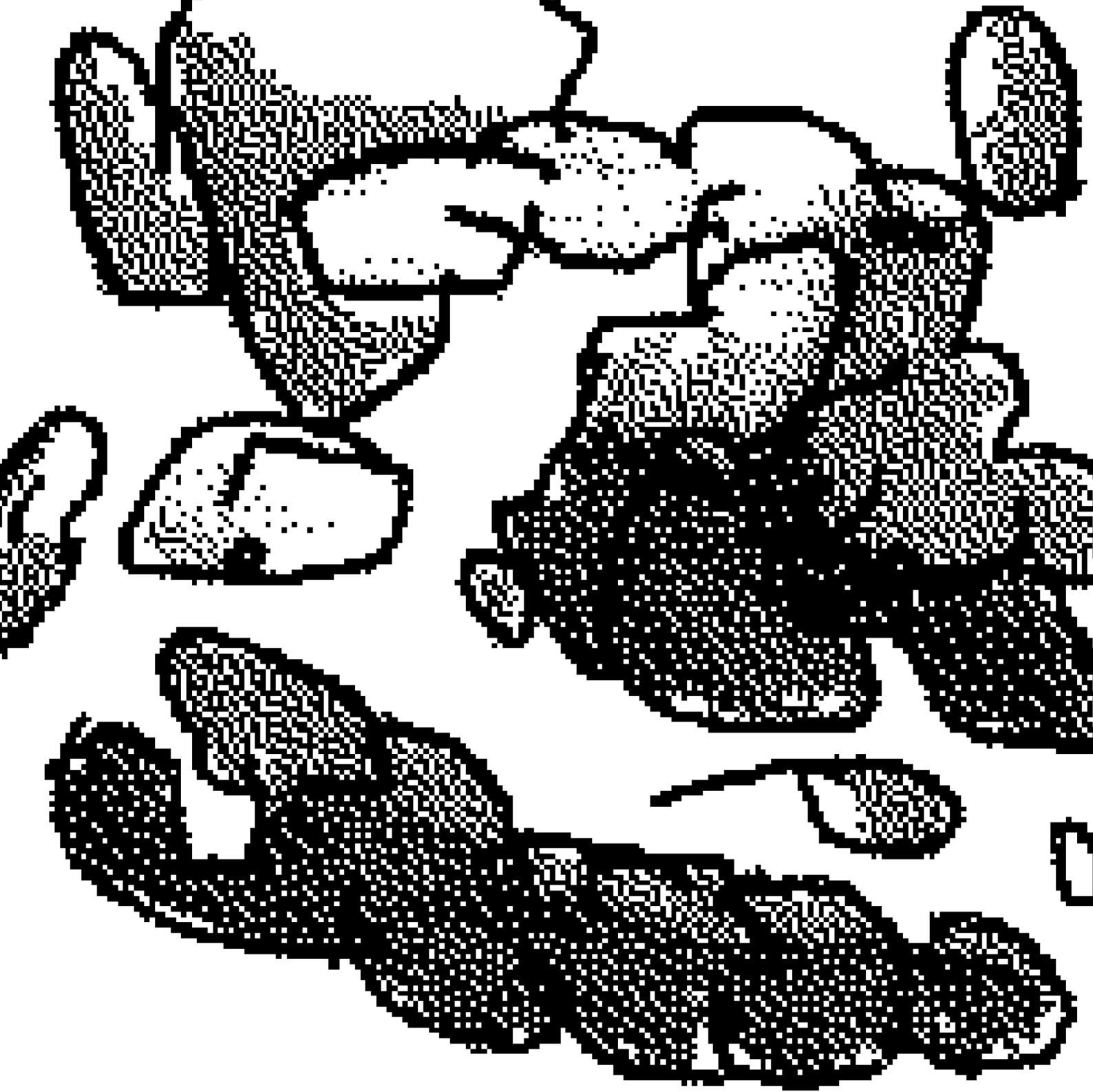


On the Path



Logical & Computational Aspects of Learning
Ana Lucía Vargas Sandoval

to the Truth

On the Path to the Truth
Logical & Computational Aspects of
Learning

Ana Lucía Vargas Sandoval

On the Path to the Truth
Logical & Computational Aspects of
Learning

ILLC Dissertation Series DS-2020-07



INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

For further information about ILLC-publications, please contact

Institute for Logic, Language and Computation
Universiteit van Amsterdam
Science Park 107
1098 XG Amsterdam
phone: +31-20-525 6051
e-mail: illc@uva.nl
homepage: <http://www.illc.uva.nl/>

The investigations for this dissertation were supported by Consejo Nacional de Ciencia y Tecnología CONACyT, México.

Copyright © 2020 by Ana Lucía Vargas Sandoval

Cover design and illustration by Moriz Oberberger.

Vevey Positive typeface by Clemens Piontek.

Printed and bound by Online-Druck GmbH & Co. KG.

ISBN: 978-90-903-3177-5

On the Path to the Truth

Logical & Computational Aspects of Learning

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor
aan de Universiteit van Amsterdam
op gezag van de Rector Magnificus
prof. dr. ir. K.I.J. Maex
ten overstaan van een door het College voor Promoties ingestelde
commissie, in het openbaar te verdedigen
op dinsdag 16 juni 2020, te 12.00 uur

door

Ana Lucía Vargas Sandoval

geboren te Morelia.

Promotiecommissie

Promotor:	Dr. A. Baltag	Universiteit van Amsterdam
Co-promotor:	Prof. dr. D.H.J. de Jongh	Universiteit van Amsterdam
Co-promotor:	Dr. A. Özgün	Universiteit van Amsterdam
Overige leden:	Prof. dr. J.F.A.K. van Benthem	Universiteit van Amsterdam
	Dr. N. Bezhanishvili	Universiteit van Amsterdam
	Dr. N. Gierasimczuk	Technical University of Denmark
	Prof. dr. Y. Venema	Universiteit van Amsterdam
	Prof. dr. T. Zeugmann	Hokkaido University

Faculteit der Natuurwetenschappen, Wiskunde en Informatica

Contents

Acknowledgments	ix
1 Introduction	1
2 Background and Technical Preliminaries	13
2.1 Modal Logics for Knowledge and Belief	14
2.1.1 Public Announcement Logic	18
2.2 Subset Space Semantics	20
2.3 Recursive sets and Recursively enumerable sets	22
2.4 Formal Learning Theory	25
2.4.1 Identification in the limit	27
2.4.2 Finite Identification	28
I Subset Space Logics for Learning	31
3 Dynamic logics for Inductive Learning from Observations	33
3.1 Introduction	33
3.2 Effort modality and knowledge	38
3.2.1 Infallible Knowledge versus Inductive Knowledge	39
3.3 A Dynamic Logic for Learning Theory	44
3.3.1 Syntax and Semantics	45
3.3.2 Axiomatization	49
3.3.3 Expressivity of \mathcal{L}_Π	55
3.3.4 Expressing belief, inductive learning and learning in the limit	56
3.3.5 Soundness of DLIT	60
3.3.6 Completeness of DLIT	63
3.4 A logic for AGM learning from partial observations	74
3.4.1 Syntax and Semantics	74

3.4.2	Expressivity of language \mathcal{L}_{Π_L}	78
3.4.3	Axiomatization	82
3.4.4	Soundness of AGML	83
3.4.5	Completeness of AGML	85
3.5	Conclusions and Open Questions	90
4	Arbitrary Public Announcement Logic with Memory	95
4.1	Introduction	95
4.2	The “issue” in APAL and our solution	98
4.3	Arbitrary Public Announcement Logic with Memory	101
4.3.1	Syntax and Semantics of APALM	101
4.3.2	An Analysis of Kuijjer’s counterexample	104
4.3.3	Expressivity of APALM	107
4.3.4	APALM Bisimulation	109
4.3.5	Axiomatization for APALM	114
4.4	Group Announcement Logic with Memory: GALM	119
4.5	Soundness of GALM and APALM	123
4.6	Completeness of GALM and APALM	136
4.7	Conclusions and Future Work	148
II	Finite Identification with positive data (pfi) and with complete data (cfi)	151
5	Structural differences between pfi & cfi	153
5.1	Introduction	153
5.2	The ignored value of negative data	155
5.3	Finite families of languages	156
5.4	Looking for maximal learnable families	157
5.4.1	Do maximal necfi/cfi families exist?	160
5.5	Counting maximal extensions	163
5.5.1	Uncountably many maximal nepfi extensions	165
5.5.2	The class of families with only singletons	166
5.5.3	The class of families with only pairs	167
5.5.4	The class of families with only n -tuples ($n \geq 3$)	174
5.5.5	Families with pairs and triples	183
5.6	Conclusions and future work	199
6	Computational differences between pfi and cfi	201
6.1	Introduction	201
6.2	Extended Characterization Theorem	204
6.3	Families of singletons and pairs	206
6.3.1	The non-existence of maximal pfi families	209

6.3.2 Non-canonical families	212
6.4 Non-pfi anti-chains of singletons and pairs	215
6.5 Fastest learning	221
6.6 Finite identification from queries	225
6.7 Conclusion and future research	231
A Technical Specifications of Part II	233
A.1 Complexity order on formulas in DLLT and in AGML	233
A.2 Complexity order on formulas in APALM and GALM	235
Bibliography	237
Samenvatting	247
Abstract	249

Acknowledgments

This thesis is the outcome of a long journey shaped by diverse thoughts, people, and experiences, resulting in one constellation of ideas.

I would like to start thanking my supervisors: Alexandru Baltag, Dick de Jongh and Aybüke Özgün. They formed the most supportive team, academically and personally, in every aspect of this process. Thank you, Alexandru, for inspiring me immensely every day with your research and your passion for it (and for nice beer and cheese), for your excitement and effort into our work, for sharing with me your endless spark, and for your always honest advice. I also thank you for genuinely caring about my wellbeing during these four years. Thank you, Aybüke, for providing your sharp intellect, your thoroughness, and commitment to our project. I am grateful for your patience and kindness in answering all my questions, for helping me structure my (often messy) drafts, for your generous reading, and most importantly, for your friendship. Last but not least, a warm thanks to you, Dick; you have been my mentor since I was a master student. Your continuous curiosity in every meeting led to new questions and encouraged me to pursue a PhD. I could not finish listing all the values that I learned from you. Thank you for your wise academic guidance with your brilliant mathematical intuition, with your trust in me and your strict methodology that positively shaped me and also our research project. Thank you, also, for writing the Dutch translation of the abstract.

I also thank my doctoral committee, Johan van Benthem, Nick Bezhanishvili, Nina Gierasimczuk, Yde Venema, and Thomas Zeugmann, for taking the time and effort to read my dissertation, especially during these extraordinary times of COVID-19. Your research was a constant inspiration for many of the ideas developed in this book.

During these four years, I had the opportunity to exchange ideas and to collaborate with brilliant minds within and beyond the ILLC. An affectionate thanks to my co-authors, Nina Gierasimczuk and Sonja Smets, for your bright insights, your acute comments, and for investing your time in the work that inspired Chap-

ter 3 of this thesis. On a more personal note: Thank you, Nina, for introducing me to Formal Learning Theory and its connection with DEL, your work strongly inspired Chapters 3, 5, and 6 of this thesis. Thank you, also, for patiently listening to my ideas, doubts and opinions (academic and personal) since I was a master student. It is always a pleasure talking to you. Thank you, Sonja, for your constant help in difficult times, for your encouraging words, and big smile when I needed it the most. You both are amazing role models for me.

A very special word of thanks to Sebastiaan Terwijn. In the initial stages of our work on anti-chains he was of essential assistance by showing us a way to obtain non-canonical indexable anti-chains of finite languages. It pointed us in the right direction for our results in Section 6.4, in particular Proposition 6.4.2.

This process would have never been the same without my ILLC friends and colleagues. I want to start by thanking all the people from the LIRa seminar, organizers and participants, for always creating an inclusive atmosphere and teaching me about many interesting topics and fresh ideas (often with a cold beer in our hands). I also want to thank all my PhD friends, both those already graduated and those still in progress, for many and various valuable exchanges, for making the PhD path less lonely, and for always showing your smiles at the one o'clock lunches. A loving thanks to Iris and Zoi: to Iris for her wise advice, to Zoi for her comforting words, and to both for being exceptional friends. I am also extremely grateful to the whole ILLC community; each member contributes to the magical and unique atmosphere that characterizes this institute. My sincere thanks to the whole ILLC office for supporting me in all the practical matters. A special word of thanks to Jenny Batson and Tanja Kassenaar who helped me tackle some complicated external difficulties with ease and care, for always having time in their busy schedules, and an open door for me.

I am very grateful to my first academic mentors, Fernando Hernandez and David Meza. Their research and fun lectures inspired me to continue learning about topology and logic during and after my bachelor. They were the first to tell me about this magical place called the ILLC.

Being away from my family and home was not so difficult due to the constant support of my (extended) Amsterdam-family. I owe special thanks to all my friends from La Banda (Adu, Aldo, Despi, Dimi, Eileen, Hari, Marianita, Nigel, Oli, Stephen, and Vane) and my Cornelis Drebbelstraat flatmates (Karin, Levin, Saga, and Zoi) who filled me with their bliss during my darkest moments. Thank you for our countless moments together, for showing me your hospitality and joy for life. You contribute deeply to the person I am today. A warm thanks to my paranymphs: Aldo and Marianita. Thank you, Aldo, for your generous friendship, for many excellent conversations, and for your useful insights and English corrections in my general introduction. Thank you, Marianita, for being my soul sister, for making me learn by questioning and for being my sun during the greyest days in Amsterdam. I also want to thank my relatives and friends from Denmark, US, Germany, Mexico, Portugal, and Switzerland for showing your honest curiosity

about my PhD project and for listening to my complicated answers. Thank you Clemens, for your beautiful typeface on my cover.

Mi querido Moriz, I thank you deeply for accompanying me in this rollercoaster. Our adventure together and my PhD started almost at the same time and I could not be more grateful for that. Thank you for respecting my long working hours, for taking care of me when I had stressful deadlines, and for cooking the most amazing Italian food when I was immersed in my writing. I thank you also for designing the cover of this book. Gracias por tu infinita paciencia y por tu colorido amor empático de cada día.

My final words of thanks are for my family, the two Marthas and the two Gerardos of my life. You were my first teachers, so this book is dedicated to you. The warmest thanks to my parents, Martha and Gerardo, for showing me their unending love and support in every step of this long journey. Gracias Mama, for illuminating me with your optimism and for teaching me perseverance. You were the light that helped me find the balance between reason and intuition. Gracias Papa, my fondness for science started in my childhood because of you. You helped me to develop an analytic voice accompanied with a curious tone. Gracias (sister) Martha and (brother) Gerardo for challenging my thoughts and for pushing me to learn from different perspectives.

