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The theory of the generalised real numbers and other topics in logic

Dissertation

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Contents

Chapter 1

Introduction

1.1 General introduction

The *real line* $\mathbb{R}$, i.e., the Dedekind completion of the rational numbers, and *Baire space* ω^ω, i.e., the topological space of countable sequences of natural numbers, are two of the most fundamental objects in set theory. These spaces have been studied extensively and are the main objects of investigation of descriptive set theory, i.e., the branch of set theory which is devoted to the study of properties of subsets of the real line. Particularly important in this context is the strict relationship between the real line and Baire space. On the one hand, one can define appropriate mappings between the two spaces which allow to transfer many properties from one space to the other; see, e.g., [73]. On the other hand, Baire space has a combinatorially simpler structure than the real line. This fact can sometimes be exploited to simplify the study of the real line. These characteristics have been used very successfully in many different contexts in set theory, and in descriptive set theory in particular.

In recent years, set theorists have become increasingly interested in *generalised Baire spaces* κ^κ, i.e., the sets of functions from κ to κ for an uncountable cardinal κ. Some of the classical results for Baire space generalise to the uncountable case (e.g., the generalised version of the Souslin-Kleene Theorem for the game theoretical characterisation of Borel sets; see [71, Corollary 35]), but others do not (e.g., the generalisation of the Blackwell equivalence between the game theoretical and the classical definitions of Borel sets; see [32, Theorem 18]). Further examples of some of these results can be found in [32, 33, 41, 42, 61, 66, 67, 71].

The fact that some classical results fail to generalise is particularly interesting: these failures frequently shed light on structures and properties hidden in the classical framework. The study of generalised Baire spaces and generalised descriptive set theory has become one of the most vibrant research topic in set theory in recent years; see [52] for an overview on the subject and a list of open questions.

Lifting the symmetry between the real line and Baire space to the generalised case would give a better generalisation of the classical framework. To achieve this, a generalised version of the real line is needed.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, leftrightarrow] \arrow[d, dashed, "\text{Generalisation}"'] & \omega^\omega \arrow[d, dashed, "\text{Generalisation}"] \\
? \arrow[r, leftrightarrow] & \kappa^\kappa
\end{tikzcd}$$

As the results in [35, 36] and in this thesis show, no generalisation of the real line can have all of the desirable properties, and different applications usually need different versions of these weakened properties. Nevertheless, in recent years only two of these generalisations have been proposed as suitable generalisations of the real line in the context of generalised descriptive set theory. The first generalisation, which we will call the *real ordinal numbers*, is due to Sikorski [89] and was recently studied by Asperó and Tsaprounis [3] who also proposed it as a suitable space for generalised descriptive set theory and generalised real analysis. The second space, which we will call *generalised real line*, is based on Conway's surreal numbers and was introduced by the author in [35] as a generalisation of the real line suitable both for generalised descriptive set theory and generalised real analysis. As shown in [35, 36], the generalised real line has many properties which make it a very suitable space to do real analysis. In particular, the author proved that appropriate versions of the *Intermediate value theorem* and of the *Extreme value theorem* hold over the generalised real line.

The discrete combinatorial nature of Baire space and as a space of infinite sequences of natural numbers makes it a natural space for studying computational processes. The area of *computable analysis* uses this fact to reduce computational properties of the reals to computational properties of Baire space, making once more use of the close connection between the two. Particularly important in this context is the theory of *Weihrauch degrees* introduced by Weihrauch as a framework to formalise a notion of computational complexity which can be used to study theorems from real analysis; see, e.g., [13] for an introduction to the theory of Weihrauch degrees.

The study of a generalisation of the theory of Weihrauch degrees to the uncountable case was sterted by the author in his Master's thesis [35] and was continued in [38]. The generalised real line was shown to be a very natural space to develop such a theory. Indeed, the generalised real line carries a natural notion of computability which, as we will see in this thesis, can be exploited to generalise to the transfinite notions of computability which, in the classical case, are based on the real number continuum.

1.2 Organisation of the thesis

In this section we will explain the organisation of the thesis, listing the main results of each chapter. Theorems and corollaries in this section are not numbered; we will provide their number in parenthesis together with the corresponding page in the thesis where they are stated and proved. Finally, note that statements of theorems and corollaries in this section use notions and notation defined later in the thesis. The reader is not expected at this point to understand these notions, they are provided here for the sake of clarity and organisation.

This thesis is organised as follows:

In the rest of this chapter we present some of the basics needed in the rest of the thesis.

In Chapter 2 we briefly introduce the two generalised versions of the real line studied in this thesis. Then, we use these spaces in the context of generalised metrisability theory and generalised descriptive set theory. In particular, we use generalised metrisability theory to define a generalised notion of Polish spaces which we will compare and combine with the game theoretical notion introduced by Coskey and Schlicht in [22]. The main results of this chapter are illustrated in the following diagram which shows that a partial generalisation of the classical equivalence between Polish spaces, G_δ spaces, and strongly Choquet spaces (see [51, Theorem 8.17.ii]) can be proved in the generalised context:

$$
\begin{tikzcd}
Y \text{ is strongly } \lambda\text{-Polish} \arrow[r] \arrow[dr, shift left] & Y \text{ is strongly } \lambda\text{-Choquet} \arrow[r, dotted, shift left] & Y \text{ is } \lambda\text{-}\mathrm{G}_\delta \text{ in } X \arrow[l, dotted, shift left] \\
& \mathrm{Y} \text{ is } \lambda\text{-Polish} \arrow[ul, shift left, "/" marking] \arrow[u, "/" marking] \arrow[ur] &
\end{tikzcd}
$$

In the previous diagram an arrow from A to B means that A implies B; a crossed arrow from A to B means that A does not imply B; and dotted arrows are used to emphasise the fact that further assumptions on Y or λ are needed. See p. 25 for a complete explanation of these results.

In Chapter 3 we study generalisations of the Bolzano-Weierstraß and Heine-Borel theorems. We consider various versions of these theorems and we fully characterise them in terms of large cardinal properties of the cardinal underlining the generalised real line. In particular we prove the following:

Corollary (Corollary 3.23, p. 53). *Let κ be an uncountable strongly inaccessible cardinal and let $(K, +, \cdot, 0, 1, \leq)$ be a Cauchy complete and κ-spherically complete totally ordered field with* $\mathrm{bn}(K) = \kappa$. *Then the following are equivalent:*

1. *κ has the tree property and*

2. *κ-wBWT_K holds.*

In particular κ has the tree property if and only κ-$\mathsf{wBWT}_{\mathbb{R}_\kappa}$ holds.

In Chapter 4 we use the generalised real line to develop two new models of transfinite computability, one generalising the so called type two Turing machines and one generalising Blum, Shub and Smale machines, i.e, a model of computation introduced by Blum, Shub and Smale in order to define notions of computation over arbitrary fields. Moreover, we use the generalised version of type two Turing machines to begin the development of a generalised version of the classical theory of Weihrauch degrees. In Chapter 4 we prove the following generalised version of a classical result in the theory of Weihrauch degrees:

Theorem (Theorem 4.24, p. 68). 1. *If there exists an effective enumeration of a dense subset of $\mathbb{R}_\kappa$, then* $\mathrm{IVT}_\kappa \leq_{\mathrm{sW}} \mathrm{B}_{\mathrm{I}}^\kappa$.

2. *We have* $\mathrm{B}_{\mathrm{I}}^\kappa \leq_{\mathrm{sW}} \mathrm{IVT}_\kappa$.

3. *We have* $\mathrm{IVT}_\kappa \leq^{\mathrm{t}}_{\mathrm{sW}} \mathrm{B}^\kappa_{\mathrm{I}}$*, and therefore* $\mathrm{IVT}_\kappa \equiv^{\mathrm{t}}_{\mathrm{sW}} \mathrm{B}^\kappa_{\mathrm{I}}$.

The last two chapters of this thesis are the result of the work of the author on topics in logic which are not directly related to generalisations of the real number continuum.

In Chapter 5 we study the possible order types of models of syntactic fragments of Peano arithmetic. The main result of this chapter is that the following arrow diagram between fragments of PA is *complete with respect to order types of their models*. By this we mean that an arrow from the theory T to the theory T' means that every order type occurring in a model of T also occurs in a model of T' and a missing arrow means that there is a model of T of an order type that cannot be an order type of a model of T'.

$$\begin{tikzcd}
 & \mathsf{SA} & \\
\mathsf{Pr}^- \arrow[ur] & & \mathsf{Pr} \arrow[ul] \arrow[ll] \\
\mathsf{PA}^- \arrow[u] & & \mathsf{PA} \arrow[u] \arrow[ll]
\end{tikzcd}$$

In Chapter 6 we study Löwenheim-Skolem theorems for logics extending first order logic. In particular, we extend the work done by Bagaria and Väänänen in [5] relating upward Löwenheim-Skolem theorems for strong logics to reflection principles in set theory. Our main result in this area is the following theorem:

Theorem (Theorem 6.49, p. 123)**.** *Let $\mathcal{L}^*$ be a logic and R be a predicate in the language of set theory such that $\mathcal{L}^*$ and R are bounded symbiotic and $\mathcal{L}^*$ has* $\mathrm{dep}(\mathcal{L}^*) = \omega$ *and is $\Delta^{\mathrm{B}}_1(R)$-finitely-definable. Moreover, let λ be a cardinal such that there is a sequence $(\delta_n)_{n\in\omega}$ of $\Delta^{\mathrm{B}}_1(R)$-definable cardinals such that $\bigcup_{n\in\omega} \delta_n = \lambda$. Then the following are equivalent:*

1. $\mathsf{ULST}_\lambda(\mathcal{L}^*) = \kappa$ *and*

2. $\mathcal{USR}_\lambda(R) = \kappa$.

In particular, the statement holds for $\lambda = \omega$ and in general for all the logics in [5, Proposition 4].

Finally, we apply the previous result to the study of the large cardinal strength of the upward Löwenheim-Skolem theorem for second order logic; we provide both upper and lower bounds.

1.3 Basics

1.3.1 Set theory

For set theoretic notions we will mostly use the usual notation; see, e.g., [48, 60].

Ordinals and cardinals

We will denote by ZF the axioms of set theory formulated with the collection schema rather than the replacement schema. As usual, ZFC denotes the theory obtained by adding to ZF the axiom of choice. The class of ordinal numbers will be denoted by Ord. As usual, the von Neumann *universe of sets* will be denoted by $\mathbf{V}$; and given an ordinal α, we will denote by $\mathbf{V}_\alpha$ the αth level of the von Neumann universe of sets. We will denote by $\mathbf{L}$ the *constructible universe* of set theory; and if α is an ordinal we will denote by $\mathbf{L}_\alpha$ the αth level of the constructible hierarchy; see, e.g., [48, Chapter 13]. The *Continuum Hypothesis* and the *Generalised Continuum Hypothesis* will be denoted by CH and GCH, respectively. Given a set X the set $\mathrm{TC}(X)$ is the *transitive closure* of X; and for every cardinal λ we will call $\mathbf{H}(\lambda) := \{X\,;\,|\mathrm{TC}(X)| < \lambda\}$ the family of *sets of hereditary cardinality* $<\lambda$. Let I and X be two sets and $f : I \to X$ be a function. We will denote by $\bigtimes_{i\in I} f(i)$ the set of functions g whose domain is I and such that for every $i \in I$ we have $g(i) \in f(i)$. Given a set X we will denote the *power set* of X by $\wp(X)$; and given a cardinal λ we will denote the collection of subsets of X of size $<\lambda$ by $[X]^{<\lambda}$. If κ is a cardinal we will denote by κ^+ its cardinal successor. Moreover, for every $n \in \omega$ we will denote by κ^{+n} the nth successor of n; and by $\kappa^{+\omega}$ the supremum of $\{\kappa^{+n}\,;\,n \in \omega\}$.

Given an ordinal α we will use X^α to denote the set of total functions from α to X and we will denote by $X^{<\alpha}$ the set of total functions whose domain is an ordinal $<\alpha$. We will call the elements of X^α *sequences of length* α *on* X or just α*-sequences on* X; and the elements of $X^{<\alpha}$ *sequences of length less than* α *on* X. We will use the word *sequences* for sequences whose length is not specified. As usual we will often use the notation $(s_\beta)_{\beta<\alpha}$ to denote the α-sequence s such that $s(\alpha) = s_\alpha$. Given an α-sequence s on X and an element $x \in X$ the *concatenation* sx of s and x is the sequence of length $\alpha + 1$ such that $sx(\beta) = s(\beta)$ if $\beta < \alpha$ and $sx(\alpha) = x$.

If α is an ordinal, we will say that α is a *delta number* if and only if it is an ordinal number closed under ordinal multiplication. Similarly, we will say that α is an *epsilon number* if and only if it an ordinal number closed under ordinal exponentiation.

The following relation is a well-ordering of the class of pairs of ordinal numbers: $(\alpha_0, \beta_0) \prec (\alpha_1, \beta_1)$ iff $(\max(\alpha_0, \beta_0), \alpha_0, \beta_0)$ is lexicographically less than $(\max(\alpha_1, \beta_1), \alpha_1, \beta_1)$. The *Gödel pairing function* is given by $\mathfrak{g}(\alpha, \beta) = \gamma$ iff (α, β) is the γ^{th} element in $\prec$.

Let κ be a regular cardinal and X be a set. Given two κ-sequences s and s' of elements in $X^{<\kappa}$ and X^κ, respectively, we define $[s_\alpha]_{\alpha<\kappa}$ to be the concatenation of the $s(\alpha)$, and $\langle s'_\alpha\rangle_{\alpha<\kappa}$ to be the sequence $p \in X^\kappa$ such that $p(\mathfrak{g}(\alpha, \beta)) = s'(\alpha)(\beta)$.

Given ordinals α and β let $\gamma_1 > \ldots > \gamma_n$ be ordinals and $m_1, \ldots m_n$ and $m'_1, \ldots, m'_n$ be two sequences of natural numbers such that: for every $0 < i < n+1$ we have $m_i + m'_i > 0$; $\alpha = \omega^{\gamma_1} m_1 + \ldots + \omega^{\gamma_n} m_n$ and $\beta = \omega^{\gamma_1} m'_1 + \ldots + \omega^{\gamma_n} m'_n$. Then, we define *Hessenberg addition* $\oplus$ as follows:

$$\alpha \oplus \beta := \omega^{\gamma_1}(m_1 + m'_1) + \ldots + \omega^{\gamma_n}(m_n + m'_n).$$

Note that $\alpha \oplus \beta$ is the polynomial addition of $\omega^{\gamma_1} m_1 + \ldots + \omega^{\gamma_n} m_n$ and $\omega^{\gamma_1} m'_1 + \ldots + \omega^{\gamma_n} m'_n$ where $\omega^{\gamma_1} m_1 + \ldots + \omega^{\gamma_n} m_n$ and $\omega^{\gamma_1} m'_1 + \ldots + \omega^{\gamma_n} m'_n$ are considered as polynomials in ω. Similarly, by using polynomial multiplication, one can define *Hessenberg multiplication* $\otimes$. These operations are sometimes called *natural operations* in the literature; and are commutative, associative, 0 is the identity for $\oplus$, 1 is the identity for $\otimes$, and they satisfy the usual distributive laws.

Given a cardinal κ and an ordinal α we say that κ is *weakly α-extendible* iff there is an ordinal β and an elementary embedding $J : \mathbf{V}_{\kappa+\alpha} \to \mathbf{V}_\beta$ such that κ is the critical point of J. If in addition, $\alpha < J(\kappa)$, then we say that κ is *α-extendible*. Note that if $\alpha < \kappa$, then the two definitions are equivalent.

We say that a cardinal κ is *extendible* if it is α-extendible for every ordinal α.

Note that the additional requirement that $J(\kappa) > \alpha$ is not relevant for full extendibility. Indeed, one can prove the following theorem:

Theorem 1.1. *Let κ be a cardinal. Then the following are equivalent:*

1. *κ is extendible;*

2. *for any $\alpha > \kappa$ the cardinal κ is weakly α-extendible.*

Proof. See [49, Proposition 23.15(b)]. □

Let κ be a cardinal and α be an ordinal. An *ultrafilter* F over $[\alpha]^{<\kappa}$ is a subset of $\wp([\alpha]^{<\kappa})$ such that

1. $[\alpha]^{<\kappa} \in F$,

2. F is closed under finite intersections,

3. if $A \in F$ and $A \subseteq B$ then $B \in F$,

4. $\varnothing \notin F$,

5. for every $A \in \wp([\alpha]^{<\kappa})$ either $A \in F$ or $\wp([\alpha]^{<\kappa}) \setminus A \in F$.

An ultrafilter F is *normal* if the following hold:

1. F is κ-complete, i.e., F is closed under intersections of size $<\kappa$,

2. for any $\beta \in \alpha$ we have $\{x \in [\alpha]^{<\kappa} ; \beta \in x\} \in F$,

3. F is closed under diagonal intersections, i.e., for every sequence $\langle X_i ; i \in [\alpha]^{<\kappa}\rangle$ of elements of F we have $\{x \in \wp([\alpha]^{<\kappa}) ; x \in \bigcap_{i\in x} X_i\} \in F$.

A cardinal κ is *γ-supercompact* if there is a normal κ-complete ultrafilter over $[\gamma]^{<\kappa}$. As usual we say that κ is $<\gamma$-supercompact if it is λ-supercompact for every $\lambda < \gamma$. Finally, a cardinal is *supercompact* if it is γ-supercompact for every $\gamma \geq \kappa$.

Supercompactness is a strengthening of the large cardinal notions of weak compactness and strong compactness ([49, Chapter 5]); weak compactness will play a role in our Chapters 2 & 3 and is defined in terms of trees in the corresponding section.

Extendible cardinals are strictly bigger than supercompact cardinals.

Theorem 1.2. *If κ is α-extendible and $\beta + 2 \leq \alpha$ then κ is $|\mathbf{V}_{\kappa+\beta}|$-supercompact. In particular every extendible is supercompact.*

Proof. See [49, Proposition 23.6]. □

Trees

As usual, a tree is a partial order $(T, \leq)$ such that for each $t \in T$, the set $\mathrm{pred}_T(t) := \{s \in T\,;\, s < t\}$ is wellordered by $<$. The height of t in T, denoted by $\mathrm{ht}_T(t)$ is the order type of $\mathrm{pred}_T(t)$. We call $\mathrm{lvl}_T(\alpha) := \{t \in T\,;\, \mathrm{ht}_T(t) = \alpha\}$ the *αth level of the tree T*. The *height of the tree* is defined by $\mathrm{ht}(T) := \sup\{\alpha + 1\,;\, \mathrm{lvl}_T(\alpha) \neq \varnothing\}$. A *branch* of T is a maximal subset of T wellordered by $<$; the length of a branch is its order type. We will denote by $[T]$ the set of *branches* of T.

A tree $(T, <)$ is called *λ-tree* if $\mathrm{ht}(T) = \lambda$ and for all α, $|\mathrm{lvl}_T(\alpha)| < \lambda$; is called *$\lambda$-Kurepa tree* if it is a λ-tree with more than λ many branches; is called *(λ, κ)-Kurepa tree* if it is a λ-Kurepa tree with exactly κ many branches; and, is called *λ-Aronszajn tree* iff it is a λ-tree with no λ-branches.

A cardinal λ has the *tree property* if every λ-tree has a branch of length λ. A cardinal λ is *weakly compact* if it is strongly inaccessible and has the tree property; see, e.g., [49, Theorem 7.8]. Equivalently, a cardinal λ is *weakly compact* if $\lambda \to (\lambda)^2_2$ holds, i.e., if for every partition of $\lambda \times \lambda$ into two sets there is a subset H of λ of cardinality λ such that all the pairs of elements of H are all in the same set of the partition.

We will call *Kőnig's lemma* the statement "Every infinite tree has an infinite branch". The following weakening of Kőnig's lemma will be called *weak Kőnig's lemma*: "Every infinite binary tree has an infinite branch". We will denote Kőnig's lemma by KL and weak Kőnig's lemma by WKL. While it is clear that KL implies WKL, one can show that the implication cannot be reversed; see, e.g., [99, Theorem 4.3]. Note that the tree property is a natural generalisation of KL to uncountable cardinals; this fact will be used several times in this thesis to generalise to uncountable cardinals classical proofs which which involve KL (see, e.g., pp. 34 & 3.1).

Hierarchies of formulas

Given a natural number $n \in \mathbb{N}$ we will denote the nth level of the *Lévy hierarchy of formulas* by Σ_n, Π_n; see, e.g., [48, p. 183]. As usual we will say that a formula φ *is* Σ_n in symbols $\varphi \in \Sigma_n$ if it is equivalent in ZFC to a Σ_n formula. Similarly for Π_n formulas. Moreover, we will denote by Δ_n the set of formulas φ that are both Σ_n and Π_n. Given a predicate R in the language of set theory we will say that a formula is $\Delta_0(R)$ if it is Δ_0 in the language of set theory augmented with the predicate R. Similarly, for every positive $n \in \mathbb{N}$, we will denote by $\Sigma_n(R)$, $\Pi_n(R)$, and $\Delta_n(R)$ the classes of formulas which are Σ_n, Π_n, or Δ_n, respectively, in the language of set theory with a new predicate symbol for R.

For each natural number n, we will denote by ZFC_n the theory obtained from ZFC by restricting the axiom schemata of separation and collection to Σ_n formulas and we will denote by ZFC_n^- the theory obtained from ZFC_n by removing the Power Set axiom. Note that these theories are all finitely axiomatizable see [98].

Topology

We will use classical notation and terminology from topology; we will follow the notation in [74] and [51]. In particular, we will denote *topological spaces* by (X, τ) where τ is the set of open subsets of X. Given a topological space (X, τ) and a subset Y of X we denote the *closure* of Y, i.e., the smallest closed set containing Y, by $\overline{Y}$. A topological space

(X, τ) is *regular* if for every $x \in X$ and every open set $U \in \tau$ such that $x \in U$ there is an open set $V \in \tau$ such that $x \in V \subseteq \overline{V} \subseteq U$. Moreover, (X, τ) is *normal* if for every pair of closed disjoint subsets C and C' of X there are $U, V \in \tau$ such that $C \subseteq U$, $C' \subseteq V$, and $V \cap U = \varnothing$. A subset Y of the topological space (X, τ) is *connected* if it can not be written as the union of disjoint open sets. Finally, a space (X, τ) is said to be *totally disconnected* if the only connected subsets of X are either empty of singletons.

In the rest we will assume familiarity with basic results in descriptive set theory; see, e.g., [51]. We will denote by ω^ω and 2^ω Baire space and Cantor space, respectively, equipped with the usual product topologies.

For every positive natural number n we will denote by $\mathbf{\Sigma}^0_n$ and $\mathbf{\Pi}^0_n$ the nth level of the *Borel hierarchy*; see, e.g., [51, § 11.B].

Descriptive set theory, and in particular its generalisation to uncountable cardinal will play a central role in this thesis. Here we will present the basic definitions and results needed in this thesis. We refer the reader to [32] for a more complete introduction to generalised descriptive set theory.

Definition 1.3. Given an uncountable cardinal λ, the *generalised Cantor space* 2^λ is the set of binary sequences of length λ endowed with the topology induced by the following basic open sets:

$$[p] = \{q \in 2^\lambda\,;\, p \subset q\}$$

where $p \in 2^{<\lambda}$.

Definition 1.4. Given an uncountable cardinal λ, the *generalised Baire space* λ^λ is the set of binary sequences of length λ endowed with the topology induced by the following basic open sets:

$$[p] = \{q \in \lambda^\lambda\,;\, p \subset q\}$$

where $p \in \lambda^{<\lambda}$.

Under the assumption that $\lambda^{<\lambda} = \lambda$, generalised Cantor and Baire space behave very similarly to their classical counterparts. In particular:

Theorem 1.5 (Folklore)**.** *Let λ be an uncountable cardinal such that $\lambda^{<\lambda} = \lambda$. Then the following hold for both 2^λ and λ^λ:*

1. *the intersection of fewer than λ basic open sets is either empty or a basic open set,*
2. *the intersection of fewer than λ open sets is open,*
3. *basic open sets are closed,*
4. $|\{U \subseteq 2^\lambda\,;\, U \text{ is basic open}\}| = |\{U \subseteq \lambda^\lambda\,;\, U \text{ is basic open}\}| = \lambda$,
5. $|\{O \subseteq 2^\lambda\,;\, O \text{ is open}\}| = |\{O \subseteq \lambda^\lambda\,;\, O \text{ is open}\}| = 2^\lambda$.

1.3.2 Totally ordered sets

Let $(X, \leq)$ be any totally ordered set; as usual, we use $<$ for the irreflexive relation associated with $\leq$ ($x < y$ if and only if $x \leq y$ and $x \neq y$). We define sets $(y, z) := \{x \in X\,;\, y < x < z\}$, $(-\infty, z) := \{x \in X\,;\, x < z\}$, and $(z, \infty) := \{x \in X\,;\, z < x\}$. We call these sets *(open) intervals*; we topologise totally ordered sets by taking the topology generated by the open intervals. Intervals of the form (y, z) for $y, z \in X$ are called *proper intervals*; as usual, we define closed intervals $[y, z] := (y, z) \cup \{y, z\}$, and half-open intervals $(x, z] := (x, z) \cup \{z\}$ and $[y, x) := (y, x) \cup \{y\}$ for $x \in X \cup \{-\infty, \infty\}$. A subset $Z \subseteq X$ is *bounded* if it is contained in a proper interval. As usual, if $Y, Z \subseteq X$, we write $Y < Z$ if for all $y \in Y$ and all $z \in Z$, we have $y < z$. In order to reduce the number of braces, we write $y < Z$ for $\{y\} < Z$ and $Y < z$ for $Y < \{z\}$. A subset $Z \subseteq X$ is called *convex* if for any $z, z' \in Z$ and x such that $z \leq x \leq z'$, we have that $x \in Z$. Clearly, every interval is convex.

We call $Z \subseteq X$ *cofinal* if for every $x \in X$ there is a $z \in Z$ such that $x \leq z$; similarly, we call $Z \subseteq X$ *coinitial* if for every $x \in X$ there is a $z \in Z$ such that $z \leq x$. The *coinitiality* and the *cofinality* of a totally ordered set $(X, \leq)$ are the sizes of coinitial or cofinal sets minimal in cardinality, respectively, and we write $\mathrm{coi}(X, \leq)$ and $\mathrm{cof}(X, \leq)$ for them. If the order $\leq$ is implicitly clear, we omit it from the notation.

Let λ be a cardinal. We say that $(X, \leq)$ is an η_λ*-set* if for any $L, R \subseteq X$ such that $L < R$ and $|L| + |R| < \lambda$, there is $x \in X$ such that $L < x < R$.

The property of η_λ-ness relates to the model theoretic property of saturation: any densely ordered set $(X, \leq)$ without endpoints is λ-saturated in the sense of model theory if and only if it is an η_λ-set [19, Proposition 5.4.2].

As a consequence, we sometimes informally refer to the fact that a totally ordered set is an η_λ-set as "saturation".

We now introduce the notion of spherical completeness which is a weakening of saturation and known from the theory of ultrametrics; see, e.g., [83, § 20]. Let $\mathcal{I} = \{I_\gamma\,;\, \gamma < \alpha\}$ be a family of closed intervals. We call such a family *nested* if for $\gamma < \gamma'$, we have $I_\gamma \supseteq I_{\gamma'}$. Let $(X, \leq)$ be a totally ordered set, λ be a regular cardinal. Then $(X, \leq)$ is λ*-spherically complete* iff for every $\alpha < \lambda$ and for every nested family $\mathcal{I} = \{I_\gamma\,;\, \gamma < \alpha\}$ of closed intervals, we have that $\bigcap \mathcal{I} \neq \varnothing$.

Proposition 1.6. *Let $(X, \leq)$ be a totally ordered set and λ be a regular cardinal. If X is an η_λ-set, then X is λ-spherically complete.*

Proof. Let $\mathcal{I} = \{I_\gamma\,;\, \gamma < \alpha\}$ be a nested family of closed intervals with $I_\gamma = [x_\gamma, y_\gamma]$ for some $\alpha < \lambda$. Then apply saturation to the pair $(\{x_\gamma\,;\, \gamma < \alpha\}, \{y_\gamma\,;\, \gamma < \alpha\})$ to obtain an element in the intersection of $\mathcal{I}$. □

Note that there are λ-spherically complete ordered sets which are not η_λ-sets: e.g., the real line $\mathbb{R}$ is $\aleph_1$-spherically complete, but not an $\eta_{\aleph_1}$-set. Indeed, the $\aleph_1$-spherical completeness of $\mathbb{R}$ is a classical theorem from real analysis; see, e.g., [96, p. 43]. Moreover, since any η_λ-set must have cofinality $\geq \lambda$ and $\mathrm{cof}(\mathbb{R}) = \omega$, we have that $\mathbb{R}$ is not an $\eta_{\aleph_1}$-set.

1.3.3 Monoids, groups, and fields

An *monoid* is a structure $(M, 0, +)$ where M is a non-empty set and $+$ is a binary operation over M satisfying the following axioms:

$$\begin{aligned}&\forall x \forall y \forall z (x+y)+z = x+(y+z),\\&\forall x \forall y x+y = y+x\\&\forall x x+0 = 0+x = x.\end{aligned}$$

If, in addition, $+$ is commutative then $(M, +, 0)$ is an *abelian monoid.*

An (abelian) monoid equipped with a total order $\leq$ such that

$$\forall x \forall y \forall z x \leq y \rightarrow x+z \leq y+z,$$

will be called a *totally ordered (abelian) monoid.* A *group* $(G, +, 0)$ is a monoid which, in addition, satisfies the following axiom:

$$\forall x \exists y x+y = y+x = 0. \tag{*}$$

As before, if $+$ is commutative, $(G, +, 0)$ is an *abelian group.* A *totally ordered (abelian) group* is a totally ordered (abelian) monoid which satisfies (*).

A *field* is a structure $(K, +, \cdot, 0, 1)$ such that $(K, +, 0)$ and $(K, \cdot, 1)$ are abelian groups, and which satisfies the following axiom:

$$\forall x \forall y \forall z (x+y) \cdot z = x \cdot z + y \cdot z.$$

A *totally ordered field* $(K, +, \cdot, 0, 1, \leq)$ is a field equipped with a total order $\leq$ such that $(K, +, 0, \leq)$ is a totally ordered abelian group, and such that

$$\forall x \forall y (0 \leq x \wedge 0 \leq y) \rightarrow 0 \leq x \cdot y.$$

1.3.4 Totally ordered groups and fields

Let $(G, +, 0, \leq)$ be a totally ordered group. We denote the *positive part of* G as $G^+ := \{x \in G\,;\, x > 0\}$. Moreover, following [24, Definition 1.19], we call $\mathrm{bn}(G) := \mathrm{coi}(G^+)$ the *base number of* G.[1]

Now let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field. As usual, we identify the element

$$\underbrace{1+\ldots+1}_{n \text{ times}}$$

with the natural number n and thus assume that $\mathbb{N} \subseteq K$. The field K is called *archimedean* if $\mathbb{N}$ is cofinal in K.

The field operations ensure that the order structure of K is homogeneous as order-theoretic phenomena can be shifted around in the field. E.g., if one considers subsets of

[1] This number was called the *degree of* G, in symbols $\deg(G)$, in [35, 36, 38]. Sikorski says that G has *character* κ if (in our notation) $\mathrm{bn}(G) \leq \kappa$ [89]. The term *base number* is due to Dales and Woodin who in [24] use the notation $\delta(G)$ for our $\mathrm{bn}(G)$.

K^+, the map $x \mapsto x^{-1}$ transforms sets that are cofinal in K^+ into sets that are coinitial in K^+ and vice versa; therefore $\mathrm{bn}(K) = \mathrm{coi}(K^+) = \mathrm{cof}(K)$.

Also, if (a, b) and (c, d) are any proper intervals in K, then the map $\pi : z \mapsto \frac{d-c}{b-a}(z - a) + c$ is a linear transformation of the one-dimensional K-vector space K such that the interval (a, b) is bijectively and order-preservingly mapped to (c, d). Clearly, this map translates subsets of (a, b) into subsets of (c, d) while preserving properties such as convergence and divergence:

Lemma 1.7. *Let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field and (a, b) and (c, d) proper intervals in K. If $s : \alpha \to (a, b)$ is a convergent or divergent sequence, then so is $\pi \circ s : \alpha \to (c, d)$.*

The following results (Lemmas 1.8, 1.9, 1.10, 1.13 and Corollary 1.11) are explaining how the characteristics of a field relates to the existence of divergent and convergent sequences of a given length. These will prove to be the main tools of Chapter 3.

Lemma 1.8. *Let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field and C be an infinite convex subset of K. Then there are strictly increasing and strictly decreasing ω-sequences inside C.*

Proof. We only construct the strictly decreasing sequence, the existence of a strictly increasing sequence follows by symmetry. Let $x, y \in C$ be such that $x < y$. Define $y_n := \frac{x+y}{n+2} \in C$ for each $n \in \omega$. Clearly, this is a strictly decreasing ω-sequence in C. □

Lemma 1.9. *Let λ be an uncountable regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field such that $\mathrm{bn}(K) = \lambda$. Then the following are equivalent:*

1. *K is λ-spherically complete,*
2. *for every $\alpha < \lambda$, every nested family $\mathcal{I} = \{I_\gamma\,;\, \gamma < \alpha\}$ of non-empty open intervals has non-empty intersection.*

Proof. Clearly, (2) implies (1). Fix $\mathcal{I} = \{I_\gamma\,;\, \gamma < \alpha\}$ with $I_\gamma =: (x_\gamma, y_\gamma)$. We only have to consider the case $\alpha \geq \omega$. By (1), we have that $\bigcap_{\gamma<\alpha}[x_\gamma, y_\gamma] \neq \varnothing$, so pick $x \in \bigcap_{\gamma<\alpha}[x_\gamma, y_\gamma]$. Since $\mathrm{bn}(K) = \lambda > \alpha$ and λ is regular, there is $\varepsilon > 0$ such $(x - \varepsilon, x + \varepsilon) \subseteq \bigcap_{\gamma<\alpha}[x_\gamma, y_\gamma]$ Note that $(x - \varepsilon, x + \varepsilon) \subseteq \bigcap_{\gamma<\alpha}(x_\gamma, y_\gamma)$ which proves the claim. □

Clearly, if K is an η_λ-set, then $\mathrm{bn}(K) \geq \lambda$. Having large base number provides us with a weaker version of η_λ-ness that is sometimes sufficient for our arguments:

Lemma 1.10. *Let λ be a regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field with $\mathrm{bn}(K) = \lambda$. Let $F \subseteq K$ be finite and $X \subseteq K$ be such that $|X| < \lambda$. Then if $X < F$, there is some $x \in K$ such that $X < x < F$. Similarly, if $F < X$, then there is some $x \in K$ such that $F < x < X$.*

Proof. Since the proofs are similar, we only deal with the case $X < F$. The case $F = \varnothing$ follows directly from $\mathrm{bn}(K) = \lambda$. Let $F = \{x_0, ..., x_n\}$ with $x_0 < x_1 < ... < x_n$, let $\mu := \mathrm{cof}(X) \leq |X| < \lambda$, and let $s : \mu \to X$ be strictly increasing and cofinal in X. If $\gamma < \mu$, let $\varepsilon_\gamma := x_0 - s(\gamma)$. Since $\mathrm{bn}(K) = \lambda > \mu$, we find $\varepsilon \in K^+$ such that for all $\gamma < \mu$, we have $x_0 - \varepsilon > s(\gamma)$. But then $X < x_0 - \varepsilon < F$. □

Corollary 1.11. *Let λ be a regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field with* $\mathrm{bn}(K) = \lambda$.

(i) If I is an open interval in K, then $\mathrm{cof}(I) = \mathrm{coi}(I) = \mathrm{bn}(K)$.

(ii) If $\mu < \lambda$, then every μ-sequence is bounded and it is either eventually constant or divergent.

(iii) Every infinite convex set C contains strictly descending and strictly increasing λ-sequences bounded in C; in particular, it contains bounded and divergent μ-sequences for every $\mu < \lambda$.

Proof. Statements (i) and (ii) are obvious from Lemma 1.10. For statement (iii), find $x, y \in C$ and apply (i) to (x, y) to find coinitial and cofinal sequences of length λ; apply (ii) to see that the initial segments of these of length μ are divergent. $\square$

As we will see, divergent sequences will have an important role in Chapter 3. For this reason we introduce the following class of fields.

Definition 1.12. We say that a totally ordered field $(K, +, \cdot, 0, 1, \leq)$ is λ-*divergent* if and only if every interval contains a strictly monotone divergent λ-sequence.

The *weight* of a totally ordered field $(K, +, \cdot, 0, 1, \leq)$ is the size of the smallest dense subset of K and is denoted by $\mathrm{w}(K)$. Since every dense set is cofinal, we have that $\mathrm{bn}(K) \leq \mathrm{w}(K)$.

Lemma 1.13. *Let λ be a regular uncountable cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field such that $\mathrm{w}(K) = \lambda$ and K is an η_λ-set. Then every interval $(x, y) \subseteq K$ contains a convex bounded subset $B \subseteq (x, y)$ without least upper or greatest lower bound such that* $\mathrm{coi}(B) = \mathrm{cof}(B) = \lambda$.

Proof. Clearly, the assumptions imply that K is non-archimedean. Pick $z \in (x, y)$ and use Lemma 1.8 to find a strictly increasing sequence $s : \omega \to (x, z)$ with $S := \mathrm{ran}(s)$ and a strictly decreasing sequence $s' : \omega \to (z, y)$ with $S' := \mathrm{ran}(s')$; in particular, $S < S'$. By Corollary 1.11 (ii), both s and s' are bounded and divergent; also, z is both an upper bound for S and a lower bound for S'. Let $B := \{b \in (x, y)\,;\, S < b < S'\}$ be the set of these elements. Clearly, B is convex; a greatest lower bound for B would be a least upper bound for S and a least upper bound for B would be a greatest lower bound for S', but since s and s' are divergent, these do not exist, so B has neither greatest lower nor least upper bound.

We will now show that $\mathrm{coi}(B) = \mathrm{cof}(B) = \lambda$. The two proofs are similar, so let us just discuss the proof for coinitiality.

Clearly, if $X \subseteq B$ with $|X| < \lambda$, then X cannot be coinitial by η_λ-ness of the field. So $\mathrm{coi}(B) \geq \lambda$. We will now construct a coinitial set of size λ. For this, let D be a dense set of size $\mathrm{w}(K) = \lambda$, let $B' := B \cap D$ and let $\sigma : \lambda \to B'$ be a surjection. We construct a strictly decreasing coinitial λ-sequence $t : \lambda \to B$: Pick any element $t(0) \in B$. Suppose $\alpha < \lambda$ and assume that $t{\restriction}\alpha$ has been defined and is a strictly descending sequence. Then $B^* := \mathrm{ran}(t{\restriction}\alpha) \cup \mathrm{ran}(\sigma{\restriction}\alpha)$ has size $|B^*| \leq |\alpha \times 2| < \lambda$. By η_λ-ness of the field, we find b such that $S < b < B^*$; then let $t(\alpha) := b$.

We claim that t is coinitial: if $b \in B$ is arbitrary, then by saturation, we find some $z \in B$ such that $S < z < b$. Now density of D means that we find some $d \in D$ with $S < z < d < b$. Clearly, $d \in B'$. Find α such that $\sigma(\alpha) = d$. Then $t(\alpha + 1) < d < b$. $\square$

The final technical result of this section will be the core of our constructions in the main proofs of Chapter 3, allowing us to split intervals:

Lemma 1.14. *Let λ be an uncountable regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field with $\mathrm{bn}(K) = \lambda$. If $I = (x, y)$ is an open interval in K with half-way point $\frac{x+y}{2}$ and $\mu < \lambda$ is a cardinal, then there is a family $\{I_\alpha\,;\,\alpha < \mu\}$ of pairwise disjoint non-empty subintervals of I with union $U := \bigcup_{\alpha<\mu} I_\alpha$ such that*

1. *there is an $\varepsilon_0 \in K^+$ such that for all $z \in U$, we have $|z - x| > \varepsilon_0$ and $|z - y| > \varepsilon_0$,*
2. *$\frac{x+y}{2} \notin U$, and*
3. *there is $\varepsilon_1 \in K^+$ such that for all $\alpha \neq \beta < \mu$, I_α and I_β are separated by a distance of at least ε_1 (i.e., for all $x_\alpha \in I_\alpha$, and $x_\beta \in I_\beta$, we have that $|x_\alpha - x_\beta| > \varepsilon_1$).*

Proof. Pick any $x', y' \in (x, \frac{x+y}{2})$ and work inside $I' := (x', y')$. Clearly, any family of subintervals contained in I' will satisfy (1) and (2). By Corollary 1.11 (i), $\mathrm{cof}(I') = \lambda$, so let $s : \lambda \to I'$ be a strictly increasing sequence cofinal in I'. Suppose that $\nu < \mu$ is a limit ordinal and $n \in \mathbb{N}$. We define

$$I_{\nu+n} := (s(\nu + 2n + 1), s(\nu + 2n + 2))$$

and claim that this sequence of intervals satisfies (3). If $\alpha < \beta = \nu + n < \lambda$, then the distance between I_α and I_β is at least

$$\delta_\beta := s(\nu + 2n + 1) - s(\nu + 2n) > 0.$$

Apply Lemma 1.10 to the sets $\{0\}$ and $\{\delta_\beta\,;\,\beta < \mu\}$ to find $\varepsilon_1 > 0$ as required by (3). $\square$

1.3.5 Completeness

A pair $\langle L, R\rangle$ of non-empty subsets of X is called a *Dedekind cut* in $(X, \leq)$ if $L \neq \varnothing \neq R$, L has no maximum, R has no minimum, $L \cup R = X$ and $L < R$. Given a totally ordered field $(K, +, \cdot, 0, 1, \leq)$, a Dedekind cut $\langle L, R\rangle$ in K, is called a *Veronese cut*[2] if for each $\varepsilon \in K^+$ there are $\ell \in L$ and $r \in R$ such that $r < \ell + \varepsilon$.

A totally ordered field $(K, +, \cdot, 0, 1, \leq)$ is called *Dedekind complete* if there are no Dedekind cuts in K and it is called *Veronese complete* if there are no Veronese cuts in K. Clearly, Dedekind completeness implies Veronese completeness, but the converse is not in general true. In fact, a totally ordered field is Dedekind complete if and only if it is isomorphic to $\mathbb{R}$ (see [20, Corollary 8.7.4] or [97, Theorem 2.4]).

We need to generalise the standard definitions from real analysis to accommodate transfinite sequences:

[2]The term "Veronese cut" is used by Ehrlich to honour the pioneering contributions of Giuseppe Veronese in the late XIXth century to theory of infinity and infinitesimals; the same concept has various other names in the literature, e.g., Cauchy cut or Scott cut.

Definition 1.15 (Cauchy sequences)**.** Let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field and α be an ordinal. A sequence $(x_\beta)_{\beta\in\alpha}$ of elements of K is called *Cauchy* if

$$\forall\varepsilon \in K^+\exists\beta < \alpha\forall\gamma, \gamma' \geq \beta(|x_{\gamma'} - x_\gamma| < \varepsilon).$$

The sequence is *convergent* if there is $x \in K$ such that

$$\forall\varepsilon \in K^+\exists\beta < \alpha\forall\gamma \geq \beta(|x_\gamma - x| < \varepsilon).$$

In this case, we will say that x is the *limit of the sequence*. The field K is called *Cauchy complete* if every Cauchy sequence of length $\mathrm{bn}(K)$ converges.

Theorem 1.16 (Folklore)**.** *A totally ordered field is Veronese complete if and only if it is Cauchy complete.*

Proof. See, e.g., [24, Proposition 3.5]. □

It is a very well-known fact that an archimedean field K is Cauchy complete if and only if it is Dedekind complete; see, e.g., [43, Theorem 3.11]. Therefore, by Theorem 1.16 if K is archimedean, then Dedekind completeness and Veronese completeness coincide.

In light of Theorem 1.16, we will from now on only use the more common term "Cauchy completeness" (even though we shall be using Veronese completeness in our proofs).

Lemma 1.17. *Let $(K, +, \cdot, 0, 1, \leq)$ be a Cauchy complete totally ordered field. For every convex set $C \subseteq K$ the following hold:*

1. *if C has no supremum, there is $\varepsilon \in K^+$ such that for every $x \in C$ we have $x+\varepsilon \in C$;*
2. *if I has no infimum, there is $\varepsilon \in K^+$ such that for every $x \in C$ we have $x - \varepsilon \in C$;*
3. *if C has neither infimum nor supremum, then there is $\varepsilon \in K^+$ such that for every $x \in C$ the interval $(x - \varepsilon, x + \varepsilon)$ is a subinterval of C.*

Proof. Clearly, (2) follows from (1) by considering $\{-c\,; c \in C\}$ and (3) follows from (1) and (2). We now prove (1). Since C is convex with no supremum $\langle C, \{y \in K\,; C < y\}\rangle$ is not a Veronese cut. Therefore there is ε such that for every $x \in C$ we have $x + \varepsilon < \{y \in K\,; C < y\}$. □

1.3.6 Surreal numbers

The surreal numbers were introduced by Conway in order to generalise both the Dedekind construction of real numbers and the ordinal numbers. In his introduction to surreal numbers, Conway proved that they form a (class) real closed field. Later, Ehrlich [54] proved that every real closed field is isomorphic to a subfield of the surreal numbers; showing that they are a universal class model for real closed fields.

The following definitions as well as most of the results in this section are due to Conway [21]. We refer the reader to [40] for a complete introduction to the subject.

A *surreal number* is a function from an ordinal α to $\{+, -\}$, i.e., a sequence of pluses and minuses of ordinal length. We denote the class of surreal numbers by No. The *length* of a surreal number x, denoted $\ell(x)$, is its domain. We define

$$\begin{aligned} \mathrm{No}_{\alpha} &:= \{x \in \mathrm{No}\,;\, \ell(x) = \alpha\}, \\ \mathrm{No}_{<\alpha} &:= \{x \in \mathrm{No}\,;\, \ell(x) < \alpha\}, \text{ and} \\ \mathrm{No}_{\leq\alpha} &:= \mathrm{No}_{\alpha} \cup \mathrm{No}_{<\alpha}. \end{aligned}$$

For surreal numbers x and y, we define $x < y$ if there exists α such that $x(\beta) = y(\beta)$ for all $\beta < \alpha$, and (i) $x(\alpha) = -$ and either $\alpha = \ell(y)$ or $y(\alpha) = +$, or (ii) $\alpha = \ell(x)$ and $y(\alpha) = +$.

In Conway's original idea, every surreal number is generated by filling some gap between shorter numbers. The following theorem connects this intuition to the surreal numbers as we have defined them.

Theorem 1.18 (Simplicity theorem)**.** *If L and R are two sets of surreal numbers such that $L < R$, then there is a unique surreal x of minimal length such that $L < \{x\} < R$, denoted by $[\,L|R\,]$. Furthermore, for every $x \in \mathrm{No}$ we have $x = [\,L|R\,]$ for $L = \{y \in \mathrm{No}\ ;\ x > y \wedge y \subset x\}$ and $R = \{y \in \mathrm{No}\ ;\ x < y \wedge y \subset x\}$. We call the cut $\langle L, R\rangle$ a representation of x.*

Proof. See, e.g., [40, Theorem 2.1]. □

Each surreal number has many different representations. For any surreal number $x \in \mathrm{No}$ we define $L_x := \{y \in \mathrm{No}\ ;\ x > y \wedge y \subset x\}$ and $R_x := \{y \in \mathrm{No}\ ;\ x < y \wedge y \subset x\}$. The sets L_x and R_x satisfy the conditions of Theorem 1.18; see, e.g., [40, Theorem 2.8]. We will call $\langle L_x, R_x\rangle$ the *canonical representation* or *canonical cut* of x.

Using the simplicity theorem Conway defined the field operations $+_{\mathrm{s}}$, $\cdot_{\mathrm{s}}$, $-_{\mathrm{s}}$, and the multiplicative inverse over No and proved that these operations satisfy the axioms of real closed fields. For any binary operation $*$, surreal z, and sets X, Y of surreals we use the notations $z * X := \{z * x\ ;\ x \in X\}$ and $X * Y := \{x * y\ ;\ x \in X \text{ and } y \in Y\}$. Similarly, if $*$ is a unary operation, we will denote by $*R$ the set $\{*x\ ;\ x \in X\}$.

Definition 1.19. Let $x = [\,L_x|R_x\,]$, $y = [\,L_y|R_y\,]$ be surreal numbers. We define

$$\begin{aligned} x +_{\mathrm{s}} y &= [\,L_x +_{\mathrm{s}} y, x +_{\mathrm{s}} L_y | R_x +_{\mathrm{s}} y, x +_{\mathrm{s}} R_y\,] \\ -_{\mathrm{s}} x &= [\,-_{\mathrm{s}} R_x | -_{\mathrm{s}} L_x\,] \\ x \cdot_{\mathrm{s}} y &= [L_x \cdot_{\mathrm{s}} y +_{\mathrm{s}} x \cdot_{\mathrm{s}} L_y -_{\mathrm{s}} L_x \cdot_{\mathrm{s}} L_y, R_x \cdot_{\mathrm{s}} y +_{\mathrm{s}} x \cdot_{\mathrm{s}} R_y -_{\mathrm{s}} R_x \cdot_{\mathrm{s}} R_y \\ &\qquad |\, L_x \cdot_{\mathrm{s}} y +_{\mathrm{s}} x \cdot_{\mathrm{s}} R_y -_{\mathrm{s}} L_x \cdot_{\mathrm{s}} R_y, R_x \cdot_{\mathrm{s}} y +_{\mathrm{s}} x \cdot_{\mathrm{s}} L_y -_{\mathrm{s}} R_x \cdot_{\mathrm{s}} L_y] \end{aligned}$$

Now let $z = [\,L_z|R_z\,]$ be a positive surreal number. Let $r_{()} := 0$ and recursively for every $z_0, \ldots, z_n \in (L_z \cup R_z) \setminus \{0\}$ let $r_{(z_0,\ldots,z_n)}$ be the solution for x of the equation $(z -_{\mathrm{s}} z_n) \cdot_{\mathrm{s}} r_{(z_0,\ldots,z_{n-1})} +_{\mathrm{s}} z_n \cdot_{\mathrm{s}} x = 1$. Then we definite $\frac{1}{z} = [\,L'|R'\,]$, where $L' = \{r_{(z_0,\ldots,z_n)}\ ;\ n \in \mathbb{N}$ and $z_i \in L_z$ for even-many $i \leq n\}$ and $R' = \{r_{(z_0,\ldots,z_n)}\ ;\ n \in \mathbb{N}$ and $z_i \in L_z$ for odd-many $i \leq n\}$.

On ordinals, the operations $+_{\mathrm{s}}$ and $\cdot_{\mathrm{s}}$ are the *Hessenberg* operations; see, e.g., [40, Theorem 4.5].

Theorem 1.20 (van den Dries & Ehrlich)**.** *If ε is an epsilon number, then* $\mathrm{No}_{<\varepsilon}$ *is a real closed field. In particular, for every cardinal λ,* $\mathrm{No}_{<\lambda}$ *is a real closed field.*

Proof. See [105, Proposition 4.7]. □

Proposition 1.21 (Folklore)**.** *Let κ be an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. Then* $|\mathrm{No}_{<\kappa}| = \mathrm{bn}(\mathrm{No}_{<\kappa}) = \mathrm{w}(\mathrm{No}_{<\kappa}) = \kappa$ *and* $\mathrm{No}_{<\kappa}$ *is an η_κ-set.*

Proof. See, e.g., [35, Propositions 3.4.3 & 3.4.4]. □

1.3.7 Computable analysis & the theory of Weihrauch degrees

The classical approach of computability theory is to define a notion of computability over ω and then extend that notion to any countable space via coding. A similar approach is taken in computable analysis, where one usually defines a notion of computability over Cantor space 2^ω or Baire space ω^ω by using the so-called *type two Turing machines* (T2TMs), and then extends that notion to spaces of cardinality at most the continuum via representations, i.e., coding functions.

Intuitively, a T2TM is a Turing machine in which a successful computation is one that runs forever (i.e., for ω steps). The hardware of a type two Turing machine is very similar to that of a classical Turing machine. A T2TM has one read-only input tape of length ω; finitely many read and write scratch tapes of length ω; and one write-only output tape of length ω. Type two Turing machines run over classical Turing machine programs; and in each step of the computation a T2TM behaves exactly as a Turing machine.

Using these machines, one can define that a function f over 2^ω is computable if there is a T2TM which, when given $p \in \mathrm{dom}(f)$ as input, writes $f(p)$ on the output tape in the long run, i.e., in ω steps.

As an example, it is a classical result of computable analysis that, given the right representation of $\mathbb{R}$, the field operations are computable; see, [107, Theorem 4.3.2]. For an introduction to computable analysis we refer the reader to [107].

Another classical application of T2TMs is the Weihrauch theory of reducibility. As we will see, the theory of Weihrauch degrees will have a central role in the sections of this thesis which deal with transfinite computability. In the rest of this section we will present basic definitions of this theory; see, e.g., [13] for a more complete introduction.

The main aim of the theory of Weihrauch degrees is the study of the computational content of theorems of real analysis. Since many of these theorems are of the form

$$\forall x \in X \exists y \in Y \varphi(x, y),$$

with $\varphi(x, y)$ a quantifier free formula, they can be thought of as their own Skolem functions. Given representations of X and Y in Cantor space, Weihrauch reducibility provides a tool for comparing the computational strength of such functions; and therefore of the theorems themselves.

Using this framework, theorems from real analysis can be arranged in a complexity hierarchy analogous to the hierarchy of problems one has in classical computability theory.

Definition 1.22 (Represented Space)**.** A *represented space* $\mathbf{X}$ is a pair (X, δ_X) where X is a set and $\delta_X : 2^\omega \to X$ is a partial surjective function. Given an element $x \in X$ we will call $y \in \delta_X^{-1}(x)$ a *δ_X-name* for x.

$$
\begin{tikzcd}
X \arrow[d, shift right, "f"'] \arrow[d, shift left] & 2^\omega \arrow[l, "\delta_X"'] \arrow[d, "F"] \\
Y & 2^\omega \arrow[l, "\delta_Y"']
\end{tikzcd}
$$

Figure 1.1: Representation of the multivalued function $f : \mathbf{X} \rightrightarrows \mathbf{Y}$.

As usual a multi-valued function between represented spaces is a multi-valued function between the underlying sets.

Definition 1.23. Let $f : \mathbf{X} \rightrightarrows \mathbf{Y}$ be a partial multi-valued function between represented spaces. We call $F : 2^\omega \to 2^\omega$ a *realiser* of f, in symbols $F \vdash f$, if for every $x \in \mathrm{dom}(\delta_X)$ we have that $\delta_Y(F(x)) \in f(\delta_X(x))$.

Using realisers one can define several notions of reducibility.

Let $\langle\ ,\ \rangle : 2^\omega \times 2^\omega \to 2^\omega$ be defined as follows: $\langle p, q\rangle(n) := p(m)$ if $n = 2m$ and $\langle p, q\rangle(n) := q(m)$ if $n = 2m + 1$. Moreover, let Id_{2^ω} be the identity function over Cantor space.

Definition 1.24 (Weihrauch Reducibility)**.** Let f and g be two multi-valued functions between represented spaces. Then we say that f is *topologically Weihrauch reducible* to g, in symbols $f \leq^{\mathrm{t}}_{\mathrm{W}} g$, if there are two continuous functions $H, K : 2^\omega \to 2^\omega$ such that $H \circ \langle \mathrm{Id}_{2^\omega}, G \circ K\rangle \vdash f$ whenever $G \vdash g$. If the functions H, K above can be taken to be computable by a type two Turing machines, then we say that f is *Weihrauch reducible* to g, in symbols $f \leq_{\mathrm{W}} g$.If $f \leq^{\mathrm{t}}_{\mathrm{W}} g$ and $g \leq^{\mathrm{t}}_{\mathrm{W}} f$ then we say that f is *topologically Weihrauch equivalent* to g and write $f \equiv^{\mathrm{t}}_{\mathrm{W}} g$. The relation $\equiv_{\mathrm{W}}$ is defined analogously.

Definition 1.25 (Strong Weihrauch Reducibility)**.** Let f and g be two multi-valued functions between represented spaces. Then we say that f is *strongly topologically Weihrauch reducible* to g, in symbols $f \leq^{\mathrm{t}}_{\mathrm{sW}} g$, if there are two continuous functions $H, K : 2^\omega \to 2^\omega$ such that $H \circ G \circ K \vdash f$ whenever $G \vdash g$. If the functions H, K above can be taken to be computable by a type two Turing machines, then we say that f is *strongly Weihrauch reducible* to g, in symbols $f \leq_{\mathrm{W}} g$.

If $f \leq^{\mathrm{t}}_{\mathrm{sW}} g$ and $g \leq^{\mathrm{t}}_{\mathrm{sW}} f$ then we say that f is *strongly topologically Weihrauch equivalent* to g and write $f \equiv^{\mathrm{t}}_{\mathrm{sW}} g$. The relation $\equiv_{\mathrm{sW}}$ is defined analogously.

As shown in, e.g., [10,11], the so-called *boundedness principles* are important building blocks in characterising the Weihrauch degrees of interest in computable analysis. These principles are formalisations of particular basic properties of the real line which, in the context of computable analysis, serve as guides in assessing the computational strength of theorems of real analysis. As we will see in § 4.2, one the boundedness principle $\mathrm{B_I}$ will play a crucial role. The principle $\mathrm{B_I}$ is the formalisation of the following informal

$$
\begin{tikzcd}
X & 2^\omega \arrow[l, "\delta_X"'] \arrow[r, dashed, "\mathrm{K}"] \arrow[d, "F"] & 2^\omega \arrow[r, "\delta_W"] \arrow[d, "G"'] & W \arrow[d, shift left, "\mathrm{g}"] \arrow[d, shift right] \\
Y \arrow[from=u, shift left] \arrow[from=u, shift right, "f"] & 2^\omega \arrow[l, "\delta_Y"] & 2^\omega \arrow[l, dashed, "\mathrm{H}"] \arrow[r, "\delta_Z"'] & Z
\end{tikzcd}
$$

Figure 1.2: Strong Weihrauch reducibility: $f \leq^{\mathrm{t}}_{\mathrm{sW}} g$.

statement: "For any two sequences $(q_n)_{n\in\omega}$ and $(q'_n)_{n\in\omega}$ of rational numbers such that $\sup_{n\in\omega} q_n \leq \inf_{n\in\omega} q'_n$, there is a real number $r \in \mathbb{R}$ such that $\sup_{n\in\omega} q_n \leq r \leq \inf_{n\in\omega} q'_n$." Note that, this informal statement is of the right form to be formalised as a multi-valued function over Cantor space. In order to do so we will need the following representations of the set of real numbers: let $\delta_< : 2^\omega \to \mathbb{R}$ be the representation such that $p \in 2^\omega$ is a $\delta_<$-name for $r \in \mathbb{R}$ if and only if p codes a sequence of rational numbers $(q_n)_{n\in\omega}$ such that $r = \sup_{n\in\omega} q_n$; similarly, we let $\delta_> : 2^\omega \to \mathbb{R}$ be the representation such that $p \in 2^\omega$ is a $\delta_>$-name for $r \in \mathbb{R}$ if and only if p codes a sequence of rational numbers $(q_n)_{n\in\omega}$ such that $r = \inf_{n\in\omega} q_n$. We will denote by $\mathbb{R}_<$ and $\mathbb{R}_>$ the the represented spaces whose underline set is the set of real numbers and whose representations are $\delta_<$ and $\delta_>$, respectively. Note that given a represented space $\mathbf{X} = (X, \delta_X)$, for every $Y \subseteq X$ we have that $(Y, \delta_X \restriction Y)$ is a represented space. In the rest of this thesis we will assume that for every $\mathbf{X} = (X, \delta_X)$ and for every $Y \subseteq X$, the space $\mathbf{Y}$ is equipped with the representation $\delta_X \restriction Y$.

