := \mathsf{int}(\psi)$ follow standardly. We here only show the cases $\varphi := [\psi]\chi$ and $\varphi := \Diamond\psi$.

Case $\varphi := [\psi]\chi$: Since ψ and χ are subformulas of φ, by induction hypothesis, there exists $\psi', \chi' \in \mathcal{L}_{K\mathsf{int}}$ such that (a) $\models \psi \leftrightarrow \psi'$ and (b) $\models \chi \leftrightarrow \chi'$. Then, by (a) and the soundness of ([!]RE), we obtain $\models [\psi]\chi \leftrightarrow [\psi']\chi$. Moreover, by (b) and the soundness of (RE), we have $\models [\psi']\chi \leftrightarrow [\psi']\chi'$. Therefore, $\models [\psi]\chi \leftrightarrow [\psi']\chi'$. Notice that $[\psi']\chi' \in \mathcal{L}^{!}_{K\mathsf{int}} \backslash \mathcal{L}_{K\mathsf{int}}$. Then, by Proposition 7.1.11, there exists $\gamma \in \mathcal{L}_{K\mathsf{int}}$ such that $\models [\psi']\chi' \leftrightarrow \gamma$. We then conclude that $\models [\psi]\chi \leftrightarrow \gamma$, i.e., $[\psi]\chi$ is semantically equivalent to $\gamma \in \mathcal{L}_{K\mathsf{int}}$ with respect to pseudo-models.

Case $\varphi := \Diamond\psi$: By induction hypothesis, there exists $\psi' \in \mathcal{L}_{K\mathsf{int}}$ such that $\models \psi \leftrightarrow \psi'$. Then, by Lemma 7.1.13, there exists a $\gamma \in \mathcal{L}_{K\mathsf{int}}$ in normal form with $\models \psi' \leftrightarrow \gamma$, hence, we also have $\models \psi \leftrightarrow \gamma$. Therefore, by the soundness of (RE), we obtain $\models \Diamond\psi \leftrightarrow \Diamond\gamma$. By Lemma 7.1.16, we can distribute $\Diamond$ over the disjunction γ. Since γ is in normal form, each disjunct of the resulting formula is of the form $\Diamond(\varphi \wedge K\varphi_0 \wedge \hat{K}\varphi_1 \wedge \hat{K}\varphi_2 \wedge \cdots \wedge \hat{K}\varphi_n)$ where $\varphi, \varphi_i \in \mathcal{L}_{\mathsf{int}}$ for all $0 \leq i \leq n$. Then, by Proposition 7.1.15, we can reduce these formulas to semantically equivalent formulas of the form $\varphi \wedge \mathsf{int}(\varphi_0) \wedge \hat{K}(\mathsf{int}(\varphi_0) \wedge \varphi_1) \wedge \cdots \wedge \hat{K}(\mathsf{int}(\varphi_0) \wedge \varphi_n)$, hence, obtain a formula in $\mathcal{L}_{K\mathsf{int}}$ that is semantically equivalent to $\Diamond\psi$ with respect to pseudo-models. □

Theorem 7.1.17 will be used in the completeness proof of $\mathsf{DTL}_{\mathsf{int}}$ for topo-models (Corollary 7.1.38). Concerning expressivity of $\mathcal{L}^!_{K\mathsf{int}\Box}$, we also obtain the following result with respect to topo-models.

7.1.18. COROLLARY. *$\mathcal{L}^!_{K\mathsf{int}\Box}$ and $\mathcal{L}_{K\mathsf{int}}$ are co-expressive with respect to topo-models.*

Proof:
This proof proceeds similarly to the proof of Theorem 7.1.17. Since every topo-model is a pseudo-model, Proposition 7.1.15 holds for topo-models as well. Moreover, recall that $\mathcal{L}_{K\mathsf{int}}$ and $\mathcal{L}^!_{K\mathsf{int}}$ are equally expressive with respect to topo-models (see Theorem 6.2.5). Therefore, we can argue along the same lines as in Theorem 7.1.17 and prove that for every formula $\varphi \in \mathcal{L}^!_{K\mathsf{int}\Box}$ there exists a formula $\psi \in \mathcal{L}_{K\mathsf{int}}$ such that φ and ψ are semantically equivalent with respect to topo-models. $\Box$

Since $\mathcal{L}_{K\Box} \subseteq \mathcal{L}^!_{K\mathsf{int}\Box}$, Corollary 7.1.18 also establishes that $\mathcal{L}_{K\mathsf{int}}$ is at least as expressive as $\mathcal{L}_{K\Box}$ on topo-models. As shown in the next theorem, $\mathcal{L}_{K\mathsf{int}}$ and $\mathcal{L}_{K\Box}$ are in fact equally expressive for topo-models.

7.1.19. THEOREM. *$\mathcal{L}_{K\Box}$ and $\mathcal{L}_{K\mathsf{int}}$ are also co-expressive with respect to topo-models.*

Proof:
Corollary 7.1.18 shows that for every $\varphi \in \mathcal{L}_{K\Box}$ there is $\psi \in \mathcal{L}_{K\mathsf{int}}$ such that $\varphi \leftrightarrow \psi$ is valid in all topo-models. We only need to show that for every $\chi \in \mathcal{L}_{K\mathsf{int}}$ there is $\theta \in \mathcal{L}_{K\Box}$ such that $\models \chi \leftrightarrow \theta$. Thus, suppose $\chi \in \mathcal{L}_{K\mathsf{int}}$. By Lemma 7.1.13, there is $\chi' \in \mathcal{L}_{K\mathsf{int}}$ in normal form such that $\models \chi \leftrightarrow \chi'$. As χ' is in normal forms, we have $\chi' := \delta_1 \vee \cdots \vee \delta_m$ where each $\delta_i := \alpha \wedge K\beta \wedge \hat{K}\gamma_1 \wedge \cdots \wedge \hat{K}\gamma_n$ with $\alpha, \beta, \gamma_i \in \mathcal{L}_{\mathsf{int}}$ for all $1 \leq i \leq n$. Now take an arbitrary $\delta_i = \alpha \wedge K\beta \wedge \hat{K}\gamma_1 \wedge \cdots \wedge \hat{K}\gamma_n$. By Proposition 6.2.10 and the soundness of (RE), we have $\models \delta_i \leftrightarrow \alpha^* \wedge K(\beta^*) \wedge \hat{K}(\gamma_1^*) \wedge \cdots \wedge \hat{K}(\gamma_n^*)$ where $* : \mathcal{L}_{\mathsf{int}} \to \mathcal{L}_{K\Box}$ is as given in Definition 6.2.7. Notice that $\alpha^* \wedge K(\beta^*) \wedge \hat{K}(\gamma_1^*) \wedge \cdots \wedge \hat{K}(\gamma_n^*) \in \mathcal{L}_{K\Box}$. Therefore, each canonical conjunction δ_i of χ' is semantically equivalent to a formula in $\mathcal{L}_{K\Box}$ with respect to topo-models. Let δ_i^* denote the formula in $\mathcal{L}_{K\Box}$ that is semantically equivalent to δ_i (this is abuse of notation since $*$ is not defined for K). Hence, we obtain (again by the soundness of (RE)) that $\models \chi' \leftrightarrow \delta_1^* \vee \cdots \vee \delta_n^*$. As $\models \chi \leftrightarrow \chi'$, we conclude $\models \chi \leftrightarrow \delta_1^* \vee \cdots \vee \delta_n^*$, where $\delta_1^* \vee \cdots \vee \delta_n^* \in \mathcal{L}_{K\Box}$. $\Box$

7.1.20. COROLLARY. *$\mathcal{L}^!_{K\mathsf{int}\Box}$, $\mathcal{L}^!_{K\mathsf{int}}$, $\mathcal{L}_{K\mathsf{int}}$ and $\mathcal{L}_{K\Box}$ are all co-expressive with respect to topo-models.*

Proof:
The proof follows easily from Corollary 7.1.18 and Theorem 7.1.19, since $\mathcal{L}_{K\mathsf{int}} \subseteq \mathcal{L}^{!}_{K\mathsf{int}} \subseteq \mathcal{L}^{!}_{K\mathsf{int}\Box}$. $\Box$

Moreover, recall that int can be defined by the public announcement modalities as $\mathsf{int}(\varphi) := \langle\varphi\rangle\top$, hence, we also obtain that $\mathcal{L}^{!}_{K}$ and $\mathcal{L}^{!}_{K\Box}$ are equally expressive as their extensions with the modality int. These results are summarized in Figure 7.1 below.

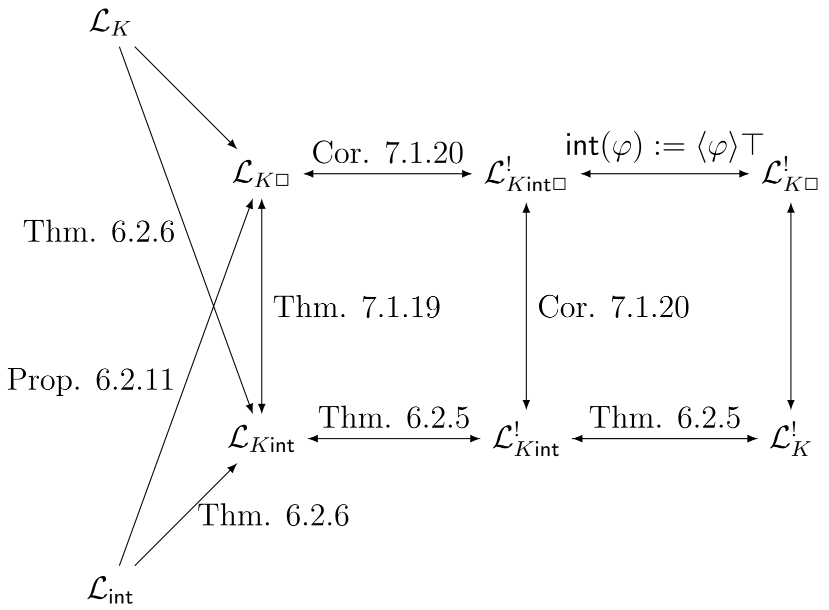


Figure 7.1: Expressivity diagram-updated with $\Box$ (Arrows point to the more expressive languages, and reflexive and transitive arrows are omitted. Arrows without tags can be obtained as easy consequences from the others.)

As a direct corollary of the above expressivity results, we obtain decidability and the finite model property for the dynamic logic of topo-models for the language $\mathcal{L}^{!}_{K\mathsf{int}\Box}$ as well as for its fragments.

7.1.21. COROLLARY. *The logic of topo-models for the language $\mathcal{L}^{!}_{K\mathsf{int}\Box}$ is decidable and has the finite model property (and thus all its fragments, including in particular* TopoLogic*, have these properties).*

Proof:
This follows from Corollary 7.1.20, together with the fact that $\mathcal{L}_{K\mathsf{int}}$ is easily shown to have these properties by a standard filtration argument (see e.g., Goranko and Passy, 1992, and Shehtman, 1999). $\Box$

7.1.3 Completeness of $\mathsf{DTL}_{\mathsf{int}}$

In this section we prove the completeness of the proof system $\mathsf{DTL}_{\mathsf{int}}$ with respect to (both pseudo and) topo-models. The plan of our proof is as follows. We first prove completeness of $\mathsf{DTL}_{\mathsf{int}}$ with respect to a *canonical pseudo-model*, consisting of *maximally consistent witnessed theories*. Roughly speaking, a maximally consistent theory is witnessed if every $\Diamond\varphi$ occurring in every "existential context" in the theory is "witnessed" by some atomic formula p meaning that $\langle p\rangle\varphi$ occurs in the same existential context in the theory. Next, we use the co-expressivity of $\mathcal{L}^{!}_{K\mathsf{int}\Box}$ and $\mathcal{L}_{K\mathsf{int}}$, as well as the fact that $\mathcal{L}_{K\mathsf{int}}$ cannot distinguish between a pseudo-model and its associated topo-model, to show that $\mathsf{DTL}_{\mathsf{int}}$ is complete with respect to the *canonical topo-model* (associated with the canonical pseudo-model).

The appropriate notion of "existential context" is represented by *possibility forms* (dual of necessity forms), in the following sense.

7.1.22. Definition. [Necessity and possibility forms for $\mathcal{L}^{!}_{K\mathsf{int}\Box}$] For any finite string $s \in (\{\varphi\!\rightarrow \mid \varphi \in \mathcal{L}^{!}_{K\mathsf{int}\Box}\} \cup \{K\} \cup \{\psi \mid \psi \in \mathcal{L}^{!}_{K\mathsf{int}\Box}\})^{*} = NF$, we define pseudo-modalities $[s]$ and $\langle s\rangle$. These pseudo-modalities are functions mapping any formula $\varphi \in \mathcal{L}^{!}_{K\mathsf{int}\Box}$ to another formula $[s]\varphi \in \mathcal{L}^{!}_{K\mathsf{int}\Box}$ (*necessity form*), respectively $\langle s\rangle\varphi \in \mathcal{L}^{!}_{K\mathsf{int}\Box}$ (*possibility form*). The necessity forms are defined recursively as $[\epsilon]\varphi = \varphi$, $[\varphi\!\rightarrow, s]\varphi = \varphi \rightarrow [s]\varphi$, $[K, s]\varphi = K[s]\varphi$, $[\psi, s]\varphi = [\psi][s]\varphi$, where ϵ is the empty string. For possibility forms, we set $\langle s\rangle\varphi := \neg[s]\neg\varphi$.

7.1.23. Lemma. *For every necessity form $[s]$, there exist formulas $\theta, \psi \in \mathcal{L}^{!}_{K\mathsf{int}\Box}$ such that for all $\varphi \in \mathcal{L}^{!}_{K\mathsf{int}\Box}$, we have*

$$\vdash [s]\varphi \textit{ iff } \vdash \psi \rightarrow [\theta]\varphi.$$

Proof:
The proof is as in (Balbiani et al., 2008, Lemma 4.8). For $s := \epsilon$, take $\psi := \top$ and $\theta := \top$. It then follows by the axiom (R$[\top]$). Otherwise, by the definition of a necessity form, $[s]\varphi$ is a formula of $\mathcal{L}^{!}_{K\mathsf{int}\Box}$ such that φ is entirely on the right (or at the bottom), and is successively bounded by finitely many implications $\chi \rightarrow$, knowledge modalities K, and announcements $[\chi']$, in arbitrary order. By rearranging the order of these symbols in a provably equivalent way, we can obtain the required form $\vdash \psi \rightarrow [\theta]\varphi$. We start with the public announcement modalities. By using the reduction laws of $\mathsf{DTL}_{\mathsf{int}}$, we can push all the public announcement modalities binding the components $\chi \rightarrow$ and K of the necessity form to the top of φ. To push them pass K, we use (R_K), and for $\chi \rightarrow$ we use ($R_\neg$) and ($R_\wedge$). We then obtain a formula that is provably equivalent to $[s]\varphi$, but in which all public announcement modalities occurring in s are stacked on top of φ. By using the axiom ($R_{[!]}$), we can write all these public announcement modalities as one announcement. We therefore obtain a formula that is provably

equivalent to $[s]\varphi$ of the following shape: a formula of the form $[\theta]\varphi$ is entirely on the right, and is successively bounded by finitely many implications $\chi \to$, and knowledge modalities K, in arbitrary order. This is still not in the required form since we might have $[\theta]\varphi$ at the bottom preceded by a knowledge modality, i.e., the resulting formula might have the shape $(\cdots \to K[\theta]\varphi)$. However, since K is of S5-type, we know that $\vdash \eta \to K\sigma$ iff $\vdash \hat{K}\eta \to \sigma$. Therefore, we can push every occurrence of the modality K bounding the consequent of an implication to the antecedent as the epistemic possibility modality $\hat{K}$. This way, we obtain a theorem of the form $\chi_1 \to (\chi_2 \to \dots (\chi_n \to [\theta]\varphi))$. Then, by classical propositional logic, we know $\vdash \chi_1 \to (\chi_2 \to \dots (\chi_n \to [\theta]\varphi)) \leftrightarrow (\bigwedge_{1\leq i\leq n} \chi_i \to [\theta]\varphi)$, thus, we have $\vdash \psi \to [\theta]\varphi$ (where $\psi := \bigwedge_{1\leq i\leq n} \chi_i$). Since every axiom used in the above argument is an equivalence, we also have $\vdash \psi \to [\theta]\varphi$ implies $\vdash [s]\varphi$. $\square$

7.1.24. Lemma. *The following rule is admissible in* $\mathsf{DTL_{int}}$*:*

$$\textit{if } \vdash [s][p]\varphi \textit{ then } \vdash [s]\Box\varphi, \textit{ where } p \notin \mathrm{P}_s \cup \mathrm{P}_\varphi.$$

Proof:
Suppose $\vdash [s][p]\varphi$. Then, by Lemma 7.1.23, there exist $\theta, \psi \in \mathcal{L}^!_{K\mathsf{int}\Box}$ such that $\vdash \psi \to [\theta][p]\varphi$. By the auxiliary reduction law ($\mathrm{R}_{[p]}$) in Proposition 7.1.2, we get $\vdash \psi \to [\theta \wedge p]\varphi$. By the construction of the formulas ψ and θ, we know that $\mathrm{P}_\psi \cup \mathrm{P}_\theta \subseteq \mathrm{P}_s$, and so $p \notin \mathrm{P}_\psi \cup \mathrm{P}_\theta \cup \mathrm{P}_\varphi$. Therefore, by ([!]$\Box$-intro)), we have $\vdash \psi \to [\theta]\Box\varphi$. Applying again Lemma 7.1.23, we obtain $\vdash [s]\Box\varphi$. $\square$

7.1.25. Definition. For every countable set of propositional variables P, let $\mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$ be the language of $\mathsf{DTL_{int}}$ based only on the propositional variables in P. Similarly, let NF^{P} denote the corresponding set of strings defined based on $\mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$.

- A P-*theory* is a *consistent* set of formulas in $\mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$, where "consistent" means consistent with respect to the axiomatization of $\mathsf{DTL_{int}}$ formulated for $\mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$.
- A *maximal* P-*theory* is a P-theory Γ that is maximal with respect to $\subseteq$ among all P-theories; in other words, Γ cannot be extended to another P-theory.
- A P-*witnessed theory* is a P-theory Γ such that, for every $s \in NF^{\mathrm{P}}$ and $\varphi \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$, if $\langle s\rangle\Diamond\varphi$ is consistent with Γ then there is $p \in \mathrm{P}$ such that $\langle s\rangle\langle p\rangle\varphi$ is consistent with Γ (i.e., if $\Gamma \vdash [s][p]\neg\varphi$ for all $p \in \mathrm{P}$, then $\Gamma \vdash [s]\Box\neg\varphi$).
- A *maximal* P-*witnessed theory* Γ is a P-witnessed theory that is not a proper subset of any P-witnessed theory.

7.1.26. Lemma. *For every maximal* P*-witnessed theory* Γ*, and every formula* $\varphi, \psi \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$*,*

1. *either* $\varphi \in \Gamma$ *or* $\neg\varphi \in \Gamma$*,*
2. $\varphi \wedge \psi \in \Gamma$ *iff* $\varphi \in \Gamma$ *and* $\psi \in \Gamma$*,*
3. $\varphi \in \Gamma$ *and* $\varphi \to \psi \in \Gamma$ *implies* $\psi \in \Gamma$*.*

7.1.27. Lemma. *For every* $\Gamma \subseteq \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$*, if* Γ *is a* P*-theory and* $\Gamma \nvdash \neg\varphi$ *for some* $\varphi \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$*, then* $\Gamma \cup \{\varphi\}$ *is a* P*-theory. Moreover, if* Γ *is* P*-witnessed, then* $\Gamma \cup \{\varphi\}$ *is also* P*-witnessed.*

Proof:
Let $\Gamma \subseteq \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$ be a P-theory and $\varphi \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$ such that $\Gamma \nvdash \neg\varphi$. It is then easy to see that $\Gamma \cup \{\varphi\}$ is consistent, and thus, is a P-theory. Now suppose that Γ is P-witnessed but $\Gamma \cup \{\varphi\}$ is not P-witnessed. By the previous statement, we know that $\Gamma \cup \{\varphi\}$ consistent. Therefore, the latter means that there is $s \in NF^{\mathrm{P}}$ and $\psi \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$ such that $\Gamma \cup \{\varphi\}$ is consistent with $\langle s\rangle\Diamond\psi$ but $\Gamma \cup \{\varphi\} \vdash \neg\langle s\rangle\langle p\rangle\psi$ for all $p \in \mathrm{P}$. This implies that $\Gamma \cup \{\varphi\} \vdash [s][p]\neg\psi$ for all $p \in \mathrm{P}$. Therefore, $\Gamma \vdash \varphi \to [s][p]\neg\psi$ for all $p \in \mathrm{P}$. This means $\Gamma \vdash [\varphi\to, s][p]\neg\psi$ for all $p \in \mathrm{P}$ (since $\varphi \to [s][p]\neg\psi := [\varphi\to, s][p]\neg\psi$). Hence, as Γ is P-witnessed and $[\varphi\to, s]$ is a necessity form, we obtain $\Gamma \vdash [\varphi\to, s]\Box\neg\psi$. By unraveling the necessity form $[\varphi\to, s]$, we get $\Gamma \vdash \varphi \to [s]\Box\neg\psi$, thus, $\Gamma \cup \{\varphi\} \vdash [s]\Box\neg\psi$, i.e., $\Gamma \cup \{\varphi\} \vdash \neg\langle s\rangle\Diamond\psi$, contradicting the assumption that $\Gamma \cup \{\varphi\}$ is consistent with $\langle s\rangle\Diamond\psi$. $\Box$

7.1.28. Lemma. *If* $\{\Gamma_i\}_{i\in\mathbb{N}}$ *is an increasing chain of* P*-theories such that* $\Gamma_i \subseteq \Gamma_{i+1}$*, then* $\bigcup_{n\in\mathbb{N}}\Gamma_n$ *is a* P*-theory.*

Proof:
The proof is standard. $\Box$

7.1.29. Lemma (Lindenbaum's Lemma). *Every* P*-witnessed theory* Γ *can be extended to a maximal* P*-witnessed theory* T_Γ*.*

Proof:
The proof proceeds by constructing an increasing chain $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots \subseteq \Gamma_n \subseteq \ldots$ of P-witnessed theories, where $\Gamma_0 := \Gamma$, and each Γ_i is recursively defined. Since we have to guarantee that each Γ_i is P-witnessed, we follow a two-fold construction, where $\Gamma_0 = \Gamma_0^+ := \Gamma$. Let $\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots$ be an enumeration of all pairs of the form $\gamma_i = (s_i, \varphi_i)$ consisting of any necessity form $s_i \in NF^{\mathrm{P}}$ and

any formula $\varphi_i \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$. Let (s_n, φ_n) be the nth pair in the enumeration. We then set

$$\Gamma_n^+ = \begin{cases} \Gamma_n \cup \{\langle s_n\rangle\varphi_n\} & \text{if } \Gamma_n \nvdash \neg\, \langle s_n\rangle\varphi_n \\ \Gamma_n & \text{otherwise} \end{cases}$$

Note that the empty string ϵ is in NF^{P}, and for every $\psi \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$ we have $\langle\epsilon\rangle\psi := \psi$ by the definition of possibility forms. Therefore, the above enumeration of pairs includes every formula ψ of $\mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$ in the form of its corresponding pair (ϵ, ψ). By Lemma 7.1.27, each Γ_n^+ is P-witnessed. Then, if φ_n is of the form $\varphi_n := \Diamond\theta$ for some $\theta \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$, there exists a $p \in \mathrm{P}$ such that Γ_n^+ is consistent with $\langle s_n\rangle\langle p\rangle\theta$ (since Γ_n^+ is P-witnessed). We then define

$$\Gamma_{n+1} = \begin{cases} \Gamma_n^+ & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\varphi_n \text{ and } \varphi_n \text{ is not of the form } \Diamond\theta \\ \Gamma_n^+ \cup \{\langle s_n\rangle\langle p\rangle\theta\} & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\varphi_n \text{ and } \varphi_n := \Diamond\theta \text{ for some } \theta{\in}\mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P}) \\ \Gamma_n & \text{otherwise} \end{cases}$$

where $p \in \mathrm{P}$ such that Γ_n^+ is consistent with $\langle s_n\rangle\langle p\rangle\theta$. Again by Lemma 7.1.27, it is guaranteed that each Γ_n is P-witnessed. Now consider the union $T_\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$. By Lemma 7.1.28, we know that T_Γ is a P-theory. To show that T_Γ is P-witnessed, let $s \in NF^{\mathrm{P}}$ and $\psi \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$ and suppose $\langle s\rangle\Diamond\psi$ is consistent with T_Γ. The pair $(s, \Diamond\psi)$ appears in the above enumeration of all pairs, thus $(s, \Diamond\psi) := (s_m, \varphi_m)$ for some $m \in \mathbb{N}$. Hence, $\langle s\rangle\Diamond\psi := \langle s_m\rangle\varphi_m$. Then, since $\langle s\rangle\Diamond\psi$ is consistent with T_Γ and $\Gamma_m \subseteq T_\Gamma$, we know that $\langle s\rangle\Diamond\psi$ is in particular consistent with Γ_m. Therefore, by the above construction, $\langle s\rangle\langle p\rangle\psi \in \Gamma_{m+1}$ for some $p \in \mathrm{P}$ such that Γ_m^+ is consistent with $\langle s\rangle\langle p\rangle\psi$. Thus, as T_Γ is consistent and $\Gamma_{m+1} \subseteq T_\Gamma$, we have that $\langle s\rangle\langle p\rangle\psi$ is also consistent with T_Γ. Hence, we conclude that T_Γ is P-witnessed. Finally, T_Γ is also maximal by construction: otherwise there would be a P-witness theory T such that $T_\Gamma \subsetneq T$. This implies that there exists $\varphi \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$ with $\varphi \in T$ but $\varphi \notin T_\Gamma$. Then, by the construction of T_Γ, we obtain $\Gamma_i \vdash \neg\varphi$ for all $i \in \mathbb{N}$. Therefore, since $T_\Gamma \subseteq T$, we have $T \vdash \neg\varphi$. Hence, since $\varphi \in T$, we conclude $T \vdash \bot$ (contradicting T being consistent). $\Box$

7.1.30. Lemma (Extension Lemma). *Let* P *be a countable set of propositional variables and* P′ *be a countable set of* fresh *propositional variables, i.e.,* $\mathrm{P}\cap\mathrm{P}' = \emptyset$. *Let* $\widetilde{\mathrm{P}} = \mathrm{P} \cup \mathrm{P}'$. *Then, every* P*-theory* Γ *can be extended to a* $\widetilde{\mathrm{P}}$*-witnessed theory* $\widetilde{\Gamma} \supseteq \Gamma$*, and hence to a maximal* $\widetilde{\mathrm{P}}$*-witnessed theory* $T_\Gamma \supseteq \Gamma$.

Proof:
Let $\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots$ an enumeration of all formulas of the form $\gamma_i := \langle s_i\rangle\Diamond\varphi_i$ consisting of any $s_i \in NF^{\widetilde{\mathrm{P}}}$, and every formula $\varphi_i \in \mathcal{L}^!_{K\mathsf{int}\Box}(\widetilde{\mathrm{P}})$ in the language. We will recursively construct a chain of $\widetilde{\mathrm{P}}$-theories $\Gamma_0 \subseteq \ldots \subseteq \Gamma_n \subseteq \ldots$ such that

1. $\Gamma_0 = \Gamma$,

2. $\mathrm{P}'_n := \{p \in \mathrm{P}' : p$ occurs in $\Gamma_n\}$ is finite for every $n \in \mathbb{N}$, and

3. for every $\gamma_n := \langle s_n\rangle\Diamond\varphi_n$ with $s_n \in NF^{\widetilde{\mathrm{P}}}$ and $\varphi_n \in \mathcal{L}^{!}_{K\mathsf{int}\Box}(\widetilde{\mathrm{P}})$, if $\Gamma_n \nvdash \neg\langle s_n\rangle\Diamond\varphi_n$ then there is p_m "fresh" such that $\langle s_n\rangle\langle p_m\rangle\varphi_n \in \Gamma_{n+1}$. Otherwise we will define $\Gamma_{n+1} = \Gamma_n$.

For every γ_n, let $\mathrm{P}'(n) := \{p \in \mathrm{P}' \mid p$ occurs either in s_n or $\varphi_n\}$. Clearly every $\mathrm{P}'(n)$ is always finite. We now construct an increasing chain of $\widetilde{\mathrm{P}}$-theories recursively. We set $\Gamma_0 := \Gamma$, and let

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\langle s_n\rangle\langle p_m\rangle\varphi_n\} & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\Diamond\varphi_n \\ \Gamma_n & \text{otherwise,} \end{cases}$$

where m is the least natural number greater than the indices in $\mathrm{P}'_n \cup \mathrm{P}'(n)$, i.e., p_m is fresh. We now show that $\widetilde{\Gamma} := \bigcup_{n\in\mathbb{N}} \Gamma_n$ is a $\widetilde{\mathrm{P}}$-witnessed theory. First show that $\widetilde{\Gamma}$ is a $\widetilde{\mathrm{P}}$-theory. By Lemma 7.1.28, it suffices to show by induction that every Γ_n is a $\widetilde{\mathrm{P}}$-theory. Clearly Γ_0 is a $\widetilde{\mathrm{P}}$-theory. For the inductive step suppose Γ_n is consistent but Γ_{n+1} is not. Hence, $\Gamma_n \neq \Gamma_{n+1}$ and moreover $\Gamma_{n+1} \vdash \bot$. Then, since $\Gamma_{n+1} = \Gamma_n \cup \{\langle s_n\rangle\langle p_m\rangle\varphi_n\}$, we have $\Gamma_n \vdash [s_n][p_m]\neg\varphi_n$. Therefore there exists $\{\theta_1, \ldots, \theta_k\} \subseteq \Gamma_n$ such that $\{\theta_1, \ldots, \theta_k\} \vdash [s_n][p_m]\neg\varphi_n$. Let $\theta = \bigwedge_{1\leq i\leq k} \theta_i$. Then $\vdash \theta \to [s_n][p_m]\neg\varphi_n$, so $\vdash [\theta\!\to, s_n][p_m]\neg\varphi_n$ with $p_m \notin \mathrm{P}_\theta \cup \mathrm{P}_{s_n} \cup \mathrm{P}_{\varphi_n}$. Thus, by the admissible rule in Lemma 7.1.24, we obtain $\vdash [\theta\!\to, s_n]\Box\neg\varphi_n$, i.e., $\vdash \theta \to [s_n]\Box\neg\varphi_n$. Therefore, $\theta \vdash \neg\langle s_n\rangle\Diamond\varphi_n$. Since $\{\theta_1, \ldots, \theta_k\} \subseteq \Gamma_n$, we therefore have $\Gamma_n \vdash \neg\langle s_n\rangle\Diamond\varphi_n$. But, this would mean $\Gamma_n = \Gamma_{n+1}$, contradicting our assumption. Therefore Γ_{n+1} is consistent and thus a $\widetilde{\mathrm{P}}$-theory. Hence, by Lemma 7.1.28, $\widetilde{\Gamma}$ is a $\widetilde{\mathrm{P}}$-theory. Condition (3) above implies that $\widetilde{\Gamma}$ is also $\widetilde{\mathrm{P}}$-witnessed. Then, by Lindenbaum's Lemma (Lemma 7.1.29), there is a maximal $\widetilde{\mathrm{P}}$-witnessed theory T_Γ such that $T_\Gamma \supseteq \widetilde{\Gamma} \supseteq \Gamma$. □

We are now ready to build the canonical pseudo-model. For a fixed countable set of propositional variables P, we let for any maximal P-witnessed theories T and S,

$$T \sim S \text{ iff } (\forall\varphi \in \mathcal{L}^{!}_{K\mathsf{int}\Box}(\mathrm{P}))(K\varphi \in T \Rightarrow \varphi \in S).$$

7.1.31. Definition. [Canonical Pseudo-Model for T_0] Let T_0 be a maximal P-witnessed theory. The *canonical pseudo-model for* T_0 is a tuple $\mathcal{X}^c = (X^c, \mathcal{O}^c, V^c)$ such that

- $X^c = \{T \subseteq \mathcal{L}^{!}_{K\mathsf{int}\Box}(\mathrm{P}) \mid T$ is a maximal P-witnessed theory with $T \sim T_0\}$,
- $\mathcal{O}^c = \{\widehat{\mathsf{int}(\varphi)} \mid \varphi \in \mathcal{L}^{!}_{K\mathsf{int}\Box}(\mathrm{P})\}$, where $\widehat{\theta} = \{T \in X^c \mid \theta \in T\}$ for any $\theta \in \mathcal{L}^{!}_{K\mathsf{int}\Box}(\mathrm{P})$,

- $V^c(p) = \{T \in X^c \mid p \in T\}$.

We let τ^c denote the topology generated by $\mathcal{O}^c$. The associated topo-model $\mathcal{X}^c_\tau = (X^c, \tau^c, V^c)$ is called the *canonical topo-model for* T_0.

In order to show that $\mathcal{X}^c = (X^c, \mathcal{O}^c, V^c)$ is indeed a pseudo-model, we need a Truth Lemma for the language $\mathcal{L}^!_{K\mathsf{int}\Box}$. We therefore postpone the proof of $\mathcal{X}^c$ being a pseudo-model until after the proof of the Truth Lemma (Lemma 7.1.35) for the completeness of $\mathsf{DTL}_{\mathsf{int}}$. For now, we show that $\mathcal{X}^c = (X^c, \mathcal{O}^c, V^c)$ is at least a pre-model, hence, it is well-defined for the language $\mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$.

7.1.32. LEMMA. *$\mathcal{X}^c = (X^c, \mathcal{O}^c, V^c)$ is a pre-model.*

Proof:
It is easy to see that $X^c, \emptyset \in \mathcal{O}^c$, since $\widehat{\mathsf{int}(\top)} = X^c$ and $\widehat{\mathsf{int}(\bot)} = \emptyset$. We need to show that $\mathcal{O}^c$ is closed under (1) finite intersections and (2) finite unions. (1) closure under finite intersection follows from the normality of int, namely from the fact that $\vdash \mathsf{int}(\varphi) \wedge \mathsf{int}(\psi) \leftrightarrow \mathsf{int}(\varphi \wedge \psi)$. (2) closure under finite union follows from the fact that $\vdash (\mathsf{int}(\varphi) \vee \mathsf{int}(\psi)) \leftrightarrow \mathsf{int}(\mathsf{int}(\varphi) \vee \mathsf{int}(\psi))$, and that $\mathsf{int}(\mathsf{int}(\varphi) \vee \mathsf{int}(\psi)) \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$. $\Box$

7.1.33. LEMMA. *For every maximal* P*-witnessed theory T, the set $\{\theta \mid K\theta \in T\}$ is a* P*-witnessed theory.*

Proof:
Observe that, by axiom (T_K), $\{\theta \mid K\theta \in T\} \subseteq T$. Therefore, as T is consistent, the set $\{\theta \mid K\theta \in T\}$ is consistent. Let $s \in NF^{\mathrm{P}}$ and $\psi \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$ such that $\{\theta \mid K\theta \in T\} \vdash [s][p]\neg\varphi$ for all $p \in \mathrm{P}$. Then, by normality of K, $T \vdash K[s][p]\neg\varphi$ for all $p \in \mathrm{P}$. Since $K[s][p]\neg\varphi := [K, s][p]\neg\varphi$ is a necessity form and T is P-witnessed, we obtain $T \vdash [K, s]\Box\neg\varphi$, i.e., $T \vdash K[s]\Box\neg\varphi$. As T is maximal, we have $K[s]\Box\neg\varphi \in T$, thus $[s]\Box\neg\varphi \in \{\theta \mid K\theta \in T\}$. $\Box$

7.1.34. LEMMA (EXISTENCE LEMMA). *Let $T \in X^c$ and $\varphi, \alpha \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$ such that $\mathsf{int}(\alpha) \in T$ and $K[\alpha]\varphi \notin T$. Then, there is $S \in X^c$ with $\mathsf{int}(\alpha) \in S$ and $[\alpha]\varphi \notin S$.*

Proof:
Let $\varphi, \alpha \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$ such that $\mathsf{int}(\alpha) \in T$ and $K[\alpha]\varphi \notin T$. The latter implies that $\{\psi \mid K\psi \in T\} \nvdash [\alpha]\varphi$, hence, $\{\psi \mid K\psi \in T\} \nvdash \neg\neg[\alpha]\varphi$. Then, by Lemma 7.1.33 and Lemma 7.1.27, we obtain that $\{\psi \mid K\psi \in T\} \cup \{\neg[\alpha]\varphi\}$ is a P-witnessed theory. Note that $\vdash \neg[\alpha]\varphi \leftrightarrow (\mathsf{int}(\alpha) \wedge [\alpha]\neg\varphi)$ (see Proposition 7.1.2-($\langle ! \rangle$)). We therefore obtain that $\{\psi \mid K\psi \in T\} \cup \{\neg[\alpha]\varphi\} \vdash \mathsf{int}(\alpha)$, thus,

$\{\psi \mid K\psi \in T\} \cup \{\neg[\alpha]\varphi\} \nvdash \neg\mathsf{int}(\alpha)$ (since $\{\psi \mid K\psi \in T\} \cup \{\neg[\alpha]\varphi\}$ is consistent). Therefore, by Lemma 7.1.27, $\{\psi \mid K\psi \in T\} \cup \{\neg[\alpha]\varphi\} \cup \{\mathsf{int}(\alpha)\}$ is also a P-witnessed theory. We can then apply Lindenbaum's Lemma (Lemma 7.1.29) and extend it to a maximal P-witnessed theory S such that $\mathsf{int}(\alpha) \in S$ and $[\alpha]\varphi \notin S$. □

7.1.35. Lemma (Truth Lemma). *Let $\mathcal{X}^c = (X^c, \mathcal{O}^c, V^c)$ be the canonical pseudo-model for a maximal* P*-witnessed theory T_0 and $\varphi \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$. Then, for all $\alpha \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$ we have*

$$[\![\varphi]\!]^{\widehat{\mathsf{int}(\alpha)}} = \widehat{\langle\alpha\rangle\varphi}.$$

Proof:
The proof follows by $<^S_d$-induction on φ (the well-founded partial order $<^S_d$ on $\mathcal{L}^!_{K\mathsf{int}\Box}$ is defined in Appendix A.1).

Base case $\varphi := p$

$$\begin{aligned}
[\![p]\!]^{\widehat{\mathsf{int}(\alpha)}} &= \widehat{\mathsf{int}(\alpha)} \cap [\![p]\!]^{X^c} && \text{(since } p \text{ is bi-persistent)}\\
&= \widehat{\mathsf{int}(\alpha)} \cap V^c(p) && \text{(by the semantics of p)}\\
&= \widehat{\mathsf{int}(\alpha)} \cap \widehat{p} && \text{(by the definition of } V^c)\\
&= \widehat{\mathsf{int}(\alpha) \wedge p} && \\
&= \widehat{\mathsf{int}(\alpha) \wedge (\mathsf{int}(\alpha) \to p)} && \text{(by propositional tautologies)}\\
&= \widehat{\mathsf{int}(\alpha) \wedge [\alpha]p} && \text{(by } (\mathrm{R}_p))\\
&= \widehat{\langle\alpha\rangle p} && \text{(Proposition 7.1.2-}(\langle!\rangle))
\end{aligned}$$

Induction Hypothesis: For $\psi <^S_d \varphi$, we have $[\![\psi]\!]^{\widehat{\mathsf{int}(\alpha)}} = \widehat{\langle\alpha\rangle\psi}$ for all $\alpha \in \mathcal{L}^!_{K\mathsf{int}}(\mathrm{P})$.