¡Gracias a todos por tanto!

Ana Lucía Vargas Sandoval

Chapter 1

Introduction

This thesis is concerned with how one learns from incoming information. It addresses learning as the process ensuing from observations, from announcements, from revising one's beliefs, and from stabilizing on a correct hypothesis. In the body of work presented here, we study various perspectives on both learning and its relation with knowledge & belief. Our approach is formal in nature, and we mostly focus on the process of inferring general conclusions from incoming data that are spread over more than one step. This process is known as *inductive inference* (or *inductive learning*). We use logic systems and mathematical/computational frameworks in order to analyze multiple mechanisms involved in said process. To be specific, our work is based on two areas that independently study dynamics of information: Dynamic Epistemic Logic (DEL) and Formal Learning Theory (FLT).

What are we if not continual learners. From the early stages of our life and throughout our adulthood we acquire information on a regular basis. Our human ancestors learned about their environment both by receiving evidence about changes in the world—in the form of *factual observations* and *truthful announcements*—and by observing actions of others. Learning successfully increased their ability to advantageously face new challenges, to avoid danger, and to survive. Acquiring truthful information plays a pivotal role in the way humans form beliefs and act on them. In fact, even with their constrained and limited capabilities to experience the *reality* of the world, humans are able to form general *theories* about the world. When not falsified, these theories (or *hypotheses*) may transform into (a certain kind of) knowledge.

The general concept of *learning* encompasses the one-step changes in the information state of a learner, together with their *long-term horizon*. To illustrate the latter, consider when children learn a mother language from a scattered sample of utterances. Having this capacity seems to be one of the trademarks of *human intelligence*. The acquired language is expressively rich and complex, and it allows children to communicate effectively with people in their community. As natural as

learning may seem to be, understanding its properties and components is not an easy task. Within the fields of artificial intelligence and cognitive science, analyzing learning has been crucial in the ongoing challenge of modelling and designing “intelligent” systems.

On the one hand, Dynamic Epistemic Logic (DEL) studies the process of incorporating new information into one’s prior epistemic/doxastic state in a step-by-step manner, from a modal logic perspective. On the other, Formal Learning Theory studies the long-term mechanism of learning, from a mathematical and computational perspective. In what follows, we present a general overview of the developments in these two areas that are relevant for this thesis. After that we point out some of the questions and issues that we solve, and then we provide a summary of the main contributions of our work.

Dynamic Epistemic Logic is a generic term for a family of modal logics of information dynamics. It employs mathematical tools to reason about knowledge, belief, and the flow of information from an agent’s current epistemic/doxastic state to the next one (Baltag et al., 1998; van Benthem, 2011; van Ditmarsch et al., 2007). Applications of DEL address issues in epistemology, economics, artificial intelligence, and theoretical computer science (for a general overview see e.g., van Ditmarsch et al., 2015a; Baltag and Renne, 2016).

Technically speaking, Dynamic Epistemic Logic incorporates *dynamics* into static modal logics for knowledge & belief, which are respectively called epistemic logic and doxastic logic. This family of modal logics originates from the seminal work of Hintikka (1962), inspired by the thoughts of von Wright (1951). In his work, Hintikka (1962) presents a precise modal language to naturally “talk about” *knowledge & belief*. These epistemic notions are modelled by interpreting them as normal modal operators in the standard possible worlds relational semantics (Kripke semantics). Roughly speaking, a relational (Kripke) model in an epistemic/doxastic context represents a “screen shot” of the actual information state of a learner. DEL was developed as an extension of the aforementioned static logic in order to capture the *step-by-step* changes of a learner’s epistemic/doxastic state brought about by new information (Baltag and Renne, 2016). The standard way to represent these changes in DEL is with the so-called dynamic modalities. These modalities encode actions (or *updates*) that transform a model into a new model called an *updated* model. Dynamic operators allow us to analyze in a simple way the epistemic/doxastic changes after an *epistemic action* has taken place.

One of the most widely studied epistemic actions is the action of incorporating a new piece of truthful information received via a public announcement into the current epistemic state of the learner. The corresponding dynamic modal logic is the so-called Public Announcement Logic (PAL) (Plaza, 1989; Gerbrandy and Groeneveld, 1997; van Ditmarsch et al., 2007; Balbiani et al., 2008). Arbitrary Public Announcement Logic (APAL) and its relatives are natural extensions of PAL. These logics involve the addition of modal operators —like the *arbitrary*

announcement modality in APAL— that quantify over public announcements of some given type. APAL and PAL are of great interest both philosophically and from the point of view of applications. Motivations range from supporting an analysis of Fitch’s paradox by modelling notions of “knowability” (van Benthem, 2004), to determining the existence of communication protocols that achieve certain goals (see e.g., van Ditmarsch, 2003), and more generally, to epistemic planning (Bolander and Andersen, 2011). One problem with APAL is that it uses an *infinitary* axiomatization. In the seminal paper on APAL, Balbiani et al. (2008) proved completeness using an infinitary rule and claimed that it can be replaced by a proposed finitary inference rule in theorem proving.¹ The finitary rule, while natural enough and seemingly suited to capture the universal quantifier implicit in arbitrary announcement modalities, has been proved to be unsound (Kuijjer, 2015). It is therefore not guaranteed that the validities of APAL are recursively enumerable.

An alternative framework that studies dynamics of doxastic states is the so-called AGM belief revision developed by Alchourrón et al. (1985). Their work also includes a formal belief revision mechanism for an agent encountering information contradicting her current beliefs. Motivated by the AGM intuitions and ideas, belief revision policies have been incorporated in the dynamic programme for epistemic & doxastic modal logics (van Benthem, 2007).

All the aforementioned logics have been primarily developed within the relational semantics approach. As it happens, using relational structures to reason about various epistemic notions naturally brings with it a simple mathematical treatment. Therefore, it is not surprising these semantics has been the most widely adopted and the most developed approach in the epistemic logic literature (van Ditmarsch et al., 2015a). However, using relational semantics to reason about learning has its limitations. Various interesting epistemic and doxastic notions that lie at the core of inductive learning cannot be accounted for. In particular, we cannot express *potential incoming information* (or *potential evidence*) that is a crucial component of the *long term* aspect of learning.

Potential evidence has been studied already in the logic literature by using structures called *topological spaces* and also by using generalizations of those called *subset spaces* (Aiello et al., 2007). In fact, the earliest links between topological spaces and logic are from the late 1930’s and 1940’s (Tarski, 1938; McKinsey, 1941; McKinsey and Tarski, 1944) (for further comments on this see e.g. Bezhanishvili et al., 2018). Basic topological notions such as *open sets* naturally represent *pieces of evidence* or, *observable properties* (see e.g., Toelstra and van Dalen, 1988). In the last decades topology and, more generally, subset spaces have been used to model information dynamics in DEL (see e.g., Özgün (2017); Bjorndahl (2018)). The reason for this is that the notion of an open set (in short, open) seems

¹This means that from any proof of a theorem from the axioms that uses the infinitary rule we can obtain a finitary proof of the same theorem, by using the finitary rule instead.

naturally suited to representing information states (Vickers, 1989). Furthermore, *shrinking an open* into a smaller one can account for any kind of evidence gathering leading to knowledge, i.e., a type of *epistemic effort* (Moss and Parikh, 1992; Dabrowski et al., 1996; Parikh et al., 2007).

Moss and Parikh (1992) were the first to have talked about epistemic effort in logic terms. They presented an epistemically motivated bimodal logic to reason about sets and points using subset space semantics. Their framework, called *Topologic*, resulted in a novel approach to model epistemic effort such as *observational effort*, or information gain via *measurement*, or *announcement* (Moss and Parikh, 1992; Dabrowski et al., 1996; Georgatos, 1994, 1997; Parikh et al., 2007). Technically speaking, epistemic effort is captured by a modal operator called the *effort* modality. This modal operator quantifies over collections of opens in a subset space which represent potential incoming evidence. Therefore, in topologic one can distinguish between the actual information state of the learner and the potential evidence that she may acquire later on. Moss and Parikh (1992) and Dabrowski et al. (1996) formalize the learning theoretic notions of *learnability with certainty*, *verifiability/falsifiability/decidability with certainty* using Topologic. Further work are the extensions of topologic that include updates with a topological interpretation used to model *evidence based knowledge* and *knowability* (Baltag et al., 2017; Özgün, 2017; Bjorndahl, 2018; Bjorndahl and Özgün, 2019).

We now shift our attention to Formal Learning Theory (FLT). Formal Learning Theory is an umbrella term for a family of mathematical and computational frameworks that study *inductive inference* (or *inductive learning*) by means of a learning function. Such a term refers to the process of conjecture change using incoming information that may result in stabilizing on an accurate hypothesis. Motivations of studying inductive learning range from modelling *children language acquisition* (inferring a grammar from inductively given examples of a language) and *scientific inquiry* (inferring a general hypothesis from an inductively given stream of empirical data).²

Inductive inference in the context of scientific inquiry has entertained and intrigued scientists and philosophers since the 16th-18th century. Bacon (1620), Mill (1843) and Whewell (1858b,a) in their work independently discuss the important role of observations in scientific discovery (Klein, 2016; Bogen, 2017). We quote (Whewell, 1858b, Aphorism XI, p. 6):

Observed facts are connected so as to produce new truths, by superinducing upon them an idea: and such truths are obtained by Induction.

²Formal learning theory often uses recursion-theoretic tools to reason about inductive inference with a computational learner, represented by a *recursive function*, an *algorithm* (or *effective procedure*) or an *inference machine*.

With the emergence of artificial intelligence and machine learning in the 1950's, the study of inductive inference gained attention in the computer science community (for a general overview see e.g., Schulte (2018); Bringsjord and Govindarajulu (2019)). More recent work on this stems from the pioneering formal studies of Putnam (1963), Solomonoff (1964a,b), and Gold (1967). With the aim of modelling children language acquisition, Gold's framework *identification in the limit* (or *learning in the limit*) marked the beginning of a mathematical and computational treatment for inductive learning.

The learning task in Gold's model consists of identifying a language (represented by a set of symbols) amidst a collection of languages on the basis of an infinite stream of examples from the language. The stream of examples consists either of positive data (an enumeration of all members of the language) or of complete data (positive and negative data, labelling all sentences as belonging to the language or not). Learning in the limit considers a learner to be *successful* if it stabilizes on a correct hypothesis after only finitely many mind changes. The fact that such a learner keeps conjecturing forever (even when she already stabilized on a correct hypothesis) suggests that she does not necessarily know when her conjecture is correct. Thus, the learner still entertains the possibility of acquiring contradictory information later on that will force her to change her conjecture (Gold, 1967). On a slightly simpler approach, *finite identification* (or *learning with certainty*) considers a more restricted notion of a successful learner (Mukouchi, 1992; Lange and Zeugmann, 1992). In this framework, a learner can produce just one conjecture that must be correct immediately.

In (Gold, 1967), a huge difference in power between learning with positive data and learning with complete data is exposed. With positive data, a family of languages containing all finite languages and at least one infinite one is not learnable. With complete data the learning task becomes almost trivial, since almost any (computationally interesting) collection of languages can be learnt in such a way. The difference in learning power between finite identification with positive information (pfi) and finite identification with complete information (cfi) has been briefly discussed and settled in Mukouchi (1992). As expected, cfi is more powerful than pfi. Still, this result and other relevant discussions in the literature do not provide a detailed analysis that make the differences between pfi and cfi explicit (for a general overview and discussion see Zeugmann and Lange (1995)).

On a different approach in inductive inference, Kelly (1996) focuses on, what he calls, *reliable* learning. In his work, Kelly (1996) addresses possible mathematical conditions guaranteeing the learner to eventually converge to the truth. He uses probability theory, topology and logic together with formal epistemology as a basis for addressing learning theoretic questions.

One can see, from what we have discussed so far, the close proximity of DEL and FLT, regardless of the distinctness of their methodologies (for a detailed discussion of this, we refer to Gierasimczuk, 2010, Chapter 3, p. 27–37). This

proximity has recently gained attention in attempts to jointly analyse scientific inquiry (inductive learnability), information processing, and other learning theoretic notions (for a detailed summary of this connection, see Gierasimczuk et al., 2014). In particular, the connection between FLT and DEL originates from their common interest concerning the process of *learning that leads to knowledge*. Some of their shared relevant notions are solvability, success, Ockham’s razor, learner, learning situations, and learning methods, just to mention some. By bringing both frameworks together we have the step-by-step information changes from DEL, together with their long-term learning horizon in FLT. A combined study gives us a way to model the two aforementioned processes in learning.

To the best of our knowledge, all the logics connecting these two areas of research have been studied with positive information only and mostly for finite identification within a relational semantics approach (see e.g., Gierasimczuk (2010)). More recent results are in Dégremont and Gierasimczuk, 2011; Bolander and Gierasimczuk, 2015, 2017, and for a brief summary see (Gierasimczuk et al., 2014). Almost nothing of the kind has been developed for learning in the limit. In more recent work, Baltag et al. (2015) use topology to reason about *learning in the limit* and the *solvability of inductive problems*. In fact, the authors provide a topological characterization for the class of inductive problems that are solvable.

Issues and open questions that motivate this dissertation. A considerable gap between DEL and FLT for a wider treatment of learning remains to be filled. In this dissertation we tackle some of the issues, difficulties, and open questions that lie in between DEL and FLT. We state some of these issues here.

- For all we know, there are no modal logics in the style of DEL that formalize learning in the limit and other relevant learning theoretic notions.³
- The long standing open question of finding a recursive axiomatization for a strong version of Arbitrary Public Announcement Logic (APAL) —as mentioned above.
- The lack of a deeper comparative analysis of the structural and computational properties of finite identification with positive and with complete data that prevents us to point out concretely the differences between these two formalisms. This complicates the task of bringing DEL and FLT closer together even more.

Contributions of this thesis. We now present a detailed description of our main contributions to the aforementioned body of work.

We develop the connection between DEL and FLT even further in Part II. Motivated by the work in (Moss and Parikh, 1992; Dabrowski et al., 1996), (Baltag

³Other frameworks using modal operators —but not in the style of DEL— have been developed in order to study inductive knowledge and inductive inference, see e.g., (Kelly, 2014).

et al., 2015) and (Baltag et al., 2017; Özgün, 2017), in Chapter 3 and Chapter 4 we present dynamic modal logics that use the topologic toolkit together with dynamic modalities to reason, on the one hand, about inductive learning via observational effort and, on the other, about truthful arbitrary public announcements in multi-agent scenarios.

