Filtration Revisited: Lattices of Stable Non-Classical Logics

Julia Ilin

Filtration Revisited: Lattices of Stable Non-Classical Logics

ILLC Dissertation Series DS-2018-12

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

For further information about ILLC-publications, please contact

Institute for Logic, Language and Computation
Universiteit van Amsterdam
Science Park 107
1098 XG Amsterdam
phone: +31-20-525 6051
e-mail: illc@uva.nl
homepage: http://www.illc.uva.nl/

Cover design by Zsuzsanna Ilijin, http://www.helloyellowstudio.com.

Printed and bound by Ipskamp, https://www.ipskampprinting.nl.

ISBN: 978-94-028-1055-4

Filtration Revisited: Lattices of Stable Non-Classical Logics

Academisch Proefschrift

ter verkrijging van de graad van doctor
aan de Universiteit van Amsterdam
op gezag van de Rector Magnificus
prof. dr. ir. K.I.J. Maex
ten overstaan van een door het College voor Promoties ingestelde commissie, in het openbaar te verdedigen in de Agnietenkapel
op vrijdag 15 juni 2018, te 12.00 uur

door

Julia Ilin

geboren te Bremen, Duitsland

Promotiecommisie

Promotor:	Prof. dr. Y. Venema	Universiteit van Amsterdam
Co-promotor:	Dr. N. Bezhanishvili	Universiteit van Amsterdam
Overige leden:	Prof. dr. J.F.A.K. van Benthem	Universiteit van Amsterdam
	Prof. dr. S. Ghilardi	Università degli Studi di Milano
	Dr. S.J. van Gool	Universiteit van Amsterdam
	Dr. R. Iemhoff	Universiteit Utrecht
	Prof. dr. S.J.L. Smets	Universiteit van Amsterdam
	Prof. dr. F. Wolter	University of Liverpool

Faculteit der Natuurwetenschappen, Wiskunde en Informatica

Gretusnak és Eta néninek

Contents

Acknowledgments

This thesis is a combined effort of many. It is hard for me to put in words how lucky and deeply grateful I feel for the people I met over the past four years. Some are teachers or colleagues, some are students or friends, and some are all of the above. I will not be able to mention the names of all the people I owe thanks to, but in case you are reading this, your name is likely to be on this list as well.

First and foremost, I would like to thank my supervisors Nick Bezhanishvili and Yde Venema. I am not a fan of exaggerations, but I can say that Nick was the best supervisor I could hope for. In fact, some of my PhD fellows envy me for having Nick as my supervisor, and I think they are right. Nick's enthusiasm and dedication for logic and research is contagious and truly motivating. Nick, thank you for pushing me right into the middle of academic life early on in my PhD project, for all your encouragements whenever they were needed. Most of all, thanks for teaching me all about logic and for not giving up on believing in me.

Yde actively took part in this project only at a later stage but his presence and actions were very much appreciated. Yde, thank you for always being so thorough and reasonable which helped me to stay sane whenever I felt very stressed. And thanks for making me feel excited about even the simple observations in the world of logic.

This thesis would have not existed without my unofficial "third supervisor", Guram Bezhanishvili. More than half of the chapters in this thesis are filled with Guram's ideas and suggestions and would have not been thinkable without his contributions. Guram's passion for logic is a true inspiration and I feel very lucky and honored that he trusted me to pursue this path. Guram, thank you for making logic, research, and my PhD project such an exciting and suspenseful journey. I also thank you for your frequent advice on academic life in general.

I would like to thank Johan van Benthem, Silvio Ghilardi, Sam van Gool, Rosalie Iemhoff, Sonja Smets, and Frank Wolter for being part of my committee, reading and evaluating this thesis.

I feel very lucky to have worked with my excellent co-authors, Alexandru Baltag, John Harding, Dick de Jongh, Frederik Möllerström Lauridsen, Aybüke Özgün, and Fan Yang who have all contributed directly and indirectly to the content of this thesis to a large extent.

Thanks, Alexandru, for your endless energy, the parties, and for teaching me about DEL. Thank you, John, for your humble and pleasantly calm optimism, for your brilliant ideas, and for taking the time to tell me about MacNeille completions. I very much enjoyed the discussion with you and Guram in Las Cruces. Thank you, Dick, for sharing your beautiful ideas with me, for your support and kindness and thank you, Fan, for letting me be part of the NNIL project that started with your Master's thesis some years ago. It was a true pleasure to work with the two of you. Thank you, Frederik, for discussing any topic that I suggested with me, for always coming so well prepared to our meetings while I was not and for being such a persistent researcher. I learned a lot from you. Thank you, Aybüke, for explaining me proofs by reduction axioms and answer all my DEL related questions. But especially I would like to thank you for being such a loyal and wonderful friend. I had always a great time with you especially in Georgia and Japan.

I would also like to thank the referees of my papers for many great suggestions that have ultimately also helped to improve this thesis substantially.

My PhD project greatly profited from many conferences and summer schools that I attended and the research visits I was allowed to make. One of my dearest memories is the conference Tolo 4 in Tbilisi, the first conference I attended during my PhD studies. The days during and after the conference were filled with engrossing whiteboard discussions and the "Esakia seminar" with David Gabelaia, Guram, Mamuka Jibladze, and Nick made me feel very welcome from the start; not to mention the unforgettable dinners with Georgian wine, dances, toasts, and *khachapuri*. The Georgian school of logic felt like logic heaven to me.

I would like to thank Ramón Jansana for hosting me in Barcelona. The discussions with Ramón and Tommaso Moraschini on abstract algebraic logic and canonical formulas were crucial for many of the formulations in Chapter 5 of this thesis. I would like to thank Nick Galatos for hosting me twice at the University of Denver. It was such pleasure to talk to you and learn from you, Nick, and even though none of the things we discussed actually made it into this thesis, I think about the weeks in Denver as one of the most instructive and productive times of my PhD studies. Thanks, Gavin and Wesley for the luxurious coffee and thanks to all the others in Denver for showing me around and letting me be part of their community. I thank Guram and John for hosting me at the NMSU and Kristina Brantley for the many discussions we had. My visits to the US were generously funded by the SYSMICS project and I would like to thank Luca Spada and Sara Lapenta for the excellent organization.

I want to thank Tom Sterkenburg for spontaneously agreeing to translate the summary of the thesis into Dutch and my cousin Zsuzsanna Ilijin for designing the cover. I would like to thank Paula Henk for her expert help with writing the introduction of this thesis. But mostly I thank you, Paula, for being a dear friend, for the many conversations and for being an inspiration and role model to me in life's many aspects. It is hard to imagine the last four years without you.

Back "home" at the ILLC I would like to thank Jenny Batson, Karine Gigengack, and Tanja Kassenaar for making all the formal matters of PhD life very easy. I would like to thank all my PhD colleagues for turning the ILLC into such a pleasant working environment. Thanks for picking me up for the lunches, for being collectively expelled from various coffee machines around the building, and for making all those jokes that I not always enjoyed.

I would like to thank Ronja Reese for those first steps in mathematics and logic that we took together. I doubt I would have desired to become a mathematician without having met you. *Ein dickes Dankeschön* to all my friends from Bremen and to my sister Anna who showed me life whenever mine started disappearing in the academic bubble. *Ich bin so froh, dass ich Euch hab, ihr Süßen!*

Thank you, Ásgeir, for being part of nearly every step in this journey, for your honesty, your stubborn serenity, for being funny and my biggest fan. I am so happy that you entered my life and that you bought cake for me that day. *Takk fyrir, elskan mín!*

Finally, I would like to thank my parents Zsuzsa Ilin-Pap and Zoltán Ilin who supported my studies unconditionally and never questioned what I would want to be(come). *Koszönöm hogy mindig és ennyire bíztatok bennem.*

Somewhere in a train — Julia Ilin
May, 2018.

Chapter 1

Introduction

This thesis studies classes of superintuitionistic and modal logics, and also touches on the areas of substructural logic and intuitionistic modal logic.

Non-classical logics

Non-classical logics are propositional logics that generalize the classical propositional calculus (CPC). In the thesis we will mostly be concerned with superintuitionistic logics and normal modal logics.

Intuitionistic propositional logic (IPC) does not validate the law of excluded middle. Thus, it is a non-classical logic weaker than CPC. *Superintuitionistic logics* (*si logics* for short)—also called *intermediate logics*—are the extensions[1] of IPC (see [40, 47]).

Normal modal logics are non-classical logics that expand CPC by an operator $\Box$ (or its dual operator $\Diamond$) (see [40, 36]). The weakest normal modal logic is called K. We will often consider *transitive modal logics*, i.e. normal extensions of the modal logic K4.

Other non-classical logics that we encounter in this thesis include *substructural logics*, i.e., logics that are even weaker than IPC, and *intuitionistic modal logics*. The latter can roughly be described as modal logics whose propositional fragment is intuitionistic as opposed to classical. In the remainder of the introduction, whenever we speak of a *logic*, we mean one of the non-classical logics just mentioned.

Algebraic vs. relational semantics. We will mostly study logics via their *semantics*. We will use both *algebraic* and *relational semantics*. A desirable feature of algebraic semantics is that non-classical logics are, in general, complete

[1] To be precise, the term *si logics* encompasses all extensions of IPC, and the term *intermediate logics* encompasses all but the inconsistent extension of IPC.

with respect to their algebraic semantics. For instance, every si logic is complete with respect to its corresponding class of *Heyting algebras* and every normal modal logic is complete with respect to its corresponding class of *modal algebras*. Another advantage of algebraic semantics is that many general results about logics can be inferred by using available tools and results from *universal algebra* [39].

Relational semantics has the advantage that it provides geometric intuition and is thus more "tangible". However, not all logics are complete with respect to their relational semantics, thus, relational semantics is not always suitable for proving general results about logics.

Algebraic and relational semantics are connected via *duality* [120, 108, 48]. A Kripke frame $\mathfrak{F}$ can be turned into an algebra $\mathfrak{F}^+$ that validates the same formulas as $\mathfrak{F}$. Conversely, an algebra $\mathfrak{A}$ can be turned into a Kripke frame $\mathfrak{A}_+$ so that $\mathfrak{A}$ is isomorphic to a subalgebra of $((\mathfrak{A})_+)^+$. In general, $\mathfrak{A}$ is not isomorphic to $((\mathfrak{A})_+)^+$, thus this correspondence does not constitute a perfect balance. However, the desired isomorphism can be achieved by extending Kripke frames with an additional topological structure. This leads to *Esakia spaces* in the intuitionistic setting [51] and *modal spaces* in the modal setting (see [40, 36]).

Duality allows us to switch smoothly between the two types of semantics and thus make use of the advantages of both types.

"Good properties" of logics. Not all logics behave equally well with respect to their semantics. The "good properties" of logics discussed most frequently in this thesis are:

- *Kripke completeness*: A logic is Kripke complete if it is complete with respect to its Kripke frames.
- *The finite model property* (*fmp*): A logic has the fmp if it is complete with respect to a class of finite frames.

Other desirable properties of logics include *finite axiomatizability* (a logic is finitely axiomatizable if it has a finite set of axioms) and *elementarity* (a logic is elementary if it is complete with respect to a first-order definable class of Kripke frames). Some of these properties are interrelated. For instance, both the fmp and elementarity implies Kripke completeness. Likewise, the fmp together with finite axiomatizability implies *decidability* (a logic is decidable if there is an algorithm determining whether or not a formula is a theorem of it). However, not all normal modal logics and not all si logics enjoy the aforementioned "good" properties.

Classes of non-classical logics. One direction in the study of non-classical logics is to identify large classes of modal and superintuitionistic logics that have some of the good properties listed in the previous paragraph. Such classes of logics can be specified in several different ways, for example:

- Via their *position in the lattice* of normal modal logics or superintuitionistic logics. For example by considering the extensions of some specific logic.
- Via *semantic properties*, such as subframe and cofinal subframe logics. (We discuss these classes in the next section.)
- Via *syntactic properties*, for instance as the logics being axiomatized by a specific class of formulas.

Standard results concerning "good properties" of large classes of logics are: Every extension of S4.3 has the fmp, is finitely axiomatizable, and is therefore decidable (results of Bull [38] and Fine [56]); every logic above K4 of finite depth has the fmp (proved by Segerberg [117]); every normal modal logic axiomatized by *Sahlqvist formulas* is elementary [116].

Subframe logics and stable logics

Subframe logics

Prime examples of classes of logics with "good" properties are transitive normal modal and superintuitionistic (cofinal) subframe logics. Transitive subframe logics were introduced by Fine [59, 58]. We recall their semantic definition. A *subframe* of a Kripke frame is what is called a *substructure* in model theory, i.e. a subset of the frame equipped with the relation restricted to that subset. A *transitive subframe logic* is a normal modal logic extending K4 whose class of frames is closed under subframes.

Fine showed that all transitive subframe logics have the fmp. The proof-method he used to show this is known under the name of *selective filtration*. Selective filtration extracts a finite countermodel from a (possibly infinite) countermodel by selecting "relevant points". Roughly speaking, since subframe logics are closed under subframes, the method of selective filtration is applicable to them. Non-transitive modal subframe logics were studied by Wolter [126, 127]. In the non-transitive case, however, subframe logics lose many of their good properties, e.g., there are non-transitive subframe logics that are Kripke incomplete.

In the intuitionistic setting, subframe logics[2] were introduced by Zakharyaschev. He also extended these results to *cofinal subframe logics* that encompass all subframe logics [133, 134, 135, 137] (see [40] for an overview). As in the transitive modal setting, all (cofinal) subframe si logics have the fmp. In the

[2]The definition of intuitionistic subframes is a little bit more involved, we discuss this in Chapter 3.

intuitionistic setting, subframe and cofinal subframe logics admit simple syntactic characterizations. In fact, subframe si logics are precisely the si logics axiomatized by $\{\wedge, \to\}$-formulas, and cofinal subframe logics are those axiomatized by $\{\wedge, \to, \bot\}$-formulas, i.e. by formulas only allowing connectives from the set $\{\wedge, \to\}$ and $\{\wedge, \to, \bot\}$, respectively.

Stable logics and filtration in non-classical logic

Another class of logics with good properties is that of *stable logics*. Stable modal logics were identified in [18] as the "filtration-analogue" of subframe logics. We explain this in more detail. A *stable map* between modal spaces is a continuous and relation-preserving map[3]. A class of modal spaces is called *stable* if it is closed under images of stable maps. A normal modal logic is called *stable* if it is complete with respect to a stable class of modal spaces.

Just like transitive subframe logics, stable modal logics have the fmp. In fact, the fmp of stable modal logics can be proved via the *filtration method*. Filtration is a standard tool for proving the fmp in modal logic. Even though the first instances of filtration were algebraic in nature [98, 100, 92, 93], in textbooks it is usually presented from a frame-theoretic perspective, going back to [117]. We recall some details of this technique. Suppose that $\mathfrak{M}$ is a model that refutes a formula φ. A *filtration* of $\mathfrak{M}$ through φ produces a finite model $\mathfrak{M}'$ that refutes φ and is a stable image of $\mathfrak{M}$. The domain of $\mathfrak{M}'$ is obtained by identifying those worlds of $\mathfrak{M}$ that agree on the subformulas of φ. This ensures that $\mathfrak{M}'$ is finite. Moreover, the right choice of the relations on $\mathfrak{M}'$ ensures that $\mathfrak{M}'$ refutes φ. The latter is the content of the filtration theorem.

To prove the fmp of a specific logic via filtration, one has to show that the underlying frame of the model $\mathfrak{M}'$ is still a frame of the logic in question. Since $\mathfrak{M}'$ is a stable image of $\mathfrak{M}$, this can always be ensured for stable logics. Thus, in the same way as transitive subframe logics admit selective filtrations (as discussed above), stable modal logics *admit filtrations*. In fact, stability is a simple and natural way to ensure that a normal modal logic admits filtrations. Thus—as a slogan—we can say that the *defining feature* of stable modal logics is that they behave well with respect to filtrations.

We continue with a discussion of instances of stability in the intuitionistic setting. Stable si logics can be seen as those si logics that admit intuitionistic filtrations. But they also have a transparent algebraic description. In fact, the algebraic view elucidates the analogies between subframe and stable si logics where—figuratively speaking—*locally finite reducts* of Heyting algebras take the role of filtrations.

[3]In model theory such maps are simply called homomorphisms but we reserve the term "homomorphism" for operation preserving maps between modal algebras. Ghilardi [64] calls such maps continuous, but we reserve the term "continuous" for structure preserving maps between topological spaces. Thus, we follow [21] in calling such maps "stable."

From an algebraic perspective, subframe si logics are the logics whose class of Heyting algebras is closed under $\{\wedge, \to\}$-subalgebras (see [40]). That such logics have the fmp was shown by McKay by building on Diego's result that meet-implicative semilattices are locally finite [97, 49].

Stable si logics were introduced in [17]. Stable si logics can be regarded as the $\{\wedge, \vee, \bot, \top\}$-analogues of subframe logics. Stable si logics are characterized by a class of Heyting algebras closed under $\{\wedge, \vee, \bot, \top\}$-subalgebras. In other words, a si logic is stable if it is complete with respect to a class of Heyting algebras $\mathcal{K}$, where $A \in \mathcal{K}$ implies that all Heyting algebras that are bounded sublattices of A also belong to $\mathcal{K}$. Using the fact that bounded lattices are locally finite, the fmp of such logics can be shown analogously to McKay's result.

Formulas based on frames and algebras

The aforementioned *semantic descriptions* already demonstrate analogies between subframe and stable logics. Further analogies arise through *syntactic characterizations* of these classes of logics. The logics of each class can be axiomatized in terms of so-called frame based or algebra based formulas. Such formulas describe—similarly to the (positive) diagrams in model theory—the *structure of finite objects*. The Jankov-de Jongh formulas [80, 81, 85] and Fine's frame formulas [57] were the first specimens of this type. Jankov-de Jongh formulas and frame formulas axiomatize all splitting si logics and transitive splitting logics, respectively.

Subframe and *cofinal subframe formulas* axiomatize precisely the subframe and cofinal subframe logics in the intuitionistic and transitive modal settings, respectively (see [40]). Similarly, *stable formulas* axiomatize stable si logics and $\mathsf{K4}$-stable logics, and non-transitive stable modal logics can be axiomatized by *stable rules* [17, 18]. Roughly, in each case, a logic is axiomatized by the formulas (rules) corresponding to its (minimal) finite countermodels (see [29, Chapter 3]).

A useful common feature of these formulas is that their validity can be translated into a tangible semantic condition. Accordingly, we can prove properties of these formulas and of the logics they axiomatize by using semantic intuitions rather than syntactic manipulations.

Canonical formulas and rules. A special category of frame-based (or algebra-based) formulas are *canonical formulas*. Canonical formulas were introduced by Zakharyaschev. They generalize subframe and cofinal subframe formulas by adding to them an additional parameter. The central property of canonical formulas is that they axiomatize *all* intermediate and transitive (normal) modal logics in a uniform way (see [40, Section 9] and [136] for an overview).

The uniform axiomatization via canonical formulas has the advantage that many properties of logics—such as the fmp or decidability—can be investigated by considering canonical formulas only. These formulas still have the "semantic feel" of subframe and cofinal subframe formulas. Due to this feature, canonical formulas can serve as important proof-tools. For instance, Zakharyschev provided an alternative proof of the celebrated Blok-Esakia theorem via canonical formulas (see [40]). Jeřábek discovered alternative proof methods for inferring the decidability of *admissible rules* of si logics and some transitive modal logics by extending canonical formulas to *canonical rules* [83].

Algebraic approaches to canonical formulas for modal and si logics have been developed more recently [13, 15, 18, 17]. The key ingredient of this work is the fact that from an algebraic perspective, the mechanism of canonical formulas is closely tied to locally finite reducts of the corresponding algebras. This idea can be traced back to [121]. Further generalizations of canonical formulas to substructural logics can be found in [32]. An alternative algebraic approach to canonical formulas via partial algebras was given in [44]. In [18, 17] it was shown that stable canonical formulas and rules can be defined as "stable analogues" of Zakharyaschev's formulas and Jeřábek's rules. This revealed yet another analogy between the subframe and stable approaches.

What this thesis is about

The overarching theme in this thesis is the notion of stability in the context of non-classical logics. In particular, we further investigate the class of stable modal logics and its generalization to M-stable logics. Moreover, we study the relations between stable logics in the modal and intuitionistic settings. Finally, we explore stability in the context of dynamic (epistemic) logic. Known properties of subframe logics are often the guiding examples in our study. In fact, a central goal of the thesis is to explore relations between stable and subframe logics by identifying common features and differences.

Below we explain the main topics and tools of the thesis in more detail.

Cofinal stable logics and $\mathcal{H}$-stable si logics

As explained above, stable si logics were introduced as the $\{\wedge, \vee, \bot, \top\}$-analogue of subframe si logics. This raises the question about the $\{\wedge, \vee, \bot, \top\}$-analogue of *cofinal* subframe logics. As an answer, we introduce the class of *cofinal stable si logics.*

Recall that the $\{\wedge, \to\}$-reduct and the $\{\wedge, \to, \bot\}$-reduct of Heyting algebras give rise to subframe and cofinal subframe si logics, respectively. Thus, from the algebraic perspective, the step from subframe logics to cofinal subframe logics is made by adding $\bot$ to the $\{\wedge, \to\}$-reduct. In the presence of implication, $\bot$ also adds negations. Therefore, we define cofinal stable logics as those si logics that the $\{\wedge, \vee, \neg, \top\}$-reduct of Heyting algebras gives rise to. The $\{\wedge, \vee, \neg, \top\}$-reducts of Heyting algebras are pseudo-complemented bounded distributive lattices. Due to the locally finiteness of the latter, all cofinal stable si logics have the fmp.

We explore cofinal stable logics by providing several equivalent characterizations of them and collect many examples from the literature that are cofinal stable. Moreover, we provide examples of si logics that distinguish the classes of cofinal stable logics from those of subframe, cofinal subframe, and stable si logics. In particular, we show that there is a continuum of cofinal stable logics that are not stable.

In addition, we take our investigations as an opportunity to give a general exposition of $\mathcal{H}$-stable si logics, where $\mathcal{H}$ is a locally finite reduct of Heyting algebras such as the $\{\wedge, \to\}$-, the $\{\wedge, \to, \bot\}$- or the $\{\wedge, \vee, \bot, \top\}$-reduct of Heyting algebras. Thus, in particular $\mathcal{H}$-stable logics include intuitionistic subframe and stable si logics. We observe that characterization results for these logics can be obtained in a uniform way. On the other hand, we will also discover subtle differences that distinguish the behavior of the various reducts.

$\mathcal{H}$-stable canonical rules and formulas

As we mentioned earlier, locally finite reducts play a crucial role in the mechanism of canonical formulas. Similarly to our uniform discussion on $\mathcal{H}$-stable si logics, we provide a uniform treatment of canonical formulas via locally finite reducts of algebras. For this purpose, we will first introduce a very general notion of $\mathcal{H}$-stable canonical rules, and then explain that in some special cases these rules can be turned into formulas. We show that our presentation covers several results from the recent literature.

Embedding si logics into intuitionistic modal logics

We already pointed out many analogies between stable and subframe si logics. By moving to the realm of intuitionistic modal logic we discover further analogies between the two classes. Specifically, we show that subframe si logics are related to the *propositional lax logic* (PLL) [54], while stable si logics are related to IS4, the intuitionistic version of the (classical) modal logic S4 [105].

We explain how to obtain these connections in a bit more detail: As was shown by Goldblatt [67], an algebraic semantics for PLL is given by Heyting algebras with nuclei. By [24], nuclei on a Heyting algebra correspond precisely to the subframes of its dual Esakia space. This allows us to translate formulas of IPC into the language of PLL via a version of the Gödel-Gentzen translation. The translation yields two natural embeddings of si logics into the lattice of extensions of PLL. We observe that subframe si logics play a special role for these embeddings. The two embeddings, in turn, provide a new characterization of subframe si logics.

In a very similar way, we relate stable logics to extensions of IS4. Algebraic semantics for IS4 is provided by Heyting algebras with interior operators. Therefore, the Gödel-McKinsey-Tarski-translation allows us to translate formulas of IPC into formulas of IS4. Analogously to the subframe case, the translation allows us to define two natural embeddings from the lattice of si logics into extension of IS4, for which stable si logics will play a special role.

Moreover, we investigate how subframe and stable si logics lie in the lattice of si logics by relating them to their "closest neighbors". We call these neighbors the downward and upward *subframizations* and *stabilization* of a si logic, respectively. Wolter [127, 126] introduced and characterized the downward and upward subframizations of modal logics in terms of relativizations, and defined the more general notion of *describable operations* on a lattice of logics. We show how *subframizations* and *stabilization* of si logics can be seen as instances of describable operations. In addition, we characterize the downwards subframization and stabilization via the aforementioned embeddings into extensions of PLL and IS4, respectively.

M-stable modal logics

In the modal case, we extend the study of stable logics started in [18]. We provide several additional characterizations of these, collect examples and non-examples of "standard" stable modal logics from the literature, and show that there is a continuum of stable modal logics. We also discuss differences and similarities between stable and subframe modal logics.

More importantly, we relativize the notion of stability to that of M-stability. The notion of K4-stability was already studied in [18]. The motivation behind the notion of K4-stability is that "standard" modal systems such as K4 and S4 do not admit *all* filtrations, but only some specific ones, namely, transitive filtrations. Hence, K4 and S4 are not stable logics. However, both K4 and S4 are K4-stable.

The notion of M-stability further generalizes K4-stability. It replaces K4 with a modal logic M that admits filtrations of a specific type. Accordingly, M-stable logics encompass an even larger range of logics whose fmp can be proved via the filtration method.

We will place a particular emphasis on *transitive* M*-stable logics*, i.e. M-stable logics where M is an extension of K4. Such logics admit particularly simple characterizations. Moreover, we explore how S4-stable logics and stable si logics behave with respect to *modal companions* and *intuitionistic fragments* of modal logics.

NNIL-formulas revisited

Another chapter of this thesis is devoted to the study of NNIL-formulas. NNIL-formulas are a class of formulas specified by a syntactic property (no nesting of implication to the left) and were introduced in [124, 125]. Since truth of NNIL-formulas is reflected by stable maps on models [125, 34], stability plays a role also in the study of NNIL-formulas. Using the aforementioned observation, we will have a fresh look at NNIL-formulas. Building on [130], among our central results is the construction of a universal model for NNIL-formulas via stable maps. Finally, we provide alternative proofs of known properties of subframe logics—the logics that NNIL-formulas axiomatize—namely the fmp and canonicity.

Stable maps as model transformations in dynamic logic

Finally, we look at instances of stability in the context of *dynamic epistemic logic* [50, 8, 6]. A central notion in dynamic logic is that of *updates*. Recall that the public announcement operator $[!\varphi]$ for some formula φ induces a model-transformation that moves from a model to a submodel, namely to the model consisting of those worlds making φ true [106, 63, 107, 8]. Hence, updating via the public announcement operator moves us from a model to a submodel. Subframe logics make their way into the context of dynamic epistemic logic as those logics that allow for updates via the public announcement operator.

In this thesis, we consider the 'stable' analogues of updates by investigating a modality whose corresponding model transformation corresponds to quotienting. Epistemically, quotienting can be thought of as an abstraction in the sense of abstracting away or disregarding irrelevant facts. We investigate technical properties of logics equipped with abstraction modalities. We also show that in some cases, such logics can be regarded as *logics of filtrations*. Stable modal logics play a similar role for the abstraction modality as subframe logics do for the public announcement operator. Roughly speaking, stable logics are the normal modal logics that allow updates via abstraction modalities.

Summary

To conclude the introduction we give a short summary of the goals and achievements of this thesis.

- We introduce *cofinal stable si logics* as the stable analogues of cofinal subframe si logics and study properties of these logics.

- We expand the study of *stable modal logics*. We investigate the behavior of stability with respect to modal companions and intuitionistic fragments. We explain similarities and differences between stable and subframe modal logics.

- We strengthen the *parallels between stable and subframe si logics* by connecting them to modal operators on Heyting algebras, namely the *lax modality* and the *interior operator*, respectively.

- We provide a unified treatment of *canonical formulas* via locally finite reducts for various non-classical logics. Moreover, we provide a unified look on $\mathcal{H}$-stable si logics, where $\mathcal{H}$ is a locally finite reduct of Heyting algebras encompassing subframe, cofinal subframe, and stable si logics.
- We take a fresh look at the class of NNIL-*formulas via stable maps*. In particular, we give full descriptions of the n-universal models for NNIL-formulas. We also provide alternative proofs of the known results that logics axiomatized by NNIL-formulas have the fmp and are canonical.
- We treat images of models under *stable maps as model-transformation operations*. These operations give rise to dynamic logics with abstraction modalities. We prove completeness results for these logics. We explain that in some special cases, these logics can be regarded as *logics of filtration*.

Content of the chapters and sources of the material

Chapter 2 discusses the technical preliminaries used in this thesis.

Chapter 3 summarizes known results on subframe and cofinal subframe si logics as well as stable si logics. Moreover, it introduces the new class of cofinal stable si logics. The new results in this chapter are joint work with G. and N. Bezhanishvili [21].

Chapter 4 discusses stable modal logics. The chapter is based on joint work with G. and N. Bezhanishvili [19].

Chapter 5 provides a summary of known results on canonical formulas.

Chapter 6 relates subframe and stable si logics to intuitionistic modal logics. The chapter is based on joint work with G. and N. Bezhanishvili [20].

Chapter 7 discusses NNIL-formulas. The chapter is based on joint work with D. de Jongh and F. Yang [79, 78].

Chapter 8 discusses stability in the dynamic epistemic context. The chapter is based on joint work with A. Baltag, N. Bezhanishvili, and A. Özgün [4].

Chapter 2

Preliminaries

In this chapter we discuss the background theory used in this thesis. We begin by recalling general notions from universal algebra. We then introduce more specific classes of algebras such as lattices, Heyting algebras and modal algebras. Next, we recall Stone, Priestley, Esakia, and Jónsson-Tarski dualities. Finally, we recall intuitionistic and modal logics, and their algebraic and relational semantics. We also explain how these logics can be axiomatized via multi-conclusion consequence relations.

Our general reference for universal algebra is [39], and for modal and intuitionistic logics we usually refer to [40, 36].

2.1 Algebras

2.1.1 Universal Algebra

We recall some basic notions and results from universal algebra that we will use throughout this thesis without explicitly referring to them. Unless stated otherwise, the results and notions from this section can be found in [39].

An *algebraic similarity type* is a set $\mathcal{F}$ of function symbols, where each $f \in \mathcal{F}$ is assigned a natural number $\sigma(f)$ called the *arity* of f. For a set A and some natural number n, let A^n denote the n-ary product of A. An *$\mathcal{F}$-algebra* $\mathfrak{A} = (A, F)$ is a pair consisting of a set A, and a set F that contains for each $f \in \mathcal{F}$ a function $f^{\mathfrak{A}} : A^{\sigma(f)} \to A$. If $f \in \mathcal{F}$ has arity 0, then $f^{\mathfrak{A}}$ is simply an element of A. Such elements are called *constants*. We say that an $\mathcal{F}$-algebra is of *finite signature* iff $\mathcal{F}$ is finite. If $\mathcal{F}$ is clear from the context we will often refer to "$\mathcal{F}$-algebras" as "algebras".

Given a similarity type $\mathcal{F}$, and a set of variables Var, the set of *$\mathcal{F}$-terms* (or simply *terms* if $\mathcal{F}$ is clear from the context) are defined straightforwardly by induction. We write $t(\bar{x})$ to indicate that t is an $\mathcal{F}$-term with variables from $\bar{x}$. If

$\mathfrak{A} = (A, F)$ is an $\mathcal{F}$-algebra and $t(\bar{x})$ is an $\mathcal{F}$-term, then t gives rise to a function $t^{\mathfrak{A}} : A^{|\bar{x}|} \to A$ in the obvious way.

A *valuation* v on an algebra $\mathfrak{A} = (A, F)$ is a function $v : \mathsf{Var} \to A$, i.e. v assigns to each variable an element of A. A pair $(\mathfrak{A}, v)$ consisting of an $\mathcal{F}$-algebra and a valuation is called a *model*. Given a model $(\mathfrak{A}, v)$ and a term $t(\bar{x})$, we write $v(t)$ for the element $t^{\mathfrak{A}}(v(x)_{x\in\bar{x}})$ of A. An *equation* is an expression of the form $t \approx s$, where $t(\bar{x})$ and $s(\bar{x})$ are terms. A model $(\mathfrak{A}, v)$ *validates* an equation $t \approx s$ iff $v(t) = v(s)$. In that case we write $(\mathfrak{A}, v) \models t \approx s$. We say that an algebra $\mathfrak{A}$ *validates* the equation $t \approx s$ iff every model on $\mathfrak{A}$ validates $t \approx s$. In that case we write $\mathfrak{A} \models t \approx s$. If Γ is a set of equations, we write $\mathfrak{A} \models \Gamma$ iff $\mathfrak{A} \models s \approx t$ for all $s \approx t \in \Gamma$.

Clearly, an algebra $\mathfrak{A} = (A, F)$ can also be seen as a first-order (FO)-structure from model theory. In a similar way, if ψ is a sentence in the FO-language of $\mathcal{F}$, we write $\mathfrak{A} \models \psi$ iff $\mathfrak{A}$ validates ψ. We will mostly be interested in the case where ψ is a universal sentence.

Varieties and universal classes

Let an algebraic similarity type $\mathcal{F}$ be fixed and let $\mathfrak{A} = (A, F_A)$, $\mathfrak{B} = (B, F_B)$, and $\mathfrak{A}_i = (A_i, F_i)$ for $i \in I$ be $\mathcal{F}$-algebras. An $(\mathcal{F})$-*homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ is a function $h : A \to B$ such that $h(f^{\mathfrak{A}}(\bar{a})) = f^{\mathfrak{B}}(h(a)_{a\in\bar{a}})$ for all $f \in \mathcal{F}$. An injective homomorphism is called an *embedding*. If there is a surjective homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$, then $\mathfrak{B}$ is called a *homomorphic image* of $\mathfrak{A}$, and $\mathfrak{B}$ is called a *subalgebra* of $\mathfrak{A}$, if $B \subseteq A$ and the identity map from B into A is a homomorphism. We say that $\mathfrak{A}$ is *isomorphic* to $\mathfrak{B}$ iff there is an injective and surjective homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. The *product* of the family $\{\mathfrak{A}_i\}_{i\in I}$ is the algebra $\prod_{i\in I} \mathfrak{A}_i = (A, F)$ consisting of $A = \{g : I \to \bigcup_{i\in I} A_i \mid g(i) \in A_i \text{ for all } i \in I\}$ and for $f \in \mathcal{F}$, the operation $f^{\prod_{i\in I}\mathfrak{A}_i}$ is defined by

$$f^{\prod_{i\in I}\mathfrak{A}_i}\left((g_1, \dots, g_{\sigma(f)})\right)(i) = f^{\mathfrak{A}_i}\left((g_1(i), \dots, g_{\sigma(f)}(i))\right),$$

for every $g_1, \dots, g_{\sigma(f)} \in A$ and $i \in I$. It is easy to see that for all $i \in I$, the *projection* $\pi_i : A \to A_i$, defined by $\pi_i(g) = g(i)$ is a homomorphism.

Finally, we recall the notion of an ultraproduct. Let A and I be as before. If U is an ultrafilter on the index set I, then U induces an equivalence relation $\sim_U$ on A by $g \sim_U g'$ iff $\{i \in I \mid g(i) = g'(i)\} \in U$. For $g \in A$, let $[g]_U$ denote the equivalence class of g with respect to $\sim_U$ and let A_U denote the set of these equivalence classes. For $f \in \mathcal{F}$ the operation

$$f^{\mathfrak{A}_U}\left(([g_1]_U, \dots, [g_{\sigma(f)}]_U)\right)(i) = [f^{\mathfrak{A}_i}\left((g_1(i), \dots, g_{\sigma(f)}(i))\right)]_U$$

is well defined for $[g_1]_U, \dots, [g_{\sigma(f)}]_U \in A_U$ and $i \in I$. The resulting algebra $\mathfrak{A}_U = (A_U, F_U)$ is called the *ultraproduct* of the family $\{\mathfrak{A}_i\}_{i\in I}$ with respect to U.

The aforementioned notions give rise to the following operations on a class $\mathcal{K}$ of $\mathcal{F}$-algebras:

$\mathbf{H}(\mathcal{K}) := \{\mathfrak{B} \mid \text{is a homomorphic image of some } \mathfrak{A} \in \mathcal{K}\}.$

$\mathbf{S}(\mathcal{K}) := \{\mathfrak{A} \mid \mathfrak{B} \text{ is a subalgebra of some } \mathfrak{A} \in \mathcal{K}\}.$

$\mathbf{I}(\mathcal{K}) := \{\mathfrak{B} \mid \mathfrak{B} \text{ is isomorphic to some } \mathfrak{A} \in \mathcal{K}\}.$

$\mathbf{P}(\mathcal{K}) := \{\mathfrak{A} \mid \mathfrak{A} \text{ is a product of some algebras from } \mathcal{K}\}.$

$\mathbf{P}_U(\mathcal{K}) := \{\mathfrak{A} \mid \mathfrak{A} \text{ is an ultraproduct of some algebras from } \mathcal{K}\}.$

A class of algebras closed under the operations **H**, **S**, and **P** is called a *variety*, and a class closed under the operations **I**, **S**, and $\mathbf{P}_U$ is called a *universal class.* If $\mathcal{K}$ is a class of algebras, then there are a least variety $\mathcal{V}(\mathcal{K})$ and a least universal class $\mathcal{U}(\mathcal{K})$ that contain $\mathcal{K}$. These can be characterized in the following way (see [39, Theorem 9.5, page 67 and Theorem 2.20, page 245]).

2.1.1. Theorem (Tarski). *Let $\mathcal{K}$ be a class of algebras, then*

(1) $\mathcal{V}(\mathcal{K}) = \mathbf{HSP}(\mathcal{K})$,

(2) $\mathcal{U}(\mathcal{K}) = \mathbf{ISP}_U(\mathcal{K})$.

If Γ is a set of equations, let $\mathcal{K}_\Gamma = \{\mathfrak{A} \mid \mathfrak{A} \models \Gamma\}$, and similarly, if Ψ is a set of universal sentences, let $\mathcal{K}_\Psi = \{\mathfrak{A} \mid \mathfrak{A} \models \Psi\}$. A class $\mathcal{K}$ of algebras is called *equationally definable* iff there is a set Γ of equations such that $\mathcal{K} = \mathcal{K}_\Gamma$; and $\mathcal{K}$ is called *universally definable* iff there is a set Ψ of universal sentences such that $\mathcal{K} = \mathcal{K}_\Psi$. Varieties and universal classes can be characterized as being equationally or universally definable (see [39, Theorem 11.9, page 83 and Theorem 2.20, page 245]).

2.1.2. Theorem (Birkhoff). *Let $\mathcal{K}$ be a class of algebras, then*

(1) $\mathcal{K}$ is a variety iff $\mathcal{K}$ is equationally definable,

(2) $\mathcal{K}$ is a universal class iff $\mathcal{K}$ is universally definable.

The above theorems entail that validity of equations is preserved by the operations **H**, **S**, and **P**. Likewise, validity of universal sentences is preserved by the operations **I**, **S**, and $\mathbf{P}_U$. We will often use the following corollary.

2.1.3. Corollary. *Let $\mathcal{K}$ be a class of algebras.*

(1) If $\mathcal{V}(\mathcal{K}) \not\models s \approx t$ for some equation $s \approx t$, then there is $\mathfrak{A} \in \mathcal{K}$ with $\mathfrak{A} \not\models s \approx t$.

(2) If $\mathcal{U}(\mathcal{K}) \not\models \psi$ for some universal sentence ψ, then there is $\mathfrak{A} \in \mathcal{K}$ with $\mathfrak{A} \not\models \psi$.

Varieties have so-called free algebras. If $\mathfrak{A}$ is an $\mathcal{F}$-algebra and $X \subseteq A$, we say that X *generates* $\mathfrak{A}$ if the least subalgebra of $\mathfrak{A}$ containing X is $\mathfrak{A}$ itself. For a proof of the following theorem see [39, Theorem 10.12, page 74].

2.1.4. Theorem (Birkhoff). *If $\mathcal{V}$ is a variety, then for every set X, there is an algebra $\mathbf{F}_{\mathcal{V}}(X) \in \mathcal{V}$ that is generated by X and has the following property. For every $\mathfrak{A} = (A, F) \in \mathcal{V}$ and every map $h : X \to A$, there is a unique homomorphism $\overline{h} : \mathbf{F}_{\mathcal{V}}(X) \to \mathfrak{A}$ extending h.*

Algebras satisfying the property of Theorem 2.1.4 are called *free algebras*. The analogous result does in general not hold if we replace varieties with universal classes.

Congruences and subdirectly irreducible algebras

An $\mathcal{F}$-algebra $\mathfrak{A} = (A, F)$ is called a *subdirect product* of a family $\{\mathfrak{A}_i\}_{i\in I}$ of $\mathcal{F}$-algebras iff $\mathfrak{A}$ is a subalgebra of the product $\prod_{i\in I} \mathfrak{A}_i$, and $\pi(A) = A_i$ for each $i \in I$, where π_i is the i-th projection from $\prod_{i\in I} \mathfrak{A}_i$ onto $\mathfrak{A}_i$.

2.1.5. Definition. An $\mathcal{F}$-algebra $\mathfrak{A} = (A, F)$ is called *subdirectly irreducible* iff whenever $\{\mathfrak{A}_i\}_{i\in I}$ is a family of $\mathcal{F}$-algebras and $h : \mathfrak{A} \hookrightarrow \prod_{i\in I} \mathfrak{A}_i$ an embedding such that the image $h(A)$ of A under h is a subdirect product of the family $\{\mathfrak{A}_i\}_{i\in I}$, then there is $i \in I$ such that $\pi_i \circ h : \mathfrak{A} \to \mathfrak{A}_i$ is an isomorphism.

Below we will see a simpler criterion characterizing subdirectly irreducible algebras.

A *congruence* on an algebra $\mathfrak{A} = (A, F)$ is an equivalence relation θ on A that respects the operations on $\mathfrak{A}$, i.e. such that for all $f \in \mathcal{F}$ and $a_1, a'_1, \ldots a_{\sigma(f)}, a'_{\sigma(f)} \in A$,

$$f^{\mathfrak{A}}((a_1, \ldots, a_{\sigma(f)}))\theta f^{\mathfrak{A}}((a'_1, \ldots, a'_{\sigma(f)})),$$

whenever $a_i\theta a'_i$ for all $1 \leq i \leq \sigma(f)$.

Given a congruence θ on $\mathfrak{A}$, let a/θ denote the equivalence class of an element $a \in A$ with respect to θ.

2.1.6. Definition. The *quotient algebra of* $\mathfrak{A}$ *by* θ is defined as $\mathfrak{A}_\theta = (A_\theta, F_\theta)$, where A_θ consists of the set of equivalence classes with respect to θ, and for $f \in \mathcal{F}$ and $a_1, \ldots, a_{\sigma(f)} \in A$,

$$f^{\mathfrak{A}_\theta}((a_1/\theta, \ldots, a_{\sigma(f)}/\theta)) = f^{\mathfrak{A}}((a_1, \ldots, a_{\sigma(f)}))/\theta.$$

The quotient map $\theta : \mathfrak{A} \to \mathfrak{A}$, given by $a \mapsto a_\theta$ is an onto homomorphism. Thus, the quotient algebra of $\mathfrak{A}$ with respect to θ is a homomorphic image of $\mathfrak{A}$. In fact, the converse of this statement is also true, i.e. every homomorphic image of $\mathfrak{A}$ gives rise to a congruence on $\mathfrak{A}$. Thus, there is a one-to-one correspondence between congruences and isomorphism classes of homomorphic images of $\mathfrak{A}$. This fact is known as the Homomorphism Theorem (see [39, Theorem 6.12, page 50]).

The set of congruences of an algebra form a bounded lattice. The least congruence is the diagonal $\Delta = \{(a, a) \mid a \in A\}$, and the largest is $\nabla = A \times A$. Many properties of an algebra can be studied via its congruence lattice. In particular, there is the following useful characterization of subdirectly irreducible algebras (see [39, Theorem 8.4, page 63]).

2.1.7. Theorem. *An algebra $\mathfrak{A}$ is subdirectly irreducible iff its congruence lattice has a second least element, i.e. there is a congruence $\theta \neq \Delta$ such that $\theta \leq \theta'$ for all congruences $\theta' \neq \Delta$.*

Every algebra can be "represented" by its subdirectly irreducible homomorphic images in the sense of the following theorem, for a proof see [39, Theorem 8.6, page 64].

2.1.8. Theorem (Birkhoff). *Every algebra $\mathfrak{A}$ is isomorphic to a subdirect product of subdirectly irreducible algebras that are homomorphic images of $\mathfrak{A}$.*

Given an algebra $\mathfrak{A}$, a subdirect product as described Theorem 2.1.8 is also called a *subdirect representation* of $\mathfrak{A}$. A simple consequence of the theorem above is that every variety is generated by its subdirectly irreducible members. Accordingly, subdirectly irreducible algebras are often described as the building blocks of a variety.

If $\mathcal{K}$ is a class of algebras, let

$$\mathcal{K}_{\mathsf{si}} = \{\mathfrak{A} \in \mathcal{K} \mid \mathfrak{A} \text{ is subdirectly irreducible}\}.$$

2.1.9. Corollary. *For every variety $\mathcal{V}$, $\mathcal{V} = \mathcal{V}(\mathcal{V}_{\mathsf{si}})$.*

A variety $\mathcal{V}$ is called *congruence distributive* iff the congruence lattice of every algebra $\mathfrak{A}$ in $\mathcal{V}$ is a distributive lattice. The following theorem is usually referred to as Jónsson's Lemma (see [39, Theorem 6.8, page 165]).

2.1.10. Theorem (Jónsson). *If $\mathcal{K}$ is a class of algebras and $\mathcal{V}(\mathcal{K})$ is congruence distributive, then*

$$\mathcal{V}(\mathcal{K})_{\mathsf{si}} \subseteq \mathbf{HSP_U}(\mathcal{K}).$$

Thus, whenever a class $\mathcal{K}$ generates a congruence distributive variety, Jónsson's Lemma guarantees that the subdirectly irreducible algebras of that variety lie already in $\mathbf{HSP_U}(\mathcal{K})$ (as opposed to $\mathbf{HSP}(\mathcal{K})$). If $\mathcal{K}$ is a finite class of finite algebras, then $\mathbf{P_U}(\mathcal{K}) = \mathcal{K}$, so we obtain the following corollary.

2.1.11. Corollary. *If $\mathcal{K}$ is a finite class of finite algebras and $\mathcal{V}(\mathcal{K})$ is congruence distributive, then*

$$\mathcal{V}(\mathcal{K})_{\mathsf{si}} \subseteq \mathbf{HS}(\mathcal{K}).$$

Reducts and locally finite algebras

Finally, we mention two more notions of universal algebra that we will often encounter, namely reducts and local finiteness.

2.1.12. DEFINITION. Let $\mathcal{F}$ be an algebraic similarity type, and let $\mathcal{H} \subseteq \mathcal{F}$. If $\mathfrak{A} = (A, F)$ is an $\mathcal{F}$-algebra, then the $\mathcal{H}$-algebra $\mathfrak{A}_{\mathcal{H}} = (A, H)$ is called the *$\mathcal{H}$-reduct of* $\mathfrak{A}$ where $H = \{f^{\mathfrak{A}} \mid f \in \mathcal{H}\}$.

Thus, the $\mathcal{H}$-reduct $\mathfrak{A}_{\mathcal{H}}$ is obtained from $\mathfrak{A}$ by "forgetting" all operations in $\mathcal{F} \setminus \mathcal{H}$. In fact, we will usually be more general with this notion by allowing $\mathcal{H}$ to be a set of $\mathcal{F}$-terms rather than a proper subset of $\mathcal{F}$.

Let $\mathfrak{A} = (A, F)$ and $\mathfrak{B} = (B, F')$ be $\mathcal{F}$-algebras and let $\mathcal{H} \subseteq \mathcal{F}$. A function $h : A \to B$ is called an *$\mathcal{H}$-homomorphism* iff $h(f^{\mathfrak{A}}(\bar{a})) = f^{\mathfrak{B}}(h(a)_{a \in \bar{a}})$ for all $f \in \mathcal{H}$.

An $\mathcal{F}$-algebra $\mathfrak{A} = (A, F)$ is called *finitely generated* iff there is a finite set $X \subseteq A$ that generates $\mathfrak{A}$. The algebra $\mathfrak{A}$ is called *n-generated* for some $n \in \mathbb{N}$ if $\mathfrak{A}$ is generated by a set of cardinality n. More generally, a subalgebra $\mathfrak{B}$ of $\mathfrak{A} = (A, F)$ is called a *finitely generated subalgebra* of $\mathfrak{A}$ iff there is a finite set $X \subseteq B$ such that $\mathfrak{B}$ is the least subalgebra of $\mathfrak{A}$ that contains X.

2.1.13. DEFINITION. Let $\mathcal{F}$ be an algebraic similarity type. An $\mathcal{F}$-algebra $\mathfrak{A} = (A, F)$ is called *locally finite* iff every finitely generated subalgebra of $\mathfrak{A}$ is finite. A class $\mathcal{K}$ of $\mathcal{F}$-algebras is called *locally finite* iff each algebra in $\mathcal{K}$ is locally finite.

There are many equivalent descriptions of locally finite varieties.

2.1.14. THEOREM. *Let $\mathcal{V}$ be a variety. Then the following conditions are equivalent.*

(1) $\mathcal{V}$ is locally finite.

(2) Every finitely generated algebra in $\mathcal{V}$ is finite.

(3) Every finitely generated free algebra of $\mathcal{V}$ is finite.

(4) For each $n \in \mathbb{N}$ there is $m(n) \in \mathbb{N}$ such that the cardinality of all n-generated algebras in $\mathcal{V}$ is at most $m(n)$.

The equivalence of (1) and (2) easily follows from the fact that $\mathcal{V}$ is closed under subalgebras. For a proof of the equivalence of (1) and (3) see [39, Theorem 10.15, page 76]. Finally, the equivalence of (3) and (4) follows from the fact each finitely n-generated algebra is a homomorphic image of the free algebra on n generators.

In fact, as shown in [12], condition (4) in the theorem above can be strengthened. This leads to another useful criterion to recognize local finiteness of a variety [12, Theorem 3.7].

2.1.15. Theorem ([12]). *A variety $\mathcal{V}$ of finite signature is locally finite iff for each $n \in \mathbb{N}$ there is $m(n) \in \mathbb{N}$ such that the cardinality of all n-generated subdirectly irreducible members of $\mathcal{V}$ is at most $m(n)$.*

In what follows we recall the definition and properties of more specific algebraic structures that we will encounter in this thesis.

2.1.2 Heyting algebras

In this section we recall the definition of Heyting algebras and other algebraic structures based on partially ordered sets.

Lattices

A *lattice* $L = (L, \wedge, \vee)$ is an algebra with two binary operations, $\wedge$ and $\vee$ satisfying the set of equations in Table 2.1.1.

$x \vee y \approx y \vee x$	$x \wedge y \approx y \wedge x$
$x \vee (y \vee z) \approx (x \vee y) \vee z$	$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$
$x \vee x \approx x$	$x \wedge x \approx x$
$x \approx x \vee (x \wedge y)$	$x \approx x \wedge (x \vee y)$.

Table 2.1.1: Equational theory of lattices

Alternatively, a lattice can be defined as a partially ordered set (*poset* for short) with binary infima and suprema. If $(P, \leq)$ is a poset with binary suprema denoted by sup and binary infima denoted by inf, then setting $p \vee q := \mathsf{sup}\{p, q\}$ and $p \wedge q := \mathsf{inf}\{p, q\}$ for all $p, q \in P$ produces a lattice. Conversely, given a lattice $(L, \wedge, \vee)$, then defining $p \leq q$ iff $p \wedge q = p$ for all $p, q \in L$ produces a poset with binary infima and suprema. (Setting $p \leq q$ iff $p \vee q = q$ produces the same partial order).

A lattice is called *distributive* iff it satisfies the equations $(x \wedge y) \vee z \approx (x \vee z) \wedge (y \vee z)$ and $(x \vee y) \wedge z \approx (x \wedge z) \vee (y \wedge z)$. A *bounded lattice* is a lattice L together with constants 0 and 1 satisfying the equations $x \vee 1 \approx 1$ and $0 \wedge x \approx 0$. From the equational definition of (bounded, distributive) lattices it immediately follows that they form a variety.

For more information on lattices the reader is referred to [3, 70].

Heyting algebras

2.1.16. Definition. A *Heyting algebra* $A = (A, \wedge, \vee, \to, 0, 1)$ is a bounded lattice $(A, \wedge, \vee, 0, 1)$ together with a binary operation $\to$, called *Heyting implication*, such that for all $a, b, c \in A$

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c. \tag{2.1}$$

In other words, $b \to c$ is the largest element x of A satisfying $b \wedge x \leq c$. The term $\neg x$ is used to abbreviate $x \to 0$, and for $a \in A$, the element $\neg a$ is called the *pseudo-complement* of a. We also write $a \leftrightarrow b$ for $(a \to b) \wedge (b \to a)$.

The lattice reduct of Heyting algebras is always distributive. Moreover, the partial order of the lattice reduct of a Heyting algebra A can be equivalently defined by setting $a \leq b$ iff $a \to b = 1$ for all $a, b \in A$.

Alternatively, Heyting algebras can be defined by the equations of Table 2.1.2 (see [84, Lemmas 1.10 and 1.11])

equational theory of lattices (Table 2.1.1)	
$x \vee 1 \approx 1$	$0 \wedge x \approx 0$
$x \to x \approx 1$	$x \to (y \wedge z) \approx (x \to y) \wedge (x \to z)$
$(x \to y) \wedge y \approx y$	$x \wedge (x \to y) \approx (x \wedge y)$

Table 2.1.2: Equational theory of Heyting algebras

The equational definition shows that Heyting algebras form a variety. It is well known that the variety of Heyting algebras is *not* locally finite. In fact, already the free Heyting algebra on one generator is infinite. It is called the *Rieger-Nishimura* lattice [113, 104].

Congruences of Heyting algebras can be characterized by filters. A *filter* of a bounded lattice L is a *non-empty* subset $F \subseteq L$ that is an *upset*, i.e. for all $a, b \in L$ with $a \leq b$ we have that $a \in F$ implies $b \in F$, and is *closed under binary meets*, i.e. we have $a \wedge b \in F$ whenever $a, b \in F$. A filter $F \subseteq L$ is called *principal* iff there is $a \in L$ such that $F = \{b \in L \mid a \leq b\}$.

2.1.17. Theorem. *For a Heyting algebra A there is a dual isomorphism between the lattice of filters of A and the congruence lattice of A.*

We describe how to obtain the correspondences, see [111, Chapter 1.13] for details. Given a filter F on the Heyting algebra A, the corresponding congruence on A is defined by $\{(a, b) \in A^2 \mid a \leftrightarrow b \in F\}$. Conversely, if θ is a congruence on A, the corresponding filter is defined by $\{a \in A \mid (a, 1) \in \theta\}$. If F is a filter on A, we sometimes write A/F to denote the quotient algebra of A with respect to the congruence corresponding to F (see Definition 2.1.6).

Recall that subdirectly irreducible algebras are those that have a second least element in their congruence lattice (Theorem 2.1.7). Using the correspondence in Theorem 2.1.17, a subdirectly irreducible Heyting algebra can be characterized via its filters.

2.1.18. Theorem. *A Heyting algebra A is subdirectly irreducible iff it has a second largest element, i.e. there is $a \in A \backslash \{1\}$, such that $b \leq a$ for all $b \in A \backslash \{1\}$.*

Using the characterization by filters, it is easy to see that the variety of Heyting algebras is congruence distributive. Thus, in particular Jónsson's Lemma (Theorem 2.1.10) applies, which we will often make use of.

We will often work with well-connected Heyting algebras: A Heyting algebra A is called *well-connected* if $a \vee b = 1$ implies $a = 1$ or $b = 1$ for all $a, b \in A$. The following theorem describes the connections between subdirectly irreducible and well-connected Heyting algebras.

2.1.19. LEMMA. *Every subdirectly irreducible algebra is well-connected. The converse is true for finite Heyting algebras.*

Other algebras based on posets

To conclude this section, we recall the definitions of further algebraic structures based on posets.

- A *pseudo-complemented bounded distributive lattice* is a bounded distributive lattice enriched with a unary operation $\neg$ such that

$$b \leq \neg a \quad \text{iff} \quad a \wedge b = 0.$$

- A *meet-semilattice* is an algebra $L = (L, \wedge)$ satisfying the first three equations on the right-hand-side of Table 2.1.1.

- A *Brouwerian semilattice* (also called *implicative meet-semilattice*) is a meet-semilattice with an implication, i.e. with a binary operation $\to$ satisfying the equation (2.1) in Definition 2.1.16. A Brouwerian semilattices A always contains a largest element given by $a \to a$ for any $a \in A$. A *bounded Brouwerian semilattices* (also called *implicative meet-semilattice with* 0) is a Brouwerian semilattices containing a constant 0 and satisfying the equation $x \wedge 0 \approx 0$.

- A *Boolean algebra* is a Heyting algebra satisfying $\neg x \vee x \approx 1$. We will typically think of Boolean algebras given in the signature $(A, \wedge, \vee, \neg, 0, 1)$. Then the Heyting implication can be recovered by $a \to b = \neg a \vee b$.

We will employ the following abbreviations:

- BDLat stands for the variety of bounded distributive lattices,
- PBDLat stands for the variety of pseudo-complemented bounded distributive lattices,
- BSLat stands for the variety of Brouwerian semilattices (aka implicative meet semilattices),
- BBSLat stands for the variety of bounded Brouwerian semilattices (aka implicative meet semilattices with 0)
- HA stands for the variety of Heyting algebras, and
- BA stands for the variety of Boolean algebras.

Table 2.1.3: Varieties of algebras based on posets

We will often make use of the fact that some of the varieties above are locally finite. The local finiteness of BA and BDLat is well known (see e.g. [70, Theorem 1(iv), page 68], or any other textbook on lattice theory). The fact that PBDLat is locally finite is also well known, we will nevertheless provide a proof of this fact in Theorem 3.2.2. The local finiteness of BSLat and BBSLat was established in [49]. Alternative proofs showing the local finiteness of the varieties BDLat, BSLat, and BBSLat can be found in [12]. Gathering the above together we obtain the following result.

2.1.20. THEOREM. *The varieties* BA*,* BDLat*,* PBDLat*,* BSLat*,* BBSLat *are locally finite.*

2.1.3 Modal algebras

In this section we recall the definition and properties of modal algebras. We pay special attention to K4-algebras.

2.1.21. DEFINITION. A *modal algebra* $\mathfrak{A} = (A, \Diamond)$ consists of a Boolean algebra A and a unary operation $\Diamond$ satisfying $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ for all $a, b \in B$ and $\Diamond 0 = 0$.

Alternatively, a modal algebra can be defined as a pair $(A, \Box)$ consisting of a Boolean algebra together with a unary operation $\Box$ satisfying

$$\Box a \wedge \Box b = \Box(a \wedge b) \quad \text{and} \quad \Box 1 = 1. \tag{2.2}$$

We can switch between the two alternative definitions of modal algebras as follows. Given an operator $\Diamond$ on a Boolean algebra A, set $\Box a = \neg\Diamond\neg a$ for all $a \in A$. Conversely, given an operator $\Box$ on A, then set $\Diamond a = \neg\Box\neg a$ for all $a \in A$.

An *atom* of a Boolean algebra A is an element $a \in A \setminus \{0\}$ such that $b \leq a$ implies that $b = 0$ or $b = a$ for all $b \in A$. In a finite Boolean algebra every element is a join of atoms. Since the $\Diamond$-operator distributes over $\vee$, a $\Diamond$-operator on a finite modal algebra is fully determined by its values on atoms. We denote the variety of modal algebras by MA.

A *modal filter* of a modal algebra $\mathfrak{A} = (A, \Diamond)$ is a filter $F \subseteq A$ of A such that $a \in F$ implies $\Box a \in F$ for all $a \in A$. Just as filters on Heyting algebras correspond to congruences, the congruences of modal algebras can be characterized by *modal filters*.

2.1.22. Theorem. *For a modal algebra $\mathfrak{A}$, there is a dual isomorphism between the lattice of modal filters of $\mathfrak{A}$ and the congruences lattice of $\mathfrak{A}$.*

Just as the variety of Heyting algebras, the variety of modal algebras is congruence distributive making Jónsson's Lemma (Theorem 2.1.10) applicable. Subdirectly irreducible modal algebras admit characterizations via oprema. An element a of a modal algebra $\mathfrak{A}$ is called an *opremum* iff $a \neq 1$ and for each $b \neq 1$ there is $n \in \mathbb{N}$ with $\blacksquare_n b \leq a$, where $\Box^0 b = b$, $\Box^{n+1} b = \Box\Box^n b$, and $\blacksquare_n b = \bigwedge_{k \leq n} \Box^k b$ (see e.g. [88, page 174])).

2.1.23. Theorem. *A modal algebra is subdirectly irreducible iff it has an opremum.*

$\mathsf{K4}$-algebras

We will often work with modal algebras satisfying stronger equations. In particular, we will work with $\mathsf{K4}$-algebras. A $\mathsf{K4}$*-algebra* is a modal algebra $\mathfrak{A} = (A, \Diamond)$ satisfying $\Diamond\Diamond a \leq \Diamond a$, or, alternatively $\Box a \leq \Box\Box a$ for all $a \in A$.

Let $\mathfrak{A} = (A, \Diamond)$ be a $\mathsf{K4}$-algebra. We use a standard convention of modal logic by setting for $a \in A$,

$$\Diamond^+ a = a \vee \Diamond a \quad \text{and} \quad \Box^+ a = a \wedge \Box a. \tag{2.3}$$

It is easy to see that an element a of a $\mathsf{K4}$-algebra $\mathfrak{A} = (A, \Diamond)$ is an opremum iff $a \neq 1$ and $\Box^+ b \leq a$ for all $b \in A$. Thus, the characterization of subdirectly irreducible modal algebras (Theorem 2.1.23) can be strengthened when applied to $\mathsf{K4}$-algebras.

2.1.24. Theorem. *A $\mathsf{K4}$-algebra $\mathfrak{A} = (A, \Diamond)$ is subdirectly irreducible iff it contains an element $a \neq 1$ with $\Box^+ b \leq a$ for all $b \neq 1 \in A$.*

Following [99, Definition 1.10], we call a $\mathsf{K4}$-algebra $\mathfrak{A} = (A, \Diamond)$ *well-connected* if $\Diamond^+ a \wedge \Diamond^+ b = 0$ implies $a = 0$ or $b = 0$ for all $a, b \in A$. Equivalently, $\mathfrak{A}$ is well-connected if $\Box^+ a \vee \Box^+ b = 1$ implies $a = 1$ or $b = 1$ for all $a, b \in A$. The following theorem describes the connections between subdirectly irreducible and well-connected $\mathsf{K4}$-algebras.

2.1.25. Theorem. *Every subdirectly irreducible* $\mathsf{K4}$*-algebra is well-connected. The converse is true for finite* $\mathsf{K4}$*-algebras.*

2.1.26. Remark. The universal algebra term encompassing well-connected Heyting algebras and well-connected $\mathsf{K4}$-algebras is *being finitely subdirectly irreducible*. Since this notion is not as standard as being subdirectly irreducible, we keep the name well-connected for Heyting algebras and $\mathsf{K4}$-algebras.

2.2 Duality

Next we recall dual descriptions of bounded distributive lattices, Heyting algebras, and modal algebras. We also recall how algebraic notions such as subalgebras, homomorphic images and subdirectly irreducible algebras translate to the dual setting.

Duality refers to *dual equivalence* of appropriate categories. Since category theory itself does not play a central role in this thesis, we skip the precise definition of dual equivalence and refer the reader to [94, 2].

2.2.1 Some notions on posets

Recall that if $(X, \leq)$ is a poset, by an *upset* of X we mean a subset $U \subseteq X$ such that $x \in U$ implies $y \in U$ whenever $x \leq y$ for $x, y \in U$, a *downset* of X is defined dually. For $U \subseteq X$ we write

$$\uparrow U = \{y \in X \mid x \leq y \text{ for some } x \in U\} \text{ and } \downarrow U = \{y \in X \mid y \leq x \text{ for some } x \in U\}.$$

Thus, $\uparrow U$ is the least upset containing U, and $\downarrow U$ is the least downset containing U, respectively. If $U = \{x\}$ is a singleton, we usually write $\uparrow x$ and $\downarrow x$ instead of $\uparrow\{x\}$ and $\downarrow\{x\}$, respectively. If $U \subseteq X$, we define the *maximum* of U by

$$\max U = \{x \in U \mid (y \in U \text{ and } x \leq y) \text{ implies } y = x\}.$$

Elements in $\max X$ are called *maximal* elements of X. If $x, y \in X$, then x is called an *immediate successor* of y iff $y < x$ and $y \leq z \leq x$ implies that $z = x$ or $z = y$. An *antichain* is a subset $A \subseteq X$ with $a \not\leq b$ and $b \not\leq a$ for all $a, b \in A$ with $a \neq b$.

A *chain* is a subset $C \subseteq X$ that is linearly ordered, i.e. for all $x, y \in C$, we have $x \leq y$ or $y \leq x$. If $C \subseteq X$ is a finite chain, we often refer to the cardinality of C as its *length*. We say that C *starts* in $x \in C$ iff $x \leq c$ for all $c \in C$.

A poset $(X, \leq)$ has *bounded depth* iff there is natural number k such that X contains a chain of length k but no chain of larger length. If $(X, \leq)$ is of bounded depth and $x \in X$, then $\mathrm{d}(x)$ denotes the cardinality of the longest chain starting from x. Thus, e.g. an element $x \in X$ is maximal iff $\mathrm{d}(x) = 1$. If $(X, \leq)$ is of bounded depth, we also write $\mathrm{d}(X)$ for the cardinality of the longest chain in $(X, \leq)$.

2.2.1. Definition. Let $(X, \leq)$ and $(Y, \leq)$ be posets and let $f : X \to Y$ be a map. Then

- f is called *order-preserving* iff $x \leq x'$ implies $f(x) \leq f(x')$ for all $x, x' \in X$.
- f is called a *p-morphism* iff $f(\uparrow x) = \uparrow f(x)$ for all $x \in X$.

It is easy to see that a p-morphism is also order-preserving. An order-preserving map $f : X \to X$ is called

- a *closure operator* iff $x \leq f(x)$ and $ff(x) \leq f(x)$, and
- an *interior operator* iff $f(x) \leq x$ and $f(x) \leq ff(x)$.

A pair (g, f) of order-preserving maps $f : X \to Y$ and $g : Y \to X$ is called an *adjoint pair* iff for all $x \in X$ and $y \in Y$,

$$y \leq f(x) \quad \text{iff} \quad g(y) \leq x.$$

In that case, f is called the *right* or *upper adjoint* of g. The pair (g, f) is also referred to as a *Galois connection.* See [94, 2], and e.g. [37, Theorem 2.7] for the connection between adjoint pairs and closure operators.

2.2.2 Priestley duality

An order-topological duality for bounded distributive lattices was developed by Priestley in [108, 109]. We here recall the basic constructions involved. For detailed expositions the reader is referred to [48, 103].

2.2.2. Definition. A *Priestley space* $(X, \leq)$ is a topological space X equipped with a partial order $\leq$ such that X is compact, zero-dimensional, and for each $x, y \in X$,

$$x \not\leq y \text{ implies that there is a clopen upset } U \text{ of } X \text{ with } x \in U \text{ and } y \notin U. \quad (2.4)$$

The property in (2.4) is called the *Priestley separation axiom.*

Properties of Priestley spaces that we will use in this thesis include the following.

2.2.3. Theorem. *Let $(X, \leq)$ be a Priestley space.*

(1) Let $\{x_1, \ldots, x_n\}$ be an antichain of X. Then there are clopen upsets $U_1, \ldots, U_n$ with $x_i \in U_i$ and $x_j \notin U_i$ for all $1 \leq i \neq j \leq n$.

(2) Let $W \subseteq X$ be a closed downset, and let $x \notin W$. Then there is a clopen upset U of X with $x \in U$ and $U \cap W = \emptyset$.

(3) X is Hausdorff.

(4) If X is finite, then X is discrete.

(5) If $V \subseteq X$ is closed, then $\downarrow V$ and $\uparrow V$ are closed.

(6) For each $x \in X$, $\uparrow x \cap \max X \neq \emptyset$, i.e. every point sees a maximal point.

(1) and (3) are simple consequences of the Priestley separation axiom. For (2) see [48, Lemma 11.25]. (4) follows from (3). For a proof of (5) see [103, Lemma 5.1] and for a proof of (6) see [29, Theorem 2.3.24].

A map between Priestley spaces is a *Priestley morphism* iff it is continuous and order-preserving. The category of Priestley spaces and Priestley morphisms is denoted by Pries. The variety BDLat of bounded distributive lattices can be seen as a category, where morphisms are bounded lattice homomorphisms.

2.2.4. Theorem (Priestley). *The categories BDLat and Pries are dually equivalent.*

We sketch the constructions involved. Let $L = (L, \wedge, \vee, 0, 1)$ be a bounded distributive lattice. A *prime filter* on L is a filter $x \subseteq L \setminus \{0\}$, such that $a \vee b \in x$ implies $a \in x$ or $b \in x$ for all $a, b \in L$. We denote the collection of prime filters of L by $\mathcal{PF}(L)$. For $a \in L$, let

$$\varphi(a) = \{x \in \mathcal{PF}(L) \mid a \in x\}.$$

The Priestley space $L_* = (\mathcal{PF}(\mathcal{L}), \leq)$ dual to L is defined as follows: L_* is a topological space with underlying set $\mathcal{PF}(\mathcal{L})$, where the topology is given by the (clopen) subbasis

$$\varphi(a) \cap (\mathcal{PF}(L) \setminus \varphi(b)) \text{ for } a, b \in L,$$

and $x \leq y$ iff $x \subseteq y$ for any $x, y \in \mathcal{PF}(\mathcal{L})$. Conversely, let $(X, \leq)$ be a Priestley space. Then the dual lattice $X^* = (\mathcal{CU}(X), \wedge, \vee, 0, 1)$ consists of the clopen upset of X, denoted by $\mathcal{CU}(X)$, and the operations $\wedge$ and $\vee$, are given by the intersection and union of clopen upsets, respectively. Moreover, the constant 0 is the empty set and 1 the whole set X.

On morphisms, the duality acts by inverse maps, i.e., if $f : L \to L'$ is a lattice homomorphism, the inverse image map f^{-1}–that sends a prime filter x of L' to $f^{-1}(x) = \{a \in L \mid f(a) \in x\}$–is a Priestley morphism from L'_* to L_*. Conversely, if $f : X \to Y$ is a Priestley morphism, the inverse image map f^{-1} is a bounded lattice homomorphism from Y^* to X^*.

The next theorem describes how to translate between some algebraic and order-topological notions. For a proof see [48, Section 11.32]. We collect the correspondences in Table 2.2.1.

2.2.5. THEOREM. *Let A a bounded distributive lattice and let $(X, \leq)$ be its dual Priestley space. There is a one-to-one correspondence between*

(1) the congruences on A and closed subsets of X,

(2) the subalgebras of A and the images of X under a Priestley morphisms.

BDLat	Pries
congruences	closed subsets
subalgebras	images of Priestley morphisms

Table 2.2.1: "Dictionary BDLat $\leftrightarrow$ Pries"

2.2.3 Esakia duality

Duality for Heyting algebras was developed by Esakia in [51].

2.2.6. DEFINITION. An *Esakia space* is a Priestley space $(X, \leq)$ in which ${\downarrow}U$ is open for each open set $U \subseteq X$.

Since an Esakia space is in particular a Priestley space, the properties of Theorem 2.2.3 obviously also hold for Esakia spaces. In addition, we have the following properties, for a proof see [52].

2.2.7. THEOREM. *Let $(X, \leq)$ be an Esakia space.*

(1) The maximum $\max X$ *is closed.*

(2) If $x_1, \ldots, x_n$ are n maximal points, then they can be separated by disjoint clopen upsets, i.e. there are clopen upsets $U_1, \ldots, U_n \subseteq X$ with $U_i \cap U_j = \emptyset$ for $i \neq j$ and $x_i \in U_i$.

A map $f : X \to Y$ between Esakia spaces is an *Esakia morphism* if it is a *continuous p-morphism*, i.e. f a p-morphism (see Definition 2.2.1) and is continuous. We denote the category of Esakia spaces and Esakia morphisms by Esa. Note that Esa is not a full subcategory of Pries. By HA we denote the category of Heyting algebras and Heyting algebra homomorphisms.

2.2.8. THEOREM (ESAKIA). *The categories* HA *and* Esa *are dually equivalent.*

The constructions of the dual algebras, spaces, and morphisms are the same as in Priestley duality. If X is an Esakia space, with dual X^*, then the Heyting implication is defined by $U \to V = X \setminus {\downarrow}(U \setminus V)$ for clopen upsets U and V of X.

2.2.9. Definition. Let $(X, \leq)$ be an Esakia space. Then X is called *rooted* iff there is $r \in X$ with $r \leq x$ for all $x \in X$. Then r is called the *root* of X. The space X is called *strongly rooted* iff r is a root of X and $\{r\}$ is a clopen subset of X.

Since a finite Esakia space has the discrete topology, a *finite* rooted Esakia space is necessarily strongly rooted.

In the next Theorem we describe how to translate between some algebraic and order-topological notions. The proofs can be found in [52]. We collect the correspondences in Table 2.2.2.

2.2.10. Theorem. *Let A be a Heyting algebra and let $(X, \leq)$ be its dual Esakia space.*

(1) There is a one-to-one correspondence between the congruences of A and the closed upsets of X.

(2) There is a one-to-one correspondence between the subalgebras of A and the p-morphic images of X.

(3) A is well-connected iff $(X, \leq)$ is rooted.

(4) A is subdirectly irreducible iff $(X, \leq)$ is strongly rooted.

HA	Esa
filters, congruences	closed upsets
subalgebras	p-morphic images
well-connected	rooted
subdirectly irreducible	strongly rooted

Table 2.2.2: "Dictionary HA $\leftrightarrow$ Esa"

2.2.4 Stone duality

Stone duality was developed by Stone in [120]. Recall that a *Stone space* is a topological space X that is zero-dimensional, compact, and Hausdorff. In other words, a Stone space is a Priestley space equipped with partial order given by the identity relation. Let Stone be the category of Stone spaces and continuous maps and let BA denote the category of Boolean algebras.

Priestley duality restricts to Stone duality, the duality between Boolean algebras and Boolean homomorphisms and Stone spaces and continuous maps.

2.2.11. Theorem (Stone). *The categories* BA *and* Stone *are dually equivalent.*

Some notions in this case can be described in simpler terms than in Priestley duality. For instance, prime filters on a Boolean algebra A are the same as *ultrafilters*, where a filter x on a Boolean algebra A is called an *ultrafilter* iff $a \in x$ or $\neg a \in x$ for every $a \in A$. Moreover, if A is a Boolean algebra, the sets of the form $\varphi(a)$ for $a \in A$ already form a basis of its dual Stone space.

2.2.12. REMARK. Of course, historically, Priestley duality was invented later than Stone duality. Accordingly, Priestley duality is seen as a generalization of Stone duality rather than Stone duality is seen as a special case of Priestley duality.

2.2.5 Jónsson-Tarski duality

Next, we recall the duality between modal spaces and modal algebras. This duality was explicitly formulated by Goldblatt [65, 66] building on the representation theorem by Jónsson and Tarski [87]. Precursors of this duality were developed for special cases in [72, 51].

Let R be a binary relation on a set X. For a subset $U \subseteq X$, we write

$$R[U] = \{y \in X \mid xRy \text{ for some } x \in U\} \text{ and } R^{-1}[U] = \{y \mid yRx \text{ for some } x \in U\}.$$

If $U = \{x\}$ is a singleton, then we usually write $R[x]$ and $R^{-1}[x]$ instead of $R[\{x\}]$ and $R^{-1}[\{x\}]$, respectively.

2.2.13. DEFINITION. A *modal space* $\mathfrak{X} = (X, R)$ consists of a Stone space X and a relation R on X that is *point-closed*, i.e. $R[x]$ is closed for every $x \in X$ and so that $R^{-1}[U]$ is clopen for each clopen subset U of X.

A *morphism* $f : \mathfrak{X} \to \mathfrak{Y}$ between modal spaces $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, S)$ is a continuous map that is a p-morphism, i.e. $f(R[x]) = S[f(x)]$ for all $x \in X$. The category of modal spaces and morphisms is denoted by MS. By MA we denote the category of modal algebras and modal algebra homomorphisms.

2.2.14. THEOREM. *There is a dual equivalence between* MA *and* MS.

We sketch the constructions establishing the dual equivalence (see e.g. [123, Theorem 5.28]). If $\mathfrak{A} = (A, \Diamond)$ is a modal algebra, then its dual modal space is $\mathfrak{X} = (X, R)$, where X is the dual Stone space of A and for $x, y \in X$,

$$xRy \quad \text{iff} \quad (a \in y \Rightarrow \Diamond a \in x) \text{ for all } a \in A.$$

Conversely, if $\mathfrak{X} = (X, R)$ is a modal space, then its dual modal algebra is $\mathfrak{A} = (A, \Diamond)$, where A is the Boolean algebra of clopen subsets of X and $\Diamond a = R^{-1}[a]$ for all $a \in A$. The duality extends to morphisms by taking preimages of the morphisms in question (similarly as in the case of Priestley duality).

MA	MS
modal filters, congruences	closed generated subframes
subalgebras	p-morphic images
subdirectly irreducible	topo-rooted

transitive MA	transitive MS
well-connected	rooted
subdirectly irreducible	strongly rooted

Table 2.2.3: "Dictionary MA $\leftrightarrow$ MS"

Let $\mathfrak{X} = (X, R)$ be a modal space. For a set $U \subseteq X$, we define $R^0[U] = U$, $R^{n+1}[U] = R[R^n[U]]$, and $R^\omega[U] = \bigcup_{n\in\mathbb{N}} R^n[U]$. An element x of X is called a *root* of $\mathfrak{X}$ if $X = R^\omega[x]$ and a *topo-root* if $R^\omega[x]$ is dense in X. We call $\mathfrak{X}$ *rooted* if it has a root, and *topo-rooted* if the set of topo-roots is not co-dense (the interior is nonempty). By [122, Theorem 2], a modal algebra $\mathfrak{A}$ is subdirectly irreducible iff its dual modal space $\mathfrak{X}$ is topo-rooted.

Duals of K4-algebras are called K4-*spaces*. These are modal spaces $\mathfrak{X} = (X, R)$ with a transitive relation R. For a binary relation R, let R^+ be the reflexive closure of R. For a K4-space, we have $R^\omega = R^+$. Thus, a K4-space is rooted iff there is $x \in X$ such that $X = R^+[x]$. Analogously as for Esakia spaces, we call a K4-space *strongly rooted* iff its set of roots is non-empty and clopen.

Further translations between the algebraic and topological side are described in the theorem below and summarized in Table 2.2.3. A *generated subspace* or *generated subframe* of a modal space $\mathfrak{X} = (X, R)$ is a subset $Y \subseteq X$ such that $y \in Y$ whenever xRy for some $x \in X$.

2.2.15. THEOREM. *Let $\mathfrak{A} = (A, \Diamond)$ be a modal algebra with dual space $\mathfrak{X} = (X, R)$.*

(1) There is a one-to-one correspondence between modal filters of $\mathfrak{A}$ and closed generated subspaces of $\mathfrak{X}$.

(2) There is a one-to-one correspondence between subalgebras of $\mathfrak{A}$ and p-morphic images of $\mathfrak{X}$.

Moreover, if $\mathfrak{A}$ is a K4-algebra, and thus $\mathfrak{X}$ a transitive modal space, then $\mathfrak{A}$ is well-connected iff $\mathfrak{X}$ is rooted.

2.3 Logics

In this section we recall the definition of intuitionistic and normal modal logics. We also discuss their algebraic and relational semantics.

2.3.1 Superintuitionistic logics

We recall the definition of IPC and its extensions, the superintuitionistic logics. For more detailed expositions the reader is referred to [40] or [47].

The *language* $\mathcal{L}_{\mathsf{IPC}}$ *of intuitionsitic propositional logic* is defined by the following grammar,

$$\varphi ::= p \mid \bot \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi,$$

where $p \in \mathsf{Prop}$ is a propositional variable. We employ the usual definitions for $\top$, $\neg$, and $\leftrightarrow$, i.e. $\neg\varphi = \varphi \to \bot$, $\top = \neg\bot$, and $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$.

2.3.1. DEFINITION. *Intuitionistic propositional logic* (IPC) is the least set of formulas containing the axioms of Table 2.3.1 and is closed under the rules (modus ponens) and (substitution) for every substitution σ.

$$\begin{array}{l}
p \to (q \to p), \quad \bot \to p, \\
p \wedge q \to p, \quad p \wedge q \to q, \\
p \to p \vee q, \quad q \to p \vee q, \\
(p \to r) \to ((q \to r) \to (p \vee q) \to r))), \\
(p \to (q \to r)) \to ((p \to q) \to (p \to r)).
\end{array}$$

Table 2.3.1: Axioms of IPC

$$\text{(modus ponens)} \ \frac{\varphi \qquad \varphi \to \psi}{\psi} \qquad\qquad \text{(substitution)} \ \frac{\varphi}{\sigma(\varphi)}$$

A *superintuitionistic logic* (*si logic* for short) is a set L of formulas containing IPC and being closed under the rules of (modus ponens) and (substitution).

If L is a si logic and $\varphi \in \mathsf{L}$, we say that φ is a *theorem of* L and often write $\mathsf{L} \vdash \varphi$. If $\Psi \subseteq \mathcal{L}_{\mathsf{IPC}}$ is a set of formulas, by $\mathsf{IPC} + \Psi$ we denote the least si logic containing Ψ. If $\Psi = \{\varphi\}$ is a singleton, then we usually write $\mathsf{IPC} + \varphi$ instead of $\mathsf{IPC} + \{\varphi\}$. *Classical propositional logic* is defined by $\mathsf{CPC} = \mathsf{IPC} + p \vee \neg p$. A set of formulas $\Gamma \subseteq \mathsf{IPC}$ has the *disjunction property* iff $\varphi \vee \psi \in \Gamma$ implies that $\varphi \in \Gamma$ or $\psi \in \Gamma$.

If for each $i \in I$, L_i is a si logics, then $\bigwedge_{i \in I} \mathsf{L}_i = \bigcap_{i \in I} \mathsf{L}_i$ is a si logic, and $\bigvee_{i \in I} \mathsf{L}_i$ is the least si logic generated by $\bigcup_{i \in I} \mathsf{L}_i$. With these operations, si logics form a complete lattice that we denote by Λ_{IPC}. The inconsistent logic Fml, the set of all formulas in $\mathcal{L}_{\mathsf{IPC}}$, is the top element of this lattice and IPC is the bottom element. It is well known that classical logic CPC is the second largest element of Λ_{IPC}.

Algebraic semantics

Algebraic semantics for superintuitionistic logics is given by Heyting algebras. If A is a Heyting algebra, a valuation on A is a map $v : \mathsf{Prop} \to \mathsf{A}$. Formulas in the language $\mathcal{L}_{\mathsf{IPC}}$ correspond precisely to the terms of the language of Heyting algebras and are also interpreted the same way.

A pair (A, v) consisting of a Heyting algebra A and a valuation v is called a *model*. We write $(A, v) \models \varphi$ iff $v(\varphi) = 1$. And we write $A \models \varphi$ provided that $(A, v) \models \varphi$ for every valuation v on A. In that case we say that A *validates* φ. If Ψ is a set of formulas, we write $A \models \Psi$ iff $A \models \varphi$ for all $\varphi \in \Psi$. If L is a si logic and $A \models \mathsf{L}$, then we call A an L-*algebra*. Finally, if $\mathcal{K}$ is a class of Heyting algebras and $\varphi \in \mathcal{L}_{\mathsf{IPC}}$, we write $\mathcal{K} \models \varphi$ iff $A \models \varphi$ for all $A \in \mathcal{K}$.

We can smoothly switch between validity of formulas and equations. In detail, if A is a Heyting algebra and $\varphi \in \mathcal{L}_{\mathsf{IPC}}$, then $A \models \varphi$ iff $A \models \varphi \approx 1$. Conversely, if s and t are terms in the language of Heyting algebras, then

$$A \models s \approx t \text{ iff } A \models s \leftrightarrow t. \tag{2.5}$$

Therefore, if Ψ is a set of formulas, in particular if Ψ is a logic, the collection of all algebras validating Ψ forms a variety (this follows from Theorem 2.1.2). If L is a si logic, by $\mathcal{V}(\mathsf{L})$, we denote its corresponding variety, i.e.

$$\mathcal{V}(\mathsf{L}) = \{A \mid A \models \mathsf{L}\}.$$

Conversely, if $\mathcal{V}$ is a variety of Heyting algebras, then $\mathsf{L}_{\mathcal{V}} := \{\varphi \in \mathcal{L}_{\mathsf{IPC}} \mid \mathcal{V} \models \varphi\}$ is a si logic. The operation $\mathcal{V}(-)$ and $-_{\mathcal{V}}$ constitute a dual isomorphism between the lattice of si logics and the lattice of subvarieties of HA.

And more generally, if $\mathcal{K}$ is a class of Heyting algebras, then $\mathsf{Log}(\mathcal{K}) = \{\varphi \in \mathcal{L}_{\mathsf{IPC}} \mid \mathcal{K} \models \varphi\}$ is a si logic.

Let $\mathcal{K}$ be a class of Heyting algebras and L a logic. We say that L is *complete* with respect to $\mathcal{K}$ iff for every $\varphi \in \mathcal{L}_{\mathsf{IPC}}$, we have $\mathsf{L} \vdash \varphi$ iff $\mathcal{K} \models \varphi$.

2.3.2. Theorem (Algebraic completeness).

(1) IPC *is complete with respect to* HA.

(2) Every si logic L *is complete with respect to* $\mathcal{V}(\mathsf{L})$.

Kripke semantics

We continue by recalling the relational semantics for IPC via Kripke frames. An (intuitionistic) *Kripke frame* is a poset $\mathfrak{F} = (W, \leq)$. A *valuation* on $\mathfrak{F}$ is a map $v : \mathsf{Prop} \to \mathsf{Up}(\mathsf{W})$, where Up denotes the set of upsets of $\mathfrak{F}$. Such valuations are also called *persistent*.

A pair $(\mathfrak{F}, v)$ consisting of a Kripke frame together with a valuation is called a *Kripke model* over $\mathfrak{F}$. We interpret formulas in a Kripke model $\mathfrak{M} = (\mathfrak{F}, v)$ at a world $w \in W$ as follows:

$w \models_v p$	iff	$w \in v(p)$
$w \models_v \varphi \wedge \psi$	iff	$w \models_v \varphi$ and $w \models \psi$
$w \models_v \varphi \vee \psi$	iff	$w \models_v \varphi$ or $w \models_v \psi$
$w \models_v \varphi \to \psi$	iff	$w' \models_v \varphi$ implies $w' \models_v \psi$ for all $w \leq w'$.

Table 2.3.2: Kripke semantics for intuitionistic logics

In order to emphasize the model in question, we often write $\mathfrak{M}, w \models \varphi$ instead of $w \models_v \varphi$. If $\varphi \in \mathcal{L}_{\mathsf{IPC}}$, let $v(\varphi) = \{w \in W \mid w \models_v \varphi\}$. It is easy to see that $v(\varphi)$ is an upset for each φ.

We say that a Kripke model $\mathfrak{M} = (\mathfrak{F}, v)$ *satisfies* a formula φ iff $w \models_v \varphi$ for every world w of $\mathfrak{M}$. In other words, $v(\varphi) = W$. In that case we write $\mathfrak{M} \models \varphi$. A Kripke frame $\mathfrak{F}$ *validates* φ iff $\mathfrak{M} \models \varphi$ for every model $\mathfrak{M}$ over $\mathfrak{F}$. In that case we write $\mathfrak{F} \models \varphi$.

IPC is complete with respect to the class of all Kripke frames, i.e. a formula φ is a theorem of IPC iff $\mathfrak{F} \models \varphi$ for every Kripke frame $\mathfrak{F}$. However, as opposed to algebraic semantics, not all si logics are complete with respect to their class of Kripke frames. Si logics that are complete with respect their class of Kripke frames are called *Kripke complete* (see Table 2.3.4).

Kripke frames vs. Esakia spaces

By Esakia duality, also Esakia spaces serve as adequate semantics for si logics. A *valuation* on an Esakia space X is a map $v : \mathsf{Prop} \to \mathcal{CU}(X)$. A formula $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ can then be interpreted at a world $w \in X$ according to the clauses of Table 6.5.1. Alternatively, φ can be interpreted in the dual Heyting algebra X^* of X. The latter is possible since the valuation v can be seen as a valuation on X^*. The two ways of evaluating φ coincide in the sense that for both cases $v(\varphi)$ will be the same clopen upset of X. Thus, Theorem 2.3.2 implies that every si logic is complete with respect to its corresponding class of Esakia spaces.

By "forgetting" the topology on an Esakia space, we obtain an intuitionistic Kripke frame. Recall that the topology on finite Esakia spaces is discrete, thus finite Esakia spaces and finite Kripke frames coincide. Accordingly, we will sometimes refer to finite Esakia spaces simply as *frames* or *posets* (since the topology is implicit in the finite case).

Note that the dual space of the one-element Heyting algebra is the empty space. Accordingly, we will in general allow our frames to be empty.

2.3.2 Substructural logics

In Chapter 5, we will briefly encounter the substructural logic FL^k_{ew}. We here collect the most important facts that we will make use of. For more details on substructural logics the reader is referred to [62].

2.3.3. Definition. A *residuated lattice* $A = (A, \wedge, \vee, \cdot, \to, 0, 1)$ consists of a lattice $(A, \wedge, \vee)$ together with a binary operations $\to$ and $\cdot$, such that $(A, \cdot, 1)$ is a monoid[1] and for all $a, b, c \in A$

$$a \cdot b \leq c \quad \text{iff} \quad a \leq b \to c,$$

i.e. "$\to$" is the *residual* of the multiplication "$\cdot$".

A residuated lattice $A = (A, \wedge, \vee, \cdot, \to, 0, 1)$ is called *commutative* iff the monoid $(A, \cdot, 1)$ is commutative, *integral* iff 1 is the top element of the lattice, and *k-potent* iff $a^{k+1} = a^k$ for all $a \in A$, where k is a fixed natural number.

In particular, every Heyting algebra is an integral commutative residuated lattice where the multiplication $\cdot$ is the meet operation. The collection of k-potent commutative bounded integral bounded residuated lattices forms a variety and is denoted by k-CIRL.

Congruences on k-CIRLs can be characterized by deductive filters, i.e. filters that are also closed under multiplication. By [32, Theorem 2.7], a k-CIRL is subdirectly irreducible iff it has a second largest element. Just like for Heyting algebras, a k-CIRL A is called *well-connected* iff $a \vee b = 1$ implies $a = 1$ or $b = 1$.

The logic FL^k_{ew} is the logic whose equivalent algebraic semantics is given by k-CIRLs.

2.3.3 Normal modal logics

We here recall the definition and semantics of normal modal logics. For more detailed expositions the reader may consult [40] or [36].

By $\mathcal{L}$ we denote the language of *basic modal logic* defined by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi,$$

[1] An algebra $(A, \cdot, 1)$ is a *monoid* iff $\cdot$ is an associative binary operation such that $a \cdot 1 = a = 1 \cdot a$ for all $a \in A$.

where $p \in \mathsf{Prop}$ is a propositional letter. We employ the usual definitions for $\vee$, $\to$, $\leftrightarrow$, $\top$, $\bot$, and $\Box$, i.e. $\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi = \neg(\varphi \wedge \psi)$, $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$, $\top = p \vee \neg p$, $\bot = p \wedge \neg p$, and $\Box\varphi = \neg\Diamond\neg\varphi$.

2.3.4. Definition. *The normal modal logic* K is the least set of formulas containing the axioms of CPC (i.e. the axioms of Table 2.3.1 together with $p \vee \neg p$), the axiom $\Box(p \to q) \to (\Box p \to \Box q)$ and is closed under the rules of (modus ponens), (substitution) for every substitution σ, and the rule of (necessiation).

$$\text{(modus ponens)}\ \frac{\varphi \qquad \varphi \to \psi}{\psi} \qquad \text{(substitution)}\ \frac{\varphi}{\sigma(\varphi)}$$

$$\text{(necessitation)}\ \frac{\varphi}{\Box\varphi}$$

A *normal modal logic* is a set of formulas L containing the axioms of K and being closed under the rules of K.

If M and L are normal modal logics with $\mathsf{M} \subseteq \mathsf{L}$, then L is called a *normal extension* of M. If M is a normal modal logic, by $\mathsf{NExt}(\mathsf{M})$ we denote the collection of all normal extensions of M. For every normal modal logic M, $\mathsf{NExt}(\mathsf{M})$ is a complete lattice.

If M is a normal modal logic and $\Psi \subseteq \mathcal{L}$ is a set of formulas, by $\mathsf{M} + \Psi$ we denote the least normal modal logic containing $\mathsf{M} \cup \Psi$.