Case $\varphi := \neg\psi$

$$\begin{aligned}
[\![\neg\psi]\!]^{\widehat{\mathsf{int}(\alpha)}} &= \widehat{\mathsf{int}(\alpha)} \backslash [\![\psi]\!]^{\widehat{\mathsf{int}(\alpha)}} && \text{(by the semantics of } \neg)\\
&= \widehat{\mathsf{int}(\alpha)} \backslash \widehat{\langle\alpha\rangle\psi} && \text{(by IH)}\\
&= \widehat{\mathsf{int}(\alpha)} \cap (X^c \backslash \widehat{\langle\alpha\rangle\psi}) && \\
&= \widehat{\mathsf{int}(\alpha)} \cap \widehat{\neg\langle\alpha\rangle\psi} && \text{(since } X^c \backslash \widehat{\langle\alpha\rangle\psi} = \widehat{\neg\langle\alpha\rangle\psi})\\
&= \widehat{\mathsf{int}(\alpha) \wedge \neg\langle\alpha\rangle\psi} && \\
&= \widehat{\mathsf{int}(\alpha) \wedge [\alpha]\neg\psi} && \\
&= \widehat{\langle\alpha\rangle\neg\psi} && \text{(Proposition 7.1.2-}(\langle!\rangle))
\end{aligned}$$

Case $\varphi := \psi \wedge \chi$

$$\begin{aligned}
[\![\psi \wedge \chi]\!]^{\widehat{\mathsf{int}(\alpha)}} &= [\![\psi]\!]^{\widehat{\mathsf{int}(\alpha)}} \cap [\![\chi]\!]^{\widehat{\mathsf{int}(\alpha)}} && \text{(by the semantics of } \wedge\text{)} \\
&= \widehat{\langle\alpha\rangle\psi} \cap \widehat{\langle\alpha\rangle\chi} && \text{(by IH)} \\
&= \widehat{\langle\alpha\rangle\psi \wedge \langle\alpha\rangle\chi} && \text{(by propositional tautologies)} \\
&= \widehat{\langle\alpha\rangle(\psi \wedge \chi)} && (\vdash (\langle\alpha\rangle\psi \wedge \langle\alpha\rangle\chi) \leftrightarrow \langle\alpha\rangle(\psi \wedge \chi))
\end{aligned}$$

Case $\varphi := K\psi$

($\Rightarrow$) Suppose $T \in [\![K\psi]\!]^{\widehat{\mathsf{int}(\alpha)}}$. This implies, by the semantic clause of K, that $T \in \widehat{\mathsf{int}(\alpha)}$ and $[\![\psi]\!]^{\widehat{\mathsf{int}(\alpha)}} = \widehat{\mathsf{int}(\alpha)}$. We want to show that $T \in \widehat{\langle\alpha\rangle K\psi}$. By Proposition 7.1.2-($\langle ! \rangle$) and the reduction axiom (R_K), we obtain $\vdash \langle\alpha\rangle K\psi \leftrightarrow \mathsf{int}(\alpha) \wedge K[\alpha]\psi$. We therefore only need to show that $T \in \widehat{\mathsf{int}(\alpha)}$ and $T \in \widehat{K[\alpha]\psi}$. We have the former by the assumption. Suppose toward contradiction that $T \notin \widehat{K[\alpha]\psi}$, i.e., $K[\alpha]\psi \notin T$. Then, by Lemma 7.1.34, there exists $S \in X^c$ such that $\mathsf{int}(\alpha) \in S$ and $[\alpha]\psi \notin S$. Since $\vdash \langle\alpha\rangle\psi \to [\alpha]\psi$, we obtain $\langle\alpha\rangle\psi \notin S$. Therefore, by IH, we have $S \notin [\![\psi]\!]^{\widehat{\mathsf{int}(\alpha)}}$. Since $S \in \widehat{\mathsf{int}(\alpha)}$, we then conclude $[\![\psi]\!]^{\widehat{\mathsf{int}(\alpha)}} \neq \widehat{\mathsf{int}(\alpha)}$. By the semantics of K, this means that $[\![K\psi]\!]^{\widehat{\mathsf{int}(\alpha)}} = \emptyset$, contradicting our first assumption. Hence, $T \in \widehat{\mathsf{int}(\alpha) \wedge K[\alpha]\psi} = \widehat{\langle\alpha\rangle K\psi}$.

($\Leftarrow$) Suppose $T \in \widehat{\langle\alpha\rangle K\psi}$. Then, by the equality $\langle\alpha\rangle K\psi \leftrightarrow \mathsf{int}(\alpha) \wedge K[\alpha]\psi$, we have $T \in \widehat{\mathsf{int}(\alpha)}$ and $T \in \widehat{K[\alpha]\psi}$. Let $S \in \widehat{\mathsf{int}(\alpha)}$. Since $S \sim T$ and $T \in \widehat{K[\alpha]\psi}$, we also have $[\alpha]\psi \in S$. Therefore, by Proposition 7.1.2-($\langle ! \rangle$), we obtain $\langle\alpha\rangle\psi \in S$. This implies, by IH, that $S \in [\![\psi]\!]^{\widehat{\mathsf{int}(\alpha)}}$. As this holds for all $S \in \widehat{\mathsf{int}(\alpha)}$, we have $[\![\psi]\!]^{\widehat{\mathsf{int}(\alpha)}} = \widehat{\mathsf{int}(\alpha)}$. Hence, $[\![K\psi]\!]^{\widehat{\mathsf{int}(\alpha)}} = \widehat{\mathsf{int}(\alpha)} \ni T$.

Case $\varphi := \mathsf{int}(\psi)$

($\Rightarrow$) Suppose $T \in [\![\mathsf{int}(\psi)]\!]^{\widehat{\mathsf{int}(\alpha)}}$. Then, by the semantics of int, there exists $U \in \mathcal{O}^c$ such that $T \in U \subseteq [\![\psi]\!]^{\widehat{\mathsf{int}(\alpha)}}$ (since $\mathcal{O}^c$ constitutes a basis for τ^c). Then, by IH, we have $U \subseteq \widehat{\langle\alpha\rangle\psi}$. By the construction of $\mathcal{O}^c$, we know that $U = \widehat{\mathsf{int}(\gamma)}$ for some $\gamma \in \mathcal{L}^{!}_{K\mathsf{int}\Box}(\mathrm{P})$. We therefore obtain that

$$T \in \widehat{\mathsf{int}(\gamma)} \subseteq \widehat{\langle\alpha\rangle\psi}.$$

This means that, for all $S \in \widehat{\mathsf{int}(\gamma)}$, we have $S \in \widehat{\langle\alpha\rangle\psi}$. Therefore, the set $\{\theta \in \mathcal{L}^{!}_{K\mathsf{int}\Box}(\mathrm{P}) \mid K\theta \in T\} \cup \{\neg(\mathsf{int}(\gamma) \to \langle\alpha\rangle\psi)\}$ is inconsistent. Otherwise, by Lemma 7.1.29, it could be extended to a maximally consistent P-witnessed theory T' such that $T \sim T'$, $\mathsf{int}(\gamma) \in T'$ and $\langle\alpha\rangle\psi \notin T'$, a contradiction. Then, there exists a formula $\chi \in \{\theta \in \mathcal{L}^{!}_{K\mathsf{int}\Box}(\mathrm{P}) \mid K\theta \in T\}$ such that $\vdash \chi \to (\mathsf{int}(\gamma) \to \langle\alpha\rangle\psi)$. Thus, by the normality of K, we have $\vdash K\chi \to K(\mathsf{int}(\gamma) \to \langle\alpha\rangle\psi)$. As $K\chi \in T$, we obtain $K(\mathsf{int}(\gamma) \to \langle\alpha\rangle\psi) \in T$. Then by axiom ($K$-$\mathsf{int}$), we have $\mathsf{int}(\mathsf{int}(\gamma) \to$

$\langle\alpha\rangle\psi) \in T$. Since int is an $\mathsf{S4}$ modality, we get $\mathsf{int}(\gamma) \to \mathsf{int}(\langle\alpha\rangle\psi) \in T$. Since $T \in \widehat{\mathsf{int}(\gamma)}$, this implies $\mathsf{int}(\langle\alpha\rangle\psi) \in T$. Moreover, we have

1. $\vdash \mathsf{int}(\langle\alpha\rangle\psi) \leftrightarrow \mathsf{int}(\mathsf{int}(\alpha) \wedge [\alpha]\psi)$ (Proposition 7.1.2-($\langle !\rangle$), (RE))
2. $\vdash \mathsf{int}(\mathsf{int}(\alpha) \wedge [\alpha]\psi) \leftrightarrow (\mathsf{int}(\alpha) \wedge \mathsf{int}([\alpha]\psi))$ ($\mathsf{S4}_{\mathsf{int}}$)
3. $\vdash (\mathsf{int}(\alpha) \wedge \mathsf{int}([\alpha]\psi)) \leftrightarrow (\mathsf{int}(\alpha) \wedge (\mathsf{int}(\alpha) \to [\alpha]\mathsf{int}(\psi)))$ (Proposition 7.1.2-($\mathrm{R}_{\mathsf{int}}$))
4. $\vdash (\mathsf{int}(\alpha) \wedge (\mathsf{int}(\alpha) \to [\alpha]\mathsf{int}(\psi))) \leftrightarrow (\mathsf{int}(\alpha) \wedge [\alpha]\mathsf{int}(\psi)))$
5. $\vdash (\mathsf{int}(\alpha) \wedge [\alpha]\mathsf{int}(\psi))) \leftrightarrow \langle\alpha\rangle\mathsf{int}(\psi)$ (Proposition 7.1.2-($\langle !\rangle$))
6. $\vdash \mathsf{int}(\langle\alpha\rangle\psi) \leftrightarrow \langle\alpha\rangle\mathsf{int}(\psi)$ (1-5, CPL)

Therefore, as T is maximal, we obtain $\langle\alpha\rangle\mathsf{int}(\psi) \in T$, i.e., $T \in \widehat{\langle\alpha\rangle\mathsf{int}(\psi)}$.

($\Leftarrow$) Suppose $T \in \widehat{\langle\alpha\rangle\mathsf{int}(\psi)}$. This implies, by the above derivation, that $T \in \widehat{\mathsf{int}(\langle\alpha\rangle\psi)}$. By the constraction of $\mathcal{O}^c$, we have $\widehat{\mathsf{int}(\langle\alpha\rangle\psi)} \in \mathcal{O}^c$. Moreover, by the axiom ($\mathrm{T}_{\mathsf{int}}$), we obtain $\widehat{\mathsf{int}(\langle\alpha\rangle\psi)} \subseteq \widehat{\langle\alpha\rangle\psi}$. By IH, we also have that $\widehat{\langle\alpha\rangle\psi} = [\![\psi]\!]^{\widehat{\mathsf{int}(\alpha)}}$. Therefore $T \in \widehat{\mathsf{int}(\langle\alpha\rangle\psi)} \subseteq \widehat{\langle\alpha\rangle\psi} = [\![\psi]\!]^{\widehat{\mathsf{int}(\alpha)}}$, i.e., $T \in Int([\![\psi]\!]^{\widehat{\mathsf{int}(\alpha)}}) = [\![\mathsf{int}(\psi)]\!]^{\widehat{\mathsf{int}(\alpha)}}$.

Case $\varphi := \langle\chi\rangle\psi$

$$\begin{aligned}
[\![\langle\chi\rangle\psi]\!]^{\widehat{\mathsf{int}(\alpha)}} &= \{T \in \widehat{\mathsf{int}(\alpha)} \mid (T, Int([\![\chi]\!]^{\widehat{\mathsf{int}(\alpha)}})) \models \psi\} \\
&= \{T \in \widehat{\mathsf{int}(\alpha)} \mid (T, [\![\mathsf{int}(\chi)]\!]^{\widehat{\mathsf{int}(\alpha)}}) \models \psi\} && \text{(by the semantics of } \mathsf{int}) \\
&= \{T \in \widehat{\mathsf{int}(\alpha)} \mid (T, \widehat{\langle\alpha\rangle\mathsf{int}(\chi)} \models \psi\} && \text{(by IH, since } \mathsf{int}(\chi) <_d^S \langle\chi\rangle\psi) \\
&= [\![\psi]\!]^{\widehat{\langle\alpha\rangle\mathsf{int}(\chi)}} && \text{(since } \widehat{\langle\alpha\rangle\mathsf{int}(\chi)} \subseteq \widehat{\mathsf{int}(\alpha)}) \\
&= \widehat{\langle\langle\alpha\rangle\mathsf{int}(\chi)\rangle\psi} && \text{(by IH, since } \psi <_d^S \langle\chi\rangle\psi) \\
&= \widehat{\langle\alpha\rangle\langle\chi\rangle\psi} && (\vdash \langle\alpha\rangle\langle\chi\rangle\psi \leftrightarrow \langle\langle\alpha\rangle\mathsf{int}(\chi)\rangle\psi)
\end{aligned}$$

Note that $\vdash \langle\alpha\rangle\langle\chi\rangle\psi \leftrightarrow \langle\langle\alpha\rangle\mathsf{int}(\chi)\rangle\psi$ follows from ($\mathrm{R}_{[!]}$) and ($\mathrm{R}[\mathsf{int}]$)).

Case $\varphi := \Box\psi$

($\Rightarrow$) Suppose $T \in [\![\Box\psi]\!]^{\widehat{\mathsf{int}(\alpha)}}$, i.e., $(T, \widehat{\mathsf{int}(\alpha)}) \models \Box\psi$. This means that for all $U \in \mathcal{O}$ with $T \in U \subseteq \widehat{\mathsf{int}(\alpha)}$, we have $(T, U) \models \psi$. This in particular implies that $(T, \widehat{\mathsf{int}(\alpha)}) \models [p]\psi$ for all $p \in \mathrm{P}$. To show, let $p \in \mathrm{P}$ and suppose $(T, \widehat{\mathsf{int}(\alpha)}) \models \mathsf{int}(p)$, i.e., $T \in Int([\![p]\!]^{\widehat{\mathsf{int}(\alpha)}}) = [\![\mathsf{int}(p)]\!]^{\widehat{\mathsf{int}(\alpha)}}$. Since $\mathsf{int}(p) <_d^S \Box\psi$ (see Lemma A.1.5-(2,4)), we know by IH that $[\![\mathsf{int}(p)]\!]^{\widehat{\mathsf{int}(\alpha)}} = \widehat{\langle\alpha\rangle\mathsf{int}(p)}$. But, as shown in the case for the modality int above, $\vdash \langle\alpha\rangle\mathsf{int}(p) \leftrightarrow \mathsf{int}(\langle\alpha\rangle p)$, hence, $[\![\mathsf{int}(p)]\!]^{\widehat{\mathsf{int}(\alpha)}} = \widehat{\mathsf{int}(\langle\alpha\rangle p)}$, thus, $[\![\mathsf{int}(p)]\!]^{\widehat{\mathsf{int}(\alpha)}} \in \mathcal{O}^c$. Hence, by the first assumption, we obtain $(T, Int([\![p]\!]^{\widehat{\mathsf{int}(\alpha)}})) \models \psi$, thus, $(T, \widehat{\mathsf{int}(\alpha)}) \models [p]\psi$. Therefore, $T \in [\![[p]\psi]\!]^{\widehat{\mathsf{int}(\alpha)}}$ for all $p \in \mathrm{P}$. Then, by IH (since $[p]\psi <_d^S \Box\psi$), we have $[\![[p]\psi]\!]^{\widehat{\mathsf{int}(\alpha)}} = \widehat{\langle\alpha\rangle[p]\psi}$, thus, $\langle\alpha\rangle[p]\psi \in T$. Hence, by Proposition 7.1.2-($\langle !\rangle$), $\mathsf{int}(\alpha) \wedge [\alpha][p]\psi \in T$ for all $p \in \mathrm{P}$. Since T is P-witnessed and maximal, we then obtain $\mathsf{int}(\alpha) \wedge [\alpha]\Box\psi \in T$. Then, by Proposition 7.1.2-($\langle !\rangle$), we conclude $\langle\alpha\rangle\Box\psi \in T$.

($\Leftarrow$) Suppose $T \in \widehat{\langle\alpha\rangle\Box\psi}$. This means (by Proposition 7.1.2-($\langle !\rangle$)) that $T \in \widehat{\mathsf{int}(\alpha) \wedge [\alpha]\Box\psi}$, i.e., that $\mathsf{int}(\alpha) \in T$ and $[\alpha]\Box\psi \in T$. Then, by axiom ($[!]\Box$-elim), we have that $[\alpha \wedge \chi]\psi \in T$ for all $\chi \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$. We want to show that $T \in [\![\Box\psi]\!]^{\widehat{\mathsf{int}(\alpha)}}$. Let $U \in \mathcal{O}^c$ such that $T \in U \subseteq \widehat{\mathsf{int}(\alpha)}$ and show $T \in [\![\psi]\!]^U$. By the construction of $\mathcal{O}^c$, we know that $U = \widehat{\mathsf{int}(\gamma)}$ for some $\gamma \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$. We therefore have that $T \in U = \widehat{\mathsf{int}(\gamma)} = \widehat{\mathsf{int}(\gamma)} \cap \widehat{\mathsf{int}(\alpha)} = \widehat{\mathsf{int}(\gamma) \wedge \mathsf{int}(\alpha)} = \widehat{\mathsf{int}(\gamma \wedge \alpha)}$. Hence, $\mathsf{int}(\alpha \wedge \gamma) \wedge [\alpha \wedge \gamma]\psi \in T$. Therefore, by Proposition 7.1.2-($\langle !\rangle$) and the fact that T is maximal, we obtain $\langle\alpha \wedge \gamma\rangle\psi \in T$. Thus, by IH (since $\psi <_d^S \Box\psi$), $T \in [\![\psi]\!]^{\widehat{\mathsf{int}(\alpha \wedge \gamma)}}$, i.e., $T \in [\![\psi]\!]^U$. $\Box$

7.1.36. Lemma. *$\mathcal{X}^c = (X^c, \mathcal{O}^c, V^c)$ is a pseudo-model.*

Proof:
Theorem 7.1.32 shows that $\mathcal{X}^c = (X^c, \mathcal{O}^c, V^c)$ is a pre-model. In order to show that it is indeed a pseudo-model, let $\varphi \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$ and $U \in \mathcal{O}^c$. We should show that $[\![\mathsf{int}(\varphi)]\!]^U \in \mathcal{O}^c$, i.e., that $[\![\mathsf{int}(\varphi)]\!]^U = \widehat{\mathsf{int}(\psi)}$ for some $\psi \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$. By the construction of $\mathcal{O}^c$, we know that $U = \widehat{\mathsf{int}(\gamma)}$ for some $\gamma \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$. By the Truth Lemma (Lemma 7.1.35), we have $[\![\mathsf{int}(\varphi)]\!]^{\widehat{\mathsf{int}(\gamma)}} = \widehat{\langle\gamma\rangle\mathsf{int}(\varphi)}$. As argued in the case for the modality int in the Truth Lemma, $\widehat{\langle\gamma\rangle\mathsf{int}(\varphi)} = \widehat{\mathsf{int}(\langle\gamma\rangle\varphi)}$. Therefore, we conclude that $[\![\mathsf{int}(\varphi)]\!]^U = [\![\mathsf{int}(\varphi)]\!]^{\widehat{\mathsf{int}(\gamma)}} = \widehat{\mathsf{int}(\langle\gamma\rangle\varphi)}$ for $\mathsf{int}(\langle\gamma\rangle\varphi) \in \mathcal{L}^!_{K\mathsf{int}\Box}(\mathrm{P})$. Hence, $\mathcal{X}^c = (X^c, \mathcal{O}^c, V^c)$ is a pseudo-model. $\Box$

The next lemma shows that the language $\mathcal{L}_{K\mathsf{int}}$ cannot distinguish a pseudo-model from its associated topo-model.

7.1.37. Lemma. *Let $\mathcal{X} = (X, \mathcal{O}, V)$ be a pseudo-model and $\mathcal{X}_\tau = (X, \tau_{\mathcal{O}}, V)$ be the associated topo-model. Then, for all $\varphi \in \mathcal{L}_{K\mathsf{int}}$ and $(x, U) \in ES(\mathcal{X})$, we have*

$$\mathcal{X}, (x, U) \models \varphi \textit{ iff } \mathcal{X}_\tau, (x, U) \models \varphi.$$

Proof:
The proof goes by subformula induction on φ and it is straightforward. We only show the case for $\varphi := \mathsf{int}(\psi)$. Note that if $U \in \mathcal{O}$ then $U \in \tau_{\mathcal{O}}$ (but not the other way around).

$$\begin{array}{rll} \mathcal{X}, (x, U) \models \mathsf{int}(\psi) & \text{iff } x \in Int([\![\varphi]\!]^U_{\mathcal{X}}) & \text{(where } Int \text{ is the interior operator of } \mathcal{X}_\tau) \\ & \text{iff } x \in Int([\![\varphi]\!]^U_{\mathcal{X}_\tau}) & \text{(by IH: } [\![\varphi]\!]^U_{\mathcal{X}_\tau} = [\![\varphi]\!]^U_{\mathcal{X}}) \\ & \text{iff } \mathcal{X}_\tau, (x, U) \models \mathsf{int}(\psi) & \end{array}$$

□

7.1.38. Corollary. $\mathsf{DTL}_{\mathsf{int}}$ *is complete for the canonical pseudo-models and canonical topo-models (and so also complete with respect to the class of all pseudo-models, as well as the class of all topo-models).*

Proof:
Let φ be an $\mathsf{DTL}_{\mathsf{int}}$-consistent formula, i.e., it is a P_φ-theory. Then, by Lemma 7.1.30, it can be extended to a maximal PROP-witnessed theory T. Let $\mathcal{X}^c = (X^c, \mathcal{O}^c, V^c)$ denote the canonical pseudo-model for T. Since $\varphi \in T$, by axiom (R[$\top$]), we obtain $\langle\top\rangle\varphi \in T$, i.e., $T \in \widehat{\langle\top\rangle\varphi}$. Thus, by Truth Lemma (Lemma 7.1.35), we have that $T \in [\![\varphi]\!]_{\mathcal{X}^c}^{\widehat{\mathsf{int}(\top)}}$, i.e., that $\mathcal{X}^c, (T, X^c) \models \varphi$ (since $\widehat{\mathsf{int}(\top)} = X^c$). This proves the first completeness claim. As for the second, by the co-expressivity of $\mathcal{L}_{K\mathsf{int}}$ and $\mathcal{L}^!_{K\mathsf{int}\Box}$ on pseudo-models (Corollary 7.1.18), there exists a $\psi \in \mathcal{L}_{K\mathsf{int}}$ such that $\varphi \leftrightarrow \psi$ is valid in all pseudo-models. We therefore have $\mathcal{X}^c, (T, X^c) \models \psi$. By Lemma 7.1.37, we obtain $\mathcal{X}^c_\tau, (T, X^c) \models \psi$ where $\mathcal{X}^c_\tau$ is the canonical topo-model. Using again the semantic equivalence of φ and ψ (applied to the model $\mathcal{X}^c_\tau$), we conclude that $\mathcal{X}^c_\tau, (T, X^c) \models \varphi$. □

This result concludes the present section. In the next section, we present a topological semantics for the so-called arbitrary announcement modality introduced by Balbiani et al. (2008), and investigate its link to the effort modality of Moss and Parikh (1992).

7.2 Topological Arbitrary Announcement Logic

Balbiani et al. (2008) proposed an extension of public announcement logic with a dynamic operator that quantifies over public announcements and expresses *what becomes true after any announcement.* More precisely, they consider the language $\mathcal{L}^!_{K\circledast}$ (in its single-agent version here)

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid [\varphi]\varphi \mid \circledast\varphi,$$

where the construct $[\varphi]\psi$ stands for the standard public announcement modality stating ‘after public announcement of φ, ψ (is true)’, and $\circledast\varphi$ represents the arbitrary (public) announcement modality which is read as “after any announcement, φ is true”. Balbiani et al. (2008) studied this modality on Kripke models with equivalence relations by using the standard semantics for public announcements in terms of model restrictions. More precisely, given a reflexive, transitive and symmetric Kripke model $\mathcal{M} = (X, R, V)$ and $x \in X$, Balbiani et al. (2008)

propose to interpret the modality $\boxast\varphi$ as

$$\begin{aligned}\mathcal{M}, x \models \boxast\varphi \text{ iff } & (\forall\psi \in \mathcal{L}^!_K)(\mathcal{M}, x \models [\psi]\varphi)\\ \text{iff } & (\forall\psi \in \mathcal{L}^!_K)(\mathcal{M}, x \models \psi \text{ implies } \mathcal{M}^\psi, x \models \varphi)\end{aligned}$$

where $\mathcal{M}^\psi = (\|\psi\|, R^\psi, V^\psi)$ is the restriction of $\mathcal{M}$ to the truth set of ψ in $\mathcal{M}$.[2] Unlike the effort modality $\Box\varphi$ which is read as "φ stays true no matter what further evidence-gathering efforts are made", the arbitrary announcement modality $\boxast\varphi$ means "φ stays true after *any epistemic announcement*". The latter therefore quantifies only over *epistemically definable* subsets ($\boxast$-free formulas of the language) of a given model.[3]

In this case, for example, $\diamondast K\varphi$ means that the agent comes to know φ, but in the interpretation that there is a $\boxast$-free formula ψ such that after announcing it the agent knows φ. What becomes true or known by an agent after an announcement can be expressed in this language without explicit reference to the announced formula. Clearly, the meaning of the effort modality $\Box\varphi$ and of the arbitrary announcement modality $\boxast\varphi$ are related in motivation, and their readings suggest that while $\boxast\varphi$ generalizes $[\psi]\varphi$, the effort modality $\Box\varphi$ seems more general than $\boxast\varphi$. However, we cannot yet see the precise connection between these modalities at the formal level as they have been studied on different semantic structures. In this section, we aim to explore the link between the Bjorndahl-style topological updates, the effort modality, and a topological version of the arbitrary announcement modality. To this end, based on (van Ditmarsch et al., 2014), we extend the language $\mathcal{L}^!_{K\mathsf{int}}$ by the arbitrary announcement operator $\boxast\varphi$ and propose a topological semantics for this modality by interpreting it as a quantification over Bjorndahl-style updates on topological spaces. We then show not only that $\mathcal{L}^!_{K\mathsf{int}\boxast}$ and $\mathcal{L}^!_{K\mathsf{int}\Box}$ are co-expressive for topo-models, but also that—quite surprisingly—*the effort modality* $\Box$ *and the topological arbitrary announcement modality* $\boxast$ *are equivalent* in the single-agent setting.

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[2]To recall, $\|\psi\| = \|\psi\|^\mathcal{M}$, $R^\psi = R \cap \|\psi\| \times \|\psi\|$, and $V^\psi(p) = V(p) \cap \|\psi\|$ for all $p \in$ PROP.

[3]To be more precise, by an "epistemically definable subset" of a model $\mathcal{M} = (X, R, V)$, we mean a subset of X that corresponds to a truth set of a formula $\psi \in \mathcal{L}^!_K$ in $\mathcal{M}$. Since the languages $\mathcal{L}_K$ and $\mathcal{L}^!_K$ are equally expressive with respect to Kripke models with equivalence relations (Plaza, 1989), quantifying over the formulas of $\mathcal{L}^!_K$ or the formulas of $\mathcal{L}_K$ in the semantic clause for $\boxast\varphi$ amounts to the same interpretation. Moreover, the reason as to why the arbitrary announcement modality quantifies only over the formulas *without* $\boxast$ is to avoid a possible circularity. Otherwise, if $\boxast\varphi$ were an announcement that plays a role in the evaluation of $\boxast\varphi$, checking the truth of $\boxast\varphi$ would require checking its truth (see Balbiani et al., 2008, Section 2.3.1 for a more detailed discussion on the semantics of $\boxast\varphi$). Van Ditmarsch et al. (2016) present an arbitrary announcement logic, called *fully arbitrary public announcement logic*, that allows $\boxast\varphi$ to quantify over formulas having arbitrary announcement operators, yet does not encounter the above mentioned circularity. This logic is defined based on a language with a proper class of auxiliary arbitrary announcement operators indexed by ordinals.

Syntax and Semantics. We consider the language $\mathcal{L}^!_{K\mathsf{int}\circledast}$ obtained by extending $\mathcal{L}^!_{K\mathsf{int}}$ with the arbitrary announcement modality $\circledast$. In other words, $\mathcal{L}^!_{K\mathsf{int}\circledast}$ is defined by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \mathsf{int}(\varphi) \mid [\varphi]\varphi \mid \circledast\varphi$$

where $p \in \textsc{prop}$. We sometimes call the formulas in $\mathcal{L}^!_{K\mathsf{int}}$ $\circledast$*-free formulas.*

Given a topo-model $\mathcal{X} = (X, \tau, V)$ and an epistemic scenario $(x, U) \in ES(\mathcal{X})$, truth of a formula in $\mathcal{L}^!_{K\mathsf{int}\circledast}$ is defined for Boolean cases, and the modalities K, int and $[!]$ as for $\mathcal{L}^!_{K\mathsf{int}}$ in Definition 6.2.1. For the modality $\circledast$, we propose the following semantic clause.

7.2.1. Definition. [Semantics of arbitrary announcement] Given a topo-model $\mathcal{X} = (X, \tau, V)$ and an epistemic scenario $(x, U) \in ES(\mathcal{X})$, the semantic clause for the arbitrary announcement modality $\circledast$ reads

$$\mathcal{X}, (x, U) \models \circledast\varphi \text{ iff } (\forall\psi \in \mathcal{L}^!_{K\mathsf{int}})(\mathcal{X}, (x, U) \models [\psi]\varphi).$$

In other words, unravelling the above semantic clause, we model $\circledast\varphi$ as

$$(x, U) \models \circledast\varphi \text{ iff } (\forall\psi \in \mathcal{L}^!_{K\mathsf{int}})((x, U) \models \mathsf{int}(\psi) \text{ implies } (x, Int([\![\psi]\!]^U)) \models \varphi)$$

We therefore work with a topological version of the arbitrary announcement modality in the sense that it quantifies over Bjorndahl's public announcements whose pre-condition is captured by the interior modality, and whose effect is modelled in terms of neighbourhood shrinking.

Expressivity of $\mathcal{L}^!_{K\mathsf{int}\circledast}$ on topo-models

We will now prove that $\mathcal{L}^!_{K\mathsf{int}\circledast}$ and $\mathcal{L}_{K\mathsf{int}}$ are equally expressive with respect to topo-models in the single-agent case (this will not be the case for the multi-agent version we present in Chapter 8). The proof of this result follows similar to the proof of Theorems 7.1.17 and 7.1.18. Thus, we first provide similar auxiliary lemmas for the language $\mathcal{L}^!_{K\mathsf{int}\circledast}$.

7.2.2. Proposition. *The rule of Replacement of Provable Equivalents* (*RE*) *is sound for $\mathcal{L}^!_{K\mathsf{int}\circledast}$ with respect to topo-models. More precisely, for all $\varphi, \psi, \chi \in \mathcal{L}^!_{K\mathsf{int}\circledast}$, if $\psi \leftrightarrow \chi$ is valid in all topo-models then so is $\varphi \leftrightarrow \varphi\{\psi/\chi\}$.*

Proof:
Let $\varphi, \psi, \chi \in \mathcal{L}^!_{K\mathsf{int}\circledast}$ and suppose $\models \psi \leftrightarrow \chi$. We want to show that $\models \varphi \leftrightarrow \varphi\{\psi/\varphi\}$, and the proof follows by subformula induction on φ, where the base case is $\varphi := \psi$. Let $\mathcal{X} = (X, \tau, V)$ be a topo-model and $(x, U) \in ES(\mathcal{X})$. For the base case $\varphi := \psi$, we then have $\varphi\{\psi/\chi\} = \chi$. Therefore, $\varphi \leftrightarrow \varphi\{\psi/\chi\}$ boils down to $\models \psi \leftrightarrow \chi$, hence follows from the assumption. Now assume inductively

that the statement holds for σ and θ. The cases for the Booleans K, int and $[!]$ are standard. We here only show the case of the new modality $\boxast$:

Case $\varphi := \boxast\theta$: Note that $(\boxast\theta)\{\psi/\chi\} = \boxast(\theta\{\psi/\chi\})$. We then have

$$
\begin{array}{ll}
(x, U) \models (\boxast\theta)\{\psi/\chi\} & \\
\text{iff } (x, U) \models \boxast(\theta\{\psi/\chi\}) & \\
\text{iff } (\forall \eta \in \mathcal{L}^{!}_{K\mathsf{int}})((x, U) \models [\eta](\theta\{\psi/\chi\})) & \text{(by the semantics of } \boxast) \\
\text{iff } (\forall \eta \in \mathcal{L}^{!}_{K\mathsf{int}})((x, U) \models \mathsf{int}(\eta) \text{ implies } (x, Int([\![\eta]\!]^U)) \models \theta\{\psi/\chi\}) & \\
 & \text{(by the semantics of } [!]) \\
\text{iff } (\forall \eta \in \mathcal{L}^{!}_{K\mathsf{int}})((x, U) \models \mathsf{int}(\eta) \text{ implies } (x, Int([\![\eta]\!]^U)) \models \theta) & \\
 & \text{(by the induction hypothesis on } \theta) \\
\text{iff } (\forall \eta \in \mathcal{L}^{!}_{K\mathsf{int}})((x, U) \models [\eta]\theta) & \text{(by the semantics of } [!]) \\
\text{iff } (x, U) \models \boxast\theta & \text{(by the semantics of } \boxast)
\end{array}
$$

□

7.2.3. PROPOSITION. *For any $\varphi, \varphi_i \in \mathcal{L}_{\mathsf{int}}$, the following is valid in all topo-models:*

$$\models \Diamond\!\!\!\ast(\varphi \wedge K\varphi_0 \wedge \bigwedge_{1 \le i \le n} \hat{K}\varphi_i) \leftrightarrow (\varphi \wedge \mathsf{int}(\varphi_0) \wedge \bigwedge_{1 \le i \le n} \hat{K}(\mathsf{int}(\varphi_0) \wedge \varphi_i)) \qquad (\mathrm{EL}^{\boxast}_n)$$

Proof:
The proof is similar to the proof of Proposition 7.1.15. For the direction from right-to-left, we take φ_0 as the witness for $\Diamond\!\!\!\ast$. □

7.2.4. LEMMA. *For all $\varphi, \psi \in \mathcal{L}^{!}_{K\mathsf{int}\boxast}$, the formula $\Diamond\!\!\!\ast(\varphi \vee \psi) \leftrightarrow (\Diamond\!\!\!\ast\varphi \vee \Diamond\!\!\!\ast\psi)$ is valid in all topo-models.*

7.2.5. THEOREM. *$\mathcal{L}^{!}_{K\mathsf{int}\boxast}$ and $\mathcal{L}_{K\mathsf{int}}$ are equally expressive with respect to topo-models.*

Proof:
Analogous to the proof of Theorem 7.1.17. □

We have therefore obtained the extended Figure 7.2 summarizing all the expressivity results we have provided on topo-models concerning the languages $\mathcal{L}^{!}_{K\mathsf{int}\Box}$, $\mathcal{L}^{!}_{K\mathsf{int}\boxast}$, and their subfragments.

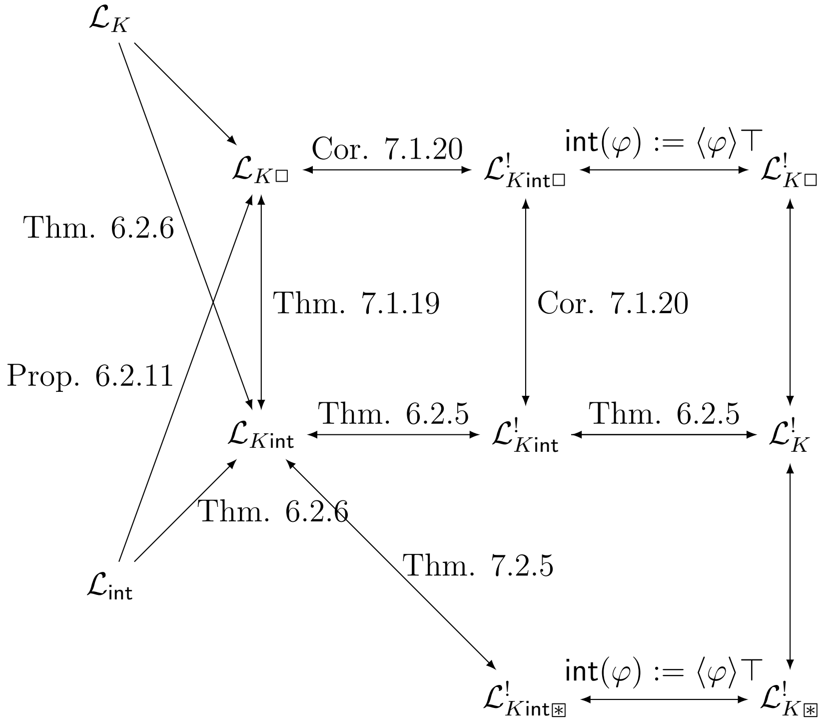


Figure 7.2: Expressivity diagram-updated with $\boxast$ (Arrows point to the more expressive languages, and reflexive and transitive arrows are omitted. Arrows without tags can be obtained as easy consequences from the others.)

We moreover prove that not only are $\mathcal{L}^!_{K\mathsf{int}\boxast}$ and $\mathcal{L}^!_{K\mathsf{int}\Box}$ co-expressive for topo-models, but also that *the effort modality* $\Box$ *and the topological arbitrary announcement modality* $\boxast$ *are equivalent*, in the following sense (Baltag et al., 2017):

7.2.6. Theorem. *Let $t : \mathcal{L}^!_{K\mathsf{int}\boxast} \to \mathcal{L}^!_{K\mathsf{int}\Box}$ be the map that replaces each instance of $\boxast$ with $\Box$. Then for every $\varphi \in \mathcal{L}^!_{K\mathsf{int}\boxast}$, we have that $\varphi \leftrightarrow t(\varphi)$ is valid in all topo-models.*

Proof:
The proof is by subformula induction on φ. We skip the proof details, which could be easily reconstructed, and provide only a sketch. The cases for the propositional variables, the Booleans, and the modalities K and int are straightforward, since $t(p) = p; t(\neg\psi) = \neg t(\psi); t(\psi \wedge \chi) = t(\psi) \wedge t(\chi); t(K\psi) = Kt(\psi); t(\mathsf{int}(\psi)) = \mathsf{int}(t(\psi))$ and $t([\psi]\chi) = [t(\psi)]t(\chi)$. The relatively complicated case is $\varphi := \diamondast\psi$, where $t(\diamondast\psi) = \Diamond t(\psi)$. The crucial part of the proof is that the elimination procedure for $\Diamond$ and $\diamondast$ are the same: they both go via normal forms and the corresponding equivalences $\mathrm{EL}^{\Box}_n$ and $\mathrm{EL}^{\boxast}_n$ (see Corollary 7.1.18 and Theorem

7.2.5). Hence, two formulas only differing in the occurrences of $\diamond$ and $\circledast$ are semantically equivalent to the same formula in $\mathcal{L}_{K\mathsf{int}}$ on topo-models. $\square$

Therefore, given the sound and complete axiomatization of $\mathsf{DTL_{int}}$ (Table 7.1) and the above link between the effort modality $\square\varphi$ and the arbitrary announcement modality $\boxast\varphi$, we immediately obtain a sound and complete axiomatization for the single-agent logic $\mathsf{APAL_{int}}$ of knowledge $K\varphi$, knowability $\mathsf{int}(\varphi)$, public announcements $[\varphi]\psi$, and arbitrary announcenements $\boxast\varphi$ with respect to the class of all topo-models. The axiomatization of $\mathsf{APAL_{int}}$ is again given by the axiom schemas in Table 7.1 defined over the language $\mathcal{L}^!_{K\mathsf{int}\boxast}$ (instead of $\mathcal{L}^!_{K\mathsf{int}\square}$). In particular, the axiom ([!]$\square$-elim) and the inference rule ([!]$\square$-intro) are replaced by ([!]$\boxast$-elim) and ([!]$\boxast$-intro) given in Table 7.2, respectively

([!]$\boxast$-elim)	$[\varphi]\boxast\theta \to [\varphi\wedge\rho]\theta$	($\rho \in \mathcal{L}^!_{K\mathsf{int}\boxast}$ arbitrary formula)
([!]$\boxast$-intro)	from $\psi \to [\varphi\wedge p]\theta$, infer $\psi \to [\varphi]\boxast\theta$	($p \notin \mathrm{P}_\psi \cup \mathrm{P}_\theta \cup \mathrm{P}_\varphi$)

Table 7.2: The axiom for $\boxast$-elimination and the rule for $\boxast$-introduction

We therefore obtain the following which, together with Theorem 7.2.6, gives us the soundness and completeness of $\mathsf{APAL_{int}}$.

7.2.7. Lemma. *For all $\varphi \in \mathcal{L}^!_{K\mathsf{int}\boxast}$, we have $\vdash_{\mathsf{APAL_{int}}} \varphi$ iff $\vdash_{\mathsf{DTL_{int}}} t(\varphi)$.*

7.2.8. Corollary. $\mathsf{APAL_{int}}$ *is sound and complete with respect to the class of all topo-models.*

Proof:
For soundness, we focus only on the new axiom schema and the inference rule, and show that ([!]$\boxast$-elim) is valid and ([!]$\boxast$-intro) preserves validity on topo-models.

([!]$\boxast$-elim): Let $\mathcal{X} = (X,\tau,V)$ and $(x,U) \in ES(\mathcal{X})$ such that $(x,U) \models [\varphi]\boxast\theta$. Then, by Theorem 7.2.6, we obtain $(x,U) \models [t(\varphi)]\square t(\theta)$. Thus, by the soundness of ([!]$\square$-elim) for topo-models (Theorem 7.1.9), we have $(x,U) \models [t(\varphi)\wedge\rho]t(\theta)$ for all $\rho \in \mathcal{L}^!_{K\mathsf{int}\square}$. Let $\rho' \in \mathcal{L}^!_{K\mathsf{int}\boxast}$. Hence, $t(\rho') \in \mathcal{L}^!_{K\mathsf{int}\square}$, therefore we have $(x,U) \models [t(\varphi)\wedge t(\rho')]t(\theta)$. Observe that $[t(\varphi)\wedge t(\rho')]t(\theta) = t([\varphi\wedge\rho']\theta)$. Therefore, by Theorem 7.2.6 again, we obtain $(x,U) \models [\varphi\wedge\rho']\theta$. As ρ' has been chosen arbitrarily from $\mathcal{L}^!_{K\mathsf{int}\boxast}$, we have the desired result.