Finally, in this thesis we will always assume that $\mathbb{R}$ is represented by *fast convergent Cauchy sequences*; i.e., by the representation $\delta_\mathbb{R} : 2^\omega \to \mathbb{R}$ such that $p \in 2^\omega$ is a $\delta_\mathbb{R}$-name for $r \in \mathbb{R}$ if and only if p codes a Cauchy sequence $(q_n)_{n\in\omega}$ of rational numbers converging to r such that for every $n \in \omega$ and for every $m > n$, we have $|q_n - q_m| < \frac{1}{2^n}$.

Now we can formally define the multivalued function $\mathrm{B_I} : \mathbb{R}_< \times \mathbb{R}_> \to \mathbb{R}$ as the function that given $p \in \mathrm{dom}(\mathbb{R}_<)$ and $p' \in \mathrm{dom}(\mathbb{R}_>)$ such that p is a $\delta_<$-name of a real r and p' is a $\delta_>$-name of a real r' with $r \leq r'$, it returns the set of $\delta_\mathbb{R}$-names of the reals $r'' \in \mathbb{R}$ such that $r \leq r'' \leq r'$.

Given an uncountable regular cardinal κ one can generalise the previous notions of reducibility using the generalised Cantor space 2^κ. To avoid an overly loaded notation we will use for these generalisations the same notation we used for their classical counterparts.

Definition 1.26 (Generalised Represented Space)**.** A *represented space* $\mathbf{X}$ is a pair (X, δ_X) where X is a set and $\delta_X : 2^\kappa \to X$ is a partial surjective function. Given an element $x \in X$ we will call $y \in \delta_X^{-1}(x)$ a δ_X*-name* for x.

Definition 1.27. Let $f : \mathbf{X} \rightrightarrows \mathbf{Y}$ be a partial multi-valued function between represented spaces. We call $F : 2^\kappa \to 2^\kappa$ a *realiser* of f, in symbols $F \vdash f$, if for every $x \in \mathrm{dom}(\delta_X)$ we have that $\delta_Y(F(x)) \in f(\delta_X(x))$.

Definition 1.28 (Generalised Weihrauch Reducibility)**.** Let f and g be two multi-valued functions between represented spaces. Then we say that f is *strongly topologically Weihrauch reducible* to g, in symbols $f \leq^{\mathrm{t}}_{\mathrm{sW}} g$, if there are two continuous functions $H, K : 2^\kappa \to 2^\kappa$ such that $H \circ G \circ K \vdash f$ whenever $G \vdash g$. If $f \leq^{\mathrm{t}}_{\mathrm{sW}} g$ and $g \leq^{\mathrm{t}}_{\mathrm{sW}} f$ then we say that f is *strongly topologically Weihrauch equivalent* to g and write $f \equiv^{\mathrm{t}}_{\mathrm{sW}} g$.

Finally note that as in the classical case it is possible to define a hierarchy of representations using the following notion of reducibility.

Definition 1.29 (Reductions)**.** Let $\delta : 2^\kappa \to X$ and $\delta' : 2^\kappa \to X$ be two representations of a space X. Then we say that δ *continuously reduces* to δ', in symbols $\delta \leq_{\mathrm{t}} \delta'$, if there is a continuous function $h : 2^\kappa \to 2^\kappa$ such that for every $p \in \mathrm{dom}(\delta)$ we have $\delta(p) = \delta'(h(p))$.

As we have seen, in the classical case each notion of reducibility comes in two versions: one topological and one computational. As we will see in Chapter 4, the surreal numbers can be used to define computational versions of generalised Weihrauch reducibility.

Chapter 2

The generalised reals: basic properties

Remarks on co-authorship. *The results of this chapter are, unless stated otherwise, all due to the author. In particular, the notions and results of § 2.2.2 were introduced in the author Master's thesis [35] and later published in [36].*

2.1 Introduction

The question of how to generalise the real number continuum to the uncountable case goes back to at least Sikorski's work from the 1940s on the *real ordinal numbers*. More recently, a much more general approach was taken by Conway [21] who developed the theory of *surreal numbers*.

Sikorski's idea was to repeat the classical Dedekind construction of the real numbers starting from an ordinal equipped with the Hessenberg operations. Unfortunately, one can prove that these fields do not have the density properties that are sometimes needed in the context of real analysis; see, e.g., [35, 36].

As every real closed field can be embedded in the surreal numbers, it is very natural to use this framework to study generalisations of the real line. This approach was the one taken by the author in his Master's thesis [35] and in [36] to define a generalisation of the real numbers: the *generalised real line*.

This chapter of the thesis is devoted to the study of the basic properties of both the real ordinal numbers and of the generalised real line and their connections to generalised descriptive set theory.

The chapter is organised as follows: in § 2.2.1 we will introduce Sikorski's construction of the real ordinal numbers and present some of the basic properties of this space. In § 2.2.2 we will shortly present the construction of the generalised real line and we will mention some of the basic results in this area. As we will see in § 2.3, having generalisations of the real line naturally leads to consider generalisations of metric spaces. In § 2.3 we will first introduce a theory of generalised metric spaces due to Sikorski. Then, we will apply this framework to generalised descriptive set theory. In particular, in § 2.3.4 we will introduce a generalised notion of Polish space based on generalised metrics; in § 2.3.5, we will compare our notion of generalised Polish space to the one based on games introduced by Coskey and Schlicht in [22]. Finally in § 2.3.6, we will begin the study of generalisations of the classical Cantor-Bendixson theorem; and we will compare our metric based results to those obtained by Väänänen in [104] using game theory.

2.2 Generalising the real line

2.2.1 The real ordinal numbers λ-$\mathbb{R}$

The real ordinal numbers were introduced by Sikorski in [89], studied by Klaua [54], and recently re-discovered by Asperó and Tsaprounis [3] as a generalisation of the real number continuum in the context of generalised descriptive set theory.

The underlying idea is to do the classical set theoretic construction of the reals, but instead of starting with the natural numbers $\mathbb{N}$, we start with an ordinal λ, considered as a total order $(\lambda, \leq)$. Since ordinal addition and multiplication are not commutative, we use the Hessenberg operations instead of the standard ordinal operations.

If λ is a delta number then $(\lambda, \oplus, \otimes, 0, 1, \leq)$ is a commutative ordered semi-ring. As in the standard construction of $\mathbb{Q}$ from $\mathbb{N}$, one can define $\lambda\text{-}\mathbb{Z} := \lambda \cup \{-\alpha\,;\, 0 < \alpha < \lambda\}$ and λ-$\mathbb{Q}$ as the $\sim$-equivalence classes of $\lambda\text{-}\mathbb{Z} \times (\lambda\backslash\{0\})$ where $(\pm\alpha, \beta) \sim (\pm\alpha', \beta')$ if and only if $\alpha \otimes \beta' = \alpha' \otimes \beta$; with the usual operations of addition and multiplication defined on λ-$\mathbb{Z}$ and λ-$\mathbb{Q}$; see, e.g., [54]. Furthermore, we let λ-$\mathbb{R}$ be the Cauchy completion of λ-$\mathbb{Q}$.

Theorem 2.1 (Sikorski). *If λ is a delta number, then λ-$\mathbb{Z}$ is a totally ordered ring, λ-$\mathbb{Q}$ is a totally ordered field, and λ-$\mathbb{R}$ is a Cauchy complete totally ordered field with* $\mathrm{bn}(\lambda\text{-}\mathbb{Q}) = \mathrm{bn}(\lambda\text{-}\mathbb{R}) = \mathrm{cof}(\lambda)$.

Furthermore, if λ is a regular uncountable cardinal, then λ-$\mathbb{Q}$ is Cauchy complete, and therefore $\lambda\text{-}\mathbb{Q} = \lambda\text{-}\mathbb{R}$ and $\mathrm{w}(\lambda\text{-}\mathbb{R}) = \lambda$.

Proof. The usual proof in which ω is substituted by λ works; see [89, pp. 72 and 73] or [53, §§ 2, 3, and 4]. □

This result was further extended by Asperó and Tsaprounis in [3, Theorem 4.6], where they showed that for every delta number λ with uncountable cofinality, $\lambda\text{-}\mathbb{Q} = \lambda\text{-}\mathbb{R}$.

The real ordinal numbers are very discontinuous:

Theorem 2.2 (Asperó & Tsaprounis). *If λ is a delta ordinal, then λ-$\mathbb{R}$ is not an $\eta_{\aleph_1}$-set.*

Proof. See [3, Corollary 4.4]. □

The classical notion of Baire spaces can be naturally generalised to uncountable cardinals.

Definition 2.3. A topological space is λ-*Baire* iff the intersection of λ-many open dense sets is dense.

The notion of λ-Baire spaces was already studied; see, e.g. [22, 89]. In particular in [22] Coskey and Schlicht showed that, similarly to the classical case, a characterisation in terms of games is possible for λ-Baire spaces.

In [3] Asperó and Tsaprounis study λ-$\mathbb{R}$ from a descriptive set theory point of view. In particular they proved that the generalisation of the Baire category theorem fails.

Theorem 2.4 (Asperó & Tsaprounis)**.** *Let $\lambda \geq \omega_1$ be a delta number. Then there are $\aleph_1$-many open dense subsets of λ-$\mathbb{R}$ whose intersection is empty. In particular, λ-$\mathbb{R}$ is not ω_1-Baire.*

Proof. See [3, Corollary 6.3]. □

2.2.2 The generalised real line $\mathbb{R}_\kappa$

Let κ be an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. We will now use the theory of surreal numbers from § 1.3.6 to define the second generalisation of the real number continuum which was introduced by the author in [35, 36, 38].

Let us call a field $K \supseteq \mathbb{R}$ a *super dense κ-real extension of* $\mathbb{R}$ if it has the following properties:

1. K is a real closed field,
2. $\mathrm{w}(K) = \kappa$,
3. K is an η_κ-set,
4. K is Cauchy complete, and
5. $|K| = 2^\kappa$.

Since the theory of real closed fields is complete [70, Corollary 3.3.16], any super dense κ-real extension of $\mathbb{R}$ has the same first order properties as $\mathbb{R}$. In [35, 36], the author argued why being a super dense κ-real extension of $\mathbb{R}$ is an adequate demand for being an appropriate κ-analogue of $\mathbb{R}$.

Theorem 1.20 and Proposition 1.21 tell us that $\mathrm{No}_{<\kappa}$ has almost all the properties that we want from $\mathbb{R}_\kappa$ except for (4) and (5) (for the failure of (4), see, e.g., [30, Lemma 1.32]). Therefore, we define

$$\mathbb{R}_\kappa := \mathrm{No}_{<\kappa} \cup \{x\,;\, x = [L|R] \text{ where } \langle L, R\rangle \text{ is a Veronese cut on } \mathrm{No}_{<\kappa}\}.$$

We will call $\mathbb{R}_\kappa$ the *generalised real line* over κ. Since $\mathrm{No}_{<\kappa}$ plays in the generalised case the role that the rational numbers play in the classical construction of $\mathbb{R}$, we will denote $\mathrm{No}_{<\kappa}$ by $\mathbb{Q}_\kappa$ and we will call the elements of $\mathbb{Q}_\kappa$ *κ-rational numbers.*

Theorem 2.5 (Galeotti 2015)**.** *Let κ be an uncountable cardinal with $\kappa^{<\kappa} = \kappa$. Then $\mathbb{R}_\kappa$ is the unique super dense κ-real extension of* $\mathbb{R}$. *Moreover,* $\mathrm{bn}(\mathbb{R}_\kappa) = \kappa$.

Proof. See [36, Theorem 4]. □

As shown by the author in [36] and [35], the field $\mathbb{R}_\kappa$ is a suitable setting for generalising results from classical analysis. We will now briefly introduce some basic results from [36] about the Intermediate value theorem.

Definition 2.6 (Folklore)**.** A *κ-topology* over a set X is a collection of subsets τ of X satisfying the following conditions:

1. $\varnothing, X \in \tau$;

2. for any $\alpha < \kappa$, if $\{A_i\}_{i\in\alpha}$ is a collection of sets in τ then $\bigcup_{i<\alpha} A_i \in \tau$;

3. and for all $A, B \in \tau$, we have $A \cap B \in \tau$.

We will call the elements of a κ-topology κ-open sets.

Intuitively, the reason to use κ-topologies is that interval topologies over real closed field extensions of $\mathbb{R}$ are too fine to do real analysis. On the other hand, κ-topologies are just coarse enough to allow us to recover some of basic result from classical real analysis.

Theorem 2.7 (Alling)**.** *Let X be a set and $\mathfrak{B}$ be a topological base over X. Moreover, let τ_κ be the smallest set such that: $\varnothing, X \in \tau_\kappa$ and τ_κ is closed under unions of less than κ many elements of $\mathfrak{B}$. Then τ_κ is a κ-topology. We will call τ_κ the κ-topology generated by $\mathfrak{B}$. Moreover we will call $\mathfrak{B}$ a base for the κ-topology.*

Proof. See [1, Theorem 2.01.0]. □

With κ-topologies one can define direct analogues of many topological notions.

Definition 2.8. Given two κ-topologies X and Y and a function $f : X \to Y$ we will say that f is κ-continuous if and only if for every κ-open set Z in Y we have that $f^{-1}[Z]$ is κ-open in X.

In this thesis we will consider the generalised real line $\mathbb{R}_\kappa$ equipped with the interval κ-topology, i.e, the κ-topology generated by intervals which have endpoints in $\mathbb{R}_\kappa \cup \{-\infty, +\infty\}$. This framework allows to prove the following version of the Intermediate value theorem:

Theorem 2.9 (IVT$_\kappa$)**.** *Let $a, b \in \mathbb{R}_\kappa$ and $f : [0,1] \to \mathbb{R}_\kappa$ be a κ-continuous function. Then for every $r \in [f(0), f(1)]$ there exists $c \in [0,1]$ such that $f(c) = r$.*

Proof. See [36, Theorem 17]. □

2.3 Generalised metrisability

2.3.1 Motivations

In this section we will introduce a generalised version of metrisability theory. As we will see, this theory leads to very natural generalisations of objects from classical descriptive set theory, serving as a tool in developing generalised descriptive set theory.

We will follow the first chapters of [51] focusing on the generalisation of basic results in the theory of metric spaces and of Polish spaces. In particular, §§ 2.3.2 and 2.3.3 will be devoted to prove some basic results in Sikorski's generalised theory of metric spaces. Some of the results of these sections are already known in the literature; moreover, in the original papers most of the results of these two sections are stated without proof or with a proof sketch. For this reason, we think that a short summary of the main results in this area with their complete proofs will be beneficial to the reader.

In § 2.3.4 we will introduce the notion of λ-Polish space (see Definition 2.43). The main aim of this section will be that of proving the generalised version of the following theorem:

Theorem 2.10 ([51, Theorem 3.11]). *A subspace of a Polish space is Polish iff it is* G_δ*.*

As we will see, we will only be able to prove the left to right direction in the generalised case; see Theorem 2.35. Note that classical proofs of the right to left direction are usually done by using infinite sums. Since infinite sums with the suitable properties cannot in general be defined over non-archimedean totally ordered groups these classical techniques cannot be used in the generalised case. Similarly, classical proofs that every open subset of a Polish space is Polish do not generalise; as a consequence, the author does not know whether the corresponding claim is true or not: every open subset of a λ-Polish space is G_δ. Note though that, as in the classical case, closed subsets of λ-Polish spaces are λ-Polish.

In § 2.3.5 our notion of λ-Polish space is compared with a game theoretical generalisation of Polish spaces introduced by Coskey and Schlicht in [22]. This section is aimed at generalising the following theorem:

Theorem 2.11 ([51, Theorem 8.17.ii]). *Let X be a Polish space Y be a subspace of X then the following are equivalent:*

1. *Y is strongly Choquet;*
2. *Y is G_δ in X;*
3. *Y is Polish.*

Let λ be a regular cardinal, X be a strongly λ-Polish space (see Definition 2.50) and Y be a subspace of X. The main results of § 2.3.5 are illustrated by the following diagram; where an arrow from A to B means that A implies B; a crossed arrow from A to B means that A does not imply B; and dotted arrows are used to emphasise the fact that further assumptions on Y or λ are needed.

$$\begin{tikzcd}
Y \text{ is strongly } \lambda\text{-Polish} \arrow[r, "1"] \arrow[dr, "4"] & Y \text{ is strongly } \lambda\text{-Choquet} \arrow[r, dotted, shift left, "2"] & Y \text{ is } \lambda\text{-}\mathrm{G}_\delta \text{ in } X \arrow[l, dotted, shift left, "3"] \\
& \text{Y is } \lambda\text{-Polish} \arrow[ul, shift left, "5", "\times" marking] \arrow[u, "6", "\times" marking] \arrow[ur, "7"'] &
\end{tikzcd}$$

Arrow 1 is Theorem 2.45; arrow 2 follows from Theorem 2.51 assuming that λ is a weakly compact cardinal; arrow 3 follows from Theorem 2.45 and Lemma 2.47 assuming that Y is a λ-topologically complete subspace of X; arrow 4 follows from the definition of strongly λ-Polish spaces; the impossibility of arrow 5 follows from the fact that λ-$\mathbb{R}$ is a λ-Polish but not strongly λ-Polish space (see p. 36); the impossibility of arrow 6 is Theorem 2.41; finally, arrow 7 is Theorem 2.35. The author does not know if, as in the classical case, the three notions of strongly λ-Polish space, λ-G_δ space and strongly λ-Choquet space coincide; see § 2.4.

Finally, in § 2.3.6 we will prove a version of the Cantor-Bendixson theorem for strong λ-Polish spaces and compare this result to the one by Väänänen in [104]. In particular, Theorem 2.53 is the generalisation of the following theorem:

Theorem 2.12 ([51, Theorem 6.2]). *Let X be a non-empty perfect Polish space. Then Cantor space can be embedded in X.*

In the following we will always assume that λ is an uncountable regular cardinal.

2.3.2 λ-metrisable spaces

As was already noted by Sikorski in [90], generalisations of the real line can be used in order to generalise the theory of metric spaces. In this section we will introduce the basic definitions and some of the basic results of this generalised theory.

Let X be a set and $(G, +, 0, \leq)$ be a totally ordered abelian group. A function $d : X \times X \to G^+ \cup \{0\}$ is a *G-metric* if for all $x, y, z \in X$, we have:

1. $d(x, y) \geq 0$,
2. $d(x, y) = 0$ if and only if $x = y$,
3. $d(x, y) \leq d(x, z) + d(z, y)$,
4. $d(x, y) = d(y, x)$.

As in the case of a metric, we can define *open balls* with respect to a G-metric d:

$$B_d(c, r) = \{x \in X \,;\, d(c, x) < r\}$$

where $c \in X$ and $r \in G^+$. If $(G, +, 0, \leq)$ is a totally ordered abelian group we say that a topological space (X, τ) is *G-metrisable* if there is a G-metric $d : X \times X \to G^+ \cup \{0\}$ such that $\{B_d(c, r) \,;\, c \in X \wedge r \in G^+\}$ is a base for τ. In this case we will say that d is a *G-metric compatible* with τ.

If λ is a regular cardinal, then we will say that a topological space (X, τ) is *λ-metrisable* if there is a totally ordered abelian group $(G, +, 0, \leq)$ with $\mathrm{bn}(G) = \lambda$ such that (X, τ) is G-metrisable. As usual a λ-metric space is a pair (X, d) where d is a G-metric on X for some totally ordered abelian group $(G, +, 0, \leq)$ with $\mathrm{bn}(G) = \lambda$. Given a topological space (X, τ),then will say that d is a *λ-metric compatible* with τ if there is totally ordered abelian group $(G, +, 0, \leq)$ with $\mathrm{bn}(G) = \lambda$ and d is a G-metric compatible with τ.

Using the fact that G is totally ordered, we can measure the distance of elements in G by

$$|x - y| := \begin{cases} x - y & \text{if } x - y \in G^+ \text{ and} \\ y - x & \text{otherwise,} \end{cases}$$

and this is a G-metric. Therefore, every totally ordered abelian group $(G, +, 0, \leq)$ with $\mathrm{bn}(G) = \lambda$ is λ-metrisable. If C and C' are two sets in a totally ordered group, we say that C and C' are *separated by a distance of at least $\varepsilon \in K^+$* if for all $x \in C$ and all $y \in C'$, we have that $|x - y| > \varepsilon$.

It follows directly from Corollary 2.25 below that ω-metrisability and classical metrisability coincide.

Since every totally ordered group of base number κ is κ-metrisable, κ-$\mathbb{R}$ and $\mathbb{R}_\kappa$ are both κ-metrisable topological fields. Moreover, Corollary 2.26 will show that the notion of $\mathbb{R}_\kappa$- and (κ-$\mathbb{R}$)-metrisability coincide[1].

As in the classical theory λ-metrisability gives a natural notion of convergence.

[1]Note that this does not imply that $\mathbb{R}_\kappa$ and κ-$\mathbb{R}$ induce the same κ-metrics.

Definition 2.13. Let G be a totally ordered group, d be a G-metric and (X,d) be a λ-metric space. A sequence $(x_i+\beta)_{\beta\in\alpha}$ of elements of X is called *Cauchy* if

$$\forall\varepsilon\in G^+\exists\beta<\alpha\forall\gamma,\gamma'\geq\beta(d(x_{\gamma'},x_\gamma)<\varepsilon).$$

The sequence is *convergent* if there is $x\in K$ such that

$$\forall\varepsilon\in G^+\exists\beta<\alpha\forall\gamma\geq\beta(d(x_\gamma,x)<\varepsilon).$$

In this case, we will say that x is the *limit of the sequence.* A λ-metric space (X,d) is called *Cauchy complete* if every Cauchy sequence of length λ converges. As usual a topological space (X,τ) is said to be *completely λ-metrisable* if it is λ-metrisable and there is a λ-metric d compatible with τ such that (X,d) is Cauchy complete.

As for the classical theory, the notions of Cauchy sequence and convergency in Definition 1.15 and Definition 2.13 coincide if we consider the field as a topological field and the λ-metric induced by the distance function. Note that the notions in Definition 2.13 behave very similarly to their classical counterpart. In particular, using the classical argument, one can show that a set is closed if and only if it contains all its limit points. Similarly, the appropriate generalisations of compactness behave as in the classical framework.

Let (X,τ) be a topological space and λ be a cardinal. Then, (X,τ) is *λ-compact* if every open cover of X of cardinality λ has a subcover of cardinality $<\lambda$; (X,τ) is *λ-sequentially compact* iff every λ-sequence has a convergent λ-subsequence.

Theorem 2.14. *Let (X,τ) be λ-metrisable and d be a λ-metric compatible with τ. Then (X,d) is λ-compact if and only if it is λ-sequentially compact.*

Proof. The standard proof of the equivalence of compactness and sequential compactness transfers directly to the case of G-metrics for a totally ordered group $(G,+,0,\leq)$ with $\mathrm{bn}(G)=\lambda$. □

Definition 2.15 (λ-additive space)**.** A topological space (X,τ) is *λ-additive* iff for every sequence of open sets $(U_\alpha)_{\alpha\in\beta}$ with $\beta<\lambda$ we have $\bigcap_{\alpha\in\beta}U_\alpha\in\tau$.

Lemma 2.16 (Sikorski [89, Theorem iii])**.** *Let (X,τ) be a regular λ-additive space. Then for every open set O and $x\in O$ there is a clopen set U such that $x\in U\subseteq O$.*

Proof. Let X be regular λ-additive. For $x\in X$ let U be an open set containing x. We define the following sequence:

1. $U_0=U$,
2. U_{n+1} is an open set such that $x\in U_{n+1}\subset\overline{U_{n+1}}\subset U_n$.

Note that by regularity $(U_n)_{n\in\omega}$ is a well defined sequence of open sets. Moreover note that $\bigcap_{n\in\omega}U_n$ is a clopen set containing x. Indeed by λ-additivity $\bigcap_{n\in\omega}U_n$ is open and since $\bigcap_{n\in\omega}U_n=\bigcap_{n\in\omega}\overline{U_n}$ then $\bigcap_{n\in\omega}U_n$ is closed. □

Corollary 2.17 (Sikorski [89, Theorem iv])**.** *Every regular λ-additive space is totally disconnected.*

Proof. This follows from Lemma 2.16. $\square$

Lemma 2.18. *Every λ-additive normal space with a base of size λ has a base of size λ of clopen sets.*

Proof. See [89, Theorem vi] or [88, Theorem 7]. $\square$

Lemma 2.19 (Sikorski [89, Theorem viii])**.** *Every λ-metrisable space is λ-additive and normal.*

Proof. Let (X,τ) be a λ-metrisable space and $d : X\times X \to G^+\cup\{0\}$ be a λ-metric over X compatible with τ. First we will prove that (X,τ) is λ-additive. We need to show that for every $\gamma < \lambda$ and every sequence $(O_\alpha)_{\alpha\in\lambda}$ of open sets we have $\bigcap_{\alpha\in\gamma} O_\alpha$ is open. If $\bigcap_{\alpha\in\gamma} O_\alpha$ is empty we are done. So, assume $\bigcap_{\alpha\in\gamma} O_\alpha \neq \varnothing$. Let $x \in \bigcap_{\alpha\in\gamma} O_\alpha$. Note that for every $\alpha < \gamma$ there are basic open sets $B_d(x_{\alpha,\beta}, r_{\alpha,\beta})_{\beta\in\lambda}$ such that $O_\alpha = \bigcup_{\beta\in\lambda} B_d(x_{\alpha,\beta}, r_{\alpha,\beta})$. For each $\alpha < \gamma$ let $\beta_\alpha < \lambda$ be such that $x \in B_d(x_{\alpha,\beta_\alpha}, r_{\alpha,\beta_\alpha})$. Since $\mathrm{bn}(G) = \lambda$ and $\gamma < \lambda$ the set $R_x := \{r_{\alpha,\beta_\alpha\,;\,\alpha<\gamma}\}$ has a lower bound. Let r_x be a lower bound of R_x. Now, it is not hard to see that $\bigcap_{\alpha\in\gamma} O_\alpha = \bigcup_{x\in\bigcap_{\alpha\in\gamma} O_\alpha} B_d(x, r_x)$. So, $\bigcap_{\alpha\in\gamma} O_\alpha$ is open and τ is λ-additive as desired. Now we want to show that X is normal. Let C and C' be disjoint closed sets. For every $c \in C$ pick r_c such that $B_d(c, r_c + r_c) \cap C' = \varnothing$ and for every $c' \in C'$ choose $r_{c'}$ such that $B_d(c', r_{c'} + r_{c'}) \cap C = \varnothing$. Note that since C is closed and $c \in C$ then there is r such that $B_d(c, r) \cap C'$, moreover there is $g \in G^+$ such that $g + g < r$, indeed let $r' \in G$ be such that $r' < r$, then there is a $g \in G$ such that $r' + g < r$ otherwise we would have $r' = r$. Therefore, $g + g < r$. Then, by the triangular inequality $U = \bigcup_{c\in C} B_d(c, r_c)$ and $V = \bigcup_{c'\in C'} B_d(c', r_{c'})$ are two open sets which separate C and C'. $\square$

Definition 2.20. A topological space (X,τ) is λ-separable iff it has a dense subset of cardinality λ.

Lemma 2.21 (Sikorski [89, Theorem ix])**.** *A λ-metrisable space (X,τ) has a base of cardinality λ iff it is λ-separable.*

Proof. If (X,τ) has a base $\mathfrak{B}$ of cardinality λ it is enough to choose for every element B of $\mathfrak{B}$ a point $p_B \in B$. Then the set $D = \{p_B \mid B \in \mathfrak{B}\}$ is dense in X and of cardinality λ. Let X be a λ-separable set and D a dense subset of X of cardinality λ. Let $d : X \times X \to G$ be a compatible metric for X and $(\varepsilon_\alpha)_{\alpha\in\lambda}$ be a strictly decreasing sequence coinitial in G^+. Consider the set $\mathfrak{B} = \{B_d(x, \varepsilon_\alpha) \mid \alpha \in \lambda \wedge x \in D\}$. We claim that $\mathfrak{B}$ is a base for X. Let U be open in X without loss of generality we can assume $U \neq X$. Assume $U = \bigcup_{\alpha\in\lambda'} B_d(x_\alpha, r_\alpha)$. Consider the set $B = D \cap U$ and for every $x \in B$ let α_x be the least such that $B_d(x, \varepsilon_{\alpha_x}) \subseteq U$. We claim that $U = \bigcup_{x\in B} B_d(x, \varepsilon_{\alpha_x})$. Let $y \in \bigcup_{x\in B} B_d(x, \varepsilon_{\alpha_x})$ then $y \in U$ since $y \in B_d(x, \varepsilon_{\alpha_x})$ for some $x \in B$ and $B_d(x, \varepsilon_{\alpha_x}) \subseteq U$ by definition. Now assume $y \in U$ and let ε_α be such that $B_d(y, \varepsilon_\alpha) \subseteq U$. Let $\varepsilon_{\alpha'}$ be such that $2\varepsilon_{\alpha'} < \varepsilon_\alpha$ and $z \in B \cap B_d(y, \varepsilon_{\alpha'})$. Then $y \in B_d(z, \varepsilon_{\alpha'}) \subseteq B_d(y, \varepsilon_\alpha) \subseteq U$. By the fact that α_z is the least such that $B_d(z, \varepsilon_{\alpha_z}) \subseteq U$ we have $\alpha' \geq \alpha_z$ and $B_d(z, \varepsilon_{\alpha'}) \subseteq B_d(z, \varepsilon_{\alpha_z})$ and $y \in \bigcup_{x\in B} B_d(x, \varepsilon_{\alpha_x})$ as desired. $\square$

Finally, using a straightforward generalisation of the proof in [74, Theorem 32.1] we have:

Lemma 2.22 ([89, Theorem vii])**.** *Every λ-additive regular space with a base of size λ is normal.*

Proof. See, e.g., [74, Theorem 32.1]. $\square$

2.3.3 Generalised Cantor spaces

In this section we will study generalisations of Cantor space to uncountable regular cardinals. See § 1.3 for basic definition and properties of these spaces.

By using the real ordinal numbers it is easy to see that the following map is a distance:

$$d(p,p') = \begin{cases} 0 & \text{if } p = p', \\ \frac{1}{\alpha} & \text{if } \alpha \text{ is the smallest ordinal such that } p(\alpha) \neq p'(\alpha). \end{cases}$$

Moreover, the λ-metric space $(2^\lambda, d)$ is complete. Let $(x_\alpha)_{\alpha\in\lambda}$ be a Cauchy sequence in $(2^\lambda, d)$. Then for every $\alpha \in \lambda$ there is β_α such that for all $\gamma > \beta_\alpha$ we have $x_{\beta_\alpha}(\alpha) = x_\gamma(\alpha)$. But then $\ell(\alpha) = x_{\beta_\alpha}(\alpha)$ is the limit for the sequence $(x_\alpha)_{\alpha\in\lambda}$. Note that, if $\lambda^{<\lambda} = \lambda$, the previous construction can be repeated with $\mathbb{R}_\lambda$ in place of λ-$\mathbb{R}$. Making Cantor space 2^λ a completely $\mathbb{R}_\lambda$-metrisable space.

Theorem 2.23 (Sikorski [89, Theorem x]). *Every regular λ-additive Hausdorff space with a base of cardinality λ is homeomorphic to a subspace of 2^λ.*

Proof. Assume X to be a regular λ-additive space with a base of cardinality λ. By Lemma 2.22, X is normal. By Lemma 2.18 X has a base $\mathfrak{B}$ of clopen sets of size λ. Let $(B_\alpha)_{\alpha\in\lambda}$ be an enumeration of $\mathfrak{B}$. Now given $x \in X$ and $\alpha \in \lambda$, we define the following map:

$$f(x)(\alpha) = \begin{cases} 1 & \text{if } x \in B_\alpha, \\ 0 & \text{if } x \notin B_\alpha. \end{cases}$$

We want to show that $f : X \to 2^\lambda$ is continuous. Let $p \in 2^{<\lambda}$. We have $f^{-1}([p]) = \bigcap_{\alpha\in\{\beta\,;\,p(\beta)=1\}} B_\alpha \cap \bigcap_{\alpha\in\{\beta\,;\,p(\beta)=0\}} X \setminus B_\alpha$ which is open by λ-additivity of X.

Moreover, note that, since X is Hausdorff, f is injective. We only need to show that f is open. Consider a basic open B_α. We have that $f(B_\alpha) = \bigcup_{p\in 2^{\alpha+1} \wedge p(\alpha)=1} [p]$. Therefore, f is an homeomorphism between X and $f[X]$ as desired. □

Corollary 2.24. *Every regular λ-additive Hausdorff space with a base of cardinality λ is λ-metrisable.*

Proof. The claim follows directly from Theorem 2.23. □

Corollary 2.25. *A space is λ-metrisable if and only if is a regular λ-additive Hausdorff space with a base of size λ.*

Proof. The claim follows directly from Theorem 2.23, Lemma 2.22, and Lemma 2.19. □

Corollary 2.26. *A space is λ-metrisable if and only if it is (λ-$\mathbb{R}$)-metrisable.*

Proof. The claim follows directly from Theorem 2.23.

□

Definition 2.27. Let G be a totally ordered group, d be a G-metric and (X, d) be a λ-metric space. Then (X, d) is λ-*totally bounded* iff for every $\varepsilon \in G^+$ there is $Y \subseteq X$ of cardinality less than λ such that $X = \bigcup_{y\in Y} B_d(y, \varepsilon)$.

Theorem 2.28. *Let λ be a strongly inaccessible cardinal. Then 2^λ is λ-totally bounded. If 2^λ is λ-totally bounded then λ is strong limit.*

Proof. Let $\varepsilon \in \lambda\text{-}\mathbb{R}^+$ be positive κ-real and α be an ordinal such that $\frac{1}{\alpha} < \varepsilon$. Consider the set $Y = \{p \in 2^\lambda \,;\, \forall\beta > \alpha(p(\beta) = 0)\}$. By the fact that λ is inaccessible we have $|Y| < \lambda$. Moreover for every $x \in 2^\lambda$ consider the sequence $x' \in 2^\lambda$ defined as follows:

$$x'(\beta) = \begin{cases} 0 & \text{if } \beta > \alpha, \\ x(\beta) & \text{otherwise.} \end{cases}$$

Then $x' \in Y$ and $x \in B_d(x', \frac{1}{\alpha}) \subset B_d(x', \varepsilon)$.

Now assume 2^λ to be λ-totally bounded and $\alpha < \lambda$. Consider the following set of basic open sets:

$$C = \{B_d(p, \frac{1}{\alpha}) \,;\, p \in 2^\lambda \wedge \forall\beta \geq \alpha(p(\beta) = 0)\}.$$

Note that $\bigcup C = 2^\lambda$. Moreover, if C has cardinality bigger than λ, then there would be no subset C' of C of cardinality strictly less than λ such that $2^\lambda = \bigcup C'$. Therefore, since $|C| = 2^\alpha$, we have $2^\alpha < \lambda$. □

2.3.4 Generalised Polish spaces

From the notion of λ-metrisable space, we can naturally define a generalised version Polish spaces.

Definition 2.29. A topological space is λ-*Polish* iff it is λ-separable and completely λ-metrisable.

In [22], Coskey and Schlicht introduce a generalised notion of Polish spaces based on a generalised version of Choquet games; see, Definition 2.38. In § 2.3.5 we will show that our notion of λ-Polish spaces and the one introduced by Coskey and Schlicht do not coincide in general; see Theorem 2.41.

In this section we will begin the study of λ-Polish spaces by proving the generalised version of results from classical descriptive set theory. In particular, we will focus on closure properties of the class of λ-Polish spaces.

Definition 2.30. Let (X, τ) be a topological space, $A \subseteq X$, G be a totally ordered group with $\mathrm{bn}(G) = \lambda$, (Y, d) be a G-metric space and $f : A \to Y$ be a function. Then we define the set of *continuity points of f* as follows:

$$C^f := \{x \in X \,;\, \forall\delta \in G^+ \exists U \in \tau(x \in U \wedge \forall y, z \in U \cap \mathrm{dom}(f)(d(f(z), f(y)) < \delta))\}$$

Definition 2.31. Let (X, τ) be a λ-metrisable space. Then $Y \subseteq X$ is λ-G_δ iff it is the intersection of λ-many open subsets of X.

Lemma 2.32. *Let (X, τ_X) be a topological space, (Y, τ_Y) be a λ-metrisable space and $f : X \to Y$ be a function. Then C^f is λ-G_δ.*

Proof. Let $d : Y \times Y \to G^+ \cup \{0\}$ be a λ-metric compatible with τ_y and $(\varepsilon_\alpha)_{\alpha\in\lambda}$ be a coinitial sequence in G^+. For every $\delta \in G^+$, we define:

$$C^f_\delta := \{x \in X\, ;\, \exists U \in \tau_X(x \in U \wedge \forall y, z \in U \cap \mathrm{dom}(f)(d(f(z), f(y)) < \delta))\}.$$

Note that C^f_δ is open in τ_X and that $C^f = \bigcap_{\alpha\in\lambda} C^f_{\varepsilon_\alpha}$. Therefore, C^f is λ-G_δ as desired. □

Lemma 2.33. *Let (X, τ) be a λ-metrisable space. Then every closed subset of X is λ-G_δ. In particular every closed subset of a λ-Polish space is λ-G_δ.*

Proof. Let $d : X \times X \to G^+ \cup \{0\}$ be a λ-metric compatible with τ. If $C \subseteq X$ is empty we are done. So, assume C to be a non-empty closed subset of X. For every $\delta \in G^+$ define the following set:

$$B_\delta := \{x \in X\, ;\, \exists y \in C(d(x, y) < \delta)\}.$$

Note that B_δ is open as a union of open sets. Indeed, let $x \in B_\delta$ and $y \in C$ be such that $d(x, y) < \delta$. Let $\varepsilon \in G^+$ such that $d(x, y) + \varepsilon < \delta$. Then $B_d(x, \varepsilon) \subseteq B_\delta$. Let $(\varepsilon_\alpha)_{\alpha\in\lambda}$ be coinitial in G^+. Then $C = \bigcap_{\alpha\in\lambda} B_{\varepsilon_\alpha}$. Indeed, if $x \in C$ then $x \in B_{\varepsilon_\alpha}$ for every $\alpha \in \lambda$. On the other hand, if $x \in B_{\varepsilon_\alpha}$ for all α then there is a sequence in C converging to x and therefore $x \in C$. □

We are now ready to generalise Kuratowski's theorem.

Theorem 2.34 (Generalised Kuratowski theorem)**.** *Let (X, τ_x) be λ-metrisable, (Y, τ_y) be completely λ-metrisable, A be a subset of X and $f : A \to Y$ be continuous. Then there are a λ-G_δ set G and a function $g : G \to Y$ such that $A \subseteq G \subseteq \overline{A}$ and g is a continuous extension of f.*

Proof. Let d be a λ-metric compatible with (Y, τ_Y) such that (Y, d) is complete. Let $G := \overline{A} \cap C^f$ and $g(x) = \lim_{\alpha\in\lambda} f(x_\alpha)$ where $(x_\alpha)_{\alpha\in\lambda}$ is a Cauchy sequence in A converging to x. By continuity of f we have that the sequence $(f(x_\alpha))_{\alpha\in\lambda}$ is also a Cauchy sequence. Therefore, by the fact that Y is complete, the function g is well defined. It is easy to see that G and g are as desired. Indeed, by Lemma 2.33, G is λ-G_δ. Moreover, let U be open in X. We have that

$$\forall x, y \in U \cap \mathrm{dom}(f) d(f(x), f(y)) < \delta \to \forall x, y \in U \cap \mathrm{dom}(g) d(g(x), g(y)) \leq \delta.$$

Without loss of generality assume $x, y \notin \mathrm{dom}(f)$; the other cases are easier. Then, there are $(x_\alpha)_{\alpha\in\lambda}$ and $(y_\alpha)_{\alpha\in\lambda}$ such that $\lim_{\alpha\in\lambda} x_\alpha = x$, $\lim_{\alpha\in\lambda} y_\alpha = y$, $g(x) = \lim_{\alpha\in\lambda} f(x_\alpha)$, and $g(y) = \lim_{\alpha\in\lambda} y_\alpha$. For every α we have

$$d(g(x), g(y)) \leq d(f(x_\alpha), g(x)) + d(f(x_\alpha), f(y_\alpha)) + d(f(y_\alpha), g(y)) < d(f(x_\alpha), g(x)) + \delta + d(f(y_\alpha), g(y)).$$

But, since by definition $\lim_{\alpha\in\lambda} d(f(x_\alpha), g(x)) = 0$ and $\lim_{\alpha\in\lambda} d(f(y_\alpha), g(y)) = 0$, we have that $d(g(x), g(y)) \leq \delta$ as desired. But then $G \subseteq C^f \subseteq C^g$ and g is continuous as desired. □

From Kuratowski's theorem we can generalise half of the classical result that subspaces of Polish spaces are Polish if and only if they are G_δ.

Theorem 2.35. *If (X, τ) is λ-metrisable and $Y \subseteq X$ is λ-Polish then Y is λ-G_δ in X.*

Proof. Consider the identity function $\mathrm{Id}_Y : Y \to Y$. This function is continuous. So by Theorem 2.34 there is are G an $g : G \to Y$ such that $Y \subseteq G \subseteq \overline{Y}$ and g is a continuous extension of Id_Y. Since Y is dense in $\overline{Y}$ and therefore in G we have that $g = \mathrm{Id}_G$ and $G = Y$. So Y is λ-G_δ in X. □

Corollary 2.36. *Every λ-Polish space is homeomorphic to a λ-G_δ subspace of the generalised Cantor space 2^λ.*

Proof. The statement follows from Theorem 2.23 and Theorem 2.35. □

2.3.5 Generalised Choquet games and Polish spaces

In [22] Coskey and Schlicht introduced the following notion:

Definition 2.37 (Coskey and Schlicht)**.** Let (X, τ) be a topological space. The *strong λ-Choquet game* λ-G_X in (X, τ) is played between two players, I and II with the following rules: on the first turn player I plays an open subset U_0 of X and a point $x_0 \in U_0$. Then II plays an open subset V_0 of U_0 such that $x_0 \in V_0$. In general, in the beginning of every turn I plays a pair (U_α, x_α) with U_α open subset of $\bigcap_{\beta \in \alpha} V_\beta$ and $x_\alpha \in U_\alpha$. Then II plays an open subset V_α of U_α such that $x_\alpha \in V_\alpha$. We say that II wins the game iff for all limit ordinals $\beta \leq \lambda$ we have $\bigcap_{\alpha \in \beta} U_\alpha \neq \varnothing$. A weaker version of the game called *weak λ-Choquet game* can be defined by dropping the requirements on the x_αs, we will denote this game with λ-$\mathrm{G}^{\mathrm{w}}_X$.

Definition 2.38. A topological space (X, τ) is said to be *strongly λ-Choquet* iff II has winning strategy in the strong λ-Choquet game λ-G_X. Similarly a topological space (X, τ) is said to be *weakly λ-Choquet* iff II has winning strategy in the weak λ-Choquet game λ-$\mathrm{G}^{\mathrm{w}}_X$.

Obviously every strongly λ-Choquet space is weakly λ-Choquet.

In [22], Coskey and Schlicht used a modified version of weak λ-Choquet games to characterise λ-Baire spaces with $\lambda^{<\lambda} = \lambda$ and such that the intersection of less than λ-many open sets has no empty interior; see [22, Proposition 2.6].

Lemma 2.39 (Coskey & Schlicht)**.** *Let (X, τ) be a topological space with a base of size $\leq\lambda$. Suppose that one of the players has a winning strategy in the strong λ-Choquet game. Then this player has a winning strategy in which she only plays basic open sets. The same is true for weak λ-Choquet games.*

Proof. See [22, Lemma 2.5]. □

Therefore, from now on, we will always assume that when players play using a winning strategy they will always play basic open sets.

Theorem 2.40. *Every λ-additive weakly λ-Choquet space (X, τ) is λ-Baire.*

Proof. Assume that X is weakly λ-Choquet and λ-additive. We want to show that II has a winning strategy for λ-G_X. Let $(U_\beta)_{\beta\in\alpha}$ be a sequence of dense subsets of X. At stage $\beta < \alpha$ player I will just play $\bigcap_{\gamma<\beta} V_\gamma \cap U_\beta$ which by λ-additivity is open, while II will play according to the winning strategy. Since II played according to the winning strategy we have $\bigcap_{\beta\in\alpha} U_\alpha \neq \varnothing$ as desired. Note that, since we could play this game starting from any open set, $\bigcap_{\beta\in\alpha} U_\alpha$ must be dense. □

The following result shows that the game theoretical notion of generalised Polish spaces introduced by Coskey and Schlicht and our metric notion of generalised Polish spaces do not coincide.

Theorem 2.41. *The space λ-$\mathbb{R}$ is λ-Polish but neither λ-Baire nor weakly λ-Choquet. In particular, λ-$\mathbb{R}$ is not strongly λ-Choquet.*

Proof. First note that λ-$\mathbb{R}$ with the absolute value metric is a λ-separable complete λ-metric space. So λ-$\mathbb{R}$ is λ-Polish. By Theorem 2.4 it is not λ-Baire. Finally from Theorem 2.40 and Lemma 2.19 it follows that λ-$\mathbb{R}$ is not weakly λ-Choquet. □

The notion of spherical completeness is connected to the following topological notion of completeness.

Definition 2.42. Let (X, d) be a λ-metric space. A sequence $(U_\alpha)_{\alpha\in\beta}$ with $0 < \beta \in \mathrm{Ord}$ of open balls of X is a *tower* iff

1. for all $\alpha \in \beta$ we have $U_\alpha \neq \varnothing$,
2. for all $\alpha, \gamma \in \beta$ if $\gamma \leq \alpha$ then $U_\alpha \subseteq U_\gamma$,
3. $\bigcap_{\alpha\in\beta} U_\alpha = \varnothing$.

We call the ordinal β the *length* of the tower.

Definition 2.43. Let (X, d) be a μ-metric space. We say that (X, d) is λ-*topologically complete* iff there are no towers of length $<\lambda$ in (X, d).

By Lemma 1.9, for ordered topological fields with base number λ equipped with the absolute value distance the notions of λ-spherical completeness and of λ-topological completeness coincide. Therefore, it follows from Theorem 2.2 that λ-$\mathbb{R}$ is not even $\aleph_1$-topologically complete.

Definition 2.44. Let (X, τ) be a topological space. We will say that (X, τ) is a *strongly completely λ-metrisable* space iff there is a λ-metric d compatible with τ such that (X, d) is Cauchy complete and λ-topologically complete.

Theorem 2.45. *Let (X, τ) be a strongly completely λ-metrisable space. Then X is strongly λ-Choquet.*

Proof. Assume that (X, τ) is strongly completely λ-metrisable and let d be a complete compatible G-metric on X for some totally ordered abelian group G with $\mathrm{bn}(G) = \lambda$ which makes (X, d) Cauchy complete and λ-topologically complete. Let $(\varepsilon_\alpha)_{\alpha\in\lambda}$ be a strictly decreasing coinitial sequence in G^+. Player II can play according to the following strategy: at stage α, given an open set U_α and a point $x_\alpha \in U_\alpha$ II will play any open ball $B_d(x_\alpha, \varepsilon_{x_\alpha}) \subset U_\alpha$ with $\varepsilon_{x_\alpha} < \varepsilon_\alpha$ and such that for every $x \notin U_\alpha$ we have $\varepsilon_{x_\alpha} < d(x, x_\alpha)$.

Note that by the fact that (X, d) is λ-topologically complete we have that for every limit ordinal $\alpha < \lambda$ we have that $\bigcap_{\beta\in\alpha} U_\beta = \bigcap_{\beta\in\alpha} B_d(x_\beta, \varepsilon_{x_\beta}) \neq \varnothing$. Also note that, by the way in which II chose the open balls, the sequence $(x_\alpha)_{\alpha\in\lambda}$ is λ-Cauchy and by completeness the sequence has a limit ℓ. Now we will show that $\ell \in U_\alpha$ for every $\alpha \in \lambda$. Assume not and let β be the smallest such that $\ell \notin U_\beta$. For every $\gamma > \beta$ we have that $d(\ell, x_\gamma) \geq d(x_\beta, \ell) - d(x_\gamma, x_\beta) \geq d(x_\beta, \ell) - \varepsilon_{x_\beta}$ which is bigger than 0 by the way II chose x_β. But this contradicts the fact that ℓ was the limit of the sequence. Therefore $\ell \in U_\alpha$ for each $\alpha \in \lambda$ and $\bigcap_{\beta\in\lambda} U_\beta \neq \varnothing$ as desired. □

Using Theorem 2.45 we can prove a generalised version of Baire Category theorem.

Corollary 2.46. *Every strongly completely λ-metrisable space is λ-Baire.*

Proof. The claim follows from Lemma 2.19 and Theorem 2.45. □

Lemma 2.47. *Every non-empty λ-topologically complete λ-G_δ subspace of a λ-metric strongly λ-Choquet space is strongly λ-Choquet.*

Proof. Let (X, d) be a λ-metric strongly λ-Choquet space; and let $\varnothing \neq Y \subseteq X$ be a λ-topologically complete λ-G_δ subspace of X. Assume that $Y = \bigcap_{\alpha\in\lambda} O_\alpha$ with O_α open in X for every $\alpha < \lambda$. Assume that II has a winning strategy σ in λ-G_X in which she only plays open balls. We want to define a winning strategy for λ-G_Y. Assume I plays (U_0, x_0) in the first turn of λ-G_Y. Then, let U'_0 be open in X and such that $U'_0 \cap Y = U_0$. Let $U^*_0 = U'_0 \cap O_0$ and V^*_0 be obtained by using σ on (U^*_0, x_0). Then in λ-G_Y II will play an open ball $V_0 \subseteq V^*_0 \cap Y$ containing x_0. In general if α and I played (U_α, x_α) in the αth turn of the game let U'_α be such that $U'_\alpha \cap Y = U_\alpha$. Let $U^*_\alpha = U'_\alpha \cap O_\alpha$, and V^*_α be obtained by using σ on (U^*_α, x_α). Player II will play an open ball $V_\alpha \subseteq V^*_\alpha \cap Y$ containing x_α. Now, we need to show that II can play this game for λ-many steps, i.e., that for every limit ordinal $\alpha \leq \lambda$ we have $\bigcap_{\beta\in\lambda} U_\beta = \bigcap_{\beta\in\lambda} V_\beta \neq \varnothing$. Since Y is λ-topologically complete then for every limit $\alpha < \lambda$ we have that $\bigcap_{\beta\in\alpha} U_\beta \neq \varnothing$.

Now, since σ is winning for II we have $\bigcap_{\beta<\lambda} U^*_\beta \neq \varnothing$. But then

$$\begin{aligned}
\bigcap_{\beta\in\lambda} U_\beta &= \bigcap_{\beta\in\lambda} (U'_\beta \cap Y) \\
&= \bigcap_{\beta\in\lambda} (U'_\beta \cap \bigcap_{\gamma\in\lambda} O_\gamma) \\
&= \bigcap_{\beta\in\lambda} (U'_\beta \cap O_\beta) \\
&= \bigcap_{\beta\in\lambda} U^*_\beta \neq \varnothing.
\end{aligned}$$

□

Classically strongly Choquet subspaces of a Polish space X are exactly the G_δ-subsets of X. The proof in [51, Theorem 8.17] of the fact that any G_δ-subsets of X is strongly Choquet relies on Kőnig's lemma. It is therefore not surprising that in the generalised case we can give a similar proof assuming that λ is weakly compact.

Lemma 2.48. *Let G be a totally ordered group, d be a G-metric and (X, d) be a λ-separable λ-metric space. Then for every non-empty open set U we can define a sequence $(U_\alpha)_{\alpha\in\lambda}$ of open sets such that*

1. *for all $\alpha' < \alpha < \lambda$ we have $U_\alpha \subseteq U_{\alpha'}$;*
2. *for all $\alpha < \lambda$ we have $\overline{U_\alpha} \subseteq U$;*
3. *$U = \bigcup_{\beta\in\lambda} U_\alpha$.*

Proof. By Lemma 2.19 X is normal and λ-additive. Moreover, by Lemma 2.33 $U := \bigcup_{\alpha\in\lambda} C_\alpha$ with C_α closed. By normality we have that for every closed subset A of U there is an open subset V of U such that $A \subseteq V$ and $\overline{V} \subseteq U$. Note that by regularity of λ and by λ-additivity of X we have that for every $\alpha \in \lambda$ the set $\bigcup_{\beta<\alpha} C_\beta$ is closed. Therefore by normality the following sequence is well defined: U_α is an open set such that $\bigcup_{\beta\leq\alpha} C_\beta \subseteq U_\alpha \subseteq \overline{U_\alpha} \subset U$. Note that the sequence has all the properties we needed. □

Lemma 2.49. *Let G be a totally ordered group, d be a G-metric, (X, d) be a λ-separable λ-metric space and $\mathcal{U}$ be a family of non empty open subsets of X. Then for every $\varepsilon \in G^+$ there is a family $\mathcal{V}$ of non-empty open sets in X such that*

1. *$\bigcup \mathcal{V} = \bigcup \mathcal{U}$;*
2. *for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subseteq U$;*
3. *for every $V \in \mathcal{V}$ for every $x, y \in V$ we have $d(x, y) < \varepsilon$;*
4. *for every $x \in X$ we have $|\{V \in \mathcal{V}\, ;\, x \in V\}| < \lambda$.*

*We will call the family $\mathcal{V}$ a $<\lambda$-*refinement *of $\mathcal{U}$ of* diameter ε.

Proof. Since X is λ-separable then there is a base of X of size λ. Therefore, since $\bigcup \mathcal{U}$ is open, we can choose a sequence of open sets $(U_\alpha)_{\alpha\in\lambda}$ such that $\bigcup \mathcal{U} = \bigcup_{\alpha\in\lambda} U_\alpha$ with U_α open such that: for every $\alpha \in \lambda$ and for every $x, y \in U_\alpha$ we have $d(x, y) < \varepsilon$ and exists $U \in \mathcal{U}$ such that $U_\alpha \subset U$.

By Lemma 2.19 X is normal and λ-additive. Therefore by Lemma 2.48 we can define a sequence $(U_\alpha^\beta)_{\beta\in\lambda}$ such that

1. for all $\beta' < \beta < \lambda$ we have $U_\alpha^{\beta'} \subseteq U_\alpha^\beta$;
2. for all $\beta < \lambda$ we have $\overline{U_\alpha^\beta} \subseteq U_\alpha$;
3. $U_\alpha = \bigcup_{\beta\in\lambda} U_\alpha^\beta$.

Now define $V_\beta := U_\beta \setminus \bigcup_{\alpha<\beta} \overline{U_\alpha^\beta}$. We claim that $\mathcal{V} := \{V_\alpha\, ;\, \alpha \in \lambda \text{ and } V_\alpha \neq \varnothing\}$ has the desired properties. Note that properties 2 and 3 are true by construction. Now, let $x \in \bigcup_{\alpha\in\lambda} U_\alpha = \bigcup \mathcal{U}$. Let β be the smallest such that $x \in U_\beta$. Then, if there is $\alpha < \beta$ such that $x \in \overline{U_\alpha^\beta}$, then $x \in U_\alpha$, contradiction. Hence, $x \notin \bigcup_{\alpha<\beta} \overline{U_\alpha^\beta}$ implies $x \in V_\beta$. Since $V_\beta \subseteq U_\beta$ for every $\beta < \lambda$, we have that $\bigcup \mathcal{V} = \bigcup \mathcal{U}$. Finally, if $x \in X$ let α be the smallest such that $x \in U_\alpha$ and β be the least such that $x \in U_\alpha^\beta$. Then $x \notin V_\gamma$ for every $\gamma > \max\{\beta, \alpha\}$ which by regularity of λ implies $|\{V \in \mathcal{V}\, ;\, x \in V\}| < \lambda$ as desired. □

Definition 2.50. A topological space is *strongly λ-Polish* iff it is λ-separable and strongly completely λ-metrisable.

Every strongly λ-Polish space is λ-Polish by definition. Moreover, by the same argument we gave on p. 33 λ-$\mathbb{R}$ is a λ-Polish space which is not strongly λ-Polish; therefore, the notion of λ-Polish is strictly weaker than that of strongly λ-Polish.

Theorem 2.51. *Let λ be weakly compact, (X, τ) be a strongly λ-Polish space and Y be a non-empty subspace of X. If Y is strongly λ-Choquet then it is λ-G_δ.*

Proof. Fix a winning strategy σ for II in Y; a totally ordered abelian group G such that there is a G-metric d compatible with τ such that (X, d) is Cauchy complete and λ-topologically complete; and fix a sequence $(\varepsilon_\alpha)_{\alpha\in\lambda}$ coinitial in G^+. We will build a sequence $(W_\alpha)_{\alpha\in\lambda}$ of open sets containing Y and a tree T of sequences of the form

$$(U_\beta, x_\beta, V_\beta, U_{\beta+1})_{\beta\in\alpha}.$$

Let S_1 be the set of sequences

$$(U_0, x_0, V_0, U_1),$$

where $U_0 = X$, $x_0 \in U_0 \subseteq V_0$, V_0 is the open set obtained by using σ on $(U_0 \cap Y, x_0)$ and U_1 is open and such that $U_1 \cap Y \subseteq V_0$. Moreover, let $\mathcal{U}_1$ be the set $\{U_1\,;\,(U_0, x_0, V_0, U_1) \in S_1\}$ and $W_1 := \bigcup \mathcal{U}_1$. Note that $Y \subseteq W_1$ and that W_1 is open. Let $\mathcal{V}_1$ be a $<\lambda$-refinement of $\mathcal{U}_1$ of diameter ε_1. Let $T_1 \subseteq S_1$ be a set of sequences such that for every $U_1 \in \mathcal{V}$ there is a unique $(U_0, x_0, V_0, U_1) \in T_1$. In general for $\alpha > 1$ we distinguish two cases: If α is a limit let $T_\alpha := \bigcup_{\beta<\alpha} T_\beta$ and $W_\alpha := \bigcap_{\beta\in\alpha} W_\beta$. Note that, by λ-additivity, W_α is open and, by the fact that for all $\beta < \lambda$ we have $Y \subseteq W_\beta$, it follows that $Y \subseteq W_\alpha$.