Algebraic semantics

Algebraic semantics for normal modal logics is given by modal algebras (see Definition 2.1.21). A *valuation* on a modal algebra $\mathfrak{A} = (A, \Diamond)$ is a map $v : \mathsf{Prop} \to \mathsf{A}$. If v is a valuation on A, formulas are interpreted as usual by induction on the structure of formulas. We will employ the exact same notations as in the intuitionstic case, for $(\mathfrak{A}, v) \models \varphi$, $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \models \Psi$.

As in the intuitionsitic case, we can translate between validity of formulas and equations. For a modal algebra $\mathfrak{A}$ and a formula $\varphi \in \mathcal{L}$, we have $\mathfrak{A} \models \varphi$ iff $\mathfrak{A} \models \varphi \approx 1$. Conversely, if s and t are terms in the language of modal algebras, then

$$\mathfrak{A} \models s \approx t \quad \text{iff} \quad \mathfrak{A} \models s \leftrightarrow t.$$

Therefore, if Ψ is a set of formulas in $\mathcal{L}$, in particular if Ψ is a logic, the collection of all modal algebras validating Ψ forms a variety. If L is a normal modal logic, by $\mathcal{V}(\mathsf{L})$, we denote its corresponding variety, i.e.

$$\mathcal{V}(\mathsf{L}) = \{\mathfrak{A} \mid \mathfrak{A} \models \mathsf{L}\}.$$

Conversely, if $\mathcal{V}$ is a variety of modal algebras, then $\mathsf{L}_{\mathcal{V}} := \{\varphi \in \mathcal{L} \mid \mathcal{V} \models \varphi\}$ is a normal modal logic. And more generally, if $\mathcal{K}$ is a class of modal algebras, then $\mathsf{Log}(\mathcal{K}) = \{\varphi \in \mathcal{L} \mid \mathcal{K} \models \varphi\}$ is a normal modal logic.

Let $\mathcal{K}$ be a class of modal algebras and let L be a normal modal logic. We say that L is *complete* with respect to $\mathcal{K}$ iff for every $\varphi \in \mathcal{L}$, we have $\mathsf{L} \vdash \varphi$ iff $\mathcal{K} \models \varphi$. We then have the following theorem.

2.3.5. Theorem (Algebraic completeness).

(1) K *is complete with respect to* MA.

(2) Every normal modal logic L *is complete with respect to its variety of modal algebras.*

Kripke semantics

A *Kripke frame* is a pair $\mathfrak{F} = (W, R)$ consisting of a set W together with a binary relation R. A valuation on $\mathfrak{F}$ is a map $v : \mathsf{Prop} \to \mathcal{P}(W)$, where $\mathcal{P}(W)$ denotes the powerset of W. A pair $\mathfrak{M} = (\mathfrak{F}, v)$ consisting of a Kripke frame $\mathfrak{F}$ and a valuation v on $\mathfrak{F}$ is called a *model*. A formula $\varphi \in \mathcal{L}$ is interpreted at a world x in a model $\mathfrak{M}$ according to the clauses of Table 2.3.3.

$x \models_v p$	iff	$x \in v(p)$
$x \models_v \varphi \wedge \psi$	iff	$x \models_v \varphi$ and $x \models_v \psi$
$x \models_v \neg\varphi$	iff	$x \not\models_v \varphi$
$x \models_v \Diamond\varphi$	iff	there is xRy with $y \models_v \varphi$.

Table 2.3.3: Kripke semantics for normal modal logics

Sometime we also write $\mathfrak{M}, x \models \varphi$ instead of $x \models_v \varphi$ in order to emphasize the model $\mathfrak{M}$ where the formula is evaluated. The normal modal logic K is complete with respect to the class of all Kripke frames. However, not all normal modal logics are complete with respect to their Kripke frames (see [40, 36]).

Given a Kripke frame $\mathfrak{F} = (W, R)$, the *complex algebra* of $\mathfrak{F}$ is the modal algebra $\mathbf{Cm}(\mathfrak{F}) = (\mathcal{P}(W), R^{-1})$, see e.g. [36, Definition 5.21].

Kripke frames vs. modal spaces

We briefly explain the relation between modal spaces and Kripke frames. A *valuation* on a modal space $\mathcal{X} = (X, R)$ is a map $v : \mathsf{Prop} \to \mathcal{C}(X)$, where $\mathcal{C}(X)$ denotes the collection of clopen subsets of $\mathcal{X}$. A formula $\varphi \in \mathcal{L}$ can then be interpreted at a world $w \in X$ according to the clauses of Table 2.3.3. Alternatively, φ can be interpreted in the dual modal algebra $\mathcal{X}^*$ of $\mathcal{X}$ by seeing the valuation v as a valuation on $\mathcal{X}^*$. Analogously to the intuitionistic case, the two ways of evaluating φ coincide in the sense that for both cases $v(\varphi)$ will be the same clopen subset of X. Thus, Theorem 2.3.5 implies that every normal modal logic L is complete with respect to its corresponding class of modal spaces, i.e. the class $\{\mathfrak{F} \mid \mathfrak{F} \models \mathsf{L}\}$.

"Forgetting" the topology on an modal space yields a Kripke frame. Since the topology on finite modal spaces is discrete, finite modal spaces and finite Kripke frames coincide. We will therefore sometimes call finite modal spaces simply *frames*.

2.3.4 Properties of logics

We list the definitions of some "good" properties of si logics and normal modal logics that we will encounter in this thesis.

A normal modal or si logic

- is called *Kripke complete* iff it is complete with respect to a class of Kripke frames.
- has *the finite model property* (*fmp*) iff it is complete with respect to a class of finite frames or algebras.
- is called *tabular* iff it is complete with respect to a single finite algebra (or frame).
- is called *elementary* iff it is complete with respect to a first-order definable class of Kripke frames.
- is called *finitely axiomatizable* iff it has a finite set of axioms.

Table 2.3.4: Properties of logics

2.3.5 Logics via consequence relations

Consequence relations provide a very abstract and general way of defining logics and relating them to their semantics (see [89, 77]). In general, consequence relations axiomatize *quasi-varieties*. The more general notion of *multi-conclusion consequence relations* was introduced in [82, 89, 83]. Multi-conclusion consequence relations define *universal classes*. In the following, we recall the general definition of multi-conclusion consequence relations and the special adjustments to intuitionistic and normal modal logics. We also explain how multi-conclusion consequence relations axiomatize logics.

2.3.6 Multi-conclusion consequence relations

Multi-conclusion consequence relations [82, 89, 83] generalize the notion of consequence relations by allowing a set of conclusions as opposed to a single conclusion in their rules (see [77] for a survey). Before defining multi-conclusion rules and consequence relations specifically for the intuitionistic and modal cases, we give a general definition.

A (*multi-conclusion*) *rule* is an expression of the form Γ/Δ, where Γ and Δ are finite sets of formulas. If $\Delta = \{\varphi\}$ is a singleton, we write Γ/φ instead of $\Gamma/\{\varphi\}$—such rules are called *single conclusion rules*—and we usually write $/\Delta$ if Γ is the empty set.

2.3.6. DEFINITION. A *multi-conclusion consequence relation* is a set $\mathcal{S}$ of multi-conclusion rules such that for every formula φ, and sets of formulas Γ, Γ', Δ, and Δ' we have that,

$$\begin{array}{ll} \varphi/\varphi \in \mathcal{S} & \text{(identity)}, \\ \text{if } \Gamma/\Delta \in \mathcal{S}, \text{ then } \Gamma, \Gamma'/\Delta, \Delta' \in \mathcal{S} & \text{(weakening)}, \\ \text{if } \Gamma/\Delta, \varphi \in \mathcal{S} \text{ and } \Gamma, \varphi/\Delta \in \mathcal{S}, \text{ then } \Gamma/\Delta \in \mathcal{S} & \text{(cut)}, \\ \text{if } \Gamma/\Delta \in \mathcal{S} \text{ and } \sigma \text{ is a substitution, then } \sigma(\Gamma)/\sigma(\Delta) \in \mathcal{S} & \text{(substitution)}. \end{array}$$

Intuitionistic case

2.3.7. DEFINITION. An *intuitionistic multi-conclusion consequence relation* is a *multi-conclusion consequence relation* $\mathcal{S}$ that satisfies

$$\begin{array}{ll} /\varphi \in \mathcal{S} \text{ for each theorem } \varphi \text{ of } \mathsf{IPC}, \text{and} & \\ \varphi, \varphi \to \psi/\psi \in \mathcal{S}. & \text{(modus ponens)}. \end{array}$$

By $\mathcal{S}_{\mathsf{IPC}}$ we denote the least intuitionistic multi-conclusion consequence relation, and the complete lattice of multi-conclusion consequence relations extending $\mathcal{S}_{\mathsf{IPC}}$ by Σ_{IPC}. For a set $\mathcal{R}$ of multi-conclusion rules, let $\mathcal{S}_{\mathsf{IPC}} + \mathcal{R}$ be the least intuitionistic multi-conclusion consequence relation containing $\mathcal{R}$. If $\mathcal{S} = \mathcal{S}_{\mathsf{IPC}} + \mathcal{R}$, then we say that $\mathcal{S}$ *is axiomatized by* $\mathcal{R}$.

If A is a Heyting algebra, we say that $A \models \Gamma/\Delta$ iff for every valuation v on A, if $(A, v) \models \gamma$ for all $\gamma \in \Gamma$, then there is $\delta \in \Delta$ with $(A, v) \models \delta$. In other words, whenever $v(\gamma) = 1$ for all $\gamma \in \Gamma$ then $v(\gamma) = 1$ for some $\delta \in \Delta$. Whereas validity of formulas is equivalent to validity of equations, validity of rules is equivalent to *universal sentences*. Namely, for every Heyting algebra A,

$$A \models \Gamma/\Delta \quad \text{iff} \quad A \models \forall \bar{x} \left(\bigwedge_{\gamma \in \Gamma} \gamma(\bar{x}) \approx 1 \to \bigvee_{\delta \in \Delta} \delta(\bar{x}) \approx 1 \right),$$

where $\bar{x}$ contains enough variables to replace the propositional variables in the formulas contained in $\Gamma \cup \Delta$. Thus, multi-conclusion rules axiomatize universal classes (see Section 2.1.1). If $\mathcal{S}$ is an intuitionistic multi-conclusion consequence relation, then we denote by $\mathcal{U}(\mathcal{S})$ the universal class corresponding to $\mathcal{S}$, i.e. $\mathcal{U}(\mathcal{S}) = \{A \mid A \models \mathcal{S}\}$.

As shown in [83], every intuitionistic multi-conclusion consequence relation is complete with respect to its corresponding class of Heyting algebras.

2.3.8. THEOREM. *Every intuitionistic multi-conclusion consequence relation $\mathcal{S}$ is complete with respect to $\mathcal{U}(\mathcal{S})$.*

Modal case

2.3.9. DEFINITION. A *normal modal multi-conclusion consequence relation* is a *multi-conclusion consequence relation $\mathcal{S}$* that satisfies

$$/\varphi \in \mathcal{S} \text{ for each theorem } \varphi \in \mathsf{K},$$

$$\varphi, \varphi \to \psi/\psi \in \mathcal{S} \qquad \text{(modus ponens)},$$

$$\varphi/\Box\varphi \in \mathcal{S} \qquad \text{(necessitation)}.$$

If $\mathcal{R}$ is a set of rules, then we denote by $\mathsf{CR}(\mathcal{R})$ the least normal modal multi-conclusion consequence relation containing $\mathcal{R}$. If $\mathcal{S} = \mathsf{CR}(\mathcal{R})$, then we say that $\mathcal{R}$ *axiomatizes* $\mathcal{S}$. More generally, if $\mathcal{S}$ is a normal modal multi-conclusion consequence relation and $\mathcal{R}$ is a set of rules, by $\mathcal{S}+\mathcal{R}$ we denote the least normal modal multi-conclusion consequence relation extending $\mathcal{S}$ and containing $\mathcal{R}$.

Just as in the intuituonistic case, validity of multi-conclusion rules corresponds to validity of universal sentences. If $\mathcal{S}$ is a normal modal multi-conclusion consequence relation, then we denote by $\mathcal{U}(\mathcal{S})$ the universal class corresponding to $\mathcal{S}$. As shown in [83, Theorem 2.2], every normal modal multi-conclusion consequence relation is complete with respect to its corresponding universal class of modal algebras.

2.3.10. THEOREM. *Every normal modal multi-conclusion consequence relation is complete with respect to its universal class of modal algebras.*

2.3.7 Axiomatizing logics via rules

Multi-conclusion consequence relations can also be used to axiomatize logics. However, as we explain below certain anomalies arise when axiomatizing logics via multi-conclusion rules as opposed to formulas. Finally, following [82, 83], we show that—under certain conditions—rules can be turned into *characteristic formulas.*

Intuitionistic case. If $\mathcal{S}$ is an intuitionistic multi-conclusion consequence relation, then $\Lambda(\mathcal{S}) = \{\varphi \mid /\varphi \in \mathcal{S}\}$ is a si logic. Conversely, if L is a si logic, then it can be turned into a multi-conclusion consequence relation by setting $\mathcal{S}_{\mathsf{L}} = \mathcal{S}_{\mathsf{IPC}} + \{/\varphi \mid \varphi \in \mathsf{L}\}$.

On the semantic level this amounts to the following. The variety corresponding to the logic $\Lambda(\mathcal{S})$ is the variety generated by the universal class $\mathcal{U}(\mathcal{S})$, i.e. $\mathcal{V}(\Lambda(\mathcal{S})) = \mathcal{V}(\mathcal{U}(\mathcal{S}))$. Conversely, the universal class corresponding to the consequence relation $\mathcal{S}_{\mathsf{L}}$ is just the variety corresponding to L, i.e. $\mathcal{U}(\mathcal{S}_{\mathsf{L}}) = \mathcal{V}(\mathsf{L})$.

Modal case. Again, we employ the same notations as in the intuitionistic case. If $\mathcal{S}$ is an normal modal multi-conclusion consequence relation, then $\Lambda(\mathcal{S}) = \{\varphi \mid /\varphi \in \mathcal{S}\}$ is a normal modal logic. Conversely, if L is a normal modal logic, then $\mathcal{S}_{\mathsf{L}} = \mathcal{S}_{\mathsf{K}} + \{/\varphi \mid \varphi \in \mathsf{L}\}$ is normal modal consequence relation. More generally, if M is a normal modal logic and $\mathcal{R}$ is a set of rules, by $\mathsf{M} + \mathcal{R}$ we denote the normal modal logic $\mathsf{L} = \Lambda(\mathcal{S}_{\mathsf{M}} + \mathcal{R})$. In that case we say that L is axiomatized by $\mathcal{R}$ over M.

On the semantic level, we have $\mathcal{V}(\Lambda(\mathcal{S})) = \mathcal{V}(\mathcal{U}(\mathcal{S}))$ and $\mathcal{U}(\mathcal{S}_{\mathsf{L}}) = \mathcal{V}(\mathsf{L})$ for a normal multi-conclusion consequence relation $\mathcal{S}$ and a normal modal logic L.

Anomalies. We point out that one must be cautious when defining logics by multi-conclusion consequence relation since certain anomalies may arise. In particular, if a set of rules $\mathcal{R}$ axiomatizes a logic L and a set of rules $\mathcal{R}'$ axiomatizes a logic L', then $\mathcal{R} \cup \mathcal{R}'$ may not axiomatize $\mathsf{L} \vee \mathsf{L}'$. The latter can be best explained from an algebraic perspective. Let $\mathcal{U}$ and $\mathcal{U}'$ be the universal classes corresponding to $\mathcal{R}$ and $\mathcal{R}'$, respectively. Then $\mathcal{V}(\mathsf{L}) = \mathcal{V}(\mathcal{U})$ and $\mathcal{V}(\mathsf{L}') = \mathcal{V}(\mathcal{U}')$ and $\mathcal{R} \cup \mathcal{R}'$ axiomatizes the variety $\mathcal{V}(\mathcal{U} \cap \mathcal{U}')$. The latter may not coincide with the variety $\mathcal{V}(\mathcal{U}) \cap \mathcal{V}(\mathcal{U}')$ corresponding to $\mathsf{L} \vee \mathsf{L}'$.

Characteristic formulas of rules. In some cases, a multi-conclusion rule Γ/Δ can be "replaced" by its characteristic formula $\chi(\Gamma/\Delta)$ as in [82, 83]. This is for instance possible in the intuitionistic setting, in the setting of FL^k_{ew}, or the transitive modal setting. If φ is a formula in the language of FL^k_{ew}, we define $\varphi^0 = 1$, and $\varphi^{k+1} = \varphi^k \cdot \varphi$ for $k \in \mathbb{N}$.

In the intuitionsitic setting,

$$\chi(\Gamma/\Delta) := \bigwedge_{\gamma \in \Gamma} \gamma \to \bigvee_{\delta \in \Delta} \delta,$$

and more generally, in the setting of FL^k_{ew},

$$\chi(\Gamma/\Delta) := \left(\bigwedge_{\gamma \in \Gamma} \gamma \right)^k \to \bigvee_{\delta \in \Delta} \delta.$$

Moreover, in the transitive modal setting,

$$\chi(\Gamma/\Delta) := \bigwedge_{\gamma\in\Gamma} \Box^{+}\gamma \to \bigvee_{\delta\in\Delta} \Box^{+}\delta,$$

where $\Box^{+}$ is defined as in (2.3) of Section 2.1.3.

The semantic relations between the rules and its characteristic formulas are described in the following lemma.

2.3.11. LEMMA. *Let A be a k-CIRL, Heyting or K4-algebra. The following are equivalent.*

(1) $A \not\models \chi(\Gamma/\Delta)$.

(2) $C \not\models \Gamma/\Delta$ for some subdirectly irreducible homomorphic image C of A.

(3) $C \not\models \Gamma/\Delta$ for some well-connected homomorphic image C of A.

Proof:
We show the case where A is a Heyting algebra in detail and then explain why the "same" proof also works for the other cases.

(1) $\Rightarrow$ (2)**:** Let v be a valuation on A with $(A, v) \not\models \chi(\Gamma/\Delta)$. Then $v(\bigwedge_{\gamma\in\Gamma}\gamma) \not\leq v(\bigvee_{\delta\in\Delta}\delta)$. Let $a = v(\bigwedge_{\gamma\in\Gamma}\gamma)$ and let $\uparrow a$ be the principal filter of A generated by a. Then $A' = A/{\uparrow a}$ is a homomorphic image of A. Let v' be the valuation on A' obtained by composing v with the quotient map. Then $v'(\bigwedge_{\gamma\in\Gamma}\gamma) = 1$ and $v'(\bigvee_{\delta\in\Delta}\delta) \neq 1$.

Let $h : A' \hookrightarrow \prod_{i\in I} A_i$ be a subdirect representation of A', where each A_i is a subdirectly irreducible homomorphic image of A' (see Theorem 2.1.8). For $i \in I$, let $f_i : A' \to A_i$ be the corresponding quotient map.

Then $h \circ v'$ is a valuation on $\prod_{i\in I} A_i$ with $h \circ v'(\bigwedge_{\gamma\in\Gamma}\gamma) = 1$ and $h \circ v'(\bigvee_{\delta\in\Delta}\delta) \neq 1$. The former implies that for all $i \in I$ the valuation $f_i \circ v'$ on A_i yields $f_i \circ v'(\bigwedge_{\gamma\in\Gamma}\gamma) = 1$ and the latter means that there is some $j \in I$ with $f_j \circ v'(\bigvee_{\delta\in\Delta}\delta) \neq 1$ in A_j. Let $v_j = f_j \circ v'$ be the valuation on some A_j that satisfies the latter.

Then $v_j(\bigwedge_{\gamma\in\Gamma}\gamma) = 1$ implies that $v_j(\gamma) = 1$ for each $\gamma \in \Gamma$. Moreover $v_j(\bigvee_{\delta\in\Delta}\delta) \neq 1$ means that $v_j(\delta) \neq 1$ for each $\delta \in \Delta$. In other words, $(A_j, v_j) \not\models \Gamma/\Delta$. Obviously, A_j is also a homomorphic image of A.

(2) $\Rightarrow$ (3)**:** Trivial.

(3) $\Rightarrow$ (1)**:** Let C be a well-connected homomorphic image of A and let v be a valuation on C with $(C, v) \not\models \Gamma/\Delta$. Then $v(\gamma) = 1$ for all $\gamma \in \Gamma$ and $v(\delta) \neq 1$ for all $\delta \in \Delta$. Since C is well-connected, the latter implies that $v(\bigvee_{\delta\in\Delta}\delta) \neq 1$ and the former obviously implies that $v(\bigwedge_{\gamma\in\Gamma}\gamma) = 1$. This yields that $(C, v) \not\models \chi(\Gamma/\Delta)$. Since C is a homomorphic image of A, also A refutes $\chi(\Gamma/\Delta)$.

By essentially the same reasoning, the above proof can be applied to k-CIRLs and modal algebras. One important point to notice in the proof of the implication (1) $\Rightarrow$ (2) is the following. If A is a k-CIRL, then the principal filter $\uparrow a$ for $a = v\left((\bigwedge_{\gamma\in\Gamma}\gamma)^k\right)$ is automatically a deductive filter and thus it corresponds to a congruence on A. Similarly, if $\mathfrak{A}$ is a $\mathsf{K4}$-algebra, then $\uparrow a$, for $a = v(\Box^+ \bigwedge_{\gamma\in\Gamma}\gamma)$ is automatically a modal filter and therefore corresponds to a congruence on $\mathfrak{A}$. $\Box$

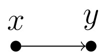
We point out that the lemma above does not imply that a well-connected algebra A refutes a rule iff it refutes its characteristic formula, since validity of rules is in general not preserved by homomorphic images. On the other hand, we clearly have the following corollary.

2.3.12. Corollary. *Let A be a well-connected k-CIRL, Heyting or $\mathsf{K4}$-algebra. Then $A \not\models \Gamma/\Delta$ implies $A \not\models \chi(\Gamma/\Delta)$.*

2.4 Conventions for drawings

We conclude by explaining our conventions when drawing finite intuitionistic or modal Kripke frames.

Intuitionistic case: By picturing partial orders we follow the convention of drawing Hasse diagrams (see [48, Section 1.15]), i.e. y is an immediate successor of x iff we draw a line between x and y, and x is below y.

Modal case: Void dots $\circ$ stand for reflexive points, i.e. points x with xRx, and filled dots $\bullet$ stand for irreflexive points, i.e. points x with $\neg xRx$. We draw 

iff xRy.

$\mathsf{K4}$-case: The same conventions as in the modal case, except that we will automatically assume that the relations are transitive, so we will omit arrows that arise in the transitive closure of the pictured relation.

Chapter 3

$\mathcal{H}$-stable si logics

3.1 Introduction

In this chapter we study $\mathcal{H}$-stable si logics. Here, $\mathcal{H}$ stands for one of the following four reducts of Heyting algebras: the $\{\wedge, \to\}$-, the $\{\wedge, \to, \bot\}$-, the $\{\wedge, \vee, \bot, \top\}$- or the $\{\wedge, \vee, \neg, \top\}$-reduct. Such reducts of Heyting algebras belong to the varieties BSLat, BBSLat, BDLat, and PBDLat, respectively (see Section 2.1.2 for the definitions). A common "good property" of the aforementioned varieties is that they are—as opposed to the variety of Heyting algebras—*locally finite*.

Each such reduct $\mathcal{H}$ gives rises to a class of si logics that we call $\mathcal{H}$-stable. We provide several semantic characterizations of $\mathcal{H}$-stable logics and axiomatize them by $\mathcal{H}$-stable formulas. The locally finiteness of the corresponding variety ensures that $\mathcal{H}$-stable logics have the fmp. The latter can be proved like McKay's theorem [97].

As a matter of fact, three of these classes of si logic were already studied in the literature. The $\mathcal{H}$-stable logics for $\mathcal{H} = \{\wedge, \to\}$ or $\mathcal{H} = \{\wedge, \to, \bot\}$, are precisely the well known classes of *subframe* and *cofinal subframe si logics*, respectively (see [40, 24, 13]). And the $\{\wedge, \vee, \bot, \top\}$-stable si logics are the *stable si logics* from [17]. Thus, most of the general characterization results that we discuss were already known to hold for these classes. Our goal is to provide a uniform treatment that sheds light on some subtle differences in the behavior of the reducts in question.

The $\mathcal{H}$-stable logics for $\mathcal{H} = \{\wedge, \vee, \neg, \top\}$ is a new class of si logics that we call *cofinal stable si logics*. As explained in the introduction of the thesis, stable si logics were introduced as the $\{\wedge, \vee\}$-analogue of subframe si logics. And we think that cofinal stable si logics can be regarded as the $\{\wedge, \vee\}$-analogue of cofinal subframe logics.

In our exposition, we play special attention to the class of cofinal stable logics. We show that there is a continuum of cofinal stable logics that are not stable and we provide examples of si logics distinguishing the classes of subframe, cofinal subframe, stable, and cofinal stable logics. In addition, we also collect many known results on subframe, cofinal subframe, and stable si logics from the literature.

Stable si logics will play a role in Chapters 4 and 6 and subframe si logics in Chapters 6 and 7 of the thesis.

The new results on cofinal stable logics are from [21].

Outline

In the following section we recall McKay's theorem [97] and explain how to adjust it to other locally finite reducts of Heyting algebras. Moreover, we define $\mathcal{H}$-filtrations that are designed to prove fmp results like McKay's theorem. In Section 3.3, we define $\mathcal{H}$-stable intuitionistic multi-conclusion consequence relations and si logics and show several characterizations of these in a uniform fashion. In Section 3.4 we recall results of subframe and cofinal subframe logics from the literature. Finally, in Section 3.5, we discuss the stable logics from [17], and some specific properties of the new class of cofinal stable logics. We conclude with a summary of the chapter in Section 3.6.

3.2 Locally finite reducts of Heyting algebras

In this preliminary section we recall how locally finite reducts of Heyting algebras can be used to prove the fmp of IPC and some of its extensions. For this purpose we fix the notion of $\mathcal{H}$-filtrations of Heyting algebras. The ideas stated below are all well known [100, 97].

By Diego's Theorem ([49], see also [12, Example 4.5]), the variety BBSLat of bounded Brouwerian semilattices, is locally finite. Based on this result McKay [97] showed that all si logics that are axiomatized by formulas in the language of BBSLat, i.e. formulas using only the connectives $\wedge, \to$, and $\bot$, have the fmp. We give a sketch of this proof (see also [40, Theorem 7.17]).

3.2.1. Theorem (McKay). *Let* L *be a si logic that is axiomatized by a set* Γ *of formulas using only connectives from the set* $\{\wedge, \to, \bot\}$. *Then* L *has the fmp.*

Proof:
Suppose L is a si logic satisfying the conditions of the theorem and suppose $\mathsf{L} \not\vdash \varphi$ for some formula $\varphi \in \mathcal{L}_{\mathsf{IPC}}$. Then there is a Heyting algebra A that validates L and a valuation v on A with $(A, v) \not\models \varphi$.

Let A' be the $\{\wedge, \to, \bot\}$-reduct of A. Then A' is a BBSLat (this can be seen immediately from the definition of BBSLat that we recalled in Section 2.1.2). Let B be the (BBSLat-)subalgebra of A' generated by the set $\{v(\psi) \mid \psi \in \mathsf{Sub}(\varphi)\}$. Since by Diego's theorem BBSLat is locally finite, B is finite. We expand B by defining

$$a \vee_B b = \bigwedge\{c \in B \mid c \geq a, b\} \text{ for each } a, b \in B. \tag{3.1}$$

It is easy to see that the expansion of B by $\vee_B$ is a Heyting algebra. We denote the Heyting algebra expansion of B by $B_\vee$. Let $\vee_A$ denote the join operation in A. Then

$$a \vee_B b = a \vee_A b \text{ whenever } a, b, \text{ and } a \vee_A b \in B. \tag{3.2}$$

So the join of elements a and b in $B_\vee$ coincides with the join $a \vee_A b$ taken in A as long as $a \vee_A b \in B_\vee$.

The valuation v on A clearly restricts to a valuation v' on $B_\vee$. A simple induction shows that $v(\psi) = v'(\psi)$ for each $\psi \in \mathsf{Sub}(\varphi)$. Indeed, for instance assume that $\psi \wedge \psi' \in \mathsf{Sub}(\varphi)$, then $v(\psi \wedge \psi') = v(\psi) \wedge_A v(\psi') = v'(\psi) \wedge_B v'(\psi') = v'(\psi \wedge \psi')$, where for the second equality we use that $B_\vee$ is a meet-sublattice of A and the induction hypothesis. The $\to$-case is similar. Finally, if $\psi \vee \psi' \in \mathsf{Sub}(\varphi)$, then $v(\psi \vee \psi') = v(\psi) \vee_A v(\psi') = v'(\psi) \vee_B v'(\psi') = v'(\psi \vee \psi')$, where for the second equality we use (3.2) and the induction hypothesis.

We conclude that $(B_\vee, v') \not\models \varphi$. Moreover, $B_\vee$ validates L, since validity of formulas in the language $\{\wedge, \to, \bot\}$ are preserved by BBSLat-subalgebras. Therefore, we found a finite L-algebra refuting φ, so L has the fmp. □

To be able to transfer McKay's proof to other locally finite reducts of Heyting algebras, we identify the important properties used in the above argument:

(i) the $\{\wedge, \to, \bot\}$-reduct of a Heyting algebra is a BBSLat,

(ii) BBSLat is locally finite, and

(iii) every *finite* BBSLat can be expanded to a Heyting algebra. In fact, we needed the slightly stronger property given in (3.2). Namely that a finite subalgebra B of a BBSLat-reduct of some Heyting algebra A can be expanded to a Heyting algebra via $\vee_B$ in such a way that the added operation $\vee_B$ coincides with $\vee_A$ on elements $a, b \in B$ whenever $a \vee_A b \in B$.

It is well known that the above properties also hold for the $\{\wedge, \to\}$-, the $\{\wedge, \vee, \bot, \top\}$-, or the $\{\wedge, \vee, \neg, \top\}$-reducts of Heyting algebras. Thus, these reducts are also suitable for McKay-like fmp-proofs. We discuss this in more detail.

Firstly, the same reasoning as in the proof of McKay's Theorem applies to logics axiomatized by $\{\wedge, \to\}$-formulas by using that BSLat is locally finite. One only has to observe that whenever B is a finite BSLat then it can be expanded to a Heyting algebra by defining $\vee_B$ as in (3.1) and defining 0_B as the least element of B. This expansion obviously satisfies the additional condition of (iii) from above.

Recall that the $\{\wedge, \vee, \top, \bot\}$-reducts of Heyting algebras are bounded distributive lattices (BDLat). Also BDLat satisfies the above conditions. In particular, BDLat is locally finite (see Theorem 2.1.20), and if B is a finite BDLat, then it can be expanded to a Heyting algebra by setting

$$a \to_B b = \bigvee_B \{c \in B \mid a \wedge c \leq b\} \text{ for all } a, b \in B. \tag{3.3}$$

This expansion satisfies the additional condition analogous to (iii), namely that

$$a \to_B b = a \to_A b \text{ whenever } a, b, \text{ and } a \to_A b \in B. \tag{3.4}$$

Hence we can prove the fmp of IPC as in McKay's theorem using BDLat instead of BBSLat (see also [17]).).

However, the fmp proof via BDLat does not have the same additional consequences as McKay's theorem. The interesting aspect in McKay's theorem is that it not only shows the fmp for IPC but also for all si logics defined by formulas in the language of BBSLat. The analogous reasoning in principle applies to BDLat, however, it is well known that no set of formulas in the language of BDLat defines a proper si logic. Thus, this reasoning only provides the fmp for IPC itself. However, the *stable logics* of [17] remedy this fact by providing a continuum of si logics whose fmp can be proved by $\{\wedge, \vee, \bot, \top\}$-filtrations. We discuss this class of si logics in Section 3.5.

By adding $\neg$ to $\{\wedge, \vee, \top, \bot\}$, we obtain $\{\wedge, \vee, \neg, \top\}$-reducts of Heyting algebras. These are pseudo-complemented bounded distributive lattices (PBDLat, see Section 2.1.2). PBDLat is another variety that satisfies the conditions (i)-(iii) above. It is folklore that this variety is locally finite, but since we could not find a proof of that in a standard textbook we recall it below. Any finite PBDLat can be expanded to a Heyting algebra as in (3.3). Contrary to the BDLat-case, there are non-trivial si logics axiomatized by formulas in the language of PBDLat, e.g. KC $=$ IPC $+ \neg p \vee \neg\neg p$ is such an example (see Table A.0.2 for more on KC). Thus, working with PBDLat we do obtain the fmp not only for IPC but also of some of its extensions.

3.2.2. Theorem (Folklore). PBDLat *is locally finite.*

Proof:
We utilize the criterion of Theorem 2.1.15. It is well known (see, e.g. [3, Theorem 5.1]) that subdirectly irreducible members of PBDLat are of the form $B \oplus 1$, where B is a Boolean algebra and $- \oplus 1$ is the operation of adjoining a new top.

We claim that the cardinality of each n-generated subdirectly irreducible $A \in$ PBDLat is bounded above by $m(n) = 2^{2^n} + 1$. Indeed, suppose $A \cong B \oplus 1$ is generated by $g_1, \ldots, g_n$. Without loss of generality we may assume that $g_1, \ldots, g_n \in B$. Therefore, $g_1, \ldots, g_n$ generate B as a Boolean algebra. Thus, the cardinality of B is bounded above by 2^{2^n}. This yields that the cardinality of A is bounded above by $2^{2^n} + 1$. Consequently, the criterion of local finiteness applies. □

What we have not yet pointed out is that in all of the above cases the Heyting algebra expansion of the finite structures are *unique*. This is because the existence of joins and Heyting implication is determined by the internal structure of a lattice rather than an external operation. Though this was not needed in the above arguments, we keep this fact for later reference.

3.2.3. FACT. If B is a finite BSLat, BBSLat, BDLat, or PBDLat, then there is a *unique* Heyting algebra expansion of B.

To be able to talk about fmp-proofs via the above method uniformly, we introduce the terminology of $\mathcal{H}$-filtrations. We will generalize this notion in Chapter 5, where we talk about $\mathcal{H}$-filtrations of arbitrary algebras with locally finite reducts.

> Unless stated otherwise, in the rest of this chapter $\mathcal{H}$ denotes either one of the languages of BSLat, BBSLat, BDLat, or PBDLat.
> That is $\mathcal{H}$ stands for $\{\wedge, \to\}$, $\{\wedge, \to, \bot\}$, $\{\wedge, \vee, \bot, \top\}$ or $\{\wedge, \vee, \neg, \top\}$, respectively.

For $\mathcal{H}$ as above, by $\mathcal{H}^c$ we denote the Heyting algebra operations "missing" in $\mathcal{H}$. In particular, if $\mathcal{H}$ is $\{\wedge, \to\}$ or $\{\wedge, \to, \bot\}$, then $\mathcal{H}^c$ is $\{\vee, \bot\}$ or $\{\vee\}$, respectively. And if $\mathcal{H}$ is $\{\wedge, \vee, \bot, \top\}$ or $\mathcal{H}$ is $\{\wedge, \vee, \neg, \top\}$, then $\mathcal{H}^c$ is $\{\to\}$. Note that if $\mathcal{H} = \{\wedge, \vee, \neg, \top\}$ we did not add $\bot$ to $\mathcal{H}^c$. It is not needed since $\bot$ can be defined by the compound operation of $\neg$ and $\top$.

Recall our notations of reducts from Definition 2.1.12. In particular, if A is a Heyting algebra, then by $A_{\mathcal{H}}$ we denote the $\mathcal{H}$-reduct of A.

3.2.4. DEFINITION. Let A be a Heyting algebra, let Σ be a finite set of formulas, and let v be a valuation on A. Let B' be the $\mathcal{H}$-subalgebra of $A_{\mathcal{H}}$ generated by

$$v(\Sigma) = \{v(\varphi) \mid \varphi \in \mathsf{Sub}(\Sigma)\},$$

and let B be the (unique) Heyting algebra expansion of B'. Let v_B be a valuation on B that coincides with v on all propositional letters in $\mathsf{Sub}(\Sigma)$. Then the pair (B, v_B) is called an *$\mathcal{H}$-filtration of* (A, v) *through* Σ.

More generally, if (B, v_B) is an $\mathcal{H}$-filtration of (A, v) through some finite subformula closed set Σ' with $\Sigma \subseteq \Sigma'$, we sometimes still call (B, v_B) an $\mathcal{H}$-filtration of (A, v) through Σ (as opposed to an $\mathcal{H}$-filtration through Σ').

The filtration theorem summarizes the important properties of $\mathcal{H}$-filtrations. See Section 2.1.2 for the definition and characterization of subdirectly irreducible Heyting algebras.

3.2.5. THEOREM (FILTRATION THEOREM). *Let (A, v) be a model and let Σ be a finite set of formulas closed under subformulas.*

(1) *There exists a model (B, v_B) that is an $\mathcal{H}$-filtration of (A, v) through Σ. Moreover, if A is subdirectly irreducible, then B can be chosen to be subdirectly irreducible, too.*

(2) If (B, v_B) is an $\mathcal{H}$-filtration of (A, v) through Σ, then $v_B(\varphi) = v(\varphi)$ for all $\varphi \in \Sigma$.

Proof:
The first statement of (1) follows immediately from our previous discussion. To see the additional statement of (1), suppose that A is subdirectly irreducible with second largest element s. Expand Σ to a set Σ' by adding a new propositional letter p to Σ. Expand the valuation v on A to a valuation $\overline{v}$ by setting $\overline{v}(p) = s$. Now let (B, v_B) be an $\mathcal{H}$-filtration of $(A, \overline{v})$ through Σ'. Then B contains s by construction and s is also the second largest element of B, so B is subdirectly irreducible. Now (2) follows from our previous discussion by a simple induction on the construction of formulas as in the proof of Theorem 3.2.1. □

3.2.6. Remark. The notion of filtration is ubiquitous in the literature on non-classical logics. The $\{\wedge, \vee\}$-filtrations come closest to what is usually referred to as (frame-theoretic) *filtrations* (see e.g. [40, Section 5.3]), though the latter is a bit more liberal. On the other hand, $\{\wedge, \to, \bot\}$-filtrations come closest to what is known as *selective filtration* (see e.g. [40, Section 5.5]). A thorough discussion on these relations can be found in [16].

3.3 $\mathcal{H}$-stable universal classes and varieties

In this section we investigate $\mathcal{H}$-stable multi-conclusion consequence relations and $\mathcal{H}$-stable si logics. Here, $\mathcal{H}$ stands for a locally finite reduct of Heyting algebras as in the previous section. Roughly speaking, $\mathcal{H}$-stable consequence relations and $\mathcal{H}$-stable logics are designed so that their fmp can be proved via $\mathcal{H}$-filtrations.

We will show that $\mathcal{H}$-stable consequence relations and logics can be characterized in many equivalent ways by properties of their corresponding universal classes and varieties, respectively. Moreover, we show that $\mathcal{H}$-stable consequence relations and si logics can be axiomatized by $\mathcal{H}$-stable rules and formulas, respectively.

In the subsequent sections of this chapter, we will discuss the specific instances of $\mathcal{H}$-stable logics separately depending on $\mathcal{H}$. As a matter of fact, $\mathcal{H}$-stable si logics for $\mathcal{H} = \{\wedge, \to\}$ or $\mathcal{H} = \{\wedge, \to, \bot\}$ are precisely the well-known classes of subframe and cofinal subframe si logics. And $\{\wedge, \vee, \bot, \top\}$-stable logics are the stable si logics of [17].

For the specific instances that we present in this section the general results were already known. Our aim is to present them in a uniform fashion.

We fix some terminology.

3.3.1. Definition. Let A and B be Heyting algebras.

(1) A map $f : B \to A$ is called an *$\mathcal{H}$-homomorphism* iff it preserves all operations from $\mathcal{H}$.

(2) The algebra B is called an *$\mathcal{H}$-subalgebra* of A iff $B_{\mathcal{H}}$ is a subalgebra of $A_{\mathcal{H}}$.

3.3.2. Definition. Let $\mathcal{K}$ and $\mathcal{V}$ be classes of Heyting algebras with $\mathcal{K} \subseteq \mathcal{V}$.

(1) We say that $\mathcal{K}$ is *$\mathcal{H}$-stable within* $\mathcal{V}$ provided that $B \in \mathcal{K}$ whenever $B \in \mathcal{V}$ and B is isomorphic to an $\mathcal{H}$-subalgebra of some $A \in \mathcal{K}$. If $\mathcal{V}$ is the variety of all Heyting algebras, we say that $\mathcal{K}$ is *$\mathcal{H}$-stable* instead of $\mathcal{H}$-stable within $\mathcal{V}$.

(2) We say that $\mathcal{K}$ is *finitely $\mathcal{H}$-stable* provided that $B \in \mathcal{K}$ whenever B is isomorphic to a finite $\mathcal{H}$-subalgebra of some $A \in \mathcal{K}$.

3.3.3. Remark. In most cases $\mathcal{V}$ will be the variety of all Heyting algebras, but the more general formulation will become handy in some notations below.

3.3.1 $\mathcal{H}$-stable universal classes and consequence relations

We define the notions of $\mathcal{H}$-stable consequence relations and provide many equivalent characterizations of these. The reader may recall the definition of intuitionistic multi-conclusion consequence relations from Section 2.3.6.

3.3.4. Definition.

(1) Let B be a finite Heyting algebra, and let for each $b \in B$, p_b be a variable. The *$\mathcal{H}$-stable rule $\delta_{\mathcal{H}}(B)$ of* B is Γ/Δ, where

$$\begin{aligned} \Gamma = &\{(p_a \circ p_b) \leftrightarrow p_{a \circ b} \mid \circ \in \mathcal{H}_2, a, b \in B\} \cup \\ &\{\neg p_a \leftrightarrow p_{\neg a} \mid \ a \in B, \neg \in \mathcal{H}\} \cup \\ &\{p_c \leftrightarrow c \mid c \in \mathcal{H}_0\}, \\ \text{and} \quad \Delta = &\{p_a \leftrightarrow p_b \mid a \neq b \in B\}, \end{aligned}$$

where $\mathcal{H}_2 := \{\wedge, \vee, \to\} \cap \mathcal{H}$ consists of the binary operations of $\mathcal{H}$ and $\mathcal{H}_0 := \{0, 1\} \cap \mathcal{H}$ consists of the constants of $\mathcal{H}$.

(2) A rule system $\mathcal{S}$ is called an *$\mathcal{H}$-stable rule system* provided it is axiomatizable by $\mathcal{H}$-stable rules over $\mathcal{S}_{\mathsf{IPC}}$.

3.3.5. Remark. By considering B above as a partial Heyting algebra where all $\mathcal{H}$-operations are defined, the rule $\delta_{\mathcal{H}}(B)$ is equivalent to the characteristic rule $\rho(B)$ from [45, Definition 4.1]. In the case where $\to \in \mathcal{H}$, the rule is also equivalent to the weak characteristic rule $\rho'(B)$ from [45].

Moreover, if $\mathcal{H} = \{\to, \wedge\}$, then $\delta_{\mathcal{H}}(B)$ is similar to the subframe rule of the dual space of B and if $\mathcal{H} = \{\to, \wedge\}$ then $\delta_{\mathcal{H}}(B)$ is similar to the cofinal subframe rule of the dual space of B from [83].

The rule $\delta_{\mathcal{H}}(B)$ of some finite Heyting algebra B corresponds to the atomic diagram of B known from model theory if we consider B to be a structure in the language $\mathcal{H}$ (see e.g. [42, page 68].) The following lemma describes the semantic content of $\delta_{\mathcal{H}}(B)$.

3.3.6. Lemma. *For every Heyting algebra A and every finite Heyting algebra B, $A \not\models \delta_{\mathcal{H}}(B)$ iff B is isomorphic to an $\mathcal{H}$-subalgebra of A.*

Proof:
For the direction from left to right, let $\delta_{\mathcal{H}}(B) = \Gamma/\Delta$ and let v be a valuation on A with $(A, v) \not\models \delta_{\mathcal{H}}(B)$. This means that

(1) $v(\gamma) = 1$ for all $\gamma \in \Gamma$, and

(2) $v(\delta) \neq 1$ for all $\delta \in \Delta$.

Define a map $g : B \to A$ by $g(b) = v(p_b)$ for all $b \in B$. Then (1) ensures that g is an $\mathcal{H}$-homomorphism. To illustrate this, we show that g preserves the connective $\wedge$ whenever $\wedge \in \mathcal{H}$. The other cases can be proved similarly. Assume that $\wedge \in \mathcal{H}$ and $b, b' \in B$. Then

$$\begin{aligned} g(b \wedge b') &= v(p_{b \wedge b'}) && \text{(by definition of } g) \\ &= v(p_b) \wedge v(p'_b) && \text{(by (1))} \\ &= g(b) \wedge g(b') && \text{(by definition of } g) \end{aligned}$$

In addition, (2) ensures that g is an embedding. Indeed, if $b \neq b'$ in B, then $p_b \leftrightarrow p'_b \in \Delta$. Then $v(p_b \leftrightarrow p'_b) \neq 1$ by (2) and so $g(b) = v(p_b) \neq v(p'_b) = g(b')$. Conversely, if $g : B \to A$ is an $\mathcal{H}$-embedding, then $v(p_b) = g(b)$ defines a valuation on A such that (A, v) refutes $\delta_{\mathcal{H}}(B)$. □

The next lemma summarizes how (finitely) $\mathcal{H}$-stable classes behave under the operation of taking universal classes. In particular, if $\mathcal{K}$ is $\mathcal{H}$-stable, or even finitely $\mathcal{H}$-stable, then so is the universal class generated by $\mathcal{K}$. As we discuss in Section 3.4.2, for $\mathcal{H} = \{\wedge, \to\}$ or $\mathcal{H} = \{\wedge, \to, \bot\}$, the same is true if we replace universal classes by varieties. However, the analogous statement for varieties is no longer true if $\mathcal{H} = \{\wedge, \vee, \bot, \top\}$ or $\mathcal{H} = \{\wedge, \vee, \neg, \top\}$ (see Section 3.5).

3.3.7. Lemma. *Let $\mathcal{K}$ be a finitely $\mathcal{H}$-stable class of Heyting algebras. Then the following properties hold.*

(1) The universal class $\mathcal{U}(\mathcal{K})$ is axiomatized by $\mathcal{H}$-stable rules.

(2) The universal class $\mathcal{U}(\mathcal{K})$ is $\mathcal{H}$-stable.

(3) $\mathcal{U}(\mathcal{K}) = \mathcal{U}(\mathcal{K}_{\mathsf{fin}})$, where $\mathcal{K}_{\mathsf{fin}}$ is the class of finite members of $\mathcal{K}$.

Proof:

(1) Suppose that $\mathcal{K}$ is finitely $\mathcal{H}$-stable. Let $\mathcal{B}$ be the set of finite non-isomorphic Heyting algebras that do not belong to $\mathcal{K}$ and let

$$\Psi = \{\delta_{\mathcal{H}}(B) \mid B \in \mathcal{B}\}.$$

We show that $\mathcal{S}_{\mathcal{K}}$ is axiomatized by Ψ over $\mathcal{S}_{\mathsf{IPC}}$. For this it is sufficient to show that $\mathcal{U}(\mathcal{K})$ consists exactly of those Heyting algebras satisfying Ψ.

First we show that each member of $\mathcal{K}$ satisfies Ψ. If there are $A \in \mathcal{K}$ and $B \in \mathcal{B}$ such that $A \not\models \delta_{\mathcal{H}}(B)$, then by Lemma 3.3.6, there is an $\mathcal{H}$-embedding $B \rightarrowtail A$. Since $\mathcal{K}$ is finitely $\mathcal{H}$-stable and B is finite, $B \in \mathcal{K}$, a contradiction to $B \in \mathcal{B}$. Thus, each member of $\mathcal{K}$ satisfies Ψ. Since $\mathcal{U}(\mathcal{K})$ is generated by $\mathcal{K}$, it follows that each member of $\mathcal{U}(\mathcal{K})$ satisfies Ψ.

Conversely, suppose that a Heyting algebra A validates Ψ, i.e. $A \models \delta_{\mathcal{H}}(B)$ for each $B \in \mathcal{B}$. If $A \notin \mathcal{U}(\mathcal{K})$, then there is a multi-conclusion rule Γ/Δ such that $\mathcal{K} \models \Gamma/\Delta$ and a valuation v on A such that $(A, v) \not\models \Gamma/\Delta$. Let (B, v') be an $\mathcal{H}$-filtration of B through $\mathsf{Sub}(\Gamma \cup \Delta)$. Then $(B, v) \not\models \Gamma/\Delta$. Since B is an $\mathcal{H}$-subalgebra of A, we have $A \not\models \delta_{\mathcal{H}}(B)$ by Lemma 3.3.6. As A satisfies $\delta_{\mathcal{H}}(B)$ for each $B \in \mathcal{B}$, we see that $B \in \mathcal{K}$, so $B \in \mathcal{U}(\mathcal{K})$. But this contradicts $B \not\models \Gamma/\Delta$. Therefore, $A \in \mathcal{U}(\mathcal{K})$.

(2) Using Lemma 3.3.6 it is easy to see that validity of $\mathcal{H}$-stable rules is preserved by $\mathcal{H}$-subalgebras. Thus, a universal class axiomatized by $\mathcal{H}$-stable rules is $\mathcal{H}$-stable and so (2) follows from (1).

(3) The inclusion $\mathcal{U}(\mathcal{K}_{\mathsf{fin}}) \subseteq \mathcal{U}(\mathcal{K})$ is obvious. To see the reverse inclusion, let Γ/Δ be a multi-conclusion rule that is refuted in $\mathcal{U}(\mathcal{K})$. Then there is $A \in \mathcal{K}$ and a valuation v on A such that (A, v) refutes Γ/Δ. Let (A', v') be a $\mathcal{H}$-filtration of (A, v) through $\mathsf{Sub}(\Gamma \cup \Delta)$. Then A' refutes Γ/Δ and $A' \in \mathcal{K}$ since A' is finite and $\mathcal{K}$ is finitely $\mathcal{H}$-stable. Thus, $A' \in \mathcal{K}_{\mathsf{fin}}$, and so $\mathcal{U}(\mathcal{K}_{\mathsf{fin}})$ refutes Γ/Δ. □

3.3.8. THEOREM. *Let $\mathcal{U}$ be a universal class of Heyting algebras. The following are equivalent:*

(1) $\mathcal{U}$ is $\mathcal{H}$-stable.

(2) $\mathcal{U}$ is generated by an $\mathcal{H}$-stable class.

(3) $\mathcal{U}$ is generated by a finitely $\mathcal{H}$-stable class.

(4) $\mathcal{U}$ is generated by an $\mathcal{H}$-stable class of finite Heyting algebras.

(5) $\mathcal{U}$ is axiomatized by $\mathcal{H}$-stable rules over $\mathcal{S}_{\mathsf{IPC}}$, i.e. $\mathcal{S}(\mathcal{U})$ is an $\mathcal{H}$-stable consequence relation.

Proof:
The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial. To see that (3) $\Rightarrow$ (4) suppose that $\mathcal{U}$ is generated by $\mathcal{K}$ and that $\mathcal{K}$ is finitely $\mathcal{H}$-stable. Then $\mathcal{U}(\mathcal{K}) = \mathcal{U}(\mathcal{K}_{\mathsf{fin}})$ by Lemma 3.3.7(3). Thus, $\mathcal{U}(\mathcal{K})$ is generated by an $\mathcal{H}$-stable class of finite algebras. The implication from (4) $\Rightarrow$ (5) follows from Lemma 3.3.7(2). Finally, (5) $\Rightarrow$ (1) can be proved the same way as Lemma 3.3.7(2). □

3.3.2 $\mathcal{H}$-stable logics

We now move to $\mathcal{H}$-stable logics. As in the previous section, our goal is to provide equivalent characterizations of these classes of logics in a uniform way.

In Section 3.4 we show that $\{\wedge, \to\}$-stable logics are nothing but subframe si logics and $\{\wedge, \to, \bot\}$-stable logics are cofinal subframe logics. In Section 3.5 we illustrate that $\{\wedge, \vee, \bot, \top\}$-stable logics are exactly the stable logics of [17].

3.3.9. Definition. A si logic is called *$\mathcal{H}$-stable* iff its corresponding variety is generated by an $\mathcal{H}$-stable class of Heyting algebras.

Note that the variety corresponding to an $\mathcal{H}$-stable logic is required to be *generated* by an $\mathcal{H}$-stable class and may itself not be $\mathcal{H}$-stable.

We define $\mathcal{H}$-stable formulas as the characteristic formulas of $\mathcal{H}$-stable rules (see Section 2.3.7).

3.3.10. Definition. Let B be a finite subdirectly irreducible Heyting algebra and let $\delta_{\mathcal{H}}(B) = \Gamma/\Delta$ as in Definition 3.3.4. The *$\mathcal{H}$-stable formula of B* is

$$\gamma_{\mathcal{H}}(B) = \bigwedge_{\gamma \in \Gamma} \gamma \to \bigvee_{\delta \in \Delta} \delta.$$

In other words, $\gamma_{\mathcal{H}}(B) = \chi(\delta_{\mathcal{H}}(B))$, where $\chi(\Gamma/\Delta)$ is the characteristic formula of Γ/Δ as in Section 2.3.7.

3.3.11. Remark. If $\mathcal{H}$ is $\{\wedge, \to\}$ or $\{\wedge, \to, \bot\}$, $\gamma_{\mathcal{H}}(B)$ is equivalent to the subframe respectively cofinal subframe formula of the dual frame of B, as we will discuss in Section 3.4. If $\mathcal{H}$ is $\{\wedge, \vee, \bot, \top\}$ then $\gamma_{\mathcal{H}}(B)$ is precisely the stable formula of B from [17, Section 6]. These formulas are also similar to the ones considered in [121, Section 4.1.4]

3.3.12. Remark. As follows from Remark 3.3.5, by considering B as a partial Heyting algebra where all operations from $\mathcal{H}$ are defined, $\gamma_{\mathcal{H}}(B)$ is equivalent to the characteristic formula $CF(B)$, and if $\to \in \mathcal{H}$, then $\gamma_{\mathcal{H}}(B)$ is equivalent to the weak characteristic formula $WCF(B)$ from [43, Section 3.2].

The following describes the semantic refutation criterion for $\mathcal{H}$-stable formulas.

3.3.13. Lemma. *Let A and B be Heyting algebras such that B is finite and subdirectly irreducible. Then*

$A \not\models \gamma_{\mathcal{H}}(B)$ iff B is isomorphic to an $\mathcal{H}$-subalgebra of a subdirectly irreducible homomorphic image of A.

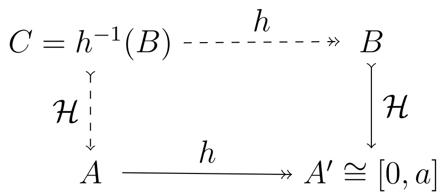
In particular, if A is subdirectly irreducible and B is isomorphic to an $\mathcal{H}$-subalgebra of A, then $A \not\models \gamma_{\mathcal{H}}(B)$.

Proof:
By Lemma 2.3.11, we have that $A \not\models \gamma_{\mathcal{H}}(B)$ iff there is a subdirectly irreducible homomorphic image C of A with $C \not\models \delta_{\mathcal{H}}(B)$. And $C \not\models \delta_{\mathcal{H}}(B)$ iff B is isomorphic to an $\mathcal{H}$-subalgebra of C by Lemma 3.3.6. □

The next lemma will become handy when proving properties of $\mathcal{H}$-stable formulas. For a class $\mathcal{K}$ of Heyting algebras, let $\mathbf{S}_{\mathcal{H}}(\mathcal{K})$ denote the collection of Heyting algebras that are $\mathcal{H}$-subalgebras of the algebras in $\mathcal{K}$. Using this terminology, the following lemma states $\mathbf{S}_{\mathcal{H}}\mathbf{H}(A) \subseteq \mathbf{H}\mathbf{S}_{\mathcal{H}}(A)$ whenever A is finite. In fact, for $\mathcal{H} = \{\wedge, \to\}$ or $\mathcal{H} = \{\wedge, \to\}$, it is well-known that $\mathbf{S}_{\mathcal{H}}\mathbf{H}(A) = \mathbf{H}\mathbf{S}_{\mathcal{H}}(A)$ for any Heyting algebra A. We included this case in the lemma below in order to stay self-contained and to demonstrate that the proofs for the different cases are analogous.

3.3.14. Lemma. *Suppose A is a finite Heyting algebra and B is isomorphic to an $\mathcal{H}$-subalgebra of a homomorphic image A' of A. Then there is an $\mathcal{H}$-subalgebra C of A such that B is a homomorphic image of C.*

Proof:
Suppose that B is an $\mathcal{H}$-subalgebra of a homomorphic image A' of A. Since A is finite, all filters on A are principal. We can thus assume that $A' \cong [0, a]$ for some $a \in A$ and the map $h : A \to [0, a]$ is defined by $h(b) = a \wedge b$ for $a \in A$. Moreover, we can identify B with an $\mathcal{H}$-subalgebra of $[0, a]$. Let $C := h^{-1}(B)$. Then C is an $\mathcal{H}$-subalgebra of A, and since C is finite, it is also a Heyting algebra. Moreover, the restriction of h to C—that we will also denote by h for simplicity—is an $\mathcal{H}$-homomorphism from C onto B. In a diagram,



We show that $h : C \to B$ is a Heyting algebra homomorphism. Since $\wedge \in \mathcal{H}$, h is order-preserving and since $\top \in \mathcal{H}$, h preserves 1. Moreover, this shows that $a \in C$. We prove that h preserves 0. This is obvious if $\bot \in \mathcal{H}$, but also if $\bot \notin \mathcal{H}$, there is $c \in C$ with $h(c) = 0_B$ since h is onto, and so $h(0_C) = 0_B$ since h is order-preserving. We further distinguish two cases depending whether $\to \in \mathcal{H}$ or $\vee \in \mathcal{H}$.

Case 1: $\to \in \mathcal{H}$. We show that h preserves joins. Let x, y be elements of C. Since C is finite and h preserves meets, we have

$$\begin{aligned} h(x \vee_C y) &= h\left(\bigwedge_C \{c \in C \mid x, y \leq c\}\right) \\ &= \bigwedge_B \{h(c) \mid c \in C \text{ and } x, y \leq c\} \\ &= \bigwedge_B \{c \wedge a \mid c \in C \text{ and } x, y \leq c\}. \end{aligned}$$

On the other hand,

$$h(x) \vee_B h(y) = \bigwedge_B \{z \in B \mid h(x), h(y) \leq z\} = \bigwedge_B \{z \in B \mid x \wedge a, y \wedge a \leq z\}.$$

To conclude that h is a Heyting algebra homomorphism it suffices to show that

$$\{c \wedge a \mid c \in C \text{ and } x, y \leq c\} = \{z \in B \mid x \wedge a, y \wedge a \leq z\}.$$

The inclusion "$\subseteq$" is easy to see. For the reverse inclusion, let $z \in B$ with $x \wedge a \leq z$ and $y \wedge a \leq z$. Since z and a are elements of C, we have $x \leq a \to_C z$ and $y \leq a \to_C z$. Moreover, $z \leq a$, so $(a \to_C z) \wedge a = z$. Thus, $(a \to_C z)$ witnesses that z is an element of $\{c \wedge a \mid c \in C \text{ and } x, y \leq c\}$. This finishes the proof.

Case 2: $\vee \in \mathcal{H}$. We show that h preserves Heyting implications. Let x, y be elements of C. Since C is finite and h preserves joins,

$$h(x \to_C y) = \bigvee_B \{h(c) \mid c \in C \text{ and } x \wedge c \leq y\} = \bigvee_B \{c \wedge a \mid c \in C \text{ and } x \wedge c \leq y\}$$

and

$$h(x) \to_B h(y) = \bigvee_B \{z \in B \mid (x \wedge a) \wedge z \leq y\}.$$

It suffices to show that

$$\{c \wedge a \mid c \in C \text{ and } x \wedge c \leq y\} = \{z \in B \mid (x \wedge a) \wedge z \leq y\}.$$

The inclusion "$\subseteq$" is easy to see. For the reverse inclusion, let $z \in B$ with $(x \wedge a) \wedge z \leq y$. Since $z \in B \subseteq [0, a]$, we see that $z = z \wedge a = h(z)$, so $z \in C$ and $x \wedge z = x \wedge (a \wedge z) \leq y$. This shows that $\{z \in B \mid (x \wedge a) \wedge z \leq y\} \subseteq \{c \wedge a \mid c \in C \text{ and } x \wedge c \leq y\}$. Therefore, $h(x \to_C y) = h(x) \to_B h(y)$. This finishes the proof.

□

3.3.15. Lemma. *If* L *is* $\mathcal{H}$*-stable and* B *is a finite subdirectly irreducible Heyting algebra, then* $B \not\models \mathsf{L}$ *iff* $\gamma_{\mathcal{H}}(B) \in \mathsf{L}$.

Proof:
For the direction from right to left, suppose $B \models \mathsf{L}$. By Lemma 3.3.13, $B \not\models \gamma_{\mathcal{H}}(B)$, so $\gamma_{\mathcal{H}}(B) \notin \mathsf{L}$. We show the converse. Since L is $\mathcal{H}$-stable, there is an $\mathcal{H}$-stable class $\mathcal{K}$ of Heyting algebras that generates $\mathcal{V}(\mathsf{L})$, i.e. $\mathcal{V}(\mathsf{L}) = \mathcal{V}(\mathcal{K})$. Suppose that $\gamma_{\mathcal{H}}(B) \notin \mathsf{L}$. By Lemma 3.3.7(3), we have $\mathcal{V}(\mathcal{K}) = \mathcal{V}(\mathcal{K}_{\mathsf{fin}})$, so there is $A \in \mathcal{K}_{\mathsf{fin}}$, with $A \not\models \gamma_{\mathcal{H}}(B)$. By Lemma 3.3.13, there is a subdirectly irreducible homomorphic image A' of A so that B is an $\mathcal{H}$-subalgebra of A'. By Lemma 3.3.14, there is an $\mathcal{H}$-subalgebra C of A such that B is a homomorphic image of C. Since $\mathcal{K}_{\mathsf{fin}}$ is $\mathcal{H}$-stable, $C \in \mathcal{K}_{\mathsf{fin}}$. Since B is a homomorphic image of C, we have $B \in \mathcal{V}(\mathcal{K}_{\mathsf{fin}})$. Therefore, $B \models \mathsf{L}$. □

3.3.16. Remark. At first sight, it may seem that [43, Theorem 3.13, Corollary 3.14] would provide a simpler proof of the lemma above. These results, however, are in general not applicable to our case since the formula $CF(B)$ used in the aforementioned theorem will in general be different from $\gamma_{\mathcal{H}}(B)$.

We are ready to prove our characterization results of $\mathcal{H}$-stable si logics. The algebraic content of the theorem describes in what way stability of a class can be "shifted". For instance, if a variety is generated by *some* $\mathcal{H}$-stable class, then the theorem tells us that also the subdirectly irreducible algebras of that variety form a $\mathcal{H}$-stable class (within the subdirectly irreducible Heyting algebras). In this sense, the stability condition can be "shifted" from an arbitrary class to the subdirectly irreducible ones.

Recall that for a class $\mathcal{K}$ of Heyting algebras, by $\mathcal{K}_{\mathsf{si}}$, we denote the subdirectly irreducible members of $\mathcal{K}$. Moreover, by $\mathcal{K}_{\mathsf{fsi}}$ we denote the class of finite subdirectly irreducible members of $\mathcal{K}$.

3.3.17. Theorem. *Let* L *be a si logic. The following are equivalent.*

(1) L *is* $\mathcal{H}$*-stable.*

(2) $\mathcal{V}(\mathsf{L})$ *is generated by a finitely* $\mathcal{H}$*-stable class.*

(3) $\mathcal{V}(\mathsf{L})$ *is generated by an* $\mathcal{H}$*-stable class of finite Heyting algebras.*

(4) $\mathcal{V}(\mathsf{L})$ *is generated by an* $\mathcal{H}$*-stable universal class of Heyting algebras.*

(5) L *is axiomatizable over* IPC *by* $\mathcal{H}$*-stable rules of finite Heyting algebras.*

(6) $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ *is* $\mathcal{H}$*-stable within* $\mathsf{HA}_{\mathsf{si}}$*.*

(7) $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ *is finitely* $\mathcal{H}$*-stable within* $\mathsf{HA}_{\mathsf{si}}$*.*

(8) $\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ *is* $\mathcal{H}$*-stable within* $\mathsf{HA}_{\mathsf{si}}$ *and generates* $\mathcal{V}(\mathsf{L})$*.*

If $\vee \in \mathcal{H}$*, then any of the above is equivalent to* $\mathcal{V}(\mathsf{L})_{\mathsf{wc}}$ *is* $\mathcal{H}$*-stable. Moreover, if* L *is* $\mathcal{H}$*-stable, then* L *is axiomatizable by* $\mathcal{H}$*-stable formulas.*

Proof:

We first show that the first five statements are equivalent. The implication (1) $\Rightarrow$ (2) is obvious since every $\mathcal{H}$-stable class is finitely $\mathcal{H}$-stable. To see (2) $\Rightarrow$ (3) suppose that $\mathcal{V}(\mathsf{L})$ is generated by the finitely $\mathcal{H}$-stable class $\mathcal{K}$. We have

$$\mathcal{V}(\mathcal{K}) = \mathcal{V}(\mathcal{U}(\mathcal{K})) = \mathcal{V}(\mathcal{U}(\mathcal{K}_{\mathsf{fin}})) = \mathcal{V}(\mathcal{K}_{\mathsf{fin}}),$$

where the second equality follows from Lemma 3.3.7(3). Since $\mathcal{K}_{\mathsf{fin}}$ is an $\mathcal{H}$-stable class of finite algebras, (3) follows. To see (3) $\Rightarrow$ (4), let $\mathcal{K}$ be an $\mathcal{H}$-stable class of finite Heyting algebras that generates $\mathcal{V}(\mathsf{L})$. Then $\mathcal{V}(\mathcal{K}) = \mathcal{V}(\mathcal{U}(\mathcal{K}))$. Now $\mathcal{U}(\mathcal{K})$ is a universal class and $\mathcal{H}$-stable by Lemma 3.3.7(2). Thus, (4) holds.

For the implication (4) $\Rightarrow$ (5), suppose that $\mathcal{V}(\mathsf{L}) = \mathcal{V}(\mathcal{U})$, where $\mathcal{U}$ is an $\mathcal{H}$-stable universal class. By Theorem 3.3.8, there is a set Ψ of $\mathcal{H}$-stable rules that axiomatize $\mathcal{U}$ over $\mathcal{S}_{\mathsf{IPC}}$. The same set of rules then axiomatizes L over IPC. Finally, to see that (5) implies (1), suppose that L is axiomatized over IPC by the set Ψ of $\mathcal{H}$-stable rules. Then the universal class of Heyting algebras that validate Ψ is $\mathcal{H}$-stable and generates $\mathcal{V}(\mathsf{L})$. Thus, L is $\mathcal{H}$-stable. This finishes the proof of the first five equivalences.

Next we show (1) $\Rightarrow$ (6). Suppose that L is $\mathcal{H}$-stable, let $A \in \mathcal{V}(\mathsf{L})_{\mathsf{si}}$, and suppose that A' is a subdirectly irreducible $\mathcal{H}$-subalgebra of A. We need to show that $A' \in \mathcal{V}(\mathsf{L})$. Suppose not. Then there is a formula φ such that $\varphi \in \mathsf{L}$, but $(A', v) \not\models \varphi$ for some valuation v on A'. By Theorem 3.2.5(1) there is an $\mathcal{H}$-filtration (B, v_B) of (A', v) through φ with B finite and subdirectly irreducible. Then $(B, v_B) \not\models \varphi$, so $B \not\models \mathsf{L}$. By Lemma 3.3.15, $\gamma_{\mathcal{H}}(B) \in \mathsf{L}$. However, since B is isomorphic to an $\mathcal{H}$-subalgebra of A, and A is subdirectly irreducible, we have $A \not\models \gamma_{\mathcal{H}}(B)$ by Lemma 3.3.13. The latter contradicts to $A \models \mathsf{L}$. Thus, $A' \in \mathcal{V}(\mathsf{L})$ and so (6) holds.

The implication (6) $\Rightarrow$ (7) is obvious. To see (7) $\Rightarrow$ (8), observe that if $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ is finitely $\mathcal{H}$-stable in $\mathsf{HA}_{\mathsf{si}}$, then $\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ is $\mathcal{H}$-stable in $\mathsf{HA}_{\mathsf{si}}$. $\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ also generates

$\mathcal{V}(\mathsf{L})$. Indeed, if $\varphi \notin \mathsf{L}$, then there is $A \in \mathcal{V}(\mathsf{L})_{\mathsf{si}}$ and a valuation with v on A with $(A, v) \not\models \varphi$. By Theorem 3.2.5(1), there is a finite $\mathcal{H}$-filtration (B, v_B) of (A, v) through $\mathsf{Sub}(\varphi)$ with B subdirectly irreducible. Then $B \in \mathcal{V}(\mathsf{L})_{\mathsf{si}}$ since the latter is finitely $\mathcal{H}$-stable. Since $B \in \mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$, this shows that $\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ generates $\mathcal{V}(\mathsf{L})$. The implication (8) $\Rightarrow$ (1) is obvious.

Finally, suppose that L is $\mathcal{H}$-stable. We show that L is axiomatized by

$$\Gamma := \{\gamma_{\mathcal{H}}(B) \mid B \in \mathsf{HA}_{\mathsf{fsi}} \setminus \mathcal{V}(\mathsf{L})\}.$$

If $B \notin \mathcal{V}(\mathsf{L})$, then $\gamma_{\mathcal{H}}(B) \in \mathsf{L}$ by Lemma 3.3.15, so $\mathsf{IPC} + \Gamma \subseteq \mathsf{L}$. Conversely, suppose that $\varphi \in \mathsf{L}$, and suppose there is a subdirectly irreducible Heyting algebra A, such that $A \models \Gamma$ but A refutes φ. By Theorem 3.2.5 there is a finite subdirectly irreducible $\mathcal{H}$-subalgebra B of A such that B refutes φ. So $B \not\models \mathsf{L}$, and therefore $\gamma_{\mathcal{H}}(B) \in \Gamma$. Since A is subdirectly irreducible, A refutes $\gamma_{\mathcal{H}}(B)$, but this contradicts to A validating Γ. □

The above considerations have two immediate consequences.

3.3.18. Corollary.

(1) $\mathcal{H}$-stable si logics have the fmp.

(2) $\mathcal{H}$-stable si logics form a complete sublattice of Λ_{IPC}.

Proof:
(1) follows from Theorem 3.3.17(8). To see (2), let $\{\mathsf{L}_i \mid i \in I\}$ be a family of $\mathcal{H}$-stable logics. Let $\mathcal{K}_i \subseteq \mathcal{V}(\mathsf{L}_i)$ be $\mathcal{H}$-stable and generate $\mathcal{V}(\mathsf{L}_i)$. Then $\bigcup_{i\in I} \mathcal{K}_i$ is $\mathcal{H}$-stable and generates $\bigwedge_i \mathsf{L}_i$, so $\bigwedge_i \mathsf{L}_i$ is $\mathcal{H}$-stable and so $\mathcal{H}$-stable logics are a $\bigwedge$-sublattice of Λ_{IPC}.

By Theorem 3.3.17, $\mathcal{V}(\mathsf{L}_i)_{\mathsf{si}}$ is $\mathcal{H}$-stable within $\mathsf{HA}_{\mathsf{si}}$ for all $i \in I$. Therefore, also $\bigcap_{i\in I} \mathcal{V}(\mathsf{L}_i)_{\mathsf{si}}$ is $\mathcal{H}$-stable within $\mathsf{HA}_{\mathsf{si}}$. Now $\bigcap_{i\in I} \mathcal{V}(\mathsf{L}_i)_{\mathsf{si}}$ is the collection of all subdirectly irreducible $\bigvee_{i\in I} \mathsf{L}_i$-algebras. Thus, $\bigvee_{i\in I} \mathsf{L}_i$ is $\mathcal{H}$-stable. So $\mathcal{H}$-stable logics are also a $\bigvee$-sublattice of Λ_{IPC}. □

We conclude by pointing out some specifics in the above theorem depending on $\mathcal{H}$. In general, we will not be able to prove that logics axiomatized by $\mathcal{H}$-stable formulas are $\mathcal{H}$-stable. As we will see in Section 3.5.2 there are si logics axiomatized by $\{\wedge, \vee, \neg, \top\}$-stable formulas that are not $\{\wedge, \vee, \neg, \top\}$-stable. However, for the other reducts under consideration, logics axiomatized by $\mathcal{H}$-stable formulas are $\mathcal{H}$-stable.

Case $\to \in \mathcal{H}$: As already mentioned, such $\mathcal{H}$-stable si logics are equivalent to subframe and cofinal subframe si logics. In that case, the conditions of Theorem 3.3.17 are equivalent to the fact that the variety $\mathcal{V}(\mathsf{L})$ is an $\mathcal{H}$-stable class. In other words, in this case, $\mathcal{H}$-stability is preserved by varieties and not only universal classes as in Theorem 3.3.8.

Case $\vee, \top \in \mathcal{H}$**:** In this case, $\mathcal{H}$-stability is in general not preserved by varieties. We discuss this in Section 3.5.

However, some other statements in Theorem 3.3.17 can be simplified. The reason is the following lemma (see Section 2.1.2 for the definition and characterization of well-connected Heyting algebras).

3.3.19. LEMMA ([17]). *If* $\vee, \top \in \mathcal{H}$*, then a finite* $\mathcal{H}$*-subalgebra of a well-connected Heyting algebra is subdirectly irreducible.*

Proof:
Let A be a well-connected Heyting algebra and let B be a finite $\mathcal{H}$-subalgebra of A. For $a, b \in B$, if $a \vee_B b = 1_B$, then $a \vee_A b = 1_A$ as $\vee, \top \in \mathcal{H}$. Since A is well-connected $a = 1_A$ or $b = 1_A$. And therefore, $a = 1_B$ or $b = 1_B$. This shows that B is also well-connected. Since B is well-connected and finite, B is subdirectly irreducible by Lemma 2.1.19. □

In light of Lemma 3.3.19, the formulations "$\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ is finitely $\mathcal{H}$-stable within $\mathsf{HA}_{\mathsf{si}}$" of (7) and "$\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ is $\mathcal{H}$-stable within $\mathsf{HA}_{\mathsf{si}}$ and generates $\mathcal{V}(\mathsf{L})$" of (8) can be simplified to "$\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ is finitely $\mathcal{H}$-stable" and "$\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ $\mathcal{H}$-stable and generates $\mathcal{V}(\mathsf{L})$", respectively. The reason is that finite $\mathcal{H}$-stable subalgebras of subdirectly irreducible ones are subdirectly irreducible. However, the formulation in (6) of Theorem 3.3.17 cannot be replaced by "$\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ is $\mathcal{H}$-stable". The latter will in general not be true since subalgebras of infinite subdirectly irreducible algebras are in general not subdirectly irreducible.

Another property that we have in this case is that $\mathcal{H}$-filtrations of well-connected Heyting algebras are automatically well-connected. Thus, with the same reasoning as in the proof of Theorem 3.3.17 we can prove the following lemma. For a class $\mathcal{K}$ of Heyting algebras, let $\mathcal{K}_{\mathsf{wc}}$ denote the class of well-connected members of $\mathcal{K}$.

3.3.20. LEMMA. *If* $\vee, \top \in \mathcal{H}$*, then a si logic* L *is* $\mathcal{H}$*-stable iff* $\mathcal{V}(\mathsf{L})_{\mathsf{wc}}$ *is* $\mathcal{H}$*-stable.*

3.3.21. REMARK. In the above, we always assumed that $\mathcal{H}$ stands for one of the reducts $\{\wedge, \to\}$, $\{\wedge, \to, \bot\}$, $\{\wedge, \vee, \bot, \top\}$ or $\{\wedge, \vee, \neg, \top\}$ of Heyting algebras. As demonstated in [90, 26] even weaker reducts of Heyting algebras give rise to interesting si logics and multi-conclusion consequence relations from a proof-theoretic and lattice-theoretic perspectives, respectively.

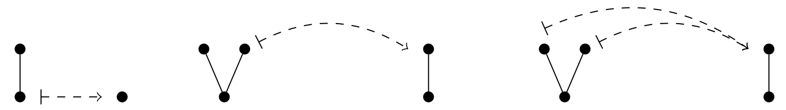


Figure 3.4.1: subred. (a); cof. subred. (b); global cof. subred.(c)

3.4 Subframe and cofinal subframe si logics

As mentioned earlier, the $\mathcal{H}$-stable si logics for $\mathcal{H} = \{\wedge, \to\}$ and $\mathcal{H} = \{\wedge, \to, \top\}$, are the well known classes of subframe and cofinal subframe si logics, respectively.

In this section we recall the original (frame-theoretic) definitions of subframe and cofinal subframe logics and properties of these classes from [133, 40]. Moreover, we discuss the algebraic perspective on subframe and cofinal subframe logics of [121], [24] and [13].

We recall the notions of (cofinal) subreductions, subframe and cofinal subframe formulas and si logics and list some of their desirable properties.

3.4.1. Definition. Let X and Y be Esakia spaces. Let $f : X \to Y$ be a partial and onto map, and let $\mathrm{dom}(f)$ denote the domain of f. The map f is called a *subreduction* iff the following conditions are satisfied for all $x, x' \in X$ and $y \in Y$:

(SR1) $x \leq x'$ implies $f(x) \leq f(x')$ (when defined).

(SR2) If $f(x) \leq y$, then there is $x \leq x' \in \mathrm{dom}(f)$ with $f(x') = y$.

(SR3) If $Y \setminus U \in \mathsf{ClUp}(Y)$, then ${\downarrow} f^{-1}(U) \in \mathsf{Clop}(X)$.

A subreduction is called a *cofinal subreduction* iff for every point $x \in X$, $x \in {\uparrow}\mathrm{dom}(f)$ implies $x \in {\downarrow}\mathrm{dom}(f)$. And a subreduction is called a *global cofinal subreduction* iff ${\downarrow}\mathrm{dom}(f) = X$. If there is a subreduction/(global) cofinal subreduction from X onto Y we call Y a *subreduct/(global) cofinal subreduct* of X.

Note that every global cofinal subreduction is a cofinal subreduction but the converse is not true in general. Using the fact that in an Esakia space every point sees a maximal point (Theorem 2.2.3(6)), one can show that a subreduction $f : X \to Y$ is cofinal iff $\max{\uparrow}\mathrm{dom}(f) \subseteq \mathrm{dom}(f)$ and it is a global cofinal subreduction iff $\max X \subseteq \mathrm{dom}(f)$. See Figure 3.4.1 for some simple examples.

3.4.2. Remark. In fact, in his original definition Zakharyaschev only uses (cofinal) subreductions (see [40]). We follow the terminology of Jeřábek [83] to call subreduction with ${\downarrow}\,\mathrm{dom}(f) = X$ global cofinal subreductions. As Jeřábek points out in [83, Remark 3.7], of these two notions, global cofinal subreductions seem to be the more fundamental concept. The latter is also reflected in the algebraic perspective.

3.4.3. DEFINITION. An Esakia space Y is a *subframe/ (global) cofinal subframe* of an Esakia space X iff Y is a subpartial order of X and the identity map is a subreduction/ (global) cofinal subreduction.

Thus, subframes and cofinal subframes are the domains of subreductions and cofinal subreductions, respectively. An alternative, sometimes more convenient, characterization of subframes can be found in [24, Lemma 2]:

3.4.4. LEMMA. *A subspace Y of an Esakia space X is a subframe iff Y is a closed subspace and $U \in \mathsf{Clop}(\mathsf{Y})$ implies ${\downarrow}U \in \mathsf{Clop}(\mathsf{X})$.*

Next we recall the definition of subframe formulas and cofinal subframe formulas.

3.4.5. DEFINITION. If Y is a finite rooted frame with root r, let p_y for $y \in Y$ be a propositional letter. The *subframe formula* $\beta(Y)$ of Y is

$$\beta(Y) = \bigwedge_{x \leq y} [(\bigwedge_{y \nleq z} p_z \to p_y) \to p_x] \to p_r,$$

and the *cofinal subframe formula* $\beta(Y, \bot)$ of Y is

$$\beta(Y, \bot) = \bigwedge_{x \leq y} [(\bigwedge_{y \nleq z} p_z \to p_y) \to p_x] \wedge [\bigwedge_{x,z} (\bigwedge_{x \nleq z} p_z \to p_x) \to \bot] \to p_r.$$

The following lemma illustrates the semantic content of subframe and cofinal subframe formulas (see [40, Theorem 9.39]).

3.4.6. LEMMA. *Let X and Y be Esakia spaces and let Y be finite and rooted.*

(1) $X \not\models \beta(Y)$ iff Y is a subreduct of X.

(2) $X \not\models \beta(Y, \bot)$ iff Y is a cofinal subreduct of X.

3.4.7. DEFINITION. A si logic L is a (*cofinal*) *subframe logic* iff it is axiomatizable by (cofinal) subframes formulas.

A proof of the following theorem can be found in [40, Section 11.3].

3.4.8. THEOREM (ZAKHARYASCHEV). *For a si logic L, the following are equivalent:*

(1) L is a (cofinal) subframe logic.

(2) L *is characterized by a class of Esakia spaces closed under (cofinal) subframes.*

(3) The class of Esakia spaces of L *is closed under (cofinal) subframes.*

(4) L *is axiomatizable by* $\{\wedge, \to\}$*-formulas* ($\{\wedge, \to, \bot\}$*-formulas*).

Subframe and cofinal subframe si logics have many desirable properties some of which we collect in the theorem below. A (stream-lined) proof of the theorem can be found in [40, Section 11]. However, many of the properties have already been proven earlier for logics axiomatized by disjunction-free formulas. In particular, as we saw in Section 3.2, the finite model property of si logics axiomatized by disjunction free axioms was proven by McKay [97]. The elementarity of such logics was proven independently by Chagrova [41] and Rodenburg [114]. In [119] Shimura provided a direct proof of canonicity of subframe and cofinal subframe logics and thus of strong Kripke completeness.

3.4.9. THEOREM. *If* L *is a subframe logic or cofinal subframe si logic, then*

(1) L *has the fmp.*

(2) L *is Kripke complete.*

(3) L *is canonical.*

(4) L *is elementary.*

There is a continuum of subframe logics and a continuum of cofinal subframe si logics that are not subframe si logics.

3.4.1 Algebraic perspective via nuclei

As shown in [24], algebraically subframes correspond to nuclei on Heyting algebras. We here recall the main results from [24].

Nuclei on Heyting algebras

We recall the definition and properties of nuclei on Heyting algebras. Nuclei are ubiquitous notion in the area of non-classical logics, for more information the reader is referred to [62] or [84].

3.4.10. DEFINITION. Let A be a Heyting algebra. A *nucleus* is a map $j : A \to A$ such that for all $a \in A$,

(1) $a \leq j(a)$,

(2) $jj(a) \leq j(a)$, and

(3) $j(a \wedge b) = j(a) \wedge j(b)$.

In other words, a nucleus is a closure operator that commutes with meets. A nucleus j is called *locally dense* if $j(\neg j(0)) = 1$, and it is called *dense* if $j(0) = 0$. Note that a dense nucleus is also locally dense.

It is well known that if j is a nucleus on the Heyting algebra A, then the j-fixed points form a Heyting algebra that is a $\{\wedge, \to\}$-subalgebra of A. We recall the details. Let

$$A_j = \{a \in A \mid j(a) = a\} = \{j(a) \mid a \in A\},$$

let $a \vee_j b = j(a \vee b)$ for $a, b \in A_j$, and $0_j = j(0)$, and let the operations $\to, \wedge, 1$ on A_j be defined as on A. Let $\mathbf{A}_j = (A_j, \wedge, \vee_j, \to, 0_j, 1)$. A proof of the following lemma can be found in [24, Proposition 7].

3.4.11. Lemma. *If A is a Heyting algebra, and j a nucleus on A, then A_j is a Heyting algebra. By definition A_j is a $\{\wedge, \to\}$-subalgebra of A that and if j is dense, then A_j is a $\{\wedge, \to, \bot\}$-subalgebra of A.*

Not every $\{\wedge, \to\}$-subalgebra of a Heyting algebra A is of the shape A_j for a nucleus j on A. In fact, a $\{\wedge, \to\}$-subalgebra B of A gives rise to a nucleus iff it is *total*, i.e. for each $a \in A$ and each $b \in B$, $a \to b \in B$, and for each $a \in A$, the set $\{b \in B \mid a \leq b\}$ has a least element [95, Theorem 5.8].

3.4.12. Definition. A class $\mathcal{K}$ of Heyting algebras is called *locally dense nuclear/dense nuclear/nuclear* iff $A_j \in \mathcal{K}$ for every $A \in \mathcal{K}$ and locally dense nucleus/ dense nucleus/ nucleus j on A.

Nuclei and subframes

As shown in [24], subframes can algebraically be captured by nuclei.

3.4.13. Theorem ([24]). *Let A be a Heyting algebra with dual Esakia space X. There are one-to-one correspondences between*

(1) subframes on X and nuclei on A,

(2) cofinal subframes on X and locally dense nuclei on A, and

(3) globally cofinal subframes on X and dense nuclei on A.

The correspondences are obtained as follows. If Y is a subframe of X, then j defined on $\mathcal{CU}(X)$ defined by

$$j(U) = X \setminus {\downarrow}(Y \setminus U)$$

for every clopen upset U of X is a nucleus on A, and every nucleus on A is given this way. Conversely, if j is a nucleus on A, then

$$Y = \{x \in \mathcal{PF}(A) \mid j^{-1}(x) = x\}$$

gives rise to a subframe of X, and every subframe is given that way. The correspondence links global cofinal subframes with dense nuclei and cofinal subframes with locally dense nuclei.

As was shown in [24, Theorem 10], a variety is dense nuclear iff it is locally dense nuclear. Then using the correspondence from Theorem 3.4.13 and Theorem 3.4.8 the following is immediate:

3.4.14. Theorem ([24]). *A logic* L *is a (cofinal) subframe logic iff* $\mathcal{V}(\mathsf{L})$ *is (dense or locally dense) nuclear.*

3.4.2 Algebraic perspective via the $\{\wedge, \to\}$-reduct

An alternative algebraic analysis of subframe and cofinal subframe si logics can also be given via the $\{\wedge, \to\}$- and $\{\wedge, \to, \bot\}$-reducts of Heyting algebras, respectively. The results below can be found in either of the references [40], [121, Sections 3.2.3 and 4.1.6], [24, Section 8], and [13, Section 5.4]. We recall the details.

First we recall the definition of subframe and cofinal subframe formulas from an algebraic perspective. Let B be a finite subdirectly irreducible Heyting algebra with second largest element s, and let for each $b \in B$, p_b be a variable. Define

$$\chi(B) = \bigwedge_{a,b \in B} (p_{a\wedge b} \leftrightarrow (p_a \wedge p_b)) \wedge \bigwedge_{a,b \in B} (p_{a\to b} \leftrightarrow (p_a \to p_b)) \to p_s, \text{ and}$$

$$\chi(B, \bot) = \bigwedge_{a,b \in B} (p_{a\wedge b} \leftrightarrow (p_a \wedge p_b)) \wedge \bigwedge_{a,b \in B} (p_{a\to b} \leftrightarrow (p_a \to p_b)) \wedge \bigwedge_{b \in B} (p_{\neg p} \leftrightarrow \neg p_a) \to p_s.$$

The semantic meaning of these formulas is described by the following lemma:

3.4.15. Lemma.

(1) $A \not\models \chi(B)$ *iff* B *is a* $\{\wedge, \to\}$*-subalgebra of a homomorphic image of* A.

(2) $A \not\models \chi(B, \bot)$ *iff* B *is a* $\{\wedge, \to, \bot\}$*-subalgebra of a homomorphic image of* A.

In light of the lemma below, $\chi(B)$ and $\chi(B, \bot)$ are the algebraic versions of $\beta(Y)$ and $\beta(Y, \neg)$ (see Definition 3.4.5), respectively, where Y is the dual frame of B. For a proof see [40, Theorem 9.7] and [40, Exercise 9.2]).

3.4.16. LEMMA. *Let X and Y be Esakia spaces with dual Heyting algebras A and B, respectively.*

(1) Y is a subreduct of X iff B is isomorphic to a $\{\wedge, \to\}$-subalgebra of A.

(2) If Y is a cofinal subreduct of X then B is isomorphic to a $\{\wedge, \to, \bot\}$-subalgebra of a homomorphic image of A.

As a consequence, an algebraic characterization of subframe and cofinal subframe logics is obtained.

3.4.17. PROPOSITION. *The following are equivalent:*

(1) L is a subframe or cofinal subframe si logic.

(2) L is axiomatizable by formulas of the shape $\chi(B)$ or $\chi(B, \bot)$, respectively.

(3) $\mathcal{V}(\mathsf{L})$ is a $\{\wedge, \to\}$-stable class or a $\{\wedge, \to, \bot\}$-stable class, respectively.

Relation to $\{\wedge, \to\}$- and $\{\wedge, \to, \bot\}$- stable logics. Finally, we show that subframe and cofinal subframe logic coincide with $\{\wedge, \to\}$- and $\{\wedge, \to, \bot\}$-stable logics from Definition 3.3.9, respectively. The formulas $\chi(B)$ and $\chi(B, \neg)$ are almost the same as the formulas $\gamma_{\{\wedge,\to\}}(B)$ and $\gamma_{\{\wedge,\to,\bot\}}(B)$ from Definition 3.3.10. The only difference is that the latter have a more complicated succedent than the former. However, as the next lemma shows the formulas are actually equivalent. The lemma is proved as [43, Proposition 3.6].

3.4.18. LEMMA. *Let B be a finite and subdirectly irreducible Heyting algebra.*

(1) $\vdash_{\mathsf{IPC}} \gamma_{\{\wedge,\to\}}(B) \leftrightarrow \chi(B)$.

(2) $\vdash_{\mathsf{IPC}} \gamma_{\{\wedge,\to,\bot\}}(B) \leftrightarrow \chi(B, \bot)$.

Proof:
We only show (2), the proof for (1) is similar. The formulas $\gamma_{\{\wedge,\to,\bot\}}(B)$ and $\chi(B, \bot)$ have a common antecedent that we call Γ. Then $\gamma_{\{\wedge,\to,\bot\}}(B) = \Gamma \to \bigvee\{p_a \leftrightarrow p_b \mid a \neq b \in B\}$ and $\chi(B, \bot) = \Gamma \to p_s$. By completeness it is enough to show that for every model $\mathfrak{M} = (\mathfrak{F}, v)$ and every world x of $\mathfrak{M}$, if $x \models_v \Gamma$ then

$$x \models_v p_s \text{ iff } x \models_v \bigvee\{p_a \leftrightarrow p_b \mid a \neq b \in B\}.$$

Suppose $x \models_v p_s$. Since $s \neq 1$ it is enough to show that $x \models_v p_s \leftrightarrow p_1$ and so it is enough to show that $x \models_v p_1$. Now $s \wedge 1 = s$ in B, thus $p_s \wedge p_1 \leftrightarrow p_s \in \Gamma$ implying that $x \models_v p_1$. Conversely, suppose that $x \models_v \bigvee\{p_a \leftrightarrow p_b \mid a \neq b \in B\}$. Let $a \neq b \in B$ with $x \models_v p_a \leftrightarrow p_b$. Then $a \to b \leq s$ or $b \to a \leq s$. Assume the former. Since $p_{a\to b} \leftrightarrow (p_a \to p_b) \in \Gamma$, $x \models_v p_{a\to b}$. Now $a \to b \leq s$ implies $a \to b \wedge s = a \to b$, thus $p_{a\to b} \wedge p_s \leftrightarrow p_{a\to b} \in \Gamma$, so $x \models_v p_s$. □

3.4.19. COROLLARY.

(1) *Subframe logics coincide with* $\{\wedge, \to\}$*-stable logics.*

(2) *Cofinal stable logics coincide with* $\{\wedge, \to, \bot\}$*-stable logics.*

Proof:
If L is a subframe logic, its corresponding variety $\mathcal{V}(\mathsf{L})$ is closed under $\{\wedge, \to\}$-subalgebras by Theorem 3.4.8. Thus L is $\{\wedge, \to\}$-stable. Conversely, if L is $\{\wedge, \to\}$-stable, then by Theorem 3.3.17 it is axiomatizable by $\{\wedge, \to\}$-stable formulas. By Lemma 3.4.18, a $\{\wedge, \to\}$-stable formula is equivalent to a subframe formula, thus, L is axiomatizable by subframe formulas. Therefore L is a subframe logic by Proposition 3.4.17. The case for cofinal subframe logics can be proved analogously. □

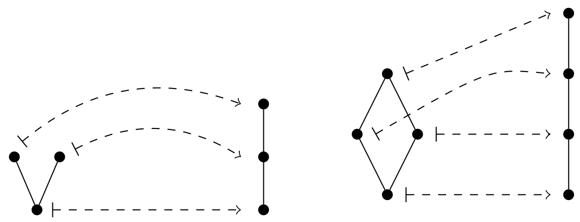
We note that from Theorem 3.3.17 and Proposition 3.4.17 we can infer that if a variety $\mathcal{V}$ is generated by some $\{\wedge, \to, \bot\}$-stable class $\mathcal{K}$ of Heyting algebras, then $\mathcal{V}$ itself is $\{\wedge, \to, \bot\}$-stable. This seems to be quite a strong property, which for instance does not hold if we replace $\{\wedge, \to, \bot\}$ by $\{\wedge, \vee, \bot, \top\}$ or $\{\wedge, \vee, \neg, \top\}$.