([!]$\boxast$-intro): Suppose $\models \psi \to [\varphi\wedge p]\theta$ for some $p \notin \mathrm{P}_\psi \cup \mathrm{P}_\theta \cup \mathrm{P}_\varphi$. Then, by Theorem 7.2.6, $\models t(\psi \to [\varphi\wedge p]\theta)$, that is, $\models t(\psi) \to [t(\varphi)\wedge p]t(\theta)$, by the definition of t. Then, by the soundness of ([!]$\square$-intro) (Theorem 7.1.9), we obtain that $\models t(\psi) \to [t(\varphi)]\square t(\theta)$. Observe that $t(\psi) \to [t(\varphi)]\square t(\theta) = t([\psi] \to [\varphi]\boxast\theta)$, therefore, $\models t([\psi] \to [\varphi]\boxast\theta)$. Thus, again by Theorem 7.2.6, we have $\models [\psi] \to [\varphi]\boxast\theta$.

For completeness, let $\varphi \in \mathcal{L}^{!}_{K\mathsf{int}\boxtimes}$ such that $\varphi \notin \mathsf{APAL_{int}}$. Hence, by Lemma 7.2.7, we obtain that $t(\varphi) \notin \mathsf{DTL_{int}}$. Then, by Corollary 7.1.38, there exists a topo-model $\mathcal{X} = (X, \tau, V)$ and an epistemic scenario $(x, U) \in ES(\mathcal{X})$ such that $\mathcal{X}, (x, U) \not\models t(\varphi)$. Therefore, by Theorem 7.2.6, we conclude $X, (x, U) \not\models \varphi$. $\square$

7.3 Conclusions and Continuation

Our work presented in this chapter uses both the interior semantics of McKinsey and Tarski (1944) (together with the global modality as knowledge), and the topological formalism introduced by Moss and Parikh (1992). Building on Bjorndahl's logic of knowledge $K\varphi$, knowability $\mathsf{int}(\varphi)$, and learning of new evidence $[\psi]\varphi$ (formalized as a "topological" public announcement modality, whose precondition is captured by $\mathsf{int}(\varphi)$), we developed the so-called Dynamic TopoLogic that is obtained by adding the effort modality to Bjorndahl's system. This way, we believe that, at the very least, the meaning of the effort modality has become more transparent as it is linked to the public announcement modalities $[\psi]\varphi$ which can be seen as a particular case of effort. This connection has been made precise in the corresponding proof system by the axiom ($[!]\square$-elim) and the inference rule ($[!]\square$-intro). In Dynamic TopoLogic the behaviour of the effort modality is described by using only the aforementioned axiom and inference rule, avoiding the complicated Union Axiom of TopoLogic. While our completeness proof of $\mathsf{DTL_{int}}$ goes by a standard canonical model construction based on maximally consistent witnessed theories, our expressivity results (Corollary 7.1.20) imply decidability and the finite model property of the logics of topological spaces over the language $\mathcal{L}^{!}_{K\mathsf{int}\square}$ and its fragments (Corollary 7.1.21), by relying on the known decidability and finite model property of $\mathcal{L}_{K\mathsf{int}}$.

We moreover study a topological semantics for the arbitrary announcement modality, and investigate its interplay with the effort modality. To the best of our knowledge, the known completeness proofs for arbitrary announcement logics (topological or relational) rely on infinitary axiomatizations formalized by using necessity forms (see, e.g., Balbiani et al., 2008, 2013; Balbiani, 2015; Balbiani and van Ditmarsch, 2015; also see Sections 8.2 and 8.3 for the multi-agent case). Although Balbiani et al. (2008) propose a finitary axiomatization similar to ours (Table 7.2), its completeness proof goes via the completeness of an infinitary system[4]. On the other hand, our completeness proof of the finitary system $\mathsf{APAL_{int}}$ does not involve a detour through an infinitary logic. Therefore, the effort modality helps to simplify and streamline the axiomatization of $\mathsf{APAL_{int}}$.

[4]The finitary axiomatization proposed in (Balbiani et al., 2008) was later proven to be unsound for the *multi-agent case* (see `http://personal.us.es/hvd/APAL_counterexample.pdf`), and the error in the complexity measure in (Balbiani et al., 2008, Truth Lemma 4.13, p. 327) is corrected in (Balbiani, 2015).

Higher-order knowledge and dynamics of information change become more interesting when more than one agent is involved. However, extending the subset space style semantics to a setting involving multiple agents comes with some challenges concerning the evaluation of higher-order knowledge. In particular, the multi-agent case requires solving the complication of "jumping out of the epistemic range". In the next chapter, we explain this problem and propose a solution for it. We then study the multi-agent versions $\mathsf{EL}^m_{\mathsf{int}}$ and $\mathsf{PAL}^m_{\mathsf{int}}$ of $\mathsf{EL}_{\mathsf{int}}$ and $\mathsf{PAL}_{\mathsf{int}}$, respectively, as well as a multi-agent logic of arbitrary public announcements, denoted by $\mathsf{APAL}^m_{\mathsf{int}}$, interpreted on topological spaces in the style of subset space semantics. The effort modality in multi-agent setting creates many challenges, both technically and conceptually. We leave investigations for the effort modality in a multi-agent setting for future.

Chapter 8

Multi-Agent Topo-Arbitrary Announcement Logic

In this chapter, we propose a *multi-agent* logic of knowledge, knowability, public and arbitrary announcements, interpreted on topological spaces in the style of subset space semantics. More precisely, we generalize the single-agent setting presented in Section 7.2 to a multi-agent setting wherein the multi-agent version of $\mathcal{L}^{!}_{K\mathsf{int}\boxtimes}$ is defined similarly but with finitely many knowledge modalities $K_i\varphi$ indexed for each agent, meaning that *agent i knows φ*.

As also recognized in (Baskent, 2007, Chapter 6) and (Wáng and Ågotnes, 2013a), a first step toward developing a multi-agent epistemic logic using topological subset space semantics requires solving the problem of "jumping out of the epistemic range" of an agent while evaluating higher-order knowledge formulas. This issue occurs independently from the dynamic extensions. The general setup is for any finite number of agents, but to demonstrate the challenges, consider the case of two agents. If we extend the setup from the single agent case in the straightforward way, then for each of two agents i and j there is an open set and the semantic primitive becomes a triple (x, U_i, U_j) instead of a pair (x, U). Now consider a formula like $K_i\hat{K}_jK_ip$, for "agent i knows that agent j considers possible that agent i knows proposition p". If this is true for a triple (x, U_i, U_j), then $\hat{K}_jK_ip$ must be true for any $y \in U_i$; but y may not be in U_j, in which case (y, U_i, U_j) is not well-defined: we cannot interpret $\hat{K}_jK_ip$. Our solution to this dilemma is to consider neighbourhoods that are not only relative to each agent, but that are also *relative to each state*. This means that, when shifting the viewpoint from x to $y \in U_i$, in (x, U_i, U_j), we simultaneously have to shift the *neighbourhood* (and not merely the point in the actual neighbourhood) for the other agent. Thus, we go from (x, U_i, U_j) to (y, U_i, V_j), where V_j may be different from U_j: while the open set U_j represents j's current evidence at x, the open V_j represents j's evidence (i.e., epistemic range) at y. Therefore, the neighbourhood shift from U_j to V_j does not mean a change of agent j's evidence set *at the actual state*. While the tuple (x, U_i, U_j) represents the actual state and the view points of both agents, the component (y, V_j) of the latter tuple merely represents agent j's epistemic state from agent i's perspective at y, a possibly different state from the actual state x.

In order to define the epistemic range of each agent with respect to the state in question, we employ a technique inspired by the standard neighbourhood semantics (see, e.g., Chellas, 1980). We use a set of *neighbourhood functions*, determining the epistemic range relative to both the given state and the corresponding agent. These functions need to be partial in order to render the semantics well-defined for the dynamic modalities in the system, namely for the public and arbitrary announcement modalities.

Moreover, using topological spaces enriched with a set of (partial) neighbourhood functions as models allows us to work with different notions of knowledge. In the standard (single-agent) subset space setting (as in Chapters 6 and 7), as the knowledge modality quantifies over the elements of a fixed neighbourhood, the $\mathsf{S5}$ type knowledge is inherent to the way the semantics defined. With the approach developed in this chapter, however, the epistemic range of an agent changes according to the neighbourhood functions when the evaluation state changes. Therefore, the valid properties of knowledge are determined by the constraints imposed on the neighbourhood functions. To this end, we work with both $\mathsf{S5}$ and $\mathsf{S4}$ types of knowledge in this chapter: while the former is the standard notion of knowledge in the subset space setting, the latter reveals a novel aspect of our approach, namely, the ability to capture different notions of knowledge.

Outline

Section 8.1 defines the syntax, structures, and semantics of our multi-agent logic of arbitrary public announcements, $\mathsf{APAL}^m_{\mathsf{int}}$, interpreted on topological spaces equipped with a set of neighbourhood functions. Without arbitrary announcements we get the logic $\mathsf{PAL}^m_{\mathsf{int}}$, and with neither arbitrary nor public announcements, the logic $\mathsf{EL}^m_{\mathsf{int}}$. In this section we also give two detailed examples illustrating the proposed semantics. In Section 8.2 we provide axiomatizations for the logics: $\mathsf{PAL}^m_{\mathsf{int}}$ extends $\mathsf{EL}^m_{\mathsf{int}}$ and $\mathsf{APAL}^m_{\mathsf{int}}$ extends $\mathsf{PAL}^m_{\mathsf{int}}$. We moreover prove their soundness and compare the expressive power of the associated multi-agent languages $\mathcal{L}^!_{K\mathsf{int}\boxast}$, $\mathcal{L}^!_{K\mathsf{int}}$ and $\mathcal{L}_{K\mathsf{int}}$ with respect to multi-agent topo-models. In Section 8.3 we demonstrate completeness for these logics. The completeness proof for the epistemic fragment, $\mathsf{EL}^m_{\mathsf{int}}$, is rather different from the completeness proof for the full logic $\mathsf{APAL}^m_{\mathsf{int}}$. Section 8.4 adapts the logics to the case of $\mathsf{S4}$ knowledge. In Section 8.5 we compare our work to that of others, and Section 8.6 provides a brief summary of the chapter while also discussing a possible interpretation of the effort modality in the current multi-agent setting.

This chapter is based on (van Ditmarsch et al., 2015b,c).

8.1 The Multi-Agent Arbitrary Announcement Logic $\mathsf{APAL}^m_{\mathsf{int}}$

We define the syntax, structures, and semantics of our multi-agent logic of knowledge, knowability, public and arbitrary announcements. From now on, $\mathcal{A}$ denotes a finite and nonempty set of agents.

8.1.1 Syntax and Semantics

The (multi-agent) language $\mathcal{L}^!_{K\mathsf{int}\boxast}$ is defined by

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid \mathsf{int}(\varphi) \mid [\varphi]\varphi \mid \boxast\varphi$$

where $p \in \textsc{prop}$, and $i \in \mathcal{A}$. Abbreviations for the connectives and the dual modalities are defined as in the previous chapters; to recall, we in particular employ $\hat{K}_i\varphi := \neg K_i\neg\varphi$, and $\diamondast\varphi := \neg\boxast\neg\varphi$. Notice that we use the same denotation $\mathcal{L}^!_{K\mathsf{int}\boxast}$ for both the single and multi-agent version of the above defined syntax. Since we study the multi-agent version in this chapter, and the single-agent language constitutes just a special case of the multi-agent extension, this should not lead to any confusions. Similarly, we let $\mathcal{L}_{K\mathsf{int}}$ and $\mathcal{L}^!_{K\mathsf{int}}$ denote the corresponding multi-agent languages.

We interpret the language $\mathcal{L}^!_{K\mathsf{int}\boxast}$ on topological spaces endowed with (partial) neighbourhood functions that *for each agent* $i \in \mathcal{A}$ *assign an open neighbourhood at a given state* x. More precisely, given a topological space (X, τ), such a neighbourhood function θ is defined from X to $\mathcal{A} \to \tau$ (i.e., the set of functions $\tau^{\mathcal{A}}$ from $\mathcal{A}$ to τ) as a *partial function*, denoted by $\theta : X \rightharpoonup \mathcal{A} \to \tau$. We let $\mathcal{D}(\theta)$ denote the *domain of* θ, that is, the set of states in X for which θ is defined.

8.1.1. Definition. [(Partial) Neighbourhood Function]
Given a topological space (X, τ), a *neighbourhood function set* Φ on (X, τ) is a set of (partial) *neighbourhood functions* $\theta : X \rightharpoonup \mathcal{A} \to \tau$ such that for all $x \in \mathcal{D}(\theta)$, for all $i \in \mathcal{A}$, and for all $U \in \tau$:

1. $x \in \theta(x)(i)$,
2. $\theta(x)(i) \subseteq \mathcal{D}(\theta)$,
3. for all $y \in X$, if $y \in \theta(x)(i)$ then $y \in \mathcal{D}(\theta)$ and $\theta(x)(i) = \theta(y)(i)$,
4. $\theta^U \in \Phi$,

where $\mathcal{D}(\theta)$ is the domain of θ, and θ^U is the *restricted/updated neighbourhood function* with $\mathcal{D}(\theta^U) = \mathcal{D}(\theta) \cap U$ and $\theta^U(x)(i) = \theta(x)(i) \cap U$.

The main role of the neighbourhood functions θ is to assign to each agent an epistemic range at a given state. It simply defines the current evidence set of each agent at the state in question. Each condition given in Definition 8.1.1 guarantees certain requirements that render the semantics well-defined and meaningful for the language $\mathcal{L}^!_{K\mathsf{int}\circledast}$. In particular, with the help of the neighbourhood functions we solve the problem of "jumping out of the epistemic range" explained in the introduction. We will provide a more detailed explanation regarding the definition of the neighbourhood functions together with our proposed semantics given in Definition 8.1.4.

8.1.2. Definition. [Multi-agent Topo-model] A *multi-agent topo-model* is a tuple $\mathcal{X} = (X, \tau, \Phi, V)$, where (X, τ) is a topological space, Φ a neighbourhood function set, and $V : \textsc{prop} \to \mathcal{P}(X)$ a valuation function. The tuple (X, τ, Φ) is called a *multi-agent topo-frame.*

Throughout this chapter, we call a multi-agent topo-model(-frame) simply a *topo-model*(*-frame*). It will be clear from the context when we consider a single-agent topo-model (X, τ, V). Similar to the case of the single-agent framework, given a topo-model $\mathcal{X} = (X, \tau, \Phi, V)$, the open sets in τ are meant to represent the evidence pieces that are *potentially* available for *all* the agents. In our multi-agent setup, all agents have the same observational power, represented by each topo-model carrying only one topology.

Formulas of $\mathcal{L}^!_{K\mathsf{int}\circledast}$ are interpreted on topo-models with respect to pairs of the form (x, θ), where $\theta \in \Phi$ and $x \in \mathcal{D}(\theta)$. Such a pair is called a *neighbourhood situation*, and $\theta(x)(i)$ corresponds to the *epistemic range of agent i at x* (with respect to θ). The epistemic range $\theta(x)(i)$ represents the *actual, current* evidence of the agent i at x and it is her only source of knowledge at state x with respect to the neighbourhood situation (x, θ). This is stipulated in the semantic clause for K_i in Definition 8.1.4 below. If (x, θ) is a neighbourhood situation in $\mathcal{X}$ we write $(x, \theta) \in \mathcal{X}$.

The following lemma shows that the domain of every neighbourhood function is open.

8.1.3. Lemma. *For any topo-frame (X, τ, Φ) and $\theta \in \Phi$, we have $\mathcal{D}(\theta) \in \tau$.*

Proof:
Let (X, τ, Φ) be a topo-frame, $\theta \in \Phi$ and $x \in \mathcal{D}(\theta)$. By Definition 8.1.1-(1) and -(2), we have $x \in \theta(x)(i) \in \tau$ and $\theta(x)(i) \subseteq \mathcal{D}(\theta)$. Therefore, $x \in Int(\mathcal{D}(\theta))$. Hence, $\mathcal{D}(\theta) = Int(\mathcal{D}(\theta))$, i.e., $\mathcal{D}(\theta) \in \tau$. $\Box$

8.1.4. Definition. [Topo-semantics for (multi-agent) $\mathcal{L}^!_{K\mathsf{int}\circledast}$]
Given a topo-model $\mathcal{X} = (X, \tau, \Phi, V)$ and a neighbourhood situation $(x, \theta) \in \mathcal{X}$,

the *truth* of a formula in the language $\mathcal{L}^!_{K\mathsf{int}\boxast}$ is defined recursively as follows:

$$
\begin{array}{lll}
\mathcal{X},(x,\theta)\models p & \text{iff} & x\in V(p)\\
\mathcal{X},(x,\theta)\models \neg\varphi & \text{iff} & \text{not } \mathcal{X},(x,\theta)\models\varphi\\
\mathcal{X},(x,\theta)\models \varphi\wedge\psi & \text{iff} & \mathcal{X},(x,\theta)\models\varphi \text{ and } \mathcal{X},(x,\theta)\models\psi\\
\mathcal{X},(x,\theta)\models K_i\varphi & \text{iff} & (\forall y\in\theta(x)(i))(\mathcal{X},(y,\theta)\models\varphi)\\
\mathcal{X},(x,\theta)\models \mathsf{int}(\varphi) & \text{iff} & x\in Int([\![\varphi]\!]^\theta)\\
\mathcal{X},(x,\theta)\models [\varphi]\psi & \text{iff} & \mathcal{X},(x,\theta)\models\mathsf{int}(\varphi) \text{ implies } \mathcal{X},(x,\theta^\varphi)\models\psi\\
\mathcal{X},(x,\theta)\models \boxast\varphi & \text{iff} & (\forall\psi\in\mathcal{L}^!_{K\mathsf{int}})(\mathcal{X},(x,\theta)\models[\psi]\varphi)
\end{array}
$$

where $p\in \textsc{prop}$, $[\![\varphi]\!]^\theta = \{y\in\mathcal{D}(\theta)\mid \mathcal{X},(y,\theta)\models\varphi\}$ and an *updated neighbourhood function* $\theta^\varphi : X \rightharpoonup \mathcal{A}\to\tau$ is defined such that $\theta^\varphi = \theta^{Int([\![\varphi]\!]^\theta)}$. More precisely, $\mathcal{D}(\theta^\varphi) = Int([\![\varphi]\!]^\theta)$ and $\theta^\varphi(x)(i) = \theta(x)(i)\cap Int([\![\varphi]\!]^\theta)$ for all $x\in\mathcal{D}(\theta^\varphi)$.

When the model is not fixed, we use subscripts and write, e.g., $[\![\varphi]\!]^\theta_{\mathcal{X}}$, to denote the model we work with. A formula $\varphi\in\mathcal{L}^!_{K\mathsf{int}\boxast}$ is *valid in a topo-model* $\mathcal{X}$, denoted $\mathcal{X}\models\varphi$, iff $\mathcal{X},(x,\theta)\models\varphi$ for all $(x,\theta)\in\mathcal{X}$; φ is *valid*, denoted $\models\varphi$, iff for all topo-models $\mathcal{X}$ we have $\mathcal{X}\models\varphi$. Soundness and completeness with respect to topo-models are defined as usual.

Let us now elaborate on the structure of topo-models and the above semantics we have proposed for $\mathcal{L}^!_{K\mathsf{int}\boxast}$. For any topo-model $\mathcal{X}=(X,\tau,\Phi,V)$, the agents' current evidence, i.e., the epistemic range of each agent at a given state x, is defined by (partial) functions $\theta\in\Phi$, where $\theta : X\rightharpoonup\mathcal{A}\to\tau$. We allow for partial functions in Φ, and close Φ under restricted functions θ^U where $U\in\tau$ (see Definition 8.1.1, condition 4), so that updated neighbourhood functions are guaranteed to be well-defined elements of Φ. As briefly mentioned in Section 6.1.1, one important feature of the subset space semantics is the local interpretation of propositions: in the single-agent case, once the epistemic scenario (x,U) has been picked, the rest of the model does not have any effect on the truth of the proposition in question. Similarly in our multi-agent setup, by choosing a neighbourhood situation (x,θ), we localize the interpretation to an open subdomain of the whole space, namely to $\mathcal{D}(\theta)$, that includes the actual state x, and embeds an epistemic range for each agent $i\in\mathcal{A}$ at every state in $\mathcal{D}(\theta)$. For every $\theta\in\Phi$ and $x\in\mathcal{D}(\theta)$, the function $\theta(x):\mathcal{A}\to\tau$ is defined to be a *total* function. It is therefore guaranteed that, given a neighbourhood situation (x,θ), the neighbourhood function θ assigns to *every agent in* $\mathcal{A}$ an open neighbourhood of x. Moreover, the conditions of neighbourhood functions given in Definition 8.1.1 make the semantics work for the multi-agent setting. To be more precise, condition 1 guarantees that θ always returns a factive evidence set for each agent at the actual state. Since the neighbourhoods given by the neighbourhood functions depend not only on the agent but also on the current state of the agent, and since $x\in\theta(x)(i)\subseteq\mathcal{D}(\theta)$ for every $x\in\mathcal{D}(\theta)$ and every $i\in\mathcal{A}$ (due to conditions 1 and 2), our semantics does not face the problem of "jumping out of the epistemic range", and thus does not end up with ill-defined evaluation pairs in the interpretation of iterated epistemic formulas such as $\hat{K}_j K_i p$. Moreover, conditions (1) and (3) of Definition 8.1.1 ensure that the $\mathsf{S5}$ axioms for each K_i are sound with respect to all topo-models: each neighbourhood function $\theta \in \Phi$ induces a partition on $\mathcal{D}(\theta)$ for each agent $i \in \mathcal{A}$. We will see in Section 8.4 that our setting can be adapted to account for the weaker $\mathsf{S4}$, $\mathsf{S4.2}$ and $\mathsf{S4.3}$ notions of knowledge by relaxing the conditions on the neighbourhood functions in Φ.

The semantics proposed for the propositional variables and the Booleans is rather standard, similar to both the relational semantics and the classical subset space semantics (see, e.g., Definition 6.1.2). Moreover, the semantics for the modality int is similar to the semantics in the single-agent case. In particular, as in the single-agent case, the truth value of the formulas in $\mathcal{L}_{\mathsf{int}}$ on multi-agent topo-models depends *only* on the actual state, *not* on the chosen neighbourhood function. In this sense, the formulas of $\mathcal{L}_{\mathsf{int}}$ are *bi-persistent* on multi-agent topo-models.

8.1.5. PROPOSITION. *Given a topo-model $\mathcal{X} = (X, \tau, \Phi, V)$, neighbourhood situations $(x, \theta_1), (x, \theta_2) \in \mathcal{X}$, and a formula $\varphi \in \mathcal{L}_{\mathsf{int}}$,*

$$(x, \theta_1) \models \varphi \textit{ iff } (x, \theta_2) \models \varphi.$$

Proof:
The proof follows along the same lines as the proof of Proposition 7.1.14 by subformula induction on φ: cases for the propositional variables and the Booleans are elementary. So assume inductively that the result holds for ψ; we must show that it holds also for $\varphi := \mathsf{int}(\psi)$.

$$\begin{aligned}(x, \theta_1) \models \mathsf{int}(\psi) &\text{ iff } x \in \mathit{Int}([\![\psi]\!]^{\theta_1}) \\ &\text{ iff } (\exists U \in \tau)(x \in U \subseteq [\![\psi]\!]^{\theta_1}) \qquad \text{(by the definition of } \mathit{Int})\end{aligned}$$

Now, consider the open set $U \cap \mathcal{D}(\theta_2)$. Since (x, θ_2) is a well-defined neighbourhood situation, $x \in \mathcal{D}(\theta_2)$. Moreover, by Lemma 8.1.3, we have $\mathcal{D}(\theta_2) \in \tau$. Hence, we obtain $x \in U \cap \mathcal{D}(\theta_2) \in \tau$. Thus, it suffices to show that $U \cap \mathcal{D}(\theta_2) \subseteq [\![\varphi]\!]^{\theta_2}$. Let $y \in U \cap \mathcal{D}(\theta_2)$. Since $U \cap \mathcal{D}(\theta_2) \subseteq U$, we have by the assumption that $(y, \theta_1) \models \psi$. Then, by IH, $(y, \theta_2) \models \psi$. As y has been chosen arbitrarily from $U \cap \mathcal{D}(\theta_2)$, we conclude that $U \cap \mathcal{D}(\theta_2) \subseteq [\![\psi]\!]^{\theta_2}$, hence, $x \in \mathit{Int}([\![\psi]\!]^{\theta_2})$, i.e., $(x, \theta_2) \models \mathsf{int}(\psi)$. □

We now take a closer look at the semantic clauses for the modalities in $\mathcal{L}^{!}_{K\mathsf{int}\boxtimes}$. Recall that the open sets in τ are meant to represent the evidence pieces that can in principle be discovered by any agent in $\mathcal{A}$. In other words, open sets of a topology can be considered as the propositions that the agents can in principle observe (but might not have observed yet). This interpretation was elaborated in Section 6.1.1, p. 106. On the other hand, $\theta(x)(i)$ serves as agent i's current (factive) evidence at the actual state x (with respect to θ). Stating the semantic clause for knowledge given in Definition 8.1.4 in a slightly different way gives us that

$$(x, \theta) \models K_i\varphi \text{ iff } \theta(x)(i) \subseteq [\![\varphi]\!]^\theta,$$

i.e, according to our proposed semantics, *agent i knows φ at x (with respect to θ) iff his current evidence entails φ*, similar to the case in the single-agent version.

As in the single agent case, the modality int serves as the precondition of an announcement that represents knowability as an existential claim over the set τ of pieces of evidence:

$$(x, \theta) \models \mathsf{int}(\varphi) \text{ iff } (\exists U \in \tau)(x \in U \subseteq [\![\varphi]\!]^\theta).$$

Therefore, whether the precondition of an announcement is fulfilled does not depend on the agents' epistemic states but depends only on the model in question. Moreover, given the semantic clause for the public announcements

$$(x, \theta) \models [\varphi]\psi \text{ iff } (x, \theta) \models \mathsf{int}(\varphi) \text{ implies } (x, \theta^\varphi) \models \psi,$$

and the definition of the updated neighbourhood function θ^φ, the effect of an announcement is again modelled as open-set-shrinkage without leading to a global change in the initial model. More precisely, a successful announcement φ transforms the initial neighbourhood function θ to θ^φ which assigns a more refined epistemic range $\theta(x)(i) \cap Int([\![\varphi]\!]^\theta) \subseteq \theta(x)(i)$ to each agent i at the actual state x, representing the effect of learning φ. We continue with some examples illustrating the above defined semantics.

8.1.2 Examples

In this section we present two examples demonstrating how our multi-agent topological semantics works. The first example is a multi-agent version of an example presented by Bjorndahl (2016) for the single-agent setting, and the second is concerned with two agents learning bit by bit (finite) prefixes of a pair of infinite binary sequences.

The Jewel in the Tomb

We illustrate our semantics by means of a multi-agent version of Bjorndahl's example in (Bjorndahl, 2016) about the jewel in the tomb. Indiana Jones (i) and Emile Belloq (e) are both scouring for a priceless jewel placed in a tomb. The tomb could either contain a jewel or not, the tomb could have been rediscovered in modern times or not, and (beyond Bjorndahl (2016)), the tomb could be in the Valley of Tombs in Egypt or not. The propositional variables corresponding to these propositions are, respectively, j, d, and t. We represent a valuation of these variables by a triple xyz, where $x, y, z \in \{0, 1\}$. Given the carrier set $X = \{xyz \mid x, y, z \in \{0, 1\}\}$, the topology τ that we consider is generated by the basis consisting of the subsets $\{000, 100, 001, 101\}$, $\{010\}$, $\{110\}$, $\{011\}$, $\{111\}$ (see Figure 8.1). The idea is that one can only conceivably know (or learn) about the jewel or the location on condition that the tomb has been discovered. Therefore, $\{000, 100, 001, 101\}$ has no strict subsets besides the empty set: if the tomb has not yet been discovered, no one can have any information about the jewel or the location. However, provided that the tomb has been discovered, the agents might know whether or not it contains a jewel, and/or whether it is the Valley of Tombs in Egypt. In this example, we stipulate that the actual state is 111.

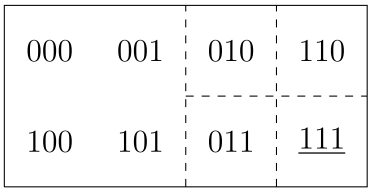


Figure 8.1: Dashed squares represent the elements of the basis generating the topology τ.

A topo-model $\mathcal{X} = (X, \tau, \Phi, V)$ for this topology (X, τ) has Φ as the set of all neighbourhood functions that are partitions of X for both agents, and restrictions of these functions to open sets. A typical $\theta \in \Phi$ describes complete ignorance of both agents and is defined as $\theta(w)(i) = \theta(w)(e) = X$ for all $w \in X$. A more interesting neighbourhood situation in this model is one wherein Indiana and Emile have different knowledge. Let us assume that Emile has the advantage over Indiana so far, as he knows the location of the tomb but Indiana does not. This is the θ' such that for all $w \in X$, $\theta'(w)(i) = X$, whereas the partition for Emile consists of sets $\{000, 100, 001, 101\}$, $\{110, 010\}$, $\{111, 011\}$, i.e., $\theta'(111)(e) = \{111, 011\}$, etc (see Figure 8.2).

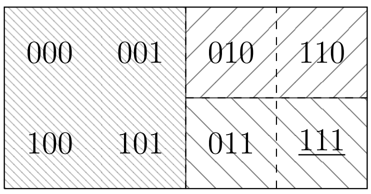


Figure 8.2: Patterned sets represent Emile's neighbourhoods defined by θ': $\theta'(111)(e) = \theta'(011)(e) = \{111, 011\}$, $\theta'(010)(e) = \theta'(110)(e) = \{010, 110\}$, $\theta'(000)(e) = \theta'(100)(e) = \theta'(001)(e) = \theta'(101)(e) = \{000, 100, 001, 101\}$.

We now can evaluate what Emile knows about Indiana at 111. Firstly, Emile knows that the tomb is in the Valley of Tombs in Egypt

$$\mathcal{X}, (111, \theta') \models K_e t \tag{8.1}$$

and he also knows that Indiana does not know that:

$$\mathcal{X}, (111, \theta') \models K_e \neg (K_i \neg t \vee K_i t) \tag{8.2}$$

The statement (8.2) involves verifying $\mathcal{X}, (w, \theta') \models \hat{K}_i t$ and $\mathcal{X}, (w, \theta') \models \hat{K}_i \neg t$ for all $w \in \theta'(111)(e) = \{111, 011\}$, which is Emile's current epistemic range. And this is true for both elements 111 and 110 of $\theta'(111)(e)$, because $\theta'(110)(i) = \theta'(111)(i) = X$, and $000, 001 \in X$, and while $\mathcal{X}, (001, \theta') \models t$, we also have $\mathcal{X}, (000, \theta') \models \neg t$. We can also check that Emile knows that Indiana considers it possible that Emile doesn't know the tomb's location:

$$\mathcal{X}, (111, \theta') \models K_e \hat{K}_i \neg (K_e t \vee K_e \neg t) \tag{8.3}$$

Evaluating this goes beyond Emile's initial epistemic range $\{111, 011\}$ because, e.g., for $111 \in \theta'(111)(e)$, we have $\mathcal{X}, (111, \theta') \models \hat{K}_i \neg (K_e t \vee K_e \neg t)$ iff there exists $y_0 \in \theta'(111)(i)$ such that $(y_0, \theta') \models \neg K_e t \wedge \neg \hat{K}_e \neg t$. Therefore, such an element y_0 cannot be in Emile's initial epistemic range $\{111, 011\}$, since $(111, \theta') \models K_e t$ and $(011, \theta) \models K_e t$. In fact, it has to be the case that $y_0 \in \{000, 001, 100, 101\}$. This situation however does not create any problems in our setting since (y_0, θ') is a well-defined neighbourhood situation, and Emile's epistemic range at y_0 is defined by θ' as $\theta'(y_0)(e) = \{000, 001, 100, 101\}$.

Given their prior knowledge, announcements will change Emile and Indiana's knowledge in different ways. Consider the announcement of j. An important point to notice is that the announcement of j does not only convey the information $[\![j]\!]^{\theta'} = \{100, 101, 110, 111\}$ but that it also leads to learning $\mathit{Int}([\![j]\!]^{\theta'}) = \{110, 111\}$. This corresponds exactly to the fact that one can know about the jewel on the condition that the tomb has already been rediscovered. Therefore, the announcement of j evidences the fact that the tomb has already been discovered, hence, it conveys more information than only j being true. This results in Emile knowing everything but Indiana still being uncertain about the location:

$$\mathcal{X}, (111, \theta') \models [j](K_e(j \wedge d \wedge t) \wedge K_i(j \wedge d) \wedge \neg (K_i t \vee K_i \neg t)) \tag{8.4}$$

Model checking this involves computing the epistemic ranges of both agents given by the updated neighbourhood function $(\theta')^j$ at 111 (see Figure 8.3). Note that $\mathit{Int}([\![j]\!]^{\theta'}) = \{111, 110\}$. Therefore, $(\theta')^j(111)(e) = \mathit{Int}([\![j]\!]^{\theta'}) \cap \theta'(111)(e) = \{111\}$, and for Indiana $(\theta')^j(111)(i) = \mathit{Int}([\![j]\!]^{\theta'}) \cap \theta'(111)(i) = \{111, 110\}$.

There is an announcement after which Emile and Indiana know everything (for example the announcement of $j \wedge t$):

$$\mathcal{X}, (111, \theta) \models \circledast (K_e(j \wedge d \wedge t) \wedge K_i(j \wedge d \wedge t)).$$

Observe that $\mathit{Int}([\![j \wedge t]\!]^{\theta'}) = \{111\}$, thus, $(\theta')^j(111)(e) = (\theta')^j(111)(j) = \{111\}$. Again, the announcement of $j \wedge t$ carries the implication that the tomb has been rediscovered. On the other hand, as long as the tomb has not been discovered,

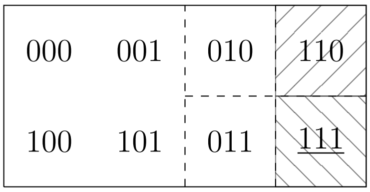


Figure 8.3: As $\mathcal{D}((\theta')^j) = Int([\![j]\!]^{\theta'}) = \{111, 110\}$, the updated neighbourhood function $(\theta')^j$ is defined only for these points. Patterned sets again represent Emile's neighbourhoods defined by $(\theta')^j$: $(\theta')^j(111)(e) = \{111\}$ and $(\theta')^j(110)(e) = \{110\}$. For Indiana, we have $(\theta')^j(111)(j) = (\theta')^j(110)(i) = \{111, 110\}$.

nothing will make Emile (or Indiana) learn that it contains a jewel or where the tomb is located:

$$\mathcal{X} \models \neg d \rightarrow \boxast(\neg(K_e j \vee K_e \neg j) \wedge \neg(K_e t \vee K_e \neg t)).$$

Binary Strings

We begin the example by defining a topology over the set of ordered pairs of binary strings, i.e., the domain of our topology is $X = \{0,1\}^\infty \times \{0,1\}^\infty$.

Note that we can consider X to be points in the unit square $[0,1] \times [0,1]$, by looking at each element of $\{0,1\}^\infty$ as the binary representation of a real number in $[0,1]$. So, for example, (01000..., 11000...) represents (.25, .75). This correspondence is not one-to-one, because many points in $[0,1]$ have more than one possible representation as binary strings. For example, 1000... and 0111... both represent 0.5. In fact, every fraction of the form $\frac{i}{2^k}$ for some $i, k \in \mathbb{N}$ with $0 < i < 2^k$ has two possible representations, while every other element of $[0,1]$ has a unique representation. Therefore, every element of $[0,1] \times [0,1]$ has either one, two, or four possible representations in $\{0,1\}^\infty \times \{0,1\}^\infty$. Hence, we can consider each element of $\{0,1\}^\infty \times \{0,1\}^\infty$ to represent one element of $[0,1] \times [0,1]$, but every element of $[0,1] \times [0,1]$ does not represent a unique element of $\{0,1\}^\infty \times \{0,1\}^\infty$.

Let us now introduce some notation. If $s \in \{0,1\}^\infty$, for $n \in \mathbb{N}^+$, we let $s|_n$ be the first n bits of s, and we let $s[n]$ be the nth bit of s. As usual, we let $\{0,1\}^*$ be the set of finite strings over $\{0,1\}$ and for $d \in \{0,1\}^*$, $|d|$ is the length of d. For $d \in \{0,1\}^*$ we define $S_d = \{x \in \{0,1\}^\infty \mid x|_{|d|} = d\}$, in other words, S_d is the set of all infinite binary strings that have d as a prefix. Note that S_ϵ is in $\{0,1\}^\infty$, since ϵ is the empty string. Note also that when we consider the elements of $\{0,1\}^\infty$ as points on the unit interval, we can think of S_d as a certain subinterval of the unit interval. More precisely, each S_d is the interval bounded by $\frac{d}{2^{|d|}}$ and $\frac{d+1}{2^{|d|}}$ when d is viewed as the binary representation of a natural number. As above, we cannot, however, go in the opposite direction and consider all such intervals to be sets of the form S_d, since there are multiple possible representations of some of the points in $[0,1]$ as binary strings.

Now consider the topology τ generated by the set

$$\mathcal{B} = \{S_d \mid d \in \{0,1\}^*\}.$$

It is not hard to see that $\mathcal{B}$ indeed constitutes a basis over the domain $\{0,1\}^\infty$:

1. Since $S_\epsilon \in \mathcal{B}$, we have $\bigcup \mathcal{B} = \{0,1\}^\infty$.
2. For any $U_1, U_2 \in \mathcal{B}$, we have either $U_1 \cap U_2 = \emptyset$, $U_1 \cap U_2 = U_1$ or $U_1 \cap U_2 = U_2$. Therefore, $\mathcal{B}$ is closed under finite intersections.

For our example, we use the product space $(\{0,1\}^\infty \times \{0,1\}^\infty, \tau \times \tau)$ and we have two agents a and b. Intuitively speaking, agent a is concerned with the bits of the first coordinate and agent b is concerned with the bits of the second coordinate encoded as infinite binary strings. Let $\theta_\epsilon((x,y))(a) = \theta_\epsilon((x,y))(b) = \{0,1\}^\infty \times \{0,1\}^\infty$, and for $i \in \mathbb{N}^+$, let $\theta_i((x,y))(a) = S_{x|_i} \times \{0,1\}^\infty$, and let $\theta_i((x,y))(b) = \{0,1\}^\infty \times S_{y|_i}$, where $\mathcal{D}(\theta_i) = \{0,1\}^\infty \times \{0,1\}^\infty$. In other words, for agent a, the neighbourhood function θ_i gives the set of pairs where the first component of the pair agrees with x in the first i bits, and any possible second component of the pair is allowed. Similarly for agent b. We note that θ_{i+1} is always more informative than θ_i. Finally, in order to obtain our neighbourhood function set Φ, we must close the set of functions described above under open domain restrictions, so we let $\Phi = \{\theta : X \rightharpoonup \{a,b\} \to \tau \mid \exists i \in \mathbb{N}^+ \cup \{\epsilon\}, U \in \tau \text{ such that } \theta = \theta_i^U\}$. It is easy to see that Φ satisfies the properties of a neighbourhood function set given in Definition 8.1.1.

In order to evaluate formulas on this topo-frame, we define atomic propositions

$$\mathit{Prop} = \{\mathbf{x}_i \mid i \in \mathbb{N}^+\} \cup \{\mathbf{y}_i \mid i \in \mathbb{N}^+\}$$

where

$$V(\mathbf{x}_i) = \{(x,y) \in \{0,1\}^\infty \times \{0,1\}^\infty \mid x[i] = 1\};$$
$$V(\mathbf{y}_i) = \{(x,y) \in \{0,1\}^\infty \times \{0,1\}^\infty \mid y[i] = 1\}.$$

Intuitively speaking, the propositional variables refer to the x- and y-coordinates of the pairs of infinite binary strings. We read $\mathbf{x}_i$ as "*the ith bit of the x-coordinate is 1*" and $\mathbf{y}_i$ as "*the ith bit of the y-coordinate is 1*".

We can now evaluate some formulas on the topo-model

$$\mathcal{X} = (\{0,1\}^\infty \times \{0,1\}^\infty, \tau \times \tau, \Phi, V)$$

at the state $(x,y) = (010000....., 110110.....)$ with respect to the neighbourhood function θ_1. In other words, we have that a knows that the first bit of x is 0,

b knows that the first bit of y is 1, and both are ignorant about the other's bits. More formally, we have

$\mathcal{X}, ((x,y),\theta_1) \models K_a\neg\mathbf{x}_1$	a knows that $x[1] = 0$
$\mathcal{X}, ((x,y),\theta_1) \models K_b\mathbf{y}_1$	b knows that $y[1] = 1$
$\mathcal{X}, ((x,y),\theta_1) \models K_a\neg(K_b\mathbf{x}_1 \vee K_b\neg\mathbf{x}_1)$	a knows that b does not know the value of $x[1]$
$\mathcal{X}, ((x,y),\theta_1) \models K_b\neg(K_a\mathbf{y}_1 \vee K_a\neg\mathbf{y}_1)$	b knows that a does not know the value of $y[1]$.