If $\alpha := \gamma + 1$ is a successor ordinal we have two cases in the definition of S_α:

Case 1. The ordinal γ is a successor: The set S_α is the collection of sequences of length α of the form

$$(U_\beta, x_\beta, V_\beta, U_{\beta+1})_{\beta<\alpha},$$

where $(U_\beta, x_\beta, V_\beta, U_{\beta+1})_{\beta\in\gamma} \in [T_\gamma]$, the set U_α is open and $U_\alpha \cap Y \subseteq V_\gamma$, as in the base case $x_\gamma \in U_\gamma$; and V_γ is obtained by playing σ on $(U_\gamma \cap Y, x_\gamma)$.

Case 2. The ordinal γ is limit: The set S_α is the collection of sequences of length α of the form

$$(U_\beta, x_\beta, V_\beta, U_{\beta+1})_{\beta<\alpha},$$

where $(U_\beta, x_\beta, V_\beta, U_{\beta+1})_{\beta<\gamma} \in [T_\gamma]$, $U_\gamma := \bigcap_{\beta<\gamma} V_\beta$; and, as before, $x_\gamma \in U_\gamma$ and V_γ is obtained by playing σ on $(U_\alpha \cap Y, x_\alpha)$, and U_α is and open such that $U_\alpha \cap Y \subseteq V_\alpha$.

Now, as for the base case let $\mathcal{U}_\alpha$ for $\alpha = \gamma + 1$ be the set:

$$\{U_{\gamma+1}\,;\,(U_\beta, x_\beta, V_\beta, U_{\beta+1})_{\beta<\alpha} \in S_\alpha\}$$

and let $W_\alpha = \bigcup \mathcal{U}_\alpha$. Note that $Y \subseteq W_\alpha$, and that W_α is open. Let $\mathcal{V}_\alpha$ be a $<\lambda$-refinement of $\mathcal{U}_\alpha$ of diameter ε_α. Let $T_\alpha \subseteq S_\alpha$ be a set of sequences such that for every $U_{\gamma+1} \in \mathcal{V}_\alpha$ there is a unique $(U_\beta, x_\beta, V_\beta, U_{\beta+1})_{\beta<\alpha} \in T_\alpha$.

Finally let $T := \bigcup_{\alpha<\lambda} T_\alpha$. We claim that $Y = \bigcap_{\alpha\in\lambda} W_\alpha$. Since $Y \subseteq W_\alpha$ for every $\alpha < \lambda$, $Y \subseteq \bigcap_{\alpha\in\lambda} W_\alpha$. Now let $x \in \bigcap_{\alpha\in\lambda} W_\alpha$ and consider the subtree

$$T_x := \{(U_\beta, x_\beta, V_\beta, U_{\beta+1})_{\beta<\alpha} \in T\,;\,\forall\beta < \alpha(x \in U_\beta)\}$$

of T. Note that T_x is a λ-tree. Since $x \in \bigcap_{\alpha\in\lambda} W_\alpha$ we have that $\mathrm{ht}(T_x) = \lambda$. Indeed, for every $\alpha < \lambda$ there is a $(U_\beta, x_\beta, V_\beta, U_{\beta+1})_{\beta<\alpha+1} \in [T_{\alpha+1}]$ with $x \in U_{\alpha+1}$. Moreover, for every $\alpha < \lambda$ we have $|\mathrm{lvl}_{T_x}(\alpha)| < \lambda$. To see this, assume that α is the smallest such that $|\mathrm{lvl}_{T_x}(\alpha)| \geq \lambda$. First note that α must be a successor. Indeed, by the fact that λ is strong limit and $|\mathrm{lvl}_{T_x}(\beta)| < \lambda$ for every $\beta < \alpha$, we have that $|\mathrm{lvl}_{T_x}(\alpha)| < \lambda$. Let $\alpha := \gamma + 1$. We have $|\mathrm{lvl}_{T_x}(\gamma)| < \lambda$ but since $|\mathrm{lvl}_{T_x}(\gamma)| = |[T_\gamma] \cap T_x|$ we must have $|\{U \in \mathcal{V}_\alpha\, ;\, x \in U\}| \geq \lambda$ contradicting the fact that $\mathcal{V}_\alpha$ is a $<\lambda$-refinement. So T_x is a λ-tree.

Now, by the tree property of λ we have $[T_x] \neq \varnothing$. Let $(U_\beta, x_\beta, V_\beta, U_{\beta+1})_{\beta<\lambda} \in [T_x]$. Since σ is winning for II we must have $\bigcap_{\alpha\in\lambda}(U_\alpha \cap Y) \neq \varnothing$ and therefore $\bigcap_{\alpha\in\lambda} U_\alpha \neq \varnothing$. But then, since the diameter of each $\mathcal{V}_\alpha$ is ε_α and $(\varepsilon_\beta)_{\beta\in\lambda}$ converges to 0, we have that $\bigcap_{\alpha\in\lambda} U_\alpha = \{x\}$ and therefore $x \in Y$. $\square$

2.3.6 The generalised Cantor-Bendixson theorem

In classical descriptive set theory, Polish spaces are very well-behaved. In particular, by the Cantor-Bendixson theorem, they satisfy a form of continuum hypothesis, viz. every Polish spaces is either countable or has cardinality $2^{\aleph_0}$ [51, Corollary 6.5]. In this section we will prove a generalised version of Cantor-Bendixson theorem; and we will begin the study of the notion of perfectness in connection to the theory of generalised metrisability. We will begin by introducing some terminology and by proving the generalised version of some classical results.

Definition 2.52. Let (X, τ) be a topological space. Then $x \in X$ is a *limit point* if for every $U \in \tau$ such that $x \in U$ there is $y \in U$ with $y \neq x$. Moreover, the space (X, τ) is a *perfect space* if all its points are limit points. Finally, a subset P of X is *perfect in* (X, τ) if it is closed and it is a perfect subspace of (X, τ).

Theorem 2.53. *Let (X, τ) be a non-empty perfect strongly λ-Polish space. Then there is an embedding of the generalised Cantor space 2^λ into X.*

Proof. Let $d : X \times X \to G^+ \cup \{0\}$ be a λ-metric compatible with (X, τ) such that (X, d) is Cauchy complete and λ-topologically complete space; and let $(\varepsilon_\alpha)_{\alpha\in\lambda}$ be a strictly decreasing coinitial sequence in G^+. We define a tree $(U_s)_{s\in 2^{<\lambda}}$ of open balls of X: let $U_\varnothing$ be any non-empty open ball in X. Assume that for $\beta < \lambda$ and $s \in 2^\beta$ the set U_s is already defined. By inductive hypothesis U_s is a non empty open subset of X. By perfectness of X we have $|U_s| \geq 2$. We let $x, y \in U_s$ be such that $x \neq y$. Let U_{s0}, U_{s1} be two open sets such that

1. $U_{s0} \cap U_{s1} = \varnothing$;
2. $U_{s0} := B_d(x, \varepsilon_x)$ with $\varepsilon_x \leq \varepsilon_{\beta+1}$;
3. $U_{s1} := B_d(y, \varepsilon_y)$ with $\varepsilon_y \leq \varepsilon_{\beta+1}$.

If α is a limit ordinal $s \in 2^\alpha$ and for all $\beta < \alpha$ we have that $U_{s\restriction\beta}$ is defined. Then note that by λ-additivity $\bigcap_{\beta<\alpha} U_{s\restriction\beta}$ is open, moreover it is not empty since X is λ-topologically complete. Let $x \in \bigcap_{\beta<\alpha} U_{s\restriction\beta}$. We let $U_s := B_d(x, \varepsilon_\beta)$.

For every $s \in 2^\lambda$ we have that $\bigcap_{\alpha\in\lambda} U_{s\restriction\alpha}$ is non empty by completeness of X. Define $f(s)$ to be the unique element in $\bigcap_{\alpha\in\lambda} U_{s\restriction\alpha}$. We need to show that f is injective and continuous. First note that if $s, s' \in 2^\lambda$ are such that $s \neq s'$ then there is $\alpha < \lambda$ such that $s(\alpha) \neq s'(\alpha)$ but then $U_s \subset U_{s\restriction\alpha+1}$ and $U_{s'} \subset U_{s'\restriction\alpha+1}$ and $U_{s\restriction\alpha+1} \cap U_{s'\restriction\alpha+1} = \varnothing$. Moreover, note that f is continuous. Indeed, let $s \in f^{-1}(B_d(x, r))$ and $f(s) = y$. Let $\beta < \lambda$ be such that $2\varepsilon_{\beta+1} < r - d(x, y)$. We claim that $f([s\restriction\beta]) \subset B_d(x, r)$. To see this, let $s' \in [s\restriction\beta]$. Then $f(s') \in U_{s\restriction\beta}$, and $d(x, f(s')) \leq d(x, f(s)) + d(f(s), f(s')) < d(x, y) + r - d(x, y) = r$. For the last inequality, note that $U_{s\restriction\beta} = B_d(z, \varepsilon_z)$ for some $z \in X$ and $\varepsilon_z \leq \varepsilon_{\beta+1} < \varepsilon_\beta$; therefore, $d(f(s), f(s')) \leq 2\varepsilon_{\beta+1} < r - d(x, y)$. □

Similarly to the classical case, it follows from Theorem 2.53 that strongly λ-Polish spaces have cardinality 2^λ.

Corollary 2.54. *Every non-empty perfect strongly λ-Polish space has cardinality 2^λ.*

Proof. Follows from Theorems 2.53 & 2.23. □

Theorem 2.55 (Generalised Cantor-Bendixson)**.** *Let (X, τ) be a λ-separable λ-metrisable space. Then X can be written as $X = P \cup C$ where $C \cap P = \varnothing$, the set P is perfect in X and C is open and has cardinality at most λ.*

Proof. Let d be a λ-metric compatible with (X, τ). We will say that $x \in X$ is a *condensation point* if every open subset of X containing x has cardinality $>\lambda$. Let P be the set of condensation points of X and $C := X \setminus P$. Note that C is open. Indeed, if $x \in C$ then there is an open subset of X containing x of cardinality $<\lambda$ and an open ball $B_d(x, r)$ which has cardinality $\leq \lambda$. Note that $B_d(x, r) \subset C$. Indeed, for $z \in B_d(x, r)$ let ε be such that $\varepsilon = r - d(x, z)$ then $B_d(z, \varepsilon) \subset B_d(x, r)$, hence $|B_d(z, \varepsilon)| \leq \lambda$ and z is not a condensation point.

Since no point in C is a condensation point and C is open then C must be of cardinality $\leq\lambda$. Being the complement of an open set $P = X \setminus C$ is closed. Now we want to show that it is perfect in X. Let $x \in P$ and U be an open subset of X containing x. Since $x \in P$ we have that U has cardinality $>\lambda$ but since $U \cap C$ is of cardinality at most λ we have that $P \cap U$ has cardinality $>\lambda$. □

In [104] Väänänen introduced a notion of $\aleph_1$-perfectness for closed subsets of $\omega_1^{\omega_1}$ in terms of games. More recently Kovachev and Schlicht extended this definition to generalised Baire spaces; see, [59, 85]. We will end this section by first generalising Väänänen's games to λ-metric spaces; and by starting the study of the relationship between the game theoretical and the topological definitions of perfectness.

Definition 2.56. Let G be a totally ordered group, d be a G-metric and (X, d) be a λ-metric space. The *λ-perfect game* λ-$\mathrm{G}^*_X(x_0)$ is played between two players I and II. Player I in his turn plays elements of G^+ and II plays elements of X. The game stars with player I choosing an element $\varepsilon_0 \in G^+$. Player II then plays $x_1 \in X$ such that $0 < d(x_0, x_1) < \varepsilon_0$. In general at the beginning of the αth turn with α successor player I starts and plays an element $\varepsilon_\alpha \in G^+$ such that for all $\beta < \alpha$ we have $\varepsilon_\alpha < \varepsilon_\beta$. Then player II plays $x_\alpha \in X$ such that for every $\beta < \alpha$ we have $0 < d(x_\beta, x_\alpha) < \varepsilon_\beta$. At a limit turn α player II starts and plays $x_\alpha \in X$ such that for every $\beta < \alpha$ we have $0 < d(x_\beta, x_\alpha) < \varepsilon_\beta$. Then player I plays an element $\varepsilon_\alpha \in G^+$ such that for all $\beta < \alpha$ we have $\varepsilon_\alpha < \varepsilon_\beta$.

Then we will say that II wins the game if he can play for λ many turns or if $(\varepsilon_\alpha)_{\alpha\in\lambda}$ is not coinitial in G^+.

Definition 2.57. Let G be a totally ordered group, d be a G-metric, (X, d) be a λ-metric space and Y be a subspace of X. Then Y is *λ-perfect* in X if and only if it is closed and for all $x_0 \in Y$ player II has a winning strategy for λ-$\mathrm{G}^*_X(x_0)$.

Note that for generalised Baire space λ^λ with the usual (λ-$\mathbb{R}$)-metric the previous definition is equivalent to those in [104, Definition 1] and [85, Definition 2.2].

Theorem 2.58. *Let G be a totally ordered group, d be a G-metric and (X, d) be a λ-metric space. Then for every $Y \subseteq X$, if Y is λ-perfect in X then it is perfect in X.*

Proof. If Y is not perfect then there are $x \in Y$ and $\varepsilon \in G^+$ such that $B_d(x, \varepsilon) \cap Y = \{x\}$ so I has a winning strategy for λ-$\mathrm{G}^*_X(x)$. □

As was already noted by Väänänen in [104, p. 189], the two notions of perfectness do not coincide in general. As for Choquet games also in this case we have that λ-topological completeness allows us to prove more.

Theorem 2.59. *Let G be a totally ordered group, d be a G-metric and (X, d) be a λ-metric space. If $Y \subseteq X$ is λ-topologically complete and perfect in X then it is λ-perfect in X.*

Proof. Let Y be perfect in X. Then Y is closed. Let $x_0 \in Y$. We need to show that player II has a winning strategy for λ-$\mathrm{G}^*_X(x_0)$. Player II plays the following strategy while building a sequence of open sets: at stage 0 by perfectness $|B_d(x_0, \varepsilon_0) \cap Y| \geq 2$. Then player II plays $x_1 \in B_d(x_0, \varepsilon_0)$ with $x_0 \neq x_1$ and sets $B_0 := B_d(x_1, \delta_0)$ with $\delta_0 < d(x_0, x_1)$. If $\alpha := \gamma + 1$ is a successor then by perfectness $|B_\gamma \cap Y| \geq 2$. Then player II plays $x_\alpha \in B_d(x_\gamma, \varepsilon_\gamma)$ with $x_\gamma \neq x_\alpha$ and sets $B_\alpha := B_d(x_\alpha, \delta_\alpha)$ with $\delta_\alpha < d(x_\gamma, x_\alpha)$. If α is limit, by λ-topological completeness of X we have that $\bigcap_{\beta<\alpha} B_\beta \cap Y$ is not empty. Let $x_\alpha \in \bigcap_{\beta<\alpha} B_\beta \cap Y$ and $B_\alpha := \bigcap_{\beta<\alpha} B_\beta$.

It is not hard to see that II will be able to play λ-$\mathrm{G}^*_Y(x_0)$ for λ-many turns for every $x_0 \in Y$ following the previous strategy. □

Corollary 2.60. *Let G be a totally ordered group, d be a G-metric and (X, d) be a λ-topologically complete λ-metric space. Then X is perfect if and only if it is λ-perfect.*

2.4 Open questions

As we have seen in § 2.3.2, generalisations of the real line naturally lead to a generalisation of metrisability. The theory that we introduced is far from being complete.

In Theorem 2.40, we have seen that every λ-additive weakly λ-Choquet space is λ-Baire. We do not know if this is actually a characterisation.

Question 2.61. Is every λ-additive λ-Baire space weakly λ-Choquet?

By Theorem 2.35 every λ-Polish subspace of a λ-metrisable space is λ-G_δ. We still do not know if this implication can be reversed.

Question 2.62. Is every λ-G_δ subset of a λ-metrisable set a λ-Polish space?

Moreover, as we mentioned in § 2.3.1, we do not even know if every open subset of a λ-Polish space is λ-Polish.

Question 2.63. Is every open subset of a λ-metrisable set a λ-Polish space?

We believe that the theory of uniform spaces used in [2, 88, 95] is going to be central in answering the previous question.

Question 2.64. Is the sum of $<\lambda$ many λ-Polish spaces λ-Polish?

Question 2.65. Is the product of $<\lambda$ many λ-Polish spaces λ-Polish?

In [22] Coskey and Schlicht show that as in the classical case strongly λ-Choquet spaces are continuous images of generalised Baire space λ^λ. We do not know if this result can be proved for λ-Polish spaces.

Question 2.66. Is every λ-Polish space a continuous image of λ^λ?

In § 2.3.1 we pointed out that we do not know the current status of the diagram on p. 25. In particular:

Question 2.67. Does Theorem 2.11 generalise to strongly λ-Polish spaces?

As we already remarked we do not know the large cardinal strength of Theorem 2.51.

Question 2.68. Can the assumption of λ being weakly compact be removed from Theorem 2.51?

In the classical theory, Theorem 2.51 is actually a characterisation of strongly Choquet spaces. We do not know if this is also true in the generalised case.

Question 2.69. Let Y be a λ-G_δ subspace of a strongly Choquet space. Is Y strongly Choquet?

Chapter 3

The generalised reals: Bolzano-Weierstraß and Heine-Borel

Remarks on co-authorship. *The results of this chapter are, unless otherwise stated, due to a collaboration of the author with Merlin Carl and Benedikt Löwe. The results in §§ 3.2 & 3.3 appear in [16]. The questions and results in § 3.4 are due solely to the author.*

3.1 Introduction

Some properties do not transfer between $\mathbb{R}$ and ω^ω. One such property is the *Bolzano-Weierstraß theorem* BWT, i.e., "every sequence with bounded range has a cluster point". The property BWT concerns the interplay between boundedness and sequential compactness, i.e., the relation between the order and the topology. Hence, the validity of BWT is not a purely topological property: it is not preserved by homeomorphisms and, moreover, BWT fails on ω^ω.[1] Another fundamental property of the real line is the *Heine-Borel theorem* HBT, i.e., "for every subset X of $\mathbb{R}$ we have that X is compact if and only if X is closed and bounded". The BWT and the HBT are closely related: for ordered fields K, K is Dedekind-complete if and only if K satisfies BWT if and only if K satisfies HBT (see, e.g., [72, Chapter 5, Theorem 7.6]). As well as the BWT, the HBT is also a property which is not preserved by homeomorphism. In particular, it does not transfer from $\mathbb{R}$ to ω^ω.[2]

As mentioned, BWT and HBT both fail on Baire space, so the natural setting for uncountable generalisations of these theorems would not be κ^κ, but rather a generalisation of the real line. We will therefore focus on the status of these properties on the real ordinal numbers and on the generalised real line.

In the classical setting, the Bolzano-Weierstraß theorem is closely related to Kőnig's lemma. This relationship was made precise by Harvey Friedman in the context of reverse mathematics. In reverse mathematics, theories in the language of second order arithmetic are used to compare the strength of classical theorems from everyday mathematics. In

[1] Let $x^{(n)}$ be the sequence $(0n0\ldots)$. The sequence $(x^{(n)}\,;\, n \in \omega)$ is bounded in ω^ω, but has no cluster point.

[2] The clopen set $[01]$ is bounded but not compact.

[31], Friedman studies extensions of the recursive comprehension axiom system (RCA); see [31, §I] for the definition of RCA. In particular, in [31, Theorem 1.1], Friedman considers systems in which RCA is augmented with KL and BWT, respectively, and proves that these two systems are equivalent.

In the setting of *Weihrauch reducibility* the relationship between BWT and WKL was studied by Brattka, Gherardi, and Marcone [12]; they introduce a purely topological version of Bolzano-Weierstraß, $\mathsf{BWT}^{\mathrm{top}}$, i.e., "every sequence whose range has compact closure has a cluster point". If a space X satisfies the BWT then it satisfies the $\mathsf{BWT}^{\mathrm{top}}$.

Lemma 3.1. *Let X be a totally ordered set and (X, τ) be the order topology on X. If the property BWT holds in X, then the property $\mathsf{BWT}^{\mathrm{top}}$ holds in (X, τ).*

Proof. Let $(x_\alpha)_{\alpha\in\lambda}$ as in the statement of the $\mathsf{BWT}^{\mathrm{top}}$. It is enough to prove that the sequence is bounded. Consider the following set of intervals $C = \{(x, y) \mid x, y \in X \wedge x < y\}$. The set C is a covering of the closure of the range of $(x_\alpha)_{\alpha\in\lambda}$. But then there are finitely many intervals $(x_0, y_0), \ldots, (x_n, y_n)$ covering the closure of the range of the sequence. But then the range of the sequence is contained in the open interval $(\inf_{0\le i\le n} x_i, \sup_{0\le i\le n} y_i)$. Therefore, the sequence is bounded. □

In contrast to BWT, the property $\mathsf{BWT}^{\mathrm{top}}$ holds in Baire space (the failure of BWT in ω^ω corresponds to the fact that not all bounded subsets of ω^ω have compact closure).

Lemma 3.2. *The $\mathsf{BWT}^{\mathrm{top}}$ property holds in Baire space.*

Proof. Let $(x_\alpha)_{\alpha\in\lambda}$ be a sequence as in the statement of $\mathsf{BWT}^{\mathrm{top}}$. We want to find a cluster point s of the range of the sequence. We will define s by recursion. Note that the set $C_0 := \{[n]\,;, n \in \omega\}$ is an open cover of the closure of the range of the sequence. Therefore there must be a finite subcover of C_0 and a natural number $n_0 \in \omega$ such that $[n_0]$ contains infinitely many points of the range of the sequence. Let $s(0) := n_0$. Now, let $C_1 := (C_0 \setminus \{[n_0]\}) \cup \{[n_0 n]\,;\, n \in \omega\}$. The set C_1 is again an open cover of the closure of the range of the sequence; and, as before, there are a finite subcover of C_1 and a natural number $n_1 \in \omega$ such that $[n_0 n_1]$ contains infinitely many points of the range of the sequence. Let $s(1) := n_1$. In general assume that we have defined the sequence s up to m. Let $C_{m+1} := (C_m \setminus \{[s{\restriction}m]\}) \cup \{[(s{\restriction}m)n]\,;\, n \in \omega\}$. As before C_{m+1} is again an open cover of the closure of the range sequence; therefore, there is a finite subcover of C_{m+1} and a natural number $n_{m+1} \in \omega$ such that $[(s{\restriction}m)n_{m+1}]$ contains infinitely many points of the sequence. It is not hard to see that the sequence $s \in \omega^\omega$ is a cluster point of $(x_\alpha)_{\alpha\in\lambda}$. □

Writing $\mathsf{BWT}^{\mathrm{top}}_X$ for the statement "every sequence in X whose range has a compact closure has a cluster point in X", Brattka, Gherardi, and Marcone proved:

$$\mathsf{BWT}^{\mathrm{top}}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{BWT}^{\mathrm{top}}_{\omega^\omega} \equiv_{\mathrm{W}} \mathsf{WKL}',$$

where WKL' denotes the jump of WKL. In the Weihrauch setting, the jump corresponds to an application of the monotone convergence theorem which allows us to do a transition from the subsequence produced by WKL to the cluster point needed by BWT; WKL is not sufficient to do that transition (in other words, $\mathsf{WKL} <_{\mathrm{W}} \mathsf{BWT}$). Note that $\mathsf{BWT}^{\mathrm{top}}_X$ and

BWT_X are not in general Weihrauch equivalent for arbitrary ordered spaces X; however, they are in the case $X = \mathbb{R}$ (because of HBT). Therefore, $\mathsf{BWT}_\mathbb{R} \equiv_\mathrm{W} \mathsf{WKL}'$.

In this chapter, we will discuss generalisations of BWT to uncountable cardinals κ. For one of these, called the κ-weak Bolzano-Weierstraß theorem, we prove that if κ is inaccessible, then the κ-weak Bolzano-Weierstraß theorem holds for the generalised reals if and only if κ has the tree property (see Corollary 3.23) and the discussion on p. 34.

The chapter is organised as follows: in § 3.2, we will study generalisations of the Bolzano-Weierstraß theorem on κ-$\mathbb{R}$ and $\mathbb{R}_\kappa$; in § 3.2.1 we will remind the reader of the classical Bolzano-Weierstraß theorem; in § 3.2.2 we will study a generalised version of the Bolzano-Weierstraß theorem introduced by Sikorski; in §§ 3.2.3 and 3.2.4 we will introduce two natural version of generalised Bolzano-Weierstraß theorem and study their status on $\mathbb{R}_\kappa$; finally, in § 3.3, we will study of a generalised version of the Heine-Borel theorem.

3.2 The Bolzano-Weierstraß theorem

3.2.1 The classical Bolzano-Weierstraß theorem

Let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field. Then the *Bolzano-Weierstraß theorem for* K, abbreviated as BWT_K, is the statement

> "every bounded sequence of elements of K has a convergent subsequence".

In this statement, by "sequence" we mean a sequence of any length.

Theorem 3.3. *Let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field. Then BWT_K holds if and only if K is Dedekind complete.*

Proof. See [72, Theorem 7.6]. □

We had seen in § 1.3 that this means that, up to isomorphism, $\mathbb{R}$ is the only field satisfying the Bolzano-Weierstraß theorem.

Corollary 3.4. *Let κ be an uncountable regular cardinal such that $\kappa^{<\kappa} = \kappa$. Then $\mathsf{BWT}_{\kappa\text{-}\mathbb{R}}$ and $\mathsf{BWT}_{\mathbb{R}_\kappa}$ do not hold.*

The reason for this is that the statement of BWT_K talks only about sequences of any length; and of ω-sequences in particular. This, together with the fact that $\mathrm{bn}(\kappa\text{-}\mathbb{R}) = \mathrm{bn}(\mathbb{R}_\kappa) = \kappa$, implies that these sequences are simply too short to have convergent subsequences (using Corollary 1.11 (ii)).

3.2.2 The generalised Bolzano-Weierstraß theorem

We identified the problem with BWT to be the length of the sequences; consequently, the following restriction due to Sikorski is natural:

Let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field and λ be a regular cardinal. Then the *λ-Bolzano-Weierstraß theorem for K*, abbreviated as λ-BWT_K, is the statement

> "every bounded λ-sequence of elements of K has a convergent λ-subsequence".

The property λ-BWT_K was studied by several authors; see [23, 86, 89].

Observation 3.5. Note that if $\lambda > \omega$ is a regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ is a totally ordered field then λ-BWT_K fails if K is λ-divergent.

For weakly compact cardinals λ, we can reformulate the λ-Bolzano-Weierstraß theorem in terms of λ-divergence.

Theorem 3.6. *Let $\lambda > \omega$ be a weakly compact cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field. Then the following are equivalent:*

1. *the field K is λ-divergent and*
2. *λ-BWT_K does not hold.*

Proof. By Observation 3.5, we only need to prove "(2)⇒(1)". By Lemma 1.7, it is enough to show that there is an interval with a monotone divergent λ-subsequence. Let s be a bounded λ-sequence which has no convergent λ-subsequence. We will show that s has a monotone subsequence. Define the following partition of $\lambda \times \lambda$: $f(\alpha, \beta) := 1$ if $\alpha < \beta$ and $s(\alpha) < s(\beta)$, $f(\alpha, \beta) := 0$ otherwise. Since κ is weakly compact there is $H \subseteq \lambda$ such that either for all $h \in H \times H$, $f(h) = 1$ or for all $h \in H \times H$, $f(h) = 0$. Without loss of generality assume the former. Now, we define recursively a subsequence s' of s. Assume we that have already defined $s \restriction \alpha$, we define: $s'(\alpha) := s(\beta)$ where β is the least ordinal in $H \setminus \{s'(\gamma) \mid \gamma \in \alpha\}$. It is easy to see that s' is strictly increasing. Indeed, if $\alpha < \beta$ then $s'(\alpha) = s(\gamma)$ and $s'(\beta) = s(\gamma')$ for some $\gamma, \gamma' \in H$ such that $\gamma < \gamma'$, but then $f(\gamma, \gamma') = 1$ which implies $s(\gamma) < s(\gamma')$ as desired. □

We do not know whether there is a non λ-divergent field K such that λ-BWT_K fails. In some cases, we can prove or refute λ-BWT_K using elementary arguments:

Theorem 3.7. *Let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field.*

1. *If $\lambda > |K|$, then λ-BWT_K holds.*
2. *If $\lambda < \mathrm{bn}(K)$, then λ-BWT_K does not hold.*
3. *If $\mathrm{w}(K) = \kappa < \lambda$, then every convergent sequence of elements of K of length λ is eventually constant. Consequently, if $|K| \geq \lambda$, λ-BWT_K does not hold.*

Proof. (1) follows from the pigeonhole principle: every λ-sequence in K contains a constant λ-subsequence. For (2), observe that by Corollary 1.11 (ii) & (iii), if $\lambda < \mathrm{bn}(K)$, then K is λ-divergent. Then Observation 3.5 implies the claim.

For (3), let D be a dense subset of K of cardinality $\kappa < \lambda$. Towards a contradiction, let $s : \lambda \to K$ be a convergent sequence with limit $\ell \in K$ that is not eventually constant. Without loss of generality, we can assume that for each $\alpha < \lambda$, $s(\alpha) \neq s(\alpha + 1)$ and furthermore that $\ell \notin \mathrm{ran}(s)$. Thus, since D is dense, for each $\alpha < \lambda$, we find some $d_\alpha \in (D \cap (s(\alpha), s(\alpha+1))) \cup (D \cap (s(\alpha+1), s(\alpha)))$ such that $d_\alpha \neq \ell$. We define $\widehat{s} : \lambda \to K$ by $\widehat{s}(\alpha) := d_\alpha$.

By construction s and $\widehat{s}$ both converge to the same limit ℓ. Since $|D| < \lambda$ there is an element $d \in D$ which appears λ many times in $\widehat{s}$. Hence, $\widehat{s}$ has a subsequence of length λ which is eventually constant (and converges to the same limit as $\widehat{s}$, i.e., ℓ). But this is a contradiction since ℓ is not an element of $\mathrm{ran}(\widehat{s})$. □

	κ-$\mathbb{R}$	$\mathbb{R}_\kappa$
$\lambda < \kappa$	No	No
$\lambda = \kappa$	Yes	No
$\kappa < \lambda \leq 2^\kappa$	Yes	No
$2^\kappa < \lambda$	Yes	Yes

Table 3.1: Does λ-BWT_K hold for $K = \kappa$-$\mathbb{R}$ and $K = \mathbb{R}_\kappa$?

Theorem 3.7 covers all cases except for $\mathrm{bn}(K) \leq \lambda \leq \mathrm{w}(K)$. It turns out that in this case, the answer depends on the saturation properties of K. We will now have a closer look at this case.

Theorem 3.8 (Sikorski)**.** *Let λ be an uncountable regular cardinal. Then λ-$\mathsf{BWT}_{\lambda\text{-}\mathbb{R}}$ holds.*

Proof. This result was proved by Sikorski in [89, Theorem V]. We give a sketch of the proof. Let $s : \lambda \to \lambda$-$\mathbb{R}$ be a bounded λ-sequence. Without loss of generality, by regularity of λ, we can assume s to be injective. By using the fact that elements of λ-$\mathbb{R}$ can be represented as finite sequences of ordinals and rational numbers, see, e.g., [3, Theorem 3.4], it is not hard to see that s has a monotone bounded λ-subsequence $b : \lambda \to \lambda$-$\mathbb{R}$. By [3, Proposition 4.2] every monotone bounded λ-sequence in λ-$\mathbb{R}$ is Cauchy. Therefore, b is Cauchy. Finally, since λ-$\mathbb{R}$ is by definition Cauchy complete, b is a convergent subsequence of s as desired. $\square$

Theorem 3.8 heavily relies on the fact that λ-$\mathbb{R}$ is not saturated (Theorem 2.2). Saturated fields behave very differently, as the following theorem shows.

Theorem 3.9. *Let λ be a regular uncountable cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field. If $\mathrm{w}(K) = \lambda$ and K is an η_λ-set, then K is λ-divergent.*

Proof. Fix any interval I; by Lemma 1.13, we find a convex set $B \subseteq I$ without least upper bound and $\mathrm{cof}(B) = \lambda$. Any cofinal λ-sequence in B must be divergent since B has no least upper bound. $\square$

Corollary 3.10. *Let λ be a regular uncountable cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field. If $\mathrm{w}(K) = \lambda$ and K is an η_λ-set, then λ-BWT_K does not hold.*

Proof. Follows directly from Observation 3.5 and Theorem 3.9. $\square$

For an uncountable cardinal κ with $\kappa^{<\kappa} = \kappa$, we will summarise the results of this section concerning the fields κ-$\mathbb{R}$ and $\mathbb{R}_\kappa$ in Table 3.1. In the table, we are using Theorems 3.7, 3.8, & 3.9, as well as the facts that $|\kappa\text{-}\mathbb{R}| = \kappa < 2^\kappa = |\mathbb{R}_\kappa|$ and that $\mathrm{bn}(\kappa\text{-}\mathbb{R}) = \mathrm{w}(\kappa\text{-}\mathbb{R}) = \mathrm{bn}(\mathbb{R}_\kappa) = \mathrm{w}(\mathbb{R}_\kappa) = \kappa$ and that $\mathbb{R}_\kappa$ is an η_κ-set (Theorems 2.1 & 2.5).

Note that in [16, Corollary 4.10], we claimed that for any successor cardinal λ^+ and any λ^+-spherically complete totally ordered field $(K, +, \cdot, 0, 1, \leq)$ with $\mathrm{w}(K) = \mathrm{bn}(K) = \lambda^+$ the property λ^+-BWT fails. However, this relied on [16, Lemma 2.8] which has a flawed proof. In [16, Lemma 2.8] we claimed that the set of lower bounds of a strictly decreasing divergent μ-sequence in a λ-spherically complete totally ordered field $(K,+,\cdot,0,1,\leq)$ with $\mu<\lambda$ must be of cardinality $\geq\lambda^+$. In the given proof we begin by considering a cofinal sequence t of length μ in the set of lower bounds of s and we considers the intersection $\bigcap_{\gamma\in\mu}[t(\gamma),s(\gamma)]$ where s is the μ-sequence in the statement of the claim. Then, from the λ^+-spherical completeness of the field, we deduce that the intersection must be non empty, contradicting the fact that s was divergent. In this argument we are tacitly assuming that the μ-sequence t is strictly increasing which may not be the case if the set of lower bounds of s has cofinality $<\mu$. Therefore, the argument only shows that the set of lower bounds does not have cofinality λ. The lemma is used in all three statements about the Bolzano-Weierstraß theorem at successor cardinals in that paper, to wit Lemma 4.9, Corollary 4.10, and Corollary 4.14. We do not know the current status of these claims. Note though that Corollary 4.10, and Corollary 4.14 are still true for $\lambda=\aleph_1$. For completeness we give here a proof of Corollary 4.10 for $\lambda=\aleph_1$. A similar proof works for Corollary 4.14 with $\lambda=\aleph_1$.

Lemma 3.11. *Let $(K,+,\cdot,0,1,\leq)$ be a $\aleph_1$-spherically complete ordered field with* $\mathrm{w}(K)=\mathrm{bn}(K)=\aleph_1$. *Then $\aleph_1$-BWT_K fails.*

Proof. By Observation 3.5 we will only prove that K is $\aleph_1$-divergent, then the corollary follows from Theorem 3.13. Let X be a subinterval of K and let $x\in X$. Consider the open interval $Y=(x,\infty)$. By Lemma 1.8 there is a strictly decreasing ω-sequence s in Y. Now, consider the set $L=\{y\in X\,;\,\forall n\in\mathbb{N}y<s(n)\}$. By the fact that K is $\aleph_1$-spherically complete of weight $\aleph_1$ we have that $\mathrm{cof}(L)=\aleph_1$. Let s' be any cofinal strictly increasing $\aleph_1$-sequence in L. Note that, since $\sup_{\alpha\in\aleph_1}(s'(\alpha))=\inf_{n\in\mathbb{N}}(s(n))$ and $\mathrm{bn}(K)=\aleph_1$, by Lemma 1.10 s' is divergent. □

3.2.3 Weakening the generalised Bolzano-Weierstraß theorem, part I: a first step.

In § 3.2.2, we have seen that the failure of the λ-Bolzano-Weierstraß theorem is closely related to the existence of bounded convex sets that are not intervals; their cofinal or coinitial sequences provide potential counterexamples to the Bolzano-Weierstraß theorem. This suggests a rather natural weakening of the Bolzano-Weierstraß theorem by restricting our attention to sequences that avoid this situation.

In this section, we will define this natural weakening. As we will see, this weakened principle, *the intermediate version of Bolzano-Weierstraß*, is still too strong to hold in $\mathbb{R}_\kappa$. Moreover, we will show that, for κ weakly compact, the intermediate version of Bolzano-Weierstraß theorem and the κ-Bolzano-Weierstraß theorem are equivalent.

Definition 3.12. Let λ be a regular cardinal and $(K,+,\cdot,0,1,\leq)$ be a totally ordered field. Let $s:\lambda\to K$ be a λ-sequence in K and $S:=\mathrm{ran}(s)$. We say that s is *weakly interval witnessed* if for every bounded convex set C in K such that $|S\cap C|=\lambda$, there is an interval $(x,y)=I\subseteq C$ such that $|S\cap I|=\lambda$.

We then say that K satisfies the *intermediate λ-Bolzano Weierstraß theorem* if every bounded weakly interval witnessed λ-sequence in K has a convergent λ-subsequence. We abbreviate this statement with λ-iBWT_K.

Theorem 3.13. *Let λ be an uncountable regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a λ-divergent totally ordered field. Then λ-iBWT_K fails.*

Proof. Fix a bounded strictly increasing λ-sequence $t : \lambda \to K$ which exists by the assumption. Let $\mathcal{S}$ be the set of strictly increasing λ-sequences in K and $T := \lambda^{<\omega}$ be the full tree of finite sequences of ordinals in λ; this is a λ-branching tree of height ω. We now recursively assign elements of $\mathcal{S}$ to the nodes of T by a function $L : T \to \mathcal{S}$. For each $p \in T$, we write $T_p := \operatorname{ran}(L(p))$ and also write $T_n := \bigcup_{p\in\lambda^n} T_p$.

We let $L(\varnothing) := t$. If $p \in \lambda^n$ and $L(p)$ is already defined, then for each $\gamma < \lambda$, $L(p)(\gamma) < L(p)(\gamma+1)$, so $(L(p)(\gamma), L(p)(\gamma+1))$ is a non-empty open interval. By the assumption, we find a strictly increasing divergent λ-sequence $t_{p,\gamma}$ in this interval and let $L(p^\frown\gamma) := t_{p,\gamma}$.

By construction, it is clear that if $x = L(p)(\gamma)$ and $y = L(p')(\gamma')$, then

$$x < y \text{ if and only if } p <_{\text{lex}} p' \text{ or } (p = p' \text{ and } \gamma < \gamma'), \tag{$*$}$$

where $<_{\text{lex}}$ is the lexicographic order.

Now fix a bijection $f : \lambda \to \lambda^{<\omega} \times \lambda$ with $f(\gamma) = (f_0(\gamma), f_1(\gamma))$ and define $s : \lambda \to K$ by

$$s(\gamma) = L(f_0(\gamma))(f_1(\gamma));$$

as usual, we write $S := \operatorname{ran}(s)$.

We claim that s is weakly interval witnessed. For this, let C be a bounded convex set such that $|S \cap C| = \lambda$. Pick any $x, y \in S \cap C$ with $L(p)(\gamma) = x < y$ for some $p \in T$ and $\gamma < \lambda$. By $(*)$ and by the construction of L, we know that $t_{p,\gamma}$ is a λ-sequence all of whose elements lie strictly between x and y, and so $|S \cap (x, y)| = \lambda$.

Finally, we claim that every λ-subsequence of s is divergent. Consider any injective $s' : \lambda \to S$ with $S' := \operatorname{ran}(s')$ and observe that since $S = \bigcup_{n\in\omega} T_n$ and λ is regular, there is some $n \in \omega$ such that $|S' \cap T_n| = \lambda$.

Case 1. There is some $p \in \lambda^n$ such that $|T_p \cap S'| = \lambda$. Then s' is a subsequence of $L(p)$ which, by construction, is a strictly increasing divergent λ-sequence and hence has no convergent subsequences.

Case 2. If that is not the case, then for every $p \in \lambda^n$, $|T_p \cap S'| < \lambda$. Define $W := \{p \in \lambda^n ; 0 < |T_p \cap S'|\}$ and for each $q \in T$, $W_q := \{p \in W ; q \subseteq p\}$. We say that q is *sparse* if $|W_q| < \lambda$ and we say that q is *cofinal* if $\{\gamma ; W_{q^\frown\gamma} \neq \varnothing\}$ is cofinal in λ.

We now claim that there is a cofinal $q \in T$:

We first observe that if $q \in \lambda^n$, then W_q has either zero or one elements, so all sequences of length n are sparse. Also, since

$$\lambda = |S' \cap T_n| = |\bigcup_{p\in W} S' \cap T_p|,$$

we know that $|W| = |W_\varnothing| = \lambda$, so $\varnothing$ is not sparse. If all immediate successors of q are sparse, then (using the regularity of λ) either q is cofinal or q is sparse. Assume now towards a contradiction that there is no cofinal sequence, then by induction, we get that $\varnothing$ is sparse. Contradiction; so there is a cofinal sequence $q \in T$.

Towards a contradiction, let us assume that s' converges to a limit ℓ. Therefore, all of its subsequences converge to ℓ as well. We now construct recursively a subsequence s''

of s': suppose that $s''\restriction\alpha$ is already defined with the property that for all $\gamma < \alpha$, there is some $p_\gamma \in W_q$ such that $s''(\gamma) \in T_{p_\gamma}$. For each such p_γ, let $\widehat{\gamma}$ be the unique ordinal such that $p_\gamma \in W_{q^\frown\widehat{\gamma}}$. Since q was cofinal, find $\beta > \sup\{\widehat{\gamma}\,;\,\gamma < \alpha\}$ such that $W_{q^\frown\beta} \neq \varnothing$. Pick $p \in W_{q^\frown\beta}$ and $x \in S' \cap T_p$ and let $s''(\alpha) := x$. As usual, we let $S'' := \mathrm{ran}(s'')$.

By construction, $L(q)(\beta) < x < L(q)(\beta+1)$, so S'' is cofinal in T_q, and therefore, $L(q)$ converges to ℓ as well. But by construction, $L(q)$ was a divergent sequence; contradiction! □

Corollary 3.14. *Let λ be a regular uncountable cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field. If $\mathrm{w}(K) = \lambda$ and K is an η_λ-set, then λ-iBWT_K does not hold.*

Proof. The proof is the same as the one of Theorem 3.9. □

Therefore, for $\kappa > \omega$ such that $\kappa^{<\kappa} = \kappa$, κ-$\mathsf{iBWT}_{\mathbb{R}_\kappa}$ fails.

Corollary 3.15. *Let λ be a weakly compact cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field. Then the following are equivalent:*

1. *λ-BWT_K and*
2. *λ-iBWT_K.*

Proof. The direction "(1)⇒(2)" is obvious, the other direction follows directly from Theorem 3.6. □

3.2.4 Weakening the generalised Bolzano-Weierstraß theorem, part II: the main result.

In this section, we will finally define the version of the Bolzano-Weierstraß theorem that can hold for $\mathbb{R}_\kappa$ and then characterise those κ for which it holds. Once more, κ is a regular uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

In § 3.2.2, we have studied counterexamples to the λ-Bolzano-Weierstraß theorem, and in the proof of Theorem 3.13, we saw how to produce a weakly interval witnessed counterexample. We implement the lessons learned from this construction and strengthen the requirement as follows:

Definition 3.16. Let λ be an uncountable regular cardinal, let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field, and let $s : \lambda \to K$ be a λ-sequence with $S := \mathrm{ran}(s)$. The sequence s is called *interval witnessed* if for every bounded convex set C in K such that $|S \cap C| = \lambda$ and every $\varepsilon \in K^+$, there is a $\mu < \lambda$ and a family of pairwise disjoint intervals of size μ, i.e., $\{I_\alpha\,;\,\alpha < \mu\} \subseteq \wp(C)$ such that

1. for each $\alpha < \mu$, the diameter of I_α is $< \varepsilon$, and
2. $|(S \cap C) \setminus \bigcup_{\alpha<\mu} I_\alpha| < \lambda$.

We say that K satisfies the *λ-weak Bolzano-Weierstraß theorem* if every bounded interval witnessed λ-sequence in K has a convergent λ-subsequence. We abbreviate this statement with λ-wBWT_K.

Theorem 3.17. *Let λ be an uncountable regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a Cauchy complete, λ-spherically complete totally ordered field with $\mathrm{bn}(K) = \lambda$. Then λ-wBWT_K implies that λ has the tree property.*

Proof. Fix a λ-tree $(T, \leq)$ and a strictly decreasing coinitial sequence $\delta : \lambda \to K^+$. For each $t \in T$, we will assign an open interval $L(t)$ in K by recursion on the level of the node t:

If $\mathrm{lvl}_T(t) = 0$, we let $L(t) := (0, 1)$. Let us assume that we have assigned intervals $L(t)$ to all nodes of level α and assign intervals to their successors: suppose $\mathrm{lvl}_T(t) = \alpha$, then since T is a λ-tree, the set of immediate successors of t has size $\mu < \lambda$ and thus can be written as $\{t_\alpha\,;\, \alpha < \mu\}$. Apply Lemma 1.14 to $L(t)$ to obtain a family $\{I_\alpha\,;\, \alpha < \mu\}$ of pairwise disjoint intervals with the additional properties (1) to (3) and assign $L(t_\alpha) := I_\alpha$.

Now let α be a limit ordinal and assume that for all $t \in T$ of level less than α, an interval $L(t)$ has been assigned. Suppose $\mathrm{lvl}_T(s) = \alpha$ and let $b_s := \mathrm{pred}_T(s)$ be the branch leading to s, a sequence of nodes of the tree of length $\alpha < \lambda$. For $\gamma < \alpha$, if $t_\gamma \in b_s$ is the uniquely defined node of level γ, we write $I_\gamma := (x_\gamma, y_\gamma)$ and $L(t_\gamma) := I_\gamma$ for the interval assigned to it. Clearly, $C := \bigcap_{\gamma<\alpha} I_\gamma$ is a convex set, and since K is λ-spherically complete, we can apply Lemma 1.9 to find $c \in C$ and then apply Lemma 1.10 to the pair $(\{c\}, \{y_\gamma\,;\, \gamma < \alpha\})$ to find a non-empty open interval (c, d) contained in C. Without loss of generality, we can find c and d such that $|d - c| < \delta(\alpha)$.

Note that two different nodes $s \neq s'$ of level α might have the same predecessors $b_s = b_{s'}$, however, since T was a λ-tree, the number of nodes sharing the same branch must be some $\mu < \lambda$. Apply Lemma 1.14 to obtain a pairwise disjoint family of subintervals that can be assigned to each of the nodes sharing the same branch.

This completes the assignment of intervals $t \mapsto L(t)$ to the nodes $t \in T$. Note that if $t < t'$, then $L(t) \supseteq L(t')$.

Claim 3.18. *For every $\alpha < \lambda$ there is $\varepsilon \in K^+$ such that if $t, t' \in \mathrm{lvl}_T(\alpha)$ and $t \neq t'$ then $L(t)$ and $L(t')$ are separated by a distance of at least ε (i.e., for every $x \in L(t)$ and $y \in L(t')$ we have $|x - y| > \varepsilon$).*

Proof. We show the claim by induction on α. For $\alpha = 0$, there is nothing to show. Fix $\alpha > 0$ and assume that for all $\beta < \alpha$, there is some ε_β such that for any $s \neq s' \in \mathrm{lvl}_T(\beta)$, the intervals $L(s)$ and $L(s')$ are separated by a distance of at least ε_β.

For each pair $(t, t') \in \mathrm{lvl}_T(\alpha)^2$ with $t \neq t'$, we will assign an $\varepsilon_{t,t'}$ such that $L(t)$ and $L(t')$ are separated by a distance of at least $\varepsilon_{t,t'}$.

Case 1. There is a $\gamma < \alpha$ with $s, s' \in \mathrm{lvl}_T(\gamma)$, $s < t$, $s' < t'$, and $s \neq s'$. Then by induction hypothesis, $L(s)$ and $L(s')$ are separated by a distance of at least ε_γ. Since $L(t) \subseteq L(s)$ and $L(t') \subseteq L(s')$, we can set $\varepsilon_{t,t'} := \varepsilon_\gamma$.

Case 2. Otherwise (i.e., the sets of predecessors of t and t' are the same). Then by construction, $L(t)$ and $L(t')$ were constructed by an application of Lemma 1.14. By property (3) in Lemma 1.14, there is some ε_1 such that $L(t)$ and $L(t')$ are separated by a distance of at least ε_1, so let $\varepsilon_{t,t'} := \varepsilon_1$.

Since T was a λ-tree, we have that $|\mathrm{lvl}_T(\alpha)| < \lambda$, and thus we can apply Lemma 1.10 to the pair $(\{0\}, \{\varepsilon_{t,t'}\,;\, t \neq t' \in \mathrm{lvl}_T(\alpha)\})$ to obtain some ε that works as a uniform bound for all intervals assigned to nodes in $\mathrm{lvl}_T(\alpha)$. □

We write $L(t) = (x_t, y_t)$ and define $r_t := \frac{x_t+y_t}{2}$. Since T was a λ-tree, there is a bijection $\pi : \lambda \to T$, and we can define a λ-sequence $r : \lambda \to K$ by $r(\alpha) := r_{\pi(\alpha)}$. Note that by construction (using Lemma 1.14 (2)), the function r is injective. As usual, we let $R := \mathrm{ran}(r)$.

Claim 3.19. *The sequence r is interval witnessed.*

Proof. Let $C \subseteq (0,1)$ be a bounded convex set such that $|C \cap R| = \lambda$ and let $\varepsilon_0 \in K^+$ be arbitrary. Without loss of generality, let us assume that C has neither a supremum nor an infimum; apply Lemma 1.17 to obtain $\varepsilon_1 \in K^+$ such that for all $x \in C$, $(x-\varepsilon_1, x+\varepsilon_1) \subseteq C$. Now let $\varepsilon := \min\{\varepsilon_0, \varepsilon_1\}$.

Since δ was coinitial in K^+, find a limit ordinal $\alpha < \lambda$ such that $\delta(\alpha) < \varepsilon$. By construction, if t is a node of level α or higher, then the interval $L(t)$ assigned to t has diameter $<\delta(\alpha) < \varepsilon$. We claim that for a node t of level α, the following are equivalent:

(i) $L(t) \subseteq C$,

(ii) $r(t) \in C$, and

(iii) $L(t) \cap C \neq \varnothing$.

The directions (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are obvious. The direction (iii)$\Rightarrow$(i) follows from the choice of ε and the fact that $L(t)$ has diameter $<\varepsilon$. Note that if t is a node of level α and $t' > t$, then $r(t') \in L(t)$. The above equivalence therefore shows that

$$\text{if } r(t') \in C, \text{ then } r(t) \in C. \tag{$\dagger$}$$

Let $X := \{t \in \mathrm{lvl}_T(\alpha)\,;\, r(t) \in C\}$. Since T was a λ-tree, the set $\mathrm{lvl}_T(\alpha)$ has size $<\lambda$ and so, there is some $\mu < \lambda$ such that $|X| = \mu$; write $X = \{t_\gamma\,;\, \gamma < \mu\}$ and write $I_\gamma := L(t_\gamma)$. By construction, each I_γ is a subset of C and the diameter of I_γ is less than ε.

We still need to show property (2) of Definition 3.16: by ($\dagger$) and the above equivalence, if t' is any node of level at least α and t its predecessor of level α, then $r(t') \in C$ if and only if there is a γ such that $t = t_\gamma$. In particular, $r(t') \in I_\gamma$ by construction. This means that

$$\begin{aligned}(R \cap C) \setminus \bigcup_{\gamma<\mu} I_\gamma &\subseteq \{r(t)\,;\, \exists\beta < \alpha(t \in \mathrm{lvl}_T(\beta)\} \\ &= \bigcup_{\beta<\alpha} \{r(t)\,;\, t \in \mathrm{lvl}_T(\beta)\}.\end{aligned}$$

Because T was a λ-tree and λ was regular, this shows that the size of this set is less than λ. □

Using Claim 3.19, we can apply λ-wBWT_K to r and obtain a convergent λ-subsequence v with $V := \mathrm{ran}(v)$. Since r was injective, we have that $|V| = \lambda$ and $|T_V| = \lambda$ for $T_V := \{t \in T\,;\, r(t) \in V\}$. We write ℓ for the limit of v, so in particular, for every ε, we have that

$$|\{t \in T_V\,;\, |\ell - r(t)| > \varepsilon\}| < \lambda. \tag{$\ddagger$}$$

Claim 3.20. *For every $\alpha < \lambda$, there is exactly one $t \in \mathrm{lvl}_T(\alpha)$ such that $\ell \in L(t)$.*

Proof. Note that since the intervals assigned to the nodes of level α are disjoint, there can be at most one such $t \in \mathrm{lvl}_T(\alpha)$. We will show by induction that each level contains such a t. By construction, we have $\ell \in (0, 1)$, which resolves the case $\alpha = 0$.

Let $\alpha > 0$ be arbitrary and assume that for each $\beta < \alpha$, there is a node $t \in \mathrm{lvl}_T(\beta)$ such that $\ell \in L(\beta)$. Note that these nodes must form a branch b through the tree of height α. Since T is a λ-tree, we let $\mathrm{lvl}_T(\alpha) = \{t_\gamma\,;\,\gamma < \mu\}$ for some $\mu < \lambda$. We write $T_{<\alpha} := \bigcup_{\beta<\alpha} \mathrm{lvl}_T(\beta)$ and $T_{\downarrow\gamma} := \{t \in T\,;\, t_\gamma \leq t\}$ and observe that we can write T as a disjoint union

$$T = T_{<\alpha} \cup \bigcup_{\gamma<\mu} T_{\downarrow\gamma}.$$

Clearly, by the fact that T was a λ-tree and by regularity of λ, $|T_{<\alpha}| < \lambda$.

We will consider $T_V \cap T_{\downarrow\gamma}$ for each $\gamma < \mu$ and observe that there are three possible cases:

Case 1. The set of predecessors of t_γ is not the branch b. That means that there is some level $\beta < \alpha$ where the path to s diverged from the branch b. Let ε_1 be the separation bound for the intervals assigned to nodes of level β. Then for every element $x \in L(t_\gamma)$ (and thus for every $x \in L(s)$ where s is a successor of t_γ), we have that $|\ell - x| > \varepsilon_1$. By (‡), we see that $|T_V \cap T_{\downarrow\gamma}| < \lambda$.

Case 2. The set of predecessors of t_γ is the branch b, but $\ell \notin L(t_\gamma)$. The intervals assigned to the immediate successors of t_γ are constructed using Lemma 1.14, and so there is an ε_2 such that for each successor s of t_γ and each $x \in L(s)$, we have $|\ell - x| > \varepsilon_2$. Once more, by (‡), we see that $|T_V \cap T_{\downarrow\gamma}| < \lambda$.

Case 3. The set of predecessors of t_γ is the branch b and $\ell \in L(t_\gamma)$. In the induction step, we need to show that there is a γ such that this case occurs.

If we now suppose towards a contradiction that *Case 3* never occurs, then

$$\begin{aligned} T_V &= T_V \cap \left(T_{<\alpha} \cup \bigcup_{\gamma<\mu} T_{\downarrow\gamma} \right) \\ &= (T_V \cap T_{<\alpha}) \cup \bigcup_{\gamma<\mu} (T_V \cap T_{\downarrow\gamma}), \end{aligned}$$

where by *Cases 1 & 2* each of the summands has size smaller than λ, so by regularity of λ, we obtain $|T_V| < \lambda$. Contradiction! □

Claim 3.20 directly gives us a branch of length λ through the tree T. □

Theorem 3.21. *Let κ be an uncountable strongly inaccessible cardinal and $(K, +, \cdot, 0, 1, \leq)$ a Cauchy complete ordered field with $\mathrm{bn}(K) = \kappa$. If κ has the tree property then K satisfies the κ-wBWT property.*

Proof. Let $s : \kappa \to K$ be an interval witnessed bounded κ-sequence (without loss of generality, s is an injective function), $S := \mathrm{ran}(s)$ and $\delta : \kappa \to K^+$ be a strictly decreasing sequence coinitial in K^+. Let (x^*, y^*) be any interval in K containing S. For each $\alpha < \kappa$, we define a set of pairwise disjoint intervals T_α by recursion. The construction will guarantee that

1. for each $\alpha < \kappa$, $|T_\alpha| < \kappa$,
2. for each $\alpha < \kappa$ and each $I \in T_\alpha$, we have that $|S \cap I| = \kappa$, and
3. for each $\alpha < \beta < \kappa$ and every $I \in T_\beta$, there is a $J \in T_\alpha$ such that I is a subinterval of J,

so in particular

$$\bigcup\{I\,;\, I \in T_\beta\} \subseteq \bigcup\{I\,;\, I \in T_\alpha\}.$$

We define $S_\alpha := S \cap \bigcup\{I\,;\, I \in T_\alpha\}$ and $M_\alpha := S \backslash S_\alpha$. Property (2) implies that for each $\alpha < \kappa$, $|S_\alpha| = \kappa$. We will furthermore check that

4. for each $\alpha < \kappa$, we have $|M_\alpha| < \kappa$.

Case $\alpha = 0$. We let $T_0 := \{(x^*, y^*)\}$. Properties (1), (2), and (3) are obviously satisfied. Note that by choice of (x^*, y^*), we have that $S_0 = S$ and so $M_0 = \varnothing$, whence (4) is satisfied as well.

Case $\alpha = \beta + 1$. If $(x, y) \in T_\beta$, define $L_{x,y} := (x, \frac{x+y}{2})$, $R_{x,y} := (\frac{x+y}{2}, y)$, and $T_\alpha := \{L_{x,y}\,;\, (x, y) \in T_\beta \text{ and } |L_{x,y} \cap S \models \kappa\} \cup \{R_{x,y}\,;\, (x, y) \in T_\beta \text{ and } |R_{x,y} \cap S \models \kappa\}$. Clearly, $|T_\alpha| \leq |2 \times T_\beta| < \kappa$, so (1) is satisfied. Properties (2) and (3) are satisfied by construction. Since $|T_\beta| < \kappa$ and κ is regular, we know that both $L_\alpha := \bigcup\{S \cap L_{x,y}\,;\, |S \cap L_{x,y}| < \kappa\}$ and $R_\alpha := \bigcup\{S \cap R_{x,y}\,;\, |S \cap R_{x,y}| < \kappa\}$ have size less than κ. Thus

$$S_\beta = S_\alpha \cup \left\{\frac{x+y}{2}\,;\, (x, y) \in T_\beta\right\} \cup L_\alpha \cup R_\alpha,$$

so using inductively property (4) for M_β, we have that $|M_\alpha| = |M_\beta| + |T_\beta| + |L_\alpha| + |R_\alpha| < \kappa$ and thus (4) is satisfied.

Case α limit ordinal. Consider the tree $T_{<\alpha} := \bigcup_{\beta<\alpha} T_\beta$ ordered by reverse inclusion and let $\mathcal{B}$ be the set of branches through this tree. The strong inaccessibility of κ implies that $|\mathcal{B}| < \kappa$. For $b \in \mathcal{B}$, the set $C_b := \bigcap\{I\,;\, I \in b\}$ is a convex set.