Our first logic, called Dynamic Logic for Learning Theory (DLLT), uses subset space semantics and the notion of a learning operator—in the spirit of FLT—to model inductive learning (learning in the limit) in a novel way. We adopt the same high degree of freedom that FLT gives to the learner, allowing the choice of *any learning method* that produces conjectures *based on the data*.

Subset space semantics and the concept of epistemic effort in (Moss and Parikh, 1992; Dabrowski et al., 1996) are precisely what we need in order to capture the notion of *potential incoming evidence* that is crucial for modelling learning in the limit. Semantically, we take intersection spaces (a type of subset spaces that are closed under finite non-empty intersections), with points interpreted as possible worlds and neighbourhoods interpreted as *observational evidence* (or information states). We endow these structures with a learning function, mapping every information state to a conjecture, representing the learner’s strongest belief in this state. The language of DLLT has simple observational variables that capture factual observations about the world. We add dynamic modalities called *observational events* built from simple observational variables and using sequential composition to represent successive observations. With this logic, we are able to formalize in a simple way other relevant learning theoretic notions such as inductive knowledge, inductive verifiability, and inductive falsifiability.

The learner in FLT (and therefore in our Dynamic Logic for Learning Theory) is assumed to satisfy only very few rationality constraints. We are also interested in studying inductive learning with a fully rational learner. For that purpose, our second logic Dynamic Logic for AGM Learning (AGML) extends DLLT in order to model learning in the limit with a *fully rational learner* in the style of AGM belief revision theory. Moreover, in AGML we model inductive learning from partial observations. Partial observations capture the idea of receiving incomplete reports from fully-determined observations or in case the learner is not sure which of (finitely) many determined observations has taken place. Such situations are very common (if not necessary) in empirical sciences. In order to capture partial observations in our semantics, our models are now with respect to lattice spaces (intersection spaces that are closed under finite unions). We also extend the syntax accordingly. We present expressivity, soundness, and completeness results for both logics. We will talk in detail about the technical results with respect to our two systems DLLT and AGML (our expressivity results and the complete axiomatization) at the end of Chapter 3.

We then shift our focus to evidence gathering via public announcements and arbitrary public announcements in scenarios with multiple learners. Our work in

Chapter 4 is strongly motivated by our aim of solving the long-standing open question of providing a recursive axiomatization for APAL. Recall the problem with the unsound finitary rule proposed for APAL in Balbiani et al. (2008) mentioned above. We analyze the reasons for this unsoundness, showing that it is due to the model's "lack of memory," namely that information is lost after updates. We fix this problem by adding to the models a "memory" (recording the initial states before any updates), and we show that this addition makes the critical inference rule sound. This results in finding a recursive axiomatization for a strong version of APAL. A similar problem occurs for APAL's relative Group Announcement Logic (GAL). Thus, we do a similar treatment for Group Announcement Logic succeeding in obtaining a finitary axiomatization for its memory equipped relative. We prove soundness and completeness for the both memory-enhanced logics.

We then put aside the dynamic logic enterprise and focus purely on finite identification in FLT. In particular, we are concerned with obtaining a more fine-grained theoretical analysis of the distinction between finite identification with positive information (pfi) and with complete (positive and negative) information (cfi). The difference between the two, if not as huge as in the limit case (for a general overview and discussion see Zeugmann and Lange (1995)) is, as we will show in Part III of this dissertation, considerable not only in power but also in character.

We first focus on the structural differences of families of languages that are pfi and families that are cfi without taking into account the computational aspects (Chapter 5). Thus, everything we do in this chapter is with respect to non-effective finite identification. In particular, we investigate whether any finitely identifiable family is contained in a maximal finitely identifiable one. Maximal learnable families are of special interest because any learner who can learn a maximal learnable family can also learn any of its subfamilies. Moreover, it turns out that we obtain more insight into the class of all learnable families if we know more about the class of the maximal ones. First, we address this in the setting of positive data. We get a positive answer for families containing only finite languages. We then address the question in the setting of complete data. We provide a strong negative result concerning maximal learnable families for effective or non-effective finite identification with complete data. Any finitely identifiable family can be extended to a larger one which is also finitely identifiable. Therefore, maximal identifiable families do not exist in the case of complete data.

Next, we study how many maximal extensions a positively identifiable family has. We concentrate mostly on families with only finite languages. Our leading conjecture is that any positively identifiable family of finite languages either has only finitely many maximal pfi extensions or uncountably many. As it happens, the conjecture is reducible to a purely combinatorial mathematical one. In particular, we address the question for families which all languages have the same

number of elements. We first solve it for families of only pairs and then for families with only n -tuples in general. We also investigate a more complex case, that is the case for families containing pairs and triples. In all these cases, families have either finitely many or uncountably many maximal extensions. The case of families containing pairs and triples is a first step on the way to establish our conjecture for families of restricted cardinality.

In Chapter 6 we are interested in the computational properties of families of languages and whether such properties allow pfi or cfi for a family. In particular, we analyze infinite anti-chains of finite languages, since for such cases it is not yet clear to what extent cfi and pfi agree or disagree. As it happens, most of the obvious examples of such anti-chains are finitely identifiable with positive data, and therefore, also with complete data. Moreover, pfi holds for many maximal anti-chains of finite languages. What can we say in general about these issues? Is every anti-chain of finite languages that is cfi also pfi? Is every maximal anti-chain of finite languages pfi (or cfi)? We will provide negative answers to all of these questions. For this purpose, we will focus on anti-chains of only singletons and pairs. The simple structure of these families makes it easy to study their computational properties. Still, their analysis is not a trivial matter. In particular, we want to analyse cases when certain subfamilies of a given pfi family are decidable or recursively enumerable. This will become handy when we construct negative answers to the questions posed above.

We also investigate a variation of finite identification that considers a learner who identifies a language as soon as it is objectively certain which language it is, called a *fastest learner*. It is clear that such a learner is closer in spirit to DEL than the standard learner in FLT. *Fastest learning* was already studied in (Gierasimczuk and de Jongh, 2013) for the case of positive information. We give a much more perspicuous example than the one in (Gierasimczuk and de Jongh, 2013) showing that fastest learning is more restrictive than pfi. Our example makes the difference between pfi and fastest learning much clearer. We also define a fastest learner for the case of complete information. We show that with fastest learners, pfi and cfi finally come closer when identifying infinite anti-chains of finite languages. Finally, we study *learning by queries*. This is a variation of cfi which considers a more active learner, namely one that can ask queries to the teacher. We show that adding this capacity to the learner does not increase her learning power.

In what follows, we give a brief overview of the structure of the thesis and a short description of each chapter. In general, our chapters start with an introduction, continuing with the developments of its content and concluding with our final remarks and connections to other relevant work.

In Chapter 2 we provide the logic and mathematical preliminaries that are needed for this dissertation.

The main contributions and original work are presented in four chapters divided in two main parts.

In Part I we present various dynamic logic systems addressing different processes involved in learning by information gathering.

In Chapter 3 we first present Dynamic Logic for Learning Theory (DLLT) and use it to formalize learning in the limit. Then we introduce Dynamic Logic for AGM Learning (AGML) that extends DLLT in order to model learning in the limit from partial observations with a *fully rational learner* in the style of AGM belief revision theory. We present expressivity, soundness, and completeness results for both logics.

In Chapter 4 we present Arbitrary Public Announcement Logic with Memory (APALM) and Group Announcement Logic with Memory (GALM), memory-enhanced variants of APAL and GAL. We give a recursive axiomatization for each of them and provide expressivity, soundness, and completeness results for both logics.

In Part II we focus purely on finite identification. We develop a fine-grained theoretical analysis of the distinction between finite identification with positive information (pfi) and with complete information (cfi).

In Chapter 5 we focus on the structural differences of families of languages that are pfi and families that are cfi without taking into account the computational aspects. We investigate whether any finitely identifiable family is contained in a maximal finitely identifiable one, first in the positive data case and then in the complete data case. We then focus on a conjecture of ours that we partially resolve: any positively identifiable family of finite languages either has only finitely many maximal positively identifiable extensions or uncountably many.

In Chapter 6 we are interested in the computational properties of families of languages. In particular, we analyze infinite anti-chains of finite languages. We provide negative answers to the following questions: is every anti-chain of finite languages that is cfi also pfi? Is every maximal anti-chain of finite languages pfi (or cfi)? We then investigate *fastest learning* and *learning by queries*.

Origin of the material

- Chapter 3 is based on three papers where the second paper is an extended version of the first paper:

Baltag, A., Gierasimczuk, N., Özgün, A., Smets, S. and Vargas-Sandoval, A. L. (2017). A dynamic logic for learning theory. In *Proceedings of the 1st Dynamic Logic: New Trends and Applications (DaLi 2017)*, pp. 35–54.

Baltag, A., Gierasimczuk, N., Özgün, A., Smets, S. and Vargas-Sandoval, A. L. (2019). A dynamic logic for learning theory. *Journal of Logical and Algebraic Methods in Programming*.

Baltag, A., Özgün, A., Vargas-Sandoval, A. L. (2020). The logic of AGM learning from partial observations. In *Proceedings of the 2nd Dynamic Logic: New Trends and Applications (DaLi 2019)*, pp. 35–52.

- Chapter 4 is based on two papers where the latter is an extended version of the former:

Baltag, A., Özgün, A., Vargas-Sandoval, A. L. (2018). APAL with memory is better. In *Proceedings of 25th Workshop on Logic, Language, Information and Computation (WOLLIC 2018)*, pp 106-129.

Baltag, A., Özgün, A., Vargas-Sandoval, A. L. (2018). APAL with memory is better. *Under review*.

- Chapter 5 is based on:

de Jongh, D., Vargas-Sandoval, A.L. (2019). Finite identification with positive and with complete data. In *International Tbilisi Symposium on Logic, Language, and Computation (TbiLLC 2017)*, pp. 42–63. Springer.

- Chapter 6 is based on the unpublished manuscript:

de Jongh, D., Vargas-Sandoval, A.L. (2020). Finite identification of anti-chains of finite languages: Positive data vs complete data.

Chapter 2

Background and Technical Preliminaries

In this chapter we present the required background and technical preliminaries that will be used in the chapters that follow.

This dissertation is divided in two main: *Subset Space Logics for Learning* and *Finite Identification with positive data and with complete data*. In the two chapters in Part I we propose logic systems to study various notions of learning, knowledge and belief. In the two chapters in Part II we investigate a mathematical and recursion-theoretic perspective on learning called finite identification. Sections 2.1 - 2.2 introduce the necessary background for Part I. Sections 2.3 - 2.3 present the necessary background for Part II.

In Section 2.1 we present some modal logics for knowledge (Epistemic Logic) and belief (Doxastic Logic) that have been widely studied in the literature. Then, in Section 2.1.1, we present Public Announcement Logic, which is a dynamic extension of the aforementioned logics with dynamic modalities acting as model transformers. In Section 2.2 we introduce a different kind of semantics named Subset Space semantics which is used in our logic frameworks. We then shift our attention to the preliminary notions for Part II. In Section 2.3, we recall recursive sets, recursively enumerable sets and briefly discuss some of its properties. We introduce Formal Learning Theory (FLT) in Section 2.4 and the relevant learning theoretic notions used in this thesis. In Section 2.4.1 we present the learning theoretic framework for inductive learning called *identification in the limit* (or, *learning in the limit*) introduced by Gold (1967). We then state in Section 2.4.2 the core notions and results of a more restricted version of a learning theoretic framework for inductive learning called *finite identification* (or *learning with certainty*), introduced by Mukouchi (1992) and independently by Lange and Zeugmann (1992).

What we assume from the reader: For Chapter 3 and Chapter 4 in Part I, we assume knowledge of core notions of classical propositional logic (CPL) (see e.g. Chagrov and Zakharyashev, 1997, Section 1.3 for an axiomatization of classical propositional logic), the basics of modal logic and relational (Kripke)

semantics (see e.g., Blackburn et al., 2001). In this dissertation we use Hilbert-style systems to formalize syntactically the definitions of the modal logics that we study. For Chapter 5 and Chapter 6 in Part III, we assume that the reader is familiar with basic notions in combinatorial mathematics, the theory of computation and core notions of recursion theory. In particular, we assume knowledge of computation-theoretic notions such as the Church-Turing Thesis, the existence of universal Turing machines, partial recursive functions, and total recursive functions. We also assume familiarity with recursive sets, recursively enumerable sets, the unsolvability of the Halting Problem, and Kleene's T-predicate.

2.1 Modal Logics for Knowledge and Belief

In this section, we briefly introduce some modal logics for knowledge and belief. In the late 1940's and 1950's, von Wright (1951) and others noticed that the properties of knowledge and belief, that were being discussed at the moment, can be expressed in an axiomatic-deductive system (Rendsvig and Symons, 2019). This idea was further developed and formalized in the pioneering work of Hintikka (1962). He used relational semantics for modal logic to formalize knowledge and belief, interpreting those as normal modal operators. As a result, various features of knowledge and belief can be formally investigated by using normal modal logics. Since then, variations of these modal logics have been studied and being further developed (Rendsvig and Symons, 2019).

We start by presenting the basic modal language. Then we shift our attention right away to an epistemic and doxastic approach where normal modal logics of knowledge and belief are discussed. Throughout this thesis, we use the term *agent* or *learner* to refer to the subject for which epistemic/doxastic state is being modelled.

2.1.1. DEFINITION. [Basic Modal Language] Let $\text{Prop} := \{p, q, \dots\}$ be a countable set of propositional variables. The language of basic modal logic \mathcal{L}_\square is defined recursively as

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \square\varphi,$$

where $p \in \text{Prop}$.

We follow the usual rules for the elimination of the parentheses and we employ the usual abbreviations for the Boolean connectives \vee , \rightarrow and \leftrightarrow . We also use \perp as an abbreviation for $p \wedge \neg p$ and $\diamond\varphi$ an abbreviation for $\neg\square\neg\varphi$. The modality \square (in a unimodal language) can be taken to be a knowledge modality K or a belief modality B (we can also have a bimodal language with both K and B , but, for illustrative purposes, it suffices to discuss the unimodal languages). Thus, when talking about knowledge or belief, instead of writing $\square\varphi$, we write $K\varphi$ and $B\varphi$. We often write \mathcal{L}_K (or $\mathcal{L}_{\text{epist}}$) to denote the epistemic unimodal language and \mathcal{L}_B (or \mathcal{L}_{dox}) to denote the doxastic unimodal language.

Later on in the Chapters [3](#) and [4](#), we use \Box explicitly to refer to the *effort modality* from [Moss and Parikh \(1992\)](#).