3.4.20. REMARK. In Section 3.4.1 we saw that in algebraic terms subframes correspond to nuclei, and that every nucleus on Heyting algebra A gives rise to a $\{\wedge, \to\}$-subalgebra of A. The converse of the latter is not true, i.e. not every $\{\wedge, \to\}$-subalgebra of A gives rise to a nucleus on A. It follows that subframes and cofinal subframes are not in one-to-one correspondence with $\{\wedge, \to\}$-subalgebras and $\{\wedge, \to, \bot\}$-subalgebras, respectively. In [13] it was shown that $\{\wedge \to\}$-, and $\{\wedge, \to, \bot\}$-homomorphisms of Heyting algebras can dually be described by *partial* and *well partial Esakia morphisms* between Esakia spaces, which give rise to a dual description of $\{\wedge, \to\}$-subalgebras and $\{\wedge, \to, \bot\}$-subalgebras, respectively.

3.5 Stable and cofinal stable si logics

In this section we investigate $\{\wedge, \vee, \bot, \top\}$-stable and $\{\wedge, \vee, \neg, \top\}$-stable si logics. As already mentioned in the introduction of this chapter, $\{\wedge, \vee, \bot, \top\}$-stable logics are precisely the *stable si logics* of [17]. Thus, from now on, we call $\{\wedge, \vee, \bot, \top\}$-stable simply stable si logic. As we also explained in the introduction of this chapter, we see the $\{\wedge, \vee, \neg, \top\}$-stable si logics as the stable analogue of cofinal subframe logics, and accordingly, call them *cofinal stable logics.*

The aim of this section is to investigate the classes of stable and cofinal stable si logics and compare their properties with those of subframe and cofinal subframe logics. In turns out that there are some subtle aspects where they behave differently.



(a) (b)

Figure 3.5.1: (a) Priestley map; (b) q-morphism

3.5.1 Dual description

The dual perspective on stable and cofinal stable logics is more transparent than in the subframe case. It is simply provided by Priestley duality. We recall how to construct the dual category of pseudocomplemented distributive lattices from [110].

Priestley duality for PBDLat. For terminology regarding Priestley spaces, the reader may consult Section 2.2.2. A Priestley space X is called a PC-*space* (short for: pseudocomplemented space) if $\downarrow U$ is open for each open upset U of X. Thus, the difference to Esakia spaces (Definition 2.2.6) is that $\downarrow U$ is required to be open for open *upsets* as opposed to arbitrary open sets. For PC-*spaces* X and Y, a PC-*morphism* $f : X \to Y$ is a Priestley morphisms that satisfies the additional condition

$$\max \uparrow f(x) = f(\max \uparrow x)$$

for each $x \in X$. Thus, PC-morphisms are not only order-preserving but satisfy a weak version of the back-condition of p-morphisms. In fact, we can say that they satisfy the back condition for maximal points. For this reason we call such maps *q-morphisms* (short for: quasi p-morphisms). See Figure 3.5.1(a) for an example of a Priestley map that is not a q-morphism and Figure 3.5.1(b) for an example of a q-morphism that is not a p-morphism.

3.5.1. Theorem ([110]). *There is a dual equivalence between the category* PBDLat *and the category of* PC-*spaces and q-morphisms.*

In the following lemma we collect many equivalent descriptions of q-morphisms. The different descriptions will become handy when we look at examples in the later sections.

3.5.2. Lemma. *Let $f : X \to Y$ be a Priestley morphism between* PC-*spaces. The following are equivalent.*

(1) *f is a q-morphism.*

(2) *For all $x \in X$, we have $\max{\uparrow} f(x) \subseteq f(\max{\uparrow} x)$.*

(3) *For all $x \in X$ and $y \in Y$, if $f(x) \leq y$ then there is $x' \in X$ with $x \leq x'$ and $y \leq f(x')$.*

(4) *For every upset $A \subseteq Y$, we have $f^{-1}({\downarrow} A) = {\downarrow} f^{-1}(A)$.*

(5) *For every $y \in Y$, we have $f^{-1}({\downarrow}{\uparrow} y) = {\downarrow} f^{-1}({\uparrow} y)$.*

Proof:
(1) ⇒ (2): This is obvious.

(2) ⇒ (3): Suppose $x \in X$, $y \in Y$, and $f(x) \leq y$. Then $y \in {\uparrow} f(x)$. Since Y is a Priestley space, there is $y' \in \max{\uparrow} f(x)$ with $y \leq y'$. By (2), $y' \in f(\max{\uparrow} x)$. Therefore, there is $x' \in X$ such that $x \leq x'$ and $f(x') = y' \geq y$. This shows (3).

(3) ⇒ (1): Let $x \in X$. We first show that $\max{\uparrow} f(x) \subseteq f(\max{\uparrow} x)$. Suppose $y \in \max{\uparrow} f(x)$. By (3), there is $x' \in X$ with $x \leq x'$ and $y \leq f(x')$. Since y is a maximal point, $y = f(x')$. Since X is a Priestley space, there is $x'' \in \max{\uparrow} x$ with $x' \leq x''$. Since f is order preserving and y is maximal, $f(x'') = y$. Thus, $y \in f(\max{\uparrow} x)$. Next we show that $f(\max{\uparrow} x) \subseteq \max{\uparrow} f(x)$. First note that every map satisfying (3) maps maximal points to maximal points. For let $x \in X$ be maximal and suppose x is mapped to a non-maximal $y \in Y$. Then there is y' in Y with $y < y'$. However, since x is maximal, there is no x' with $x \leq x'$ that is mapped to y'. So condition (3) is violated for x. Now suppose $y \in f(\max{\uparrow} x)$. Then there is $x' \in \max{\uparrow} x$ such that $f(x') = y$. Since f is order preserving, $f(x) \leq f(x')$. By the above, x' is mapped to a maximal point, so $f(x') = y \in \max{\uparrow} f(x)$. Thus, f is a q-morphism.

(3) ⇒ (4): Let $A \subseteq Y$ be an upset. Since f is order preserving, ${\downarrow} f^{-1}(A) \subseteq f^{-1}({\downarrow} A)$. We show $f^{-1}({\downarrow} A) \subseteq {\downarrow} f^{-1}(A)$. Suppose $x \in f^{-1}({\downarrow} A)$. Then there is $y \in A$ with $f(x) \leq y$. By (3), there is $x' \geq x$ with $y \leq f(x')$. Since A is an upset, $f(x') \in A$, yielding $x \in {\downarrow} f^{-1}(A)$.

(4) ⇒ (5): This is obvious since ${\uparrow} y$ is an upset.

(5) ⇒ (3): Let $x \in X$, $y \in Y$, and $f(x) \leq y$. Then $f(x) \in {\downarrow}{\uparrow} y$ so $x \in f^{-1}({\downarrow}{\uparrow} y)$. By (5), $x \in {\downarrow} f^{-1}({\uparrow} y)$. This implies that there is $x' \geq x$ with $y \leq f(x')$, which proves (3). □

3.5.3. Remark. Conditions (1) and (2) are not *locally* equivalent; that is, it is not true that for a given $x \in X$ we have $\max{\uparrow} f(x) = f(\max{\uparrow} x)$ iff $\max{\uparrow} f(x) \subseteq f(\max{\uparrow} x)$.

Dual description of stable and cofinal stable si logics. Using the above and Priestley duality for BDLat we obtain a dual description of the results concerning stable and cofinal stable logics.

We employ the terminology of [17] by calling a Priestley map $f : X \to Y$ between two Esakia spaces X and Y a *stable map*. If Y is an image of X under a stable map, then we call Y a *stable image* of X. By Priestly duality, stable images are dual to $\{\wedge, \vee, \bot, \top\}$-subalgebras (see Table 2.2.1). To match this terminology, we call an image under a q-morphism a *cofinal stable image*. Then, cofinal stable images are dual to $\{\wedge, \vee, \neg, \top\}$-subalgebras. Moreover, we call a class of Esakia spaces (*finitely*) *stable* or *cofinal stable* iff it is closed under (finite) stable or cofinal stable images of Esakia spaces, respectively.

For simplicity, if B is a finite and subdirectly irreducible Heyting algebra, we write $\gamma(B)$ and $\gamma(B, \neg)$ instead of $\gamma_{\{\wedge,\vee,\bot,\top\}}(B)$ and $\gamma_{\{\wedge,\vee,\neg,\top\}}(B)$, respectively. We refer to the aforementioned formulas as *stable* and *cofinal stable* formulas, respectively. Also if Y is a finite rooted Esakia space with dual B, we sometimes write $\gamma(Y)$ and $\gamma(Y, \neg)$ instead of $\gamma(B)$ and $\gamma(B, \neg)$, respectively.

The refutation criterion from Lemma 3.3.13 then translates to the following.

3.5.4. Lemma. *Let X and Y be Esakia spaces such that Y is finite and rooted.*

(1) $X \not\models \gamma(Y)$ iff Y is isomorphic to a stable image of a strongly rooted generated subframe of X.

(2) $X \not\models \gamma(Y, \neg)$ iff Y is isomorphic to a cofinal stable image of a strongly rooted generated subframe of X.

In the stable case this was already described in [17]. In the following theorem we summarize some of the characterizations of Theorem 3.3.17 in dual terms.

3.5.5. Theorem. *Let L be a si logic. The following are equivalent.*

(1) L is (cofinal) stable.

(2) L is characterized by a finitely (cofinal) stable class of Esakia spaces.

(3) If X, Y are strongly rooted Esakia spaces, then $X \models \mathsf{L}$ implies $Y \models \mathsf{L}$ provided that Y is a (cofinal) stable image of X.

(4) The class of rooted L-Esakia spaces is (cofinal) stable.

(5) The class of finite rooted L-Esakia spaces is (cofinal) stable and characterizes L.

3.5.2 Stable and cofinal stable formulas

In this part, we say a bit more about the behavior of the formulas $\gamma(B)$ and $\gamma(B, \neg)$. Recall that all $\mathcal{H}$-stable logics are axiomatizable by $\mathcal{H}$-stable formulas by Theorem 3.3.17. As we saw in Proposition 3.4.17 and the discussion thereafter, the converse is also true if $\to \in \mathcal{H}$, i.e. for subframe and cofinal subframe si logics. As shown in [17], the analogous result also holds for stable logics and stable formulas, i.e. a si logic is stable iff it is axiomatizable by stable formulas. However, cofinal stable formulas do not behave that well; we will see that not all logics axiomatized by cofinal stable formulas are cofinal stable.

First we recall from [17] that stable formulas axiomatize stable si logics. The key property is the lemma below ([17, Lemmas 6.1 and 6.2]). Recall that in Lemma 3.3.14 we showed $\mathbf{S}_{\mathcal{H}}\mathbf{H}(A) \subseteq \mathbf{H}\mathbf{S}_{\mathcal{H}}(A)$ for every finite A and $\mathcal{H}$ any of our reducts. In short, the lemma below is a version of this for $\mathcal{H} = \{\wedge, \vee, \bot, \top\}$, stating that $(\mathbf{S}_{\mathcal{H}}\mathbf{H}(A))_{\mathsf{si}} \subseteq \mathbf{S}_{\mathcal{H}}(A)$, whenever A is finite.

3.5.6. Lemma ([17]). *Suppose A is a finite Heyting algebra and B is a subdirectly irreducible $\{\wedge, \vee, \top, \bot\}$-subalgebra of a homomorphic image C of A. Then B is a $\{\wedge, \vee, \top, \bot\}$-subalgebra of A.*

Using the lemma above, we obtain a stronger refutation criterion than the one in Lemma 3.3.13 for stable formulas when applied to well-connected algebras or rooted spaces (see [17, Theorem 6.3]).

3.5.7. Lemma ([17]). *Let A, B be Heyting algebras with duals X, Y, respectively. Let A be well-connected, i.e. X is rooted, and let B be finite and subdirectly irreducible, i.e. Y is finite and rooted. Then*

$A \not\models \gamma(B)$ iff B is isomorphic to a $\{\wedge, \vee, \bot, \top\}$-subalgebra of A.
$X \not\models \gamma(Y)$ iff Y is a stable image of X.

Proof:
If $A \not\models \gamma(B)$, then by $\{\wedge, \vee, \bot, \top\}$-filtration, there is a finite $\{\wedge, \vee, \bot, \top\}$-subalgebra A' of A with $A' \not\models \gamma(B)$. By Lemma 3.3.13, there is a subdirectly irreducible homomorphic image C of A' and B is isomorphic to a $\{\wedge, \vee, \bot, \top\}$-subalgebra of C. By Lemma 3.5.7, B is isomorphic to a $\{\wedge, \vee, \bot, \top\}$-subalgebra of A' and therefore of A. Conversely, if B is isomorphic to a $\{\wedge, \vee, \bot, \top\}$-subalgebra of A, then $A \not\models \rho_{\{\wedge,\vee,\bot,\top\}}(B)$ by Lemma 3.3.6. Since A is well-connected, $A \not\models \gamma(B)$ by Corollary 2.3.12. □

The lemma above shows that validity of stable formulas is preserved by $\{\wedge, \vee, \top, \bot\}$-subalgebras of well-connected Heyting algebras. This implies that stable formulas axiomatize stable logics.

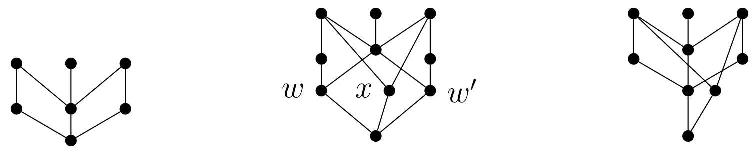
3.5.8. Corollary ([17]). *A si logic is stable iff it can be axiomatized by stable formulas.*

Proof:
If L is stable, then it can be axiomatized by stable formulas as we already saw in Theorem 3.3.17. Conversely, suppose that L is axiomatized by stable formulas. By Lemma 3.5.7, validity of stable formulas is preserved by $\{\wedge, \vee, \top, \bot\}$-subalgebras of well-connected Heyting algebras. Thus, the well-connected L-Heyting algebras form a $\{\wedge, \vee, \top, \bot\}$-stable class. So L is stable. $\square$

The next lemma shows that the analogous statement does not hold for cofinal stable logics.

3.5.9. Lemma. *There is a finite subdirectly irreducible Heyting algebra B such that the logic $\mathsf{IPC} + \gamma(B, \neg)$ is not cofinal stable.*

Proof:
Let X, Y, Z be the finite rooted frames drawn below.



Y X Z

Let A, B, C be the dual Heyting algebras of X, Y, Z, respectively, and let $\mathsf{L} = \mathsf{IPC} + \gamma(B, \neg)$. We show that L is not cofinal stable. First observe that Y is not a q-morphic image of X. Indeed, if there were an onto q-morphism $f : X \to Y$, then f would map the maximal points of X onto the maximal points of Y. But $x \in X$ sees exactly two maximal points, while X has no point with this property. This violates the q-morphism condition at x, a contradiction. Since each rooted upset of X has smaller cardinality than Y, it follows that there is no rooted upset U of X such that Y is a q-morphic image of U. From this we conclude by Lemma 3.3.13 that $A \models \gamma(B, \neg)$, so $A \models \mathsf{L}$. On the other hand, Z is a q-morphic image of X, witnessed by the map that identifies w and w' in X. Therefore, C is isomorphic to a pseudocomplemented sublattice of A. It is also obvious that Y is isomorphic to an upset of Z, so by Lemma 3.3.13, $C \not\models \gamma(B, \neg)$. Thus, $C \not\models \mathsf{L}$. Since $A \models \mathsf{L}$, C is isomorphic to a $\{\wedge, \vee, \neg, \top\}$-algebra of A, and $C \not\models \mathsf{L}$, we conclude that L is not a cofinal stable logic. $\square$

Another difference between the subframe and stable cases is of a semantic nature. If L is a (cofinal) subframe, then as we saw in Proposition 3.4.17, the variety $\mathcal{V}(\mathsf{L})$ is closed under $\{\wedge, \to\}$-, respectively $\{\wedge, \to, \bot\}$-subalgebras. The analogous statement is not true in the stable and cofinal stable cases. In fact, the next proposition shows that there is no non-trivial stable logic with the property that its variety is stable class.

3.5.10. PROPOSITION. *For a si logic* L, $\mathcal{V}(\mathsf{L})$ *is a stable class iff* L *is* IPC *or the inconsistent logic.*

Proof:
Suppose $\mathcal{V}(\mathsf{L})$ is a variety of Heyting algebras and a stable class. Suppose $\mathcal{V}(\mathsf{L})$ does not correspond to the inconsistent logic. Then $\mathcal{V}(\mathsf{L})$ contains all Boolean algebras. Now every Heyting algebra A is a bounded sublattice of its free Boolean extension $\mathcal{B}(A)$ ([100], see e.g. [40]). Thus, if $\mathcal{V}(\mathsf{L})$ is a stable class, it contains all Heyting algebras. Therefore, $\mathsf{L} = \mathsf{IPC}$. □

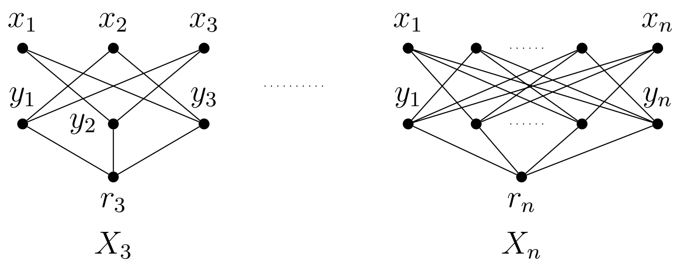
The analogue of the above lemma does not hold in the cofinal stable case. For instance, KC is cofinal stable and is corresponding variety is a $\{\wedge, \vee, \neg, \top\}$-stable class. Of course, all si logics axiomatized by formulas in the signature $\{\wedge, \vee, \bot, \top, \neg\}$ have this property (see also our discussion in Section 3.2). However, in general cofinal stable logics do not have the property that their variety is a $\{\wedge, \vee, \neg, \top\}$-stable class.

3.5.3 Cardinality

Clearly, every stable logic is cofinal stable. In [17] it was shown that there is a continuum of stable si logics which implies that there is a continuum of cofinal stable logics. We show here that there is a continuum of cofinal stable logics that are not stable. The proof method that we are using is a standard method due to Jankov [81]. However, we have to be a little bit more careful in our arguments because, as we have just seen in Lemma 3.5.9, si logics axiomatized by cofinal stable formulas may not be cofinal stable.

Consider the sequence of finite posets drawn below. Formally, for each $n \geq 3$, $X_n = \{x_i \mid 1 \leq i \leq n\} \cup \{y_i \mid 1 \leq i \leq n\} \cup \{r_n\}$ and the order $\leq$ on X_n is described by $x_i \leq x_j$ iff $i = j$, and $y_i \leq y_j$ iff $i = j$, and $x_i \leq y_j$ iff $i \neq j$ for all $1 \leq i, j \leq n$. Moreover, r is the root of X_n, i.e. $r \leq w$ for all $w \in X_n$.

It is well known that no member of this sequence is a p-morphic image of an upset of some other member. We will show that the same result holds if we replace p-morphisms with q-morphisms.



3.5.11. LEMMA. *For $n, m \geq 3$, if $n \neq m$, then X_n is not a cofinal stable image of any upset of X_m.*

Proof:
Let $m, n \geq 3$ with $n \neq m$. For simplicity, we denote the elements of X_n by x_i, y_i for $1 \leq i, j, \leq n$ and the elements of X_m by x'_i, y'_i for $1 \leq i, j, \leq m$. Moreover, we denote the root of X_n by r_n and the root of X_m by r_m.

If $m < n$, then for cardinality reasons, X_n cannot be a q-morphic image of any upset of X_m. So suppose $n < m$ and suppose there is an onto q-morphism f from an upset U of X_m onto X_n. Since f is a q-morphism, f maps maximal elements to maximal elements. Thus, each $x'_i \in U$ is mapped to some x_j of X_n.

Next show that $\max X_m \subseteq U$. Since $n \geq 3$, X_n contains at least three non-maximal elements. The preimages of these elements cannot be maximal elements of U, since this would violate the q-morphism condition. Thus, U contains at least three non-maximal elements of X_m. This implies that U contains y'_i, y'_j for $i \neq j$ (since the third non-maximal element might be the root r_m). But then $\max X_m \subseteq \uparrow\{y'_i, y'_j\} \subseteq U$.

Next we observe that no $y'_i \in U$ is mapped to some x_j of X_n. For suppose there is $y'_i \in U$ with $f(y'_i) = x_j$. Now y'_i sees all but one maximal elements of X_m. Since f is order-preserving, f needs to map all those maximal elements to x_j. As X_n has at least three maximal elements, this contradicts to the fact that $f(\max X_m) = \max X_n$. We conclude that f maps each $y'_i \in U$ to some y_j or to r_n.

Suppose that f maps two maximal elements x'_i and x'_j, $i \neq j$ to the same maximal element x_k of X_n. If $y'_i \in U$, then $\max \uparrow f(y'_i) = f(\max \uparrow y'_i) = \max X_n$. This means that f needs to map y'_i to the root r_n. Thus, whenever some $y'_i \in U$ and x'_i is not a unique preimage of some $x_k \in X_n$, then f sends y'_i to the root r_n of X_n. In other words, f can only map y'_i to some y_k if x'_i is a unique preimage of some x_k.

Since $m > n$, by the pigeonhole principle, there is at least one maximal element of X_n that has two f-preimages. Therefore, there are at most $n-1$ maximal elements of X_n that have a unique f-preimage. This in turn means that there are at most $n-1$ candidates $y'_i \in U$ that f could map to some element y_i of X_n. But then f cannot be onto. The obtained contradiction proves that there is no q-morphism from an upset of X_m onto X_n. □

On the other hand, we have the following lemma.

3.5.12. LEMMA. *If $m \geq 2n$, then X_n is a stable image of X_m.*

Proof:
Using the same notation from the proof of the previous lemma, define $f : X_m \to X_n$ as follows. For $i \leq n$, send x'_i to x_i, x'_{n+i} to y_i, and send the rest of the points in X_m to the root r_n of X_n. It is straightforward to see that such f is onto and stable. □

We are ready to prove the main result of this section.

3.5.13. PROPOSITION. *There are continuum many cofinal stable si logics that are not stable.*

Proof:
For $n \geq 3$, let $\mathcal{K}_n$ be the class of cofinal stable images of X_n. Let $I = \{n \in \mathbb{N} \mid n \geq 3\}$, and for $J \subseteq I$, let L_J be the logic of $\bigcup_{n \in J} \mathcal{K}_n$. Since $\bigcup_{n \in J} \mathcal{K}_n$ is closed under cofinal stable images, L_J is a cofinal stable logic.

3.5.1. CLAIM. *For $J, K \subseteq I$, if $J \neq K$, then $\mathsf{L}_J \neq \mathsf{L}_K$.*

Proof:
Without loss of generality we assume that there is $n \in K \setminus J$. Then $X_n \models \mathsf{L}_K$, which implies that $\gamma(X_n, \neg) \notin \mathsf{L}_K$ since $X_n \not\models \gamma(X_n, \neg)$. We prove that $\gamma(X_n, \neg) \in \mathsf{L}_J$. If $\gamma(X_n, \neg) \notin \mathsf{L}_J$, then there is $m \in J$ and $Y \in \mathcal{K}_m$ with $Y \not\models \gamma(X_n, \neg)$. By Lemma 3.5.4(2) there is a rooted upset Z of Y such that X_n is a cofinal stable image of Z. Now $Y \in \mathcal{K}_m$ means that Y is a q-morphic image of X_m. Let $f : X_m \to Y$ be an onto q-morphism, and let $U = f^{-1}(Z)$. Then U is an upset of X_m and the restriction of f to U is a q-morphism onto Z. Therefore, Z is a cofinal stable image of an upset of X_m, and hence X_n is a cofinal stable image of an upset of X_m. Since $n \notin J$, we have $m \neq n$. This contradicts Lemma 3.5.11. Thus, $\gamma(X_n, \neg) \in \mathsf{L}_J$, and so $\mathsf{L}_J \neq \mathsf{L}_K$. □

Let $\Delta := \{J \subseteq I \mid J \text{ and } I \setminus J \text{ are infinite}\}$. By Claim 3.5.1, the collection $\{\mathsf{L}_J \mid J \in \Delta\}$ provides continuum many cofinal stable logics. It remains to be shown that L_J is not stable for each $J \in \Delta$. Let $J \in \Delta$. Then there is $n \in I \setminus J$. By the proof of Claim 3.5.1, $\gamma(X_n, \neg) \in \mathsf{L}_J$. Therefore, since $X_n \not\models \gamma(X_n, \neg)$, we see that $X_n \not\models \mathsf{L}_J$. Since J is infinite, there is $m \in J$ with $m \geq 2n$. As $m \in J$, we have $X_m \models \mathsf{L}_J$. By Lemma 3.5.12, X_n is a stable image of X_m. Because $X_n \not\models \mathsf{L}_J$, we conclude that L_J is not stable. □

3.5.4 Examples

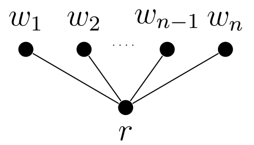
We conclude this section by considering standard examples of stable and cofinal stable logics. In particular, we provide examples of si logics separating the class of cofinal stable logics from those of stable, subframe, and cofinal subframe si logics.

For the definition of the si logics mentioned but not defined in this section, the reader is referred to Table A.0.2.

In the previous section, we constructed a continuum of cofinal stable logics that are not stable. A more natural example of this kind is the logic BD_2, the logic of finite frames of depth at most two (see Section 2.2.1 for the definition of depth). It was shown in [17, Theorem 7.4] that BD_n is not stable, for $n \geq 2$.

3.5.14. PROPOSITION. *Let* L *be an extension of* BD_2. *Then* L *is cofinal stable, and* L *is stable iff* $\mathsf{LC}_2 \subseteq \mathsf{L}$.

Proof:
It is well known (see, e.g., [75]) that BD_2 is the logic of all finite forks F_n, as drawn below. Moreover, LC_2 is the logic of the two element chain F_1, and each L in the interval $[\mathsf{BD}_2, \mathsf{LC}_2)$ is the logic of an n-fork F_n for some $n \geq 1$.



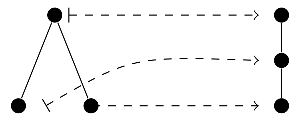
F_n

It follows from the proof of [17, Theorem 7.4] that no si logic in the interval $[\mathsf{BD}_2, \mathsf{LC}_2)$ is stable. The only other si logics extending BD_2 that are not in the interval $[\mathsf{BD}_2, \mathsf{LC}_2)$ are $\mathsf{LC}_2, \mathsf{CPC}$, and the inconsistent logic. It is easy to verify that these are all stable.

Now let $\mathsf{BD}_2 \subseteq \mathsf{L}$. We show that L is cofinal stable. Since q-morphisms map maximal elements to maximal elements, it is immediate to see that every cofinal stable image of an n-fork is either the one-point poset or an m-fork for some $m \leq n$. Therefore, the class of finite rooted L-frames is closed under cofinal stable images and characterizes L. Thus, by Theorem 3.5.5, L is cofinal stable. □

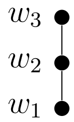
3.5.15. LEMMA. *The variety* $\mathcal{V}(\mathsf{BD}_2)$ *is not cofinal stable.*

Proof:
Consider the posets drawn below. The poset of the left is a (non-rooted) BD_2-frame, but the poset on the right side is not a BD_2-frame.



The map indicated is a q-morphism. Thus, the frames of BD_2 are not closed under cofinal stable images. So the variety $\mathcal{V}(\mathsf{BD}_2)$ is not cofinal stable. $\square$

We next show that BD_2 can be axiomatized by the cofinal stable formula of the three-element chain $C_3 = \{w_1, w_2, w_3\}$, where $w_1 < w_2 < w_3$.



C_3

Recall from Section 2.2.1 that by $\mathrm{d}(X)$ we denote the depth of a poset.

3.5.16. Lemma. *Let X be a finite rooted poset. Then $\mathrm{d}(X) \leq 2$ iff C_3 is not a q-morphic image of X.*

Proof:
We prove that $\mathrm{d}(X) \geq 3$ iff C_3 is a q-morphic image of X. First suppose that $\mathrm{d}(X) \geq 3$. Then X contains a chain $x_1 < x_2 < x_3$. Define $f : X \to C_3$ by setting

$$f(y) = \begin{cases} w_1 & \text{if } y \in {\downarrow} x_1 \\ w_2 & \text{if } y \in {\downarrow} x_2 \setminus {\downarrow} x_1, \\ w_3 & \text{otherwise.} \end{cases}$$

It is easy to see that f is an onto q-morphism. (In fact, f is moreover an onto p-morphism). Conversely, suppose $f : X \to C_3$ is an onto q-morphism. We show that X contains a chain of three elements. Since f is onto, the root r of X is mapped by f to w_1. Using again that f is onto, we find $x > r$ with $f(x) = w_2$. Since $f(x) = w_2 < w_3$ and $w_3 \in \max C_3$, there is $y > x$ such that $f(y) = w_3$. Thus, $r < x < y$ is a three element chain in X. $\square$

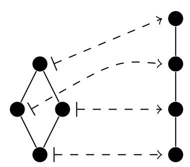
3.5.17. Proposition. $\mathsf{BD}_2 = \mathsf{IPC} + \gamma(C_3, \neg)$.

Proof:
Let F_n be the n-fork for some $n \in \mathbb{N}$. Since $\mathrm{d}(F_n) \leq 2$, by Lemma 3.5.16, C_3 is not a cofinal stable image of a rooted upset of F_n. By Lemma 3.5.4(2), $F_n \models \gamma(C_3, \neg)$.

Since BD_2 is the logic of all n-forks, we conclude that $\mathsf{IPC}+\gamma(C_3,\neg) \subseteq \mathsf{BD}_2$. Conversely, suppose X is an Esakia space such that $X \not\models \mathsf{BD}_2$. Since $\mathsf{BD}_2 = \mathsf{IPC}+\mathsf{bd}_2$, where $\mathsf{bd}_2 = q \vee (q \to (p \vee \neg p))$, we see that $X \not\models \mathsf{bd}_2$. By $\{\wedge, \vee, \neg, \top\}$-filtration, there is a finite rooted poset Y such that Y is a continuous q-morphic image of X and $Y \not\models \mathsf{bd}_2$. It follows that $\mathrm{d}(Y) \geq 3$. By Lemma 3.5.16, C_3 is a q-morphic image of Y. Therefore, C_3 is a continuous q-morphic image of X. Thus, by Lemma 3.5.4(2), $X \not\models \gamma(C_3,\neg)$. As every si logic L is complete with respect to the Esakia spaces validating L, we conclude that $\mathsf{BD}_2 \subseteq \mathsf{IPC} + \gamma(C_3,\neg)$. □

3.5.18. Lemma. *For $n \geq 3$, the logics BD_n are not cofinal stable.*

Proof:
The proof is the same as to show that these logics are not stable which was proved in [17]. The following picture shows an onto q-morphism from a rooted poset of depth three to a rooted poset of depth four. We infer that the class of finite rooted posets of depth three is not closed under cofinal stable images, which entails that BD_3 is not cofinal stable.



Clearly similar examples can be constructed to show that BD_n is not cofinal stable for all $n \geq 3$. □

Next we consider the logics ND_k introduced in [96] (see also [69, Section 2.4]). For $k \in \mathbb{N}$, let

$$\mathsf{nd}_k = (\neg p \to \bigvee_{1\leq i\leq k} \neg q_i) \to \bigvee_{1\leq i\leq k} (\neg p \to \neg q_i),$$

and let $\mathsf{ND}_k = \mathsf{IPC} + \mathsf{nd}_k$. As shown in [96], each ND_k has the fmp. On finite frames, nd_k characterizes the property of having divergence k, where a finite poset X is of *divergence* k if for all $x \in X$ and $W \subseteq \max X$ satisfying $|W| \leq k$ and $W \subseteq {\uparrow}x$, there is $y \geq x$ with $\max{\uparrow}y = W$. Therefore, ND_k is the logic of the finite frames of divergence k.

We show that each ND_k is a cofinal stable logic, but that for $k \geq 2$, ND_k is neither stable nor cofinal subframe logic (and hence not a subframe logic).

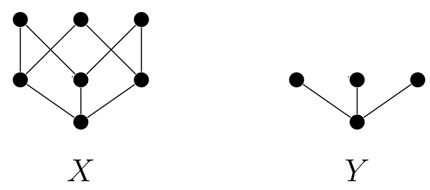
3.5.19. Lemma. *Each ND_k is a cofinal stable logic.*

Proof:
Since ND_k has the fmp, it is sufficient to check that the finite frames of ND_k are closed under cofinal stable images. Let X be a finite poset of divergence k and

let X' be a q-morphic image of X under the q-morphism $f : X \to X'$. We show that X' is of divergence k. Let $x' \in X'$, $W' \subseteq \max X'$, $|W'| \leq k$, and $W' \subseteq {\uparrow}x'$. Since f is onto, there is $x \in X$ with $f(x) = x'$. As f is a q-morphism, for each $w' \in W'$ we can pick one $w \in \max X$ with $x \leq w$ and $f(w) = w'$. We denote the set of these elements by W. Then $|W| = |W'| \leq k$, $W \subseteq \max X$, and $W \subseteq {\uparrow}x$. Since X is of divergence k, there is $y \geq x$ with $\max{\uparrow}y = W$. Because f is order-preserving, $x' \leq' f(y)$, so by the q-morphism condition, $\max{\uparrow}f(y) = W'$. Thus, X' is of divergence k. □

3.5.20. Lemma. *For $k \geq 2$, the logic ND_k is neither stable nor a cofinal subframe logic. Consequently, ND_k is not a subframe logic.*

Proof:
Let $k \geq 2$, and consider the posets X and Y depicted below.

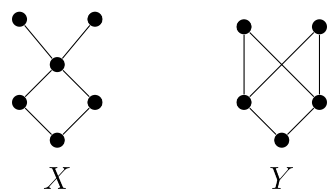


It is easy to see that X is of divergence k, while Y is not of divergence 2. But Y is both a stable image and a cofinal subframe of Y. □

Finally, we construct a tabular (see Section 2.3.4) stable logic that is not a cofinal subframe logic. Let X, Y be the posets drawn below. We let

$$\mathsf{Stab}(X) = \{Z \mid Z \text{ is a stable image of } X\} \tag{3.5}$$

and $\mathsf{L}_{\mathsf{Stab}(X)}$ be the logic of $\mathsf{Stab}(X)$. Since the class of Heyting algebras corresponding to the class $\mathsf{Stab}(X)$ is stable, $\mathsf{L}_{\mathsf{Stab}(X)}$ is a stable logic.



3.5.21. Lemma. *The logic $\mathsf{L}_{\mathsf{Stab}(X)}$ is not a cofinal subframe logic.*

Proof:
For contradiction, suppose that $\mathsf{L}_{\mathsf{Stab}(X)}$ was a cofinal subframe logic. It is easy to see that Y is a cofinal subframe of X, so $Y \models \mathsf{L}_{\mathsf{Stab}(X)}$. Since Y is rooted

and $\mathsf{Stab}(X)$ is finite, the dual statement of Jónsson's Lemma yields that Y is isomorphic to an upset of a p-morphic image of a member of $\mathsf{Stab}(X)$. But $\mathsf{Stab}(X)$ is closed under stable images, hence it is closed under p-morphic images. Therefore, there is some stable image Z of X and such that Y is an upset of Z.

Since Z is an order preserving image of X, we see that Z has at most two maximal points. But Y is an upset of Z, so Z has precisely two maximal points. Thus, if $f : X \to Z$ is onto and stable, then f maps the maximal points of X onto the maximal points of Z. Note that X has only one element of depth 2, which is a lower cover of the maximal elements of X. It follows that Z also has only one element of depth 2. However, since Y has two elements of depth 2, so does Z. Thus, Y cannot be an upset of Z. The obtained contradiction proves that $\mathsf{L}_{\mathsf{Stab}(X)}$ is not a cofinal subframe logic. □

The following table contains the examples of stable, cofinal stable, subframe, and cofinal subframe logics mentioned above.

	subframe	**cof. subframe**	**stable**	**cof. stable**
LC_n	✓	✓	✓	✓
BD_2	✓	✓	-	✓
$\mathsf{BD}_n, n \geq 3$	✓	✓	-	-
$\mathsf{ND}_k, k \geq 2$	-	-	-	✓
$\mathsf{L}_{\mathsf{Stab}(X)}$	-	-	✓	✓

Table 3.5.1: "✓" means "belongs to the class"; "-" means "does not belong to the class"

3.6 Summary and related results

We summarize some of the main properties of subframe, cofinal subframe, stable and cofinal stable si logics in Table 3.6.1. Among them are some properties that we have not yet discussed but that fit here very well. We will see those in action in other chapters of the thesis.

We saw that $\mathcal{H}$-stable logics can be semantically characterized in many different ways, both from algebraic and frame theoretical perspectives. Roughly speaking, the characterizations in Table 3.6.1 are those corresponding to the "largest" class of algebras/frames that we can assume to be $\mathcal{H}$-stable for a given $\mathcal{H}$-stable logic. If L is a subframe and cofinal subframe logics, the class of *all* L-frames is closed under subframes and cofinal subframe, respectively. And similarly for L-algebras. In contrast, if L is stable or cofinal stable, then the class of *all* L-

frames is general not closed under stable and cofinal stable images, respectively (see Proposition 3.5.10 and Lemma 3.5.15). However, we can always assume that *rooted* frames (algebras) are closed under stable and cofinal stable images, respectively.

Concerning the syntactic characterizations of these classes (third row of Table 3.6.1), as follows from Theorem 3.3.17, $\mathcal{H}$-stable logics are always axiomatizable by $\mathcal{H}$-stable formulas. However, $\{\wedge, \vee, \neg, \bot\}$-stable formulas do in general not axiomatize $\{\wedge, \vee, \neg, \bot\}$-stable logics (Lemma 3.5.9). So $\{\wedge, \vee, \neg, \bot\}$-stable formulas do not provide a syntactic characterization of cofinal stable logics. We conclude that $\mathcal{H}$-stable formulas do not in general provide a syntactic characterization of $\mathcal{H}$-stable si logics.

As explained, another syntactic characterization of subframe and cofinal subframe logics is provided by the $\{\wedge, \to\}$- and $\{\wedge, \to, \bot\}$-formulas, respectively. Moreover, in [29] it was shown that subframe logics are syntactically characterized by NNIL-formulas (see Section 7.2). NNIL-formulas are the main topic of Chapter 7, where we will, among other things, provide alternative proofs of properties of subframe si logics from the NNIL-perspective.

In [34], it was shown that stable logics can be syntactically characterized as the logics axiomatized by ONNILLI-formulas. These formulas can be seen as the stable analogue of NNIL-formulas.

A uniform proof that all $\mathcal{H}$-stable logics have the fmp, and are thus Kripke complete, and that $\mathcal{H}$-stable si logics form a complete sublattice of Λ_{IPC} is the content of Corollary 3.3.18(2). (As we already explained, these results were of course known before).

That the classes of subframe si logics, cofinal subframe si logics, and that of stable si logics are that of continuum was known. In Section 3.5.3 we provided a continuum of cofinal stable non-stable logics.

In Theorem 3.4.9 we pointed out that subframe and cofinal subframe logics are elementary, and thus canonical. The analogue of these results is open for stable and cofinal stable logics. Thus, we conclude this chapter by stating the following two open problems.

Problem 1: Are all (cofinal) stable logics canonical?

Problem 2: Are all (cofinal) stable logics elementary?

	Subframe logics	Cofinal subframe logics	Stable logics	Cofinal stable logics
Locally finite reduct	$\{\wedge, \to\}$	$\{\wedge, \to, \bot\}$	$\{\wedge, \vee, \bot, \top\}$	$\{\wedge, \vee, \neg, \top\}$
Algebra-based semantic characterization	Heyting algebras are closed under $\{\wedge, \to\}$-subalgebras	Heyting algebras are closed under $\{\wedge, \to, \bot\}$-subalgebras	*Well-connected* Heyting algebras are closed under $\{\wedge, \vee, \bot, \top\}$-subalgebras	*Well-connected* Heyting algebras are closed under $\{\wedge, \vee, \neg, \top\}$-subalgebras
Frame-based semantic characterization	Kripke frames are closed under subframes	Kripke frames are closed under cofinal subframes	*Rooted* Kripke frames are closed under stable images	*Rooted* Kripke frames are closed under cofinal stable images
Syntactic characterization	Subframe formulas, $\{\wedge, \to\}$-formulas, NNIL-formulas	Cofinal subframe formulas, $\{\wedge, \to, \bot\}$-formulas	Stable formulas, ONNILLI-formulas	?
Sublattice of Λ_{IPC}	✓	✓	✓	✓
fmp	✓	✓	✓	✓
Kripke complete	✓	✓	✓	✓

Continuum-sized	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
Elementary	$\checkmark$	$\checkmark$	?	?
Canonical	$\checkmark$	$\checkmark$	?	?

Table 3.6.1: "$\checkmark$" means "property holds "; "?" means "we do not know whether the property holds in general"

Chapter 4

Stable modal logics

4.1 Introduction

In this chapter we move to the realm of normal modal logics. Analogously to the previous chapter we study properties of particular classes of normal modal logics having the fmp. One of the most standard techniques for proving the fmp in modal logic is the filtration method. If a model $\mathfrak{N}$ is a filtration of a model $\mathfrak{M}$, then $\mathfrak{N}$ is an image of $\mathfrak{M}$ under a relation-preserving map. Following [18], we call such maps *stable maps*. Thus, if a normal modal logic is characterized by a class of modal spaces closed under images of stable maps, its fmp can be proved via filtration. Such logics were called *stable modal logics* in [18].

Examples of stable logics are the basic modal logic K, the logic T of all reflexive frames, the logic D of all serial frames, the epistemic logic $\mathsf{S5}$, the logic KMT of the frames where each point sees a reflexive point, etc. (see Table A.0.5 for the definition of these logics). Stable logics enjoy the following strong property: they admit *all filtrations*[1].

There are modal logics that are not stable but still admit particular filtrations. For example, the well-known modal systems $\mathsf{K4}$ and $\mathsf{S4}$ admit transitive filtrations, but they do not admit all filtrations, hence are not stable. This generates the problem of how to deal with logics that only admit *some* filtrations. As a solution, we weaken the notion of stability by parameterizing it over a ground logic.

If a normal modal logic M admits a particular filtration, we define M-stable logics as logics that are stable over M (meaning that they are characterized by a class of modal spaces closed under those stable images that validate M). A stable logic is then simply a K-stable logic. $\mathsf{K4}$-stable logics were also studied in [18].

The aim of this chapter is to develop the theory of M-stable modal logics. Our main results include several characterizations of M-stable normal modal logics similar to those for $\mathcal{H}$-stable si logics in the previous chapter. Moreover, we collect

[1]In Definition 4.2.7, we define two versions of "admitting filtration". In that terminology, stable logics *admit all filtrations in a weak sense*.

many examples and prove cardinality results. We find that M-stable logics behave particularly well if M-spaces are closed under adding *sharp roots* (see Definition 4.4.3).

Since logics above K4 and S4 play an important role in modal logic, we pay special attention to K4-stable and S4-stable logics. Over K4, we get the advantage that we can turn every stable rule into a stable formula which behaves similarly to its corresponding rule on rooted frames (see Section 2.3.7). As a consequence, every K4-stable logic is axiomatizable by stable formulas. The converse does in general not hold for logics above K4, but we prove that it is true for logics above S4; that is, every logic axiomatized by S4-stable formulas is S4-stable.

Furthermore, we investigate the connection between S4-stable logics and stable si logics[2]. We prove that the *intuitionistic fragment* of every S4-stable logic is a stable si logic. Conversely, stability is preserved by the *least modal companion* of a si logic. However, stability is in general not preserved by the greatest modal companion of a si logic. We also demonstrate how to translate axiomatizations in terms of stable formulas from the intuitionistic to the modal case and vice versa.

Finally, we discuss the relation between stable modal logics and modal subframe logics. Our study can in fact be understood as an attempt to investigate how similar stable logics are to subframe logics. As we will see, in many ways stable logics parallel subframe logics, however, there are also essential differences. For instance, whereas stable logic "behave well" also in the non-transitive case, non-transitive subframe logics may not enjoy "good" properties, e.g. there exists a transfinite chain of Kripke-incomplete subframe logics [126]. This is explained by the fact that the method of filtration works well in the non-transitive case, whereas selective filtration is only available for transitive modal logics.

This chapter is largely based on [19].

Outline

In the next section we recall filtrations for modal logics. In Section 4.3 we recall the terminology of stable maps, discuss M-stable classes and study properties of M-stable universal classes. In Section 4.4 we move to M-stable logics. In Section 4.5 we study M-stable logics for the case that M is a normal extension of K4. In Section 4.6 we investigate relations between S4-stable modal logics and stable si logics. In Section 4.7 we provide many examples and non-examples of stable, K4-stable and S4-stable modal logics along with their axiomatizations in terms of stable rules and formulas. In Section 4.8 we compare properties of stable modal logics with those of modal subframe logics and summarize the findings of this chapter.

[2]Recall that stable si logics are the $\{\wedge, \vee, \top, \bot\}$-stable logics in the terminology of Section 3.3.2.

4.2 Filtrations for modal logic

As discussed in detail in the introduction of the thesis, filtrations are ubiquitous in the study of modal logic. In this section we recall the algebraic and frame-theoretic characterization of filtrations. Our presentation is taken from [18]. For other modern accounts on filtrations connecting the algebraic and frame-theoretic perspectives we refer the reader to [64] and [46].

Filtrations frame-theoretically

We recall the frame-theoretic definition of a filtration as it can be found in modern textbooks of modal logics (see e.g. [36, Section 2.3] or [40, Section 5.2]).

Recall from Section 2.3.3 that by $\mathcal{L}$ we denote the language of modal logic. In the following let $\Sigma \subseteq \mathcal{L}$ be a subformula closed set of formulas. Let $\mathfrak{X} = (X, R)$ be a modal space and let v be a valuation on $\mathfrak{X}$. The set Σ induces an equivalence relation on X by

$$x \sim_\Sigma y \quad \text{iff} \quad (x \models_v \varphi \text{ iff } y \models_v \varphi, \text{ for all } \varphi \in \Sigma).$$

Thus, two worlds of X are $\sim_\Sigma$-equivalent iff they agree on the truth value of all formulas in Σ. For an element $x \in X$, by $[x]_\Sigma$ we denote the equivalence class of x with respect to Σ, and by X_Σ we denote the set of equivalence classes with respect to $\sim_\Sigma$. It is clear that whenever Σ is finite, then X_Σ can have at most $2^{|\Sigma|}$-many elements since each equivalence class is determined by a subset of Σ.

4.2.1. Definition. Let $\mathfrak{X} = (X, R)$ be a modal space and let v be a valuation on $\mathfrak{X}$. A *filtration of* $(\mathfrak{X}, v)$ *through* Σ is a modal space $\mathfrak{X}_\Sigma = (X_\Sigma, R')$ together with a valuation v' satisfying the following conditions for all $x, y \in X$:

(F1) $v'(p) = \{[x]_\Sigma \mid x \in v(p)\}$ for all propositional letters $p \in \Sigma$,

(F2) xRy implies $[x]_\Sigma R' [y]_\Sigma$, and

(F3) if $[x]_\Sigma R' [y]_\Sigma$ then ($x \models_v \Diamond\varphi$ whenever $y \models_v \varphi$ for $\Diamond\varphi \in \Sigma$).

Note that the definition of filtration does *not* determine a *unique* model on the set X_Σ of equivalence classes. Instead, the conditions (F2) and (F3) determine lower and upper bounds on the relation R', respectively. Also note that the condition (F1) determines the valuation v' only for propositional letters that are included in the set Σ.

We often write $[x]$ instead of $[x]_\Sigma$ if Σ is clear from the context. Sometimes, we omit the valuation v', and refer to the modal space $\mathfrak{X}_\Sigma = (X_\Sigma, R')$ as a filtration.

We list a few standard ways to define filtrations. Let $\mathfrak{X} = (X, R)$ be a modal space and let v a valuation on $\mathfrak{X}$. For $x, y \in X$ define the relations R^s, R^l, R^t and R^{st} on X_Σ as follows:

smallest: $[x]R^s[y]$ iff (there are $x' \in |x|$ and $y' \in |y|$ with $x'Ry'$).

largest: $[x]R^l[y]$ iff ($x \models \Diamond\varphi$ whenever $y \models \varphi$ for $\Diamond\varphi \in \Sigma$).

transitive: $[x]R^t[y]$ iff ($x \models \Diamond\varphi$ whenever $y \models \varphi \vee \Diamond\varphi$ for $\Diamond\varphi \in \Sigma$).

smallest transitive: R^{st} is the transitive closure of R^s.

The transitive filtration is also known as the *Lemmon filtration*. The following well-known fact is a routine check.

4.2.2. FACT. Let $\mathfrak{X} = (X, R)$ be a modal space, let v be a valuation on $\mathfrak{X}$, and let $\Sigma \subseteq \mathcal{L}$ be a subformula closed set. Then

(1) (X_Σ, R^s) and (X_Σ, R^l) are filtrations.

(2) If R is transitive, then (X_Σ, R^t) and (X_Σ, R^{st}) are filtrations. Moreover, R^t and R^{st} define transitive relations on X_Σ.

Filtrations are important because of the following theorem.

4.2.3. THEOREM (FILTRATION THEOREM). *Let $\mathfrak{X} = (X, R)$ be a modal space, let v be a valuation on $\mathfrak{X}$, and let $\Sigma \subseteq \mathcal{L}$ be a subformula closed set. If $(\mathfrak{Y}, v')$ is a filtration of $(\mathfrak{X}, v)$ through Σ, then for every $\varphi \in \Sigma$, and every $x \in X$,*

$$x \models_v \varphi \quad \textit{iff} \quad [x] \models_{v'} \varphi.$$

Filtrations algebraically

Here we recall the algebraic perspective on filtration as presented in [18, Section 4]. Recall that the variety of Boolean algebras is locally finite (Theorem 2.1.20), thus, every finitely generated subalgebra of a Boolean algebra is finite.

4.2.4. DEFINITION. Let $\mathfrak{B} = (B, \Diamond)$ be a modal algebra, let v be a valuation on $\mathfrak{B}$, and let Σ be a finite set of formulas closed under subformulas. Let B' be the Boolean subalgebra of B generated by the set

$$v(\Sigma) = \{v(\varphi) \mid \varphi \in \Sigma\}.$$

Then B' is finite because Σ is finite. Set $D_\Sigma = \{v(\varphi) \mid \Diamond\varphi \in \Sigma\}$. Let $\Diamond'$ be a modal operator on B' and let v' be a valuation on $\mathfrak{B}' = (B', \Diamond')$ satisfying the following conditions.

(A1) $v'(p) = v(p)$ for all propositional letters $p \in \Sigma$,

(A2) $\Diamond b \leq \Diamond' b$ for all $b \in B'$, and

(A3) $\Diamond' b \leq \Diamond b$ for all $b \in D_\Sigma$.

Then $(\mathfrak{B}', v')$ is called a *filtration of* $(\mathfrak{B}, v)$ *through* Σ.

Recall that in the frame-theoretic definition of filtrations the relation R' is not fully determined by the axioms (F2) and (F3). Likewise, the modality $\Diamond'$ on B' is not fully determined by (A2) and (A3). In fact, (A2) constitutes a lower bound on $\Diamond' b$ for all $b \in B'$ and (A2) and (A3) together imply that $\Diamond' b = \Diamond b$ for all $b \in D_\Sigma$. As in the frame-theoretic case, we will sometimes omit referring to the valuation v' and say that a modal algebra $\mathfrak{B}'$ is a filtration of $(\mathfrak{B}, v)$.

The specific filtrations discussed in the previous section can be described in purely algebraic terms. Let $\mathfrak{B} = (B, \Diamond)$ be a modal algebra, let v be a valuation on $\mathfrak{B}$, and let Σ be a finite set of formulas closed under subformulas. Let $D_\Sigma \subseteq B$ and B' be as in Definition 4.2.4. Recall from Section 2.1.3 that $\Diamond^+ a$ stands for $\Diamond a \vee a$. Consider the following ways to define $\Diamond$-modalities on B'. For all $a \in B'$,

smallest: $\Diamond^s a = \bigwedge\{b \in B' \mid \Diamond a \leq b\}$,

largest: $\Diamond^g a = \bigwedge\{\Diamond b \in B' \mid a \leq b \text{ and } a = \bigvee F, \text{ for some } F \subseteq D_\Sigma\}$,

smallest transitive: $\Diamond^{st} a = \bigwedge\{\Diamond b \in B' \mid \Diamond a \leq \Diamond b \text{ and } b \in B'\}$,

transitive: $\Diamond^t a = \bigwedge\{\Diamond b \in B' \mid \Diamond^+ a \leq \Diamond^+ b \text{ and } b = \bigvee F, \text{ for some } F \subseteq D_\Sigma\}$.

The following proposition is proved in [18, Lemmas 4.6 and 6.3].

4.2.5. PROPOSITION ([18]). *Let $\mathfrak{B} = (B, \Diamond)$ be a modal algebra with valuation v. Let $\mathfrak{Y} = (Y, R)$ be the modal space dual to $\mathfrak{B}$ and let v_Y be a valuation on $\mathfrak{Y}$ corresponding to v. Let $\Sigma \subseteq \mathcal{L}$ be finite and subformula closed and let D_Σ be as in Definition 4.2.4. Then for all $f \in \{s, l, t, st\}$, the modal algebra $(B', \Diamond^f)$ is isomorphic to the dual algebra of (Y_Σ, R^f).*

The algebraic version of the filtration theorem can be formulated as follows:

4.2.6. THEOREM (FILTRATION THEOREM ALGEBRAICALLY). *Let $(\mathfrak{B}, v)$ and Σ be as in Definition 4.2.4. Let $(\mathfrak{B}', v')$ be a filtration of $(\mathfrak{B}, v)$ through Σ. Then*

$$v'(\varphi) = v(\varphi) \text{ for all } \varphi \in \Sigma.$$

Admitting filtration

Filtrations can be used to prove the fmp of normal modal logics whenever they 'admit filtrations'. We will distinguish between two ways how to define this notion.

4.2.7. Definition. Let M be a normal modal logic.

(1) The logic M *admits filtration in the weak sense* if for every $\varphi \notin \mathsf{M}$, there is a model $(\mathfrak{B}, v)$ based on an M-algebra $\mathfrak{B}$ that refutes φ and a filtration $(\mathfrak{B}', v')$ of $(\mathfrak{B}, v)$ through some finite subformula closed set Σ with $\varphi \in \Sigma$ such that $\mathfrak{B}'$ is an M-algebra.

(2) The logic M *admits filtration* or (M *admits filtration in the strong sense*) if for every M-algebra $\mathfrak{B}$, every valuation v on $\mathfrak{B}$, and every finite set Σ of formulas closed under subformulas, there is a filtration $(\mathfrak{B}', v')$ of $(\mathfrak{A}, v)$ through Σ such that $\mathfrak{B}'$ is an M-algebra.

The above can be formulated in frame-theoretic terms in the obvious way. The definition of admitting filtration in the weak sense follows [40, page 142], and admitting filtration in the strong sense follows [64, page 201]. Clearly the latter is stronger than the former, but the former is sufficient for proving the fmp. Indeed, as an immediate consequence of the filtration theorem we obtain:

4.2.8. Corollary. *If a normal modal logic* M *admits filtration in the weak sense, then* M *has the finite model property.*

For instance, the modal logics K, T, and D admit the smallest and the largest filtrations. The logics $\mathsf{K4}$ and $\mathsf{S4}$ admit the transitive and the smallest transitive filtrations. If a normal modal logic M admits filtration and $(\mathfrak{B}, v)$ is a model based on an M-algebra $\mathfrak{B}$, then a filtration $(\mathfrak{B}', v')$ of $(\mathfrak{B}, v)$ where $\mathfrak{B}'$ is also an M-algebra is called an M-*filtration*.

4.3 Stable rules and M-stable universal classes

In this section we discuss M-stable multi-conclusion consequence relations and prove similar characterization results to those of $\mathcal{H}$-stable stable rule systems from Section 3.3.1. For the most part, we will work with modal algebras and state our results in algebraic terms. However, we will sometimes switch to modal spaces. It should always be clear how to translate definitions and results into frame-theoretic terms even if we do not explicitly state them.

4.3.1. Definition. Let $\mathfrak{A} = (A, \Diamond_A)$ and $\mathfrak{B} = (B, \Diamond_B)$ be modal algebras.

(1) A Boolean homomorphism $h : B \to A$ is called *stable* provided $\Diamond_A h(b) \leq h(\Diamond_B b)$ for all $b \in B$.

(2) If B is a Boolean subalgebra of A and the embedding $h : B \rightarrowtail A$ is stable, then $\mathfrak{B}$ is called a *stable subalgebra* of $\mathfrak{A}$. In other words, $\mathfrak{B}$ is a stable subalgebra of $\mathfrak{A}$ if B is a Boolean subalgebra of A and $\Diamond_A b \leq \Diamond_B b$ for all $b \in B$.

Thus, stable homomorphisms between modal algebras are not full modal algebra homomorphisms, but preserve $\Diamond$ only "half-way." Such maps were studied in [28] under the name of semi-homomorphisms and in [64] under the name of continuous morphisms. We follow [18] in calling them stable homomorphisms. Note that if a modal algebra $\mathfrak{B}'$ is a filtration of $\mathfrak{B}$, then $\mathfrak{B}'$ is a stable subalgebra of $\mathfrak{B}$ by condition (A1).

4.3.2. FACT. If $(\mathfrak{B}', v')$ is a filtration of $(\mathfrak{B}, v)$, then $\mathfrak{B}'$ is a stable subalgebra of $\mathfrak{B}$.

Stable homomorphism and subalgebras can easily be expressed in frame-theoretic terms:

4.3.3. DEFINITION. Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, R')$ be modal spaces.

(1) A continuous map $f : X \to Y$ is called *stable* provided xRy implies $f(x)R'f(y)$ for all $x, y \in X$.

(2) If there is a stable and onto map $f : X \to Y$, then $\mathfrak{Y}$ is called a *stable image* of $\mathfrak{X}$.

We refer to [18, Lemma 3.3] for a proof that stable maps between modal spaces and algebras are dual to each other. We already state the following two lemmas that we will often make use of in the later sections of this chapter. The reader may recall the definition of subdirectly irreducible modal algebras, the definition of well-connected $\mathsf{K4}$-algebras and the definition of the operation $\Box^+$ from Section 2.1.3. A proof of the following lemma can be found in [18, Proposition 6.4].

4.3.4. LEMMA ([18]). *Finite stable subalgebras of subdirectly irreducible modal algebras are subdirectly irreducible.*

4.3.5. LEMMA. *Suppose* $\mathfrak{A} = (A, \Diamond_A)$ *and* $\mathfrak{B} = (B, \Diamond_B)$ *are* $\mathsf{K4}$*-algebras. If* $\mathfrak{A}$ *is well-connected and* $\mathfrak{B}$ *is a stable subalgebra of* $\mathfrak{A}$*, then* $\mathfrak{B}$ *is well-connected.*

Proof:
Since $\mathfrak{B}$ is a stable subalgebra of $\mathfrak{A}$, we have that $\Diamond_A b \leq \Diamond_B b$ for all $b \in B$. Therefore, $\Diamond_A^+ b \leq \Diamond_B^+ b$ for all $b \in B$. Now, let $a, b \in B$ with $\Diamond_B^+ a \wedge \Diamond_B^+ b = 0$. Then $\Diamond_A^+ a \wedge \Diamond_A^+ b = 0$. As $\mathfrak{A}$ is well-connected, $a = 0$ or $b = 0$. Thus, $\mathfrak{B}$ is well-connected. □

4.3.6. DEFINITION. Let M be a normal modal logic.

(1) Suppose $\mathcal{K}$ and $\mathcal{V}$ are two classes of modal algebras with $\mathcal{K} \subseteq \mathcal{V}$. We say that $\mathcal{K}$ is $\mathcal{V}$*-stable* (or *stable within* $\mathcal{V}$) provided for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{V}$, if $\mathfrak{A} \in \mathcal{K}$ and there is a stable embedding $\mathfrak{B} \rightarrowtail \mathfrak{A}$, then $\mathfrak{B} \in \mathcal{K}$.

(2) Let $\mathcal{K}$ be a class of M-algebras. We say that the class $\mathcal{K}$ is M-*stable* if $\mathcal{K}$ is stable within $\mathcal{V}(\mathsf{M})$. We say that the class $\mathcal{K}$ is *finitely* M-*stable* provided for every finite M-algebra $\mathfrak{B}$ and any $\mathfrak{A} \in \mathcal{K}$, whenever there is a stable embedding $\mathfrak{B} \rightarrowtail \mathfrak{A}$, then $\mathfrak{B} \in \mathcal{K}$.

We recall the *stable rules* of [18, Section 7]. Let $\mathfrak{B} = (B, \Diamond)$ be a finite modal algebra. For every $a \in B$, let p_a be a propositional letter such that $a \neq b$ implies $p_a \neq p_b$ for all $a, b \in B$. The *stable* (*multi-conclusion*) *rule* $\rho(\mathfrak{B})$ is the rule Γ/Δ, where

$$\begin{aligned} \Gamma = &\{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid a, b \in B\} \cup \\ &\{p_{\neg a} \leftrightarrow \neg p_a \mid a \in B\} \cup \\ &\{\Diamond p_a \rightarrow p_{\Diamond a} \mid a \in B\} \\ \text{and} \quad \Delta = &\{p_a \mid a \in B, a \neq 1\}. \end{aligned}$$

Stable rules generalize the Jankov rules of [83], which in model theory correspond to diagrams of finite modal algebras [42, page 68]. Recall that a (model-theoretic) structure $\mathfrak{A}$ satisfies the diagram of a finite structure $\mathfrak{B}$ iff $\mathfrak{B}$ is isomorphically embeddable into $\mathfrak{A}$ (see e.g., [42, Propostion 2.1.8]). Similarly, refutation of the stable rule of a finite modal algebra $\mathfrak{B}$ is equivalent to $\mathfrak{B}$ being stably embeddable as was proved in [18, Proposition 7.1].

4.3.7. Lemma ([18]). *For every modal algebra $\mathfrak{A}$ and finite modal algebra $\mathfrak{B}$, $\mathfrak{A} \not\models \rho(\mathfrak{B})$ iff $\mathfrak{B}$ is isomorphic to a stable subalgebra of $\mathfrak{A}$.*

Thus, in particular, every finite modal algebra refutes its own stable rule.

The next lemma is the analogue of Lemma 3.3.7 applied to M-stable universal classes. The first part of the lemma essentially says that "stability is preserved under generating universal classes", i.e. if $\mathcal{K}$ is an M-stable class, then the universal class generated by $\mathcal{K}$ is M-stable. The analogous statement does not hold for varieties, i.e. if $\mathcal{K}$ is M-stable, then the variety $\mathcal{V}(\mathcal{K})$ may not be an M-stable class. The second part of the lemma says that M-stable universal classes satisfy a very strong version of the fmp.

The proof of the lemma below is virtually the same as that of Lemma 3.3.7. For the reader's convenience we still spell out its proof in detail. However, in the following sections we will skip some proofs that are almost the same as in the $\mathcal{H}$-stable case.

4.3.8. Lemma. *Let* M *be a normal modal logic that admits filtration. Let $\mathcal{K}$ be a finitely* M*-stable class of* M*-algebras. Then the following conditions hold.*

(1) The universal class $\mathcal{U}(\mathcal{K})$ is axiomatized over $\mathcal{S}_{\mathsf{M}}$ by stable rules of finite M*-algebras.*

(2) The universal class $\mathcal{U}(\mathcal{K})$ *is* M*-stable.*

(3) $\mathcal{U}(\mathcal{K}) = \mathcal{U}(\mathcal{K}_{\mathsf{fin}})$*, where* $\mathcal{K}_{\mathsf{fin}}$ *is the class of finite members of* $\mathcal{K}$*.*

Proof:

(1) Let $\mathcal{K}$ be finitely M-stable. Let $\mathcal{B}$ be the set of finite non-isomorphic M-algebras that do not belong to $\mathcal{K}$ and let

$$\Psi = \{\rho(\mathfrak{B}) \mid \mathfrak{B} \in \mathcal{B}\}.$$

We show that $\mathcal{S}_{\mathcal{K}}$ is axiomatized by Ψ over $\mathcal{S}_{\mathsf{M}}$. For this it is sufficient to show that $\mathcal{U}(\mathcal{K})$ consists exactly of those M-algebras satisfying Ψ.

First we show that each member of $\mathcal{K}$ satisfies Ψ. If there are $\mathfrak{A} \in \mathcal{K}$ and $\mathfrak{B} \in \mathcal{B}$ such that $\mathfrak{A} \not\models \rho(\mathfrak{B})$, then by Lemma 4.3.7, there is a stable embedding $\mathfrak{B} \rightarrowtail \mathfrak{A}$. Since $\mathcal{K}$ is finitely M-stable and $\mathfrak{B}$ is finite, $\mathfrak{B} \in \mathcal{K}$, which contradicts to $\mathfrak{B} \in \mathcal{B}$. Thus, each member of $\mathcal{K}$ satisfies Ψ. Since $\mathcal{U}(\mathcal{K})$ is generated by $\mathcal{K}$, it follows that each member of $\mathcal{U}(\mathcal{K})$ satisfies Ψ.

Conversely, suppose that an M-algebra $\mathfrak{A}$ validates Ψ, i.e. $\mathfrak{A} \models \rho(\mathfrak{B})$ for each $\mathfrak{B} \in \mathcal{B}$. If $\mathfrak{A} \notin \mathcal{U}(\mathcal{K})$, then there is a multi-conclusion rule Γ/Δ such that $\mathcal{K} \models \Gamma/\Delta$ and a valuation v on $\mathfrak{A}$ such that $(\mathfrak{A}, v) \not\models \Gamma/\Delta$. Let $(\mathfrak{B}, v')$ be an M-filtration of $\mathfrak{B}$ through $\mathsf{Sub}(\Gamma \cup \Delta)$. Then $(\mathfrak{B}, v) \not\models \Gamma/\Delta$. Since $\mathfrak{B}$ is a stable subalgebra of $\mathfrak{A}$, we have $\mathfrak{A} \not\models \rho(\mathfrak{B})$ by Lemma 4.3.7. As $\mathfrak{A}$ satisfies $\rho(\mathfrak{B})$ for each $\mathfrak{B} \in \mathcal{B}$ and $\mathfrak{B}$ is an M-algebra, we see that $\mathfrak{B} \in \mathcal{K}$, so $\mathfrak{B} \in \mathcal{U}(\mathcal{K})$. But this contradicts to $\mathfrak{B} \not\models \Gamma/\Delta$. Therefore, $\mathfrak{A} \in \mathcal{U}(\mathcal{K})$.

(2) Using Lemma 4.3.7 it is easy to see that validity of stable rules is preserved by stable subalgebras. Thus, a universal class axiomatized over $\mathcal{S}_{\mathsf{M}}$ by stable rules of finite M-algebras is M-stable. So, (2) follows from (1).

(3) The inclusion $\mathcal{U}(\mathcal{K}_{\mathsf{fin}}) \subseteq \mathcal{U}(\mathcal{K})$ is obvious. To see the reverse inclusion, let Γ/Δ be a multi-conclusion rule that is refuted in $\mathcal{U}(\mathcal{K})$. Then there is $\mathfrak{A} \in \mathcal{K}$ and a valuation v on $\mathfrak{A}$ such that $(\mathfrak{A}, v)$ refutes Γ/Δ. Let $(\mathfrak{A}', v')$ be an M-filtration of $(\mathfrak{A}, v)$ through $\mathsf{Sub}(\Gamma \cup \Delta)$. Then $\mathfrak{A}'$ refutes Γ/Δ and $\mathfrak{A}' \in \mathcal{K}$ since $\mathfrak{A}'$ is a finite M-algebra and $\mathcal{K}$ is finitely M-stable. Thus, $\mathfrak{A}' \in \mathcal{K}_{\mathsf{fin}}$, and so $\mathcal{U}(\mathcal{K}_{\mathsf{fin}})$ refutes Γ/Δ. □

The following theorem provides many equivalent descriptions of M-stable universal classes and extends [18, Theorem 7.4]. By using Lemma 4.3.8 it can be proved just like Theorem 3.3.8, so we skip the details.

4.3.9. Theorem. *Let* M *be a normal modal logic that admits filtration. Let* $\mathcal{U}$ *be a universal class of modal algebras. The following are equivalent:*

(1) $\mathcal{U}$ *is* M*-stable.*

(2) $\mathcal{U}$ is generated by an M*-stable class.*

(3) $\mathcal{U}$ is generated by a finitely M*-stable class.*

(4) $\mathcal{U}$ is generated by an M*-stable class of finite* M*-algebras.*

(5) $\mathcal{U}$ is axiomatizable over $\mathcal{S}_{\mathsf{M}}$ by stable rules of finite M*-algebras.*

4.4 M-stable modal logics

We introduce M-stable logics and prove many equivalent characterizations.

4.4.1. DEFINITION. We say that the logic L is M-*stable* if the variety $\mathcal{V}(\mathsf{L})$ is generated by an M-stable class.

The K-stable logics are simply called *stable logics* as in [18]. Note that the definition of an M-stable logic L requires its corresponding variety $\mathcal{V}(\mathsf{L})$ to be generated by an M-stable class, but the variety $\mathcal{V}(\mathsf{L})$ may itself not be an M-stable class.

M-stable logics are of interest to us only when M admits filtration (in the strong sense). In that case, every M-stable logic L has the fmp. Roughly speaking, whenever L is M-stable and M admits filtration, then the fmp of L can be shown with the "same proof" as the fmp for M.

4.4.2. PROPOSITION. *If* M *is a normal modal logic that admits filtration, then every* M*-stable logic admits filtration in the weak sense and thus has the fmp.*

Proof:
Let L be M-stable. Then $\mathcal{V}(\mathsf{L})$ is generated by an M-stable class $\mathcal{K}$. If $\mathsf{L} \nvdash \varphi$, then there is $\mathfrak{B} \in \mathcal{K}$ and a valuation v on $\mathfrak{B}$ such that $(\mathfrak{B}, v) \not\models \varphi$. Let $\mathsf{Sub}(\varphi)$ be the set of subformulas of φ. Since M admits filtration, there is a finite M-algebra $\mathfrak{B}'$ and a valuation v' on $\mathfrak{B}'$ such that $(\mathfrak{B}', v')$ is a filtration of $(\mathfrak{B}, v)$ through $\mathsf{Sub}(\varphi)$. Because $\mathcal{K}$ is M-stable, $\mathfrak{B}' \in \mathcal{K}$. By the Filtration Theorem, $(\mathfrak{B}', v') \not\models \varphi$. Thus, L has the fmp. □

As in the case of $\mathcal{H}$-stable si logics, we are aiming for a characterization of M-stable modal logics that tells us more about specific properties of the generating M-stable class of M-stable logics. For this purpose, we single out the property of M being closed under adding *sharp roots* to its frames. If M satisfies that property, we show that the M-stable generating set of an M-stable logic can be assumed to consist of subdirectly irreducible algebras only.

4.4.3. Definition.

(1) Let $\mathfrak{F} = \langle W, R \rangle$ be a finite Kripke frame. We call $r \in W$ a *sharp root of* $\mathfrak{F}$ if rRw for all $w \in W$.

(2) Let $\mathfrak{F} = (W, R)$ be a finite Kripke frame and let $r \notin W$. We set $\mathfrak{F}^r = (W', R')$ where $W' = W \cup \{r\}$ and $R' = R \cup \{(r, w) \mid w \in W'\}$. Figuratively speaking, $\mathfrak{F}^r$ is obtained by adding a sharp root below $\mathfrak{F}$.

(3) We say that a normal modal logic M has the $(*)$*-property* if for each finite M-frame $\mathfrak{F}$ we have that $\mathfrak{F}^r$ is also an M-frame.

Note that by definition, if r is a sharp root, then it is reflexive. In algebraic terms, a sharp root corresponds to an atom a of a finite modal algebra $\mathfrak{A}$ such that $a \leq \Diamond b$ for all $0 \neq b \in \mathfrak{A}$. Consequently, a normal modal logic M has the $(*)$-property if for every finite M-algebra $\mathfrak{A} = (A, \Diamond)$, the algebra $\mathfrak{A}' = (A', \Diamond')$ is an M-algebra, where A' is the Boolean algebra generated by A and a fresh atom a with $\Diamond' a = a$ and $\Diamond' b = \Diamond b \vee a$ for every atom $b \in A$.

Examples of normal modal logics satisfying the $(*)$-property are K, D, T, $\mathsf{K4}$, $\mathsf{S4}$, etc. On the other hand, the logics KB, $\mathsf{S5}$, and GL do not satisfy the $(*)$-property (see Table A.0.5 for the definition of these logics).

The following theorem provides a characterization of M-stable logics. Some of the equivalences have already been proved in [18, Theorem 7.6]. It is very similar to Theorem 3.3.17, but with some drawbacks. As explained in Section 3.3.2, Theorem 3.3.17 allowed us to "shift" the $\mathcal{H}$-stability condition from an arbitrary $\mathcal{H}$-stable subclass of an $\mathcal{H}$-stable variety $\mathcal{V}(\mathsf{L})$ to $\mathcal{H}$-stability of the subdirectly irreducible L-algebras. The latter for instance entails that $\mathcal{H}$-stable si logics are a complete sublattice of Λ_{IPC}.

Note that Theorem 4.4.4 is not as strong. Even if M has the $(*)$-property, the theorem below only shows that an M-stable logic L is generated by *some* M-stable class of subdirectly irreducible L-algebras. It does not show that the class of *all* subdirectly irreducible L-algebras is M-stable.

4.4.4. Theorem. *Suppose* M *is a normal modal logic that admits filtration and* L *is a normal extension of* M*. The following are equivalent.*

(1) L *is* M*-stable.*

(2) $\mathcal{V}(\mathsf{L})$ *is generated by a finitely* M*-stable class.*

(3) $\mathcal{V}(\mathsf{L})$ *is generated by an* M*-stable class of finite* M*-algebras.*

(4) $\mathcal{V}(\mathsf{L})$ *is generated by an* M*-stable universal class of* M*-algebras.*

(5) L *is axiomatizable over* M *by stable rules of finite* M*-algebras.*

Moreover, if M *has the* $(*)$*-property, then the above conditions are equivalent to the following ones:*

(6) $\mathcal{V}(\mathsf{L})$ *is generated by an* M*-stable class of finite subdirectly irreducible* M*-algebras.*

(7) $\mathcal{V}(\mathsf{L})$ *is generated by a class* $\mathcal{K}$ *of subdirectly irreducible* M*-algebras that is stable within* $\mathcal{V}(\mathsf{M})_{\mathsf{si}}$*.*

(8) $\mathcal{V}(\mathsf{L})$ *is generated by a finitely* M*-stable class of subdirectly irreducible algebras.*

Proof:
The equivalence of the first five statements can be proved exactly as in Theorem 3.3.17.

So we assume that M has the $(*)$-property. The implication (6) $\Rightarrow$ (7) follows from the fact that an M-stable class of finite subdirectly irreducible M-algebras is stable within $\mathcal{V}(\mathsf{M})_{\mathsf{si}}$. This is because stable subalgebras of finite subdirectly irreducible algebras are subdirectly irreducible by Lemma 4.3.4.

The implication (7) $\Rightarrow$ (8) is immediate since if a class $\mathcal{K}$ is stable within $\mathcal{V}(\mathsf{M})_{\mathsf{si}}$, then $\mathcal{K}$ is finitely M-stable. The implication (8) $\Rightarrow$ (2) is obvious.

To finish the proof, we show (3) $\Rightarrow$ (6). Suppose $\mathcal{K}$ is a stable class of finite M-algebras that generates $\mathcal{V}(\mathsf{L})$. Let $\mathcal{K}_{\mathsf{si}}$ be the class of all subdirectly irreducible members of $\mathcal{K}$. It is sufficient to show that $\mathcal{K}_{\mathsf{si}}$ generates $\mathcal{V}(\mathsf{L})$, and for this it is sufficient to show that $\mathcal{K}$ is contained in the variety generated by $\mathcal{K}_{\mathsf{si}}$, i.e. $\mathcal{K} \subseteq \mathcal{V}(\mathcal{K}_{\mathsf{si}})$. Suppose $\mathfrak{A} \in \mathcal{K}$. If $\mathfrak{A}$ is subdirectly irreducible, then $\mathfrak{A} \in \mathcal{K}_{\mathsf{si}}$, and there is nothing to prove.

So assume that $\mathfrak{A}$ is not subdirectly irreducible. By Theorem 2.1.8, we have that $\mathfrak{A}$ is a subdirect product of its subdirectly irreducible homomorphic images. Therefore, to conclude that $\mathfrak{A} \in \mathcal{V}(\mathcal{K}_{\mathsf{si}})$, it is sufficient to show that every subdirectly irreducible homomorphic image of $\mathfrak{A}$ belongs to $\mathcal{V}(\mathcal{K}_{\mathsf{si}})$.

Let $\mathfrak{B}$ be a subdirectly irreducible homomorphic image of $\mathfrak{A}$. Since $\mathfrak{A}$ is finite, so is $\mathfrak{B}$. Therefore, the dual $\mathfrak{Y} = (Y, R)$ of $\mathfrak{B}$ is a finite rooted M-frame. Let $\mathfrak{Y}^r = (Y', R')$ be obtained from $\mathfrak{Y}$ by adding a new sharp root r below $\mathfrak{Y}$. Since M has the $(*)$-property, $\mathfrak{Y}^r$ is an M-frame. Since $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$, $\mathfrak{Y}$ is a generated subframe of $\mathfrak{X}$, where $\mathfrak{X} = (X, R)$ is the dual of $\mathfrak{A}$.

Since $\mathfrak{A}$ is not subdirectly irreducible $\mathfrak{X}$ is not rooted, and since $\mathfrak{Y}$ is rooted, we have $\mathfrak{Y} \neq \mathfrak{X}$. Define $f : X \to Y'$ by mapping the points of Y to themselves and the points of $X \setminus Y$ to r. It is easy to see that f is onto and stable.

Let $\mathfrak{B}'$ be the dual algebra of $\mathfrak{Y}^r$. Then the dual of f is a stable embedding from $\mathfrak{B}'$ into $\mathfrak{A}$. Since $\mathfrak{A} \in \mathcal{K}$ and $\mathcal{K}$ is M-stable, we conclude that $\mathfrak{B}' \in \mathcal{K}$. Since $\mathfrak{Y}^r$ is finite and rooted, $\mathfrak{B}'$ is subdirectly irreducible, and hence $\mathfrak{B}' \in \mathcal{K}_{\mathsf{si}}$. Now, $\mathfrak{Y}$ is a generated subframe of $\mathfrak{Y}^r$, so $\mathfrak{B}$ is a homomorphic image of $\mathfrak{B}'$, and hence

$\mathfrak{B}$ belongs to $\mathcal{V}(\mathcal{K}_{\mathsf{si}})$ as desired. □

We conclude with some small observations on how M-stable logics lie in the lattice of normal modal logics. Recall that for a normal modal logic M, by NExtM we denote the sublattice of the lattice of all normal modal logics consisting of normal extensions of M.

4.4.5. PROPOSITION. *Suppose* $\mathsf{M}, \mathsf{L}, \mathsf{N}$ *are normal modal logics with* $\mathsf{M} \subseteq \mathsf{L} \subseteq \mathsf{N}$.

(1) If N *is* M*-stable, then* N *is* L*-stable.*

(2) The converse of (1) *is not true in general, i.e. if* N *is* L*-stable, then* N *may not be* M*-stable.*

(3) If $\mathcal{V}(\mathsf{L})$ *is a* $\mathcal{V}(\mathsf{M})$*-stable class, then* N *is* L*-stable iff* N *is* M*-stable.*

(4) The M*-stable logics form a* $\bigwedge$*-subsemilattice of* NExtM*.*

Proof:

(1) Since N is M-stable, $\mathcal{V}(\mathsf{N})$ is generated by an M-stable class $\mathcal{K}$. As $\mathcal{K}$ is M-stable, it is obviously L-stable. Thus, N is L-stable.

(2) We will see in Section 4.7 that taking $\mathsf{M} = \mathsf{K}$, $\mathsf{L} = \mathsf{K4}$, and $\mathsf{N} = \mathsf{S4}$ provides the desired counterexample.

(3) One implication follows from (1). For the other, suppose that N is L-stable. Then $\mathcal{V}(\mathsf{N})$ is generated by an L-stable class $\mathcal{K}$. Since $\mathcal{V}(\mathsf{L})$ is $\mathcal{V}(\mathsf{M})$-stable, $\mathcal{K}$ is also $\mathcal{V}(\mathsf{M})$-stable. Therefore, N is M-stable.

(4) Suppose that $\{\mathsf{L}_i \mid i \in I\}$ is a family of M-stable logics. Then every L_i is generated by some M-stable class $\mathcal{K}_i$. Clearly the class $\bigcup\{\mathcal{K}_i \mid i \in I\}$ is also M-stable, and generates $\mathcal{V}(\bigwedge\{\mathsf{L}_i \mid i \in I\})$. □

Note that by (4) in the above proposition, M-stable logics form a $\bigwedge$-subsemilattice of NExtM and thus form a complete lattice. It is however not clear whether M-stable logics form a complete sublattice of NExtM.

4.4.1 Tabular M-stable logics

In this section we show that tabular M-stable logics satisfy stronger properties than those of Theorem 4.4.4 and Proposition 4.4.5 (see Section 2.3.4 for the definition of tabularity). In particular, we show that tabular M-stable logics have the property that the class of all their subdirectly irreducible algebras is a stable class (whenever M admits filtration and has the $(*)$-property). This property implies that tabular M-stable logics form a $\bigvee$-subsemilattice of NExtM. The latter was left open in the general case as we remarked at the end of the previous section. Finally, we show that every extension of $\mathsf{S5}$ is a stable logic. Therefore, the proper extensions of $\mathsf{S5}$ constitute an infinite family of tabular stable modal logics.

4.4.6. PROPOSITION. *Let* M *be a normal modal logic admitting filtration and satisfying the* $(*)$*-property.*

(1) If L *is a tabular* M*-stable normal extension of* M*, then* $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ *is* M*-stable.*

(2) The tabular M*-stable normal modal logics form a* $\bigvee$*-subsemilattice of* NExtM*.*

Proof:

(1) Since L is M-stable, by Theorem 4.4.4, there is an M-stable class $\mathcal{K}$ of subdirectly irreducible algebras that generates $\mathcal{V}(\mathsf{L})$. Since L is tabular, we may assume that $\mathcal{K}$ is a finite class of finite subdirectly irreducible algebras. Let $\mathfrak{B} \in \mathcal{V}(\mathsf{L})_{\mathsf{si}}$ and let $\mathfrak{C}$ be a stable subalgebra of $\mathfrak{B}$. By Jónsson's Lemma—or rather its Corollary 2.1.11—$\mathfrak{B} \in \mathbf{HS}(\mathcal{K})$, so there is $\mathfrak{A}$ in $\mathbf{S}(\mathcal{K})$ such that $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$. Since $\mathfrak{C}$ is finite, it is subdirectly irreducible by Lemma 4.3.4. Therefore, it is sufficient to show that $\mathfrak{C}$ is an L-algebra. Let $\mathfrak{X}$ be the dual of $\mathfrak{A}$, let $\mathfrak{Y}$ be the dual of $\mathfrak{B}$, and let $\mathfrak{Z}$ be the dual of $\mathfrak{C}$. Then $\mathfrak{Y}$ is a generated subframe of $\mathfrak{X}$ and $\mathfrak{Z}$ is a stable image of $\mathfrak{Y}$. Since $\mathcal{K}$ is M-stable, so is $\mathbf{S}(\mathcal{K})$. Thus, all stable images of $\mathfrak{X}$ are L-frames. If $\mathfrak{X} = \mathfrak{Y}$, then $\mathfrak{Z}$ is a stable image of $\mathfrak{X}$, and so $\mathfrak{Z}$ is an L-frame. If $\mathfrak{X} \neq \mathfrak{Y}$, then since M has the $(*)$-property, the frame $\mathfrak{Z}^r$, obtained from $\mathfrak{Z}$ by adding a sharp root below $\mathfrak{Z}$, is an M-frame. As we observed in the proof of Theorem 4.4.4, $\mathfrak{Z}^r$ is a stable image of $\mathfrak{X}$. Therefore, $\mathfrak{Z}^r$ is an L-frame, and hence so is $\mathfrak{Z}$ as a generated subframe of $\mathfrak{Z}^r$. Thus, $\mathfrak{C} \in \mathcal{V}(\mathsf{L})_{\mathsf{si}}$.

(2) Suppose $\{\mathsf{L}_i \mid i \in I\}$ is a family of tabular M-stable logics. By (1), $\mathcal{V}(\mathsf{L}_i)_{\mathsf{si}}$ is M-stable for all $i \in I$. Therefore, $\mathcal{V}(\bigvee\{\mathsf{L}_i \mid i \in I\})_{\mathsf{si}} = \bigcap\{\mathcal{V}(\mathsf{L}_i)_{\mathsf{si}} \mid i \in I\}$ is M-stable. Thus, $\bigvee\{\mathsf{L}_i \mid i \in I\}$ is M-stable, and it is clearly tabular. □

The proof of the proposition above uses essentially that subdirectly irreducible L-algebras are finite, and does not extend directly to non-tabular logics. Next we show how to create examples of tabular stable logics. For a finite modal algebra $\mathfrak{A}$, let $\mathbf{Stable}(\mathfrak{A})$ be the class of modal algebras that are isomorphic to stable subalgebras of $\mathfrak{A}$, and let $\mathsf{L}(\mathbf{Stable}(\mathfrak{A}))$ be the normal modal logic of $\mathbf{Stable}(\mathfrak{A})$.

4.4.7. THEOREM.

(1) For a finite modal algebra $\mathfrak{A}$*, the logic* $\mathsf{L}(\mathbf{Stable}(\mathfrak{A}))$ *is a stable modal logic.*

(2) Every extension of $\mathsf{S5}$ *is a stable modal logic. Thus, there are infinitely many tabular stable logics.*

Proof:

(1) This is obvious since **Stable**($\mathfrak{A}$) is a stable class of modal algebras.

(2) It is well known that an S5-algebra is subdirectly irreducible iff its dual is a cluster (see Table A.0.4 for the definition of a cluster). It is easy to see that the class of finite clusters is a stable class. Since S5 is the logic of this class, S5 is a stable logic. It is also well known that for every extension L of S5 there is n such that L is the logic of clusters of cardinality $\leq n$. This class is stable by the same reasoning. Thus, every extension of S5 is stable. As every genuine extension of S5 is tabular and there are infinitely many such, there are infinitely many tabular stable logics. □

Note that since tabular M-stable logics form a $\bigvee$-subsemilattice of NExtM, tabular M-stable logics form a complete lattice. However, by the theorem above this lattice is—in general—*not* a *complete* sublattice of NExtM. Indeed, recall that S5 is the meet of all its (tabular) extensions but is itself not tabular. By the above, S5 and all its extensions are stable logics, thus, tabular stable logics do not form a complete $\bigwedge$-subsemilattice of NExtK.

4.4.2 A continuum of stable logics

Finally, we show that there are continuum many stable logics. In fact, we show that there are continuum many stable logics above the normal modal logic wK4 of weakly transitive frames, where a frame $\mathfrak{F} = (X, R)$ is *weakly transitive* provided xRy, yRz, and $x \neq z$ imply xRz, for all $x, y, z \in X$.

For our proof we will make use of Jankov formulas for finite wK4-algebras (see [112] or [15, Section 7.2]). For a finite subdirectly irreducible wK4-algebra $\mathfrak{A}$, let $\chi(\mathfrak{A})$ be the Jankov formula of $\mathfrak{A}$. Then for a wK4-algebra $\mathfrak{B}$, we have:

$$\mathfrak{B} \not\models \chi(\mathfrak{A}) \text{ iff } \mathfrak{A} \text{ is a subalgebra of a homomorphic image of } \mathfrak{B}.$$

See e.g. [15, Proposition 7.5] for a proof of the above. Dually, if $\mathfrak{F}$ is a finite rooted weakly transitive frame and $\mathfrak{X}$ is an arbitrary weakly transitive space, then we have:

$$\mathfrak{X} \not\models \chi(\mathfrak{F}) \text{ iff } \mathfrak{F} \text{ is a p-morphic image of a generated subframe of } \mathfrak{X}.$$

In the following we will often switch back and forth between modal algebras and their duals. If $\mathfrak{A}$ is a finite modal algebra and $\mathfrak{F}$ is its dual, then we often write $\rho(\mathfrak{F})$ instead of $\rho(\mathfrak{A})$. As usual, we denote a reflexive point by $\circ$ and an irreflexive point by $\bullet$.

4.4.8. Theorem. *There is a continuum of weakly transitive non-transitive stable modal logics.*

Proof:
For $n \geq 2$ let $\mathfrak{C}_n = (X_n, R_n)$ be the irreflexive n-point cluster depicted in Figure 4.4.1; that is, $X_n = \{x_1, \ldots x_n\}$ and $R_n = \{(x_i, x_j) \in X_n \times X_n \mid i \neq j\}$.

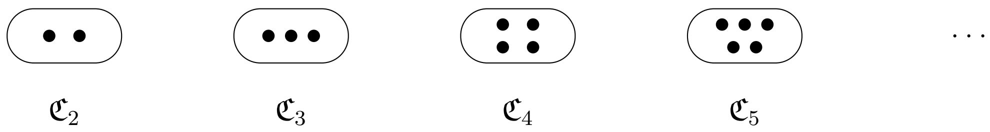


Figure 4.4.1

Let $\mathbb{N}_{\geq 2} = \{n \in \mathbb{N} \mid n \geq 2\}$. For $I \subseteq \mathbb{N}_{\geq 2}$ set

$$\mathcal{K}_I = \{\mathfrak{X} \mid \exists n \in I \text{ such that } \mathfrak{X} \text{ is a stable image of } \mathfrak{C}_n\}.$$

It is clear that $\mathcal{K}_I$ is a stable class of modal spaces. Let L_I be the logic of $\mathcal{K}_I$. Since $\mathcal{K}_I$ is stable, L_I is a stable modal logic. Thus, we need to show that if $I \neq J$, then $\mathsf{L}_I \neq \mathsf{L}_J$. First we show that $n \in I$ iff $\chi(\mathfrak{C}_n) \notin \mathsf{L}_I$.

If $n \in I$, then $\mathfrak{C}_n \in \mathcal{K}_I$, so $\mathfrak{C}_n \models \mathsf{L}_I$. Clearly $\mathfrak{C}_n \not\models \chi(\mathfrak{C}_n)$, which implies that $\chi(\mathfrak{C}_n) \notin \mathsf{L}_I$. Conversely, suppose that $\chi(\mathfrak{C}_n) \notin \mathsf{L}_I$. Since L_I is the logic of $\mathcal{K}_I$, there is $\mathfrak{X} \in \mathcal{K}_I$ such that $\mathfrak{X} \not\models \chi(\mathfrak{C}_n)$. Therefore, $\mathfrak{C}_n$ is a p-morphic image of a generated subframe of $\mathfrak{X}$. But the only generated subframe of $\mathfrak{X}$ is $\mathfrak{X}$, so $\mathcal{K}_I$ is closed under generated subframes. Also a p-morphic image of $\mathfrak{X}$ is a stable image of $\mathfrak{X}$, and $\mathcal{K}_I$ is closed under stable images. Thus, $\mathfrak{C}_n \in \mathcal{K}_I$.

If $n \notin I$, then there is $m \in I$ and an onto stable map $f : \mathfrak{C}_m \twoheadrightarrow \mathfrak{C}_n$. Since $m = |\mathfrak{C}_m| > |\mathfrak{C}_n| = n$, we see that f must identify at least two points of $\mathfrak{C}_m$. Therefore, there are distinct $x, y \in \mathfrak{C}_m$ with $f(x) = f(y)$. Thus, $xR_m y$ and $f(x) \not R_n f(y)$, which is a contradiction because f is stable. Consequently, $n \in I$, and so $n \in I$ iff $\chi(\mathfrak{C}_n) \notin \mathsf{L}_I$.

Now, if $I \neq J$, then without loss of generality we may assume that there is $n \in I \setminus J$. By the above, $\chi(\mathfrak{C}_n) \in \mathsf{L}_J \setminus \mathsf{L}_I$, and hence $\mathsf{L}_I \neq \mathsf{L}_J$.

Since each $\mathfrak{C}_n$ is weakly transitive and non-transitive, we conclude that all logics in the family $\{\mathsf{L}_I \mid I \subseteq \mathbb{N}_{\geq 2}\}$ are weakly transitive non-transitive stable logics. Obviously, the cardinality of this family is that of the continuum. $\square$

4.5 Transitive stable modal logics

We next study M-stability when M is a normal extension of $\mathsf{K4}$ that admits filtration and has the $(*)$-property. Moving to the transitive case has the advantage that—similarly to the intuitionistic case—stable rules can be turned into stable formulas (see Section 2.3.7). We show that for such M, M-stable logics are axiomatizable by stable formulas (building on [18]).