Claim 3.22. *We have that $S \backslash \bigcup_{\beta<\alpha} M_\beta = S \cap \bigcup_{b\in\mathcal{B}} C_b$.*

Proof. "$\subseteq$": If x is not in any M_β, then for every $\beta < \alpha$, there is an $I_\beta \in T_\beta$ such that $x \in I_\beta$. By construction, these intervals form a branch $b := \{I_\beta\,;\, \beta < \alpha\}$ in the tree $T_{<\alpha}$ and $x \in C_b$. "$\supseteq$": If $x \in S \cap C_b$, then the elements of the branch b witness that $x \notin M_\beta$ for any $\beta < \alpha$. □

By regularity of κ and the inductive assumption that all earlier levels satisfy property (4), we know that $\bigcup_{\beta<\alpha} M_\beta$ has size less than κ, so by Claim 3.22, we know that $|S \cap \bigcup_{b\in\mathcal{B}} C_b| = \kappa$. Since $|\mathcal{B}| < \kappa$, we know that there are branches $b \in \mathcal{B}$ such that $|S \cap C_b| = \kappa$.

Consequently, we can apply the fact that s was interval witnessed to such a convex set C_b and find a set $\mathcal{I}_b$ of fewer than κ many subintervals of C_b with diameter $<\delta(\alpha)$ such that $|S \cap (C_b \backslash \bigcup \mathcal{I}_b)| < \kappa$. Now let $T_\alpha := \{I\,;\, \text{there is a } b \in \mathcal{B} \text{ such that } |S \cap C_b| = \kappa$ and $I \in \mathcal{I}_b$ and $|S \cap I| = \kappa\}$.

Property (1) follows from the facts that κ is regular, $|\mathcal{B}| < \kappa$, and for each $b \in \mathcal{B}$, $|\mathcal{I}_b| < \kappa$. Property (2) and (3) are clear by construction. Let $W_0 := \bigcup\{S \cap C_b\,;\, |S \cap C_b| <$

$\kappa\}$; once more, by regularity of κ and $|\mathcal{B}| < \kappa$, we get that $|W_0| < \kappa$. Furthermore, let $W_1 := \bigcup\{S \cap (C_b \setminus \bigcup \mathcal{I}_b)\,;\, |S \cap C_b \models \kappa\}$; again, regularity of κ and the choice of $\mathcal{I}_b$ implies that $|W_1| < \kappa$. But $M_\alpha = \bigcup_{\beta<\alpha} M_\beta \cup W_0 \cup W_1$, so it has size less than κ, and thus we checked that property (4) holds as well.

This finishes the recursive construction. From property (1), it follows that the resulting tree $T := \bigcup_{\alpha<\kappa} T_\alpha$ is a κ-tree, so by the tree property, T has a branch $b = \{I_\alpha\,;\, \alpha < \kappa\}$. For each $\alpha < \kappa$, pick some $r_\alpha \in S \cap I_\alpha$. By the choice of the diameter of the intervals at the limit levels, the sequence $\alpha \mapsto r_\alpha$ is a Cauchy subsequence of s, thus by Cauchy completeness of K, it is convergent. □

Corollary 3.23. *Let κ be an uncountable strongly inaccessible cardinal and let $(K, +, \cdot, 0, 1, \leq)$ be a Cauchy complete and κ-spherically complete totally ordered field with* $\mathrm{bn}(K) = \kappa$. *Then the following are equivalent:*

1. *κ has the tree property and*
2. *κ-wBWT_K holds.*

In particular κ has the tree property if and only κ-$\mathsf{wBWT}_{\mathbb{R}_\kappa}$ holds.

As we have seen in § 3.2.3, if κ is weakly compact then the κ-Bolzano-Weierstraß theorem and the κ-intermediate-Bolzano-Weierstraß theorem are equivalent. In this case, as Corollary 3.23 shows, the κ-weak-Bolzano-Weierstraß theorem becomes a natural generalisation of the classical Bolzano-Weierstraß theorem.

3.3 The generalised Heine-Borel theorem

We end this chapter by considering a generalised version of the Heine-Borel theorem. First recall that the Heine-Borel theorem for $\mathbb{R}$ can be stated as follows:

Theorem 3.24. *For every set $X \subseteq \mathbb{R}$, the following are equivalent:*

(i) X is closed and bounded,

(ii) every open cover of X has a finite subcover, i.e., X is compact.

In order to generalise the Heine-Borel theorem to uncountable cardinals, we use the notion of λ-metrisable spaces from Chapter 2 and remind the reader of Theorem 2.14 stating that in λ-metrisable spaces, the notions of λ-compactness and λ-sequential compactness are equivalent.

The following natural generalisation of the Heine-Borel theorem is due to Cowles and LaGrange [23]:

Definition 3.25. Let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field and λ be a cardinal. Then we will say that K satisfies the *λ-Heine-Borel theorem* if for every $X \subseteq K$ the following are equivalent:

1. X is closed and bounded,
2. X is λ-compact.

We abbreviate this statement as λ-HBT_K.

Theorem 3.26 (Cowles & LaGrange)**.** *Let K be ordered field with* $\mathrm{bn}(K) = \lambda$. *Then λ-BWT_K holds if and only if λ-HBT_K holds.*

Proof. See [23, p. 136]. □

Corollary 3.27. *For every regular cardinal λ, we have that λ-$\mathsf{HBT}_{\lambda-\mathbb{R}}$ holds.*

Proof. Follows from Theorems 3.26 & 3.8. □

Corollary 3.28. *Let λ be an uncountable regular cardinal and let $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field with* $\mathrm{w}(K) = \lambda$ *which is an η_λ-set. Then λ-HBT_K does not hold.*

Proof. Follows from Theorems 3.26 & 3.9. □

In particular, if κ is such that $\kappa^{<\kappa} = \kappa$, then $\mathbb{R}_\kappa$ does not satisfy the κ-Heine-Borel theorem. The underlying reason for this is that closed intervals in $\mathbb{R}_\kappa$ are not κ-compact.

Proposition 3.29. *Let κ be an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. Then closed intervals in $\mathbb{R}_\kappa$ are not κ-compact.*

Proof. Let I be a closed interval; we use the proof of Lemma 1.13 to find a strictly increasing ω-sequence $s : \omega \to I$ such that the set B of its upper bounds has coinitiality κ. Take a coinitial sequence $t : \kappa \to B$ and two elements $x < I$ and $y > I$. Then the family

$$\{(x, s(n))\,;\, n \in \omega\} \cup \{(t(\alpha), y)\,;\, \alpha < \kappa\}$$

is an open cover of I that has no subcover of size less than κ. □

In line with the definitions from § 3.2.4, we say that a topological space (X, τ) is called *interval witnessed κ-sequentially compact* if every interval witnessed κ-sequence in X has a convergent subsequence. If $(K, +, \cdot, 0, 1, \leq)$ is a totally ordered field and κ be a cardinal, we will say that K satisfies the *κ-weak Heine-Borel theorem* (in symbols: κ-wHBT_K) if for every $X \subseteq K$, the following are equivalent:

1. X is closed and bounded,

2. X is interval witnessed κ-sequentially compact.

As for the classical case it turns out that for ordered fields of base number κ, κ-wHBT and κ-wBWT are equivalent.

Theorem 3.30. *Let λ be an uncountable regular cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a totally ordered field with* $\mathrm{bn}(K) = \lambda$. *Then λ-wBWT_K holds if and only if λ-wHBT_K holds.*

Proof. Clearly, if λ-wHBT_K, then λ-wBWT_K. Also, if X is bounded and closed and λ-wBWT_K holds, then X is interval witnessed λ-sequentially compact.

So, let us now assume that λ-wBWT_K holds and that X is interval witnessed λ-sequentially compact. If $s : \lambda \to X$ is a sequence converging in K, then this is a Cauchy sequence, and hence interval witnessed. Thus by interval witnessed λ-sequential compactness, s must converge to an element of X; hence, X is closed. Finally, assume towards a contradiction that X is unbounded in K, so there is a strictly increasing sequence $t : \lambda \to X$ cofinal in K. But then, no bounded convex set contains λ many elements of $\mathrm{ran}(t)$ and therefore, t is interval witnessed by definition. By interval witnessed λ-sequential compactness, t converges contradicting the assumption that it is cofinal in K. □

We combine Corollary 3.23 with Theorem 3.30:

Corollary 3.31. *Let κ be an uncountable strongly inaccessible cardinal and $(K, +, \cdot, 0, 1, \leq)$ be a Cauchy complete κ-spherically complete ordered field with* $\mathrm{bn}(K) = \kappa$. *Then the following are equivalent:*

1. *κ has the tree property,*
2. *κ-wBWT_K holds, and*
3. *κ-wHBT_K holds.*

In particular, κ has the tree property if and only if κ-$\mathsf{wBWT}_{\mathbb{R}_\kappa}$ holds if and only if κ-$\mathsf{wHBT}_{\mathbb{R}_\kappa}$ holds.

3.4 Open questions

In this section we present some open questions which are of particular interest in the area of Bolzano-Weierstraß theorem and non-archimedean real closed fields.

3.4.1 Trees and non-archimedean fields

As we have seen in § 3.2, properties of fields such as the Bolzano-Weierstraß theorem are strongly connected to existence of certain trees with particular properties. In [23] Cowles and LaGrange started a systematic study of these connections.

Definition 3.32. Let K be ordered field and κ be a cardinal. Then $F \subset K^+$ is said to be *separated* if there is $r \in K^+$ such that for all $x, y \in F$ we have $|x - y| > r$. We say that K is κ-archimedean iff it has a separated family of size κ and if no such family is bounded.

Note that according to the previous definition K is archimedean if and only if it is $\aleph_0$-archimedean.

In the following, we will use the notions of κ-Kurepa tree, (κ, λ)-Kurepa tree, and κ-Aronzajn tree from § 1.3.1.

In [23] Cowles and LaGrange proved the following results:

Theorem 3.33 (Cowles and LaGrange)**.** *If there is κ^+-archimedean ordered field of size at least κ^+ then there is a κ^+-Kurepa tree.*

Proof. See [23, p. 138]. □

Corollary 3.34 (Cowles and LaGrange)**.** *For each infinite cardinal κ, it is consistent with* ZFC+ *"there is an inaccessible cardinal" that κ^+-archimedean fields of cardinality larger than κ^+ do not exist.*

Proof. See [23, p. 139]. □

In [89], Sikorski asked whether a totally ordered field K of size $>\lambda$ such that λ-BWT_K holds exists or not. In [86], Schmerl proved the following:

Theorem 3.35 (Schmerl)**.** *Suppose $\lambda \geq \kappa > \aleph_0$, are cardinals with κ regular. Then the following are equivalent:*

1. *There is an ordered field K of cardinality λ such that κ-BWT_K.*
2. *There is a (κ, λ)-Kurepa with no κ-Aronszajn subtrees.*

Proof. See, [86, p. 145]. □

This result reduces Sikorski's question to the existence of (κ, λ)-Kurepa with no κ-Aronszajn subtrees with $\lambda > \kappa > \aleph_0$. We list some known facts about (κ, λ)-Kurepa trees.

1. If $\mathbf{V} = \mathbf{L}$ then for every successor cardinal κ there is a (κ, κ^+)-Kurepa tree; see, e.g., [27, Theorem 3.3];
2. if ZFC+ "there is a proper class of inaccessible cardinals" is consistent so is ZFC+ "there are no κ^+-Kurepa trees for every regular κ"; see, e.g., [85, Theorem 2.20].

Lücke (personal communication) reports that the techniques of [65, Theorem 4.5] were used for results on Kurepa trees (e.g., [91, Theorem 3.5]) and that this method yields the following:

Definition 3.36. A cardinal κ is *indestructibly weakly compact* if it is weakly compact and in any forcing extension obtained by a $<\kappa$-closed notion of forcing κ is weakly compact.

Theorem 3.37 (Lücke, private communication)**.** *If* ZFC+ *"there is an indestructibly weakly compact cardinal" is consistent so is* ZFC+ *"there are $\kappa < \lambda$ such that there are (κ, λ)-Kurepa trees with no κ-Aronszajn subtrees".*

Corollary 3.38. *If there is an indestructibly weakly compact cardinal then it is consistent with* ZFC *that there is a totally ordered field K of size larger than λ such that λ-BWT_K holds.*

Proof. Follows from Theorems 3.37 & 3.35. □

It is then natural to ask the following questions:

Question 3.39. Is the indestructibly weakly compact cardinal needed in Corollary 3.38?

Question 3.40. Let $f \in 2^\omega$ be a countable binary sequence. Is it consistent, relative to large cardinals, to have κ with $2^\kappa \geq \kappa^{+\omega}$ and for all $n \in \omega$, $f(n) = 1$ iff there is a (κ, κ^{+n})-Kurepa T_n tree which has no κ-Aronszajn subtrees?

We do not even know the answer to the simplified version of Question 3.40 in which we do not require that T_n does not have κ-Aronszajn subtrees.

3.4.2 The Bolzano-Weierstraß theorem at successor cardinals

As we have seen in § 3.2.4, our proof of the existence of fields where the κ-wBWT holds strongly depends on the assumption that κ is a strong limit cardinal. It is therefore very natural to ask if this assumption can be removed.

Note the following facts:

Theorem 3.41 (Specker [94])**.** *If $2^{<\lambda} = \lambda$ then λ^+ does not have the tree property.*

Theorem 3.42 (Jensen [26, Theorem 5.2])**.** *If $\mathbf{V} = \mathbf{L}$ then for every uncountable cardinal λ we have that λ^+ does not have the tree property.*

From the previous facts and our results in § 3.2.4 it is easy to see the following:

Theorem 3.43. *For every infinite cardinal λ, if $2^{<\lambda} = \lambda$ then for every Cauchy complete, λ^+-spherically complete totally ordered field $(K, +, \cdot, 0, 1, \leq)$ with $\mathrm{bn}(K) = \lambda$ the weak λ^+-Bolzano-Weierstraß theorem λ^+-wBWT_K fails.*

Proof. The claim follows from Theorems 3.17 & 3.41. □

In particular:

Corollary 3.44. *If GCH holds then for every cardinal λ which is successor of a regular cardinal and for every Cauchy complete, λ-spherically complete totally ordered field $(K, +, \cdot, 0, 1, \leq)$ with $\mathrm{bn}(K) = \lambda$ the weak λ-Bolzano-Weierstraß theorem λ-wBWT_K fails.*

Theorem 3.45. *Assume GCH. Let $(K, +, \cdot, 0, 1, \leq)$ be a Cauchy complete, λ-spherically complete totally ordered field with $\mathrm{bn}(K) = \lambda$. Then we have that each element of the following list implies the subsequent:*

1. *λ is weakly compact;*
2. *λ-wBWT_K;*
3. *λ is weakly compact or $\lambda = \kappa^+$ for some singular cardinal κ and has the tree property.*

Proof. The first implication is Corollary 3.23. For the second implication, note that by Theorem 3.17 λ must have the tree property. Moreover, it follows from GCH that λ-wBWT_K implies λ limit or λ successor of a singular cardinal. Indeed, if λ is successor of a regular cardinal then by Theorem 3.41 we have that λ does not have the tree property; and therefore, by Theorem 3.17, λ-wBWT_K must fail. Finally, if λ is limit, by GCH it is a strong limit and therefore weakly compact. □

Note that the reverse of the second implication of Theorem 3.45 cannot be proved. Indeed, it follows from [39, Theorem 1.2] that if ZFC+"there are κ^+-many supercompact cardinals for a supercompact cardinal κ" is consistent so is $\mathsf{ZFC} + \mathsf{GCH}$+"$\aleph_{\omega+1}$ has the tree property".

Corollary 3.46. *If $\mathbf{V} = \mathbf{L}$ then for every uncountable cardinal λ and every Cauchy complete, λ-spherically complete totally ordered field $(K, +, \cdot, 0, 1, \leq)$ with $\mathrm{bn}(K) = \lambda$ the following are equivalent:*

1. *λ is weakly compact and*

2. λ-wBWT_K.

Proof. It follows from Theorems 3.45 & 3.42. $\square$

The following questions are therefore natural:

Question 3.47. Can the assumption that $\mathbf{V} = \mathbf{L}$ be weakened in Corollary 3.46? In particular, let $(K, +, \cdot, 0, 1, \leq)$ be a Cauchy complete, λ-spherically complete totally ordered field with $\mathrm{bn}(K) = \lambda$; are the weak compactness of λ and λ-wBWT_K equivalent under GCH?

Chapter 4

The generalised reals: transfinite computability

Remarks on co-authorship. *The results of this chapter are partially due to a collaboration of the author with Hugo Nobrega. In particular all results in § 4.2 are, unless otherwise stated, due jointly to the author and Hugo Nobrega. These results were mostly developed when the collaborators were Visiting Fellows at the* Isaac Newton Institute for Mathematical Sciences *for the program* Mathematical, Foundational and Computational Aspects of the Higher Infinite*. The outcomes of this collaboration have also been published in a joint paper [38] for which the authors won the* Best Student Paper *award at the conference* Computability in Europe 2017 *held in Turku, Finland in 2017. Lemma 4.7 and Lemma 4.25 were not included in [38] and were proved later solely by the author.*

The results in § 4.3 are due solely to the author and will be published in the proceedings volume of the conference Computability in Europe 2019 as an invited paper.

4.1 Introduction

In classical computability theory computations are thought as *finite* and *discrete* processes carried out by (idealised) machines. Although these assumptions are quite natural, since the beginning of the research in this area, researchers have been developing theories in which these assumptions are weakened; see, e.g., [100].

Particularly interesting for us are those notions of computability in which the finiteness of the process is relaxed. The idea is to allow computations to "go on forever". Different formalisations of this abstract notion gave rise to different models of computability. In this chapter we will consider models of *transfinite computability*.

The modern approach to the study of transfinite computability began with the seminal paper [44] in which Hamkins and Lewis introduced the notion of *infinite time Turing machine* (ITTM). These machines are Turing machines whose clock runs over ordinal numbers rather than just natural numbers. Therefore, infinite time Turing machines have the same hardware as classical Turing machines and run classical programs; but, contrary to their classical counterpart, they can run for an amount of time corresponding to a transfinite ordinal. An ITTM behaves as a normal Turing machine at successor stages, while at limit stages the head goes to the first cell of the tape, the content of the tape is computed by using pointwise inferior limits, and the state of the machine is set to a special limit state.

Time and space are treated asymmetrically in infinite time Turing machines. Indeed, while tapes have length ω the machine is allowed to run for an arbitrary transfinite amount of steps. This asymmetry is the source of behaviour of ITTMs that is different from that of classical Turing machines.

The theory of infinite time Turing computability is very rich and deeply connected to set theory; see, e.g., [15, 44–46, 84, 108].

In [55], Koepke started the study of *ordinal Turing machines* (OTMs) which are meant to repair the asymmetry between space and time introduced by ITTMs. An ordinal Turing machine is a machine with an infinite tape whose length is the supremum of all the ordinals; and which, as an ITTM, can run for a transfinite amount of time. As for ITTMs, OTMs run classical Turing machine programs and behave as standard Turing machines at successor stages. At limit stages, the content of the tape is computed by taking the point-wise inferior limit, the position of the head is set to the inferior limit of the head positions at previous stages, and the state of the machine is computed using the inferior limit of the states at previous stages.

The generalised version of many classical results from computability theory can be proved to hold for ordinal Turing machines; see, e.g., [17, 18, 25, 55, 57, 80, 87].

As we mentioned in § 1.1, the fact that the construction of the generalised real line $\mathbb{R}_\kappa$ is carried out within the framework of surreal numbers gives us a natural notion of computability. Indeed, looking at the definitions of surreal numbers and surreal operations in § 1.3.6, it is not hard to see that they come with an intrinsically computational flavour.

In this chapter, we will exploit the computational nature of surreal numbers and of $\mathbb{R}_\kappa$ to generalise notions of computability which are based on real numbers in the classical framework.

The chapter is organised as follows: in § 4.2 we will first use the generalised real line and ordinal time Turing machines to generalise the classical notion of type two Turing machines; then, we will use this new model of transfinite computability to start the generalisation of the classical theory of Weihrauch degrees; in § 4.3, we will use the generalised real line to define a generalisation of Blum-Shub-Smale machines; we will compare this new model with the main models of transfinite computability; and we will show that this new notion of computability can serve as a very general type of transfinite register machines.

4.2 Generalised computable analysis

4.2.1 Introduction

In [36], the author provided the foundational basis for the study of *generalised computable analysis*, namely the generalisation of computable analysis to generalised Baire and Cantor spaces.

The work in this section is a continuation of [35, 36], strengthening their results and answering in the positive the open question from [36] of whether a natural notion of computability exists for 2^κ. We generalise the framework of type two computability to uncountable cardinals κ such that $\kappa^{<\kappa} = \kappa$. Then we use this framework to induce a notion of computability over the generalised real line $\mathbb{R}_\kappa$, showing that, as in the classical case, by using suitable representations, the field operations are computable. Finally we will generalise Weihrauch reducibility to spaces of cardinality 2^κ and extend a classical result by showing that the generalised version of the Intermediate value theorem introduced in [36] is Weihrauch equivalent to a generalised version of the boundedness principle $\mathrm{B_I}$.

Throughout this section, κ will be a fixed uncountable cardinal, as usual assumed to satisfy $\kappa^{<\kappa} = \kappa$, which in particular implies that κ is a regular cardinal.

4.2.2 Generalised type two Turing machines

In this section we define a generalised version of type two Turing machines based on the notion of κ-Turing machines.

We will only sketch the definition of κ-Turing machines, which were developed by several people (e.g., [25, 57, 80]); we are going to follow the definition of Koepke and Seyfferth [57, § 2]. We refer the reader to the original paper for the full details.

Definition 4.1. A *κ-Turing machine* has the following tapes of length κ: finitely many read-only tapes for the input, finitely many read and write scratch tapes and one write-only tape for the output. Each cell of each tape has either 0 or 1 written in it at any given time, with the default value being 0. These machines can run for infinite time of ordinal type κ; at successor stages of a computation a κ-Turing machine behaves exactly like a classical Turing machine, while at limit stages the contents of each cell of each tape and the positions of the heads is computed using inferior limits.

As in the classical case $\kappa = \omega$, the difference between κ-Turing machines and type two κ-Turing machines is *not* on the hardware level, but rather on the notion of what it means for a machine to compute a function.

Definition 4.2. A partial function $f : 2^{<\kappa} \to 2^{<\kappa}$ is *computed* by a κ-Turing machine M if whenever M is given $x \in \mathrm{dom}(f)$ as input, its computation halts after fewer than κ steps with $f(x)$ written on the output tape. A partial function $f : 2^\kappa \to 2^\kappa$ is *type two-computed* by a κ-Turing machine M, or *computed by the type two κ-Turing machine M*, or simply *computed* by M, if whenever M is given $x \in \mathrm{dom}(f)$ as input, for every $\alpha < \kappa$ there exists a stage $\beta < \kappa$ of the computation at which $f(x)\restriction\alpha$ is written on the output tape. We abbreviate the phrase "*type two κ-Turing machine*" by T2κTM.

An *oracle* T2κTM is a T2κTM with an additional read-only input tape of length κ, called its *oracle tape*. A partial function $f : 2^\kappa \to 2^\kappa$ is *computable with an oracle* if there exists an oracle T2κTM M and $x \in 2^\kappa$ such that M computes f when x is written on the oracle tape. Note that by minor modifications of classical proofs one can prove that T2κTMs are closed under recursion and composition, and that there is a universal T2κTM. In what follows, the term *computable* will mean *computable by a T2κTM*, unless specified otherwise.

By a straightforward generalisation of the classical proofs one can prove the following theorems:

Theorem 4.3. *If a partial function $f : 2^\kappa \to 2^\kappa$ is computable with an oracle, then it is continuous.*

Theorem 4.4. *A partial function $f : 2^\kappa \to 2^\kappa$ is continuous iff it is computable with an oracle.*

In [57, Theorem 7], Koepke and Seyfferth proved the following:

Theorem 4.5. *If α is an epsilon ordinal and $A \subseteq \alpha$. Then A is computable by an α-Turing machine with finitely many ordinal parameters if and only if A is $\Delta_1(\mathbf{L}_\alpha)$.*

Proof. See [57, Theorem 7.a]. □

Corollary 4.6 (Koepke [55, Theorem 6.2])**.** *A set of ordinals A is computable by an OTM with finitely many ordinal parameters if and only if $A \in \mathbf{L}$.*

For T2κTM we have the following:

Lemma 4.7. *Let $A \subset \kappa$. Then,*

1. *if A is T2κTM computable then $A \in \Delta_1(\mathbf{L}_{\kappa+1})$;*
2. *if $A \in \Delta_1(\mathbf{L}_\kappa)$ then it is T2κTM computable with finitely many ordinal parameters.*

Proof. For the first claim, note that, if δ is a limit ordinal, computations of κ-Turing machines of length δ are uniformly $\Delta_1(\mathbf{L}_\delta)$; see [57, Lemma 3.b]. Therefore A is $\Delta_1(\mathbf{L}_{\kappa+1})$. Indeed, $a \in A$ if and only if there is a computation of length κ of the program computing A with output a; and similarly, $a \in A$ if and only if every computation of length κ of the program computing A has output a. The second claim follows from the fact that every κ-Turing machine computable set is T2κTM computable and Theorem 4.5. □

4.2.3 Represented spaces

In classical computability the notion of computability on Cantor space can be naturally extended to arbitrary countable spaces via codings. This is particularly important in classical computable analysis where codings play a very important role (see [75,107]). In this section, we will present the computational version of the definitions in § 1.3. In this chapter, unless explicitly stated we will always refer to the generalised versions of the notions in § 1.3.

Given a class Γ of functions between 2^κ and 2^κ, and two represented spaces (X, δ_X) and (Y, δ_Y); we say that a function $f : X \to Y$ is (δ_X, δ_Y)-Γ if f has a realiser in Γ. In the case that $\delta_X = \delta_Y$, we will say that f is δ_X-Γ. For example, a function $f : X \to Y$ is (δ_X, δ_Y)-computable if it has a computable realiser.

Definition 4.8 (Generalised Weihrauch Reducibility)**.** Let f and g be two multi-valued functions between represented spaces. Then we say that f is *strongly Weihrauch reducible* to g, in symbols $f \leq_{\mathrm{sW}} g$, if there are two computable functions $H, K : 2^\kappa \to 2^\kappa$ such that $H \circ G \circ K \vdash f$ whenever $G \vdash g$.[1] As usual, if $f \leq_{\mathrm{sW}} g$ and $g \leq_{\mathrm{sW}} f$ then we say that f is *strongly Weihrauch equivalent* to g and write $f \equiv^{\mathrm{t}}_{\mathrm{sW}} g$.

[1]Carl has also introduced a notion of generalised (strong) Weihrauch reducibility in [14]. Because his goal is to investigate multi-valued (class) functions on V, the space of codes he uses is the class of ordinal numbers, considered with the ordinal Turing machines of Koepke [55]. Therefore his approach is significantly different from ours, and we do not know of any connections between the two.

As in the classical case, given two represented spaces one can naturally define representations for composed spaces.

Definition 4.9 (Computable Reductions)**.** Let $\delta : 2^\kappa \to X$ and $\delta' : 2^\kappa \to X$ be two representations of a space X. Then we say that δ *computably reduces* to δ', in symbols $\delta \leq \delta'$, if there is a computable function $h : 2^\kappa \to 2^\kappa$ such that for every $p \in \mathrm{dom}(\delta)$ we have $\delta(p) = \delta'(h(p))$. If $\delta \leq \delta'$ and $\delta' \leq \delta$ we say that δ and δ' are *computably equivalent* and write $\delta \equiv \delta'$.

Note that as in classical computable analysis if $\delta \leq \delta'$ and f is δ-computable then f is also δ'-computable. Finally, as in the classical case, given two represented spaces $\mathbf{X}$ and $\mathbf{Y}$, we can define canonical representations for the product space $\mathbf{X} \times \mathbf{Y}$, the union space $\mathbf{X}+\mathbf{Y}$ and the space of continuous functions $[\mathbf{X} \to \mathbf{Y}]$. In particular, as in classical computable analysis $[\mathbf{X} \to \mathbf{Y}]$ can be represented as follows: $\delta_{[X\to Y]}(p) = f$ iff $p = 0^n 1 p'$ with $p' \in 2^\kappa$ and $n \in \mathbb{N}$ is a code for an oracle T2κTM which (δ_X, δ_Y)-computes f when given the oracle p'.

We fix the following representations of κ and κ^κ: $\delta_\kappa(p) = \alpha$ iff $p = 0^\alpha 1\mathbf{0}$, where $\mathbf{0}$ is the constant 0 κ-sequence, $\delta_{\kappa^\kappa}(p) = x$ iff $p = [0^{\alpha_\beta+1}1]_{\beta<\kappa}$ and $x = (\alpha_\beta)_{\beta<\kappa}$. It is straightforward to see that a function $f : \kappa \to \kappa$ is δ_κ-computable iff it is computable by a κ-machine as in [57, Definition 2].

Lemma 4.10. *The restriction of $\mathfrak{g}$ to $\kappa \times \kappa$ is a δ_κ-computable bijection between $\kappa \times \kappa$ and κ, and has a δ_κ-computable inverse.*

Proof. It is a standard fact of the theory of cardinals and ordinals that $\mathfrak{g}\restriction(\mu \times \mu)$ is a bijection between $\mu \times \mu$ and μ whenever μ is an infinite cardinal.

For the computability of $\mathfrak{g}\restriction(\kappa \times \kappa)$, note that it is enough to prove that the inverse $\mathfrak{g}^{-1}\restriction\kappa$ of $\mathfrak{g}\restriction(\kappa\times\kappa)$ is δ_κ-computable, since then $\mathfrak{g}\restriction(\kappa\times\kappa)$ can be δ_κ-computed by simulating the program for the inverse with each ordinal in increasing order as input until the correct output is found (note that whether an output is correct or not can be recognised in time less than κ, and thus this whole process also takes time less than κ). To compute $\mathfrak{g}^{-1}\restriction\kappa$, given γ the idea is to enumerate the first γ pairs of ordinals less than κ in the order $\prec$. This can be done computably as follows. At successor stages, having listed (α, β) in the previous stage, the next pair to be listed is $(\alpha+1, \beta)$, if $\alpha + 1 < \beta$; $(\beta, 0)$, if $\alpha + 1 = \beta$; and $(\alpha, \beta+1)$, if $\beta+1 \leq \alpha$. At limit stages, the counters keeping track of the values of α and β along the computation get set to $\liminf$ of those values. This information allows us to decide the next pair to be listed by a straightforward case distinction. For example, suppose the $\liminf$ of the values of α is α' and the $\liminf$ of the values of β is also α'. If it is the first time that we have reached the pair (α', α') in this way, then the next pair to be listed is $(\alpha', 0)$; otherwise the next pair is indeed (α', α').

□

Proposition 4.11. *The representation δ_{κ^κ} is $\leq$-maximal among the continuous representations of κ^κ.*

Proof. Let δ be a continuous representation of κ^κ. We want to show that there is a continuous function $f : 2^\kappa \to 2^\kappa$ such that $\delta_{\kappa^\kappa}(f(p)) = \delta(p)$ for every $p \in \mathrm{dom}(\delta)$. Since δ is continuous there is a monotone function $\vartheta : 2^{<\kappa} \to \kappa^{<\kappa}$ such that $\delta(p) =$

$\bigcup_{\alpha<\mathrm{dom}(p)} \vartheta(p\restriction\alpha)$. For every $\alpha<\kappa$ let $\beta_\alpha<\kappa$ be the smallest ordinal such that $\vartheta(p\restriction\beta_\alpha)$ is of length $\alpha+1$. Define f as follows: $f(p)=[0^{\xi_\alpha+1}1]_{\alpha<\kappa}$ where $\xi_\alpha=\vartheta(p\restriction\beta_\alpha)(\alpha)$ for every $p\in\mathrm{dom}(\delta)$. By the monotonicity of ϑ the function f is well defined. Moreover the function f is continuous. Now we need to show that $\delta_{\kappa^\kappa}(f(p))=\delta(p)$ for every $p\in\mathrm{dom}(\delta)$. Let $p\in\mathrm{dom}(\delta)$. Then $f(p)=[0^{\xi_\alpha+1}1]_{\alpha<\kappa}$ where $\xi_\alpha=\vartheta(p\restriction\beta_\alpha)(\alpha)$ and by definition $\delta_{\kappa^\kappa}(f(p))=(\xi_\alpha)_{\alpha<\kappa}=\bigcup_{\alpha\in\mathrm{dom}(p)}\vartheta(p\restriction\alpha)=\delta(p)$ as desired. □

4.2.4 Representing the generalised real line $\mathbb{R}_\kappa$

In classical computable analysis one can show that many of the natural representations of $\mathbb{R}$ are well behaved with respect to type two computability. In this section we show that some of these results naturally extend to the uncountable case. First we introduce representations for generalised rational numbers, which will serve as a starting point to representing $\mathbb{R}_\kappa$. As we have seen in the introduction, surreal numbers can be expressed as binary sequences and, because of the simplicity theorem, as cuts. It is then natural to introduce two representations which reflect this fact.

Definition 4.12 (Representation of $\mathbb{Q}_\kappa$)**.** Let $p\in 2^\kappa$ and $q\in\mathbb{Q}_\kappa$. We define $\delta_{\mathbb{Q}_\kappa}(p)=q$ iff $p=[w_\alpha]_{\alpha<\kappa}$ where $w_\alpha:=00$ if $\alpha\in\mathrm{dom}(q)$ and $q(\alpha)=-$, $w_\alpha:=01$ if $\alpha\notin\mathrm{dom}(q)$, and finally $w_\alpha:=11$ if $\alpha\in\mathrm{dom}(q)$ and $q(\alpha)=+$.

It is not hard to see that since every κ-rational is a sequence of $+$ and $-$ of length less than κ the function $\delta_{\mathbb{Q}_\kappa}$ is indeed a representation of $\mathbb{Q}_\kappa$. Now we define a representation based on cuts by recursion on the simplicity structure of the surreal numbers.

Definition 4.13 (Cut Representation of $\mathbb{Q}_\kappa$)**.** We define $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}:2^\kappa\to\mathbb{Q}_\kappa$ as follows: We define $\delta^{\mathrm{c},0}_{\mathbb{Q}_\kappa}(p)=0$ iff $p=\langle p_\alpha\rangle_{\alpha<\kappa}$ and $p_\alpha=[10]_{\beta<\kappa}$ for every $\alpha<\kappa$. For $\alpha>0$ we define $\delta^{\mathrm{c},\alpha}_{\mathbb{Q}_\kappa}(p)=[\,L|R\,]$ where $p=\langle p_\alpha\rangle_{\alpha<\kappa}$ and

1. $p_\alpha\in\mathrm{dom}(\bigcup_{\gamma<\alpha}\delta^{\mathrm{c},\gamma}_{\mathbb{Q}_\kappa})\cup\{[10]_{\beta<\kappa}\}$ for every $\alpha<\kappa$,
2. for all even[2] $\alpha<\kappa$, if $p_\alpha=[10]_{\beta<\kappa}$ then for all even $\beta>\alpha$ we have $p_\beta=[10]_{\beta<\kappa}$,
3. for all odd $\alpha<\kappa$, if $p_\alpha=[10]_{\beta<\kappa}$ then for all odd $\beta>\alpha$ we have $p_\beta=[10]_{\beta<\kappa}$,
4. finally: $L=\{\delta^{\mathrm{c},\gamma}_{\mathbb{Q}_\kappa}(p_\beta)\ ;\ \ \gamma<\alpha,\ \beta<\kappa$ is even and $p_\beta\in\mathrm{dom}(\delta^{\mathrm{c},\gamma}_{\mathbb{Q}_\kappa})\}$ and
   $R=\{\delta^{\mathrm{c},\gamma}_{\mathbb{Q}_\kappa}(p_\beta)\ ;\ \gamma<\alpha,\ \beta<\kappa$ is odd and $p_\beta\in\mathrm{dom}(\delta^{\mathrm{c},\gamma}_{\mathbb{Q}_\kappa})\}$.

Then we define $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}:=\bigcup_{\gamma<\kappa}\delta^{\mathrm{c},\gamma}_{\mathbb{Q}_\kappa}$.

Note that $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}$ is surjective, since for every $x\in\mathbb{Q}_\kappa$ there exists $p\in\mathrm{dom}(\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa})$ such that $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}(p)=[\,L|R\,]$ and $\langle L,R\rangle$ is the canonical cut of x. Therefore $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}$ is indeed a representation of $\mathbb{Q}_\kappa$.

Lemma 4.14. *Let $\delta_{\mathbb{Q}_\kappa}$ and $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}$ be defined as before. Then $\delta_{\mathbb{Q}_\kappa}\equiv\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}$.*

[2] We call an ordinal α *even* if $\alpha=\lambda+2n$ for some limit λ and natural n, *odd* otherwise.

Proof. First we show that $\delta_{\mathbb{Q}_\kappa} \leq \delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}$. Let $p \in \mathrm{dom}(\delta_{\mathbb{Q}_\kappa})$. The conversion can be done recursively. If p is a code for the empty sequence[3] we just return a representation for $\langle \varnothing, \varnothing \rangle$. Otherwise we compute two subsets $L_s := \{p'01 \,;\, p'11 \subset p\}$ and $R_s := \{p'01 \,;\, p'00 \subset p\}$. Then we compute recursively the cuts for the elements of L_s and R_s and return them as the left and right sets of the cut representation of p, respectively. It easy to see that the algorithm computes a code for the canonical cut of $\delta_{\mathbb{Q}_\kappa}(p)$.

Now we will show that $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa} \leq \delta_{\mathbb{Q}_\kappa}$. Let $p \in \mathrm{dom}(\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa})$. If p is a code for the cut $\langle \varnothing, \varnothing \rangle$ we return a representation of the empty sequence. Now, assume that p is the code for the cut $\langle L, R \rangle \neq \langle \varnothing, \varnothing \rangle$. We first recursively compute the sequences for the element of L and R, call the sets of these sequences L_s and R_s. Now suppose $\alpha < \kappa$ is even and we want to compute the value at α and $\alpha + 1$ of the output sequence. We first compute M_L and m_R, the minimal and maximal in $\{00, 01, 11\}$, respectively, such that for every $p' \in L_s$ and $p'' \in R_s$ we have $p'(\alpha)p'(\alpha+1) \leq M_L$ and $m_R \leq p''(\alpha)p''(\alpha+1)$. Then, by a case distinction on M_L and m_R, we can decide the i^{th} sign of the output. For example, if the output is already smaller than R_s, $M_L = 00$ (i.e. $-$) and $m_R = 00$ (i.e., $-$) then we can output the sequence 01 (i.e., undefined). All the other combinations can be treated similarly. $\square$

Lemma 4.15. *The operations $+_{\mathrm{s}}$, $-_{\mathrm{s}}$, $\cdot_{\mathrm{s}}$, $x \mapsto \frac{1}{x}$ and the characteristic function of the order $<$ are $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}$-computable.*

Proof. We will only prove the lemma for $+_{\mathrm{s}}$. Given $q, q' \in \mathbb{Q}_\kappa$ we want to $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}$-compute $q +_{\mathrm{s}} q'$. The algorithm is given by recursion. If $q = 0$ (similarly for $q' = 0$)[4] copy the code of q' on the output tape. If neither q nor q' are 0 then by using Definition 1.19 we compute a representation for $q +_{\mathrm{s}} q'$ (note that this involves the computation of less than κ many rational sums of shorter length). Finally, since the resulting code would not in general be in $\mathrm{dom}(\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa})$, we use the algorithms of the previous lemma to convert $q +_{\mathrm{s}} q'$ to a sign sequence code and than we convert it back to an element in $\mathrm{dom}(\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa})$. By using the second algorithm from the previous proof we can convert every element in $L_{q+_{\mathrm{s}}q'}$ and in $R_{q+_{\mathrm{s}}q'}$ into a sequence (note that by induction the codes of these cuts are in $\mathrm{dom}(\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa})$ so we can use the algorithm). Then, by the same method used in the previous lemma, we can compute the code of the sequence representation for $q +_{\mathrm{s}} q'$. Once we have the code of the sequence representation for $q +_{\mathrm{s}} q'$ we can convert it to a code of the cut representation by using the first algorithm from the previous lemma. $\square$

Given that $\mathbb{R}_\kappa$ is the Cauchy completion of $\mathbb{Q}_\kappa$, the following is a natural representation of $\mathbb{R}_\kappa$.

Definition 4.16 (Cauchy representation of $\mathbb{R}_\kappa$)**.** We let $\delta_{\mathbb{R}_\kappa}(p) = x$ iff $p = \langle p_\alpha \rangle_{\alpha<\kappa}$, where for each $\alpha < \kappa$ we have $p_\alpha \in \mathrm{dom}(\delta_{\mathbb{Q}_\kappa})$, $\delta_{\mathbb{Q}_\kappa}(p_\alpha) < x +_{\mathrm{s}} \frac{1}{\alpha+1}$, and $x < \delta_{\mathbb{Q}_\kappa}(p_\alpha) +_{\mathrm{s}} \frac{1}{\alpha+1}$.

It is routine to check the following.

Theorem 4.17. *The field operations $+_{\mathrm{s}}$, $-_{\mathrm{s}}$, $\cdot_{\mathrm{s}}$, and $x \mapsto \frac{1}{x}$ are $\delta_{\mathbb{R}_\kappa}$-computable.*

[3] Note that this can be checked just by looking at the first two bits of p.

[4] Note that this is easily computable, it is in fact enough to check that L and R are empty, and this can be done just by checking the first two bits of the first sequence in the left and in the first sequence on the right.

Proof. Let us give the proof for $\cdot_{\mathrm{s}}$, the others being similar. Given codes $p = \langle p_\alpha \rangle_{\alpha<\kappa}$ and $q = \langle q_\alpha \rangle_{\alpha<\kappa}$ for $x, y \in \mathbb{R}_\kappa$, respectively, let $x_\alpha = \delta_{\mathbb{Q}_\kappa}(p_\alpha)$ and $y_\alpha = \delta_{\mathbb{Q}_\kappa}(q_\alpha)$. Note that for each α we can compute some α' such that $\frac{1}{\alpha'+1}(x_0 +_{\mathrm{s}} y_0 +_{\mathrm{s}} 3) \leq \frac{1}{\alpha+1}$. We then output $r = (r_\alpha)_{\alpha<\kappa}$, where r_α is a $\delta_{\mathbb{Q}_\kappa}$-name for $x_{\alpha'} y_{\alpha'}$.

We have $xy -_{\mathrm{s}} x_{\alpha'} y_{\alpha'} = x(y -_{\mathrm{s}} y_{\alpha'}) +_{\mathrm{s}} y_{\alpha'}(x -_{\mathrm{s}} x_{\alpha'}) < (x_0 +_{\mathrm{s}} 1)\frac{1}{\alpha'+1} +_{\mathrm{s}} (y_0 +_{\mathrm{s}} 2)\frac{1}{\alpha'+1} \leq \frac{1}{\alpha+1}$, as desired, and likewise we can prove $x_{\alpha'} y_{\alpha'} -_{\mathrm{s}} xy < \frac{1}{\alpha+1}$. □

On the other hand, the following is suggested by the definition of $\mathbb{R}_\kappa$ as the collection of Veronese cuts over $\mathbb{Q}_\kappa$.

Definition 4.18 (Veronese representation of $\mathbb{R}_\kappa$)**.** We let $\delta^{\mathrm{V}}_{\mathbb{R}_\kappa}(p) = x$ iff $p = \langle p_\alpha \rangle_{\alpha<\kappa}$, where for each $\alpha < \kappa$ we have $p_\alpha \in \mathrm{dom}(\delta_{\mathbb{Q}_\kappa})$ and $x = [\,L|R\,]$, with $L = \{\delta_{\mathbb{Q}_\kappa}(p_\alpha)\,;\, \alpha < \kappa \text{ is even}\}$; $R = \{\delta_{\mathbb{Q}_\kappa}(p_\alpha)\,;\, \alpha < \kappa \text{ is odd}\}$; and for each even $\alpha < \kappa$ we have $\delta_{\mathbb{Q}_\kappa}(p_{\alpha+1}) < \delta_{\mathbb{Q}_\kappa}(p_\alpha) +_{\mathrm{s}} \frac{1}{\alpha+1}$.

Theorem 4.19. *The representation $\delta_{\mathbb{R}_\kappa}$ and $\delta^{\mathrm{V}}_{\mathbb{R}_\kappa}$ are equivalent.*

Proof. To reduce $\delta^{\mathrm{V}}_{\mathbb{R}_\kappa}$ to $\delta_{\mathbb{R}_\kappa}$, given $p = \langle p_\alpha \rangle_{\alpha<\kappa}$, we output $q = \langle q_\alpha \rangle_{\alpha<\kappa}$ by making q_α equal to p_β, where β is the α^{th} even ordinal. It is now easy to see that q is a $\delta_{\mathbb{R}_\kappa}$-name for $\delta^{\mathrm{V}}_{\mathbb{R}_\kappa}(p)$.

For the reduction in the other direction, given $p = \langle p_\alpha \rangle_{\alpha<\kappa}$, we output $q = \langle q_\alpha \rangle_{\alpha<\kappa}$ where for each even α we let q_α be a $\delta_{\mathbb{Q}_\kappa}$-name for $\delta_{\mathbb{Q}_\kappa}(p_{2\cdot_{\mathrm{s}}\alpha+2}) -_{\mathrm{s}} \frac{1}{2\cdot_{\mathrm{s}}\alpha+3}$ and $q_{\alpha+1}$ be a $\delta_{\mathbb{Q}_\kappa}$-name for $\delta_{\mathbb{Q}_\kappa}(p_{2\cdot_{\mathrm{s}}\alpha+2}) +_{\mathrm{s}} \frac{1}{2\cdot_{\mathrm{s}}\alpha+3}$. Then, letting $L := \{\delta_{\mathbb{Q}_\kappa}(p_\alpha)\,;\, \alpha < \kappa \text{ is even}\}$, and $R := \{\delta_{\mathbb{Q}_\kappa}(p_\alpha)\,;\, \alpha < \kappa \text{ is odd}\}$, we have $L < \{x\} < R$ and for each even $\alpha < \kappa$ we have $\delta_{\mathbb{Q}_\kappa}(q_{\alpha+1}) = \delta_{\mathbb{Q}_\kappa}(p_{2\cdot_{\mathrm{s}}\alpha+2}) +_{\mathrm{s}} \frac{1}{2\cdot_{\mathrm{s}}\alpha+3} = \delta_{\mathbb{Q}_\kappa}(q_\alpha) +_{\mathrm{s}} \frac{2}{2\cdot_{\mathrm{s}}\alpha+3} < \delta_{\mathbb{Q}_\kappa}(q_\alpha) +_{\mathrm{s}} \frac{1}{\alpha+1}$, as desired. □

Note that the results in this section can be easily extended to show that the surreal operations are computable by OTMs. Let $\delta_{\mathrm{No}} : 2^{<\mathrm{On}} \to \mathrm{No}$ be the class function that maps each surreal number to a binary sequence as follows: $\delta_{\mathrm{No}}(p) = q$ iff $p = [w_\alpha]_{\alpha\in\mathrm{dom}(q)+1}$ where $w_\alpha := 00$ if $\alpha \in \mathrm{dom}(q)$ and $q(\alpha) = -$, $w_\alpha := 01$ if $\alpha = \mathrm{dom}(q)$, and finally $w_\alpha := 11$ if $\alpha \in \mathrm{dom}(q)$ and $q(\alpha) = +$. Then, similarly to what we have seen before, we can define a notion of computability over No using OTMs. In particular, we will say that a function $F : \mathrm{No} \to \mathrm{No}$ is δ_{No}-computable if and only if there is an OTM computable function $G : 2^{<\mathrm{On}} \to 2^{<On}$ such that $F = \delta_{\mathrm{No}} \circ G \circ \delta_{\mathrm{No}}^{-1}$.

Theorem 4.20. *The surreal operations $+_{\mathrm{s}}$, $-_{\mathrm{s}}$, $\cdot_{\mathrm{s}}$, and $x \mapsto \frac{1}{x}$ are δ_{No}-computable.*

Proof. An inessential modification of the algorithms we presented in Lemmas 4.14 & 4.15 work. □

4.2.5 Generalised boundedness principles and the Intermediate value theorem

In this section we focus on the study of IVT and its relationship with the boundedness principle $\mathrm{B_I}$; see § 1.3.7. In particular, we generalise a classical result from Brattka and Gherardi [11, Theorem 6.2], proving that IVT_κ is Weihrauch equivalent to a generalised version of $\mathrm{B_I}$. This strengthens a result from [36], namely that $\mathrm{B_I}$ is continuously equivalent to IVT_κ.

The theorem IVT_κ as stated in Theorem 2.9 can be considered as the partial multi-valued function $\mathrm{IVT}_\kappa : \mathrm{C}_{[0,1]} \rightrightarrows [0,1]$ defined as follows: $\mathrm{IVT}_\kappa(f) = \{c \in [0,1] \; ; \; f(c) = 0\}$, where $[0,1]$ is represented by $\delta_{\mathbb{R}_\kappa}\restriction[0,1]$ and $\mathrm{C}_{[0,1]}$ is endowed with the standard representation of $[[0,1] \to \mathbb{R}_\kappa]$ restricted to $\mathrm{C}_{[0,1]}$. By lifting the proof in [107, Theorem 6.3.2] to κ it is easy to show that this version of IVT_κ is not continuous, and thus also not computable, relative to these representations.

To introduce the boundedness principle $\mathrm{B}_{\mathrm{I}}^\kappa$, we will need the following represented spaces. Let $\mathbf{S}_{\mathrm{b}}^{\uparrow}$ be the space of bounded increasing sequences of κ-rationals, represented by letting p be a name for $(x_\alpha)_{\alpha<\kappa}$ iff $p = \langle p_\alpha \rangle_{\alpha<\kappa}$ where $p_\alpha \in \mathrm{dom}_{\mathbb{Q}_\kappa}$ and $\delta_{\mathbb{Q}_\kappa}(p_\alpha) = x_\alpha$ for each $\alpha < \kappa$. The represented space $\mathbf{S}_{\mathrm{b}}^{\downarrow}$ is defined analogously, with bounded decreasing sequences of κ-rationals. Similarly one can define cuts representations $\delta_{\mathbb{R}_<}$ and $\delta_{\mathbb{R}_>}$ representing a real r has a list of all the rational numbers q such that $q < r$ and a list of all the rational numbers q such that $q > r$, respectively. Note that, unlike the classical case of the real line, not all limits of bounded monotone sequences of length κ exist in $\mathbb{R}_\kappa$; see Chapter 3. Therefore, although for the real line the spaces $\mathbf{S}_{\mathrm{b}}^{\uparrow}$ and $\mathbf{S}_{\mathrm{b}}^{\downarrow}$ naturally correspond to the spaces of $\mathbb{R}_<$ and $\mathbb{R}_>$, respectively, in our generalised setting the correspondence fails. We define $\mathrm{B}_{\mathrm{I}}^\kappa$ as the principle which, given an increasing sequence $(q_\alpha)_{\alpha<\kappa}$ and decreasing sequence $(q'_\alpha)_{\alpha<\kappa}$ in $\mathbb{Q}_\kappa$ for which there exists $x \in \mathbb{R}_\kappa$ such that $\{q_\alpha \; ; \; \alpha < \kappa\} \leq \{x\} \leq \{q'_\alpha \; ; \; \alpha < \kappa\}$, picks one such x. Formally we have the partial multi-valued function $\mathrm{B}_{\mathrm{I}}^\kappa : \mathbf{S}_{\mathrm{b}}^{\uparrow} \times \mathbf{S}_{\mathrm{b}}^{\downarrow} \rightrightarrows \mathbb{R}_\kappa$ with $x \in \mathrm{B}_{\mathrm{I}}^\kappa(s, s')$ iff $\{s(\alpha) \; ; \; \alpha < \kappa\} \leq \{x\} \leq \{s'(\alpha) \; ; \; \alpha < \kappa\}$.

Lemma 4.21. *Let $f : [0,1] \to \mathbb{R}_\kappa$ and $x \in \mathbb{R}_\kappa$. Suppose there exists a sequence $(x_\alpha)_{\alpha<\kappa}$ of pairwise distinct elements of $[0,1]$ such that $f(x_\alpha) = x$ if $\alpha < \kappa$ is even and $f(x_\alpha) \neq x$ otherwise, and such that for any odd $\alpha, \beta < \kappa$ there exists an even $\gamma < \kappa$ such that x_γ is between x_α and x_β. Then f is not κ-continuous.*

Proof. If such a sequence exists, then either the preimage of the κ-open set $(x, +\infty)$ or of the κ-open set $(-\infty, x)$ under f must contain x_α for κ-many of the odd $\alpha < \kappa$, and thus cannot be κ-open. □

Lemma 4.22. *Let $f : [0,1] \to \mathbb{R}_\kappa$ be κ-continuous an let $\beta, \beta' < \kappa$, $y \in \mathbb{R}_\kappa$ and let $(r_\alpha)_{\alpha<\beta}$ and $(r'_\alpha)_{\alpha<\beta'}$ be two sequences in $[0,1]$ such that $\{r_\alpha \; ; \; \alpha < \beta\} < \{r'_\alpha \; ; \; \alpha < \beta'\}$ and $\{f(r_\alpha) \; ; \; \alpha < \beta\} < \{y\} < \{f(r'_\alpha) \; ; \; \alpha < \beta'\}$. Then there is $x \in [0,1]$ such that $\{r_\alpha \; ; \; \alpha < \beta\} < \{x\} < \{r'_\alpha \; ; \; \alpha < \beta'\}$ and $f(x) = y$.*

Proof. Assume not. Without loss of generality we can assume that for every x such that $\{r_\alpha \; ; \; \alpha < \beta\} < \{x\} < \{r'_\alpha \; ; \; \alpha < \beta'\}$ we have $f(x) > y$ (a similar proof works for $f(x) < y$). Note that the set $\{r_\alpha \; ; \; \alpha < \beta\}$ has cofinality at most $\beta < \kappa$ and, since $\mathbb{R}_\kappa$ is an η_κ-set, it follows that $R = \{r \in [0,1] \; ; \; \forall \alpha < \beta. \, r_\alpha < r\}$ has coinitiality κ. Therefore R is not κ-open. Now since f is κ-continuous we have that $f^{-1}[(y, +\infty)]$ is κ-open. Therefore $f^{-1}[(y, +\infty)] = \bigcup_{\alpha \in \gamma}(y_\alpha, b_\alpha)$ with $\gamma < \kappa$ and $y_\alpha, b_\alpha \in [0,1]$ for every $\alpha < \gamma$. Now consider the set $I := \{\alpha \in \gamma \; ; \; (y_\alpha, b_\alpha) \cap R \neq \varnothing\}$. We have that $R \subset \bigcup_{\alpha \in I}(y_\alpha, b_\alpha)$. Note that since R is not κ-open we have $R \neq \bigcup_{\alpha \in I}(y_\alpha, b_\alpha)$. Now assume $r \in \bigcup_{\alpha \in I}(y_\alpha, b_\alpha) \setminus R$, so that there is $\alpha \in I$ such that $r \in (y_\alpha, b_\alpha)$. Take $r' \in (y_\alpha, b_\alpha) \cap R$. By the fact that $r \notin R$, there is $\alpha' < \beta$ such that $r < r_{\alpha'}$ and by IVT_κ there is a root of f between $r_{\alpha'}$ and r', but this is a contradiction because $(y_\alpha, b_\alpha) \subset f^{-1}[(y, +\infty)]$. □

Corollary 4.23. *Let $f : [0,1] \to \mathbb{R}_\kappa$ be κ-continuous, and let $x \in [0,1]$, $(r_\alpha)_{\alpha<\kappa}$ and $(r'_\alpha)_{\alpha<\kappa}$ be increasing and decreasing sequences in $[0,1]$, respectively, such that for all $\alpha < \kappa$ we have $f(r_\alpha) < x$ and $f(r'_\alpha) > x$. Then there exists $y \in [0,1]$ such that $f(y) = x$ and $\{r_\alpha \ ; \ \alpha < \kappa\} < \{y\} < \{r'_\alpha \ ; \ \alpha < \kappa\}$.*

Proof. Construct a sequence $(x_\alpha)_{\alpha<\gamma}$ for some $\gamma \leq \kappa$ as follows. First let $\delta_0 = 1$. Having constructed $(x_\beta)_{\beta<\alpha}$ for some even $\alpha < \kappa$, by Lemma 4.22 there exists $x_\alpha \in [0,1]$ such that $f(x_\alpha) = x$ and $\{r_\beta \ ; \ \beta < \sup_{\nu<\alpha} \delta_\nu\} < \{x_\alpha\} < \{r'_\beta \ ; \ \beta < \sup_{\nu<\alpha} \delta_\nu\}$. If $\{r_\beta \ ; \ \beta < \kappa\} < \{x_\alpha\} < \{r'_\beta \ ; \ \beta < \kappa\}$, then we are done and $\gamma = \alpha$. Otherwise there exists $\beta < \kappa$ such that $r_\beta > x$ or $r'_\beta < x$, so we let $x_{\alpha+1} = r_\beta$ or $x_{\alpha+1} = r'_\beta$ accordingly, and let $\delta_\alpha = \beta + 1$. If the construction goes on for κ steps, then $(x_\alpha)_{\alpha<\kappa}$ is as in Lemma 4.21, a contradiction. Hence the construction ends at some stage $\gamma < \kappa$, and therefore $\{r_\beta \ ; \ \beta < \kappa\} < \{x_\gamma\} < \{r'_\beta \ ; \ \beta < \kappa\}$. □

As in the classical theory, we will say that a set X is *effectively enumerable* if there it is the image of a computable function f; and we will call the function f an *enumeration* of X.

Theorem 4.24. *1. If there exists an effectively enumerable dense subset of $\mathbb{R}_\kappa$, then $\mathrm{IVT}_\kappa \leq_{\mathrm{sW}} \mathrm{B}_\mathrm{I}^\kappa$.*

2. We have $\mathrm{B}_\mathrm{I}^\kappa \leq_{\mathrm{sW}} \mathrm{IVT}_\kappa$.

3. We have $\mathrm{IVT}_\kappa \leq_{\mathrm{sW}}^{\mathrm{t}} \mathrm{B}_\mathrm{I}^\kappa$, and therefore $\mathrm{IVT}_\kappa \equiv_{\mathrm{sW}}^{\mathrm{t}} \mathrm{B}_\mathrm{I}^\kappa$.