Now consider announcements of the following form: given $((x,y),\theta_n)$ (wherein a and b know up to the nth bit of x and y, respectively), the announcement φ_x^{n+1} is of the form 'if the nth bit of x is 1, then the $(n+1)$st bit is j, and if the nth bit of x is 0, then $(n+1)$st bit of x is $1-j$' with the restriction that the announcement is indeed truthful and where $j \in \{0,1\}$. So it can only be announced for $j = 0$ or $j = 1$ but not for both. In other words, φ_x^{n+1} is either of the form 'the nth bit of x is equal to its $(n+1)$st bit' or of the form 'the nth bit of x is different from its $(n+1)$st bit' but they cannot be announced at the same time as only one of them can be truthful. Then, this announcement informs a *but not* b of the value of the $(n+1)$st digit of x. For b it is merely an extension of the initial sequences (that he is unable to distinguish anyway, as we will see) with either 1 or 0. But he does not know which is the real one. Then, the next announcement φ_y^{n+1} informs b of the $(n+1)$st bit of y in the same way. We can go on in the same way, and successively announce the first n bits of both sequences by public announcements in such a way that a learns every prefix of x and b learns every prefix of y up to length n, as desired; but a remains uncertain about every bit in the y-prefix that b learnt, and b remains uncertain about every bit in the x-prefix that a learnt. For example, given that the agents a and b only learned their first bits and that $x = 010000\ldots$ and $y = 110110\ldots$, the next two announcements are now:

$$\begin{aligned}\varphi_x^2 &= (\neg\mathbf{x}_1 \to \mathbf{x}_2) \wedge (\mathbf{x}_1 \to \neg\mathbf{x}_2)\\ \varphi_y^2 &= (\mathbf{y}_1 \to \mathbf{y}_2) \wedge (\neg\mathbf{y}_1 \to \neg\mathbf{y}_2)\end{aligned}$$

where φ_x^2 truthfully states that "the first bit of the sequence x is different from its second bit", and φ_y^2 truthfully states that "the first and the second bit of y are the same". We then have that

$$\begin{aligned}Int([\![\varphi_x^2]\!]^{\theta_1}) &= S_{01} \times \{0,1\}^\infty \cup S_{10} \times \{0,1\}^\infty\\ Int([\![\varphi_y^2]\!]^{\theta_1}) &= \{0,1\}^\infty \times S_{11} \cup \{0,1\}^\infty \times S_{00}.\end{aligned}$$

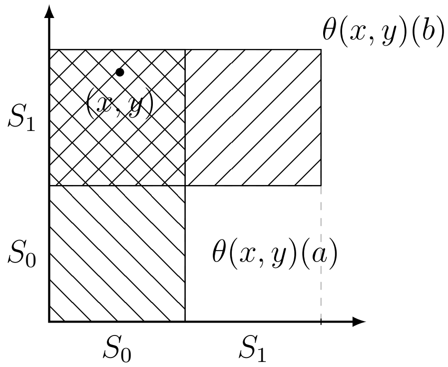


Figure 8.4: Initial situation where a knows the 1st bit of x is 0 and b knows the first bit of y is 1, and both are ignorant about the other's bit. We have $\theta((x,y))(a) = S_0 \times \{0,1\}^\infty$ and $\theta((x,y))(b) = \{0,1\}^\infty \times S_1$.

$\Downarrow \langle \varphi_x^2 \rangle$

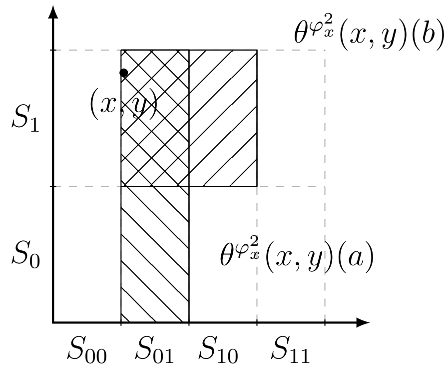


Figure 8.5: After the announcement of φ_x^2, we obtain the following smaller neighbourhoods given by the updated function $\theta^{\varphi_x^2}$: $\theta^{\varphi_x^2}((x,y))(a) = S_{01} \times \{0,1\}^\infty$, and $\theta^{\varphi_x^2}((x,y))(b) = (S_{01} \cup S_{10}) \times S_1$.

$\Downarrow \langle \varphi_y^2 \rangle$

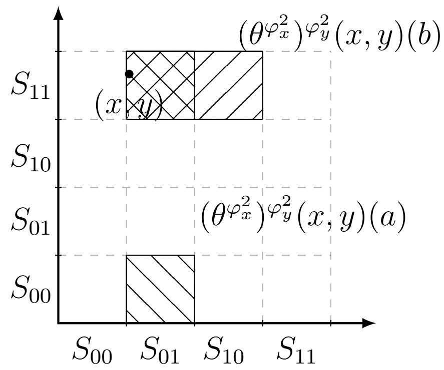


Figure 8.6: After further announcing φ_y^2, the updated function $(\theta^{\varphi_x^2})^{\varphi_y^2}$ gives the neighbourhoods: $(\theta^{\varphi_x^2})^{\varphi_y^2}(x,y)(a) = S_{01} \times (S_{00} \cup S_{11})$, and $(\theta^{\varphi_x^2})^{\varphi_y^2}(x,y)(b) = (S_{01} \cup S_{10}) \times S_{11}$

Figures 8.4-8.6 depict the neighbourhood transformations that result from the announcement φ_x^2 and, after that, the announcement of φ_y^2, consecutively. One can show (details omitted) that

$$\begin{array}{l}\mathcal{X}, ((x,y),\theta_1) \models \circledast K_a\mathbf{x}_2\\ \mathcal{X}, ((x,y),\theta_1) \models \langle\varphi_x^2\rangle(K_a\mathbf{x}_2 \wedge \neg(K_b\mathbf{x}_2 \vee K_b\neg\mathbf{x}_2))\\ \mathcal{X}, ((x,y),\theta_1) \models \langle\varphi_x^2\rangle\langle\varphi_y^2\rangle(K_b\mathbf{y}_2 \wedge \neg(K_a\mathbf{y}_2 \vee K_a\neg\mathbf{y}_2))\\ \mathcal{X}, ((x,y),\theta_2) \models K_a\mathbf{x}_2.\end{array}$$

After every finite sequence of such announcements, a knows a prefix of x and b knows a prefix of y, and a is uncertain between two dual prefixes of y and b is uncertain between two prefixes of x. So, for example, after 10 announcements, a is uncertain whether y starts with 110110 or 001001, etc.

8.2 Axiomatizations, Soundness and Expressivity

We now provide the axiomatizations for multi-agent $\mathsf{EL}^m_{\mathsf{int}}$, $\mathsf{PAL}^m_{\mathsf{int}}$, and $\mathsf{APAL}^m_{\mathsf{int}}$ (in Table 8.1), and prove their soundness with respect to the proposed semantics. The axiomatization of $\mathsf{APAL}^m_{\mathsf{int}}$ involves an infinitary rule, denoted by ($\boxast^\omega$-intro), that is formalized using necessity forms. To this end, we first define necessity forms for the language $\mathcal{L}^!_{K\mathsf{int}\boxast}$. These necessity forms are defined similarly as in Definition 7.1.22, but involve a recursive clause for int and each K_i.

8.2.1. Definition. [Necessity and possibility forms for $\mathcal{L}^!_{K\mathsf{int}\boxast}$] For any finite string $s \in (\{\varphi\!\rightarrow \mid \varphi \in \mathcal{L}^!_{K\mathsf{int}\boxast}\} \cup \{K_i, \mathsf{int} \mid i \in \mathcal{A}\} \cup \{\psi \mid \psi \in \mathcal{L}^!_{K\mathsf{int}\boxast}\})^* = NF_{\boxast}$, we define pseudo-modalities $[s]$ and $\langle s\rangle$. These pseudo-modalities are functions mapping any formula $\varphi \in \mathcal{L}^!_{K\mathsf{int}\boxast}$ to another formula $[s]\varphi \in \mathcal{L}^!_{K\mathsf{int}\boxast}$ (*necessity form*), respectively $\langle s\rangle\varphi \in \mathcal{L}^!_{K\mathsf{int}\boxast}$ (*possibility form*). The necessity forms are defined recursively as $[\epsilon]\varphi = \varphi$, $[\varphi\!\rightarrow, s]\varphi = \varphi \rightarrow [s]\varphi$, $[K_i, s]\varphi = K_i[s]\varphi$, $[\mathsf{int}, s]\varphi = \mathsf{int}([s]\varphi)$, $[\psi, s]\varphi = [\psi][s]\varphi$, where ϵ is the empty string. For possibility forms, we set $\langle s\rangle\varphi := \neg[s]\neg\varphi$.

The system $\mathsf{APAL}^m_{\mathsf{int}}$ is the smallest subset of the language $\mathcal{L}^!_{K\mathsf{int}\boxast}$ that contains the axioms, and is closed under the inference rules given in Table 8.1. The system $\mathsf{EL}^m_{\mathsf{int}}$ is defined in a similar way over the language $\mathcal{L}_{K\mathsf{int}}$ by the axioms and inference rules in group (I) of Table 8.1, and $\mathsf{PAL}^m_{\mathsf{int}}$ is defined over the language $\mathcal{L}^!_{K\mathsf{int}}$ by the axioms and inference rules in groups (I) and (II).

Let us now elaborate on these axiomatizations. The axiomatizations of multi-agent $\mathsf{EL}^m_{\mathsf{int}}$ and $\mathsf{PAL}^m_{\mathsf{int}}$ are straightforward generalizations of their single-agent versions presented in Table 7.1.[1] The axiom scheme ($\boxast$-elim) is similar to ($\Box$-elim) of $\mathsf{DTL}_{\mathsf{int}}$, directly reflecting the semantics of the arbitrary announcement

[1] The axiom scheme R[$\top$] given in Table 7.1 is derivable in $\mathsf{PAL}^m_{\mathsf{int}}$ for the multi-agent language $\mathcal{L}^!_{K\mathsf{int}}$. This can be proven easily by $\leq^S$-induction on φ using the reduction axioms. R[$\top$] is also derivable in $\mathsf{APAL}^m_{\mathsf{int}}$ for the language $\mathcal{L}^!_{K\mathsf{int}\boxast}$: its proof follows by $<^S_d$-induction on φ using ($\boxast$-elim) and ($\boxast^\omega$-intro) (see Appendix A.1 for the definition of $<^S_d$).

	(I) Axioms and rules of system $\mathsf{EL}^m_{\mathsf{int}}$:
(CPL)	all classical propositional tautologies and Modus Ponens
($\mathrm{S5}_K$)	all $\mathsf{S5}$ axioms and rules for the knowledge modality K_i
($\mathrm{S4}_{\mathsf{int}}$)	all $\mathsf{S4}$ axioms and rules for the interior modality int
(K-int)	*Knowledge implies knowability*: $K_i\varphi \to \mathsf{int}(\varphi)$
	(II) Additional axioms and rules for $\mathsf{PAL}^m_{\mathsf{int}}$:
($\mathrm{K}_!$)	$[\varphi](\psi \to \theta) \to ([\varphi]\psi \to [\varphi]\theta)$
($\mathrm{Nec}_!$)	from θ, infer $[\varphi]\theta$
([!]RE)	from $\varphi \leftrightarrow \psi$, infer $[\varphi]\theta \leftrightarrow [\psi]\theta$
	Reduction axioms:
(R_p)	$[\varphi]p \leftrightarrow (\mathsf{int}(\varphi) \to p)$
($\mathrm{R}_\neg$)	$[\varphi]\neg\psi \leftrightarrow (\mathsf{int}(\varphi) \to \neg[\varphi]\psi)$
(R_K)	$[\varphi]K_i\psi \leftrightarrow (\mathsf{int}(\varphi) \to K_i[\varphi]\psi)$
($\mathrm{R}_{[!]}$)	$[\varphi][\psi]\chi \leftrightarrow [\langle\varphi\rangle\psi]\chi$
	(III) Axioms and rules of $\boxast$ for $\mathsf{APAL}^m_{\mathsf{int}}$:
($\boxast$-elim)	$\boxast\varphi \to [\chi]\varphi$ ($\chi \in \mathcal{L}^!_{K\mathsf{int}}$ arbitrary formula)
($\boxast^\omega$-intro)	from $[s][\psi]\chi$ for all $\psi \in \mathcal{L}^!_{K\mathsf{int}}$, infer $[s]\boxast\chi$

Table 8.1: The axiomatizations for multi-agent $\mathsf{EL}^m_{\mathsf{int}}$, $\mathsf{PAL}^m_{\mathsf{int}}$ and $\mathsf{APAL}^m_{\mathsf{int}}$.

modality $\boxast$. On the other hand, the inference rule ($\boxast^\omega$-intro) is infinitary, thus making the multi-agent logic $\mathsf{APAL}^m_{\mathsf{int}}$ quite different from the other logics studied in this dissertation. In an infinitary proof system the notion of a derivation is non-standard since a derivation of a formula can involve infinitely many premises, in particular within the axiomatic system of $\mathsf{APAL}^m_{\mathsf{int}}$, an application of the rule ($\boxast^\omega$-intro) requires infinitely many premises. We can think of a *derivation involving an infinitary inference rule* as a finite-depth tree with possibly *infinite branching*, where the leaves are axioms or premises, the root is the derived formula, and a step in the tree from child nodes to parent node corresponds to the application of a derivation rule. Note that, due to the infinitary derivation rule ($\boxast^\omega$-intro) of $\mathsf{APAL}^m_{\mathsf{int}}$, the set of formulas Γ deriving φ within this system can be infinite, hence, the set of all theorems of $\mathsf{APAL}^m_{\mathsf{int}}$ cannot be defined by using the usual notion of a derivation as a *finite* sequence of formulas where each element of the sequence is either an axiom or obtained from the previous formulas in the sequence by a rule of inference. The set of all theorems of $\mathsf{APAL}^m_{\mathsf{int}}$ is then defined as the smallest subset of $\mathcal{L}^!_{K\mathsf{int}\boxast}$ that contains all the axioms, and is closed under the inference rules given in Table 8.1. In this case, we write $\varphi \in \mathsf{APAL}^m_{\mathsf{int}}$. We refer to (Goldblatt, 1982, Chapter 2.4) for a more detailed discussion of infinitary proof systems, and to (Balbiani and van Ditmarsch, 2015, p. 70) for a discussion on the axiomatizations of arbitrary announcement logics (see also Rybakov, 1997, Chapter 5.4 for a precise treatment of infinitary calculi).[2] On the other hand, derivations in $\mathsf{EL}^m_{\mathsf{int}}$ and $\mathsf{PAL}^m_{\mathsf{int}}$ are of the form of finite-depth trees with *finite* branching, since their axiomatizations contain only finitary derivation rules.

8.2.2. Proposition. *The following reduction schemas and the rule (RE) are provable both in* $\mathsf{PAL}^m_{\mathsf{int}}$ *and* $\mathsf{APAL}^m_{\mathsf{int}}$ *(for languages* $\mathcal{L}^!_{K\mathsf{int}}$ *and* $\mathcal{L}^!_{K\mathsf{int}\boxast}$*, respectively).*

1. $(\mathrm{R}_\bot)$ $\quad [\varphi]\bot \leftrightarrow \neg\mathsf{int}(\varphi)$
2. $(\mathrm{R}_\wedge)$ $\quad [\varphi](\psi \wedge \theta) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\theta)$
3. $(\mathrm{R}_{\mathsf{int}})$ $\quad [\varphi]\mathsf{int}(\psi) \leftrightarrow (\mathsf{int}(\varphi) \to \mathsf{int}([\varphi]\psi))$
4. (RE) $\quad$ from $\psi \leftrightarrow \chi$, infer $\varphi \leftrightarrow \varphi\{\psi/\chi\}$

Proof:
See Proposition 7.1.2: for (RE), we use ($\boxast$-elim) and ($\boxast^\omega$-intro) to prove the K-axiom and the Necessitation rule for $\boxast$. $\square$

We next provide some semantic results that will be helpful in the validity proof of $(\mathrm{R}_{[!]})$.

8.2.3. Lemma. *For any topo-model* $\mathcal{X} = (X, \tau, \Phi, V)$, $\theta \in \Phi$ *and* $\varphi, \psi \in \mathcal{L}^!_{K\mathsf{int}\boxast}$, *we have*

1. $[\![\psi]\!]^{\theta^\varphi} = [\![\langle\varphi\rangle\psi]\!]^\theta$,
2. $(\theta^\varphi)^\psi = \theta^{\langle\varphi\rangle\psi}$, *and*
3. $[\![\langle\varphi\rangle\mathsf{int}(\chi)]\!]^\theta = [\![\mathsf{int}(\langle\varphi\rangle\chi)]\!]^\theta$.

Proof:
Let $\mathcal{X} = (X, \tau, \Phi, V)$ be a topo-model, $\theta \in \Phi$ and $\varphi, \psi \in \mathcal{L}^!_{K\mathsf{int}\boxast}$. For (1) we have:

$$\begin{aligned}
[\![\psi]\!]^{\theta^\varphi} &= \{y \in \mathcal{D}(\theta^\varphi) \mid (y, \theta^\varphi) \models \psi\} \\
&= \{y \in \mathit{Int}([\![\varphi]\!]^\theta) \mid (y, \theta^\varphi) \models \psi\} && (\mathcal{D}(\theta^\varphi) = \mathit{Int}([\![\varphi]\!]^\theta)) \\
&= \{y \in \mathcal{D}(\theta) \mid y \in \mathit{Int}([\![\varphi]\!]^\theta) \text{ and } (y, \theta^\varphi) \models \psi\} && (\mathit{Int}([\![\varphi]\!]^\theta) \subseteq \mathcal{D}(\theta)) \\
&= \{y \in \mathcal{D}(\theta) \mid (y, \theta) \models \langle\varphi\rangle\psi\} && (\text{by the semantics of } [!]) \\
&= [\![\langle\varphi\rangle\psi]\!]^\theta
\end{aligned}$$

[2]Finitary alternatives for the axiomatizations of the multi-agent arbitrary announcement logic (without the interior modality) based on Kripke models with equivalence relations were considered in (Balbiani et al., 2008, Section 4). They, for example, proposed an axiomatization with a ($[!]\Box$-intro)-like rule from Table 7.1. However, it was later proven that their inference rule was not sound in the multi-agent setting (see `http://personal.us.es/hvd/APAL_counterexample.pdf` for the resounding counterexample). This counterexample also applies in our setting as a special case. We are therefore not aware of a sound and complete *finitary* axiomatization of a multi-agent logic of arbitrary announcements, neither for Kripke models nor for topo-models.

For (2): By Definition 8.1.4, we have that $\mathcal{D}(\theta^{\langle\varphi\rangle\psi}) = Int(\llbracket\langle\varphi\rangle\psi\rrbracket^{\theta})$ and $\mathcal{D}((\theta^{\varphi})^{\psi}) = Int(\llbracket\psi\rrbracket^{\theta^{\varphi}})$. We then obtain

$$\mathcal{D}((\theta^{\varphi})^{\psi}) = Int(\llbracket\psi\rrbracket^{\theta^{\varphi}}) = Int(\llbracket\langle\varphi\rangle\psi\rrbracket^{\theta}) = \mathcal{D}(\theta^{\langle\varphi\rangle\psi}),$$

where the second equality follows by part (1). Therefore, $(\theta^{\varphi})^{\psi}$ and $\theta^{\langle\varphi\rangle\psi}$ are defined for the same states. Moreover, for any $x \in \mathcal{D}((\theta^{\varphi})^{\psi})$ and $i \in \mathcal{A}$, we have

$$\begin{aligned}
&(\theta^{\varphi})^{\psi}(x)(i) \\
&= \theta^{\varphi}(x)(i) \cap Int(\llbracket\psi\rrbracket^{\theta^{\varphi}}) \\
&= \theta(x)(i) \cap Int(\llbracket\varphi\rrbracket^{\theta}) \cap Int(\llbracket\psi\rrbracket^{\theta^{\varphi}}) && \text{(since } Int(\llbracket\psi\rrbracket^{\theta^{\varphi}}) \subseteq Int(\llbracket\varphi\rrbracket^{\theta})\text{)} \\
&= \theta(x)(i) \cap Int(Int(\llbracket\varphi\rrbracket^{\theta})) \cap Int(\llbracket\psi\rrbracket^{\theta^{\varphi}}) && \text{(by the properties of } Int\text{)} \\
&= \theta(x)(i) \cap Int(\llbracket\mathsf{int}(\varphi)\rrbracket^{\theta}) \cap Int(\llbracket\psi\rrbracket^{\theta^{\varphi}}) && \text{(by the semantics of } \mathsf{int}\text{)} \\
&= \theta(x)(i) \cap Int(\llbracket\mathsf{int}(\varphi)\rrbracket^{\theta} \cap \llbracket\psi\rrbracket^{\theta^{\varphi}}) && \text{(by the properties of } Int\text{)} \\
&= \theta(x)(i) \cap Int(\llbracket\mathsf{int}(\varphi)\rrbracket^{\theta} \cap \llbracket\langle\varphi\rangle\psi\rrbracket^{\theta}) && \text{(by Proposition 8.2.3-(1))} \\
&= \theta(x)(i) \cap Int(\llbracket\mathsf{int}(\varphi) \wedge \langle\varphi\rangle\psi\rrbracket^{\theta}) && \text{(by the semantics of } \wedge\text{)} \\
&= \theta(x)(i) \cap Int(\llbracket\langle\varphi\rangle\psi\rrbracket^{\theta}) && \text{(by the semantics of } \mathsf{int}\text{)} \\
&= \theta^{\langle\varphi\rangle\psi}(x)(i) && \text{(by the definition of } \theta^{\langle\varphi\rangle\psi}\text{)}
\end{aligned}$$

For (3):

$$\begin{aligned}
\llbracket\langle\varphi\rangle\mathsf{int}(\chi)\rrbracket^{\theta} &= Int(\llbracket\varphi\rrbracket^{\theta}) \cap \llbracket\mathsf{int}(\chi)\rrbracket^{\theta^{\varphi}} \\
&= Int(Int(\llbracket\varphi\rrbracket^{\theta})) \cap \llbracket\mathsf{int}(\chi)\rrbracket^{\theta^{\varphi}} && \text{(by the properties of } Int\text{)} \\
&= Int(\llbracket\mathsf{int}(\varphi)^{\theta})\rrbracket) \cap Int(\llbracket\chi\rrbracket^{\theta^{\varphi}}) && \text{(by the semantics of } \mathsf{int}\text{)} \\
&= Int(\llbracket\mathsf{int}(\varphi)^{\theta})\rrbracket) \cap Int(\llbracket\langle\varphi\rangle\chi\rrbracket^{\theta}) && \text{(by Proposition 8.2.3-(1))} \\
&= Int(\llbracket\mathsf{int}(\varphi)\rrbracket^{\theta} \cap \llbracket\langle\varphi\rangle\chi\rrbracket^{\theta}) && \text{(by the properties of } Int\text{)} \\
&= Int(\llbracket\langle\varphi\rangle\chi\rrbracket^{\theta}) && \text{(by the semantics of [!])} \\
&= \llbracket\mathsf{int}(\langle\varphi\rangle\chi)\rrbracket^{\theta} && \text{(by the semantics of } \mathsf{int}\text{)}
\end{aligned}$$

□

8.2.4. PROPOSITION. $\mathsf{APAL}^{m}_{\mathsf{int}}$ *is sound with respect to the class of all topo-models.*

Proof:
The soundness of $\mathsf{APAL}^{m}_{\mathsf{int}}$ is, as usual, shown by proving that all axioms are validities and that all derivation rules preserve validities. Having proved that, soundness follows by induction on the depth of the derivation tree.

We prove the following cases: the first two cases shows the validity of the reduction axioms (R_K) and ($R_{[!]}$), the next two illustrate the need for the constraint in Definition 8.1.1-(3), the fifth shows the validity of the axiom (K-int)

which connects the modalities K_i and int, and the last two prove validity of the axiom ($\boxast$-elim) and validity preservation of the inference rule ($\boxast^{\omega}$-intro). Let $\mathcal{X} = (X, \tau, \Phi, V)$ be a topo-model, $(x, \theta) \in \mathcal{X}$ and $\varphi, \psi, \chi \in \mathcal{L}^{!}_{K\mathsf{int}\boxast}$.

(R_K): Suppose $(x,\theta) \models [\varphi]K_i\psi$. This means that if $(x,\theta) \models \mathsf{int}(\varphi)$ then $(x,\theta^{\varphi}) \models K_i\psi$. We want to show that $(x,\theta) \models \mathsf{int}(\varphi) \to K_i[\varphi]\psi$. Hence, suppose also that $(x,\theta) \models \mathsf{int}(\varphi)$ and let $z \in \theta(x)(i)$ such that $(z,\theta) \models \mathsf{int}(\varphi)$, i.e., that $z \in Int(\llbracket\varphi\rrbracket^{\theta})$. Then, by assumption, $(x,\theta) \models \mathsf{int}(\varphi)$ implies that $(x,\theta^{\varphi}) \models K_i\psi$. In other words, $(y,\theta^{\varphi}) \models \psi$ for all $y \in \theta^{\varphi}(x)(i)$. Recall, by Definition 8.1.4, that $\theta^{\varphi}(x)(i) = \theta(x)(i) \cap Int(\llbracket\varphi\rrbracket^{\theta})$. Thus, since $z \in \theta(x)(i) \cap Int(\llbracket\varphi\rrbracket^{\theta}) = \theta^{\varphi}(x)(i)$, we obtain $(z,\theta^{\varphi}) \models \psi$, implying together with the assumption $z \in Int(\llbracket\varphi\rrbracket^{\theta})$ that $(z,\theta) \models [\varphi]\psi$. Since z has been chosen arbitrarily from $\theta(x)(i)$, the results holds for every element of $\theta(x)(i)$. Therefore, $(x,\theta) \models K_i[\varphi]\psi$. Since we also have $(x,\theta) \models \mathsf{int}(\varphi)$, we conclude $(x,\theta) \models \mathsf{int}(\varphi) \to K_i[\varphi]\psi$. The converse direction follows similarly.

($\mathrm{R}_{[!]}$):

$$
\begin{aligned}
&(x,\theta) \models [\varphi][\psi]\chi \\
&\text{iff } ((x,\theta) \models \mathsf{int}(\varphi) \text{ and } (x,\theta^{\varphi}) \models \mathsf{int}(\psi)) \text{ implies } (x,(\theta^{\varphi})^{\psi}) \models \chi \\
&\text{iff } (x,\theta) \models \langle\varphi\rangle\mathsf{int}(\psi) \text{ implies } (x,(\theta^{\varphi})^{\psi}) \models \chi \\
&\text{iff } (x,\theta) \models \mathsf{int}(\langle\varphi\rangle\psi) \text{ implies } (x,\theta^{\langle\varphi\rangle\psi}) \models \chi && \text{(Proposition 8.2.3-(2-3))} \\
&\text{iff } (x,\theta) \models [\langle\varphi\rangle\psi]\chi && \text{(by the semantics of [!])}
\end{aligned}
$$

(4_K): Suppose $(x,\theta) \models K_i\varphi$. This means, $(y,\theta) \models \varphi$ for all $y \in \theta(x)(i)$. Let $y \in \theta(x)(i)$ and $z \in \theta(y)(i)$. By Definition 8.1.1-(3), $\theta(y)(i) = \theta(x)(i)$ and Definition 8.1.1-(1) guarantees that $\theta(y)(i) \neq \emptyset$. Therefore, by assumption, $(z,\theta) \models \varphi$. As z has been chosen from $\theta(y)(i)$ arbitrarily, we obtain $(y,\theta) \models K_i\varphi$. For the similar reason, we also obtain $(x,\theta) \models K_iK_i\varphi$.

(5_K): Suppose $(x,\theta) \models \neg K_i\varphi$. This means, $(y_0,\theta) \not\models \varphi$ for some $y_0 \in \theta(x)(i)$. Let $y \in \theta(x)(i)$. By Definition 8.1.1-(3), $\theta(x)(i) = \theta(y)(i)$. Therefore, as $y_0 \in \theta(y)(i)$, by assumption, we have that there is a $z \in \theta(y)(i)$, namely $z = y_0$, such that $(z,\theta) \not\models \varphi$. Thus, $(y,\theta) \models \neg K_i\varphi$. As y has been chosen from $\theta(x)(i)$ arbitrarily, we conclude $(x,\theta) \models K_i\neg K_i\varphi$.

(K-int): Suppose $(x,\theta) \models K_i\varphi$. This means, $(y,\theta) \models \varphi$ for all $y \in \theta(x)(i)$. Hence, $\theta(x)(i) \subseteq \llbracket\varphi\rrbracket^{\theta}$. By Definition 8.1.1, $\theta(x)(i)$ is an open neighbourhood of x, therefore, we obtain that $x \in Int(\llbracket\varphi\rrbracket^{\theta})$, i.e., $(x,\theta) \models \mathsf{int}(\varphi)$.

($\boxast$-elim): Suppose $(x,\theta) \models \boxast\varphi$ and let $\chi \in \mathcal{L}^{!}_{K\mathsf{int}}$. By the semantics, we have $(x,\theta) \models \boxast\varphi$ iff $(\forall\psi \in \mathcal{L}^{!}_{K\mathsf{int}})((x,\theta) \models [\psi]\varphi)$. Therefore, in particular, $(x,\theta) \models [\chi]\varphi$.

($\boxast^{\omega}$-intro): The proof follows by induction on the structure of the necessity form s. We here show the base case $[s] := [\epsilon]$ and the inductive case $[s] := [\mathsf{int}, s]$. All other inductive cases follow similarly.

Base case $s := \epsilon$: In this case, we have $[s][\psi]\chi = [\epsilon][\psi]\chi = [\psi]\chi$, and $[s]\boxast\chi = [\epsilon]\boxast\chi = \boxast\chi$. Suppose $[\psi]\chi$ is valid for all $\psi \in \mathcal{L}^{!}_{K\mathsf{int}}$. This means $\mathcal{X},(x,\theta) \models [\psi]\chi$ for all $\psi \in \mathcal{L}^{!}_{K\mathsf{int}}$, all topo-models $\mathcal{X}$, and $(x,\theta) \in \mathcal{X}$. Therefore, by the semantics, $\mathcal{X},(x,\theta) \models \boxast\chi$ for all topo-models $\mathcal{X}$, and $(x,\theta) \in \mathcal{X}$. Hence, we conclude $\models \boxast\chi$.

Induction Hyposthesis: $\models [s'][\psi]\chi$ for all $\psi \in \mathcal{L}^{!}_{K\mathsf{int}}$, implies $\models [s']\boxast\chi$

Case $[s] := [\mathsf{int}, s']$: In this case, we have $[s][\psi]\chi = [\mathsf{int}, s'][\psi]\chi = \mathsf{int}([s'][\psi]\chi)$. Suppose that $\mathsf{int}([s'][\psi]\chi)$ is valid for all $\psi \in \mathcal{L}^{!}_{K\mathsf{int}}$. This implies that $[s'][\psi]\chi$ is valid for all $\psi \in \mathcal{L}^{!}_{K\mathsf{int}}$. Otherwise, there is a topo-model $\mathcal{X} = (X, \tau, \Phi, V)$ and $(x,\theta) \in \mathcal{X}$ such that $\mathcal{X},(x,\theta) \not\models [s'][\psi]\chi$ for some $\psi \in \mathcal{L}^{!}_{K\mathsf{int}}$. This means $x \notin [\![[s'][\psi]\chi]\!]^{\theta}$. Since $\mathit{Int}([\![[s'][\psi]\chi]\!]^{\theta}) \subseteq [\![[s'][\psi]\chi]\!]^{\theta}$, we also obtain that $x \notin \mathit{Int}([\![[s'][\psi]\chi]\!]^{\theta})$, i.e., $\mathcal{X},(x,\theta) \not\models \mathsf{int}([s'][\psi]\chi)$ contradicting validity of $\mathsf{int}([s'][\psi]\chi)$. Then, by IH, we have $[s']\boxast\chi$ valid. This means that $[\![[s']\boxast\chi]\!]^{\theta} = \mathcal{D}(\theta)$ for every topo-model $\mathcal{X} = (X, \tau, \Phi, V)$ and all $\theta \in \Phi$. Since $\mathcal{D}(\theta) \in \tau$ (by Lemma 8.1.3), we have $\mathcal{D}(\theta) = \mathit{Int}(\mathcal{D}(\theta)) = \mathit{Int}([\![[s']\boxast\chi]\!]^{\theta}) = [\![\mathsf{int}([s']\boxast\chi)]\!]^{\theta}$. We can then conclude that $\mathsf{int}([s']\boxast\chi)$ is valid. □

8.2.5. Corollary. *$\mathsf{EL}^{m}_{\mathsf{int}}$ and $\mathsf{PAL}^{m}_{\mathsf{int}}$ are sound with respect to the class of all topo-models.*

8.2.6. Corollary. *$\mathcal{L}_{K\mathsf{int}}$ and $\mathcal{L}^{!}_{K\mathsf{int}}$ are co-expressive with respect to topo-models.*

Proof:
The proof follows similarly to the proof of Proposition 7.1.11. □

On the other hand, unlike the case in the single-agent setting (see Theorem 7.2.5), multi-agent $\mathcal{L}^{!}_{K\mathsf{int}\boxast}$ is strictly more expressive than $\mathcal{L}_{K\mathsf{int}}$. This is analogous to the case in the setting of Balbiani et al. (2008) based on Kripke semantics. The counterexample given in (Balbiani et al., 2008, Proposition 3.13) can be adapted for our framework based on a discrete topology, as shown below. To make the expressivity argument clearer, we first define a notion of *partial bisimulation* that induces a modal invariance result for the language $\mathcal{L}_K$. This is the natural analogue of the usual notion of bisimulation defined on multi-relational Kripke models (see, e.g., Blackburn et al., 2001, Chapter 2.2).

8.2.7. Definition. [Partial Bisimulation (for $\mathcal{L}_K$)] Let two topo-models $\mathcal{X} = (X, \tau, \Phi, V)$ and $\mathcal{X}' = (X', \tau', \Phi', V)$ be given. A relation $\rightleftharpoons$ between the set of neighbourhood situations of $\mathcal{X}$ and $\mathcal{X}'$ is a *partial bisimulation* between $\mathcal{X}$ and $\mathcal{X}'$ iff for all $(x,\theta) \in \mathcal{X}$ and $(x',\theta') \in \mathcal{X}'$ with $(x,\theta) \rightleftharpoons (x',\theta')$ the following conditions are satisfied:

- **Base**: for all $p \in \textsc{prop}$, $x \in V(p)$ iff $x' \in V'(p)$.

- **Forth**: for all $i \in \mathcal{A}$ and all $y \in \theta(x)(i)$, there exists $y' \in \theta'(x')(i)$ such that $(y, \theta) \rightleftharpoons (y', \theta')$
- **Back**: for all $i \in \mathcal{A}$ and all $y' \in \theta'(x')(i)$, there exists $y \in \theta(x)(i)$ such that $(y, \theta) \rightleftharpoons (y', \theta')$.

8.2.8. Proposition. *Let $\rightleftharpoons$ be a partial bisimulation between topo-models $\mathcal{X}$ and $\mathcal{X}'$ with $(x, \theta) \rightleftharpoons (x', \theta')$, where $(x, \theta) \in \mathcal{X}$ and $(x', \theta') \in \mathcal{X}'$. Then for all $\varphi \in \mathcal{L}_K$,*

$$\mathcal{X}, (x, \theta) \models \varphi \text{ iff } \mathcal{X}', (x', \theta') \models \varphi.$$

Proof:
The proof follows standardly by subformula induction on φ. □

8.2.9. Proposition. *(Multi-agent) $\mathcal{L}^!_{K\mathsf{int}\circledast}$ is strictly more expressive than $\mathcal{L}_{K\mathsf{int}}$ with respect to topo-models.*

Proof:
The proof follows the same argument as (Balbiani et al., 2008, Proposition 3.13). It is not hard to see that the modality int becomes redundant on topo-models based on discrete spaces. More precisely, given a topo-model $\mathcal{X} = (X, \mathcal{P}(X), \Phi, V)$ where $\mathcal{P}(X)$ is a the set of all subsets of X, i.e., $(X, \mathcal{P}(X))$ is the discrete space, for all $\varphi \in \mathcal{L}^!_{K\mathsf{int}\circledast}$, we have $\mathcal{X} \models \varphi \leftrightarrow \mathsf{int}(\varphi)$. This fact and the modal invariance result for the language $\mathcal{L}_K$ given in Proposition 8.2.8 help us to adapt the counterexample in (Balbiani et al., 2008, Proposition 3.13) to our setting based on a discrete space in a straightforward way. The proof follows by contradiction: suppose that $\varphi \in \mathcal{L}^!_{K\mathsf{int}\circledast}$ and $\mathcal{L}_{K\mathsf{int}}$ are equally expressive for (multi-agent) topo-models, i.e., for all $\varphi \in \mathcal{L}^!_{K\mathsf{int}\circledast}$ there exists $\psi \in \mathcal{L}_{K\mathsf{int}}$ such that $\models \varphi \leftrightarrow \psi$. Now consider the formula $\langle\circledast\rangle(K_a p \wedge \neg K_b K_a p)$. By the assumption, there must be $\psi \in \mathcal{L}_{K\mathsf{int}}$ such that $\models \varphi \leftrightarrow \psi$. To reach the desired contradiction, we now construct two models which agree on ψ at the actual neighbourhood situations but disagree on $\langle\circledast\rangle(K_a p \wedge \neg K_b K_a p)$. For this argument, it is crucial to observe that any such ψ contains only finitely many propositional variables. As we have countably many propositional variables, there is a propositional variable q that does not occur in ψ (that is also different from p). Without loss of generality, suppose ψ is built using only one variable p. Consider the topo-models $\mathcal{X}' = (\{1, 0\}, 2^{\{1,0\}}, \Phi', V')$ and $\mathcal{X} = (\{10, 00, 11, 01\}, 2^{\{10,00,11,01\}}, \Phi, V)$ such that $V'(p) = \{1\}$, and $V(p) = \{10, 11\}$ and $V(q) = \{01, 11\}$. We compare $\mathcal{X}', (1, \theta')$ with $\mathcal{X}, (10, \theta)$, where θ' and θ partition the corresponding models in such a way that a cannot distinguish p-states from $\neg p$-states, while agent b can. More precisely, we set $\theta'(1)(a) = \theta'(0)(a) = \{1, 0\}$ and $\theta'(1)(b) = \{1\}, \theta'(0)(b) = \{0\}$ (see Figure 8.7a). For $\mathcal{X}$, we have θ partitioning the space in the way shown in Figure 8.7b: $\theta(00)(a) = \theta(10)(a) = \{10, 00\}$ and $\theta(01)(a) = \theta(11)(a) = \{11, 01\}$, whereas $\theta(10)(b) = \theta(11)(b) = \{10, 11\}$ and $\theta(00)(b) = \theta(01)(b) = \{00, 01\}$.

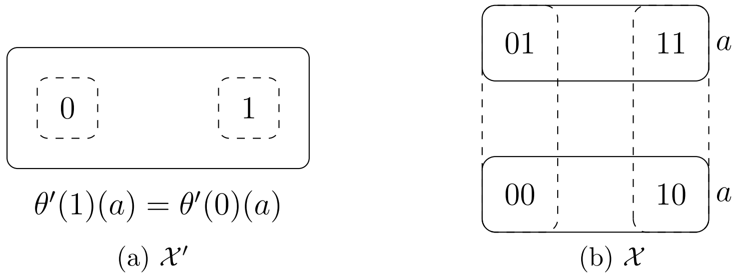


Figure 8.7: The straight round circles show the neighbourhoods of agent a, and the dashed ones are for agent b.