However, logics axiomatized by M-stable formulas are not always M-stable. But if M is a normal extension of $\mathsf{S4}$ then M-stable logics are exactly the normal extensions of M axiomatizable by stable formulas.

We also improve on the characterizations from Theorem 4.4.4. As a corollary we derive that transitive M-stable logics form a complete sublattice of NextM whenever M admits filtration.

As in [18, Section 6.2], if $\mathfrak{B}$ is a finite subdirectly irreducible $\mathsf{K4}$-algebra, then the stable rule $\rho(\mathfrak{B}) = \Gamma/\Delta$ can be rewritten into a stable formula by taking its characteristic formula (see Section 2.3.7).

4.5.1. Definition. Let $\mathfrak{B}$ be a finite subdirectly irreducible $\mathsf{K4}$-algebra, the *stable formula of* $\mathfrak{B}$ is defined as

$$\gamma(\mathfrak{B}) = \bigwedge\{\Box^+\gamma \mid \gamma \in \Gamma\} \to \bigvee\{\Box^+\delta \mid \delta \in \Delta\},$$

where $\rho(\mathfrak{B}) = \Gamma/\Delta$ is the stable rule of $\mathfrak{B}$. If $\mathfrak{G}$ is a finite rooted $\mathsf{K4}$-frame, then we write $\gamma(\mathfrak{G})$ for the stable formula of the dual algebra of $\mathfrak{G}$.

Then Lemmas 2.3.11 and 4.3.7 together entail:

4.5.2. Lemma ([18]). *Let $\mathfrak{B}$ be a finite subdirectly irreducible $\mathsf{K4}$-algebra and let $\mathfrak{A}$ be an arbitrary $\mathsf{K4}$-algebra. The following are equivalent:*

(1) $\mathfrak{A} \not\models \gamma(\mathfrak{B})$.

(2) There is a subdirectly irreducible homomorphic image $\mathfrak{C}$ of $\mathfrak{A}$ and a stable embedding from $\mathfrak{B}$ into $\mathfrak{C}$.

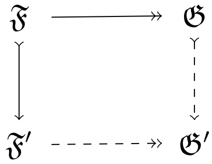
(3) There is a well-connected homomorphic image $\mathfrak{C}$ of $\mathfrak{A}$ and a stable embedding from $\mathfrak{B}$ into $\mathfrak{C}$.

In particular, if $\mathfrak{A}$ is well-connected and $\mathfrak{B}$ is isomorphic to a stable subalgebra of $\mathfrak{A}$, then $\mathfrak{A} \not\models \gamma(\mathfrak{B})$

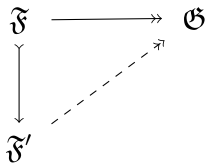
Next we prove two lemmas that will help us to find convenient characterizations of M-stable logics (cf. Lemma 3.3.14). Note that for finite $\mathsf{K4}$-frames, the sharp roots from Definition 4.4.3 are the same as reflexive roots.

4.5.3. Lemma. *Let $\mathfrak{G} = (Y, Q)$, $\mathfrak{F} = (X, R)$, and $\mathfrak{F}' = (X', R')$ be finite $\mathsf{K4}$-frames such that $\mathfrak{G}$ is a stable image of $\mathfrak{F}$ and $\mathfrak{F}$ is a generated subframe of $\mathfrak{F}'$.*

(1) There is a finite $\mathsf{K4}$-frame $\mathfrak{G}' = (Y', Q')$ such that $\mathfrak{G}$ is a generated subframe of $\mathfrak{G}'$, $\mathfrak{G}'$ is a stable image of $\mathfrak{F}'$, and the following diagram commutes.



(2) *If in addition $\mathfrak{G}$ has a sharp root, then $\mathfrak{G}$ is a stable image of $\mathfrak{F}'$ and the following diagram commutes.*



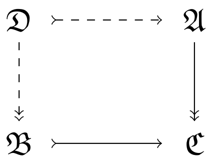
Proof:
(1). If $\mathfrak{F} = \mathfrak{F}'$, then there is nothing to show as we can take $\mathfrak{G}'$ to be $\mathfrak{G}$. Otherwise we let $\mathfrak{G} = \mathfrak{G}^r$ be obtained by adding a sharp root r below $\mathfrak{G}$. It is easy to see that $\mathfrak{G}^r$ is a K4-frame and that $\mathfrak{G}$ is a generated subframe of $\mathfrak{G}^r$. Moreover, the same argument as in the proof of Theorem 4.4.4 yields that $\mathfrak{G}^r$ is a stable image of $\mathfrak{F}'$. Furthermore, it follows from the definition that the diagram commutes.

(2). Let $f : \mathfrak{F} \to \mathfrak{G}$ be an onto stable map. Define $g : \mathfrak{F}' \to \mathfrak{G}$ so that the restriction of g to X is f and g maps $X' \setminus X$ to the reflexive root r of $\mathfrak{G}$ (provided $X' \setminus X \neq \varnothing$). Then it is easy to see that g is an onto stable map, and that the diagram commutes. □

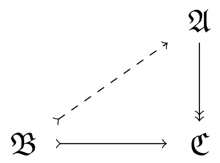
We can reformulate Lemma 4.5.3 in algebraic terms as follows.

4.5.4. Lemma. *Let $\mathfrak{B}$, $\mathfrak{A}$, and C be finite* K4*-algebras such that there is a stable embedding of $\mathfrak{B}$ into $\mathfrak{C}$ and $\mathfrak{C}$ is a homomorphic image of $\mathfrak{A}$.*

(1) *There is a finite* K4*-algebra $\mathfrak{D}$ such that $\mathfrak{B}$ is a homomorphic image of $\mathfrak{D}$, $\mathfrak{D}$ is isomorphic to a stable subalgebra of $\mathfrak{A}$, and the following diagram commutes.*



(2) *If in addition $\mathfrak{B}$ has an atom a such that $a \leq \Diamond b$ for all $0 \neq b \in \mathfrak{B}$, then there is a stable embedding of $\mathfrak{B}$ into $\mathfrak{A}$ and the following diagram commutes.*



4.5.5. Lemma. *Let* M *be a normal extension of* $\mathsf{K4}$ *that admits filtration and has the* $(*)$*-property and let* L *be* M*-stable. For any finite subdirectly irreducible* M*-algebra* $\mathfrak{B}$*,* $\mathfrak{B} \not\models \mathsf{L}$ *iff* $\gamma(\mathfrak{B}) \in \mathsf{L}$.

Proof:
If $\mathfrak{B} \models \mathsf{L}$, then $\gamma(\mathfrak{B}) \notin \mathsf{L}$ since $\mathfrak{B}$ refutes $\gamma(\mathfrak{B})$ by Lemma 4.5.2. Conversely, suppose that $\gamma(\mathfrak{B}) \notin \mathsf{L}$. We show that $\mathfrak{B} \models \mathsf{L}$. Since L is M-stable, by Theorem 4.4.4, $\mathcal{V}(\mathsf{L})$ is generated by an M-stable class $\mathcal{K}$ of finite M-algebras. Then there is $\mathfrak{A} \in \mathcal{K}$ such that $\mathfrak{A} \not\models \gamma(\mathfrak{B})$. By Lemma 4.5.2, there is a subdirectly irreducible homomorphic image $\mathfrak{C}$ of $\mathfrak{A}$ and a stable embedding of $\mathfrak{B}$ into $\mathfrak{C}$.

By Lemma 4.5.4(1), there is a finite $\mathsf{K4}$-algebra $\mathfrak{D}$ such that $\mathfrak{D}$ is isomorphic to a stable subalgebra of $\mathfrak{A}$ and $\mathfrak{B}$ is a homomorphic image of $\mathfrak{D}$. Since M has the $(*)$-property, it follows from the proof of Lemma 4.5.3(1) that $\mathfrak{D}$ is an M-algebra. As $\mathcal{K}$ is M-stable and $\mathfrak{A} \in \mathcal{K}$, we have that $\mathfrak{D} \in \mathcal{K}$. Because $\mathcal{V}(\mathsf{L})$ is closed under homomorphic images, $\mathfrak{B} \in \mathcal{V}(\mathsf{L})$. Therefore, $\mathfrak{B} \models \mathsf{L}$. $\square$

We next build on Theorem 4.4.4 and obtain several more convenient characterizations of M-stability when M is a normal extension of $\mathsf{K4}$ that admits filtration and satisfies the $(*)$-property. The proof is a straightforward adaption of the proof of Theorem 3.3.17. Nevertheless we spell it out for the reader's convenience. For a class $\mathcal{K}$ of $\mathsf{K4}$-algebras we employ the following notations:

- By $\mathcal{K}_{\mathsf{wc}}$ we denote the class of well-connected members of $\mathcal{K}$.
- By $\mathcal{K}_{\mathsf{fsi}}$ we denote the class of finite subdirectly irreducible members of $\mathcal{K}$.

4.5.6. Theorem. *Let* M *be a normal extension of* $\mathsf{K4}$ *that admits filtration and has the* $(*)$*-property. For a normal extension* L *of* M*, the following are equivalent.*

(1) L *is* M*-stable.*

(2) $\mathcal{V}(\mathsf{L})_{\mathsf{wc}}$ *is* M*-stable.*

(3) $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ *is* M*-stable within* $\mathcal{V}(\mathsf{M})_{\mathsf{si}}$.

(4) $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ *is finitely* M*-stable.*

(5) $\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ *is* M*-stable and generates* $\mathcal{V}(\mathsf{L})$.

Moreover, each M*-stable logic is axiomatizable by stable formulas.*

Proof:
First we show (1) $\Rightarrow$ (2). Suppose that L is M-stable, let $\mathfrak{A} \in \mathcal{V}(\mathsf{L})_{\mathsf{wc}}$, and suppose that $\mathfrak{A}'$ is an M-algebra that is a stable subalgebra of $\mathfrak{A}$. By Lemma 4.3.5, $\mathfrak{A}'$ is well-connected. Thus, it remains to show that $\mathfrak{A}' \models \mathsf{L}$. Suppose not. Then there is a formula φ with $\varphi \in \mathsf{L}$, but $(\mathfrak{A}', v) \not\models \varphi$ for some valuation v on $\mathfrak{A}'$. Let $(\mathfrak{B}, v_B)$ be an M-filtration of $\mathfrak{A}'$ through φ. As a finite stable subalgebra of a well-connected $\mathsf{K4}$-algebra, $\mathfrak{B}$ is subdirectly irreducible by Lemmas 4.3.5 and 2.1.25. Moreover, $(\mathfrak{B}, v_B) \not\models \varphi$, so $\mathfrak{B} \not\models \mathsf{L}$. Thus, by Lemma 4.5.5, $\gamma(\mathfrak{B}) \in \mathsf{L}$. However, $\mathfrak{B}$ a stable subalgebra of $\mathfrak{A}$ and $\mathfrak{A}$ is well-connected, $\mathfrak{A} \not\models \gamma(\mathfrak{B})$ by Lemma 4.5.2. The latter contradicts to $\mathfrak{A} \models \mathsf{L}$. Thus, $\mathfrak{A}' \in \mathcal{V}(\mathsf{L})$ and so (2) holds.

The implication (2) $\Rightarrow$ (3) follows immediately from the fact that subdirectly irreducible $\mathsf{K4}$-algebras are well-connected (Theorem 2.1.25). The implication (3) $\Rightarrow$ (4) is obvious.

To see (4) $\Rightarrow$ (5), observe that if $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ is finitely M-stable, then $\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ is M-stable. The collection $\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ also generates $\mathcal{V}(\mathsf{L})$. Indeed, if $\varphi \notin \mathsf{L}$, then there is $\mathfrak{A} \in \mathcal{V}(\mathsf{L})_{\mathsf{si}}$ and a valuation v on $\mathfrak{A}$ with $(\mathfrak{A}, v) \not\models \varphi$. Let $(\mathfrak{B}, v)$ be an M-filtration of $(\mathfrak{A}, v)$ through $\mathsf{Sub}(\varphi)$. Then $(\mathfrak{B}, v) \not\models \varphi$. Moreover, as a stable subalgebra of $\mathfrak{A}$, $\mathfrak{B}$ is subdirectly irreducible by Lemma 4.3.4. Since $\mathfrak{B}$ is an M-algebra, we have $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ and since $\mathfrak{B}$ is finite, we conclude that $\mathfrak{B} \in \mathcal{V}(\mathsf{L})_{\mathsf{si}}$. Thus, $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ generates $\mathcal{V}(\mathsf{L})$. The implication (5) $\Rightarrow$ (1) is trivial.

Finally, suppose that L is M-stable. We show that L is axiomatized by

$$\Gamma := \{\gamma(\mathfrak{B}) \mid \mathfrak{B} \in \mathcal{V}(\mathsf{M})_{\mathsf{fsi}} \setminus \mathcal{V}(\mathsf{L})\}.$$

If $\mathfrak{B} \notin \mathcal{V}(\mathsf{L})$, then $\gamma(\mathfrak{B}) \in \mathsf{L}$ by Lemma 4.5.5, so $\mathsf{M} + \Gamma \subseteq \mathsf{L}$. Conversely, suppose that $\varphi \in \mathsf{L}$, and suppose there is a subdirectly irreducible M-algebra $\mathfrak{A}$, such that $\mathfrak{A} \models \Gamma$ but $(\mathfrak{A}, v) \not\models \varphi$ for some valuation v on $\mathfrak{A}$. Let $(\mathfrak{B}, v_B)$ be an M-filtration of $(\mathfrak{A}, v)$ through $\mathsf{Sub}(\varphi)$. So $\mathfrak{B}$ refutes L and therefore $\gamma(\mathfrak{B}) \in \Gamma$. Since $\mathfrak{A}$ is subdirectly irreducible, $\mathfrak{A}$ refutes $\gamma(\mathfrak{B})$ by Lemma 4.5.2. The latter contradicts to the fact that $\mathfrak{A}$ validates Γ. □

4.5.7. Corollary. *If* M *is a normal extension of* $\mathsf{K4}$ *that admits filtration and has the* $(*)$*-property, then the* M*-stable logics form a complete sublattice of* NExtM.

Proof:
Let $\{\mathsf{L}_i \mid i \in I\}$ be a family of M-stable logics. Then $\bigvee\{\mathsf{L}_i \mid i \in I\}$ is generated by $\bigcap_{i \in I} \mathcal{V}(\mathsf{L})_{\mathsf{si}}$. This class is M-stable by Theorem 4.5.6. Thus, $\bigvee\{\mathsf{L}_i \mid i \in I\}$ is M-stable. That $\bigwedge\{\mathsf{L}_i \mid i \in I\}$ is M-stable follows from Proposition 4.4.5(4). □

In particular, since $\mathsf{K4}$ admits filtration and has the $(*)$-property, we obtain:

4.5.8. Corollary. *Let* L *be a normal extension of* $\mathsf{K4}$. *The following are equivalent.*

(1) L *is* $\mathsf{K4}$*-stable.*

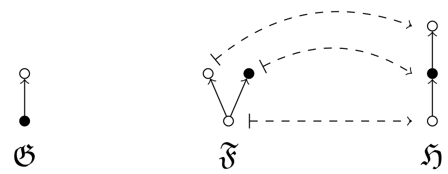
(2) $\mathcal{V}(\mathsf{L})_{\mathsf{wc}}$ *is* $\mathsf{K4}$*-stable.*

(3) $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ *is finitely* $\mathsf{K4}$*-stable.*

(4) $\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ *is* $\mathsf{K4}$*-stable and generates* $\mathcal{V}(\mathsf{L})$*.*

Moreover, each $\mathsf{K4}$*-stable logic is axiomatizable by stable formulas, and hence the stable* $\mathsf{K4}$*-logics form a complete sublattice of* $\mathsf{NExtK4}$*.*

4.5.9. Example. On the other hand, there exist logics above $\mathsf{K4}$ that are axiomatizable over $\mathsf{K4}$ by stable formulas, but are not $\mathsf{K4}$-stable logics. To see this, consider the $\mathsf{K4}$-frames $\mathfrak{F}$, $\mathfrak{G}$, and $\mathfrak{H}$ shown below.



We set $\mathsf{L} = \mathsf{K4} + \gamma(\mathfrak{G})$. Clearly $\mathfrak{F}$ is the only non-singleton rooted upset of $\mathfrak{F}$ and $\mathfrak{G}$ is not a stable image of $\mathfrak{F}$ since $\mathfrak{F}$ has a reflexive root and $\mathfrak{G}$ has an irreflexive root. Therefore, $\mathfrak{F} \models \gamma(\mathfrak{G})$, and so $\mathfrak{F} \models \mathsf{L}$. Next consider the map from $\mathfrak{F}$ to $\mathfrak{H}$ indicated in the picture above. This is clearly a stable map from $\mathfrak{F}$ onto $\mathfrak{H}$. If L was $\mathsf{K4}$-stable, Theorem 4.5.6 would yield $\mathfrak{H} \models \gamma(\mathfrak{G})$. However, $\mathfrak{H} \not\models \gamma(\mathfrak{G})$ since $\mathfrak{G}$ is a rooted upset of $\mathfrak{H}$. Thus, L is not $\mathsf{K4}$-stable.

In Example 4.5.9 it was essential that the root of $\mathfrak{G}$ was irreflexive. We show below that whenever $\mathfrak{G}$ has an irreflexive root, the formula $\gamma(\mathfrak{G})$ yields a $\mathsf{K4}$-stable logics only in the trivial case where $\mathfrak{G}$ is a singleton. On the hand, we show that every logic that is axiomatizable over $\mathsf{K4}$ by stable formulas of finite $\mathsf{K4}$-frames with sharp roots is $\mathsf{K4}$-stable.

In algebraic terms we will show that a logic is $\mathsf{K4}$-stable if it is axiomatizable over $\mathsf{K4}$ by stable formulas of finite $\mathsf{K4}$-algebras $\mathfrak{B}$ that have an atom a such that $a \leq \Diamond b$ for each $b \neq 0 \in \mathfrak{B}$. For convenience, we call such algebras *strongly subdirectly irreducible.*

4.5.10. Proposition. *Let* $\mathfrak{B} = (B, \Diamond)$ *be a finite subdirectly irreducible* $\mathsf{K4}$*-algebra.*

(1) If $\mathfrak{B}$ *is strongly subdirectly irreducible, then for every well-connected* $\mathsf{K4}$*-algebra* $\mathfrak{A}$ *we have* $\mathfrak{A} \not\models \gamma(\mathfrak{B})$ *iff there is a stable embedding of* $\mathfrak{B}$ *into* $\mathfrak{A}$*.*

(2) The logic $\mathsf{K4} + \gamma(\mathfrak{B})$ *is* $\mathsf{K4}$*-stable iff* $|B| = 2$ *or* $\mathfrak{B}$ *is strongly subdirectly irreducible.*

(3) *Suppose* $\mathsf{L} = \mathsf{K4} + \{\gamma(\mathfrak{B}_i) \mid i \in I\}$, *where each* $\mathfrak{B}_i$ *is a finite strongly subdirectly irreducible* K4*-algebra. Then* L *is* K4*-stable.*

Proof:

(1) The right to left direction was already part of Lemma 4.5.2. For the left to right direction, let $\mathfrak{A}$ be a K4-algebra such that $\mathfrak{A} \not\models \gamma(\mathfrak{B})$. (Note that for this direction it is not needed that $\mathfrak{B}$ is well-connected.) Since K4 admits filtration, there is a finite K4-algebra $\mathfrak{C}$ that is a stable subalgebra of $\mathfrak{A}$ and $\mathfrak{C} \not\models \gamma(\mathfrak{B})$. By Lemma 4.5.2, there is a subdirectly irreducible homomorphic image $\mathfrak{D}$ of $\mathfrak{C}$ and a stable embedding of $\mathfrak{B}$ into $\mathfrak{D}$. Since $\mathfrak{B}$ is strongly subdirectly irreducible, by Lemma 4.5.4(2), there is a stable embedding of $\mathfrak{B}$ into $\mathfrak{C}$, and hence a stable embedding of $\mathfrak{B}$ into $\mathfrak{A}$.

(2) Suppose that $\mathfrak{B}$ is a strongly subdirectly irreducible. By (1), validity of $\gamma(\mathfrak{B})$ is preserved under stable subalgebras of well-connected algebras. Thus, by Corollary 4.5.8, the logic $\mathsf{K4} + \gamma(\mathfrak{B})$ is K4-stable.

If $|B| = 2$ then the dual frame $\mathfrak{G}$ of $\mathfrak{B}$ is a singleton. If $\mathfrak{G}$ is a reflexive singleton, then $\mathfrak{B}$ is strongly subdirectly irreducible and so $\mathsf{K4} + \gamma(\mathfrak{B})$ is K4-stable by the above argument. Otherwise, $\mathfrak{G}$ is an irreflexive singleton and we will see in Section 4.7.3 that it axiomatizes the K4-stable logic D4.

To see the converse, we generalize the construction in Example 4.5.9. Suppose that $\mathfrak{B}$ is neither strongly subdirectly irreducible nor the dual of a singleton.

Then the dual $\mathfrak{G} = (Y, R)$ of $\mathfrak{B}$ has an irreflexive root r and contains at least one other point. In particular, $\mathfrak{G}$ contains a quasi-maximal point $m \neq r$. Note that since r is an irreflexive root it does not have any incoming R-edges.

Let $\mathfrak{F} = (X, R')$ be the frame with domain $X = Y \cup \{r'\}$ where r' is a fresh point and define

$$R' = (R \setminus \{(y, m) \mid y \in Y\}) \cup \{(r', y) \mid y \in X\}.$$

In other words, $\mathfrak{F}$ is obtained from $\mathfrak{G}$ by first deleting all incoming R-edges into m and then adding a fresh sharp root r'.

The only point-generated upset of $\mathfrak{F}$ that has at least the cardinality of Y is the whole domain X. Indeed, r and m are the only upper covers of r' in $\mathfrak{F}$. Since $(r, m), (m, r) \notin R'$ the point generated upsets of r and m have cardinality less than $|Y|$.

Since the root r' of $\mathfrak{F}$ is reflexive and the root r of $\mathfrak{G}$ is irreflexive, there is no stable map from $\mathfrak{F}$ onto $\mathfrak{G}$. Thus, $\mathfrak{F} \not\models \gamma(\mathfrak{G})$.

Let $\mathfrak{H}$ be obtained from $\mathfrak{G}$ by adding a sharp root below $\mathfrak{G}$. Then the map that sends every point of Y to itself and r' to the root of $\mathfrak{H}$ is stable and onto. However, $\mathfrak{H} \not\models \gamma(\mathfrak{G})$ since $\mathfrak{G}$ is a point-generated subframe of $\mathfrak{H}$. Thus, $\mathsf{K4} + \gamma(\mathfrak{G})$ is not stable.

(3) Immediately follows from (2). □

In contrast to Proposition 4.5.10(2), if a logic L is axiomatized over $\mathsf{K4}$ by more than one stable formula, e.g. $\mathsf{L} = \mathsf{K4} + \{\gamma(\mathfrak{A}_i) \mid i \in I\}$, $|I| > 1$, then L may be $\mathsf{K4}$-stable even if the $\mathfrak{A}_i$'s are not (all) strongly subdirectly irreducible (see e.g. the axiomatization of $\mathsf{S4}$ given in Table 4.7.1). Since every finite subdirectly irreducible $\mathsf{S4}$-algebra is strongly subdirectly irreducible, Proposition 4.5.10 yields:

4.5.11. Corollary. *Let $\mathfrak{B}$ be a finite subdirectly irreducible $\mathsf{S4}$-algebra. For every well-connected $\mathsf{S4}$-algebra $\mathfrak{A}$ we have $\mathfrak{A} \not\models \gamma(\mathfrak{B})$ iff there is a stable embedding of $\mathfrak{B}$ into $\mathfrak{A}$.*

This immediately yields that if M is a normal extension of $\mathsf{S4}$ that admits filtration and has the $(*)$-property, then all logics axiomatizable over M by stable formulas of finite subdirectly irreducible M-algebras are M-stable. Thus, we obtain the following improvement of Theorem 4.5.6.

4.5.12. Corollary. *Let M be a normal extension of $\mathsf{S4}$ that admits filtration and has the $(*)$-property. For a normal extension L of M, the following are equivalent.*

(1) L is M-stable.

(2) $\mathcal{V}(\mathsf{L})_{\mathsf{wc}}$ is M-stable.

(3) $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ is M-stable within $\mathcal{V}(\mathsf{M})_{\mathsf{si}}$.

(4) $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ is finitely M-stable.

(5) $\mathcal{V}(\mathsf{L})_{\mathsf{fsi}}$ is M-stable and generates $\mathcal{V}(\mathsf{L})$.

(6) L is axiomatizable over M by stable formulas of finite subdirectly irreducible M-algebras.

In particular, since $\mathsf{S4}$ admits filtration and has the $(*)$-property, Corollary 4.5.12 holds for $\mathsf{S4}$.

4.6 Stable modal logics and stable si logics

In this section we will study the relationship between $\mathsf{S4}$-stable logics and stable si logics. We show that the intuitionistic fragment of an $\mathsf{S4}$-stable logic is a stable si logic, and that the least modal companion of a stable si logic is $\mathsf{S4}$-stable. We also translate axiomatizations of stable si logics to axiomatizations of $\mathsf{S4}$-stable logics and vice versa.

4.6.1 Modal companions of si logics

We recall a few facts about intuitionistic fragments of normal extensions of $\mathsf{S4}$ and modal companions of si logics. We follow the notation of [40, Section 9.6]. Recall that the Gödel-McKinsey-Tarski translation maps a formula φ of IPC to the formula $t(\varphi)$ of modal logic as follows:

- $t(p) = \Box p$ for a propositional letter p,
- $t(\bot) = \Box\bot$,
- $t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi)$,
- $t(\varphi \vee \psi) = t(\varphi) \vee t(\psi)$,
- $t(\varphi \to \psi) = \Box(t(\varphi) \to t(\psi))$.

The effect of the semantics of the translation t can be captured via skeletons that turn $\mathsf{S4}$-spaces into Esakia spaces obtained by "modding out clusters". In detail, for an $\mathsf{S4}$-frame $\mathfrak{F} = (X, R)$, define $x \sim y$ iff xRy and yRx for $x, y \in X$. The *skeleton* of $\mathfrak{F}$ is $\rho\mathfrak{F} = (\rho X, \rho R)$, where $\rho(X)$ consists of the equivalence classes of the relation $\sim$ and is topologized by the quotient topology and $[x]\rho R[y]$ iff xRy for $[x], [y] \in \rho(X)$. It is easy to see that $\rho\mathfrak{F}$ is an Esakia space. It is well known (see, e.g., [40, Lemma 9.67]) that for every $\mathsf{S4}$-frame $\mathfrak{F}$ and formula φ of IPC we have

$$\mathfrak{F} \models t(\varphi) \quad \text{iff} \quad \rho(\mathfrak{F}) \models \varphi.$$

If $\mathfrak{F} = (X, R)$ is an $\mathsf{S4}$-space and R is a partial order, then we will formally not distinguish between $\mathfrak{F}$ being an $\mathsf{S4}$- or Esakia space, respectively. Let M be a normal extension of $\mathsf{S4}$ and let L be a si logic. The *intuitionistic fragment* of M is defined as $\rho\mathsf{M} := \{\varphi \mid t(\varphi) \in \mathsf{M}\}$. If $\mathsf{L} = \rho(\mathsf{M})$, then M is called a *modal companion* of L. It is well known that every si logic L has a *least* modal companion that we denote by $\tau\mathsf{L}$. Then for every $\mathsf{S4}$-space $\mathfrak{F}$, we have

$$\mathfrak{F} \models \tau\mathsf{L} \quad \text{iff} \quad \rho\mathfrak{F} \models \mathsf{L},$$

and if $\mathfrak{F}$ is a poset, then

$$\mathfrak{F} \models \mathsf{M} \quad \text{iff} \quad \mathfrak{F} \models \rho\mathsf{M}.$$

We will often use the fact that the fmp is preserved by least modal companions as well as by intuitionistic fragments , i.e. if a si logic L has the fmp, then so does $\tau\mathsf{L}$ and if M extends $\mathsf{S4}$ and has the fmp, then so does $\rho\mathsf{M}$. A proof of these facts can be found in [40, p. 328].

4.6.2 Modal companions and stability

Recall that stable si logics are the $\{\wedge, \vee, \bot, \top\}$-stable logics in the terminology of Section 3.3.2. If $\mathfrak{F}$ is a rooted intuitionistic frame, we denote by $\gamma(\mathfrak{F})$ the stable formula of $\mathfrak{F}$ and recall from Lemma 3.5.4 that if $\mathfrak{G}$ is a rooted Esakia space, then $\mathfrak{G} \models \gamma(\mathfrak{F})$ iff $\mathfrak{F}$ is a stable image of $\mathfrak{G}$. Moreover, a si logic L is stable iff L is axiomatizable by stable formulas of some finite rooted frames.

Note that we used the name *stable formula* and notation $\gamma(\mathfrak{F})$ in both, the modal and intuitionistic cases, even though they are syntactically different (cf. Definition 4.5.1 and Definition 3.3.10). This is justified by the similar semantic property of these formulas. It should always be clear from the context which formula we are referring to.

Next we show that stability is preserved by least modal companions, allowing us to translate axiomatizations of stable si logics to axiomatizations of their least modal companions. We will use these results in Section 4.7 to axiomatize $\mathsf{S4}$-stable logics. We point out that the greatest modal companion of a stable si logic is not necessarily $\mathsf{S4}$-stable. For instance, the Grzegorczyk logic $\mathsf{S4.Grz}$ is the greatest modal companion of IPC, and we will see in Section 4.7 that it is not $\mathsf{S4}$-stable

4.6.1. Theorem.

(1) Let $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, R)$ be finite rooted $\mathsf{S4}$-frames. If $\mathfrak{G}$ is a stable image of $\mathfrak{F}$, then $\rho\mathfrak{G}$ is a stable image of $\rho\mathfrak{F}$.

(2) If L is a stable si logic, then $\tau\mathsf{L}$ is $\mathsf{S4}$-stable.

(3) If $\mathsf{L} = \mathsf{IPC} + \{\gamma(\mathfrak{G}_i) \mid i \in I\}$, then $\tau\mathsf{L} = \mathsf{S4} + \{\gamma(\mathfrak{G}_i) \mid i \in I\}$.

Proof:

(1) Let $f : X \to Y$ be an onto stable map. Since the quotient map $\pi_Y : Y \to \rho Y$ is an onto p-morphism, the composition $\pi_Y \circ f : X \to \rho Y$ is onto and stable. Define $g : \rho X \to \rho Y$ by $g(\pi_X(x)) = \pi_Y(f(x))$. By stability of f, the map g is well-defined on equivalence classes. In detail, if for $x, y \in X$ we have $\pi_X(x) = \pi_X(y)$ in $\rho(\mathfrak{F})$, then xRy and yRx, thus, $f(x)Rf(y)$ and $f(y)Rf(x)$. Therefore $\pi_Y(f(x)) = \pi_Y(f(y))$. It is easy to see that g is onto and stable. Therefore, $\rho\mathfrak{G}$ is a stable image of $\rho\mathfrak{F}$.

(2) Let L be a stable si logic. By Corollary 3.3.18, L has the fmp. Then also $\tau\mathsf{L}$ has the fmp. Thus, $\tau\mathsf{L}$ is the logic of its finite rooted frames. We show that this class is $\mathsf{S4}$-stable. Let $\mathfrak{F}$ be a finite rooted $\tau\mathsf{L}$-frame and $\mathfrak{G}$ be a finite rooted $\mathsf{S4}$-frame that is a stable image of $\mathfrak{F}$. Since $\mathfrak{F}$ is a $\tau\mathsf{L}$-frame, $\rho\mathfrak{F}$ is an L-frame. By (1), $\rho\mathfrak{G}$ is a stable image of $\rho\mathfrak{F}$. As L is stable, $\rho\mathfrak{G} \models \mathsf{L}$. Therefore, $\mathfrak{G} \models \tau\mathsf{L}$, and hence the class of finite rooted $\tau\mathsf{L}$-frames is $\mathsf{S4}$-stable. Thus, by Corollary 4.5.12, $\tau\mathsf{L}$ is an $\mathsf{S4}$-stable logic.

(3) Let $\mathsf{M} = \mathsf{S4} + \{\gamma(\mathfrak{G}_i) \mid i \in I\}$. By Corollary 4.5.12 and (2), both $\tau\mathsf{L}$ and M are $\mathsf{S4}$-stable. Therefore, to see that $\tau\mathsf{L} = \mathsf{M}$, it is sufficient to check that the two logics have the same finite rooted frames. Let $\mathfrak{F}$ be a finite rooted $\mathsf{S4}$-frame. If $\mathfrak{F} \not\models \tau\mathsf{L}$, then $\rho\mathfrak{F} \not\models \mathsf{L}$, so $\mathfrak{G}_i$ is a stable image of $\rho\mathfrak{F}$ for some $i \in I$. Since $\rho\mathfrak{F}$ is a stable image of $\mathfrak{F}$, we conclude that $\mathfrak{G}_i$ is a stable image of $\mathfrak{F}$. Thus, $\mathfrak{F} \not\models \gamma(\mathfrak{G}_i)$, and hence $\mathfrak{F} \not\models \mathsf{M}$. Conversely, if $\mathfrak{F} \not\models \mathsf{M}$, then $\mathfrak{G}_i$ is a stable image of $\mathfrak{F}$ for some $i \in I$. From (1) it follows that $\rho\mathfrak{G}_i$ is a stable image of $\rho\mathfrak{F}$. Since $\mathfrak{G}_i$ is partially ordered, $\mathfrak{G}_i \cong \rho\mathfrak{G}_i$, implying that $\mathfrak{G}_i$ is a stable image of $\rho\mathfrak{F}$. Thus, $\rho\mathfrak{F} \not\models \mathsf{L}$, and so $\mathfrak{F} \not\models \tau\mathsf{L}$. □

Next we will show that stability is preserved by intuitionistic fragments, which will allow us to translate axiomatizations of $\mathsf{S4}$-stable logics to axiomatizations of their intuitionistic fragments.

For a finite rooted $\mathsf{S4}$-frame $\mathfrak{F} = (X, R)$, let $\overline{\mathfrak{F}} = (X, \overline{R})$ be the partially ordered $\mathsf{S4}$-frame that is obtained from $\mathfrak{F}$ by unraveling each n-cluster into an n-chain (see Figure 4.6.1); that is, if $X = C_1 \cup \cdots \cup C_k$ is the division of $\mathfrak{F}$ into clusters, with $C_i = \{x_1^i, \ldots, x_{n_i}^i\}$, then for all $x_l^i, x_m^j \in X$, we have

$$x_l^i \; \overline{R} \; x_m^j \quad \text{iff} \quad \begin{cases} i = j \text{ and } l \geq m \text{ or} \\ i \neq j \text{ and } xRy, \end{cases}$$

where $1 \leq i, j \leq k$ and $1 \leq l \leq n_i$, $1 \leq m \leq n_j$. Note that $x_{n_i}^i$ is the root of the chain C_i in $\overline{\mathfrak{F}}$.

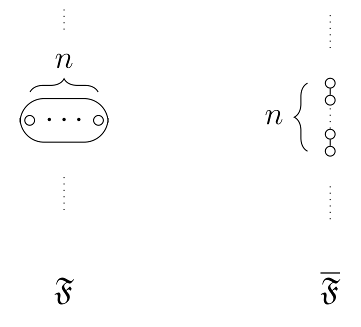


Figure 4.6.1

4.6.2. Theorem.

(1) Let $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, R)$ be finite rooted $\mathsf{S4}$-frames, with $\mathfrak{G}$ being partially ordered. Then $\mathfrak{F}$ is a stable image of $\mathfrak{G}$ iff $\overline{\mathfrak{F}}$ is a stable image of $\mathfrak{G}$.

(2) If M is $\mathsf{S4}$-stable, then $\rho\mathsf{M}$ is stable.

(3) If $\mathsf{M} = \mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$, *then* $\rho\mathsf{M} = \mathsf{IPC} + \{\gamma(\overline{\mathfrak{F}}_i) \mid i \in I\}$.

Proof:

(1) Since $\mathfrak{F}$ is easily seen to be a stable image of $\overline{\mathfrak{F}}$, the implication from right to left is obvious. Conversely, suppose that $f : \mathfrak{G} \to \mathfrak{F}$ is an onto stable map. We transform f into a stable map $\overline{f} : \mathfrak{G} \to \overline{\mathfrak{F}}$ by shuffling the values of f belonging to some cluster of $\mathfrak{F}$. Let C_i be a cluster of $\mathfrak{F}$ and let $Y' = f^{-1}(C_i)$. We view Y' as a subframe of $\mathfrak{G}$, and define $\overline{f} : Y' \to C_i$ by induction on the depth of points in Y'. We map the points of the smallest depth injectively onto the first $n_i - 1$ points of C_i and all the other points of Y' to the root $x^i_{n_i}$. More precisely, suppose $\{y_1, \ldots, y_m\} \subseteq Y'$ are the points of depth d and we have mapped all the points of Y' of smaller depth injectively onto $\{x^i_1, \ldots, x^i_l\}$. If $m \leq n_i - l$, then set $\overline{f}(y_h) = x^i_{l+h}$ for all $1 \leq h \leq m$. If $m \not\leq n_i - l$, then define $\overline{f}$ as before for all y_l with $l \leq m - (n_i - l)$ and map all the other points of Y' to $x^i_{n_i}$. It is straightforward to check that $\overline{f}$ is stable.

(2) Since M is $\mathsf{S4}$-stable, it has the fmp and therefore, so does $\rho\mathsf{M}$. So it suffices to show that the finite rooted $\rho\mathsf{M}$-frames form a stable class. Suppose $\mathfrak{G}$ is a stable image of a finite rooted $\rho\mathsf{M}$-frame $\mathfrak{F}$. From $\mathfrak{F} \models \rho\mathsf{M}$ it follows that $\mathfrak{F} \models \mathsf{M}$. Since M is $\mathsf{S4}$-stable, $\mathfrak{G} \models \mathsf{M}$. Consequently, $\mathfrak{G} \models \rho\mathsf{M}$.

(3) Since M is $\mathsf{S4}$-stable, $\rho\mathsf{M}$ is stable by (2). Let $\mathsf{L} = \mathsf{IPC} + \{\gamma(\overline{\mathfrak{F}}_i) \mid i \in I\}$. By Corollary 3.5.8, L is stable. Therefore, both $\rho\mathsf{M}$ and L have the fmp, and hence it suffices to show that the two logics have the same finite rooted frames. Suppose $\mathfrak{G}$ is a finite rooted partially ordered frame. If $\mathfrak{G} \not\models \mathsf{L}$, then there is $i \in I$ such that $\mathfrak{G} \not\models \gamma(\overline{\mathfrak{F}}_i)$. Therefore, $\overline{\mathfrak{F}}_i$ is a stable image of $\mathfrak{G}$. By (1), $\mathfrak{F}_i$ is a stable image of $\mathfrak{G}$. Thus, $\mathfrak{G} \not\models \gamma(\mathfrak{F}_i)$, and so $\mathfrak{G} \not\models \mathsf{M}$. Since $\mathfrak{G}$ is a partially ordered frame, we conclude that $\mathfrak{G} \not\models \rho\mathsf{M}$. Conversely, if $\mathfrak{G} \not\models \rho\mathsf{M}$, then $\mathfrak{G} \not\models \mathsf{M}$, and hence $\mathfrak{G} \not\models \gamma(\mathfrak{F}_i)$ for some $i \in I$. Therefore, $\mathfrak{F}_i$ is a stable image of $\mathfrak{G}$. By (1), $\overline{\mathfrak{F}}_i$ is a stable image of $\mathfrak{G}$. Thus, $\mathfrak{G} \not\models \gamma(\overline{\mathfrak{F}}_i)$, yielding that $\mathfrak{G} \not\models \mathsf{L}$. □

4.6.3. COROLLARY.

(1) A si logic L *is stable iff* $\tau\mathsf{L}$ *is* $\mathsf{S4}$*-stable.*

(2) An $\mathsf{S4}$*-stable logic is the least modal companion of a si logic iff it can be axiomatized by stable formulas of finite rooted partially ordered* $\mathsf{S4}$*-frames.*

Proof:

(1) It is well known that $\mathsf{L} = \rho\tau\mathsf{L}$ (see, e.g. [40, Theorem 9.57]). Now apply Theorems 4.6.1(3) and 4.6.2(3).

(2) Suppose M is the least modal companion of a si logic L. Then $\mathsf{M} = \tau\mathsf{L}$, and so $\mathsf{L} = \rho\mathsf{M}$. Since M is $\mathsf{S4}$-stable, L is stable by Theorem 4.6.2(2). Therefore, by Corollary 3.5.8 there are finite rooted partially ordered frames $\{\mathfrak{F}_i \mid i \in I\}$ such that $\mathsf{L} = \mathsf{IPC} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$. Thus, $\mathsf{M} = \mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$ by Theorem 4.6.1(3). Conversely, if $\mathsf{M} = \mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$ for some finite rooted partially ordered $\mathsf{S4}$-frames $\{\mathfrak{F}_i \mid i \in I\}$, then $\rho\mathsf{M} = \mathsf{IPC} + \{\gamma(\overline{\mathfrak{F}}_i) \mid i \in I\}$ by Theorem 4.6.2(3). Since $\overline{\mathfrak{F}}_i = \mathfrak{F}_i$ for all $i \in I$, we conclude that $\tau\rho\mathsf{M} = \mathsf{IPC} + \{\gamma(\overline{\mathfrak{F}}_i) \mid i \in I\} = \mathsf{M}$, and hence M is the least modal companion of $\rho\mathsf{M}$. □

4.6.3 K4-stable and S4-stable logics

Next we discuss connections between $\mathsf{S4}$-stable and $\mathsf{K4}$-stable logics. For a formula φ, let φ^+ be obtained from φ by replacing each subformula of φ of the form $\Box\psi$ by $\psi\wedge\Box\psi$. The formula φ^+ obtained in this way is also called the *splitting translation* of φ. Semantically, the effect of the splitting translation can be captured via reflexivizations. For a binary relation R on a set X, let $R^+ := R\cup\{(x,x) \mid x \in X\}$ be the *reflexive closure* of R. For a $\mathsf{K4}$-space $\mathfrak{F} = (X, R)$, define the *reflexivization* of $\mathfrak{F}$ as $\mathfrak{F}^+ = (X, R^+)$. Then $\mathfrak{F}^+$ is an $\mathsf{S4}$-space and for every formula φ of the basic modal language

$$\mathfrak{F} \models \varphi^+ \quad \text{iff} \quad \mathfrak{F}^+ \models \varphi.$$

If $\mathsf{L} = \mathsf{S4} + \Gamma$ is a normal extension of $\mathsf{S4}$, let $\mathsf{L}^+ = \mathsf{K4} + \Gamma^+$, where $\Gamma^+ = \{\varphi^+ \mid \varphi \in \Gamma\}$. By the above, $\mathfrak{F} \models \mathsf{L}^+$ iff $\mathfrak{F}^+ \models \mathsf{L}$. Therefore, L^+ is the logic of $\{\mathfrak{F} \mid \mathfrak{F}^+ \models \mathsf{L}\}$ (see, e.g., [40, Section 3.9]).

4.6.4. Lemma.

(1) Let $\mathfrak{F}$ be a finite $\mathsf{S4}$-frame and let $\mathfrak{G}$ be a $\mathsf{K4}$-space. Then $\mathfrak{F}$ is a stable image of $\mathfrak{G}$ iff $\mathfrak{F}$ is a stable image of $\mathfrak{G}^+$.

(2) If $\mathsf{L} = \mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$, where the $\mathfrak{F}_i$ are $\mathsf{S4}$-frames, then $\mathsf{L}^+ = \mathsf{K4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$.

(3) If L is $\mathsf{S4}$-stable, then L^+ is $\mathsf{K4}$-stable.

Proof:

(1) Immediate since $\mathfrak{F}$ is reflexive.

(2) By (1) and Corollary 4.5.11, if $\mathfrak{G}$ is a rooted $\mathsf{K4}$-space, then $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ iff $\mathfrak{G}^+ \models \gamma(\mathfrak{F}_i)$. Therefore, $\mathfrak{G} \models \mathsf{L}^+$ iff $\mathfrak{G}^+ \models \mathsf{L}$ iff $\mathfrak{G}^+ \models \{\gamma(\mathfrak{F}_i) \mid i \in I\}$ iff $\mathfrak{G} \models \{\gamma(\mathfrak{F}_i) \mid i \in I\}$. Thus, L^+ and $\mathsf{K4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$ have the same $\mathsf{K4}$-spaces, and hence the two logics coincide.

(3) If L is $\mathsf{S4}$-stable, then L is axiomatizable by stable formulas of $\mathsf{S4}$-frames. By (2), L^+ is axiomatized by the same stable formulas. In particular, L^+ is axiomatizable by stable formulas of frames with reflexive roots. Thus, L^+ is $\mathsf{K4}$-stable by Proposition 4.5.10. □

For two normal modal logics L and M, let $\mathsf{L} \vee \mathsf{M}$ denote the join of these logics in the lattice of normal modal logics.

4.6.5. LEMMA. *Let L be a normal extension of $\mathsf{K4}$.*

(1) If $\mathsf{S4} \subseteq \mathsf{L}$, then L is $\mathsf{K4}$-stable iff L is $\mathsf{S4}$-stable.

(2) If L is $\mathsf{K4}$-stable, then $\mathsf{S4} \vee \mathsf{L}$ is $\mathsf{S4}$-stable.

(3) If $\mathsf{L} = \mathsf{K4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$, then $\mathsf{S4} \vee \mathsf{L} = \mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid \mathfrak{F}_i = \mathfrak{F}_i^+\}$.

(4) If $\mathsf{L} = \mathsf{K4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$, then $\mathsf{L} \subseteq \mathsf{S4}$ iff each $\mathfrak{F}_i$ contains an irreflexive point.

Proof:

(1) Observe that $\mathcal{V}(\mathsf{S4})$ is a $\mathcal{V}(\mathsf{K4})$-stable class and apply Proposition 4.4.5(3).

(2) By Theorem 4.5.6, the rooted L-spaces are $\mathsf{K4}$-stable. Therefore, the rooted $(\mathsf{S4} \vee \mathsf{L})$-spaces are $\mathsf{S4}$-stable. Thus, $\mathsf{S4} \vee \mathsf{L}$ is $\mathsf{S4}$-stable by Corollary 4.5.12.

(3) Let $\mathfrak{G}$ be a rooted $\mathsf{S4}$-space. We have $\mathfrak{G} \models \mathsf{S4} \vee \mathsf{L}$ iff $\mathfrak{G} \models \mathsf{L}$ iff $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ for all $i \in I$. It is obvious that $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ for every $\mathfrak{F}_i$ that contains an irreflexive point because no such $\mathfrak{F}_i$ can be a stable image of a reflexive space. Therefore, $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ for all $i \in I$ is equivalent to $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ for all $\mathfrak{F}_i$ with $\mathfrak{F}_i = \mathfrak{F}_i^+$. Thus, $\mathsf{S4} \vee \mathsf{L} = \mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid \mathfrak{F}_i = \mathfrak{F}_i^+\}$.

(4) First suppose that each $\mathfrak{F}_i$ contains an irreflexive point. Then $\mathfrak{F}_i \neq \mathfrak{F}_i^+$ for all $i \in I$. Therefore, (3) implies that $\mathsf{S4} \vee \mathsf{L} = \mathsf{S4}$, and hence $\mathsf{L} \subseteq \mathsf{S4}$. Conversely, suppose that some $\mathfrak{F}_i$ is reflexive. Since $\mathfrak{F}_i \not\models \mathsf{L}$ and $\mathfrak{F}_i$ is an $\mathsf{S4}$-frame, we see that $\mathsf{L} \not\subseteq \mathsf{S4}$. □

4.6.4 A continuum of S4-stable and K4-stable logics

Recall that there is a continuum of stable logics. By slightly modifying the proof and using the results from the previous section, we show that there are continuum many $\mathsf{S4}$-stable logics, and continuum many $\mathsf{K4}$-stable logics between $\mathsf{K4}$ and $\mathsf{S4}$.

4.6.6. THEOREM.

(1) There are continuum many $\mathsf{K4}$-stable logics above $\mathsf{S4}$.

(2) There are continuum many K4*-stable logics between* K4 *and* S4*.*

Proof:

(1) By [17, Theorem 6.13], there are continuum many stable si logics. The least modal companions of these logics are S4-stable by Lemma 4.6.1. Thus, these yield continuum many S4-stable logics. By Lemma 4.6.5(1), these logics are also K4-stable. Thus, there are continuum many K4-stable logics above S4.

(2) Consider the sequence $\{\mathfrak{F}_n \mid n \in \mathbb{N}_{\geq 1}\}$, shown in Figure 4.6.2, where $\mathbb{N}_{\geq 1} = \{n \in \mathbb{N} \mid n \geq 1\}$. By [17, Lemma 6.12], $\mathfrak{F}_n$ is not a stable image of $\mathfrak{F}_m$ for $n \neq m$. We slightly modify the sequence. For $n \in \mathbb{N}_{\geq 1}$, let $\mathfrak{G}_n$ be the K4-frame that is obtained from $\mathfrak{F}_n$ by making x_1 irreflexive. The proof of [17, Lemma 6.12] shows that $\mathfrak{G}_n$ is not a stable image of $\mathfrak{G}_m$ for $n \neq m$.

For $I \subseteq \mathbb{N}_{\geq 1}$ let $\mathsf{L}_I = \mathsf{K4} + \{\gamma(\mathfrak{G}_n) \mid n \in I\}$. Since each $\mathfrak{G}_n$ has a reflexive root, by Proposition 4.5.10, every L_I is K4-stable. As each $\mathfrak{G}_n$ has a an irreflexive point, by Lemma 4.6.5(4), $\mathsf{L}_I \subseteq \mathsf{S4}$ for every $I \subseteq \mathbb{N}_{\geq 1}$. Thus, every L_I is a K4-stable logic between K4 and S4. Finally, if $n \in I \setminus J$, then $\gamma(\mathfrak{G}_n) \in \mathsf{L}_J \setminus \mathsf{L}_I$, so the cardinality of $\{\mathsf{L}_I \mid I \subseteq \mathbb{N}_{\geq 1}\}$ is that of continuum, completing the proof. □

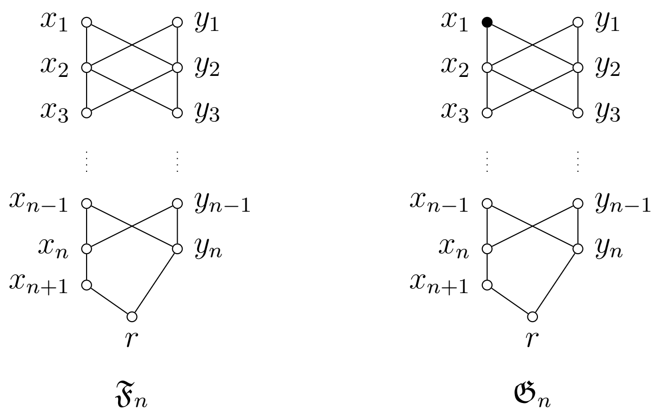


Figure 4.6.2

4.6.5 Summary

We summarize the main results of this section in Table 4.6.1.

First we recollect the sources of the results mentioned in the first two rows.

- That τ preserves and reflects stability is the content of Corollary 4.6.3(1).

- That ρ preserves stability follows from Theorem 4.6.2(3). That ρ does not reflect stability follows from the fact that IPC is stable, $\mathsf{S4.Grz}$ is not $\mathsf{S4}$-stable (see the next section), and that $\rho\mathsf{S4.Grz} = \mathsf{IPC}$.
- That $\mathsf{S4} \vee -$ preserves stability follows from Lemma 4.6.5(2). It does not reflect stability because $\mathsf{GL} \vee \mathsf{S4}$ is the inconsistent logic, which is $\mathsf{S4}$-stable, but as we will see in the next section, GL is not $\mathsf{K4}$-stable.
- That $(-)^+$ preserves stability follows from Lemma 4.6.4(3). It also reflects stability because $\mathsf{S4} \vee -$ preserves stability and for every normal extension M of $\mathsf{S4}$ we have $\mathsf{S4} \vee \mathsf{M}^+ = \mathsf{M}$.

We explain how to read the bottom three rows on the table on the basis of an example. The third row says that if a si logic is axiomatized over IPC by the stable formulas $\{\gamma(\mathfrak{F}_i) \mid i \in I\}$, then its τ-translation is axiomatized over $\mathsf{S4}$ by $\{\gamma(\mathfrak{F}_i) \mid i \in I\}$ (where we consider the intuitionistic frame $\mathfrak{F}_i$ as an $\mathsf{S4}$-frame). The axiomatization results follow from Theorems 4.6.1(3) and 4.6.2(3) and Lemmas 4.6.5(3) and 4.6.4(2).

	τ	ρ	$\mathsf{S4} \vee -$	$(-)^+$
preserves stability	✓	✓	✓	✓
reflects stability	✓	-	-	✓
$\mathsf{IPC} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$	$\mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$	×	×	×
$\mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$	×	$\mathsf{IPC} + \{\gamma(\overline{\mathfrak{F}}_i) \mid i \in I\}$	×	$\mathsf{K4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$
$\mathsf{K4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$	×	×	$\mathsf{S4} + \{\gamma(\mathfrak{F}_i) \mid \mathfrak{F}_i = \mathfrak{F}_i^+\}$	×

"✓" means yes; "-" means no; "×"means not applicable.

Table 4.6.1

4.7 Examples and axiomatizations

In this section we provide many examples (and non-examples) of stable, $\mathsf{K4}$-stable, and $\mathsf{S4}$-stable logics. Moreover, we look at the concept of stability from the model-theoretic perspective, especially in relation to Lyndon's theorem. Note that the definition of all modal and si logics occurring in this section can be found in Tables A.0.5 and A.0.2, respectively.

4.7.1 Stable logics and Lyndon's theorem

In this section we make some observations concerning the elementarity of M-stable logics. While it remains an open problem whether all stable logics are elementary, we connect elementary stable logics with Lyndon's theorem.

We mostly work with Kripke semantics. As usual, Kripke frames can be regarded as binary relational structures from model theory and from this perspective stable maps between Kripke frames correspond precisely to homomorphisms from model theory. To keep our terminology consistent, we refer to homomorphisms between Kripke frames as stable maps.

As the next propositions shows, seen from a model-theoretic perspective, stable logics are precisely the ones that can be characterized by a class of Kripke frames closed under homomorphic (or *stable*) images. Even though we have not explicitly mentioned it yet, M-stable logics are of course Kripke complete whenever M admits filtration. This follows immediately from the fact that they have the fmp. By an M-*stable class of Kripke frames* we mean a class of M-frames that is closed under stable images which are M-frames themselves.

4.7.1. PROPOSITION. *Let* L *and* M *be normal modal logics with* M *admitting filtration.*

(1) L *is stable iff* L *is the logic of a class of Kripke frames closed under stable images.*

(2) If $\mathsf{M} \subseteq \mathsf{L}$*, then* L *is* M*-stable iff* L *is the logic of an* M*-stable class of Kripke frames.*

Proof:
We only show (1), the proof of (2) is an easy adaption. To see the left to right implication assume that L is stable. By Theorem 4.4.43, $\mathcal{V}(\mathsf{L})$ is generated by a stable class of finite modal algebras. Dually, these finite algebras correspond to Kripke frames. Therefore, L is the logic of a class of Kripke frames closed under stable images.

To see the implication from right to left, suppose L is the logic of a class $\mathcal{K}$ of Kripke frames closed under stable images. We show that the corresponding class $\mathbf{Cm}(\mathcal{K}) := \{\mathbf{Cm}(\mathfrak{F}) \mid \mathfrak{F} \in \mathcal{K}\}$ of complex algebras (see Section 2.3.3) is finitely stable. Let $\mathfrak{A} \in \mathbf{Cm}(\mathcal{K})$ and let $\mathfrak{B}$ be a finite stable subalgebra of $\mathfrak{A}$. Then $\mathfrak{A} = \mathbf{Cm}(\mathfrak{F})$ for some $\mathfrak{F} \in \mathcal{K}$ and $\mathfrak{B} = \mathbf{Cm}(\mathfrak{G})$ for some finite frame $\mathfrak{G}$. Since $\mathfrak{B}$ is a finite stable subalgebra of $\mathfrak{A}$, we see that $\mathfrak{G}$ is a finite stable image of $\mathfrak{F}$. As $\mathcal{K}$ is closed under stable images, $\mathfrak{G} \in \mathcal{K}$, and hence $\mathfrak{B} \in \mathbf{Cm}(\mathcal{K})$. Therefore, L is the logic of a finitely stable class of modal algebras. Thus, L is stable by Theorem 4.4.4. □

We recall that a first-order formula is *positive* iff it is built from atomic formulas via the connectives $\wedge, \vee$ and quantifiers $\forall, \exists$. Recall the following theorem due to Lyndon (see, e.g. [42, Theorem 3.2.4]):

4.7.2. Theorem (Lyndon). *A consistent first-order theory is preserved under homomorphisms iff it can be defined by a set of positive axioms.*

If $\mathcal{C}$ is a class of Kripke frames, we say that a class $\mathcal{C}' \subseteq \mathcal{C}$ is *first-order definable within* $\mathcal{C}$ iff there is a set Γ of first-oder sentences such that $\mathcal{C}' = \{\mathfrak{F} \in \mathcal{C} \mid \mathfrak{F} \text{ satisfies } \Gamma\}$. From (the simple direction of) Lyndon's theorem and Proposition 4.7.1 we immediately obtain the following corollary.

4.7.3. Corollary. *Suppose* L *and* M *are normal modal logics,* M *admits filtration and is characterized by a class* $\mathcal{C}$ *of Kripke frames.*

(1) If L *is the logic of a class of Kripke frames definable by positive formulas, then* L *is stable.*

(2) If L *is the logic of a class of Kripke frames definable by positive formulas within* $\mathcal{C}$*, then* L *is* M*-stable.*

Before we extract what the more interesting (left-to-right) direction of Lyndon theorem tells us about stable logics, we look at a few examples of stable logics. Examples of positive first-order formulas are:

reflexivity: $\forall x\,(xRx)$,

seriality: $\forall x \exists y\,(xRy)$,

universality: $\forall x \forall y\,(xRy)$,

every world sees a reflexive world: $\forall x \exists y\,(xRy \wedge yRy)$.

The logics of the corresponding classes of Kripke frames are T, D, $\mathsf{S5}$, and KMT, respectively (see Table A.0.5). We discuss how Corollary 4.7.3 applies to these logics. Recall that *all* T-frames are reflexive and *all* D-frames are serial. In particular, both logics have the property that the class of all their Kripke frames is first-order definable.[3] Since reflexivity and seriality are expressible by positive formulas, both T and D are stable logics according to Corollary 4.7.3.

The case of S5 is slightly different from that of T and D. The universality formula defines clusters and as we already discussed in the proof of Theorem 4.4.7, S5 is complete with respect to clusters. Since universality is a positive first-order formula, S5 is the logic of a class of frames definable by a positive formula, and hence S5 is stable by Corollary 4.7.3. On the other hand, S5-frames are precisely those with an equivalence relation and equivalence relations are not preserved by stable images. Thus, the class of all S5-frames is not stable.

The logic KMT is yet of a different type. As shown in [76], KMT is the logic of the class of frames in which every world sees a reflexive world. However, not all KMT-frames satisfy this condition. In fact, it is shown in [76] that the class of all KMT-frames is not definable by any first-order formula. Still, it is proved in [76] that a Kripke frame is a KMT-frame iff the successors of any world form a non-finitely colorable subframe (note that this is a second-order condition). The latter condition is preserved under stable images, and hence all KMT-frames form a stable class.

Given a normal modal logic L, let **FR**(L) denote the class of Kripke frames of L. In this terminology, we have that **FR**(T) and **FR**(D) are stable and first-order definable. The class **FR**(S5) is first-order definable but is not stable, and **FR**(KMT) is stable but not first-order definable (even though KMT is elementary). These examples indicate that even if a stable logic L is elementary, the stable class characterizing L and the first-order definable class characterizing L may be different. Thus, we can only infer this—rather weak—corollary from the full strength of Lyndon's theorem.

4.7.4. Corollary. *If* L *is a stable and elementary normal modal logic such that* **FR**(L) *is stable, then* L *is characterized by a positively definable class of Kripke frames.*

We next turn our attention to examples of K4-stable and S4-stable logics and illustrate that K4- and S4-stability is in a way "more frequent" than stability. Roughly speaking, the reason is that some first-order properties become positively definable modulo transitivity and rootedness.

Consider the following normal extensions of K4 together with the first-order description of their Kripke frames.

K4B is the logic of symmetric K4-frames,

K4.2 is the logic of directed K4-frames,

K4.3 is the logic of upward connected K4-frames,

[3] Logics axiomatizable by Sahlqvist formulas always have this property.

$\mathsf{K4Alt}_n$ is the logic of $\mathsf{K4}$-frames such that each point has $\leq n$ successors.

The definitions of the first-order properties mentioned above are:

symmetry: $\forall xy\,(xRy \to yRx)$,

directedness: $\forall xuv\,((xRu \wedge xRv \wedge u \neq v) \to \exists y\,(uRy \wedge vRy))$,

upward connectedness: $\forall xuv\,((xRu \wedge xRv \wedge u \neq v) \to (uRv \vee vRu))$,

bounded alternatives: $\forall xx_1 \ldots x_{n+1}\,(\bigwedge_{1\leq i\leq n+1} xRx_i \to \bigvee_{1\leq i<j\leq n+1} x_i = x_j)$.

Clearly none of these formulas is positive. It is not hard to see that none of the properties is preserved by stable maps, and hence is not definable by positive formulas. In fact, the classes of transitive frames of the logics just described are not stable. However, $\mathsf{K4.B}$ and $\mathsf{K4.3}$ are $\mathsf{K4}$-stable. One way to see this is that in these cases the classes of the corresponding *transitive rooted frames* are definable by positive formulas.

$\mathsf{K4B}$ is characterized by frames satisfying $\forall xy\,(xRy) \vee \forall xy\,(x = y)$.

$\mathsf{K4.3}$ is characterized by transitive frames satisfying

$$\exists r \forall x\,(r = x \vee rRx) \wedge \forall xy\,(x = y \vee xRy \vee yRx).$$

Note that for frames characterizing $\mathsf{K4B}$ the additional condition of transitivity is not needed since the clause $\forall xy\,(xRy) \vee \forall xy\,(x = y)$ implies transitivity.

As we will see in Theorem 4.7.11, $\mathsf{K4Alt}_n$ and $\mathsf{K4.2}$ are not $\mathsf{K4}$-stable. Thus, $\mathsf{K4.2}$ and $\mathsf{K4Alt}_n$ cannot be characterized by positive formulas over transitive frames. On the other hand,

$\mathsf{S4.2}$ is characterized by transitive frames satisfying

$$\forall x(xRx) \wedge \exists r \forall x\,(rRx) \wedge \forall uv\,(\exists y\,(uRy \wedge vRy)).$$

$\mathsf{S4Alt_n}$ is characterized by transitive frames satisfying

$$\forall x(xRx) \wedge \exists r \forall x\,(rRx) \wedge \forall x_1 \ldots x_{n+1}\,(\bigvee_{1\leq i<j\leq n+1} x_i = x_j),$$

implying that $\mathsf{S4.2}$ and $\mathsf{S4Alt_n}$ are $\mathsf{S4}$-stable.

4.7.5. Remark. In the above, we followed the definitions of $\mathsf{K4.2}$ and $\mathsf{K4Alt}_n$ from [40]. There may, however, not be a consensus in the literature on how to define the $\mathsf{K4}$-version of $\mathsf{S4.2}$ and $\mathsf{S4Alt}_n$. The intricacy is that some definition of these logics produce different logics over $\mathsf{K4}$ but become equivalent over $\mathsf{S4}$. For instance, with the above definition $\mathsf{K4.2} \neq (\mathsf{S4.2})^+$—recall the $(-)^+$-operation from Section 4.6.3—which may alternatively be used as a definition of $\mathsf{K4.2}$.

4.7.2 Axiomatization of stable logics

Next we show how to axiomatize some stable logics by stable rules. Given a modal space $\mathfrak{F}$ with dual algebra $\mathfrak{A}$, we write $\rho(\mathfrak{F})$ instead of $\mathfrak{A}$. By [18, Theorem 8.3], T is axiomatized by the stable rules $\rho(\bullet)$ and $\rho(\bullet\!\!\leftrightarrow\!\!\circ)$, and D is axiomatized by the stable rules $\rho(\bullet)$ and $\rho(\bullet\!\!\longrightarrow\!\!\circ)$. As further examples, we give axiomatizations of $\mathsf{S5}$ and KMT.

As in the proof of Theorem 4.4.8, by $\mathfrak{C}_n$ we denote the irreflexive n-cluster and by $\mathfrak{C}'_n$ the frame that arises by adding a sharp root r_n below $\mathfrak{C}_n$ so that $x_i R r_n$ for all $2 \leq i \leq n$; in other words, the sharp root r_n is seen by all elements of $\mathfrak{C}'_n$ except by x_1. Observe that x_1 does not see a reflexive world neither in $\mathfrak{C}_n$ nor in $\mathfrak{C}'_n$, and hence $\Psi := \forall x \exists y\, (xRy \wedge yRy)$ is refuted in both $\mathfrak{C}_n$ and $\mathfrak{C}'_n$.

4.7.6. THEOREM.

(1) $\mathsf{S5}$ *is axiomatized by* $\Gamma := \{\rho(\bullet), \rho(\bullet\!\!\leftrightarrow\!\!\circ), \rho(\begin{smallmatrix}\circ\\ \uparrow\\ \circ\end{smallmatrix}), \rho(\overset{\circ}{\circ\!\!\leftrightarrow\!\!\circ})\}$.

(2) KMT *is axiomatized by* $\Delta := \{\rho(\mathfrak{C}_n) \mid n \geq 1\} \cup \{\rho(\mathfrak{C}'_n) \mid n \geq 1\}$.

Proof:

(1) First we show that a finite rooted frame validates Γ iff it is a cluster. Since none of the frames $\bullet$, $\bullet\!\!\leftrightarrow\!\!\circ$, $\begin{smallmatrix}\circ\\ \uparrow\\ \circ\end{smallmatrix}$, and $\overset{\circ}{\circ\!\!\leftrightarrow\!\!\circ}$ is a cluster, and hence neither is a stable image of a cluster, every finite cluster validates Γ.

Conversely, suppose that $\mathfrak{F} = (X, R)$ is a finite rooted frame that is not a cluster. If $\mathfrak{F}$ is a singleton, then it must be irreflexive, so $\bullet$ is a stable image of $\mathfrak{F}$, and hence $\mathfrak{F} \not\models \rho(\bullet)$. Suppose that $\mathfrak{F}$ has at least two points. If $\mathfrak{F}$ contains an irreflexive point x, then $\bullet\!\!\leftrightarrow\!\!\circ$ is a stable image of $\mathfrak{F}$ as mapping x to the irreflexive point of $\bullet\!\!\leftrightarrow\!\!\circ$ and the rest to the reflexive point of $\bullet\!\!\leftrightarrow\!\!\circ$ is an onto stable map. Therefore, $\mathfrak{F} \not\models \rho(\bullet\!\!\leftrightarrow\!\!\circ)$. Suppose that $\mathfrak{F}$ is reflexive. If $\mathfrak{F}$ contains exactly two points x and y, then without loss of generality we may assume that xRy and $y \not\!R x$. Thus, mapping x to the root of $\begin{smallmatrix}\circ\\ \uparrow\\ \circ\end{smallmatrix}$ and y to the other point of $\begin{smallmatrix}\circ\\ \uparrow\\ \circ\end{smallmatrix}$ is stable and onto, and hence $\mathfrak{F} \not\models \rho(\begin{smallmatrix}\circ\\ \uparrow\\ \circ\end{smallmatrix})$. Suppose $\mathfrak{F}$ has at least three points. Since $\mathfrak{F}$ is not a cluster, without loss of generality we may assume that there are $x, y \in \mathfrak{F}$ with $x \not\!R y$. Then mapping x to the top node, y to the bottom right node, and all the other points to the bottom left node of $\overset{\circ}{\circ\!\!\leftrightarrow\!\!\circ}$ provides an onto stable map. This yields $\mathfrak{F} \not\models \rho(\overset{\circ}{\circ\!\!\leftrightarrow\!\!\circ})$.

Now, let L be the logic axiomatized over K by Γ. Since $\mathsf{S5}$ is the logic of finite clusters and each such validates Γ, we see that $\mathsf{L} \subseteq \mathsf{S5}$. Conversely, by Theorem 4.4.4, L is the logic of a stable class of finite rooted frames. Each such must be a cluster. Therefore, $\mathsf{S5} \subseteq \mathsf{L}$, and hence $\mathsf{S5}$ is axiomatized over K by Γ.

(2) First we show that a finite frame validates Δ iff it satisfies the positive formula Ψ. Suppose that the finite frame $\mathfrak{F}$ refutes Δ. Then there are $n \geq 1$ and a stable onto map $f : \mathfrak{F} \to \mathfrak{C}_n$ or a stable onto map $g : \mathfrak{F} \to \mathfrak{C}'_n$. Since $\mathfrak{C}_n$ and $\mathfrak{C}'_n$ refute Ψ, we conclude that $\mathfrak{F}$ refutes Ψ. For the converse, suppose $\mathfrak{F}$ refutes Ψ. Then $\mathfrak{F}$ has a node u_1 such that all successors of u_1 are irreflexive. Let $u_2, \ldots, u_n$ be the successors of u_1. If $\mathfrak{F}$ consists only of $u_1, u_2, \ldots u_n$, then define $f : \mathfrak{F} \to \mathfrak{C}_n$ by $f(u_i) = x_i$ for all $1 \leq i \leq n$. If $\mathfrak{F}$ contains at least one other node, then define $g : \mathfrak{F} \to \mathfrak{C}'_n$ by

$$g(x) = \begin{cases} x_i & \text{if } x = u_i \text{ for } 1 \leq i \leq n, \\ r_n & \text{otherwise.} \end{cases}$$

In both cases it is easy to see that the defined map is stable and onto. Thus, $\mathfrak{F}$ refutes Δ.

Let L be the normal modal logic axiomatized over K by Δ. It is shown in [76] that KMT has the fmp and a finite frame is a KMT-frame iff it satisfies Ψ. Therefore, a finite frame is a KMT-frame iff it validates Δ. Thus, since both KMT and L have the fmp and have the same finite frames, the two logics coincide. Consequently, KMT is axiomatized over K by Δ. $\square$

4.7.3 Axiomatization of K4-stable and S4-stable logics

Next we calculate axiomatizations of some $\mathsf{K4}$-stable logics in terms of stable formulas. Note that the $\mathsf{K4}$-stability of $\mathsf{D4} = \mathsf{K4} \vee \mathsf{D}$ and $\mathsf{S4} = \mathsf{K4} \vee \mathsf{T}$ can, for example, be inferred from the stability of T and D and Proposition 4.4.5.

4.7.7. THEOREM. *The following are axiomatizations of the* $\mathsf{K4}$*-stable logics* $\mathsf{D4}$*,* $\mathsf{S4}$*, and* $\mathsf{K4B}$ *in terms of stable formulas:*

(1) $\mathsf{D4} = \mathsf{K4} + \gamma(\bullet)$.

(2) $\mathsf{S4} = \mathsf{K4} + \gamma(\bullet) + \gamma(\begin{smallmatrix}\circ\\ \uparrow\\ \bullet\end{smallmatrix})$.

(3) $\mathsf{K4B} = \mathsf{K4} + \gamma(\begin{smallmatrix}\circ\\ \uparrow\\ \circ\end{smallmatrix})$.

Proof:

(1) Let $\mathfrak{X}$ be a $\mathsf{K4}$-space. It is sufficient to show that $\mathfrak{X} \models \Box p \to \Diamond p$ iff $\mathfrak{X} \models \gamma(\bullet)$. If $\mathfrak{X} \not\models \Box p \to \Diamond p$, then there is an $x \in X$ such that $x \not R y$ for all $y \in X$. Therefore, $\{x\}$ is a closed generated subframe of X, and $\mathfrak{Y} = (\{x\}, \varnothing)$ is a finite rooted $\mathsf{K4}$-frame. The unique map from $\mathfrak{Y}$ onto $\bullet$ is stable, and so we conclude that $\mathfrak{X} \not\models \gamma(\bullet)$. Conversely, suppose that $\mathfrak{X} \not\models \gamma(\bullet)$. Then there is a stable map from a topo-rooted closed generated subframe $\mathfrak{Y}$ of $\mathfrak{X}$ onto $\bullet$. This implies that $\mathfrak{Y}$ is a singleton with no R-successors, and hence $\mathfrak{X}$ contains a point with no R-successors. Thus, $\mathfrak{X} \not\models \Box p \to \Diamond p$.

(2) Let $\mathfrak{X}$ be a $\mathsf{K4}$-space. It is sufficient to show that $\mathfrak{X} \models p \to \Diamond p$ iff $\mathfrak{X} \models \gamma(\bullet), \gamma(\overset{\circ}{\underset{\bullet}{|}})$. Suppose $\mathfrak{X} \not\models \gamma(\bullet)$ or $\mathfrak{X} \not\models \gamma(\overset{\circ}{\underset{\bullet}{|}})$. Then there is a topo-rooted closed generated subframe $\mathfrak{Y}$ of $\mathfrak{X}$ and a stable map from $\mathfrak{Y}$ onto $\bullet$ or $\overset{\circ}{\underset{\bullet}{|}}$. Observe that under a stable map a preimage of an irreflexive point has to be irreflexive. Now both of the latter frames contain an irreflexive point, so in either case $\mathfrak{Y}$ contains an irreflexive point. Therefore, so does $\mathfrak{X}$. Thus, $\mathfrak{X}$ is not reflexive, and so $\mathfrak{X} \not\models p \to \Diamond p$. For the converse, suppose that x is an irreflexive point of $\mathfrak{X}$. Consider the closed generated subframe $Y := R^+[x]$ of $\mathfrak{X}$, and let $\mathfrak{Y}$ be the corresponding $\mathsf{K4}$-space. Clearly x is a unique root of $\mathfrak{Y}$. Since $x \notin R[x]$, there is a clopen subset of $\mathfrak{X}$ separating x from $R[x]$. Therefore, x is an isolated point of Y. Thus, $\mathfrak{Y}$ is topo-rooted. If $Y = \{x\}$, then the unique map from $\mathfrak{Y}$ onto $\bullet$ is stable, and so $\mathfrak{X} \not\models \gamma(\bullet)$. Otherwise mapping x to the root of $\overset{\circ}{\underset{\bullet}{|}}$ and the rest of Y to the top point of $\overset{\circ}{\underset{\bullet}{|}}$ gives rise to a stable map, and hence $\mathfrak{X} \not\models \gamma(\overset{\circ}{\underset{\bullet}{|}})$.

(3) As we already pointed out, $\mathsf{K4B}$ has the fmp. Also, since $\mathsf{K4} + \gamma(\overset{\circ}{\underset{\circ}{|}})$ is axiomatized over $\mathsf{K4}$ by the stable formula of a finite rooted $\mathsf{K4}$-frame with a reflexive root, it has the fmp by Proposition 4.5.10. Therefore, it is sufficient to show that for any finite rooted $\mathsf{K4}$-frame $\mathfrak{F} = (X, R)$, we have $\mathfrak{F} \models p \to \Box\Diamond p$ iff $\mathfrak{F} \models \gamma(\overset{\circ}{\underset{\circ}{|}})$. Suppose $\mathfrak{F} \not\models p \to \Box\Diamond p$. Then $\mathfrak{F}$ is not symmetric, and so there are $x, y \in X$ such that xRy but $y\not R x$. Define $f : \mathfrak{F} \to \overset{\circ}{\underset{\circ}{|}}$ by mapping $R^+[y]$ to the top node of $\overset{\circ}{\underset{\circ}{|}}$ and the rest to the root of $\overset{\circ}{\underset{\circ}{|}}$. It is easy to see that f is an onto stable map. Therefore, $\mathfrak{F} \not\models \overset{\circ}{\underset{\circ}{|}}$. Conversely, if $\mathfrak{F} \not\models \gamma(\overset{\circ}{\underset{\circ}{|}})$, then since $\mathfrak{F}$ is rooted, by Proposition 4.5.10(1), there is a stable map from $\mathfrak{F}$ onto $\overset{\circ}{\underset{\circ}{|}}$. Let x be a root of $\mathfrak{F}$ and let $y \in X$ be such that $f(y)$ is the top point of $\overset{\circ}{\underset{\circ}{|}}$. Since f is stable, we have xRy but $y\not R x$. Thus, $\mathfrak{F}$ is not symmetric. This yields that $\mathfrak{F} \not\models p \to \Box\Diamond p$. □

We also provide axiomatization of some $\mathsf{S4}$-stable logics.

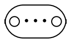
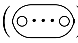
4.7.8. PROPOSITION. *The logics* $\mathsf{S5}$ *and* $\mathsf{S4Alt}_n$ *are* $\mathsf{S4}$*-stable. They are axiomatized over* $\mathsf{S4}$ *by the following stable formulas:*

(1) $\mathsf{S5} = \mathsf{S4} + \gamma(\overset{\circ}{\underset{\circ}{|}})$.

(2) $\mathsf{S4Alt}_n = \mathsf{S4} + \gamma(\boxed{\circ \cdots \circ})$.

Proof:

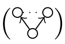
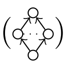
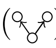
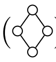
(1) Since $\mathsf{S5} = \mathsf{S4} \vee \mathsf{K4B}$, this follows from Lemma 4.6.5 connecting $\mathsf{S4}$-stability and $\mathsf{K4}$-stability.

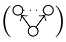
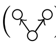
(2) Observe that there is a stable map from a finite rooted S4-frame $\mathfrak{F}$ onto the $(n+1)$-cluster  iff the cardinality of $\mathfrak{F}$ is greater than n. The result follows since both $\mathsf{S4Alt}_n$ and $\mathsf{S4} + \gamma($$)$ have the fmp.

□

Next we provide axiomatizations of some more standard S4-stable logics. This time we use our results connecting S4-stability and stability of si logics from [17]. Recall that the intuitionistic fragments of the logics S4.2, S4.3, $\mathsf{S4BW}_n$, and $\mathsf{S4BTW}_n$ are LC, KC, BW_n, and BTW_n, respectively (see Table A.0.2). In fact, the modal logics above are the respective least modal companions of their intuitionistic fragments, i.e. $\mathsf{S4.2} = \tau(\mathsf{LC})$, $\mathsf{S4.3} = \tau(\mathsf{KC})$, and more generally, $\mathsf{S4BW}_n = \tau(\mathsf{BW}_n)$ and $\mathsf{S4BTW}_n = \tau(\mathsf{BTW}_n)$ for every n. Lemma 4.6.1 together with the axiomatizations provided in [17, Theorem 7.5] yield the following.

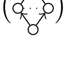
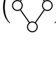
4.7.9. Proposition. *The logics* S4.2 *and* S4.3 *are* S4*-stable. More generally,* $\mathsf{S4BW}_n$ *and* $\mathsf{S4BTW}_n$ *are* S4*-stable for every n. These logics are axiomatized by the following stable formulas:*

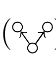
(1) $\mathsf{S4BW}_n = \mathsf{S4} + \gamma($$) + \gamma($$)$. *In particular,* $\mathsf{S4.3} = \mathsf{S4} + \gamma($$) + \gamma($$)$.

(2) $\mathsf{S4BTW}_n = \mathsf{S4} + \gamma($$)$. *In particular,* $\mathsf{S4.2} = \mathsf{S4} + \gamma($$)$.

Since $\mathsf{K4.3} = \mathsf{S4.3}^+$, and more generally, $\mathsf{K4BW}_n = (\mathsf{S4BW}_n)^+$, Proposition 4.7.9 together with Lemma 4.6.4 yield:

4.7.10. Proposition.

(1) $\mathsf{K4BW}_n = \mathsf{K4} + \gamma($$) + \gamma($$)$. *In particular,* $\mathsf{K4.3} = \mathsf{K4} + \gamma($$) + \gamma($$)$.

(2) $(\mathsf{S4BTW}_n)^+ = \mathsf{K4} + \gamma($$)$. *In particular,* $(\mathsf{S4.2})^+ = \mathsf{K4} + \gamma($$)$.

In the following table we summarize the axiomatizations of K4-stable and S4-stable logics obtained above.

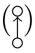
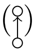
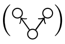
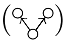
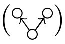
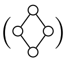
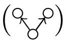
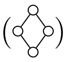
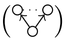
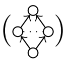
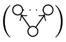
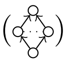
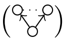
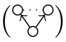
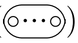
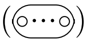
D4	=	$\mathsf{K4} + \gamma(\bullet)$	S4	=	$\mathsf{K4} + \gamma(\bullet) + \gamma($$)$
K4B	=	$\mathsf{K4} + \gamma($$)$	S5	=	$\mathsf{S4} + \gamma($$)$
$(\mathsf{S4.2})^+$	=	$\mathsf{K4} + \gamma($$)$	S4.2	=	$\mathsf{S4} + \gamma($$)$
K4.3	=	$\mathsf{K4} + \gamma($$) + \gamma($$)$	S4.3	=	$\mathsf{S4} + \gamma($$) + \gamma($$)$
$\mathsf{K4BW}_n$	=	$\mathsf{K4} + \gamma($$) + \gamma($$)$	$\mathsf{S4BW}_n$	=	$\mathsf{S4} + \gamma($$) + \gamma($$)$
$(\mathsf{S4BTW}_n)^+$	=	$\mathsf{K4} + \gamma($$)$	$\mathsf{S4BTW}_n$	=	$\mathsf{S4} + \gamma($$)$
$(\mathsf{S4Alt}_n)^+$	=	$\mathsf{K4} + \gamma($$)$	$\mathsf{S4Alt}_n$	=	$\mathsf{S4} + \gamma($$)$

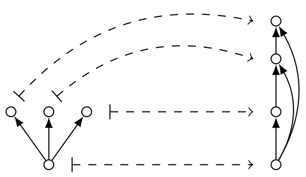
Table 4.7.1: Axiomatizations of some K4-stable and S4-stable logics

4.7.4 Non-examples

Finally, as promised, we show that several well-known logics are not stable. We point out that to prove that a given logic L is not stable it is not sufficient to show that the class of all finite L-frames is not stable. The difficulty is in proving that L is not characterized by *any* stable class of finite L-frames. The definition and frame properties of the logics mentioned below can be found in Table A.0.5.

4.7.11. THEOREM. *None of the logics* K4, S4, KB, *and* K5 *is stable. Neither are the logics* GL, S4.Grz, K4.1, S4.1, K4.2, *and* $\mathsf{K4Alt}_n$. *In fact,* GL, K4.1, K4.2, *and* $\mathsf{K4Alt}_n$ *are not* K4*-stable and* S4.Grz *and* S4.1 *are neither* K4*-stable nor* S4*-stable.*

Proof:
We start by showing that K4 is not stable. If K4 were stable, then by Theorem 4.4.4, there would exist a stable class $\mathcal{K}$ of finite rooted K4-frames whose logic is K4. Consider the finite rooted frames $\mathfrak{F}, \mathfrak{G}$ and an onto stable map $\mathfrak{G} \twoheadrightarrow \mathfrak{F}$ shown below.

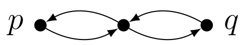


Note that $\mathfrak{G}$ is transitive, but $\mathfrak{F}$ is not, since the maximal point is not an immediate successor of the point of depth 3. Since $\mathfrak{G}$ is a $\mathsf{K4}$-frame and $\mathfrak{G} \not\models \gamma(\mathfrak{G})$, we see that $\mathsf{K4} \not\models \gamma(\mathfrak{G})$. Therefore, there is $\mathfrak{H} \in \mathcal{K}$ such that $\mathfrak{H} \not\models \gamma(\mathfrak{G})$. As $\mathfrak{G}$ has a reflexive root, by Proposition 4.5.10(1), $\mathfrak{G}$ is a stable image of $\mathfrak{H}$. Thus, since $\mathcal{K}$ is stable, $\mathfrak{G} \in \mathcal{K}$. Since $\mathfrak{F}$ is a stable image $\mathfrak{G}$, we have $\mathfrak{F} \in \mathcal{K}$. But this is a contradiction since $\mathfrak{F}$ is not transitive. Consequently, $\mathsf{K4}$ is not a stable logic.

A similar reasoning gives that $\mathsf{S4}$ is not a stable logic. We next show that KB is not a stable logic. If it were, then by Theorem 4.4.4, there would exist a stable class $\mathcal{K}$ of finite rooted KB-frames whose logic is KB.

4.7.1. Claim. *There is $\mathfrak{F} \in \mathcal{K}$ containing distinct x, y that are not R-related to each other.*

Proof:
Clearly the KB-model



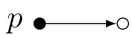
refutes $\mathsf{bw}_1 = \Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond^+ q) \vee \Diamond(q \wedge \Diamond^+ p)$. Therefore, $\mathsf{KB} \not\vdash \mathsf{bw}_1$. Thus, there is $\mathfrak{F} \in \mathcal{K}$ such that $\mathfrak{F} \not\models \mathsf{bw}_1$. It is easy to see that $\mathfrak{F}$ has the desired property. □

For $\mathfrak{F} = (X, R)$ and x, y as in Claim 4.7.1, define $\mathfrak{F}' = (X, R')$, where $R' = R \cup \{(x, y)\}$. Then the identity map is a stable map from $\mathfrak{F}$ onto $\mathfrak{F}'$. Since $\mathcal{K}$ is stable, $\mathfrak{F}' \in \mathcal{K}$. But this is a contradiction as $\mathfrak{F}'$ is not symmetric. Thus, KB is not a stable logic.

Next we show that $\mathsf{K5}$ is not a stable logic. If $\mathsf{K5}$ were stable, then there would be a stable class $\mathcal{K}$ of finite rooted $\mathsf{K5}$-frames whose logic is $\mathsf{K5}$.

4.7.2. Claim. *There is $\mathfrak{F} \in \mathcal{K}$ containing x, y such that xRy and $x \not{R} x$.*

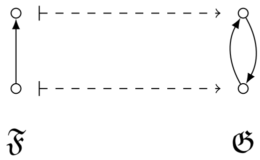
Proof:
Clearly the $\mathsf{K5}$-model



refutes the formula $\varphi := p \to \Diamond p \vee \Box\bot$. Therefore, $\mathsf{K5} \not\vdash \varphi$. Thus, there is $\mathfrak{F} \in \mathcal{K}$ such that $\mathfrak{F} \not\models \varphi$. It is easy to see that $\mathfrak{F}$ has the desired property. □

For such an $\mathfrak{F} = (X, R)$ define $\mathfrak{F}' = (X, R')$, where $R' = R \cup \{(y, x)\}$. Then the identity map is a stable map from $\mathfrak{F}$ onto $\mathfrak{F}'$. Since $\mathcal{K}$ is stable, $\mathfrak{F}' \in \mathcal{K}$. But this is a contradiction as $\mathfrak{F}'$ is not Euclidean because in an Euclidean frame every successor is reflexive. Thus, $\mathsf{K5}$ is not a stable logic.

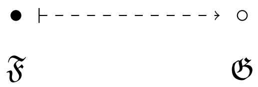
Next we show that $\mathsf{S4.Grz}$ is not a stable logic. By Proposition 4.4.5(1), it is sufficient to show that $\mathsf{S4.Grz}$ is not $\mathsf{S4}$-stable. It is easy to see that the map $\mathfrak{F} \twoheadrightarrow \mathfrak{G}$ between finite rooted $\mathsf{S4}$-frames depicted below is stable.



Note that $\mathfrak{F}$ is a S4.Grz-frame, while $\mathfrak{G}$ is not. Therefore, by Corollary 4.5.12(6), S4.Grz is not S4-stable. Thus, by Lemma 4.6.5(1), S4.Grz is not K4-stable.

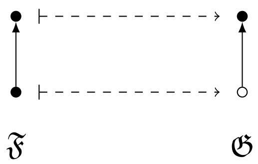
The same argument yields that S4.1 is not S4-stable. So by Lemma 4.6.5(1), S4.1 is not K4-stable. Since S4.1 $=$ S4 $\vee$ K4.1, Lemma 4.6.5(2) yields that K4.1 is not K4-stable. Thus, neither S4.1 nor K4.1 is stable by Proposition 4.4.5(1).

We show that GL is not stable. For this it is sufficient to show that GL is not K4-stable. It is easy to see that the map depicted below is a stable map from a finite rooted GL-frame $\mathfrak{F}$ onto a finite rooted K4-frame $\mathfrak{G}$, which is not a GL-frame.

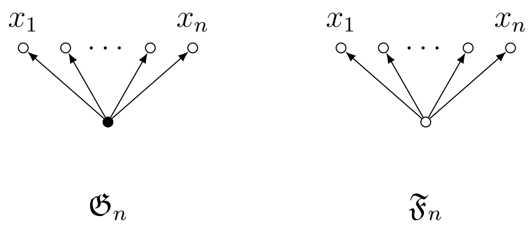


The rest of the argument is the same as in the case of S4.Grz.

Next, we show that K4.2 is not stable and not K4-stable. Again it is sufficient to show that K4.2 is not K4-stable. The frame $\mathfrak{F}$ below is directed and thus validates K4.2. On the other hand, the frame $\mathfrak{G}$ below is not directed since the root has itself and the top node as successors but the two nodes do not have a common successor. It is easy to see that the map depicted below is a stable map from $\mathfrak{F}$ onto $\mathfrak{G}$. As above, we conclude that K4.2 is not K4-stable.



Next, we show that K4Alt$_n$ is not K4-stable for any n. Consider the frames $\mathfrak{G}_n$ and $\mathfrak{F}_n$ shown below.



Note that the only difference between $\mathfrak{F}_n$ and $\mathfrak{G}_n$ is that $\mathfrak{F}_n$ has a reflexive root while $\mathfrak{G}_n$ does not. Because of this, $\mathfrak{G}_n$ is a K4Alt$_n$-frame while $\mathfrak{F}_n$ is not since its root has $n+1$ successors. The identity map from $\mathfrak{G}_n$ onto $\mathfrak{F}_n$ is obviously stable. Thus, as before, we can infer that K4Alt$_n$ is not K4-stable. □

4.8 Stable modal logics compared to subframe modal logics

We conclude this chapter by comparing the class of stable modal logics to the class of modal subframe logics. Since both classes of logics behave differently in the transitive case, we distinguish between the general and transitive cases.

Let $\mathfrak{F} = (X, R)$ be a modal space. A (*modal*) *subframe* of $\mathfrak{F}$ is a modal space of the shape $\mathfrak{G} = (U, R \cap (U \times U))$ where U is a clopen subset of X. If $\mathfrak{F} = (X, R)$ is a Kripke frame, then a (modal) subframe of $\mathfrak{F}$ is a substructure of $\mathfrak{F}$ as known from model theory, i.e. a subset of X with the relation restricted to that subset.

From an algebraic perspective, modal subframes correspond to relativizations of modal algebras (see e.g. [40, Section 9.1]). If $\mathfrak{A} = (A, \Diamond)$ is a modal algebra and $a \in A$, the *relativization of* $\mathfrak{A}$ *to* a is the modal algebra $\mathfrak{A}_a = (A_a, \Diamond_a)$ with domain $A_a = \{c \in A \mid c \leq a\}$ and with operations defined by $1_a = a$, $c \wedge_a b = c \wedge b$, $\neg c = \neg c \wedge a$, and $\Diamond_a c = \Diamond c \wedge a$ for all $c, d \in A_a$.

4.8.1. Definition. ([59, 126]) A normal modal logic L is called a *subframe* logic iff its modal spaces are closed under subframes.

In many ways stable logics parallel subframe logics. First of all, both classes of logics are defined by imposing a closure property on their classes of modal spaces (or algebras).

In the transitive case, subframe logics admit selective filtration, and hence have the fmp [59]. Similarly, $\mathsf{K4}$-stable logics admit transitive filtrations and hence have the fmp. In the transitive case, analogies can also be found from a syntactic perspective. Transitive subframe logics admit uniform axiomatizations via the so-called subframe formulas [59] and also by the subframe rules of [83]. In parallel, $\mathsf{K4}$-stable logics can be axiomatized by stable formulas and also by stable rules.

Similarities, but also differences arise in the behavior under intuitionistic fragments and modal companions. Subframe logics are preserved by least and greatest modal companions, and also intuitionistic fragments (see [40, Section 9.6]). As we saw in Section 4.6.2, intuitionistic fragments of $\mathsf{S4}$-stable logics are stable, and least modal companions of stable si logics are $\mathsf{S4}$-stable. Stability is however not preserved by least modal companions.

From a model-theoretic perspective, subframe logics are complete with respect to a class of frames closed under subframes, whereas stable modal logics are complete with respect to a class closed under stable images. Further analogies from the model-theoretic perspective can be found between elementary subframe logics and elementary stable logics.

As we discussed in Section 4.7.1 by Lyndon's theorem, a first-order sentence is preserved by stable maps iff it is equivalent to a set of positive sentences. In particular, if a modal logic L is characterized by a class of frames that is definable by positive sentences, then L is stable.