Proof. For item 1, let the κ-continuous function $f : [0,1] \to \mathbb{R}_\kappa$ be given, D be a dense subset of $\mathbb{R}_\kappa$ and $(d_\gamma)_{\gamma<\kappa}$ be an effective enumeration of $[0,1] \cap D$. Without loss of generality we can assume $f(0) < 0$ and $f(1) > 0$, and start setting $r_0 = 0$ and $r'_0 = 1$. Now assume that for $0 < \alpha < \kappa$ we have already defined an increasing sequence $(r_\beta)_{\beta<\alpha}$ and a decreasing sequence $(r'_\beta)_{\beta<\alpha}$ of elements of $[0,1] \cap \mathbb{D}$ with $\{r_\beta \ ; \ \beta < \alpha\} < \{r'_\beta \ ; \ \beta < \alpha\}$ and $\{f(r_\beta) \ ; \ \beta < \alpha\} < \{0\} < \{f(r'_\beta) \ ; \ \beta < \alpha\}$. By Lemma 4.22 there is still a root of f between the two sequences. Note that, since $\mathbb{R}_\kappa$ is an η_κ-set and again by applying Lemma 4.22, there exist $r_L, r_R \in \mathbb{D}$ such that $\{r_\beta \ ; \ \beta < \alpha\} < \{r_L\} < \{r_R\} < \{r'_\beta \ ; \ \beta < \alpha\}$ and $f(r_L) < 0$, $f(r_R) > 0$. Therefore, by searching in the sequence $(d_\gamma)_{\gamma<\kappa}$ and running the corresponding algorithms in parallel, we can find such a pair r_L, r_R in fewer than κ computation steps. Let β, γ, δ be such that $\mathfrak{g}(\beta, \mathfrak{g}(\gamma, \delta)) = \alpha$, where $\mathfrak{g}$ is the Gödel pairing function, which has a computable inverse by Lemma 4.10. If $r_L < d_\gamma < d_\delta < r_R$, $f(d_\gamma) < 0$, and $f(d_\delta) > 0$, where the last two comparisons are decided in fewer than β steps of computation, then let $r_\alpha = d_\gamma$ and $r'_\alpha = d_\delta$; otherwise let $r_\alpha = r_L$ and $r'_\alpha = r_R$.

By Corollary 4.23 we have that there exists $x \in [0,1]$ such that $\{r_\alpha \ ; \ \alpha < \kappa\} < \{x\} < \{r'_\alpha \ ; \ \alpha < \kappa\}$. It remains to be proved that $f(x) = 0$ for any such x. Suppose not, say $f(x) > 0$ for some such x. Then also $f(y) > 0$ for some $y \in \mathbb{D}$ such that $\{r_\alpha \ ; \ \alpha < \kappa\} < \{y\} < \{r'_\alpha \ ; \ \alpha < \kappa\}$. Now let $\beta, \gamma, \delta < \kappa$ be such that $d_\gamma = y$, $d_\delta = r_\nu$ for some ν such that $\{y -_{\mathrm{s}} r_\nu\} < \{r'_\alpha -_{\mathrm{s}} r_\beta \ ; \ \alpha, \beta < \kappa\}$ and $f(y) < 0$, $f(r_\nu) > 0$ are decided in fewer than β computation steps. Then at stage $\alpha = \mathfrak{g}(\beta, \mathfrak{g}(\gamma, \delta))$ of the computation we define a pair r_α, r'_α such that $r'_\alpha -_{\mathrm{s}} r_\alpha \leq y -_{\mathrm{s}} r_\nu$, a contradiction. This ends the proof of 1.

Item 2 is a straightforward generalisation of [11, Theorem 6.2], and the proof of item 3 is the same as that of item 1 without the requirement that the enumeration $(d_\gamma)_{\gamma<\kappa}$ of the dense subset of $[0,1] \cap \mathbb{D}$ be effective. $\square$

Note that condition of item 1 of Theorem 4.24 is satisfied, e.g., in the constructible universe $\mathbf{L}$. We end this section showing that condition 1 of Theorem 4.24 can be easily destroyed by using forcing techniques. In the following we are going to use classical notions in set theory, for a detailed introduction to the subject see [48, Chapters 13 and 14].

Lemma 4.25. *It is consistent with the axioms of* ZFC *that no dense subset of* $\mathbb{R}_\kappa$ *is effectively enumerable.*

Proof. Assume $\mathbf{V} = \mathbf{L}$. Note that, by Lemma 4.7 and by the fact that $\mathbf{L}$ is absolute between transitive models of set theory, no new T2κTM computable function is added in any forcing extension of $\mathbf{L}$. So it is enough to show that the density of every constructible subset of $\mathbb{R}_\kappa$ is destroyed by some forcing notion.

Any notion of forcing that is not $<\kappa$-closed will do. In particular, let $\mathbb{P}$ be notion of forcing that adds one Cohen real. Let G be generic over $\mathbb{P}$. Then, $r = \bigcup G$ is a new countable sequence in Cantor space. Let $s \in (\mathrm{No}_{\leq\omega})^{\mathbf{L}[G]}$ be the surreal of countable length such that $s(n) = +$ if $r(n) = 1$ and $s(n) = -$ if $r(n) = 0$. Then, consider the open interval $(s-, s+)$ in $(\mathbb{R}_\kappa)^{\mathbf{L}[G]}$. There is no surreal s' in $\mathbf{L}$ such that $s' \in (s-, s+)$ since $s' \restriction \omega = s$. Therefore, no subset of $(\mathbb{R}_\kappa)^{\mathbf{L}}$ is dense in $(\mathbb{R}_\kappa)^{\mathbf{L}[G]}$. $\square$

4.3 Generalised Blum-Shub-Smale machines

4.3.1 Introduction

In 1989 Blum, Shub and Smale introduced a model of computation to study computability over rings; see [7]. Of particular interest for us is the notion of computability that Blum-Shub-Smale machines (BSSM) induce over the real numbers. A BSSM for the real numbers is a register based machine in which each register contains a real number. A program for such a machine is a finite list of commands. Each command can be either a computation or branch command. The execution of a computation command allows the machine to apply a rational function to update the content of the registers. A branch command, on the other hand, leaves the content of the registers unchanged and allows the machine to apply a rational function to some register and execute a jump based on the result of this operation, i.e., to jump to a different point of the code if the result is 0 and to continue the normal execution otherwise. Note that this notion of computability is very different from the one used by Weihrauch in [107]. While Weihrauch notion of computability is based on representations, i.e., type two Turing machines work on binary sequences and we need to choose a representation to transfer this notion of computability from 2^ω to $\mathbb{R}$, BSSMs work directly on $\mathbb{R}$ and have all the basic operations over the real line as primitives. As we have seen, the notion of computability induced by type two Turing machines over $\mathbb{R}$ is very dependent on the representation we choose. Indeed, there are choices of representations δ and δ' of the reals such that, the notions of δ-computable and δ'-computable differs. Therefore, depending on this choice we can get very different notions of computability. This is not true for BSSMs; where there is no coding involved, and we only have one notion of computability over the real line.

In [58, 87], Koepke and Seyfferth defined the notion of *infinite time Blum-Shub-Smale machine* that is a generalised version of Blum-Shub-Smale machines which can carry out transfinite computations over the real numbers.

Infinite time Blum-Shub-Smale machines work essentially as standard BSSMs at successor times apart from the fact that, contrary to classical BSSMs, they can only apply rational functions with rational coefficients[5]. At limit stages an infinite time Blum-Shub-Smale machine computes the content of each register by taking the limit over the real line of the values that the register assumed at previous stages (if this exists); and updates the program counter to the inferior limit of its values at previous stages. The theory of infinite time Blum-Shub-Smale machine was further studied in [56].

Similarly to ITTMs, infinite time Blum-Shub-Smale machines provide an asymmetric generalisation of BSSMs. In particular, while infinite time Blum-Shub-Smale machines are allowed to run for arbitrary transfinite time, they are using real numbers, a set that can be very small compared to the running times. It is then natural to ask whether a symmetric notion can be defined.

In this section, we will introduce a generalised version of Blum-Shub-Smale machines based on surreal numbers and on the generalised real line and we will show some preliminary results of the theory of these machines.

4.3.2 Surreal Blum-Shub-Smale machines

A surreal Blum-Shub-Smale machine (SBSSM) is a register machine. Since, as we will see, the formal definition of SBSSMs is quite involved, let us start by giving a brief informal explanation of how they work. There are two different types of registers in our machines: *normal registers* and *Dedekind registers*. Normal registers are just registers that contain surreal numbers; as we will see, the machine can write and read normally from these registers. Dedekind registers on the other hand are a new piece of hardware. Each Dedekind register R can be thought of as to have three different components S^{L}, S^{R}, and R. The components S^{L} and S^{R} called left and right stack of R, respectively, can be thought of as two possibly infinite stacks of surreal numbers. The last component R of the register can be thought as a normal register whose content is automatically updated by the machine to the surreal $[S^{\mathrm{L}}|S^{\mathrm{R}}]$. Note that it could be that $S^{\mathrm{L}} \not< S^{\mathrm{R}}$; in this case we will assume that the machine crashes.

A SBSSM is just a finite set of normal and Dedekind registers. A program for such a machine will be a finite linear sequence of commands. As for BSSMs there are two types of commands:

> *Computation:* the machine can apply a rational function to a normal register or to a Dedekind register and save the result in a register (either Dedekind or normal) or in a stack.
>
> *Branch:* the machine can check if the content of a normal register or of a Dedekind register is bigger than 0 and perform a jump based on the result.

[5]A stronger version of infinite time BSSMs could be obtained by allowing infinite time Blum-Shub-Smale machines to use rational functions with real coefficients, but this was not done in [87]

In each program we should specify two subsets of the set of normal registers one that will contain the input of the program and the other that will contain the output of the program.

A surreal Blum-Shub-Smale machine will behave as follows: at successor stages our machine just executes the current command and updates content of stacks, registers, and program counter accordingly. At limit stage α, the program counter is set using $\liminf$ as for infinite time Blum-Shub-Smale machines; the content of each normal register is updated as follows: if the content of the register is eventually constant with value x, then we set the value of the register to x; otherwise we set it to 0. For Dedekind registers we proceed as follows: if from some point on the content of the stacks is constant, we leave the content of the stacks, and therefore the content of the register, unchanged. If the content of the stacks is not eventually constant but from some point $\beta < \alpha$ on there is no computation instruction whose result is saved in the register, then we set the value of each stack to the union of its values from β on, and we set the content of the register accordingly. If none of the previous cases occurs, then we set the content of the register to 0 and empty the stacks.

We are now ready to give a formal definition of surreal Blum-Shub-Smale machines.

Definition 4.26. Given two polynomials $p, q \in \mathrm{No}[X_0, \ldots, X_n]$, we will call $\frac{p(X_0,\ldots,X_n)}{q(X_0,\ldots,X_n)}$ a *formal polynomial quotient* over No in $n+1$ variables.

Definition 4.27. Let $n \in \mathbb{N}$ and $F : \mathrm{No}^{n+1} \to \mathrm{No}$ be a partial class function. Then, we say that F is a *rational map* over No if there are polynomials in $n+1$ variables $p, q \in \mathrm{No}[X_0, \ldots, X_n]$ such that $F(s_0, \ldots, s_n) = \frac{p(s_0,\ldots,s_n)}{q(s_0,\ldots,s_n)}$ for each $s_0, \ldots, s_n \in \mathrm{No}$. In this case, we will say that $\frac{p(X_0,\ldots,X_n)}{q(X_0,\ldots,X_n)}$ is a formal polynomial quotient *defining* F.

Definition 4.28. Denote by $\vec{X}$ the set of finite tuples of variables of any length. Then, we will denote by $\mathrm{No}(\vec{X})$ the class of formal polynomials quotients over No in any number of variables. Given a subclass K of $\mathrm{No}(\vec{X})$ and a partial class function $F : \mathrm{No}^{m+1} \to \mathrm{No}$ with $m \in \mathbb{N}$, we will say that F is in the class K, in symbols $F \in K$, if there is a formal polynomial quotient in K defining F. Finally, given a subclass K of No we will denote by $K(\vec{X})$ the the class of *formal polynomial quotients with coefficients* in K.

Definition 4.29. Let $\mathfrak{N}$ and $\mathfrak{D}$ be two disjoint sets of natural numbers, I and O be two disjoint subsets of $\mathfrak{N}$, and K be a subclass of No.
A $(\mathfrak{N}, \mathfrak{D}, I, O, K)$-*SBSSM program* P is a finite sequence $(C_0, \ldots, C_n)$ with $n \in \mathbb{N}$ such that for every $0 \leq m \leq n$ the command C_m is of one of the following types

Computation $\mathtt{R}_i{:=}f(\mathtt{R}_{j_0}, \ldots, \mathtt{R}_{j_m})$ were $f : \mathrm{No}^{n+1} \to \mathrm{No}$ is a map in $K(\vec{X})$ and $i \in (\mathfrak{N} \setminus I) \cup \mathfrak{D}$ and $j_0, \ldots, j_m \in \mathfrak{N} \cup \mathfrak{D}$.

Stack Computation $\mathtt{Push}_d(\mathtt{R}_i, \mathtt{R}_j)$ were $i \in \mathfrak{D}$, $j \in \mathfrak{N} \cup \mathfrak{D}$ and $d \in \{\mathrm{L}, \mathrm{R}\}$.

Branch `if` $\mathtt{R}_i$ `then` j were $i \in \mathfrak{N} \cup \mathfrak{D}$ and $j \leq n$.

The sets $\mathfrak{N}$ and $\mathfrak{D}$ are the sets of normal and Dedekind registers of our program, respectively; and, I and O are the sets of input and output registers, respectively. When the registers are irrelevant for the argument we will omit, $\mathfrak{N}$, $\mathfrak{D}$, I, and O and call P a $K(\vec{X})$-SBSSM program.

Definition 4.30. Let $\mathfrak{N}$ and $\mathfrak{D}$ be two disjoint sets of natural numbers, $I = (i_0, \ldots, i_m)$ and $O = (i_0, \ldots, i_{m'})$ be two disjoint subsets of $\mathfrak{N}$, K be a subclass of No, and $P = (C_0, \ldots, C_n)$ be a $(\mathfrak{N}, \mathfrak{D}, I, O, K)$- SBSSM program. Given $x \in \mathrm{No}^{m+1}$ the SBSSM *computation* of P with input x is the transfinite sequence[6]

$$(R^{\mathrm{N}}(t), S^{\mathrm{L}}(t), S^{\mathrm{R}}(t), \mathrm{PC}(t))_{t\in\vartheta} \in (\mathrm{No}^{\mathfrak{N}} \times \wp(\mathrm{No})^{\mathfrak{D}} \times \wp(\mathrm{No})^{\mathfrak{D}} \times \omega)^{\vartheta}$$

where

1. ϑ is a successor ordinal or $\vartheta = \mathrm{On}$;

2. $\mathrm{PC}(0) = 0$;

3. $R^{\mathrm{N}}(0)(i_j) = x(j)$ if $i_j \in I$ and $R^{\mathrm{N}}(0)(i) = 0$ otherwise;

4. for all $i \in \mathfrak{D}$ we have $S^{\mathrm{L}}(0)(i) = S^{\mathrm{R}}(0)(i) = \varnothing$;

5. if $\vartheta = \mathrm{On}$ then for every $t < \vartheta$ we have $0 \leq \mathrm{PC}(t) \leq n$. If ϑ is a successor ordinal $\mathrm{PC}(\vartheta - 1) > n$ and for every $t < \vartheta - 1$ we have $0 \leq \mathrm{PC}(t) \leq n$;

6. for all $t < \vartheta$ for all $j \in \mathfrak{D}$ we have $S^{\mathrm{L}}(t)(j) < S^{\mathrm{R}}(t)(j)$;

7. for every $t < \vartheta$ if $0 \leq \mathrm{PC}(t) \leq n$ and $C_{\mathrm{PC}(t)} = \mathtt{R}_i{:=}f(\mathtt{R}_{j_0}, \ldots, \mathtt{R}_{j_n})$ then $\mathrm{PC}(t+1) = \mathrm{PC}(t)+1$ and: $R^{\mathrm{N}}(t+1)(i) = f(c(j_0), \ldots, c(j_n))$ if $i \in \mathfrak{N}$, $(S^{\mathrm{L}}(t+1)(i), S^{\mathrm{R}}(t+1)(i))$ is the canonical representation of $f(c(0), \ldots, c(n))$ if $i \in \mathfrak{D}$, where for every $m < n$ $c(m) := \begin{cases} [S^{\mathrm{L}}(t)(j_m) | S^{\mathrm{R}}(t)(j_m)] & \text{if } j_m \in \mathfrak{D}; \\ R_{j_m} & \text{otherwise.} \end{cases}$

8. for every $t < \vartheta$ if $0 \leq \mathrm{PC}(t) \leq n$ and $C_{\mathrm{PC}(t)} = \mathtt{Push}_d(\mathtt{R}_i, \mathtt{R}_j)$ then $\mathrm{PC}(t+1) = \mathrm{PC}(t)+1$ and

$$S^d(t+1)(i) = \begin{cases} R^{\mathrm{N}}(t)(j) & \text{if } j \in \mathfrak{N}; \\ [S^{\mathrm{L}}(t)(j) | S^{\mathrm{R}}(t)(j)] & \text{if } j \in \mathfrak{D}. \end{cases}$$

   The rest is left unchanged in $t+1$;

9. for every $t < \vartheta$ if $0 \leq \mathrm{PC}(t) \leq n$ and $C_{\mathrm{PC}(t)} = \mathtt{if}\ \mathtt{R}_i\ \mathtt{then}\ j$ then:

$$\mathrm{PC}(t+1) = \begin{cases} j & \text{if } i \in \mathfrak{N} \text{ and } R^{\mathrm{N}}(t)(i) > 0; \\ j & \text{if } i \in \mathfrak{D} \text{ and } [S^{\mathrm{L}}(t)(i) | S^{\mathrm{R}}(t)(i)] > 0; \\ \mathrm{PC}(t)+1 & \text{if } i \in \mathfrak{N} \text{ and } R^{\mathrm{N}}(t)(i) \leq 0; \\ \mathrm{PC}(t)+1 & \text{if } i \in \mathfrak{D} \text{ and } [S^{\mathrm{L}}(t)(i) | S^{\mathrm{R}}(t)(i)] \leq 0. \end{cases}$$

   The rest is left unchanged in $t+1$;

[6] By abuse of notation we write $\wp(\mathrm{No})$ for the class of subsets of No.

10. for every $t < \vartheta$ if t is a limit ordinal then: $\mathrm{PC}(t) = \liminf_{s<t} \mathrm{PC}(s)$, for every $i \in \mathfrak{N}$

$$R^{\mathrm{N}}(t)(i) = \begin{cases} R^{\mathrm{N}}(t')(i) & t' \text{ is such that } \forall t > t'' > t' R^{\mathrm{N}}(t')(i) = R^{\mathrm{N}}(t'')(i); \\ 0 & \text{if there is no such a } t'. \end{cases}$$

For all $i \in \mathfrak{D}$, if there are t'_L and t'_R smaller than t such that for every $t'_L < t''_L < t$ and $t'_R < t''_R < t$ we have $S^{\mathrm{L}}(t''_L)(i) = S^{\mathrm{L}}(t'_L)(i)$ and $S^{\mathrm{R}}(t''_R)(i) = S^{\mathrm{R}}L(t'_R)(i)$ we have $S^{\mathrm{L}}(t)(i) = S^{\mathrm{L}}(t'_L)(i)$ and $S^{\mathrm{R}}(t)(i) = S^{\mathrm{R}}(t'_R)(i)$. Otherwise, let

$$U_{t,i} := \{t'' < t \mid \forall t'' \leq t' < t(C_{\mathrm{PC}(t')} = \mathtt{R}_j\mathtt{:=}f\mathtt{(R}_{j_0}, \ldots, \mathtt{R}_{j_n}\mathtt{)} \to i \neq j)\}.$$

Then $S^{\mathrm{L}}(t)(i) = \bigcup_{t' \in U_{t,i}} S^{\mathrm{L}}(t')(i)$ and $S^{\mathrm{R}}(t)(i) = \bigcup_{t' \in U_{t,i}} S^{\mathrm{R}}(t')(i)$.

If ϑ is a successor ordinal we say that P *halts* on x with output $y := (R^{\mathrm{N}}(\vartheta - 1)(i))_{i \in O}$ and write $P(x) = y$.

In the previous definition, for each $\alpha \in \vartheta$ and $i \in \mathfrak{N}$, $R^{\mathrm{N}}(\alpha)(i)$ is the content of the normal register i at the αth step of the computation; similarly, $S^{\mathrm{L}}(\alpha)(i)$ and $S^{\mathrm{R}}(\alpha)(i)$ are the sets representing the left and the right stack of the Dedekind register i; moreover, $\mathrm{PC}(\alpha)$ is the value of the *program counter*. Items 2, 3, and 4 in the previous definition describe the initialisation of the machine. In particular, the program counter is set to 0, each normal register but the input registers are initialised to 0, the input registers are initialised to x, and each stack is emptied. In item 5 we make sure that the program counter is a value between 0 and n during the computation and that the machine *stops*, i.e., the program counter is set to a number bigger than the number of commands during in last step. Items 7, 8, and 9 describe the semantics of the instructions. In 7 the execution of $\mathtt{R}_i\mathtt{:=}f\mathtt{(R}_{j_0}, \ldots, \mathtt{R}_{j_n}\mathtt{)}$ is described. If i is a normal register the content of the register i is updated to the value of the rational function f applied to the content of the registers $j_0, \ldots, j_n$. If i is a Dedekind register the stacks of the register i are emptied and filled with the canonical representation of the surreal obtained by applying f to the content of the registers $j_0, \ldots, j_n$. In 8 we describe the execution of $\mathtt{Push}_d\mathtt{(R}_i, \mathtt{R}_j\mathtt{)}$ with $d \in \{\mathrm{L}, \mathrm{R}\}$ which results in pushing the content of the register j in the stack S^d of register i. In 9 we describe the execution of the conditional statement $\mathtt{if\ R}_i\mathtt{\ then\ }j$ which changes the program counter to its successor if the content of the register i is ≤ 0 and to j otherwise. Finally, item 10 describes the behaviour of the machine at limit stages according to the description we gave before.

Definition 4.31. Let $n, m \in \mathbb{N}$ and $F : \mathrm{No}^n \to \mathrm{No}^m$ be a (partial) class function over the surreal numbers and K a subclass of No. Then we say that F is $K(\vec{X})$*-SBSSM computable* iff there are $\mathfrak{N}, \mathfrak{D}, I, O \subset \mathbb{N}$ with $|I| = n$, $|O| = m$ and there is a $(\mathfrak{N}, \mathfrak{D}, I, O, K)$-SBSSM program P such that for every n-tuple x of surreal numbers we have that: if $F(x) = y$ then $P(x) = y$, and if $x \notin \mathrm{dom}(F)$ then $P(x)$ does not halt.

Moreover, we say that F is *SBSSM computable* if it is $\mathrm{No}(\vec{X})$-SBSSM computable.

Note that, since at successor stages our machines behave exactly like normal BSSMs, and since $\mathbb{R}$ is a subfield of No, we can easily simulate every BSSM with a $\mathbb{R}(\vec{X})$-SBSSM. In particular, note that every real is $\mathbb{R}(\vec{X})$-SBSSM computable. Therefore, since they can only compute reals in $\mathbf{L}_{\omega^\omega}$ (see [56]), infinite time Blum-Shub-Smale machines are weaker than $\mathbb{R}(\vec{X})$-SBSSMs.

Note that the hardware of our machines in principle does not allow a direct access to the sign sequence representing a surreal number, e.g., there is no instruction which allows us to read the αth sign of a surreal in the register i. The following lemma tells us that, if our machines can compute rational functions with coefficients in $\{-1, 0, 1\}$, then they are actually capable of computing sign sequences of a surreal numbers and to modify them.

Lemma 4.32. *Let K be a subclass of* No *such that $\{-1, 0, 1\} \subseteq K$. Then, the following functions are $K(\vec{X})$-SBSSM computable:*

1. *The function Lim that given an ordinal number α returns 1 if α is a limit ordinal and 0 otherwise;*
2. *Gödel's pairing function $\mathfrak{g} : \mathrm{On} \times \mathrm{On} \to \mathrm{On}$;*
3. *The function $\mathrm{sgn} : \mathrm{No} \times \mathrm{On} \to \{0, 1, 2\}$ that for every $\alpha \in \mathrm{On}$ and $s \in \mathrm{No}$ returns 0 if the $1 + \alpha$th[7] sign in the sign expansion of s is $-$, 1 if the $1 + \alpha$th sign in the sign expansion of s is $+$ and 2 if the sign expansion of s is shorter than $1 + \alpha$;*
4. *the function $\mathrm{seg} : \mathrm{No} \times \mathrm{On} \to \mathrm{No}$ that given a surreal s and an ordinal $\alpha \in \mathrm{dom}(s)$ returns the surreal whose sign sequence is the initial segment of s of length α.*
5. *The function $\mathrm{cng} : \mathrm{No} \times \mathrm{On} \times \{0, 1\} \to \mathrm{No}$ that given a surreal $s \in \mathrm{No}$, $\mathrm{sgn} \in \{0, 1\}$ and $\alpha \in \mathrm{On}$ such that $\alpha < \mathrm{dom}(s)$ returns a surreal $s' \in \mathrm{No}$ whose sign expansion is obtained by substituting the $1 + \alpha$th sign in the expansion of s with $-$ if $\mathrm{sgn} = 0$ and with $+$ if $\mathrm{sgn} = 1$;*

Proof. For the first item, the algorithm is illustrated in Algorithm 1.

For the second item of the lemma, note that there is an algorithm that, given an ordinal γ, computes the $1 + \gamma$th pair (α, β) in the ordering given by the Gödel's map, see Algorithm 2. Now, to compute the value of the Gödel's map for the pair (α, β) our algorithm can just start generating pairs of ordinals in the order given by the Gödel's map using the algorithm in Algorithm 2 until the pair (α, β) is generated.

For the third item, it is enough to note that there is a program that can go through the surreal tree No using s as a guide. The pseudo algorithm for such a program is illustrated in Algorithm 3.

For fourth item, note that in Algorithm 3 at each step α, the register $Curr$ contains the surreal whose sign sequence is the prefix of the sign sequence of s of length α.

Finally for fifth item, note that, by using fourth item of the lemma, one can easily compute s' by using a Dedekind register. The algorithm is illustrated in Algorithm 4. □

By interpreting 0 as $-$ and 1 as $+$, every binary sequence corresponds naturally to a surreal number. Therefore, we can represent the content of a tape of Turing machines, T2TMs, ITTMs, and OTMs as a surreal number. Lemma 4.32 tells us that we can actually access this representation and modify it.

[7] In this sentence $1 + \alpha$ should be read as the ordinal addition so that for $\alpha \geq \omega$ we have $1 + \alpha = \alpha$.

Algorithm 1: Limit Ordinal $Lim(\alpha)$

```
  Input: Input in R_1
  Output: Output in R_0
  Data: Dedekind registers: Step
1 if R_1 = Step then
2 |  R_0 := 1
3 |  Stop
4 if R_1 = Step + 1 then
5 |  R_0 := 0
6 |  Stop
7 Push_L(Step, Step)
8 Jump 1
```

4.3.3 Computational power of surreal Blum-Shub-Smale machines

Now that we introduced a notion of computability over No, we will compare our new model of computation with classical and transfinite models of computation.

We start by fixing a representation of binary sequences in No. Let $2^{<\mathrm{On}}$ be the class of binary sequences of ordinal length. Let $\Delta : \mathrm{No} \to 2^{<\mathrm{On}}$ be such that for all $s \in \mathrm{No}$, $\Delta(s)$ is the binary sequence of length $\mathrm{dom}(s)$ obtained by substituting each $+$ in s by a 1 and each $-$ by a 0.

Definition 4.33. Given a partial function $f : 2^{<\mathrm{On}} \to 2^{<\mathrm{On}}$ and a class of rational functions $K(\vec{X})$ we say that f is $K(\vec{X})$-SBSSM computable if there is a $K(\vec{X})$-SBSSM program which computes the surreal function F such that $f = \Delta \circ F \circ \Delta^{-1}$.

As we will see, if K is a subclass of No containing $\{-1, 0, 1\}$ then $K(\vec{X})$-SBSSMs are very powerful. In order to show this, we will now begin by proving their capability of simulating all the most important classical models of transfinite computation.

Corollary 4.34. *Let K be a subclass of* No *such that $\{-1, 0, 1\} \subseteq K$. Then, every function computable by an ordinary Turing machine is $K(\vec{X})$-SBSSM computable. Moreover, the classical halting problem is $K(\vec{X})$-SBSSM computable.*

Proof. First note that we can code the content of a tape of a Turing machine as the sign sequence of an element of No_ω. Then, using Lemma 4.32, we can easily see that $K(\vec{X})$-SBSSM programs can simulate a Turing machines. Moreover, a $K(\vec{X})$-SBSSM program can keep track of the number of Turing machine steps simulated; and can therefore recognise if the Turing machine has run for ω-many steps. If this happens, the $K(\vec{X})$-SBSSM program can just halt, recognising that the Turing machine did not halt. $\square$

In the classical theory BSSMs and type 2-Turing machines are very different models. In fact, they are in some sense incomparable. Indeed, there are BSSM-computable functions which are not T2TM-computable and vice versa.

Corollary 4.35. *Let K be a subclass of* No *such that $\{-1, 0, 1\} \subseteq K$. Then, every function computable by an ordinary type 2-Turing machine is $K(\vec{X})$-SBSSM computable.*

Algorithm 2: Gödel's Map

Input: Input in R_0
Output: Output in R_1 and R_2
Data: Dedekind Registers: Max, $Alpha$, $Beta$, $Count$

```
1  Beta := 0
2  Alpha := 0
3  Push_L(Beta, Max − 1)
4  if Count = R_0 then
5  |   R_1 := Alpha
6  |   R_2 := Beta
7  |   Stop
8  if Alpha < Max then
9  |   Push_L(Count, Count)
10 |   if Count = R_0 then
11 |   |   R_1 := Alpha
12 |   |   R_2 := Beta
13 |   |   Stop
14 |   Push_L(Alpha, Alpha)
15 |   Jump to 8
16 0 → Beta
17 if Beta < Max then
18 |   if Count = R_0 then
19 |   |   R_1 := Alpha
20 |   |   R_2 := Beta
21 |   |   Stop
22 |   Push_L(Beta, Beta)
23 |   Push_L(Count, Count)
24 |   Jump to 17
25 if Count = R_0 then
26 |   R_1 := Alpha
27 |   R_2 := Beta
28 |   Stop
29 Push_L(Max, Max)
30 Push_L(Count, Count)
31 Jump to 1
```

Algorithm 3: Sign Sequence $\operatorname{sgn}(s, \alpha)$

Input: Input in R_1, R_2
Output: Output in R_0
Data: Dedekind registers: Step, Curr

if $\mathit{Curr} = R_1$ **then**
    $R_0 := 2$
    Stop
if $R_1 < \mathit{Curr}$ **then**
    $R_0 := 0$
    $\mathit{Push}_R(\mathit{Curr}, \mathit{Curr})$
if $\mathit{Curr} < R_1$ **then**
    $R_0 := 1$
    $\mathit{Push}_L(\mathit{Curr}, \mathit{Curr})$
$\mathit{Push}_L(\mathit{Step}, \mathit{Step})$
if $\mathit{Step} < R_2$ **then**
    GoTo 1

Proof. Since $K(\vec{X})$-SBSSMs set normal registers which are not eventually constant to 0 at limit stages, we will need to use a Dedekind register to deal with the output tape. To do so, it is enough to keep two copies of the output tape O_1 and O_2. The first one filled with 0s at the beginning and the other filled with 1s. Now, each time we would have to modify the output tape we do so in both copies. Moreover, we put the surreal O_1 in the left stack of a Dedekind register O and the surreal O_2 in the right stack of the Dedekind register O. At stage ω, the register O will contain the shortest sequence in between its left and right stacks. But note that for each cell of the output tape this sequence will have exactly 0 or 1 according to what we wrote; this because, from a certain point on, both stacks will agree on that value. Finally, note that we can keep track with a Dedekind register of the fact that the type two algorithm wrote on every cell of the output. If it did we are done. If not, then the Dedekind register will contain a dyadic number and our program will enter an infinite loop. Otherwise it will stop. □

We can even go further and prove that, if K is a subclass of No such that $\{-1, 0, 1\} \subseteq K$, then $K(\vec{X})$-SBSSM programs can also be used to simulate ITTMs and decide the halting problem for ITTMs. The following notion was introduced by Hamkins and Lewis in [44] and further studied by several authors; see, e.g., [109].

Definition 4.36. An ordinal α is *clockable* if there is an ITTM which runs on empty input for exactly α-many steps. We will denote by Λ the supremum of the clockable ordinals.[8]

Theorem 4.37. *Let K be a subclass of* No *such that $\{-1, 0, 1\} \subseteq K$. Then, every ITTM-computable function is $K(\vec{X})$-SBSSM computable. Moreover, if $\Lambda \in K$, then the halting problem for ITTMs is $K(\vec{X})$-SBSSM computable.*

[8] The supremum of the clockable ordinals is usually denoted by λ. We decided not to use this notation to avoid confusion.

Algorithm 4: Bit Change $\operatorname{cng}(s, \alpha, \operatorname{sgn})$

Input: Input in R_1, R_2, R_3
Output: Output in R_0
Data: Dedekind registers: *Step*, *Curr*

```
R_0 := Curr
if R_2 = Step then
    if R_3 = 0 then
        Push_R(Curr, Curr)
    else
        Push_L(Curr, Curr)
if sgn(R_1, Step) = 0 ∧ R_2 ≠ Step then
    Push_R(Curr, Curr)
if sgn(R_1, Step) = 1 ∧ R_2 ≠ Step then
    Push_L(Curr, Curr)
Push_L(Step, Step)
if sgn(R_1, Step) ≠ 2 then
    GoTo 1
```

Proof. We will assume that our ITTM has only one tape; a similar proof works in the general case. We call a *snapshot* of an execution of an ITTM at time α a tuple $(T(\alpha), I(\alpha), H(\alpha)) \in \{0,1\}^\omega \times \omega \times \omega$ where $T(\alpha)$ is a function representing the tape content of the ITTM at time α, $I(\alpha)$ is the state of the machine at time α, and $H(\alpha)$ is the position of the head at time α. We know that we can code $T(\alpha)$ as a sign sequence of length ω. Moreover, at the successor stages, by Lemma 4.32, we can modify this sequence in such a way that the result is a sign sequence in No_ω coding the ITTM tape after that the operation is performed. Moreover, we know that there is a bound, Λ, to the possible halting times of an ITTM. Therefore, we can code the list of the $T(\alpha)$ in the snapshots of an ITTM as a sequence of pluses and minuses length Λ; hence, as a surreal number of the same length. Consider the $K(\vec{X})$-SBSSM program that uses two Dedekind registers T and S, and two normal registers I and H. The first Dedekind register is used to keep track of the tapes in the snapshots, the second Dedekind register is used to keep track of how many ITTM instructions have been executed, the register I is used to keep track of the current state of the ITTM, and the register H to keep track of the current head position.

At each step α, if S is a successor ordinal, the program first copies the last ω-many bits of T into a normal register R; then, executes the instruction I with head position[9] $(\omega \times S) + H$ on the string sequence of T writing the result in R. Then, the program computes the concatenation s_α of T and R; and uses Algorithm 5 to push the canonical representation of s_α into the stacks of T. Since for all $\beta < \alpha$, the sign sequence of s_β is an initial segment of s_α, T will contain $\bigcup_{\beta \in \alpha} s_\alpha$ at limit stages.

Now, if S is a limit, the program first computes the content of R as the point-wise $\liminf$ of the snapshots in T. Note that this is computable. Indeed, suppose that the program needs to compute the $\liminf$ of the bit in position i; then it can just look

[9]Once again the operations in $(\omega \times S) + H$ must be interpreted as ordinal operations.

sequentially at the values of the snapshots at i and if it finds a 0 at i in the αth snapshot it pushes $\alpha - 1$ into the left stack of a Dedekind register R'. Once the program has looked through all the snapshots, it will compute the lim inf of the cell in position i as 0 if $R' = S$ and as 1 otherwise. Then, the program will set H to 0 and I to the special limit state and continue the normal execution. This ends the first part of the proof.

Now, assume that $\Lambda \in K$. Note that the $K(\vec{X})$-SBSSM program we have just introduced can simulate the ITTM and check after the execution of every ITTM step that $S < \Lambda$. If at some point the program simulates Λ-many steps of the ITTM, i.e., $S \geq \Lambda$, the program will just halt knowing that the ITTM can not halt. $\square$

Algorithm 5: *CanonicalRep* Subroutine

Input: Input in R_1

Data: Dedekind registers: *Step*, H

```
if sgn(R_1, Step) ≠ ⊥ then
    if sgn(R_1, Step) = + then
        Push_L(H, seg(R_1, Step))
    if sgn(R_1, Step) = − then
        Push_R(H, seg(R_1, Step))
    Push_L(Step, Step)
    Jump 1
```

Corollary 4.38. *Let K be a subclass of* No *such that $\{-1, 0, 1\} \subseteq K$. Then, every function computable by an infinite time Blum-Shub-Smale machine is $K(\vec{X})$-SBSSM computable and the halting problem for infinite time Blum-Shub-Smale machine is $K(\vec{X})$-SBSSM computable.*

Proof. This follows from Theorems 4.37 and from the fact that ITTM can simulate and decide the halting problem of infinite time Blum-Shub-Smale machines; see [58, Lemma 5]. $\square$

Lemma 4.39. *If* ZFC *is consistent, so is* ZFC+ *"there is a function that is $\mathbb{R}(\vec{X})$-SBSSM computable but not OTM computable".*

Proof. Let $\mathbf{V}[G]$ be the forcing extension of $\mathbf{V}$ obtained by adding a Cohen real r. Then the constant function $F : x \mapsto r$ is $\mathbb{R}(\vec{X})$-SBSSM computable. But, since by Corollary 4.6 OTMs only compute elements of $\mathbf{L}$, we have that F is not OTMs computable. $\square$

Theorem 4.40. *Let K be a subclass of* No *such that $\{-1, 0, 1\} \subseteq K$. Then, every OTM computable partial function $f : 2^{<\mathrm{On}} \to 2^{<\mathrm{On}}$ is $K(\vec{X})$-SBSSM computable.*

Proof. We will assume that our machine has two tapes, one read-only input tape and an output tape; the general case follows.

Our program will be very similar to the one we used for ITTMs. For this reason, we will mostly focus on the differences.

The main difference is that, while for ITTM we can just save the sequence of tape snapshots, for OTM we cannot simply do that because the tape has class length. The problem can be solved by padding. Given a binary sequence $b := [b_\beta]_{\beta\in\alpha}$ where $b_\beta \in \{-,+\}$ for each $\beta < \alpha$, let b^p be the sequence obtained by concatenating the sequence $[+b_\beta+]_{\beta\in\alpha}$ with the sequence $--$. We call b^p the padding of b. With this operation, we can now save the initial meaningful part of the OTM tape in a register.

The program has four Dedekind registers T, S, H_i, I_i, and two normal registers H and I. As for ITTMs, the Dedekind register T is used to keep track of the tapes in the snapshots; the Dedekind register S is used to keep track of how many OTM instructions have been executed; the register I is used to keep track of the current state of the OTM; and the register H to keep track of the current head position. Note that, since at limit stages the head position and the state of the machine need to be set to the lim inf of their previous contents, we added the Dedekind registers H_i and I_i to keep track of the histories of H and I, respectively.

The registers T, H_i and I_i are really the main difference between this program and the one we used to simulate ITTM. At each stage, T will contain the concatenation of the paddings of the previous configurations of the OTM tape. Note that the sequence $--$ works as a delimiter between one snapshot and the next one. Also, since we cannot save all the OTM tape, each time we will just record the initial segment of the OTM tape of length S, i.e., the maximum portion we could have modified.

If $S := \alpha + 1$, the program first copies the last snapshot in T to a normal register s_α removing the padding.

At this point, the program can just simulate one step of OTM and then compute the padding s^p_α of s_α, and push the standard representation of s^p_α in T.

Now, the program will take the content of H_i, and will compute the surreal number h_α whose sign sequence is H_i followed by H minuses and one plus. Then, the program will push the canonical representation of h_α into the stacks of H_i. Similarly for I, the program will take the content of I_i, and will compute the surreal number i_α whose sign sequence is I_i followed by I_i minuses and one plus. Then, the program will push the canonical representation of i_α into the stacks of I_i.

Again, note that, as for ITTMs, at limit stages T, H_i and I_i will contain the concatenation of the padded snapshots of the tape, H and I, respectively.

If S is a limit ordinal, with a bit of overhead due to padding, the program can compute the pointwise lim inf of the tape. It is not hard to see that this operation is a minor modification of the one used for ITTMs. Note that, in this case, not all the bits will be present in every snapshot; if we want to compute the ith bit of the limit snapshot we will have to start computing the lim inf from the ith snapshot in T. The rest is essentially the same as what we did for ITTM case.

Then, the program will use Algorithm 7 to compute the content of I; and, with a minor modification of the same algorithm, using H_i and I_i, it can compute the lim inf of H only considering the stages where I was the current state. Then, the program can proceed exactly as in the successor case. □

As we have seen so far, if K is a subclass of No such that $\{-1, 0, 1\} \subseteq K$ then $K(\vec{X})$-SBSSMs are at least as powerful as OTMs. It turns out that, if $K = \{-1, 0, 1\}$, the two models of computation are actually equivalent; see Theorem 4.44. Note that this

Algorithm 6: *Pluses* Subroutine

Input: Input in H_i
Data: Dedekind registers: *Plus*, *Step*

```
if sgn(H_i, Step) ≠ ⊥ then
    if sgn(H_i, Step) = + then
        Push_L(Plus, Plus)
    Push_L(Step, Step)
    Jump 1
```

is analogous to the equivalence between Turing Machines and the restricted version of BSSMs which are allowed only to use rational functions with coefficients in $\{-1, 0, 1\}$.

As we have seen in Section § 4.2, via representations, it is possible to use OTMs to induce a notion of computability over the surreal numbers. We will take the same approach here.

To avoid unnecessary complications, in the following we will only deal with unary surreal functions. The theory can be easily generalised to functions of arbitrary arity.

Definition 4.41. Given a partial function $F : \mathrm{No} \to \mathrm{No}$, we say that F is *OTM computable* if there is an OTM program that computes the function G such that $F = \delta_{\mathrm{No}} \circ G \circ \delta_{\mathrm{No}}^{-1}$.

Because of the fact that, as we have seen in section § 4.2, OTMs are capable of computing surreal operations and convert back and forth from cut representation to sign sequences, it is therefore easy to see that OTMs and $\{-1, 0, 1\}(\vec{X})$-SBSSM have the same computational strength.

Theorem 4.42. *Let K be a subclass of OTM computable elements of* No*, i.e., such that for every $s \in K$ the sequence $\delta_{\mathrm{No}}(s)$ is computable by an OTM with no input. Then, every $K(\vec{X})$-SBSSM computable function is OTM computable.*

Proof. As we proved in Theorem 4.20, the surreal operations are δ_{No}-computable.

Moreover, using the algorithms in Lemma 4.15 that convert $\delta_{\mathbb{Q}_\kappa}$ into $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}$ and vice versa, OTMs can simulate the behaviour of Dedekind registers.

Therefore, since by assumptions the (codes for) the elements of K are computable, every $K(\vec{X})$-SBSSM computable function is OTM computable. □

Corollary 4.43. *Every $\{-1, 0, 1\}(\vec{X})$-computable function is OTM computable.*

So, $\{-1, 0, 1\}(\vec{X})$-SBSSM have the same computational power as OTMs. Note that, if we enlarge the class of rational functions our machine is allowed to use we obtain progressively stronger models of computations. Moreover, it is easy to see that the class of coefficients allowed in the class of rational functions acts as a set of parameters on the OTMs side.

Theorem 4.44. *Let K be a subclass of* No*. Then a partial function $F : \mathrm{No} \to \mathrm{No}$ is $K(\vec{X})$-SBSSM computable iff it is computable by an OTM with parameters in K.*

Algorithm 7: *Liminf* Subroutine

Input: Input in H_i
Data: Dedekind registers: Inf, Aus, $Step$, $Step_2$, Lim, $Zero$

$Step := 0$
$Inf := 0$
$Zero := 0$
if $Zero < Step_2 \wedge \operatorname{sgn}(H_i, Step) \neq \perp$ **then**
    if $\operatorname{sgn}(H_i, Step) = +$ **then**
        $Push_L(Zero, Zero)$
    $Push_L(Step, Step)$
    Jump 4
if $Pluses(H_i) = Step_2$ **then**
    Stop
if $\operatorname{sgn}(H_i, Step) = -$ **then**
    $Push_L(Aus, Aus)$
    $Push_L(Step, Step)$
    Jump 11
if $\operatorname{sgn}(H_i, Step) = +$ **then**
    $Push_L(Step, Step)$
    $Push_R(Inf, Aus + 1)$
    $Aus := 0$
    Jump 11
$Push_L(Lim, Inf - 1)$
$Push_L(Step_2, Step_2)$
Jump 1

Proof. For the right to left direction, note that each element of K is $K(\vec{X})$-SBSSM computable. Therefore, by using the algorithm in the proof of Theorem 4.40 we have that, if F is OTM computable with parameters in K, then it is $K(\vec{X})$-SBSSM computable. For the other direction, note that, as we have just showed in Theorem 4.42, surreal operations and operations of SBSSM which involve computable coefficients are computable. Therefore, it is enough to input to the OTM the coefficients of the rational functions involved in the $K(\vec{X})$-SBSSM algorithm in order to make the OTM capable of computing F. Therefore, $F : \mathrm{No} \to \mathrm{No}$ will be OTM computable with parameters in K as desired. $\square$

Corollary 4.45. *Every partial function* $F : \mathrm{No} \to \mathrm{No}$ *which is a set is* $\mathrm{No}(\vec{X})$*-SBSSM computable.*

Proof. Note that F is a sequence of pairs of surreal numbers $\{(s^\ell_\beta, s^r_\beta) \mid \beta \in \alpha\}$ for some $\alpha \in \mathrm{On}$. Consider the function $G := \{(\Delta(s^\ell_\beta), \Delta(s^r_\beta)) \mid \beta \in \alpha\}$. As usual, using some padding bits, we can code each pair in G as a binary sequence. Then, by using the Gödel function $\mathfrak{g}$ we can code G as a binary sequence. Therefore, using Δ again, G can be coded a surreal number s.

Now, given a surreal s', our program can just go through the coding of G using the functions in Lemma 4.32 looking for a pair of the form (s', s''). Then, the program will return s'' in case of success or will diverge otherwise. $\square$

In his Master's thesis [62] (co-sepervised by the author of this dissertation), Ethan Lewis defines a notion of computability based on OTMs which allows for infinite programs. We will call these machines *infinite program machines* (IPMs) and refer the reader to [62, Chapter 3] for the formal definition. Theorem 4.44 tells us that $\mathrm{No}(\vec{X})$-SBSSM are a register model for IPMs. As for OTMs we say that a partial function $F : \mathrm{No} \to \mathrm{No}$ is *IPM computable* if there is an IPM program that computes the function G such that $F = \delta_{\mathrm{No}} \circ G \circ \delta_{\mathrm{No}}^{-1}$.

Corollary 4.46. *A partial function* $F : \mathrm{No} \to \mathrm{No}$ *is* $\mathrm{No}(\vec{X})$*-SBSSM computable iff it is IPM computable.*

Proof. It follows from Theorem 4.44 and the fact that IPM computable functions are exactly those computable by OTM with a parameter in $2^{<\mathrm{Ord}}$; see [62, p. 19]. $\square$

We end this section by introducing halting sets and universal programs for our new model of computation. Note that, using classical coding techniques, given a class of rational functions, every K-SBSSM program can be coded as one (possibly infinite) binary sequence, i.e., a surreal number.

Given two natural numbers n and m, and a subclass K of surreal numbers we will denote by $\mathfrak{P}_K^{n,m}$ the class of $(\mathfrak{N}, \mathfrak{D}, I, O, K)$-SBSSM programs with $|I| = n$, $|O| = m$.

Definition 4.47. Let K be a class of the surreal numbers. We define the following class[10]:

$$H_K^{n,m} := \{(p, s) \in \mathrm{No} \mid \ p \text{ is a } K(\vec{X})\text{-SBSSM program in } \mathfrak{P}_K^{n,m} \text{ halting with input } s\}.$$

As usual, we say that a set of surreal numbers is decidable if its characteristic function is computable.

If we assume that K contains $\{-1, 0, 1\}$ we can use the code of a program, together with the fact that OTMs can simulate $K(\vec{X})$-SBSSM and can be simulated by $K(\vec{X})$-SBSSM, to define a universal SBSSM program.

Definition 4.48. Let $\mathfrak{N}$ and $\mathfrak{D}$ be two disjoint sets of natural numbers, I and O be two disjoint subsets of $\mathfrak{N}$, and K be a class of surreal numbers. A $(\mathfrak{N}, \mathfrak{D}, I, O, K)$-SBSSM program P with $|I| = n + 1$ and $|O| = m$ is called *universal* if for every code p' of a program in $\mathfrak{P}_K^{n,m}$ and for every $x \in \mathrm{No}^n$ we have that $P(p', x) = P(x)$.

Theorem 4.49. *Let K be a subclass of* No *containing* $\{-1, 0, 1\}$*, and* $\mathfrak{N}, \mathfrak{D}, I, O \subset \mathbb{N}$ *be such that:* $\mathfrak{N}$ *and* $\mathfrak{D}$ *are disjoint; and,* I *and* O *are disjoint subsets of* $\mathfrak{N}$*. Then, there is a universal* $(\mathfrak{N}, \mathfrak{D}, I, O, K)$*-SBSSM program.*

Proof. It is enough to note that any reasonable coding of a $K(\vec{X})$-SBSSM program can be computably translated into a code for an OTM computing the same function. In particular, note that all the functions in $K(\vec{X})$ used in the program will be in the code

[10]Note that if K is a set $H_K^{n,m}$ is also a set.

and all the parameters will also be encoded. So our universal program can start by taking the code of the program and converting it into an OTM simulation of the corresponding SBSSM program with the coefficients of the rational functions in $K(\vec{X})$-SBSSM program as parameters as in Theorem 4.40. Then, by using the algorithms in Lemma 4.32, the universal program will compute δ_{No} of the input of the $K(\vec{X})$-SBSSM program; and, as in Theorem 4.40, it will simulate the OTM obtained with the translation on the δ_{No} of the input. Finally, the universal program will, again by using the algorithms in Lemma 4.32, translate back the output of the OTM using δ_{No}. □

Note that, since the coefficients of the polynomials needed in each program will be coded in the program, the universal machine in the Theorem 4.49 is actually a $\{-1, 0, 1\}$-SBSSM machine. This is not very surprising in view of the fact that the universal program for IPMs is a classical OTM program; see [62, Theorem 3.13].

Corollary 4.50. *Let K be a subclass of* No *containing* $\{-1, 0, 1\}$. *Then $H_K^{1,1}$ is not $K(\vec{X})$-SBSSM computable.*

Proof. Assume that $H_K^{1,1}$ is computable. Then, there is a program P that computes it. Now, consider the program P' that converges on x only if $(x, x) \notin H_K^{1,1}$. This program is computable by Theorem 4.49 and by the assumptions. Now, let p' be a code for P'. We have that, $P'(p')$ converges if and only if $(p', p') \notin H_K$ diverges if and only if $P'(p')$ diverges. □

4.4 Open questions

Our results from § 4.2 are just the beginning of a more systematic application of generalised analysis to transfinite computability. The results in § 4.2 lead naturally to many interesting questions.

As we mentioned at the end of § 4.2.5, our proof of the fact that $\mathrm{IVT}_\kappa \leq_{\mathrm{sW}} \mathrm{B}_{\mathrm{I}}^\kappa$ strongly depends on the surrounding universe of set theory. Particularly important in this context is the assumption that a computable enumerable dense subset of $\mathbb{R}_\kappa$ exists.

Question 4.51. Can the assumption that an computably enumerable dense subset of $\mathbb{R}_\kappa$ exists in Theorem 4.24 be removed or weakened?

Similarly we showed that in Lemma 4.25 that it is not hard to destroy this assumption. It is therefore natural to ask:

Question 4.52. How sensitive is T2κTM computability to a change in the surrounding set theory?

Question 4.53. Are there natural set theoretic assumptions that make the generalised theory of Weihrauch degrees closer to the classical theory?

The results in § 4.2.5 are just an instance of the generalisation of classical theorems from the theory of Weihrauch degrees. In Chapter 3 we showed that, sometimes, under some large cardinal assumptions, it is possible to prove versions of classical theorems from real analysis over $\mathbb{R}_\kappa$.

Question 4.54. Assume that κ is weakly compact. What is the Weihrauch degree of the $\mathsf{wBWT}_{\mathbb{R}_\kappa}$?

Question 4.55. Assume that κ is weakly compact. What is the relation of the $\mathsf{wBWT}_{\mathbb{R}_\kappa}$ and the tree property of κ from a computable analysis prospective?

In § 4.3 we introduced the a new model of computation which generalises classical Blum-Shub-Smale machines. As we have seen, these new machines generalise the register machines counterpart of OTMs and even more. Our results in § 4.3 are very preliminary and a full theory of surreal Blum-Shub-Smale machines is still missing. This would be a worthwhile topic for future research.

Note that our definition of SBSSM can be easily modified to work with any real closed field K. Indeed, as shown by Ehrlich in [28, Theorem 19], every real closed field is isomorphic to an initial subtree of No; therefore, given a real closed field K, and adding to the Definition 4.30 the requirement that the left and right stacks S^{L} and S^{R} of every Dedekind register are such that $[S^{\mathrm{L}} \mid S^{\mathrm{R}}]$ is in K, we can induce a notion of computability over K.

Question 4.56. Let κ be a cardinal such that $\kappa^{<\kappa} = \kappa$. How does the notion of computability induced by SBSSM on $\mathbb{R}_\kappa$ compare to the classical one induced by BSSM on $\mathbb{R}$?

Chapter 5

Order types of models of arithmetic

Remarks on co-authorship. *The results of this chapter are due to a collaboration of the author and his supervisor Benedikt Löwe. The results of this section have been submitted for publication and are currently under review [37].*

5.1 Introduction

5.1.1 Motivations & results

The incompleteness phenomenon for arithmetic is due to the interaction of addition and multiplication: the theory of the natural numbers in the full language of arithmetic with addition and multiplication is essentially incomplete whereas its syntactic fragments in the language with only addition (known as *Presburger arithmetic*; see [77]) and the language with only multiplication (known as *Skolem arithmetic*; see [92]) are complete and decidable [79, § 1.2.3]. Addition and multiplication combined make theories *sequential*, i.e., they can encode the notion of finite sequence; this in turn paves the path to Gödel's incompleteness argument.

Non-standard models of arithmetic naturally split into archimedean classes (Definition 5.3) of elements with finite distance; a standard argument using only very basic properties of arithmetic shows that the order type of a non-standard model of arithmetic is of the form $\mathbb{N} + \mathbb{Z} \cdot D$ where D is a dense linear order without first or last element (see [50, Theorem 6.4]). In general, it is not known which (uncountable) dense linear orders D give rise to an order type of a non-standard model of arithmetic (see [8, 9] for an overview of what is known).

The three basic properties used in the standard argument mentioned in the last paragraph are (a) that the model is linearly ordered, (b) that addition is well-behaved with respect to that order, and (c) that every element is either even or odd. Given any standard axiomatisation of PA, properties (a) and (b) do not need induction to be proved, while property (c) does. An inspection of the argument reveals that property (c) is important for the density argument; so, we have linked induction to the density of the order D in the order type of the model.

It is the aim of this chapter to study in which ways properties of systems of arithmetic constrain the possible order types occurring as order types of non-standard models of these systems.

We consider three operations, the unary successor operation and the binary addition and multiplication operations and their associated languages: $\mathcal{L}_{<,\mathrm{s}} := \{0, <, \mathrm{s}\}$, the language with an order relation and the successor operation, $\mathcal{L}_{<,\mathrm{s},+} := \{0, <, \mathrm{s}, +\}$, the language augmented with addition, and $\mathcal{L}_{<,\mathrm{s},+,\cdot} := \{0, <, \mathrm{s}, +, \cdot\}$, the full language of arithmetic. For each of the languages, we will define the appropriate arithmetical axiom systems and the corresponding axiom schemes of induction, resulting a total of six theories,

$$\begin{array}{ccc} \mathsf{SA}^- & \subseteq & \mathsf{SA} \\ \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\ \mathsf{Pr}^- & \subseteq & \mathsf{Pr} \\ \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\ \mathsf{PA}^- & \subseteq & \mathsf{PA}, \end{array}$$

where the theories in the left column are without induction and the theories in the right column are with the axiom scheme of induction (for definitions, see § 5.1.2).

As usual, we use the following syntactic abbreviations: for $n \in \mathbb{N}$ and a variable x, we write

$$\mathrm{s}^n(x) := \underbrace{\mathrm{s}(\ldots(\mathrm{s}}_{n \text{ times.}}(x))\ldots) \text{ and}$$
$$nx := \underbrace{x + \ldots + x}_{n \text{ times.}}.$$

We will show that SA^- proves the axiom scheme of induction (Theorem 5.10) and hence SA^- and SA are the same theory, reducing our diagram to five theories. The main result of this chapter is the separation of the remaining five theories in terms of order types: in the following diagram, an arrow from a theory T to a theory S means "every order type that occurs in a model of T occurs in a model of S". In § 5.6, we will show that the diagram is complete in the sense that if there is no arrow from T to S, then there is an order that is the order type of a model of T that cannot be the order type of a model of S.

$$\begin{tikzcd} & \mathsf{SA} & \\ \mathsf{Pr}^- \arrow[ur] & & \mathsf{Pr} \arrow[ul] \arrow[ll] \\ \mathsf{PA}^- \arrow[u] & & \mathsf{PA}. \arrow[u] \arrow[ll] \end{tikzcd}$$

5.1.2 Definitions

In this section, we will introduce the axiomatic systems whose order type we will study. The axioms come in four groups corresponding to the order, the successor function, addition, and multiplication.

The order axioms O1 to O4 express that $<$ describes a linear order with least element 0 (O1 is trichotomy, O2 is transitivity, and O3 is antisymmetry):

$$x < y \vee x = y \vee x > y, \tag{O1}$$
$$(x < y \wedge y < z) \to x < z, \tag{O2}$$
$$\neg(x < x), \tag{O3}$$
$$x = 0 \vee 0 < x. \tag{O4}$$

The successor axioms S1 to S4 express that $<$ is discrete and that s is the successor operation with respect to $<$:

$$x = 0 \leftrightarrow \neg\exists y x = \mathrm{s}(y), \tag{S1}$$
$$x < y \to y = \mathrm{s}(x) \vee \mathrm{s}(x) < y, \tag{S2}$$
$$x < y \to \mathrm{s}(x) < \mathrm{s}(y), \tag{S3}$$
$$x < \mathrm{s}(x). \tag{S4}$$

Taken together, the axioms O1 to O4 and S1 to S4 (later called SA^-) constitute the theory of discrete linear orders with a minimum and a strictly increasing successor function.