In [Table 2.1](#) we present some of the most widely studied and discussed axioms and inference rules for knowledge and belief.

Axioms	
$(K_{\Box}) \quad \Box(\varphi \rightarrow \psi) \rightarrow \Box\varphi \rightarrow \Box\psi$	Normality (Kripke's axiom)
$(D_{\Box}) \quad \Box\varphi \rightarrow \neg\Box\neg\varphi$	Consistency
$(T_{\Box}) \quad \Box\varphi \rightarrow \varphi$	Factivity
$(4_{\Box}) \quad \Box\varphi \rightarrow \Box\Box\varphi$	Positive Introspection
$(.2_{\Box}) \quad \neg\Box\neg\Box\varphi \rightarrow \Box\neg\Box\neg\varphi$	Directedness
$(.3_{\Box}) \quad \Box(\Box\varphi \rightarrow \psi) \vee \Box(\Box\psi \rightarrow \varphi)$	Connectedness
$(5_{\Box}) \quad \neg\Box\varphi \rightarrow \Box\neg\Box\varphi$	Negative Introspection
Rules	
$(MP) \quad \text{from } \vdash \varphi \rightarrow \psi \text{ and } \vdash \varphi, \text{ infer } \vdash \psi$	Modus Ponens
$(Nec_{\Box}) \quad \text{from } \vdash \varphi, \text{ infer } \vdash \Box\varphi$	

Table 2.1: Some axiom schemes and an inference rule for \Box

With [Table 2.1](#) in hand, we can talk about logics that are characterized by some of these axioms and inference rules. In this dissertation, (CPL) denotes all instances of classical propositional tautologies (see e.g. [Chagrov and Zakharyashev, 1997](#), Section 1.3). The least subset of \mathcal{L}_{\Box} that contains (CPL) and (K_{\Box}) , and is closed under uniform substitution and under the inference rules (MP) and (Nec_{\Box}) is the weakest (or smallest) *normal modal logic*, denoted by \mathbf{K}_{\Box} . In what follows, $\mathbf{L} + (\varphi)$ denotes the smallest modal logic that contains \mathbf{L} and φ . Below in [Table 2.2](#), we present some of the most common logics to represent knowledge and belief. Having a variety of logics to represent knowledge and belief captures the idea that we can study agents (or learners) with different reasoning power. For instance, the systems $\mathbf{S4}_{\Box}$ and $\mathbf{S5}_{\Box}$ are logics of knowledge of widely different strength.

$$\begin{aligned}
 \mathbf{S4}_{\Box} &= \mathbf{KT}_{\Box} + (4_{\Box}) \\
 \mathbf{S5}_{\Box} &= \mathbf{S4}_{\Box} + (5_{\Box}) \\
 \mathbf{KD45}_{\Box} &= \mathbf{K}_{\Box} + (D_{\Box}) + (4_{\Box}) + (5_{\Box})
 \end{aligned}$$

Table 2.2: Some normal epistemic/doxastic modal logics

Since we work with these logics in an epistemic and doxastic setting, we write the subscript K instead of \square when a logic represents knowledge (with formulas in \mathcal{L}_K) and we write the subscript B when it represents belief (with formulas in \mathcal{L}_B).

In his work, [Hintikka \(1962\)](#) argued for the thesis that $\mathbf{S4}_K$ is the logic of knowledge. The system $\mathbf{S5}_K$ is often used to reason about knowledge in computer science and logics relevant to it ([Fagin et al., 1995](#); [Meyer and Hoek, 1995](#); [van Ditmarsch et al., 2007](#)).¹ The standard and most often adopted system for belief is the so-called $\mathbf{KD45}_B$ (see e.g., [Baltag et al., 2008](#); [van Ditmarsch et al., 2007](#)). In this dissertation, all the logics we work with are based on the system $\mathbf{S5}_K$ for knowledge and $\mathbf{KD45}_B$ for belief.²

Before continuing, let us recall the following usual definitions from basic modal logic (see e.g., [Blackburn et al., 2001](#)). An \mathbf{L} -*derivation/proof* is a finite sequence of formulas (from a language \mathcal{L}) such that each element of the sequence is either an axiom of \mathbf{L} , or obtained from the previous formulas in the sequence by one of the inference rules. We call a formula $\varphi \in \mathcal{L}$ *provable* in \mathbf{L} , or, equivalently, a *theorem* of \mathbf{L} , if it is the last formula of some \mathbf{L} -proof. In this case, we write $\vdash \varphi$ (or, equivalently, $\varphi \in \mathbf{L}$). For any set of formulas $\Gamma \subseteq \mathcal{L}$ and any formula $\varphi \in \mathcal{L}$, we write $\Gamma \vdash \varphi$ if there exist finitely many formulas $\varphi_1, \dots, \varphi_n \in \Gamma$ such that $\vdash \varphi_1 \wedge \dots \wedge \varphi_n \rightarrow \varphi$. We say that Γ is \mathbf{L} -*consistent* if $\Gamma \not\vdash \perp$, and \mathbf{L} -*inconsistent* otherwise. A formula φ is *consistent with* Γ if $\Gamma \cup \{\varphi\}$ is \mathbf{L} -consistent (or, equivalently, if $\Gamma \not\vdash \neg\varphi$). We drop mention of \mathbf{L} when it is contextually clear. A rule of inference R is *admissible*³ in \mathbf{L} if the set of theorems of \mathbf{L} does not change when R is added to \mathbf{L} . In other words, R is admissible in \mathbf{L} if every formula that can be derived using R is already derivable in \mathbf{L} , so, in a sense, having R in \mathbf{L} is redundant.

Formulas in the language \mathcal{L}_\square can be interpreted in a relational model (also called Kripke model), defined as follows.

2.1.2. DEFINITION. [Relational frame/model] A *relational frame* is a pair $F = (X, R)$ where X is a nonempty set, called *set of states* (or, *set of possible worlds*), and $R \subseteq X \times X$ is a binary relation on X called *accessibility relation* (or, *indistinguishability relation*). A *relational model* is a tuple $M = (X, R, \|\cdot\|)$ where (X, R) is a relational frame and $\|\cdot\| : \text{Prop} \rightarrow \mathcal{P}(X)$ is a valuation map.

¹Other logics for knowledge have been considered in the literature, see e.g., [Lenzen, 1978](#) and [Stalnaker, 2006](#), who argued in favour of $\mathbf{S4.2}_K = \mathbf{S4}_K + (.2_K)$, and [van der Hoek, 1993](#) and [Baltag and Smets, 2008](#) where $\mathbf{S4.3}_K = \mathbf{S4}_K + (.3_K)$ was studied.

²The system $\mathbf{S5}_K$ has been criticised by some philosophers for being *too strong* (see e.g., [Hintikka, 1962](#)). In this dissertation, we are not going to engage in any kind of discussion supporting one logic instead of another. Our choice for $\mathbf{S5}_K$ is to reason about certain learning theoretic notions.

³The concept of *admissible rule* was introduced by [Lorenzen \(1955\)](#), for a detailed discussion see e.g., [Jeřábek, 2005](#).

We say $M = (X, R, \|\cdot\|)$ is a relational model based on the relational frame $F = (X, R)$.

Often we also use W to denote the nonempty set of states. We use the letters x, y, w, u, v to talk about states (possible worlds) in X (in W).

In formal epistemology, a relational frame represents the agent's current uncertainty about the actual situation via the truth conditions defined recursively as follows.

2.1.3. DEFINITION. [Relational semantics for \mathcal{L}_\square] Given a relational model $M = (X, R, \|\cdot\|)$ and a state $x \in X$, *truth* of a formula in \mathcal{L}_\square (also called, *satisfaction relation*) is defined recursively as follows:

$$\begin{aligned} M, x \models p & \quad \text{iff} \quad x \in \|p\|, \quad p \in \text{Prop}, \\ M, x \models \neg\varphi & \quad \text{iff} \quad M, x \not\models \varphi, \\ M, x \models \varphi \wedge \psi & \quad \text{iff} \quad M, x \models \varphi \text{ and } M, x \models \psi, \\ M, x \models \square\psi & \quad \text{iff} \quad \forall y \in X, xRy \text{ implies } M, y \models \psi. \end{aligned}$$

From the definition above we obtain the following,

$$M, x \models \diamond\psi \text{ iff } \exists y \in X \text{ such that } xRy \text{ and } M, y \models \psi.$$

We write $\llbracket\varphi\rrbracket_M$ to denote the set of states in a model M that satisfy $\varphi \in \mathcal{L}_\square$ and we call it *truth set of φ* (or *interpretation of φ*). In the chapters that follow in the thesis, we will leave out the subscript M and only write $\llbracket\varphi\rrbracket$ when the model we are referring to is clear from the context. Given a formula $\varphi \in \mathcal{L}_\square$, φ is *satisfiable in a model M* iff there is a state x in M such that $M, x \models \varphi$. We say that a formula φ is *valid in a relational model M* , denoted by $M \models \varphi$, if for every $x \in X$, $M, x \models \varphi$, and is *valid in a relational frame F* , denoted by $F \models \varphi$, if $M \models \varphi$ for every model M based on F . A formula φ is *valid in a class of models \mathcal{M}* , denoted by $\models_{\mathcal{M}} \varphi$ if for every model $M \in \mathcal{M}$, $M \models \varphi$. More generally, a formula φ is *valid in a class of frames \mathcal{F}* , denoted by $\models_{\mathcal{F}} \varphi$ if for every $F \in \mathcal{F}$, $F \models \varphi$. We omit the subscripts whenever the class of models (or the class of frames) is clear from the context. Given a set of formulas $\Gamma \subseteq \mathcal{L}_\square$, a class of models \mathcal{M} and a formula $\varphi \in \mathcal{L}_\square$, $\Gamma \models_{\mathcal{M}} \varphi$ denotes that for all $M \in \mathcal{M}$ and all states x in M , if every $\gamma \in \Gamma$ is satisfied at x , i.e., $M, x \models \gamma$, then $M, x \models \varphi$.

We now state some frame conditions and some notions in the theory of order.

2.1.4. DEFINITION. [Pre/Partial order, Equivalence relation] Given a relational frame (X, R) ,

- R is a *pre-order* if it is reflexive and transitive (see Table [2.3](#)),
- R is a *partial order* if it is reflexive, transitive and antisymmetric (see Table [2.3](#)), and

$\forall x (xRx)$	Reflexivity
$\forall x, y, z (xRy \wedge yRz \rightarrow xRz)$	Transitivity
$\forall x, y (xRy \rightarrow yRx)$	Symmetry
$\forall x, y (xRy \wedge yRx \rightarrow x = y)$	Antisymmetry
$\forall x \exists y (xRy)$	Seriality
$\forall x, y, z (xRy \wedge xRz \rightarrow yRz)$	Euclideaness
$\forall x, y (xRy \vee yRx)$	Totality(Connected)

Table 2.3: Frame conditions in relational frames

- R is an *equivalence relation* (often denoted as \sim) if it is reflexive, transitive and symmetric (see Table 2.3).

We call the relational model (X, \sim) an *epistemic frame* and we call the relational model $M = (X, \sim, \|\cdot\|)$ an *epistemic model*.

In the following theorem we state some general relational (Kripke) soundness and completeness results. For a detailed presentation of these results and a further discussion, we refer to (Chagroff and Zakharyashev, 1997; Blackburn et al., 2001).

2.1.5. THEOREM (RELATIONAL COMPLETENESS).

- The logic $\mathbf{S4}_\square$ is sound and complete with respect to the class of preordered sets.
- The logic $\mathbf{S5}_\square$ is sound and complete with respect to the class of frames with equivalence relations.
- The logic $\mathbf{KD45}_\square$ is sound and complete with respect to the class of serial, transitive and Euclidean frames.

As an instance of Theorem 2.1.5 we have that the logic $\mathbf{S5}_K$ is sound and complete with respect to the class of epistemic frames.

2.1.1 Public Announcement Logic

In this section we present a logic that models knowledge and information change by public, truthful announcements. Public Announcement Logic (PAL) (Plaza, 1989; Gerbrandy and Groeneveld, 1997; van Ditmarsch et al., 2007) is a dynamic epistemic logic that formalizes knowledge and knowledge update via public, truthful announcements.

Dynamic Epistemic Logic (DEL) in the study of *dynamic* modal logics for knowledge and belief. These logics result from extending the *static* epistemic

language presented in Definition 2.1.1 (or extensions of it) with *dynamic modalities*. A dynamic modality (or, *update modality*) encodes and describes a model transforming action (or, *update mechanism*). In the context of DEL, a model transforming action is an epistemic action that transforms the initial model into an *updated* model. The updated model represents what is known or believed after the corresponding epistemic action has been performed.

We focus on a special type of dynamic modalities, namely the public announcement modalities and a variation present in an extension of PAL called Arbitrary Announcement Logic (Balbiani et al., 2008). A public announcement modality encodes the change in the knowledge and belief states of an agent (or agents) after a completely trustworthy, truthful public announcement has happened. In other words, public announcement modalities capture the epistemic action of learning *factual information* that has been made public. The basic language of PAL extends the basic epistemic language \mathcal{L}_{epi} (Definition 2.1.1) as follows.

2.1.6. DEFINITION. [PAL Language] Let $\text{Prop} := \{p, q, \dots\}$ be a countable set of propositional variables. The language of PAL, denoted by $\mathcal{L}_{K!}$, is defined recursively as

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K\varphi \mid [\psi]\varphi$$

where $p \in \text{Prop}$.

The formula $[\psi]\varphi$ reads “after ψ is truthfully announced, φ holds”. The standard notation of formulas with public announcement modalities is $[\psi!]\varphi$, but to keep the notation simpler we decided to leave out the symbol (!). We only use the aforementioned symbol as a subscript in $\mathcal{L}_{K!}$.

PAL formulas are interpreted as follows.

2.1.7. DEFINITION. [Relational semantics for $\mathcal{L}_{K!}$] Given a relational model $M := (X, R, \|\cdot\|)$ and the language $\mathcal{L}_{K!}$, the semantic definition of PAL extends Definition 2.1.3 with the following clause:

$$M, x \models [\psi]\varphi \text{ iff } M, x \models \psi \text{ implies } M|[\psi]_M, x \models \varphi,$$

where $M|[\psi]_M := ([\psi]_M, R \cap ([\psi]_M \times [\psi]_M), \|\cdot\|_{[\psi]_M})$ such that $\|p\|_{[\psi]_M} = \|p\| \cap [\psi]_M$ for any $p \in \text{Prop}$. We call $M|[\psi]_M$, an *updated model*.