On the other hand, it is a well known result of Łoś and Tarski that a first-order sentence is preserved by substructures iff it is equivalent to a set of universal sentences (see, e.g., [42, Theorem 3.2.2]). Consequently, if a modal logic L is characterized by a class of frames that is definable by universal sentences, then L is a subframe logic. However, for Kripke-complete subframe logics the converse is also true: If a Kripke-complete subframe logic is elementary, then its class of frames is universal [126] (see also [40, Theorem 11.31]).

An essential difference in the non-transitive case is that stable logics admit filtration and thus have the fmp. On the other hand, subframe logics may fail to have this property. Even worse, as shown in [126], there is an infinite chain of Kripke incomplete subframe logics.

Another good proof-theoretic property—that we have not yet mentioned—that stable modal logics enjoy is the bounded proof property (bpp) [33]. To the best of our knowledge it is open whether subframe logics enjoy the bpp.

We summarize the results from the above discussion in Table 4.8.1.

Open problems

We conclude by explicitly stating some open problems that we have encountered in this chapter. In the following, let M be a normal modal logic that admits filtration.

In Proposition 4.4.5 we showed that M-stable logics form a $\bigwedge$-subsemilattice of $\mathsf{NExt}(\mathsf{M})$. However, we could not prove that they also form a $\bigvee$-semilattice of $\mathsf{NExt}(\mathsf{M})$. We therefore state the following open problem.

Problem 1: Do M-stable modal logics form a complete sublattice of $\mathsf{NExt}(\mathsf{M})$? In particular, do stable modal logics form a complete sublattice of $\mathsf{NExt}(\mathsf{K})$?

Recall that in some cases we answered Problem 1 affirmatively. Indeed, in Corollary 4.5.7 we proved that whenever M is transitive and has the $(*)$-property, then M-stable logics form a complete sublattice of $\mathsf{NExt}(\mathsf{M})$. Moreover, in Proposition 4.4.6 we showed that tabular M-stable logics form a $\bigvee$-semilattice of $\mathsf{NExt}(\mathsf{M})$ whenever M has the $(*)$-property. Both results immediately follow from the fact that in the corresponding cases, M-stable logics have the property that their subdirectly irreducible algebras form a $\mathcal{V}(\mathsf{M})_{\mathsf{si}}$-stable class. Thus, a positive answer to the following problem would solve Problem 1 affirmatively.

Problem 2: If L is M-stable, is $\mathcal{V}(\mathsf{L})_{\mathsf{si}}$ stable within $\mathcal{V}(\mathsf{M})_{\mathsf{si}}$? What if M has the $(*)$-property?

Concerning elementary stable logics, we saw in Corollary 4.7.3 that if L is characterized by a positively definable class of Kripke frames, then L is stable.

Whether the converse of this fact holds was left open. We therefore formulate the following problem.

Problem 3: Suppose L is an elementary stable logic. Is L the logic of a class of frames definable by positive formulas?

An affirmative answer to Problem 3 would constitute an analogy with Kripke complete subframe logics that have the property of being elementary iff they are definable by universal formulas (as discussed above). It is, however, well-known that there are non-elementary subframe logics such as GL and S4.Grz. On the other hand, we lack an example of a non-elementary stable logic. This leads us to the following problem:

Problem 4: Is every stable normal modal logic elementary?

Obviously, the last two problems can be adjusted suitably to M-stable logics (e.g. Problem 4 only makes sense for elementary M).

	Subframe logics over K	subframe logics over K4	Stable logics (over K)	K4-stable logics	S4-stable logics
Frame-based characterization	Modal spaces are closed under subframes	Kripke frames are closed under subframes	Complete with respect to a class of *rooted* Kripke frames closed under stable images	*Rooted* Kripke frames are closed under stable images	*Rooted* Kripke frames are closed under stable images
Algebra-based characterization	Modal algebras are closed under relativizations	Complex algebras are closed under relativization	Generated by a stable class of *subdirectly irreducible* modal algebras	*Subdirectly irreducible* modal algebras closed under stable subalgebras	*Subdirectly irreducible* modal algebras closed under stable subalgebras
Syntactic characterization		Subframe formulas and subframe rules	Stable rules	Stable rules of finite K4-algebras	Stable formulas of finite S4-algebras
Kripke-complete	-	✓	✓	✓	✓
fmp	-	✓	✓	✓	✓
Continuum-sized	✓	✓	✓	✓	✓

Elementary	-	-	?	?	?
Canonical	-	-	?	?	?

Table 4.8.1: "✓" means "property holds "; "?" means "we do not know whether the property holds in general" "-" means " property does not hold in general"

Chapter 5

Canonical rules and formulas

5.1 Introduction

As explained in the introduction of this thesis, canonical rules and formulas can serve as powerful tools for studying modal and si logics. The central property of canonical rules and formulas is that they axiomatize rule systems and logics in a uniform way.

Whereas Zakharyaschev's original approach to canonical formulas was frame-theoretic, algebraic approaches to canonical formulas have been investigated more recently [121, 13, 15, 18, 17].

In this chapter, we aim to provide a very general—and partially unifying—perspective on canonical rules and formulas. Our treatment applies to classes of algebras with *expandable locally finite reducts of algebras*. These have been identified in the aforementioned papers as the crucial ingredient of the method of canonical formulas.

Our aim is to summarize known results via a unified treatment by using tools from universal algebra. This has the advantage that the common root of various results becomes clear and some particularities get highlighted.

Apart from giving a general exposition, we also discuss how particular instances of canonical rules and formulas from the literature fit into our framework. Stable canonical rules for normal modal logics and stable canonical formulas for transitive modal logics, si logics and extension of k-CIRL [13, 17, 18, 32] are covered by our account (modulo some minor adjustements). However, Zakharyaschev's canonical formulas and Jeřábek's canonical rules for NExtK4 require a different technique and lie outside of our scope. We will come back to this point in this chapter.

Similar uniform accounts of canonical formulas appeared in [121, 44]. The main difference between those accounts and ours is that [121] and [44] work with formulas of partial algebras whereas we always assume our algebras to be total. The main advantage of working with total algebras as opposed to partial algebras is the immediate availability of duality theory, which often provides a nice geometric intuition of canonical formulas as refutation patterns.

5.2 Expandable locally finite reducts and filtrations

In this section we define the notion of an expandable locally finite reduct of $\mathcal{F}$-algebras. This leads to a very general notion of $\mathcal{H}$-filtrations. The latter is a generalization of the notion of $\mathcal{H}$-filtrations for Heyting algebras from Section 3.2 and that of filtrations in modal logic (see Section 4.2).

The reader may consult Section 2.1.1 for the terminology of universal algebra that we are using. Let $\mathcal{F}$ be an algebraic similarity type and let $\mathcal{H} \subseteq \mathcal{F}$. Recall that if $\mathfrak{A} = (A, F)$ is an $\mathcal{F}$-algebra, by $\mathfrak{A}_{\mathcal{H}} = (A, H)$ we denote the $\mathcal{H}$-reduct of $\mathfrak{A}$. If $\mathcal{K}$ is a class of $\mathcal{F}$-algebras, let

$$\mathcal{K}_{\mathcal{H}} = \{\mathfrak{A}_{\mathcal{H}} \mid \mathfrak{A} \in \mathcal{K}\},$$

denote the collection of all $\mathcal{H}$-reducts of algebras in $\mathcal{K}$.

5.2.1. Definition. Let $\mathcal{F}$ be a finite algebraic similarity type. A set $\mathcal{H} \subseteq \mathcal{F}$ is called an *expandable locally finite reduct of $\mathcal{F}$ in $\mathcal{V}$* if the following conditions hold.

(1) There is a locally finite variety $\mathcal{V}^{\mathcal{H}}$ that contains the collection of $\mathcal{H}$-reducts of all $\mathcal{F}$-algebras in $\mathcal{V}$, i.e. $\mathcal{V}_{\mathcal{H}} \subseteq \mathcal{V}^{\mathcal{H}}$.

(2) If $\mathfrak{A} \in \mathcal{V}$, then every finite $\mathcal{H}$-subalgebra $\mathfrak{B}' = (B, H)$ of $\mathfrak{A}_{\mathcal{H}}$ can be expanded to an $\mathcal{F}$-algebra $\mathfrak{B} = (B, F) \in \mathcal{V}$ so that

$$f_{\mathfrak{B}}(\bar{b}) = f_{\mathfrak{A}}(\bar{b}) \text{ for every } f \in \mathcal{F} \setminus \mathcal{H},\ \bar{b} \in B^{\sigma(f)} \text{ and } f_{\mathfrak{A}}(\bar{b}) \in B.$$

We make a few remarks about this definition. As explained right after Definition 2.1.12, we will be a bit more liberal when it comes to the notion of a reduct. In particular, we allow $\mathcal{H}$ to be a set of $\mathcal{F}$-terms as opposed to a proper subset of the signature $\mathcal{F}$.

We did not require $\mathcal{V}^{\mathcal{H}}$ to be unique, but whenever $\mathcal{H}$ is an expandable locally finite reduct of some $\mathcal{F}$, we will assume that it comes with a variety $\mathcal{V}^{\mathcal{H}}$ satisfying (1) and (2) from above. The fact that $\mathcal{V}^{\mathcal{H}}$ is a *variety* ensures that the size of each finitely generated $\mathcal{V}^{\mathcal{H}}$-algebra is bounded by some $m(n)$ which depends on the number n of its generators (see Theorem 2.1.14).

In words, condition (2) requires that f-values of elements in B coincide in $\mathfrak{B}$ and $\mathfrak{A}$ whenever the $f^{\mathfrak{A}}$-value of these elements is already in B. As a matter of fact, we have already seen several examples of expandable locally finite varieties in this thesis.

Variety $\mathcal{V}$	reduct $\mathcal{H}$	$\mathcal{V}^{\mathcal{H}}$	"missing" operation
HA	$\{\wedge, \to\}$	BSLat	$\dot{1} = \bigvee B$; $\dot{0} = \bigwedge B$; $a \dot{\vee} b = \bigwedge\{c \mid c \geq a, b\}$;
"-"	$\{\wedge, \to, \bot\}$	BBSLat	$\dot{1} = \bigvee B$; $a \dot{\vee} b = \bigwedge\{c \mid c \geq a, b\}$;
"-"	$\{\wedge, \vee, \bot, \top\}$	BDLat	$a \dot{\to} b = \bigvee\{c \mid a \wedge c \leq b\}$
"-"	$\{\wedge, \vee, \neg, \top\}$	BPDLat	$a \dot{\to} b = \bigvee\{c \mid a \wedge c \leq b\}$
MA	$\{0, 1, \wedge, \vee, \neg\}$	BA	Many options, e.g. filtrations (see Section 4.2)
K4-algebras	$\{0, 1, \wedge, \vee, \neg\}$	BA	Many options, e.g. transitive filtrations (see Section 4.2)
k-CIRL	$\{\cdot, \vee, 1\}$	commutative i-semirings	$a \dot{\to} b = \bigvee\{c \mid a \cdot c \leq b\}$; $a \dot{\wedge} b := \bigvee\{c \mid c \leq a$ and $c \leq b\}$

Table 5.2.1: Examples of varieties expandable locally finite reducts

In Table 5.2.1 we collect examples of expandable locally finite reducts with corresponding varieties $\mathcal{V}^{\mathcal{H}}$. The last row of the table describes how algebra reducts can be expanded to full algebras.

Recall that in Section 3.2 we discussed in detail the expandable locally finite reducts of Heyting algebras that are mentioned in Table 5.2.1. Condition (2) in Definition 5.2.1 is simply a generalization of condition (3.2) in the proof of McKay's theorem (Theorem 3.2.1), and condition (3.4) of the same section. In Fact 3.2.3 we also discussed that these reducts allow for a unique expansion into a Heyting algebra. The latter may not be the case for arbitrary expandable locally finite reducts.

Boolean algebras are expandable locally finite reducts of modal algebras. In contrast to the Heyting case, Boolean algebras allow several expansions into modal algebras. In fact, the modal operators arising from filtrations (see Section 4.2) indicate ways to build modal algebra expansions. Likewise, transitive filtrations indicate ways to build expansions of Boolean algebras into K4-algebras.

Some varieties of *residuated lattices* (see Section 2.3.2) also have expandable locally finite reducts. The case of k-CIRL, the variety of k-potent commutative integral residuated lattices is discussed in [32], where it is proved that commutative i-semirings are an expandable locally finite reduct of k-CIRLs. We summarize the results from that paper at the end of Section 5.4.1.

Unless stated otherwise, in the remaining part of this section, $\mathcal{H}$ denotes a fixed expandable locally finite reduct of $\mathcal{F}$ in a class $\mathcal{V}$ of $\mathcal{F}$-algebras.

Our next goal is to define a general algebraic notion of filtration. For this we need the notion of an $\mathcal{H}$-homomorphism satisfying the closed domain condition.

5.2.2. Definition. Let $\mathfrak{A} = (A, F_A)$ and $\mathfrak{B} = (B, F_B)$ be $\mathcal{F}$-algebras. For each $f \in \mathcal{F} \setminus \mathcal{H}$, let $D_f \subseteq B^{\sigma(f)}$. We say that an $\mathcal{H}$-homomorphism $h : B \to A$ satisfies the *closed domain condition* (CDC) for $\{D_f\}_{f \in \mathcal{F} \setminus \mathcal{H}}$ iff

$$h(f_{\mathfrak{B}}(\bar{b})) = f_{\mathfrak{A}}\left(h(\bar{b})\right) \text{ for all } f \in \mathcal{F} \setminus \mathcal{H} \text{ and } \bar{b} \in D_f,$$

where for $\bar{b} \in D_f$, $h(\bar{b})$ denotes the $|\bar{b}|$-tuple $(h(b))_{b \in \bar{b}}$.

Thus, the CDC condition simply ensures that the $\mathcal{H}$-homomorphism h preserves additional operations on the specified sets $\mathcal{D}_f$ for $f \in \mathcal{F} \setminus \mathcal{H}$.

5.2.3. Remark. The name CDC was used by Zakharyschev in his frame-theoretic definition of canonical formulas. From the frame theoretic perspective, the condition refers to a specific geometric condition. As shown in [13], in its algebraic version for Heyting algebras, Zakharyaschev's CDC condition translates to the condition of Definition 5.2.2. Thus, we kept the name CDC even though the geometric intuition of closed domains is not available in the algebraic setting.

Next we define $\mathcal{H}$-filtrations. The notion below obviously extends the notions of $\mathcal{H}$-filtrations for si logics (Definition 3.2.4) and the standard notion of filtration in modal logic (see Section 4.2). In fact, $\mathcal{H}$-filtrations in the intuitionistic setting are just instances of the definition below. On the other hand, the definition below cannot capture condition (A2) of definition of filtrations in modal logic (Definition 4.2.4) that corresponds to the fact that diamonds are preserved "half-ways". We will get back to this at a later point.

5.2.4. Definition. Let $\mathfrak{A} = (A, F)$ be a $\mathcal{V}$-algebra and let Σ be a finite set of terms closed under subterms and let v be a valuation on $\mathfrak{A}$. Let $\mathfrak{B}' = (B, H)$ be the $\mathcal{H}$-subalgebra of $\mathfrak{A}_{\mathcal{H}}$ generated by $v(\Sigma) = \{v(t) \mid t \in \Sigma\}$. For $f \in \mathcal{F} \setminus \mathcal{H}$, let

$$D_f = \{(v(t_1), \dots, v(t_n)) \mid f(t_1, \dots, t_n) \in \Sigma\}.$$

If $\mathfrak{B} = (B, F)$ is a $\mathcal{V}$-algebra with $\mathfrak{B}_{\mathcal{H}} = \mathfrak{B}'$, and $v_{\mathfrak{B}}$ a valuation on $\mathfrak{B}$ such that

(1) $v(x) = v_{\mathfrak{B}}(x)$ for all variables $x \in \Sigma$, and

(2) the $\mathcal{H}$-embedding $B \hookrightarrow A$ satisfies CDC for $\{D_f\}_{f \in \mathcal{F} \setminus \mathcal{H}}$,

then $(\mathfrak{B}, v_{\mathfrak{B}})$ is called an *$\mathcal{H}$-filtration of* $(\mathfrak{A}, v)$ *through* Σ *in* $\mathcal{V}$. If the reduct $\mathcal{H}$ and the variety $\mathcal{V}$ are clear from the context, we will refer to $(\mathfrak{B}, v_{\mathfrak{B}})$ as a filtration of $(\mathfrak{A}, v)$ through Σ.

By a simple induction on the structure of terms we obtain an analogue of the standard filtration theorem. By design, we have that "filtrations exist" and that the cardinality of filtrated models can be bound by a number depending only on the cardinality of Σ. More precisely, filtrations exist since $\mathcal{V}$ is an expandable locally finite reduct and the latter condition follows from the fact that $\mathcal{V}^{\mathcal{H}}$ is a variety.

5.2.5. Theorem (Filtration Theorem). *Let $\mathfrak{A}$ be a $\mathcal{V}$-algebra, let Σ be a finite set of terms closed under subterms, and let v be a valuation on $\mathfrak{A}$.*

(1) If $(\mathfrak{B}, v')$ is an $\mathcal{H}$-filtration of $(\mathfrak{A}, v)$ through Σ in $\mathcal{V}$, then $v'(t) = v(t)$ for all $t \in \Sigma$.

(2) There exists a finite model $(\mathfrak{B}, v')$ that is an $\mathcal{H}$-filtration of $(\mathfrak{A}, v)$ through Σ in $\mathcal{V}$.

(3) There is a natural number $\kappa(|\Sigma|)$ that bounds the cardinality of all $\mathcal{H}$-filtrations through Σ in $\mathcal{V}$.

Proof:
The proof of (1) is a simple induction on the structure of terms.

The proof of (2) follows easily from the definition of expandable locally finite reducts. Indeed, let $\mathfrak{B}'$ be the $\mathcal{H}$-subalgebra of $\mathfrak{A}_{\mathcal{H}}$ generated by $v(\Sigma)$. Since $\mathfrak{B}' \in \mathcal{V}^{\mathcal{H}}$, $\mathfrak{B}'$ is finite. Let $\mathfrak{B}$ be an expansion of $\mathfrak{B}'$ to an $\mathcal{F}$-algebra in $\mathcal{V}$ that satisfies condition (2) of Definition 5.2.1. Moreover, let a valuation on $\mathfrak{B}$ be defined by $v_{\mathfrak{B}}(x) = v(x)$ for all variables $x \in \Sigma$. It is easy to verify that $(\mathfrak{B}, v_{\mathfrak{B}})$ satisfies the conditions of Definition 5.2.4 and is therefore an $\mathcal{H}$-filtration of $(\mathfrak{A}, v)$ through Σ in $\mathcal{V}$.

To see (3) recall that by the local finiteness of $\mathcal{V}^{\mathcal{H}}$, there is a natural number $\kappa(|\Sigma|)$ that bounds the cardinality of each $|\Sigma|$-generated algebra in $\mathcal{V}^{\mathcal{H}}$ (Theorem 2.1.14). Then $\kappa(|\Sigma|)$ bounds the cardinality of all $\mathcal{H}$-filtrations through Σ. □

5.3 $\mathcal{H}$-stable canonical rules

Next we define $\mathcal{H}$-stable canonical rules of finite $\mathcal{V}$-algebras. Then we show that every multi-conclusion rule in the signature $\mathcal{F}$ can be "replaced" by a finite number of $\mathcal{H}$-canonical rules. Roughly speaking, a rule can be replaced by the $\mathcal{H}$-canonical rules corresponding to "minimal" refutation algebras.

We then explain how the aforementioned general result—which is using merely universal algebra—relates to more specific ones from the literature. We will also see the connections to the $\mathcal{H}$-stable rules for si logics from Definition 3.3.4 and the stable rules for modal logics from Section 4.3.

We start by introducing a few technical notions. The reader may consult Section 2.1.1 for our notational conventions in universal algebra.

By an *equational multi-conclusion rule* in the signature $\mathcal{F}$, we mean an expression of the form Γ/Δ, where Γ and Δ are finite sets of $\mathcal{F}$-equations (see also [44]).

A model $(\mathfrak{A}, v)$ satisfies a rule Γ/Δ iff $(\mathfrak{A}, v) \models \Gamma$ implies $(\mathfrak{A}, v) \models s \approx s'$ for some $s \approx s' \in \Delta$. In that case we write $(\mathfrak{A}, v) \models \Gamma/\Delta$. If $(\mathfrak{A}, v) \models \Gamma/\Delta$ for every valuation, we write $\mathfrak{A} \models \Gamma/\Delta$ and say that the rule is valid on $\mathfrak{A}$. It is easy to see that the validity of a rule Γ/Δ is equivalent to the truth of the universal first order sentence

$$\forall \bar{x} \left(\bigwedge_{t \approx t' \in \Gamma} t(\bar{x}) \approx t'(\bar{x}) \to \bigvee_{s \approx s' \in \Delta} s(\bar{x}) \approx s'(\bar{x}) \right), \tag{5.1}$$

where $\bar{x}$ contains the variables occurring in the terms of $\Gamma \cup \Delta$. Conversely, by using normal forms it is easy to see that every universal sentence can be replaced by finitely many universal sentences of the shape of equation (5.1). Therefore, by Birkhoff's theorem (Theorem 2.1.2(2)), equational multi-conclusion rules axiomatize precisely the universal classes of $\mathcal{F}$-algebras.

We follow the notational convention from the previous section, i.e.

$\mathcal{H}$ denotes a fixed expandable locally finite reduct of $\mathcal{F}$ in a class $\mathcal{V}$ of $\mathcal{F}$-algebras.

We are ready to define $\mathcal{H}$-stable canonical rules of finite $\mathcal{V}$-algebras.

5.3.1. Definition. Let $\mathfrak{B} = (B, F)$ be a finite $\mathcal{V}$-algebra, and for each $f \in \mathcal{F} \setminus \mathcal{H}$, let a finite $D_f \subseteq B^{\sigma(f)}$ be fixed. For each $b \in B$ let x_b be a variable. The *$\mathcal{H}$-canonical rule* of $(\mathfrak{B}, \{D_f\}_{f \in \mathcal{F} \setminus \mathcal{H}})$ is Γ/Δ, where

$$\begin{aligned} \Gamma &= \{f(x_{\bar{b}}) \approx x_{f^{\mathfrak{B}}(\bar{b})} \mid f \in \mathcal{H}, \bar{b} \in B^*\} \cup \\ & \{f(x_{\bar{b}}) \approx x_{f^{\mathfrak{B}}(\bar{b})} \mid f \in \mathcal{F} \setminus \mathcal{H}, \bar{b} \in D_f\}, \text{ and} \\ \Delta &= \{x_b \approx x'_b \mid b \neq b' \in B\}, \end{aligned}$$

where $x_{\bar{b}} := (x_b)_{b \in \bar{b}}$ for some $\bar{b} \in B^{\sigma(f)}$.

Obviously, the $\mathcal{H}$-canonical rule of an algebra $\mathfrak{B}$ is an adaption of the positive diagram of $\mathfrak{B}$ known from model theory. Similarly to diagrams, $\mathcal{H}$-canonical rules have a semantic refutation criterion.

5.3.2. LEMMA. *Let $(\mathfrak{B}, \{D_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$ be as in Definition 5.3.1 and let $\mathfrak{A} = (A, F)$ be a $\mathcal{V}$-algebra The following are equivalent:*

(1) $\mathfrak{A} \not\models \rho(\mathfrak{B}, \{D_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$.

(2) There is an $\mathcal{H}$-embedding $h : \mathfrak{B} \to \mathfrak{A}$ satisfying CDC for $\{D_f\}_{f\in\mathcal{F}\setminus\mathcal{H}}$.

Proof:
To see that (1) implies (2), let v be a valuation on $\mathfrak{A}$ such that $(\mathfrak{A}, v)$ refutes $\rho(\mathfrak{B}, \{D_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$. Define $h : B \to A$ by $h(b) = v(p_b)$. It is easy to see that h defines an embedding as required. In fact, if $\rho(\mathfrak{B}, \{D_f\}_{f\in\mathcal{F}\setminus\mathcal{H}}) = \Gamma/\Delta$, then the fact that h is an $\mathcal{H}$-homomorphism satisfying CDC is ensured since $(\mathfrak{A}, v)$ satisfies all equations in Γ, and h is an embedding since $(\mathfrak{A}, v)$ refutes all equations in Δ.

Conversely, let $h : B \to A$ be as in (2). Then define a valuation $v(p_b) = h(b)$ on $\mathfrak{A}$. It is easy to see that $(\mathfrak{A}, v)$ refutes $\rho(\mathfrak{B}, \{D_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$. □

Now we are ready to show that every rule can be replaced by a finite set of canonical rules. In the next section we discuss more specific instances of these rules for Heyting and modal algebras and also explain how they compare to other results from the literature.

The proof of the following theorem is a straightforward adaption of [18, Theorem 5.1].

5.3.3. THEOREM. *For every multiple-conclusion rule Γ/Δ, there is $n \in \mathbb{N}$ and a collection $\{(\mathfrak{B}_i, \{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})\}_{1\le i\le n}$ of finite $\mathcal{V}$-algebras $\mathfrak{B}_i = (B_i, F_i)$, and $D^i_f \subseteq B^{\sigma(f)}$, for $f \in \mathcal{F} \setminus \mathcal{H}$ and $1 \le i \le n$ such that for every $\mathcal{V}$-algebra $\mathfrak{A} = (A, F)$ the following are equivalent:*

(1) $\mathfrak{A} \not\models \Gamma/\Delta$,

(2) $\mathfrak{A} \not\models \rho(\mathfrak{B}_i, \{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$ *for some* $1 \le i \le n$.

Proof:
Let Σ be the collection of all subterms of the equations in $\Gamma \cup \Delta$. By Theorem 5.2.5(3), there is a natural number $\kappa(|\Sigma|)$ that limits the size of every $\mathcal{H}$-filtration through Σ in $\mathcal{V}$. Let $\{(\mathfrak{B}_i, v_i)\}_{1\le i\le n}$ be a collection of all $\mathcal{V}$-models up to isomorphism that satisfy the following conditions:

- $|B_i| \le \kappa(|\Sigma|)$,
- $(\mathfrak{B}_i, v_i) \not\models \Gamma/\Delta$,

where two such models $(\mathfrak{B}, v)$ and $(\mathfrak{B}, v')$ are isomorphic iff there is an $\mathcal{F}$-algebra isomorphism $\iota : B \to B'$ with $\iota \circ v = v'$.

For $f \in \mathcal{F} \setminus \mathcal{H}$, define

$$D^i_f = \{(v_i(t_1), \ldots v_i(t_n)) \mid f(t_1, \ldots t_n) \in \Sigma\}.$$

We show that the collection $\{(\mathfrak{B}_i, \{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})\}_{1\leq i\leq n}$ satisfies the above requirements. Note that it is finite since the size of the $\mathfrak{B}_i$ is bounded by $\kappa(|\Sigma|)$ and $\mathcal{F}$ is finite.

We show the equivalence between (1) and (2).

(1) ⇒ (2): Suppose $\mathfrak{A} \in \mathcal{V}$ and let v be a valuation on $\mathfrak{A}$ with $(\mathfrak{A}, v) \not\models \Gamma/\Delta$. By Theorem 5.2.52, there is an $\mathcal{H}$-filtration $(\mathfrak{B}, v_{\mathfrak{B}})$ of $(\mathfrak{A}, v)$ through Σ in $\mathcal{V}$. Then $(\mathfrak{B}, v_{\mathfrak{B}})$ is isomorphic to some $(\mathfrak{B}_i, v_i)$ for some $1 \leq i \leq n$. Then $\mathfrak{A} \not\models \rho(\mathfrak{B}_i, \{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$ by Lemma 5.3.2.

(2) ⇒ (1): Suppose $\mathfrak{A} \not\models \rho(\mathfrak{B}_i, \{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$ for some $1 \leq i \leq n$. By Lemma 5.3.2 there is an $\mathcal{H}$-embedding h from $\mathfrak{B}_i$ into $\mathfrak{A}$ that satisfies CDC for $\{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}}$. Define a valuation $v = h \circ v_i$ on $\mathfrak{A}$. A simple induction shows that $(\mathfrak{A}, v)$ refutes Γ/Δ.

□

5.3.1 $\mathcal{H}$-stable canonical rules for si logics and normal modal logics

We now move to $\mathcal{H}$-stable canonical rules in the intuitionistic and modal setting. In particular, we explain how to axiomatize intuitionistic and modal multi-conclusion consequence relations via $\mathcal{H}$-stable canonical rules using the expandable locally finite reducts from Table 5.2.1.

In this section, $\mathcal{V}$ stands for the variety of Heyting-, modal or K4-algebras and $\mathcal{H}$ denotes a corresponding expandable locally finite reduct from Table 5.2.1.

As we explained in Sections 2.3.1 and 2.3.3, equations over Heyting or modal algebras can be turned into bi-implications. By replacing the equations in the $\mathcal{H}$-stable canonical rules from Definition 5.3.1 with bi-implications, we obtain multi-conclusion rules of the usual shapes as in Section 2.3.6. We will still refer to these as $\mathcal{H}$-stable canonical rules.

By replacing every (rule-)axiom of a multi-conclusion consequence relation by its corresponding set of $\mathcal{H}$-canonical rules according to Theorem 5.3.3, and by using the completeness theorem for multi-conclusion consequence relations (Theorems 2.3.8 and 2.3.10), we obtain:

5.3.4. Corollary.

(1) Every (intuitionistic or normal modal) multi-conclusion consequence relation can be axiomatized over $\mathcal{S}_{\mathcal{V}}$ by $\mathcal{H}$-stable canonical rules.

(2) Every (intuitionistic or normal modal) logic can be axiomatized over $\mathsf{L}_{\mathcal{V}}$ by $\mathcal{H}$-stable canonical rules.

Finally, we explain some specifics of the modal and intuitionistic cases and in particular relate the aforementioned general result to those in the literature.

Intuitionistic case The expandable locally finite reducts of Heyting algebras from Table 5.2.1 offer four options to define $\mathcal{H}$-stable canonical rules for Heyting algebras. For $\mathcal{H} = \{\wedge, \vee, \bot, \top\}$ and $\mathcal{H} = \{\wedge, \vee, \neg, \top\}$, we obtain the stable and cofinal stable canonical rules of [21, 31], respectively.

On the one hand, the $\{\wedge, \to\}$- and $\{\wedge, \to, \bot\}$-stable canonical rules are related to Jeřábek's canonical rules for intuitionistic multi-conclusion consequence relations [83]. As explained in the introduction of this thesis, Jeřábek defined canonical rules as an adaptation of Zakharyaschev's canonical formulas and used them—among other things—as a tool in an alternative proof of decidability of the admissibility problem for IPC. Just like for Zakharyaschev's formulas, Jeřábek's account on rules was frame-theoretic. The $\{\wedge, \to\}$- and $\{\wedge, \to, \bot\}$-stable canonical rules are similar but not precisely dual to Jeřábek's rules. The subtle differences are of the same nature as the fact that subframes do not correspond algebraically to $\{\wedge, \to\}$-subalgebras but to nuclei (as we discussed in Section 3.4). In particular, the semantic refutation criterion of Jeřábek's rules is formulated in terms of subreductions. The latter do not precisely correspond algebraically to $\{\wedge, \to\}$- and $\{\wedge, \to, \bot\}$-subalgebras.

We also point out that [31] shows that Jeřábek's analysis of admissible rules for IPC can also be accomplished via stable canonical rules. We also note that the $\mathcal{H}$-stable rules we saw in Definition 3.3.4 are a special case of $\mathcal{H}$-stable canonical rules where the D_fs are empty.

Modal case We explain how the general $\mathcal{H}$-stable canonical rules can be used in the modal case and how the resulting rules relate to the stable canonical rules of [18] and [31] and Jeřábek's rules of [83].

For the modal case, we take as $\mathcal{V}$ the variety of modal algebras, and $\mathcal{H}$ and $\mathcal{V}^{\mathcal{H}}$ are the signature and variety of Boolean algebras, respectively. As explained, the Boolean algebra reducts can be expanded to modal algebras via filtrations. For simplicity, we call the resulting rules BA-canonical rules for a moment.

The stable canonical rules of [18] and [31] are based on filtrations. Modulo a slight modification, stable canonical rules fit well into our framework. We explain this in detail. As we remarked earlier, the general definition of filtration (Definition 5.2.4) does not capture the stability condition of filtrations in modal logic (condition (A2) in Definition 4.2.4). Thus, in the theory of BA-canonical rules, it is not captured that filtrations produce stable embeddings as opposed to Boolean algebra homomorphism satisfying the CDC condition.

Thus, in order to obtain stable canonical rules, we need to "replace" the Boolean algebra homomorphism in the theory of BA-canonical rules with *stable homomorphisms*. Since this needs to be reflected syntactically in the corresponding rules, also the rules need to be modified. To illustrate, we recall the definition of stable canonical rules from [18, Definition 5.2]. For a finite modal algebra $\mathfrak{B} = (B, \Diamond)$, the *stable canonical rule* is defined as Γ/Δ, where Γ and Δ are defined below. As usual $\{p_a \mid a \in B\}$ is a collection of propositional letters and $D \subseteq B$.

$$\begin{aligned} \Gamma = &\{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid a, b \in B\} \cup \\ &\{p_{\neg a} \leftrightarrow \neg p_a \mid a \in B\} \cup \\ &\{\Diamond p_a \to p_{\Diamond a} \mid a \in B\} \\ &\{p_{\Diamond a} \to \Diamond p_a \mid a \in D\} \\ \text{and} \quad \Delta = &\{p_a \mid a \in A, a \neq 1\}. \end{aligned}$$

The difference to the BA-stable canonical rules lies in the third and forth lines of the definition of Γ. In the BA-stable canonical rule of (B, D) these two lines would be replaced by $\{p_{\Diamond a} \leftrightarrow \Diamond p_a \mid a \in D\}$. The third line in Γ forces the refutation criterion of the corresponding rule to be a stable as opposed to a Boolean algebra homomorphism (cf. stable rules from Section 4.3).

In the modal case, Jeřábek's rules for axiomatizing normal extensions of $\mathsf{K4}$ [83] lie, however, outside of our scope. A rough explanation is that Zakharyaschev's formulas and Jeřábek's rules make use of a *selective* filtration method that is in a way more sophisticated than filtration via expandable locally finite reducts. In [14, 15] such formulas are explored from an algebraic perspective.

5.4 Canonical formulas for non-classical logics

In this section we discuss canonical formulas for non-classical logics for special cases of expandable locally finite reducts of Table 5.2.1. As in the previous section, we state a very general axiomatization result and then relate it to more specific results from the literature.

The general exposition follows a two-step process. We first define $\mathcal{H}$-stable canonical formulas as the characteristic formulas of $\mathcal{H}$-stable canonical formulas of finite algebras. It will be an easy consequence of Theorem 5.3.3 that an arbitrary formula can be "replaced" by a set of $\mathcal{H}$-stable canonical formulas. However, $\mathcal{H}$-stable canonical formulas of finite *non-subdirectly irreducible* algebras loose some of their expected properties. Thus, a second step is to show that a formula can be "replaced" by a set $\mathcal{H}$-stable canonical formulas of subdirectly irreducible algebras. We will sketch the latter for some cases.

The intuitionistic case is treated in a separate section, since we will use those formulas in other parts of this thesis.

5.4.1 $\mathcal{H}$-stable canonical formulas

In this section L stands for one of the logics $\mathsf{K4}$, FL^k_{ew} or IPC. Moreover, $\mathcal{H}$ stands for a corresponding reduct from Table 5.2.1.

In Section 2.3.6 we discussed how rules can be turned into their characteristic formulas. For the reader's convenience we collect these definitions in Table 5.4.1 below.

Logic	$\chi(\Gamma/\Delta)$
IPC	$\bigwedge_{\gamma\in\Gamma}\gamma \to \bigvee_{\delta\in\Delta}\delta$
$\mathsf{K4}$	$\bigwedge_{\gamma\in\Gamma}\Box^+\gamma \to \bigvee_{\delta\in\Delta}\Box^+\delta$
FL^k_{ew}	$\left(\bigwedge_{\gamma\in\Gamma}\gamma\right)^k \to \bigvee_{\delta\in\Delta}\delta$

Table 5.4.1: Characteristic formulas of rules

5.4.1. Remark. We hint towards the more abstract cause for the existence of characteristic formulas. From an algebraic perspective it lies in the existence of *equationally definable principal congruences* (EDPC) (see [62]). From the logical perspective, this corresponds to having a deduction theorem. It lies beyond the scope of this thesis to elaborate on these correspondences in detail. See also [44] where characteristic formulas are defined from rules via ternary deductive terms (td-terms).

Now we simply define $\mathcal{H}$-stable canonical formulas as the characteristic formulas of $\mathcal{H}$-stable canonical rules from Definition 5.3.1 except that we assume that the equations in the $\mathcal{H}$-stable canonical rules are replaced by bi-implications (as we already did in Section 5.3.1).

5.4.2. Definition. Let $\mathfrak{B} = (B, F)$ be a finite L-algebra, and for each $f \in \mathcal{F}\setminus\mathcal{H}$, let a finite $D_f \subseteq B^{\sigma(f)}$ be fixed. The *$\mathcal{H}$-stable canonical formula* is defined as

$$\gamma(\mathfrak{B}, \{D_f\}_{f\in\mathcal{F}\setminus\mathcal{H}}) = \chi(\rho(\mathfrak{B}, \{D_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})).$$

Also recall the definitions of subdirectly irreducible L-algebras from Sections 2.1.2, 2.1.3, and 2.3.2, respectively. An immediate consequence of Lemma 2.3.11 is the following correspondence between the characteristic formulas and rules of a finite algebra.

5.4.3. LEMMA. *Let $\mathfrak{A}$ be an L-algebra. Then*

$\mathfrak{A} \not\models \gamma(\mathfrak{B}, \{D_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$ *iff there is a subdirectly irreducible homomorphic image* $\mathfrak{C}$ *of* $\mathfrak{A}$ *such that* $\mathfrak{C} \not\models \rho(\mathfrak{B}, \{D_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$.

In particular, if $\mathfrak{A}$ is subdirectly irreducible and $\mathfrak{A} \not\models \rho(\mathfrak{B}, \{D_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$, then $\mathfrak{A} \not\models \gamma(\mathfrak{B}, \{D_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$.

As we already mentioned in the introduction of this chapter, the $\mathcal{H}$-stable canonical formulas do not behave as expected on non-subdirectly irreducible algebras. In particular, by spelling out the refutation criteria of Lemmas 5.3.2 and 5.4.3, it immediately follows that a finite algebra will refute its corresponding formula only if it is subdirectly irreducible.

Nevertheless, we show that just as we replaced arbitrary rules by $\mathcal{H}$-stable canonical rules in Theorem 5.3.3, we can replace formulas by $\mathcal{H}$-stable canonical formulas.

5.4.4. THEOREM. *For a formula φ, there is a collection $\{(\mathfrak{B}_i, \{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})\}_{1\leq i\leq n}$ of pairs $(\mathfrak{B}_i, \{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$ consisting of an L-algebra $\mathfrak{B}_i$ and a sets $D^i_f \subseteq B^{\sigma(f)}$ for $f \in \mathcal{F} \setminus \mathcal{H}$ and $1 \leq i \leq n$, such that for every sudirectly irreducible L-algebra $\mathfrak{A}$, the following are equivalent:*

(1) $\mathfrak{A} \not\models \varphi$.

(2) $\mathfrak{A} \not\models \gamma(\mathfrak{B}_i, \{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$ *for some* $1 \leq i \leq n$.

Proof:
By Theorem 5.3.3, the rule $/\varphi$ can be expressed via $\mathcal{H}$-stable canonical rules, thus, there is a finite collection $\{(\mathfrak{B}_i, \{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})\}_{1\leq i\leq n}$ such that for any L-algebra $\mathfrak{A}$ we have that $\mathfrak{A} \not\models /\varphi$ iff $\mathfrak{A} \not\models \rho(\mathfrak{B}_i, \{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$ for some $1 \leq i \leq n$. We show that the same collection $\{(\mathfrak{B}_i, \{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})\}_{1\leq i\leq n}$ can be used for the current theorem.

(1) ⇒ (2): Suppose $\mathfrak{A} \not\models \varphi$. Then $\mathfrak{A}$ also refutes the rule $/\varphi$, so there is $1 \leq i \leq n$ such that $\mathfrak{A} \not\models \rho(\mathfrak{B}_i, \{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$. Since $\mathfrak{A}$ is well-connected, $\mathfrak{A} \not\models \gamma(\mathfrak{B}_i, \{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$ by Lemma 5.4.3.

(2) ⇒ (1): Suppose $\mathfrak{A} \not\models \gamma(\mathfrak{B}_i, \{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$ for some $1 \leq i \leq n$. Then by Lemma 5.4.3 there is a subdirectly irreducible homomorphic image $\mathfrak{C}$ of $\mathfrak{A}$ such that $\mathfrak{C} \not\models \rho(\mathfrak{B}, \{D_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$. This implies that $\mathfrak{C} \not\models /\varphi$. Since validity of the rule $/\varphi$ is equivalent to validity of the formula φ, we conclude that $\mathfrak{C} \not\models \varphi$. Since $\mathfrak{C}$ is a homomorphic image of $\mathfrak{A}$, also $\mathfrak{A} \not\models \varphi$.

□

A simple consequence is the following axiomatization theorem in terms of $\mathcal{H}$-stable canonical rules.

5.4.5. Corollary.

(1) Every si logic can be axiomatized by $\mathcal{H}$-stable canonical formulas.

(2) Every normal extension of $\mathsf{K4}$ can be axiomatized over $\mathsf{K4}$ by $\mathcal{H}$-stable canonical formulas.

(3) Every extension of FL_{ew}^k can be axiomatized over FL_{ew}^k by $\mathcal{H}$-stable canonical formulas.

Moreover, in each case, whenever the logic in question is finitely axiomatizable, then it is axiomatizable by finitely many $\mathcal{H}$-stable canonical formulas.

The shape of Theorem 5.4.6 and Corollary 5.4.5 come close in the intuitionistic case to Zakharyaschev's original result (see [40, Theorem 9.44]), and in particular to the algebraic versions of [13, 17]. In the case of k-CIRL, the above is similar to [32, Proposition 3.9, Theorem 3.11], and to [18, Corollary 6.9, Theorem 6.10] in the $\mathsf{K4}$-case.

A first evident difference that distinguishes Theorem 5.4.6 and Corollary 5.4.5 from the aforementioned results is that we did not ensure that the finite L-algebra $\mathfrak{B}$ in the $\mathcal{H}$-stable canonical formulas is subdirectly irreducible.

Ensuring that the finite algebras in the collection$\{(\mathfrak{B}_i, \{D_f^i\}_{f\in\mathcal{F}\setminus\mathcal{H}})\}_{1\leq i\leq n}$ are subdirectly irreducible requires an extra argument. We sketch suitable adaption of Theorem 5.4.6 for some cases in the corollary below.

5.4.6. Corollary. *Suppose that L stands for IPC or FL_{ew}^k, and $\mathcal{H}$ one of the corresponding reducts of Table 5.2.1.*

For a formula φ, there is a collection $\{(\mathfrak{B}_i, \{D_f^i\}_{f\in\mathcal{F}\setminus\mathcal{H}})\}_{1\leq i\leq n}$ consisting of pairs $(\mathfrak{B}_i, \{D_f^i\}_{f\in\mathcal{F}\setminus\mathcal{H}})$ of a subdirectly irreducible L-algebra $\mathfrak{B}_i$ and sets $D_f^i \subseteq B^{\sigma(f)}$ for $f \in \mathcal{F}\setminus\mathcal{H}$ and $1 \leq i \leq n$, such that for every sudirectly irreducible L-algebra $\mathfrak{A}$, the following are equivalent:

(1) $\mathfrak{A} \not\models \varphi$.

(2) $\mathfrak{A} \not\models \gamma(\mathfrak{B}_i, \{D_f^i\}_{f\in\mathcal{F}\setminus\mathcal{H}})$ for some $1 \leq i \leq n$.

Proof:

$\vee, \top \in \mathcal{H}$: Then a finite (non-trivial) $\mathcal{H}$-stable subalgebra of subdirectly irreducible L-algebras is subdirectly irreducible. For a proof in the intuitionistic setting see Lemma 3.3.19 and for the case of k-CIRL see [32, Lemma 2.9].

Without loss of generality, assume that $\{(\mathfrak{B}_i, \{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})\}_{1\leq i\leq k}$ is the subcollection of those pairs of the collection in the proof of Theorem 5.4.6, where $\mathfrak{B}_i$ is subdirectly irreducible.

It suffices to show that the implication (1) ⇒ (2) in the proof of Theorem 5.4.6 holds for the restricted class. For that, it is enough to observe that if $\mathfrak{A} \not\models \rho(\mathfrak{B}_i, \{D^i_f\}_{f\in\mathcal{F}\setminus\mathcal{H}})$ for some $1 \leq i \leq k$, then $\mathfrak{B}_i$ is isomorphic to an $\mathcal{H}$-stable subalgebra of $\mathfrak{A}$ and thus $\mathfrak{B}$ is subdirectly irreducible by the above.

$\to, \wedge \in \mathcal{H}$: For this case, we need a different argument than in the previous case, since the $\{\to, \wedge\}$-subalgebras of subdirectly irreducible Heyting algebras may not be subdirectly irreducible. However, similarly to the proof of Theorem 3.2.5, we can ensure that $\{\to, \wedge\}$-filtrations of subdircetly irreducible algebras are subdirectly irreducible.

We sketch the slightly modified proof. We concentrate on the case of BBSLat, the case of BSLat is similar. As before, we also use A, B etc. to denote Heyting algebras.

Let $\Sigma := \mathsf{Sub}(\varphi) \cup \{p\}$, where p is a fresh variable. Let $\kappa(|\Sigma|)$ bound the cardinality of all $|\Sigma|$-generated BBSLats. Let $\{(B_i, v_i)\}_{1\leq i\leq n}$ be up to isomorphism the collection of all finite Heyting algebras B_i and valuation v_i such that

- $|B_i| \leq \kappa(|\Sigma|)$,
- $(B_i, v_i) \not\models \varphi$,
- B_i is subdirectly irreducible.

For each $1 \leq i \leq n$, define

$$D^i_\vee = \{((v_i(\varphi), v_i(\psi)) \in B_i^2 \mid \varphi \vee \psi \in \mathsf{Sub}(\varphi)\}.$$

We show that the collection $\{(B_i, D^i_\vee)\}_{1\leq i\leq n}$ satisfies our requirements. The proof of the implication (2) ⇒ (1) is exactly as in the proof of Theorem 5.4.6. To see that (1) ⇒ (2), suppose that $(A, v) \not\models \varphi$. Then let v' be the valuation on A that is extending v by $v'(p) = s$, where s is the second largest element of A. Let (B, v_B) be an $\{\wedge, \to\}$-filtration of (A, v') through Σ. Then (B, v_B) satisfies the above requirements, note in particular that B is subdirectly irreducible because s is the second largest element of B. Thus, there is $1 \leq i \leq n$ such that (B_i, v_i) is isomorphic to (B, v_B). By the construction of filtration, there is an embedding from B_i into A satisfying the CDC for $D^i_\vee$. Since A is subdirectly irreducible, $A \not\models \gamma(B_i, D^i_\vee)$. □

In the next section of this chapter we are going to discuss how $\mathcal{H}$-stable canonical formulas for si logics relate to canonical formulas for si logics from the literature. We conclude with two paragraphs explaining the specifics of the modal and substructural cases.

Modal case Just as in the case of Jeřábek's rules (that we discussed at the end of the previous section), Zakharyaschev's canonical formulas for $\mathsf{K4}$ are not covered by the framework of $\mathcal{H}$-stable canonical formulas. The same holds for the canonical formulas for $\mathsf{wK4}$ from [15]. As we will see in the next section, Zakharyaschev's canonical formulas for si logics do, however, fit into our framework.

The *stable canonical formulas* for normal extensions of $\mathsf{K4}$ from [18, Theorem 6.10] are precisely the characteristic formulas of the stable canonical rules that we discussed at the end of the previous section. Thus, stable canonical formulas neatly fit into our framework. Note that in the above we have, however, not covered the proof of [18, Theorem 6.10] which shows that every normal extension of $\mathsf{K4}$ can be axiomatized by stable canonical formulas of *subdirectly irreducible* algebras.

Of course, stable canonical formulas are generalizations of the stable formulas that we saw in Definition 4.5.1. In fact, the stable formulas can be seen as those instances of stable canonical formulas where the "additional parameter" $D_\diamond$ is empty. The other extreme case, namely when the parameter $D_\diamond$ is the full algebra, leads to the well-known *Jankov formulas*.

Substructural case The $\mathcal{H}$-stable canonical formulas for k-CIRL for $\mathcal{H} = \{\vee, \cdot, \top\}$ correspond to the $\{\vee, \cdot, \top\}$-canonical formulas of [32]. The only difference is that in the latter the succedent of the formulas looks slightly simpler, but the two formulas are easily seen to be equivalent. In [32] it is also shown that—in parallel to the si and modal cases— the $\{\vee, \cdot, \top\}$-canonical formulas can be restricted to $\{\vee, \cdot, \top\}$-stable formulas that axiomatize precisely the $\{\vee, \cdot, \top\}$-stable extensions of k-CIRL.

5.4.2 Canonical formulas for si logics

In this section we recall the definition of Zakharyaschev's canonical formulas for si logics and the definition of stable canonical formulas for si logics from [17]. In both cases we present the frame-theoretic and algebraic versions. We also compare them to the $\mathcal{H}$-stable canonical formulas from above. We will make use of these formulas in the next chapter.

Zakharyaschev's canonical formulas for si logics

We recall the frame-theoretic definition of Zakharyaschev's canonical formulas [132, 133, 40].

Recall the definition of a subreduction from Section 3.4. Let X and Y be Esakia spaces and let $\mathfrak{D}$ be a set of upsets of Y. A subreduction $f : X \to Y$ satisfies the *closed domain condition* (CDC) for $\mathfrak{D}$ provided

$$(x \in {\uparrow}\,\mathrm{dom}(f) \text{ and } f({\uparrow}x) \in \mathfrak{D}) \text{ imply } x \in \mathrm{dom}(f).$$

Let Y be a finite rooted frame and $\mathfrak{D}$ be a family of upsets of $\mathfrak{Y}$, called *closed domains*. For each $y \in Y$ we introduce a new propositional variable p_y. The *canonical formula* of the pair $(Y, \mathfrak{D})$ is defined as

$$\beta(Y,\mathfrak{D}) = \bigwedge_{x\leq y}[(\bigwedge_{y\not\leq z} p_z \to p_y) \to p_x] \wedge \bigwedge_{d\in\mathfrak{D}}[\bigwedge_{x\notin d}(\bigwedge_{x\not\leq z} p_z \to p_x) \to \bigvee_{w\in d} p_w] \to p_r,$$

where $x, y, z \in Y$, and r is the root of Y. Then

$$X \not\models \beta(Y,\mathfrak{D}) \text{ iff there is a subreduction from } X \text{ onto } Y \text{ satisfying CDC for } \mathfrak{D}.$$

Note that if $\mathfrak{D} = \emptyset$, then $\beta(Y, \mathfrak{D})$ is precisely the subframe formula $\beta(Y)$ as discussed in Section 3.4.

5.4.7. Theorem (Zakharyaschev). *Every si logic is axiomatizable by canonical formulas of the shape $\beta(Y, \mathfrak{D})$. Moreover, every axiomatization of a si logic can be transformed effectively into an axiomatization in terms of canonical formulas.*

Next we recall the algebraic perspective on canonical formulas from [13]. Let B be a finite subdirectly irreducible Heyting algebra with second largest element s and let D be a subset of $B^2 \cup \{*\}$. For each $a \in B$, introduce a new variable p_a, and set

$$\begin{aligned}\beta(B,D) = \quad & [\bigwedge \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a,b \in B\} \wedge \\ & \bigwedge \{p_{a\to b} \leftrightarrow p_a \to p_b : a,b \in B\} \wedge \\ & \bigwedge \{p_{a\vee b} \leftrightarrow p_a \vee p_b : (a,b) \in D\} \wedge \\ & \bigwedge \{p_0 \leftrightarrow 0 : * \in D\}\,] \to p_s.\end{aligned}$$

As shown in [13], for any Heyting algebra A, we have

$A \not\models \beta(B, D)$ iff there is a homomorphic image C of A and a $(\wedge, \to)$-embedding $h : B \to C$ such that $h(a \vee b) = h(a) \vee h(b)$ for all $(a, b) \in D$ and $h(0) = 0$ if $* \in D$.

Although the formulas $\beta(B, D)$ and $\beta(Y, \mathfrak{D})$ look syntactically quite different, they are equivalent as discussed in [13, Remark 5.6]:

5.4.8. THEOREM ([13]). *If B is a finite subdirectly irreducible Heyting algebra with dual Y, and $D \subseteq B^2 \cup \{*\}$, then there is a collection $\mathfrak{D}$ of upsets of Y such that for any Esakia space X we have*

$$X \models \beta(B, D) \quad \text{iff} \quad X \models \beta(Y, \mathfrak{D}).$$

5.4.9. REMARK. The formula $\beta(B, D)$ looks almost the same as the $\mathcal{H}$-stable canonical formula for $\mathcal{H} = \{\wedge, \to\}$. The difference is that $\gamma(\mathfrak{B}, \{D_f\}_{f \in \mathcal{F} \setminus \mathcal{H}})$ has slightly more complicated succedent than $\beta(B, D)$. However, it can be shown that the two formulas are provably equivalent by an argument similar to that of Lemma 3.4.18.

5.4.10. REMARK. When presenting the canonical formulas $\beta(Y, \mathfrak{D})$ and $\beta(B, D)$ we slightly deviated from the original presentations. For $\beta(Y, \mathfrak{D})$ we follow Jeřábek's account [83, Section 3]. Namely our closed domains are upsets rather than antichains. Also, closed domains may be empty, which allows us to work with subframes rather than cofinal subframes (see [83, Remark 3.7]). In order to match this on the algebraic side, we altered the presentation of the canonical formula $\beta(B, D)$, which now combines the formulas $\beta(B, D, \bot)$ and $\beta(B, D)$ of [13].

Stable canonical formulas

The stable canonical formulas of [17] are an alternative to Zakharyaschev's canonical formulas. Stable canonical formulas are precisely the $\mathcal{H}$-stable canonical formulas for $\mathcal{H} = \{\wedge, \vee, \bot, \top\}$. For the sake of completeness, we spell out their definition and also recall their frame-theoretic characterization from [17].

Given a pair (B, D), consisting of a finite subdirectly irreducible Heyting algebra B and a set $D \subseteq B^2$, we introduce a new variable p_a for each $a \in B$ and define the stable canonical formula of (B, D) as follows:

$$\begin{aligned}\gamma(B, D) = \quad & [\,(p_0 \leftrightarrow \bot) \wedge (p_1 \leftrightarrow \top) \wedge \\ & \bigwedge \{p_{a \wedge b} \leftrightarrow p_a \wedge p_b : a, b \in B\} \wedge \\ & \bigwedge \{p_{a \vee b} \leftrightarrow p_a \vee p_b : a, b \in B\} \wedge \\ & \bigwedge \{p_{a \to b} \leftrightarrow p_a \to p_b : (a, b) \in D\}\,] \to \bigvee \{p_a \to p_b : a, b \in B \text{ with } a \not\leq b\}.\end{aligned}$$

Then for any Heyting algebra A we have,

$A \not\models \gamma(B, D)$ iff there is a well-connected homomorphic image C of A and a bounded lattice embedding $h : B \to C$ such that $h(a \to b) = h(a) \to h(b)$, for all $(a, b) \in D$.

As discussed in [17], the above can easily be translated into frame-theoretic terms. Let Z and Y be Esakia spaces, where Y is finite. Let also $\mathfrak{D}$ be a set of subsets of Y. We say that a stable onto map $f : Z \to Y$ satisfies the *stable closed domain condition* (SCDC) for $\mathfrak{D}$ provided

$$\uparrow f(x) \cap d \neq \emptyset \;\Rightarrow\; f[\uparrow x] \cap d \neq \emptyset \text{ for all } d \in \mathfrak{D}.$$

Suppose Y is a finite Esakia space, and $\mathfrak{D}$ is a set of subsets of Y, write $\gamma(Y, \mathfrak{D})$ for the canonical formula $\gamma(B, D)$, where B is the dual Heyting algebra of Y and $D = \{(U, V) \mid U \setminus V \in \mathfrak{D}\}$ for upsets U, V of Y. Then

$$X \not\models \gamma(Y, \mathfrak{D}) \quad \text{iff} \quad \begin{array}{l}\text{there are a point-generated subframe } Z \text{ of } Y \text{ and a}\\ \text{stable onto map } f : Z \to Y \text{ satisfying SCDC for } \mathfrak{D}.\end{array}$$

5.4.11. Theorem ([17]). *Every si logic is axiomatizable by stable canonical formulas. Moreover, every axiomatization of a si logic can be transformed effectively into an axiomatization in terms of stable canonical formulas.*

5.5 Conclusion

We provided a rather general view on canonical formulas via the notion of *expandable locally finite reducts*. We saw that the resulting $\mathcal{H}$-stable canonical rules and formulas cover some instances of canonical rules and formulas from the recent literature. We also explained that other instances are not covered, thus, those instances required more sophisticated techniques.

In our presentation we aimed to point out the levels of abstraction and difficulty that allow the definition of canonical formulas and rules. In particular, we aimed to separate the universal algebra content from the more specific features of reducts and properties of canonical formulas in more special contexts. To conclude this chapter we summarize the different abstraction levels used to build canonical rules and formulas in Table 5.5.1 below.

Abstraction level	universal algebra	algebraizable logics	algebraizable logics with EDPC	specific properties of $\mathcal{V}$ and $\mathcal{H}$
Axiomatization tools	$\mathcal{H}$-stable equational canonical rules	$\mathcal{H}$-stable canonical rules	$\mathcal{H}$-stable canonical formulas	specific canonical formulas
Discussed in	Section 5.3	Section 5.3.1	Section 5.4.1	[40, 83, 14, 15]

Table 5.5.1

Chapter 6

Subframization and stabilization for si logics

6.1 Introduction

In this chapter we further study stable and subframe si logics that we encountered in Chapter 3. We focus on how these logics lie in the lattice of all si logics. In our investigations we aim to strengthen the parallels between subframe and stable si logics.

The classes of subframe and stable si logics both form a complete sublattice of the lattice of all si logics. This entails that every si logic has closest subframe- and stable neighbors. More precisely, for a si logic L there is a least subframe (or stable) logic above and a greatest subframe (stable) logic below L. We call these the *upward* and *downward subframizations* (*stabilizations*) of L, respectively.

The notion of subframization has been investigated by Wolter in the modal case [126, 127]. Finding the subframizations of a logic is for instance relevant in the context of dynamic epistemic logic. Indeed, the public announcement update can only be executed on subframe logics (see [5]). Thus, when defining an epistemic logic, one might need to move to its closest subframe neighbor in order to make it "updatable" by the *public announcement operator*. We note that analogously, the stabilization of a logic is relevant to make it updatable by the *abstraction modality* which we discuss in Chapter 8.

In the first part of this chapter our goal is to find characterizations of subframizations and stabilizations of si logics similar to those provided by Wolter in the modal case. Recall that in the modal case, subframes correspond to relativizations on algebras (see Section 4.8), and thus a simple syntactic characterization of the upward subframization is provided via *relativizations* of formulas [126, 127]. Subframes in the intuitionistic case are, on the other hand, a bit more complicated (see Definition 3.4.3). In particular, subframes do not correspond to relativizations and so we need to have a different approach to subframization in the intuitionistic case.

We observe that we can, nevertheless, mimic relativizations in the intuitionsitic case by restriction to Zakharyachev's *canonical formulas*. This approach provides a syntactic characterization of the upward subframization. Similarly, we get a syntactic characterization of the upward stabilizations via *stable canonical formulas* (see Section 5.4.2 for canonical formulas).

In the second part of the chapter we explore connections of subframe and stable si logics with intuitionistic modal logics. First, we investigate connections of subframe logics and subframization with the propositional lax logic (PLL) (see [67, 54] for details on PLL). In particular, by embedding the lattice of si logics into extensions of PLL we obtain a new characterization of subframe logics and the downward subframization.

We then find analogous results for stable si logics and the downward stabilization by embedding the lattice of si logics into extensions of intuitionistic S4 (IS4) (see [105] for details on IS4).

This chapter is largely based on [20].

Outline

In the next section we define the notion of subframization and show how to calculate the downward subframization of a si logic via Zakharyachev's canonical formulas. We also calculate the subframizations of many standard si logics. The second section mirrors the results of the first but for stable logics. The second part of the chapter starts with Section 6.4, where we recall the basic definition of intuitionistic modal logic. We then introduce PLL and discuss its algebraic and frame-based semantics via S-frames. We define the translation τ from formulas of IPC into formulas of PLL. This gives rise to two embeddings of the lattice of si logics into extensions of PLL. In Section 6.5.5 we relate our findings to subframe logics and subframization. Section 6.6 mirrors the results of Section 6.5 for stable logics and IS4. In Section 6.7 we summarize our findings.

6.2 Subframization

In this section we define the notion of subframization, gather some elementary properties and provide characterizations via canonical formulas which allow us to effectively calculate the upward subframization of a si logic. We also discuss how our findings relate to Wolter's describable operations of [126]. Finally, we will calculate the upward and downward subframization of a number of si logics.

Subframe logics form a complete sublattice of Λ_{IPC}, the lattice of all si logics. (We gave a proof of this well known fact in Corollary 3.3.18(2)). We denote the lattice of all subframe logics by Λ_{Subf}. (The analogue of the following definition for the modal case can be found in [126, 127]).

6.2.1. Definition. For a si logic L, define the *downward subframization of* L as

$$\mathsf{Subf}_{\downarrow}(\mathsf{L}) := \bigvee\{\mathsf{L}' \in \Lambda_{\mathsf{Subf}} \mid \mathsf{L}' \subseteq \mathsf{L}\}$$

and the *upward subframization of* L as

$$\mathsf{Subf}_{\uparrow}(\mathsf{L}) := \bigwedge\{\mathsf{L}' \in \Lambda_{\mathsf{Subf}} \mid \mathsf{L} \subseteq \mathsf{L}'\}.$$

Clearly, the downward subframization of L is the largest subframe logic below L and its upward subframization is the least subframe logic above L. We summarize some rather obvious facts about the downward and upward subframizations that we will often make use of.

6.2.2. Lemma.

(1) $\mathsf{Subf}_{\downarrow}$ *is an interior operator and* $\mathsf{Subf}_{\uparrow}$ *is a closure operator on the lattice of si logics.*

(2) $\mathsf{Subf}_{\downarrow}(\mathsf{L}) = \mathsf{IPC} + \{\varphi \mid \varphi$ *is a* $\{\wedge, \to\}$*-formula and* $\mathsf{L} \vdash \varphi\}$.

(3) $\mathsf{Subf}_{\downarrow}(\mathsf{L}) = \mathsf{IPC}$ *iff for every* $\{\wedge, \to\}$*-formula* φ*,* $\mathsf{L} \vdash \varphi$ *iff* $\mathsf{IPC} \vdash \varphi$*.*

Proof:
(1) is straightforward from the definition. We show (2). By Theorem 3.4.8, every subframe logic is axiomatizable by $\{\wedge, \to\}$-formulas. Therefore, every subframe logic contained in L is axiomatizable by a set of $\{\wedge, \to\}$-formulas that are provable in L. Thus, the set $\{\varphi \mid \varphi$ is a $\{\wedge, \to\}$-formula and $\mathsf{L} \vdash \varphi\}$ axiomatizes the largest subframe logic contained in L. Finally, (3) follows from (2). □

We next give a semantic characterization of the downward and upward subframizations of a si logic L. Recall that if $\mathcal{K}$ is a class of Esakia spaces, then $\mathsf{Log}(\mathcal{K}) = \{\varphi \in \mathcal{L}_{\mathsf{IPC}} \mid \mathcal{K} \models \varphi\}$ is a si logic that we refer to as the si logic of $\mathcal{K}$.

6.2.3. Proposition. *Suppose* L *is a si logic and* $\mathsf{L} = \mathsf{Log}(\mathcal{K})$ *for some class* $\mathcal{K}$ *of Esakia spaces.*

(1) $\mathsf{Subf}_{\downarrow}(\mathsf{L}) = \mathsf{Log}\left(\{Y \mid Y \text{ is a subframe of some } X \text{ of } \mathcal{K}\}\right)$.

(2) $\mathsf{Subf}_{\uparrow}(\mathsf{L}) = \mathsf{Log}\left(\{X \mid Y \models \mathsf{L} \text{ for all subframes } Y \text{ of } X\}\right)$.

Proof:
(1). Let $\mathcal{K}' = \{Y \mid Y$ is a subframe of some $X \in \mathcal{K}\}$. Then $\mathcal{K} \subseteq \mathcal{K}'$, so $\mathsf{Log}(\mathcal{K}') \subseteq \mathsf{Log}(\mathcal{K}) = \mathsf{L}$. Since $\mathcal{K}'$ is closed under subframes, $\mathsf{Log}(\mathcal{K}')$ is a subframe logic by Theorem 3.4.8. If L' is a subframe logic contained in L, then $\mathcal{K} \models \mathsf{L}'$, so $\mathcal{K}' \models \mathsf{L}'$ as L' is a subframe logic. Therefore, $\mathsf{L}' \subseteq \mathsf{Log}(\mathcal{K}')$. Thus, $\mathsf{Log}(\mathcal{K}')$ is the largest subframe logic contained in L, and hence $\mathsf{Subf}_{\downarrow}(\mathsf{L}) = \mathsf{Log}(\mathcal{K}')$.

(2). Let $\mathcal{K}' = \{X \mid Y \models \mathsf{L}$ for all subframes Y of $F\}$. It is clear that $\mathcal{K}'$ is closed under subframes, so $\mathsf{Log}(\mathcal{K}')$ is a subframe logic by Theorem 3.4.8. Moreover, $\mathcal{K}' \models \mathsf{L}$, so $\mathsf{L} \subseteq \mathsf{Log}(\mathcal{K}')$. Let L' be a subframe logic containing L. If $X \models \mathsf{L}'$, then since L' is a subframe logic, $Y \models \mathsf{L}'$ for every subframe Y of X. But then $Y \models \mathsf{L}$ as $\mathsf{L} \subseteq \mathsf{L}'$, so $X \in \mathcal{K}'$. Therefore, every L'-space is contained in $\mathcal{K}'$, and so $\mathsf{Log}(\mathcal{K}') \subseteq \mathsf{L}'$. Thus, $\mathsf{Log}(\mathcal{K}')$ is the smallest subframe logic containing L, and hence $\mathsf{Subf}_{\uparrow}(\mathsf{L}) = \mathsf{Log}(\mathcal{K}')$. □

Next, we use Proposition 6.2.3 and Zakharyaschev's canonical formulas to give a syntactic characterization of the downward and upward subframizations of a si logic L. The reader may recall the definition and properties of Zakharyaschev's canonical formulas from Section 5.4.2. Since by Zakharyaschev's theorem (Theorem 5.4.7) all si logics are axiomatizable by canonical formulas, we can assume that a si logic is given by such an axiomatization without imposing any restrictions on L.

6.2.4. THEOREM. *Let* $\mathsf{L} = \mathsf{IPC} + \{\beta(Z_i, \mathfrak{D}_i) \mid i \in I\}$ *be a si logic.*

(1) $\mathsf{Subf}_{\downarrow}(\mathsf{L}) = \mathsf{IPC} + \{\beta(Z) \mid \mathsf{L} \vdash \beta(Z)\}$.

(2) $\mathsf{Subf}_{\uparrow}(\mathsf{L}) = \mathsf{IPC} + \{\beta(Z_i) \mid i \in I)\}$.

Proof:

(1). By Theorem 3.4.8, every subframe logic is axiomatizable by subframe formulas. Therefore, every subframe logic contained in L is axiomatizable by a set of subframe formulas that are provable in L. Thus, $\mathsf{IPC} + \{\beta(Z) \mid \mathsf{L} \vdash \beta(Z)\}$ is the largest subframe logic contained in L.

(2). Let $\mathsf{M} = \mathsf{IPC} + \{\beta(Z_i) \mid i \in I\}$. If X is an M-space, then $X \models \beta(Z_i)$ for all $i \in I$. By comparing the semantic refutation criteria of $\beta(Z_i)$ and $\beta(Z_i, \mathfrak{D}_i)$ it immediately follows that $X \models \beta(Z_i, \mathfrak{D}_i)$ for all $i \in I$. Thus, X is an L-space, and so $\mathsf{L} \subseteq \mathsf{M}$. Since M is axiomatized by subframe formulas, M is a subframe logic. It remains to show that M is the least subframe logic containing L. If not, then there is a subframe logic $\mathsf{L}' \supseteq \mathsf{L}$ and an L'-space X such that $X \not\models \mathsf{M}$. Therefore, $X \not\models \beta(Z_i)$ for some $i \in I$. By Lemma 3.4.6, Z_i is a subreduction of Z_i, thus Z_i is a p-morphic image of a subframe Y of X. Since L' is a subframe logic, Y is an L'-space. Thus, Z_i is also an L'-space. But $Z_i \not\models \beta(Z_i, \mathfrak{D}_i)$ because the identity map is a p-morphism from Z_i onto itself that satisfies CDC for any set of closed domains. Consequently, Z_i is not an L-space, which is a contradiction since $\mathsf{L}' \supseteq \mathsf{L}$. □

6.2.5. Remark.

(1) It follows from Theorem 6.2.4(2) that if L is a si logic axiomatized by a set of formulas Γ, then the upward subframization $\mathsf{Subf}_{\uparrow}(\mathsf{L})$ of L can be calculated effectively from Γ as follows: First use Zakharyaschev's theorem to transform Γ into an equivalent set of canonical formulas; then delete the additional parameters $\mathfrak{D}_i$ in the resulting canonical formulas; and finally apply Theorem 6.2.4(2).

(2) On the other hand, Theorem 6.2.4(1) does not provide an effective axiomatization of the downward subframization $\mathsf{Subf}_{\downarrow}(\mathsf{L})$ of L. We will come back to this issue at the end of Section 6.5.5.

6.2.6. Remark. In [126] Wolter studied *describable operations* on varieties of modal algebras. These translate to Esakia spaces as follows. A map $\mathbf{C}$ that associates with each Esakia space X a class $\mathbf{C}(X)$ of Esakia spaces is called *describable* if there is a map $(\cdot)^c$ on $\mathcal{L}_{\mathsf{IPC}}$ such that for each Esakia space X and each formula φ,

$$X \models \varphi^c \text{ iff } \mathbf{C}(X) \models \varphi.$$

Roughly speaking, the map $(\cdot)^c$ *describes* the (result of) the operation $\mathbf{C}$. As follows from [126, page 23], if L is the logic of a class $\mathcal{K}$ of Esakia spaces, then the logic of $\mathbf{C}(\mathcal{K})$ is axiomatized by $\{\varphi^c \mid L \vdash \varphi^c\}$, and the logic of $\{X \in \mathcal{K} \mid \mathbf{C}(X) \subseteq \mathcal{K}\}$ is axiomatized by $\{\varphi^c \mid L \vdash \varphi\}$.

Now let $\mathbf{C}(X) = \{Y \mid Y \text{ is a subframe of } X\}$ for some Esakia space X. We show that the operation $\mathbf{C}$ is *describable*—in a slightly weaker sense than Wolter's—if we restrict the map $(\cdot)^c$ to canonical formulas. Since canonical formulas axiomatize every si logic this does in fact not impose any major restriction and the above result still holds. For an Esakia space X, every formula $\beta(Z, \mathfrak{D})$ satisfies

$$X \models \beta(Z) \text{ iff } \mathbf{C}(X) \models \beta(Z, \mathfrak{D}). \tag{6.1}$$

The left to right direction follows from the refutation criteria for the formulas. If $Y \in \mathbf{C}(X)$ and $Y \not\models \beta(Z, \mathfrak{D})$, then $Y \not\models \beta(Z)$, and so $X \not\models \beta(Z)$ since validity of subframe formulas is preserved by subframes. For the right to left direction, suppose $X \not\models \beta(Z)$. Then there is a subframe Y of X which is p-morphically mapped onto Z. Since $Z \not\models \beta(Z, \mathfrak{D})$, we have $Y \not\models \beta(Z, \mathfrak{D})$. Therefore, we found $Y \in \mathbf{C}(X)$ such that $Y \not\models \beta(Z, \mathfrak{D})$.

From (7.1) we deduce that setting $\beta(Z, \mathfrak{D})^c = \beta(Z)$ defines a map on canonical formulas describes the operation $\mathbf{C}$. Thus, applying Wolter's result to Proposition 6.2.3 yields an alternative proof of Theorem 6.2.4.

6.2.1 Examples

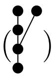
We conclude this section by calculating the upward and downward subframizations of many well-known si logics. For the definition of the si logics below see Table A.0.2 and for their axiomatization in terms of canonical formulas see Table A.0.3.

6.2.7. PROPOSITION.

(1) $\mathsf{Subf}_{\downarrow}(\mathsf{KC}) = \mathsf{IPC}$ *and* $\mathsf{Subf}_{\uparrow}(\mathsf{KC}) = \mathsf{LC}$.

(2) $\mathsf{Subf}_{\downarrow}(\mathsf{BTW}_n) = \mathsf{IPC}$ *and* $\mathsf{Subf}_{\uparrow}(\mathsf{BTW}_n) = \mathsf{BW}_n$ *for every* $n \geq 2$.

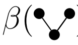
(3) $\mathsf{Subf}_{\downarrow}(\mathsf{T}_n) = \mathsf{IPC}$ *and* $\mathsf{Subf}_{\uparrow}(\mathsf{T}_n) = \mathsf{BW}_n$ *for every* $n \geq 2$.

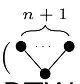
(4) $\mathsf{Subf}_{\downarrow}(\mathsf{RN}) = \mathsf{KG}$ *and* $\mathsf{Subf}_{\uparrow}(\mathsf{RN}) = \mathsf{KG} + \beta($$)$.

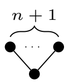
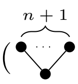
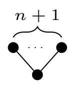
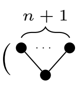
(5) $\mathsf{Subf}_{\downarrow}(\mathsf{KP}) = \mathsf{IPC}$ *and* $\mathsf{Subf}_{\uparrow}(\mathsf{KP}) = \mathsf{BW}_2$.

(6) $\mathsf{Subf}_{\downarrow}(\mathsf{ND}_n) = \mathsf{IPC}$ *and* $\mathsf{Subf}_{\uparrow}(\mathsf{ND}_n) = \mathsf{BW}_2$ *for every* $n \geq 2$.

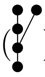
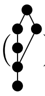
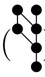
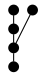
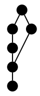
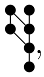
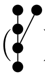
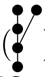
Proof:

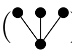
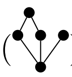
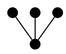
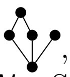
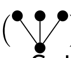
(1). Since KC is axiomatized by $\beta($$, \{\varnothing\})$, it follows from Theorem 6.2.4(2) that $\mathsf{Subf}_{\uparrow}(\mathsf{KC}) = \mathsf{IPC} + \beta($$) = \mathsf{LC}$. To calculate the downward subframization of KC, we utilize Proposition 6.2.3(1). It is well known that IPC is the logic of all finite frames and that KC is the logic of all finite directed frames. Moreover, adding a new top to a finite frame X results in a finite directed frame Y containing X as a subframe. In other words, every finite frame is a subframe of a KC-frame. Therefore, by Proposition 6.2.3(1), $\mathsf{Subf}_{\downarrow}(\mathsf{KC}) = \mathsf{IPC}$.

(2). From the axiomatization of BTW_n in Table A.0.3 and Theorem 6.2.4(2) it follows that $\mathsf{Subf}_{\uparrow}(\mathsf{BTW}_n) = \mathsf{IPC} + \beta($$) = \mathsf{BW}_n$. In order to see that $\mathsf{Subf}_{\downarrow}(\mathsf{BTW}_n) = \mathsf{IPC}$, observe that $\mathsf{BTW}_n \subseteq \mathsf{KC}$ and apply (1) and Lemma 6.2.2(1).

(3). It follows from Table A.0.3 that T_n is axiomatized by the negation-free Jankov-formula $\beta^\sharp($$)$ which we view as the canonical formula $\beta($$, \mathfrak{D})$, where $\mathfrak{D}$ is the set of all nonempty upsets of . Therefore, $\mathsf{Subf}_{\uparrow}(\mathsf{T}_n) = \mathsf{IPC} + \beta($$) = \mathsf{BW}_n$. To determine the downward subframization, since

T_n has the disjunction property [60] and every si logic with the disjunction property proves the same disjunction-free formulas as IPC [101, 131], we conclude that T_n proves the same $\{\wedge, \to\}$-formulas as IPC. Thus, by Lemma 6.2.2(3), $\mathsf{Subf}_{\downarrow}(\mathsf{T}_n) = \mathsf{IPC}$.

(4). Since KG is a subframe logic contained in RN (see [22, Section 3]), it follows from the axiomatization of RN in Table A.0.3 and Theorem 6.2.4(2) that the upward subframization of RN is $\mathsf{KG} + \beta($$) + \beta($$) + \beta($$)$. Since  is a subframe of both  and , the latter logic is equal to $\mathsf{KG} + \beta($$)$. Therefore, $\mathsf{Subf}_{\uparrow}(\mathsf{RN}) = \mathsf{KG} + \beta($$)$. To determine the downward subframization, $\mathsf{KG} \subseteq \mathsf{Subf}_{\downarrow}(\mathsf{RN})$ since KG is a subframe logic contained in RN. For the reverse inclusion, since KG is the logic of its finite rooted frames, by Proposition 6.2.3(1), it is sufficient to show that every finite rooted KG-frame is a subframe of the Rieger-Nishimura ladder $\mathfrak{L}$. First note that the subframe of $\mathfrak{L}$ obtained by deleting the first k layers of $\mathfrak{L}$ is isomorphic to $\mathfrak{L}$. Using this it is easy to see that every finite generated subframe of $\mathfrak{L}$ can be realized as a subframe of $\mathfrak{L}$ at an arbitrary depth, i.e., as a subframe of $\mathfrak{L}$ that does not contain the first k-layers of $\mathfrak{L}$ for any $k \in \mathbb{N}$. Therefore, a finite rooted KG-frame $\bigoplus_{i=1}^{n} X_i$ can be realized as a subframe of $\mathfrak{L}$ by embedding $X_1, \dots, X_n$ below each other so that the two subsequent points in $\mathfrak{L}$ between the embeddings of X_i and X_{i+1} are skipped.

(5). The axiomatization of KP in Table A.0.3 and Theorem 6.2.4(2) yield that $\mathsf{Subf}_{\uparrow}(\mathsf{KP})$ is axiomatized by $\beta($$)$ and $\beta($$)$. But  is a subframe of , so $\mathsf{Subf}_{\uparrow}(\mathsf{KP})$ is axiomatized by $\beta($$)$, and hence $\mathsf{Subf}_{\uparrow}(\mathsf{KP}) = \mathsf{BW}_2$. Since KP has the disjunction property, $\mathsf{Subf}_{\downarrow}(\mathsf{KP}) = \mathsf{IPC}$ by the same argument as in (3).

(6). Since the 3-fork is a subframe of the n-fork for $n \geq 3$, it follows from the axiomatization of ND_n in Table A.0.3 and Theorem 6.2.4(2) that $\mathsf{Subf}_{\uparrow}(\mathsf{ND}_n) = \mathsf{BW}_2$ for $n \geq 2$. Since ND_n has the disjunction property, $\mathsf{Subf}_{\downarrow}(\mathsf{ND}_n) = \mathsf{IPC}$ by the same argument as in (3). □

6.3 Stabilization

This section mirrors the previous section in content and structure for stable si logics (see Section 3.5). As we discussed in Section 3.5, a si logic L is stable iff its *rooted* Esakia spaces are closed under stable images[1], but even if L is stable, the collection of *all* its spaces may *not be stable*—this happens only in trivial cases. This is in contrast to subframe logics since the spaces of a subframe logic are closed under subframes. Thus, we will have to slightly modify some of the characterizations by restricting ourselves to rooted spaces. However, this will have no major effect on the proofs.

By Λ_{Stab} we denote the lattice of all stable si logics. It is a complete sublattice of Λ_{IPC}.

6.3.1. DEFINITION. For a si logic L, define the *downward stabilization of* L as

$$\text{Stab}_{\downarrow}(\mathsf{L}) := \bigvee\{\mathsf{L}' \in \Lambda_{\text{Stab}} \mid \mathsf{L}' \subseteq \mathsf{L}\}$$

and the *upward stabilization of* L as

$$\text{Stab}_{\uparrow}(\mathsf{L}) := \bigwedge\{\mathsf{L}' \in \Lambda_{\text{Stab}} \mid \mathsf{L} \subseteq \mathsf{L}'\}.$$

We next give a semantic characterization of upward and downward stabilizations. If X is an Esakia space, by $\mathsf{Stab}(X)$ we denote the collection of stable images of X (as in (3.5) from Section 3.5.4).

6.3.2. PROPOSITION. *Let* L *be a si logic.*

(1) $\text{Stab}_{\downarrow}(\mathsf{L}) = \mathsf{Log}\left(\{Y \mid Y \text{ is a stable image of a rooted } \mathsf{L}\text{-space } X\}\right)$.

(2) $\text{Stab}_{\uparrow}(\mathsf{L}) = \mathsf{Log}(\{X \mid X \text{ is finite and rooted and } \mathsf{Stab}(X) \models \mathsf{L}\})$.

Proof:

(1). Let $\mathcal{K} = \{Y \mid Y$ is a stable image of a rooted L-space $X\}$. Then $\mathcal{K}$ is closed under stable images, so $\mathsf{Log}(\mathcal{K})$ is a stable logic. Since $\mathcal{K}$ contains the class of rooted L-spaces, $\mathsf{Log}(\mathcal{K}) \subseteq \mathsf{L}$. Let L' be a stable logic contained in L. Then the class $\mathcal{K}'$ of rooted L'-spaces contains the class of rooted L-spaces and is closed under stable images. Therefore, $\mathcal{K} \subseteq \mathcal{K}'$, and so $\mathsf{L}' \subseteq \mathsf{Log}(\mathcal{K})$. Thus, $\mathsf{Log}(\mathcal{K})$ is the largest stable logic contained in L.

(2). Let $\mathcal{K} = \{X \mid X$ is finite rooted and $\mathsf{Stab}(X) \models \mathsf{L}\})$. Then $\mathcal{K}$ is closed under stable images, so $\mathsf{Log}(\mathcal{K})$ is a stable logic. Since every frame in $\mathcal{K}$ validates L, $\mathsf{L} \subseteq \mathsf{Log}(\mathcal{K})$. Let L' be a stable logic extending L, and let X be a finite rooted L'-space. Since L' is stable, all stable images of X are L'-spaces, and hence also L-spaces. Therefore, $X \in \mathcal{K}$. Since L' is stable, L' is the logic of its finite rooted frames. Thus, $\mathsf{Log}(\mathcal{K}) \subseteq \mathsf{L}'$, and so $\mathsf{Log}(\mathcal{K})$ is the least stable extension of L.

[1] Recall that a stable image of an Esakia space X is an image of X under a Priestley morphism.

□

We can characterize the upward and downward subframization via stable canonical formulas that we discussed in Section 5.4.2. Since stable canonical formulas axiomatize all si logics (Theorem 5.4.11), imposing an axiomatization in terms of stable canonical formulas does not impose a restriction on a si logic.

6.3.3. THEOREM. *Let* $\mathsf{L} = \mathsf{IPC} + \{\gamma(X_i, \mathfrak{D}_i) \mid i \in I\}$ *be a si logic.*

(1) $\mathrm{Stab}_{\downarrow}(\mathsf{L}) = \mathsf{IPC} + \{\gamma(X) \mid \mathsf{L} \vdash \gamma(X)\}$.

(2) $\mathrm{Stab}_{\uparrow}(\mathsf{L}) = \mathsf{IPC} + \{\gamma(X_i) \mid i \in I\}$.

Proof:

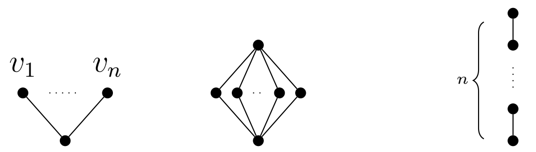
(1). Since every stable si logic is axiomatizable by stable formulas, $\mathsf{IPC}+\{\gamma(X) \mid \mathsf{L} \vdash \gamma(X)\}$ is the largest stable logic contained in L. Therefore, $\mathrm{Stab}_{\downarrow}(\mathsf{L}) = \mathsf{IPC} + \{\gamma(X) \mid \mathsf{L} \vdash \gamma(X)\}$.

(2). Let $\mathsf{M} = \mathsf{IPC} + \{\gamma(X_i) \mid i \in I\}$, and let Y be a rooted M-space. Then $Y \models \gamma(X_i)$ for all $i \in I$. Thus, $Y \models \gamma(X_i, \mathfrak{D}_i)$ for all $i \in I$ as can easily be seen by the semantic description of the formulas. Therefore, Y is an L-space, and so $\mathsf{L} \subseteq \mathsf{M}$. Since M is axiomatized by stable formulas, M is a stable logic (Corollary 3.5.8). Suppose L' is a stable extension of L, and Y is a rooted L'-space. If $Y \not\models \gamma(X_i)$ for some $i \in I$, then X_i is a stable image of some point-generated subframe Z of Y. Therefore, X_i is an L'-space. But X_i is not an L-space, which contradicts to L' being an extension of L. Thus, $Y \models \gamma(X_i)$ for all $i \in I$, and so $\mathsf{M} \subseteq \mathsf{L}'$. Consequently, M is the least stable extension of L, and hence $\mathrm{Stab}_{\uparrow}(\mathsf{L}) = \mathsf{M}$. □

6.3.4. REMARK. If a si logic L is axiomatized by a set of formulas Γ, then $\mathrm{Stab}_{\uparrow}(\mathsf{L})$ can be calculated effectively as follows: First use Theorem 5.4.11 to transform Γ into an equivalent set of stable canonical formulas; then delete the additional parameters $\mathfrak{D}_i$ in the resulting canonical formulas; and finally apply Theorem 6.3.3(2).

6.3.5. REMARK. In Remark 6.2.6 we recalled Wolter's describable operations. We also explained that it causes no harm to restrict the scope of the (describing) function $(\cdot)^c$ to canonical formulas. If in addition, we restrict the operation $\mathbf{C}$ to *rooted* spaces, the operation $\mathbf{C}(X) = \{Y \mid Y \text{ is a stable image of } X\}$ for some rooted X is describable. Since every si logic is generated by its rooted spaces, this does not cause any major restriction so we can obtain an alternative proof of Theorem 6.3.3. We show that

$$X \models \gamma(Z) \text{ iff } \mathbf{C}(X) \models \gamma(Z, \mathfrak{D}).$$



X_n — n-fork; Y_n — n-fork with top; C_n — n-chain

Figure 6.3.1

The left to right direction is obvious. For the right to left direction, suppose $X \not\models \gamma(Z)$. Since X is rooted, it follows from Lemma 3.5.7 that Z is a stable image of X. Therefore, $Z \in \mathbf{C}(X)$. Thus, since $X \not\models \gamma(X, \mathfrak{D})$, we conclude that $\mathbf{C}(X) \not\models \gamma(X, \mathfrak{D})$. Set $(\gamma(Z, \mathfrak{D}))^c = \gamma(Z)$. Because every logic is characterized by its rooted Esakia spaces, Wolter's result applied to Proposition 6.3.2 yields an alternative proof of Theorem 6.3.3.

6.3.1 Examples

We conclude this section by calculating stabilizations of some si logics. For the definition of the si logics below see Table A.0.2.

6.3.6. Proposition.

(1) $\mathrm{Stab}_{\downarrow}(\mathsf{BD}_n) = \mathsf{IPC}$ *and* $\mathrm{Stab}_{\uparrow}(\mathsf{BD}_n) = \mathsf{BC}_n$ *for all* $n \geq 2$.

(2) If L *is consistent and has the disjunction property, then* $\mathsf{Stab}_{\downarrow}(\mathsf{L}) = \mathsf{IPC}$.

(3) $\mathrm{Stab}_{\downarrow}(\mathsf{T}_n) = \mathsf{IPC}$ *and* $\mathrm{Stab}_{\uparrow}(\mathsf{T}_n) = \mathsf{BW}_n$ *for all* $n \geq 2$.

Proof:

(1). First we show that $\mathrm{Stab}_{\downarrow}(\mathsf{BD}_n) = \mathsf{IPC}$ for all $n \geq 2$. Since $\mathsf{BD}_n \subseteq \mathsf{BD}_2$ for all $n \geq 2$, it suffices to show that $\mathrm{Stab}_{\downarrow}(\mathsf{BD}_2) = \mathsf{IPC}$. Let X be a finite rooted frame. Suppose X has at most $n+1$ elements, and X_n is the n-fork shown in Figure 6.3.1. Mapping the root of X_n to the root of X and the top nodes of X_n surjectively onto the other nodes of X defines a stable map from X_n onto X. Since X_n is a BD_2-frame, by Proposition 6.3.2(1), $X \models \mathrm{Stab}_{\downarrow}(\mathsf{BD}_2)$ for every finite rooted frame X. Thus, $\mathrm{Stab}_{\downarrow}(\mathsf{BD}_2) = \mathsf{IPC}$.

Next we show that $\mathrm{Stab}_{\uparrow}(\mathsf{BD}_n) = \mathsf{BC}_n$ for all $n \geq 2$. Since BC_n has the fmp, by by Proposition 6.3.2(2) it suffices to show that a finite rooted frame is a BC_n-frame, i.e. has cardinality at most n, iff all its stable images are BD_n-frames, i.e. have depth at most n.

So let X be a finite rooted frame. If X has no more than n elements, then every stable image of X also has no more than n elements. Thus, every stable image of X is in particular a BD_n-frame. On the other hand, if X has at least $n+1$ elements, then we can define a stable map from X on the $(n+1)$-chain C_{n+1} (see Figure 6.3.1) as follows: map the root r of X to the root of C_{n+1}; map the immediate successors of r on top of each other; continue this process with the immediate successors of the immediate successors of r, and so on; if all points in C_{n+1} are used, then map the remaining points to the top node of C_{n+1}. Since C_{n+1} is not a BD_n-frame, X has a stable image refuting BD_n. Thus, we conclude that $\mathrm{Stab}_{\uparrow}(\mathsf{BD}_n) = \mathsf{BC}_n$.

(2). Suppose L is consistent and has the disjunction property. By [40, Theorem 15.5], if X_1, X_2 are rooted L-spaces, then their disjoint union $X_1 \sqcup X_2$ is a generated subframe of some rooted L-frame. This implies that for every n, there is a rooted L-frame X containing at least n maximal points. To see this, since L is consistent, the one-point frame X_1 is an L-frame. Therefore, $X_1 \sqcup X_1$ is a generated subframe of some rooted L-frame X_2. Clearly X_2 has at least 2 maximal points. By the same argument, $X_2 \sqcup X_2$ is a generated subframe of some rooted L-frame X_3 that has at least 4 maximal points. Continuing this process yields a rooted L-frame X with at least n maximal points, say $\{x_1, x_2, \dots x_n\}$. We show that the n-fork X_n is a stable image of X. Separate $x_1, \dots, x_n$ by disjoint clopen upsets $U_1, \dots, U_n$ with $x_i \in U_i$ for $1 \le i \le n$ (see Theorem 2.2.7), and define a map $f : X \to X_n$ by

$$f(x) = \begin{cases} v_i & \text{if } x \in U_i \text{ for some } 1 \le i \le n, \\ r & \text{otherwise,} \end{cases}$$

where r is the root of X_n. It is straightforward to see that f is an onto stable map. Thus, $\mathrm{Stab}_{\downarrow}(\mathsf{L}) \subseteq \mathsf{BD}_2$. Now apply (1) to conclude that $\mathrm{Stab}_{\downarrow}(\mathsf{L}) = \mathsf{IPC}$.

(3). Since T_n is consistent and has the disjunction property for all $n \ge 2$, by (2), $\mathrm{Stab}_{\downarrow}(\mathsf{T}_n) = \mathsf{IPC}$ for all $n \ge 2$.

Next we show that $\mathrm{Stab}_{\uparrow}(\mathsf{T}_n) = \mathsf{BW}_n$ for all $n \ge 2$. Let $\mathcal{K} = \{X \mid X$ is finite rooted and $Y \models \mathsf{T}_n$ for every stable image Y of $X\}$. By Proposition 6.3.2(2), $\mathrm{Stab}_{\uparrow}(\mathsf{T}_n) = \mathsf{Log}(\mathcal{K})$. Let $\mathcal{K}'$ be the class of finite rooted frames of width $\le n$. We show that $\mathcal{K} = \mathcal{K}'$. Let X be finite and rooted. If X is of width $\le n$, then so are all its stable images (see [17, Theorem 7.3(2)]). Therefore, $\mathcal{K}' \subseteq \mathcal{K}$. Conversely, if X has width greater than n, then by [17, Theorem 7.5(3)], either the $(n+1)$-fork or the $(n+1)$-fork with top (see Figure 6.3.1) is a stable image of X. Since neither of these is a T_n-frame, $X \notin \mathcal{K}$. Thus, $\mathcal{K} = \mathcal{K}'$, and as BW_n is the logic of $\mathcal{K}'$, we conclude that $\mathrm{Stab}_{\uparrow}(\mathsf{T}_n) = \mathsf{BW}_n$. □

6.4 Intuitionistic modal logics

In the second part of this chapter we aim to relate the subframization and stabilization to intuitionistic modal logics. More precisely, we will relate subframe logics and subframization to the intuitionistic modal logic PLL and stable logics and stabilization to intuitionistic S4.