The addition axioms P1 to P5 express the fact that the $+$ and $<$ satisfy the axioms of ordered abelian monoids:

$$(x + y) + z = x + (y + z), \tag{P1}$$
$$x + y = y + x, \tag{P2}$$
$$x + 0 = x, \tag{P3}$$
$$x < y \to x + z < y + z, \tag{P4}$$
$$x + \mathrm{s}(y) = \mathrm{s}(x + y). \tag{P5}$$

The axiom $\divideontimes$ expresses the fact that if $x < y$, then the difference between them exists:

$$x < y \to \exists z x + z = y. \tag{$\divideontimes$}$$

The multiplicative axioms M1 to M6 express that $\cdot$ and $+$ are commutative semiring operations respecting $<$:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \tag{M1}$$
$$x \cdot y = y \cdot x, \tag{M2}$$
$$(x + y) \cdot z = x \cdot z + y \cdot z, \tag{M3}$$
$$x \cdot \mathrm{s}(0) = x, \tag{M4}$$
$$x \cdot \mathrm{s}(y) = (x \cdot y) + x, \tag{M5}$$
$$x < y \wedge z \neq 0 \to x \cdot z < y \cdot z. \tag{M6}$$

Finally we have a schema of induction axioms.

$$(\varphi(0,\bar{y}) \wedge \forall x(\varphi(x,\bar{y}) \to (x+1,\bar{y})) \to \forall x\varphi((x,\bar{y}). \qquad (\mathrm{Ind}_\varphi)$$

When considering subsystems of these axioms, we will denote the axiom schema of induction restricted to the formulas of a language $\mathcal{L}$ by $\mathrm{Ind}(\mathcal{L})$. We will consider the following systems of axioms:

$$\begin{aligned}
\mathsf{SA}^- &= \mathrm{O1} + \mathrm{O2} + \mathrm{O3} + \mathrm{O4} + \mathrm{S1} + \mathrm{S2} + \mathrm{S3} + \mathrm{S4},\\
\mathsf{SA} &= \mathsf{SA}^- + \mathrm{Ind}(\mathcal{L}_{<,\mathrm{s}}),\\
\mathsf{Pr}^- &= \mathsf{SA}^- + \divideontimes + \mathrm{P1} + \mathrm{P2} + \mathrm{P3} + \mathrm{P4} + \mathrm{P5},\\
\mathsf{Pr} &= \mathsf{Pr}^- + \mathrm{Ind}(\mathcal{L}_{<,\mathrm{s},+}),\\
\mathsf{PA}^- &= \mathsf{Pr}^- + \mathrm{M1} + \mathrm{M2} + \mathrm{M3} + \mathrm{M4} + \mathrm{M5} + \mathrm{M6},\\
\mathsf{PA} &= \mathsf{PA}^- + \mathrm{Ind}(\mathcal{L}_{<,\mathrm{s},+,\cdot});
\end{aligned}$$

standing for 'Successor Arithmetic', 'Presburger Arithmetic', and 'Peano Arithmetic', respectively. Note that SA should not be confused with the theory $\mathrm{Th}(\mathbb{Q},+)$ called SA in [47] and [93] (the 'S' there stands for 'Skolem').

In his original paper [77], Presburger uses a different axiomatisation of Presburger Arithmetic that we will call Pr^{D}. The axioms of Pr^{D} are the axioms for discretely ordered abelian additive monoids with smallest non-zero element 1 (i.e., axioms O1 to O4, S1 to S4, and P1 to P4), and the following axiom schema:

$$\forall x \exists y x = ny \vee x = \mathrm{s}(ny) \vee \ldots \vee x = \mathrm{s}^{n-1}(ny), \qquad (\mathrm{D}_n)$$

for $0 < n \in \mathbb{N}$. (Note that D_2 is the statement "every number is either even or odd" called *property (c)* in our informal argument in § 5.1.1.)

Theorem 5.1 (Presburger [77]). *The theory* Pr^{D} *axiomatises the complete theory* $\mathrm{Th}(\mathbb{N},+)$.

Since our Pr clearly implies Pr^{D}, it also axiomatises $\mathrm{Th}(\mathbb{N},+)$.

We do not take into consideration Skolem arithmetic SK, i.e., the multiplicative fragment of PA. This is due to the fact that SK, usually defined as $\mathrm{Th}(\mathbb{N},\cdot)$, does not carry an order structure, i.e., the order is not definable in $\mathcal{L}_\cdot$. Moreover, adding the order to Skolem arithmetic makes it much more expressive.

Theorem 5.2 (Robinson [81, Theorem 1.1]). *The theories* $\mathrm{Th}(\mathbb{N},<,\cdot)$, $\mathrm{Th}(\mathbb{N},\mathrm{s},\cdot)$, *and* $\mathrm{Th}(\mathbb{N},<,\mathrm{s},+,\cdot)$ *are equal.*

Therefore, an analysis of Skolem arithmetic in terms of order types is

5.1.3 Order types

As usual, order types are the isomorphism classes of partial orders. If $\mathcal{L}$ is any language containing $<$ and M is an $\mathcal{L}$-structure, by a slight abuse of language, we refer to the $\{<\}$-reduct of M as its *order type*. In situations where the order structure is clear from the context, we do not explicitly include it in the notation: e.g., the notation $\mathbb{Z}$ refers to both the set of integers and the ordered structure $(\mathbb{Z}, <)$ with the natural order $<$ on $\mathbb{Z}$.

Let $(A, <)$ be a linearly ordered set and $(B, 0, <)$ be linearly ordered set with a least element 0. Given a function f from A to B, we will call the set

$$\operatorname{supp}(f) = \{b \in B\,;\, b = 0 \vee f(b) \neq 0\}$$

the *support of* f. As usual, we say that a subset $S \subseteq A$ is *reverse well-founded* if it has no strictly increasing infinite sequences. Given a function $f : A \to B$ whose support is reverse well-founded, we call the maximum element of the support of f the *leading term of* f and denote it by $\textsc{lt}(f)$.

If A and B are two linear orders, then A^* is the inverse order of A, $A + B$ is the order sum, and $A \cdot B$ is the product order. Moreover, if A has a least element 0 then A^B is the set of functions with finite support from B to A ordered anti-lexicographically. Note that in the case that A and B are ordinal numbers, then the above operations correspond to the classical ordinal operations.

If $a \in A$, we denote the *initial segment defined by* a as $\mathrm{IS}(a) := \{b \in A\,;\, b < a\}$ and the *final segment defined by* a as $\mathrm{FS}(a) := \{b \in A\,;\, a < b\}$.

If $(G, 0, <, +)$ is an ordered abelian group, then we define $G^+ := \{g \in G\,;\, 0 < g\} = \mathrm{FS}(0)$ to be the *positive part of* G. We call linear orders *groupable* if and only if there is an ordered abelian group with the same order type.

Let G be an ordered additive group. We define the *standard monoid over* G as the ordered monoid $(\mathbb{N} + \mathbb{Z} \cdot G^+, <, +)$ where $<$ is the order relation of $\mathbb{N} + \mathbb{Z} \cdot G^+$ and $+$ is defined point-wise, i.e.,

$$x + y = \begin{cases} n + m & \text{if } x = n,\ y = m \text{ and } m, n \in \mathbb{N}, \\ \langle z + x, g\rangle & \text{if } x \in \mathbb{N} \text{ and } y = \langle z, g\rangle \in \mathbb{Z} \cdot G^+, \\ \langle z + y, g\rangle & \text{if } y \in \mathbb{N} \text{ and } x = \langle z, g\rangle \in \mathbb{Z} \cdot G^+, \\ \langle z_x + z_y, g_x + g_y\rangle & \text{if } x = \langle z_x, g_x\rangle \in \mathbb{Z} \cdot G^+ \text{ and } y = \langle z_y, g_y\rangle \in \mathbb{Z} \cdot G^+. \end{cases}$$

It is easy to see that for each ordered group G the standard monoid over G is indeed a positive monoid.

If $(B, <, +)$ is any ordered group and X is a variable, we can consider the set $B[X]$ of polynomials in the variable X over B, consisting of terms $f = b_n X^n + \ldots + b_1 X + b_0$ where if $n \neq 0$ then $b_n \neq 0$, the *degree* of a polynomial is the highest occurring exponent, i.e., $\deg(f) = n$. We order polynomials as follows:

$$b_n X^n + \ldots + b_1 X + b_0 < c_m X^m + \ldots c_1 X + c_0$$

if either $n < m$ or $n = m$ and $b_i < c_i$ where i is the largest index such that $b_i \neq c_i$. This order respects addition and multiplication of polynomials in the sense of axioms P4 and M6, respectively. A polynomial is called *positive* if it is larger than the zero-polynomial in this order. If we define

$$\begin{aligned} O_0 &= \varnothing, \\ O_{\gamma+1} &= O_\gamma + \mathbb{Z}^\gamma \cdot \mathbb{N} \\ O_\lambda &= \bigcup_{\gamma \in \lambda} O_\gamma \text{ for } \lambda \text{ limit,} \end{aligned}$$

then for every natural number $n > 0$, the linear order O_n is the order type of non-negative polynomials with integer coefficients of degree at most $n - 1$ and thus O_ω is the order type of all non-negative polynomials with integer coefficients.

5.1.4 Basic properties

In this section, we will remind the reader about basic tools of model theory of PA. We refer the reader to [50] for a comprehensive introduction to the theory of non-standard models of PA. One of the main tools in studying the order types of models of PA is the concept of *archimedean class*.

Definition 5.3. Let M be a model of SA^-. Given $x, y \in M$ we say that *x and y are of the same magnitude*, in symbols $x \sim y$, if there are $m, n \in \mathbb{N}$ such that $\mathrm{s}^n(y) \geq x$ and $y \leq \mathrm{s}^m(x)$. The relation $\sim$ is an equivalence relation. For every $x \in M$, we will denote by $[x]$ the equivalence class of x with respect to $\sim$ called the *archimedean class of x*.

The archimedean classes of a model of SA^- partition the model into convex blocks: if $y, w \in [x]$ and $y < z < w$, then $z \in [x]$ (the reader can check that only the axioms of SA^- are needed for this).

Proposition 5.4. *Let M be a model of SA^-. The quotient structure $M/\!\sim$ of archimedean classes is linearly ordered by the relation $<$ defined by $[x] < [y]$ if and only if $x < y$ and $[x] \neq [y]$. Furthermore, $[0]$ is the least element of the quotient structure.*

Proof. the claim follows directly from the linearity of the order on M. $\square$

We refer to the classes that are different from $[0]$ as the *non-zero archimedean classes*. In particular, if A is the order type of the non-zero archimedean classes of M, then the order type of M is $\mathbb{N} + \mathbb{Z} \cdot A$.

So far, we worked entirely in the language $\mathcal{L}_{<,\mathrm{s}}$ with just the axioms of SA^-. If we also have addition in our language, we observe:

Lemma 5.5. *Let M be a non-standard model of Pr^- and $a \in M$ be a non-standard element of M. Then for every $n, m \in \mathbb{N}$ such that $n < m$ we have $[na] < [ma]$. In particular, if $\mathbb{N} + \mathbb{Z} \cdot A$ is the order type of M, then A does not have a largest element.*

Proof. Assume that $n < m$. We want to prove that $[na] < [ma]$. Let $n' > 0$ be such that $m = n + n'$. Let $i \in \mathbb{N}$ we want to show that $na + \mathrm{s}^i(0) < ma$. By definition $ma = (n + n')a = na + n'a$. Now by monotonicity of $+$ and by the fact that a is non-standard and $n' > 0$ we have $na + \mathrm{s}^i(0) < na + a = (n+1)a \leq (n+n')a = ma$. Therefore $[na] < [ma]$ as desired. $\square$

Another important tool in the classical study of order types of models of PA is the *overspill* property:

Definition 5.6. Let M be a model of SA^-. Then $I \subseteq M$ is a *cut* of M if it is an initial segment of M with respect to $<$ and it is closed under s, i.e., for every $i \in I$ we have $\mathrm{s}(i) \in I$. A cut of M is *proper* if it is neither empty nor M itself.

Definition 5.7. Let $\mathcal{L} \supseteq \mathcal{L}_{<,\mathrm{s}}$ be a language. A theory $T \supseteq \mathsf{SA}^-$ has the *$\mathcal{L}$-overspill property* if for every model $M \models T$ there are no $\mathcal{L}$-definable proper cuts of M.

Overspill is essentially a notational variant of induction:

Theorem 5.8. *Let $\mathcal{L} \supseteq \mathcal{L}_{<,\mathrm{s}}$ be a language and $T \supseteq \mathsf{SA}^-$ be any theory. Then the following are equivalent:*

(i) $\mathrm{Ind}(\mathcal{L}) \subseteq T$ *and*

(ii) T has the $\mathcal{L}$-overspill property.

Proof. "(i)⇒(ii)". Let $M \models T$ and I be a proper cut of M. Then $0 \in I$. Suppose towards a contradiction that I is definable by an $\mathcal{L}$-formula φ. Then Ind_φ implies that $I = M$, so I was not proper.

"(ii)⇒(i)". Assume that $\mathrm{Ind}_\varphi \notin T$ for some $\mathcal{L}$-formula φ and find $M \models T$ such that $M \models \neg\mathrm{Ind}_\varphi$. Define the formula $\varphi'(x) := \varphi(x) \wedge \forall y(y < x \to \varphi(y))$. Then φ' defines a proper cut in M, and thus, T does not have the $\mathcal{L}$-overspill property. □

In particular, SA, Pr, and PA have the overspill property for their respective languages $\mathcal{L}_{<,\mathrm{s}}$, $\mathcal{L}_{<,\mathrm{s},+}$, and $\mathcal{L}_{<,\mathrm{s},+,\cdot}$.

5.2 Successor arithmetic

We begin our study by considering the two subsystems obtained by restricting our language to $\mathcal{L}_{<,\mathrm{s}}$, viz. SA^- and SA. The theory SA^- the theory of discrete linear orders with a minimum and a strictly increasing successor function.

Lemma 5.9. *The theory SA^- satisfies quantifier elimination.*

Proof. It is enough to prove that for every quantifier free formula $\chi(\overline{x}, y)$ there is a quantifier free formula φ such that

$$\mathsf{SA}^- \models \exists y \chi(\overline{x}, y) \leftrightarrow \varphi(\overline{x})$$

where y does not appear in φ. We prove this claim by induction over χ. The only interesting cases are the atomic formulas.

If $\chi(x, y) \equiv \mathrm{s}^n(x) < \mathrm{s}^m(y)$: let $\varphi \equiv x = x$. Let $M \models \mathsf{SA}^-$, we want to show $M \models \exists y \chi(x, y)$. First assume $m \geq n$. Since $\mathsf{SA}^- \vdash \forall x \mathrm{s}^n(x) < \mathrm{s}^{m+1}(x)$ we have $M \models \exists y \mathrm{s}^n(x) < \mathrm{s}^m(y)$ as desired. Otherwise if $n > m$ since $\mathsf{SA}^- \vdash \forall x x < \mathrm{s}^{(n-m)+1}(x)$ then $M \models \exists y \chi(\overline{x}, y)$. Hence:

$$\mathsf{SA}^- \models \exists y \chi(\overline{x}, y) \leftrightarrow \varphi(\overline{x})$$

as desired.

If $\chi(x, y) \equiv \mathrm{s}^n(y) < \mathrm{s}^m(x)$: first assume $m > n$ then since $\mathsf{SA}^- \vdash \forall x \mathrm{s}^n(x) < \mathrm{s}^m(x)$ we have $\mathsf{SA}^- \vdash \exists y \chi(x, y) \leftrightarrow x = x$. If $m \leq n$ then $\mathsf{SA}^- \vdash \exists y \chi(x, y) \leftrightarrow \mathrm{s}^n(0) < \mathrm{s}^m(x)$. Indeed, let $M \models \mathsf{SA}^-$ be a model such that there is a $y \in M$ such that $M \models \mathrm{s}^n(y) < \mathrm{s}^m(x)$ and $M \models \neg \mathrm{s}^n(0) < \mathrm{s}^m(x)$. We have two cases: if $M \models \mathrm{s}^n(0) = \mathrm{s}^m(x)$ then we would have $M \models \mathrm{s}^n(y) < \mathrm{s}^m(x) = \mathrm{s}^n(0)$ but since $M \models \forall x \mathrm{s}^n(x) < \mathrm{s}^n(y) \to x < y$ then we would have

$M \models y < 0$. If $M \models \mathrm{s}^m(x) < \mathrm{s}^n(0)$ again we would have $M \models \mathrm{s}^n(y) < \mathrm{s}^m(x) < \mathrm{s}^n(0)$ which implies $M \models y < 0$. On the other hand if $M \models \mathrm{s}^n(0) < \mathrm{s}^m(x)$ then trivially $M \models \exists y \chi(\overline{x}, y)$ as desired.

If $\chi(\overline{x}, y)$ does not have occurrences of y: then $\exists y \chi(\overline{x}, y)$ is either equivalent to $0 = 0$ or $\neg(0 = 0)$.

If $\chi(x, y) \equiv \mathrm{s}^n(x) = \mathrm{s}^m(y)$: similar to the second case. □

Note that this proof is essentially in [29, Theorem 32A] where Enderton shows quantifier elimination for a theory he calls A_L which is essentially the conjunction of our O1 to O4, S1, S3, and S4. [29, Corollary 32B(b)] claims that $A_L = \mathrm{Th}(\mathbb{N}, <, \mathrm{s}, 0)$, but his theory cannot prove our axiom S2 (the discreteness of the order).

By using quantifier elimination, it is not hard to see that SA^- proves the induction schema.

Theorem 5.10. *For every formula φ in the language $\mathcal{L}_{<,\mathrm{s}}$ we have*

$$\mathsf{SA}^- \vdash \mathrm{Ind}_\varphi.$$

Proof. We will prove that for every model M of SA^-, the only definable set which contains 0 and is closed under s is M itself. We say that $I \subseteq M$ is an *open interval* if there are $a, b \in M \cup \{\infty\}$ such that $I = \{x \in M\,;\, a < x < b\}$ and a set $X \subseteq M$ is called *basic* if it is a finite union of open intervals and singletons. As usual, an $\mathcal{L}$-theory T is called *o-minimal* or *order-minimal* if every $\mathcal{L}$-definable subset is basic.

We claim that SA^- is an o-minimal theory: Let $(M, 0, <, \mathrm{s}) \models \mathsf{SA}^-$ and $X \subseteq M$ be $\mathcal{L}_{<,\mathrm{s}}$-definable; by Lemma 5.9, SA^- has quantifier elimination and therefore, X is definable by a quantifier-free $\mathcal{L}_{<,\mathrm{s}}$-formula. We observe that sets definable by atomic formulae are either open intervals or points, hence basic; we furthermore observe that the basic sets are closed under finite intersections and complements. Thus all sets definable by quantifier-free formulae are basic.

By Theorem 5.8, in order to show induction, it is enough to show that the only non-empty $\mathcal{L}_{<,\mathrm{s}}$-definable cut of M is M itself. Suppose X is an $\mathcal{L}_{<,\mathrm{s}}$-definable cut in M. By o-minimality, we have that $X = I_0 \cup \ldots \cup I_n$ where for every $0 \leq i \leq n$, the set I_i is either an open interval (a_j, b_j) or a singleton $\{b_j\}$. Towards a contradiction, let $y \in M$ be such that $y \notin X$. We define $L := X \cap \mathrm{IS}(y)$ and $R := X \cap \mathrm{FS}(y)$, i.e., $X = L \cup R$. Note that there is $J \subseteq \{0, \ldots, n\}$ such that $L = \bigcup_{j \in J} I_j$ and that for $j \in J$, we have that $b_j \in M$. Let $m := \max\{b_j\,;\, j \in J\}$.

Case 1. $m \in L$. Then, since $L \subseteq X$, $m \in X$, but X is closed under successors, and so $\mathrm{s}(m) \in R$. But then $m < y < \mathrm{s}(m)$ which contradicts axiom S2.

Case 2. $m \notin L$. Then there is some $j \in J$ with $I_j = (a_j, m)$. By axiom S1, we find $m' \in I_j \subseteq X$ such that $\mathrm{s}(m') = m$. Once more, since X is closed under successors, $m \in R$, but this yields $m < y < \mathrm{s}(m)$ which contradicts axiom S2. □

In particular this means that SA and SA^- axiomatise the same theory:

Corollary 5.11. *Let M be a structure in the language $\mathcal{L}_{<,\mathrm{s}}$. Then $M \models \mathsf{SA}$ if and only if $M \models \mathsf{SA}^-$.*

Visser asked whether there is a reasonable finitely axiomatised theory that satisfies full induction (preferably in the full language of arithmetic); it is known that such a theory cannot be sequential (see [78, 106] for more on sequentiality). By Corollary 5.11, SA is a finitely axiomatised theory that satisfies full induction (and is not sequential).

Corollary 5.12. *A linear order L is the order type of a model of SA if and only if there is a linear order A such that $L \cong \mathbb{N} + \mathbb{Z} \cdot A$.*

Proof. By Corollary 5.11, it is enough to show that a model satisfies SA^- in order to get full SA. We already proved in Proposition 5.4 that the forward direction holds. For the other direction, if A is a linear order then $\mathbb{N}+\mathbb{Z}\cdot A$ can be easily made into an SA^- model by defining $\mathrm{s}(n) := n + 1$ and $\mathrm{s}(z, a) := (z + 1, a)$. $\square$

5.3 Models based on generalised formal power series

Generalised formal power series, introduced by Levi-Civita, are a generalisation of polynomials over a ring: while polynomials only have natural number exponents, generalised formal power series allow exponents from any ordered additive abelian group. For an introduction to the theory of generalised formal power series, see [34]. In this section, we will adapt the classical theory of generalised formal power series to our context. In particular, we will show how generalised power series can be used as a tool in building non-standard models of Pr^- and PA^-, and even Pr.

Definition 5.13. Let $(\Gamma, 0, <)$ be a linear order with a minimum and $(B, 0, <, +)$ be an ordered group. A function $f : \Gamma \to B \cup \mathbb{Z}$ is a *non-negative formal power series* on B with exponents in Γ if $\mathrm{supp}(f)$ is reverse well-founded, for all $a \in \Gamma \setminus \{0\} f(a) \in B$, $f(0) \in \mathbb{Z}$, and $f(\mathrm{LT}(f)) \geq 0$. We will denote by $B(X^\Gamma)$ the set of *non-negative formal power series* with *base* B and *exponent* Γ.

We think of $f \in B(X^\Gamma)$ as the formal sum $\sum_{a \in \mathrm{supp}(f)} f(a)X^a$ and define order and additive structure on $B(X^\Gamma)$ according to this algebraic intuition:

Definition 5.14. Let $(\Gamma, 0, <)$ be a linear order with a minimum and $(B, 0, <, +)$ be an ordered group. We define

$$(B(X^\Gamma), 0, <, \mathrm{s}, +)$$

to be the structure where $<$ is the anti-lexicographic order, i.e., $f < g$ if and only if $f \neq g$ and the biggest $a \in \Gamma$ such that $f(a) \neq g(a)$ is such that $f(a) < g(a)$, given $f, g \in \mathbb{Z}(X^\Gamma)$, we define $(f + g)(a) = f(a) + g(a)$, we interpret 0 as the constant 0 function and finally we define $\mathrm{s}(f)$ as $f + \mathbf{1}$ where $\mathbf{1}(0) = 1$ and $\mathbf{1}(a) = 0$ if $a \neq 0$.

Theorem 5.15. *Let $(\Gamma, 0, <)$ be a linear order with a minimum and $(B, 0, <, +)$ be an ordered abelian group. Then $(B(X^\Gamma), 0, <, \mathrm{s}, +)$ is a model of Pr^-.*

Proof. We want to show that $(B(X^\Gamma), 0, <, \mathrm{s}, +)$ is a model of Pr^-. We will first prove the closure of $B(X^\Gamma)$ under $+$. Let $f, g \in B(X^\Gamma)$. First of all note that by definition of $+$ we have $\mathrm{supp}(f + g) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(g)$ since $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ are reverse well-ordered so is $\mathrm{supp}(f) \cup \mathrm{supp}(g)$ (any chain in $\mathrm{supp}(f) \cup \mathrm{supp}(g)$ contains a cofinal chain in $\mathrm{supp}(f)$ or $\mathrm{supp}(g)$). Therefore, $\mathrm{supp}(f+g)$ is reverse well-ordered. Moreover, $\mathrm{LT}(f+g) = \max\{\mathrm{LT}(f), \mathrm{LT}(g)\}$. Indeed, if $\mathrm{LT}(f) < \mathrm{LT}(g)$ then trivially $\mathrm{LT}(f+g) = \mathrm{LT}(g)$, similarly for $\mathrm{LT}(f) > \mathrm{LT}(g)$ and $\mathrm{LT}(f) = \mathrm{LT}(g)$. Note that we have $f+g(\mathrm{LT}(f+g)) \geq 0$. Again we have three cases $\mathrm{LT}(f) < \mathrm{LT}(g)$, $\mathrm{LT}(f) > \mathrm{LT}(g)$ and $\mathrm{LT}(f) = \mathrm{LT}(g)$. If $\mathrm{LT}(f) < \mathrm{LT}(g)$ then

$$f+g(\mathrm{LT}(f+g)) = f+g(\mathrm{LT}(g)) = \\ f(\mathrm{LT}(g)) + g(\mathrm{LT}(g)) = 0 + g(\mathrm{LT}(g)) = g(\mathrm{LT}(g)) \geq 0,$$

similarly for $\mathrm{LT}(f) > \mathrm{LT}(g)$. If $\mathrm{LT}(f) = \mathrm{LT}(g)$ then

$$f+g(\mathrm{LT}(f+g)) = f+g(\mathrm{LT}(g)) = f(\mathrm{LT}(f)) + g(\mathrm{LT}(g)) \geq 0.$$

Finally, it is routine to check that all the axioms of Pr^- are satisfied by $(B(X^\Gamma), 0, <, \mathrm{s}, +)$. $\square$

Let us consider a few instructive examples: If $\Gamma = \{0\} = 1$ and $B = \mathbb{Z}$ then $B(X^\Gamma) = \mathbb{Z}(X^1)$ and $(\mathbb{Z}(X^1), 0, <, \mathrm{s}, +)$ is isomorphic to the natural numbers. If $\Gamma = \{0, 1\} = 2$ and $B = \mathbb{Z}$, then $B(X^\Gamma) = \mathbb{Z}(X^2)$ and $(\mathbb{Z}(X^2), 0, <, \mathrm{s}, +)$ is isomorphic to the non-negative polynomials of degree at most 1 on $\mathbb{Z}$ with the standard order and operations. Similarly, if $\Gamma = \{0, 1, 2\} = 3$ and $B = \mathbb{Z}$, then $B(X^\Gamma) = \mathbb{Z}(X^3)$ and $(\mathbb{Z}(X^\Gamma), 0, <, \mathrm{s}, +)$ is isomorphic to the non-negative polynomials of degree at most 2 over $\mathbb{Z}$ with the standard order and operations, and, more generally for every $0 < n \in \mathbb{N}$, if $\Gamma = n$ and $B = \mathbb{Z}$ then $(\mathbb{Z}(X^n), 0, <, \mathrm{s}, +)$ is isomorphic to the non-negative polynomials of degree at most $n-1$ over $\mathbb{Z}$ with the standard order and operations. Finally, by taking $\Gamma = \mathbb{N}$ and $B = \mathbb{Z}$ we have that $(\mathbb{Z}(X^\mathbb{N}), 0, <, \mathrm{s}, +)$ is isomorphic to the non-negative polynomials over $\mathbb{Z}$ with the standard order and operations. As mentioned in § 5.1.3, this means that the order type of $\mathbb{Z}(X^n)$ is O_n and the order type of $\mathbb{Z}(X^\mathbb{N})$ is O_ω.

Let $(\Gamma, 0, <, +)$ be an ordered commutative additive positive monoid and $(B, 0, 1, <, +, \cdot)$ be an ordered ring. We define a multiplicative structure over $B(X^\Gamma)$ as follows: for $f, g \in B(X^\Gamma)$ let $f \cdot g$ be the following function: if $a \in \Gamma$, then

$$(f \cdot g)(a) := \sum_{b+c=a} f(b) \cdot g(c).$$

We need to prove that this operation is well-defined:

Lemma 5.16. *Let $(\Gamma, 0, <, +)$ be an ordered commutative additive positive monoid and $(B, 0, 1, <, +, \cdot)$ be an ordered commutative ring. The multiplication over $B(X^\Gamma)$ is well-defined.*

Proof. It is enough to show that for each $a \in \Gamma$, there are only finitely many pairs $c, b \in \Gamma$ such that $c + b = a$ and $f(b) > 0$ and $g(c) > 0$. This follows from the fact that $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ are reverse well-founded: Assume that there is an infinite sequence $\langle c_n, b_n \rangle_{n \in \mathbb{N}}$ such that $c_n + b_n = a$, $f(b_n) \neq 0$, $g(c_n) \neq 0$, $c_n \neq c_{n+1}$ and $b_n \neq b_{n+1}$ for all $n \in \mathbb{N}$. We can build strictly increasing sequence either in $\mathrm{supp}(f)$ or in $\mathrm{supp}(g)$. Given a sequence $(s_n)_{n \in \mathbb{N}}$ we call an element s_n of the sequence a *spike* if for all $m > n$ we have $s_n > s_m$. Now consider the sequence $(c_n)_{n \in \mathbb{N}}$ either it has infinitely many spikes or there is n such that there are no spikes after n. If there are infinitely many spikes

$(c_{n_m})_{m\in\mathbb{N}}$ in $(c_n)_{n\in\mathbb{N}}$ then they form an infinite strictly decreasing subsequence of $(c_n)_{n\in\mathbb{N}}$. Therefore, since $c_{n_m} + b_{n_m} = a$ and $c_{n_m} < c_{n_{m+1}}$, the sequence $(b_{n_m})_{m\in\mathbb{N}}$ is a strictly increasing sequence in $\mathrm{supp}(g)$. If there are only finitely many spikes there is trivially a strictly increasing subsequence in $(c_m)_{m\in\mathbb{N}}$. In both cases we obtain a contradiction since $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ are reverse well-founded. $\square$

The following theorem is the PA^--analogue of Theorem 5.15:

Theorem 5.17. *Let $(\Gamma, 0, <, +)$ be an ordered commutative additive positive monoid and $(B, 0, 1, <, +, \cdot)$ be an ordered commutative ring. Then $(B(X^\Gamma), 0, <, \mathrm{s}, +, \cdot)$ is a model of PA^-.*

Proof. Since $(B(X^\Gamma), 0, <, \mathrm{s}, +)$ is a model Pr^-, we only need to prove that $B(X^\Gamma)$ is closed under $\cdot$ and that it satisfies the axioms M1 to M6. Let f and g be two functions in $B(X^\Gamma)$. We want to show $f \cdot g \in B(X^\Gamma)$. First of all note that since $\mathrm{supp}(f \cdot g) = \{a+b\,;\, a \in \mathrm{supp}(f) \text{ and } b \in \mathrm{supp}(g)\}$ then $\mathrm{supp}(f\cdot g)$ is reverse well-founded (by a similar argument as the one in the proof of Lemma 5.16). Note that since $\textsc{lt}(f\cdot g) = \textsc{lt}(f)+\textsc{lt}(g)$, we have that $(f \cdot g)(\textsc{lt}(f \cdot g)) = f(\textsc{lt}(b)) \cdot g(\textsc{lt}(c)) > 0$, since B is an ordered ring where products of positive elements are positive. Thus, $f \cdot g \in B(X^\Gamma)$.

It is again routine to check that the axioms M1 to M6 are satisfied by $B(X^\Gamma)$. $\square$

We end this section by showing that if we require that B is divisible, then the resulting formal power series construction will give a non-standard model of Pr. This matches with Llewellyn-Jones's Theorem 5.19(ii) discussed in the next section.

Theorem 5.18. *Let $(\Gamma, 0, <)$ be a linearly ordered set with a minimum and $(B, 0, <, +)$ be a ordered divisible abelian group. Then $(B(X^\Gamma), 0, <, \mathrm{s}, +)$ is a model of Pr.*

Proof. We already know that $(B(X^\Gamma), 0, <, \mathrm{s}, +)$ is a model of Pr^-. We will show that $(B(X^\Gamma), 0, <, \mathrm{s}, +)$ is a model of Pr^D, i.e., that for every natural number $n > 0$, the structure $(B(X^\Gamma), 0, <, \mathrm{s}, +)$ satisfies D_n.

Let $f \in B(X^\Gamma)$ and $0 < n \in \mathbb{N}$. First note that $\mathbb{Z}$ satisfies D_n for every $n > 0$ therefore there is $z \in \mathbb{Z}$ and a natural number $0 < m < n$ such that $f(0) = zn + m$. Moreover by divisibility of B for every $a \in \Gamma$ there is $b_a \in B$ such that $f(a) = b_a n$. Now, define $g \in B(X^\Gamma)$ as follows:

$$g(x) = \begin{cases} z & \text{if } x = 0, \\ b_x & \text{if } x > 0. \end{cases}$$

It is not hard to see that $f = \mathrm{s}^m(g \cdot n)$ as desired. $\square$

In particular note that if $B = \mathbb{Q}$ and $\Gamma = 2$, then $\mathbb{Q}(X^2)$ is a model of Pr of order type $\mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$. This model is well-known in the literature, see, e.g., [110].

5.4 Presburger arithmetic

Presburger arithmetic, the additive fragment of arithmetic, is closely related to ordered abelian groups. [64] considers an integer version of Presburger arithmetic, allowing for additive inverses and gives an axiomatisation for this theory that we will call $\mathsf{Pr}^\mathbb{Z}$. If

$(M, 0, <, \mathrm{s}, +) \models \mathsf{Pr}^{\mathbb{Z}}$, then $(M, 0, <, +)$ is an ordered abelian group; Llewellyn-Jones calls these groups *Presburger groups.* Llewellyn-Jones proves in his integer setting that G is a Presburger group if and only if G is isomorphic to $\mathbb{Z} \cdot H$ where H is an ordered divisible abelian group [64, §§ 3.1 & 3.2]. In the following, we reformulate Llewellyn-Jones's approach in the standard setting of arithmetic (i.e., without additive inverses).

Theorem 5.19. *Let M be an $\mathcal{L}_{<,\mathrm{s},+}$-structure.*

(i) The structure M is a model of Pr^- if and only if there is an ordered abelian group G such that M is isomorphic to the standard monoid over G, and

(ii) the structure M is a model of Pr if and only if there is an ordered divisible abelian group G such that M is isomorphic to the standard monoid over G.

Proof. This proof is a reformulation of the characterisation of Presburger groups by [64] to the standard setting.

For the forward direction of (i), it is enough to see that in $\mathbb{N} + \mathbb{Z} \cdot G^+$ all the axioms of Pr^- are trivially satisfied. For the other direction, if $M \models \mathsf{Pr}^-$ then by (the proof of) Corollary 5.12, the order type of M is $\mathbb{N} + \mathbb{Z} \cdot A$ for a linear order A consisting of the non-zero archimedean classes of M. For each $a \in A$, we define a formal *negative element* $-a$ such that the negative elements are all distinct from the elements of A and pairwise distinct. Then we define $-A := \{-a\,;\, a \in A\}$ and $G := -A \cup \{[0]\} \cup A$. For notational convenience, we define $-[0] := [0]$. We define an abelian group structure on G as follows:

1. For any $g \in G$, $g + [0] := [0] + g := g$.
2. If $a, b \in A$ are non-zero archimedean classes of M, then there is a unique $c \in A$ such that for all $x \in a$ and $y \in b$, we have that $x + y \in c$; define $a + b := b + a := c$ and $(-a) + (-b) := (-b) + (-a) := -c$.
3. If $a, b \in A$, $x \in A$, and $y \in b$ with $x < y$, then by $\divideontimes$, we find z such that $x + z = y$. Let c be the archimedean class of z, i.e., $c \in A \cup \{[0]\}$. Then $(-a) + b := b + (-a) := c$ and $a + (-b) := (-b) + a := -c$.

It is routine to check that $(G, 0, <, +)$ is an ordered abelian group and that M isomorphic to $\mathbb{N} + \mathbb{Z} \cdot G^+$. For (ii), all that is left to show that that divisibility of the group corresponds to the additional axioms D_n of Pr^{D}. □

Corollary 5.20 (Folklore)**.** *There is a model of Pr with order type $\mathbb{N} + \mathbb{Z} \cdot \mathbb{R}$.*

Proof. The real numbers $\mathbb{R}$ are an ordered divisible abelian group, so by Theorem 5.19 (i), there is a model of Pr with order type $\mathbb{N} + \mathbb{Z} \cdot \mathbb{R}^+$. The claim follows from the fact that $\mathbb{R}^+$ and $\mathbb{R}$ have the same order type. □

Corollary 5.21. *Let M be a non-standard model of Pr. Then M has order type $\mathbb{N} + \mathbb{Z} \cdot A$ where A is a dense linear order without endpoints.*

Proof. It is enough to observe that divisibility implies density and use Theorem 5.19. □

We can use Theorem 5.19 and the general theory of groupable linear orders to get a characterisation theorem for the order types of models of Pr^-. First let us recall a classical result about groupable linear orders; see, e.g., [82, Theorem 8.14]:

Theorem 5.22. *A linear order $(L,<)$ is groupable if and only if there is an ordinal α and a densely ordered abelian group $(D,0,<,+)$ such that L has order type $\mathbb{Z}^\alpha \cdot D$.*

Corollary 5.23. *A structure M is a model of Pr^- if and only if there is an ordinal α and a densely ordered abelian group $(D,0,<,+)$ such that M has order type $\mathbb{N}+\mathbb{Z}\cdot(\mathbb{Z}^\alpha \cdot D)^+$.*

Proof. Follows from Theorems 5.19 & 5.22. □

As we have seen in § 5.3, the non-negative formal power series on $\mathbb{Z}$ with exponent 2 are isomorphic to the ordered abelian monoid of polynomials of degree at most 1 with integer coefficients. Moreover, by Theorem 5.15 (or Theorem 5.19), $(\mathbb{Z}(X^2),0,<,\mathrm{s},+) \models \mathsf{Pr}^-$. The next theorem shows that, in terms of order types, this is a lower bound for non-standard models of Pr^-.

Theorem 5.24. *Let M be a non-standard model of Pr^-. Then M has a submodel isomorphic to $(\mathbb{Z}(X^2),0,<,\mathrm{s},+)$.*

Proof. Let M be a non-standard model of Pr^- and $a \in M$ be a non-standard element of M. define the following mapping $\varphi : \mathbb{Z}(X^2) \to M$:

$$\varphi(f) = \begin{cases} \mathrm{s}^n(0) & \text{if } \textsc{lt}(f) = 0 \text{ and } f(0) = n, \\ \mathrm{s}^m(na) & \text{if } \textsc{lt}(f) = 1 \text{ and } f(1) = n,\ f(0) = m \geq 0, \\ b & \text{if } \textsc{lt}(f) = 1 \text{ and } f(1) = n,\ f(0) = m < 0 \text{ and } \mathrm{s}^{-m}(b) = na. \end{cases}$$

It is easy to see that φ is an order-preserving injection. □

Corollary 5.25. *Let M be a non-standard model of Pr^- then the order $\mathbb{N}+\mathbb{Z}\cdot\mathbb{N}$ can be embedded in the order type of M.*

Proof. As mentioned, $\mathbb{Z}(X^2)$ is the set of non-negative polynomials of degree at most 1 over $\mathbb{Z}$ and clearly has order type $\mathbb{N}+\mathbb{Z}\cdot\mathbb{N}$. The result then follows from Theorem 5.24. □

Corollary 5.26. *Every non-standard model of Pr^- has a proper non-standard submodel.*

Proof. By Theorem 5.24, it is enough to show that $\mathbb{Z}(X^2)$ has a non-standard submodel. Consider all polynomials with degree at most 1 and even leading terms, i.e.,

$$M := \{2nX + z \in \mathbb{Z}(X^2)\,;\, n \in \mathbb{N}, z \in \mathbb{Z}\}.$$

Clearly, this set is closed under s and $+$, so it is a substructure of $\mathbb{Z}(X^2)$. Since the only existential axiom of Pr^- is $\circledast$, it is enough to prove that M satisfies it. Let $f, g \in M$ such that $f < g$. Define $h(a) = g(a) - f(a)$. We want to show that $h \in M$. If $\textsc{lt}(f) = 0$ this is trivially true since $h(1) = g(1)$. If $\textsc{lt}(f) = 1$ then $f(1) = 2n$ and $g(1) = 2n'$ for some $n, n' \in \mathbb{N}$ such that $n < n'$. Then $h(1) = 2n' - 2n = 2(n'-n)$, therefore $h \in M$. The fact that $f + h = g$ follows trivially by the definition of $+$ in $\mathbb{Z}(X^2)$. □

5.5 Peano arithmetic

Theorem 5.19 tells us that every model $M \models \mathsf{PA}^-$ ($M \models \mathsf{PA}$) must have the order type $\mathbb{N}+\mathbb{Z}\cdot G^+$ where G is an ordered (divisible) abelian group. However, in the case of Peano Arithmetic, this cannot be a complete characterisation since [76] proved that no model of PA can have the order type $\mathbb{N}+\mathbb{Z}\cdot\mathbb{R}$. The proof of Potthoff's theorem given by [9, p. 5] easily generalises to PA^-:

Theorem 5.27. *Let M be a non-standard model of PA^- with order type $\mathbb{N}+\mathbb{Z}\cdot A$. If A is dense then there are $|M|$ many non empty disjoint intervals in A. In particular $A \neq \mathbb{R}$.*

Proof. Let $a \in M$ be non-standard. Consider the set $\{a_m\,;\, m \in M\}$ where $a_m = a \cdot m$ for every $m \in M$. By M6, this set has cardinality $|M|$. We will now show that $\{(a_m, a_{\mathrm{s}(m)})\,;\, m \in M\}$ forms a collection of non-empty disjoint intervals of size $|M|$:

By Lemma 5.5, $[a\cdot m] < [a\cdot \mathrm{s}(m)]$ for every $m \in M$. By density of A, the interval $([a_m],[a_{\mathrm{s}(m)}])$ is not empty in A. Now if $m < m'$, then by M6 we have $a\cdot\mathrm{s}(m) \leq a\cdot m'$ and $[a\cdot\mathrm{s}(m)] \leq [a\cdot m']$. Therefore $([a_m],[a_{\mathrm{s}(m)}]) \cap ([a_{m'}],[a_{\mathrm{s}(m')}]) = \varnothing$ as desired.

If $A = \mathbb{R}$, then the order type of M is $\mathbb{N} = \mathbb{Z}\cdot\mathbb{R}$ and hence $|M| = 2^{\aleph_0}$. Now the main claim of the theorem gives us an uncountable family of pairwise disjoint intervals in $\mathbb{R}$ which contradicts the countable chain condition of the real line. □

Theorem 5.27 shows that the closure under multiplication adds more requirements on the order type of models of PA^-. One natural such requirement is the following:

Definition 5.28. Let L be a linear order. We say that L is *closed under finite products of initial segments* if for every $\ell \in L$ the order $\mathrm{IS}(\ell)^\omega$ embeds into $\mathrm{FS}(\ell)$.

Theorem 5.29. *Let M be a non-standard model of PA^- with order type $\mathbb{N}+\mathbb{Z}\cdot L$. Then L is closed under finite products of initial segments.*

Proof. As before, we assume that L is the set of non-zero archimedean classes of M. For every $\ell \in L$ choose a representative $r_\ell \in M$ such that $r_\ell \in \ell$ and $r_\ell > 0$. Let $\ell \in L$ be an element of the linear order L. We want to define an order embedding of $\mathrm{IS}(\ell)^\omega$ into $\mathrm{FS}(\ell)$. Fix some non-standard $a \in M$ such that $\ell \leq [a]$.

Clearly, $\mathrm{IS}(\ell)^\omega$ is order isomorphic to the functions from ω to $\mathrm{IS}(\ell)$ with finite support ordered anti-lexicographically. Consider the following function:

$$\varphi(f) = [\sum_{i \leq \mathrm{LT}(f)} r_{f(i)} \cdot a^{i+1}],$$

for every $f \in \mathrm{IS}(\ell)^\omega$. Note that since f has finite support, φ is well defined. Now we want to prove that φ is order-preserving. First we prove the following claim:

Claim 5.30. *For every $n > 0$ and every finite sequence $\langle \ell_0, \ldots, \ell_{n-1}\rangle$ of elements of $\mathrm{IS}(\ell)$ we have*

$$\sum_{i<n} r_{\ell_i} \cdot a^{i+1} < a^{n+1}.$$

Proof. By induction on n. For $n = 1$ we have $r_{\ell_0} \cdot a < a \cdot a$. For $n = n' + 1 > 1$ we have

$$\begin{aligned}\sum_{i<n'+1} \cdot r_{\ell_i} \cdot a^{i+1} &= \sum_{i<n'} r_{\ell_i} \cdot a^{i+1} + r_{\ell_{n'}} \cdot a^{n'+1} \\ &< a^{n'+1} + r_{\ell_{n'}} \cdot a^{n'+1} \\ &= a^{n'+1} \cdot (\mathrm{s}(0) + r_{\ell_{n'}}) < a^{n'+2}.\end{aligned}$$

□

We want to prove that if $f < f'$ are two elements of $\mathrm{IS}(\ell)^\omega$ then $\varphi(f) < \varphi(f')$. Let $n \in \mathbb{N}$ be the biggest natural number such that $f(n) \neq f'(n)$. Since $f < f'$ we have $f(n) < f'(n)$, then $[r_{f(n)}] < [r_{f'(n)}]$.

Moreover since $n \leq \textsc{lt}(f')$ we have

$$\sum_{n<i\leq \textsc{lt}(f')} r_{f(i)} \cdot a^{i+1} = \sum_{n<i\leq \textsc{lt}(f')} r_{f'(i)} \cdot a^{i+1}.$$

Therefore, by monotonicity of $+$ it is enough to prove that for every $n' \in \mathbb{N}$ we have

$$\sum_{i\leq n} r_{f(i)} \cdot a^{i+1} + \mathrm{s}^{n'}(0) < r_{f'(n)} \cdot a^{n+1}.$$

For $n = 0$ it is trivially true. For $n > 0$, we have

$$\begin{aligned}\sum_{i\leq n} r_{f(i)} \cdot a^{i+1} + \mathrm{s}^{n'}(0) &= \sum_{i<n} r_{f(i)} \cdot a^{i+1} + r_{f(n)} \cdot a^{n+1} + \mathrm{s}^{n'}(0) \\ &< a^{n+1} + r_{f(n)} \cdot a^{n+1} + \mathrm{s}^{n'}(0) \\ &< a^{n+1} \cdot (r_{f(n)} + \mathrm{s}^{n'+1}(0)) \\ &< a^{n+1} \cdot r_{f'(n)},\end{aligned}$$

where we used Claim 5.30 in the first inequality. Therefore φ is order-preserving as desired. □

Theorem 5.17 showed that the non-negative polynomials with integer coefficients $\mathbb{Z}(X^{\mathbb{N}})$ are a model of PA^-. In analogy to Theorem 5.24, we show that this provides a lower bound for the order type of non-standard models of PA^-:

Theorem 5.31. *Let M be a non-standard model of PA^-. Then there is a submodel of M isomorphic to $(\mathbb{Z}(X^{\mathbb{N}}), 0, <, \mathrm{s}, +, \cdot)$.*

Proof. Let M be a non-standard model of PA^- and $a \in M$ be a non-standard element of M. Let $f \in \mathbb{Z}(X^{\mathbb{N}})$; remember that if $\mathrm{supp}(f) \subseteq \{0, \ldots, n\}$ and $\textsc{lt}(f) = n$, then f can be thought of as a polynomial

$$f_n X^n + f_{n-1} X^{n-1} + \ldots + f_0$$

where $f_n > 0$ and $f_i \in \mathbb{Z}$ (for $0 \leq i < n$). We define the function

$$\varphi : \mathbb{Z}(X^{\mathbb{N}}) \to M : f \mapsto f_n a^n + f_{n-1} a^{n-1} + \ldots + f_0$$

where negative terms are unique interpreted by the fact that we have axiom $\divideontimes$. It is routine to check that φ is an embedding of $(\mathbb{Z}(X^{\mathbb{N}}), 0, <, \mathrm{s}, +, \cdot)$ into M. □

Corollary 5.32. *Let M be a non-standard model of* PA^-. *Then the order type O_ω can be embedded in the order type of M. In particular $\mathbb{Z}(X^2)$ is not a model of* PA^-.

Proof. Since O_ω is the order type of the non-negative polynomials on $\mathbb{Z}$, this follows directly from Theorem 5.31. □

Corollary 5.33. *Every non-standard model of* PA^- *has a proper non-standard submodel.*

Proof. As in the proof of Corollary 5.26, by Theorem 5.31, it is enough to check that $\mathbb{Z}(X^{\mathbb{N}})$ has a proper non-standard submodel. Consider the polynomials in which only terms with even exponent occur and observe that they are closed under addition and multiplication and that the structure satisfies ⁕. □

We end this section by showing that our methods give an insight in the number of non-isomorphic order types of models of PA^- of a given cardinality. As we will see, at least in the countable case, the situation is quite different from the one for the theories with induction, Pr and PA.

Lemma 5.34. *Let α and β be two positive ordinals. If $\mathbb{Z}(X^\alpha)$ is order isomorphic to $\mathbb{Z}(X^\beta)$ then $\alpha = \beta$.*

Proof. An easy induction shows that for every ordinal $\gamma > 0$, $\mathbb{Z}(X^\gamma)$ is order isomorphic to O_γ. Now we want to prove that if $0 < \alpha < \beta$ then O_β cannot be order embedded into O_α. First note that for every ordinal $0 < \alpha$ and for every order embedding φ of ω^α into $\mathbb{Z}^\alpha$ we have that φ is cofinal in $\mathbb{Z}^\alpha$. By induction on α. If $\alpha = 1$ or α is limit, the claim is trivially true. For $\alpha = \beta + 1$, let $\varphi : \omega^\beta \cdot \omega \to \mathbb{Z}^\alpha$ be an order embedding. Assume that there is $f \in \mathbb{Z}^\beta \cdot \mathbb{Z}$ such that for every $\gamma \in \omega^\beta \cdot \omega$ we have $\varphi(\gamma) < f$. Then $f = \langle g, z\rangle$ for some $g \in \mathbb{Z}^\beta$ and $z \in \mathbb{Z}$. Without loss of generality we can assume that z is the minimum such that f is an upper bound of φ. For every $\langle \gamma, n\rangle \in \omega^\beta \cdot \omega$ let us denote by $\langle g_{\langle \gamma,n\rangle}, z_{\langle \gamma,n\rangle}\rangle$ the image of $\langle \gamma, n\rangle$ under φ. Note that since for every $n \in \mathbb{N}$, the sequence $\{(\langle \gamma, n\rangle)\,;\,_\gamma \in \omega^\beta\}$ is strictly increasing of order type ω^β, so it is its image. Moreover, since $z \in \mathbb{Z}$ and it is the minimum such that f is an upper bound of φ, there are $n \in \mathbb{N}$ and $\gamma \in \omega^\beta$ such that for every $\gamma' \in \omega^\beta$ if $\gamma < \gamma'$ we have $z_{\langle \gamma,n\rangle} = z_{\langle \gamma',n\rangle} = z$. Finally, since ω^β is additively indecomposable we have that $\{(g_{\langle \gamma',n\rangle})\,;\,\gamma < \gamma' \in \omega^\beta\}$ is a strictly increasing bounded sequence of order type ω^β in $\mathbb{Z}^\beta$. But this contradicts the inductive hypothesis.

Given what we have just proved, it is a routine induction to prove that for every $\alpha > 0$, α is the biggest ordinal such that ω^α can be embedded in O_α.

Therefore, for every $0 < \beta < \alpha$ we have that the order type of $\mathbb{Z}(X^\beta)$ is not isomorphic to the order type of $\mathbb{Z}(X^\alpha)$. □

Theorem 5.35. *There are at least λ^+ non-isomorphic order types of models of* PA^- *of cardinality λ. Therefore, under* GCH *there are exactly 2^λ non isomorphic order types of models of* PA^- *of cardinality λ.*

Proof. Note that for every additively indecomposable ordinal α the structure $(\alpha, <, 0, \oplus)$ where $\oplus$ is the natural addition of ordinals, is an ordered commutative positive monoid. Since for every $\lambda < \alpha < \lambda^+$ we have $\omega^\alpha < \lambda^+$ then there are λ^+ many additively indecomposable ordinals smaller than λ^+. But then, since for every such ordinal ω^α we have that $(\mathbb{Z}(X)^{\omega^\alpha}, 0, <, \mathrm{s}, +, \cdot)$ is a model of PA^- of cardinality λ. Hence there are at least λ^+ non-isomorphic order types of models of PA^- of cardinality λ as desired. □

In particular, note that for $\lambda = \omega$, Theorem 5.35 gives us uncountably many non-isomorphic countable models of PA^- in stark contrast with the two order types of countable models of PA (by Cantor's theorem, $\mathbb{N}$ and $\mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$ are the only possible order types). Moreover, note that none of the order types generated by the proof of Theorem 5.35 satisfy the requirements of Corollary 5.21, and so they cannot be order types of models of Pr (nor of PA). Therefore, we have:

Corollary 5.36. *There are at least λ^+ non-isomorphic order types of models of PA^- of cardinality λ which are not order types of models of Pr or PA.*

5.6 Summary

As mentioned, one of the major open questions in this area is a complete characterisation of the order types of models of PA. For the theories SA and Pr^-, we were able to give complete characterisations in Corollaries 5.12 and 5.23; for the theories Pr and PA^-, we were able to give necessary conditions in Corollary 5.21 and Theorems 5.27 and 5.29, respectively. In particular, the negative results from §§ 5.3 & 5.4 imply:

Corollary 5.37. *There is no model of Pr (and hence, no model of PA) with order type O_2 or O_ω.*

Proof. We have that $O_2 = \mathbb{N} + \mathbb{Z} \cdot \mathbb{N}$ and $O_\omega = \mathbb{N} + \mathbb{Z} \cdot O_\omega$. Clearly, $\mathbb{N}$ and O_ω are not the positive parts of a densely ordered abelian group, so by Corollary 5.21, no model of Pr can have these order types. $\square$

We are now in the position to combine our results to show the separation of the five theories mentioned in § 5.1.1 in terms of order types. In the following diagram, an arrow from a theory T to a theory S means "every order type that occurs in a model of T occurs in a model of S". The diagram is complete in the sense that the absence of an arrow means that no arrow can be drawn, i.e., "there is an order type of a model of T that cannot be an order type of a model of S".

$$\begin{tikzcd}
& \mathsf{SA} & \\
\mathsf{Pr}^- \arrow[ur] & & \mathsf{Pr} \arrow[ul] \arrow[ll] \\
\mathsf{PA}^- \arrow[u] & & \mathsf{PA} \arrow[u] \arrow[ll]
\end{tikzcd}$$

$\mathsf{SA} \not\twoheadrightarrow \mathsf{Pr}^-$ follows from Corollary 5.12 and Corollary 5.25: $\mathbb{N} + \mathbb{Z}$ is an order type witnessing the separation.

$\mathsf{Pr}^- \not\twoheadrightarrow \mathsf{Pr}$ follows from Theorem 5.15 and Corollary 5.37: $\mathbb{N} + \mathbb{Z} \cdot \mathbb{N}$ is an order type witnessing the separation.

$\mathsf{Pr}^- \not\twoheadrightarrow \mathsf{PA}^-$ follows from Theorem 5.15 and Corollary 5.32: $\mathbb{N} + \mathbb{Z} \cdot \mathbb{N}$ is an order type witnessing the separation.

$\mathsf{PA}^- \nrightarrow \mathsf{Pr}$ follows from Theorem 5.17 and Corollary 5.37: $\mathbb{N} + \mathbb{Z} \cdot O_\omega$ is an order type witnessing the separation.

$\mathsf{Pr} \nrightarrow \mathsf{PA}^-$ follows from Theorem 5.27 and Corollary 5.20: $\mathbb{N} + \mathbb{Z} \cdot \mathbb{R}$ is an order type witnessing the separation.

5.7 Open questions

While the results in § 5.6 are a complete description of which theories can be separated by order types, there is no complete description of which order types can occur as order types of models of each theory.

As we have seen, a full characterisation as the one of Lemma 5.12 is still missing for most of the fragments of PA that we have seen in this chapter. The order types of PA and Pr were widely studied in the literature; see, e.g., [8, 63].

Question 5.38. Is there a full characterisation of the class of order types that may occur in models of PA^-?

Finally, as we have seen in § 5.3, models based on generalised power series can be used to generate many order types of models of PA^- which are not models of PA or Pr.

Question 5.39. Let μ be the number of non-isomorphic order types of models of PA^-. Theorem 5.35 tells us that $\lambda^+ \leq \mu \leq 2^\lambda$; and thus that GCH implies $\mu = \lambda^+$. What can we say in general about μ?

In Corollary 5.26 and Corollary 5.33 we proved that non-standard models of Pr^- and PA^- have proper non-standard submodels. The following questions by Anand Pillay (personal communication) are interesting as well:

Question 5.40. What can be said about the strengthening of Corollary 5.26 and Corollary 5.33 in which the submodels are required to be elementary?

Chapter 6

The large cardinal strength of Löwenheim-Skolem theorems

Remarks on co-authorship. *The results of this chapter are due to an ongoing collaboration of the author with Yurii Khomskii and Jouko Väänänen. Therefore, unless otherwise stated, the results of this chapter are due jointly to the author, Yurii Khomskii and Jouko Väänänen.*

6.1 Introduction

The meta-logical properties of first order logic (such as completeness and compactness) imply that there are limits to its expressivity in terms of axiomatisations: many natural classes of structures cannot be axiomatised by first order formulas; e.g., there is no first order formula φ such that a structure $\mathcal{A}$ is a well-order if and only if $\mathcal{A} \models \varphi$, and similarly for the class complete orders, or the real number line.

Of course, all of these classes can easily be defined in first order set theory. For instance, there is a Δ_1 formula Φ such that $\mathcal{A}$ is a well-order if and only if $\Phi(\mathcal{A})$.

If we have a class of models axiomatisable in first order logic by a formula φ, i.e., $\mathrm{Mod}(\varphi) = \{\mathcal{A}\,;\, \mathcal{A} \models \varphi\}$, then, because the satisfaction relation for first order structures is Δ_1, this class is definable by a Δ_1 formula Φ_φ:

$$\mathcal{A} \models \varphi \iff \Phi_\varphi(\mathcal{A}).$$

The example of well-orders above shows that this cannot be reversed: the class of well-orders is Δ_1-definable, but not first order axiomatisable. It is natural to ask whether there is a logic $\mathcal{L}$ stronger than first order logic such that $\mathcal{L}$-axiomatisability of a model class corresponds exactly to its Δ_1-definability.

This question was studied by Väänänen in [101] who introduced the notion of *symbiosis* to study the relationship between axiomatisability and definability: informally, a logic $\mathcal{L}$ and a class of formulas Δ are said to be symbiotic if $\mathcal{L}$-axiomatisability coincides with Δ-definability (for a mathematically precise definition, see Definition 6.16). The concept of symbiosis was studied, e.g., in [4, 5, 68, 102].