In what follows, we always refer to the update mechanism for the public announcement modalities as *update*⁴

A natural extension of PAL is Arbitrary Public Announcement Logic (APAL) (Balbiani et al., 2008) that results from extending PAL with the *arbitrary announcement modality*, \blacksquare , quantifying over public announcements.

⁴An update mechanism corresponding to a public announcement modality can also be found in the relevant literature as *hard update* or *update for hard information* (van Benthem, 2011).

2.1.8. DEFINITION. [APAL Language] Let $\text{Prop} := \{p, q, \dots\}$ be a countable set of propositional variables. The language of APAL, denoted by $\mathcal{L}_{K! \blacksquare}$, is defined recursively as

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K\varphi \mid [\psi]\varphi \mid \blacksquare\varphi$$

where $p \in \text{Prop}$.

The formula $\blacksquare\varphi$ reads “after any truthful public announcement, φ holds”. In simple words, the modality \blacksquare expresses what is true after *any* truthful announcement. The corresponding dual $\blacklozenge\varphi := \neg\blacksquare\neg\varphi$ reads “there is a truthful public announcement after which φ holds”. The resulting logic provides an account to reason about what is *knowable after an announcement*, captured by the formula $\blacklozenge K\varphi$. We will say more about APAL in Chapter 4 thus, for now, we just present the semantics of \blacksquare .

2.1.9. DEFINITION. [Relational semantics for $\mathcal{L}_{K! \blacksquare}$] Given an epistemic model $M := (X, \sim, \|\cdot\|)$ and the language $\mathcal{L}_{K! \blacksquare}$, the semantic definition of APAL extends Definition 2.1.7 with the following clause:

$$M, x \models \blacksquare\varphi \text{ iff } \forall \psi \in \mathcal{L}_{\text{epis}}, M, x \models [\psi]\varphi.$$

2.2 Subset Space Semantics

The logics that we consider in Chapters 3 and 4 are interpreted on the so-called subset space semantics first introduced and investigated by Moss and Parikh (1992). In this section we present the core notions concerning the subset space semantics. We briefly state some basic notions and results in topology. For further explanations on these, we refer to (Engelking, 1989; Kelley, 1991).

2.2.1. DEFINITION. [Subset/Intersection/Lattice/Topological space]

- A *subset space* is a pair (X, \mathcal{O}) where X is a nonempty set of states and $\mathcal{O} \subseteq \mathcal{P}(X)$ is a collection of sets of X .
- An *intersection space* is a subset space (X, \mathcal{O}) where \mathcal{O} is closed under finite intersections, i.e., if $\mathcal{F} \subseteq \mathcal{O}$ is finite then $\bigcap \mathcal{F} \in \mathcal{O}$.
- A *lattice space* is an intersection space (X, \mathcal{O}) where \mathcal{O} is closed under finite unions, i.e., if $\mathcal{F} \subseteq \mathcal{O}$ is finite then $\bigcup \mathcal{F} \in \mathcal{O}$.
- A *topological space* is a subset space (X, τ) where τ has the following properties:

- $X, \emptyset \in \tau$, and

- τ is closed under finite intersections and arbitrary unions, i.e., if $\mathcal{F} \subseteq \tau$ is finite then $\bigcap \mathcal{F} \in \tau$, and if $\mathcal{F} \subseteq \tau$ then $\bigcup \mathcal{F} \in \tau$.

The set X is a *space* and we call the family τ a *topology* on X . The elements in τ are called *open sets* (or *opens*) in the space. We refer to the elements in \mathcal{O} (in an intersection space or a lattice space) also as opens. Given $x \in X$ and $U \in \mathcal{O}$ such that $x \in U$, we call U an *open neighbourhood* of x (or simply, a *neighbourhood* of x) and is often denoted as U_x .

Given $C \subseteq X$, the set C is *closed* iff $C = X - U$ for some $U \in \tau$.⁵ We say that a set A is *clopen* iff it is both closed and open.

We say that an $x \in X$ is an *interior point* of a set $Y \subseteq X$ if there is an open neighbourhood U of x such that $U \subseteq Y$. We denote the set of all interior points of Y by $Int(Y)$ and call it the *interior* of Y . For every $Y \subseteq X$, $Int(Y)$ is the largest open set subset of Y . Note that $Int : \mathcal{P}(X) \rightarrow \tau$ is a map such that for any $A \subseteq X$, $Int(A) = \bigcup \{U \in \tau : U \subseteq A\}$ and it is called the *interior operator*. Dually, for every $Y \subseteq X$ we denote the smallest closed set that contains Y by $Cl(Y)$. It is easy to see that $Cl(Y) = X - Int(X - Y)$. We let $\bar{\tau} := \{X - U : U \in \tau\}$ denote the family of all closed sets of (X, τ) . Note that $Cl : \mathcal{P}(X) \rightarrow \bar{\tau}$ is a map such that for any $A \subseteq X$, $Cl(A) = \bigcap \{C \in \bar{\tau} : A \subseteq C\}$ and it is called the *closure operator*.

2.2.2. DEFINITION. [Topological Basis] A family $\mathcal{B} \subseteq \tau$ is called a *basis* for a topological space (X, τ) if every $U \in \tau - \{\emptyset\}$ can be written as a union of elements of \mathcal{B} . The elements of \mathcal{B} are called *basic opens*.

Given any family $\mathcal{Y} = \{Y_\alpha : \alpha \in I\}$ where $Y_\alpha \subseteq X$ for all $\alpha \in I$, there is a unique, smallest topology $\tau(\mathcal{Y})$ such that $\mathcal{Y} \subseteq \tau(\mathcal{Y})$. The topology $\tau(\mathcal{Y})$ consists of all finite intersections of the Y_α , all arbitrary unions of such sets, \emptyset and X . The topology $\tau(\mathcal{Y})$ is said to be *generated* by \mathcal{Y} and the family \mathcal{Y} is called a *subbasis* for $\tau(\mathcal{Y})$. The set $\{\bigcap \mathcal{F} : \mathcal{F} \subseteq \mathcal{Y} \text{ finite}\}$ constitutes a basis for $\tau(\mathcal{Y})$.

2.2.3. DEFINITION. [Subspace] Given a subset space (X, \mathcal{O}) and a nonempty subset $Y \subseteq X$, the subset space (Y, \mathcal{O}_Y) is called a *subspace* of (X, \mathcal{O}) where $\mathcal{O}_Y := \{U \cap Y : U \in \mathcal{O}\}$.

Now we define the subset space semantics as in (Moss and Parikh, 1992; Dabrowski et al., 1996). For that, we consider a bi-modal extension of the epistemic language presented in (Moss and Parikh, 1992; Dabrowski et al., 1996), that we denote by $\mathcal{L}_{K\Box}$, and interpret it in subset spaces. We discuss this language and the semantic definition that follows extensively in Chapter 3.

⁵For every $Y \subseteq X$, the set $X - Y$ is the set theoretic complement of Y contained in X .

2.2.4. DEFINITION. [Language $\mathcal{L}_{K\Box}$] Let $\text{Prop} := \{p, q, \dots\}$ be a countable set of propositional variables. The language $\mathcal{L}_{K\Box}$, is defined recursively as

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid K\varphi \mid \Box\varphi$$

where $p \in \text{Prop}$.

2.2.5. DEFINITION. [Subset Space Model] A *subset space model* is a triple, $(X, \mathcal{O}, \|\cdot\|)$, where (X, \mathcal{O}) is a subset space and $\|\cdot\| : \text{Prop} \rightarrow \mathcal{P}(X)$ is a valuation map.

- If (X, \mathcal{O}) is an intersection space, we call $(X, \mathcal{O}, \|\cdot\|)$ an *intersection model*.
- If (X, \mathcal{O}) is a lattice space, we call $(X, \mathcal{O}, \|\cdot\|)$ a *lattice model*.
- If (X, τ) is a topological space, we call $(X, \tau, \|\cdot\|)$ a *topological model*.

2.2.6. DEFINITION. [Subset Space Semantics for $\mathcal{L}_{K\Box}$] Given a subset space model $M = (X, \mathcal{O}, \|\cdot\|)$ and a pair (x, U) with $x \in X$ and $U \in \mathcal{O}$, *truth* of a formula in $\mathcal{L}_{K\Box}$ (also called, *satisfaction relation*) is defined recursively as:

$$\begin{aligned} (x, U) \models p & \quad \text{iff} & \quad x \in \|p\|, \\ (x, U) \models \neg\varphi & \quad \text{iff} & \quad (x, U) \not\models \varphi, \\ (x, U) \models \varphi \wedge \psi & \quad \text{iff} & \quad (x, U) \models \varphi \text{ and } (x, U) \models \psi, \\ (x, U) \models K\varphi & \quad \text{iff} & \quad (\forall y \in U) ((y, U) \models \varphi), \\ (x, U) \models \Box\varphi & \quad \text{iff} & \quad (\forall O \in \mathcal{O}) (x \in O \subseteq U \text{ implies } (x, O) \models \varphi), \\ & \quad \text{i.e.,} & \quad (\forall O \in \mathcal{O}) (x \in O \text{ implies } (x, U \cap O) \models \varphi). \end{aligned}$$

It is useful to note the definition for the semantics of $\langle K \rangle$ and \diamond :

$$\begin{aligned} (x, U) \models \langle K \rangle\varphi & \quad \text{iff} & \quad (\exists y \in U) ((y, U) \models \varphi), \\ (x, U) \models \diamond\varphi & \quad \text{iff} & \quad (\exists O \in \mathcal{O}) (x \in O \subseteq U \text{ and } (x, O) \models \varphi). \end{aligned}$$

Truth set, satisfiability and *validity* of a formula $\varphi \in \mathcal{L}_{K\Box}$ are defined as for the relational semantics. Therefore, we apply the same conventions and notation as in Section [2.1](#).

2.3 Recursive sets and Recursively enumerable sets

In this short section we briefly recall some core notions and results concerning recursive sets and recursively enumerable sets that will become handy especially in Chapter [6](#).

2.3.1. DEFINITION. [Recursive/Recursively Enumerable set]

- A set $A \subseteq \mathbb{N}$ is *recursive* (*computable*) if its characteristic function,

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

is recursive.

- We can generalize the above notion as follows, a predicate (or relation) $R \subseteq \mathbb{N}^n$ with $n \geq 1$ is recursive if its characteristic function χ_R is recursive.
- A set $X \subseteq \mathbb{N}$ is recursively enumerable (r.e.) if X is the domain of some partial recursive function.

The classes Σ_n and Π_n with $n \geq 0$ are defined in the recursion-theoretic literature as part of the *Arithmetical hierarchy* (this hierarchy should not be confused with the Borel hierarchy in Topology that uses a similar notation). Σ_1 is the class of predicates that have the form of one or more quantifier \exists followed by a recursive predicate, i.e., predicates of the form $\exists y_1 \dots \exists y_n R(x, y_1, \dots, y_n)$ for some recursive predicate $R(x, y_1, \dots, y_n)$. Π_1 is the class of predicates that have the form of one or more quantifier \forall followed by a recursive predicate, i.e., predicates of the form $\forall y_1 \dots \forall y_n R(x, y_1, \dots, y_n)$ for some recursive predicate $R(x, y_1, \dots, y_n)$. We also define $\Delta_n := \Sigma_n \cap \Pi_n$. Δ_1 is the class of recursive (computable) predicates, thus $R(x, y_1, \dots, y_n)$ is recursive iff $R(x, y_1, \dots, y_n)$ is in Δ_1 . For a detailed discussion about the Arithmetical Hierarchy, see e.g. (Soare, 1999, Chapter IV, p. 60).

The definition above of r.e. set is known to be equivalent to the more intuitive following definitions:

- A set $X \subseteq \mathbb{N}$ is r.e. if there is an algorithm that enumerates the members of X , i.e., $X = \{f(n) : n \in \mathbb{N}\}$ for some recursive function f .
- A set $X \subseteq \mathbb{N}$ is r.e. if $X = \{x : \exists y R(x, y)\}$ for some recursive predicate $R \subseteq \mathbb{N} \times \mathbb{N}$. This generalizes to the fact that if $X = \{x : \exists y_1 \dots \exists y_n R(x, y_1, \dots, y_n)\}$ for some recursive predicate $R \subseteq \mathbb{N}^n$, X is r.e. We then say that X is in Σ_1 -form (in short, X is Σ_1).

The following theorem is an instance of Post's theorem where the notion of recursive set is characterized in terms of recursively enumerable sets, for a detailed discussion on this see e.g., (Soare, 1999, Theorem 2.2, p. 64–65).

2.3.2. THEOREM. *A set $X \subseteq \mathbb{N}$ is recursive if and only if X and $\mathbb{N} - X$ are recursively enumerable, i.e., X and $\mathbb{N} - X$ are Σ_1 .*

2.3.3. EXAMPLE. Some examples of recursive sets are the following.

1. \mathbb{N} and \emptyset are recursive sets.
2. The set of even natural numbers $EVEN := \{0, 2, 4, \dots\}$ and the set of odd natural numbers, $ODD := \mathbb{N} - EVEN$, are recursive.
3. Any finite set is recursive.
4. For every partial recursive function λ_x , the set W_x denotes the domain of λ_x . The set $K := \{x : \lambda_x(x) \text{ is convergent}\} = \{x : x \in W_x\}$ is recursively enumerable and not recursive.

We say informally that e is a *Turing machine* if e is an integer that codes a Turing machine. We define informally Kleene's T predicate since this is enough for our purposes. For a detailed definition of the predicate see [Kleene \(1943\)](#) and for further relevant results ([Soare, 1999](#), p.15, 28, 41).

2.3.4. DEFINITION. [Kleene's T Predicate] $Texy$ holds iff e is a Turing machine that on input x performs computation y .

In the standard version of Kleene's predicate we have that, if $Texy$ then $x, e < y$.

The question of existence of a computation y performed by e on input x is the so-called *Halting Problem* which was shown to be undecidable by [Turing \(1937\)](#). A simpler undecidable question is the question of whether Turing machine e stops on the input e . This question can be defined in terms of the T predicate as follows:

$$\exists y (Teey) \text{ iff } \lambda_e(e) \downarrow.$$

Considering the standard version of Kleene's predicate, we have the following:

if $Teey$ exists then e can be computed from the y and $e \leq y$.

In this thesis we always consider the standard version of Kleene's predicate. In this way, the T predicate can be used to generate undecidable sets in Σ_1 , for instance $K = \{x : \exists y (Txy)\}$ which we already know is recursively enumerable.

The following notion is the one of *disjoint pairs of recursively enumerable sets*.

2.3.5. DEFINITION. [Recursively inseparable sets] We say that $A, B \subseteq \mathbb{N}$, $A \cap B = \emptyset$, are recursively inseparable iff there is no recursive set $C \subseteq \mathbb{N}$ such that $(C \supseteq A \text{ and } C \cap B = \emptyset)$.