The notion of intuitionistic modal logic is more ambiguous than in the classical case. This is primarily due to the fact that by adding a $\Box$-like operation—i.e. an operation commuting with meets and preserving $\top$ — intuitionistic logic cannot prove that its dual behaves like $\Diamond$. We choose as our starting point a very weak notion of intuitionistic modal logic (as e.g. in [128, 129]) by defining it to be intuitionistic propositional logic with a single unary operation that satisfies the congruence rule, and thus allowing for algebraic semantics. Accordingly, we set the following definition.

6.4.1. Definition.

(1) The *language of intuitionistic modal logic* $\mathcal{L}_{\mathsf{IM}}$ is an expansion of the language $\mathcal{L}_{\mathsf{IPC}}$ by a unary operator $\bigcirc$.

(2) An *intuitionistic modal logic* M is set of formulas of $\mathcal{L}_{\mathsf{IM}}$ that contains (the axioms of) IPC, and is closed under the rules of substitution, modus ponens, and the congruence rule $\varphi \leftrightarrow \psi / \bigcirc\varphi \leftrightarrow \bigcirc\psi$

In the following we will however be interested in intuitionistic modal logics satisfying more specific axioms. For more details on intuitionistic modal logic the reader may consult [61].

6.5 Subframe logics and PLL

In this section we investigate relations between subframe logics and the intuitionistic modal logic PLL. We will define a translation—similar to the Gödel-Gentzen double negation translation—from the language of IPC to the language of PLL. The translation allows us to embed the lattice of si logics into the lattice of extensions of PLL in two different ways. We observe that via the interaction of the two embeddings we obtain a new characterization of subframe logics and a new description of the downward subframization.

6.5.1 The intuitionistic modal logic PLL

The intuitionistic modal logic PLL was studied by Goldblatt in [67], where it was called $\mathscr{J}$, and further investigated by Fairtlough and Mendler [54].

6.5.1. Definition. Propositional lax logic (PLL) is the intuitionistic modal logic satisfying the following axioms

(1) $p \to \bigcirc p$,

(2) $\bigcirc\bigcirc p \to \bigcirc p$, and

(3) $\bigcirc(p \wedge q) \leftrightarrow (\bigcirc p \wedge \bigcirc q)$.

A modality satisfying the axioms above is called a *lax modality.* For a classical modal logician the axioms for $\bigcirc$ may seem quite peculiar as they show features of $\Box$- and $\Diamond$-modalities from classical modal logic. In particular, axioms (1) and (2) of the above definition are the classical S4-axioms formulated in terms of $\Diamond$, but commutativity with meets, i.e. axiom (3), is an axiom reserved for $\Box$-modalities. Indeed, in the classical setting such a modality occurs only trivially, i.e. satisfying $\bigcirc p \leftrightarrow p$ or $\bigcirc p \leftrightarrow 1$ (see e.g. [54]). However, in the intuitionistic setting lax modalities occur naturally. For instance, Goldblatt initially studied PLL as the logic of a geometric modality, where the intended meaning of $\bigcirc p$ is "p is locally true". The motivation for Fairtlough and Mendler [54], on the other hand, stems from verification of computer hardware, where $\bigcirc p$ is meant to be read as "p is true under a constraint". In [54, Section 1] various contexts where the lax modality has occurred are identified.

As the reader may have observed, the axioms of PLL correspond precisely to the axioms of nuclei (see Section 3.4.1). Heyting algebras with nuclei thus provide algebraic semantics for PLL. This was observed by Goldblatt [67]. Goldblatt calls a Heyting algebra with a nucleus a *local algebra*, we will, however, follow the terminology of [24] calling it a nuclear Heyting algebra.

6.5.2. Definition. A *nuclear Heyting algebra* is a pair (A, j) where A is a Heyting algebra and j is a nucleus on A.

A *valuation* on a nuclear Heyting algebra is the same as a valuation on the underlying Heyting algebra, and a formula of PLL is evaluated in a nuclear Heyting algebra in the obvious way, by extending the valuation on Heyting algebras with $v(\bigcirc\varphi) = j(v(\varphi))$. In general, we use the same notations as in the Heyting case, e.g. we write $(A, j) \models \varphi$ iff $v(\varphi) = 1$ for all valuations on (A, j), etc. It was shown in [67] that nuclear Heyting algebras provide an adequate semantics for PLL.

6.5.3. Theorem (Goldblatt). PLL *is sound and complete with respect to nuclear Heyting algebras.*

6.5.2 Nuclear Heyting algebras and S-spaces

In this section we introduce S-spaces and explain that they provide frame-theoretic semantics for PLL.

Recall the definition of a subframe from Definition 3.4.3.

6.5.4. Definition. An *S-space* (short for: subframe space) is a pair (X, S), where X is an Esakia space and S is a subframe of X.

As we already explained in Section 3.4.1, nuclei on a Heyting algebra correspond to subframes of its dual space. Thus, S-spaces are in one-to-one correspondence to nuclear Heyting algebras as shown in [24]. Below we provide a proof[2] sketch of this fact in a slightly different way than in [24]. The advantage of this presentation is that the analogy between this correspondence and the correspondence between St-spaces and interior Heyting algebras (Theorem 6.6.4)—which we discuss in the next section—becomes very prominent.

6.5.5. Theorem ([24]). *There is a one-to-one correspondence between nuclear Heyting algebras and S-spaces.*

Proof:
We outline the main constructions. Let (X, S) be an S-frame and let A be the Heyting algebra dual to X. As we already discussed in Section 3.4.1, the map $j_S : \mathcal{CU}(X) \to \mathcal{CU}(X)$ defined by $j_S(U) = X \setminus {\downarrow}(S \setminus U)$ for each clopen upset U of X gives a nucleus on A.

Conversely, let (A, j) be a nuclear Heyting algebra. Then $A_j = (A_j, \wedge, \vee_j, \to_j j(0), 1)$ is a Heyting algebra (see Section 3.4.1). It is easy to see that the map $g : A \to A_j$, given by $a \mapsto j(a)$ is an onto bounded lattice homomorphism and that the inclusion map $g^+ : A_j \to A$ is its right adjoint[3].

Let $(X, \leq)$ be a the dual Esakia space of A and let $(S, \leq)$ be the dual Esakia space of A_j. By Priestely duality, we can identify S with a closed subset of X (see Section 2.2.2). Thus, to conclude that S is a subframe of X, by Lemma 3.4.4 it suffices to show that ${\downarrow}U \in \mathcal{CU}(X)$ whenever $U \in \mathcal{CU}(S)$.

By Priestley duality, we may assume that the map $g : \mathcal{CU}(X) \to \mathcal{CU}(S)$ is described as $U \mapsto U \cap S$. That this map has a right adjoint means that for each $V \in \mathcal{CU}(S)$ the set

$$\{U \in \mathcal{CU}(X) \mid U \cap S \subseteq V\} \tag{6.2}$$

has a largest element. Now, $U \cap S \subseteq V$ holds iff $S \setminus V \subseteq X \setminus U$. Thus, in other words, there is a least clopen downset U' of X containing $S \setminus V$.

6.5.1. Claim. *${\downarrow}(S \setminus V)$ is clopen in X.*

[2] This proof was done with the help of M. Jibladze and T. Litak.

[3] See Section 2.2.1 for the definition of adjoint pairs.

Suppose $\downarrow(S \setminus V)$ is not clopen and let U' be the least clopen downset containing $S \setminus V$. Then $U' \neq \downarrow(S \setminus V)$. So there is $u \in U' \setminus \downarrow(S \setminus V)$. Since V is clopen in S we have that $S \setminus V$ is closed in S. Since S is closed in X, the latter implies that $S \setminus V$ is also closed in X. Then also $\downarrow(S \setminus V)$ is closed in X (see Theorem 2.2.3). Thus, we can separate u from $\downarrow(S \setminus V)$, i.e. there is $W \in \mathcal{CU}(X)$ with $u \in W$ and such that $W \cap \downarrow(S \setminus V) = \emptyset$ (see Theorem 2.2.3).

Then $X \setminus W$ is a clopen downset containing $S \setminus V$. But now the clopen downset $U' \cap X \setminus W$ also contains $S \setminus V$ but is properly contained in U'. This contradicts to the fact that U' is the smallest such. Thus, $\downarrow(S \setminus V)$ is clopen and so the claim holds.

Since every clopen downset is the complement of a clopen upset, the claim above shows that for every clopen downset Z of $(S, \leq)$, we have that $\downarrow Z$ is clopen in X. Now suppose that Z is an arbitrary clopen subset of $(S, \leq)$. Since $(S, \leq)$ is an Esakia space, $\downarrow_S Z$ is a clopen downset in S. Then $\downarrow(\downarrow_S Z) = \downarrow Z$ is clopen in X. □

Frame-based semantics for PLL

The correspondence in Theorem 6.5.5 and the algebraic semantics of PLL, S-spaces provide frame-based semantics for PLL. We spell out the details.

A valuation on an S-space (X, S) corresponds to a valuation on the underlying Esakia-space X, thus, a valuation maps a propositional letter to a clopen upset of X. The value of a formula of PLL in (X, S) can then calculated in the dual algebra of (X, S). However, as in Esakia spaces, we can evaluate a formula φ of PLL locally at a world $x \in X$. In detail, if $x \in X$ and v is a valuation on (X, S) then $x \models_v \varphi$ can be calculated according to Table 6.5.1.

$x \models_v p$	iff	$x \in v(p)$
$x \models_v \varphi \wedge \psi$	iff	$x \models_v \varphi$ and $x \models_v \psi$
$x \models_v \varphi \vee \psi$	iff	$x \models_v \varphi$ or $x \models_v \psi$
$x \models_v \varphi \to \psi$	iff	$y \models_v \varphi$ implies $y \models_v \psi$ for every $x \leq y$.
$x \models_v \bigcirc\varphi$	iff	$y \models_v \varphi$ for all $y \in \uparrow x \cap S$.

Table 6.5.1: Semantics for PLL via S-spaces

It is easy to see that $v(\varphi) = \{x \in X \mid x \models_v \varphi\}$, where $v(\varphi)$ is evaluated in the dual algebra of (X, S). We will use the similar notations as in the intuitionistic case, e.g. we write $(X, S) \models \varphi$ iff $x \models_v \varphi$ for every valuation v on (X, S) and $x \in X$.

As a consequence of Theorem 6.5.3, we obtain that PLL is complete with respect to the above semantics.

6.5.6. COROLLARY. PLL *is sound and complete with respect to the class of S-spaces.*

6.5.7. REMARK. The semantics via S-spaces is closely related to the frame-based semantics of PLL developed by Goldblatt [67] and Fairtlough and Mendler [54] (see also [27]). We do not discuss the precise connections here but refer the interested reader to [20].

6.5.3 Translating $\mathcal{L}_{\mathsf{IPC}}$ into $\mathcal{L}_{\mathsf{PLL}}$

In this section we define a translation τ from $\mathcal{L}_{\mathsf{IPC}}$ into the language of PLL. Even though the language of PLL coincides with that of intuitionistic modal logic, we will give it the specific name $\mathcal{L}_{\mathsf{PLL}}$. This will make it easier to differentiate the results of this section from the results in the section thereafter.

6.5.8. DEFINITION. Define a translation $\tau : \mathcal{L}_{\mathsf{IPC}} \to \mathcal{L}_{\mathsf{PLL}}$ by

- $\tau(p) = \bigcirc p$ for a propositional letter p,
- $\tau(\bot) = \bigcirc\bot$,
- $\tau(\varphi \wedge \psi) = \tau(\varphi) \wedge \tau(\psi)$,
- $\tau(\varphi \to \psi) = \tau(\varphi) \to \tau(\psi)$,
- $\tau(\varphi \vee \psi) = \bigcirc(\tau(\varphi) \vee \tau(\psi))$.

Obviously, we can regard $\mathcal{L}_{\mathsf{IPC}}$ as a sublanguage of $\mathcal{L}_{\mathsf{PLL}}$, giving rise to a (trivial) translation from $\mathcal{L}_{\mathsf{IPC}}$ into $\mathcal{L}_{\mathsf{PLL}}$ that maps a formula of $\mathcal{L}_{\mathsf{IPC}}$ to itself. Semantically, the trivial and the τ translations have the following effect on S-spaces:

6.5.9. PROPOSITION. *Let $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ and let (X, S) be an S-frame. Then*

(1) $(X, S) \models \varphi$ iff $X \models \varphi$.

(2) $(X, S) \models \tau(\varphi)$ iff $S \models \varphi$.

To prove the above proposition we need the lemma below. Note that in the statement of this lemma, by $\models_{v_S}$ we refer to the truth relation in the intuitionistic model (S, v_S) and by $\models_v$ we refer to the truth relation in the S-space (X, S) with valuation v.

6.5.10. LEMMA. *Let v be a valuation on an S-space (X, S). Define a valuation v_S on S by $v_S(p) = v(p) \cap S$. Then for every $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ and $x \in X$,*

$$x \models_v \tau(\varphi) \text{ iff } y \models_{v_S} \varphi \text{ for all } y \in {\uparrow}x \cap S.$$

Proof:
The proof is by induction on the complexity of $\varphi \in \mathcal{L}_{\mathsf{IPC}}$.

If $\varphi = p$, then $\tau(\varphi) = \bigcirc p$. We have,

$$\begin{aligned} & x \models_v \bigcirc p \\ \Leftrightarrow\quad & y \models_v p \text{ for all } y \in {\uparrow}x \cap S && \text{(Table 6.5.1)} \\ \Leftrightarrow\quad & y \models_{v_S} p \text{ for all } y \in {\uparrow}x \cap S. && \text{(by definition of } v_S\text{)} \end{aligned}$$

If $\varphi = \bot$, then $\tau(\varphi) = \bigcirc\bot$. Therefore, $x \models_v \bigcirc\bot$ iff ${\uparrow}x \cap S = \varnothing$. Thus, $x \models_v \bigcirc\bot$ iff $y \models_{v_S} \bot$ for all $y \in {\uparrow}x \cap S$.

If $\varphi = \psi \wedge \chi$, then $\tau(\psi \wedge \chi) = \tau(\psi) \wedge \tau(\chi)$. Therefore,

$$\begin{aligned} & x \models_v \tau(\psi \wedge \chi) \\ \Leftrightarrow\quad & x \models_v \tau(\psi) \text{ and } x \models_v \tau(\chi) && \text{(Table 6.5.1)} \\ \Leftrightarrow\quad & y \models_{v_S} \psi \text{ and } y \models_{v_S} \chi \text{ for all } y \in {\uparrow}x \cap S && \text{(by the I.H.)} \\ \Leftrightarrow\quad & y \models_{v_S} \psi \wedge \chi \text{ for all } y \in {\uparrow}x \cap S. \end{aligned}$$

If $\varphi = \psi \to \chi$, then $\tau(\psi \to \chi) = \tau(\psi) \to \tau(\chi)$. We prove the two implications separately. First suppose that $x \models_v \tau(\varphi) \to \tau(\psi)$. Let $y \in {\uparrow}x \cap S$. In order to show that $y \models_{v_S} \varphi \to \psi$, let $y \leq w \in S$ with $w \models_{v_S} \varphi$. Then

$$\begin{aligned} & w' \models_{v_S} \varphi \text{ for all } w' \in {\uparrow}w \cap S && \text{(since } v_S \text{ is persistent)} \\ \Rightarrow\quad & w \models_v \tau(\varphi) && \text{(by I.H.)} \\ \Rightarrow\quad & w \models_v \tau(\psi) && \text{(since } x \leq y \leq w \text{ and } x \models_v \tau(\varphi) \to \tau(\psi)\text{)} \\ \Rightarrow\quad & w \models_{v_S} \psi. && \text{(by I.H.)} \end{aligned}$$

We conclude that $y \models_{v_S} \varphi \to \psi$.

For the converse, suppose that $y \models_{v_S} \varphi \to \psi$ for all $y \in {\uparrow}x \cap S$. To see that $x \models_v \tau(\varphi) \to \tau(\psi)$, let $x \leq z$ with $z \models_v \tau(\varphi)$. Since ${\uparrow}z \cap S \subseteq {\uparrow}x \cap S$, we have

$$w \models_{v_S} \varphi \to \psi \text{ for all } w \in {\uparrow}z \cap S. \tag{6.3}$$

Therefore,

$$\begin{aligned} & w \models_{v_S} \varphi \text{ for all } w \in {\uparrow}z \cap S && \text{(by I.H.)} \\ \Rightarrow\quad & w \models_{v_S} \psi \text{ for all } w \in {\uparrow}z \cap S && \text{(use (6.3))} \\ \Rightarrow\quad & z \models_v \tau(\varphi). && \text{(by I.H.)} \end{aligned}$$

Finally, let $\varphi = \psi \vee \chi$, then $\tau(\psi \vee \chi) = \bigcirc(\tau(\psi) \vee \tau(\chi))$. We prove the two implications separately. For the direction from left to right, assume that

$x \models_v \bigcirc(\tau(\psi) \vee \tau(\chi))$ and let $y \in {\uparrow}x \cap S$. Then

$$
\begin{array}{lll}
 & x \models_v \bigcirc(\tau(\psi) \vee \tau(\chi)) & \\
\Rightarrow & y \models_v \tau(\psi) \vee \tau(\chi) & (\text{Table 6.5.1, } y \in {\uparrow}x \cap S) \\
\Rightarrow & y \models_v \tau(\psi) \text{ or } y \models_v \tau(\chi) & (\text{Table 6.5.1}) \\
\Rightarrow & (w \models_{v_S} \psi \text{ for all } w \in {\uparrow}y \cap S) \text{ or } (w \models_{v_S} \chi \text{ for all } w \in {\uparrow}y \cap S) & (\text{by I.H.}) \\
\Rightarrow & y \models_{v_S} \psi \text{ or } y \models_{v_S} \chi & (\text{since } y \in S) \\
\Rightarrow & y \models_{v_S} \psi \vee \chi. &
\end{array}
$$

Thus, $y \models_{v_S} \psi \vee \chi$ for all $y \in {\uparrow}x \cap S$. Conversely, suppose that $y \models_{v_S} \psi \vee \chi$ for all $y \in {\uparrow}x \cap S$. In order to see that $x \models \bigcirc(\tau(\psi) \vee \tau(\chi))$, let $y \in {\uparrow}x \cap S$. Then $y \models_{v_S} \psi \vee \chi$ by assumption. Without loss of generality we may assume that $y \models \psi$. Since v_S is persistent, $z \models_{v_S} \psi$ for all $z \in {\uparrow}y \cap S$. Therefore, $y \models_v \tau(\psi)$ by I.H., and so $y \models_v \tau(\psi) \vee \tau(\chi)$. Thus, $x \models \bigcirc(\tau(\psi) \vee \tau(\chi))$.

□

Proof of Proposition 6.5.9:

(1). This is obvious since φ contains no occurrences of $\bigcirc$.

(2). For the right to left direction, suppose v is a valuation on (X, S) that refutes $\tau(\varphi)$. Define a valuation v' on S by $v'(p) = v(p) \cap S$. By Lemma 6.5.10, the valuation v' refutes φ on S. For the left to right direction, suppose v' is a valuation on S that refutes φ. Define a valuation v on X by $v(p) = X \setminus {\downarrow}(S \setminus v'(p))$. Note that the latter is clopen since $S \setminus v'(p)$ is a clopen in S, so ${\downarrow}(S \setminus v'(p))$ is clopen in X by the subframe condition.

Then $v'(p) = v(p) \cap S$ for every propositional letter p. Indeed, for $z \in X$,

$$
\begin{array}{lll}
 & z \in v(p) \cap S & \\
\Leftrightarrow & z \in X \setminus {\downarrow}(S \setminus v'(p)) \cap S & (\text{by definition of } v) \\
\Leftrightarrow & z \notin {\downarrow}(S \setminus v'(p)) \text{ and } z \in S & \\
\Leftrightarrow & {\uparrow}z \cap S \subseteq S \cap v'(p) \text{ and } z \in S & \\
\Leftrightarrow & z \in v'(p). & (\text{since } v'(p) \text{ is persistent})
\end{array}
$$

Then $v'(p) = v(p) \cap S$ for every propositional letter p. Applying Lemma 6.5.10 again yields that v refutes $\tau(\varphi)$ on (X, S).

6.5.11. Remark. An algebraic reformulation of Proposition 6.5.9 is as follows. If $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ and (A, j) is a nuclear Heyting algebra, then

(1) $(A, j) \models \varphi$ iff $A \models \varphi$,

(2) $(A, j) \models \tau(\varphi)$ iff $A_j \models \varphi$.

6.5.12. Remark. The translation τ is a version of the Gödel-Gentzen translation (see, e.g. [55]). It has been pointed out by Aczel [1] that every lax modality which is definable within IPC (for example, the lax modality given by double negation) provides a translation from IPC to itself. The translation τ can be seen as a generalization of this, where the lax modality is not necessarily definable within IPC.

6.5.4 Two embeddings from Λ_{IPC} to Λ_{PLL}.

An intuitionistic modal logic $\mathsf{M} \subseteq \mathcal{L}_{\mathsf{PLL}}$ is called an *extension of* PLL iff $\mathsf{PLL} \subseteq \mathsf{M}$. Note that extensions of PLL are automatically normal, since they contain the axiom $p \to \bigcirc p$. Extensions of PLL form a lattice that we denote by Λ_{PLL}.

We employ the same notations as for si logics, e.g. if $\Gamma \subseteq \mathcal{L}_{\mathsf{PLL}}$, by $\mathsf{PLL} + \Gamma$ we denote the least extension M of PLL with $\Gamma \subseteq \mathsf{M}$. Also, if $\mathcal{K}$ is a class of nuclear Heyting algebras or a class of S-spaces then $\mathsf{Log}(\mathcal{K}) = \{\varphi \in \mathcal{L}_{\mathsf{PLL}} \mid \mathcal{K} \models \varphi\} \in \Lambda_{\mathsf{PLL}}$.

By the general algebraic completeness result and the correspondence between nuclear Heyting algebras and S-spaces (Theorem 6.5.5), it follows that every $\mathsf{M} \in \Lambda_{\mathsf{PLL}}$ is sound and complete with respect to its collection of nuclear Heyting algebras as well as its collection of S-spaces.

6.5.13. Corollary. *Let* $\mathsf{M} \in \Lambda_{\mathsf{PLL}}$.

(1) M *is sound and complete with respect to its corresponding class of S-spaces.*

(2) M *is sound and complete with respect to its corresponding variety of nuclear Heyting algebras.*

The two translations from $\mathcal{L}_{\mathsf{IPC}}$ into $\mathcal{L}_{\mathsf{PLL}}$ from the previous section, i.e. the trivial translation and τ, give rise to two non-modal fragments of an extension of PLL.

6.5.14. Definition. Let $\mathsf{M} \in \Lambda_{\mathsf{PLL}}$.

(1) We say that $\mathsf{L} \in \Lambda_{\mathsf{IPC}}$ is the *intuitionistic fragment* of M if for all $\varphi \in \mathcal{L}_{\mathsf{IPC}}$,

$$\varphi \in \mathsf{L} \text{ iff } \varphi \in \mathsf{M}.$$

(2) We say that $\mathsf{L} \in \Lambda_{\mathsf{IPC}}$ is the *lax fragment* of M if for all $\varphi \in \mathcal{L}_{\mathsf{IPC}}$,

$$\varphi \in \mathsf{L} \text{ iff } \tau(\varphi) \in \mathsf{M}.$$

Lemma 6.5.16 shows how to semantically characterize these fragments. As we will see below, thinking of an S-space (X, S) as consisting of two components, the component X and the component S, the intuitionistic fragment of an $\mathsf{M} \in \Lambda_{\mathsf{PLL}}$ is the logic of all X-components of the S-spaces validating M, whereas the lax fragment is the logic of all S-components of those spaces.

6.5.15. Definition. For $\mathsf{M} \in \Lambda_{\mathsf{PLL}}$, we define

$$\rho_1(\mathsf{M}) = \{\varphi \in \mathcal{L}_{\mathsf{IPC}} \mid \varphi \in \mathsf{M}\},$$
$$\rho_2(\mathsf{M}) = \{\varphi \in \mathcal{L}_{\mathsf{IPC}} \mid \tau(\varphi) \in \mathsf{M}\}.$$

6.5.16. Lemma. *Let* $\mathsf{M} \in \Lambda_{\mathsf{PLL}}$.

(1) $\rho_1(\mathsf{M})$ *is the intuitionistic fragment of* M *and*

$$\rho_1(\mathsf{M}) = \mathsf{Log}\left(\{X \mid (X,S) \models \mathsf{M} \text{ for some subframe } S \text{ of } X\}\right).$$

(2) $\rho_2(\mathsf{M})$ *is the lax fragment of* M *and*

$$\rho_2(\mathsf{M}) = \mathsf{Log}\left(\{S \mid (X,S) \models \mathsf{M}\}\right).$$

Proof:
We first show (2). For $\varphi \in \mathcal{L}_{\mathsf{IPC}}$, using Proposition 6.5.9(1), we have

$$\begin{aligned}\varphi \in \mathsf{Log}\left(\{S \mid (X,S) \models \mathsf{M}\}\right) &\Leftrightarrow S \models \varphi \text{ for all } (X,S) \models \mathsf{M} \\ &\Leftrightarrow (X,S) \models \tau(\varphi) \text{ for all } (X,S) \models \mathsf{M} \\ &\Leftrightarrow \tau(\varphi) \in \mathsf{M}.\end{aligned}$$

Therefore, $\rho_2(\mathsf{M}) = \mathsf{Log}\left(\{S \mid (X,S) \models \mathsf{M}\}\right)$. Thus, $\rho_2(\mathsf{M})$ is a si logic, and hence is the lax fragment of M. This shows (2). (1) is proved similarly but uses Proposition 6.5.9(1) instead. □

6.5.17. Remark. An algebraic reformulation of Lemma 6.5.16 is as follows:

(1) $\rho_1(\mathsf{M}) = \mathsf{Log}\left(\{A \mid (A,j) \models \mathsf{M} \text{ for some nucleus } j \text{ on } A\}\right)$.

(2) $\rho_2(\mathsf{M}) = \mathsf{Log}\left(\{A_j \mid (A,j) \models \mathsf{M}\}\right)$.

The translations also give rise to two embeddings σ_1 and σ_2 from Λ_{IPC} into Λ_{PLL}. We will show that for a si logic L, the logics $\sigma_1(\mathsf{L})$ and $\sigma_2(\mathsf{L})$ are the minimal extensions of PLL having L as their intuitionistic and lax fragments, respectively.

6.5.18. Definition. For a si logic L, we define

$$\sigma_1(\mathsf{L}) = \mathsf{PLL} + \{\varphi \mid \varphi \in \mathsf{L}\},$$
$$\sigma_2(\mathsf{L}) = \mathsf{PLL} + \{\tau(\varphi) \mid \varphi \in \mathsf{L}\}.$$

6.5.19. Lemma. *Let* L *be a si logic. Then*

(1) $\sigma_1(\mathsf{L}) = \mathsf{Log}\left(\{(X,S) \mid X \models \mathsf{L}\}\right)$.

(2) $\sigma_2(\mathsf{L}) = \mathsf{Log}\,(\{(X,S) \mid S \models \mathsf{L}\})$.

Proof:
We first show (2). Suppose (X,S) is an S-space. By Proposition 6.5.9(2), $S \models \mathsf{L}$ iff $(X,S) \models \{\tau(\varphi) \mid \varphi \in \mathsf{L}\}$. Thus, $\sigma_2(\mathsf{L}) = \mathsf{Log}\,(\{(X,S) \mid S \models \mathsf{L}\})$. (1) is proved similarly but uses Proposition 6.5.9(1) instead. □

6.5.20. Remark. In algebraic terms, Lemma 6.5.19 can be expressed as follows:

(1) $\sigma_1(\mathsf{L}) = \mathsf{Log}\,(\{(A,j) \mid A \models \mathsf{L}\})$.

(2) $\sigma_2(\mathsf{L}) = \mathsf{Log}\,(\{(A,j) \mid A_j \models \mathsf{L}\})$.

6.5.21. Lemma. *Let* L *be a si logic.*

(1) $\mathsf{L} = \rho_1\sigma_1(\mathsf{L})$. *In fact,* $\sigma_1(\mathsf{L})$ *is the least element of* $\rho_1^{-1}(\mathsf{L})$.

(2) $\mathsf{L} = \rho_2\sigma_2(\mathsf{L})$. *In fact,* $\sigma_2(\mathsf{L})$ *is the least element of* $\rho_2^{-1}(\mathsf{L})$.

Proof:

(1). Let $\varphi \in \mathcal{L}_{\mathsf{IPC}}$. Then $\varphi \in \mathsf{L}$ implies $\varphi \in \sigma_1(\mathsf{L})$, which implies $\varphi \in \rho_1\sigma_1(\mathsf{L})$. Therefore, $\mathsf{L} \subseteq \rho_1\sigma_1(\mathsf{L})$. If $\varphi \notin \mathsf{L}$, then there is an L-frame X such that $X \not\models \varphi$. Consider the S-space (X,X). By Lemma 6.5.19(1), $(X,X) \models \sigma_1(\mathsf{L})$, and by Proposition 6.5.9(1), $(X,X) \not\models \varphi$. Thus, $\varphi \notin \sigma_1(L)$, and so by Lemma 6.5.16(1), $\varphi \notin \rho_1\sigma_1(\mathsf{L})$. This shows that $\mathsf{L} = \rho_1\sigma_1(\mathsf{L})$. If $\mathsf{M} \in \rho_1^{-1}(\mathsf{L})$, then for every $\varphi \in \mathsf{L}_{\mathsf{IPC}}$, we have $\varphi \in \mathsf{L}$ iff $\varphi \in \mathsf{M}$. Consequently, $\sigma_1(\mathsf{L}) \subseteq \mathsf{M}$, and hence $\sigma_1(\mathsf{L})$ is the least element of $\rho_1^{-1}(\mathsf{L})$.

(2). Let $\varphi \in \mathcal{L}_{\mathsf{IPC}}$. Then $\varphi \in \mathsf{L}$ implies $\tau(\varphi) \in \sigma_2(\mathsf{L})$, which implies $\varphi \in \rho_2\sigma_2(\mathsf{L})$. Therefore, $\mathsf{L} \subseteq \rho_2\sigma_2(\mathsf{L})$. If $\varphi \notin \mathsf{L}$, then there is an L-frame X such that $X \not\models \varphi$. By Lemma 6.5.19(2), the S-space (X,X) is a $\sigma_2(\mathsf{L})$-frame, and by Proposition 6.5.9(2), $(X,X) \not\models \tau(\varphi)$. Thus, $\tau(\varphi) \notin \sigma_2(\mathsf{L})$, and so by Lemma 6.5.16(2), $\varphi \notin \rho_2\sigma_2(\mathsf{L})$. This shows that $\mathsf{L} = \rho_2\sigma_2(\mathsf{L})$. If $\mathsf{M} \in \rho_2^{-1}(\mathsf{L})$, then for every $\varphi \in \mathcal{L}_{\mathsf{IPC}}$, we have $\varphi \in \mathsf{L}$ iff $\tau(\varphi) \in \mathsf{M}$. Consequently, $\sigma_2(\mathsf{L}) \subseteq \mathsf{M}$, and hence $\sigma_2(\mathsf{L})$ is the least element of $\rho_2^{-1}(\mathsf{L})$. □

Obviously, if L is a si logic, then $\rho_1^{-1}(\mathsf{L})$ is the collection of all extensions of PLL having L as their intuitionistic fragment and similarly $\rho_2^{-1}(\mathsf{L})$ is the collection of all extensions of PLL having L as their lax fragment. As follows from Lemma 6.5.21, for a si logic L, both $\rho_1^{-1}(\mathsf{L})$ and $\rho_2^{-1}(\mathsf{L})$ have least elements, namely $\sigma_1(\mathsf{L})$ and $\sigma_2(\mathsf{L})$. In the last part of this section, we will show that in general neither $\rho_1^{-1}(\mathsf{L})$ nor $\rho_2^{-1}(\mathsf{L})$ has a largest element. To see this we first prove several lemmas.

6.5.22. LEMMA. *Let (X, S) be an S-space.*

(1) $(X, S) \models \bigcirc p \leftrightarrow p$ *iff* $X = S$.

(2) $(X, S) \models \bigcirc p$ *iff* $S = \varnothing$.

Proof:

(1). First suppose that $X = S$. Then it is clear that $(X, S) \models \bigcirc p \leftrightarrow p$. Next suppose that $X \neq S$. Let $x \in X \setminus S$. Then $x \notin {\uparrow}x \cap S$, so $x \notin {\uparrow}({\uparrow}x \cap S)$. Therefore, since ${\uparrow}({\uparrow}x \cap S)$ is a closed upset of X, there is a clopen upset U of X with ${\uparrow}({\uparrow}x \cap S) \subseteq U$ and $x \notin U$ (see Section 2.2.3).

Let v be a valuation on (X, Y) such that $v(p) = U$. Clearly $x \not\models_v p$. On the other hand, $x \models_v \bigcirc p$ by the choice of v and the semantics in Table 6.5.1. Thus, $(X, S) \not\models \bigcirc p \leftrightarrow p$.

(2). If $S = \varnothing$, then it is clear that $(X, S) \models \bigcirc p$. If $S \neq \varnothing$, then let v be a valuation on (X, S) such that $v(p) = \varnothing$. For $x \in S$, we then have $x \not\models_v \bigcirc p$, so $(X, S) \not\models \bigcirc p$. □

For $\psi \in \mathcal{L}_{\mathsf{PLL}}$, let ψ^- be the formula obtained from ψ by deleting all occurrences of the $\bigcirc$ modality and let ψ^* be the formula obtained from ψ by replacing all subformulas of the form $\bigcirc\chi$ with $\top$. Clearly $\psi^-, \psi^* \in \mathcal{L}_{\mathsf{IPC}}$. Both ψ^- and ψ^* were considered in [54, Section 3].

6.5.23. LEMMA. *Let* $\mathsf{M} \in \Lambda(\mathsf{PLL})$.

(1) If $\bigcirc p \leftrightarrow p \in \mathsf{M}$*, then* $\psi \in \mathsf{M}$ *iff* $\psi^- \in \mathsf{M}$ *for every formula* $\psi \in \mathcal{L}_{\mathsf{PLL}}$.

(2) If $\bigcirc p \in \mathsf{M}$*, then* $\psi \in \mathsf{M}$ *iff* $\psi^* \in \mathsf{M}$ *for every formula* $\psi \in \mathcal{L}_{\mathsf{PLL}}$.

Proof:

(1). Suppose that $\bigcirc p \leftrightarrow p \in \mathsf{M}$ and let $\psi \in \mathcal{L}_{\mathsf{PLL}}$. By Lemma 6.5.22(1), M is the logic of the class of S-spaces of the shape (X, X). For (X, X), a valuation v on X, and $x \in X$, we have $x \models_v \bigcirc\varphi$ iff $x \models_v \varphi$. Therefore, an induction on the structure of ψ yields $x \models_v \psi$ iff $x \models_v \psi^-$. Thus, $(X, X) \models \psi$ iff $(X, X) \models \psi^-$, and therefore, $\psi \in \mathsf{M}$ iff $\psi^- \in \mathsf{M}$.

(2). Let $\bigcirc p \leftrightarrow \top \in \mathsf{M}$ and let $\psi \in \mathcal{L}_{\mathsf{PLL}}$. By Lemma 6.5.22(2), M is the logic of the class of S-spaces of the shape $(X, \varnothing)$. For $(X, \varnothing)$, a valuation v on X, and $x \in X$, we have $x \models_v \bigcirc\varphi$. Therefore, an induction on the structure of ψ yields $x \models_v \psi$ iff $x \models_v \psi^*$. Therefore, $(X, \varnothing) \models \psi$ iff $(X, \varnothing) \models \psi^*$ and so $\psi \in \mathsf{M}$ iff $\psi^* \in \mathsf{M}$. □

6.5.24. Lemma. *Let* L *be a si logic.*

(1) $\sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow p$ *is a maximal element of both* $\rho_1^{-1}(\mathsf{L})$ *and* $\rho_2^{-1}(\mathsf{L})$*.*

(2) $\sigma_1(\mathsf{L}) + \bigcirc p$ *is a maximal element of* $\rho_1^{-1}(\mathsf{L})$*.*

Proof:

(1). Let $\mathsf{M} = \sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow p$. First we show that M is a maximal element of $\rho_1^{-1}(\mathsf{L})$. By Lemma 6.5.22(1), an S-space (X, S) validates M iff X is an L-frame and $X = S$. Therefore, by Lemma 6.5.16(1), $\rho_1(\mathsf{M}) = \mathsf{L}$, so $\mathsf{M} \in \rho_1^{-1}(\mathsf{L})$. To see that M is maximal in $\rho_1^{-1}(\mathsf{L})$, suppose that $\mathsf{M} \subseteq \mathsf{M}' \in \rho_1^{-1}(\mathsf{L})$. We show that $\mathsf{M} = \mathsf{M}'$. Let $\psi \in \mathcal{L}_{\mathsf{PLL}}$. If $\psi \notin \mathsf{M}$, then by Lemma 6.5.23(1), $\psi^- \notin \mathsf{M}$, and so $\psi^- \notin \mathsf{L}$ as $\psi^- \in \mathcal{L}_{\mathsf{IPC}}$. Since $\rho_1(\mathsf{M}') = \mathsf{L}$, we see that $\psi^- \notin \mathsf{M}'$. Because $\mathsf{M} \subseteq \mathsf{M}'$, we have $\bigcirc p \leftrightarrow p \in \mathsf{M}'$, so $\psi \notin \mathsf{M}'$ by Lemma 6.5.23(1). Thus, $\mathsf{M} = \mathsf{M}'$, and hence M is maximal in $\rho_1^{-1}(\mathsf{L})$.

Next we show that M is a maximal element of $\rho_2^{-1}(\mathsf{L})$. By Lemma 6.5.16(2), $\rho_2(\mathsf{M}) = \mathsf{L}$, so $\mathsf{M} \in \rho_2^{-1}(\mathsf{L})$. Suppose $\mathsf{M} \subseteq \mathsf{M}' \in \rho_2^{-1}(\mathsf{L})$. We show that $\mathsf{M} = \mathsf{M}'$. Let $\psi \in \mathcal{L}_{\mathsf{PLL}}$. If $\psi \notin \mathsf{M}$, then $\psi^- \notin \mathsf{M}$ by Lemma 6.5.23(1). Therefore, $\tau(\psi^-) \notin \mathsf{M}$ because $(\tau(\psi^-))^- = \psi^-$. Thus, $\psi^- \notin \mathsf{L}$, and so $\tau(\psi^-) \notin \mathsf{M}'$. Since $\mathsf{M} \subseteq \mathsf{M}'$, we have $\bigcirc p \leftrightarrow p \in \mathsf{M}'$, and hence $\psi^- = (\tau(\psi^-))^- \notin \mathsf{M}'$ by Lemma 6.5.23(1). Consequently, $\psi \notin \mathsf{M}'$, and so $\mathsf{M} = \mathsf{M}'$, which yields that M is maximal in $\rho_2^{-1}(\mathsf{L})$.

(2). Let $\mathsf{M} = \sigma_1(\mathsf{L}) + \bigcirc p$. By Lemma 6.5.22(2), an S-space (X, S) validates M iff X is an L-frame and $S = \varnothing$. Therefore, by Lemma 6.5.16(1), $\rho_1(\mathsf{M}) = \mathsf{L}$, so $\mathsf{M} \in \rho_1^{-1}(\mathsf{L})$. To see that M is maximal in $\rho_1^{-1}(\mathsf{L})$, suppose that $\mathsf{M} \subseteq \mathsf{M}' \in \rho_1^{-1}(\mathsf{L})$. We show that $\mathsf{M} = \mathsf{M}'$. Let $\psi \in \mathcal{L}_{\mathsf{PLL}}$. If $\psi \notin \mathsf{M}$, then by Lemma 6.5.23(2), $\psi^* \notin \mathsf{M}$, and so $\psi^* \notin \mathsf{L}$ as $\psi^* \in \mathcal{L}_{\mathsf{IPC}}$. Since $\rho_1(\mathsf{M}') = \mathsf{L}$, we see that $\psi^* \notin \mathsf{M}'$. Because $\mathsf{M} \subseteq \mathsf{M}'$, we have $\bigcirc p \in \mathsf{M}'$, so $\psi \notin \mathsf{M}'$ by Lemma 6.5.23(2). Thus, $\mathsf{M} = \mathsf{M}'$, and hence M is maximal in $\rho_1^{-1}(\mathsf{L})$. □

6.5.25. Proposition.

(1) If L *is a consistent si logic, then* $\rho_1^{-1}(\mathsf{L})$ *does not have a largest element.*

(2) Suppose $\mathsf{KC} \not\subseteq \mathsf{L}$*, then* $\rho_2^{-1}(\mathsf{L})$ *does not have a largest element.*

Proof:

(1). Let L be a consistent si logic. Then $\{x\}$ is an L-frame. We show that $\sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow p$ and $\sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow \top$ are different. Indeed, the S-space $(\{x\}, \varnothing)$ validates $\sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow \top$ but refutes $\sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow p$. Therefore, by Lemma 6.5.24, $\rho_1^{-1}(\mathsf{L})$ has at least two maximal elements and therefore does not have a largest element.

(2). Let L be a si logic with $\mathsf{KC} \nsubseteq \mathsf{L}$. Suppose L is axiomatized by the set of formulas Γ.

Set $\mathsf{M} = \sigma_1(\mathsf{KC}) + \{\tau(\gamma) \mid \gamma \in \Gamma\}$. By Lemma 6.5.19(1) and Proposition 6.5.9(2), an S-space (X, S) validates M iff X is a KC-frame and S is an L-frame. Therefore, by Lemma 6.5.16(2), $\mathsf{L} \subseteq \rho_2(\mathsf{M})$.

To see the reverse inclusion, suppose that $\varphi \notin \mathsf{L}$. Then there is a L-frame S with $S \not\models \varphi$. Let X be the Esakia frame that is obtained from S by adding a new isolated top node t. Algebraically, this corresponds to the operation of adding a new minimal element to the Heyting dual of S. Then X is a KC-frame, and since the domain of S is a clopen subset of X, S is a subframe of X. Thus, (X, S) validates M, but refutes $\tau(\varphi)$ by Proposition 6.5.9(2). Thus, $\varphi \notin \rho_2(\mathsf{M})$. Consequently, $\rho_2(\mathsf{M}) = \mathsf{L}$, and so $\mathsf{M} \in \rho_2^{-1}(\mathsf{L})$.

On the other hand, since $\mathsf{KC} \nsubseteq \mathsf{L}$, there is an L-frame X that is not a KC-frame. Then $(S, S) \models \sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow p$ but refutes M. This shows that $\mathsf{M} \nsubseteq \sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow p$. If $\rho_2^{-1}(\mathsf{L})$ were to have a largest element it would need to be $\sigma_1(\mathsf{L}) + \bigcirc p \leftrightarrow p$, since this is maximal by Lemma 6.5.24(1) but since M is not contained in that logic, $\rho_2^{-1}(\mathsf{L})$ does not have a largest element. □

Figure 6.5.1 illustrates the embeddings σ_1 and ρ_1. Recall that by CPC we denote the classical propositional logic and by Fml the inconsistent logic. The picture is similar for σ_2 and ρ_2.

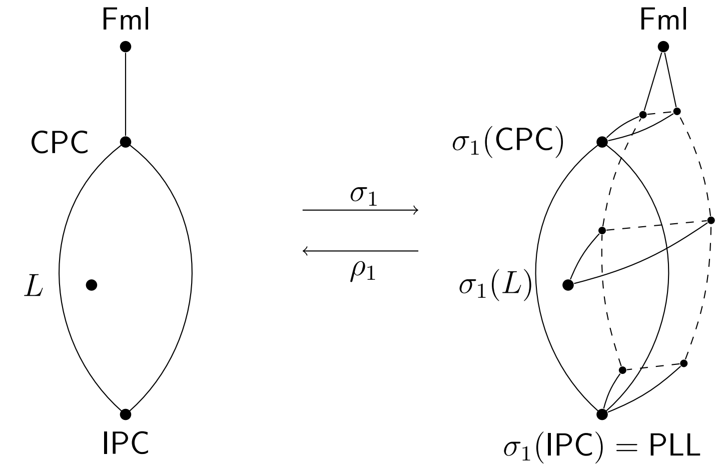


Figure 6.5.1

6.5.5 Subframe logics and subframization via PLL

We utilize the embeddings from the previous section to obtain a new characterization of subframe si logics and of the downward subframization of a si logic.

6.5.26. THEOREM. *For a si logic* L*, the following are equivalent:*

(1) L *is a subframe logic.*

(2) $\sigma_2(\mathsf{L}) \subseteq \sigma_1(\mathsf{L})$.

(3) $\sigma_2(\mathsf{L}) + \{\varphi \mid \varphi \in \mathsf{L}\} = \sigma_1(\mathsf{L})$.

(4) $\rho_2\sigma_1(\mathsf{L}) = \mathsf{L}$.

(5) $\sigma_1(\mathsf{L})$ *is closed under the rule* $\varphi/\tau(\varphi)$ *for every* $\varphi \in \mathcal{L}_{\mathsf{IPC}}$.

Proof:

(1)⇒(2). Suppose (X, S) is an S-space such that $(X, S) \models \sigma_1(\mathsf{L})$. By Lemma 6.5.19(1), $X \models \mathsf{L}$. Since L is a subframe logic, $S \models \mathsf{L}$. Therefore, by Lemma 6.5.19(2), $(X, S) \models \sigma_2(\mathsf{L})$. Thus, $\sigma_2(\mathsf{L}) \subseteq \sigma_1(\mathsf{L})$.

(2)⇒(3). This is obvious.

(3)⇒(4). By Lemmas 6.5.21(2) and 6.5.16(2), $\mathsf{L} = \rho_2\sigma_2(\mathsf{L}) = \mathsf{Log}(\{S \mid (X, S) \models \sigma_2(\mathsf{L})\})$ and $\rho_2\sigma_1(\mathsf{L}) = \mathsf{Log}(\{S \mid (X, S) \models \sigma_1(\mathsf{L})\})$. Therefore, it is sufficient to show that $\{S \mid (X, S) \models \sigma_2(\mathsf{L})\} = \{S \mid (X, S) \models \sigma_1(\mathsf{L})\}$. The inclusion $\supseteq$ is immediate from (3). For the reverse inclusion, suppose that $(X, S) \models \sigma_2(\mathsf{L})$. By Lemma 6.5.19(2), $S \models \mathsf{L}$, so $(S, S) \models \sigma_1(\mathsf{L})$ by Lemma 6.5.19(1). Thus, $S \in \{S \mid (X, S) \models \sigma_1(\mathsf{L})\}$.

(4)⇒(5). Suppose that there is $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ such that $\varphi \in \sigma_1(\mathsf{L})$ but $\tau(\varphi) \notin \sigma_1(\mathsf{L})$. Then there is an S-space (X, S) with $(X, S) \models \sigma_1(\mathsf{L})$ and $(X, S) \not\models \tau(\varphi)$. By Lemma 6.5.16(2), $(X, S) \models \sigma_1(\mathsf{L})$ implies $S \models \rho_2\sigma_1(\mathsf{L}) = \mathsf{L}$, and by Proposition 6.5.92, $(X, S) \not\models \tau(\varphi)$ implies $S \not\models \varphi$. Therefore, $\varphi \notin \mathsf{L}$, contradicting $\varphi \in \sigma_1(\mathsf{L})$.

(5)⇒(1). Let X be an L-space and S be a subframe of X. By Lemma 6.5.19(1), $(X, S) \models \sigma_1(\mathsf{L})$. By (5), $(X, S) \models \tau(\varphi)$ for each $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ with $\varphi \in \sigma_1(\mathsf{L})$. Therefore, $(X, S) \models \tau(\varphi)$ for each $\varphi \in \mathsf{L}$. Thus, $S \models \mathsf{L}$ by Proposition 6.5.9(2), and we conclude that L is a subframe logic. □

6.5.27. REMARK. In general, $\sigma_1(\mathsf{L}) \not\subseteq \sigma_2(\mathsf{L})$. In fact, for any consistent si logic L, from $\sigma_1(\mathsf{L}) \subseteq \sigma_2(\mathsf{L})$ it follows that $\mathsf{L} = \mathsf{IPC}$. To see this, suppose $\mathsf{L} \neq \mathsf{IPC}$. Then there is a finite frame X that refutes L. Pick a point in X and let S be the subframe of X consisting of that point. Clearly S is an L-frame. Therefore, by Lemma 6.5.19(2), $(X, S) \models \sigma_2(\mathsf{L})$. On the other hand, by Lemma 6.5.19(1), $(X, S) \not\models \sigma_1(\mathsf{L})$. Thus, $\sigma_1(\mathsf{L}) \not\subseteq \sigma_2(\mathsf{L})$.

As a consequence of Theorem 6.5.26, we obtain the following characterization of the downward subframization of a si logic.

6.5.28. Theorem. *Let* L *be a si logic. Then* $\mathsf{Subf}_{\downarrow}(\mathsf{L}) = \rho_2\sigma_1(\mathsf{L})$.

Proof:
Let S be an Esakia space. By Lemma 6.5.16(2), $S \models \rho_2\sigma_1(\mathsf{L})$ iff there is an Esakia space X such that $(X, S) \models \sigma_1(\mathsf{L})$. By Lemma 6.5.19(1), $(X, S) \models \sigma_1(\mathsf{L})$ iff $X \models \mathsf{L}$. Therefore, $S \models \rho_2\sigma_1(\mathsf{L})$ iff S is a subframe of some $X \models \mathsf{L}$. Thus, by Proposition 6.2.3(1), $\rho_2\sigma_1(\mathsf{L}) = \mathsf{Subf}_{\downarrow}(\mathsf{L})$. □

6.5.29. Remark.

(1) Let L be a si logic and $\varphi \in \mathcal{L}_{\mathsf{IPC}}$. By Theorem 6.5.28, $\varphi \in \mathsf{Subf}_{\downarrow}(\mathsf{L})$ iff $\tau(\varphi) \in \sigma_1(\mathsf{PLL})$. Therefore, if $\sigma_1(\mathsf{PLL})$ is decidable, then so is $\mathsf{Subf}_{\downarrow}(\mathsf{L})$.

(2) In contrast to Theorem 6.5.28, for every si logic L, we have $\rho_1\sigma_2(\mathsf{L}) = \mathsf{IPC}$. Indeed, suppose L is a si logic and X is an Esakia space. By Lemma 6.5.16(1), $X \models \rho_1\sigma_2(\mathsf{L})$ iff there is a subframe S of X such that $(X, S) \models \sigma_2(\mathsf{L})$. By Lemma 6.5.19(2), $(X, S) \models \sigma_2(\mathsf{L})$ iff $S \models \mathsf{L}$. Therefore, $X \models \rho_1\sigma_2(\mathsf{L})$ iff $S \models \mathsf{L}$ for some subframe S of X. Now every space contains the empty frame as a subframe and since the empty frame is an L-frame, we conclude that every Esakia space validates $\rho_1\sigma_2(\mathsf{L})$. Thus, $\rho_1\sigma_2(\mathsf{L}) = \mathsf{IPC}$.

6.5.30. Remark. Recall that a subframe S of an Esakia space X is *cofinal* provided it contains the maximum of X. Cofinal subframes of an Esakia space X correspond to *dense* nuclei on the dual Heyting algebra A of X, where we recall that a nucleus j is dense if $j0 = 0$ (see Section 3.4.1). Since being a dense nucleus can be expressed by adding $\bigcirc\neg\bot$ to PLL, the correspondence between subframe logics and extensions of PLL discussed in this section extends to the correspondence between cofinal subframe logics and extensions of $\mathsf{PLL} + \bigcirc\neg\bot$.

6.6 Stable logics and IS4

In this section we will explore the connection between stable si logics and the extensions of the intuitionistic modal logic $\mathsf{IS4}$. We aim to parallel the results from the previous section. The first part of this section is standard, e.g. the analogue of the translation τ is the well-known Gödel-McKinsey-Tarski translation, so we will be a bit quicker than in the previous section.

As before, there will be some discrepancies between the stable and the subframe cases. These are caused by the fact that only the rooted Esakia spaces of a stable si logic form a stable class (and the class of all spaces of a stable si logic is in general not stable). Thus, instead of embedding stable si logic into logics extending $\mathsf{IS4}$ we are forced to move the embeddings to multi-conclusion consequence relations extending $\mathsf{IS4}$ with the disjunction rule $p \vee q/p, q$ that expresses rootedness. Modulo this adjustment, we provide a characterization of stable logics and the downward stabilization analogous to the previous section.

6.6.1 The intuitionistic modal logic IS4

In this section we recall the definition of the intuitionistic modal logic IS4, its algebraic and frame-based semantics and the definition of the Gödel-McKinsey-Tarski translation.

6.6.1. Definition. IS4 is the intuitionistic modal logic with modal operator $\Box$ satisfying the following axioms

(1) $\Box p \to p$,

(2) $\Box p \to \Box\Box p$, and

(3) $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$,

and is closed under rule of necessitation $\varphi/\Box\varphi$.

We will denote the language of IS4 by $\mathcal{L}_{\mathsf{IS4}}$. As was observed by Ono in [105], algebraic semantics for IS4 is provided by *interior Heyting algebras*, which are pairs $(A, \Box)$, consisting of a Heyting algebra A and a unary operation $\Box$ that is an *interior operator* on A; that is, $\Box$ satisfies $\Box a \leq a$, $\Box a \leq \Box\Box a$, $\Box(a \wedge b) = \Box a \wedge \Box b$, and $\Box 1 = 1$. It is clear how to interpret formulas of $\mathcal{L}_{\mathsf{IS4}}$ in interior Heyting algebras. Interior Heyting algebras provide an adequate algebraic semantics for IS4.

6.6.2. Theorem (Ono). IS4 *is sound and complete with respect to interior Heyting algebras.*

6.6.2 Interior Heyting algebras and St-spaces

In this section, we introduce St-spaces and show that they are in one-to-one correspondence with interior Heyting algebras. This correspondence can be seen as the stable analogue of the correspondence between S-spaces and nuclear Heyting algebras from Section 6.5.2.

First, we recall some properties of interior Heyting algebras. If $(A, \Box)$ is an interior Heyting algebra, the fixed points $A_\Box := \{a \in A \mid \Box a = a\}$ form a bounded sublattice of A. In fact $A_\Box = (A_\Box, \wedge, \vee, \to_\Box, 0, 1)$ is also a Heyting algebra, where $a \to_\Box b = \Box(a \to b)$. Not every bounded sublattice of a Heyting algebra is of this shape. In fact, interior Heyting algebras correspond to pairs $(A, A_\Box)$ of Heyting algebras such that $A_\Box$ is a bounded sublattice of A and the embedding $A_\Box \rightarrowtail A$ has a right adjoint. This is a well-known fact in residuation theory, see e.g. [37]. (The reader may also consult [11, Section 3] where this correspondence is spelled out for *monadic* Heyting algebras, but the results can easily be adjusted to interior Heyting algebras.)

As Theorem 6.6.4 below reveals, interior Heyting algebras correspond to what we call St-spaces.

6.6.3. DEFINITION. An *St-space* (short for: stable space) is a pair (X, Y) of Esakia spaces X and Y together with a map $\pi : X \to Y$ which is onto and stable map and such that $\downarrow\pi(U)$ is clopen in Y for each clopen U in X.

To simplify notation we will usually not explicitly mention the stable map π of an St-space (X, Y).

6.6.4. THEOREM. *There is a one-to-one correspondence between interior Heyting algebras and St-spaces.*[4]

Proof:
We only sketch the proof. Note that it is entirely analogous to the proof of Theorem 6.5.5. Let (X, Y) together with $\pi : X \to Y$ be an St-space. Let A be a dual Heyting algebra of X. A routine check shows that $\Box : A \to A$ defined by

$$\Box U = X \setminus \pi^{-1}(\downarrow_Y(\pi(X \setminus U)))$$

for each $U \in \mathcal{CU}(X)$ is an interior operator on A.

Conversely, let $(A, \Box)$ be an interior Heyting algebra. As discussed above, the set of fixed points $A_\Box$ is also a Heyting algebra and the map $g^+ : A \to A_\Box$, $a \mapsto \Box a$ is the right adjoint of the inclusion $g : A_\Box \to A$.

Let X be the Esakia space dual to A, let Y be the Esakia space dual to $A_\Box$ and let $\pi : X \to Y$ be the dual of g. Then π is a stable and onto. Moreover, g can then be described by $g : \mathcal{CU}(Y) \to \mathcal{CU}(X)$, given by $V \mapsto \pi^{-1}(V)$ for each clopen upset U of X. That g has a right adjoint, in dual terms means that for each clopen upset U of X, the set

$$\{V \in \mathcal{CU}(Y) \mid \pi^{-1}(V) \subseteq U\} \tag{6.4}$$

has a largest element. Now

$$\begin{array}{ll} & \pi^{-1}(V) \subseteq U \\ \text{iff} & X \setminus U \subseteq X \setminus \pi^{-1}(V) = \pi^{-1}(Y \setminus V) \\ \text{iff} & \pi(X \setminus U) \subseteq Y \setminus V \\ \text{iff} & \downarrow_Y \pi(X \setminus U) \subseteq Y \setminus V. \end{array}$$

Thus, in other words (6.4) means that there is a smallest clopen downset W of Y that contains $\downarrow_Y \pi(X \setminus U)$.

6.6.1. CLAIM. *$\downarrow_Y \pi(X \setminus U)$ is clopen in Y.*

[4] This proof was done with the help of M. Jibladze and T. Litak.

Suppose that $\downarrow_Y \pi(X \setminus U)$ is not clopen in Y. This means that there is a smallest clopen downset $W \subseteq Y$ containing $\downarrow_Y \pi(X \setminus U)$ and $W \neq \downarrow_Y \pi(X \setminus U)$. Then there is $w \in W$ with $w \notin \downarrow_Y \pi(X \setminus U)$. Since π maps closed sets onto closed sets, and downsets of closed sets are closed in Y (see Theorem 2.2.3), $\downarrow_Y \pi(X \setminus U)$ is closed in Y. Thus, there is a clopen downset W' containing $\downarrow_Y \pi(X \setminus U)$ but not w. Then $W' \cap W$ is strictly contained in W and still contains $\downarrow_Y \pi(X \setminus U)$ which shows that W was not the smallest set with that property. Thus, $\downarrow_Y \pi(X \setminus U)$ is clopen in Y and the claim follows.

The claim shows that $\downarrow_Y \pi(W)$ is clopen in Y for each clopen downset W of X. Now if U is an arbitrary clopen subset of X, then $\downarrow_X U$ is clopen in X, and therefore $\downarrow_Y \pi(\downarrow_X U) = \downarrow_Y \pi(U)$ is clopen in Y. This shows that (X, Y) is an St-frame. □

Frame-based semantics via St-spaces

The correspondence between interior Heyting algebras and St-spaces allows us to interpret formulas of IS4 in St-spaces. Let (X, Y) be an St-space, where $X = (X, \leq)$ and $Y = (Y, \leq)$. We interpret propositional letters as clopen upsets of X and intuitionistic connectives as the corresponding operations in the Heyting algebra of clopen upsets of X. In addition, $\Box$ is interpreted as the corresponding unary function on the clopen upsets of X; that is, $\Box U = \pi^{-1}(Y \setminus \downarrow\pi(X \setminus U))$.

To locally evaluate formulas at a world x, observe that, $x \notin \Box v(\varphi)$ iff $\pi(x) \in \downarrow\pi(X \setminus v(\varphi))$, which happens iff there is $z \in X \setminus v(\varphi)$ with $\pi(x) \leq \pi(z)$. This gives rise to the semantics described in Table 6.6.1.

$x \models_v p$	iff	$x \in v(p)$
$x \models_v \varphi \wedge \psi$	iff	$x \models_v \varphi$ and $x \models_v \psi$
$x \models_v \varphi \vee \psi$	iff	$x \models_v \varphi$ or $x \models_v \psi$
$x \models_v \varphi \to \psi$	iff	$y \models_v \varphi$ implies $y \models_v \psi$ for every $x \leq y$.
$x \models_v \Box\varphi$	iff	$z \models_v \varphi$ for all $z \in X$ with $\pi(x) \leq \pi(z)$.

Table 6.6.1: Semantics for IS4 via St-spaces

6.6.5. Corollary. IS4 *is sound and complete with respect to the class of St-spaces.*

The Gödel-McKinsey-Tarski translation

We map formulas φ of IPC into formulas $t(\varphi)$ of IS4 according to the Gödel-McKinsey-Tarski translation (see Section 4.6.1).

It is clear that for every $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ and every interior Heyting algebra $(A, \Box)$, we have:

(1) $(A, \Box) \models \varphi$ iff $A \models \varphi$.

(2) $(A, \Box) \models t(\varphi)$ iff $A_\Box \models \varphi$.

In dual terms we have:

6.6.6. Lemma. *For every $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ and every St-space (X, Y), we have*

(1) $(X, Y) \models \varphi$ iff $X \models \varphi$,

(2) $(X, Y) \models t(\varphi)$ iff $Y \models \varphi$.

6.6.3 Two embeddings from Λ_{IPC} into Σ_{IS4}

Recall the notion of a multi-conclusion consequence relation and rules from Section 2.3.6. By $\mathcal{S}_{\mathsf{IS4}}$ we denote $\mathsf{IS4}$ seen as a multi-conclusion consequence relation that additionally contains the disjunction rule $p \vee q/p, q$. The multi-conclusion consequence relations extending $\mathcal{S}_{\mathsf{IS4}}$ form a complete lattice that we denote by Σ_{IS4}.

Consequence relations in Σ_{IS4} correspond to universal classes of interior Heyting algebras whose underlying Heyting algebras are *well-connected* ($a \vee b = 1$ implies $a = 1$ or $b = 1$, see Section 2.1.2). Thus, dually they are characterized by classes of St-spaces (X, Y) such that X is rooted. We call such St-spaces *rooted.* For a class $\mathcal{K}$ of rooted St-spaces, let $\mathsf{Con}(\mathcal{K})$ be the set of multi-conclusion rules that are valid in $\mathcal{K}$. Then $\mathsf{Con}(\mathcal{K}) \in \Sigma_{\mathsf{IS4}}$. We then have the following theorem.

6.6.7. Corollary. *Let $\mathcal{S} \in \Sigma_{\mathsf{IS4}}$.*

(1) $\mathcal{S}$ is sound and complete with respect to its corresponding class of St-spaces.

(2) $\mathcal{S}$ is sound and complete with respect to its corresponding universal class of interior Heyting algebras.

For the rest of this Section all St-spaces are assumed to be rooted.

6.6.8. Definition. Let $\mathsf{L} \in \Lambda_{\mathsf{IPC}}$ and $\mathcal{S} \in \Sigma_{\mathsf{IS4}}$.

(1) We say that L is the *intuitionistic fragment* of $\mathcal{S}$ if for all formulas $\varphi \in \mathcal{L}_{\mathsf{IPC}}$,

$$\varphi \in \mathsf{L} \text{ iff } /\varphi \in \mathcal{S}.$$

(2) We say that L is the *stable fragment* of $\mathcal{S}$ if for all formulas $\varphi \in \mathcal{L}_{\mathsf{IPC}}$,

$$\varphi \in \mathsf{L} \text{ iff } /t(\varphi) \in \mathcal{S}.$$

For $\mathcal{S} \in \Sigma_{\mathsf{IS4}}$, we define

$$\zeta_1(\mathcal{S}) = \{\varphi \in \mathcal{L}_{\mathsf{IPC}} \mid /\varphi \in \mathcal{S}\},$$
$$\zeta_2(\mathcal{S}) = \{\varphi \in \mathcal{L}_{\mathsf{IPC}} \mid /t(\varphi) \in \mathcal{S}\}.$$

6.6.9. LEMMA. *Let $\mathcal{S} \in \Sigma_{\mathsf{IS4}}$.*

(1) $\zeta_1(\mathcal{S})$ is the intuitionistic fragment of $\mathcal{S}$ and

$$\zeta_1(\mathcal{S}) = \mathsf{Log}\left(\{X \mid \exists Y : (X,Y) \textit{ is an St-space and } (X,Y) \models \mathcal{S}\}\right).$$

(2) $\zeta_2(\mathcal{S})$ is the stable fragment of $\mathcal{S}$ and

$$\zeta_2(\mathcal{S}) = \mathsf{Log}\left(\{Y \mid \exists X : (X,Y) \textit{ is an St-space and } (X,Y) \models \mathcal{S}\}\right).$$

Proof:
We only show (1) since the proof of (2) is similar. For $\varphi \in \mathcal{L}_{\mathsf{IPC}}$, we have

$$\begin{aligned} & \varphi \in \mathsf{Log}\left(\{X \mid \exists Y : (X,Y) \text{ is an St-space and } (X,Y) \models \mathcal{S}\}\right) \\ \Leftrightarrow\quad & X \models \varphi \text{ for all } (X,Y) \models \mathcal{S} \\ \Leftrightarrow\quad & X \models /\varphi \text{ for all } (X,Y) \models \mathcal{S} \\ \Leftrightarrow\quad & (X,Y) \models /\varphi \text{ for all } (X,Y) \models \mathcal{S} \\ \Leftrightarrow\quad & /\varphi \in \mathcal{S} \\ \Leftrightarrow\quad & \varphi \in \zeta_1(\mathcal{S}). \end{aligned}$$

Therefore, $\zeta_1(\mathcal{S}) = \mathsf{Log}\left(\{X \mid \exists Y : (X,Y) \text{ is an St-space and } (X,Y) \models \mathcal{S}\}\right)$. This shows that $\zeta_1(\mathcal{S})$ is a si logic, and so it is the intuitionistic fragment of $\mathcal{S}$. □

Conversely, for a si logic L, define:

$$\eta_1(\mathsf{L}) = \mathcal{S}_{\mathsf{IS4}} + \{/\varphi \mid \varphi \in \mathsf{L}\},$$
$$\eta_2(\mathsf{L}) = \mathcal{S}_{\mathsf{IS4}} + \{/t(\varphi) \mid \varphi \in \mathsf{L}\}.$$

6.6.10. LEMMA. *For every si logic L, we have:*

(1) $\eta_1(\mathsf{L}) = \mathsf{Con}\left(\{(X,Y) \mid X \textit{ is a rooted } \mathsf{L}\textit{-space}\}\right)$,

(2) $\eta_2(\mathsf{L}) = \mathsf{Con}\left(\{(X,Y) \mid Y \textit{ is a rooted } \mathsf{L}\textit{-space})\right\}$.

Proof:
We prove (2), the proof of (1) is similar. For an St-space (X,Y) we have $Y \models \mathsf{L}$ iff $(X,Y) \models \{t(\varphi) \mid \varphi \in \mathsf{L}\}$, which happens iff $(X,Y) \models \{/t(\varphi) \mid \varphi \in \mathsf{L}\}$. Thus, $\eta_2(\mathsf{L}) = \mathsf{Con}\left(\{(X,Y) \mid Y \text{ is an } \mathsf{L}\text{-space}\}\right)$. □

6.6.11. LEMMA. *Let L be a si logic.*

(1) $\mathsf{L} = \zeta_1\eta_1(\mathsf{L})$*, and* $\eta_1(\mathsf{L})$ *is the least multi-conclusion consequence relation in* $\zeta_1^{-1}(\mathsf{L})$*.*

(2) $\mathsf{L} = \zeta_2\eta_2(\mathsf{L})$*, and* $\eta_2(\mathsf{L})$ *is the least multi-conclusion consequence relation in* ${\zeta_2}^{-1}(\mathsf{L})$*.*

Proof:

(1). Let $\varphi \in \mathcal{L}_{\mathsf{IPC}}$. Then $\varphi \in \mathsf{L}$ implies $/\varphi \in \eta_1(\mathsf{L})$, which implies $\varphi \in \zeta_1\eta_1(\mathsf{L})$. Therefore, $\mathsf{L} \subseteq \rho_1\eta_1(\mathsf{L})$. If $\varphi \notin \mathsf{L}$, then there is a rooted L-space X such that $X \not\models \varphi$. Consider the St-space (X, X), where π is the identity map. Then $(X, X) \not\models /\varphi$, and $(X, X) \models \eta_1(\mathsf{L})$ by Lemma 6.6.10(1). Therefore, by Lemma 6.6.9(1), $\varphi \notin \zeta_1\eta_1(\mathsf{L})$. This shows that $\mathsf{L} = \zeta_1\eta_1(\mathsf{L})$. If $\mathcal{S} \in \zeta_1^{-1}(\mathsf{L})$, then for every $\varphi \in \mathcal{L}_{\mathsf{IPC}}$, we have $\varphi \in \mathsf{L}$ iff $/\varphi \in \mathcal{S}$. Thus, $\eta_1(\mathsf{L}) \subseteq \mathcal{S}$, and hence $\eta_1(\mathsf{L})$ is the least element of $\zeta_1^{-1}(\mathsf{L})$.

(1). Let $\varphi \in \mathcal{L}_{\mathsf{IPC}}$. Then $\varphi \in \mathsf{L}$ implies $/t(\varphi) \in \eta_2(\mathsf{L})$, which implies $\varphi \in \zeta_2\eta_2(\mathsf{L})$. Therefore, $\mathsf{L} \subseteq \zeta_2\eta_2(\mathsf{L})$. If $\varphi \notin \mathsf{L}$, then there is a rooted L-space X such that $X \not\models \varphi$. Then $(X, X) \not\models /t(\varphi)$, and (X, X) is a $\eta_2(\mathsf{L})$-space by Lemma 6.6.10(2). Thus, by Lemma 6.6.9(2), $\varphi \notin \zeta_2\eta_2(\mathsf{L})$. This shows that $\mathsf{L} = \zeta_2\eta_2(\mathsf{L})$. If $\mathcal{S} \in \zeta_2^{-1}(\mathsf{L})$, then for every $\varphi \in \mathcal{L}_{\mathsf{IPC}}$, we have $\varphi \in \mathsf{L}$ iff $/t(\varphi) \in \mathcal{S}$. Consequently, $\eta_2(\mathsf{L}) \subseteq \mathcal{S}$, and hence $\eta_2(\mathsf{L})$ is the least element of $\zeta_2^{-1}(\mathsf{L})$. □

As follows from Lemma 6.6.11, for a si logic L, both $\zeta_1^{-1}(\mathsf{L})$ and $\zeta_2^{-1}(\mathsf{L})$ have least elements, but they may not have largest elements. To see this we require the following lemma.

6.6.12. LEMMA. *Let* (X, Y) *be an St-space. Then* $(X, Y) \models /\Box p \leftrightarrow p$ *iff* π *is an isomorphism.*

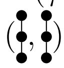
Proof:

Let $X = (X, \leq)$ and $Y = (Y, \leq)$. First suppose that π is an isomorphism. Then using that valuations are persistent and the semantics of Table 6.6.1, we see that $(X, Y) \models \Box p \leftrightarrow p$. Next suppose that π is not an isomorphism. Then there are $x \not\leq y$ with $\pi(x) \leq \pi(y)$. Let U be a clopen upset of X, with $x \in U$ but $y \notin U$. Define a valuation v on (X, Y) with $v(p) = U$. Then $x \models_v p$ but $x \not\models_v \Box p$. Thus, $(X, Y) \not\models \Box p \leftrightarrow p$. □

For $\psi \in \mathcal{L}_{\mathsf{IS4}}$, let ψ^- be the formula obtained from ψ by deleting all occurrences of $\Box$. Similarly to Lemmas 6.5.23 and 6.5.24, we can show that for every $\mathcal{S} \in \Sigma_{\mathcal{S}_{\mathsf{IS4}}}$, if $/\Box p \leftrightarrow p \in \mathcal{S}$, then $/\psi \in \mathcal{S}$ iff $/\psi^- \in \mathcal{S}$. From this we can infer that $\eta_1(\mathsf{L}) + /\Box p \leftrightarrow p$ is maximal in both $\zeta_1^{-1}(\mathsf{L})$ and $\zeta_2^{-1}(\mathsf{L})$. On the other hand, neither of $\zeta_1^{-1}(\mathsf{L})$ and $\zeta_2^{-1}(\mathsf{L})$ has to have a largest element, as the next example shows.

6.6.2. Example. Let γ abbreviate $(p \to q) \vee (q \to p)$ and let $\mathcal{S} = \eta_1(\mathsf{BD}_2) + /t(\gamma)$. By Lemma 6.6.6, an St-space (X, Y) is an $\mathcal{S}$-space iff X is a BD_2-space and Y is an LC-space.

(1) We show that $\zeta_1(\mathcal{S}) = \mathsf{BD}_2$. By Lemma 6.6.9(1), $\mathsf{BD}_2 \subseteq \zeta_1(\mathcal{S})$. Conversely, suppose $\varphi \notin \mathsf{BD}_2$. Then there is a finite rooted BD_2-space X refuting φ. Let $n = |X|$ and let Y be the n-chain. As we saw in the proof of Proposition 6.3.6(1), Y is a stable image of X. Therefore, (X, Y) is an $\mathcal{S}$-space refuting φ. Thus, $\zeta_1(\mathcal{S}) = \mathsf{BD}_2$. On the other hand, $\mathcal{S} \not\subseteq \eta_1(\mathsf{BD}_2) + /\Box p \leftrightarrow p$ because  validates $\eta_1(\mathsf{BD}_2) + /\Box p \leftrightarrow p$ but refutes $\mathcal{S}$. Consequently, $\zeta_1^{-1}(\mathsf{BD}_2)$ does not have a largest element.

(2) We show that $\zeta_2(\mathcal{S}) = \mathsf{LC}$. By Lemma 6.6.9(2), $\mathsf{LC} \subseteq \zeta_2(\mathcal{S})$. Conversely, suppose $\varphi \notin \mathsf{LC}$. Then there is a finite chain Y refuting φ. Let $n = |Y|$. As follows from the proof of Proposition 6.3.6(1), Y is a stable image of the $(n-1)$-fork X. Therefore, (X, Y) is an $\mathcal{S}$-space and $(X, Y) \not\models t(\varphi)$. Thus, $\varphi \notin \zeta_2(\mathcal{S})$. On the other hand, $\mathcal{S} \not\subseteq \eta_1(\mathsf{LC}) + \Box p \leftrightarrow p$ because  satisfies $\eta_1(\mathsf{LC}) + /\Box p \leftrightarrow p$ but refutes $\mathcal{S}$. Consequently, $\zeta_2^{-1}(\mathsf{LC})$ does not have a largest element.

6.6.4 Stable logics and stabilization via $\mathcal{S}_{\mathsf{IS4}}$

Finally, we use the embeddings η_1 and η_2 to provide characterizations of stable logics and of the downward stabilization.

6.6.3. Theorem. *For a si logic* L*, the following are equivalent.*

(1) L *is a stable logic.*

(2) $\eta_2(\mathsf{L}) \subseteq \eta_1(\mathsf{L})$.

(3) $\eta_2(\mathsf{L}) + \{/\varphi \mid \varphi \in \mathsf{L}\} = \eta_1(\mathsf{L})$.

(4) $\zeta_2\eta_1(\mathsf{L}) = \mathsf{L}$.

(5) For every $\varphi \in \mathcal{L}_{\mathsf{IPC}}$*, from* $/\varphi \in \eta_1(\mathsf{L})$ *it follows that* $/t(\varphi) \in \eta_1(\mathsf{L})$.

Proof:

(1)⇒(2). Suppose that $(X, Y) \models \eta_1(\mathsf{L})$. By Lemma 6.6.10(1), $X \models \mathsf{L}$. Since L is a stable logic, $Y \models \mathsf{L}$. Therefore, by Lemma 6.6.10(1), $(X, Y) \models \eta_2(\mathsf{L})$. Thus, $\eta_2(\mathsf{L}) \subseteq \eta_1(\mathsf{L})$.

(2)⇒(3). This is obvious.

(3)⇒(5). By Lemmas 6.6.11(1) and 6.6.9(2), $\mathsf{L} = \zeta_2\eta_2(\mathsf{L}) = \mathsf{Log}(\{Y \mid (X,Y) \models \eta_2(\mathsf{L})\})$ and $\zeta_2\eta_1(\mathsf{L}) = \mathsf{Log}(\{Y \mid (X,Y) \models \eta_1(\mathsf{L})\})$. Therefore, it is sufficient to show that $\{Y \mid (X,Y) \models \eta_2(\mathsf{L})\} = \{Y \mid (X,Y) \models \eta_1(\mathsf{L})\}$. The inclusion $\supseteq$ is immediate from (3). For the reverse inclusion, suppose that $(X,Y) \models \eta_2(\mathsf{L})$. By Lemma 6.6.10(2), $Y \models \mathsf{L}$, so $(Y,Y) \models \eta_1(\mathsf{L})$ by Lemma 6.6.10(1). Thus, $Y \in \{Y \mid (X,Y) \models \eta_1(\mathsf{L})\}$.

(5)⇒(6). Suppose that there is $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ such that $/\varphi \in \eta_1(\mathsf{L})$ but $/t(\varphi) \notin \eta_1(\mathsf{L})$. Then there is an St-space (X,Y) with $(X,Y) \models \eta_1(\mathsf{L})$ and $(X,Y) \not\models t(\varphi)$. By Lemma 6.6.9(2), $(X,Y) \models \eta_1(\mathsf{L})$ implies $Y \models \zeta_2\eta_1(\mathsf{L}) = \mathsf{L}$. Also, $(X,Y) \not\models t(\varphi)$ implies $Y \not\models \varphi$. Therefore, $\varphi \notin \mathsf{L}$, contradicting $/\varphi \in \eta_1(\mathsf{L})$.

(6)⇒(1). Suppose that X is a rooted L-space and Y is a stable image of X. Then (X,Y) is an St-space, and by Lemma 6.6.10(1), $(X,Y) \models \eta_1(\mathsf{L})$. By (6), $(X,Y) \models t(\varphi)$ for each $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ such that $/\varphi \in \eta_1(\mathsf{L})$. Therefore, $(X,Y) \models t(\varphi)$ for each $\varphi \in \mathsf{L}$. Thus, $Y \models \mathsf{L}$, and we conclude that L is a stable logic. □

6.6.4. THEOREM. *Let* L *be a si logic. Then* $\mathsf{Stab}_{\downarrow}(\mathsf{L}) = \zeta_2\eta_1(\mathsf{L})$.

Proof:
By Lemma 6.6.9(2),

$$\zeta_2\eta_1(\mathsf{L}) = \mathsf{Log}\left(\{Y \mid \exists X : (X,Y) \text{ is an St-space and } X \models \mathsf{L}\}\right).$$

Let

$$\begin{aligned}\mathcal{K} &= \{Y \mid \exists X : (X,Y) \text{ is an St-space and } X \models \mathsf{L}\},\\ \mathcal{K}' &= \{Y \mid Y \text{ is a stable image of a rooted } \mathsf{L}\text{-space } X\}.\end{aligned}$$

By Proposition 6.3.2(1), $\mathsf{Stab}_{\downarrow}(\mathsf{L}) = \mathsf{Log}(\mathcal{K}')$. Clearly $\mathcal{K} \subseteq K'$, so $\mathsf{Stab}_{\downarrow}(\mathsf{L}) = \mathsf{Log}(\mathcal{K}') \subseteq \mathsf{Log}(\mathcal{K}) = \zeta_2\eta_1(\mathsf{L})$. Suppose that $\varphi \notin \mathsf{Stab}_{\downarrow}(\mathsf{L})$. Then there is $Y \in \mathcal{K}'$ refuting φ. Therefore, there is an L-space X such that Y is a stable image X. Filtration then yields a finite stable image Y' of Y refuting φ. Since Y' is finite, (X,Y') is an St-space (because the topological condition of Definition 6.6.3 trivializes), so $Y' \in \mathcal{K}$. Thus, $\varphi \notin \zeta_2\eta_1(\mathsf{L})$. □

6.6.13. REMARK.

(1) Let L be a si logic and $\varphi \in \mathcal{L}_{\mathsf{IPC}}$. By Theorem 6.6.4, $\varphi \in \mathsf{Stab}_{\downarrow}(\mathsf{L})$ iff $t(\varphi) \in \mathcal{S}_{\mathsf{IS4}} + \{/\varphi \mid \varphi \in \mathsf{L}\}$. In particular, if $\mathcal{S}_{\mathsf{IS4}} + \{/\varphi \mid \varphi \in \mathsf{L}\}$ is decidable, then so is $\mathsf{Stab}_{\downarrow}(\mathsf{L})$.

(2) In contrast to Theorem 6.6.4, if L is consistent, then $\zeta_1\eta_2(\mathsf{L}) = \mathsf{IPC}$. Indeed, suppose X is a nonempty Esakia space. Let Y be the one-point frame. Then (X,Y) is an St-space. Since L is consistent, Y is an L-space, so $(X,Y) \models \eta_2(\mathsf{L})$ by Lemma 6.6.10(2), and hence $X \models \zeta_1\eta_2(\mathsf{L})$ by Lemma 6.6.9(1). Thus, $\zeta_1\eta_2(\mathsf{L}) = \mathsf{IPC}$.

6.7 Summary

We conclude by highlighting the parallels between stable and subframe si logics that we found in this chapter. First, we showed how to obtain a characterization of the upward subframization of a si logic via Zakharyaschev's canonical formulas, and similarly a characterization of the upward stabilization of a si logic via stable canonical formulas.

As shown in [24], subframes correspond to nuclei, that constitute the modal operators of the intuitionistic modal logic PLL. We obtained a characterization of the downward subframization and of subframe si logics using the interplay of two embeddings into extensions of PLL.

Similarly, stable images are related to the interior operator on Heyting algebras which constitute the modal operator of the intuitionistic modal logic IS4. We obtained a characterization of the downward stabilization and of stable si logics using the interplay of two embeddings into (rule-) extensions of $\mathcal{S}_{\mathsf{IS4}}$. We summarize these analogies in Table 6.7.1.

	Subframe si logics	**Stable si logics**
Corresponding modal operator	nuclei	interior
Characterization via Wolter's describable operations	Zakharyaschev's canonical formulas	Stable canonical formulas
Characterized by translations into	PLL via a version of the Gödel-Gentzen translation	IS4 via the Gödel-McKinsey-Tarski translation

Table 6.7.1: Connections between subframe and stable si logics

Chapter 7

NNIL-formulas revisited

7.1 Introduction

NNIL-formulas are formulas in the language of IPC defined by a syntactic condition: they allow **n**o **n**esting of **i**mplication to the **l**eft of implication. They were introduced in [124] and it was shown in [125] that NNIL-formulas are precisely the formulas whose validity is preserved under submodels. The main tools used in [125] are *subsimulations*. The latter can be regarded as the "stable version" of bisimulations, since a stable map is a total and functional subsimulation.

Validity of NNIL-formulas is also preserved by subframes as was observed in [29] (see also [130]), thus NNIL-formulas axiomatize subframe si logics. In fact, NNIL-formulas axiomatize precisely all subframe si logics. This was shown in [29] by defining subframe formulas in NNIL-form. These formulas are frame equivalent to the "usual" subframe formulas (see Section 3.4) but are in NNIL-form. Accordingly, si logics axiomatized by NNIL-formulas have "good properties" such as fmp, elementarity and canonicity (see Theorem 3.4.9).

In this chapter we have a fresh look at NNIL-formulas and via them at subframe si logics. Instead of working with subsimulations (as in [125]), our main tools for this study are stable maps, i.e. order-preserving maps, between Kripke models and the observation made in [125] and [34] that validity of NNIL-formulas is reflected by stable maps on Kripke models.

Our central result is a full description of the n-universal model $\mathcal{T}(n)$ for NNIL-formulas. This will complete the work started in [130] where the 2-universal model for NNIL-formulas was constructed. In fact, it turns out that $\mathcal{T}(n)$ is also the universal model for *stably reflective formulas* (SR-formulas), the class of formulas whose validity is reflected by stable maps. This implies that every stably reflective formula is equivalent to a NNIL-formula, a result[1] that was already proved in [125]. The facts that logics axiomatized by NNIL-formulas have the fmp and are

[1] In [125], stably reflective formulas are called $\preceq_1$-*robust*.

canonical will be simple consequences of our investigations. As logics axiomatized by NNIL-formulas correspond to subframe logics, these results are not new.

Thus, many of the results presented were already known, but our presentation provides an independent—and we believe at times simpler—treatment than in earlier works.

This chapter is largely based on [78]. The proof of the fmp (Theorem 7.6.8) has already appeared in [79].

Outline

The chapter is organized as follows: In the next section we recall the definition of NNIL-formulas and summarize some of their properties. Moreover, we recall some notions often used in the study of Kripke models such as colors of points and tree unravellings. In Section 7.3 we define subframe formulas for finite models in NNIL-form. In Section 7.4 we define the notion of a color-preserving submodel and show that every infinite model can be reduced to a finite color-preserving submodel. Section 7.5 constitutes the main part of this chapter, where we construct the n-universal models for NNIL (and SR). Finally, in Section 7.6 we show that all logics axiomatized by NNIL- or SR-formulas have the fmp and are canonical.

7.2 Preliminaries

In this chapter we often work with Kripke models as opposed to Kripke frames or Esakia spaces. We recall a few notions that we often use.

By an *n-model* we mean a model for a set of n propositional letters. When we speak of an n-model we always assume that the set of propositional letters $\{p_1, \ldots, p_n\}$ is fixed in advance. Also, by an *n-formula* we mean a formula in the language of IPC containing only propositional letters from the set $\{p_1, \ldots, p_n\}$. If x is an element of a model $\mathfrak{M}$, by $\mathfrak{M}_x$, we denote the submodel of $\mathfrak{M}$ generated by x. If x is an element in an n-model $\mathfrak{M}$, then $col(x)$—the *color of x*—is the n-sequence $(a_1, \ldots, a_n)$, where $a_i = 1$ iff $\mathfrak{M}, x \models p_i$ and $a_i = 0$ otherwise. By an *n-color* we simply mean a binary n-sequence. The set of n-colors can be ordered by the product order that we denote by $\leq$, i.e. for n-colors $c = (a_1, \ldots, a_n)$ and $c' = (a'_1, \ldots, a'_n)$ we have $c \leq c'$ iff $a_i \leq a'_i$ for all $1 \leq i \leq n$. We write $c < c'$ iff $c \leq c'$ and $c \neq c'$.

The reader may recall the notions of successor, immediate successor, and depth from Section 2.2.1. We say that a model $\mathfrak{M}$ is of *bounded depth* iff the underlying poset is of bounded depth, and we write $\mathrm{d}(\mathfrak{M})$ for the depth of the poset underlying $\mathfrak{M}$. A model $\mathfrak{M}$ is of *bounded branching* iff there is natural number k such that each point in $\mathfrak{M}$ has at most k immediate successors.

A rooted model $\mathfrak{M} = (W, \leq, v)$ is called a *tree* iff for all $w \in W$, $\downarrow w$ is a finite linear order. We will also use the following lemma that is standard.

7.2.1. LEMMA. *Let n and k be natural numbers. There are only finitely many non-isomorphic n-trees of branching $\leq n$ and depth $\leq k$.*

Proof:
It is easy to see that the 'full' tree T of branching n and depth $\leq k$ has $s := \Sigma_{0 \leq i \leq k} n^i$-many nodes. Now every tree of branching n and depth $\leq k$ is a subtree of T. Thus, disregarding the valuation, there at most 2^s-many non-isomorphic trees of branching $\leq n$ and depth $\leq k$. Now each of these trees admits only a finite number of possible valuations. □

We will often unravel models into trees: Recall that the *standard unraveling* of a rooted model $\mathfrak{M} = (W, \leq, v)$ with root r is a tree $\mathfrak{M}_{\mathsf{tree}} = (W_t, \leq_t, v_t)$ consisting of finite paths starting from the root, i.e. we have

- $W_t = \{\langle r, w_1, \ldots, w_k \rangle \mid w_1, \ldots, w_k \in W$ and $r \leq w_1 \cdots \leq w_k\}$,
- $\sigma \leq_t \tau$ iff σ is an initial segment of τ,
- $\langle r, w_1, \ldots, w_k \rangle \in v_t(p)$ iff $w_k \in v(p)$.

We identify the root r of $\mathfrak{M}$ and the root $\langle r \rangle$ of $\mathfrak{M}_{\mathsf{tree}}$, which we often denote simply as r.

Note that by reflexivity of the relation $\leq$, the model $\mathfrak{M}_{\mathsf{tree}}$ is always infinite even if $\mathfrak{M}$ is finite. Thus, if $\mathfrak{M}$ is finite, we will define its unraveling using paths along immediate successors instead. This will produce finite models whenever $\mathfrak{M}$ is finite. In this case, we write $\mathfrak{T}_{\mathfrak{M}}$ instead of $\mathfrak{M}_{\mathsf{tree}}$.

It is easy to see that $\mathfrak{M}_{\mathsf{tree}}$ and $\mathfrak{T}_{\mathfrak{M}}$ (for finite $\mathfrak{M}$) are trees. Moreover, the natural map α that sends a path σ in $\mathfrak{M}_{\mathsf{tree}}$ or $\mathfrak{T}_{\mathfrak{M}}$ to its final point $\alpha(\sigma)$ is a surjective p-morphism. Thus, for any $\varphi \in \mathcal{L}_{\mathsf{IPC}}$ and $\sigma \in \mathfrak{M}_{\mathsf{tree}}$,

$$\mathfrak{M}_{\mathsf{tree}}, \sigma \models \varphi \text{ iff } \mathfrak{M}, \alpha(\sigma) \models \varphi$$

and similarly for $\mathfrak{T}_{\mathfrak{M}}$.

As mentioned in the introduction, NNIL-formulas allow no nesting of implication to the left of implication. NNIL-formulas in *normal form* are defined by the following grammar:

$$\varphi ::= \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid p \to \varphi$$

As observed in [125], using the IPC-equivalences $((\varphi \vee \psi) \to \chi) \leftrightarrow ((\varphi \to \chi) \wedge (\psi \to \chi))$ and $((\varphi \wedge \psi) \to \chi) \leftrightarrow (\varphi \to (\psi \to \chi))$, every NNIL-formula is provably equivalent to a NNIL-formula in normal form. Thus, we will usually assume that a NNIL-formula is given in normal form.

Moreover, NNIL-formulas are *locally finite* in the sense that for every natural number n, there are up to provable equivalence only finitely many n-formulas in NNIL ([125, Theorem 2.2], see also [130, Theorem 6.1.7]).

7.2.2. Lemma ([125, 34]). *NNIL-formulas are* backwards preserved by stable maps*, i.e., for $\varphi \in$ NNIL, and two models $\mathfrak{M} = (W, \leq, v)$ and $\mathfrak{N} = (W', \leq', v')$, and a stable map $f : W' \to W$, we have that for any $w \in W'$,*

$$\mathfrak{M}, f(w) \models \varphi \quad \textit{implies} \quad \mathfrak{N}, w \models \varphi.$$

The above gives rise to the class SR (short for *stably reflective*) of formulas that are backwards preserved, i.e. reflected, by stable maps. To be precise, SR is the class of formulas that satisfy the property of Lemma 7.2.2. We then obviously have

$$\mathsf{NNIL} \subseteq \mathsf{SR}. \tag{7.1}$$

7.2.3. Remark. The "converse" of the above statement is also true, i.e. every formula that is stably reflective is equivalent to a NNIL-formula as follows from Theorems 7.4.1 and 6.11 in [125]. We provide an alternative proof of this fact in Corollary 7.6.1.

7.3 NNIL-subframe formulas on trees

We observe that the definition of NNIL-subframe formulas from [29] also makes sense for arbitrary finite rooted models (not only for finite rooted frames). These formulas allow a simple refutation criterion on models via stable maps. The criterion can also be extended to frames as we will show in Section 7.6. As a simple consequence we obtain that NNIL- and SR-formulas distinguish the same finite pointed models (in the sense of Definition 7.3.4).

7.3.1. Definition. Let $\mathfrak{N} = (W, \leq, v)$ be a finite n-model. For $w \in W$ let $prop(w) = \{p \mid \mathfrak{N}, w \models p\}$ and $notprop(w) = \{p \mid \mathfrak{N}, w \not\models p\}$. Define a NNIL-formula $\beta(w)$ by induction on the depth of w: for a maximal point w let

$$\beta(w) = \bigwedge prop(w) \to \bigvee notprop(w),$$

and if $w \in W$ is not maximal, and $w_1, \ldots, w_m$ are the immediate successors of w, let

$$\beta(w) = \bigwedge prop(w) \to \bigvee notprop(w) \vee \bigvee_{i=1}^{m} \beta(w_i).$$

If $\mathfrak{N}$ is rooted with root r, then define $\beta(\mathfrak{N}) = \beta(r)$.