In search of our logic $\mathcal{L}$ that is symbiotic with Δ_1, we could consider the logic $\mathcal{L}_{\mathrm{I}} =$

$\mathcal{L}_{\omega\omega}(\mathrm{I})$, obtained from first order logic $\mathcal{L}_{\omega\omega}$ adding the *Härtig quantifier* I defined by

$$\mathcal{A} \models \mathrm{I}xy\ \varphi(x)\psi(y) \text{ if and only if } |\{a \in A\,;\, \mathcal{A} \models \varphi[a]\}| = |\{b \in A\,;\, \mathcal{A} \models \psi[b]\}|.$$

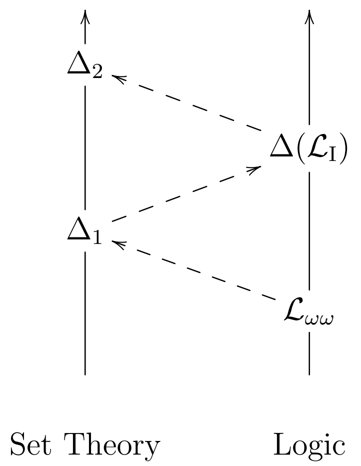
For technical reasons, we will consider a slight strengthening of $\mathcal{L}_\mathrm{I}$, the abstract logic $\Delta(\mathcal{L}_\mathrm{I})$ (see Definition 6.14). Väänänen proved that every Δ_1-definable class closed under isomorphisms is $\Delta(\mathcal{L}_\mathrm{I})$-axiomatisable [102, Examples 2.3]. However, $\Delta(\mathcal{L}_\mathrm{I})$-axiomatisability is now too strong to be symbiotic with Δ_1: the class

$$\{(A, P)\ ;\ |\{x \in A\,;\, P(x)\}| = |\{x \in A\,;\, \neg P(x)\}|\}$$

is not Δ_1, but it is axiomatisable in $\mathcal{L}_\mathrm{I}$, and hence in $\Delta(\mathcal{L}_\mathrm{I})$, by the sentence

$$\mathrm{I}xy(P(x))(\neg P(y)).$$

One can observe that all $\Delta(\mathcal{L}_\mathrm{I})$-axiomatisable classes are Δ_2-definable, but once more, there are Δ_2-definable classes that are not $\Delta(\mathcal{L}_\mathrm{I})$-axiomatisable, yielding the following picture:



Thus, a class of formulas that has a chance to be symbiotic with $\Delta(\mathcal{L}_\mathrm{I})$ will have to lie somewhere strictly between Δ_1 and Δ_2, e.g., $\Delta_1(R)$ for a fixed Π_1 predicate R or finitely many such predicates. Bagaria and Väänänen have provided such an analysis for the classes and logics mentioned in the above diagram: let $\mathcal{L}^2$ be full second order logic and $\mathcal{L}_\mathrm{WO}$ be the logic obtained from $\mathcal{L}_{\omega\omega}$ adding the quantifier WO defined by

$$\mathcal{A} \models_{\mathcal{L}_\mathrm{WO}} \mathrm{WO}xy\varphi(x,y) \text{ if and only if } \{(x,y) \in A \times A\,;\, \mathcal{A} \models_{\mathcal{L}_\mathrm{WO}} \varphi(x,y)\} \text{ is a well-order.}$$

Furthermore, let $\mathrm{Cd}(x)$ be the predicate "x is a cardinal" and $\mathrm{PwSt}(x,y)$ be the predicate "$y = \wp(x)$". Then the following symbiosis relations hold (proved in [5, Lemma 8], [5, Proposition 4], and [5, § 4.1], respectively):

$$\begin{tikzcd}
{} & {} \\
\Delta_2 = \Delta_1(\mathrm{PwSt}) \arrow[u] \arrow[r, dotted, no head] & \Delta(\mathcal{L}^2) \arrow[u] \\
\Delta_1(\mathrm{Cd}) \arrow[u, no head] \arrow[r, dotted, no head] & \Delta(\mathcal{L}_{\mathrm{I}}) \arrow[u, no head] \\
\Delta_1 \arrow[u, no head] \arrow[r, dotted, no head] & \Delta(\mathcal{L}_{\mathrm{WO}}) \arrow[u, no head] \\
{} \arrow[u, no head] & \mathcal{L}_{\omega\omega} \arrow[u, no head] \\
{} \arrow[u, no head] & {} \arrow[u, no head] \\
\text{Set Theory} & \text{Logic}
\end{tikzcd}$$

Symbiosis allows to gauge the precise level of set-theoretic complexity corresponding to axiomatisability and it connects meta-logical properties of the logic (such as the Löwenheim-Skolem theorems) with set-theoretic properties. The following concepts were defined and studied in [4, 5, 68, 102]:

Definition 6.1. The *downward Löwenheim-Skolem-Tarski number* of a logic $\mathcal{L}^*$, in symbols $\mathsf{LST}(\mathcal{L}^*)$, is the smallest cardinal κ such that for all $\varphi \in \mathcal{L}^*$, if $\mathcal{A} \models_{\mathcal{L}^*} \varphi$ then there exists a substructure $\mathcal{B} \subseteq \mathcal{A}$ such that $|B| \leq \kappa$ and $\mathcal{B} \models_{\mathcal{L}^*} \varphi$. If such a number does not exists, then we assume that $\mathsf{LST}(\mathcal{L}^*)$ is undefined.

Definition 6.2. Let R be a Π_1 formula in the language of set theory. The *structural reflection number* $\mathcal{SR}(R)$ is the smallest cardinal κ such that for every $\Sigma_1(R)$-definable class $\mathcal{K}$ of models (in a fixed vocabulary), for every $\mathcal{A} \in \mathcal{K}$ there exists a substructure $\mathcal{B} \subseteq \mathcal{A}$ such that $|B| \leq \kappa$ and $\mathcal{B} \in \mathcal{K}$. If such a number does not exists, then we assume that $\mathcal{SR}(R)$ is undefined.

Theorem 6.3 (Bagaria & Väänänen)**.** *Suppose $\mathcal{L}^*$ and $\Delta_1(R)$ are symbiotic. Then $\mathsf{LST}(\mathcal{L}^*) = \kappa$ if and only if $\mathcal{SR}(R) = \kappa$.*

Proof. See [5, Theorem 6]. □

Theorem 6.3 links a meta-logical property to a reflection principle; usually, in set theory, reflection principles correspond to large cardinals. In fact, structural reflection can be viewed as a particular kind of *Vopěnka's principle.*

Definition 6.4. *Vopěnka's principle* is the statement "for every proper class $\mathcal{K}$ of models of a fixed vocabulary, there are distinct $\mathcal{M}, \mathcal{N} \in \mathcal{K}$ with an elementary embedding $e : \mathcal{M} \preccurlyeq_{\mathcal{L}_{\omega\omega}} \mathcal{N}$."

Variants of Vopěnka's principle have also been considered; e.g., restriction to subsets of $\mathbf{V}_\kappa$ or to classes of a specific complexity. See [4, Section 4] for more on the connection between $\mathcal{SR}$ and Vopěnka principles. The main application of Theorem 6.3 is the computation of the large cardinal strength of the corresponding statements $\mathcal{SR}(R)$ and $\mathsf{LST}(\mathcal{L}^*)$.

In this last chapter of the thesis, we continue the systematic investigation of the concept of *symbiosis* and apply it to *upward* versions of structural reflection numbers and *upward* Löwenheim-Skolem numbers.

6.2 Preliminaries

In this chapter we consider *abstract logics* $\mathcal{L}^*$, i.e., logics with a well-defined model-satisfaction relation but without an effective notion of a syntactic calculus. In this section we will introduce some basic definitions. For a detailed introduction to the subject see [6, Chapter II].

Definition 6.5. A *vocabulary* τ is a tuple $(\mathcal{S}, \mathcal{R}, \mathcal{F}, \mathcal{C}, \mathfrak{A}, \mathfrak{T})$ where $\mathcal{S}$ is a finite set of sort symbols, $\mathcal{R}$ is a set of relation symbols, $\mathcal{F}$ a set of function symbols, $\mathcal{R}$, $\mathcal{F}$, and $\mathcal{C}$ are pairwise disjoint, $\mathfrak{A} : \mathcal{R} \cup \mathcal{F} \cup \mathcal{C} \to \mathbb{N}$ is the arity function, and $\mathfrak{T} : \mathcal{R} \cup \mathcal{F} \cup \mathcal{C} \to \mathcal{S}^{<\omega}$ is the type function. Moreover, $\mathfrak{A}$ and $\mathfrak{T}$ are such that: $\mathfrak{A}(c) = 0$ for each $c \in \mathcal{C}$; $\mathrm{dom}(\mathfrak{T}(x)) = \mathfrak{A}(x) + 1$ for each $x \in \mathcal{C} \cup \mathcal{F}$; and $\mathrm{dom}(\mathfrak{T}(x)) = \mathfrak{A}(x)$ for each $x \in \mathcal{R}$.

We will use the type function to determine the sorts of relations and functions. In particular, given a function symbol $f \in \mathcal{F}$, we will interpret it as a function whose input is a tuple of sorts $\bigtimes_{n<\mathfrak{A}(f)} \mathfrak{T}(f)(n)$, and whose output is of sort $\mathfrak{T}(f)(\mathfrak{A}(f))$. Similarly for relations.

When dealing with vocabularies we will use the standard notation; treating every vocabulary τ as a set of symbols, i.e., we will identify τ with $\mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$.

Of course, since there are no restrictions on the sets occurring in a vocabulary, vocabularies can be arbitrarily complex. Even if one assumes that $\mathcal{S} = 1$, and that $\mathcal{R}$, $\mathcal{F}$, and $\mathcal{C}$ are (disjoint copies of) cardinals, vocabularies can encode complicated sets, as the following example shows:

Let $X \subseteq \omega$ be arbitrary and let $\{x_i\,;\, i \in \omega\}$ be its increasing enumeration. Let $\mathcal{R} = \omega$ be the set of relation symbols with $\mathfrak{A}(i) = x_i$ and $\mathfrak{T}(i)(n) = 0$ for each $n < \mathfrak{A}(i)$. Then the vocabulary $\tau_X := (1, \mathcal{R}, \varnothing, \varnothing, \mathfrak{A}, \mathfrak{T})$ is essentially the set X.

This is a non-desirable side effect which, as we will see, has consequences on the main theorems of this chapter.

Definition 6.6. Given a vocabulary $\tau = (\mathcal{S}, \mathcal{R}, \mathcal{F}, \mathcal{C}, \mathfrak{A}, \mathfrak{T})$ a tuple

$$\mathcal{M} := ((M_s)_{s\in\mathcal{S}}, R^{\mathcal{M}}, F^{\mathcal{M}}, C^{\mathcal{M}})$$

is a τ*-structure* if and only if

1. for every $s \in \mathcal{S}$, the set M_s is non-empty, we will call M_s a *domain* of $\mathcal{M}$;
2. $R^{\mathcal{M}}$ is a function which associates to every $R \in \mathcal{R}$ the set $R^{\mathcal{M}}(R) \subset \bigtimes_{n<\mathfrak{A}(R)} M_{\mathfrak{T}(R)(n)}$;
3. $F^{\mathcal{M}}$ is a function with domain $\mathcal{F}$ such that for every $G \in \mathcal{F}$ we have that $F^{\mathcal{M}}(G)$ is a function from $\bigtimes_{n<\mathfrak{A}(G)} M_{\mathfrak{T}(G)(n)}$ to $M_{\mathfrak{T}(G)(\mathfrak{A}(G))}$;

4. $C^{\mathcal{M}}$ is a function that associates to every $c \in \mathcal{C}$ an element of $M_{\mathfrak{I}(c)(0)}$.

Given a vocabulary τ we will denote by $\mathrm{Str}(\tau)$ the class of *τ-structures.*

If τ is a finite language, we will adopt the usual convention of writing τ-structures just as tuples listing the interpretation of each symbol of τ. E.g, given $\tau := \{s_0, s_1, c, P, G\}$ with c a constant symbol of sort s_0, P a relation symbol of type (s_0, s_1), and G a unary function of type (s_0, s_1). If M and N are non empty sets, $c^{\mathcal{M}}$ is an element of M, $G^{\mathcal{M}}$ is a function in N^M, and $P^{\mathcal{M}}$ is a relation in $M \times N$, we will write $\mathcal{M} = (M, N, c^{\mathcal{M}}, G^{\mathcal{M}}, P^{\mathcal{M}})$ meaning that $\mathcal{M}$ is the τ-structure such that $M_{s_0} = M$, $M_{s_1} = N$, $C^{\mathcal{M}}(c) = c^{\mathcal{M}}$, $R^{\mathcal{M}}(P) = P^{\mathcal{M}}$, and $F^{\mathcal{M}}(G) = G^{\mathcal{M}}$. Moreover, as usual, we will often omit the superscript $\mathcal{M}$ in $(M, N, c^{\mathcal{M}}, G^{\mathcal{M}}, P^{\mathcal{M}})$ writing (M, N, c, G, P).

Given a structure $\mathcal{M}$ in a vocabulary τ we will denote by $|\mathcal{M}|$ the *cardinality of* $\mathcal{M}$, i.e., the cardinality of the union of the domains of $\mathcal{M}$.

Definition 6.7. Given two vocabularies τ and τ' we say that τ is a *finite extension* of τ' if and only if $\tau' \subseteq \tau$ and $\tau \setminus \tau'$ is a finite set.

Definition 6.8. Suppose that $\tau \subseteq \tau'$ are many-sorted vocabularies and that $\mathcal{M}$ is a τ'-structure. The *reduct* or *projection* $\mathcal{M}{\restriction}\tau$ is the structure whose domains are those whose sorts are available in τ, and the interpretation of all symbols not in τ are ignored.

Note that, since we have chosen to have many-sorted vocabularies, a projection $\mathcal{M}{\restriction}\tau$ can have much smaller cardinality than $\mathcal{M}$.

Definition 6.9. Given two vocabularies τ and τ'. A *renaming* from τ onto τ' is a bijection $\varrho : \tau \to \tau'$ which maps sort symbols to sort symbols, relation symbols to relation symbols of the same arity, function symbols to function symbols of the same arity and constant symbols to constant symbols and such that sorts are preserved, i.e., any symbol is mapped to a symbol whose sorts correspond via ϱ.

Every renaming induces a *renaming of structures.* In particular, given a renaming $\varrho : \tau \to \tau'$ and a τ-structure $\mathcal{M}$ we will denote by $\mathcal{M}^{\varrho}$ the τ'-structure obtained by interpreting every symbol with the interpretation of its translation under ϱ.

Definition 6.10. A *pre-logic* $\mathcal{L}^*$ is a pair $(\mathrm{Ste}_{\mathcal{L}^*}, \models_{\mathcal{L}^*})$ where, $\mathrm{Ste}_{\mathcal{L}^*}$ is a (class) function that takes a vocabulary τ and returns a class $\mathcal{L}^*[\tau]$ of sets that we call τ-sentences, and $\models_{\mathcal{L}^*}$ is (class) relation between structures (of all vocabularies) to sentences such that

1. if $\tau \subseteq \tau'$ then $\mathcal{L}^*[\tau] \subseteq \mathcal{L}^*[\tau']$,

2. if $\mathcal{M} \models_{\mathcal{L}^*} \varphi$ and $\mathcal{M}$ is a τ-structure then $\varphi \in \mathcal{L}^*[\tau]$,

3. $\models_{\mathcal{L}^*}$ respects isomorphisms of structures, i.e., isomorphic structures satisfy exactly the same sentences,

4. if $\mathcal{M}$ is a τ-structure and $\tau' \subset \tau$ then for all $\varphi \in \mathcal{L}^*[\tau']$:

$$\mathcal{M} \models_{\mathcal{L}^*} \varphi \Leftrightarrow \mathcal{M}{\restriction}\tau' \models_{\mathcal{L}^*} \varphi$$

5. $\vDash_{\mathcal{L}^*}$ respects renaming of vocabularies, i.e., for every renaming $\varrho : \tau \to \tau'$, every $\varphi \in \mathcal{L}^*[\tau]$ there is $\varphi^\varrho \in \mathcal{L}^*[\tau']$ such that for every every τ-structure $\mathcal{M}$:

$$\mathcal{M} \models_{\mathcal{L}^*} \varphi \text{ iff } \mathcal{M}^\varrho \models_{\mathcal{L}^*} \varphi^\varrho.$$

We remark that the last requirement ensures that in what follows we can ignore the specific coding of vocabularies.

As we will see in Definition 6.15, sentences in a pre-logic need not to have any syntactic structure in the classical sense.

As usual, given a pre-logic $\mathcal{L}^*$, a vocabulary τ and a τ-sentence $\varphi \in \mathcal{L}^*[\tau]$ we will denote by $\mathrm{Mod}_\tau^{\mathcal{L}^*}(\varphi)$ the class of τ-*structures* which satisfy φ. When the pre-logic is clear from the context will write $\mathrm{Mod}_\tau(\varphi)$ and $\mathcal{M} \models \varphi$ instead of $\mathrm{Mod}_\tau^{\mathcal{L}^*}(\varphi)$ and $\mathcal{M} \models_{\mathcal{L}^*} \varphi$.

Definition 6.11. We say that a pre-logic $\mathcal{L}^* = (\mathrm{Ste}_{\mathcal{L}^*}, \models_{\mathcal{L}^*})$ is an *abstract logic* if the following apply:

1. First order logic $\mathcal{L}_{\omega\omega}$ is a sublogic of $\mathcal{L}^*$, i.e., for every vocabulary τ and every τ-sentence $\Phi \in \mathcal{L}_{\omega\omega}[\tau]$, there are τ' and $\psi \in \mathcal{L}^*[\tau']$ such that $\mathrm{Mod}_\tau^{\mathcal{L}_{\omega\omega}}(\varphi) = \mathrm{Mod}_{\tau'}^{\mathcal{L}^*}(\psi)$.

2. For all vocabularies τ and $\varphi \in \mathrm{Ste}_{\mathcal{L}^*}(\tau)$ there is $\psi \in \mathrm{Ste}_{L^*}(\tau)$ such that $\mathrm{Mod}_\tau(\psi) = \mathrm{Str}(\tau) \setminus \mathrm{Mod}_\tau(\varphi)$ (i.e., existence of negation).

3. For all vocabularies τ and τ-sentences $\varphi_0, \varphi_1 \in \mathrm{Ste}_{\mathcal{L}^*}(\tau)$ there are $\psi, \psi' \in \mathrm{Ste}_{\mathcal{L}^*}(\tau)$ such that $\mathrm{Mod}_\tau(\psi) = \mathrm{Mod}_\tau(\varphi_0) \cap \mathrm{Mod}_\tau(\varphi_0)$ and $\mathrm{Mod}_\tau(\psi') = \mathrm{Mod}_\tau(\varphi_0) \cup \mathrm{Mod}_\tau(\varphi_0)$, i.e., the logic is closed under conjunction and disjunction.

4. For all vocabularies τ, sentences $\varphi \in \mathrm{Ste}_{\mathcal{L}^*}(\tau)$, and constant symbols $c \in \tau$, there is a sentence $\psi \in \mathrm{Ste}_{\mathcal{L}^*}(\tau \setminus \{c\})$ such that for every $(\tau \setminus \{c\})$-structure $\mathcal{M} := ((M_s)_{s\in\mathcal{S}}, R^\mathcal{M}, F^\mathcal{M}, C^\mathcal{M})$, we have:

   $$\mathcal{M} \models_{\mathcal{L}^*} \psi \text{ iff } ((M_s)_{s\in\mathcal{S}}, R^\mathcal{M}, F^\mathcal{M}, C_0^\mathcal{M}) \models_{\mathcal{L}^*} \varphi,$$

   where $C_0^\mathcal{M}$ is such that $C_0^\mathcal{M}(c') = C^\mathcal{M}(c')$ for all $c' \neq c$. I.e., the logic is closed under existential quantification.

In the rest of this chapter we will call abstract logics simply *logics*.

Typical examples of logics are *infinitary logics* $\mathcal{L}_{\kappa\lambda}$, *second-order logic* $\mathcal{L}^2$, logics with *generalised quantifiers*, and various combinations thereof.

As we mentioned, infinite vocabularies can code arbitrarily complex sets. Similarly, infinitary logics one can encode complex sets as sentences: e.g., let $A \subseteq \mathbb{N}$ be a subset of natural numbers, τ be a vocabulary, and let $\varphi_0, \varphi_1, \ldots$ be a Δ_0 enumeration of non equivalent $\mathcal{L}_{\omega\omega}[\tau]$ sentences. Take for example φ_n to be a sentence saying "there are exactly $n + 1$ elements in the model". Then, we can define a τ-sentence in $\mathcal{L}_{\omega_1\omega}[\tau]$ as follows: $\psi := \bigwedge_{n\in A} \varphi_n$. Note that, as our example shows the problem arises even if we restrict ourself to finite vocabularies. It is easy to see that ψ encodes A. As for vocabularies we will want to avoid this phenomenon.

Definition 6.12. Let $\mathcal{L}^*$ be any logic. The *dependence number* of $\mathcal{L}^*$, $\mathrm{dep}(\mathcal{L}^*)$, is the least λ such that for any vocabulary τ and any formula $\varphi \in \mathcal{L}^*[\tau]$, there is another vocabulary $\sigma \subseteq \tau$ with $|\sigma| < \lambda$ such that, $\varphi \in \mathcal{L}^*[\sigma]$. If such number does not exists we will assume the dependence number for the logic is undefined.

In Theorem 6.49, we will restrict ourself to logics with dependence number at most ω. This ensures that sentences in these logics are essentially finite objects.

We end this section by defining a version of the upward Löwenheim-Skolem number for abstract logics.

Definition 6.13 (Upward Löwenheim-Skolem number)**.** Let $\mathcal{L}^*$ be a logic.

1. The *upward Löwenheim-Skolem number of $\mathcal{L}^*$ for $<\lambda$-vocabularies*, denoted by $\mathsf{ULST}_\lambda(\mathcal{L}^*)$, is the smallest cardinal κ such that

    for every vocabulary τ with $|\tau| < \lambda$ and every φ in $\mathcal{L}^*[\tau]$, if there is a model $\mathcal{A} \models \varphi$ with $|\mathcal{A}| \geq \kappa$, then for every $\kappa' > \kappa$, there is a model $\mathcal{B} \models \varphi$ such that $|\mathcal{B}| \geq \kappa'$ and $\mathcal{A} \preccurlyeq_{\mathcal{L}_{\omega\omega}} \mathcal{B}$.

    If there is no cardinal satisfying this requirement, we will just assume that $\mathsf{ULST}_\lambda(\mathcal{L}^*)$ is not defined.

2. The *upward Löwenheim-Skolem number* of $\mathcal{L}^*$, denoted by $\mathsf{ULST}_\infty(\mathcal{L}^*)$ is the smallest cardinal κ such that $\mathsf{ULST}_\lambda(\mathcal{L}^*) \leq \kappa$ for all cardinals λ. Once more, if there is no such cardinal we assume that $\mathsf{ULST}_\infty(\mathcal{L}^*)$ is not defined.

In the classical theory of abstract logics, one usually defines the Hanf-number of a logic as the smallest cardinal κ such that

> for every vocabulary τ and every φ in $\mathcal{L}^*[\tau]$, if there is a model $\mathcal{A} \models \varphi$ with $|\mathcal{A}| \geq \kappa$, then for every $\kappa' > \kappa$, there is a model $\mathcal{B} \models \varphi$ such that $|\mathcal{B}| \geq \kappa'$.

One can prove that this number always exists for logics with a dependence number; see, e.g., [6, Theorem 6.4.1]. As we will see in §6.5, the existence of upward Löwenheim-Skolem number for logics, even just for logics with dependence number ω, implies the existence of large cardinals.

6.3 Symbiosis and bounded symbiosis

In this section we will introduce notions of *symbiosis* which, as we will see, will allow us to connect set theoretic upward refelction principles and upward Löwenheim-Skolem theorems. For technical reasons, symbiosis only works with logics that are closed under the Δ-operation; see, e.g., [5, 69, 102].

Definition 6.14. Let $\mathcal{L}^*$ be a logic. A class $\mathcal{K}$ of τ-structures is $\Sigma(\mathcal{L}^*)$-*axiomatisable* if there are a *finite* extension $\tau' \supseteq \tau$ and a $\varphi \in \mathcal{L}^*[\tau']$ such that

$$\mathcal{K} = \{\mathcal{A}\,;\, \exists\mathcal{B}\ (\mathcal{B} \models \varphi \text{ and } \mathcal{A} = \mathcal{B}{\upharpoonright}\tau)\}.$$

A class $\mathcal{K}$ is $\Delta(\mathcal{L}^*)$-*axiomatisable* if both $\mathcal{K}$ and its complement (i.e., the class of τ-structures not in $\mathcal{K}$) are $\Sigma(\mathcal{L}^*)$-axiomatisable.

It is easy to see that these classes are closed under union, intersection, and projection. Therefore, one can define the following abstract logic:

Definition 6.15. Given a logic $\mathcal{L}^*$, we define the abstract logic $\Delta(\mathcal{L}^*)$ as follows: for each vocabulary τ, the class $\Delta(\mathcal{L}^*)[\tau]$ of τ-sentences consists of the $\Delta(\mathcal{L}^*)$-axiomatisable classes in τ, and for every τ-structure and every $\mathcal{K} \in \Delta(\mathcal{L}^*)[\tau]$ we define $\mathcal{M} \models \mathcal{K}$ if and only if $\mathcal{M} \in \mathcal{K}$.

It is clear that $\Delta(\Delta(\mathcal{L}^*)) = \Delta(\mathcal{L}^*)$. For classical logic, $\Delta(\mathcal{L}^*)$-axiomatisability coincides with $\mathcal{L}^*$-axiomatisability, and in general this holds for any logic satisfying the Craig Interpolation theorem see [69, Lemma 2.7].

Definition 6.16 (Symbiosis). Let $\mathcal{L}^*$ be a logic and R be a predicate in the language of set theory. Then $\mathcal{L}^*$ and R are *symbiotic* if and only if

1. the relation $\models_{\mathcal{L}^*}$ is $\Delta_1(R)$-definable and
2. every $\Delta_1(R)$ class of τ-structures closed under isomorphisms is $\Delta(\mathcal{L}^*)$-axiomatisable.

As Lemma 6.17 below shows, the second condition of symbiosis is equivalent to a statement which, in practice, is easier both to verify and to apply. Let R be an n-ary predicate in the language of set theory. We say that a transitive model of set theory M is *R-correct* if for all $m_1, \dots, m_n \in M$ we have $M \models R(m_1, \dots, m_n)$ iff $R(m_1, \dots, m_n)$. Note that the statement "M is R-correct" is $\Delta_1(R)$.

Lemma 6.17. *Let R be an n-ary predicate in the language of set theory. The following are equivalent:*

1. *Every $\Delta_1(R)$ class of τ-structures closed under isomorphisms is $\Delta(\mathcal{L}^*)$-axiomatisable.*
2. *The class $\mathcal{Q}_R := \{\mathcal{A}\,;\, \mathcal{A} = (A, E, a_1, \dots, a_n)$ is isomorphic to a transitive model $(M, \in, m_1, \dots, m_n)$ and $R(m_1, \dots, m_n)\}$ is $\Delta(\mathcal{L}^*)$-axiomatisable.*

Proof. See [5, Proposition 3]. $\square$

Although symbiosis is stated as a property of $\mathcal{L}^*$, it is really a property of $\Delta(\mathcal{L}^*)$. For many applications, this is irrelevant: for example, downward Löwenheim-Skolem theorems are all preserved by the Δ-operation. However, in [103, Theorem 4.1] it was shown that the Hanf-number may not be preserved, and the *bounded* Δ-operation was introduced as a closely related operation which still fulfils most of the properties of the Δ-operation but, in addition, preserves Hanf-numbers (see [103]), and coincides with the original Δ in many but not all cases. As we will see in Theorem 6.49, similarly to the case of Hanf-numbers, the bounded Δ-operation will make symbiosis work with upward Löwenheim-Skolem theorems.

Definition 6.18 ([103, p. 45])**.** A class $\mathcal{K}$ of τ-structures is $\Sigma^{\mathrm{B}}(\mathcal{L}^*)$-*axiomatisable* if there is a finite extension $\tau' \supseteq \tau$ and a $\varphi \in \mathcal{L}^*[\tau']$ such that

$$\mathcal{K} = \{\mathcal{A}\,;\, \exists \mathcal{B}\ (\mathcal{B} \models \varphi \text{ and } \mathcal{A} = \mathcal{B}{\restriction}\tau)\},$$

and for all $\mathcal{A}$ there exists a cardinal $\lambda_{\mathcal{A}}$ such that for any τ'-structure $\mathcal{B}$: if $\mathcal{B} \models \varphi$ and $\mathcal{A} = \mathcal{B}{\restriction}\tau$ then $|\mathcal{B}| \leq \lambda_{\mathcal{A}}$. $\mathcal{K}$ is $\Delta^{\mathrm{B}}(\mathcal{L}^*)$-*axiomatisable* if both $\mathcal{K}$ and its complement are $\Sigma^{\mathrm{B}}(\mathcal{L}^*)$-axiomatisable.

Lemma 6.19. *Let $\mathcal{K}$ be a $\Sigma^{\mathrm{B}}(\mathcal{L}^*)$-axiomatisable class of τ-structure. Then there is a non-decreasing function $h:$ Ord $\to$ Ord such that:*

$$\forall \mathcal{A} \in \mathcal{K} \forall \mathcal{B} (\mathcal{A} = \mathcal{B}{\restriction}\tau \to |\mathcal{B}| \leq h(|\mathcal{A}|)).$$

Proof. Define h as follows:

$$h(|\mathcal{A}|) := \sup\{\lambda_{\mathcal{A}'}\,;\, \mathcal{A}' = \mathcal{B}{\restriction}\tau \ \wedge \ |\mathcal{A}'| \leq |\mathcal{A}|\}.$$

Where $\lambda_{\mathcal{A}'}$ is as in Definition 6.18. Note that since there are only set-many non-isomorphic models of any cardinality h is well-defined. Moreover it is an easy to see that h is indeed non-decreasing and has the desired property. □

The fact that we have a notion of bounded definability for abstract logics requires a corresponding change on the set-theoretic side as well. We will now define two notions of bounded definability and we will associate a notion of symbiosis to each of them. The first definition of bounded $\Sigma_1(R)$ formulas that we will present was introduced by Väänänen in [102, Definition 3.1].

Definition 6.20. Given a Π_1 predicate R, a formula $\psi(x_1, \ldots, x_n)$ is $\Sigma_1^{\mathrm{SB}}(R)$ if there is a $\Delta_0(R)$ formula $\varphi(x_1, \ldots, x_n, y)$ such that:

$$\forall x_1, \ldots, x_n \, (\psi(x_1, \ldots, x_n) \leftrightarrow \exists y \, (\varrho_{\mathbf{H}}(y) \leq \varrho_{\mathbf{H}}(x_1, \ldots, x_n) \wedge \varphi(x_1, \ldots, x_n, y)))$$

where $\varrho_{\mathbf{H}}(x_1, \ldots, x_n) := \max\{\aleph_0, |\mathrm{TC}(\{x_1, \ldots, x_n\})|\}$. A formula is $\Pi_1^{\mathrm{SB}}(R)$ if its negation is $\Sigma_1^{\mathrm{SB}}(R)$, and $\Delta_1^{\mathrm{SB}}(R)$ if it is both $\Sigma_1^{\mathrm{SB}}(R)$ and $\Pi_1^{\mathrm{SB}}(R)$.

Definition 6.21. A non-decreasing function $F : \mathrm{Ord} \to \mathrm{Ord}$ such that for every cardinal $F(\mu) \geq \max\{2^{\aleph_0}, 2^{\mu}\}$ is *definably bounding* iff the class of structures

$$\mathcal{K} := \{(A, B)\,;\, F(|A|) \geq |B|\}$$

in the vocabulary with only two sorts symbols and no other symbol, is $\Sigma^{\mathrm{B}}(\mathcal{L}_{\omega\omega})$-axiomatisable.

Definition 6.22. Given a function $F : \mathrm{Ord} \to \mathrm{Ord}$ we recursively define a family of functions as follows:

$$F^1(x) := F(x) \text{ for every } x \in \mathrm{Ord};$$
$$F^{n+1}(x) := F(F^n(x)) \text{ for every } x \in \mathrm{Ord}.$$

Lemma 6.23. *Let F be a definably bounding function. Then, for every $n > 0$, F^n is definably bounding.*

Proof. We prove the claim by induction on n. The statement is trivial for $n = 1$. For the successor case, assume that F^n is a definably bounding function. Note that $F(F^n(\mu)) \geq \max\{2^{\aleph_0}, 2^{F^n(\mu)}\} \geq \max\{2^{\aleph_0}, 2^{\mu}\}$. Now, by assumption

$$\mathcal{K}_1 := \{(A, C)\,;\, F(|A|) \geq |C|\}$$

and

$$\mathcal{K}_n := \{(C, B)\,;\, F^n(|C|) \geq |B|\}$$

are $\Sigma^{\mathrm{B}}(\mathcal{L}_{\omega\omega})$. We need to show that

$$\mathcal{K}_{n+1} := \{(A, B)\,;\, F(F^n(|A|)) \geq |B|\}$$

is also $\Sigma^{\mathrm{B}}(\mathcal{L}_{\omega\omega})$. Consider the following class of structures:

$$\mathcal{K}' := \{(A, B, C)\,;\, F(|A|) \geq |C| \wedge F^n(|C|) \geq |B|\}.$$

We claim that this class is also $\Sigma^{\mathrm{B}}(\mathcal{L}_{\omega\omega})$-axiomatisable. Indeed, let τ_1 and τ_n be the vocabularies needed to $\Sigma^{\mathrm{B}}(\mathcal{L}_{\omega\omega})$-axiomatise $\mathcal{K}_1$ and $\mathcal{K}_n$, respectively. Without loss of generality we can assume that $\tau_1 \cap \tau_n = \{C\}$. Moreover, let φ_1 and φ_n be the formulas needed to $\Sigma^{\mathrm{B}}(\mathcal{L}_{\omega\omega})$-axiomatise $\mathcal{K}_1$ and $\mathcal{K}_n$, respectively. Then, it is easy to see that $\mathcal{K}'$ is the projection of the $\tau := \tau_1 \cup \tau_n$ class $\mathcal{K}^* := \{\mathcal{A}\,;\, \mathcal{A} \models \varphi_1 \wedge \varphi_n\}$. Finally note that the class $\mathcal{K}_{n+1}$ the projection of $(A, B, C) \in \mathcal{K}'$, and therefore is also $\Sigma^{\mathrm{B}}(\mathcal{L}_{\omega\omega})$-axiomatisable. $\square$

The following instances of definably bounding functions are enough for many interesting applications. For every cardinal λ we define:

$$\begin{aligned}
\beth_0(\lambda) &:= \max(\lambda, \aleph_0);\\
\beth_{n+1}(\lambda) &:= 2^{\beth_n(\lambda)};\\
\beth_\omega(\lambda) &:= \sup\{\beth_n(\lambda)\,;\, n < \omega\}.
\end{aligned}$$

Corollary 6.24. *For every $n \in \mathbb{N}^+$, the class function $\beth_n$ is definably bounding.*

Proof. By Lemma 6.23 it is enough to prove that $\beth_1$ is definably bounding. Note that $\beth_1(\mu) \geq \max\{2^{\aleph_0}, 2^\mu\}$ for every cardinal μ. We only need to prove that $\mathcal{K}_0 := \{(A, B)\,;\, \beth_1(|A|) \geq |B|\}$ is $\Sigma^{\mathrm{B}}(\mathcal{L}_{\omega\omega})$-axiomatisable.

Let τ be the vocabulary with only two sorts and no other symbol. Consider the class $\mathcal{K}^*$ of structures in the vocabulary $\tau_0 := \tau \cup \{R_0\}$ where $R_0 \subset A \times B$ is a binary relation satisfying the $\mathcal{L}_{\omega\omega}$ formula:

$$\forall a, a' \in A((\forall b \in B(bR_0a \leftrightarrow bR_0a')) \to a = a').$$

It is easy to see that for every $(A, B, R_0) \in \mathcal{K}^*$ we have $|A| \leq 2^{|B|}$ and that $\mathcal{K}_0$ is then the projection of $\mathcal{K}^*$ over τ. $\square$

Definition 6.25. Let R be a Π_1 predicate in set theory. A formula $\psi(x_1, \ldots, x_n)$ in set theory is $\Sigma_1^{\mathrm{B}}(R)$ if there exists a $\Delta_0(R)$ formula $\varphi(x_1, \ldots, x_n, y)$ and a definably bounding function F such that:

$$\forall x_1, \ldots, x_n\, (\psi(x_1, \ldots, x_n, y) \leftrightarrow \exists y\, (\varrho_{\mathbf{H}}(y) < F(\varrho_{\mathbf{H}}(x_1, \ldots, x_n)) \wedge \varphi(x_1, \ldots, x_n, y))).$$

A formula is $\Pi_1^{\mathrm{B}}(R)$ if its negation is $\Sigma_1^{\mathrm{B}}(R)$, and $\Delta_1^{\mathrm{B}}(R)$ if it is both $\Sigma_1^{\mathrm{B}}(R)$ and $\Pi_1^{\mathrm{B}}(R)$.

Note that, as for their unbounded counterpart, it is not hard to see that $\Delta_1^{\mathrm{B}}(R)$ formulas are closed under $\wedge$, $\vee$, $\neg$, and bounded quantification, while $\Sigma_1^{\mathrm{B}}(R)$ formulas are closed under $\wedge$, $\vee$ and bounded quantification.

This leads us to introduce two new notions of symbiosis. The first of these definitions is due to Väänänen, see [102, Definition 3.3].

Definition 6.26 (**Strongly Bounded Symbiosis**)**.** Let $\mathcal{L}^*$ be a logic and R a set theoretic predicate. We say that $\mathcal{L}^*$ and R are *strongly boundedly symbiotic* if

1. the satisfaction relation $\models_{\mathcal{L}^*}$ is $\Delta_1^{\mathrm{SB}}(R)$, and
2. the class $\overline{\mathcal{Q}}_R := \{\mathcal{A};\, \mathcal{A} = (A, E) \cong (M, \in)$ with M transitive and R-correct$\}$ is $\Delta^{\mathrm{B}}(\mathcal{L}^*)$-axiomatisable.

We will now introduce a weaker version of strongly bounded symbiosis, called bounded symbiosis. Before we do so, we need some preliminary lemmas.

Lemma 6.27 (Lévy)**.** *Let R be a Π_1 set-theoretic predicate and κ be an uncountable cardinal. Then $\mathbf{H}_\kappa$ is R-correct.*

Proof. We need to show that for every $a_1, \dots, a_n \in \mathbf{H}_\kappa$, $R(a_1, \dots, a_n)$ if and only if $\mathbf{H}_\kappa \models R(a_1, \dots, a_n)$, or equivalently that $\neg R(a_1, \dots, a_n)$ if and only if $\mathbf{H}_\kappa \models \neg R(a_1, \dots, a_n)$. Thus, it is enough to show that Σ_1-formulas are absolute between $\mathbf{V}$ and $\mathbf{H}_\kappa$. Let $\varphi(x_1, \dots, x_n, y)$ be a Δ_0 predicate and $\psi(x_1, \dots, x_n) := \exists y \varphi(x_1, \dots, x_n, y)$. We want to show that $\psi(x_1, \dots, x_n)$ is downward-absolute. Let $a_1, \dots, a_n \in \mathbf{H}_\kappa$ be such that $\psi(a_1, \dots, a_n)$ holds. Let α be such that $\mathbf{V}_\alpha \models \exists y \varphi(a_1, \dots, a_n, y)$ and $\mathcal{M}$ be elementary sub $\mathbf{V}_\alpha$ containing $a_1, \dots, a_n$ of size $\max\{\aleph_0, |\mathrm{TC}(\{a_1, \dots, a_n\})|\} < \kappa$. Then, the transitive collapse of $\mathcal{M}$ is in $\mathbf{H}_\kappa$. But since $\mathbf{V}_\alpha \models \psi(a_1, \dots, a_n)$ then $\mathcal{M} \models \psi(a_1, \dots, a_n)$ and by upward-absoluteness of Σ_1-formulas $\mathbf{H}_\kappa \models \psi(a_1, \dots, a_n)$ as desired. $\square$

Lemma 6.28. *Let R be a Π_1 set-theoretic predicate and κ be an uncountable cardinal. Let M be a transitive set which is R-correct. Then every $\Sigma_1(R)$ formula is upward-absolute between M and the universe. In particular, every $\Sigma_1^{\mathrm{B}}(R)$ formula is upward-absolute between M and the universe.*

Proof. Since by assumptions $\Delta_0(R)$ formulas are absolute, the usual proof of Σ_1 upward-absoluteness works for $\Sigma_1(R)$. $\square$

Lemma 6.29. *Let R be a Π_1 set-theoretic predicate, φ be a $\Sigma_1^{\mathrm{B}}(R)$ formula and F be the definably bounding function for φ. Then for every tuple $a_1, \dots, a_n$ such that $\varphi(a_1, \dots, a_n)$ and for every $\kappa > F(|\mathrm{TC}(\{a_1, \dots, a_n\})|)$ we have $\mathbf{H}_\kappa \models \varphi(a_1, \dots, a_n)$.*

Proof. By definition φ is equivalent to a formula $\exists y\ \psi(x_1, \dots, x_n, y)$ where ψ is $\Delta_0(R)$. So ψ is downward-absolute between transitive sets. Note that $a_1, \dots, a_n \in \mathbf{H}_\kappa$ and that, since $\exists y\ \psi(a_1, \dots, a_n, y)$, there is b such that

$$\varrho_{\mathbf{H}}(b) \leq F(\varrho_{\mathbf{H}}(a_1, \dots, a_n)) = F(|\mathrm{TC}(\{a_1, \dots, a_n\})|) < \kappa,$$

and $\psi(a_1, \dots, a_n, b)$. Then $b \in \mathbf{H}_\kappa$. Moreover, since $\psi(a_1, \dots, a_n, b)$ and $\Delta_0(R)$ formulas are downward-absolute, we have that $\mathbf{H}_\kappa \models \psi(a_1, \dots, a_n, b)$ and therefore $\mathbf{H}_\kappa \models \varphi(a_1, \dots, a_n)$ as desired. $\square$

Lemma 6.30. *Let $\mathcal{L}^*$ be a logic and R be a Π_1 predicate. Then the following are equivalent:*

1. *every $\Sigma_1^{\mathrm{B}}(R)$ class of τ-structures closed under isomorphisms is $\Sigma^{\mathrm{B}}(\mathcal{L}^*)$-axiomatisable,*

2. *every* $\Sigma_1^{\mathrm{SB}}(R)$ *class of* τ*-structures closed under isomorphisms is* $\Sigma^{\mathrm{B}}(\mathcal{L}^*)$*-axiomatisable,*

3. *the class* $\mathcal{Q}_R := \{\mathcal{A}\,;\, \mathcal{A} = (A, E, a_1, \ldots, a_n) \cong (M, \in, m_1, \ldots, m_n)$ *with* M *transitive and* $R(m_1, \ldots, m_n)\}$ *is* $\Delta^{\mathrm{B}}(\mathcal{L}^*)$*-axiomatisable,*

4. *the class* $\overline{\mathcal{Q}}_R := \{\mathcal{A}\,;\, \mathcal{A} = (A, E) \cong (M, \in)$ *with* M *transitive and* R*-correct*$\}$ *is* $\Delta^{\mathrm{B}}(\mathcal{L}^*)$*-axiomatisable.*

Proof. The implication $1 \to 2$ is immediate. To see that $2 \to 3$ it is enough to observe that $\mathcal{Q}_R$ is actually $\Delta_1^{\mathrm{SB}}(R)$-definable. Indeed, note that $\mathcal{A} \in \mathcal{Q}_R$ iff

$$\exists (M, \in, a_1, \ldots, a_n)\, (M \text{ transitive } \wedge \mathcal{A} \cong (M, \in, a_1, \ldots, a_n) \wedge R(a_1, \ldots, a_n)).$$

Which, since $(M, \in, a_1, \ldots, a_n)$ can be chosen in $\mathbf{H}_{\varrho_{\mathbf{H}}(\mathcal{A})^+}$, is $\Sigma_1^{\mathrm{SB}}(R)$. Similarly, $\mathcal{A} \notin \mathcal{Q}_R$ iff

$$\begin{aligned}&\mathcal{A} \text{ is not a well-founded extensional structure } \vee \\ &\quad \exists (M, \in, a_1, \ldots, a_n)\, (M \text{ transitive } \wedge \mathcal{A} \cong (M, \in, a_1, \ldots, a_n) \wedge \neg R(a_1, \ldots, a_n)).\end{aligned}$$

The displayed formula is again $\Sigma_1^{\mathrm{SB}}(R)$.

To show $3 \to 4$ let R be of arity $n+1$. Note that the same proof works for 1-ary relations. Let τ be the vocabulary consisting of one sort s_0, a binary relation symbol E and $n+1$ constant symbols $a_0, \ldots, a_n$. Assume that $\mathcal{Q}_R := \{\mathcal{A}\,;\, \mathcal{A} = (A, E, a_0^{\mathcal{A}}, \ldots, a_n^{\mathcal{A}}) \cong (M, \in, m_0, \ldots, m_n)$ with M transitive and $R(m_0, \ldots, m_n)\}$ is $\Delta^{\mathrm{B}}(\mathcal{L}^*)$-axiomatisable. This means that there is a finite extension τ' of the language τ of $\mathcal{Q}_R$ and a τ'-sentence Ψ such that

$$\mathcal{A} \in \mathcal{Q}_R \text{ iff there is a } \tau'\text{-structure } \mathcal{B} \text{ such that } \mathcal{B} \models \Psi \text{ and } \mathcal{A} = \mathcal{B}{\restriction}\tau.$$

Let $\Psi'(x_0, \ldots, x_n)$ be the τ'-formula obtained by substituting[1] in Ψ the constants $a_0, \ldots, a_n$ with fresh variables $x_0, \ldots, x_n$ of sort s_0. Let $\tau'_{\{a_0,\ldots,a_n\}}$ be the vocabulary $\tau' \setminus \{a_0, \ldots, a_n\}$. Note that the class $\mathcal{K}$ of $\tau'_{\{a_0,\ldots,a_n\}}$-structures $\mathcal{M}$ such that

$$\mathcal{M} \models \forall x_0, \ldots, x_n R(x_0, \ldots, x_n) \leftrightarrow \Psi'(x_0, \ldots, x_n)$$

with R written with E instead of $\in$ is $\mathcal{L}^*$-axiomatisable.

Now, we have

$$\begin{aligned}\mathcal{M} = (M, E, \ldots) \in \mathcal{K} &\Leftrightarrow \mathcal{M} \models \forall x_0, \ldots, x_n R(x_0, \ldots, x_n) \leftrightarrow \Psi'(x_0, \ldots, x_n) \\ &\Leftrightarrow \forall a_0^{\mathcal{M}}, \ldots, a_n^{\mathcal{M}} \in M (\mathcal{M} \models R(a_0^{\mathcal{M}}, \ldots, a_n^{\mathcal{M}}) \\ &\qquad\qquad \Leftrightarrow \mathcal{M} \models \Psi'(a_0^{\mathcal{M}}, \ldots, a_n^{\mathcal{M}})) \\ &\Leftrightarrow \forall a_0^{\mathcal{M}}, \ldots, a_n^{\mathcal{M}} \in M ((M, E) \models R(a_0^{\mathcal{M}}, \ldots, a_n^{\mathcal{M}}) \Leftrightarrow \\ &\qquad\qquad (M, \ldots, E, \ldots, a_0^{\mathcal{M}}, \ldots, a_n^{\mathcal{M}}) \in \mathcal{Q}_R) \\ &\Leftrightarrow (M, E) \cong (N, \in) \text{ and } N \text{ is } R\text{-correct.}\end{aligned}$$

Therefore, $\overline{\mathcal{Q}}_R$ is the projection of $\mathcal{K}$, and is therefore $\Delta^{\mathrm{B}}(\mathcal{L}^*)$-axiomatisable.

[1]Note that all the manipulations of sentences in this proof can be performed because $\mathcal{L}^*$ is a logic and is therefore closed under existential quantification, negation and disjunction. By abuse of notation we will work as if the logic $\mathcal{L}^*$ as a syntax analogous to the one of first order logic.

Finally, we will show $4 \to 1$. Let $\mathcal{K}$ be a $\Sigma_1^{\mathrm{B}}(R)$-definable class over the vocabulary τ which is closed under isomorphism, $\Phi(x)$ be the $\Sigma_1^{\mathrm{B}}(R)$ formula defining $\mathcal{K}$, and F be a definably bounding function for Φ. Without loss of generality we will assume that τ consists only of one binary predicate P and one sort; a similar proof works in the general case. Let τ' be the language in two sorts s_0 and s_1, with E a binary relation symbol of sort s_1, G a function symbol from s_0 to s_1, c a constant symbol of sort s_1, and P a predicate in s_0. Let $\mathcal{K}'$ be the class of all structures $\mathcal{M} := \left(M, N, E^{\mathcal{M}}, c^{\mathcal{M}}, G^{\mathcal{M}}, P^{\mathcal{M}}\right)$ satisfying the following:

1. $(M, E) \in \overline{\mathcal{Q}}_R$, i.e., is isomorphic to a transitive model which is R-correct,
2. $(M, E) \models \mathsf{ZFC}_n^-$ for n big enough so that the argument will go through,
3. $(M, E) \models \Phi(c)$,
4. $|M| \leq F^3(|N|)$
5. $(M, E) \models$ "$c = (a, b)$ and $b \subset a \times a$"
6. $(M, E) \models$ "G is an isomorphism between (N, P) and (a, b)"

Note that, by our assumption and by the fact that F is definably bounding, we have that $\mathcal{K}'$ is $\Sigma^{\mathrm{B}}(\mathcal{L}^*)$-axiomatisable.

Moreover, $\mathcal{K}$ is the projection of $\mathcal{K}'$. Indeed, let $\mathcal{M} \in \mathcal{K}'$. Then by (1) in the definition of $\mathcal{K}'$ we have that $(M, E^{\mathcal{M}})$ is isomorphic to a transitive model $(\overline{M}, \in)$ which is R-correct. Let $\overline{c^{\mathcal{M}}}$ be the image of $c^{\mathcal{M}}$ under the isomorphism between M and $\overline{M}$. Then $(\overline{M}, \in) \models \Phi(\overline{c^{\mathcal{M}}})$. Moreover $\overline{M}$ is R-correct and since Φ is $\Sigma_1^{\mathrm{B}}(R)$, by Lemma 6.28, we have that Φ is upward-absolute. Hence, $\overline{c^{\mathcal{M}}} \in \mathcal{K}$. Now, by (6) in the definition of $\mathcal{K}'$ we have $\overline{c^{\mathcal{M}}} \cong c^{\mathcal{M}} \cong (N, P^{\mathcal{M}})$. Finally, since $\mathcal{K}$ is closed under isomorphism, we have $(N, P^{\mathcal{M}}) \in \mathcal{K}$.

On the other hand, let $\mathcal{A} \in \mathcal{K}$. We want to find a structure $\mathcal{N} \in \mathcal{K}'$ such that $\mathcal{A} = \mathcal{N} \restriction \tau$. Let $\bar{\mathcal{A}}$ be isomorphic to $\mathcal{A}$ and such that its domain is the cardinal $\mu := |A|$. Moreover, let $f : \bar{\mathcal{A}} \to \mathcal{A}$ be the isomorphism between $\bar{\mathcal{A}}$ and $\mathcal{A}$. Note that $\varphi(\bar{\mathcal{A}})$ holds since $\mathcal{K}$ is closed under isomorphism. Let ϑ be the cardinal $\vartheta := F^2(\mu)$. Now, let M be such that $\mathcal{A} \in M$ and $|M| = |\mathbf{H}_\vartheta|$, and let f' be a bijection between $\mathbf{H}_\vartheta$ and M such that $f' \restriction A = f$. Define $E^{\mathcal{M}}$ as follows: $\forall a, b \in M \; aEb \Leftrightarrow f'(a) \in f'(b)$.

It is easy to see that $(M, E^{\mathcal{M}})$ is isomorphic to $(\mathbf{H}_\vartheta, \in)$. Therefore, $|M| = |\mathbf{H}_\vartheta| \leq F^3(|A|)$. Moreover, by Lemma 6.29, we have that $\mathbf{H}_\vartheta \models \Phi(\bar{\mathcal{A}})$. So $(M, \mathcal{A}, E) \models \Phi(\mathcal{A})$. Let $\mathcal{N} := (M, A, E^{\mathcal{M}}, \mathcal{A}, \mathrm{id}, P^{\mathcal{A}})$. Then, by what we have just proved $\mathcal{N}$ satisfies (1)-(6) in the definition of $\mathcal{K}'$. Moreover, by Corollary 6.27 $\mathcal{N}$ satisfies (1) which means that $\mathcal{N} \in \mathcal{K}'$ as desired. $\square$

Corollary 6.31. *Let $\mathcal{L}^*$ be a logic and R be a Π_1 predicate. Then the following are equivalent:*

1. *every $\Delta_1^{\mathrm{B}}(R)$ class of τ-structures closed under isomorphisms is $\Delta^{\mathrm{B}}(\mathcal{L}^*)$-axiomatisable,*
2. *every $\Delta_1^{\mathrm{SB}}(R)$ class of τ-structures closed under isomorphisms is $\Delta^{\mathrm{B}}(\mathcal{L}^*)$-axiomatisable,*
3. *every $\Sigma_1^{\mathrm{B}}(R)$ class of τ-structures closed under isomorphisms is $\Sigma^{\mathrm{B}}(\mathcal{L}^*)$-axiomatisable,*

4. every $\Sigma_1^{\mathrm{SB}}(R)$ class of τ-structures closed under isomorphisms is $\Sigma^{\mathrm{B}}(\mathcal{L}^*)$-axiomatisable,

5. the class $\overline{\mathcal{Q}}_R$ is $\Delta^{\mathrm{B}}(\mathcal{L}^*)$-axiomatisable.

Proof. This follows from Lemma 6.30. □

We are now ready to define our new notion of symbiosis. Given the result we have just proved we will define bounded symbiosis requiring the $\Delta^{\mathrm{B}}(\mathcal{L}^*)$-axiomatisability of $\overline{\mathcal{Q}}_R$ rather than that of $\Delta_1^{\mathrm{B}}(R)$ classes of τ-structures closed under isomorphisms.

Definition 6.32 (Bounded Symbiosis). Let $\mathcal{L}^*$ be a logic and R a set theoretic predicate. We say that $\mathcal{L}^*$ and R are *boundedly symbiotic* if

I the satisfaction relation $\models_{\mathcal{L}^*}$ is $\Delta_1^{\mathrm{B}}(R)$, and

II the class $\overline{\mathcal{Q}}_R$ is $\Delta^{\mathrm{B}}(\mathcal{L}^*)$-axiomatisable.

In [5, Proposition 4], symbiosis for many interesting pairs $\mathcal{L}^*$ and R was established. In all these cases, a slight variation of the proof would give us bounded symbiosis for the same pairs as well. For completeness, we will provide a detailed proof of bounded symbiosis for the pair $\mathcal{L}_{\mathrm{I}}$ and Cd (see Section 6.1). Note that this is a non-trivial result since by [103, § 4], it is consistent that $\Delta(\mathcal{L}_{\mathrm{I}}) \neq \Delta^{\mathrm{B}}(\mathcal{L}_{\mathrm{I}})$.

Definition 6.33. Let $(A, <_A)$ be partial order. Given an element a of A we will denote by $a{\downarrow}$ the set $\{a' \in A\,;\, a' <_A a\}$ of *predecessors* of a in A. Given a cardinal κ, an element $a \in A$ is said to be *κ-like* if $|a{\downarrow}| = \kappa$ and for every $a' \in a{\downarrow}$ we have $|a'{\downarrow}| < \kappa$.

Lemma 6.34. *The class $\mathcal{K}$ of well-orders $(A, <)$ is $\Sigma(\mathcal{L}_{\mathrm{I}})$-axiomatisable by a class $\mathcal{K}'$ such that*

$$\forall (A,<) \in \mathcal{K} \forall M' \in \mathcal{K}'(M' \restriction \{<\} = (A,<) \to |M'| \leq \aleph_{\mathrm{OT}(A,<)}).$$

where $\mathrm{OT}(A,<)$ is the unique ordinal isomorphic to $(A,<)$. So the class of well-orders is $\Sigma^{\mathrm{B}}(\mathcal{L}_{\mathrm{I}})$-axiomatisable.

Proof. Consider the class $\mathcal{K}'$ of structures of type $(A, B, <_A, <_B, f)$, where $<_A \subset A \times A$, $<_B \subset B \times B$, and $f : A \to B$ is a function. First we define the following formulas:

$$\mathrm{Inf}(x) := \forall b <_B x (\mathrm{I}y, z(y <_B x \wedge y \neq b)(z <_B x)),$$

i.e., x has infinitely many predecessors.

$$\mathrm{Like}(x) := \forall b <_B x \neg(\mathrm{I}y, z(y <_B x)(z <_B b)),$$

i.e., x is $|x{\downarrow}|$-like.

Let φ be the following $\mathcal{L}_{\mathrm{I}}$ conjunction of the following sentences:

(i) $(\forall a, a' \in A(f(a) = f(a') \to a = a')) \wedge (\forall a, a' \in A(a <_A a' \to f(a) <_B f(a')))$, i.e., f is injective order preserving;

(ii) $<_A$ and $<_B$ are linear orders;

(iii) $\forall b \in B \neg(\mathrm{I}x, y(x <_B b)(y = y \wedge y \in B))$, i.e., B is $|B|$-like;

(iv) $\forall a \in A(\mathrm{Inf}(f(a)) \wedge \mathrm{Like}(f(a)))$, i.e., every b in the image of A under f is $|b{\downarrow}|$-like and has infinitely many predecessors;

(v) $\forall b \in B((\mathrm{Inf}(b) \wedge \mathrm{Like}(b)) \to \exists! a \in A(f(a) = b))$, i.e., the image of f is exactly the set of elements $b \in B$ which are $|b{\downarrow}|$-like and have infinite predecessors;

(vi) $\forall b \in B(\mathrm{Inf}(b) \to \exists b' \in B(\mathrm{I}x, y(x <_B b)(y <_B b') \wedge \mathrm{Like}(b')))$, i.e., "no infinite cardinals are jumped".

First note that if $(A, <_A) \cong (\alpha, \in)$ is a well-order, $<_B = \in$, and $B = \aleph_\alpha$, then by putting $f(\beta) := \aleph_\beta$ we have $(A, B, <_A, <_B, f)$ is a model of φ.

Now assume that $(A, B, <_A, <_B, f)$ is a model of φ.

Claim 6.35. *If $a <_A a'$ then $|f(a){\downarrow}| < |f(a'){\downarrow}|$.*

Proof. By (i) we have that $|f(a){\downarrow}| \leq |f(a'){\downarrow}|$. Moreover, by (iv) we have $|f(a){\downarrow}| \neq |f(a'){\downarrow}|$ and $|f(a){\downarrow}| < |f(a'){\downarrow}|$ as desired. □

Claim 6.36. *The structure $(A, <_A)$ is a well-order.*

Proof. Note that if $s : \omega \to A$ is a strictly decreasing sequence in A, then by the previous claim $f \circ s$ is a strictly decreasing sequence of cardinals which is a contradiction. □

So there is some ordinal α such that $(A, <_A) \cong (\alpha, \in)$.