It is then easy to see that, for the language $\mathcal{L}_K$ build from the only propositional variable p, we have $(1, \theta') \rightleftharpoons (10, \theta)$, hence, $\mathcal{X}', (1, \theta') \models \psi$ iff $\mathcal{X}, (10, \theta) \models \psi$. However, while $\mathcal{X}', (1, \theta') \not\models \Diamond\!\!\!\!\ast\,(K_a p \wedge \neg K_b K_a p)$, we have $\mathcal{X}, (10, \theta) \models \langle p \vee q \rangle (K_a p \wedge \neg K_b K_a p)$, hence, $\mathcal{X}, (10, \theta) \models \Diamond\!\!\!\!\ast\,(K_a p \wedge \neg K_b K_a p)$. $\square$

8.3 Completeness

We now show completeness for $\mathsf{EL}^m_{\mathsf{int}}$, $\mathsf{PAL}^m_{\mathsf{int}}$, and $\mathsf{APAL}^m_{\mathsf{int}}$ with respect to the class of all topo-models. Completeness of $\mathsf{EL}^m_{\mathsf{int}}$ is shown in a standard way via a canonical model construction and a Truth Lemma that is proved by subformula induction. Completeness for $\mathsf{PAL}^m_{\mathsf{int}}$ is shown by reducing each formula in $\mathcal{L}^!_{K\mathsf{int}}$ to a provably and semantically equivalent formula of $\mathcal{L}_{K\mathsf{int}}$. The proof of the completeness for $\mathsf{APAL}^m_{\mathsf{int}}$ becomes more involved. Reduction axioms for public announcements no longer suffice in the $\mathsf{APAL}^m_{\mathsf{int}}$ case, and the inductive proof needs a subinduction where announcements are considered. Moreover, the proof system of $\mathsf{APAL}^m_{\mathsf{int}}$ has an infinitary derivation rule, namely the rule ($\boxast^\omega$-into), and given the requirement of closure under this rule, the maximally consistent sets for that case are defined to be maximally consistent *theories* (see Section 8.3.2). Lastly, the Truth Lemma requires the more complicated complexity measure on formulas defined in Appendix A.1. There, we need to adapt the completeness proof in (Balbiani and van Ditmarsch, 2015) to our setting.

8.3.1 Completeness of $\mathsf{EL}^m_{\mathsf{int}}$ and $\mathsf{PAL}^m_{\mathsf{int}}$

Recall that the logic $\mathsf{EL}^m_{\mathsf{int}}$ is the familiar multi-modal normal system whose axiomatization consists of the $\mathsf{S4}$-type modality int, the $\mathsf{S5}$-type modalities K_i and the connecting axioms (K-int). Therefore, proofs of Lemma 8.3.1 and Lemma 8.3.2 below are standard (see, e.g., Blackburn et al., 2001, Proposition 4.16 and Lemma 4.17, respectively).

8.3.1. Lemma. *For every maximally consistent set x of formulas in $\mathsf{EL}^m_{\mathsf{int}}$ and every formula $\varphi, \psi \in \mathcal{L}_{K\mathsf{int}}$*

1. $\mathsf{EL}^m_{\mathsf{int}} \subseteq x$,
2. $\varphi \in x$ *and* $\varphi \to \psi \in x$ *implies* $\psi \in x$,
3. $\varphi \in x$ *or* $\neg\varphi \in x$,
4. $\varphi \wedge \psi \in x$ *iff* $\varphi \in x$ *and* $\psi \in x$.

8.3.2. Lemma (Lindenbaum's Lemma). *Each consistent set of $\mathsf{EL}^m_{\mathsf{int}}$ can be extended to a maximally consistent set.*

Let X^c be the set of all maximally consistent sets of $\mathsf{EL}^m_{\mathsf{int}}$. We define relations $\sim_i$ on X^c as

$$x \sim_i y \text{ iff } \forall\varphi \in \mathcal{L}_{K\mathsf{int}}(K_i\varphi \in x \text{ implies } \varphi \in y).$$

Notice that the latter is equivalent to saying $\forall\varphi \in \mathcal{L}_{K\mathsf{int}}(K_i\varphi \in x \text{ iff } K_i\varphi \in y)$ since K_i is an $\mathsf{S5}$ modality. As each K_i is of $\mathsf{S5}$ type, every $\sim_i$ is an equivalence relation, hence, it induces equivalence classes on X^c. Let $[x]_i$ denote the equivalence class of x induced by the relation $\sim_i$. Moreover, we again set $\widehat{\varphi} = \{y \in X^c \mid \varphi \in y\}$.

Our canonical model construction is similar to the one for the single-agent case in (Bjorndahl, 2016). We give a comparison in Section 8.5.

8.3.3. Definition. [Canonical Model for $\mathsf{EL}^m_{\mathsf{int}}$] We define the canonical model $\mathcal{X}^c = (X^c, \tau^c, \Phi^c, V^c)$ as follows:

- X^c is the set of all maximally consistent sets of $\mathsf{EL}^m_{\mathsf{int}}$;
- τ^c is the topological space generated by the subbasis

$$\Sigma = \{[x]_i \cap \widehat{\mathsf{int}(\varphi)} \mid x \in X^c, \varphi \in \mathcal{L}_{K\mathsf{int}} \text{ and } i \in \mathcal{A}\};$$

- $x \in V^c(p)$ iff $p \in x$, for all $p \in \text{PROP}$;
- $\Phi^c = \{(\theta^c)^U \mid U \in \tau^c\}$, where we define $\theta^c : X^c \to \mathcal{A} \to \tau^c$ as $\theta^c(x)(i) = [x]_i$, for $x \in X^c$ and $i \in \mathcal{A}$.

We first need to show that (X^c, τ^c, Φ^c) is indeed a topo-frame.

8.3.4. Lemma. (X^c, τ^c, Φ^c) *is a topo-frame.*

Proof:
In order to show the above statement, we need to show that (X^c, τ^c) is a topological space, and Φ^c satisfies the conditions in Definition 8.1.1. For the former, we only need to show that Σ covers X^c, i.e., that $\bigcup \Sigma = X^c$, since τ^c is generated by a *subbasis*, namely by Σ (in the way described in Chapter 2.2). Since every element of Σ is a subset of X^c, we obviously have $\bigcup \Sigma \subseteq X^c$. Observe moreover that, since $\widehat{\mathsf{int}(\top)} = X^c$, we have $[x]_i \cap \widehat{\mathsf{int}(\top)} = [x]_i \in \Sigma$ for each $x \in X^c$ and $i \in \mathcal{A}$. Now let $x \in X^c$. Since every $\sim_i$ is an equivalence relation, in particular, each $\sim_i$ is reflexive, we have $x \in [x]_i$. Therefore, we obtain $\bigcup_{x \in X^c}[x]_i = X^c \subseteq \bigcup \Sigma$ for any $i \in \mathcal{A}$. Hence, we conclude $\bigcup \Sigma = X^c$ implying that (X^c, τ^c) is a topological space. We now show that Φ^c satisfies the conditions in Definition 8.1.1. Let $\theta \in \Phi^c$. Thus, by definition of Φ^c, we have $\theta = (\theta^c)^U$ for some $U \in \tau^c$ (in particular, note that $\theta^c = (\theta^c)^{X^c}$). Therefore, we have that $\mathcal{D}(\theta) = \mathcal{D}(\theta^c) \cap U = X^c \cap U = U \subseteq X^c$ and $\theta(x)(i) = \theta^c(x)(i) \cap U = [x]_i \cap U$ for any $x \in \mathcal{D}(\theta)$ and $i \in \mathcal{A}$. As argued above, $[x]_i \in \Sigma$ for all $x \in X^c$ and each $i \in \mathcal{A}$. We therefore obtain that function θ is defined as a partial function such that $\theta : X^c \rightharpoonup \mathcal{A} \to \tau^c$. For condition (1), let $x \in \mathcal{D}(\theta)$. Since $\mathcal{D}(\theta) = U$ and $\theta(x)(i) = [x]_i \cap U$, we also have $x \in [x]_i \cap U = \theta(x)(i)$ for all $i \in \mathcal{A}$. Moreover, since $\theta(x)(i) = [x]_i \cap U \subseteq U = \mathcal{D}(\theta)$, we also satisfy condition (2). For condition (3), let $y \in \theta(x)(i)$. As $\theta(x)(i) = [x]_i \cap U$, we have $y \in [x]_i$ and $y \in \mathcal{D}(\theta)$. While the latter proves the first consequent of condition (3), the former implies $[y]_i = [x]_i$ since $[x]_i$ is an equivalence class. We therefore obtain $\theta(y)(i) = [y]_i \cap U = [x]_i \cap U = \theta(x)(i)$. Condition (4) is satisfied by definition of Φ^c. ☐

8.3.5. LEMMA (TRUTH LEMMA). *For every $\varphi \in \mathcal{L}_{K\mathsf{int}}$ and for each $x \in X^c$,*

$$\varphi \in x \textit{ iff } \mathcal{X}^c, (x, \theta^c) \models \varphi.$$

Proof:
The proof follows by subformula induction on φ. The case for the propositional variables follows from the definition of V^c and the cases for the Booleans are straightforward. We only show the cases $\varphi := K_i\psi$ and $\varphi := \mathsf{int}(\psi)$.

Case $\varphi = K_i\psi$

($\Rightarrow$) Suppose $K_i\psi \in x$ and let $y \in \theta^c(x)(i)$. Since $y \in \theta^c(x)(i) = [x]_i$, by definition of $\sim_i$, we have $K_i\psi \in y$. Then, by axiom (T_K), we obtain $\psi \in y$. Thus, by IH, $\mathcal{X}^c, (y, \theta^c) \models \psi$. Therefore $\mathcal{X}^c, (x, \theta^c) \models K_i\psi$.

($\Leftarrow$) Suppose $K_i\psi \notin x$. Then, $\{K_i\gamma \mid K_i\gamma \in x\} \cup \{\neg\psi\}$ is a consistent set. By Lemma 8.3.2, we can then extend it to a maximally consistent set y. As $\{K_i\gamma \mid K_i\gamma \in x\} \subseteq y$, we have $y \in [x]_i$ meaning that $y \in \theta^c(x)(i)$. Moreover, since $\neg\psi \in y$, we obtain $\psi \notin y$. Therefore, we have a maximally consistent set $y \in \theta^c(x)(i)$ such that $\psi \notin y$. By IH, $\mathcal{X}^c, (y, \theta^c) \not\models \psi$. Hence, $\mathcal{X}^c, (x, \theta^c) \not\models K_i\psi$.

Case $\varphi = \mathsf{int}(\psi)$

($\Rightarrow$) Suppose $\mathsf{int}(\psi) \in x$. Consider the set $[x]_i \cap \widehat{\mathsf{int}(\psi)}$ for some $i \in \mathcal{A}$. Obviously, $x \in [x]_i \cap \widehat{\mathsf{int}(\psi)} \in \tau^c$ (in fact, $[x]_i \cap \widehat{\mathsf{int}(\psi)} \in \Sigma$). Now let $y \in [x]_i \cap \widehat{\mathsf{int}(\psi)}$. Since $y \in \widehat{\mathsf{int}(\psi)}$, we have $\mathsf{int}(\psi) \in y$. Then, by ($\mathrm{T}_{\mathsf{int}}$) and since y is maximally consistent, we have $\psi \in y$. Thus, by IH, we obtain $(y, \theta^c) \models \psi$. Therefore, $y \in [\![\psi]\!]^{\theta^c}$. This implies $[x]_i \cap \widehat{\mathsf{int}(\psi)} \subseteq [\![\psi]\!]^{\theta^c}$. And, since $x \in [x]_i \cap \widehat{\mathsf{int}(\psi)} \in \tau^c$, we have $x \in Int([\![\psi]\!]^{\theta^c})$, i.e., $(x, \theta^c) \models \mathsf{int}(\psi)$.

($\Leftarrow$) Suppose $(x, \theta^c) \models \mathsf{int}(\psi)$, i.e., $x \in Int([\![\psi]\!]^{\theta^c})$. Recall that the set of finite intersections of the elements of Σ forms a basis, which we denote by $\mathcal{B}_\Sigma$, for τ^c. The assumption $x \in Int([\![\psi]\!]^{\theta^c})$ implies that there exists an open $U \in \mathcal{B}_\Sigma$ such that $x \in U \subseteq [\![\psi]\!]^{\theta^c}$. Given the construction of $\mathcal{B}_\Sigma$, U is of the form

$$U = \bigcap_{i \in I_1} [x_1]_i \cap \cdots \bigcap_{i \in I_n} [x_k]_i \cap \bigcap_{\eta \in \mathrm{Form_{fin}}} \widehat{\mathsf{int}(\eta)}$$

where $I_1, \ldots, I_n$ are finite subsets of $\mathcal{A}$, $x_1 \ldots x_k \in X^c$ and $\mathrm{Form_{fin}}$ is a finite subset of $\mathcal{L}_{K\mathsf{int}}$. Since int is a normal modality, we can simply write

$$U = \bigcap_{i \in I_1} [x_1]_i \cap \cdots \bigcap_{i \in I_n} [x_k]_i \cap \widehat{\mathsf{int}(\gamma)},$$

where $\bigwedge_{\eta \in \mathrm{Form_{fin}}} \eta := \gamma$. Since x is in each $[x_j]_i$ with $1 \leq j \leq k$, we have $[x_j]_i = [x]_i$ for all such j. Therefore, we have

$$x \in U = (\bigcap_{i \in I} [x]_i) \cap \widehat{\mathsf{int}(\gamma)} \subseteq [\![\psi]\!]^{\theta^c},$$

where $I = I_1 \cup \cdots \cup I_n$. This implies, for all $y \in (\bigcap_{i \in I} [x]_i)$, if $y \in \widehat{\mathsf{int}(\gamma)}$ then $\psi \in y$. From this, we can say $\bigcup_{i \in I} \{K_i \sigma \mid K_i \sigma \in x\} \vdash \mathsf{int}(\gamma) \to \psi$. Then, there is a finite subset $\Gamma \subseteq \bigcup_{i \in I} \{K_i \sigma \mid K_i \sigma \in x\}$ such that $\vdash \bigwedge_{\lambda \in \Gamma} \lambda \to (\mathsf{int}(\gamma) \to \psi)$. It then follows by the normality of int that

$$\vdash (\bigwedge_{\lambda \in \Gamma} \mathsf{int}(\lambda)) \to \mathsf{int}(\mathsf{int}(\gamma) \to \psi)).$$

Observe that each $\lambda \in \Gamma$ is of the form $K_j \alpha$ for some $K_j \alpha \in \bigcup_{i \in I} \{K_i \sigma \mid K_i \sigma \in x\}$ and we have $\vdash K_i \varphi \leftrightarrow \mathsf{int}(K_i \varphi)$. Therefore, $\vdash (\bigwedge_{\lambda \in \Gamma} \lambda) \to \mathsf{int}(\mathsf{int}(\gamma) \to \psi))$. Thus, since $\bigwedge_{\lambda \in \Gamma} \lambda \in x$ (by $\Gamma \subseteq x$ and x being maximal), we have $\mathsf{int}(\mathsf{int}(\gamma) \to \psi)) \in x$. Then, by ($\mathrm{K}_{\mathsf{int}}$) and Lemma 8.3.1-(2), we obtain $\mathsf{int}(\mathsf{int}(\gamma)) \to \mathsf{int}(\psi) \in x$. Moreover, since $\vdash \mathsf{int}(\mathsf{int}(\gamma)) \leftrightarrow \mathsf{int}(\gamma)$ and $x \in \widehat{\mathsf{int}(\gamma)}$ (i.e., $\mathsf{int}(\gamma) \in x$), we conclude $\mathsf{int}(\psi) \in x$. $\square$

8.3.6. THEOREM. $\mathsf{EL}^m_{\mathsf{int}}$ *is complete with respect to the class of all topo-models.*

Proof:
Let φ be a $\mathsf{EL}^m_{\mathsf{int}}$-consistent formula. Then, by Lemma 8.3.2, the singleton $\{\varphi\}$ can be extended to a maximally consistent set x of $\mathsf{EL}^m_{\mathsf{int}}$ with $\varphi \in x$. Therefore, by Lemma 8.3.5, we obtain $\mathcal{X}^c, (x, \theta^c) \models \varphi$, where $\mathcal{X}^c = (X^c, \tau^c, \Phi^c, V^c)$ is the canonical model. □

8.3.7. Theorem. $\mathsf{PAL}^m_{\mathsf{int}}$ *is complete with respect to the class of all topo-models.*

Proof:
This follows from Theorem 8.3.6 by reduction in a standard way: using the size measure $S(\varphi)$ given in Definition A.1.1 for the language $\mathcal{L}^!_{K\mathsf{int}}$ provides the desired result via Lemma A.1.5 (note that the strict orders $<^S$ and $<^S_d$ given in Definition A.1.3 are equivalent on the language $\mathcal{L}^!_{K\mathsf{int}}$ as given in Lemma A.1.4-(2)). We refer to (van Ditmarsch et al., 2007, Chapter 7.4) for a detailed presentation of the completeness method via reduction, and in particular to (Wang and Cao, 2013, Theorem 10, p. 111) for an analogous proof. A similar proof for single-agent $\mathsf{EL}^m_{\mathsf{int}}$ is also presented in (Bjorndahl, 2016, Section 4). □

8.3.2 Completeness of $\mathsf{APAL}^m_{\mathsf{int}}$

We now reuse the technique of Balbiani and van Ditmarsch (2015) in the setting of topological semantics. Given the closure requirement under the derivation rule ($\boxast^\omega$-intro), it seems more proper to call maximally consistent sets of $\mathsf{APAL}^m_{\mathsf{int}}$ *maximally consistent theories*, as further explained below.

8.3.8. Definition. [Theory of $\mathsf{APAL}^m_{\mathsf{int}}$]

- A set x of formulas is called a *theory of* $\mathsf{APAL}^m_{\mathsf{int}}$ (or simply, a *theory*) iff $\mathsf{APAL}^m_{\mathsf{int}} \subseteq x$ and x is closed under Modus Ponens and ($\boxast^\omega$-intro).
- A theory x is said to be *consistent* iff $\bot \notin x$.
- A theory x is *maximally consistent* iff x is consistent and any set of formulas properly containing x is inconsistent.

The logic $\mathsf{APAL}^m_{\mathsf{int}}$ constitutes the smallest theory. Moreover, maximally consistent theories of $\mathsf{APAL}^m_{\mathsf{int}}$ possess the usual properties of maximally consistent sets:

8.3.9. Lemma. *For any maximally consistent theory x of $\mathsf{APAL}^m_{\mathsf{int}}$, and $\varphi, \psi \in \mathcal{L}^!_{K\mathsf{int}\boxast}$*

1. *$\varphi \notin x$ iff $\neg\varphi \in x$, and*

2. $\varphi \wedge \psi \in x$ *iff* $\varphi \in x$ *and* $\psi \in x$.

In the setting of our axiomatization based on the infinitary rule ($\boxtimes^\omega$-intro), we will say that a *set* x of formulas is consistent iff there exists a consistent *theory* y such that $x \subseteq y$. Obviously, maximal consistent theories are maximal consistent sets of formulas. Under the given definition of consistency for sets of formulas, maximal consistent sets of formulas are also maximal consistent theories.

8.3.10. Definition. Let $\varphi \in \mathcal{L}^!_{K\mathsf{int}\boxtimes}$ and $i \in \mathcal{A}$. Then $x+\varphi := \{\psi \mid \varphi \to \psi \in x\}$, $K_i x := \{\varphi \mid K_i\varphi \in x\}$, and $\mathsf{int}(x) := \{\varphi \mid \mathsf{int}(\varphi) \in x\}$.

8.3.11. Lemma. *For every theory x of $\mathsf{APAL}^m_{\mathsf{int}}$, $\varphi \in \mathcal{L}^!_{K\mathsf{int}\boxtimes}$ and agent $i \in \mathcal{A}$,*

1. *$x + \varphi$ is a theory that contains x and φ,*
2. *$K_i x$ is a theory,*
3. *$\mathsf{int}(x)$ is a theory, and*
4. *$\mathsf{int}(x) \subseteq x$.*

Proof:
The proof is similar to the proof of Balbiani et al. (2008, Lemma 4.11) and here we only prove items 3 and 4. Suppose x is a theory of $\mathsf{APAL}^m_{\mathsf{int}}$ and $\varphi \in \mathcal{L}^!_{K\mathsf{int}\boxtimes}$.

(3): Suppose $\varphi \in \mathsf{APAL}^m_{\mathsf{int}}$. Since φ is a theorem, by ($\mathrm{Nec}_{\mathsf{int}}$), $\mathsf{int}(\varphi)$ is a theorem of $\mathsf{APAL}^m_{\mathsf{int}}$ as well. Therefore, $\mathsf{int}(\varphi) \in x$ meaning that $\varphi \in \mathsf{int}(x)$. Hence, $\mathsf{APAL}^m_{\mathsf{int}} \subseteq \mathsf{int}(x)$. Let us now show that $\mathsf{int}(x)$ is closed under (MP). Suppose $\varphi, \varphi \to \psi \in \mathsf{int}(x)$. This means, by the definition of $\mathsf{int}(x)$, that $\mathsf{int}(\varphi), \mathsf{int}(\varphi \to \psi) \in x$. By axiom ($\mathrm{K}_{\mathsf{int}}$), we have $\mathsf{int}(\varphi) \to (\mathsf{int}(\varphi \to \psi) \to \mathsf{int}(\psi)) \in \mathsf{APAL}^m_{\mathsf{int}}$. Thus, since $\mathsf{APAL}^m_{\mathsf{int}} \subseteq x$ and x is closed under (MP), we obtain $\mathsf{int}(\psi) \in x$, i.e., $\psi \in \mathsf{int}(x)$. Finally we show that $\mathsf{int}(x)$ is closed under ($\boxtimes^\omega$-intro). Let $s \in NF_{\boxtimes}$ and $\chi \in \mathcal{L}^!_{K\mathsf{int}\boxtimes}$ such that $[s][\psi]\chi \in \mathsf{int}(x)$ for all $\psi \in \mathcal{L}^!_{K\mathsf{int}}$. This means $\mathsf{int}([s][\psi]\chi) \in x$ for all $\psi \in \mathcal{L}^!_{K\mathsf{int}}$. As $\mathsf{int}([s][\psi]\chi)$ is also a necessity form and x is closed under ($\boxtimes^\omega$-intro), we obtain $\mathsf{int}([s]\boxtimes\chi) \in x$ meaning that $[s]\boxtimes\chi \in \mathsf{int}(x)$. We therefore conclude that $\mathsf{int}(x)$ is a theory.

(4): Suppose $\varphi \in \mathsf{int}(x)$. This means $\mathsf{int}(\varphi) \in x$. By ($\mathrm{T}_{\mathsf{int}}$) and the fact that $\mathsf{APAL}^m_{\mathsf{int}} \subseteq x$, we have $\mathsf{int}(\varphi) \to \varphi \in x$. Therefore, since x is closed under (MP), we obtain $\varphi \in x$. As φ has been taken arbitrarily from $\mathsf{int}(x)$, we conclude that $\mathsf{int}(x) \subseteq x$. □

8.3.12. Lemma. *Let $\varphi \in \mathcal{L}^!_{K\mathsf{int}\boxtimes}$. For all theories x, $x+\varphi$ is consistent iff $\neg\varphi \notin x$.*

Proof:
Let x be a theory of $\mathsf{APAL}^m_{\mathsf{int}}$. Then $\neg\varphi \in x$ iff $\varphi \to \bot \in x$ (as $\neg\varphi \leftrightarrow \varphi \to \bot$ is a theorem, and x is closed under (MP)) iff $\bot \in x + \varphi$. Therefore, $x + \varphi$ is inconsistent iff $\neg\varphi \in x$, i.e., $x + \varphi$ is consistent iff $\neg\varphi \notin x$. □

8.3.13. LEMMA (LINDENBAUM'S LEMMA). *Each consistent theory x can be extended to a maximal consistent theory y such that $x \subseteq y$.*

Proof:
The proof is the same as the proof of (Balbiani et al., 2008, Lemma 4.12). We here recapitulate it in our notation to render the chapter self-contained. The proof proceeds by constructing an increasing chain

$$y_0 \subseteq y_1 \subseteq \ldots \subseteq y_n \subseteq \ldots,$$

of consistent theories where $y_0 := x$, and each y_i will be recursively defined. At each step, we have to guarantee that y_i is consistent, $\mathsf{APAL}^m_{\mathsf{int}}$ is included in y_i and it is closed under (MP) and ($\circledast^\omega$-intro). Let $\psi_0, \psi_1, \ldots$ be the enumeration of all formulas in $\mathcal{L}^!_{K\mathsf{int}\circledast}$, and set $y_0 = x$. Now suppose we are at the (n+1)st step of the construction, that is, y_n has already been defined as a consistent theory containing x. We first observe that either $y_n + \psi_n$ is consistent, or $y_n + \neg\psi_n$ is consistent (but not both). Suppose otherwise, i.e., suppose both $y_n + \psi_n$ and $y_n + \neg\psi_n$ are inconsistent. Then, by Lemma 8.3.12, we have both $\neg\psi \in y_n$ and $\neg\neg\psi \in y_n$. However, since $\neg\psi_n \to (\neg\neg\psi_n \to \bot) \in \mathsf{APAL}^m_{\mathsf{int}} \subseteq y_n$ and y_n is closed under (MP), we obtain $\bot \in y_n$, contradicting consistency of y_n. If $y_n + \psi_n$ is consistent, we define $y_{n+1} = y_n + \psi_n$. By Lemma 8.3.11-(1), it is guaranteed that $y_n + \psi_n$ is a theory. If $y_n + \psi_n$ is inconsistent, we have $\neg\psi_n \in y_n$ (by Lemma 8.3.12). We then have two cases: (a) ψ_n is *not* a consequence of ($\circledast^\omega$-intro) (b) ψ_n is a consequence of ($\circledast^\omega$-intro). If (a) is the case, we let $y_{n+1} = y_n$. For (b), let $[s_1]\circledast\chi_1, [s_2]\circledast\chi_2, \ldots, [s_k]\circledast\chi_k$ be the enumeration of all possible representations of ψ_n as a consequence of ($\circledast^\omega$-intro). We now define another sequence $y^0_n, \ldots, y^k_n$ of consistent theories such that $y^0_n = y_n$ and each y^i_n with $i \leq k$ is recursively defined and includes y_n. Now suppose we are at the (i+1)st step of the construction, that is, y^i_n has already been defined as a consistent theory containing y_n. This means, $\neg[s_i]\circledast\chi_i \in y^i_n$ (as $\neg[s_i]\circledast\chi_i := \neg\psi_n \in y_n \subseteq y^i_n$). Since y^i_n is closed under ($\circledast^\omega$-intro), there exists $\varphi_i \in \mathcal{L}^!_{K\mathsf{int}}$ such that $[s_i][\varphi_i]\chi_i \notin y^i_n$. Then we define $y^{i+1}_n = y^i_n + \neg[s_i][\varphi_i]\chi_i$ (by Lemmas 8.3.11-(1) and 8.3.12, y^{i+1}_n is guaranteed to be a consistent theory). Then, we set $y_{n+1} = y^k_n$. Now define $y = \bigcup_{i\in\mathbb{N}} y_i$. We then show that y is in fact a maximally consistent theory. Since $\mathsf{APAL}^m_{\mathsf{int}} \subseteq x = y_0 \subseteq y$, we have $\mathsf{APAL}^m_{\mathsf{int}} \subseteq y$. It is also easy to see that y is consistent (since every element of the chain is consistent). Second, we prove y is closed under (MP). Let $\varphi, \varphi \to \psi \in y$. Then, by the construction of y, there is y_n and y_m in the above chain such that $\varphi \in y_n$ and $\varphi \to \psi \in y_m$. W.l.o.g, we can assume $n \leq m$, thus, $y_n \subseteq y_m$. Hence, $\varphi \in y_m$. Since y_m is closed under (MP), we obtain $\psi \in y_m$, thus, $\varphi \in y$ (since $y_m \subseteq y$). Third, we show y is closed under ($\circledast^\omega$-intro). Let $s \in NF_\circledast$ such that $[s][\varphi]\chi \in y$ for all $\varphi \in \mathcal{L}^!_{K\mathsf{int}}$, and suppose toward contradiction that $[s]\circledast\chi \notin y$. This implies $[s]\circledast\chi \notin y_i$ for all y_i in the above chain, since $y_i \subseteq y$ for all $i \in \mathbb{N}$. Moreover, observe that $[s]\circledast\chi$ appears in the enumeration of all formulas. Let $[s]\circledast\chi := \psi_m$. Since $\psi_m \notin y_{m+1}$, we know that $y_{m+1} \neq y_m + \psi_m$. This means, by the definition of y_{m+1}, that $y_m + \psi_m$ is inconsistent, thus, $\neg\psi_m \in y_m$ (by Lemma 8.3.12). Then, by the construction of y_{m+1}, it is guaranteed that there is a $\eta \in \mathcal{L}^!_{K\mathsf{int}}$ such that $\neg[s][\eta]\chi \in y_{m+1}$. As $y_{m+1} \subseteq y$, we obtain $\neg[s][\eta]\chi \in y$, contradicting consistency of y (since we assumed $[s][\varphi]\chi \in y$ for all $\varphi \in \mathcal{L}^!_{K\mathsf{int}}$). It remains to show that y is maximal. Suppose otherwise, i.e., suppose that there is a consistent theory y' such that $y \subsetneq y'$. This implies that there is $\varphi \in y'$ but $\varphi \notin y$. Hence, $\varphi \notin y_i$ for all $i \in \mathbb{N}$. W.l.o.g, assume $\varphi = \psi_m$. Therefore, in particular, $y_m + \varphi$ is inconsistent, hence, $\neg\varphi \in y_m$. This implies $\neg\varphi \in y'$ (since $y_m \subseteq y'$), hence, both φ and $\neg\varphi$ are in y'. Then, by Lemma 8.3.9-(2) and y' being closed under (MP), we obtain $\bot \in y'$, contradicting consistency of y'. Therefore, y is a maximally consistent theory. □

8.3.14. LEMMA (EXISTENCE LEMMA FOR K_i). *Let $\varphi \in \mathcal{L}^!_{K\mathsf{int}\boxast}$ and $i \in \mathcal{A}$. For every theory x, if $K_i\varphi \notin x$, then there is a maximally consistent theory y such that $K_i x \subseteq y$ and $\varphi \notin y$.*

Proof:
Let x be a theory of $\mathsf{APAL}^m_{\mathsf{int}}$ such that $K_i\varphi \notin x$. Thus, $\varphi \notin K_i x$. This implies that $\neg\neg\varphi \notin K_i x$: otherwise, since $K_i x$ is a theory (Lemma 8.3.11-(2)), thus, closed under (MP), and $\neg\neg\varphi \leftrightarrow \varphi \in \mathsf{APAL}^m_{\mathsf{int}}$, we would obtain $\varphi \in K_i x$, contradicting the assumption. Hence, by Lemma 8.3.12, $K_i x + \neg\varphi$ is consistent. Then, by Lemma 8.3.13, there exists a maximally consistent theory y such that $K_i x + \neg\varphi \subseteq y$. By Lemma 8.3.11-(1), we know that $K_i x \subseteq K_i x + \neg\varphi$ and $\neg\varphi \in K_i x$. Hence, we conclude $K_i x \subseteq y$ and $\varphi \notin y$. □

8.3.15. LEMMA. *Let $\varphi \in \mathcal{L}^!_{K\mathsf{int}\boxast}$ and x be a theory. Then, $\boxast\varphi \in x$ iff for all $\psi \in \mathcal{L}^!_{K\mathsf{int}}$, $[\psi]\varphi \in x$.*

Proof:
For the direction left-to-right, suppose $\boxast\varphi \in x$. Then, by ($\boxast$-elim) and (MP), we have $[\psi]\varphi \in x$ for all $\psi \in \mathcal{L}^!_{K\mathsf{int}}$. For the other direction, suppose $[\psi]\varphi \in x$ for all $\psi \in \mathcal{L}^!_{K\mathsf{int}}$. Consider the necessity form $[s] := \epsilon$. We know that $[\epsilon][\psi]\varphi := [\psi]\varphi$. Thus, by assumption, $[\epsilon][\psi]\varphi$ for all $\psi \in \mathcal{L}^!_{K\mathsf{int}}$. Then, since x is closed under ($\boxast^\omega$-intro), $[\epsilon]\boxast\varphi \in x$, i.e., $\boxast\varphi \in x$ as well. □

The definition of *the canonical model for* $\mathsf{APAL}^m_{\mathsf{int}}$ is the same as for $\mathsf{EL}^m_{\mathsf{int}}$, except that the maximally consistent sets are maximally consistent theories of $\mathsf{APAL}^m_{\mathsf{int}}$. We now come to the Truth Lemma for the logic $\mathsf{APAL}^m_{\mathsf{int}}$. Here we use the complexity measure $\psi <^S_d \varphi$ (see Appendix A.1), and we recall that $\theta^c : X^c \to \mathcal{A} \to \tau^c$ is defined as $\theta^c(x)(i) = [x]_i$, for $x \in X^c$ and $i \in \mathcal{A}$.

8.3.16. LEMMA (TRUTH LEMMA). *For every $\varphi \in \mathcal{L}^{!}_{K\mathsf{int}\circledast}$ and for each $x \in X^c$,*

$$\varphi \in x \textit{ iff } \mathcal{X}^c, (x, \theta^c) \models \varphi.$$

Proof:
Let $\varphi \in \mathcal{L}^{!}_{K\mathsf{int}\circledast}$ and $x \in \mathcal{X}^c$. The proof is by $<^S_d$-induction on φ, where the case $\varphi = [\psi]\chi$ is proved by a subinduction on χ. We therefore consider 13 cases, where the base case $\varphi := p$ as usual follows from the definition of V^c.

Induction Hypothesis: for all $\psi \in \mathcal{L}^{!}_{K\mathsf{int}\circledast}$ and $x \in \mathcal{X}^c$, if $\psi <^S_d \varphi$, then $\psi \in x$ iff $\mathcal{X}^c, (x, \theta^c) \models \psi$.

The Boolean cases follow standardly, where we observe that the subformula order is subsumed under the $<^S_d$ order (see Lemma A.1.5-(1)). We proceed with the cases $\varphi = K_i\psi$ and $\varphi = \mathsf{int}(\psi)$ respectively, and then with the subinduction on χ for case announcement $\varphi = [\psi]\chi$, and finally with the case $\varphi = \circledast\psi$.

Case $\varphi := K_i\psi$

For the direction from left-to-right, see (Truth) Lemma 8.3.5. For the opposite direction, suppose $K_i\psi \notin x$. Then, by Lemma 8.3.14, there exists a maximally consistent theory y such that $K_i x \subseteq y$ and $\psi \notin y$. Then, by $\psi <^S_d K_i\psi$ and induction hypothesis (IH), we obtain $(y, \theta^c) \not\models \psi$. Since $K_i x \subseteq y$, we have $y \in [x]_i$ meaning that $y \in \theta^c(x)(i)$. Therefore, by the semantics, $\mathcal{X}^c, (x, \theta^c) \not\models K_i\psi$.

Case $\varphi := \mathsf{int}(\psi)$

For the direction from left-to-right, see (Truth) Lemma 8.3.5. For the opposite direction, suppose $\mathsf{int}(\psi) \notin x$. We want to show that $x \notin Int([\![\psi]\!]^{\theta^c})$, i.e., show that for all $U \in \mathcal{B}_\Sigma$ with $x \in U$, we obtain $U \not\subseteq [\![\psi]\!]^{\theta^c}$, where $\mathcal{B}_\Sigma$ is the basis of $\mathcal{X}^c$ constructed by closing Σ under finite intersections (as in the proof of Lemma 8.3.5). Let $U \in \mathcal{B}_\Sigma$ such that $x \in U$. Given the construction of $\mathcal{B}_\Sigma$, U is of the form

$$U = (\bigcap_{i \in I} [x]_i) \cap \widehat{\mathsf{int}(\gamma)},$$

where I and $\mathsf{int}(\gamma)$ are as in the proof of Lemma 8.3.5, case for the modality int. In order to complete the proof, we need to construct a maximally consistent theory $y \in U$ such that $y \notin [\![\psi]\!]^{\theta^c}$. Therefore, this maximally consistent theory y should satisfy the following properties:

1. $\bigcup_{i \in I}\{K_i\sigma \mid K_i\sigma \in x\} \subseteq y$, i.e., $y \in \bigcap_{i \in I}[x]_i$,
2. $\mathsf{int}(\gamma) \in y$, i.e., $y \in \widehat{\mathsf{int}(\gamma)}$,
3. $\neg\psi \in y$, or equivalently, $\psi \notin y$.

Toward the goal of finding this maximal consistent theory y, we first construct a consistent theory z (that we later expand to the maximal consistent theory y).

Consider the set of formulas

$$z_0 := \bigcup_{i\in I}\{K_i\sigma \mid K_i\sigma \in x\} \cup \{\mathsf{int}(\gamma)\} \cup \mathsf{APAL}^m_{\mathsf{int}},$$

and close z_0 under (MP) and ($\boxast^\omega$-intro) to obtain z. It is guaranteed that z is a theory since it includes $\mathsf{APAL}^m_{\mathsf{int}}$ and it is closed under (MP) and ($\boxast^\omega$-intro). Moreover, $z_0 \subseteq x$, since (1) $\bigcup_{i\in I}\{K_i\sigma \mid K_i\sigma \in x\} \subseteq x$ and (2) $\mathsf{int}(\gamma) \in x$ because $x \in U = (\bigcap_{i\in I}[x]_i) \cap \widehat{\mathsf{int}(\gamma)}$, and thus, $x \in \widehat{\mathsf{int}(\gamma)}$. Therefore, $z_0 \subseteq x$ and since z is the smallest theory containing z_0 (by construction), we obtain $z \subseteq x$. It follows that z is consistent since x is consistent. We now consider the set $\mathsf{int}(z)$. Similarly, $\mathsf{int}(z)$ is a consistent theory such that $\mathsf{int}(z) \subseteq z \subseteq x$ (by Lemma 8.3.11-(3,4) and x being a maximally consistent theory). Furthermore, $\bigcup_{i\in I}\{K_i\sigma \mid K_i\sigma \in x\} \cup \{\mathsf{int}(\gamma)\} \subseteq \mathsf{int}(z)$, since $K_i\sigma \leftrightarrow \mathsf{int}(K_i\sigma) \in \mathsf{APAL}^m_{\mathsf{int}}$ and $K_i\sigma \in z$ for each $i \in I$, and similarly since $\mathsf{int}(\gamma) \leftrightarrow \mathsf{int}(\mathsf{int}(\gamma)) \in \mathsf{APAL}^m_{\mathsf{int}}$ and $\mathsf{int}(\gamma) \in z$. In fact, given that z is the smallest theory constructed from z_0 by closing z_0 under (MP) and ($\boxast^\omega$-intro), and $\mathsf{int}(z)$ is also a consistent theory such that $z_0 \subseteq \mathsf{int}(z) \subseteq z$, we obtain $\mathsf{int}(z) = z$. Observe that, since $\mathsf{int}(\psi) \notin x$ and $z \subseteq x$, we have $\mathsf{int}(\psi) \notin z$. Therefore, the fact that $\mathsf{int}(\psi) \notin \mathsf{int}(z) = z$ implies that $\psi \notin z$. Finally, we extend the consistent theory z to the set of formulas $z + \neg\psi$. By Lemma 8.3.11-(1), we know that $z + \neg\psi$ is a theory such that $z \subseteq z + \neg\psi$ and $\neg\psi \in z + \neg\psi$. Moreover, since $\psi \notin z$, Lemma 8.3.12 implies that $z + \neg\psi$ is a consistent theory. Thus, by Lemma 8.3.13, there exists a maximally consistent theory y such that $z + \neg\psi \subseteq y$. Hence, we have the maximally consistent theory y with:

1. $\bigcup_{i\in I}\{K_i\sigma \mid K_i\sigma \in x\} \subseteq y$, since $\bigcup_{i\in I}\{K_i\sigma \mid K_i\sigma \in x\} \subseteq z \subseteq y$,
2. $\mathsf{int}(\gamma) \in y$, since $\mathsf{int}(\gamma) \in z \subseteq y$, and
3. $\neg\psi \in y$, since $\neg\psi \in z + \neg\psi \subseteq y$.

Therefore, $y \in (\bigcap_{i\in I}[x]_i) \cap \widehat{\mathsf{int}(\gamma)} = U$ (by (1) and (2) above) such that $y \notin [\![\psi]\!]^{\theta^c}$ (by (3) and IH). Thus, $U \not\subseteq [\![\psi]\!]^{\theta^c}$. Since the basic open neighbourhood U of x has been chosen from $\mathcal{B}_\Sigma$ arbitrarily, we obtain $x \notin Int([\![\psi]\!]^{\theta^c})$.

Case $\varphi := [\psi]\chi$: This case follows from a subinduction on χ.

Subcase $\varphi := [\psi]p$

$$\begin{aligned} [\psi]p \in x & \text{ iff } \mathsf{int}(\psi) \to p \in x && (\mathrm{R}_p)\\ & \text{ iff } (x, \theta^c) \models \mathsf{int}(\psi) \to p && (\text{Lemma A.1.5-(5) and (IH)})\\ & \text{ iff } (x, \theta^c) \models [\psi]p && (\mathrm{R}_p) \end{aligned}$$

Subcase $\varphi := [\psi]\neg\eta$
Use ($\mathrm{R}_\neg$), (IH), and Lemma A.1.5-(6) stating $\mathsf{int}(\psi) \to \neg[\psi]\eta <^S_d [\psi]\neg\eta$.