Let B denote the collection of all NNIL-subframe formulas of finite models, i.e.

$$\mathsf{B} = \{\beta(w) \mid w \text{ is a node in some finite model}\}.$$

7.3.2. Lemma. *For any finite model* $\mathfrak{N}$ *and* $w \in \mathfrak{N}$*, we have* $\mathfrak{N}, w \not\models \beta(w)$*.*

Proof:
We prove the lemma by induction on $\mathrm{d}(w)$. If $\mathrm{d}(w) = 1$, clearly,

$$\mathfrak{N}, w \models \bigwedge prop(w) \text{ and } \mathfrak{N}, w \not\models \bigvee notprop(w), \tag{7.2}$$

which gives $\mathfrak{N}, w \not\models \beta(w)$. Suppose $\mathrm{d}(w) = k + 1$ and the lemma holds for all points with depth k. Assume that $w_1, \ldots, w_m$ are immediate successors of w. By induction hypothesis, we have $\mathfrak{N}, w_i \not\models \beta(w_i)$ for all $1 \leq i \leq m$. Thus, we obtain $\mathfrak{N}, w \not\models \bigvee_{i=1}^{m} \beta(w_i)$ by persistence. Since (7.2) also holds for w in this case, we conclude $\mathfrak{N}, w \not\models \beta(w)$. □

7.3.3. Lemma. *Let* $\mathfrak{N}$ *be a finite rooted model, let* $\mathfrak{M}$ *be an arbitrary model, and let* $x \in \mathfrak{M}$*. If* $\mathfrak{M}, x \not\models \beta(\mathfrak{N})$ *then there exists a stable map* f *from* $\mathfrak{T}_{\mathfrak{N}}$ *into* $\mathfrak{M}_x$*.*

Proof:
It is not hard to see that $\beta(\mathfrak{T}_{\mathfrak{N}}) = \beta(\mathfrak{N})$ since nodes in $\mathfrak{N}$ and $\mathfrak{T}_{\mathfrak{N}}$ have essentially the same immediate successors. Let r be the root of $\mathfrak{T}_{\mathfrak{N}}$. The function f is defined stepwise upwards from the root r in such a way that for every w in $\mathfrak{T}_{\mathfrak{N}}$, $f(w)$ has the color of w and $\mathfrak{M}, f(w) \not\models \beta(w_i)$ for all immediate successors w_i of w. Suppose f has been defined already for some w and suppose u is an immediate successor of w. By the fact that $\mathfrak{M}, f(w) \not\models \beta(u)$ and the form of $\beta(u)$ it is clear that $f(w)$ has a successor u' that has the same color as u and $\mathfrak{M}, u' \not\models \beta(u_i)$ for all immediate successors u_i of u. Define $f(u) = u'$. Note that the unraveling makes sure that we never assign different values to the same node since we can arrive there from the root in only one way. □

Obviously, the collection B of NNIL-subframe formulas, is contained in NNIL and recall that by (7.1) all NNIL-formulas are in SR. The results above imply that all these formula classes distinguish the same finite pointed models in the following sense.

7.3.4. Definition. Let Φ be a set of formulas and let $\mathfrak{M}$ and $\mathfrak{N}$ be models, $w \in \mathfrak{M}$ and $u \in \mathfrak{N}$. Then $(\mathfrak{M}, w)$ and $(\mathfrak{N}, u)$ are called Φ-*equivalent* (or Φ-*indistinguishable*), written $(\mathfrak{M}, w) \simeq_\Phi (\mathfrak{N}, u)$, iff for each $\varphi \in \Phi$,

$$\mathfrak{M}, w \models \varphi \iff \mathfrak{N}, u \models \varphi.$$

We say that Φ *distinguishes* $(\mathfrak{M}, w)$ and $(\mathfrak{N}, u)$ iff $(\mathfrak{M}, w)$ and $(\mathfrak{N}, u)$ are not Φ-equivalent.

7.3.5. Proposition. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be finite* n*-models and let* $w \in \mathfrak{M}$ *and* $u \in \mathfrak{N}$*. The following are equivalent:*

(1) $(\mathfrak{M}, w) \simeq_{\mathsf{B}} (\mathfrak{N}, u)$.

(2) $(\mathfrak{M}, w) \simeq_{\mathsf{NNIL}} (\mathfrak{N}, u)$.

(3) $(\mathfrak{M}, w) \simeq_{\mathsf{SR}} (\mathfrak{N}, u)$.

Thus, B, NNIL, and SR distinguish the same pointed models.

Proof:
The implications (3) ⇒ (2) ⇒ (1) are obvious since $\mathsf{B} \subseteq \mathsf{NNIL} \subseteq \mathsf{SR}$. We show that (1) implies (3). Assume that $(\mathfrak{M}, w) \simeq_{\mathsf{B}} (\mathfrak{N}, u)$. Let $\varphi \in \mathsf{SR}$ be such that $\mathfrak{M}, w \models \varphi$. We show that also $\mathfrak{N}, u \models \varphi$. By Lemma 7.3.2, $\mathfrak{N}, u \not\models \beta(u)$, which by B-equivalence implies $\mathfrak{M}, w \not\models \beta(u)$. Now, by Lemma 7.3.3, there is a stable map f from $\mathfrak{T}_{\mathfrak{N}_u}$ into $\mathfrak{M}_w$. By assumption, $\mathfrak{M}_w, f(u) \models \varphi$, so $\mathfrak{T}_{\mathfrak{N}_u}, u \models \varphi$ since $\varphi \in \mathsf{SR}$. This implies that $\mathfrak{N}, u \models \varphi$. The other implication follows from symmetry. □

In Corollary 7.6.1, we show that the statement of the above proposition also holds for infinite models. As explained at the end of the previous section, this fact already follows from the results in [125].

7.4 Finite color-preserving submodels

In this section we provide a way to reduce an infinite model to a finite one via a version of the selective filtration method. In case the infinite model is a tree, the finite (reduced) model is SR- and NNIL-indistinguishable from the infinite model.

7.4.1. Definition. A submodel $\mathfrak{M}'$ of a model $\mathfrak{M}$ is called *color-preserving* if, for any $x \in \mathfrak{M}'$, any $y \in \mathfrak{M}$, $x \leq y$ implies that there exists $y' \in \mathfrak{M}'$ such that $x \leq y'$ and $col(y') = col(y)$.

The next lemma shows that color-preserving submodels are *preserved under images of p-morphisms* in the sense that the image of a color-preserving submodel in the domain of a p-morphism f leads a color-preserving submodel in the co-domain of f. Moreover, being a color-preserving submodel is a transitive relation.

7.4.2. Lemma. *Let $\mathfrak{M}$ be a model.*

(1) Let $f : \mathfrak{M} \to \mathfrak{N}$ be a p-morphism onto a model $\mathfrak{N}$. If $\mathfrak{M}'$ is a color-preserving submodel of $\mathfrak{M}$, then the image $f[\mathfrak{M}']$ of $\mathfrak{M}'$ under f is a color-preserving submodel of $\mathfrak{N}$.

(2) If $\mathfrak{M}'$ is a color-preserving submodel of $\mathfrak{M}$ and $\mathfrak{M}''$ is a color-preserving submodel of $\mathfrak{M}'$, then $\mathfrak{M}''$ is a color-preserving submodel of $\mathfrak{M}$.

Proof:

(1) Let $x \in f[\mathfrak{M}']$ and $x \leq y$ for some $y \in \mathfrak{N}$. Let $u \in \mathfrak{M}'$ with $f(u) = x$. Since f is a p-morphism, there is $u \leq w \in \mathfrak{M}$ with $f(w) = y$. Since $\mathfrak{M}'$ is color-preserving, there is $u \leq w' \in \mathfrak{M}'$ with $col(w) = col(w')$. Then $f(u) \leq f(w') \in f[\mathfrak{M}']$ and clearly $col(f(w')) = col(y)$.

(2) Suppose $x \in \mathfrak{M}''$ and $x \leq y$ for some $y \in \mathfrak{M}$. Then $x \in \mathfrak{M}'$ and since $\mathfrak{M}'$ is a color-preserving submodel of $\mathfrak{M}$, there is $y' \in \mathfrak{M}'$ with $x \leq y'$ and $col(y') = col(y)$. Thus, since $\mathfrak{M}''$ is a color-preserving submodel of $\mathfrak{M}'$ there is $y'' \in \mathfrak{M}''$ with $x \leq y''$ and $col(y'') = col(y)$. □

7.4.3. LEMMA. *Every n-tree $\mathfrak{M}$ has a color-preserving subtree $\mathfrak{N}$ of bounded depth with the same root.*

Proof:
Let $\mathfrak{M} = (W, \leq, v)$ and let r be the root of $\mathfrak{M}$. Let $\mathfrak{M}'$ be the submodel of $\mathfrak{M}$ on the set

$$W' = \{r\} \cup \{w \in W \mid col(u) < col(w) \text{ for the immediate predecessor } u \text{ of } w\}.$$

Since $\mathfrak{M}$ is a tree, $\downarrow x$ is finite for each $x \in \mathfrak{M}$, thus every point has a predecessor, and so W' is well defined. The model $\mathfrak{M}'$ has bounded depth since all chains in $\mathfrak{M}'$ are strictly increasing in color and there are only finitely many colors. It remains to check that $\mathfrak{M}$ is a color-preserving submodel of $\mathfrak{M}$. Let $x \in W'$, any $y \in W$ with $x \leq y$. Since $\downarrow y$ is a finite, there exists a predecessor y' of y in W' with $col(y) = col(y')$. Since $\downarrow y$ is linear, we obtain $x \leq y'$. □

7.4.4. LEMMA. *Every n-tree $\mathfrak{M}$ of bounded depth has a finite color-preserving subtree $\mathfrak{N}$ with the same root.*

Proof:
Assume that $\mathfrak{M}$ is of depth k. Putting $\mathfrak{N}_0 = \mathfrak{M}$, we inductively select a sequence of submodels $\mathfrak{N}_k \subseteq \mathfrak{N}_{k-1} \subseteq \cdots \subseteq \mathfrak{N}_1 \subseteq \mathfrak{N}_0$ and a sequence $n_k \geq n_{k-1} \geq \cdots \geq n_0$ of natural numbers satisfying the following two conditions:

(1) The model $\mathfrak{N}_i$ is a color-preserving submodel of $\mathfrak{N}_{i-1}$ and is obtained by deleting points from $\mathfrak{N}_{i-1}$ that have depth at most $i-1$ in $\mathfrak{M}$, and

(2) every point in $\mathfrak{N}_i$ of depth $\leq i$ has at most n_i-many immediate successors.

Thus, by Lemma 7.4.2(2), $\mathfrak{N}_k$ will be a color-preserving submodel of $\mathfrak{M}$. Moreover, every point in $\mathfrak{N}_k$ has at most n_k immediate successors and since $\mathfrak{N}_k$ has depth k, the model $\mathfrak{N}_k$ is finite.

Suppose $\mathfrak{N}_{i-1}$ and n_{i-1} have already been constructed satisfying (1) and (2) above. By Lemma 7.2.1, there is a number $n_i \geq n_{i-1}$ binding the number of non-isomorphic trees of branching at most n_{i-1} and depth i.

Consider the points of depth $\mathfrak{N}_{i-1}$ that have depth $i+1$ in $\mathfrak{M}$. For each such point w, consider all the subtrees generated by the immediate successors of w in $\mathfrak{N}_{i-1}$. All these trees have branching at most n_{i-1} and depth $\leq i$. Thus, there are at most n_i-many non-isomorphic such trees. For each isomorphism type we keep exactly one tree in $\mathfrak{N}_i$ and delete all the others.

By construction it is clear that all points of depth $\leq i$ in $\mathfrak{N}_i$ have at most n_i-many successors, thus (2) from above is satisfied. It is also clear that we obtained $\mathfrak{N}_i$ from $\mathfrak{N}_{i-1}$ by deleting points from $\mathfrak{N}_{i-1}$ that have depth $\leq i$ in $\mathfrak{M}$. To see that $\mathfrak{N}_i$ is a color-preserving submodel of $\mathfrak{N}_{i-1}$, suppose that $w \in \mathfrak{N}_i$, $u \in \mathfrak{N}_{i-1}$ and $w \leq u$. If $u \in \mathfrak{N}_i$, then we are done. Otherwise, u is in a subtree T generated in $\mathfrak{N}_{i-1}$ by an immediate successor of w that was deleted by building $\mathfrak{N}_i$. By the construction, there remains an isomorphic copy of T in $\mathfrak{N}_i$ above w and the point corresponding to u in this isomorphic copy will have the same color as u. Thus, $\mathfrak{N}_i$ is a color-preserving submodel of $\mathfrak{N}_{i-1}$. This shows that also (1) is satisfied for $\mathfrak{N}_i$. □

7.4.5. THEOREM. *Every rooted n-model $\mathfrak{M}$ has a finite color-preserving submodel $\mathfrak{N}$ with the same root r. Moreover, if $\mathfrak{M}$ is a tree then $\mathfrak{N}$ is also a stable image of $\mathfrak{M}$ and so $(\mathfrak{M}, r) \simeq_{\mathsf{SR}} (\mathfrak{N}, r)$ and $(\mathfrak{M}, r) \simeq_{\mathsf{NNIL}} (\mathfrak{N}, r)$.*

Proof:
We construct $\mathfrak{N}$ in stages. First unravel $\mathfrak{M}$ to obtain a tree $\mathfrak{M}_{\mathsf{tree}}$ with the same root. Second, apply Lemma 7.4.3 to $\mathfrak{M}_{\mathsf{tree}}$ to obtain a color-preserving submodel $\mathfrak{M}_1$ of bounded depth with the same root. The model $\mathfrak{M}_1$ is obviously a tree. Lemma 7.4.4 provides a finite color-preserving subtree $\mathfrak{M}_2$ of $\mathfrak{M}_1$ with the same root. By Lemma 7.4.2(2), $\mathfrak{M}_2$ is a color-preserving submodel of $\mathfrak{M}_{\mathsf{tree}}$.

Let α be the natural p-morphism from $\mathfrak{M}_{\mathsf{tree}}$ onto $\mathfrak{M}$, and let $\alpha[\mathfrak{M}_2] = \mathfrak{N}$ be the image of $\mathfrak{M}_2$ under α. By Lemma 7.4.2, the model $\mathfrak{N}$ is a color-preserving submodel of $\mathfrak{M}$. Moreover, since $\mathfrak{M}_2$ is finite, so is $\mathfrak{N}$. Finally, since α maps the root of $\mathfrak{M}_{\mathsf{tree}}$ to the root of $\mathfrak{M}$, the models $\mathfrak{M}$ and $\mathfrak{N}$ have the same root.

Now suppose in addition that $\mathfrak{M}$ is a tree. Then $\mathfrak{N}$ can be obtained by subsequently applying Lemmas 7.4.3 and 7.4.4. We show that $\mathfrak{N}$ is also a stable image of $\mathfrak{M}$. Let $\mathfrak{M}'$ be obtained from $\mathfrak{M}$ as in Lemma 7.4.3. Then the map

$$f(w) = \min\{u \leq w \mid col(u) = col(w)\}$$

is easily seen to be a stable map from $\mathfrak{M}$ onto $\mathfrak{M}'$. Now let $k = d(\mathfrak{M}')$ and let $\mathfrak{N} = \mathfrak{N}_k \subseteq \cdots \subseteq N_0 = \mathfrak{M}'$ be the sequence of models constructed in the proof of Lemma 7.4.4. For $1 \leq i \leq k$ define a family of maps $g_i : \mathfrak{N}_{i-1} \to \mathfrak{N}_i$ as follows: the map g_i sends a subtree that is removed in the construction to its isomorphic copy that is kept in $\mathfrak{N}_i$. Then g_i is obviously stable, in fact it is a p-morphism. The composition g of the g_is – note that these are only finitely many – gives a stable map from $\mathfrak{M}'$ onto $\mathfrak{N}$. Then $g \circ f$ is a stable map from $\mathfrak{M}$ onto $\mathfrak{N}$. □

7.5 Universal models

We construct n-universal models for NNIL- and SR-formulas. The results in this section are a continuation of [130], where the 2-universal model for NNIL was constructed (although slightly differently than here, namely via subsimulations as opposed to stable maps). Roughly, an n-universal model is the "minimal" model that contains witnesses for all "non-implications", i.e. for all pairs of n-formulas φ and ψ with $\varphi \not\vdash \psi$, the n-universal model contains a point that satisfies φ but refutes ψ. In detail:

7.5.1. Definition. We say that an n-model $\mathfrak{M} = (W, \leq, v)$ is an *n-universal model* for a class Ψ of formulas iff the following conditions are satisfied:

(1) For any n-formulas $\varphi, \psi \in \Psi$, if $\varphi \nvdash \psi$, then there is $w \in \mathfrak{M}$ with $w \models \varphi$ and $w \not\models \psi$.

(2) For each finite point-generated upset U of $\mathfrak{M}$ there is an n-formula $\psi \in \Psi$ such that $v(\psi) = U$.

If, in addition, for all upsets U of an n-universal model $\mathfrak{M}$ of Ψ, there is an n-formula $\psi \in \Psi$ such that $v(\psi) = U$, we call $\mathfrak{M}$ an *exact model* of Ψ.

Thus, condition (1) in the n-universal model makes sure that "there are enough counterexamples" and (2) ensures that "there are no superfluous" points. Note that (2) ensures that no distinct worlds in the n-universal model for Ψ are Ψ-equivalent.

Results on the n-universal models for IPC can be found in [118, 7, 71, 115, 86], see e.g. [29, Chapter 3] for an overview. Results on n-universal models for fragments of IPC have also been investigated by a number of authors, see [74, 91, 30, 35].

On the first sight, our construction of the n-universal model for NNIL may appear a bit "non-standard", as we will define it as a collection of models, i.e. the worlds in our n-universal model are models themselves. These models are then ordered via *stable maps*. On the other hand, also the n-universal model for IPC can be regarded as a collection of models–namely the collection of its point-generated upsets—that are ordered by the generated submodel relation, i.e. via *injective p-morphisms* (which was in fact also the presentation of [85]).

The elements of the n-universal model $\mathcal{T}(n)$ for NNIL will be finite rooted trees. If T and T' are finite rooted trees, we write

$$T \leq T' \text{ iff there is stable map } f : T' \to T$$

and we write $T \equiv T'$ iff $T \leq T'$ and $T' \leq T$. It is clear that the relation $\leq$ is a preorder. Before we give the definition of $\mathcal{T}(n)$, we collect a few obvious facts about the relation $\leq$:

7.5.2. Lemma. *Let T and T' be finite n-trees with roots r and r', respectively.*

(1) If $T \equiv T'$, then $(T, r) \simeq_{\mathsf{SR}} (T', r')$ and $(T, r) \simeq_{\mathsf{NNIL}} (T', r')$.

(2) If $T' \not\leq T$ then $T \models \beta(T')$.

7.5.3. Definition. The model $\mathcal{T}(n) = (W, \leq v)$ is defined as follows:

- The domain of $\mathcal{T}(n)$ is inductively defined in layers:

  The first layer consists of the 2^n distinct n-colors.

  Assume that the l-th layers for $l \leq m$ have been defined already. We define the $(m+1)$-th layer as follows:

  Let $\mathcal{X} = \{T_1, \ldots, T_k\}$ be a set of $\leq$-incomparable trees from the layers $\leq m$ containing at least one member of layer m and let c be a color strictly smaller than all the color occurring in the trees of $\mathcal{X}$. Build the tree $T_{\mathcal{X},c}$ by taking the disjoint union of the members of $\mathcal{X}$ and adding a fresh root of color c. Add $T_{\mathcal{X},c}$ to $\mathcal{T}(n)$.

- Order $\mathcal{T}(n)$ by the $\leq$ relation.

- For each propositional letter p, set $v(p) = \{T \in \mathcal{T}(n) \mid T \models p\}$.

The valuation v on $\mathcal{T}(n)$ is easily seen to be persistent and (2) of Lemma 7.5.4 implies that the relation $\leq$ is a partial order on $\mathcal{T}(n)$, thus, $\mathcal{T}(n)$ is an intuitionistic Kripke model. Note that the m-th layer of $\mathcal{T}(n)$ contains trees of depth m. However, the depth of a tree regarded as a point in $\mathcal{T}(n)$ is often larger. It is also easy to see that $\mathcal{T}(n)$ is finite: indeed, because the color c in some $T_{\mathcal{X},c}$ is strictly smaller than all the color occurring in the trees of the corresponding set $\mathcal{X}$, every tree in $\mathcal{T}(n)$ has at most depth $n+1$. Thus, the construction of $\mathcal{T}(n)$ terminates after $n+1$ rounds, i.e. $\mathcal{T}(n)$ is constructed in $n+1$-many layers and it is clear that each layer is finite.

Figure 7.5.1 shows $\mathcal{T}(2)$, which has three layers. The dashed lines indicate the order in $\mathcal{T}(n)$ and the solid lines indicate the order in the respective trees.

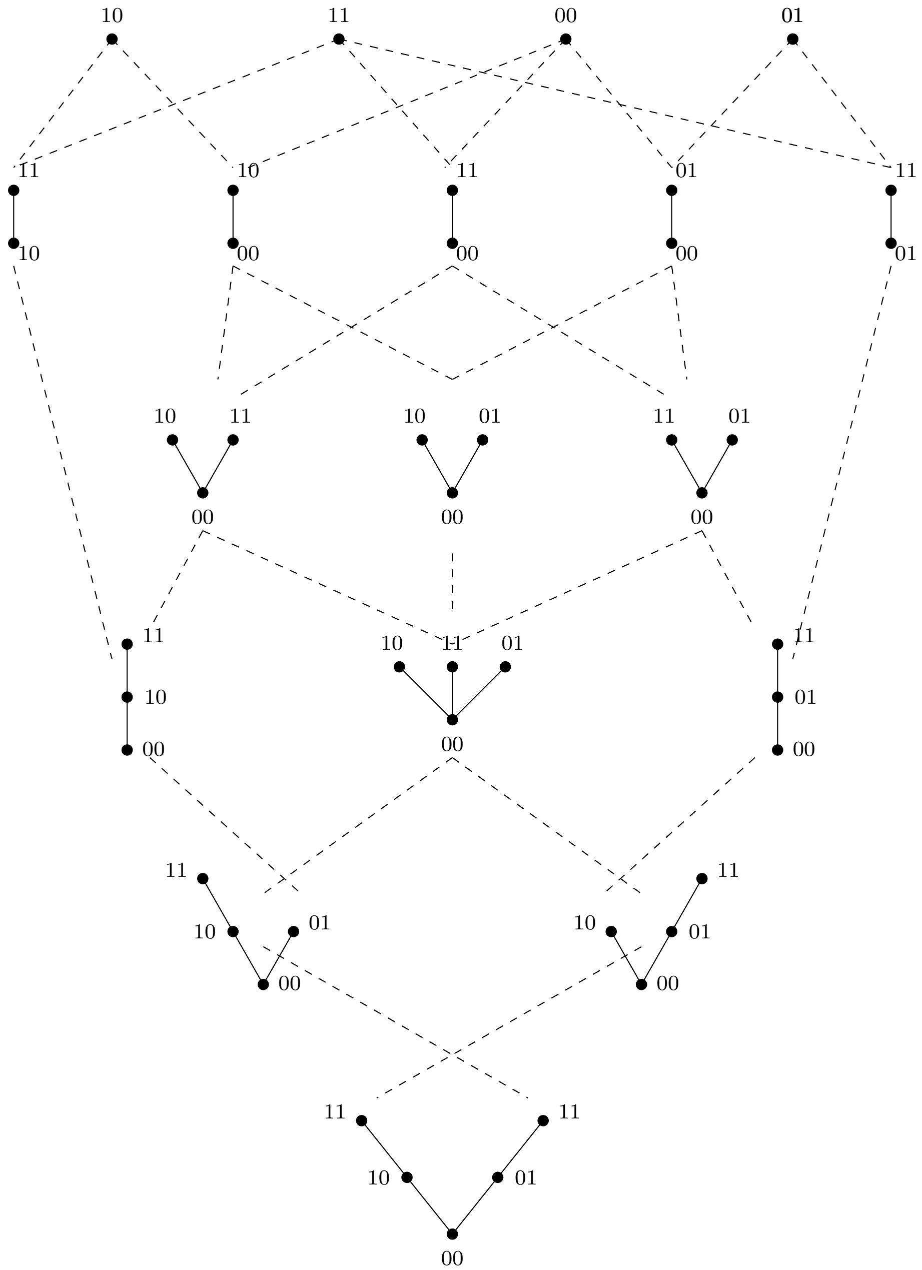


Figure 7.5.1: $\mathcal{T}(2)$

7.5.4. Lemma. *Let $T, T' \in \mathcal{T}(n)$.*

(1) If $f : T \to T$ is stable, then f is the identity map on T.

(2) If $T \neq T'$, then $T \not\equiv T'$.

(3) If $T \in \mathcal{T}(n)$, then $T_x \in \mathcal{T}(n)$ for every $x \in T$.

Proof:

(1) We show this by induction on $\mathrm{d}(T)$, the depth of the tree T. If $\mathrm{d}(T) = 1$ the claim is obvious. So let $\mathrm{d}(T) = l$ and suppose we have shown (1) for all $T' \in \mathcal{T}(n)$ with $\mathrm{d}(T') < l$. Let $f : T \to T$ be stable. Suppose T is constructed from the set $\mathcal{X} = \{T_1, \ldots T_k\}$ by adding the root r. Let $1 \leq i \leq k$ and let r_i be the root of T_i. Then $f(r_i) \in T_j$ for some $1 \leq j \leq k$ since by construction of $\mathcal{T}(n)$ no element in $\mathcal{X}$ has a node of the color of r. Then $f[T_i] \subseteq T_j$ since f is stable. Thus, the restriction $f\restriction_{T_i}$ of f to T_i is a stable map from T_i to T_j. Since the elements of $\mathcal{X}$ are pairwise incomparable, this implies that $T_i = T_j$, so $f\restriction_{T_i} : T_i \to T_i$ is a stable map. Since $\mathrm{d}(T_i) < l$, $f\restriction_{T_i}$ is the identity map on T_i. We have thus shown that f restricted to all elements in $\mathcal{X}$ is the identity. Since no element of T has the same color as r, $f(r) = r$. This finishes the proof of (1).

(2) This is a simple consequence of (1). We show the contrapositive of the claim. Suppose $T \equiv T'$, i.e. there are stable maps $f : T \to T'$ and $g : T' \to T$. Then $g \circ f : T \to T$ and $f \circ g : T' \to T'$ are stable and thus the identity on T, respectively, T' by (1). So g is a bijective stable map with a stable inverse. It is a well known property of partial orders that in this case T is isomorphic to T', so $T = T'$.

(3) This follows by construction. □

Since the points in $\mathcal{T}(n)$ are models themselves, a formula φ can be evaluated at a point T of $\mathcal{T}(n)$ in two ways: Either φ can be evaluated at the root r of the model T, or φ can be evaluated in $\mathcal{T}(n)$ at point T. The next lemma shows that the resulting truth value of NNIL- (and even SR-) formulas coincides for both ways of evaluating. In other words, $(\mathcal{T}(n), T)$ and (T, r) are NNIL- and even SR-equivalent.

7.5.5. LEMMA. *Let $T \in \mathcal{T}(n)$ and let r be the root of T. Then*

(1) $(T, r) \simeq_{\mathsf{NNIL}} (\mathcal{T}(n), T)$,

(2) $(T, r) \simeq_{\mathsf{SR}} (\mathcal{T}(n), T)$.

Proof:

(1) Let φ be a NNIL-formula. The proof is by induction on $c(\varphi)$, where $c(\varphi)$ is the number of symbols of φ.

If $\mathrm{d}(T) = 1$, the claim is trivial and if $c(\varphi) = 1$, the claim follows by definition of the valuation. So suppose $c(\varphi) > 1$. The inductions steps for $\wedge$ and $\vee$ are easy. So suppose $\varphi = p \to \psi$.

First suppose that $T, r \models \varphi$ for some $T \in \mathcal{T}(n)$. We show that $\mathcal{T}(n), T \models \varphi$. To this end, suppose that $T \leq T'$ in $\mathcal{T}(n)$ with $\mathcal{T}(n), T' \models p$. We aim to show that $\mathcal{T}(n), T' \models \psi$. Since $T \leq T'$, there is a stable map $f : T' \to T$. L. Since φ is stably reflective, $T', r' \models p \to \psi$, where r' is the root of T'. Moreover, $T', r' \models p$ by definition of the valuation on $\mathcal{T}(n)$, thus, $T', r' \models \psi$. Since $c(\psi) < c(\varphi)$, we can apply the induction hypothesis to get $\mathcal{T}(n), T' \models \psi$. This shows that $\mathcal{T}(n), T \models \varphi$.

Conversely, suppose that $\mathcal{T}(n), T \models \varphi$. To see that $T, r \models \varphi$, let $x \in T$ and suppose that $T, x \models p$. Then $T_x \in \mathcal{T}(n)$ by Lemma 7.5.4(3). Moreover, $T \leq T_x$ since the identity on T_x is a stable map from T_x into T. Since $T, x \models p$, also $T_x, x \models p$ and so $\mathcal{T}(n), T_x \models p$. Since $T \leq T_x$ it follows that $\mathcal{T}(n), T_x \models \psi$ and so $T_x, x \models \psi$ by induction hypothesis. Thus, $T, x \models \psi$ and so $T, r \models \varphi$.

(2) This follows from the fact that T and $\mathcal{T}(n)$ are finite, using item (1) and Proposition 7.3.5.

□

Our next goal is to show that every finite rooted n-tree is SR-equivalent to a point in $\mathcal{T}(n)$. To this end, we first show a lemma:

7.5.6. Lemma. *For each finite rooted n-tree $\mathfrak{T}$, there exists T in $\mathcal{T}(n)$ such that*

(1) there is a stable map from $\mathfrak{T}$ onto T, and

(2) T is isomorphic to a submodel of $\mathfrak{T}$ that has the same root as $\mathfrak{T}$.

Proof:
We prove this by induction on $\mathrm{d}(\mathfrak{T})$. The case $\mathrm{d}(\mathfrak{T}) = 1$ is trivial. So, assume $\mathrm{d}(\mathfrak{T}) > 1$ and the root of $\mathfrak{T}$ is r. Let $w_1, \ldots, w_h$ be the minimal elements in $\mathfrak{T}$ with a color different from $col(r)$. The induction hypothesis applies to the trees $\mathfrak{T}_1, \ldots, \mathfrak{T}_h$ generated by $w_1, \ldots, w_h$. So, there are $\mathcal{T}(n)$-members $T_1, \ldots, T_h$ and stable maps $f_1, \ldots, f_h$ with the properties (1) and (2). Assume without loss of generality that $\mathcal{X} = \{T_1, \ldots, T_k\}$ are the minimal elements among $T_1, \ldots, T_h$ with respect to $\leq$ (and are therefore incomparable). Then for each $k+1 \leq i \leq h$, there is $m(i) \leq k$ such that $T_{m(i)} \leq T_i$ and thus for each such i there is a stable map $g_i : T_i \to T_{m(i)}$.

Let T be the tree that is formed by taking the disjoint union of the trees in $\mathcal{X}$ and adding a fresh root r' of color $col(r)$ below. We show that T serves our purpose.

First note that T is a member of $\mathcal{T}(n)$. This is because the elements of $\mathcal{X}$ are $\leq$-incomparable and $col(r')$ does not occur in them. Next we define a map $f \colon \mathfrak{T} \to T$ as follows: for each $w \in \mathfrak{T}$,

$$f(w) = \begin{cases} r' & \text{if } col(w) = col(r), \\ f_i(w) & \text{if } w \in \mathfrak{T}_i \text{ for some } i \leq k, \\ g_{m(i)} \circ f_i(w) & \text{if } w \in \mathfrak{T}_i \text{ for some } k+1 \leq i. \end{cases}$$

It is easy to see that f is stable. Since $f_1, \dots, f_k$ map $\mathfrak{T}_1, \dots \mathfrak{T}_k$ onto $T_1, \dots, T_k$ and $f(r) = r'$, the map f is surjective. Finally, since by induction hypothesis, T_i is isomorphic to a submodel of $\mathfrak{T}_i$, for each for each $1 \leq i \leq k$, the tree T is isomorphic to a submodel of $\mathfrak{T}$. □

7.5.7. Theorem. *For every finite rooted n-tree $\mathfrak{T}$, there is a unique $T \in \mathcal{T}(n)$ with $\mathfrak{T} \equiv T$. In particular, $\mathfrak{T} \simeq_{\mathsf{SR}} (\mathcal{T}(n), T)$.*

Proof:
Let T be as in Lemma 7.5.6. Then obviously $\mathfrak{T} \equiv T$. By Lemma 7.5.4(2), $T \in \mathcal{T}(n)$ is unique with respect to this property. Then $\mathfrak{T} \simeq_{\mathsf{SR}} T$ by Lemma 7.5.2(1) and thus $\mathfrak{T} \simeq_{\mathsf{SR}} (\mathcal{T}(n), T)$ by Lemma 7.5.5. □

7.5.8. Lemma.

(1) For n-formulas $\varphi, \psi \in \mathsf{SR}$, if $\varphi \nvdash \psi$, then there exists an element of $\mathcal{T}(n)$ verifying φ and falsifying ψ.

(2) For each T in $\mathcal{T}(n)$ there exists a NNIL-formula $\beta^+(T)$ such that for each $T' \in \mathcal{T}(n)$, $T' \models \beta^+(T)$ iff $T \leq T'$.

(3) For each upset U of $\mathcal{T}(n)$ there exists a NNIL-formula $\beta^+(U)$ such that for each $T' \in \mathcal{T}(n)$, $T \models \beta^+(U)$ iff $T \in U$.

Proof:

(1) If $\varphi \nvdash \psi$, then there is a finite rooted tree $\mathfrak{M}$ that satisfies φ and refutes ψ in its root. By Theorem 7.5.7, there is a T in $\mathcal{T}(n)$ with the same property.

(2) Define $\beta^+(T) = \bigwedge\{\beta(S) \mid S \in \mathcal{T}(n) \text{ and } T \not\leq S\}$. Let $T' \in \mathcal{T}(n)$ with $T \leq T'$. To see that $T' \models \beta^+(T)$, let $S \in \mathcal{T}(n)$ with $T \not\leq S$. Then $T' \not\leq S$. So $T \models \beta(S)$ by Lemma 7.5.2(1). Thus, $T' \models \beta^+(T)$.

Conversely, suppose that $T \not\leq T'$. Then $\beta(T')$ is one of the conjuncts in $\beta^+(T)$ and since $T' \not\models \beta(T')$ by Lemma 7.3.2, we have $T' \not\models \beta^+(T)$.

(3) Define $\beta^+(U) = \bigwedge\{\beta(u) \mid u \notin U\}$. The proof of (3) is similar to the one of (2). □

7.5.9. Theorem. *$\mathcal{T}(n)$ is an exact n-universal model for NNIL and SR in the sense of Definition 7.5.1*

Proof:
Lemma 7.5.8(1) shows that $\mathcal{T}(n)$ satisfies (1) from Definition 7.5.1 for SR. This implies that $\mathcal{T}(n)$ also satisfies (1) for NNIL. Point (2) of the same lemma shows that $\mathcal{T}(n)$ satisfies (2) for NNIL and thus for SR. Finally, (3) of that lemma shows that $\mathcal{T}(n)$ is exact for NNIL and thus for SR. □

We finish this section with two additional observations on the universal model. Using the results from Section 7.4, we can show that Theorem 7.5.7 also holds in the infinite case:

7.5.10. Theorem. *For every n-tree $\mathfrak{M}$, there is a unique $T \in \mathcal{T}(n)$ with $\mathfrak{M} \equiv T$. In particular, for every rooted n-model $\mathfrak{M}$ there is $T \in \mathcal{T}(n)$ with $\mathfrak{M} \simeq_{\mathsf{SR}} (\mathcal{T}(n), T)$.*

Proof:
Let $\mathfrak{M}$ be an n-tree. By Theorem 7.4.5, there is a finite subtree $\mathfrak{M}'$ of $\mathfrak{M}$ that is a stable image of $\mathfrak{M}$. In particular, $\mathfrak{M} \equiv \mathfrak{M}'$. By applying Theorem 7.5.7 to $\mathfrak{M}'$, we obtain $T \in \mathcal{T}(n)$ with $\mathfrak{M}' \equiv T$, then also $\mathfrak{M} \equiv T$. Thus, T is as desired. If $\mathfrak{M}$ is not a tree, then first unravel $\mathfrak{M}$ into a tree $\mathfrak{M}_{\mathsf{tree}}$ and obtain T as before. Then $\mathfrak{M}_{\mathsf{tree}}$ and T are SR-equivalent. Since $\mathfrak{M}$ and $\mathfrak{M}_{\mathsf{tree}}$ satisfy the same formulas, $\mathfrak{M}$ and T are also SR-equivalent. □

Finally, we show that $\mathcal{T}(n)$ is isomorphic to what we call the *n-canonical model for* NNIL*-formulas*: By NNIL_n we denote the collection of all n-formulas in NNIL. The elements of the n-canonical model for NNIL-formulas are the consistent theories of NNIL_n-formulas with the disjunction property and these are ordered by inclusion. A theory Ψ in this model satisfies a propositional letter p iff $p \in \Psi$. This obviously provides a persistent valuation. We denote the resulting model by $\mathfrak{M}_n^{\mathsf{NNIL}}$. Since NNIL is locally finite, this model is finite and its members are nothing but the theories generated by the (consistent) NNIL-formulas with the disjunction property.

7.5.11. Theorem. *$(\mathfrak{M}_n^{\mathsf{NNIL}}, \subseteq)$ is isomorphic to $(\mathcal{T}(n), \leq)$.*

Proof:
Let $\{\varphi_1, \dots, \varphi_k\}$ be the collection of NNIL_n-formulas up to provable equivalence.

First we define a map σ from $\mathcal{T}(n)$ to $\mathfrak{M}_n$. If $T \in \mathcal{T}(n)$ let $\sigma(T) = \{\psi \in \mathsf{NNIL}_n \mid \mathcal{T}(n), T \models \psi\}$. Then $\sigma(T)$ is a theory of NNIL_n-formulas with the disjunction property. Moreover, if $T \leq T'$ in $\mathcal{T}(n)$, then $\sigma(T) \subseteq \sigma(T')$ by persistence, thus σ is order preserving.

Conversely, we define a map η from $\mathfrak{M}_n$ to $\mathcal{T}(n)$. Let $\Psi \in \mathfrak{M}_n^{\mathsf{NNIL}}$. Then there is a set $s \subseteq \{1, \dots, k\}$ such that Ψ is generated by $\{\varphi_i \mid i \in s\}$. In particular, $\bigwedge_{i \in s} \varphi_i \nvdash \bigvee_{i \notin s} \varphi_i$ since Ψ has the disjunction property. By property (1) in Definition 7.5.1, there is some $T \in \mathcal{T}(n)$ satisfying $\bigwedge_{i \in s} \varphi_i$ and refuting

$\bigvee_{i \notin s} \varphi_i$. In fact, $T \in \mathcal{T}(n)$ is unique with this property. Indeed, suppose T' satisfies φ_i for all $i \in s$ and refutes all φ_i with $i \in s$, then T and T' agree on all NNIL_n-formulas and thus coincide by property (2) of the n-universal model. So we define $\eta(\Psi) = T$. It is easy to see that $\sigma\eta(\Psi) = \Psi$ for all $\Psi \in \mathfrak{M}_n$ also that $\eta\sigma(T) = T$ for every $T \in \mathfrak{M}_n$.

To conclude that η and σ provide an isomorphism between $\mathfrak{M}_n$ and $\mathcal{T}(n)$, it remains to show that η is order-preserving. For $\Psi \subseteq \Psi'$ in $\mathfrak{M}_n$, let $\eta(\Psi) = T$ and $\eta(\Psi') = T'$. Then $\beta^+(T) \in \sigma(T) = \Psi$ and therefore $\beta^+(\eta(\Psi)) \in \Psi' = \sigma(T')$. Thus, $T' \models \beta^+(T)$ and so $T \leq T'$ by Lemma 7.5.8(2). □

7.6 Subframe si logics via NNIL-formulas

We show that si logics axiomatized by B-, NNIL- or SR-formulas have the fmp and are canonical. As already discussed, these results are known, since the above mentioned logics are precisely the subframe si logics. However, since fmp and canonicity follow smoothly from our previous considerations we include these results here.

An immediate consequence of the construction of the universal model is that logics axiomatized by B-, NNIL-, or SR-formulas coincide:

7.6.1. Corollary. *If $\varphi \in \mathsf{NNIL}$ or $\varphi \in \mathsf{SR}$, then there is a finite collection $\mathcal{C}$ of B-formulas, such that $\vdash_{\mathsf{IPC}} \varphi \leftrightarrow \bigwedge_{\beta(r)\in\mathcal{C}} \beta(r)$. In particular, every $\varphi \in \mathsf{SR}$ is provably equivalent to a NNIL-formula.*

Proof:
Suppose that φ is an n-formula. Consider the upset $v(\varphi)$ of the universal model $\mathcal{T}(n)$. By (the proof of) Lemma 7.5.8(3) there is a finite collection $\mathcal{C}$ of B-formulas such that for each $T \in \mathcal{T}(n)$, $T \models \varphi$ iff $T \models \bigwedge_{\beta(r)\in\mathcal{C}} \beta(r)$. Then $\vdash_{\mathsf{IPC}} \varphi \leftrightarrow \bigwedge_{\beta(r)\in\mathcal{C}} \beta(r)$ by the properties of the universal model. □

7.6.2. Corollary ([125], [29]). *For a si logic L, the following are equivalent:*

(1) L is axiomatizable by SR-formulas.

(2) L is axiomatizable by NNIL-formulas.

(3) L is axiomatizable by B-formulas.

Proof:
Since $\mathsf{B} \subseteq \mathsf{NNIL} \subseteq \mathsf{SR}$, the implications (3) ⇒ (2) ⇒ (1) are obvious. To see that (1) implies (3), suppose that $\mathsf{L} = \mathsf{IPC} + \Gamma$, where Γ is a set of SR-formulas. For

each $\varphi \in \Gamma$, let $\mathcal{K}_\varphi$ be as in Corollary 7.6.1. Then $\mathsf{L} = \mathsf{IPC} + \bigcup_{\varphi \in \Gamma}\{\mathcal{K}_\varphi\}$. □

In light of Corollary 7.6.2, it will be enough to show our intended results for logics axiomatized by B-formulas. The results in Section 7.3 imply the following simple refutation criterion for the B-formulas on Esakia spaces and frames. However, Theorem 7.6.5 shows a more useful criterion via color-consistent maps.

7.6.3. Theorem. *Let $\mathfrak{N}$ be a finite rooted model and let $\mathfrak{F}$ be an Esakia space or Kripke frame. Then, $\mathfrak{F} \not\models \beta(\mathfrak{N})$ iff there is a stable map from $\mathfrak{T}_{\mathfrak{N}}$ into some model $\mathfrak{M}$ on $\mathfrak{F}$.*

Proof:
If $\mathfrak{F} \not\models \beta(\mathfrak{N})$, then $\mathfrak{M} \not\models \beta(\mathfrak{N})$ for some model $\mathfrak{M}$ on $\mathfrak{F}$. By Lemma 7.3.3, there exists a stable map f from $\mathfrak{T}_{\mathfrak{N}}$ into $\mathfrak{M}$. Conversely, assume that f is a stable map from $\mathfrak{T}_{\mathfrak{N}}$ into a model $\mathfrak{M}$ on $\mathfrak{F}$. By Lemma 7.3.2, we have $\mathfrak{T}_{\mathfrak{N}}, r \not\models \beta(\mathfrak{N})$ where r is the root of $\mathfrak{N}$. Since $\beta(\mathfrak{N}) \in \mathsf{NNIL}$, we obtain $\mathfrak{M}, f(r) \not\models \beta(\mathfrak{N})$ by Lemma 7.2.2. □

7.6.4. Definition. A stable map f from a model $\mathfrak{N}$ into a frame $\mathfrak{F} = (W, \leq)$ is said to be *color-consistent* iff for all w, u in $\mathfrak{N}$,

$$f(w) \leq f(u) \Longrightarrow \mathit{col}(w) \leq \mathit{col}(u).$$

7.6.5. Theorem. *Let $\mathfrak{N}$ be a finite rooted model and let $\mathfrak{F}$ be a Esakia space or Kripke frame. Then, $\mathfrak{F} \not\models \beta(\mathfrak{N})$ iff there is a stable color-consistent map from $\mathfrak{T}_{\mathfrak{N}}$ into $\mathfrak{F}$.*

Proof:
The left to right direction follows from Theorem 7.6.3, as a stable map into a model is clearly color-consistent. For the other direction assume that f is a stable color-consistent map from $\mathfrak{T}_{\mathfrak{N}}$ into $\mathfrak{F} = (W, \leq)$. We assume that $\mathfrak{F}$ is an Esakia space. (The proof where $\mathfrak{F}$ is a Kripke frame is even simpler). Let $w_1, \ldots, w_k$ be the elements of $\mathfrak{T}_{\mathfrak{N}}$ and let $f(w_1) = x_1, \ldots, f(w_k) = x_k$. For each i, j such that $x_i \not\leq x_j$, by Priestley separation there is a clopen upset $U_{(i,j)}$ containing x_i but not x_j. Let U_i be the intersection of all $U_{(i,j)}$ with $1 \leq j \leq k$. Then it is easy to see that for all $1 \leq i, j \leq k$,

$$x_j \in U_i \quad \text{iff} \quad x_i \leq x_j. \tag{7.3}$$

Now define a valuation v on $\mathfrak{F}$ by setting for each propositional letter p,

$$v(p) = \bigcup\{U_i \mid 1 \leq i \leq k,\ \mathfrak{T}_{\mathfrak{N}}, w_i \models p\}.$$

We check that f preserves the valuation. If $\mathfrak{T}_{\mathfrak{N}}, w_i \models p$ for some $1 \leq i \leq k$, then U_i appears in the union defining $v(p)$ and $x_i \in U_i$, thus $x_i \in v(p)$. Conversely, if $x_j \in v(p)$ for some $1 \leq i \leq k$, then there exists $1 \leq i \leq k$ such that

$x_j \in U_i$ and $\mathfrak{T}_{\mathfrak{N}}, w_i \models p$. Then $x_i \leq x_j$ by (7.6.5), i.e. $f(w_i) \leq f(w_j)$. Since f is color-consistent, $col(w_i) \leq col(w_j)$ implying that $\mathfrak{T}_{\mathfrak{N}}, w_j \models p$, as required. Thus, the function f is a stable map from $\mathfrak{T}_{\mathfrak{N}}$ into the model $(\mathfrak{F}, v)$, and therefore $\mathfrak{F} \not\models \beta(\mathfrak{N})$ by Theorem 7.6.3. □

An immediate consequence of the theorem above is that $\beta(\mathfrak{N})$-formulas are preserved by substructures: Let $\mathfrak{F} = (W, \leq)$ and $\mathfrak{G} = (W', \leq')$ be Esakia spaces or Kripke frames. We say that $\mathfrak{G}$ is a *substructure* of $\mathfrak{F}$ iff $W' \subseteq W$ and $x \leq y$ iff $x \leq' y$ for all $x, y \in W'$. Note that even if $\mathfrak{F}$ and $\mathfrak{G}$ are Esakia spaces, the former definition does not impose any relation between the topological structure of $\mathfrak{G}$ and $\mathfrak{F}$ (in contrast to the definition of subframes in Definition 3.4.3).

7.6.6. Corollary. *Let $\mathfrak{F} = (X, \leq)$ and $\mathfrak{G} = (Y, \leq')$ be Esakia spaces or Kripke frames and let $\mathfrak{G}$ be a substructure of $\mathfrak{F}$. If $\mathfrak{F} \models \beta(\mathfrak{N})$ for some finite rooted model $\mathfrak{N}$, then $\mathfrak{G} \models \beta(\mathfrak{N})$. In other words, validity of* B*-formulas is preserved by substructures.*

Proof:
Suppose for contraposition that $\mathfrak{G} \not\models \beta(\mathfrak{N})$. By Theorem 7.6.5, there is a color-consistent stable map $f : \mathfrak{T}_{\mathfrak{N}} \to \mathfrak{G}$. The map f composed with the embedding from $\mathfrak{G}$ into $\mathfrak{F}$ is easily seen to be color-consistent. Thus, $\mathfrak{F} \not\models \beta(\mathfrak{N})$. □

An easy consequence of the above is all logics axiomatized by SR or NNIL-formulas are canonical (which corresponds to Theorem 3.4.9(3)).

7.6.7. Corollary. *Suppose that* L *is axiomatized by* SR- *or* NNIL*-formulas, then* L*-spaces and* L*-frames are closed under substructures. In particular,* L *is canonical.*

Proof:
The first part is immediate by Corollaries 7.6.1 and 7.6.6. The underlying Kripke frame of an L-space is obviously a substructure. Thus, L is canonical. □

Moreover, we obtain that si logics axiomatized by NNIL or SR-formulas have the fmp (which corresponds to Theorem 3.4.9(1)).

7.6.8. Theorem. *If* L *is axiomatized by* NNIL *or* SR*-formulas then* L *has the fmp.*

Proof:
Assume that $\mathsf{L} \nvdash \varphi$ for some formula φ. Without loss of generality we can assume that φ is a (stable) canonical formula (see Section 5.4.2). Then φ is of the shape $\bigwedge_{i \in I} \varphi_i \to \bigvee_{j \in J} p_j$, where for $j \in J$, p_j is a propositional letter and almost all of the conjuncts φ_i are in NNIL-form except for those that are of the shape $(p_1 \to$

$q) \to p_2$ where p_1, p_2 are propositional letters and q stands for a propositional letter or $\bot$.

By completeness, there is a model $\mathfrak{M}$ based on a rooted L-space $\mathfrak{F}$ with root r and such that $\mathfrak{M}, r \models \varphi_i$ for all $i \in I$ and $\mathfrak{M} \not\models p_j$ for all $j \in J$. By Corollary 7.4.5, there is a color-preserving submodel $\mathfrak{M}'$ of $\mathfrak{M}$ with root r. Clearly $\mathfrak{M}', r \models \varphi_i$ whenever φ_i is of NNIL-shape and $\mathfrak{M}', r \not\models p_j$ for all $j \in J$. It remains to show that $\mathfrak{M}', r \models (p_1 \to q) \to p_2$. Let $x \in \mathfrak{M}'$ with $\mathfrak{M}', x \not\models p_2$. Then $\mathfrak{M}, x \not\models p_2$, thus $\mathfrak{M}, x \not\models p_1 \to q$, so there is $x \leq y$ with $\mathfrak{M}, y \models p_1$ and $\mathfrak{M}, y \not\models q$. Then there is $x \leq y' \in \mathfrak{M}'$ with $col(y) = col(y')$. Then $\mathfrak{M}', y' \not\models p_1 \to q$.

Finally, the underlying frame $\mathfrak{F}'$ of $\mathfrak{M}'$ is obviously a substructure on $\mathfrak{F}$, thus $\mathfrak{F}'$ is an L-space by Corollary 7.6.7. □

Chapter 8

Quotient dynamics

8.1 Introduction

In this chapter we move to the realm of dynamic logics. Dynamic logics are modal logics containing modalities associated with *actions*. Semantically, such action modalities are often interpreted via *model transformations*, i.e. executing an action modality (at a world) in a (Kripke) model $\mathfrak{M}$ requires a transformation of $\mathfrak{M}$ into another model. Modalities that do not induce *model transformations* are referred to being part of the *static* language.

We place our work in the area of *dynamic epistemic logic* [50, 8, 6], where Kripke models represent the beliefs or knowledge states of a collection of agents. Action modalities capture the belief or knowledge changes of these agents that occur for instance after receiving some new information.

The *public announcement operator* [106, 107] is a prime example of an action modality. It captures the action of a true statement φ being publicly announced and causing all agents to simultaneously learn φ. The corresponding modality—usually denoted as $[!\varphi]$—is interpreted in a Kripke model $\mathfrak{M}$ by deleting all those worlds of $\mathfrak{M}$ that refute φ. In other words, the model transformation corresponding to the public announcement of φ produces the *submodel* of $\mathfrak{M}$ consisting of all the worlds where φ is true.

In this chapter we study model transformations producing *quotients models* as opposed to submodels. More precisely, we will introduce a modality $[\Sigma]$, where Σ is a (finite) set of formulas, that is interpreted in a model $\mathfrak{M}$ by moving to a quotient model of $\mathfrak{M}$ that is obtained by identifying all worlds that are equivalent from the point of view of Σ. From a technical point of view, this process is similar to *filtrations* in modal logic (see Section 4.2).

From an epistemic point of view, we aim to formalize the process of *abstraction*, in the specific sense of "abstracting away", i.e. disregarding all 'irrelevant' distinctions. Abstraction is essential and ubiquitous in scientific modeling: in principle, a model should represents *all* the facts, but in practice the model is always tailored to the relevant issues under discussion. The size and complexity of a model is reduced to manageable proportions by identifying situations that are indistinguishable from the point of view of the relevant issues.

A well-known example from the formal epistemology literature is the Muddy Children puzzle [53]. A standard relational model for the n-children puzzle has 2^n states, where the states represent all possibilities of the children's faces being dirty or clean. In this model many (irrelevant) facts, such as the age of the children or the color of their clothes, are disregarded.

From this perspective the set Σ contains the *relevant facts* and the modality $[\Sigma]$ can be thought of as the *abstraction modality.* The set Σ induces a relation on models that is similar to the so-called *issue relation* of [10, 102].

In this chapter we mainly study technical aspects of abstraction modalities. First we provide a formal framework for quotient dynamics. While the *states* of the quotient model can be defined in a natural and canonical way (as *equivalence classes* with respect to the relevant equivalence relation like in filtrations) we will see that there are several natural choices to make regarding the specific definition of the *valuation* and especially of the *relation* on quotient models. Accordingly, we do not single out one specific definition of the relation on quotient models, but instead work with several relations that are described by programs in the language of propositional dynamic logic (see Section 8.2). This is in the style of van Benthem and Liu [9] who describe several notions of epistemic upgrades via PDL-programs. Each program π gives rise to a modal logic whose semantics require quotients determined by π. In this way we obtain a family of logics, indexed by the programs of PDL. We show—in the style of *public announcement logic* [106, 63, 107, 8]—that these logics are complete by reduction to basic modal logic with the universal modality.

We will also explain how exactly filtrations fit into our framework and argue that in special cases our logics can be seen as logics of filtrations. The stable and M-stable logics from Chapter 4 will play a special role here; roughly speaking, whenever L is a stable logic, then the abstraction modality can be safely added to L without losing soundness and completeness.

Finally, we will generalize our framework and introduce a single dynamic logic—the logic of abstraction—that encompasses all previously considered logics as it allows for different ways of quotienting. We prove soundness and completeness of this logic by reducing its validities to PDL.

This chapter is largely based on [4].

Outline

The following section contains the definition of quotient models with respect to a PDL-program, the definition of the associated abstraction modality and some results concerning its expressive power. In Section 8.4, we introduce a family of logics for quotient dynamics and show that they are sound and complete. In Section 8.5 we explain the precise connection to filtrations. Finally, in Section 8.6 we define the logic of abstraction and provide a detailed proof of its soundness and completeness.

8.2 Preliminaries

We recall notions around propositional dynamic logic (PDL) and the universal modality that we will use in this section. In this section we mostly work with Kripke models as opposed to frames. The logics in this section are often not closed under the rule of substitution, as it is often the case in dynamic logic. Therefore, in this section, we understand the axioms and rules of modal logics in Definition 2.3.4 as schemes and disregard the substitution rule.

Adding the universal modality

Typically, the logics in this section will contain the universal modality (see [68]). Recall that by $\mathcal{L}$ we denote the language of normal modal logic. By $\mathcal{L}_E$ we denote the language $\mathcal{L}$ enriched with the *universal modality*, i.e. the language obtained by extending the grammar of $\mathcal{L}$ with $E\varphi$. The formulas of $\mathcal{L}_E$ are interpreted at a world x of a model $\mathfrak{M}$ by extending the clauses of Table 2.3.3 by

$$x \models_v E\varphi \quad \text{iff} \quad \text{there is } y \in \mathfrak{M} \text{ with } y \models_v \varphi.$$

The logic K_E is the normal modal logic K together with the $\mathsf{S5}$-axioms for E, i.e. $\varphi \to E\varphi$, $EE\varphi \to E\varphi$, and $E \to \neg E\neg E\varphi$ together with the (K)-axiom for E and $\diamond\varphi \to E\varphi$. Then K_E is sound and complete with respect to the semantics described above (see [68]).

Propositional dynamic logic

For more on PDL, the reader is referred to [36, 73]. The language of PDL is defined as follows. First, a set Π_0 of basic programs is fixed, then programs and formulas of PDL are defined in a parallel recursion according to the grammar in Table 8.2.1.

$$\pi ::= r \mid ?\varphi \mid 1 \mid \pi;\pi \mid \pi \cup \pi \mid \pi^*,$$

where $r \in \Pi_0$ and $\varphi \in \mathsf{PDL}$, where

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle\pi\rangle\varphi.$$

Table 8.2.1: Syntax of PDL

The programs $?\varphi$, 1, $\pi;\pi$, $\pi \cup \pi$, and π^* are referred to as *test*, *universal*, *composition*, *choice*, and *iteration*, respectively. As in the language of basic modal logic, formulas including the connectives $\vee$, $\rightarrow$, $\leftrightarrow$, $\top$, $\bot$, and $[\pi]$ are considered as abbreviations.

Formulas of PDL are evaluated in Kripke models of the shape $\mathfrak{M} = (\mathfrak{F}, v)$, where $\mathfrak{F} = (W, (R_r)_{r\in\Pi_0})$ is a (multi-relational) Kripke frame containing a binary relation R_r for each basic program $r \in \Pi_0$, and v is a valuation on $\mathfrak{F}$, according to Table 8.2.2.

$$
\begin{array}{lll}
x \models_v p & \text{iff} & x \in v(p) \\
x \models_v \varphi \wedge \psi & \text{iff} & x \models_v \varphi \text{ and } x \models_v \psi \\
x \models_v \neg\varphi & \text{iff} & x \not\models_v \varphi \\
x \models_v \langle\pi\rangle\varphi & \text{iff} & xR_\pi y \text{ for some } y \in \mathfrak{M} \text{ with } y \models_v \varphi,
\end{array}
$$

where the relation R_π on W is defined as follows:

$$
\begin{array}{l}
R_{?\varphi} = \{(x,x) \mid \mathfrak{M}, x \models \varphi\} \\
R_1 = W \times W \\
R_{\pi;\pi'} = \{(x,y) \mid \exists z \in W, xR_\pi z \text{ and } zR_{\pi'}y\} \\
R_{\pi\cup\pi'} = \{(x,y) \mid xR_\pi y \text{ or } xR_{\pi'}y\} \\
R_{\pi^*} = \{(x,y) \mid \text{there is a finite } R\text{-path from } x \text{ to } y\}
\end{array}
$$

Table 8.2.2: Semantics of PDL

The logic PDL is defined by the axioms and rules given in Table 8.2.3.

(CPC)	Axioms and rules of classical propositional logic
(Ax-K_π)	$[\pi](\varphi \rightarrow \psi) \rightarrow ([\pi]\varphi \rightarrow [\pi]\psi)$
(Ax-$\langle ?\varphi\rangle$)	$\langle ?\varphi\rangle\psi \leftrightarrow \varphi \wedge \psi$
(Ax-$\langle 1\rangle$)	S5-axioms for $\langle 1\rangle$, $\langle\pi\rangle\varphi \rightarrow \langle 1\rangle$
(Ax-$\langle\pi;\pi'\rangle$)	$\langle\pi;\pi'\rangle\varphi \leftrightarrow \langle\pi\rangle\langle\pi'\rangle\varphi$
(Ax-$\langle\pi\cup\pi'\rangle$)	$\langle\pi\cup\pi'\rangle\varphi \leftrightarrow (\langle\pi\rangle\varphi \vee \langle\pi'\rangle\varphi)$
(Ax-$\langle\pi^*\rangle$-1)	$\langle\pi^*\rangle\varphi \leftrightarrow (\varphi \vee \langle\pi\rangle\langle\pi^*\rangle\varphi)$
(Ax-$\langle\pi^*\rangle$-2)	$[\pi^*](\varphi \rightarrow [\pi^*]\varphi) \rightarrow (\varphi \rightarrow [\pi^*]\varphi)$
(Nec$_{[\pi]}$)	From φ infer $[\pi]\varphi$

Table 8.2.3: Axioms and rules of PDL

8.2.1. THEOREM (FISCHER-LADNER). PDL *is sound and complete with respect to the semantics described in Table 8.2.2.*

By PDL_{-*} we denote the language of *star-free* PDL that is obtained by the grammar of Table 8.2.1 by omitting the operator π^*. In fact, PDL_{-*} is equally expressive to $\mathcal{L}_E$, the basic modal language with the universal modality, via the following reductions: $\langle r\rangle\psi := \Diamond\psi$, $\langle 1\rangle\psi := E\psi$, $\langle ?\varphi\rangle\psi := \psi \wedge \varphi$, $\langle \pi;\pi'\rangle\psi := \langle\pi\rangle\langle\pi'\rangle\psi$, and $\langle \pi \cup \pi'\rangle\psi := \langle\pi\rangle\psi \vee \langle\pi'\rangle\psi$ for formulas $\psi \in \mathcal{L}_{E,[\Sigma]}$, $\varphi \in \mathcal{L}_E$. Thus, PDL_{-*} can be seen as an abbreviation for formulas in $\mathcal{L}_E$.

8.2.2. Definition. Let $\mathfrak{M} = (W, (R_r)_{r\in\Pi_0}, v)$ be a Kripke model. A binary relation Q on W is called PDL_{-*} -definable *iff* $Q = R_\pi$ *for some program* π *of* PDL_{-*}.

8.3 Quotient-taking as a model transformer

In this section we explain the main ideas behind our formalism of quotient dynamics. In particular, we provide a detailed description of our *quotient models* and introduce the so-called *abstraction modalities.* Our quotient models are similar to *filtrations* in modal logic (see Section 4.2), but the notion considered here is a bit more general as we will show in Section 8.5. We also discuss some results concerning the expressivity of abstraction modalities.

Recall from Section 2.3.3 that by $\mathcal{L}$ we denote the language of basic modal logic and by $\mathcal{L}_E$ we denote the language of basic modal logic enriched with the universal modality.

In the following let a Kripke model $\mathfrak{M} = (W, R, v)$ be fixed. Our aim is to define a *quotient model* $\mathfrak{M}_\Sigma$ of $\mathfrak{M}$ with respect to a finite[1] set of formulas $\Sigma \subseteq \mathcal{L}_E$ of the following shape

$$\mathfrak{M}_\Sigma = (W_\Sigma, R_\Sigma, v_\Sigma).$$

While introducing the concrete definition of $\mathfrak{M}_\Sigma$ we discuss and explain the particular choices we made regarding the domain W_Σ, the valuation v_Σ and the relation R_Σ.

We start with the definition of the domain W_Σ of the quotient model. W_Σ coincides with the definition of models obtained by filtrations (see Section 4.2) except that we do not require our set Σ to be subformula closed. In detail, the set Σ induces an *equivalence relation* $\sim_\Sigma$ on W: for $w, v \in W$

$$w \sim_\Sigma v \quad \text{iff} \quad \text{for all } \varphi \in \Sigma \ (\mathfrak{M}, w \models \varphi \text{ iff } \mathfrak{M}, v \models \varphi)\,. \tag{8.1}$$

In other words, two worlds are Σ-*equivalent* iff they satisfy the same formulas from Σ. We denote by $|w|_\Sigma$ the equivalence class of w with respect to $\sim_\Sigma$, i.e. $|w|_\Sigma := \{v \in W \mid w \sim_\Sigma v\}$. If Σ is clear from the context, we will sometimes write $|w|$ instead of $|w|_\Sigma$. The domain of our quotient model is the set of equivalence classes with respect to $\sim_\Sigma$, i.e. $W_\Sigma = \{|w|_\Sigma \mid w \in W\}$.

[1] The finiteness of Σ is in fact irrelevant for the definition of quotient models. However, this will be required to provide reduction axioms for our new dynamic modalities introduced in the next section. This is why we keep the setting simple by always assuming Σ to be finite.

Concerning the valuation v_Σ, for any propositional letter p, we set

$$v_\Sigma(p) := \{|w|_\Sigma \mid \text{ there is } w' \in |w|_\Sigma \text{ with } w' \in v(p)\}.$$

Clearly, there are other ways to define v_Σ that may be preferable depending on a particular context. We chose to work with this definition since this generalizes the definition of the valuation used in filtrations. Moreover, v_Σ constitutes the minimal[2] valuation that preserves the truth value of *true* propositional variables in each world, in the sense that if $w \models p$ then $|w|_\Sigma \models p$. Note however that *false* propositional variables may change their truth value and become true in the quotient model, i.e. a world $|w|_\Sigma$ may satisfy a propositional variable even if w does not. This is because two Σ-equivalent worlds may disagree on the propositional variables that are not in the set Σ.

Finally, we get to the most important definition, namely, the definition of the relation R_Σ. We think about the relation R_Σ as being determined by two factors: the first factor is a prescription on how to transfer a relation on W to a relation on W_Σ. We refer to such a prescription as a *lifting* of the relation R from W to W_Σ (similar to *relation liftings* studied in theoretical computer science). As an example consider the definition

$$|w|_\Sigma R_\Sigma |v|_\Sigma \text{ iff } \text{ there exists } w' \in |w|_\Sigma, \text{ and there exists } v' \in |v|_\Sigma \text{ such that } w'Rv'$$

which is in fact known under the name of *smallest filtration* (see Section 4.2). We can think about this definition as the $(\exists, \exists)$-lifting of the relation R for obvious reasons. In a similar manner, we could also define $(\exists, \forall)$-, $(\forall, \exists)$- and $(\forall, \forall)$-liftings of R or combinations of these. However, we here work with the $(\exists, \exists)$-lifting only.

The second factor to characterize R_Σ consists in deciding which relation to lift from W to W_Σ. For example, in the definition above, the relation R is lifted (as maybe the most obvious choice). In our framework though, we will allow more flexibility by considering liftings of all PDL_{-*}-*definable relations* (see Definition 8.2.2) just as in the work of van Benthem and Liu [9] on upgrades. This will allow us to treat different quotient upgrades simultaneously in a unified way.

In detail, in our framework each program π of PDL_{-*} leads to a model transformation function that takes a Kripke model $\mathfrak{M}$ and a finite $\Sigma \subseteq \mathcal{L}_E$, and returns the quotient model $\mathfrak{M}_\Sigma$ *whose relation* R_Σ *is determined by the* $(\exists, \exists)$*-lifting of the relation* R_π. As a consequence, each program π will lead to a π-dependent dynamic logic. We are ready to provide the formal definition of quotient models.

8.3.1. Definition. [Quotient model with respect to π] Let $\mathfrak{M} = (W, R, v)$ be a Kripke model. For every finite $\Sigma \subseteq \mathcal{L}_E$, the quotient model of $\mathfrak{M}$ with respect to Σ is $\mathfrak{M}_\Sigma = (W_\Sigma, R_\Sigma, v_\Sigma)$, where

[2] The valuation is minimal in the sense that it makes the least amount of propositional letters true.

- $W_\Sigma := \{|w|_\Sigma \mid w \in W\}$,
- $v_\Sigma(p) := \{|w|_\Sigma \mid \text{ there is } w' \in |w|_\Sigma \text{ with } w' \in v(p)\}$, and
- $|w|_\Sigma R_\Sigma |v|_\Sigma$ iff there are $w' \in |w|_\Sigma$ and $v' \in |v|_\Sigma$ such that $w' R_\pi v'$.

To illustrate the concept we consider a few examples of quotient models depending on a program π. For instance, if π is the universal program, then the relation on the corresponding quotient models is the total relation and if π is the program $?\bot$, then the relation on the corresponding quotient models is the empty relation no matter what Σ is. In Table 8.3.1 we calculate various quotient models of the following model $\mathfrak{M}$ in Figure 8.3.1 for varying Σ. As usual, black dots denote irreflexive worlds and void dots denote reflexive worlds. The examples indicate that—in particular by using tests in the program π— quotienting leads to a wide range of models.

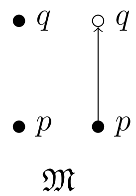


Figure 8.3.1: The model $\mathfrak{M}$.

As usual in dynamic epistemic logics [50, 8, 6], we introduce dynamic modalities, denoted by $[\Sigma]$, capturing the model change of moving to the quotient model induced by Σ. We call these modalities *abstraction modalities* and the intended semantics of formulas of the shape $[\Sigma]\varphi$ is given by

$$\mathfrak{M}, x \models [\Sigma]\varphi \quad \text{iff} \quad \mathfrak{M}_\Sigma, |x|_\Sigma \models \varphi.$$

Before we formally define the dynamic language and the semantics of the abstraction modalities in the next section, we point out some observations concerning their expressive power. Unlike e.g. the public announcement operator (see, e.g. [106, 107]), the abstraction modality adds expressivity to the basic modal language $\mathcal{L}$.

8.3.2. PROPOSITION. *Suppose that π is the basic program r. Then the abstraction modality adds expressivity to the basic modal language $\mathcal{L}$.*

Proof:
Recall that $\pi = r$ means that the relation R_Σ on the quotient model $\mathfrak{M}_\Sigma$ with respect to π is defined as for the smallest filtration. Consider the formula $[\{\top\}]\Diamond\top$. It is easy to see that for every Kripke model $\mathfrak{M}$,

$$\mathfrak{M} \models [\{\top\}]\Diamond\top \quad \text{iff} \quad \text{there are } x, y \in \mathfrak{M} \text{ with } xRy.$$

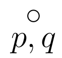
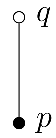
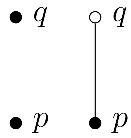
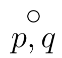
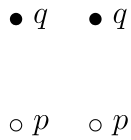
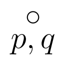
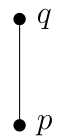
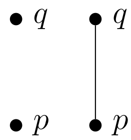
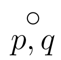
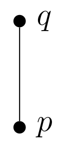
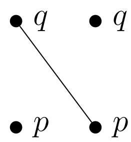
	$\Sigma = \emptyset$	$\Sigma = \{p\}$	$\Sigma = \{p, \Diamond q\}$
$\pi = r$			
$\pi = ?p$			
$\pi = ?p; r$			
$\pi = ?(\Diamond q \wedge p); 1;$ $?(q \wedge \neg\Diamond q)$			

Table 8.3.1: Examples of quotient models of $\mathfrak{M}$

It is well known that the latter statement is not expressible in basic modal logic. As an alternative example consider the formula $[\{\top\}]p$. The validity of this can be expressed by "there is x in $\mathfrak{M}$ with $\mathfrak{M}, x \models p$" . The latter is known not to be expressible in the basic modal language. □

Note, however, that the statements in the proof above *are* expressible in $\mathcal{L}_E$, that is, when the universal modality is added to $\mathcal{L}$. On the other hand, the universal modality can express statements that are not expressible via the abstraction modality.

8.3.3. Proposition. *The universal modality and the abstraction modality are in general not equally expressive.*

Proof:
For example, the statement $\chi :=$ "$\exists x \in W$ with $\mathfrak{M}, x \models \neg p$" for some propositional letter p is not expressible with the abstraction modality if $\pi = r$. To illustrate this, consider the two models $\mathfrak{M}$ and $\mathfrak{M}'$ drawn below.

Then $\mathfrak{M}, x$ satisfies χ but $\mathfrak{M}', x'$ does not satisfy χ. Since x and x' are bisimilar for $\mathcal{L}$, they satisfy the same formulas in the language $\mathcal{L}$. Now for every finite $\Sigma \subseteq \mathcal{L}$, either ($\mathfrak{M}_\Sigma = \mathfrak{M}$ and $\mathfrak{M}'_\Sigma = \mathfrak{M}'$) or $\mathfrak{M}_\Sigma = \mathfrak{M}'_\Sigma = \mathfrak{M}'$. Therefore, x and x' agree on all formulas in the language $\mathcal{L}$ extended by the abstraction modalities. Thus, χ is not expressible via $[\Sigma]$. □

8.4 Logics of quotient dynamics

In this section we introduce logics corresponding to the dynamics just defined. In fact, we introduce a family of logics $\mathsf{K}_{E,\Sigma}(\pi)$, one for each program π of PDL_{-*}. The above expressivity results imply that the basic modal language with the abstraction modality is not reducible to basic modal language. This motivates why we work with the language $\mathcal{L}_E$ (but not with the simpler basic modal language $\mathcal{L}$) as our static language. In fact, we will show that $\mathcal{L}_E$ together with the abstraction modality is co-expressive with $\mathcal{L}_E$. Thus we obtain completeness of our logics via so-called reduction axioms. We omit all the proofs in this section since in the later Section 8.6 we will prove the statements in a more general setting, so the proofs in this section are only simplifications of those.

8.4.1. Definition. The *dynamic language* $\mathcal{L}_{E,[\Sigma]}$ is defined by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid E\varphi \mid \Diamond\varphi \mid [\Sigma]\varphi$$

where Σ is a finite subset of $\mathcal{L}_E$.

For a fixed program π of PDL_{-*}, we evaluate formulas of $\mathcal{L}_{E,[\Sigma]}$ as follows.

8.4.2. Definition. [Semantics for $[\Sigma]\varphi$ with respect to π] Given a Kripke model $\mathfrak{M} = (X, V, R)$ and a state $w \in W$, the truth of $\mathcal{L}_{E,[\Sigma]}$-formulas is defined by extending the usual semantics for modal logics by

$$\mathfrak{M}, w \models [\Sigma]\varphi \quad \text{iff} \quad \mathfrak{M}_\Sigma, |w|_\Sigma \models \varphi,$$

where $\mathfrak{M}_\Sigma$ is the quotient model built with respect to the program π.

We will define a family of logics $\mathsf{K}_{E,\Sigma}(\pi)$—one for each program π of PDL_{-*}—and show their soundness and completeness with respect to our semantics. While the soundness proof is standard, the completeness is established via reducing the dynamic logic to its underlying static base through a set of so-called *reduction axioms*. The reduction axioms (given in Table 8.4.1) describe a recursive rewriting algorithm that converts the formulas in $\mathcal{L}_{E,[\Sigma]}$ to semantically and provably equivalent formulas in $\mathcal{L}_E$. The key property that allows us to obtain reduction axioms in this particular setting is that—by finiteness of Σ and the presence of the universal modality—the equivalence relation $\sim_\Sigma$ becomes definable in our language.

We fix the following notations: for every finite $\Sigma \subseteq \mathcal{L}_E$, let

$$\widehat{\Psi} = \bigwedge \Psi \wedge \bigwedge \neg(\Sigma \setminus \Psi), \tag{8.2}$$

and for every formula $\chi \in \mathcal{L}_{E,[\Sigma]}$ let

$$\langle\sim_\Sigma\rangle\chi := \bigvee_{\Psi\subseteq\Sigma} \left(\widehat{\Psi} \wedge E\left(\widehat{\Psi} \wedge \chi\right)\right). \tag{8.3}$$

The modality $\langle\sim_\Sigma\rangle$ is the diamond modality of the equivalence relation induced by Σ, i.e., $\sim_\Sigma$ is definable in $\mathcal{L}_{E,[\Sigma]}$ in the sense that for any Kripke model $\mathfrak{M}$,

$$\mathfrak{M}, x \models \langle\sim_\Sigma\rangle\chi \text{ iff there is } x' \sim_\Sigma x \text{ with } \mathfrak{M}, x' \models \chi.$$

We will prove this fact in Lemma 8.6.4. The reduction axioms and rules of the logic $\mathsf{K}_{E,\Sigma}(\pi)$ can be found in Table 8.4.1. Note that the axiom (Ax-$\Diamond_\pi$) contains the symbol $\langle\pi\rangle$ which is *not* part of the language $\mathcal{L}_{E,[\Sigma]}$. Recall that the programs used to build π do not contain the star-operator. Since the language of star-free-PDL (with the universal program) is as expressive as the language $\mathcal{L}_E$,

(K)	Axioms and rules of the basic modal logic K
(E)	S5-axioms and rules for E, $\Diamond\varphi \to E\varphi$
(Ax-p)	$[\Sigma]\, p \leftrightarrow \langle\sim_\Sigma\rangle p$
(Ax-$\neg$)	$[\Sigma]\neg\varphi \leftrightarrow \neg[\Sigma]\varphi$
(Ax-$\wedge$)	$[\Sigma](\varphi \wedge \psi) \leftrightarrow [\Sigma]\varphi \wedge [\Sigma]\psi$
(Ax-E)	$[\Sigma]E\varphi \leftrightarrow E[\Sigma]\varphi$
(Ax-$\Diamond_\pi$)	$[\Sigma]\Diamond\varphi \leftrightarrow \bigvee_{\Psi\subseteq\Sigma} \left(\widehat{\Psi} \wedge E\left(\widehat{\Psi} \wedge \langle\pi\rangle[\Sigma]\varphi\right)\right)$
(Nec$_{[\Sigma]}$)	From φ infer $[\Sigma]\varphi$

Table 8.4.1: The logic $\mathsf{K}_{E,\Sigma}(\pi)$

it is legitimate to use the axiom (Ax-$\Diamond_\pi$) as an abbreviation for a formula in the language $\mathcal{L}_{E,[\Sigma]}$ (we discussed this in Section 8.2). We also remark that—as it often happens in dynamic epistemic logic—the logic $\mathsf{K}_{E,\Sigma}(\pi)$ is not closed under uniform substitution. Thus, the axioms in rules under (K) should be understood as schemes using formulas of the language $\mathcal{L}_{E,[\Sigma]}$.

Completeness of $\mathsf{K}_{E,\Sigma}(\pi)$ is shown by defining a translation $t_\pi : \mathcal{L}_{E,[\Sigma]} \to \mathcal{L}_E$ that transforms each formula in the language $\mathcal{L}_{E,[\Sigma]}$ to a $\mathsf{K}_{E,\Sigma}(\pi)$-provably equivalent formula in the language $\mathcal{L}_E$. We will skip the details of this translation since we will later discuss a similar translation in Section 8.6. We then obtain:

8.4.3. Theorem (Expressivity). *Let π be a PDL_{-*}-program. For every $\varphi \in \mathcal{L}_{E,[\Sigma]}$, we have $\vdash_{\mathsf{K}_{E,\Sigma}(\pi)} \varphi \leftrightarrow t_\pi(\varphi)$.*

We can now derive completeness results by standard arguments from the completeness of the basic modal logic with the universal modality K_E (see [68] for the completeness of K_E) and the soundness of $\mathsf{K}_{E,\Sigma}(\pi)$.

8.4.4. Theorem (Completeness). *If π is a PDL_{-*}-program, then the logic $\mathsf{K}_{E,\Sigma}(\pi)$ is sound and complete with respect to the class of all Kripke models, where the quotient models are taken with respect to the program π.*

8.5 Logics of filtrations

In this section we explain that in some special cases the logics of quotient dynamics can be seen as logics of filtrations. We also discuss the possibility of adding axioms to our systems without losing soundness and completeness. The M-stable logics from Chapter 4 (see Section 4.4) play a special role here. Finally, we explain the meaning of the filtration theorem in our context.

The reader is invited to recall the definition of the smallest, largest and smallest transitive filtration that we discussed in Section 4.2. As in Section 4.2 we employ the following abbreviations:

By s, l, t and st we denote the smallest, largest, transitive, and smallest transitive filtration, respectively.

To show that filtrations are special quotient models, for each $f \in \{s, l, t, st\}$ we define a program π_f in the language PDL_{-*} whose corresponding quotient models coincide with f-filtrations. Let Σ be a finite set of formulas in the language $\mathcal{L}_E$. For $\Psi \subseteq \Sigma$, we set

- $\Psi_\Diamond := \bigwedge_{\Diamond\varphi\in\Sigma,\varphi\in\Psi} \Diamond\varphi$,
- $\Psi_{\Diamond,\vee} := \bigwedge_{\Diamond\varphi\in\Sigma,\varphi\in\Psi}(\Diamond\varphi \vee \varphi)$, and
- $\neg\Psi := \{\neg\varphi \mid \varphi \in \Psi\}$.

Also recall the notation for $\widehat{\Psi}$ from (8.2). We define the following programs:

$$\pi_s = r, \quad \pi_l = \bigcup_{\Psi\subseteq\Sigma} (?\Psi_\Diamond; 1; ?\widehat{\Psi}), \quad \pi_t = \bigcup_{\Psi\subseteq\Sigma} (?\Psi_{\Diamond,\vee}; 1; ?\widehat{\Psi}).$$

Moreover, let $\pi_\Sigma = \bigcup_{\Psi\subseteq\Sigma}(?\widehat{\Psi}; 1; ?\widehat{\Psi})$, and for $k \in \mathbb{N}$, let $\pi_1 = r$ and $\pi_{k+1} = r; \pi_\Sigma; \pi_k$, and define

$$\pi_{st} = \bigcup_{1\leq k\leq 2^{|\Sigma|}} \pi_k.$$

As stated in the next lemma quotient models with respect to the programs above correspond precisely to filtrations. This justifies referring to the corresponding logics $\mathsf{K}_{E,\Sigma}(\pi_f)$ as *logics of filtrations.*

8.5.1. Lemma. *Let $f \in \{s, l, t, st\}$, let Σ be a finite subformula closed set and let $\mathfrak{M}$ be a Kripke model. The quotient model $\mathfrak{M}_\Sigma$ with respect to the program π_f corresponds to an f-filtration of $\mathfrak{M}$ via Σ.*

Proof:
It is clear that the worlds W_Σ of the model $\mathfrak{M}_\Sigma$ are in line with those of a filtration as in Definition 4.2.1 and that the valuation v_Σ on $\mathfrak{M}_\Sigma$ satisfies (F1) from the respective definition. Thus, it remains to show that for worlds $|x|, |v| \in \mathfrak{M}_\Sigma$, we have

$$|x|\, R_\Sigma\, |v| \quad \text{iff} \quad |x|\, R^f\, |v|, \tag{8.4}$$

where R_Σ is defined as in Definition 8.3.1 with respect to π_f. Equation (8.4) is easily seen to be true for case $f = s$. We show the case $f = l$ in detail: Following Definition 8.3.1, we have $|x|R_\Sigma|v|$ iff there are $v' \sim_\Sigma v$ and $x' \sim_\Sigma x$ with $x'R_{\pi_l}v'$. Thus, to establish the equivalence of (8.4) for $f = s$, it suffices to show that

$$x'R_{\pi_l}v' \quad \text{iff} \quad \text{for all } \Diamond\varphi \in \Sigma \ (v \models \varphi \text{ implies } x \models \Diamond\varphi). \tag{8.5}$$

Also, a moment's thought shows that $x' R_{\pi_l} v'$ iff there is $\Psi \subseteq \Sigma$ with $x' \models \Psi_\Diamond$ and $v' \models \widehat{\Psi}$. For the left to right direction, suppose that $x' R_{\pi_l} v'$ and let $\Psi \subseteq \Sigma$ with $x' \models \Psi_\Diamond$ and $v' \models \widehat{\Psi}$. Moreover, let $\Diamond\varphi \in \Sigma$ and suppose that $v \models \varphi$. Since $\Diamond\varphi \in \Sigma$, also $\varphi \in \Sigma$, so $v' \models \varphi$. It follows that $\varphi \in \Psi$ and so $x' \models \Diamond\varphi$ implying that $x \models \Diamond\varphi$. For the converse implication, suppose the right hand side of (8.5). It is easy to check that $x' \models \Psi_\Diamond$ and $v' \models \widehat{\Psi}$ for $\Psi = \{\varphi \in \Sigma \mid v \models \varphi\}$. Thus, $x' R_{\pi_l} v'$.

The proof for the case $f = t$ is similar. To prove (8.4) for $f = st$, first observe that the program π_k induces the relation that connects two equivalence classes iff they are connected via k-many iterations of the relation induced by π_s. Now by finiteness of Σ, the size of W_Σ is bounded by $2^{|\Sigma|}$. Thus, the transitive closure of a relation on W_Σ is reached by at most $2^{|\Sigma|}$ many iterations of the relation induced by π_s. □

Next we move to stronger logics by adding modal axioms to our systems. First, we define what we mean by extensions of the logic $\mathsf{K}_{E,\Sigma}(\pi)$. If $\mathsf{L} = \mathsf{K} + \Psi$ is a normal modal logic, by $\mathsf{K}_{E,\Sigma}(\pi) + \Psi$ we mean the least collection of formulas containing all axioms of Table 8.4.1 and Ψ, and being closed under all rules of Table 8.4.1. To be precise, we add the axioms of Ψ as axiom-schemas in the language $\mathcal{L}_{E,[\Sigma]}$. This is important since the logics in this chapter are in general not closed under uniform substitution.

In the following, we provide several conditions that are sufficient to keep the extended systems sound and complete.

8.5.2. Definition. If $\mathsf{L} = \mathsf{K} + \Psi$ is a normal modal logic and $f \in \{s, t, l, st\}$ is a filtration type, by $\mathsf{L}_{E,\Sigma}(\pi_f)$ we denote the logic $\mathsf{K}_{E,\Sigma}(\pi_f) + \Psi$.

Note that axiom (Ax-$\Diamond_\pi$) of $\mathsf{K}_{E,\Sigma}(\pi)$ in Table 8.4.1 depends on π, thus the logics $\mathsf{L}_{E,\Sigma}(\pi_f)$ differ for varying f. We next discuss how the logics $\mathsf{L}_{E,\Sigma}(\pi_f)$ behave regarding soundness and completeness.

Recall that by Theorem 8.4.4, the logic $\mathsf{K}_{E,\Sigma}(\pi)$ is sound and complete with respect to Kripke models. In general, this is not automatically true when we move to stronger systems than K, meaning that even if a normal modal logic L is Kripke complete, the logic $\mathsf{L}_{E,\Sigma}(\pi_f)$ may not be complete with respect to the corresponding semantics on models based on L-frames. The reason is that quotient models with respect to π_f may not preserve the frame conditions determined by L and thus violate the soundness of the necessitation rule ($\mathrm{Nec}_{[\Sigma]}$).

On the other hand, whenever the frames underlying quotient models with respect to π_f are L-frames, then the logic $\mathsf{L}_{E,\Sigma}(\pi_f)$ is sound and complete with respect to models based on L-frames. For instance the logics $\mathsf{T}_{E,\Sigma}(\pi_f)$ and $\mathsf{D}_{E,[\Sigma]}(\pi_f)$ are sound and complete with respect to reflexive respectively serial Kripke models for $f \in \{s, l\}$, and the logic $\mathsf{KB}_{E,[\Sigma]}(\pi_s)$ is sound and complete with respect to symmetric Kripke models.

In other words, whenever a modal logic L *admits filtrations of type* f (in the strong sense) (see Definition 4.2.7(2)), then soundness and completeness of $\mathsf{L}_{E,\Sigma}(\pi_f)$ with respect to models based on L-frames is guaranteed.

Something more general is true whenever L admits filtrations of type f in the weak sense (see Definition 4.2.7(1)). As we already thoroughly discussed in Section 4.2, often a particular filtration type produces a model based on an L-frame *not for all* L-models but only for models of a specific generating set. For instance, the largest filtration produces $\mathsf{S5}$-models for rooted $\mathsf{S5}$-models (clusters) but not necessarily for all $\mathsf{S5}$-models. In such cases soundness and completeness of $\mathsf{L}_{E,\Sigma}(\pi_f)$ is guaranteed for a restricted class of L-models, namely for those that are closed under f-filtration. The following theorem summarizes the results from our discussion.

8.5.3. THEOREM. *Let $f \in \{s, l, t, st\}$ be a filtration type and let L be a normal modal logic.*

(1) If L admits f-filtrations (in the strong sense), then $\mathsf{L}_{E,\Sigma}(\pi_f)$ is sound and complete with respect the class of models based on L-frames.

(2) If L admits f-filtrations in the weak sense, and L is sound and complete with respect to the class $\mathcal{K}$ of models that is closed under f-filtrations, then $\mathsf{L}_{E,\Sigma}(\pi_f)$ is sound and compete with respect to $\mathcal{K}$.

In particular, the modal logics mentioned in Table 8.5.1 are sound and complete with respect to the corresponding class of Kripke models.

Obviously, stable and M-stable logics from Chapter 4 play a special role for logics of filtrations as they fall into the class of logics described in the theorem above. Together with Theorem 4.5.6 from Chapter 4 we can infer:

8.5.4. THEOREM. *Let M be a logic that admits a filtrations of type f for $f \in \{s, l, t, st\}$.*

(1) If L is M-stable and characterized by the M-stable class $\mathcal{K}$ of Kripke frames, then $\mathsf{L}_{E,\Sigma}(\pi_f)$ is sound and complete with respect to the class

$$\{\mathfrak{M} \mid \mathfrak{M} \text{ is a model based on a frame from } \mathcal{K}\}.$$

(2) If $\mathsf{K4} \subseteq \mathsf{M}$, and L is M-stable, then $\mathsf{L}_{E,\Sigma}(\pi_f)$ is sound and complete with respect to the class

$$\{\mathfrak{M} \mid \mathfrak{M} \text{ is a model based on a rooted } \mathsf{L}\text{-frame}\}.$$

Finally, we comment on the meaning of the filtration theorem in our context. We note that the universal modality behaves well with respect to filtrations, in the sense that the statement of the filtration theorem extends to formulas containing E ([68, Theorem 5.9]). The filtration theorem can be internalized in our logic and thus gets a syntactic shape.

Modal logic	sound and complete with respect to ... models
$\mathsf{D}_{E,[\Sigma]}(\pi_f)$, $f \in \{s,l\}$	serial
$\mathsf{T}_{E,[\Sigma]}(\pi_f)$, $f \in \{s,l\}$	reflexive
$\mathsf{KB}_{E,[\Sigma]}(\pi_s)$	symmetric
$\mathsf{K4}_{E,[\Sigma]}(\pi_f)$, $f \in \{t,st\}$	transitive
$\mathsf{D4}_{E,[\Sigma]}(\pi_f)$, $f \in \{t,st\}$	transitive and serial
$\mathsf{K4.2}_{E,[\Sigma]}(\pi_{st})$	rooted, directed and transitive
$\mathsf{K4.3}_{E,[\Sigma]}(\pi_{st})$	rooted, connected and transitive
$\mathsf{S4}_{E,[\Sigma]}(\pi_f)$, $f \in \{t,st\}$	transitive and reflexive
$\mathsf{S4.2}_{E,[\Sigma]}(\pi_{st})$	rooted directed quasi-ordered
$\mathsf{S4.3}_{E,[\Sigma]}(\pi_{st})$	rooted linear quasi-ordered
$\mathsf{S5}_{E,[\Sigma]}(\pi_f)$, $f \in \{s,l,t,st\}$	clustered

Table 8.5.1: Sound and complete extensions of $\mathsf{K}_{E,\Sigma}(\pi)$

8.5.5. Theorem (Internalized Filtration Theorem). *For a finite subformula closed set $\Sigma \subseteq \mathcal{L}_E$ and all $\varphi \in \Sigma$,*

(1) $\vdash_{\mathsf{K}_{E,\Sigma}(\pi_f)} [\Sigma]\varphi \leftrightarrow \varphi$, for $f \in \{s,l\}$;

(2) $\vdash_{\mathsf{K4}_{E,\Sigma}(\pi_f)} [\Sigma]\varphi \leftrightarrow \varphi$, for $f \in \{t,st\}$.

Proof:
This is an easy consequence of the standard filtration theorem and the completeness of the logics in question. □

8.6 The logic of abstraction

This section generalizes the setting presented in Section 8.4 in several ways. We define a logic QPDL, the *logic of abstraction*, that involves all previously discussed quotient dynamics simultaneously. We achieve this by allowing the PDL-program π to be a parameter of the abstraction modality. (Recall that in the previous setting the program π was fixed in advance). We provide a detailed proof of soundness and completeness of the logic QPDL via reduction to propositional dynamic logic PDL.

In more detail, the generalizations we take can be summarized as follows:

- We move to a *multi-agent setting* allowing for many basic programs in a given PDL-language.
- We *generalize the abstraction modalities* in such a way that the PDL-programs become a component of these modalities thus they get the shape $[\overrightarrow{\pi}/\Sigma]$, where Σ is a finite set of formulas of PDL and $\overrightarrow{\pi}$ is a sequence of programs indexed by the set of agents.
- We allow programs in the (full) PDL-language in particular *including the star-operator.*

The language of the logic of abstraction is defined by extending the language of propositional dynamic logic PDL with the abstraction modalities $[\overrightarrow{\pi}/\Sigma]\varphi$. More precisely:

8.6.1. Definition. The *dynamic language* $\mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$ is defined by the grammar:

$$\pi := r \mid ?\psi \mid 1 \mid \pi;\pi \mid \pi \cup \pi \mid \pi^* \quad \text{and}$$
$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle\pi\rangle\varphi \mid [\overrightarrow{\pi}/\Sigma]\varphi,$$

where r is an element of the set of basic programs Π_0, $\psi \in \mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$, $\overrightarrow{\pi} = (\pi_r)_{r\in\Pi_0}$ is a sequence of PDL-programs, and Σ is a *finite*[3] subset of PDL.

Next we define the *(multi-relational) quotient models*. Recall that in Definition 8.3.1 we defined quotient models for a fixed program π. In the current setting, the sequence of programs $\overrightarrow{\pi}$ becomes a parameter of the quotient models, thus receives a similar status as the set Σ. This is reflected in the shape of the abstraction modalities $[\overrightarrow{\pi}/\Sigma]\varphi$.