Claim 6.37. *For every $\beta \in \alpha$ we have $|f(\beta){\downarrow}| = \aleph_\beta$*

Proof. By induction on β. If $\beta = 0$ we want to prove $|f(0){\downarrow}| = \aleph_0$. Assume $|f(0){\downarrow}| > \aleph_0$. We have $|f(0){\downarrow}| = |\bigcup_{b <_B f(0)} b{\downarrow}|$. Since $|f(0){\downarrow}| > \aleph_0$, there is $b <_B f(0)$ such that $|f(0){\downarrow}| > |b{\downarrow}| \geq \aleph_0$. By (vi) there is $b' \in B$ such that $|b'{\downarrow}| = |b{\downarrow}|$ and b' is $|b'|$-like. Now, note that by construction $b' < f(0)$, and that by (v) there must be $\beta \in \alpha$ such that $f(\beta) = b'$. But this contradicts (i), since $0 < \beta$, and $|f(\beta){\downarrow}| < |f(0){\downarrow}|$. So, by (iv) $|f(0){\downarrow}| = \aleph_0$. The case for $\beta > 0$ is analogous to the case $\beta = 0$. □

Finally, we have that $|B| \leq \aleph_\alpha$. This follows from (vi) and the previous claim. Indeed,

$$|B| = |\bigcup_{b \in B} b{\downarrow}| \leq \sup\{|f(\gamma){\downarrow}|\,;\, \gamma < \alpha\} = \sup\{\aleph_\gamma\,;\, \gamma < \alpha\} \leq \aleph_\alpha.$$

□

Lemma 6.38. *The class $\mathcal{K}$ of non well-ordered sets $(A, <)$ is $\Delta^{\mathrm{B}}(\mathcal{L}_{\mathrm{I}})$-axiomatisable.*

Proof. By Lemma 6.34 it is enough to show that $\mathcal{K}$ is $\Sigma^{\mathrm{B}}(\mathcal{L}_{\mathrm{I}})$-axiomatisable. Note that $(A, <) \in \mathcal{K}$ iff it is not a linear order or it is not well-founded. Not being a linear order is $\mathcal{L}_{\omega\omega}$-axiomatisable and therefore $\Sigma^{\mathrm{B}}(\mathcal{L}_{\mathrm{I}})$-axiomatisable. Finally, the class of non well-founded linear orders is the class of linear orders $(A, <_A)$, satisfying $\forall x \exists y(y <_A x)$. □

A similar proof works for well-founded orders. Given a well-founded partial order $(A,<)$ we will call *chain* a totally ordered subset of O. Note that every chain C in $(A,<)$ is a well-order. We will call the *height of the chain* the unique ordinal $\mathrm{ht}(C)$ such that $(C,<) \cong (\mathrm{ht}(C),<)$. The *height of the well-founded partial order* $(A,<)$ is defined by $\mathrm{ht}(A,<) := \sup\{\mathrm{ht}(C)+1\,; C \text{ is a maximal chain of } (O,<)\}$.

Theorem 6.39. *The class $\mathcal{K}$ of well-founded partial orders $(A,<)$ is $\Sigma(\mathcal{L}_{\mathrm{I}})$-axiomatisable by a class $\mathcal{K}'$ such that*

$$\forall (A,<) \in \mathcal{K} \forall M' \in \mathcal{K}' (M' \restriction \{<\} = (A,<) \to |M'| \leq \aleph_{\mathrm{ht}(A,<)}).$$

In particular $\mathcal{K}$ is $\Sigma^{\mathrm{B}}(\mathcal{L}_{\mathrm{I}})$-axiomatisable.

Proof. To show that $\mathcal{K}$ is $\Sigma^{\mathrm{B}}(\mathcal{L}_{\mathrm{I}})$, consider the class $\mathcal{K}'$ of structures of type

$$(A, B, <_A, <_B, f)$$

which satisfy the formula conjunction φ of the following sentences:

1. $(\forall a, a' \in A(a <_A a' \to f(a) <_B f(a')))$ i.e., f order preserving;
2. $<_A$ is a partial order and $<_B$ is linear order;
3. $\forall b \in B \neg(\mathrm{I}x, y(x <_B b)(y = y \wedge y \in B))$, i.e., B is $|B|$-like;
4. $\forall a \in A \exists b \in B(f(a) = b \wedge \mathrm{Inf}(b) \wedge \mathrm{Like}(b))$ i.e., every b in the image of A under f is $|b{\downarrow}|$-like and has infinitely many predecessors;
5. $\forall b \in B((\mathrm{Inf}(b) \wedge \mathrm{Like}(b)) \to \exists! a \in A(f(a) = b))$ i.e., the image of f is exactly the set of elements $b \in B$ which are $|b{\downarrow}|$-like and have infinite predecessors;
6. $\forall b \in B(\mathrm{Inf}(b) \to \exists b' \in B(\mathrm{I}x, y(x <_B b)(y <_B b') \wedge \mathrm{Like}(b')))$, i.e., "no infinite cardinals are jumped";
7. $\forall a \in A \forall b <_B f(a)(\mathrm{Like}(b) \to \exists a' \in A(f(a') = b \wedge a' <_A a))$ "chains do not jump cardinals".

Note that, if $(A,<)$ is a well-founded partial order, taking $B = \aleph_{\mathrm{ht}(A,<)}$, $<_B = \in$, and $f(a) := \aleph_{ht(a\downarrow)}$, we get a model which is in $\mathcal{K}'$ and whose projection is $(A,<)$.

Claim 6.40. *If $a <_A a'$, then $|f(a){\downarrow}| < |f(a'){\downarrow}|$.*

Proof. By (1) we have that $|f(a){\downarrow}| \leq |f(a'){\downarrow}|$. Moreover, by (iv) we have $|f(a){\downarrow}| \neq |f(a'){\downarrow}|$, and $|f(a){\downarrow}| < |f(a'){\downarrow}|$ as desired. ☐

Claim 6.41. *The structure $(A,<_A)$ is a well-founded.*

Proof. Note that if $s : \omega \to A$ is a strictly decreasing sequence in A, then by the previous claim $f \circ s$ is a strictly decreasing sequence of cardinals which is a contradiction. ☐

Claim 6.42. *For each $a \in A$ $|f(a){\downarrow}| = \aleph_{\mathrm{ht}(a\downarrow)}$.*

Proof. Let $a \in A$ be minimal such that

$$|f(a){\downarrow}| \neq \aleph_{\mathrm{ht}(a\downarrow)}.$$

Because f is order preserving, we have $|f(a){\downarrow}| > \aleph_{\mathrm{ht}(a\downarrow)}$. Now, $\aleph_{\mathrm{ht}(a\downarrow)} < |f(a){\downarrow}| = |\bigcup_{b<_B f(a)} b{\downarrow}|$. Then, there is $b <_B f(a)$ such that $|b{\downarrow}| \geq \aleph_{\mathrm{ht}(a\downarrow)}$. By (6), there is $b' <_B f(a)$ which is $|b{\downarrow}|$-like. Furthermore, by (7) there is $a' <_A a$ such that $|f(a'){\downarrow}| = |b'{\downarrow}|$. But by minimality of a we have $|f(a'){\downarrow}| = \aleph_{\mathrm{ht}(a'\downarrow)}$ and $\mathrm{ht}(a{\downarrow}) \leq \mathrm{ht}(a'{\downarrow})$ which is a contradiction since $a' <_A a$. □

By the previous claim for every chain C in $(A, <)$ we have $|f[C]| \leq \aleph_{\mathrm{ht}(C)}$.

Finally, we have that $|B| \leq \aleph_{\mathrm{ht}(A,<)}$. Indeed, by (6) we have that

$$|B| = |\bigcup\{f[C]\,;\, C \text{ is a chain in } (A, <)\}|.$$

And by the previous claim:

$$|B| \leq \aleph_{\mathrm{ht}(A,<)}.$$

□

Corollary 6.43. *The class of well-founded orders $(A, <)$ is $\Delta^{\mathrm{B}}(\mathcal{L}_{\mathrm{I}})$-axiomatisable.*

Proof. By Lemma 6.39 it is enough to show that the class of non well-founded orders $\mathcal{K}$ is $\Sigma^{\mathrm{B}}(\mathcal{L}_{\mathrm{I}})$-axiomatisable. Note that $(A, <) \in \mathcal{K}$ iff it is not a partial order or it is not well-founded. Not being a partial order is $\mathcal{L}_{\omega\omega}$-axiomatisable and therefore $\Sigma^{\mathrm{B}}(\mathcal{L}_{\mathrm{I}})$-axiomatisable. And, as we have seen in the proof of Lemma 6.38, not being well-founded is $\mathcal{L}_{\omega\omega}$-axiomatisable. □

Theorem 6.44. *The logic $\mathcal{L}_{\mathrm{I}}$ and* Cd *are boundedly symbiotic.*

Proof. First we show I of Definition 6.32. Note first that the statement "$\mathcal{A} \models_{\mathcal{L}_{\mathrm{I}}} \varphi$" is absolute for Cd-correct models of set theory. Therefore $\mathcal{A} \models_{\mathcal{L}_{\mathrm{I}}} \varphi$ iff for some n big enough

1. $\exists M$ (M is a transitive model of $\mathsf{ZFC}_n \wedge M$ is correct for cardinals $\wedge\, \mathcal{A} \in M \wedge M \models (\mathcal{A} \models_{\mathcal{L}_{\mathrm{I}}} \varphi)$), iff
2. $\forall M$ (M is a transitive model of $\mathsf{ZFC}_n \wedge M$ is correct for cardinals $\wedge\, \mathcal{A} \in M \rightarrow M \models (\mathcal{A} \models_{\mathcal{L}_{\mathrm{I}}} \varphi)$).

Since the classical satisfaction relation is Δ^{B}_1, and "M is Cd-correct" is $\Delta_1(\mathrm{Cd})$, we obtain a $\Delta_1(\mathrm{Cd})$ statement.

To verify II of Definition 6.32, consider the class $\mathcal{Q}_{\mathrm{Cd}} := \{\mathcal{A}\,;\, \mathcal{A}$ is isomorphic to a Cd-correct transitive model $\}$. Then $\mathcal{A} = (A, E) \in \mathcal{Q}_{\mathrm{Cd}}$ iff

1. E is well-founded
2. $(A, E) \models \mathsf{ZFC}_n$ for n big enough so that the argument will go through
3. For all $\alpha \in \bar{A}$ if $\bar{A} \models \mathrm{Cd}(\alpha)$ then $\mathrm{Cd}(\alpha)$, where $\bar{A}$ is the transitive collapse of (A, E).

Clause (2) is a statement in $\mathcal{L}_{\omega\omega}$, and (3) holds iff

$$\bar{A} \models \forall\alpha\forall x < \alpha\neg(\mathrm{I}yz(y \in x)\,(z \in \alpha))$$

Written using E instead of $\in$, this gives an $\mathcal{L}_{\mathrm{I}}$-statement. Finally, by Corollary 6.43, we have that (1) is $\Delta^{\mathrm{B}}(\mathcal{L}_{\mathrm{I}})$-axiomatisable as desired. □

6.4 Upward Löwenheim-Skolem numbers and upward reflection numbers

In this section we will finally introduce a reflection number analogous to the one in Definition 6.2 which, as in [5], will allow to connect the strength of existence of upward Löwenheim-Skolem numbers for strong logics to large cardinals.

Definition 6.45. Let R be a Π_1 predicate in the language of set theory and $\mathcal{K}$ be a $\Sigma_1^{\mathrm{B}}(R)$-definable class of structures in some vocabulary τ. Then we say that $\mathcal{K}$ is *transitive* iff

$$\forall \mathcal{A} \in \mathcal{K} \exists \mathcal{B} \in \mathcal{K} (\mathcal{A} \cong \mathcal{B} \ \wedge \ \text{the domains of } \mathcal{B} \text{ are transitive sets}).$$

Note that very class $\mathcal{K}$ which is closed under isomophisms is transitive.

Definition 6.46. Let R be a Π_1 predicate in the language of set theory and λ be a cardinal. The *bounded upwards structural reflection number* $\mathcal{USR}_\lambda(R)$ is the least κ such that

> For every vocabulary τ with $|\tau| < \lambda$, for every $\Sigma_1^{\mathrm{B}}(R)$-definable transitive class of τ-structures $\mathcal{K}$, if there is $\mathcal{A} \in \mathcal{K}$ with $|\mathcal{A}| \geq \kappa$, then for every $\kappa' > \kappa$ there is a $\mathcal{B} \in \mathcal{K}$ with $|\mathcal{B}| \geq \kappa'$ and an elementary embedding $e : \mathcal{A} \preccurlyeq_{\mathcal{L}_{\omega\omega}} \mathcal{B}$.

If there is no such cardinal, we will assume that $\mathcal{USR}_\lambda(R)$ is not defined.

One might be tempted to relax the restriction on the size of vocabularies and define the number $\mathcal{USR}_\infty(R) = \kappa$ iff κ is the least such that for all λ we have $\mathcal{USR}_\lambda(R) \leq \kappa$. As before, if such cardinal does not exists as usual we assume that $\mathcal{USR}_\infty(R)$ is not defined.

However, this number is never defined. Assume that $\mathcal{USR}_\infty(R) = \kappa$, fix a vocabulary with κ-many constants, and consider the class $\mathcal{K} := \{\mathcal{A}\,;\, \mathcal{A} \text{ is a } \tau\text{-structure and for every } a \in A \text{ there exists a constant } c \in \tau \text{ such that } a = c^{\mathcal{A}}\}$. The class $\mathcal{K}$ is clearly $\Sigma_1(|\tau|)$-definable and contains a model $\mathcal{A}$ with $|\mathcal{A}| = \kappa$, but does not contain any models $\mathcal{B}$ with $|\mathcal{B}| > \kappa$.

As we mentioned in § 6.2, if we allow infinite vocabularies or arbitrarily complex logics then we can code very complex sets. For this reason in proving our main theorem we will require that our logic and our vocabularies are well-behaved. In particular we will require that the logic has dependence number ω, i.e., sentences are not too complex, moreover, we will require that the cardinality of the vocabularies we take into consideration is bounded by a $\Delta_1^{\mathrm{R}}(R)$-definable cardinal.

Definition 6.47. Let R be a Π_1 predicate in the language of set theory and m be a set. Then we say that m is $\Delta_1^{\mathrm{B}}(R)$-*definable* if and only if there is a $\varphi \in \Delta_1^{\mathrm{B}}(R)$ such that $\forall x(\varphi(x) \leftrightarrow x = m)$.

Definition 6.48. Suppose that R is a Π_1 predicate in the language of set theory. Then we say that $\mathcal{L}^*$ is $\Delta_1^{\mathrm{B}}(R)$-*finitely-definable* iff for every *finite* vocabulary τ and τ-sentence $\varphi \in \mathcal{L}^*[\tau]$ we have that φ is $\Delta_1^{\mathrm{B}}(R)$-definable with parameter τ.

Note that, all the finitary logics extending $\mathcal{L}_{\omega\omega}$ with finitely many logical symbols are $\Delta_1^{\mathrm{B}}(\varnothing)$-finitely-definable. In particular, all the logics in [5, Proposition 4] are $\Delta_1^{\mathrm{B}}(\varnothing)$-finitely-definable.

We are finally ready to prove the main theorem of this chapter.

Theorem 6.49. *Let $\mathcal{L}^*$ be a logic and R be a predicate in the language of set theory such that $\mathcal{L}^*$ and R are boundedly symbiotic and $\mathcal{L}^*$ has $\mathrm{dep}(\mathcal{L}^*) = \omega$ and is $\Delta_1^{\mathrm{B}}(R)$-finitely-definable. Moreover, let λ be a cardinal such that there is a sequence $(\delta_n)_{n\in\omega}$ of $\Delta_1^{\mathrm{B}}(R)$-definable cardinals such that $\bigcup_{n\in\omega} \delta_n = \lambda$. Then the following are equivalent:*

1. *$\mathsf{ULST}_\lambda(\mathcal{L}^*) = \kappa$ and*
2. *$\mathcal{USR}_\lambda(R) = \kappa$.*

In particular, the statement holds for $\lambda = \omega$ and in general for all the logics in [5, Proposition 4].

Proof. For 2 $\to$ 1: assume that $\mathcal{USR}_\lambda(R) = \kappa$. We will prove that $\mathsf{ULST}_\lambda(\mathcal{L}^*) \leq \mathcal{USR}_\lambda(R)$. Let τ be a vocabulary of size $<\lambda$ and $\mathcal{K}$ be the class of τ-structures which satisfy $\varphi \in \mathcal{L}^*[\tau]$. By the fact that $\mathrm{dep}(\mathcal{L}^*) = \omega$ there is a finite vocabulary $\tau' \subseteq \tau$ such that $\varphi \in \mathcal{L}^*[\tau']$. Let τ'_ϱ be a finite Δ_0-definable vocabulary such that $\tau'_\varrho \cap \tau = \varnothing$ and there is a renaming $\varrho : \tau' \to \tau'_\varrho$. By the fact that $\tau'_\varrho \cap \tau = \varnothing$, the renaming ϱ can be extended to a renaming from τ to $\tau_\varrho := \tau'_\varrho \cup (\tau \setminus \tau')$. Let δ_n be the least such that $|\tau| \leq \delta_n$. Define τ'' to be a $\Delta_1^{\mathrm{B}}(R, \delta_n)$-definable vocabulary of size δ_n extending τ_ϱ with the same sorts of τ. Note that φ^ϱ is $\Delta_1^{\mathrm{B}}(R)$-definable by assumptions and the fact that τ'_ϱ is Δ_0-definable. Moreover, τ'' is also $\Delta_1^{\mathrm{B}}(R)$-definable by construction.

For every τ-structure $\mathcal{M}$, the τ_ϱ structure $\mathcal{M}^\varrho$ can be easily extended to a τ''-structure. Moreover, every τ''-structure induces by projection a τ_ϱ-structure which can be renamed back via ϱ^{-1} to a τ-structure. Therefore, $\mathcal{K}$ is the renaming of a class which is the projection of the class $\mathcal{K}' := \{\mathcal{M}\ ;\ \mathcal{M} \text{ is a } \tau''\text{-structure and } \mathcal{M} \models_{\mathcal{L}^*} \varphi^\varrho\}$.

Now, by bounded symbiosis I, the class $\mathcal{K}'$ is $\Delta_1^{\mathrm{B}}(R)$-definable. Let $\mathcal{M} \in \mathcal{K}$ be of size $\geq\kappa$ and $\mathcal{M}'$ be an extension of $\mathcal{M}^\varrho$ in $\mathcal{K}'$. Let $\kappa' > \kappa$. By the fact that $\mathcal{USR}_\lambda(R) = \kappa$ and since $\mathcal{K}'$ is closed under isomorphisms, there is $\mathcal{N}' \in \mathcal{K}'$ of size $>\kappa'$ and $\mathcal{M}' \preccurlyeq_{\mathcal{L}_{\omega\omega}} \mathcal{N}'$. Then $\mathcal{N}'^{\varrho^{-1}} \restriction \tau \in \mathcal{K}$ and, since no sorts were added to τ, we have $|(\mathcal{N}')^{\varrho^{-1}} \restriction \tau| = |\mathcal{N}'| > \kappa'$ as desired.

1 $\to$ 2: suppose $\mathsf{ULST}_\lambda(\mathcal{L}^*) = \kappa$. We will prove that $\mathcal{USR}_\lambda(R) \leq \mathsf{ULST}_\lambda(\mathcal{L}^*)$. Let $\mathcal{K}$ be a $\Sigma_1^{\mathrm{B}}(R)$-definable transitive class of τ-structures with $\Phi(x)$ a defining $\Sigma_1^{\mathrm{B}}(R)$-formula for $\mathcal{K}$, and $|\tau| < \lambda$. Without loss of generality we assume that τ is in one sort; a similar proof works in the general case. Let F be the definably bounding function for $\Phi(x)$.

Define a vocabulary τ' with two sorts: s_0 and s_1, with all of the symbols occurring in τ written in sort s_0 and with: E a binary relation symbol of sort s_1, a function symbol G from s_0 to s_1, and constant symbol c of sort s_1. Let $\mathcal{K}^*$ be the class of all structures $\mathcal{N} := \big(N, B, E^{\mathcal{N}}, G^{\mathcal{N}}, c^{\mathcal{N}}\big)$ such that

1. $(N, E^{\mathcal{N}}) \models \mathsf{ZFC}_n$ for n big enough so that the argument will go through,
2. $(N, E^{\mathcal{N}}) \in \overline{\mathcal{Q}}_R$, i.e., it is isomorphic to a transitive model which is R-correct,
3. $|N| \leq F^3(|B|)$,

4. $\mathcal{N} \models \Phi^E(c)$, where Φ^E is Φ rewritten with E instead of $\in$,

5. $\mathcal{N} \models$ "G is a bijection between B and the domain of c" .

Part (1), (4) and (5) of the definition are clearly in $\mathcal{L}_{\omega\omega}$ therefore $\Sigma^{\mathrm{B}}(R)$-axiomatisable. Part (2) is axiomatisable in $\Sigma^{\mathrm{B}}(\mathcal{L}^*)$ by Lemma 6.23. By II of bounded symbiosis, (1) has a $\Delta^{\mathrm{B}}(\mathcal{L}^*)$-axiomatisation. Thus the class $\mathcal{K}^*$ is $\Sigma^{\mathrm{B}}(\mathcal{L}^*)$-axiomatisable.

Let $\mathcal{A} \in \mathcal{K}$ with $|A| \geq \kappa$, and let $\kappa' > |A| \geq \kappa$ be arbitrary. Since $\mathcal{K}$ is transitive we can assume that A is transitive. The aim is to find $\mathcal{A}' \in \mathcal{K}$ such that $\mathcal{A} \preccurlyeq_{\mathcal{L}_{\omega\omega}} \mathcal{A}'$.

Let $\vartheta := F^2(|A|)$. Consider the structure $\mathcal{M} := (\mathbf{H}_\vartheta, A, \in, \mathrm{id}, \mathcal{A})$.

Note that, (1) and (5) in the definition of $\mathcal{K}^*$ are satisfied by $\mathcal{M}$. Since Φ is $\Sigma_1^{\mathrm{B}}(R)$, by Lemma 6.29, we have $H_\vartheta \models \Phi(\mathcal{A})$. Therefore, (4) in the definition of $\mathcal{K}^*$ is also satisfied by $\mathcal{M}$. By Corollary 6.27, $\mathbf{H}_\vartheta$ is R-correct, which means that (2) in the definition of $\mathcal{K}^*$ is satisfied by $\mathcal{M}$. Finally, note that $|\mathbf{H}_\vartheta| = 2^{<\vartheta} = 2^{<F^2(|A|)} \leq F^3(|A|)$. Therefore, $(\mathbf{H}_\vartheta, A, \in, \mathrm{id}, \mathcal{A}) \in \mathcal{K}^*$.

Let $\mathcal{K}_1^* = \mathrm{Mod}(\chi)$ be an $\mathcal{L}^*$-axiomatisable class such that $\mathcal{K}^*$ is a bounded projection of $\mathcal{K}_1^*$. Let $h : \mathrm{Ord} \to \mathrm{Ord}$ as in Lemma 6.19.

Let $\mathcal{M}_1 \in \mathcal{K}_1^*$ be such that $\mathcal{M}$ is the projection of $\mathcal{M}_1$. Since $|\mathcal{M}_1| \geq |\mathcal{M}| \geq |A| \geq \kappa$, we can apply the fact that $\mathsf{ULST}_\lambda(\mathcal{L}^*) = \kappa$ to find an $\mathcal{N}_1 \in \mathcal{K}_1^*$, such that $|\mathcal{N}_1| \geq h(F^3(\kappa'))$, and $\mathcal{M}_1 \preccurlyeq_{\mathcal{L}_{\omega\omega}} \mathcal{N}_1$. Let $\mathcal{N}$ be the projection of $\mathcal{N}_1$. Note that $\mathcal{N} \in \mathcal{K}^*$. We write $\mathcal{N} = (N, B, E, \dots)$ for this structure.

Let $(\bar{N}, \in)$ be the transitive collapse of the (N, E) and $\bar{c}$ be the image of $c^{\mathcal{N}}$ under this collapse. Since $\mathcal{N} \in \mathcal{K}^*$ we know that $\bar{N} \models \Phi(\bar{c})$ and that $\bar{N}$ is R-correct. Since Φ is $\Sigma_1^{\mathrm{B}}(R)$ it is upwards-absolute by Corollary 6.27, $\Phi(\bar{c})$ holds. Hence $\bar{c} \in \mathcal{K}$.

Claim 6.50. $\kappa' \leq |\bar{c}|$.

Proof. Recall that $|\mathcal{N}_1| \geq h(F^3(\kappa'))$. This implies that $|\overline{\mathcal{N}}| = |\mathcal{N}| \geq F^3(\kappa')$. Indeed, if $|\mathcal{N}| < F^3(\kappa')$ then by monotonicity of h we have $h(|\mathcal{N}|) \leq h(F^3(\kappa')) < |\mathcal{N}_1|$ which contradicts the fact that $\mathcal{N}$ is the projection of $\mathcal{N}_1$. Finally, note that, since $\mathcal{N} \in \mathcal{K}^*$, by 3 and 5 we have that $F^3(\kappa') \leq |\mathcal{N}| \leq F^3(|B|) = F^3(|\bar{c}|)$. Hence, by monotonicity of F, we have $\kappa' \leq |\bar{c}|$ as desired. □

Claim 6.51. *There is an $\mathcal{L}_{\omega\omega}$ elementary embedding from $\mathcal{A}$ to $\bar{c}$.*

Proof. Let $\pi : \mathcal{N} \to \overline{\mathcal{N}}$ be the collapsing map. Since the first-order satisfaction relation is Δ_1, for every first-order ψ and for every $a_1, \dots a_n \in \mathcal{A}$ we have $\mathcal{A} \models \psi(a_1, \dots a_n)$ iff $\mathbf{H}_\vartheta \models (\mathcal{A} \models \psi(a_1, \dots a_n))$ iff $\mathcal{M} \models (c \models \psi(a_1, \dots a_n))$ iff $\mathcal{M}_1 \models (c \models \psi(a_1, \dots a_n))$ iff $\mathcal{N}_1 \models (c \models \psi(a_1, \dots a_n))$ iff $\mathcal{N} \models (c \models \psi(a_1, \dots a_n))$ iff $\bar{\mathcal{N}} \models (\bar{c} \models \psi(\pi(a_1), \dots \pi(a_n)))$ iff $\bar{c} \models \psi(\pi(a_1), \dots \pi(a_n))$. Hence $\pi \restriction A : \mathcal{A} \preccurlyeq_{\mathcal{L}_{\omega\omega}} \bar{c}$. □

□

6.5 One application: second order logic

In this section study of the large cardinal strength of the existence of upwards structural reflection numbers. In particular, we will give upper and lower bounds for the strength of the existence of $\mathcal{USR}_\omega(\mathrm{PwSt})$.

We begin by proving that the power set predicate PwSt is in some sense an upper bound for Π_1 predicates.

Definition 6.52. A class function $G(x,z) : \mathbf{V}^{<\mathrm{Ord}} \times \mathbf{V} \to \mathbf{V}$ is said to be $\Sigma_1^{\mathrm{B}}(\mathrm{PwSt})$-*definable* if and only if there is a $\Sigma_1^{\mathrm{B}}(\mathrm{PwSt})$ formula $\psi(x,y,z)$ such that for every x, y, and z we have $G(x,z) = y$ if and only if $\psi(x,y,z)$. Moreover, we will say that $G(x,z)$ is *normal on x* iff

1. G is monotone on x, i.e., for every pair of ordinals $\alpha \leq \beta$, every pair of functions $f \in \mathbf{V}^\alpha$ and $g \in \mathbf{V}^\beta$, and for every z we have that, if $f \subseteq g$ then $G(f,z) \subseteq G(g,z)$;
2. G is continuous on x, i.e., for all limit ordinal λ, for every function $f \in \mathbf{V}^\lambda$, and for every z we have $G(f,z) = \bigcup_{\alpha\in\lambda} G(f{\restriction}\alpha, z)$.

Lemma 6.53. *Let $G(x,z)$ be a $\Sigma_1^{\mathrm{B}}(\mathrm{PwSt})$-definable function on $\mathbf{V}$ that is normal on x, and let $F : \mathrm{Ord} \times \mathbf{V} \to \mathbf{V}$ be defined recursively as follows:*

$$F(\alpha,z) = G(F \restriction \alpha, z) \text{ for every successor ordinal } \alpha;$$
$$F(\lambda,z) = \bigcup_{\alpha\in\lambda} F(\alpha,z) \text{ for } \lambda \text{ limit.}$$

Then F is also $\Sigma_1^{\mathrm{B}}(\mathrm{PwSt})$-definable.

Proof. One can define F as follows:

$$y = F(\alpha,z) \Leftrightarrow \exists f(\ f \text{ is a function } \wedge\ \mathrm{dom}(f) = \alpha \\ \wedge \forall\beta \in \alpha f(\beta) = G(f \restriction \beta, z) \wedge y = G(f,z)).$$

This definition is $\Sigma_1(\mathrm{PwSt})$ since G was assumed to be $\Sigma_1^{\mathrm{B}}(\mathrm{PwSt})$. To see that the formula is $\Sigma_1^{\mathrm{B}}(\mathrm{PwSt})$ it is enough to note that, since G is normal on x, the function f is a subset of $\alpha \times \wp(y)$ and therefore $\varrho_{\mathbf{H}}(f) \leq 2^{\varrho_{\mathbf{H}}(\alpha,y)}$. $\square$

Corollary 6.54. *The functions $\alpha \mapsto \mathbf{V}_\alpha$ is $\Sigma_1^{\mathrm{B}}(\mathrm{PwSt})$-definable. Moreover the function* H *that maps every infinite successor cardinal κ^+ to $\mathbf{H}_{\kappa^+}$ is $\Sigma_1^{\mathrm{B}}(\mathrm{PwSt})$-definable.*

Proof. The normal operation $\alpha \mapsto F(\alpha) := \mathbf{V}_\alpha$ is defined by recursion using the normal operation G defined as follows: $G(f) = \wp(f(\alpha))$ if $\mathrm{dom}(f) = \alpha + 1$ is a successor ordinal and $G(f) = \bigcup_{\beta<\lambda} \wp(f(\beta))$ if $\mathrm{dom}(f) = \lambda$ is a limit ordinal. Thus, by Lemma 6.53, F is $\Sigma_1^{\mathrm{B}}(\mathrm{PwSt})$. Now, we claim that H is $\Sigma_1^{\mathrm{B}}(\mathrm{PwSt})$-definable. For every cardinal κ, define the following function F:

$$F(\alpha+1,\kappa^+) = \{x \subset F(\alpha,\kappa^+)\,;\, |x| \leq \kappa\};$$
$$F(\lambda,\kappa^+) = \bigcup_{\alpha\in\lambda} F(\alpha,\kappa^+) \text{ for } \lambda \leq \kappa^+ \text{ limit.}$$

Note that $F(\kappa^+,\kappa^+) = \mathbf{H}_{\kappa^+}$. Moreover, by Lemma 6.53, F is $\Sigma_1^{\mathrm{B}}(\mathrm{PwSt})$-definable. Finally, by what we have just said $\mathrm{H}(\kappa^+) = F(\kappa^+,\kappa^+)$. $\square$

Lemma 6.55. *For every Π_1 predicate R in the language of set theory and every sentence φ in the language of set theory we have*

1. *if φ is $\Sigma_1(R)$ then φ is $\Sigma_1(\mathrm{PwSt})$;*
2. *if φ is $\Pi_1(R)$ then φ is $\Pi_1(\mathrm{PwSt})$.*

Therefore, every $\Delta_1(R)$ sentence φ is $\Delta_1(\mathrm{PwSt})$. The same holds if we substitute Σ_1 with Σ_1^{B}.

Proof. We will only prove the bounded version of item 1.; all of the other claims can be proved with a similar proof.

By Corollary 6.54 the function H that maps every infinite successor cardinal κ^+ to $\mathbf{H}_{\kappa^+}$ is $\Sigma_1^{\mathrm{B}}(\mathrm{PwSt})$-definable.

Let $\varphi(x)$ be a $\Sigma_1(R)$ formula. Then we will show that $\varphi(x)$ is equivalent to the following $\Sigma_1(\mathrm{PwSt})$ formula:

$$\exists\kappa\exists y(\ \kappa \text{ is a successor cardinal } \wedge \mathrm{H}(\kappa) = y \wedge x \in y \wedge y \models \varphi(x)).$$

If $\varphi(a)$ for some a then by Lemma 6.29 there is κ such that $a \in \mathbf{H}_\kappa$ and $\mathbf{H}_\kappa \models \varphi(a)$. Similarly, if $\exists\kappa\exists y(\mathrm{H}(\kappa) = y \wedge x \in y \wedge y \models \varphi(a))$ then by Lemma 6.28 we have that $\varphi(a)$ as desired. □

Theorem 6.56. *If $\mathcal{USR}_\omega(\mathrm{PwSt})$ is defined then there is an n-extendible cardinal for every natural number $n > 0$.*

Proof. Assume that $\mathcal{USR}_\omega(\mathrm{PwSt}) = \kappa$. Define the class $\mathcal{K}$ of structures of the type (M, a, E), where

1. a is an ordinal,
2. $M = \mathbf{V}_{a+n}$,
3. $E = \in\restriction\mathbf{V}_{a+n}$.

The class $\mathcal{K}$ is $\Sigma_1^{\mathrm{B}}(\mathrm{PwSt})$-definable. Moreover, for every cardinal μ we have that

$$(\mathbf{V}_{\mu+n}, \mu, \in\restriction\mathbf{V}_{\mu+n})$$

is in $\mathcal{K}$. If $\mu \geq \kappa$ then there is $(\mathbf{V}_{\beta+n}, \beta, \in\restriction\mathbf{V}_{\beta+n})$ in $\mathcal{K}$ such that $\beta > \mu$, and $J : (\mathbf{V}_{\mu+n}, \mu, \in\restriction\mathbf{V}_{\mu+n}) \preccurlyeq_{\mathcal{L}_{\omega\omega}} (\mathbf{V}_{\beta+n}, \beta, \in\restriction\mathbf{V}_{\beta+n})$. Note that $J(\mu) = \beta > \mu$. So the critical point λ of J is smaller or equal to μ. If $\lambda = \mu$ then we are done. Now, assume that $\lambda < \mu$. Note that $J\restriction\mathbf{V}_{\lambda+n} : \mathbf{V}_{\lambda+n} \preccurlyeq_{\mathcal{L}_{\omega\omega}} \mathbf{V}_{J(\lambda)+n}$. Indeed, for every formula φ and $x_1, \dots, x_n \in \mathbf{V}_{\lambda+n}$ we have

$$\begin{aligned}
\mathbf{V}_{\lambda+n} \models \varphi(x_1, \dots, x_n) &\Leftrightarrow \mathbf{V}_{\mu+n} \models \varphi^{\mathbf{V}_{\lambda+n}}(x_1, \dots, x_n) \text{ by absoluteness of } \models, \\
&\Leftrightarrow \mathbf{V}_{\beta+n} \models \varphi^{J(\mathbf{V}_{\lambda+n})}(J(x_1), \dots, J(x_n)) \text{ by elementarity}, \\
&\Leftrightarrow \mathbf{V}_{\beta+n} \models \varphi^{\mathbf{V}_{J(\lambda)+n}}(J(x_1), \dots, J(x_n)) \text{ since } J(\mathbf{V}_{\lambda+n}) = \mathbf{V}_{J(\lambda+n)}, \\
&\Leftrightarrow \mathbf{V}_{J(\lambda)+n} \models \varphi(J(x_1), \dots, J(x_n)) \text{ by absoluteness of } \models.
\end{aligned}$$

Finally, since $J(\lambda) > n$, we have that λ is n-extendible as desired. □

Corollary 6.57. *If* $\mathsf{ULST}_\omega(\mathcal{L}^2)$ *is defined, then there is an* n*-extendible cardinal for every natural number* $n > 0$.

Note that the proof of Theorem 6.56 can be easily adapted to prove the following:

Theorem 6.58. *If* $\mathcal{USR}_\omega(\mathrm{PwSt})$ *is defined then for every* $\Sigma_1^{\mathrm{B}}(\mathrm{PwSt})$*-definable ordinal* η *there is a weakly* η*-extendible cardinal.*

Proof. Let $\mathcal{USR}_\omega(\mathrm{PwSt}) = \kappa$ and η be as in the statement. Define the class $\mathcal{K}$ of structures of the type (M, a, E), where

1. a is an ordinal,
2. $M = \mathbf{V}_{a+\eta}$,
3. $E = \in\restriction\mathbf{V}_{a+\eta}$.

The class is $\Sigma_1^{\mathrm{B}}(\mathrm{PwSt})$-definable. Now, the argument is the same as the one in Theorem 6.56. For every cardinal μ we have that

$$(\mathbf{V}_{\mu+\eta}, \mu, \in\restriction\mathbf{V}_{\mu+\eta})$$

is in $\mathcal{K}$. If $\mu \geq \kappa$ then there is $(\mathbf{V}_{\beta+\eta}, \beta, \in\restriction\mathbf{V}_{\beta+\eta})$ in $\mathcal{K}$ such that $\beta > \mu$, and $J : (\mathbf{V}_{\mu+\eta}, \mu, \in\restriction\mathbf{V}_{\mu+\eta}) \preccurlyeq_{\mathcal{L}_{\omega\omega}} (\mathbf{V}_{\beta+\eta}, \beta, \in\restriction\mathbf{V}_{\beta+\eta})$. Note that $J(\mu) = \beta > \mu$. So the critical point λ of J is smaller or equal to μ. Note that, as in the proof of Theorem 6.56, $J\restriction\mathbf{V}_{\lambda+\eta} : \mathbf{V}_{\lambda+\eta} \preccurlyeq_{\mathcal{L}_{\omega\omega}} \mathbf{V}_{J(\lambda)+J(\eta)}$. Therefore λ is weakly η-extendible as desired. $\square$

Note that in the previous theorem, if η happens to be smaller than the critical point λ of J, then we get that λ is actually η-extendible. This is, e.g., the case for all absolutely definable ordinals η.

Theorem 6.59. *If there is an extendible cardinal, then for every* Π_1 *predicate* R *in the language of set theory* $\mathcal{USR}_\omega(R)$ *is defined and* $\mathcal{USR}_\omega(R) \leq \kappa$ *where* κ *is the least extendible cardinal.*

Proof. Assume the κ is the least extendible cardinal. By Lemma 6.55 it is enough to prove the claim for $R = \mathrm{PwSt}$.

Let $\mathcal{K}$ be $\Sigma_1^{\mathrm{B}}(\mathrm{PwSt})$-definable by some formula Φ. We want to show that $\mathcal{USR}_\omega(\mathrm{PwSt})$ is defined and $\mathcal{USR}_\omega(\mathrm{PwSt}) \leq \kappa$.

Let $\mathcal{M} \in \mathcal{K}$ be such that $|M| \geq \kappa$. Let κ' be a cardinal bigger than κ. Let $\eta > \kappa'$ be such that $\mathcal{M} \in \mathbf{V}_\eta$ and $\mathbf{V}_\eta \models \Phi(\mathcal{M}) \wedge (|M| > \kappa')$. Then there is an elementary embedding J from $(\mathbf{V}_\eta, \in)$ to $(\mathbf{V}_\vartheta, \in)$ for some ordinal ϑ with $J(\kappa) > \eta > \kappa'$. But then by elementarity we have that $\mathbf{V}_\vartheta \models \Phi(J(\mathcal{M})) \wedge (|J(\mathcal{M})| > \eta)$. Since $\mathbf{V}_\vartheta$ is PwSt-correct by Lemma 6.28 $\Phi(J(\mathcal{M}))$ holds. Finally note that since κ' is a cardinal and η is an ordinal bigger than κ we have that $\mathbf{V}_\vartheta \models |J(\mathcal{M})| > \eta > \kappa'$ implies $|J(\mathcal{M})| \geq \kappa'$ as desired. $\square$

Corollary 6.60. *If* κ *is the least extendible cardinal then* $\mathcal{USR}_\omega(\mathrm{PwSt})$ *is defined and* $\mathcal{USR}_\omega(\mathrm{PwSt}) \leq \kappa$. *Therefore,* $\mathsf{ULST}_\omega(\mathcal{L}^2) \leq \kappa$.

Proof. The first part of the claim follows from Theorem 6.59. The second part follows from Theorem 6.49. $\square$

Corollaries 6.57 & 6.60 give a lower and an upper bound to the large cardinal strength of the statement "$\mathsf{ULST}_\omega(\mathcal{L}^2)$ is defined". The author does not know the exact large cardinal strength of this statement.

Finally, note that in the proof of Theorem 6.59 we only use the fact that $\mathcal{K}$ is $\Sigma_1(R)$-definable rather than the fact that it is $\Sigma_1^{\mathrm{B}}(R)$-definable. Therefore, the same proof shows that extendible cardinals imply the following stronger version of upward reflection number:

Definition 6.61. Let R be a Π_1 predicate in the language of set theory and λ be a cardinal. The *unbounded upwards structural reflection number* $\mathcal{UUSR}_\lambda(R)$ is the least κ such that

> For every vocabulary τ with $|\tau| < \lambda$, for every $\Sigma_1(R)$-definable transitive class of τ-structures $\mathcal{K}$, if there is $\mathcal{A} \in \mathcal{K}$ with $|\mathcal{A}| \geq \kappa$, then for every $\kappa' > \kappa$ there is a $\mathcal{B} \in \mathcal{K}$ with $|\mathcal{B}| \geq \kappa'$ and an elementary embedding $e : \mathcal{A} \preccurlyeq_{\mathcal{L}_{\omega\omega}} \mathcal{B}$.

If there is no such cardinal, we will assume that $\mathcal{UUSR}_\lambda(R)$ is not defined.

Theorem 6.62. *If there is an extendible cardinal, then for every Π_1 predicate R in the language of set theory $\mathcal{UUSR}_\omega(R)$ is defined and $\mathcal{UUSR}_\omega(R) \leq \kappa$ where κ is the least extendible cardinal.*

6.6 Open questions

In [5] Bagaria and Väänänen studied various version of downward Löwenheim-Skolem theorems for strong logics, giving a full characterisation of their large cardinal strength. In this chapter we have only looked at the large cardinal strength of the upward Löwenheim-Skolem theorem for second order logic. These questions are therefore natural:

Question 6.63. What is the large cardinal strength of the existence of upward Löwenheim-Skolem numbers of the logics in [5] for vocabularies of size ω?

In particular:

Question 6.64. What is the large cardinal strength of the existence of the upward Löwenheim-Skolem number of $\mathcal{L}_{\mathrm{I}}$ for vocabularies of size ω?

In § 6.5 we gave upper and lower bounds for the large cardinal strength of the existence of the upward Löwenheim-Skolem number of $\mathcal{L}^2$ for vocabularies of size ω; see p. 128.

Question 6.65. What is the large cardinal strength of the existence of the upward Löwenheim-Skolem number of $\mathcal{L}^2$ for vocabularies of size ω?

In § 6.4, we focused on logics with dependence number ω and languages of definable size. As we have seen, these assumptions are crucial in our proof of Theorem 6.49. It is therefore natural to ask what happens in the more general case.

Question 6.66. Which conditions on $\mathcal{L}^*$ and R do we need to assume in order to generalise Theorem 6.49 to arbitrarily big vocabularies?

In the classical theory of first order logic it is a well-known fact that the Compactness theorem and the upward Löwenheim-Skolem theorem are strongly connected. It is indeed easy to see that in general if a logic satisfies the Compactness theorem then it satisfies the upward Löwenheim-Skolem theorem and if a logic satisfies upward Löwenheim-Skolem theorem then it satisfies the Compactness theorem restricted to countable theories. Even though many strong logics do not satisfy the classical Compactness theorem, they may satisfy a weaker versions of it.

Definition 6.67. A logic $\mathcal{L}^*$ is (α, β)-compact iff for every vocabulary τ a set $T \subset \mathcal{L}^*[\tau]$ of size $\leq \beta$ is consistent if every $T_0 \in [T]^{<\alpha}$ is consistent. If a logic is (α, β)-compact for every β, then we will say that the logic is (α, ∞)-compact.

The classical Compactness theorem is the statement saying that first order logic is (ω, ∞)-compact.

As we said, most strong logics are not (ω, ∞)-compact; but sometimes, under some large cardinal assumptions, they can be (κ, ∞)-compact for some cardinal κ. A famous example of the connection between large cardinals and compactness properties is given by Magior's famous result that $\mathcal{L}^2$ is (κ, ∞)-compact if and only if κ is the first extendible cardinal.

By using the proof in [70, Theorem 2.3.4] one can show the following theorem:

Theorem 6.68. *If the logic $\mathcal{L}^*$ is (κ, ∞)-compact then $\mathsf{ULST}_\infty(\mathcal{L}^*) \leq \kappa$.*

So, half of the usual connection can be lifted to the general case. It is therefore natural to ask under which conditions the second part of the classical relationship between upward Löwenheim-Skolem theorem and Compactness theorem can also be generalised.

Question 6.69. Assume that κ is a regular cardinal. For which logics $\mathsf{ULST}_\infty(\mathcal{L}^*) \leq \kappa$ implies that $\mathcal{L}^*$ is (κ, κ)-compact?

Finally, given the results in this chapter and in the literature it is natural to ask if set theoretic reflection principles can be used to study compactness properties of strong logics:

Question 6.70. Is there a reflection principle with an associated concept of a R-reflection number for which we can show the following result: If R and $\mathcal{L}^*$ are symbiotic, then the R-reflection number is κ if and only if $\mathcal{L}^*$ is (κ, ∞)-compact.

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Summary

In Chapter 2 we briefly introduce the two generalised versions of the real line studied in this thesis. Then, we use these spaces in the context of generalised metrisability theory and generalised descriptive set theory. In particular, we use generalised metrisability theory to define a generalised notion of Polish spaces which we will compare and combine with the game theoretical notion introduced by Coskey and Schlicht in [22]. The main results of this chapter are illustrated in the following diagram which shows that a partial generalisation of the classical equivalence between Polish spaces, G_δ spaces, and strongly Choquet spaces (see [51, Theorem 8.17.ii]) can be proved in the generalised context:

$$\begin{tikzcd}
Y \text{ is strongly } \lambda\text{-Polish} \arrow[r] \arrow[dr, shift left] & Y \text{ is strongly } \lambda\text{-Choquet} \arrow[r, dotted, shift left] & Y \text{ is } \lambda\text{-}G_\delta \text{ in } X \arrow[l, dotted, shift left] \\
& \text{Y is } \lambda\text{-Polish} \arrow[ul, shift left, "\times" marking] \arrow[u, "\times" marking] \arrow[ur] &
\end{tikzcd}$$

In the previous diagram an arrow from A to B means that A implies B; a crossed arrow from A to B means that A does not imply B; and dotted arrows are used to emphasise the fact that further assumptions on Y or λ are needed. See p. 25 for a complete explanation of these results.

In Chapter 3 we study generalisations of the Bolzano-Weierstraß and Heine-Borel theorems. We consider various versions of these theorems and we fully characterise them in terms of large cardinal properties of the cardinal underlining the generalised real line. In particular we prove the following:

Corollary (Corollary 3.23, p. 53)**.** *Let κ be an uncountable strongly inaccessible cardinal and let $(K, +, \cdot, 0, 1, \leq)$ be a Cauchy complete and κ-spherically complete totally ordered field with* $\mathrm{bn}(K) = \kappa$*. Then the following are equivalent:*

1. *κ has the tree property and*

2. *κ-wBWT_K holds.*

In particular κ has the tree property if and only κ-$\mathsf{wBWT}_{\mathbb{R}_\kappa}$ holds.

In Chapter 4 we use the generalised real line to develop two new models of transfinite computability, one generalising the so called type two Turing machines and one generalising Blum, Shub and Smale machines, i.e, a model of computation introduced by Blum, Shub and Smale in order to define notions of computation over arbitrary fields. Moreover, we use the generalised version of type two Turing machines to begin the development of a generalised version of the classical theory of Weihrauch degrees. In Chapter 4 we prove the following generalised version of a classical result in the theory of Weihrauch degrees:

Theorem (Theorem 4.24, p. 68)**.**

1. *If there exists an effective enumeration of a dense subset of $\mathbb{R}_\kappa$, then* $\mathrm{IVT}_\kappa \leq_{\mathrm{sW}} \mathrm{B}_{\mathrm{I}}^\kappa$.

2. *We have* $\mathrm{B}_{\mathrm{I}}^\kappa \leq_{\mathrm{sW}} \mathrm{IVT}_\kappa$.

3. *We have* $\mathrm{IVT}_\kappa \leq^{\mathrm{t}}_{\mathrm{sW}} \mathrm{B}^\kappa_{\mathrm{I}}$*, and therefore* $\mathrm{IVT}_\kappa \equiv^{\mathrm{t}}_{\mathrm{sW}} \mathrm{B}^\kappa_{\mathrm{I}}$.

The last two chapters of this thesis are the result of the work of the author on topics in logic which are not directly related to generalisations of the real number continuum.

In Chapter 5 we study the possible order types of models of syntactic fragments of Peano arithmetic. The main result of this chapter is that the following arrow diagram between fragments of PA is *complete with respect to order types of their models*. By this we mean that an arrow from the theory T to the theory T' means that every order type occurring in a model of T also occurs in a model of T' and a missing arrow means that there is a model of T of an order type that cannot be an order type of a model of T'.

$$\begin{tikzcd}
& \mathsf{SA} & \\
\mathsf{Pr}^- \arrow[ur] & & \mathsf{Pr} \arrow[ul] \arrow[ll] \\
\mathsf{PA}^- \arrow[u] & & \mathsf{PA} \arrow[u] \arrow[ll]
\end{tikzcd}$$

In Chapter 6 we study Löwenheim-Skolem theorems for logics extending first order logic. In particular, we extend the work done by Bagaria and Väänänen in [5] relating upward Löwenheim-Skolem theorems for strong logics to reflection principles in set theory. Our main result in this area is the following theorem:

Theorem (Theorem 6.49, p. 123)**.** *Let* $\mathcal{L}^*$ *be a logic and* R *be a predicate in the language of set theory such that* $\mathcal{L}^*$ *and* R *are bounded symbiotic and* $\mathcal{L}^*$ *has* $\mathrm{dep}(\mathcal{L}^*) = \omega$ *and is* $\Delta^{\mathrm{B}}_1(R)$*-finitely-definable. Moreover, let* λ *be a cardinal such that there is a sequence* $(\delta_n)_{n\in\omega}$ *of* $\Delta^{\mathrm{B}}_1(R)$*-definable cardinals such that* $\bigcup_{n\in\omega} \delta_n = \lambda$*. Then the following are equivalent:*

1. $\mathsf{ULST}_\lambda(\mathcal{L}^*) = \kappa$ *and*

2. $\mathcal{USR}_\lambda(R) = \kappa$.

In particular, the statement holds for $\lambda = \omega$ *and in general for all the logics in [5, Proposition 4].*

Finally, we apply the previous result to the study of the large cardinal strength of the upward Löwenheim-Skolem theorem for second order logic; we provide both upper and lower bounds.

Zusammenfassung

In Kapitel 2 führen wir zwei Räume ein, welche die reellen Zahlen verallgemeinern. Diese Räume gebrauchen wir dann im Zusammenhang mit verallgemeinerter Metrisierbarkeitstheorie und verallgemeinerter deskriptiver Mengenlehre. Insbesondere benutzen wir verallgemeinerte Metrisierbarkeitstheorie um eine verallgemeinerte Version von polnischen Räumen zu definieren, welche wir mit den spieltheoretischen Ideen von Coskey und Schlicht in [22] vergleichen. Die Hauptergebnisse dieses Kapitels sind im folgenden Diagramm illustriert, das zeigt, dass wir eine Verallgemeinerung der klassischen Äquivalenz zwischen polnischen Räumen, G_δ-Räumen und starken Choquet-Räumen (siehe [51, Theorem 8.17.ii]) beweisen können.

$$
\begin{tikzcd}
Y \text{ ist stark } \lambda\text{-Polnisch} \arrow[rr] \arrow[drr, shift left] & & Y \text{ ist stark } \lambda\text{-Choquet} \arrow[rr, dotted, shift left] & & Y \text{ ist } \lambda\text{-G}_\delta \text{ in } X \arrow[ll, dotted, shift left] \\
& & \text{Y ist } \lambda\text{-Polnisch} \arrow[ull, "/" marking, shift left] \arrow[u, "/" marking] \arrow[urr] & &
\end{tikzcd}
$$

Dass die Aussage A die Aussage B impliziert wird im obigen Diagramm durch einen Pfeil von A nach B dargestellt; ein durchgestrichener Pfeil von A nach B bedeutet, dass B nicht durch A impliziert wird; gepunktete Pfeile werden gebraucht, um die Tatsache zu betonen, dass weitere Annahmen bezüglich Y oder λ notwendig sind, um die entsprechende Aussage zu beweisen. Siehe Seite 25 für eine vollständige Erklärung dieser Resultate.

In Kapitel 3 studieren wir Verallgemeinerung der Sätze von Bolzano-Weierstraß und Heine-Borel. Wir betrachten verschiedene Varianten dieser Sätze und charakterisieren diese vollständig bezüglich der großen Kardinalzahleigenschaften der verallgemeinerten reellen Zahlen. Insbesondere beweisen wir das folgende Resultat:

Corollary (Corollary 3.23, p. 53)**.** *Sei κ eine überabzählbare stark unerreichbare Kardinalzahl und $(K, +, \cdot, 0, 1, \leq)$ ein Cauchy-vollständiger und κ-sphärisch-vollständiger, vollständig geordneter Körper mit $\mathrm{bn}(K) = \kappa$. Dann sind die folgenden Aussagen äquivalent:*

1. *κ hat die Baumeigenschaft und*

2. *κ-wBWT_K gilt.*

Insbesondere hat κ die Baumeigenschaft genau dann, wenn κ-$\mathsf{wBWT}_{\mathbb{R}_\kappa}$ gilt.

In Kapitel 4 benutzen wir die verallgemeinerten reellen Zahlen um zwei neue Modelle der transfiniten Berechenbarkeit zu entwickeln. Wir verallgemeinern sowohl die sogenannten Typ-Zwei Turingmaschinen als auch Blum, Shub und Smale Maschinen. Letztere Maschinen sind ein von Blum, Shub und Smale eingeführtes Modell der Berechenbarkeit, das es erlaubt, Berechenbarkeit über beliebigen Körpern zu definieren. Darüber hinaus gebrauchen wir die verallgemeinerte Version der Typ-Zwei Turingmaschinen um die Entwicklung verallgemeinerter Weihrauchränge zu beginnen. In diesem Kapitel beweisen wir die folgende Verallgemeinerung eines klassischen Resultats in der Theorie der Weihrauchränge.

Theorem (Theorem 4.24, p. 68)**.**

1. *Wenn eine effektive Aufzählung einer dichten Teilmenge von $\mathbb{R}_\kappa$ existiert, dann gilt $\mathrm{IVT}_\kappa \leq_{\mathrm{sW}} \mathrm{B}_{\mathrm{I}}^\kappa$.*

2. *Es gilt, dass $\mathrm{B}_{\mathrm{I}}^\kappa \leq_{\mathrm{sW}} \mathrm{IVT}_\kappa$.*

3. *Es gilt, dass $\mathrm{IVT}_\kappa \leq_{\mathrm{sW}}^{\mathrm{t}} \mathrm{B}_{\mathrm{I}}^\kappa$, und somit folgt, dass $\mathrm{IVT}_\kappa \equiv_{\mathrm{sW}}^{\mathrm{t}} \mathrm{B}_{\mathrm{I}}^\kappa$.*

Die letzten beiden Kapitel dieser Arbeit sind das Resultat von Arbeiten des Autors, die sich nicht direkt mit Verallgemeinerungen der reellen Zahlen beschäftigen.

In Kapitel 5 studieren wir die mögliche Struktur der Ordnungstypen von Modellen syntaktischer Fragmente der Peanoarithmetik. Das Hauptresultat dieses Kapitels ist die Vollständigkeit des folgenden Pfeildiagramms zwischen Fragmenten von PA bezüglich der Ordnungstypen der Modelle der Fragmente. Das bedeutet, dass ein Pfeil von der Theorie T zur Theorie T' angibt, dass jeder Ordnungstyp eines Modells von T auch Ordnungstyp eines Modells von T' ist. Ein ausgelassener Pfeil bedeutet, dass es ein Modell T eines Ordnungstyps gibt, das nicht der Ordnungstyp eines Modells von T' sein kann.

$$\begin{tikzcd}
 & \mathsf{SA} & \\
\mathsf{Pr}^- \arrow[ur] & & \mathsf{Pr} \arrow[ul] \arrow[ll] \\
\mathsf{PA}^- \arrow[u] & & \mathsf{PA} \arrow[u] \arrow[ll]
\end{tikzcd}$$

In Kapitel 6 studieren wir Löwenheim-Skolem Sätze für Logiken, welche die Logik der ersten Stufe erweitern. Insbesondere setzen wir die Arbeit von Bagaria und Väänänen in [5] fort, in der aufwärts Löwenheim-Skolem Sätze für starke Logiken mit Reflexionsprinzipien in der Mengenlehre verknüpft werden.

Theorem (Theorem 6.49, p. 123)**.** *Sei $\mathcal{L}^*$ eine Logik und R ein Prädikat in der Sprache der Mengenlehre, sodass $\mathcal{L}^*$ und R beschränkt symbiotisch sind. Für $\mathcal{L}^*$ gelte $\mathrm{dep}(\mathcal{L}^*) = \omega$ und $\mathcal{L}^*$ sei $\Delta_1^{\mathrm{B}}(R)$-endlich-definierbar. Ferner sei λ eine Kardinalzahl sodass eine Folge $(\delta_n)_{n\in\omega}$ von $\Delta_1^{\mathrm{B}}(R)$-definierbaren Kardinalzahlen mit $\bigcup_{n\in\omega}\delta_n = \lambda$ existiert. Dann sind die folgenden Aussagen äquivalent:*

1. $\mathsf{ULST}_\lambda(\mathcal{L}^*) = \kappa$ *und*
2. $\mathcal{USR}_\lambda(R) = \kappa$.

Insbesondere gilt diese Aussage für $\lambda = \omega$ und im Allgemeinen für alle Logiken in [5, Proposition 4].

Schließlich wenden wir das obige Resultat zum Studium der Kardinalzahlstärke des aufwärts Löwenheim-Skolem Satzes für die Prädikatenlogik der zweiten Stufe an; wir bestimmen sowohl eine untere als auch eine obere Schranke.

List of Publications

- Chapter 3: M. Carl, L. Galeotti, and B. Löwe. The Bolzano-Weierstrass theorem in genralised analysis. *Houston Journal of Mathematics*, 44(4):1081–1109, 2018.
- Chapter 4: L. Galeotti and H. Nobrega. Towards computable analysis on the generalised real line. In J. Kari, F. Manea, and I. Petre, editors, *Unveiling Dynamics and Complexity: 13th Conference on Computability in Europe, CiE 2017, Turku, Finland, June 12–16, 2017, Proceedings*, volume 10307 of *Lecture Notes in Computer Science*, pages 246–257. Springer, 2017.

Results in § 4.3 will be published in the proceedings volume of the conference Computability in Europe 2019: L. Galeotti Surreal Blum-Shub-Smale Machines.

- Chapter 5: L. Galeotti and B. Löwe. Order types of models of reducts of Peano Arithmetic and their fragments. ILLC Publications PP-2017-10, 2017.