Subcase $\varphi := [\psi](\eta \wedge \sigma)$
Use ($R_\wedge$), (IH), and Lemma A.1.5-(7) stating $[\psi]\eta \wedge [\psi]\sigma <^S_d [\psi](\eta \wedge \sigma)$.
Subcase $\varphi := [\psi]\mathsf{int}(\eta)$
Use (R_{int}), (IH), and Lemma A.1.5-(8) stating $\mathsf{int}(\psi) \to \mathsf{int}([\psi]\eta) <^S_d [\psi]\mathsf{int}(\eta)$.
Subcase $\varphi := [\psi]K_i\eta$
Use (R_K), (IH), and Lemma A.1.5-(9) stating $\mathsf{int}(\psi) \to K_i[\psi]\eta <^S_d [\psi]K_i\eta$.
Subcase $\varphi := [\psi][\eta]\sigma$
Use ($R_{[!]}$), (IH), and Lemma A.1.5-(10) stating $[\langle\psi\rangle\eta]\sigma <^S_d [\psi][\eta]\sigma$.
Subcase $\varphi := [\psi]\boxast\sigma$
For all $\eta \in \mathcal{L}^!_{K\mathsf{int}}$, we have $[\psi][\eta]\sigma <^S_d [\psi]\boxast\sigma$ since $[\psi]\boxast\sigma$ has one more $\boxast$ than $[\psi][\eta]\sigma$ (see Lemma A.1.5-(3-4)). Therefore, it suffices to show $[\psi]\boxast\sigma \in x$ iff $\forall\eta \in \mathcal{L}^!_{K\mathsf{int}}([\psi][\eta]\sigma \in x)$. For the direction from right-to-left, assume that for all $\eta \in \mathcal{L}^!_{K\mathsf{int}}$, $[\psi][\eta]\sigma \in x$. Notice that each $[\psi][\eta]\sigma$ is a necessity form of the shape $[s][\eta]\sigma$ where $s = \psi$. Therefore, since x is closed under ($\boxast^\omega$-intro), we obtain $[\psi]\boxast\sigma \in x$. For the opposite direction, suppose $[\psi]\boxast\sigma \in x$. Observe that $[\psi]\boxast\sigma \to [\psi][\eta]\sigma \in \mathsf{APAL}^m_{\mathsf{int}}$ for all $\eta \in \mathcal{L}^!_{K\mathsf{int}}$ (this can easily be proven by ($\boxast$-elim, Nec$_!$, K$_!$ and MP). Therefore, for all $\eta \in \mathcal{L}^!_{K\mathsf{int}}$, $[\psi][\eta]\sigma \in x$, since x is closed under (MP). We can then obtain

$$\begin{aligned}
[\psi]\boxast\sigma \in x &\text{ iff } \forall\eta \in \mathcal{L}^!_{K\mathsf{int}}([\psi][\eta]\sigma \in x) && \text{(by the above argument)}\\
&\text{ iff } (\forall\eta \in \mathcal{L}^!_{K\mathsf{int}})((x,\theta^c) \models [\psi][\eta]\sigma) && (\text{IH, } [\psi][\eta]\sigma <^S_d [\psi]\boxast\sigma)\\
&\text{ iff } (\forall\eta \in \mathcal{L}^!_{K\mathsf{int}})((x,\theta^c) \models \mathsf{int}(\psi) \text{ implies } (x,(\theta^c)^\psi) \models [\eta]\sigma)\\
&\text{ iff } (x,\theta^c) \models \mathsf{int}(\psi) \text{ implies } (\forall\eta \in \mathcal{L}^!_{K\mathsf{int}})((x,(\theta^c)^\psi) \models [\eta]\sigma)\\
&\text{ iff } (x,\theta^c) \models \mathsf{int}(\psi) \text{ implies } (x,(\theta^c)^\psi) \models \boxast\sigma\\
&\text{ iff } (x,\theta^c) \models [\psi]\boxast\sigma
\end{aligned}$$

This completes the case $\varphi := [\psi]\chi$.

Case $\varphi := \boxast\psi$
Again note that for all $\eta \in \mathcal{L}^!_{K\mathsf{int}}$, $[\eta]\psi <^S_d \boxast\psi$, since $\boxast\psi$ has one more $\boxast$ than $[\eta]\psi$ (see Lemma A.1.5-(3-4)). Therefore, we obtain

$$\begin{aligned}
\boxast\psi \in x &\text{ iff } (\forall\eta \in \mathcal{L}^!_{K\mathsf{int}\boxast})([\eta]\psi \in x) && \text{(Lemma 8.3.15)}\\
&\text{ iff } (\forall\eta \in \mathcal{L}^!_{K\mathsf{int}\boxast})((x,\theta^c) \models [\eta]\psi) && \text{(IH)}\\
&\text{ iff } (x,\theta^c) \models \boxast\psi && \text{(by the semantics of } \boxast)
\end{aligned}$$

$\square$

8.3.17. Theorem. $\mathsf{APAL}^m_{\mathsf{int}}$ *is complete with respect to the class of all topo-models, i.e., for all* $\varphi \in \mathcal{L}^!_{K\mathsf{int}\boxast}$*, if* φ *is valid, then* $\varphi \in \mathsf{APAL}^m_{\mathsf{int}}$.

Proof:
Let $\varphi \in \mathcal{L}^{!}_{K\mathsf{int}\circledast}$ such that $\varphi \notin \mathsf{APAL}^m_{\mathsf{int}}$ (recall that $\mathsf{APAL}^m_{\mathsf{int}}$ is the smallest theory). Then, by Lemma 8.3.12, $\mathsf{APAL}^m_{\mathsf{int}} + \neg\varphi$ is a consistent theory, and, by Lemma 8.3.11-(1), we have $\neg\varphi \in \mathsf{APAL}^m_{\mathsf{int}} + \neg\varphi$. By Lemma 8.3.13, the consistent theory $\mathsf{APAL}^m_{\mathsf{int}} + \neg\varphi$ can be extended to a maximally consistent theory y such that $\mathsf{APAL}^m_{\mathsf{int}} + \neg\varphi \subseteq y$. Since y is maximally consistent and $\neg\varphi \in y$, we obtain $\varphi \notin y$ (by Proposition 8.3.9). Then, by Lemma 8.3.16 (Truth Lemma), we conclude $\mathcal{X}^c, (y, \theta^c) \not\models \varphi$. □

8.4 S4 knowledge on multi-agent topo-models

As mentioned earlier, some of our results generalize to weaker versions of $\mathsf{EL}^m_{\mathsf{int}}$, $\mathsf{PAL}^m_{\mathsf{int}}$, and $\mathsf{APAL}^m_{\mathsf{int}}$ that have knowledge modalities of different strength, such as S4, S4.2 and S4.3. More precisely, we can weaken the conditions on the neighbourhood functions given in Definition 8.1.1 in a way that the corresponding logics on such weaker models embed only $\mathsf{S4}_K$, $\mathsf{S4.2}_K$ or $\mathsf{S4.3}_K$ types of knowledge. In this section, we focus on the case $\mathsf{S4}_K$, and briefly state the required adjustments for $\mathsf{S4.2}_K$ and $\mathsf{S4.3}_K$.

Since the S4 type of knowledge does not satisfy the axiom (5_K): $\neg K_i\varphi \to K_i\neg K_i\varphi$ and the key property that makes the axiom (5_K) sound on topo-models is Definition 8.1.1-(3), we weaken exactly this clause to obtain topo-models for logics for knowledge of different strength.

8.4.1. Definition. [Weak Topo-Model] A *weak multi-agent topological model* (weak topo-model) is a topo-model $\mathcal{X} = (X, \tau, \Phi, V)$ as in Definition 8.1.1 with clause 3 replaced by

3. for all $y \in X$, if $y \in \theta(x)(i)$ then $y \in \mathcal{D}(\theta)$ and $\theta(y)(i) \subseteq \theta(x)(i)$.

A weak topo-frame is defined analogously to Definition 8.1.2.

8.4.2. Definition. The axiomatization of $w\mathsf{EL}^m_{\mathsf{int}}$ is that of $\mathsf{EL}^m_{\mathsf{int}}$ minus the axiom (5_K). The axiomatizations for $w\mathsf{PAL}^m_{\mathsf{int}}$ and $w\mathsf{APAL}^m_{\mathsf{int}}$ are the obvious further extensions as in Table 8.1.

Soundness of $w\mathsf{EL}^m_{\mathsf{int}}$, $w\mathsf{PAL}^m_{\mathsf{int}}$, and $w\mathsf{APAL}^m_{\mathsf{int}}$ with respect to weak topo-models follow as in Proposition 8.2.4 and Corollary 8.2.5. As for completeness, we again use a canonical model construction similar to the one for the stronger logics, however, adapted for the S4-type knowledge. Let us first introduce some notation and basic concepts.

Let X^c be the set of all maximally consistent sets of $w\mathsf{EL}^m_{\mathsf{int}}$. We define relations R^c_i on X^c as

$$xR^c_iy \text{ iff } \forall\varphi \in \mathcal{L}_{K\mathsf{int}}(K_i\varphi \in x \text{ implies } \varphi \in y).$$

Let $R_i^c(x)$ denote the upward-closed set generated by x with respect to the relation R_i^c, i.e., $R_i^c(x) = \{y \in X^c \mid xR_i^c y\}$. Note that, since K_i is of $\mathsf{S4}$-type, the canonical relations R_i^c are reflexive and transitive. As usual, we set $\widehat{\varphi} = \{y \in X^c \mid \varphi \in y\}$.

8.4.3. Definition. [Canonical Model for $w\mathsf{EL}^m_{\mathsf{int}}$] We define the (weak) canonical model $\mathcal{X}^c = (X^c, \tau^c, \Phi^c, V^c)$ as follows:

- X^c is the set of all maximally consistent sets of $w\mathsf{EL}^m_{\mathsf{int}}$;
- τ^c is the topological space generated by the subbasis
$$\Sigma = \{R_i^c(x) \cap \widehat{\mathsf{int}(\varphi)} \mid x \in X^c, \varphi \in \mathcal{L}_{K\mathsf{int}} \text{ and } i \in \mathcal{A}\};$$
- $x \in V^c(p)$ iff $p \in x$, for all $p \in \textsc{prop}$;
- $\Phi^c = \{(\theta^c)^U \mid U \in \tau^c\}$, where we define $\theta^c : X^c \to \mathcal{A} \to \tau^c$ as $\theta^c(x)(i) = R_i^c(x)$, for $x \in X^c$ and $i \in \mathcal{A}$.

Observe that (X^c, τ^c, Φ^c) is a weak topo-frame. This can be shown as in the proof of Lemma 8.3.4. As in the previous case we have $\widehat{\mathsf{int}(\top)} = X^c$, thus, each $R_i^c(x)$ is an open set in τ^c. Moreover, Φ^c satisfies the required properties of the elements of Φ given in Definition 8.4.1. Observe that $\mathcal{D}(\theta^c) = X^c$ and $\mathcal{D}((\theta^c)^U) = U$ for all $U \in \tau^c$. Moreover, $(\theta^c)^U(x)(i) = R_i^c(x) \cap U$ when $x \in U$.

8.4.4. Lemma (Truth Lemma). *For every $\varphi \in \mathcal{L}_{K\mathsf{int}}$ and for each $x \in X^c$*

$$\varphi \in x \textit{ iff } \mathcal{X}^c, (x, \theta^c) \models \varphi.$$

Proof:
Proof is similar to the proof of Lemma 8.3.5 except that we replace each $[x]_i$ by $R_i^c(x)$. □

8.4.5. Theorem. *$w\mathsf{EL}^m_{\mathsf{int}}$, $w\mathsf{PAL}^m_{\mathsf{int}}$, and $w\mathsf{APAL}^m_{\mathsf{int}}$ are complete with respect to the class of all weak topo-models.*

Proof:
For completeness of $w\mathsf{EL}^m_{\mathsf{int}}$, let $\varphi \in \mathcal{L}_{K\mathsf{int}}$ such that $w\mathsf{EL}^m_{\mathsf{int}} \not\vdash \varphi$. This implies that $\{\neg\varphi\}$ is a consistent set. Then, by Lindenbaum's Lemma, it can be extended to a maximally consistent set x such that $\neg\varphi \in x$. Therefore, by (Truth) Lemma 8.4.4, $X^c, (x, \theta^c) \not\models \varphi$. For completeness of $w\mathsf{PAL}^m_{\mathsf{int}}$, see proof of Theorem 8.3.7. The completeness proof of $\mathsf{APAL}^m_{\mathsf{int}}$ follows similarly as in Theorem 8.3.17, however, the canonical model is the same as for $w\mathsf{EL}^m_{\mathsf{int}}$, except that the maximally consistent sets are maximally consistent theories of $w\mathsf{APAL}^m_{\mathsf{int}}$. □

Moreover, by adding the following condition to Definition 8.4.1, we obtain topo-models for $\mathsf{S4.3}_K$:

- for all $y, z \in X$, if $y, z \in \theta(x)(i)$ then $y, z \in \mathcal{D}(\theta)$ and either $\theta(y)(i) \subseteq \theta(z)(i)$ or $\theta(z)(i) \subseteq \theta(y)(i)$.

The logics based on $\mathsf{S4.2}_K$ on the other hand demand a more careful treatment if dynamics are involved (as in Section 4.2.2). In particular, the condition on neighbourhood functions that makes the axiom $(.2_K)$: $\hat{K}_i K_i \varphi \to K_i \hat{K}_i \varphi$ valid on topo-models is

- for all $y, z \in X$, if $y, z \in \theta(x)(i)$ then $y, z \in \mathcal{D}(\theta)$ and $\theta(y)(i) \cap \theta(z)(i) \neq \emptyset$.

However, the $(.2_K)$-axiom may no longer hold after an update, as the intersection of updated open neighbourhoods $\theta^{\varphi}(y)(i) \cap \theta^{\varphi}(z)(i)$ may have become empty after the refinement. This is analogous to the problem presented in Section 4.2.2. Therefore, in order to work with $\mathsf{S4.2}_K$ in the present setting, we should drop condition (4) of Definition 8.1.1, and confine ourselves to the epistemic fragment $w\mathsf{EL}^m_{\mathsf{int}} + \hat{K}_i K_i \varphi \to K_i \hat{K}_i \varphi$.

8.5 Comparison to other work

In this section we compare our work in greater detail to some of the prior literature that we already referred to. In this comparison, a prominent position is taken by an embedding from single-agent topological semantics to multi-agent topological semantics and vice versa, wherein the (single-agent) work of Bjorndahl (2016) and van Ditmarsch et al. (2014) play a large role. Bjorndahl's use of the interior operator and topological semantics motivated our own approach: our semantics for $\mathcal{L}_{K\mathsf{int}}$ and $\mathcal{L}^{!}_{K\mathsf{int}}$ are essentially multi-agent extensions of Bjorndahl's semantics for the single-agent versions of these languages. This is the topic of the first half of this section. The second contains a review of other related works.

From multi-agent to single-agent. Throughout this section, we denote single-agent topo-models (X, τ, V) by $\mathcal{M}$ in order to distinguish them from multi-agent topo-models $\mathcal{X} = (X, \tau, \Phi, V)$ with neighbourhood functions. We moreover focus on the single agent case, i.e., assume that $\mathcal{A} = \{i\}$.

In the single-agent case, it is clear that a neighbourhood situation (x, θ) of a given topo-model $\mathcal{X} = (X, \tau, \Phi, V)$ reverts to an epistemic scenario (x, U) of $\mathcal{X}^- = (X, \tau, V)$, where $U = \theta(x)(i)$ and $\mathcal{X}^-$ denotes $\mathcal{X} = (X, \tau, \Phi, V)$ without the Φ component. For the other direction, given a single-agent model (without a neighbourhood function set) $\mathcal{M} = (X, \tau, V)$, for each epistemic scenario $(x, U) \in ES(\mathcal{M})$, we define a neighbourhood function $\theta_U : X \rightharpoonup \{i\} \to \tau$ such that $\mathcal{D}(\theta_U) = U$ and $\theta_U(x)(i) = U$ for all $x \in U$. We therefore define the neighbourhood function set for $\mathcal{M}$ as

$$\Phi_{\mathcal{M}} := \{\theta_U \mid (x, U) \in ES(\mathcal{M})\}.$$

It is not hard to see that $\Phi_{\mathcal{M}}$ satisfies the properties given in Definition 8.1.2, and thus is indeed a neighbourhood function set on the underlying topological space of $\mathcal{M}$. Therefore, $\mathcal{M}^+ = (X, \tau, \Phi_{\mathcal{M}}, V)$ constitutes a topo-model as described in Definition 8.1.2, and it is constructed from $\mathcal{M} = (X, \tau, V)$.

In the following theorem, $\models_s$ refers to the satisfaction relation defined for (single-agent) $\mathcal{L}^!_{K\mathsf{int}\circledast}$ on topo-models $\mathcal{M} = (X, \tau, V)$ with respect to epistemic scenarios (x, U), as given in Definitions 6.2.1 and 7.2.1. The usual notation $\models$ is reserved for the satisfaction relation defined on (X, τ, Φ, V) with respect to neighbourhood situations as given in Definition 8.1.4.

8.5.1. THEOREM.

1. *For every $\mathcal{M} = (X, \tau, V)$, epistemic scenario $(x, U) \in ES(\mathcal{M})$ and $\varphi \in \mathcal{L}^!_{K\mathsf{int}\circledast}$,*
$$\mathcal{M}, (x, U) \models_s \varphi \textit{ iff } \mathcal{M}^+, (x, \theta_U) \models \varphi.$$
2. *For every $\mathcal{X} = (X, \tau, \Phi, V)$, neighbourhood situation $(x, \theta) \in \mathcal{X}$, and $\varphi \in \mathcal{L}^!_{K\mathsf{int}\circledast}$,*
$$\mathcal{X}, (x, \theta) \models \varphi \textit{ iff } \mathcal{X}^-, (x, \theta(x)(i)) \models_s \varphi.$$

Proof:
The proofs for both items follow similarly by $<^S_d$-induction on the formulas in single-agent $\mathcal{L}^!_{K\mathsf{int}\circledast}$. The cases for the propositional variables, Booleans and the modalities K and int are standard. The case $\varphi := [\psi]\chi$ for the public announcement modality follows by subinduction on χ, by using the soundness of the reduction axioms with respect to both single and multi-agent topo-models. Here we present only the subcase for $\chi = p$ and $\chi := \circledast\sigma$ of item (1). The other cases are similar.

Subcase $\varphi := [\psi]p$

$$\begin{aligned}
\mathcal{M}, (x, U) \models_s [\psi]p \text{ iff } & \mathcal{M}, (x, U) \models_s \mathsf{int}(\psi) \to p && \text{(the validity (R}_p\text{) for } \models_s) \\
\text{iff } & \mathcal{M}^+, (x, \theta_U) \models \mathsf{int}(\psi) \to p && \text{(Lemma A.1.5-(5) and (IH))} \\
\text{iff } & \mathcal{M}^+, (x, \theta_U) \models [\psi]p && \text{(the validity (R}_p\text{) for } \models)
\end{aligned}$$

Subcase $\varphi := [\psi]\circledast\sigma$

$$\begin{aligned}
& \mathcal{M}^+, (x, \theta_U) \models [\psi]\circledast\sigma \\
& \text{iff } (\forall \eta \in \mathcal{L}^!_{K\mathsf{int}})(\mathcal{M}^+, (x, \theta_U) \models [\psi][\eta]\sigma) && (*) \\
& \text{iff } (\forall \eta \in \mathcal{L}^!_{K\mathsf{int}})(\mathcal{M}, (x, U) \models_s [\psi][\eta]\sigma) && \text{(Lemma A.1.5-(4) and (IH))} \\
& \text{iff } \mathcal{M}, (x, U) \models_s [\psi]\circledast\sigma && \text{(similar to (*))}
\end{aligned}$$

*: see Lemma 8.3.16, subcase $\varphi := [\psi]\circledast\sigma$. □

Theorem 8.5.1-(1) therefore states that $\mathcal{M}, (x, U)$ and $\mathcal{M}^+, (x, \theta_U)$ are modally equivalent with respect to $\mathcal{L}^!_{K\mathsf{int}\boxast}$. Moreover, for all $\varphi \in \mathcal{L}^!_{K\mathsf{int}\boxast}$, $\mathcal{M} \models_s \varphi$ iff $\mathcal{M}^+ \models \varphi$, i.e., $\mathcal{M}$ and $\mathcal{M}^+$ are (globally) modally equivalent with respect to the same language. Furthermore, Theorem 8.5.1-(2) shows that $\mathcal{X}, (x, \theta)$ and $\mathcal{X}^-, (x, \theta(x)(i))$ are modally equivalent with respect to $\mathcal{L}^!_{K\mathsf{int}\boxast}$. However, $\mathcal{X}$ is not necessarily (globally) modally equivalent to $\mathcal{X}^-$, as the following example demonstrates.

8.5.2. Example. The reason why $\mathcal{X}$ and $\mathcal{X}^-$ are not necessarily modally equivalent is that while $\mathcal{X}^-$ reverts to using the full topology τ, the view on that in $\mathcal{X}$ is restricted by Φ. For a counterexample, consider the topo-model $\mathcal{X} = (X, \tau, \Phi, V)$ where $X = \{1, 2\}$ and τ is the discrete topology on X. We set $\Phi = \{\theta\}$ where $\mathcal{D}(\theta) = \{2\}$ and $\theta(2) = \{2\}$. Hence, the only neighbourhood situation of $\mathcal{X}$ is $(2, \theta)$. Finally we let $V(p) = \{1\}$. Therefore, $\mathcal{X}, (2, \theta) \models \neg Kp$ and as $(2, \theta)$ is the only neighbourhood situation of the model, we obtain $\mathcal{X} \models \neg Kp$. On the other hand, $(1, \{1\})$ is an epistemic scenario in $\mathcal{X}^-$, and $\mathcal{X}^-, (1, \{1\}) \models Kp$, therefore, $\mathcal{X}^- \not\models \neg Kp$.

In the remainder of this section, we compare mainly three aspects of our work to that of others in the relevant literature.

Multi-agent epistemic systems. Multi-agent epistemic systems with subset space-like semantics have been proposed in (Heinemann, 2008, 2010; Baskent, 2007; Wáng and Ågotnes, 2013a), however, none of these are concerned with public or arbitrary public announcements. An unorthodox approach to multi-agent knowledge is proposed in (Heinemann, 2008, 2010). Roughly speaking, instead of having a knowledge modality K_i for each agent as a primitive operator in his syntax, Heinemann uses additional operators to define K_i and his semantics only validates the $\mathsf{S4}$-axioms for K_i. The necessitation rule for K_i does not preserve validity under the proposed semantics (Heinemann, 2008, 2010). On the other hand, we follow the methods of dynamic epistemic logic in our multi-agent generalization by extending the single-agent case with a knowledge modality K_i for each agent and propose a multi-agent topological semantics for this language general enough to model both $\mathsf{S4}$ and $\mathsf{S5}$ types of knowledge, and flexible enough for further generalizations as shown in Section 8.4. Another multi-agent logic of subset spaces is developed in (Wáng and Ågotnes, 2013a). This setting uses multi-agent versions of both knowledge K_i and effort $\Box_i$, where, for example, $\Diamond_1 K_2 p$ is read as "agent 1 comes up with evidence so that agent 2 gets to know p" (Wáng and Ågotnes, 2013a, p. 1160). They have left the question of how to model an agent-independent effort operator open, while pointing out its connection to the arbitrary announcement modality of Balbiani et al. (2008). Besides, no announcements or further generalizations (unlike in their other, single-agent, work Wáng and Ågotnes, 2013b) are considered in (Wáng and Ågotnes, 2013a), and a purely topological case is left for future research. To this end, we believe our work in this chapter at least partially answers some of their open questions. Their use of partitions for each agent instead of a single neighbourhood is compatible with our requirement that all neighbourhoods for a given agent be disjoint. A further difference from the existing literature is that we restrict our attention to topological spaces and prove our results by means of topological tools. For example, our completeness proofs employ direct topological canonical model constructions without a detour referring to different types of semantics and completeness results therein.

Completeness proof. We applied the new completeness proof for arbitrary public announcement logic of Balbiani and van Ditmarsch (2015) to a topological setting. The modality int in our system demands a different complexity measure in the Truth Lemma of the completeness proof of $\mathsf{APAL}^m_{\mathsf{int}}$ than in (Balbiani and van Ditmarsch, 2015). Moreoever, we modified the complexity measure given in (van Ditmarsch et al., 2015b) to make it work for both the completeness of $\mathsf{APAL}^m_{\mathsf{int}}$ and of $\mathsf{PAL}^m_{\mathsf{int}}$. The canonical modal construction is as in (Bjorndahl, 2016) with some multi-agent modifications: we defined the set Σ from which the topology of the canonical model is generated in a similar way as in (Bjorndahl, 2016), however, having multiple agents renders this set weaker in the sense that while it constitutes a basis in the single-agent case, it becomes a subbasis in the multi-agent setting.

Single agent case. In standard (single-agent) subset space semantics (Moss and Parikh, 1992; Dabrowski et al., 1996) and in the later extensions (Wáng and Ågotnes, 2013a; Bjorndahl, 2016; Balbiani et al., 2013; van Ditmarsch et al., 2014), the modality K quantifies over the elements of a given open neighbourhood U that is fixed from the beginning of the evaluation. This makes K behave like a universal modality within U, therefore, $\mathsf{S5}_K$ as an underlying epistemic system becomes intrinsic to the semantics. However, in our proposal, the soundness of the epistemic axioms (i.e., axioms involving only the modality K) depends on the constraints posed on the neighbourhood functions and relaxing these constraints enables us to work with weaker notions of knowledge as shown in Section 8.4. In this sense, our approach generalizes the epistemic aspect of the aforementioned literature. Moreover, Balbiani et al. (2013) proposed subset space semantics for arbitrary announcements. However, their approach does not go beyond the single-agent case and the semantics provided is in terms of model restriction.

8.6 Conclusions and Future Work

In this chapter, we proposed a multi-agent topological semantics for knowledge, knowability, public and arbitrary announcements in the style of subset space semantics. We in particular provided a multi-agent semantic framework, based on topological spaces, that eliminates the so-called problem of "jumping out of the epistemic range" in the evaluation of higher-order knowledge formulas involving different agents. In our setup all agents have the same observational power in the sense that they have access to exactly the same collection of potential evidence, represented by each topo-model carrying only one topology. In order to model the informational attitudes of a group of agents with different observational powers, one could associate a possibly different topology with each agent together with a "common" topology representing all potential evidence. Moreover, the studied notions of dynamics of "learning new evidence" brought about by announcements were of public nature, and the information source was assumed to be external. Van Ditmarsch et al. (2017) generalizes the topological public announcement semantics of this chapter for semi-private announcement, again assuming the information source to be external.

Unsurprisingly, working with $\mathsf{S5}$-type of knowledge required a partitioning of the (sub)domain of a topological space. This might seem like a restrictive requirement since it rules out working with more familiar spaces such as the natural topology of open intervals on the real line or the Euclidean space. However, as long as multiple $\mathsf{S5}$-type agents are concerned, we believe it is hard to avoid such a restriction, if possible at all. We then axiomatized the multi-agent logic of knowledge and knowability $\mathsf{EL}^m_{\mathsf{int}}$, its extension with public announcements $\mathsf{PAL}^m_{\mathsf{int}}$, and also with arbitrary public announcements $\mathsf{APAL}^m_{\mathsf{int}}$. The arbitrary announcement modality $\boxast\varphi$ capturing "stability of the truth of φ after any announcement" comes closer to the intuition behind the effort modality $\Box\varphi$ as "stability of the truth of φ after any evidence-acquisition". These two modalities are proven to be equivalent in the single-agent setting (see Theorem 7.2.6). However, the appropriate interpretation of effort in the multi-agent setting and its connection to the arbitrary announcement modality still remain elusive and deserve a closer look.

The connection between the effort modality and the arbitrary announcement modality has also been observed in (Wáng and Ågotnes, 2013a), however, providing a formal analysis regarding the link between these two modalities in a multi-agent setting is not straightforward: there is not yet agreement on how to interpret the effort modality in a multi-agent framework. The existing proposals neither agree on the general framework, nor are they entirely compatible with each other or with our multi-agent topological setting (see Section 8.5 for a comparison with other work on multi-agent subset space semantics). This was *not* the case in the single-agent version, since the effort modality originated in a single-agent framework, and once we have a semantics for the public announcement modalities, it is obvious how to generalize it for arbitrary announcements, namely by following the intuitive reading of the arbitrary announcement modality as in (Balbiani et al., 2008). In the following, we propose a semantics for the effort modality on multi-agent topo-models that, we believe, fits well with the underlying dynamic epistemic setting developed in this chapter. More precisely, we consider the below semantic clause for the effort modality on multi-agent topo-models

$$\mathcal{X}, (x, \theta) \models \Box\varphi \text{ iff } (\forall U \in \tau)(x \in U \subseteq \mathcal{D}(\theta) \text{ implies } \mathcal{X}, (x, \theta^U) \models \varphi) \qquad (\Box\text{-sem})$$

This interpretation fits well with and generalizes the arbitrary announcement modality $\boxast\varphi$, to recall, interpreted as

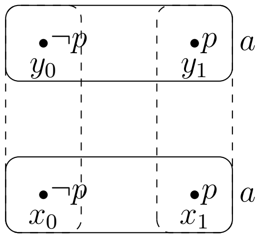
$$\mathcal{X}, (x, \theta) \models \boxast\varphi \text{ iff } (\forall \psi \in \mathcal{L}^{!}_{K\mathsf{int}})(x \in Int(\llbracket\psi\rrbracket^{\theta}) \text{ implies } \mathcal{X}, (x, \theta^{\psi}) \models \varphi),$$

To elaborate, $\boxast\varphi$ quantifies over all announceable formulas in $\mathcal{L}^{!}_{K\mathsf{int}}$, and in turn, quantifies over all epistemically definable open subsets of $\mathcal{D}(\theta)$, and checks whether φ is true with respect to the corresponding updated functions θ^{ψ} that is obtained by restricting θ with the open set $Int(\llbracket\psi\rrbracket^{\theta})$. On the other hand, the effort modality $\Box\varphi$ simply quantifies over *all* open subsets of $\mathcal{D}(\theta)$, and checks whether φ remains true with respect to the restricted neighbourhood functions θ^U. These two modalities are proven to be equivalent in the single-agent case (see Theorem 7.2.6), however, this result does not carry over to the multi-agent setting. In fact, the above semantic clause for $\Box\varphi$ is analogous to what is called "structural semantics for $\boxast\varphi$", which is stated as a possible alternative for the interpretation of the arbitrary announcement modality in (Balbiani et al., 2008, Section 2.3.1). We can then use the example presented in (Balbiani et al., 2008, p. 310), which was based on a multi-agent Kripke model, to show that $\Box\varphi$ and $\boxast\varphi$ do not coincide in our multi-agent setting either.

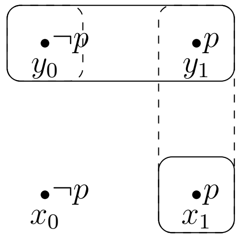
8.6.1. Example. (Balbiani et al., 2008, p. 310) We consider the following two-agent example with agent a and b based on a discrete space. The topo-model we use in this example is the same as $\mathcal{X}$ in Proposition 8.2.9 except for its valuation. Let $\mathcal{X} = (X, \mathcal{P}(X), \Phi, V)$ our topo-model where $X = \{x_0, y_0, x_1, y_1\}$, the topology $\mathcal{P}(X)$ is the set of all subsets of X and $V(p) = \{x_1, y_1\}$. We stipulate that the actual state is x_1 and the neighbourhood function θ defining the epistemic ranges of the agents induces a partition for each agent exactly as in Proposition 8.2.9, also see Figure 8.8a. Now consider the sentences $\Diamond(K_a p \wedge \neg K_b K_a p)$ and $\diamondast(K_a p \wedge \neg K_b K_a p)$. We have $(x_1, \theta) \models \Diamond(K_a p \wedge \neg K_b K_a p)$ since $(x_1, \theta^U) \models K_a p \wedge \neg K_b K_a p$ for $U = \{y_0, y_1, x_1\}$. Moreover, observe that U is the only open in $(X, \mathcal{P}(X))$ such that $(x_1, \theta^U) \models K_a p \wedge \neg K_b K_a p$.

On the other hand, we have $(x_1, \theta) \rightleftharpoons (y_1, \theta)$ and $(x_0, \theta) \rightleftharpoons (y_0, \theta)$. Therefore, since $\mathcal{X}$ is based on a discrete topology, we obtain by Proposition 8.2.8 that (1) $x_1 \in \llbracket\psi\rrbracket^{\theta}$ iff $y_1 \in \llbracket\psi\rrbracket^{\theta}$, and (2) $x_0 \in \llbracket\psi\rrbracket^{\theta}$ iff $y_0 \in \llbracket\psi\rrbracket^{\theta}$, for all $\psi \in \mathcal{L}_{K\mathsf{int}}$. Hence, $U \neq \llbracket\psi\rrbracket^{\theta}$ for all $\psi \in \mathcal{L}_{K\mathsf{int}}$ (since the underlying space is discrete, we have $\llbracket\psi\rrbracket^{\theta} = Int(\llbracket\psi\rrbracket^{\theta})$). Thus, there is no $\psi \in \mathcal{L}_{K\mathsf{int}}$ such that $(x_1, \theta) \models \langle\psi\rangle(K_a p \wedge \neg K_b K_a p)$, thus, $(x_1, \theta) \not\models \diamondast(K_a p \wedge \neg K_b K_a p)$.

Therefore, unlike in the case of the single-agent setting, the effort and the arbitrary announcement modalities behave very differently in the multi-agent



(a) $\mathcal{X}, (x_1, \theta)$



(b) $\mathcal{X}, (x_1, \theta^U)$

Figure 8.8: The straight round circles show the neighbourhoods of agent a, and the dashed ones are for agent b.

case. This should not be surprising. In essence, the arbitrary announcement modality is quite "syntactical" as it quantifies over a set of formulas in a given language, whereas the effort modality is comparatively very "semantical" as it quantifies over subsets of a given domain regardless of whether the subsets are epistemically definable or not. This difference disappears in the single-agent case since both languages $\mathcal{L}^{!}_{K\mathsf{int}\Box}$ and $\mathcal{L}^{!}_{K\mathsf{int}\circledast}$ were co-expressive with the epistemic language $\mathcal{L}_{K\mathsf{int}}$. It falls outside of the scope of this dissertation, and so we leave for a future work, to systematically investigate possible interpretations of the effort modality and its behaviour in a multi-agent setting.

Appendix A

Technical Specifications

A.1 Complexity Measure for $\mathcal{L}^!_{K\mathsf{int}\Box}$ and $\mathcal{L}^!_{K\mathsf{int}\boxast}$

In several proofs in Chapters 7 and 8 (such as Proposition 7.1.11, Lemmas 7.1.35 and 8.3.16), we need a complexity measure of the formulas of $\mathcal{L}^!_{K\mathsf{int}\Box}$ (as well as of $\mathcal{L}^!_{K\mathsf{int}\boxast}$ and its multi-agent extension studied in Chapter 8) that induces a well-founded strict partial order on the formulas of these languages satisfying certain properties (that are given in Lemma A.1.5). For example, in Lemma 7.1.35, we need a complexity measure for which $[p]\varphi$ is less complex than $\Box\varphi$ for an arbitrary propositional variable p; while Lemma 8.3.16 requires $[\psi]\varphi$ to be less complex than $\boxast\varphi$ for arbitrary $\psi \in \mathcal{L}^!_{K\mathsf{int}}$. For this reason, subformula complexity does not suffice. In this appendix, we define a complexity measure that has these properties. Since the languages $\mathcal{L}^!_{K\mathsf{int}\Box}$ and $\mathcal{L}^!_{K\mathsf{int}\boxast}$ are defined in the same way, the proposed complexity measure and the subsequent lemmas also hold for $\mathcal{L}^!_{K\mathsf{int}\boxast}$, as well as for the multi-agent $\mathcal{L}^!_{K\mathsf{int}\boxast}$.

The appropriate complexity measure is composed of a measure $S(\varphi)$ that is a weighted count of the number of symbols and a measure $d(\varphi)$ that counts the number of the $\Box$-modalities occurring in a formula. Although somewhat simpler complexity measures would work for some of the lemmas mentioned above, we here define one complexity measure, based on S and d, that induces a well-founded partial order $<^S_d$ on $\mathcal{L}^!_{K\mathsf{int}\Box}$ (and on $\mathcal{L}^!_{K\mathsf{int}\boxast}$) which works in every relevant proof. The relation $<^S_d$ introduced below was first defined in (van Ditmarsch et al., 2015c) for the language $\mathcal{L}^!_{K\mathsf{int}\boxast}$ (by adapting similar notions introduced before in, e.g., Balbiani and van Ditmarsch, 2015; van Ditmarsch et al., 2015b).

A.1.1. Definition. [Size of formulas in $\mathcal{L}^!_{K\mathsf{int}\Box}$] The *size* $S(\varphi)$ of formula $\varphi \in \mathcal{L}^!_{K\mathsf{int}\Box}$ is defined as:

$$\begin{array}{rcl} S(p) & = & 1, \\ S(\neg\varphi) & = & S(\varphi)+1, \\ S(\varphi\wedge\psi) & = & S(\varphi)+S(\psi)+1, \\ S(K\varphi) & = & S(\varphi)+1, \\ S(\mathsf{int}(\varphi)) & = & S(\varphi)+1, \\ S([\varphi]\psi) & = & 4(S(\varphi)+4)S(\psi), \\ S(\Box\varphi) & = & S(\varphi)+1. \end{array}$$

A.1.2. Definition. [Depth of formulas in $\mathcal{L}^!_{K\mathsf{int}\Box}$] The $\Box$-*depth* $d(\varphi)$ of formula $\varphi \in \mathcal{L}^!_{K\mathsf{int}\Box}$ is defined as:

$$\begin{array}{rcl} d(p) & = & 0, \\ d(\neg\varphi) & = & d(\varphi), \\ d(\varphi\wedge\psi) & = & max\{d(\varphi), d(\psi)\}, \\ d(K\varphi) & = & d(\varphi), \\ d(\mathsf{int}(\varphi)) & = & d(\varphi), \\ d([\varphi]\psi) & = & max\{d(\varphi), d(\psi)\}, \\ d(\Box\varphi) & = & d(\varphi)+1 \end{array}$$

Finally, we define our intended complexity relation $<^S_d$ as lexicographic merge of $\Box$-depth and size, exactly as in (van Ditmarsch et al., 2015c) (adapted from Balbiani and van Ditmarsch, 2015; van Ditmarsch et al., 2015b):

A.1.3. Definition. For any $\varphi, \psi \in \mathcal{L}^!_{K\mathsf{int}\Box}$, we put

- $\varphi <^S \psi$ iff $S(\varphi) < S(\psi)$
- $\varphi <_d \psi$ iff $d(\varphi) < d(\psi)$
- $\varphi <^S_d \psi$ iff (either $d(\varphi) < d(\psi)$, or $d(\varphi) = d(\psi)$ and $S(\varphi) < S(\psi)$)

A.1.4. Lemma.

1. *$<^S, <_d, <^S_d$ are well-founded strict partial orders between formulas in $\mathcal{L}^!_{K\mathsf{int}\Box}$,*
2. *if $\varphi, \psi \in \mathcal{L}^!_{K\mathsf{int}}$, then $\varphi <^S_d \psi$ iff $\varphi <^S \psi$.*

A.1.5. Lemma. *For all $\varphi, \psi \in \mathcal{L}^!_{K\mathsf{int}\Box}$,*

1. *$\varphi \in Sub(\psi)$ implies $\varphi <^S_d \psi$,*
2. *$\mathsf{int}(\varphi) <^S_d [\varphi]\psi$,*
3. *$\varphi \in \mathcal{L}^!_{K\mathsf{int}}$ iff $d(\varphi) = 0$,*
4. *$\varphi \in \mathcal{L}^!_{K\mathsf{int}}$ implies $[\varphi]\psi <^S_d \Box\psi$.*
5. *$\mathsf{int}(\varphi) \to p <^S_d [\varphi]p$,*
6. *$\mathsf{int}(\varphi) \to \neg[\varphi]\psi <^S_d [\varphi]\neg\psi$,*
7. *$[\varphi]\psi \wedge [\varphi]\chi <^S_d [\varphi](\psi \wedge \chi)$,*
8. *$\mathsf{int}(\varphi) \to \mathsf{int}([\varphi]\psi) <^S_d [\varphi]\mathsf{int}(\psi)$,*
9. *$\mathsf{int}(\varphi) \to K[\varphi]\psi <^S_d [\varphi]K\psi$,*
10. *$[\langle\varphi\rangle\psi]\chi <^S_d [\varphi][\psi]\chi$.*

Proof:
The proof of this lemma follows from simple arithmetic calculations and many items are obvious. We here prove the items (7), (8) and (10). Recall that we define $\varphi \to \psi$ as $\neg(\varphi \wedge \neg\psi)$, so that $S(\varphi \to \psi) = S(\varphi) + S(\psi) + 3$.