For a detailed discussion on recursively inseparable sets, see e.g. [Rogers, 1967](#), p. 93). It is well-known that r.e. sets A, B exist which are recursively inseparable. A standard example is the following, let \mathbf{PA} be Peano's arithmetic axiomatic system and consider the sets $A := \{\varphi : \vdash_{\mathbf{PA}} \varphi\}$ and $B := \{\varphi : \not\vdash_{\mathbf{PA}} \varphi\}$. The sets A and B are recursively inseparable ([Smullyan, 1958](#)). Another example can be found in [Soare, 1999](#), p. 23).

2.4 Formal Learning Theory

Formal Learning Theory (FLT) addresses mathematically and computationally the process of *inductive inference* (or, *inductive learning*). It focuses on the question of how a learner should use partial inductively given information (finite strings of symbols) to infer systematically general and correct conclusions (see e.g., [Osherson et al., 1986; Jain et al., 1999]). Its origins go back to the seminal papers of [Putnam (1963)], [Solomonoff (1964a,b)], and [Gold (1967)].

In FLT, a learner is modelled by a (recursive) function that receives elements from a sequence of data and stabilizes on an appropriate value fitting the data. Technically, a learner is conceived as a system that transforms finite segments of data into hypotheses.

To illustrate, consider a game between a learner and nature (or teacher) where the learner needs to identify the current state of the world. We assume that the incoming information is readable and that all the data that are consistent with the actual world are eventually presented to the learner. The source of data is also taken to be truthful (since nature never lies). This game is described as follows. Initially, there is a *class of concepts* (or *class of realities*). Intuitively, this class represents the uncertainty range of the learner. Nature chooses at the beginning of the game one of these concepts to be the *target concept* and starts providing to the learner pieces of data concerning the target concept. The learner's aim is to guess correctly which concept from the class is the one chosen by nature. If the learner succeeds, we say that the learner *identifies* (or *learns*) the target concept. If the learner identifies every concept of the class, we say that the learner identifies the class of concepts.

It is worth mentioning that the connection between data and concept is like the one between truthful evidence and hypothesis, observed sentences and grammar.

The concepts we focus on are so-called *languages* i.e., sets of strings of symbols. Since we can represent strings of symbols by natural numbers, we always refer to \mathbb{N} as our *universal set*. Thus, languages are sets of natural numbers, i.e., $S \subseteq \mathbb{N}$. Thus, *families of languages* are collections of subsets of \mathbb{N} . Given this definition of a language, sequences of data are infinite sequences of natural numbers.

In what follows, a concept will be always called a *language* and a class of concepts will be called a *family of languages*.

Often in this dissertation, we focus on a special kind of families called *indexed families* first introduced and investigated by [Angluin (1980)]. For a survey on learning indexed families and further developments, see [Lange et al., 2008].

2.4.1. DEFINITION. [Indexed/Canonical Family]

- An *indexed family* is a collection of non-empty recursive languages $\mathcal{S} := \{S_i : i \in \mathbb{N}\}$ for which a computable function $g : \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$ exists such

that,

$$g(i, x) = \begin{cases} 1 & \text{if } x \in S_i, \\ 0 & \text{if } x \notin S_i. \end{cases}$$

In other words, $\mathcal{S} = \{S_i : i \in \mathbb{N}\}$ is indexed if for every $S_i \in \mathcal{S}$ the two-place predicate $y \in S_i$ is recursive.

- If a family \mathcal{S} can be represented as an indexed family, we say that \mathcal{S} is *indexable*.
- In case all languages are finite and there is a recursive function f such that for each i , $f(i)$ is the canonical index for S_i i.e., $S_i = F_{f(i)}$, then we call \mathcal{S} a *canonical family*. We often write F_n for the finite set with canonical index n .

2.4.2. DEFINITION. [Data (information) sequence]

- ϵ is the empty sequence.
- A *positive data presentation* (or, *positive sequence*) of a language S is an infinite sequence $\sigma^+ := x_0, x_1, \dots$ of elements of \mathbb{N} such that $\{x_0, x_1, \dots\} = S$.
- A *complete data presentation* (or, *complete sequence*) of a language S is an infinite sequence of pairs $\sigma := (x_0, t_0), (x_1, t_1), \dots$ of $\mathbb{N} \times \{0, 1\}$ such that $\{x_n \in \mathbb{N} : t_n = 1, n \geq 0\} = S$ and $\{x_m \in \mathbb{N} : t_m = 0, m \geq 0\} = \mathbb{N} - S$.
- An *initial segment* of length n of σ and σ^+ is indicated by $\sigma[n]$ and $\sigma^+[n]$, respectively. Given a family of languages \mathcal{S} , we call Seg^+ the *set of all initial segments of all positive sequences* of languages in \mathcal{S} ; and we call Seg the *set of all initial segments of all complete sequences* of languages in \mathcal{S} .
- Let σ^+ be any positive sequence and σ be any complete sequence (finite or infinite), we denote as $set(\sigma^+)$ the set of elements that occur in σ^+ . Similarly, $set^1(\sigma) := \{x \in \mathbb{N} : (x, 1) \in \sigma\}$ and $set^0(\sigma) := \{x \in \mathbb{N} : (x, 0) \in \sigma\}$.
- We say that an initial segment $\sigma^+[n]$ in Seg^+ is *consistent with a language* S if $set(\sigma^+[n]) \subseteq S$. Similarly, we say that $\sigma[n] := (x_0, t_0), \dots, (x_{n-1}, t_{n-1})$ in Seg is *consistent with a language* S , if $set^1(\sigma[n]) := \{x \in \mathbb{N} : (x, 1) \in \sigma[n]\} \subseteq S$ and $set^0(\sigma[n]) := \{x \in \mathbb{N} : (x, 0) \in \sigma[n]\} \subseteq \mathbb{N} - S$.

Note that even for finite languages, the sequences of data are infinite. The reason for this is that we want to consider situations when the learner does not know the size of the language she investigates (or if it is finite or not).

2.4.3. DEFINITION. [Positive/Complete/Non-effective learner]

- A *learning function with positive data* (in short, *positive learner*) $\lambda := \text{Seg}^+ \rightarrow \mathbb{N}$ is a recursive map from finite positive sequences to indices of languages in a given family.
- A *learning function with complete data* (in short, *complete learner*) $\lambda := \text{Seg} \rightarrow \mathbb{N}$ is a recursive map from finite complete sequences to indices of languages in a given family.
- We can also allow the learner to abstain from producing a natural number output. Such a learner is defined as $\lambda := D \rightarrow \mathbb{N} \cup \{\uparrow\}$ where $D \in \{\text{Seg}^+, \text{Seg}\}$ and \uparrow stands for *undefined*.
- We occasionally relax the condition of the recursivity of the learner, in this case we say that the learner is *non-effective*.

2.4.1 Identification in the limit

The model in (Gold, 1967), *identification in the limit*, has extensively been studied for learning recursive functions, recursively enumerable languages, and recursive languages with *positive data* and with *complete data*. The learning function outputs infinitely many conjectures, and for a successful learning function these are required to stabilize into one permanent right one.

Given a learner φ , we assume that the domain of φ is Seg^+ in the case of positive data, and Seg in the case of complete data.

2.4.4. DEFINITION. [Identification in the Limit] Given an indexed family \mathcal{S} , a learning function λ ,

- identifies in the limit with positive data (complete data) a language $S_i \in \mathcal{S}$ on a sequence σ^+ (on a sequence σ) iff for all but finitely many $n \in \mathbb{N}$, $\lambda(\sigma^+[n]) = j$ with $S_j = S_i$ ($\lambda(\sigma[n]) = j$ with $S_j = S_i$),
- identifies in the limit with positive data (complete data) a language $S_i \in \mathcal{S}$ iff it identifies in the limit with positive data (complete data) S_i on every σ^+ (on every σ) for S_i ,
- identifies in the limit with positive data (complete data) a family \mathcal{S} iff identifies in the limit with positive data (complete data) every language $S_i \in \mathcal{S}$.
- A family \mathcal{S} of languages is said to be *identifiable in the limit from positive data* (or *identifiable in the limit from complete data*) if there exists a recursive learner λ which identifies in the limit with positive data (complete data) the family \mathcal{S} .

- We occasionally relax the condition of the recursivity of the learner λ or the indexicality of \mathcal{S} . In such cases λ is said to be a *non-effective learner* and \mathcal{S} is said to be *non-effectively identifiable in the limit from positive data* (or *non-effectively identifiable in the limit from complete data*).

A characterization of the indexed families that are identifiable in the limit with positive data was given by [Angluin \(1980\)](#) in terms of tell-tale sets. A finite tell-tale set is like a “birth mark” for a language and its supersets.

2.4.5. DEFINITION. [Tell-Tale sets] Let \mathcal{S} be a family of languages, and let $S_i \in \mathcal{S}$. A finite set D_i is a *finite tell-tale set* for S_i if $D_i \subseteq S_i$ and for all $S_j \in \mathcal{S}$ if $D_i \subseteq S_j$, then $S_j \subseteq S_i$.

2.4.6. THEOREM (CHARACTERIZATION IDENTIFIABILITY IN THE LIMIT). *An indexed family of recursive languages $\mathcal{S} = \{S_i : i \in \mathbb{N}\}$ is identifiable in the limit from positive data iff there is an effective procedure Φ , that on input i enumerates all elements of a finite tell-tale set of S_i .*

In simple words, a family is identifiable in the limit with positive data if for every language there is a finite set which distinguishes the language from all its subsets in the family. For an effective identification it is necessary and sufficient that there is a recursive procedure that enumerates some finite tell-tales for all the languages in the family.

2.4.2 Finite Identification

Finite identification of a family of languages is defined with respect to a learner that, on a sequence consistent with some of the languages in the family, can produce just one conjecture which must be correct immediately. In other words, the learner abstains from making a conjecture until she makes an immediately decisive one.

Given a learner φ , we assume that the domain of φ is Seg^+ in the case of positive data, and Seg in the case of complete data.

2.4.7. DEFINITION. [Finite identification] Given a family \mathcal{S} , a learning function λ ,

- finitely identifies with positive data a language $S_i \in \mathcal{S}$ on a sequence σ^+ iff for some $n \in \mathbb{N}$, $\lambda(\sigma^+[n]) = j$ with $S_j = S_i$ and $\lambda(\sigma^+[m]) = \uparrow$ for all $m < n$.
- finitely identifies with complete data a language $S_i \in \mathcal{S}$ on a sequence σ iff for some $n \in \mathbb{N}$, $\lambda(\sigma[n]) = j$ with $S_j = S_i$ and $\lambda(\sigma[m]) = \uparrow$ for all $m < n$.
- finitely identifies with positive data (complete data) a language $S_i \in \mathcal{S}$ iff it finitely identifies with positive data (complete data) S_i on every σ^+ (on every σ) for S_i ,

- finitely identifies with positive data (complete data) a family \mathcal{S} iff finitely identifies with positive data (complete data) every language $S_i \in \mathcal{S}$.
- A family \mathcal{S} of languages is said to be *finitely identifiable from positive data* (in short, *pf*) (or *finitely identifiable from complete data* (in short, *cf*)) if there exists a recursive learner λ which finitely identifies with positive data (complete data) the family \mathcal{S} .
- We occasionally relax the condition of the recursivity of the learner λ or the indexicality of \mathcal{S} . In such cases λ is said to be a *non-effective learner* and \mathcal{S} is said to be *non-effectively finitely identifiable from positive data* (in short, *nepfi*) (or *non-effectively finitely identifiable from complete data* (in short, *necfi*)).

Clearly a family that is *pf* is also *non-effectively pf*. Similarly for *cf*.

A characterization theorem for finitely identifiable families with positive and with complete data has been provided by Mukouchi (1992) and simultaneously by Lange and Zeugmann (1992). For this, we give the definition of a definite tell-tale set and a definite tell-tale pair in Mukouchi's style. A definite tell-tale set is like a "definite birth mark" for the language, namely a birth mark that no other language in the family has. So the definite tell-tale set distinguishes the set from the other languages in the family.

2.4.8. DEFINITION. [Definite tell-tale sets/pairs]

- Let \mathcal{S} be a family of languages, and let $S_i \in \mathcal{S}$. A finite set D_i is a *definite tell-tale set* (DFTT) for S_i if $D_i \subseteq S_i$ and for all $S_j \in \mathcal{S}$ if $D_i \subseteq S_j$, then $S_j = S_i$.
- A language S_i is said to be *consistent* with a pair of finite sets (B, C) if $B \subseteq S_i$ and $C \subseteq \mathbb{N} \setminus S_i$. A pair of finite sets (D_i, \overline{D}_i) is a *definite tell-tale pair* (in short, *d-tell-tale pair*) for S_i if S_i is consistent with (D_i, \overline{D}_i) , and for all $S_j \in \mathcal{S}$, if S_j is consistent with (D_i, \overline{D}_i) , then $S_j = S_i$. We refer to D_i as the positive member of the definite tell-tale pair and to \overline{D}_i as its negative member.

2.4.9. THEOREM (CHARACTERIZATION FINITE IDENTIFIABILITY).

- A family \mathcal{S} of languages is *finitely identifiable with positive data (pf)* iff for every $S \in \mathcal{S}$ there is a DFTT set D_S obtainable in a uniformly computable way. That is, there exists an effective procedure Φ that on input i , index of S , produces the canonical index $\Phi(i)$ of some definite finite tell-tale set of S .

- A family \mathcal{S} of languages is finitely identifiable with complete data (cfi) iff for every $S \in \mathcal{S}$ there is a definite tell-tale pair $(D_S, \overline{D_S})$ in a uniformly computable way.

Proof:

See (Mukouchi, 1992, Theorem 7, p. 262) or (Lange and Zeugmann, 1992, Theorem 3, p. 382) for pfi and (Mukouchi, 1992, Theorem 10, p. 264) for cfi. \square

Clearly if a family is pfi then it is cfi. A completely analogous theorem holds for non-effective learners and non-effective procedures for pfi and cfi.

2.4.10. COROLLARY. *If a family \mathcal{S} has two languages such that $S_i \subset S_j$, then \mathcal{S} is not nepfi.*

2.4.11. THEOREM. *If \mathcal{S} is a canonical family where no $S_i \in \mathcal{S}$ is a proper subset of any other $S_j \in \mathcal{S}$, then \mathcal{S} is pfi.*

Proof:

For every $S_i \in \mathcal{S}$, simply take $D_i = S_i$ as the DFTT. \square

Similarly, if \mathcal{S} is any family of finite languages that is an anti-chain with respect to \subset , then \mathcal{S} is non-effectively pfi.