8.6.2. Definition. [Quotient model] Let $\mathfrak{M} = (W, (R_r)_{r\in\Pi_0}, V)$ be a Kripke model. For any finite $\Sigma \subseteq \mathsf{PDL}$ and any sequence $\overrightarrow{\pi} = (\pi_r)_{r\in\Pi_0}$ of programs, the *quotient model* $\mathfrak{M}^{\overrightarrow{\pi}}_{\Sigma}$, is $\mathfrak{M}^{\overrightarrow{\pi}}_{\Sigma} = (W_\Sigma, (R^{\pi_r}_{\Sigma})_{r\in\Pi_0}, v_\Sigma)$, where

- $W_\Sigma := \{|w|_\Sigma \mid w \in W\}$,
- $v_\Sigma(p) := \{|w|_\Sigma \mid \text{there is } w' \sim_\Sigma w \text{ with } w' \in V(p)\}$, and
- for each $r \in \Pi_0$,

$$|w|_\Sigma R^{\pi_r}_{\Sigma} |v|_\Sigma \text{ iff there is } w' \sim_\Sigma w \text{ and there is } v' \sim_\Sigma v \text{ with } w' R_{\pi_r} v'.$$

where the relation $\sim_\Sigma$ is the equivalence relation induced by Σ defined just as in (8.1) of Section 8.3 and R_π denotes the relation induced by the program π just as in Table 8.2.2.

[3] Similar to the case in Section 8.5, the sets Σ being finite is essential in order to obtain reduction axioms for the corresponding dynamic logic.

In other words, using the terminology of Section 8.3, the quotient model $\mathfrak{M}_\Sigma^{\vec{\pi}}$ arises from $\mathfrak{M}$ by interpreting a basic program $r \in \Pi_0$ via the $(\exists, \exists)$-lifting of the relation R_{π_r} from W to W_Σ.

8.6.3. Definition. [Semantics for $\mathsf{PDL}_{[\vec{\pi}/\Sigma]}$] Let $\mathfrak{M} = (W, (R_r)_{r \in \Pi_0}, v)$ be a Kripke model and w in W. The truth of $\mathsf{PDL}_{[\vec{\pi}/\Sigma]}$-formulas is defined recursively as for PDL (see Table 8.2.2) with the additional clause:

$$\mathfrak{M}, w \models [\vec{\pi}/\Sigma]\varphi \quad \text{iff} \quad \mathfrak{M}_\Sigma^{\vec{\pi}}, |w|_\Sigma \models \varphi$$

where $\mathfrak{M}_\Sigma^{\vec{\pi}}$ is as in Definition 8.6.2.

Next we introduce reduction axioms that axiomatize $\mathsf{PDL}_{[\vec{\pi}/\Sigma]}$ and allow us to convert a formula of $\mathsf{PDL}_{[\vec{\pi}/\Sigma]}$ to a provably equivalent formula in PDL. In the current setting, there are two key properties that allow us to obtain reduction axioms. Firstly, the equivalence relation $\sim_\Sigma$ is *definable* in the language $\mathsf{PDL}_{[\pi/\Sigma]}$ as we have been discussed in Section 8.4. Secondly, Σ being finite ensures that the model $\mathfrak{M}_\Sigma^{\pi}$ is not only finite but its size is bounded in terms of the size of Σ. In fact, the size of $\mathfrak{M}_\Sigma^{\pi}$ is at most $2^{|\Sigma|}$. For this reason we can obtain reduction axioms for the star-operator. As in (8.3) in Section 8.4, for every formula $\chi \in \mathsf{PDL}_{[\vec{\pi}/\Sigma]}$ and finite $\Sigma \subseteq \mathsf{PDL}$ we fix the following notation:

$$\langle \sim_\Sigma \rangle \chi := \bigvee_{\Psi \subseteq \Sigma} \left(\widehat{\Psi} \wedge \langle 1 \rangle \left(\widehat{\Psi} \wedge \chi \right) \right).$$

The following lemma shows that the modality $\langle \sim_\Sigma \rangle$ is in fact the diamond modality of the relation $\sim_\Sigma$.

8.6.4. Lemma. *For a Kripke model $\mathfrak{M}$ and world $x \in \mathfrak{M}$, we have*

$$\mathfrak{M}, x \models \langle \sim_\Sigma \rangle \chi \textit{ iff there is } x' \sim x \textit{ with } \mathfrak{M}, x' \models \chi.$$

Proof:
Observe that for any $x, y \in \mathfrak{M}$, ($x \models \widehat{\Psi}$ and $y \models \widehat{\Psi}$ for some $\Psi \subseteq \Sigma$) iff $x \sim_\Sigma y$. The rest of the proof is a routine check. □

The logic QPDL is determined by the axioms and rules in Table 8.6.1.

(PDL)	Axiom-schemes and rules of PDL (see Table 8.2.3)
(Ax-$\mathsf{K}_{[\overrightarrow{\pi}/\Sigma]}$)	$[\overrightarrow{\pi}/\Sigma](\varphi \to \psi) \to ([\overrightarrow{\pi}/\Sigma]\varphi \to [\overrightarrow{\pi}/\Sigma]\psi)$
(Ax-p)	$[\overrightarrow{\pi}/\Sigma]\, p \leftrightarrow \langle\sim_\Sigma\rangle p$
(Ax-$\neg$)	$[\overrightarrow{\pi}/\Sigma]\neg\varphi \leftrightarrow \neg[\overrightarrow{\pi}/\Sigma]\varphi$
(Ax-$\wedge$)	$[\overrightarrow{\pi}/\Sigma](\varphi \wedge \psi) \leftrightarrow [\overrightarrow{\pi}/\Sigma]\varphi \wedge [\overrightarrow{\pi}/\Sigma]\psi$
(Ax-$\langle 1\rangle$)	$[\overrightarrow{\pi}/\Sigma]\langle 1\rangle\varphi \leftrightarrow \langle 1\rangle[\overrightarrow{\pi}/\Sigma]\varphi$
(Ax-$\langle r\rangle$)	$[\overrightarrow{\pi}/\Sigma]\langle r\rangle\varphi \leftrightarrow \langle\sim_\Sigma\rangle\langle\pi_r\rangle[\overrightarrow{\pi}/\Sigma]\varphi$ for all $r \in \Pi_0$
(Ax-$*$)	$[\overrightarrow{\pi}/\Sigma]\langle\alpha^*\rangle\varphi \leftrightarrow [\overrightarrow{\pi}/\Sigma]\bigvee_{0\leq n\leq 2^{\lvert\Sigma\rvert}}\langle\alpha\rangle^n\varphi$
(Nec$_{[\overrightarrow{\pi}/\Sigma]}$)	From φ infer $[\overrightarrow{\pi}/\Sigma]\varphi$

Table 8.6.1: The logic QPDL

8.6.5. Theorem (Soundness). *The axioms in Table 8.6.1 are sound with respect to the semantics of Definition 8.6.3.*

Proof:
The proof is a routine check. For example, the validity of the axiom (Ax-p) follows immediately from the definition of the valuation in quotient models and using Lemma 8.6.4. Similarly, the validity of the axiom of the axiom (Ax-$\langle r\rangle$) is an easy consequence of the definition of R_r in the quotient models and Lemma 8.6.4. Finally, the validity of the axiom (Ax-$*$) follows from the fact that the size of quotient models with respect to Σ are bounded by $2^{|\Sigma|}$. Thus, the reflexive-transitive closure of a relation R is obtained by at most $2^{|\Sigma|}$-many iterations of R. □

The reduction axioms enable us to show that every formula in $\mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$ is provably equivalent (in the system QPDL) to a formula in the language PDL. We provide a detailed proof of the following expressivity result below.

8.6.6. Theorem (Expressivity). *For every $\varphi \in \mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$ there is a $\psi \in \mathsf{PDL}$ such that $\vdash_{\mathsf{QPDL}} \varphi \leftrightarrow \psi$.*

Using Theorem 8.6.6, the completeness of QPDL can be obtained from the completeness theorem for PDL and the soundness of the system QPDL in a standard way:

8.6.7. Theorem (Completeness). QPDL *is complete with respect to the semantics described in Definition 8.6.3.*

Proof:
Suppose φ is not a theorem of QPDL for some formula $\varphi \in \mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$. By Theorem 8.6.6, there is a formula $\psi \in \mathsf{PDL}$ with $\vdash_{\mathsf{QPDL}} \varphi \leftrightarrow \psi$. Thus, ψ is not a

theorem of QPDL. Since every theorem of PDL is a theorem of QPDL, ψ is not a theorem of PDL. By completeness of PDL, there is a model $\mathfrak{M}$ that refutes ψ. By soundness of QPDL, $\mathfrak{M}$ validates $\varphi \leftrightarrow \psi$, thus, $\mathfrak{M}$ refutes φ as desired. □

Proof of Theorem 8.6.6

We provide a detailed proof of Theorem 8.6.6. The proof is done is several steps and for this purpose we define two intermediate languages $\mathsf{PDL}^0_{[\overrightarrow{\pi}/\Sigma]}$ and $\mathsf{PDL}^1_{[\overrightarrow{\pi}/\Sigma]}$ where

$$\mathsf{PDL} \subseteq \mathsf{PDL}^0_{[\overrightarrow{\pi}/\Sigma]} \subseteq \mathsf{PDL}^1_{[\overrightarrow{\pi}/\Sigma]} \subseteq \mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}.$$

In each language, Σ is a finite subset of PDL and $\overrightarrow{\pi}$ is a finite sequence of PDL-programs. The language $\mathsf{PDL}^0_{[\overrightarrow{\pi}/\Sigma]}$ allows only one nesting of the abstraction modality and only formulas of the static language in tests. To be precise, it is defined recursively by

$$\begin{aligned} \pi &:= r \mid ?\chi \mid 1 \mid \pi;\pi \mid \pi \cup \pi \mid \pi^* \quad \text{and} \\ \varphi &:= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle\pi\rangle\varphi \mid [\overrightarrow{\pi}/\Sigma]\psi, \end{aligned}$$

where $\psi, \chi \in \mathsf{PDL}$, $\varphi \in \mathsf{PDL}^0_{[\overrightarrow{\pi}/\Sigma]}$. The language $\mathsf{PDL}^1_{[\overrightarrow{\pi}/\Sigma]}$ allows arbitrary nesting of the abstraction modality, but allows only formulas of the static language in tests. In precise terms, it is recursively defined as

$$\begin{aligned} \pi &:= r \mid ?\chi \mid 1 \mid \pi;\pi \mid \pi \cup \pi \mid \pi^* \quad \text{and} \\ \varphi &:= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle\pi\rangle\varphi \mid [\overrightarrow{\pi}/\Sigma]\varphi, \end{aligned}$$

where $\chi \in \mathsf{PDL}$ and $\varphi \in \mathsf{PDL}^1_{[\overrightarrow{\pi}/\Sigma]}$. The proof of Theorem 8.6.6 proceeds in the following steps:

Step 1: We define a translation $t : \mathsf{PDL}^0_{[\overrightarrow{\pi}/\Sigma]} \to \mathsf{PDL}$ and show that $\vdash_{\mathsf{QPDL}} \varphi \leftrightarrow t(\varphi)$ for all $\varphi \in \mathsf{PDL}^0_{[\overrightarrow{\pi}/\Sigma]}$.

Step 2: We show that for every $\chi \in \mathsf{PDL}^1_{[\overrightarrow{\pi}/\Sigma]}$ there is a formula $\chi' \in \mathsf{PDL}$ with $\vdash_{\mathsf{QPDL}} \chi \leftrightarrow \chi'$.

Step 3: We show that for every $\psi \in \mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$ there is a formula $\psi' \in \mathsf{PDL}$ with $\vdash_{\mathsf{QPDL}} \psi \leftrightarrow \psi'$.

In Table 8.6.2 we define the complexity measure c for formulas of $\mathsf{PDL}^0_{[\overrightarrow{\pi}/\Sigma]}$ that we use for inductive reasoning in the later proofs. Roughly, it is designed so that the formulas on the right-hand side of the (reduction) axioms in Table 8.6.1 are less complex than the ones on the left-hand side of the equivalences.

In order to simplify the definition of c and not needing to deal with too many intricacies of the Boolean language, we assume that $\vee$ is a fixed symbol in our language. Moreover, the formulas $\bigvee \Psi$ or $\bigwedge \Psi$ stand for some string of binary disjunction or conjunctions of the formulas in Ψ, respectively, i.e. we assume that some bracketing is arbitrary but fixed in advanced.

We comment on a few particularities of the definition of c and argue that it is well-defined. First note that the complexity for programs cannot be separated from the complexity of formulas since programs involving tests make use of the complexity of formulas. Formulas of the shape $[\overrightarrow{\pi}/\Sigma]\langle\beta\rangle\varphi$ need a special treatment, since the complexity of a formula $[\overrightarrow{\pi}/\Sigma]\langle\beta\rangle\varphi$ necessarily depends on the size of Σ. In particular, it cannot be bound by a fixed number because of the reduction clause (Ax-$*$) in Table 8.6.1. Note that the complexity of the formula $[\overrightarrow{\pi}/\Sigma]\langle\alpha^*\rangle\varphi$ makes use of the complexity of the formula $[\overrightarrow{\pi}/\Sigma]\langle\alpha\rangle^{2^{|\Sigma|}}\varphi$ even though the formula $\langle\alpha\rangle^{2^{|\Sigma|}}\varphi$ is more complex than $\langle\alpha^*\rangle\varphi$ whenever $|\Sigma| > 2$. This, however, does not cause the measure to be ill-defined since in each step in the definition of $c([\overrightarrow{\pi}/\Sigma]\langle\beta\rangle\varphi)$ the complexity of the program in the outermost diamond gets reduced. This guarantees that the complexity of formulas of the shape $[\overrightarrow{\pi}/\Sigma]\langle\beta\rangle\varphi$ can eventually be calculated.

The next lemma introduces a few short-cuts for the complexity of abbreviated formulas.

8.6.8. Lemma. *For all finite $\Psi \subseteq \mathsf{PDL}$ and all $[\overrightarrow{\pi}/\Sigma]\chi$, $[\overrightarrow{\pi}/\Sigma]\langle\beta\rangle\varphi \in \mathsf{PDL}^0_{[\overrightarrow{\pi}/\Sigma]}$ we have:*

(1) $c(\bigwedge \Psi) = c(\bigvee \Psi) \leq \max\{c(\psi) \mid \psi \in \Psi\} + |\Psi| - 1,$

(2) $c([\overrightarrow{\pi}/\Sigma] \bigwedge \Psi) \leq \max\{c([\overrightarrow{\pi}/\Sigma]\psi) \mid \psi \in \Psi\} + |\Psi| - 1,$

(3) $c(\langle\sim_\Sigma\rangle\chi) \leq \max\{c(\psi) \mid \psi \in \Sigma \cup \{\chi\}\} + 2^{|\Sigma|} + |\Sigma| + 1,$

(4) $c(\psi) \leq c([\overrightarrow{\pi}/\Sigma]\chi)$ *for all* $\psi \in \Sigma$,

(5) $c([\overrightarrow{\pi}/\Sigma]\varphi) < c([\overrightarrow{\pi}/\Sigma]\langle\beta\rangle\varphi)$.

Proof:
For (1) use that $\bigwedge \Psi$ includes $|\Psi| - 1$-many conjunctions, and each such conjunction increases the complexity by at most 1. Analogously for disjunctions. The proof of (2) is similar. We show (3). Recall that

$$\langle\sim_\Sigma\rangle\chi = \bigvee_{\Psi\subseteq\Sigma} \left(\widehat{\Psi} \wedge \langle 1\rangle \left(\widehat{\Psi} \wedge \chi\right)\right).$$

$$c(r) = 1$$

$$c(1) = 1$$

$$c(?\varphi) = c(\varphi) + 1$$

$$c(\alpha_1; \alpha_2) = c(\alpha_1) + c(\alpha_2) + 3$$

$$c(\alpha_1 \cup \alpha_2) = \max\{c(\alpha_1), c(\alpha_2)\} + 1$$

$$c(\alpha^*) = c(\alpha) + 1$$

$$c(p) = 1$$

$$c(\neg\varphi) = c(\varphi) + 1$$

$$c(\varphi \wedge \psi) = c(\varphi \vee \psi) = \max\{c(\varphi), c(\psi)\} + 1$$

$$c(\langle\alpha\rangle\varphi) = c(\varphi) + c(\alpha)$$

$$c([\overrightarrow{\pi}/\Sigma]p) = \max\{c(\psi) \mid \psi \in \Sigma\} + 2^{|\Sigma|} + |\Sigma| + 3$$

$$c([\overrightarrow{\pi}/\Sigma]\neg\varphi) = c([\overrightarrow{\pi}/\Sigma]\varphi) + 1$$

$$c([\overrightarrow{\pi}/\Sigma](\varphi \wedge \psi)) = c([\overrightarrow{\pi}/\Sigma](\varphi \vee \psi)) = \max\{c([\overrightarrow{\pi}/\Sigma]\varphi), c([\overrightarrow{\pi}/\Sigma]\psi)\} + 1$$

$$c([\overrightarrow{\pi}/\Sigma]\langle\beta\rangle\varphi) = \begin{cases} c([\overrightarrow{\pi}/\Sigma]\varphi) + 3 & \text{if } \beta = 1 \\ c([\overrightarrow{\pi}/\Sigma]\varphi) + c(\pi_r) + 2^{|\Sigma|} + |\Sigma| + 2 & \text{if } \beta = r \\ c([\overrightarrow{\pi}/\Sigma]\langle\alpha_1\rangle\langle\alpha_2\rangle\varphi) + 1 & \text{if } \beta = \alpha_1; \alpha_2 \\ \max\{c([\overrightarrow{\pi}/\Sigma]\langle\alpha_1\rangle\varphi, c([\overrightarrow{\pi}/\Sigma]\langle\alpha_2\rangle\varphi)\} + 2 & \text{if } \beta = \alpha_1 \cup \alpha_2 \\ \max\{c([\overrightarrow{\pi}/\Sigma]\psi, c([\overrightarrow{\pi}/\Sigma]\varphi)\} + 2 & \text{if } \beta =?\psi \\ c([\overrightarrow{\pi}/\Sigma]\langle\alpha\rangle^{2^{|\Sigma|}}\varphi) + 2^{|\Sigma|} & \text{if } \beta = \alpha^* \end{cases}$$

Table 8.6.2: Complexity measure for formulas in $\mathsf{PDL}^0_{[\pi/\Sigma]}$

Therefore,

$$
\begin{aligned}
& c(\langle \sim_\Sigma \rangle \chi) \\
& = c(\bigvee_{\Psi \subseteq \Sigma} \left(\widehat{\Psi} \wedge \langle 1 \rangle \left(\widehat{\Psi} \wedge \chi\right)\right) \\
& \leq \max\{c\left(\widehat{\Psi} \wedge \langle 1 \rangle \left(\widehat{\Psi} \wedge \chi\right)\right) \mid \Psi \subseteq \Sigma\} + (2^{|\Sigma|} - 1) && \text{(by (1))} \\
& = \max\{c\left(\widehat{\Psi}\right), c\left(\langle 1 \rangle \left(\widehat{\Psi} \wedge \chi\right)\right) \mid \Psi \subseteq \Sigma\} + 2^{|\Sigma|} && \text{(by def. of } c) \\
& = c\left(\langle 1 \rangle ((\bigwedge \neg \Sigma) \wedge \chi)\right) + 2^{|\Sigma|} && (8.6) \\
& = c((\bigwedge \neg \Sigma) \wedge \chi) + 2^{|\Sigma|} + 1 && \text{(by def. of } c) \\
& \leq \max\{c(\psi) \mid \psi \in \neg \Sigma \cup \{\chi\}\} + 2^{|\Sigma|} + |\Sigma| && \text{(by (1))} \\
& \leq \max\{c(\psi) \mid \psi \in \Sigma \cup \{\chi\}\} + 2^{|\Sigma|} + |\Sigma| + 1.
\end{aligned}
$$

The equality (8.6) is obtained as follows. Firstly, for every $\Psi \subseteq \Sigma$ we have $c(\langle 1 \rangle (\widehat{\Psi} \wedge \chi)) > c(\widehat{\Psi})$, thus, the maximum of the set

$$\{c(\widehat{\Psi}), c(\langle 1 \rangle (\widehat{\Psi} \wedge \chi) \mid \Psi \subseteq \Sigma\}$$

is of the shape $c(\langle 1 \rangle (\widehat{\Psi} \wedge \chi)$ for some $\Psi \subseteq \Sigma$. Moreover, since $\neg$ adds complexity, the maximum of the set $\{c(\langle 1 \rangle (\widehat{\Psi} \wedge \chi)) \mid \Psi \subseteq \Sigma\}$ is reached when every formula of Σ is negated, i.e. for $\Psi = \neg \Sigma$. Finally, (4) and (5) are simple consequence of the definitions of $c([\overrightarrow{\pi}/\Sigma]p)$ and $c([\overrightarrow{\pi}/\Sigma]\langle \beta \rangle \varphi)$, respectively. $\square$

We define a translation $t : \mathsf{PDL}^0_{[\overrightarrow{\pi}/\Sigma]} \to \mathsf{PDL}$ according to the clauses in Table 8.6.3. In Lemma 8.6.9 we show that the formulas on the right-hand side of the clauses in Table 8.6.3 are less complex—in the complexity measure c—than the formulas on the left-hand side of the respective equations. This ensures that the translation t is well-defined.

$$
\begin{aligned}
t(p) &= p \\
t(\neg \varphi) &= \neg t(\varphi) \\
t(\varphi \wedge \psi) &= t(\varphi) \wedge t(\psi) \\
t(\varphi \vee \psi) &= t(\varphi) \vee t(\psi) \\
t(\langle \alpha \rangle \varphi) &= \langle \alpha \rangle t(\varphi) \\
t([\overrightarrow{\pi}/\Sigma]p) &= t(\langle \sim_\Sigma \rangle p) \\
t([\overrightarrow{\pi}/\Sigma]\neg \varphi) &= t(\neg [\overrightarrow{\pi}/\Sigma]\varphi) \\
t([\overrightarrow{\pi}/\Sigma](\varphi \wedge \psi)) &= t([\overrightarrow{\pi}/\Sigma]\varphi \wedge [\overrightarrow{\pi}/\Sigma]\psi)
\end{aligned}
$$

$t([\overrightarrow{\pi}/\Sigma](\varphi \vee \psi))$	$=$	$t([\overrightarrow{\pi}/\Sigma]\varphi \vee [\overrightarrow{\pi}/\Sigma]\psi)$
$t([\overrightarrow{\pi}/\Sigma]\langle 1\rangle\varphi)$	$=$	$t(\langle 1\rangle[\overrightarrow{\pi}/\Sigma]\varphi)$
$t([\overrightarrow{\pi}/\Sigma]\langle r\rangle\varphi)$	$=$	$t(\langle\sim_\Sigma\rangle\langle\pi_r\rangle[\overrightarrow{\pi}/\Sigma]\varphi)$
$t([\overrightarrow{\pi}/\Sigma]\langle ?\psi\rangle\varphi)$	$=$	$t([\overrightarrow{\pi}/\Sigma](\psi \wedge \varphi))$
$t([\overrightarrow{\pi}/\Sigma]\langle\alpha_1;\alpha_2\rangle\varphi)$	$=$	$t([\overrightarrow{\pi}/\Sigma]\langle\alpha_1\rangle\langle\alpha_2\rangle\varphi)$
$t([\overrightarrow{\pi}/\Sigma]\langle\alpha_1 \cup \alpha_2\rangle\varphi)$	$=$	$t([\overrightarrow{\pi}/\Sigma]\langle\alpha_1\rangle\varphi \vee [\overrightarrow{\pi}/\Sigma]\langle\alpha_2\rangle\varphi)$
$t([\overrightarrow{\pi}/\Sigma]\langle\alpha^*\rangle\varphi)$	$=$	$t([\overrightarrow{\pi}/\Sigma]\bigvee_{n\leq 2^{\lvert\Sigma\rvert}}\langle\alpha\rangle^n\varphi)$

Table 8.6.3: Translation $t : \mathsf{PDL}^0_{[\bar{\pi}/\Sigma]} \to \mathsf{PDL}$

8.6.9. LEMMA. *In Table 8.6.3, the formulas in the range of t on the right-hand sides are less complex than the formulas in the range of t on the left-hand sides. In particular,*

(1) $c([\overrightarrow{\pi}/\Sigma]\, p) > c(\langle\sim_\Sigma\rangle p)$

(2) $c([\overrightarrow{\pi}/\Sigma]\neg\varphi) > c(\neg[\overrightarrow{\pi}/\Sigma]\varphi)$

(3) $c([\overrightarrow{\pi}/\Sigma](\varphi \wedge \psi)) > c([\overrightarrow{\pi}/\Sigma]\varphi \wedge [\overrightarrow{\pi}/\Sigma]\psi)$

(4) $c([\overrightarrow{\pi}/\Sigma](\varphi \vee \psi)) > c([\overrightarrow{\pi}/\Sigma]\varphi \vee [\overrightarrow{\pi}/\Sigma]\psi)$

(5) $c([\overrightarrow{\pi}/\Sigma]\langle 1\rangle\varphi) > c(\langle 1\rangle[\overrightarrow{\pi}/\Sigma]\varphi)$

(6) $c([\overrightarrow{\pi}/\Sigma]\langle r\rangle\varphi) > c(\langle\sim_\Sigma\rangle\langle\pi_r\rangle[\overrightarrow{\pi}/\Sigma]\varphi)$

(7) $c([\overrightarrow{\pi}/\Sigma]\langle ?\psi\rangle\varphi) > c([\overrightarrow{\pi}/\Sigma](\psi \wedge \varphi))$.

(8) $c([\overrightarrow{\pi}/\Sigma]\langle\pi_1;\pi_2\rangle\varphi) > c([\overrightarrow{\pi}/\Sigma]\langle\pi_1\rangle\langle\pi_2\rangle\varphi)$

(9) $c([\overrightarrow{\pi}/\Sigma]\langle\pi_1 \cup \pi_2\rangle\varphi) > c\,([\overrightarrow{\pi}/\Sigma]\langle\pi_1\rangle\varphi \vee [\overrightarrow{\pi}/\Sigma]\langle\pi_2\rangle\varphi)$

(10) $c([\overrightarrow{\pi}/\Sigma]\langle\alpha^*\rangle\varphi) > c([\overrightarrow{\pi}/\Sigma]\bigvee_{n\leq 2^{\lvert\Sigma\rvert}}\langle\alpha\rangle^n\varphi)$

Proof:
The proofs of (2)-(5) and (7)-(9) follow immediately from the definition of c. We show (1), (6), and (10).

For (1):

$$\begin{aligned}
&c([\overrightarrow{\pi}/\Sigma]\, p) \\
&= \max\{c(\psi) \mid \psi \in \Sigma\} + 2^{|\Sigma|} + |\Sigma| + 3 && \text{(by definition of } c) \\
&> \max\{c(\psi) \mid \psi \in \Sigma\} + 2^{|\Sigma|} + |\Sigma| + 2 \\
&\geq \max\{c(\psi) \mid \psi \in \Sigma \cup \{p\}\} + 2^{|\Sigma|} + |\Sigma| + 1 \\
&\geq c(\langle\sim_\Sigma\rangle p). && \text{(by Lemma 8.6.8(3))}
\end{aligned}$$

Next we show (6):

$$\begin{aligned}
&c([\overrightarrow{\pi}/\Sigma]\langle r\rangle\varphi) \\
&= c([\overrightarrow{\pi}/\Sigma]\varphi) + c(\pi_r) + 2^{|\Sigma|} + |\Sigma| + 2 && \text{(by definition of } c) \\
&> c([\overrightarrow{\pi}/\Sigma]\varphi) + c(\pi_r) + 2^{|\Sigma|} + |\Sigma| + 1 \\
&= c(\langle\pi_r\rangle[\overrightarrow{\pi}/\Sigma]\varphi) + 2^{|\Sigma|} + |\Sigma| + 1 && \text{(by definition of } c) \\
&\geq \max\{c(\psi) \mid \psi \in \Sigma \cup \{\langle\pi_r\rangle[\overrightarrow{\pi}/\Sigma]\varphi\}\} + 2^{|\Sigma|} + |\Sigma| + 1 && (8.7) \\
&\geq c(\langle\sim_\Sigma\rangle\langle\pi_r\rangle[\overrightarrow{\pi}/\Sigma]\varphi). && \text{(by Lemma 8.6.8(3))}
\end{aligned}$$

To see the inequality in (8.7), observe that by Lemma 8.6.8(4), $c(\psi) \leq c([\overrightarrow{\pi}/\Sigma]\varphi)$ for all $\psi \in \Sigma$. Thus, also $c(\psi) \leq c(\langle\pi_r\rangle[\overrightarrow{\pi}/\Sigma]\varphi)$, implying that the maximum of the set $\{c(\psi) \mid \psi \in \Sigma \cup \{\langle\pi_r\rangle[\overrightarrow{\pi}/\Sigma]\varphi\}$ is reached for $\psi = \langle\pi_r\rangle[\overrightarrow{\pi}/\Sigma]\varphi$.

Finally, we show (10).

$$\begin{aligned}
&c([\overrightarrow{\pi}/\Sigma]\langle\alpha^*\rangle\varphi) \\
&= c([\overrightarrow{\pi}/\Sigma]\langle\alpha\rangle^{2^{|\Sigma|}}\varphi) + 2^{|\Sigma|} && \text{(by definition of } c) \\
&= \max\{c([\overrightarrow{\pi}/\Sigma]\ \langle\alpha\rangle^n\varphi) \mid n \leq 2^{|\Sigma|}\} + 2^{|\Sigma|} && (8.8) \\
&> \max\{c([\overrightarrow{\pi}/\Sigma]\ \langle\alpha\rangle^n\varphi) \mid n \leq 2^{|\Sigma|}\} + 2^{|\Sigma|} - 1 \\
&\geq c([\overrightarrow{\pi}/\Sigma] \bigvee_{n\leq 2^{|\Sigma|}} \langle\alpha\rangle^n\varphi). && \text{(by Lemma 8.6.8(2))}
\end{aligned}$$

For (8.8), observe that by Lemma 8.6.8(5), $c([\overrightarrow{\pi}/\Sigma]\langle\alpha\rangle^{2^{|\Sigma|}}\varphi) \geq c([\overrightarrow{\pi}/\Sigma]\langle\alpha\rangle^n\varphi)$ for all $n \leq 2^{|\Sigma|}$. Therefore, the maximum of the set $\{c([\overrightarrow{\pi}/\Sigma]\ \langle\alpha\rangle^n\varphi) \mid n \leq 2^{|\Sigma|}\}$ is reached for $c([\overrightarrow{\pi}/\Sigma]\ \langle\alpha\rangle^{2^{|\Sigma|}}\varphi)$. □

Using the rule ($\text{Nec}_{[\overrightarrow{\pi}/\Sigma]}$) and the axiom ($\text{Ax-}\mathsf{K}_{[\overrightarrow{\pi}/\Sigma]}$), it is easy to see that the logic QPDL admits the following *congruence rules* for $[\overrightarrow{\pi}/\Sigma]$ and $\langle\sim_\Sigma\rangle$:

8.6.10. Lemma. *For every φ and ψ in the language $\mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$, we have:*

(1) if $\vdash_{\mathsf{QPDL}} \varphi \leftrightarrow \psi$ then $\vdash_{\mathsf{QPDL}} [\overrightarrow{\pi}/\Sigma]\varphi \leftrightarrow [\overrightarrow{\pi}/\Sigma]\psi$, and

(2) if $\vdash_{\mathsf{QPDL}} \varphi \leftrightarrow \psi$ *then* $\vdash_{\mathsf{QPDL}} \langle\sim_\Sigma\rangle\varphi \leftrightarrow \langle\sim_\Sigma\rangle\psi$.

Now we are ready to finish the proof of step 1 of our agenda.

8.6.11. PROPOSITION. *For all* $\varphi \in \mathsf{PDL}^0_{[\overrightarrow{\pi}/\Sigma]}$, $\vdash_{\mathsf{QPDL}} \varphi \leftrightarrow t(\varphi)$.

Proof:
The proof is by induction on the complexity c of formulas. We distinguish several cases depending on the structure of ψ. The different cases coincide with the case distinctions made in the definition of c. Roughly, each induction step is shown by first using the induction hypothesis and then applying an axiom from QPDL from Table 8.6.1. We show a few examples. First assume that $\psi = \langle\alpha\rangle\varphi$, then

$$\begin{aligned} &\vdash_{\mathsf{QPDL}} \varphi \leftrightarrow t(\varphi) && \text{(by induction hypothesis)}\\ &\vdash_{\mathsf{QPDL}} \langle\alpha\rangle\varphi \leftrightarrow \langle\alpha\rangle t(\varphi) && \text{(congruence rule for } \langle\alpha\rangle)\\ &\vdash_{\mathsf{QPDL}} \langle\alpha\rangle\varphi \leftrightarrow t(\langle\alpha\rangle\varphi). && \text{(definition of } t)\end{aligned}$$

If $\psi = [\overrightarrow{\pi}/\Sigma]\varphi$, then we further distinguish cases depending on the structure of φ. For instance if $\varphi = \langle r\rangle\chi$, then $c([\overrightarrow{\pi}/\Sigma]\chi) < c([\overrightarrow{\pi}/\Sigma]\langle r\rangle\chi)$ by Lemma 8.6.8(5), so we can apply the induction hypothesis to $[\overrightarrow{\pi}/\Sigma]\chi$. We get

$$\begin{aligned} &\vdash_{\mathsf{QPDL}} [\overrightarrow{\pi}/\Sigma]\chi \leftrightarrow t([\overrightarrow{\pi}/\Sigma]\chi) && \text{(by induction hypothesis)}\\ &\vdash_{\mathsf{QPDL}} \langle\sim_\Sigma\rangle\langle\pi_r\rangle[\overrightarrow{\pi}/\Sigma]\chi \leftrightarrow \langle\sim_\Sigma\rangle\langle\pi_r\rangle t([\overrightarrow{\pi}/\Sigma]\chi) && (8.9)\\ &\vdash_{\mathsf{QPDL}} [\overrightarrow{\pi}/\Sigma]\langle r\rangle\chi \leftrightarrow \langle\sim_\Sigma\rangle\langle\pi_r\rangle[\overrightarrow{\pi}/\Sigma]\chi && \text{(axiom Ax-}\langle r\rangle)\\ &\vdash_{\mathsf{QPDL}} [\overrightarrow{\pi}/\Sigma]\langle r\rangle\chi \leftrightarrow \langle\sim_\Sigma\rangle\langle\pi_r\rangle t([\overrightarrow{\pi}/\Sigma]\chi) && \text{(modus ponens)}\\ &\vdash_{\mathsf{QPDL}} t(\langle\sim_\Sigma\rangle\langle\pi_r\rangle[\overrightarrow{\pi}/\Sigma]\chi) \leftrightarrow \langle\sim_\Sigma\rangle\langle\pi_r\rangle t([\overrightarrow{\pi}/\Sigma]\chi) && \text{(definition of } t)\\ &\vdash_{\mathsf{QPDL}} [\overrightarrow{\pi}/\Sigma]\langle r\rangle\chi \leftrightarrow t([\overrightarrow{\pi}/\Sigma]\langle r\rangle\chi). && \text{(definition of } t\text{, modus ponens)}\end{aligned}$$

The inference in (8.9) follows from the congruence rules for $\langle\sim_\Sigma\rangle$ and $\langle\pi_r\rangle$. As a final example we show the case where $\varphi = \langle\alpha^*\rangle\chi$. Recall that by Lemma 8.6.9(10), we have $c([\overrightarrow{\pi}/\Sigma]\langle\alpha^*\rangle\varphi) > c([\overrightarrow{\pi}/\Sigma]\bigvee_{n\le 2^{|\Sigma|}}\langle\alpha\rangle^n\varphi)$. Thus, the induction hypothesis is applicable for $[\overrightarrow{\pi}/\Sigma]\bigvee_{n\le 2^{|\Sigma|}}\langle\alpha\rangle^n\varphi)$ and we have:

$$\begin{aligned} &\vdash_{\mathsf{QPDL}} [\overrightarrow{\pi}/\Sigma]\bigvee_{n\le 2^{|\Sigma|}}\langle\alpha\rangle^n\chi \leftrightarrow t([\overrightarrow{\pi}/\Sigma]\bigvee_{n\le 2^{|\Sigma|}}\langle\alpha\rangle^n\chi) && \text{(induction hypothesis)}\\ &\vdash_{\mathsf{QPDL}} [\overrightarrow{\pi}/\Sigma]\langle\alpha^*\rangle\chi \leftrightarrow [\overrightarrow{\pi}/\Sigma]\bigvee_{n\le 2^{|\Sigma|}}\langle\alpha\rangle^n\chi && \text{(axiom Ax-}*)\\ &\vdash_{\mathsf{QPDL}} [\overrightarrow{\pi}/\Sigma]\langle\alpha^*\rangle\chi \leftrightarrow t([\overrightarrow{\pi}/\Sigma]\bigvee_{n\le 2^{|\Sigma|}}\langle\alpha\rangle^n\chi) && \text{(modus ponens)}\\ &\vdash_{\mathsf{QPDL}} [\overrightarrow{\pi}/\Sigma]\langle\alpha^*\rangle\chi \leftrightarrow t([\overrightarrow{\pi}/\Sigma]\langle\alpha^*\rangle\chi). && \text{(definition of } t)\end{aligned}$$

□

Step 2 and 3 are now easy consequences:

8.6.12. LEMMA. *For every* $\chi \in \mathsf{PDL}^1_{[\overrightarrow{\pi}/\Sigma]}$ *there is a formula* $\chi' \in \mathsf{PDL}$ *with* $\vdash_{\mathsf{QPDL}} \chi \leftrightarrow \chi'$.

Proof:
The proof is by induction on the number of nestings of $[\overrightarrow{\pi}/\Sigma]$. If $\chi = [\overrightarrow{\pi}/\Sigma]\psi$, then ψ has less nestings than χ, so by induction hypothesis there is $\psi' \in \mathsf{PDL}$ such that

$$\vdash_{\mathsf{QPDL}} \psi \leftrightarrow \psi'.$$

Then $\vdash_{\mathsf{QPDL}} [\overrightarrow{\pi}/\Sigma]\psi \leftrightarrow [\overrightarrow{\pi}/\Sigma]\psi'$ by the congruence rule for $[\overrightarrow{\pi}/\Sigma]$ and so $\vdash_{\mathsf{QPDL}}$ $[\overrightarrow{\pi}/\Sigma]\psi' \leftrightarrow t([\overrightarrow{\pi}/\Sigma]\psi')$ by Proposition 8.6.11 which is applicable since $[\overrightarrow{\pi}/\Sigma]\psi' \in \mathsf{PDL}^0_{[\overrightarrow{\pi}/\Sigma]}$. Thus,

$$\vdash_{\mathsf{QPDL}} [\overrightarrow{\pi}/\Sigma]\psi \leftrightarrow t([\overrightarrow{\pi}/\Sigma]\psi')$$

by modus ponens. Therefore, $\chi' = t([\overrightarrow{\pi}/\Sigma]\psi')$ gives the desired formula. □

8.6.13. COROLLARY. *For every* $\chi \in \mathsf{PDL}_{[\pi/\Sigma]}$ *there is a formula* $\chi' \in \mathsf{PDL}$ *with* $\vdash_{\mathsf{QPDL}} \chi \leftrightarrow \chi'$.

Proof:
Recall that the difference between formulas in $\mathsf{PDL}^1_{[\overrightarrow{\pi}/\Sigma]}$ and $\mathsf{PDL}_{[\pi/\Sigma]}$ is that in the formulas of $\mathsf{PDL}_{[\pi/\Sigma]}$ we allow abstraction modalities in tests. Thus, in light of Lemma 8.6.12, it remains to show that each formula of $\mathsf{PDL}_{[\pi/\Sigma]}$ is equivalent to a formula in $\mathsf{PDL}^1_{[\overrightarrow{\pi}/\Sigma]}$.

Suppose that $\chi \in \mathsf{PDL}_{[\pi/\Sigma]}$ is a formula containing a test that involves the abstraction modality. In particular, suppose that the test $?\psi$ occurs in χ. Using the above, we can infer that ψ in the test $?\psi$ is QPDL-equivalent to a formula ψ' not containing the abstraction modality. We can then replace ψ with ψ' in the test of χ. To prove this formally, we need to do an induction on the nesting of the abstraction modality within tested formulas. We omit the details. □

8.7 Further directions

In this final section, we outline some further results and alternatives to our framework.

Other Liftings: Recall that we used the $(\exists, \exists)$-lifting to build the quotient models in Definitions 8.3.1 and 8.6.2. As we already indicated in Section 8.3, we can use other liftings to build relations in the quotient models. However, we conjecture that reduction axioms for the logics resulting from the $(\forall, \forall)$- and the

$(\exists,\forall)$-lifts are *not available* in our setting —though such reduction axioms might become available if we extend the base language by nominals as in hybrid logics. On the other hand, the setting using the $(\forall,\exists)$-lift of the relation R_π admits reduction axioms, obtained by replacing Ax-$\langle r\rangle$ from Table 8.6.1 by:

$$(\text{Ax-}\langle r\rangle) \quad [\overrightarrow{\pi}/\Sigma]\langle r\rangle\varphi \leftrightarrow \bigvee_{\Psi\subseteq\Sigma}\bigvee_{\Phi\subseteq\Sigma}\left(\hat{\Psi}\wedge\langle\pi_r\rangle\left(\hat{\Phi}\wedge[\overrightarrow{\pi}/\Sigma]\varphi\right)\wedge[1]\left(\hat{\Psi}\to\langle\pi_r\rangle\hat{\Phi}\right)\right).$$

Other Multi-Agent Settings: We can also vary options in the multi-agent setting. Recall that in the setting of QPDL, a sequence of program $\overrightarrow{\pi}$ determined the relations for the agents in the quotient model (Definition 8.6.2). Here, each agent a came with an individual 'prescription', i.e. her own program π_a that determines how to build the relation corresponding to a in the quotient model. This setting could be more restricted, for instance, instead of having a sequence of $\overrightarrow{\pi}$ of programs in the abstraction modality, there might only be one program that is used to build the relations for all agents. Moreover, syntactic restrictions on the programs can be made, e.g. by allowing only the basic program r_a in the program π_a corresponding to agent a.

The 'Semantic Option': Finally, we can also generalize the status of Σ. We can think of the set Σ as *syntactically* specified issues, that induce the equivalence relation $\sim_\Sigma$, and thus determine the worlds in the quotient models. Instead of working with a set Σ, we can work with models that come together with their own equivalence relation or "issue" relation Q. Thus, as opposed to having a syntactically defined equivalence relation, we have a *semantically* given equivalence relation.

In this set-up, models have the shape $\mathfrak{M} = (W,(R_r)_{r\in\Pi_0},Q,v)$, consisting of a Kripke model $(W,(R_r)_{r\in\Pi_0},v)$ and an equivalence relation Q on W. We then define a language $\mathsf{PDL}_{Q,\overrightarrow{\pi}/Q}$ as:

$$\begin{aligned}\pi &:= r \mid Q \mid ?\psi \mid 1 \mid \pi;\pi \mid \pi\cup\pi \mid \pi^*, \text{ and}\\ \varphi &:= p \mid \neg\varphi \mid \varphi\wedge\varphi \mid \langle\pi\rangle\varphi \mid [\overrightarrow{\pi}/Q]\varphi,\end{aligned}$$

where r is an element of the set of the basic programs Π_0 and $\psi \in \mathsf{PDL}$ (the language $\mathsf{PDL}_{Q,\overrightarrow{\pi}/Q}$ without $[\overrightarrow{\pi}/Q]\varphi$). Note that we add a symbol Q to the basic programs whose intended interpretation is the equivalence relation Q. Its modality $[Q]$ is the so-called *issue modality* from [10]. For a model $\mathfrak{M} = (W,(R_r)_{r\in\Pi_0},Q,v)$ and a sequence of programs $\overrightarrow{\pi}$, we define a model $\mathfrak{M}_Q^{\overrightarrow{\pi}} := (W_Q,(R_Q^{\pi_r})_{r\in\Pi_0},\mathsf{Id},v_Q)$, where $W_Q := \{|w| \mid \text{there is } w'Qw \text{ with } w'\in v(p)\}$, $v_Q(p) := \{|w| \mid w\in v(p)\}$, Id denotes the identity relation, and

$$|w|R_Q^{\pi_r}|v| \text{ iff there is } w'Qw \text{ and there is } v'Qv \text{ such that } w'R_{\pi_r}v',$$

where $|w|$ is the equivalence class of w wrt Q. The crucial step in the semantics is:

$$\mathfrak{M}, x \models [\overrightarrow{\pi}/Q]\varphi \quad \text{iff} \quad \mathfrak{M}_Q^{\overrightarrow{\pi}}, |x| \models \varphi.$$

To get a convenient representation of the reduction axioms, we define functions $f_{Q,\overrightarrow{\pi}}$ on programs by $f_{\overrightarrow{\pi},Q}(Q) = ?\top$, $f_{\overrightarrow{\pi},Q}(r) = Q; \overrightarrow{\pi}$, $f_{\overrightarrow{\pi},Q}(\alpha_1 \circ \alpha_2) = f_{\overrightarrow{\pi},Q}(\alpha_1) \circ f_{\overrightarrow{\pi};Q}(\alpha_2)$ for $\circ \in \{\cup, ;\}$ and $f_{\overrightarrow{\pi},Q}(\pi^*) = \left(f_{\overrightarrow{\pi},Q}(\pi)\right)^*$. Here is the full list of reduction axioms:

(**PDL**)	Axiom-schemes and rules of **PDL**
(**Q**)	**S5**-axioms and rules for Q
(Ax-p)	$[\overrightarrow{\pi}/Q]p \leftrightarrow \langle Q \rangle p$
(Ax-$\neg$)	$[\overrightarrow{\pi}/Q]\neg\varphi \leftrightarrow \neg[\overrightarrow{\pi}/Q]\varphi$
(Ax-$\wedge$)	$[\overrightarrow{\pi}/Q](\varphi \wedge \psi) \leftrightarrow [\overrightarrow{\pi}/Q]\varphi \wedge [\overrightarrow{\pi}/Q]\psi$
(Ax-$\langle\alpha\rangle$)	$[\overrightarrow{\pi}/Q]\langle\alpha\rangle\varphi \leftrightarrow \langle f_{Q,\overrightarrow{\pi}}(\alpha)\rangle[\overrightarrow{\pi}/Q]\varphi$
(Ax-$\langle Q\rangle$)	$[\overrightarrow{\pi}/Q]\langle Q\rangle\varphi \leftrightarrow [\overrightarrow{\pi}/Q]\varphi$
(DR-Nec)	From φ infer $[\overrightarrow{\pi}/Q]\varphi$

Table 8.7.1: The logic $\mathbf{PDL}_Q$

Note that in our earlier versions, the analogue of the modality $\langle Q \rangle$ was *definable* in the language $\mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$ (cf. Lemma 8.6.4), thus was not needed in the syntax.

Appendix A

Some standard logics

In this appendix we recall definitions, axiomatizations, and characterizations of some standard superintuitionistic and normal modal logics.

A partially ordered set $(X, \leq)$

is directed iff for all $z, x, y \in X$ if $z \leq x, y$, then there is $u \in X$ with $x, y \leq u$.

is upward linear iff for all $z, x, y \in X$, if $z \leq x, y$ then ($x \leq y$ or $y \leq z$).

has top-width $\leq n$ iff for all $z, x_1, \ldots x_{n+1} \in X$, if $z \leq x_i$ for all $1 \leq i \leq n+1$, then there is $y \in X$ with $x_i \leq y$ and $x_j \leq y$ for some $i \neq j$, $1 \leq i, j \leq n+1$.

has width $\leq n$ iff for all $z, x_1, \ldots x_{n+1} \in X$, if $z \leq x_i$ for all $1 \leq i \leq n+1$, then $x_i \leq x_j$ for some $i \neq j$, $1 \leq i, j \leq n+1$.,

has depth $\leq n$ iff for all $x_1, \ldots, x_{n+1} \in X$, if $x_1 \leq \cdots \leq x_{n+1}$, then $x_i = x_j$ for some $i \neq j$, $1 \leq i, j \leq n+1$,

has divergence $\leq n$ iff for all $x \in X$ and $W \subseteq \max X$ satisfying $|W| \leq k$ and $W \subseteq {\uparrow}x$, there is $y \geq x$ with $\max{\uparrow}y = W$.

Table A.0.1: Properties of partial orders

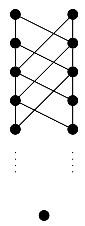


Figure A.0.1: The Rieger-Nishimura ladder $\mathfrak{L}$

Logic	Axiomatization IPC+	Frame-theoretic characterization
KC	$\neg p \vee \neg\neg p$	logic of finite directed frames
LC (Gödel-Dummett logic)	$(p \to q) \vee (q \to p)$	logic of finite upward linear frames
BTW_n	$\bigwedge_{0 \leq i \leq j \leq n} \neg(\neg p_i \wedge \neg p_j) \to \bigvee_{i=0}^{n}(\neg p_i \to \bigvee_{j \neq i} \neg p_j)$	logic of finite frames of top width $\leq n$
BW_n	$\bigvee_{i=0}^{n}(p_i \to \bigvee_{j \neq i} p_j)$	logic of finite frames of width $\leq n$
BD_n	bd_n, where $\mathsf{bd}_1 = p_1 \vee \neg p_1$, and $\mathsf{bd}_{n+1} = p_{n+1} \vee (p_{n+1} \to \mathsf{bd}_n)$	logic of finite rooted frames of depth $\leq n$
BC_n	$p_0 \vee (p_0 \to p_1) \vee \cdots \vee (\bigwedge_{i=0}^{n-1} p_i \to p_n)$	finite rooted frames of cardinality $\leq n$
T_n (Gabbay-de Jongh-logics)	$\bigwedge_{i=0}^{n}((p_i \to \bigvee_{i \neq j} p_j) \to \bigvee_{i \neq j} p_j) \to \bigvee_{i=0}^{n} p_i$	logic of finite rooted trees of branching $\leq n$
RN	see Table A.0.3	logic of the Rieger-Nishimura ladder $\mathfrak{L}$

KG (Kuznetsov-Gerciu logic)	$(p \to q) \vee (q \to r) \vee ((q \to r) \to r) \vee (r \to (p \vee q))$	logic of $\bigoplus_{i=1}^{n} \mathfrak{F}_i$[1], where each $\mathfrak{F}_i$ is a generated subframe of $\mathfrak{L}$
KP (Kreisel-Putnam logic)	$(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$	See [40, Exercise 2.10]
ND_n (Maksimova's logics)	$(\neg p \to \bigvee_{i=1}^{n} \neg q_i) \to \bigvee_{i=1}^{n} (\neg p \to \neg q_i)$	logic of finite frames of divergence $\leq n$

Table A.0.2: Some standard si logics (see [40])

[1]For Esakia spaces $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$, we denote their *ordered sum* by $\bigoplus_{i=1}^{n} \mathfrak{F}_i$ [22, Section 2.2].

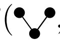
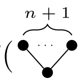
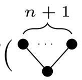
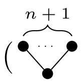
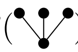
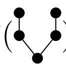
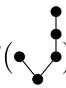
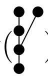
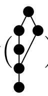
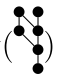
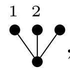
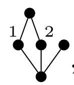
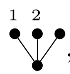
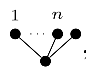
Logic	Axiomatization in terms of canonical formulas
KC	IPC + β(, $\{\varnothing\}$)
LC	IPC + β()
BTW_n	IPC + β(, $\{\varnothing\}$)
BW_n	IPC + β()
T_n	IPC + $\beta^\sharp$()
KG	IPC + β() + β() + β()
RN	KG + χ() + χ() + χ()
KP	IPC + β(, $\{\varnothing, \{1,2\}\}$) + β(, $\{\varnothing, \{1,2\}\}$)
ND_n	IPC + β(, $\{\varnothing, \{1,2\}\}$) + $\cdots$ + β(, $\{\varnothing, \{1,\ldots,n\}\}$)

Table A.0.3: Axiomatizations in terms of canonical formulas (see [22, Theorems 3.4 and 4.33] for the axiomatization of KG ans RN, and [40, Table 9.7] for the other cases). The formula $\beta^\sharp(X)$ denotes the canonical formula with all but the empty domain, and $\chi(X)$ denotes the Jankov-de Jongh formula of X, i.e., the canonical formula with all domains.

A binary relation R on a set X is

reflexive iff xRx for all $x \in X$,

serial iff for all $x \in X$ there is $y \in X$ with xRy,

symmetric iff xRy implies yRx for all $x, y \in X$,

Eucledian iff (xRy and xRz) imply yRz for all $x, y, z \in X$,

transitive iff (xRy and yRz) imply xRz for all $x, y, z \in X$,

weakly transitive iff (xRy and yRz and $x \neq z$) imply xRz for all $x, y, z \in X$,

upward connected iff (xRu and xRv and $u \neq v$) imply (uRv or vRu) for all $x, u, v, \in X$,

directed iff (xRu and xRv and $u \neq v$) imply there is $y \in X$ with (uRy and vRy), for all $x, u, v, \in X$,

an equivalence relation iff R is transitive, reflexive, and symmetric,

Noetherian iff there is no infinite ascending R-chains of distinct points,

quasi-order iff R is transitive and reflexive,

a cluster/universal iff xRy for all $x, y \in X$,

a degenerate cluster iff $X = \{x\}$ and $\neg xRx$,

of width n iff every point has at most n successors that do not see each other.

Table A.0.4: Properties of relations

Logic	Axiomatization	Sound and complete wrt frames that are
T	$\Box p \to p$	reflexive
D	$\Box p \to \Diamond p$	serial
B	$p \to \Box\Diamond p$	symmetric
K5	$\Diamond\Box p \to \Box p$	Eucledian
K4	$\Box p \to \Box\Box p$	transitive
wK4	$\Diamond\Diamond p \to (p \vee \Diamond p)$	weakly transitive
S4	$\mathsf{K4} \vee \mathsf{T}$	transitive and reflexive
S5	$\mathsf{S4} \vee \mathsf{B}$	equivalence relations
KMT	$\{\Diamond ((\Box p_1 \to p_1) \wedge \cdots \wedge (\Box p_n \to p_n)) \mid n \geq 1\}$	every point sees a reflexive point

$\mathsf{K4Alt}_n$	$\mathsf{K4}+\Box p_1 \vee \Box(p_1 \to p_2) \vee \cdots \vee \Box(p_1 \wedge \cdots \wedge p_n \to p_{n+1})$	K4-frames such that every point has at most n successors
$\mathsf{S4Alt}_n$	$\mathsf{S4} \vee \mathsf{K4Alt}_n$	K4-frames such that every point has at most n successors
$\mathsf{K4BW}_n$	$\mathsf{K4} + \bigwedge_{i=0}^{n} \Diamond p_i \to \bigvee_{0 \leq i \neq j \leq n} \Diamond(p_i \wedge \Diamond p_j)$	S4-frames such that rooted subframes have width at most n
$\mathsf{S4BW}_n$	$\mathsf{S4} \vee \mathsf{K4.BW}_n$	S4-frames such that rooted subframes have width at most n
GL	$\mathsf{K4} + \Box(\Box p \to p) \to \Box p$	Noetherian strict partial order
S4Grz	$\mathsf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$	Noetherian partial order
K4.1	$\mathsf{K4} + \Box\Diamond p \to \Diamond\Box p$	finite K4-frames with degenerate final clusters
S4.1	$\mathsf{S4} \vee \mathsf{K4.1}$	finite S4-frames with degenerate final clusters
K4.2	$\mathsf{K4} + \Diamond(\Box p \wedge q) \to \Box(\Diamond p \vee q)$	directed K4-frames

S4.2	S4 ∨ K4.2	directed S4-frames
K4.3	$\Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p)$	connected K4-frames
S4.3	S4 ∨ K4.3	connected S4-frames

Table A.0.5: Some standard normal modal logics

Most of the above logics are discussed in the monographs [36, 40]. For more on wK4 see [23, 25], and for KMT see [76].

Bibliography

[1] P. Aczel. The Russell-Prawitz modality. *Math. Structures Comput. Sci.*, 11(4):541–554, 2001. Modalities in type theory (Trento, 1999).

[2] S. Awodey. *Category theory*, volume 52 of *Oxford Logic Guides*. Oxford University Press, Oxford, second edition, 2010.

[3] R. Balbes and P. Dwinger. *Distributive lattices*. University of Missouri Press, Columbia, Mo., 1974.

[4] A. Baltag, N. Bezhanishvili, J. Ilin, and A. Özgün. Quotient dynamics: The logic of abstraction. In *Logic, Rationality, and Interaction - 6th International Workshop, LORI 2017, Proceedings*, pages 181–194, 2017.

[5] A. Baltag, L. Moss, and S. Solecki. The logic of public announcements, common knowledge and private suspicions. In *Proc. TARK '98*, pages 43–56, 1998.

[6] A. Baltag and B. Renne. Dynamic epistemic logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2016 edition, 2016.

[7] F. Bellissima. Finitely generated free Heyting algebras. *J. of Symbolic Logic*, 51:152–165, 1986.

[8] J. van Benthem. *Logical Dynamics of Information and Interaction*. Cambridge University Press, 2011.

[9] J. van Benthem and F. Liu. Dynamic logic of preference upgrade. *J. Appl. Non-Classical Logics*, 17(2):157–182, 2007.

[10] J. van Benthem and Ş. Minică. Toward a dynamic logic of questions. *J. Philos. Logic*, 41(4):633–669, 2012.

[11] G. Bezhanishvili. Varieties of monadic Heyting algebras. I. *Studia Logica*, 61(3):367–402, 1998.

[12] G. Bezhanishvili. Locally finite varieties. *Algebra Universalis*, 46(4):531–548, 2001.

[13] G. Bezhanishvili and N. Bezhanishvili. An algebraic approach to canonical formulas: Intuitionistic case. *Rev. Symb. Log.*, 2(3):517–549, 2009.

[14] G. Bezhanishvili and N. Bezhanishvili. An algebraic approach to canonical formulas: Modal case. *Studia Logica*, 99(1-3):93–125, 2011.

[15] G. Bezhanishvili and N. Bezhanishvili. Canonical formulas for wK4. *Rev. Symb. Log.*, 5(4):731–762, 2012.

[16] G. Bezhanishvili and N. Bezhanishvili. An algebraic approach to filtrations for superintuitionistic logics. In *Liber Amicorum Albert Visser*, volume 30 of *Tributes Series*, pages 47–56. College Publications, 2016.

[17] G. Bezhanishvili and N. Bezhanishvili. Locally finite reducts of Heyting algebras and canonical formulas. *Notre Dame J. Form. Log.*, 58(1):21–45, 2017.

[18] G. Bezhanishvili, N. Bezhanishvili, and R. Iemhoff. Stable canonical rules. *J. Symbolic Logic*, 81(1):284–315, 2016.

[19] G. Bezhanishvili, N. Bezhanishvili, and J. Ilin. Stable modal logics. To appear in Rev. Symb. Log.

[20] G. Bezhanishvili, N. Bezhanishvili, and J. Ilin. Subframization and stabilization. Submitted.

[21] G. Bezhanishvili, N. Bezhanishvili, and J. Ilin. Cofinal stable logics. *Studia Logica*, 104(6):1287–1317, 2016.

[22] G. Bezhanishvili, N. Bezhanishvili, and D. de Jongh. The Kuznetsov-Gerčiu and Rieger-Nishimura logics. The boundaries of the finite model property. *Logic Log. Philos.*, 17(1-2):73–110, 2008.

[23] G. Bezhanishvili, L. Esakia, and D. Gabelaia. Spectral and T_0-spaces in d-semantics. In N. Bezhanishvili, S. Löbner, K. Schwabe, and L. Spada, editors, *8th International Tbilisi Symposium on Logic, Language, and Computation. Selected papers.*, Lecture Notes in Artificial Intelligence, pages 16–29. Springer, 2011.

[24] G. Bezhanishvili and S. Ghilardi. An algebraic approach to subframe logics. intuitionistic case. *Ann. Pure Appl. Logic*, 147:84–100, 2007.

[25] G. Bezhanishvili, S. Ghilardi, and M. Jibladze. An algebraic approach to subframe logics. Modal case. *Notre Dame J. Form. Log.*, 52(2):187–202, 2011.

[26] G. Bezhanishvili, J. Harding, J. Ilin, and F. M. Lauridsen. MacNeille transferability and stable classes of Heyting algebras. Submitted. Available as ILLC Prepublication Series Report PP-2017-19.

[27] G. Bezhanishvili and W. H. Holliday. Locales, nuclei, and Dragalin frames. In Lev Beklemishev, Stéphane Demri, and András Máté, editors, *Advances in Modal Logic*, volume 11, pages 177–196. College Publications, London, 2016.

[28] G. Bezhanishvili, R. Mines, and P. J. Morandi. Topo-canonical completions of closure algebras and Heyting algebras. *Algebra Universalis*, 58(1):1–34, 2008.

[29] N. Bezhanishvili. *Lattices of Intermediate and Cylindric Modal Logics*. PhD thesis, University of Amsterdam, 2006.

[30] N. Bezhanishvili, D. Coumans, S. van Gool, and D. de Jongh. Duality and universal models for the meet-implication fragment of IPC. In M. Aher, D. Hole, E. Jeřábek, and C. Kupke, editors, *10th International Tbilisi Symposium on Logic, Language, and Computation, TbiLLC 2013, revised selected papers*, volume 8984 of *Lecture Notes in Computer Science*, pages 97–116. Springer-Verlag, 2015.

[31] N. Bezhanishvili, D. Gabelaia, S. Ghilardi, and M. Jibladze. Admissible bases via stable canonical rules. *Studia Logica*, 104(2):317–341, 2016.

[32] N. Bezhanishvili, N. Galatos, and L. Spada. Canonical formulas for k-potent commutative, integral, residuated lattices. *Algebra universalis*, 77(3):321–343, 2017.

[33] N. Bezhanishvili and S. Ghilardi. Multiple-conclusion rules, hypersequents syntax and step frames. In R. Gore, B. Kooi, and A. Kurucz, editors, *Advances in Modal Logic (AiML 2014)*, pages 54–61. College Publications, 2014. An extended version available as ILLC Prepublication Series Report PP-2014-05.

[34] N. Bezhanishvili and D. de Jongh. *Notre Dame J. Form. Log.* To appear, Available as ILLC Prepublication Series Report PP-2014-19.

[35] N. Bezhanishvili, D. de Jongh, A. Tzimoulis, and Z. Zhao. Universal models for the positive fragment of intuitionistic logic. In *Logic, Language, and Computation - 11th International Tbilisi Symposium, TbiLLC 2015, Revised Selected Papers*, pages 229–250, 2015.

[36] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, 2001.

[37] T. S. Blyth and M. F. Janowitz. *Residuation theory*. Pergamon Press, Oxford-New York-Toronto, Ont., 1972. International Series of Monographs in Pure and Applied Mathematics, Vol. 102.

[38] R. A. Bull. That all normal extensions of S4.3 have the finite model property. *Z. Math. Logik Grundlagen Math.*, 12:341–344, 1966.

[39] R. Burris and H. Sankappanavar. *A Course in Universal Algebra (The Millennium Edition)*. Springer, 2012.

[40] A. Chagrov and M. Zakharyaschev. *Modal Logic*. The Clarendon Press, New York, 1997.

[41] L.A. Chagrova. On the first order definability of intuitionsitic formulas with restrictions on occureneces of the connectives. In M.I. Kanovich, editor, *Logical Methods for Constructing Effective Algorithms*, pages 135–136. Kalinin State University, 1986.

[42] C. C. Chang and H. J. Keisler. *Model theory*, volume 73 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[43] A. Citkin. Characteristic formulas of partial Heyting algebras. *Log. Univers.*, 7(2):167–193, 2013.

[44] A. Citkin. Characteristic formulas 50 years later: an algebraic account. Available on ArXiv, 2014.

[45] A. Citkin. Characteristic inference rules. *Log. Univers.*, 9(1):27–46, 2015.

[46] W. Conradie, W. Morton, and C. van Alten. An algebraic look at filtrations in modal logic. *Log. J. IGPL*, 21(5):788–811, 2013.

[47] D. van Dalen. *Logic and structure*. Universitext. Springer, London, fifth edition, 2013.

[48] B. A. Davey and H. A. Priestley. *Introduction to lattices and order*. Cambridge University Press, 2 edition, 2002.

[49] A. Diego. *Sur les Algèbres de Hilbert*. Translated from the Spanish by Luisa Iturrioz. Collection de Logique Mathématique, Sér. A, Fasc. XXI. Gauthier-Villars, Paris, 1966.

[50] H. van Ditmarsch, W. van der Hoek, and B. Kooi. *Dynamic Epistemic Logic*. Springer Publishing Company, Incorporated, 1st edition, 2007.

[51] L. Esakia. Topological Kripke models. *Soviet Math. Dokl.*, 15:147–151, 1974.

[52] L. Esakia. *Heyting Algebras I. Duality Theory (Russian).* "Metsniereba", 1985. Engl. translation, forthcoming.

[53] Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning About Knowledge.* MIT Press, 1995.

[54] M. Fairtlough and M. Mendler. Propositional lax logic. *Information and computation*, 137:1–33, 1997.

[55] G. Ferreira and P. Oliva. On the relation between various negative translations. In *Logic, construction, computation*, volume 3 of *Ontos Math. Log.*, pages 227–258. Ontos Verlag, Heusenstamm, 2012.

[56] K. Fine. The logics containing S4.3. *Z. Math. Logik Grundlagen Math.*, 17:371–376, 1971.

[57] K. Fine. An ascending sequence of S4 logics. *Theoria*, 40(2):110–116, 1974.

[58] K. Fine. Logics containing K4. I. *J. Symbolic Logic*, 39:31–42, 1974.

[59] K. Fine. Logics containing K4. II. *J. Symbolic Logic*, 50(3):619–651, 1985.

[60] D. M. Gabbay and D. H. J. de Jongh. A sequence of decidable finitely axiomatizable intermediate logics with the disjunction property. *J. Symbolic Logic*, 39(1):67–78, 1974.

[61] D. M. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-dimensional modal logics: theory and applications*, volume 148 of *Studies in Logic and the Foundations of Mathematics.* North-Holland Publishing Co., Amsterdam, 2003.

[62] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated lattices: an algebraic glimpse at substructural logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics.* Elsevier B. V., Amsterdam, 2007.

[63] J. Gerbrandy and W. Groeneveld. Reasoning about information change. *Journal of Logic, Language and Information*, 6(2):147–169, Apr 1997.

[64] S. Ghilardi. Continuity, freeness, and filtrations. *J. Appl. Non-Classical Logics*, 20(3):193–217, 2010.

[65] R. I. Goldblatt. Metamathematics of modal logic. *Rep. Math. Logic*, (6):41–77, 1976.

[66] R. I. Goldblatt. Metamathematics of modal logic. II. *Rep. Math. Logic*, (7):21–52, 1976.

[67] R. I. Goldblatt. Grothendieck topology as geometric modality. *Z. Math. Logik Grundlag. Math.*, 27(6):495–529, 1981.

[68] V. Goranko and S. Passy. Using the universal modality: gains and questions. *J. Logic Comput.*, 2(1):5–30, 1992.

[69] J. Goudsmit. *Intuitionistic Rules. Admissible Rules of Intermediate Logics.* PhD thesis, Utrecht University, 2015.

[70] G. Grätzer. *General lattice theory*, volume 75 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.

[71] R. Grigolia. *Free Algebras of Non-Classical Logics.* "Metsniereba", Tbilisi, 1987. (Russian).

[72] P. R. Halmos. *Algebraic logic.* Chelsea Publishing Co., New York, 1962.

[73] D. Harel, D. Kozen, and J. Tiuryn. *Dynamic Logic.* MIT Press Cambridge, MA, USA, 2000.

[74] A. Hendriks. *Computations in Propositional Logic.* PhD thesis, University of Amsterdam, 1996. Available at *http://www.illc.uva.nl/Research/Publications/Dissertations/DS-1996-01.text.ps.gz*.

[75] T. Hosoi and H. Ono. The intermediate logics on the second slice. *J. Fac. Sci. Univ. Tokyo Sect. I A Math.*, 17:457–461, 1970.

[76] G. E. Hughes. Every world can see a reflexive world. *Studia Logica*, 49(2):175–181, 1990.

[77] R. Iemhoff. Consequence relations and admissible rules. *J. Philos. Logic*, 45(3):327–348, 2016.

[78] J. Ilin, D. de Jongh, and F. Yang. NNIL formulas revisited: fmp and universal model. In preparation.

[79] J. Ilin, D. de Jongh, and F. Yang. NNIL axioms have the finite model property. In J. van Eijck, R. Iemhoff, and J. Joosten, editors, *Liber Amicorum Alberti*, volume 30 of *Tributes*, pages 177–185. College publications, 2016.

[80] V. Jankov. On the relation between deducibility in intuitionistic propositional calculus and finite implicative structures. *Dokl. Akad. Nauk SSSR*, 151:1293–1294, 1963. (Russian).

[81] V. Jankov. The construction of a sequence of strongly independent superintuitionistic propositional calculi. *Soviet Math. Dokl.*, 9:806–807, 1968.

[82] E. Jeřábek. Admissible rules of modal logics. *J. Logic Comput.*, 15(4):411–431, 2005.

[83] E. Jeřábek. Canonical rules. *J. Symbolic Logic*, 74(4):1171–1205, 2009.

[84] P.T. Johnstone. *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Reprint of the 1982 edition.

[85] D. de Jongh. *Investigations on the Intuitionistic Propositional Calculus*. PhD thesis, University of Wisconsin, 1968.

[86] D. de Jongh and F. Yang. Jankov's theorems for intermediate logics in the setting of universal models. In N. Bezhanishvili et al., editor, *Proceedings of TbiLLC'09: International Conference on Logic, Language and Computation*, volume 6618 of *LNAI*, pages 53–76. Springer-Verlag, 2011.

[87] B. Jónsson and A. Tarski. Boolean algebras with operators. I. *Amer. J. Math.*, 73:891–939, 1951.

[88] M. Kracht. *Tools and Techniques in Modal Logic*. North-Holland Publishing Co., Amsterdam, 1999.

[89] M. Kracht. Modal consequence relations. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of modal logic*, pages 491–545. Elsevier, 2007.

[90] F. M. Lauridsen. Intermediate logics admitting a structural hypersequent calculus. Studia Logica, to appear. Available as ILLC Prepublication Series Report PP-2018-05.

[91] G.R. Renardel de Lavalette, A. Hendriks, and D. de Jongh. Intuitionistic Implication without Disjunction. *J. Log. Comp.*, 22(3):375–404, 2012.

[92] E. J. Lemmon. Algebraic semantics for modal logics. I. *J. Symbolic Logic*, 31:46–65, 1966.

[93] E. J. Lemmon. Algebraic semantics for modal logics. II. *J. Symbolic Logic*, 31:191–218, 1966.

[94] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[95] D. S. Macnab. *An algebraic study of modal operators on Heyting algebras with applications to topology and sheafification.* PhD thesis, University of Aberdeen, 1976.

[96] L. L. Maksimova. On maximal intermediate logics with the disjunction property. *Studia Logica*, 45(1):69–75, 1986.

[97] C. McKay. The decidability of certain intermediate propositional logics. *J. Symbolic Logic*, 33:258–264, 1968.

[98] J. C. C. McKinsey. A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology. *J. Symbolic Logic*, 6:117–134, 1941.

[99] J. C. C. McKinsey and A. Tarski. The algebra of topology. *Annals of Mathematics*, 45:141–191, 1944.

[100] J. C. C. McKinsey and A. Tarski. On closed elements in closure algebras. *Ann. of Math. (2)*, 47:122–162, 1946.

[101] P. Minari. Intermediate logics with the same disjunctionless fragment as intuitionistic logic. *Studia Logica*, 45(2):207–222, 1986.

[102] Ş. Minică. *Dynamic Logic of Questions.* PhD thesis, ILLC, University of Amsterdam, 2011.

[103] P. J. Morandi. Dualities in lattice theory. available at *http://sierra.nmsu.edu/morandi/notes/Duality.pdf*, 2005.

[104] I. Nishimura. On formulas of one variable in intuitionistic propositional calculus. *J. Symbolic Logic*, 25:327–331, 1960.

[105] H. Ono. On some intuitionistic modal logics. *Publ. Res. Inst. Math. Sci.*, 13(3):687–722, 1977/78.

[106] J. Plaza. Logics of public communications. In *Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems*, pages 201–216, 1989.

[107] J. Plaza. Logics of public communications. *Synthese*, 158(2):165–179, 2007.

[108] H. A. Priestley. Representation of distributive lattices by means of ordered Stone spaces. *Bull. London Math. Soc.*, 2:186–190, 1970.

[109] H. A. Priestley. Ordered topological spaces and the representation of distributive lattices. *Proc. London Math. Soc.*, 24:507–530, 1972.

[110] H. A. Priestley. The construction of spaces dual to pseudocomplemented distributive lattices. *Quart. J. Math. Oxford Ser. (2)*, 26(102):215–228, 1975.

[111] H. Rasiowa and R. Sikorski. *The Mathematics of Metamathematics*. Monografie Matematyczne, Tom 41. Państwowe Wydawnictwo Naukowe, Warsaw, 1963.

[112] W. Rautenberg. Splitting lattices of logics. *Arch. Math. Logik Grundlag.*, 20(3-4):155–159, 1980.

[113] L. Rieger. On the lattice theory of Brouwerian propositional logic. *Acta fac. rerum nat. Univ. Car.*, 189:1–40, 1949.

[114] P. H. Rodenburg. *Intuitionistic correspondence theory*. PhD thesis, University of Amsterdam, 1986.

[115] V. V. Rybakov. *Admissibility of Logical Inference Rules*. North-Holland Publishing Co., Amsterdam, 1997.

[116] H. Sahlqvist. Completeness and correspondence in the first and second order semantics for modal logic. In Stig Kanger, editor, *Proceedings of the Third Scandinavian Logic Symposium*, volume 82 of *Studies in Logic and the Foundations of Mathematics*, pages 110 – 143. Elsevier, 1975.

[117] K. Segerberg. *An Essay in Classical Modal Logic. Vols. 1, 2, 3.* Filosofiska Föreningen och Filosofiska Institutionen vid Uppsala Universitet, Uppsala, 1971.

[118] V. Shehtman. Rieger-Nishimura lattices. *Soviet Mathematics Doklady*, 19:1014–1018, 1978.

[119] T. Shimura. Kripke completeness of some intermediate predicate logics with the axiom of constant domain and a variant of canonical formulas. *Studia Logica*, 52(1):23–40, 1993.

[120] M. H. Stone. The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, 40(1):37–111, 1936.

[121] E. Tomaszewski. *On Sufficiently Rich Sets of Formulas*. PhD thesis, Institute of Philosophy, Jagellonian University, Kraków, 2003.

[122] Y. Venema. A dual characterization of subdirectly irreducible BAOs. *Studia Logica*, 77(1):105–115, 2004.

[123] Y. Venema. Algebras and coalgebras. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of modal logic*, pages 331–426. Elsevier, 2007.

[124] A. Visser. Evaluation, provably deductive equivalence in Heyting's Arithmetic. *Technical Report 4, Dept. of Philosophy, Utrecht University*, 1985.

[125] A. Visser, D. de Jongh, J. van Benthem, and G. Renardel de Lavalette. NNIL a study in intuitionistic logic. In A. Ponse, M. de Rijke, and Y. Venema, editors, *Modal logics and Process Algebra: a bisimulation perspective*, pages 289–326, 1995.

[126] F. Wolter. *Lattices of modal logics.* PhD thesis, FU Berlin, 1993.

[127] F. Wolter. The structure of lattices of subframe logics. *Ann. Pure Appl. Logic*, 86(1):47–100, 1997.

[128] F. Wolter and M. Zakharyaschev. Intuitionistic modal logic. In *Logic and foundations of mathematics (Florence, 1995)*, volume 280 of *Synthese Lib.*, pages 227–238. Kluwer Acad. Publ., Dordrecht, 1999.

[129] F. Wolter and M. Zakharyaschev. Intuitionistic modal logics as fragments of classical bimodal logics. In *Logic at work*, volume 24 of *Stud. Fuzziness Soft Comput.*, pages 168–183. Physica, Heidelberg, 1999.

[130] F. Yang. Intuitionistic subframe formulas, NNIL-formulas and n-universal models. Master's Thesis, MoL-2008-12, ILLC, University of Amsterdam, 2008.

[131] M. Zakharyaschev. The disjunction property of superintuitionistic and modal logics. *Mat. Zametki*, 42(5):729–738, 763, 1987.

[132] M. Zakharyaschev. Syntax and semantics of modal logics that contain S4. *Algebra and Logic*, 27(6):408–428, 1988.

[133] M. Zakharyaschev. Syntax and semantics of superintuitionistic logics. *Algebra and Logic*, 28(4):262–282, 1989.

[134] M. Zakharyaschev. Canonical formulas for K4. I. Basic results. *J. Symbolic Logic*, 57(4):1377–1402, 1992.

[135] M. Zakharyaschev. Canonical formulas for K4. II. Cofinal subframe logics. *J. Symbolic Logic*, 61(2):421–449, 1996.

[136] M. Zakharyaschev. Canonical formulas for modal and superintuitionistic logics: a short outline. In *Advances in intensional logic*, volume 7 of *Appl. Log. Ser.*, pages 195–248. Kluwer Acad. Publ., Dordrecht, 1997.

[137] M. Zakharyaschev. Canonical formulas for K4. III. The finite model property. *J. Symbolic Logic*, 62(3):950–975, 1997.

Samenvatting

Het overkoepelende thema in dit proefschrift is de notie van stabiliteit in de context van superintuïtionistische (s.i.) logica's en normale modale logica's. Stabiliteit verwijst hier naar een klasse van (Kripke)frames of modellen die gesloten is onder relatiebewarende beelden. Een logica wordt *stabiel* genoemd als deze gekarakteriseerd wordt door een klasse van frames die gesloten is onder relatiebewarende beelden. Omdat de standaard filtratiemethode relatiebewarende beelden produceert, kan filtratie toegepast worden om de eindige-modeleigenschap (e.m.e.) van stabiele logica's te bewijzen. Stabiele logica's kunnen dus gezien worden als de "filtratie"-analoog van de bekende transitieve modale en s.i. subframelogica's, welke de logica's zijn waarvan de e.m.e. bewezen kan worden via selectieve filtratie.

In dit proefschrift onderzoeken we de klasse van stabiele modale logica's en veralgemeniseringen naar M-stabiele logica's. Daarnaast bestuderen we de relaties tussen stabiele logica's in de modale en de intuïtionistische setting. Tot slot bekijken we stabiliteit in de context van dynamische (epistemische) logica. Als leidend voorbeeld in onze studie dienen vaak bekende eigenschappen van deelframelogica's. Inderdaad is een centraal doel van dit proefschrift het verkennen van relaties tussen stabiele en deelframelogica's door het identificeren van gedeelde eigenschappen en verschillen.

We beschrijven nu in detail de inhoud van de belangrijkste hoofdstukken. In Hoofdstuk 3 identificeren we *cofinale stabiele s.i. logica's* als de stabiele analogen van cofinale s.i. deelframelogica's, en bestuderen we eigenschappen van deze logica's. Verder presenteren we een unificerend perspectief op $\mathcal{H}$-stabiele s.i. logica's, waar $\mathcal{H}$ een lokaal eindig reduct van Heytingalgebra's is die subframe-, cofinale subframe-, en stabiele s.i. logica's beslaat. In Hoofdstuk 4 breiden we onze studie uit naar *stabiele modale logica's*. In het bijzonder onderzoeken we het gedrag van stabiliteit met betrekking tot modale metgezellen en intuïtionistische fragmenten. Ook verklaren we overeenkomsten en verschillen tussen stabiele en subframe modale logica's. Hoofdstuk 5 behandelt *canonieke formules* vanuit een algebraïsch perspectief in de vorm van een bespreking en samenvatting van resultaten uit de literatuur. In Hoofdstuk 6 versterken we de *parallellen tussen stabiele en subframe s.i. logica's* door deze te verbinden met modale operatoren op Heytingalgebra's, namelijk de *lakse modaliteit* en de *inwendige operator*. In Hoofdstuk 7 werpen we een frisse blik op de klasse van NNIL-*formules* via stabiele afbeeldingen. In het bijzonder geven we volledige beschrijvingen van de n-universele modellen voor NNIL-formules. Ook geven we alternatieve bewijzen voor de bestaande resultaten dat logica's die geaxiomatiseerd zijn door NNIL-formules de e.m.e. hebben en canoniek zijn. In Hoofdstuk 8, ten slotte, onderzoeken we stabiliteit in de context van dynamisch-epistemische logica. We behandelen beelden van modellen onder *stabiele afbeeldingen als model-transformatieoperaties*. Deze operaties leiden tot dynamische logica's met abstractiemodaliteiten. We bewijzen volledigheidsresultaten voor deze logica's via reductie. We leggen uit dat in sommige bijzondere gevallen deze logica's gezien kunnen worden als *filtratielogica's*.

Abstract

The overarching theme in this thesis is the notion of stability in the context of superintuitionistic (si logic) and normal modal logics. Stability here refers to a class of (Kripke) frames or models that is closed under relation preserving images. A logic is called *stable* if it is characterized by a class of frames closed under relation preserving images. Since the standard filtration method produces relation preserving images, filtration can be applied to prove the finite model property (fmp) of stable logics. Thus, stable logics can be seen as the "filtration"-analogues of the well known transitive subframe modal and subframe si logics that are the logics whose fmp can be proved via selective filtration.

In this thesis we investigate the class of stable modal logics and its generalization to M-stable logics. Moreover, we study the relations between stable logics in the modal and intuitionistic settings. Finally, we explore stability in the context of dynamic (epistemic) logic. Known properties of subframe logics are often the guiding examples in our study. In fact, a central goal of the thesis is to explore relations between stable and subframe logics by identifying common features and differences.

We describe the content of the main chapters in detail. In Chapter 3 we introduce *cofinal stable si logics* as the stable analogues of cofinal subframe si logics and study properties of these logics. Moreover, we provide a unified look on $\mathcal{H}$-stable si logics, where $\mathcal{H}$ is a locally finite reduct of Heyting algebras encompassing subframe, cofinal subframe, and stable si logics. In Chapter 4 we expand the study of *stable modal logics*. In particular, we investigate the behavior of stability with respect to modal companions and intuitionistic fragments. We also explain similarities and differences between stable and subframe modal logics. Chapter 5 discuss *canonical formulas* from an algebraic perspective by reviewing and summarizing results from the literature. In Chapter 6 we strengthen the *parallels between stable and subframe si logics* by connecting them to modal operators on Heyting algebras, namely the *lax modality* and the *interior operator*, respectively. In Chapter 7 we take a fresh look at the class of NNIL-*formulas* via stable maps. In particular, we give full descriptions of the n-universal models for NNIL-formulas. We also provide alternative proofs of the known results that logics axiomatized by NNIL-formulas have the fmp and are canonical. Finally, in Chapter 8 we investigate stability in the context of dynamic epistemic logic. We treat images of models under *stable maps as model-transformation operations*. These operations give rise to dynamic logics with abstraction modalities. We prove completeness results for these logics via reduction. We explain that in some special cases, these logics can be regarded as *logics of filtration*.

Titles in the ILLC Dissertation Series:

ILLC DS-2009-01: **Jakub Szymanik**
Quantifiers in TIME and SPACE. Computational Complexity of Generalized Quantifiers in Natural Language

ILLC DS-2009-02: **Hartmut Fitz**
Neural Syntax

ILLC DS-2009-03: **Brian Thomas Semmes**
A Game for the Borel Functions

ILLC DS-2009-04: **Sara L. Uckelman**
Modalities in Medieval Logic

ILLC DS-2009-05: **Andreas Witzel**
Knowledge and Games: Theory and Implementation

ILLC DS-2009-06: **Chantal Bax**
Subjectivity after Wittgenstein. Wittgenstein's embodied and embedded subject and the debate about the death of man.

ILLC DS-2009-07: **Kata Balogh**
Theme with Variations. A Context-based Analysis of Focus

ILLC DS-2009-08: **Tomohiro Hoshi**
Epistemic Dynamics and Protocol Information

ILLC DS-2009-09: **Olivia Ladinig**
Temporal expectations and their violations

ILLC DS-2009-10: **Tikitu de Jager**
"Now that you mention it, I wonder...": Awareness, Attention, Assumption

ILLC DS-2009-11: **Michael Franke**
Signal to Act: Game Theory in Pragmatics

ILLC DS-2009-12: **Joel Uckelman**
More Than the Sum of Its Parts: Compact Preference Representation Over Combinatorial Domains

ILLC DS-2009-13: **Stefan Bold**
Cardinals as Ultrapowers. A Canonical Measure Analysis under the Axiom of Determinacy.

ILLC DS-2010-01: **Reut Tsarfaty**
Relational-Realizational Parsing

ILLC DS-2010-02: **Jonathan Zvesper**
Playing with Information

ILLC DS-2010-03: **Cédric Dégremont**
The Temporal Mind. Observations on the logic of belief change in interactive systems

ILLC DS-2010-04: **Daisuke Ikegami**
Games in Set Theory and Logic

ILLC DS-2010-05: **Jarmo Kontinen**
Coherence and Complexity in Fragments of Dependence Logic

ILLC DS-2010-06: **Yanjing Wang**
Epistemic Modelling and Protocol Dynamics

ILLC DS-2010-07: **Marc Staudacher**
Use theories of meaning between conventions and social norms

ILLC DS-2010-08: **Amélie Gheerbrant**
Fixed-Point Logics on Trees

ILLC DS-2010-09: **Gaëlle Fontaine**
Modal Fixpoint Logic: Some Model Theoretic Questions

ILLC DS-2010-10: **Jacob Vosmaer**
Logic, Algebra and Topology. Investigations into canonical extensions, duality theory and point-free topology.

ILLC DS-2010-11: **Nina Gierasimczuk**
Knowing One's Limits. Logical Analysis of Inductive Inference

ILLC DS-2010-12: **Martin Mose Bentzen**
Stit, Iit, and Deontic Logic for Action Types

ILLC DS-2011-01: **Wouter M. Koolen**
Combining Strategies Efficiently: High-Quality Decisions from Conflicting Advice

ILLC DS-2011-02: **Fernando Raymundo Velazquez-Quesada**
Small steps in dynamics of information

ILLC DS-2011-03: **Marijn Koolen**
The Meaning of Structure: the Value of Link Evidence for Information Retrieval

ILLC DS-2011-04: **Junte Zhang**
System Evaluation of Archival Description and Access

ILLC DS-2011-05: **Lauri Keskinen**
Characterizing All Models in Infinite Cardinalities

ILLC DS-2011-06: **Rianne Kaptein**
Effective Focused Retrieval by Exploiting Query Context and Document Structure

ILLC DS-2011-07: **Jop Briët**
Grothendieck Inequalities, Nonlocal Games and Optimization

ILLC DS-2011-08: **Stefan Minica**
Dynamic Logic of Questions

ILLC DS-2011-09: **Raul Andres Leal**
Modalities Through the Looking Glass: A study on coalgebraic modal logic and their applications

ILLC DS-2011-10: **Lena Kurzen**
Complexity in Interaction

ILLC DS-2011-11: **Gideon Borensztajn**
The neural basis of structure in language

ILLC DS-2012-01: **Federico Sangati**
Decomposing and Regenerating Syntactic Trees

ILLC DS-2012-02: **Markos Mylonakis**
Learning the Latent Structure of Translation

ILLC DS-2012-03: **Edgar José Andrade Lotero**
Models of Language: Towards a practice-based account of information in natural language

ILLC DS-2012-04: **Yurii Khomskii**
Regularity Properties and Definability in the Real Number Continuum: idealized forcing, polarized partitions, Hausdorff gaps and mad families in the projective hierarchy.

ILLC DS-2012-05: **David García Soriano**
Query-Efficient Computation in Property Testing and Learning Theory

ILLC DS-2012-06: **Dimitris Gakis**
Contextual Metaphilosophy - The Case of Wittgenstein

ILLC DS-2012-07: **Pietro Galliani**
The Dynamics of Imperfect Information

ILLC DS-2012-08: **Umberto Grandi**
Binary Aggregation with Integrity Constraints

ILLC DS-2012-09: **Wesley Halcrow Holliday**
Knowing What Follows: Epistemic Closure and Epistemic Logic

ILLC DS-2012-10: **Jeremy Meyers**
Locations, Bodies, and Sets: A model theoretic investigation into nominalistic mereologies

ILLC DS-2012-11: **Floor Sietsma**
Logics of Communication and Knowledge

ILLC DS-2012-12: **Joris Dormans**
Engineering emergence: applied theory for game design

ILLC DS-2013-01: **Simon Pauw**
Size Matters: Grounding Quantifiers in Spatial Perception

ILLC DS-2013-02: **Virginie Fiutek**
Playing with Knowledge and Belief

ILLC DS-2013-03: **Giannicola Scarpa**
Quantum entanglement in non-local games, graph parameters and zero-error information theory

ILLC DS-2014-01: **Machiel Keestra**
Sculpting the Space of Actions. Explaining Human Action by Integrating Intentions and Mechanisms

ILLC DS-2014-02: **Thomas Icard**
The Algorithmic Mind: A Study of Inference in Action

ILLC DS-2014-03: **Harald A. Bastiaanse**
Very, Many, Small, Penguins

ILLC DS-2014-04: **Ben Rodenhäuser**
A Matter of Trust: Dynamic Attitudes in Epistemic Logic

ILLC DS-2015-01: **María Inés Crespo**
Affecting Meaning. Subjectivity and evaluativity in gradable adjectives.

ILLC DS-2015-02: **Mathias Winther Madsen**
The Kid, the Clerk, and the Gambler - Critical Studies in Statistics and Cognitive Science

ILLC DS-2015-03: **Shengyang Zhong**
Orthogonality and Quantum Geometry: Towards a Relational Reconstruction of Quantum Theory

ILLC DS-2015-04: **Sumit Sourabh**
Correspondence and Canonicity in Non-Classical Logic

ILLC DS-2015-05: **Facundo Carreiro**
Fragments of Fixpoint Logics: Automata and Expressiveness

ILLC DS-2016-01: **Ivano A. Ciardelli**
Questions in Logic

ILLC DS-2016-02: **Zoé Christoff**
Dynamic Logics of Networks: Information Flow and the Spread of Opinion

ILLC DS-2016-03: **Fleur Leonie Bouwer**
What do we need to hear a beat? The influence of attention, musical abilities, and accents on the perception of metrical rhythm

ILLC DS-2016-04: **Johannes Marti**
Interpreting Linguistic Behavior with Possible World Models

ILLC DS-2016-05: **Phong Lê**
Learning Vector Representations for Sentences - The Recursive Deep Learning Approach

ILLC DS-2016-06: **Gideon Maillette de Buy Wenniger**
Aligning the Foundations of Hierarchical Statistical Machine Translation

ILLC DS-2016-07: **Andreas van Cranenburgh**
Rich Statistical Parsing and Literary Language

ILLC DS-2016-08: **Florian Speelman**
Position-based Quantum Cryptography and Catalytic Computation

ILLC DS-2016-09: **Teresa Piovesan**
Quantum entanglement: insights via graph parameters and conic optimization

ILLC DS-2016-10: **Paula Henk**
Nonstandard Provability for Peano Arithmetic. A Modal Perspective

ILLC DS-2017-01: **Paolo Galeazzi**
Play Without Regret

ILLC DS-2017-02: **Riccardo Pinosio**
The Logic of Kant's Temporal Continuum

ILLC DS-2017-03: **Matthijs Westera**
Exhaustivity and intonation: a unified theory

ILLC DS-2017-04: **Giovanni Cinà**
Categories for the working modal logician

ILLC DS-2017-05: **Shane Noah Steinert-Threlkeld**
Communication and Computation: New Questions About Compositionality

ILLC DS-2017-06: **Peter Hawke**
The Problem of Epistemic Relevance

ILLC DS-2017-07: **Aybüke Özgün**
Evidence in Epistemic Logic: A Topological Perspective

ILLC DS-2017-08: **Raquel Garrido Alhama**
Computational Modelling of Artificial Language Learning: Retention, Recognition & Recurrence

ILLC DS-2017-09: **Miloš Stanojević**
Permutation Forests for Modeling Word Order in Machine Translation

ILLC DS-2018-01: **Berit Janssen**
Retained or Lost in Transmission? Analyzing and Predicting Stability in Dutch Folk Songs

ILLC DS-2018-02: **Hugo Huurdeman**
Supporting the Complex Dynamics of the Information Seeking Process

ILLC DS-2018-03: **Corina Koolen**
Reading beyond the female: The relationship between perception of author gender and literary quality

ILLC DS-2018-04: **Jelle Bruineberg**
Anticipating Affordances: Intentionality in self-organizing brain-body-environment systems

ILLC DS-2018-05: **Joachim Daiber**
Typologically Robust Statistical Machine Translation: Understanding and Exploiting Differences and Similarities Between Languages in Machine Translation

ILLC DS-2018-06: **Thomas Brochhagen**
Signaling under Uncertainty

ILLC DS-2018-07: **Julian Schlöder**
Assertion and Rejection

ILLC DS-2018-08: **Srinivasan Arunachalam**
Quantum Algorithms and Learning Theory

ILLC DS-2018-09: **Hugo de Holanda Cunha Nobrega**
Games for functions: Baire classes, Weihrauch degrees, transfinite computations, and ranks

ILLC DS-2018-10: **Chenwei Shi**
Reason to Believe

ILLC DS-2018-11: **Malvin Gattinger**
New Directions in Model Checking Dynamic Epistemic Logic