(7) On the left-hand-side, we have $S([\varphi]\psi \wedge [\varphi]\chi) = 1 + 4(S(\varphi)+4)(S(\psi)+S(\chi))$. However, $S([\varphi](\psi \wedge \chi)) = 4(S(\varphi) + 4)(1 + S(\psi) + S(\chi)) = 4(S(\varphi) + 4) + 4(S(\varphi)+4)(S(\psi)+S(\chi))$. Thus, $S([\varphi]\psi \wedge [\varphi]\chi) < S([\varphi](\psi \wedge \chi))$. Moreover, $d([\varphi]\psi \wedge [\varphi]\chi) = max\{d(\varphi), d(\psi), d(\chi)\} = d([\varphi](\psi \wedge \chi))$ (This is similar in the other items). Therefore, by Definition A.1.3, we obtain $[\varphi]\psi \wedge [\varphi]\chi <^S_d [\varphi](\psi \wedge \chi)$.

(8) On the left-hand-side, we obtain $S(\mathsf{int}(\varphi) \to \mathsf{int}([\varphi]\psi)) = S(\mathsf{int}(\varphi)) + S(\mathsf{int}([\varphi]\psi)) + 3 = 1 + S(\varphi) + 1 + S([\varphi]\psi) + 3 = 5 + S(\varphi) + 4S(\varphi)S(\psi) + 16S(\psi)$. However, $S([\varphi]\mathsf{int}(\psi)) = 4(S(\varphi) + 4)S(\mathsf{int}(\psi)) = 4(S(\varphi) + 4)(S(\psi) + 1) = 16 + 4S(\varphi) + 4S(\varphi)S(\psi) + 16S(\psi)$. Therefore, $S(\mathsf{int}(\varphi) \to \mathsf{int}([\varphi]\psi)) < S([\varphi]\mathsf{int}(\psi))$. As in item (7) the $\Box$-depth of both formulas is the same. Therefore, $\mathsf{int}(\varphi) \to \mathsf{int}([\varphi]\psi) <^S_d [\varphi]\mathsf{int}(\psi)$.

(10) We have that $S([\langle\varphi\rangle\psi]\chi) = S([\neg[\varphi]\neg\psi]\chi) = 4(S(\neg[\varphi]\neg\psi) + 4)S(\chi) = 4(5 + 4(S(\varphi) + 4)(1 + S(\psi)))S(\chi) = 4S(\chi)(21 + 4S(\varphi) + 16S(\psi) + 4S(\varphi)S(\psi))$. On the other hand, $S([\varphi][\psi]\chi) = 4(S(\varphi)+4)4(S(\psi)+4)S(\chi) = 4S(\chi)(64 + 16S(\varphi) + 16S(\psi) + 4S(\varphi)S(\psi))$. Thus, $S([\langle\varphi\rangle\psi]\chi) < S([\varphi][\psi]\chi)$. Further, we observe that $d([\langle\varphi\rangle\psi]\chi) = \max\{d(\varphi), d(\psi), d(\chi)\} = d([\varphi][\psi]\chi)$. Therefore, $[\langle\varphi\rangle\psi]\chi <^S_d [\varphi][\psi]\chi$.

$\Box$

A.2 Proof of Lemma 7.1.13

Recall that a formula $\psi \in \mathcal{L}_{K\mathsf{int}}$ is said to be in normal form if it is a disjunction of conjunctions of the form

$$\delta := \alpha \wedge K\beta \wedge \hat{K}\gamma_1 \wedge \cdots \wedge \hat{K}\gamma_n$$

where $\alpha, \beta, \gamma_i \in \mathcal{L}_{\mathsf{int}}$ for all $1 \leq i \leq n$. Following the naming convention in (Meyer and van der Hoek, 1995), we call the formula δ *canonical conjunction* and the subformulas $K\beta$ and $\langle K\rangle\gamma_i$ *prenex formulas.*

In the following, we present the steps used in the proof of Lemma 7.1.13. This proof was first presented in (van Ditmarsch et al., 2014) in a slightly different way.

A.2.1. LEMMA. *If $\psi \in \mathcal{L}_{K\mathsf{int}}$ is in normal form and contains a prenex formula σ, then ψ can be written as $\pi \vee (\lambda \wedge \sigma)$, where π, λ and σ are all in normal form.*

Proof:
See (Meyer and van der Hoek, 1995, Lemma 1.7.6.2). □

A.2.2. LEMMA. *The following equivalence is a propositional tautology:*

$$(\varphi_1 \vee \cdots \vee \varphi_n) \wedge (\psi_1 \vee \cdots \vee \psi_m) \leftrightarrow ((\varphi_1 \wedge \psi_1) \vee \dots (\varphi_1 \wedge \psi_m)) \vee ((\varphi_2 \wedge \psi_1) \vee \dots \vee (\varphi_2 \wedge \psi_m)) \vee \cdots \vee ((\varphi_n \wedge \psi_1) \vee \cdots \vee (\varphi_n \wedge \psi_m)).$$

We will show the following results for pseudo-models, however, it is not hard to see that they all follow also for topo-models.

A.2.3. LEMMA. *For all $\varphi, \sigma, \beta \in \mathcal{L}_{K\mathsf{int}}$, we have the following equivalences valid in all pseudo-models:*

1. $\mathsf{int}(\varphi \vee K\beta) \leftrightarrow \mathsf{int}(\varphi) \vee K\beta$
2. $\mathsf{int}(\varphi \vee \langle K\rangle\beta) \leftrightarrow \mathsf{int}(\varphi) \vee \langle K\rangle\beta$
3. $\mathsf{int}(\varphi \vee (\sigma \wedge K\beta)) \leftrightarrow (\mathsf{int}(\varphi \vee \sigma) \wedge (\mathsf{int}(\varphi) \vee K\beta))$
4. $\mathsf{int}(\varphi \vee (\sigma \wedge \langle K\rangle\beta)) \leftrightarrow (\mathsf{int}(\varphi \vee \sigma) \wedge (\mathsf{int}(\varphi) \vee \hat{K}\beta))$
5. $K(\varphi \vee (\sigma \wedge K\beta)) \leftrightarrow ((K(\varphi \vee \sigma) \wedge K\beta) \vee (K\varphi \wedge \neg K\beta))$
6. $K(\varphi \vee (\sigma \wedge \hat{K}\beta)) \leftrightarrow ((K(\varphi \vee \sigma) \wedge \hat{K}\beta) \vee (K\varphi \wedge \neg \hat{K}\beta))$

Proof:
Let $\mathcal{X} = (X, \mathcal{O}, V)$ be a pseudo-model, $(x, U) \in ES(\mathcal{X})$ and $\varphi, \sigma, \beta \in \mathcal{L}_{K\mathsf{int}}$. The key step in each item consists in the fact that K acts as the global modality within the given epistemic range, i.e., that for any $\varphi \in \mathcal{L}_{K\mathsf{int}}$, $[\![K\varphi]\!]^U = U$ or $[\![K\varphi]\!]^U = \emptyset$, and $[\![\hat{K}\varphi]\!]^U = U$ or $[\![\hat{K}\varphi]\!]^U = \emptyset$.

1. ($\Rightarrow$) Suppose $(x, U) \models \mathsf{int}(\varphi \vee K\beta)$, i.e., $x \in Int([\![\varphi \vee K\beta]\!]^U)$. This means, $x \in Int([\![\varphi]\!]^U \cup [\![K\beta]\!]^U)$. We then have two cases:
   (a) If $[\![K\beta]\!]^U = U$, then $(x, U) \models K\beta$, hence, $(x, U) \models \mathsf{int}(\varphi) \vee K\beta$.

(b) If $[\![K\beta]\!]^U = \emptyset$, then $Int([\![\varphi]\!]^U \cup [\![K\beta]\!]^U) = Int([\![\varphi]\!]^U)$, therefore, $(x, U) \models \mathsf{int}(\varphi)$ implying that, $(x, U) \models \mathsf{int}(\varphi) \vee K\beta$.

($\Leftarrow$) Suppose $(x, U) \models \mathsf{int}(\varphi) \vee K\beta$, i.e., $(x, U) \models \mathsf{int}(\varphi)$ or $(x, U) \models K\beta$. We again have two cases:

(a) If $(x, U) \models \mathsf{int}(\varphi)$, i.e., $x \in Int([\![\varphi]\!]^U)$, we obtain $x \in Int([\![\varphi \vee K\beta]\!]^U)$ (since $[\![\varphi]\!]^U \subseteq [\![\varphi \vee K\beta]\!]^U$, and thus $Int([\![\varphi]\!]^U) \subseteq Int([\![\varphi \vee K\beta]\!]^U)$). Thus, $(x, U) \models \mathsf{int}(\varphi \vee K\beta)$.

(b) If $(x, U) \models K\beta$, then $[\![K\beta]\!]^U = U$. Thus, $Int([\![\varphi \vee K\beta]\!]^U) = U$. Hence, $x \in Int([\![\varphi \vee K\beta]\!]^U)$, i.e., $(x, U) \models \mathsf{int}(\varphi \vee K\beta)$.

2. Follows similar to item (1), using the fact that for any $\varphi \in \mathcal{L}_{K\mathsf{int}}$, either $[\![\hat{K}\varphi]\!]^U = U$ or $[\![\hat{K}\varphi]\!]^U = \emptyset$.

3.

$$
\begin{aligned}
&(x, U) \models \mathsf{int}(\varphi \vee (\sigma \wedge K\beta)) \\
&\text{ iff } x \in Int([\![\varphi \vee (\sigma \wedge K\beta)]\!]^U) && \text{(by the semantics of } \mathsf{int}) \\
&\text{ iff } (\exists O \in \tau_{\mathcal{O}})(x \in O \subseteq [\![\varphi \vee (\sigma \wedge K\beta)]\!]^U) && \text{(by the definition of } Int) \\
&\text{ iff } (\exists O \in \tau_{\mathcal{O}})(x \in O \subseteq [\![\varphi]\!]^U \cup ([\![\sigma]\!]^U \cap [\![K\beta]\!]^U)) \\
&\text{ iff } (\exists O \in \tau_{\mathcal{O}})(x \in O \subseteq ([\![\varphi]\!]^U \cup [\![\sigma]\!]^U) \cap ([\![\varphi]\!]^U \cup [\![K\beta]\!]^U)) \\
&\text{ iff } (\exists O \in \tau_{\mathcal{O}})(x \in O \subseteq [\![\varphi]\!]^U \cup [\![\sigma]\!]^U \text{ and } x \in O \subseteq [\![\varphi]\!]^U \cup [\![K\beta]\!]^U) \\
&\text{ iff } x \in Int([\![\varphi]\!]^U \cup [\![\sigma]\!]^U) \text{ and } x \in Int([\![\varphi]\!]^U \cup [\![K\beta]\!]^U) \\
&\text{ iff } (x, U) \models \mathsf{int}(\varphi \vee \sigma) \text{ and } (x, U) \models \mathsf{int}(\varphi \vee K\beta) \\
& && \text{(by the semantics of } \mathsf{int}) \\
&\text{ iff } (x, U) \models \mathsf{int}(\varphi \vee \sigma) \text{ and } (x, U) \models \mathsf{int}(\varphi) \vee K\beta && \text{(by item (1))} \\
&\text{ iff } (x, U) \models \mathsf{int}(\varphi \vee \sigma) \wedge (\mathsf{int}(\varphi) \vee K\beta)
\end{aligned}
$$

4. Follows similarly to item (3), by using item (2).

5. ($\Rightarrow$) Suppose $(x, U) \models K(\varphi \vee (\sigma \wedge K\beta))$. This means, by the semantics of K, that $[\![\varphi \vee (\sigma \wedge K\beta)]\!]^U = U$. Therefore, we have

$$[\![\varphi \vee (\sigma \wedge K\beta)]\!]^U = [\![(\varphi \vee \sigma) \wedge (\varphi \vee K\beta)]\!]^U = [\![\varphi \vee \sigma]\!]^U \cap [\![\varphi \vee K\beta]\!]^U = U.$$

We then have two cases:

(a) If $[\![K\beta]\!]^U = U$, then $[\![\varphi \vee (\sigma \wedge K\beta)]\!]^U = [\![\varphi \vee \sigma]\!]^U = U$. Therefore, $(x, U) \models K(\varphi \vee \sigma) \wedge K\beta$.

(b) If $[\![K\beta]\!]^U = \emptyset$, then $[\![\varphi \vee (\sigma \wedge K\beta)]\!]^U = [\![\varphi]\!]^U = U$. Moreover, $[\![\neg K\beta]\!]^U = U$. Therefore, $(x, U) \models K\varphi \wedge \neg K\beta$. Therefore, by (a) and (b), we conclude that $(x, U) \models (K(\varphi \vee \sigma) \wedge K\beta) \vee (K\varphi \wedge \neg K\beta)$.

($\Leftarrow$) Suppose $(x, U) \models (K(\varphi \vee \sigma) \wedge K\beta) \vee (K\varphi \wedge \neg K\beta)$. We then have two cases:

(a) If $(x, U) \models K(\varphi \vee \sigma) \wedge K\beta$, then $[\![\varphi \vee \sigma]\!]^U = U$ and $[\![K\beta]\!]^U = U$. The latter implies that $[\![\sigma \wedge K\beta]\!]^U = [\![\sigma]\!]^U \cap [\![K\beta]\!]^U = [\![\sigma]\!]^U$. We therefore obtain $[\![\varphi \vee (\sigma \wedge K\beta)]\!]^U = [\![\varphi \vee \sigma]\!]^U = U$. Hence, $(x, U) \models K(\varphi \vee (\sigma \wedge K\beta))$.

(b) $(x, U) \models K\varphi \wedge \neg K\beta$, then $[\![\varphi]\!]^U = U$ and $[\![K\beta]\!]^U = \emptyset$. Therefore, $[\![\varphi \vee (\sigma \wedge K\beta)]\!]^U = [\![\varphi]\!]^U = U$. Hence, $(x, U) \models K(\varphi \vee (\sigma \wedge K\beta))$.

6. Follows similarly to item (5), using the fact that for any $\varphi \in \mathcal{L}_{K\mathsf{int}}$, either $[\![\hat{K}\varphi]\!]^U = U$ or $[\![\hat{K}\varphi]\!]^U = \emptyset$.

$\square$

Proof of Lemma 7.1.13 (Normal Form Lemma): The proof follows by subformula induction on φ, using Proposition 7.2.2 several times. The base case $\varphi := p$ follows easily since $p \in \mathcal{L}_{\mathsf{int}}$, thus, it is already in normal form. Now assume inductively that the statement holds for ψ and χ, and show the cases for the Booleans, K and int:

Case $\varphi := \neg\psi$: By induction hypothesis, we can assume w.l.o.g. that ψ is in normal form. Thus, in particular, $\psi := \delta_1 \vee \cdots \vee \delta_m$ where each δ_i is a canonical conjunction. Hence, $\models \varphi \leftrightarrow (\neg\delta_1 \wedge \cdots \wedge \neg\delta_m)$. We can then distribute $\neg$ of each δ_i over the conjuncts. In other words, for each $\delta_i := \alpha \wedge K\beta \wedge \hat{K}\gamma_1 \wedge \cdots \wedge \hat{K}\gamma_n$, we have

$$\models \neg\delta_i \leftrightarrow (\neg\alpha \vee \hat{K}\neg\beta \vee K\neg\gamma_1 \vee \cdots \vee K\neg\gamma_n)$$

where $\alpha, \beta, \gamma_i \in \mathcal{L}_{\mathsf{int}}$ for all $1 \leq i \leq n$. Let us call $\neg\delta_i$ *canonical disjunction.* Notice that each disjunct of $\neg\delta_i$ is still in the required form, i.e., each disjunct is either a prenex formula or in $\mathcal{L}_{\mathsf{int}}$. By using Lemma A.2.2 repeatedly, we can write φ in normal form, i.e., as disjunctions of canonical conjuncts.

Case $\varphi := \psi \wedge \chi$: By induction hypothesis, w.l.o.g, we assume that ψ and χ are in normal form. Therefore $\psi := \delta_1 \vee \cdots \vee \delta_m$ and $\chi := \delta'_1 \vee \cdots \vee \delta'_k$ where each δ_i and δ'_j is a canonical conjunct. Therefore, $\models \varphi \leftrightarrow ((\delta_1 \vee \cdots \vee \delta_m) \wedge (\delta'_1 \vee \cdots \vee \delta'_k))$. Then, by Lemma A.2.2, we easily obtain a formula θ in normal form such that $\models \varphi \leftrightarrow \theta$.

Case $\varphi := \mathsf{int}(\psi)$: By induction hypothesis, w.l.o.g, assume ψ is in normal. We also assume that ψ includes some prenex formula, otherwise we are done. By Lemma A.2.1, we can consider ψ to be of the form $\psi := \pi \vee (\lambda \wedge \sigma)$ where σ is a prenex formula occurring in ψ, and π and λ are in normal form. Then, we have $\models \mathsf{int}(\psi) \leftrightarrow \mathsf{int}(\pi \vee (\lambda \wedge \sigma))$, and by Lemma A.2.3-(3) or (4) (depending on the form of the prenex formula σ), we have $\models \mathsf{int}(\pi \vee (\lambda \wedge \sigma)) \leftrightarrow (\mathsf{int}(\pi \vee \lambda) \wedge (\mathsf{int}(\pi) \vee \sigma))$. By repeating this procedure, we can push every prenex formula in the scope of int to the top level, hence, obtain a semantically equivalent formula in normal form.

Case $\varphi := K\psi$: Proof of this case is quite similar to the case for int, and follows by using Lemma A.2.3-(5) and (6) instead. A similar argument is presented also in (Meyer and van der Hoek, 1995, Theorem 1.7.6.4, p.37).

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Samenvatting

Dit proefschrift gaat over logica's van kennis, geloof en informatieverandering in topologische ruimtes. Wij onderzoeken de formele representatie van bewijsmateriaal/aanwijzingen in relatie tot rechtvaardiging, gerechtvaardigd geloof, kennis, en gemotiveerde informatieverandering. Topologische ruimtes zijn geschikt om deze epistemische noties te formaliseren vanwege de wiskundige elegantie en epistemisch rijke gegevensstructuren. We vervolgen nu met een overzicht van de inhoud van het proefschrift.

Deel I onderzoekt de rol van bewijsmateriaal bij het vormen van gefundeerd geloof en kennis door een rationeel handelende persoon. Bewijsmateriaal wordt semantisch gerepresenteerd als een verzameling van mogelijke werelden en syntactisch door middel van zogenaamde bewijsmodaliteiten in de logische taal.

Hoofdstuk 3 geeft een overzicht van de achtergrondliteratuur en motiveert Deel I. Het definiëert voor de taal van de modale logica een topologische semantiek die derhalve is gebaseerd op open deelverzamelingen, met het oog op epistemische interpretaties. In dit hoofdstuk wordt het gebruik van topologische ruimtes voor de modellering van kennis toegelicht, en bovendien geeft het een gedetailleerd overzicht van bekende resultaten uit de literatuur over het gebruik van topologische ruimtes voor het modelleren van geloof.

Hoofdstuk 4 gaat voornamelijk over een topologische interpretatie van de notie 'geloof': wat kunnen topologische modellen doen voor de semantiek van reeds bestaande epistemische en doxastische logica's? Met name onderzoeken we de notie van geloof als *mogelijke kennis*, waarvan de sematiek is afgeleid van de op open deelverzamelingen gebaseerde semantiek voor kennis uit hoofdstuk 3. We tonen correctheid en volledigheid aan voor de logica $\mathsf{KD45}_B$ geïnterpreteerd op zekere topologische modelklassen, namelijk de zogenaamde onvergelijkbare ruimtes en de erfelijk-onvergelijkbare gescheiden ruimtes. Een uitbreiding van de logica met openbare aankondigingen voor de laatste van deze twee modelklassen wordt ook onderzocht. Het begrip bewijsvoering wordt beschreven op een puur semantisch niveau, omdat de logische taal hier geen operatoren voor bevat.

Hoofdstuk 5 is de belangrijkste bijdrage van Deel I van het proefschrift. Het geeft een topologische semantiek voor de begrippen *bewijsvoering, gemotiveerde verantwoording, geloof*, en *kennis*, inclusief verbanden tussen al deze epistemische begrippen. De bijbehorende logische taal heeft nu wel modaliteiten voor bewijsvoering, zodat alle verschillende aan bewijsvoering geliëerde begrippen expliciet deel uitmaken van de logica. De resultaten in dit hoofdstuk blijven niet beperkt tot een statische situatie en we presenteren eveneens dynamische noties, namelijk acties zoals *het toevoegen van bewijsvoering*, *het veranderen van de plausibiliteit van bewijsvoering*, *het combineren van bewijsvoering uit verschillende bronnen*, en *het verwerken van (onfeilbaar geachte) informatie uit openbare aankondigingen*. De belangrijkste technische resultaten zijn de volledigheid, beslisbaarheid en eindige-modeleigenschap van de hiermee verbonden logica's. Deze resultaten zijn relevant voor de wijsbegeerte, omdat hiermee noties van kennis en geloof zijn te formalizeren die zijn gebaseerd op de literatuur naar aanleiding van het werk van Gettier.

Deel II gaat vooral over kennis en kennisverandering. Het onderzoekt de noties *absoluut zekere kennis* en *leerbaarheid als mogelijke kennis*, evenals de wisselwerking tussen de epistemische notie van *inspanning* (moeite) voor bewijsvergaring. Tevens komt het verband aan bod met uit de logische dynamiek welbekende begrippen als *openbare aankondigingen* and kwantificatie over dergelijke openbare aankondigingen.

Hoofdstuk 6 geeft het achtergrondmateriaal voor Deel II. Het definiëert de zogenaamde *deelverzamelingsruimte-semantiek* en een *topologische* versie van de eerder genoemde *openbare aankondigingen*.

Hoofdstuk 7 presenteert een formeel raamwerk om de relatie te onderzoeken tussen de belangrijke dynamische noties inspanning, openbare aankondiging, en gekwantificeerde openbare aankondiging. De resultaten over het verband tussen inspanning en openbare aankondiging verduidelijken wat 'inspanning' bij kennisvergaring eigenlijk betekent. De technische resultaten voor expressiviteit en volledigheid in dit hoofdstuk zijn eenvoudiger dan in eerder werk over deze materie en geven daar, in zekere zin, beter inzicht in.

In hoofdstuk 8 wordt de logica van hoofdstuk 7 die was geformuleerd voor een handelende persoon gegeneraliseerd naar een logica voor meerdere handelende personen. We presenteren nu een logica waarin de kennis en de leerbaarheid van meerdere personen wordt gemodelleerd, inclusief uitbreidigen hiervan met openbare aankondingen en kwantificatie daarover; steeds geïnterpreerd op topologische ruimtes. We tonen correctheid en volledigheid aan van deze logica's.

Wij concluderen dat dit proefschrift aan de ene kant verschillende bekende epistemische en doxastische logica's, inclusief dynamische uitbreidingen daarvan, herinterpreteert vanuit topologisch perspectief en voorziet van een interpretatie van verschillende noties van bewijsvoering, terwijl aan de andere kant dit proefschrift topologische technieken gebruikt de verdere ontwikkeling en uitbreiding van bestaande logische analyses, resulterend in nieuwe logica's voor bewijsvoering en voor informatieverandering.

Abstract

This dissertation studies logics of knowledge, belief and information dynamics using topological spaces as models. It is concerned with the formal representation of evidence and its link to justification, justified belief, knowledge, and evidence-based information dynamics. Topological spaces emerge naturally as mathematically elegant and epistemically rich information structures to formalize these epistemic notions. In the following, we give an overview of the content of this thesis.

Part I investigates the role of evidence in forming justified belief and knowledge of a rational idealized agent, where evidence is represented semantically as sets of possible worlds, as well as syntactically via evidence modalities.

Chapter 3 provides background material and motivation for Part I. It introduces the interior-based topological semantics for the basic modal language, focusing on its epistemic interpretation. In this chapter, we motivate the use of topological spaces as models for knowledge, and discuss the *status quo* of the use of topological spaces as belief models.

Chapter 4 focuses primarily on a topological interpretation of belief: how topological models can contribute to the semantics of existing epistemic/doxastic logics. In particular, we study a notion of belief defined as *epistemic possibility of knowledge*, whose topological semantics is derived from the interior semantics for knowledge presented in Chapter 3. We provide soundness and completeness results for the belief logic $\mathsf{KD45}_B$ with respect to the class of extremally and hereditarily extremally disconnected spaces, and study public announcements based on topological models in the latter class. The notion of evidence in this setting is described at a purely semantic level as the corresponding syntax does not have any components representing evidence.

Chapter 5 presents the main contribution of Part I. We propose a topological semantics for various notions of *evidence, evidence-based justification, belief*, and *knowledge*, and explore the connections between these epistemic notions. The corresponding syntax bears evidence modalities, making various notions of evidence

an explicit part of the logic. Our investigations in this chapter are not limited to a static setting. We discuss evidence-based actions such as *evidence addition, upgrade,* and *feasible evidence combination* as well as receiving information from infallible truthful sources via *public announcements*. Our main technical results are concerned with completeness, decidability and the finite model property for the associated logics. These investigations have philosophical consequences, as they allow us to formalize some post-Gettier debates surrounding justified belief and knowledge.

Part II focuses on knowledge and knowledge change. More precisely, it studies the notions of *absolutely certain knowledge* and *knowability as potential knowledge*, as well as the interplay between the notion of epistemic *effort* encompassing any method of evidence acquisition and the well-studied dynamic attitudes such as *public* and *arbitrary public announcements*.

Chapter 6 provides the background material of Part II, introducing *subset space semantics* and a *topological* version of *public announcements*.

Chapter 7 designs a formal framework elucidating the relationship between three dynamic notions of interest: effort, public announcements, and arbitrary announcements. While the established link between effort and public announcements makes the meaning of the intended notion of effort more transparent, our technical results concerning expressivity and completeness simplify, and in a sense, improve on some of the earlier approaches.

Finally, in Chapter 8, we generalize the single-agent setting presented in Chapter 7 to a multi-agent setting. We present a multi-agent logic of knowledge and knowability, as well as its extensions with public and arbitrary announcements, interpreted on topological spaces. We provide soundness and completeness results for the corresponding systems.

To sum up, this dissertation on one hand re-interprets some existing epistemic and doxastic logics and their dynamic extensions from a topological perspective, providing an evidence-based interpretation. On the other hand, it uses topological tools to refine and extend earlier analysis, leading to novel logics of evidence and information dynamics.

Résumé

Cette dissertation réunit logique épistémique et topologie. Elle étudie les représentations formelles de la notion d'évidence[1] et ses liens avec la justification, les croyances justifiées, la connaissance, et la dynamique de l'information basée sur évidence, en utilisant des outils venant de la topologie et de la logique épistémique (dynamique).

La logique épistémique est un terme englobant une grande variété de logiques modales dont les principaux objets d'étude sont la connaissance et la croyance. En tant que champ d'investigation, la logique épistémique utilise la logique modale et les mathématiques pour formaliser, clarifier et résoudre les questions qui motivent l'épistémologie (formelle), et ses applications s'étendent non seulement à la philosophie, mais aussi à l'informatique fondamentale, l'intelligence artificielle et l'économie (voir van Ditmarsch et al., 2015a pour un aperçu). Hintikka (1962) est considéré comme le père fondateur de la logique épistémique moderne. Dans son livre *Knowledge and Belief: An Introduction to the Logic of the Two Notions* (1962)—inspiré par des idées de (von Wright, 1951)—Hintikka formalise connaissance et croyance comme des opérateurs modaux basiques, dénotés respectivement par K et B, et les interprète en utilisant la sémantique des mondes possibles standard, basée sur les structures de Kripke (relationnelles). Depuis lors—la sémantique de Kripke fournissant une façon naturelle et relativement aisée de modéliser la logique épistémique—cela a été une des structures sémantiques les plus proéminentes et fréquemment utilisées en logique épistémique, et la recherche dans ce domaine a en grande partie avancé sur les bases formelles de la sémantique de Kripke.

Cependant, la sémantique de Kripke standard possède certaines caractéristiques qui rendent trop fortes les notions de connaissance et de croyance qu'elle

[1]Faute d'une meilleure traduction pour le mot anglais "evidence", nous utilisons le terme "évidence" pour désigner les éléments, indices, informations sur la base de quoi les croyances sont construites; on pourrait dire aussi que l'évidence est la substance de ce qui constitue une preuve.

implémente—menant des problèmes d'omniscience logique—et il lui manque les ingrédients qui permettent de parler de la nature et des bases de la connaissance et croyance acquises. C'est ce dernier problème qui est l'origine du travail présenté dans cette dissertation : non seulement nous cherchons une façon simple de modéliser la connaissance et la croyance, mais nous étudions aussi l'émergence, l'usage, et la transformation d'*évidence* comme une composante inséparable des croyances justifiées et de la connaissance d'un agent *rationnel* et *idéal*.

Dans ce but, nous montrons que les espaces topologiques sont des objets mathématiques naturels pour formaliser les notions épistémiques mentionnées ci-dessus, ainsi que la dynamique de l'information basée sur évidence : tout en fournissant une compréhension plus profonde de l'interprétation à base d'évidence de la connaissance et de la croyance, la sémantique topologique généralise aussi la sémantique relationnelle standard de la logique épistémique. Schématiquement parlant, les notions topologiques telles que ouverts, fermés, espaces denses et denses nulle part encodent qualitativement et naturellement des notions telles que pa *mesure/observation*, la *proximité*, la *petitesse*, la *grandeur* et la *consistance*, qui toutes reviendront régulièrement dans cette dissertation avec une interprétation épistémique. De plus, les espaces topologiques sont équipés d'opérateurs basiques bien connus tels que les opérateurs d'intérieur et de clôture qui—seuls ou combinés—interprètent de manière succincte différentes modalités épistémiques, apportant une meilleur compréhension de leurs propriétés axiomatiques. Dans ce but, nous voyons les espaces topologiques comme des structures d'information équipées d'une théorie mathématique forte et élégante qui aide à éclairer les débats philosophiques entourant la connaissance et la croyance justifiée, et à mieux comprendre le phénomène d'apprentissage par acquisition d'évidence.

L'usage épistémique des espaces topologiques comme structures d'information remonte aux années 1930 et 1940, quand les espaces topologiques servaient de modèles aux langages intuitionnistes, et les ensembles ouverts sont considérés comme des 'éléments d'évidence', des 'propriétés observables' concernant l'état actuel (voir, e.g., Troelstra and van Dalen, 1988). Cette interprétation assignée aux ensembles ouverts constitue la motivation épistémique basique derrière notre usage des modèles topologiques, et elle reviendra souvent à divers endroits (sous des formes modifiées) dans le corps principal de cette dissertation. Des variantes de cette idée peuvent aussi être trouvées dans la théorie des domaines en informatique (Abramsky, 1987, 1991; Vickers, 1989), guidant le programme de recherche de la théorie formelle "topologique" de l'apprentissage initiée entre autres par Kelly (Kelly, 1996; Schulte and Juhl, 1996; Kelly et al., 1995; Kelly and Lin, 2011; Baltag et al., 2015c) en épistémologie formelle.

La littérature reliant la logique épistémique (modale) et la topologie est organisée autour de deux cadres topologiques distincts, quoique fortement liés. Notre travail dans cette dissertation profite des deux approches. La première direction vient de la sémantique topologique basée sur l'intérieur, de McKinsey (1941) et McKinsey and Tarski (1944), pour le langage de la logique modale basique (certaines idées peuvent déjà être trouvées dans Tarski, 1938 et Tsao-Chen, 1938). Dans cette sémantique l'opérateur modal $\Box$ est interprété sur les espaces topologiques comme l'opérateur d'intérieur. Ces recherches eurent lieu dans un contexte mathématique abstrait, indépendant de considérations épistémiques/doxastiques. McKinsey and Tarski (1944) non seulement prouvèrent que le système modal $\mathsf{S4}$ est la logique de tous les espaces topologiques (sous l'interprétation mentionnée ci-dessus), mais ils montrèrent aussi que c'est la logique de tout espace métrique séparable dense dans lui-même, tel que la ligne rationnelle $\mathbb{Q}$, la ligne réelle $\mathbb{R}$ et l'espace de Cantor, parmi d'autres. Cette approche initia un tout nouveau domaine de logiques spatiales, établissant une connection persistante entre logique modale et topologie (voir, e.g., Aiello et al., 2007 pour une vue d'ensemble sur le sujet, en particulier, voir van Benthem and Bezhanishvili, 2007). De plus, les résultats de complétude concernant le système épistémique $\mathsf{S4}$ ont naturellement attiré les logiciens épistémiques, menant à une réévaluation *épistémique* de la sémantique de l'intérieur, voyant les topologies comme des modèles pour l'information. Une branche de la connexion logique épistémique-topologie a donc été bâtie sur la sémantique topologique basée sur l'intérieur, où la notion épistémique centrale est la connaissance (voir, e.g., van Benthem and Sarenac, 2004). Ce que nous ajoutons à cet ensemble de travaux, dans la Partie I de cette dissertation, ce sont les composants épistémiques manquants d'*évidence* et de *croyance*, ainsi que la *dynamique d'apprentissage de nouvelle évidence*, renforçant ainsi la connection entre logique épistémique et topologie. Pour ce faire nous réanalysons les modèles d'évidence à base de voisinages de van Benthem and Pacuit (2011) d'un point de vue topologique. La façon dont nous représentons l'évidence et comment elle se connecte avec la croyance justifiée sont inspirés pas l'approche de (van Benthem and Pacuit, 2011), et les actions de transformation d'évidence considérées sont adaptées de ce travail de grande importance.

La seconde approche topologique pour la logique épistémique fut initiée par Moss and Parikh (1992). Ils introduisirent la *topologique*, un cadre bimodal pour formaliser le raisonnement à propos d'ensembles et de points dans un unique système modal. Leurs recherches topologiques sont fortement motivées par la logique épistémique, suggérant que "des aspects simples du raisonnement topologique sont aussi connectés avec des logiques spécialisées de la *connaissance*" (Moss and Parikh, 1992, p. 95). L'élément clef que Moss and Parikh (1992) introduisent dans le paradigme de la logique épistémique est la notion abstraite d'*effort* épistémique. L'effort peut, pour parler simplement, être décrit comme n'importe quel type de collecte d'évidence—via, e.g., mesure, calcul, approximation, expérimentation ou annonce—qui peut mener à une connaissance accrue. Le formalisme de la topologique combine donc la notion statique de connaissance avec la notion dynamique d'effort, et est par conséquent fortement lié à la *logique épistémique dynamique* (Baltag et al., 1998; van Ditmarsch et al., 2007; van Benthem, 2011; Baltag and Renne, 2016). Dans la Partie II de cette thèse, nous établissons une connection entre les deux formalismes, et nous en tirons des bénéfices à la fois conceptuels et techniques. Alors que la logique épistémique dynamique étend le domaine des attitudes dynamiques qu'elle étudie, le cadre de la topologique obtient des axiomatisations épistémiquement plus intuitives, clarifiant ainsi la signification de l'*effort* en le connectant à des exemples bien compris tels que les *annonces arbitraires* et *publiques*.

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Les contributions de cette thèse sont présentées en deux parties. Nous donnons ci-dessous un aperçu approfondi de chaque chapitre.

Le chapitre 2 fournit les préliminaires techniques essentiels aux deux parties de cette dissertation. Cela inclue, dans la première moitié, une très brève introduction à la sémantique de Kripke standard pour la logique modale basique. Nous rappelons les systèmes statiques habituellement étudiés pour les logiques épistémiques/doxastiques et les propriétés relationnelles correspondantes qui rendent ces logiques correctes et complètes. Le cadre relationnel sert seulement d'outil technique utilisé dans les parties I et II dans le but d'atteindre des résultats techniques dans le cadre topologique. Dans la seconde partie, nous introduisons les notions topologiques élémentaires que nous utiliserons à travers cette dissertation.

Plan. La section 2.1 discute brièvement la sémantique relationnelle standard pour le langage de la logique modale basique, et les systèmes épistémiques et doxastiques unimodaux qui seront étudiés dans les chapitres ultérieurs. La section 2.2 introduit les préliminaires purement topologiques qui seront utilisés dans toute la thèse. De plus, ce chapitre sert aussi à fixer les notations pour le corps principal de cette dissertation. Les lecteurs familiers avec les sujets mentionnés ci-dessus devraient pouvoir passer ce chapitre sans problème.

PARTIE I : De la Sémantique de l'Intérieur aux Modèles de Faits

La partie I concerne les interprétations à base d'évidence de la croyance justifiée et de la connaissance. En commençant par une interprétation topologique maintenant standard de l'opérateur d'intérieur, nous développons graduellement un cadre topologique qui (1) peut parler d'évidence non seulement sémantiquement, mais aussi au niveau syntactique, rendant ainsi la notion d'évidence plus explicite; (2) prend l'évidence comme notion la plus primitive, sur laquelle croyance et connaissance sont définies, reliant ainsi ces deux notions épistémologiques cruciales de manière plus profonde et plus basique. Ces investigations ont des conséquences philosophiques considérables puisqu'elles nous permettent de discerner, d'isoler, et d'étudier divers aspects de la notion d'évidence, et ses relations avec la justification, la connaissance et la croyance.

Le chapitre 3 fournit les bases formelles de la sémantique topologique basée sur l'intérieur pour la logique modale basique, qui remonte aux travaux de McKinsey (1941) et McKinsey and Tarski (1944). Dans cette sémantique l'opérateur modal $\Box$ est interprété sur des espaces topologiques comme l'opérateur d'intérieur. L'une des raisons en est que le système épistémique $\mathsf{S4}$ est la logique de tous les espaces topologiques. Une autre est que l'interprétation des ensembles ouverts comme 'propriétés observables' ou 'éléments d'évidence' met la sémantique topologique à base d'intérieurs sur le radar des logiciens épistémiques. Dans ce chapitre, nous introduisons brièvement la *sémantique topologique d'intérieurs*, nous concentrant particulièrement sur ses idées épistémiques, et nous expliquons comment et pourquoi elle constitue une interprétation satisfaisante pour la connaissance (basée sur évidence), et, par conséquent, pourquoi—dans certains contextes—elle forme une sémantique plus riche que la sémantique relationnelle. Nous discutons ensuite une sémantique topologique de la croyance de la littérature, basée sur l'opérateur d'ensemble dérivé, et nous argumentons qu'elle ne constitue pas une sémantique satisfaisante pour la croyance, en particulier quand on la considère conjointement avec la connaissance comme intérieur. Notre contribution dans la partie I est inspirée de, et développée sur, les bases de ce cadre. Dans les chapitres suivants, nous étendons et enrichissons la sémantique d'intérieur afin de formaliser différentes notions de connaissance (basée sur évidence) et de croyance justifiée, ainsi que différentes notions de possession d'évidence.

Plan. La section 3.1 est une section technique qui introduit la sémantique d'intérieur ainsi que ses connections avec la sémantique relationnelle (section 3.1.2). Dans la section 3.1.3, nous listons les résultats topologiques généraux de correction et complétude pour les systèmes $\mathsf{S4}$, $\mathsf{S4.2}$ et $\mathsf{S4.3}$ qui seront utilisés dans les chapitres suivants. La section 3.2 explique ensuite les motivations derrière l'utilisation de l'opérateur d'intérieur comme modalité de connaissance, en mettant l'accent sur l'interprétation à base d'évidence sous-jacente.

Dans le chapitre 4 notre attention passe de l'interprétation topologique de la connaissance à l'interprétation topologique de la croyance, et nous présentons le premier pas vers le développement d'une théorie topologique de la croyance qui fonctionne bien combinée avec la connaissance comme opérateur d'intérieur. Plus précisément, la première partie de ce chapitre présente un examen de la sémantique de croyance topologique de (Özgün, 2013; Baltag et al., 2013), traitant les questions suivantes :

- *Étant donnée la sémantique topologique d'intérieur pour la connaissance, comment peut-on construire une sémantique topologique pour la croyance* *qui réponde aussi au problème de comprendre la relation entre connaissance et croyance ?*

- *Dans quelle mesure les notions topologiques capturent-elles la signification intuitive de la notion de croyance en question ?*

Comprendre la relation entre connaissance et croyance est un problème central en épistémologie. Tout spécialement après que Gettier (1963) a fait voler en éclat la vision traditionnelle de la connaissance comme *croyance vraie et justifiée*, de nombreux épistémologues ont tenté de renforcer cette dernière notion pour obtenir une notion satisfaisante de la première. Dans cette approche, on commence avec une notion faible de croyance (qui est au moins justifiée et vraie) et on essaye d'atteindre la connaissance en renforçant la notion de croyance choisie de manière à obtenir une notion de connaissance qui ne soit plus sujette aux contre-exemples de type Gettier (Gettier, 1963). Plus récemment l'approche inverse—dériver la croyance à partir de la connaissance—ou, du moins, mettre la “connaissance en premier” (Williamson, 2000) a aussi été considérée. Dans cet esprit, Stalnaker (2006) a proposé un cadre formel dans lequel la croyance est réalisée comme une forme affaiblie de connaissance. Plus précisément, en commençant avec le système logique donné en Table A.1, dans lequel la croyance et la connaissance

(K)	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	Normalité de la connaissance
(T)	$K\varphi \to \varphi$	Factualité de la connaissance
(PI)	$K\varphi \to KK\varphi$	Introspection positive
(D_B)	$B\varphi \to \neg B\neg\varphi$	Consistance de la croyance
(sPI)	$B\varphi \to KB\varphi$	Introspection positive forte
(sNI)	$\neg B\varphi \to K\neg B\varphi$	Introspection négative forte
(KB)	$K\varphi \to B\varphi$	Connaissance implique croyance
(FB)	$B\varphi \to BK\varphi$	Croyance complète

Table A.1: Schéma axiomatique de Stalnaker

sont toutes deux représentées comme primitives, Stalnaker formalise quelques relations d'apparence naturelle entre les deux, et prouve sur la base de ces relations que la croyance peut être *définie* à partir de la connaissance comme la *possibilité épistémique de connaissance* :

$$B\varphi := \neg K \neg K\varphi.$$

Dans ce but, la formalisation syntactique de Stalnaker semble être analogue au *status quo* de la sémantique de l'intérieur pour la connaissance et de l'interprétation topologique de la croyance élaborée dans le chapitre 3, où nous donnons l'interprétation de la connaissance et dévoilons une bonne sémantique pour la croyance.