Part I

Subset Space Logics for Learning

Chapter 3

Dynamic logics for Inductive Learning from Observations

3.1 Introduction

Learning from observations is one of the most primitive forms of learning that we as humans perform. It is a basic action we must do in order to understand the world that surrounds us. Learning by observing the changes in the world and also observing others' actions is how we could avoid danger at the beginning of humankind which lead us to survive. Even now in modern times, this kind of learning is crucial for the dangerous art of living. Learning plays a vital role in which we act and form our beliefs. These are important for choosing our future steps and planning ahead. Moreover, forming beliefs allows us to form general theories that, when not falsified, transform into (a certain kind of) empirical knowledge. This kind of knowledge, that we will call *inductive knowledge*, is essential for us to make sense of this world. Studying inductive knowledge (and other complex information processes involved in learning from observations) is certainly not a trivial matter. It is not surprising that studying some of these processes has been entertaining scientists from centuries (for an overview, see e.g., [Bogen, 2017](#) and [Schulte, 2018](#)). As it happens, not much has been done about inductive knowledge, as described above, from a modal logic perspective. Thus, we are interested in reasoning about some of the aforementioned learning processes using epistemic logic, in a way that some answers and new insights can be obtained.

The process of learning consists of incorporating new information into one's prior information state. Dynamic epistemic logic (DEL) studies such one-step information changes from a logical perspective ([Baltag et al., 1998](#); [van Benthem, 2011](#); [van Ditmarsch et al., 2007](#)), but the general concept of *learning* encompasses not only one-step revisions, but also their *long-term horizon*. In the long run, learning should lead to *knowledge*, an epistemic state of a particular value. Examples include language learning (inferring the underlying grammar

from examples of correct sentences), and scientific inquiry (inferring a theory on the basis of observations). Our goal in this chapter is to provide simple logics for reasoning about this process of *inductive learning* from successful (fully determined or partial) observations. We do this with respect to two kinds of learners: a minimally rational learner (as in Formal Learning Theory, satisfying only two basic rationality constraints) and a fully rational learner (an AGM learner who satisfies the postulates of belief revision). Understanding inductive inference is of course an infamously difficult open problem. There are many different approaches in the literature, from probabilistic and statistical formalisms based on Bayesian reasoning, Popper-style measures of corroboration, through default and non-monotonic logics, Carnap-style “inductive logic”, to AGM-style rational belief revision and theory change. However, in the work presented here we *do not try to solve* the problem of induction, but only to *reason about* a (rational) inductive learner.

In our first framework “A Dynamic Logic for Learning Theory (DLLT)” (Section 3.3), we adopt the more flexible and open-ended approach of Formal Learning Theory (FLT) where inductive inference is known as *identifiability in the limit*, first introduced and studied by Gold (1967). While most other approaches adopt a normative stance, aimed at prescribing “the” correct algorithm for forming and changing rational beliefs from observations (e.g., Bayesian conditioning), or at least at prescribing some general rationality constraints that any such algorithm should obey (e.g., the AGM postulates for belief revision), FLT gives the learner a high degree of freedom, allowing the choice of *any learning method* that produces conjectures *based on the data* (no matter how “crazy” or unjustified are these conjectures, or how erratic is the process of belief change). In FLT the only criterion of success is to track the truth in the limit. In other words, the only thing that matters is whether or not the iterated belief revision process will eventually stabilise on a conjecture which matches the truth about some given issue. Essentially, the learner should only obey two requirements: consistency of conjectures, and that the conjectures fit the evidence. We call such a learner an *unrestricted learner*. We are of course not interested in cases of convergence to the truth “by accident”, but in determining whether or not a given learner is *guaranteed* to eventually track the truth; hence, the focus is on “The Logic of Reliable Inquiry”¹.

Our framework DLLT, combines ideas from: Subset Space Logics, as introduced by Moss and Parikh (1992), investigated further in (Dabrowski et al., 1996) and already merged with the DEL tradition in prior work (Wáng and Ågotnes, 2013a; Balbiani et al., 2013; van Ditmarsch et al., 2014, 2019; Bjorndahl, 2018; Baltag et al., 2017); the topological approach to Formal Learning Theory (FLT) initiated by Kelly (1996), also studied and developed in (Baltag et al., 2015); and

¹“The Logic of Reliable Inquiry” is the title of a classic text in FLT-based epistemology (Kelly, 1996).

the general agenda of bridging DEL and FLT in (Gierasimczuk, 2010). Semantically, we take *intersection spaces* (a type of subset spaces that are closed under finite non-empty intersections), with points interpreted as possible worlds and neighbourhoods interpreted as *observations* (or *information states*) (for a survey on subset space logics see, e.g., Parikh et al., 2007). We endow these structures with a *learner* \mathbb{L} , mapping every information state to a *conjecture*, representing the learner’s strongest belief in this state.

As in Subset Space Logics, our language features an S5-type *knowledge-with-certainty modality*, capturing the learner’s *hard information*, as well as the so-called *effort modality*, which we interpret as “stable truth” (i.e., truth immune to further observations). To capture *observations*, we add to the language simple observational variables o that capture factual observations about the world. Moreover, we add dynamic modalities called *observational events* that are like PDL programs, built from simple observations $!o$ and using sequential composition $e; e'$ to represent successive observations. We also add a *learning operator* $L(e)$, which encodes the learner’s conjecture after having observed an observational event e . The learner forms a conjecture $L(e)$ with respect to the event’s informational content $pre(e)$ (its *precondition* captures the informational content of event e , defined recursively by taking conjunctions of the preconditions in a sequential composition $e; e'$). This can be used to give a natural definition of *belief*: a learner believes P iff she knows that P is entailed by her current conjecture.

We succeed in using this logic to characterise various interesting learning theoretic notions in Section 3.3.4. In particular, we are able to model inductive learning as *coming to stably believe*² *a true fact after observing an incoming sequence of true data* (corresponding to the key concept in FLT introduced in (Gold, 1967), *identifiability in the limit*). We discuss the expressive power of the language and some of its fragments in Section 3.3.3. In particular, we show that the dynamic observational modalities are in principle eliminable via reduction laws. In Sections 3.3.2, 3.3.5 and 4.6 we present a sound and complete axiomatisation of DLLT with respect to our learning models. The completeness proof uses a neighbourhood version of the standard canonical model construction.

The learner in Formal Learning Theory (and therefore in our Dynamic Logic for Learning Theory) is assumed to satisfy only very few rationality constraints. We are also interested in studying inductive inference with a fully rational learner. This is a learner that follows a certain “rational procedure” when she is confronted with a new piece of truthful evidence that contradicts her prior beliefs. There are various ways in the literature to define what a rational learner is (for instance, Bayesian conditioning, for an overview, see e.g., Talbott (2016), or truth maintenance systems studied by Doyle (1979) in computer science). Our choice is with the so-called AGM postulates for belief revision (the process of changing

²By “*stably believing* (a proposition)” we here mean that the learner believes that proposition and her belief in that proposition cannot be defeated by further truthful observations.

beliefs after receiving a new piece of information). The theory for belief revision was introduced and studied in the pioneering work of Alchourrón, Gärdenfors and Makinson (1985). The AGM postulates are strongly desirable rationality constraints, governing the way in which a rational learner forms and revises her beliefs. In simple words, they express that a learner should accept the new fact, and, instead of dropping all her prior beliefs, she should maintain as many as possible without getting a contradiction.

The AGM postulates became some of the most influential accounts for belief revision in the dynamic logic literature (van Benthem, 2007; van Benthem and Smets, 2015). Moreover, (Baltag et al., 2015) proved that AGM conditioning is a “universal learning method”, meaning that: any questions that can be inductively solved (or solved with certainty) by some learner can also be solved by an AGM learner. One of the technical advantages of the AGM postulates, when we are working in a logical framework, is that we can capture them, mathematically, in various ways. For instance, they can be captured in terms of the properties we impose on a *set-selection function*, with a (certain) *system of spheres*, or with the so-called *plausibility relation* defined as a total pre-order (a reflexive and transitive order) (Grahne, 1998). This flexibility will be useful for our completeness result of the aforementioned logic.

In Section 3.4 we present a dynamic logic for inductive inference that focuses on an AGM learner. A learner whose conjectures obey all the AGM postulates for belief revision (Alchourrón et al., 1985). Semantically, such an “AGM learner” comes with a family of nested Grove spheres (encoding the learner’s defaults and her belief-revision policy), or equivalently with a total plausibility (pre)order on the set of possible worlds. After observing some evidence, the learner forms a conjecture by applying “AGM conditioning”: essentially, her conjecture encompasses the most plausible worlds that fit the evidence. This belief dynamics is non-monotonic, but only minimally so: it respects the principle of Rational Monotonicity (equivalent to the so-called “subexpansion” and “superexpansion” AGM postulates in (Alchourrón et al., 1985)). This principle requires that the dynamics is just monotonic (putting together the old conjecture with the new evidence) as long as the old conjecture is consistent with the new evidence.

Our aim is to realize the same program for AGM learners as the one we achieved for the unrestricted learners: a sound and complete axiomatization, and to capture the aforementioned learning theoretic notions above. In order to obtain our completeness result, we had to extend the domain of our learning functions to *partial observations*. These type of observations capture the intuition of receiving *partial information*. For instance, incomplete reports of a fully-determined observation or when the learner is not sure which of (finitely) many observations has taken place. Making conjectures from partial information is very common (if not necessary) in empirical sciences. In reality most of the time an empirical scientist can only acquire partial information of her object of study. This is due to many factors, by human perception limitations, time limitations, language limi-

tations for describing what is observed; and, due to technical and technological constraints.

Partial observations will be represented in the framework by finite disjunctions of observations. Technically, we have to move from the framework of *intersection spaces* adopted in DLLT (in which the observable properties were closed under finite intersections) to the one of *lattice spaces* (in which closure under finite unions is also required). At the syntactic level, this leads us to extend the recursive definition of *observational events* (recall that these are like PDL programs, built from simple observations $!o$ and using sequential composition $e; e'$) with non-deterministic program executions (in short, epistemic non-determinism), $e \sqcup e'$, to capture the receipt of *partial information* (after which the learner is not sure which of the two observations e, e' has taken place). The language is defined as before, featuring the S5 knowledge modality, updates, the effort modality and an *AGM learning operator* $L(e)$, which encodes the AGM learner's conjecture (her *strongest belief*) given an observational event e that now can also be of the form $e'' \sqcup e'$. Thus, after an observational event e , the learner forms a conjecture $L(e)$ obtained by applying AGM conditioning, with respect to her plausibility order \leq and the event's informational content $pre(e)$.

We present some expressivity results and apply our logic to an example, showing how various concrete properties can be learnt with certainty or inductively by such an AGM learner. We present a sound and complete axiomatization (which is a proper extension of the axiomatization for DLLT). However, our completeness proof is not a straightforward modification of the completeness proof for DLLT. We implemented some standard techniques in non-monotonic and conditional logics for the canonical model construction, although with important differences. First, since we do not allow conditioning on arbitrary formulas, but only on those corresponding to (preconditions of) observational events, the proof is more subtle than those for conditional logics. In particular, it shows that AGM has no need for conditioning on negated formulas. Second, the completeness proof uses a mixture of relational and neighbourhood versions of the standard canonical model construction, with further complications due to the presence of the effort modality.

Outline

This chapter is organized as follows. In the introductory Sections [3.2](#) and [3.2.1](#) we start with an example (originally from [Moss and Parikh \(1992\)](#)) that illustrates *inductive learning* which has not been addressed in the dynamic logic literature. In Section [3.3](#) we present our sound and complete Dynamic Logic for Learning Theory for unrestricted learners. We present a recursive axiomatization for this logic in Section [3.3.2](#) whose soundness and completeness proofs are given in Sections [3.3.5](#) and [4.6](#), respectively. We present some expressivity results in Section [3.3.3](#) and Section [3.3.4](#). In Section [3.4](#) we present our sound and complete Logic for

AGM learning from partial observations. Expressivity results can be found in Section 3.4.2. A recursive axiomatization is given in Section 3.4.3, soundness and completeness can be found in Sections 3.4.4 and 3.4.5, respectively. In Section 3.5 we conclude and connect our work with other work in the literature.

This Chapter is based on (Baltag et al., 2018a) and (Baltag et al., 2020).

3.2 Effort modality and knowledge

In this section we briefly revisit some learning theoretic notions presented in previous work in the literature that inspired our questions, ideas and results in the sections that follow. In particular, we discuss these notions on an example that was first introduced in (Parikh et al., 2007). We then use the example to reflect about the missing element to capture *inductive knowledge* (a *softer* notion of knowledge than the one of *knowledge with certainty*) and a *learning function* (unrestricted or restricted). Based on the definition of inductive knowledge, we are able to define inductive learning. For simplicity and to illustrate the basic intuitions, our analysis will be for an unrestricted learner. We can provide a similar analysis with respect to a fully rational learner by adapting Example 3.2.1 appropriately.

First let us briefly present some of the previous ideas concerning the use of basic topological notions, such as *open neighborhoods*, in logics of information. Vickers (1989) reconstructed general topology as a logic of observation, in which the points of the space represent possible states of the world, while basic open neighborhoods of a point are interpreted as *information states* produced by accumulating finitely many observations. Moss and Parikh (1992) gave an account of learning in terms of *observational effort*. Making the epistemic effort to obtain more information about the actual world has a natural topological interpretation: it can be seen as *shrinking* the open neighborhood which represents the current information state, thus providing a more accurate approximation of the actual state of the world (Moss and Parikh, 1992; Dabrowski et al., 1996; Georgatos, 1994, 1997; Parikh et al., 2007). A similar idea was proposed in Formal Epistemology (Kelly, 1996; Baltag et al., 2015), where it was combined with more sophisticated notions of learning borrowed from FLT.

The following example relates the effort modality with knowledge.

3.2.1. EXAMPLE. [(Parikh et al., 2007, p. 309)] Let us consider some *measurement*, say of a vehicle’s velocity. Suppose a policeman uses radar gun to determine whether a car is speeding in a 50-mile speed-limit zone. The property *speeding* can be identified with the interval $(50, \infty)$. Suppose the radar gun shows 51 mph, but its accuracy is ± 2 mph. The meaning of a speed measurement of 51 ± 2 is that the car’s true speed v is in the *open interval* $(49, 53)$. According to Parikh et al. (2007), “anything which we *know* about v must hold not only of v itself,

but also of any v' in the same interval” (Parikh et al., 2007, p. 300). Since the interval $(49, 53)$ is not fully included in the “speeding” interval $(50, \infty)$, the policeman does *not know* that the car is speeding. But suppose that he does another measurement, using a more accurate radar gun with an accuracy of ± 1 mph, which shows 51.5 mph. Then he will *come to know* that the car is speeding: the open interval $(50.5, 52.5)$ is included in $(50, \infty)$.