Baltag et al. (2013) et Özgün (2013), en partant du formalisme de Stalnaker, proposèrent d'interpréter la croyance, en particulier la croyance de Stalnaker, comme *certitude subjective*, en termes de *la clôture de l'opérateur d'intérieur* sur des espaces extrêmement discontinus. Tandis que ce cadre statique fournit une réponse satisfaisante aux questions ci-dessus, l'extension dynamique par des modalités d'annonces publiques rencontre des problèmes dûs aux propriétés structurelles des espaces extrêmement discontinus. Cela mène à la quête d'une logique de la connaissance et de la croyance adaptée aux annonces publiques. La seconde partie de ce chapitre est dévolue à la résolution de ce problème, et la solution que nous proposons consiste en une interprétation de la connaissance et de la croyance sur des *espaces héréditairement extrêmement discontinus*.

Alors que cette sémantique pour la croyance fonctionne bien pour la notion de croyance forte de Stalnaker comme *certitude subjective*, d'un point de vue plus général elle peut être vue comme quelque peu restrictive pour deux raisons. Elle est basée sur des classes d'espaces topologiques assez exotiques, et les logiques correspondantes n'incluent pas réellement l'évidence car elles n'en ont pas de représentation syntaxique. Cela constitue une partie de la motivation pour le travail présenté dans le chapitre 5, menant aux questions plus générales et fondamentales que nous y traitons.

Plan. La section 4.1 présente le système combiné de Stalnaker pour la connaissance et la croyance, et liste les aspects importants de son travail qui inspirèrent (Özgün, 2013; Baltag et al., 2013). Dans la section 4.2, nous passons en revue la sémantique topologique de croyance de (Özgün, 2013; Baltag et al., 2013), et la section 4.2.2 rappelle pourquoi les mises à jour ne fonctionnent pas sur les espaces extrêmement discontinus. Dans la section 4.3, nous introduisons ce qui va au-delà de (Özgün, 2013; Baltag et al., 2013), nous modélisons croyance, croyance conditionnelle et annonces publiques sur les espaces héréditairement extrêmement déconnectés, et nous présentons plusieurs résultats de complétude concernant $\mathsf{KD45}_B$ et ses extensions avec croyances conditionnelles et annonces publiques.

Le chapitre 5 contient la contribution principale de la partie I. En s'appuyant sur l'hypothèse que la croyance rationnelle d'un agent est basée sur l'évidence disponible, nous essayons de révéler la relation concrète entre l'évidence à disposition d'un agent, ses croyances et sa connaissance, et nous étudions la dynamique de l'évidence supportée par la représentation statique mise au point. Ce projet est motivé par des questions à la fois philosophiques et techniques, ainsi que par les inconvénients susmentionnés de notre propre travail du chapitre 4. Plus précisément, nous considérons entre autres les questions suivantes :

- *Comment un agent en possession d'éléments d'évidence possiblement faux, possiblement mutuellement contradictoires, réunit de manière consistante son évidence et forme des croyances consistantes ?*

- *Quelles sont les conditions nécessaires et suffisantes pour qu'un élément d'évidence constitue une justification pour une croyance ?*
- *Quelles propriétés devrait posséder un élément d'évidence pour entraîner une connaissance (défaisable) ?*
- *Comment notre formalisation des notions susmentionnées aide-t-elle à comprendre les discussions en épistémologie formelle quant aux liens entre croyance justifiée et connaissance ?*
- *Quelles sont les axiomatisations complètes des logiques associées de croyance justifiée, connaissance et évidence ? Ont-elles la propriété du modèle fini ? Sont-elles décidables ?*

Ces questions guident aussi l'approche de van Benthem and Pacuit (2011); van Benthem et al. (2012, 2014), qui inspira considérablement notre travail. Les travaux influents de (van Benthem and Pacuit, 2011; van Benthem et al., 2012, 2014) représentent l'évidence sémantiquement—pour faire simple, comme des ensembles de mondes possibles—en se basant sur des structures de voisinage, ainsi que syntaxiquement en introduisant des modalités d'évidence. Leur cadre va au-delà et généralise le traitement formel des notions sémantiques citées précédemment en termes de structures relationnelles, telles que modèles de Kripke ou de plausibilité, et modèles non-relationnels, tels que les modèles de sphère de Grove. Dans ce chapitre nous franchissons une étape de plus dans l'amélioration du traitement théorique formel modal de l'évidence, de la croyance justifiée et de la connaissance en révélant la structure topologique cachée des modèles d'évidence de van Benthem and Pacuit (2011). La perspective topologique permet des représentations mathématiques plus précises et raffinées de diverses notions d'évidence telles que l'*évidence basique*, l'*évidence combinée*, l'*évidence vraie* et l'*évidence non-trompeuse*, ainsi que de notions épistémiques pertinentes telles que l'*argument* et la *justification* (basée sur évidence) et, enfin, la *croyance justifiée* et la *connaissance indéfectible*. En conséquence, nous obtenons un cadre sémantiquement et syntaxiquement riche qui fournit une analyse logique plus profonde quant au rôle de l'évidence dans l'atteinte d'un état épistémique/doxastique par un agent. Nous examinons aussi plusieurs types de dynamiques d'évidence introduits dans (van Benthem and Pacuit, 2011) et nous appliquons ce cadre pour analyser et aborder des problèmes clefs en épistémologie tels que les exemples de Gettier de type "pas de lemme faux", les contradicteurs trompeurs, et la justification non contredite face à la croyance non contredite. Nos résultats techniques principaux traitent de complétude, de décidabilité et de propriété du modèle fini pour les logiques associées. Dans ce qui suit, nous fournissons un aperçu détaillé des notions épistémiques étudiées dans ce chapitre, nous introduisons les modalités que nous considérons, et nous expliquons où notre travail se situe par rapport à la littérature concernée.

Une raison cruciale pour laquelle notre approche présentée dans le chapitre 5 fait mieux que celles des chapitres 3 et 4 est qu'ici nous introduisons des modalités d'évidence afin de fournir aussi des représentations syntactiques des notions d'évidence, et finalement pour construire des *logiques d'évidence*. En particulier, nous étudions l'opérateur "avoir un élément d'évidence pour une proposition P" proposé pas van Benthem and Pacuit (2011), mais nous étudions aussi des variantes intéressantes de ce concept : "avoir une évidence (combinée) pour P", "avoir un élément d'évidence *vrai* pour P" et "avoir une évidence (combinée) vraie pour P". La table A.2 ci-dessous liste les modalités d'évidence correspondantes ainsi que leur lecture intuitive.

$E_0\varphi$	l'agent a un élément d'évidence pour φ
$E\varphi$	l'agent a une évidence combinée pour φ
$\Box_0\varphi$	l'agent a un élément d'évidence vrai pour φ
$\Box\varphi$	l'agent a une évidence combinée vraie pour φ

Table A.2: Modalités d'évidence et leur lecture intuitive

Les *éléments basiques d'évidence* possédés par un agent sont modélisés comme des ensembles non vides de mondes possibles. Une *évidence combinée* (ou simplement "évidence") est une intersection non vide d'un nombre fini d'éléments d'évidence. Cette notion d'évidence n'est pas nécessairement vraie, puisque les éléments d'évidence sont potentiellement faux et possiblement inconsistants entre eux. Par *évidence vraie* nous entendons évidence qui est vraie dans le monde actuel. En épistémologie il est commun de réserver le terme "évidence" pour l'évidence vraie. Cependant nous suivons ici l'usage plus libéral fait de ce terme dans (van Benthem and Pacuit, 2011), qui est en accord avec l'acception commune de la vie de tous les jours, e.g. quand on parle d' "évidence incertaine", de "fausse évidence", d'"évidence trompeuse" etc.[2] La famille des ensembles d'évidence (combinée) forme une base topologique qui engendre ce que nous appelons la *topologie évidentielle*. Il s'agit de la plus petite topologie dans laquelle tous les éléments basiques d'évidence sont ouverts, et elle jouera un rôle important dans notre formalisme. En fait, la modalité $\Box\varphi$ qui capture le concept de "avoir une évidence vraie pour φ" coïncide avec l'*opérateur d'intérieur* dans la topologie évidentielle (voir section 5.2.2). Nous utilisons donc la sémantique d'intérieur de McKinsey and Tarski (1944) pour interpréter une notion d'évidence *vraie* (à la différence de ce qui est fait dans le chapitre 4, où l'opérateur d'intérieur était traité comme connaissance). Nous montrons aussi que deux variantes vraies d'opérateurs de possession d'évidence ($\Box_0$ et $\Box$) sont plus expressives que celles non vraies (E_0 et E) : lorsqu'elles interagissent avec la modalité globale, les

[2]Bien sûr cela ne passe pas bien la traduction. En français on pourrait penser à "indice incertain", "faux indice", "indice trompeur" etc.

deux modalités d'évidence vraie $\Box_0\varphi$ et $\Box\varphi$ peuvent définir les variantes non vraies $E_0\varphi$ et $E\varphi$, respectivement, ainsi que de nombreux autre opérateurs doxastiques/épistémiques.

La notion de *croyance justifiée* que nous étudions dans ce chapitre sera définie purement à travers les notions d'évidence mentionnées ci-dessus. Nous proposons une *sémantique "cohérentiste" de la justification et de la croyance justifiée*, que nous obtenons en étendant, en généralisant et (dans une certaine mesure) en profilant le cadre formel des modèles d'évidence pour les croyances introduit par van Benthem and Pacuit (2011). L'idée principale derrière la définition de la croyance de van Benthem and Pacuit (2011) semble être que l'agent rationnel essaye de former des croyances consistantes en regardant toutes les *plus fortes collections d'évidence finiment consistantes*, et elle croît tout ce qui est impliqué par l'ensemble de ces dernières.[3]

Leur définition de la croyance dépend donc crucialement de la notion de "plus forte" évidence, et elle fonctionne bien dans le cas fini (quand l'agent a un nombre fini d'éléments d'évidence de base) ainsi que dans *certains* cas infinitaires. Mais, comme déjà remarqué dans (van Benthem et al., 2014), ce cadre formel présente l'inconvénient qu'il peut produire des *croyances inconsistantes* dans le cas infinitaire général. Un défaut plus technique de ce cadre est que la logique doxastique correspondante ne possède pas la propriété du modèle fini (voir van Benthem et al., 2012, Corollary 2.7 ou van Benthem et al., 2014, Corollary 1). Dans ce chapitre, nous proposons une sémantique "améliorée" pour la croyance basée sur évidence obtenue en affaiblissant, en un sens, la définition de (van Benthem and Pacuit, 2011). Selon nous, une proposition P est crue si P est impliquée par *des collections d'évidence finiment consistantes suffisamment fortes.* Cette définition coïncide avec celle de van Benthem and Pacuit (2011) pour les modèles portant des collections d'évidence finies, mais elle fait appel à une généralisation différente de leur notion dans le cas infinitaire. En fait, notre sémantique *assure toujours la consistance des croyances*, même lorsque les éléments d'évidence disponibles sont mutuellement inconsistants. Nous fournissons aussi une formalisation de l'*argument* et une *vue "cohérentiste" des justifications.* Un *argument* est essentiellement constitué d'un ou plusieurs ensembles d'évidence supportant la même proposition (fournissant donc de multiples chemins potentiels vers une conclusion commune); une *justification* est un argument qui n'est contredit par aucune autre évidence disponible. Notre définition de la croyance équivaut à demander que P *soit vraie ssi il y a quelque justification* (*basée sur évidence*) *pour* P. Elle capture donc correctement le concept de "croyance justifiée". Notre proposition est aussi très naturelle d'un point de vue topologique : elle est équivalente à dire que P

[3]Ceci est encore vague puisque nous n'avons pas encore spécifié ce que signifie "plus fortes collections d'évidence finiment consistante" (nous formalisons ces notions dans la section 5.2.1. Cependant ce niveau de précision devrait être suffisant pour expliquer l'idée derrière la définition de croyance de van Benthem and Pacuit (2011), et notre notion de croyance justifiée étudiée dans ce chapitre).

est crue ssi elle est vraie dans "presque tous" les états épistémiquement possibles, où "presque tous" est interprété topologiquement comme "tous sauf pour un ensemble dense nulle part". De plus, nous généralisons cette croyance sémantique pour les croyances conditionnelles. La table A.3 ci-dessous liste les modalités de croyance que nous étudions dans ce chapitre.

$B\varphi$	l'agent a une croyance justifiée de φ
$B^\varphi\psi$	l'agent croit ψ à condition que φ

Table A.3: Modalités de croyance et leurs lectures intuitives

Quant à la *connaissance*, il y a un certain nombre de différentes notions qui peuvent être considérées. Premièrement, il y a la connaissance "absolument certaine" ou "infaillible", proche du concept de connaissance partitionnelle d'Aumann (Aumann, 1999) ou du concept d'information dure de van Benthem (van Benthem, 2007). Dans notre cadre mono-agent, cela peut être défini simplement comme la modalité globale (qui quantifie universellement sur les états épistémiquement possibles). Il y a très peu de propositions qui peuvent être connues de cette manière infaillible (e.g., celles connues par introspection ou par preuve logique). La plupart des faits en science ou dans la vraie vie sont inconnus dans ce sens. Il est donc plus intéressant de considérer des notions de connaissance moins qu'absolument certaine, telle que la *connaissance défaisable.* Dans notre cadre, nous considérons à la fois la connaissance absolument certaine et la connaissance défaisable, mais nous nous intéressons plus particulièrement à cette dernière. Voir la table A.4 ci-dessous pour les modalités de connaissance correspondantes et leur signification.

$[\forall]\varphi$	l'agent a la connaissance infaillible de φ
$K\varphi$	l'agent sait φ de manière faillible (ou défaisable)

Table A.4: Modalités de connaissance et leurs significations intuitives

Les célèbres contre-exemples de Gettier (Gettier, 1963) montrent que simplement ajouter la "véracité" la croyance ne nous donnera pas de "bonne" notion de connaissance défaisable : la croyance vraie (justifiée) est extrêmement fragile (i.e., elle peut être perdue trop facilement), et elle est consistante avec le fait de n'avoir que des justifications erronées pour une conclusion accidentellement vraie. Nous formalisons ici une notion de connaissance défaisable disant que "*P est connue (de manière faillible) s'il y a une justification vraie pour P*". Nous étudions par conséquent une notion de connaissance définie comme une *croyance correctement justifiée.* Comme nous le développons en section 5.5.1, cette notion de connaissance moins qu'absolument certaine trouve sa place dans la littérature post-Gettier en étant plus forte que celle caractérisée par le "pas de lemme faux" de Clark (1963) et plus faible que la conception de la connaissance décrite par la théorie de la défaisabilité de la connaissance défendue par Lehrer and Paxson (1969); Lehrer (1990); Klein (1971, 1981).

Encore un autre chemin menant à notre cadre formel dans ce chapitre passe par notre travail précédent (Baltag et al., 2013, 2015a), présenté dans le chapitre 4, sur une topologie sémantique pour les axiomes doxastiques/épistémiques de Stalnaker (2006). Rappelons que le système de Stalnaker présenté dans la table A.1 est fait pour capturer une notion de connaissance faillible, en forte interaction avec une notion de "croyance forte" définie comme une *certitude subjective.* La principale idée spécifique à ce système était que "croire implique croire que l'on sait", idée capturée par l'axiome de Croyance complète ($B\varphi \rightarrow BK\varphi$). La sémantique topologique que nous avons proposée pour ces concepts dans (Özgün, 2013; Baltag et al., 2013, 2015a) était trop restrictive (car limitée à la classe peu familière des topologies extrêmement discontinues et héréditairement extrêmement discontinues). Dans le chapitre 5 nous montrons que ces notions peuvent être interprétées sur des espaces topologiques arbitraires, sans changer leur logique. En effet, nos définitions de croyance et de connaissance peuvent être vues comme les généralisations naturelles aux topologies arbitraires des notions de (Özgün, 2013; Baltag et al., 2013, 2015a).

Nous axiomatisons complètement les différentes logiques d'évidence, de connaissance et de croyance que nous obtenons, et nous établissions des résultats de décidabilité et de propriété du modèle fini. De plus, nous étudions quelques extensions dynamiques, en encodant différents types de dynamique d'évidence. Techniquement, notre résultat le plus difficile est la complétude de la logique de l'évidence vraie la plus riche $\mathsf{Log}_{\forall\Box\Box_0}$, qui contient les deux modalités d'évidence vraie $\Box_0\varphi$ et $\Box\varphi$, ainsi que la modalité globale $[\forall]\varphi$. L'axiomatisation de $\mathsf{Log}_{\forall\Box\Box_0}$ est donnée par les schémas d'axiomes et les règles d'inférence de la table A.5.

(CPL)	toutes les tautologies propositionnelles classiques et Modus Ponens (MP)
($\mathsf{S5}_{[\forall]}$)	tous les axiomes de $\mathsf{S5}$ et les règles pour la modalité $[\forall]$
($\mathsf{S4}_{\Box}$)	tous les axiomes de $\mathsf{S4}$ et les règles pour la modalité $\Box$
($4_{\Box_0}$)	$\Box_0\varphi \rightarrow \Box_0\Box_0\varphi$
Universalité (U)	$[\forall]\varphi \rightarrow \Box_0\varphi$
Évidence Vraie (FE)	$\Box_0\varphi \rightarrow \Box\varphi$
Retrait	$(\Box_0\varphi \wedge [\forall]\psi) \rightarrow \Box_0(\varphi \wedge [\forall]\psi)$
Monotonicité de $\Box_0$	de $\varphi \rightarrow \psi$, on infère $\Box_0\varphi \rightarrow \Box_0\psi$

Table A.5: Axiomatisation de $\mathsf{Log}_{\forall\Box\Box_0}$

Cette logique peut définir tous les opérateurs modaux que nous étudions dans ce chapitre. Tandis que les autres preuves sont plus ou moins de l'ordre de la routine, les résultats techniques mentionnés pour $\mathsf{Log}_{\forall\Box\Box_0}$ font intervenir une combinaison non triviale de méthodes connues.

Plan. La section 5.1 sert d'introduction semi-formelle et de résumé du chapitre comme ce qui précède, mettant l'accent sur les caractéristiques importantes de son contenu. Dans la section 5.2 nous introduisons les modèles d'évidence de van Benthem and Pacuit (2011) ainsi que nos modèles d'évidence *topologiques*, et nous donnons la sémantique pour les notions d'élément d'évidence, d'évidence combinée et d'évidence vraie. De plus, nous donnons des définitions topologiques pour l'argument et la justification. Dans la section 5.3, nous proposons une sémantique topologique pour une notion de croyance justifiée tout en comparant notre système à celui de van Benthem and Pacuit (2011). Nous généralisons ensuite notre sémantique de croyance (simple) à la croyance conditionnelle. La section 5.4 définit les modèles de transformations induits par les dynamiques d'information basées sur évidence telles que les annonces publiques, l'addition d'évidence, l'amélioration d'évidence et la combinaison faisable d'évidence. Dans la section 5.5 nous proposons une interprétation topologique pour une notion de connaissance faillible et nous relions notre formalisme à certaines discussions importantes qui ont émergé dans la littérature de l'épistémologie post-Gettier, telles que les théories de stabilité/défaisabilité de la connaissance, contradicteurs trompeurs contre contradicteurs sincères, etc. Finalement, la section 5.6 présente tous nos résultats techniques. Le lecteur intéressé uniquement par les aspects techniques peut sauter directement à la section 5.6.

PARTIE II : Des Annonces Publiques aux Efforts

Dans la Partie II de cette dissertation nous ne parlons plus de croyances, mais nous nous concentrons sur certaines notions de connaissance ainsi que différents types de dynamique d'information qui incluent l'apprentissage de nouvelle évidence. Cette partie prend comme point de départ le cadre de l'espace des sous-ensembles de Moss and Parikh (1992), et elle tourne autour des notions de *connaissance absolument certaine* et *connaissabilité* comme "*connaissance potentielle*", ainsi que des connections entre la notion abstraite d'*effort* épistémique, qui recouvre toute méthode d'acquisition d'évidence, et certaines attitudes dynamiques bien connues telles que les *annonces publiques* et les *annonces publiques arbitraires.*

Le chapitre 6 fournit les bases pour la Partie II et motive le changement de paradigme entre les deux parties de cette thèse. En particulier, il introduit la sémantique d'espace des sous-ensembles de Moss and Parikh (1992) et la logique topologique d'annonces publiques de Bjorndahl (2016). Dans ce chapitre, nous mettons aussi en relief les connections et différences entre les utilisations épistémiques des espaces topologiques dans les parties I et II de cette thèse, en particulier en ce qui concerne les types d'évidence représentés et les notions de connaissance étudiées.

Plan. La section 6.1 présente le cadre de l'espace des sous-ensembles, fournissant sa syntaxe et sémantique ainsi que les axiomatisations complètes des logiques associées pour les espaces de sous-ensembles et les espaces topologiques. La section 6.2 introduit la logique topologique d'annonces publiques de Bjorndahl (2016) et présente plusieurs résultats d'expressivité pour les langages étudiés dans les cadres formels mentionnés ci-dessus.

Le chapitre 7 étudie les extensions de la logique topologique d'annonces publiques de Bjorndahl (2016) avec la modalité d'*effort* de Moss and Parikh (1992), ainsi qu'avec une version topologique de la modalité d'annonce arbitraire de Balbiani et al. (2008). Ce travail présente un intérêt tant conceptuel que technique, en visant à clarifier la connection, intuitivement évidente mais difficile à saisir formellement, entre les notions dynamiques d'*effort* et ce qui semble en être des instanciations : les annonces publiques et arbitraires. Ces modalités sont données en Table A.6 avec leurs lectures intuitives.

$\Box\varphi$	φ reste vraie après tout effort
$[\psi]\varphi$	après l'annonce publique de ψ, φ devient vraie
$\boxed{*}\varphi$	φ reste vraie après toute annonce épistémique

Table A.6: Modalités dynamiques étudiées dans le chapitre 7 et leurs lectures intuitives

En particulier, nous nous intéressons aux questions suivantes, et y répondons par l'affirmative :

- *Peut-on clarifier la signification de l'effort modal en le reliant aux modalités dynamiques citées ci-dessus ?*
- *Traiter ensemble dans un même cadre topologique la modalité d'effort et les annonces publiques fournit-il quelque avantage technique quant à l'axiomatisation complète, la décidabilité et la propriété du modèle fini de ses logiques associées ?*

Nous donnons l'axiomatisation complète de la *logique topologique dynamique de l'effort et des annonces publiques* (appelée TopoLogique Dynamique) donnée dans la table A.7 ci-dessous, et nous défendons l'idée qu'elle est plus intuitive et, dans un sens, plus simple que les axiomes standards de la logique topologique

(CPL)	toutes les tautologies propositionnelles classiques et le (MP)
$(S5_K)$	tous les axiomes de $\mathsf{S5}$ et les règles pour la modalité K
$(S4_{\mathsf{int}})$	tous les axiomes $\mathsf{S4}$ et les règles pour la modalité int
(K-int)	*Connaissance implique connaissabilité*: $K\varphi \to \mathsf{int}(\varphi)$
$(K_!)$	$[\varphi](\psi \to \theta) \to ([\varphi]\psi \to [\varphi]\theta)$
$(Nec_!)$	de θ, on infère $[\varphi]\theta$
([!]RE)	*Remplacement d'Équivalents pour* [!]: de $\varphi \leftrightarrow \psi$, on infère $[\varphi]\theta \leftrightarrow [\psi]\theta$
	Axiomes de réduction:
$(R[\top])$	$[\top]\varphi \leftrightarrow \varphi$
(R_p)	$[\varphi]p \leftrightarrow (\mathsf{int}(\varphi) \to p)$
$(R_\neg)$	$[\varphi]\neg\psi \leftrightarrow (\mathsf{int}(\varphi) \to \neg[\varphi]\psi)$
(R_K)	$[\varphi]K\psi \leftrightarrow (\mathsf{int}(\varphi) \to K[\varphi]\psi)$
$(R_{[!]})$	$[\varphi][\psi]\chi \leftrightarrow [\langle\varphi\rangle\psi]\chi$
([!]$\Box$-elim)	$[\varphi]\Box\theta \to [\varphi \wedge \rho]\theta$ ($\rho \in \mathcal{L}^!_{K\mathsf{int}\Box}$ une formula arbitraire)
([!]$\Box$-intro)	de $\psi \to [\varphi \wedge p]\theta$, on infère $\psi \to [\varphi]\Box\theta$ ($p \notin P_\psi \cup P_\theta \cup P_\varphi$)

Table A.7: Les axiomatisations des $\mathsf{TopoLogiques}$ Dynamiques. Noter que P_φ dénote l'ensemble de toutes les variables propositionnelles qui apparaissent dans φ.

(Georgatos, 1993, 1994; Dabrowski et al., 1996). Notre preuve de complétude est aussi plus directe, utilisant une construction de modèle canonique standard. De plus, nous étudions les relations entre cette extension et d'autres formalismes logiques connus, montrant en particulier qu'elle est co-expressive avec la logique d'intérieur et de modalité globale (Goranko and Passy, 1992; Bennett, 1996; Shehtman, 1999; Aiello, 2002), plus simple et plus ancienne.

Nous considérons aussi une sémantique topologique pour la modalité d'annonce arbitraire, et nous étudions ses interactions avec la modalité d'effort. A notre connaissance, les preuves de complétude connues pour les logiques d'annonces arbitraires (topologiques ou relationnelles) reposent sur des axiomatisations infinitaires formalisées en ayant recours à des formes de nécessité (voir, e.g., Balbiani et al., 2008, 2013; Balbiani, 2015; Balbiani and van Ditmarsch, 2015; voir aussi les sections 8.2 et 8.3 pour le cas multi-agent). Bien que Balbiani et al. (2008) propose une axiomatisation finitaire similaire à la nôtre, sa preuve de complétude passe par la complétude d'un système infinitaire.[4] A l'inverse, notre preuve de

[4] L'axiomatisation finitaire proposée dans (Balbiani et al., 2008) a par la suite été prouvée incorrecte pour le cas *multi-agent* (voir `http://personal.us.es/hvd/APAL_counterexample.`

complétude pour le système finitaire de la logique topologique d'annonces arbitraires ne fait pas de détour par une logique infinitaire. La modalité d'effort aide donc à simplifier et profiler l'axiomatisation de la logique d'annonces arbitraires.

Plan. La section 7.1 présente la TopoLogique Dynamique qui combine le formalisme topologique avec les annonces publiques de Bjorndahl présentées dans le chapitre 6. Alors que la section 7.1.2 présente plusieurs résultats d'expressivité, la section 7.1.3 se concentre sur la preuve de complétude de l'axiomatisation pour la TopoLogique Dynamique que nous proposons. Dans la section 7.2, nous étudions les annonces arbitraires sur des topo-modèles et nous démontrons que, en fait, l'annonce arbitraire et la modalité d'effort sont équivalentes dans notre cadre mono-agent.

Le chapitre 8 étudie la généralisation au cas multi-agent du cadre formel présenté dans le chapitre précédent. Nous rappelons brièvement que Moss and Parikh (1992), dans le cadre *mono-agent* original des espaces de sous-ensembles, évalue les formules du langage bimodal avec connaissance et modalités d'effort, dénotées respectivement par K et $\Box$, sur des *espaces de sous-ensembles* $(X, \mathcal{O})$, où X est un domaine non vide et $\mathcal{O}$ est un ensemble non vide de sous-ensembles de X. Les formules sont interprétées non seulement par rapport à l'état courant, mais aussi par rapport à un élément d'évidence vraie. L'unité d'évaluation est donc une paire (x, U) telle que $x \in U \in \mathcal{O}$, où le point x représente le véritable état des choses, et l'ensemble U représente tous les points que l'agent considère possible, i.e., sa portée épistémique.

Modéliser des systèmes multi-épistémiques multi-agents dans le style de la sémantique des sous-espaces n'est pas chose facile. Comme (Baskent, 2007, Chapitre 6) et (Wáng and Ågotnes, 2013a) le reconnaissent, développer une logique épistémique multi-agents utilisant une sémantique topologique basée sur les espaces de sous-ensembles requière d'abord de résoudre le problème du "saut hors de la portée épistémique" d'un agent lors de l'évaluation de formules de connaissance d'ordre supérieur. Ce problème est indépendant d'éventuelles extensions dynamiques. Le cadre général que nous considérons traite n'importe quel nombre fini d'agents, mais afin d'illustrer le problème considérons ici le cas de deux agents. Si nous étendons le cadre mono-agent de manière naïve, alors nous avons un ensemble ouvert pour chacun des agents i et j, et la sémantique primitive devient un triplet (x, U_i, U_j) au lieu d'une paire (x, U). Considérons maintenant une formule telle que $K_i\hat{K}_jK_ip$, pour "l'agent i sait que l'agent j considère possible que l'agent i sache la proposition p". Si cela est vrai pour un triplet (x, U_i, U_j), alors $\hat{K}_jK_ip$ doit être vraie pour tout $y \in U_i$; mais y pourrait ne pas être dans U_j, auquel cas (y, U_i, U_j) n'est pas bien défini : nous ne pouvons y interpréter

`pdf`), et l'erreur dans l'analyse de complexité dans (Balbiani et al., 2008, Truth Lemma 4.13, p. 327) est corrigée dans (Balbiani, 2015).

$\hat{K}_j K_i p$. Notre solution à ce dilemme est de considérer des voisinages qui ne sont pas seulement relatifs à chaque agent, mais sont aussi *relatifs à chaque état*. Cela signifie que, lorsque dans (x, U_i, U_j) on change de point de vue de x à $y \in U_i$, simultanément nous changeons aussi de *voisinage* (et non uniquement de point dans le voisinage courant) pour l'autre agent. Par conséquent, nous passons de (x, U_i, U_j) à (y, U_i, V_j), où V_j peut être différent de U_j : tandis que l'ouvert U_j représente l'évidence courante (i.e., sa portée épistémique) de j en x, l'ouvert V_j représente l'évidence de j en y. Ainsi, le changement de voisinage de U_j à V_j ne signifie pas un changement d'ensemble d'évidence pour l'agent j *dans l'état courant*. Tandis que le tuple (x, U_i, U_j) représente l'état courant et les points de vue des deux agents, la composante (y, V_j) du second tuple (y, U_i, V_j)ne représente que l'état épistémique de l'agent j du point de vue de l'agent i dans l'état y, qui est un état possiblement différent de l'état actuel x.

Afin de définir la portée épistémique de chaque agent par rapport à l'état en question, nous employons une technique inspirée de la sémantique des voisinages standard (voir, e.g., Chellas, 1980). Nous utilisons un ensemble de *fonctions de voisinage*, déterminant la portée épistémique relative à la fois à l'état donné et l'agent correspondant. Ces fonctions doivent être partielles afin d'obtenir une sémantique bien fondée pour les modalités dynamiques du système, à savoir les modalités d'annonce publique et d'annonce arbitraire.

De plus, utiliser des espaces topologiques enrichis par un ensemble de fonctions (partielles) de voisinage comme modèles nous permet de travailler avec différentes notions de connaissance. Dans le cadre standard (mono-agent) des espaces de sous-ensembles (comme dans les chapitres 6 et 7), puisque la modalité de connaissance quantifie sur les éléments d'un voisinage fixé, la connaissance de type $\mathsf{S5}$ est inhérente à la façon dont la sémantique est définie. En revanche, avec l'approche développée dans ce chapitre, la portée épistémique d'un agent change selon les fonctions de voisinage quand l'état d'évaluation change. Par conséquence, les validités de la connaissance sont déterminées par les contraintes imposées sur les fonctions de voisinage. Dans ce but, nous travaillons dans ce chapitre à la fois avec la connaissance de type $\mathsf{S5}$ et celle de type $\mathsf{S4}$: alors que la première est la notion de connaissance standard dans le cadre des espaces de sous-ensembles, la seconde révèle un nouvel aspect de notre approche, à savoir, la possibilité de capturer différentes notions de connaissance. Cela contraste avec et enrichit les approches existantes pour les sémantiques de connaissance basées sur les espaces de sous-ensembles, car à notre connaissance, les autres approches ne peuvent fonctionner qu'avec la connaissance $\mathsf{S5}$.

Sans surprise, travailler avec la connaissance de type $\mathsf{S5}$ dans notre cadre multi-agents requière un partitionnement du (sous-)domaine d'un espace topologique. Cela pourrait sembler être une restriction puisque cela exclue de travailler avec des espaces plus familiers tels que la topologie naturelle des intervalles ouverts sur la ligne réelle ou l'espace Euclidien. Cependant, lorsqu'on considère de multiples agents de type $\mathsf{S5}$, nous croyons qu'il est difficile voire impossible d'éviter une telle restriction. Nous axiomatisons la logique multi-agents de connaissance et connaissabilité $\mathsf{EL}^m_{\mathsf{int}}$, ses extensions avec annonces publiques $\mathsf{PAL}^m_{\mathsf{int}}$, ainsi qu'avec annonces publiques arbitraires $\mathsf{APAL}^m_{\mathsf{int}}$ (voir la table A.8 ci-dessous). La modalité d'annonce arbitraire $\boxast\varphi$, qui capture la "stabilité de la véracité de φ après toute annonce" se rapproche de l'intuition derrière la modalité d'effort $\Box\varphi$ signifiant "stabilité de la véracité de φ après toute acquisition d'évidence". Nous prouvons que ces deux modalités sont équivalentes dans le cas mono-agent (Théorème 7.2.6). Cependant, l'interprétation appropriée de l'effort dans le cas multi-agents et ses liens avec la modalité d'annonce arbitraire demeurent dures à saisir. Cela sort du cadre de cette dissertation, et nous laissons donc pour de futurs travaux la tâche d'étudier de manière systématique les interprétations possibles de la modalité d'effort et son comportement dans un cadre multi-agents.

	(I) Axiomes et règles du système $\mathsf{EL}^m_{\mathsf{int}}$:
(CPL)	toutes les tautologies propositionnelles classiques et le (MP)
($S5_K$)	tous les axiomes et règles $\mathsf{S5}$ pour la modalité K_i
($S4_{\mathsf{int}}$)	tous les axiomes et règles $\mathsf{S4}$ pour la modalité int
(K-int)	*Connaissance implique connaissabilité*: $K_i\varphi \to \mathsf{int}(\varphi)$
	(II) Axiomes et règles supplémentaires pour $\mathsf{PAL}^m_{\mathsf{int}}$:
($K_!$)	$[\varphi](\psi \to \theta) \to ([\varphi]\psi \to [\varphi]\theta)$
($Nec_!$)	de θ, on infère $[\varphi]\theta$
([!]RE)	de $\varphi \leftrightarrow \psi$, on infère $[\varphi]\theta \leftrightarrow [\psi]\theta$
	Axiomes de réduction:
(R_p)	$[\varphi]p \leftrightarrow (\mathsf{int}(\varphi) \to p)$
($R_\neg$)	$[\varphi]\neg\psi \leftrightarrow (\mathsf{int}(\varphi) \to \neg[\varphi]\psi)$
(R_K)	$[\varphi]K_i\psi \leftrightarrow (\mathsf{int}(\varphi) \to K_i[\varphi]\psi)$
($R_{[!]}$)	$[\varphi][\psi]\chi \leftrightarrow [\langle\varphi\rangle\psi]\chi$
	(III) Axiomes et règles de $\boxast$ pour $\mathsf{APAL}^m_{\mathsf{int}}$:
($\boxast$-elim)	$\boxast\varphi \to [\chi]\varphi$ ($\chi \in \mathcal{L}^!_{K\mathsf{int}}$ une formule arbitraire)
($\boxast^\omega$-intro)	de $[s][\psi]\chi$ pour tout $\psi \in \mathcal{L}^!_{K\mathsf{int}}$, on infère $[s]\boxast\chi$

Table A.8: Axiomatisations pour les logiques multi-agents $\mathsf{EL}^m_{\mathsf{int}}$, $\mathsf{PAL}^m_{\mathsf{int}}$ et $\mathsf{APAL}^m_{\mathsf{int}}$.

Plan. La section 8.1 définit la syntaxe, les structures et la sémantique de notre logique multi-agents d'annonces publiques arbitraires, $\mathsf{APAL}^m_{\mathsf{int}}$, interprétée sur des espaces topologiques équipés d'un ensemble de fonctions de voisinages. Sans annonces arbitraires nous obtenons la logique $\mathsf{PAL}^m_{\mathsf{int}}$, et sans annonces arbitraires ni publiques, la logique $\mathsf{EL}^m_{\mathsf{int}}$. Dans cette section nous donnons aussi deux exemples détaillés illustrant les sémantiques proposées. Dans la section 8.2 nous fournissons des axiomatisations pour nos logiques: $\mathsf{PAL}^m_{\mathsf{int}}$ étend $\mathsf{EL}^m_{\mathsf{int}}$ et $\mathsf{APAL}^m_{\mathsf{int}}$ étend $\mathsf{PAL}^m_{\mathsf{int}}$. De plus, nous prouvons leur correction et comparons les pouvoirs expressifs des langages multi-agent associés, $\mathcal{L}^!_{K\mathsf{int}\boxast}$, $\mathcal{L}^!_{K\mathsf{int}}$ et $\mathcal{L}_{K\mathsf{int}}$, par rapport aux topo-modèles multi-agents. En section 8.3 nous démontrons la complétude pour ces logiques. La preuve de complétude pour le fragment épistémique, $\mathsf{EL}^m_{\mathsf{int}}$, est assez différente de la preuve de complétude pour la logique complète $\mathsf{APAL}^m_{\mathsf{int}}$. La section 8.4 adapte les logiques au cas de la connaissance $\mathsf{S4}$. Dans la section 8.5 nous comparons notre travail avec la littérature, et la section 8.6 contient un bref résumé du chapitre et une discussion sur une interprétation possible de la modalité d'effort dans le système multi-agents actuel.

Origine de la matière de cette dissertation

- Le chapitre 4 est basé sur :

  Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. (2015a). The topological theory of belief. *En cours d'évaluation.* Disponible en ligne à *http://www.illc.uva.nl/Research/Publications/Reports/PP-2015-18.text.pdf.*

  La partie I du chapitre 4 (sections 4.1-4.2.1) présente un aperçu de (Özgün, 2013; Baltag et al., 2013), tandis que le reste du chapitre est basé sur des idées absentes de (Özgün, 2013; Baltag et al., 2013) mais présentées dans (Baltag et al., 2015a).

- Le chapitre 5 se base sur deux articles, dont le second est une version étendue du premier :

  Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. (2016a). Justified belief and the topology of evidence. In *Proceedings of 23rd Workshop on Logic, Language, Information and Computation* (*WoLLIC 2016*), pp. 83-103.

  Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. (2016b). Justified belief and the topology of evidence–Version étendue. Disponible en ligne à *http://www.illc.uva.nl/Research/Publications/Reports/PP-2016-21.text.pdf.*

- Le chapitre 7 est basé sur :

  van Ditmarsch, H., Knight, S., and Özgün, A. (2014). Arbitrary announcements on topological subset spaces. In *Proceedings of the 12th European Conference on Multi-Agent Systems* (*EUMAS 2014*), pp. 252-266.

  Baltag, A., Özgün, A., and Vargas-Sandoval, A. L. (2017). Topo-Logic as dynamic epistemic logic. In *Proceedings of the 6th International Workshop on Logic, Rationality and Interaction* (*LORI 2017*). A paraître.

- Le chapitre 8 est basé sur :

  van Ditmarsch, H., Knight, S., and Özgün, A. (2015b). Announcements as effort on topological spaces. In *Proceedings of the 15th Conference on Theoretical Aspects of Rationality and Knowledge* (*TARK 2015*), pp. 283-297.

  van Ditmarsch, H., Knight, S., and Özgün, A. (2015c). Announcements as effort on topological spaces–Version étendue. Accept pour publication dans *Synthese*.

De plus, bien que les résultats principaux des articles suivants ne soient pas inclus dans cette dissertation, la discussion concernant leur contenu conceptuel contribue dans une large mesure au présent travail.

van Ditmarsch, H., Knight, S., and Özgün, A. (2017). Private announcements on topological spaces. *Studia Logica.* A paraître.

Bjorndahl, A., and Özgün, A. (2017). Logic and Topology for Knowledge, Knowability, and Belief. In *Proceedings of the 16th Conference on Theoretical Aspects of Rationality and Knowledge* (*TARK 2017*), pp. 88-101.

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