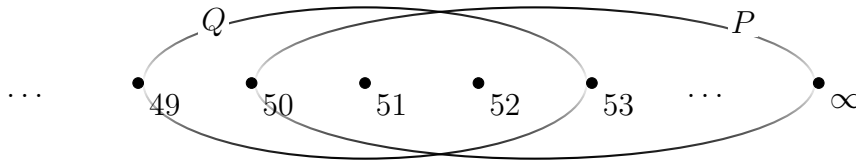


Figure 3.1: Example 3.2.1; $P :=$ “the car is speeding”, $Q :=$ “the reading of the radar is 51 mph”

3.2.1 Infallible Knowledge versus Inductive Knowledge

Let us now extend this picture with learning as understood in FLT. We start by briefly introducing *learning frames*, the underlying structures of learning. We will return to it later, with complete definitions, in Section 3.3.1. Our DLLT is interpreted over such frames and, using them, we will be able to explain and model various epistemic notions.

First, consider a pair (X, \mathcal{O}) , where X is a non-empty set of *possible worlds*; $\mathcal{O} \subseteq \mathcal{P}(X)$ is a non-empty set of *information states* (or “observables”, or “evidence”). We take \mathcal{O} to be closed under finite intersections, i.e., for any $O_1, O_2 \in \mathcal{O}$, we have $O_1 \cap O_2 \in \mathcal{O}$ (for the fully rational learners, we will consider \mathcal{O} to be also closed under finite unions). The resulting (X, \mathcal{O}) is called an *intersection space* (a *lattice space* when we consider a fully rational learner). A *learning frame* is a triplet $(X, \mathcal{O}, \mathbb{L})$, where $\mathbb{L} : \mathcal{O} \rightarrow \mathcal{P}(X)$ is a *learning function* (in short, a *learner*), i.e., a map associating to every $O \in \mathcal{O}$ some “conjecture” $\mathbb{L}(O) \subseteq X$ (see Definition 3.3.1 for the full description of \mathbb{L} and how it can be extended to range over finite sequences of observations).

3.2.2. DEFINITION. [Certain (Infallible) Knowledge] Given a learning frame $(X, \mathcal{O}, \mathbb{L})$ and an information state $U \in \mathcal{O}$, the learner is said to:

1. *infallibly know a proposition* $P \subseteq X$ *conditional on observation* O if her conditional information state entails P , i.e, if $U \cap O \subseteq P$.
2. *(unconditionally) know* P if $U \subseteq P$.

The possibility of achieving certain knowledge about a proposition $P \subseteq X$ in a possible world $x \in X$ by a learner \mathbb{L} if given enough evidence (true at x) is called *learnability with certainty*.³

3.2.3. DEFINITION. [Learnability with certainty] Given a learning frame $(X, \mathcal{O}, \mathbb{L})$ and an information state $U \in \mathcal{O}$, we say that:

1. P is *learnable with certainty in $x \in U$* if there exists some observable property $O \in \mathcal{O}$ (with $x \in O \subseteq U$) such that the learner unconditionally knows P in information state O .
2. Learnability can be used to define verifiability and falsifiability. A proposition $P \subseteq X$ is *verifiable with certainty (by the learner)* if it is learnable with certainty by the learner whenever it is true. In other words, if P is learnable with certainty at all worlds $x \in P$ with respect to all information states $U \in \mathcal{O}$ that contain x . Dually, a proposition $P \subseteq X$ is *falsifiable with certainty (by the learner)* if its negation $X - P$ is learnable with certainty by the learner whenever P is false. In other words, if $X - P$ is learnable with certainty at all worlds $x \notin P$ with respect to all $U \in \mathcal{O}$ that contain x .
3. Finally, a proposition $P \subseteq X$ is *decidable with certainty (by the learner)* if it is both verifiable and falsifiable with certainty (by the learner).

We can now reconstruct Example 3.2.1 as an intersection space to talk about the certain knowledge of the policeman. We take $X = (0, \infty)$ as the set of possible worlds (representing possible velocities of the car, where we assume the car is known to be *moving*); $\mathcal{O} = \{(a, b) \in \mathbb{Q} \times \mathbb{Q} : 0 < a < b < \infty\}$ is the set of all open intervals with positive rational endpoints (representing possible measurement results by arbitrarily accurate radars). The pair (X, \mathcal{O}) is an intersection space, and the smallest topology generated by \mathcal{O} is the standard topology on real numbers (restricted to X).

Let us consider the certain knowledge of the policeman. In the information state $U = (49, 53)$, the learner/policeman does not know the proposition $P = (50, \infty)$, so he cannot be certain that the car is speeding. However, the speeding property P is verifiable with certainty: whenever P is actually true, he could perform a more accurate speed measurement, by which he can get to an information state in which P is infallibly known. In our example, the policeman refined his measurement and reached the information state $O = (50.5, 52.5)$, thus come to know P . In contrast, the property $X - P = (0, 50]$ (“not speeding”) is not verifiable with certainty: if by some kind of miraculous coincidence, the speed

³When we quantify over learners, learnability with certainty (by *some* learners) matches the concept of “finitely identifiable” from Formal Learning Theory defined in Section 2.4.2 from Chapter 2. See also the PhD thesis of Gierasimczuk (2010).

of the car is exactly 50 mph, then the car is not speeding, but the policeman will never know that for certain (since every speed measurement, of any degree of accuracy, will be consistent both with P and with $X - P$). Nevertheless, $X - P$ is always falsifiable with certainty: if false (i.e., if the speed is in P , so that car is speeding), then as we saw the policeman will come to infallibly know that (by some more accurate measurement).

Let us now consider epistemic states weaker than knowledge with certainty, namely *belief*.

3.2.4. DEFINITION. [Undefeated Belief] Given a learning frame $(X, \mathcal{O}, \mathbb{L})$ and an information state $U \in \mathcal{O}$, the learner \mathbb{L} is said to:

1. *unconditionally believe* (in short, *believe*) $P \subseteq X$ if $\mathbb{L}(U) \subseteq P$.⁴
2. *believe a proposition* $P \subseteq X$ *conditional on observation* O if $\mathbb{L}(U \cap O) \subseteq P$;
3. *have undefeated belief in a proposition* $P \subseteq X$ *at world* x if she believes P in every information state $O \in \mathcal{O}$ that is true at x (i.e., $x \in O$) and is at least as strong as U (i.e., $O \subseteq U$). In other words, \mathbb{L} has undefeated belief in a proposition $P \subseteq X$ at world x if

for every $O \in \mathcal{O}$ such that $x \in O \subseteq U$, $\mathbb{L}(O) \subseteq P$.

This means that, once she reaches information state U , no further evidence can defeat the learner's belief in P .

One of the central problems in epistemology is to define a realistic notion of knowledge that fits the needs of empirical sciences. It should allow fallibility, while requiring a higher standards of evidence and robustness than simple belief. One of the main contenders is the so-called Defeasibility Theory of Knowledge, which defines *defeasible (fallible) knowledge* as true undefeated belief. In the learning-theoretic context, this gives us an evidence-based notion of *inductive knowledge*. This is the kind of knowledge that can be gained by *empirical induction*, based on experimental evidence (that is usually partial and incomplete). In other words, the knowledge obtained in the long run when acquiring an adequate hypothesis that is consistent with an ongoing stream of truthful empirical data. To illustrate, one can think of the knowledge a biologist acquires based on her experimental results and measurements on the processes of a certain bacteria. As in the case of learnability with certainty, inductive learnability of a proposition corresponds to achieving inductive knowledge of the proposition. Inductive verifiability and falsifiability are defined in terms of inductive learnability.

3.2.5. DEFINITION. [Inductive learning theoretic notions] Given a learning frame $(X, \mathcal{O}, \mathbb{L})$ and an information state $U \in \mathcal{O}$, the learner \mathbb{L} is said to:

⁴In the tautological information state X , the learner believes P iff $\mathbb{L}(X) \subseteq P$.

1. *track the truth of P* if the following holds: For every $x \in U$, if given enough evidence \mathbb{L} will come to have undefeated belief in P (in the sense defined above in item 3 from Definition 3.2.4) iff P is true at x . In other words, \mathbb{L} tracks the truth of P if the following holds,

$$\begin{aligned} &\text{for every } x \in U \text{ there is } V \in \mathcal{O} \text{ with } x \in V \subseteq U \text{ such that,} \\ &\quad \text{for every } O \in \mathcal{O} \text{ with } x \in O \subseteq V, \mathbb{L}(O) \subseteq P \\ &\quad \text{iff} \\ &\quad x \in P. \end{aligned}$$

2. Dually, the learner \mathbb{L} is said to *track the falsehood of P* iff \mathbb{L} tracks the truth of $\neg P$.
3. The learner \mathbb{L} is said to *inductively know* a proposition P at world x in U if \mathbb{L} has undefeated belief in P at x in U and \mathbb{L} tracks the truth of P in U .
4. A proposition $P \subseteq X$ is *inductively learnable* (or *learnable in the limit*) by \mathbb{L} at world x in U if given enough evidence (true at x), \mathbb{L} will come to inductively know P in U ; i.e., if there exists some observable property $O \in \mathcal{O}$ of world x (i.e., with $x \in O \subseteq U$) such that \mathbb{L} inductively knows P in information state O .
5. A proposition $P \subseteq X$ is *inductively verifiable* by \mathbb{L} , if P is inductively learnable by \mathbb{L} whenever it is true. In other words, $P \subseteq X$ is inductively verifiable by \mathbb{L} if and only if for every world $x \in X$ and every $U \in \mathcal{O}$ such that $x \in U$, P is inductively learnable by \mathbb{L} at world x in U whenever $x \in P$. Dually, a proposition $P \subseteq X$ is *inductively falsifiable* by \mathbb{L} , if its negation $X - P$ is inductively learnable by \mathbb{L} whenever P is false. In other words, $P \subseteq X$ is inductively falsifiable by \mathbb{L} if and only if for every world $x \in X$ and every $U \in \mathcal{O}$ such that $x \in U$, $X - P$ is inductively learnable by \mathbb{L} at world x in U whenever $x \notin P$. A proposition $P \subseteq X$ is *inductively decidable* by \mathbb{L} if it is both inductively verifiable and inductively falsifiable by \mathbb{L} .

In the context of Example 3.2.1, let us now turn to the inductive knowledge of the policeman. Both speeding (P) and non-speeding ($X - P$) are inductively decidable (and thus both inductively verifiable and inductively falsifiable): for instance, they are inductively decidable by the learner \mathbb{L} , defined by putting

$$\mathbb{L}((a, b)) = \begin{cases} (a, b), & \text{if } (a, b) \subseteq P, \\ (a, b) \cap (X - P), & \text{otherwise (i.e., } (a, b) \cap (X - P) \neq \emptyset). \end{cases}$$

Intuitively, such a learner is like a fair “judge” who assumes innocence until proven guilty: she conjectures that the car is not speeding as long as her measurement is consistent with $(X - P)$. The dual learner, the “suspicious cop”,

$$\mathbb{L}((a, b)) = \begin{cases} (a, b) \cap P, & \text{if } (a, b) \cap P \neq \emptyset, \\ (a, b), & \text{otherwise (i.e., if } (a, b) \subseteq X - P), \end{cases}$$

on the other hand can not inductively decide the speeding issue. Intuitively, this learner believes the car to be *speeding* whenever the available evidence cannot settle the issue, and keeps this conjecture until it is disproven by some more accurate measurement. In some cases, this policeman will be right “in the limit”: after doing enough accurate measurements, he will *eventually settle on the correct belief* (about speeding or not); though of course (in case the car’s speed is exactly 50 mph) he will never be certain of this. This is because any measurement (a, b) that contains 50 will intersect P , therefore the policeman will believe that the car is speeding when in reality it is not. An example of a property which is *inductively decidable but neither verifiable with certainty nor falsifiable with certainty* is the proposition $S = [50, 51)$. It is not verifiable with certainty, since if the car’s speed is exactly 50 mph, then S is true but the learner will never be certain of this; and it is not falsifiable with certainty, since if the car’s speed is exactly 51 mph, then S is false but the learner will never be certain of that. Nevertheless, S is *inductively decidable*, e.g., by the learner defined by:

$$\mathbb{L}((a, b)) = \begin{cases} (a, b) \cap S, & \text{if } a < 50 < b, \\ (a, b), & \text{if } (a, b) \subseteq S \text{ or } (a, b) \cap S = \emptyset, \\ [51, b), & \text{if } 50 < a < 51 < b. \end{cases}$$

Dependence on the Learner. It is easy to see that learnability (verifiability, falsifiability, decidability) with certainty are *learner-independent* notions (since they are directed towards achieving infallible knowledge), so they do not depend on \mathbb{L} , but only on the underlying intersection space. In contrast, *the corresponding inductive notions above are learner-dependent*. As a consequence, the interesting concepts in Formal Learning Theory are obtained from them by *quantifying existentially over learners*: given a learning frame, a proposition P is *inductively learnable* (verifiable, falsifiable, decidable) if there exists *some* learner \mathbb{L} such that P is respectively inductively learnable (verifiable, falsifiable, decidable) by \mathbb{L} . This definition can be extended to families of propositions of a given learning frame: a family of propositions \mathcal{P} is inductively learnable (verifiable, falsifiable, decidable) if there exists *some* learner \mathbb{L} such that every $P \in \mathcal{P}$ is respectively inductively learnable (verifiable, falsifiable, decidable) by \mathbb{L} . These two notions concerning inductive learnability (when quantifying over learners) match with the standard concepts of *non-effective*⁵ *inductive learning* from Formal Learning Theory as defined in Section 2.4.1 (see also e.g. Jain et al., 1999, Section 1.4.3, p. 10).

⁵Recall from Section 2.4.1 from Chapter 2 that the notion of *non-effective inductive learning* refers to inductive learning without computational features on the learners.

Topological Characterisations. As it is well-known in learning theory and formal epistemology (Vickers, 1989; Kelly, 1996), the above notions are *topological* in nature:

- P is learnable with certainty at world x iff x is in the *interior* of P with respect to the smallest topology generated by \mathcal{O} , i.e., $x \in \text{Int}(P)$ and $\text{Int}(P) \in \tau(\mathcal{O})$;
- P is verifiable with certainty iff P is *open* in the topology $\tau(\mathcal{O})$;
- P is falsifiable with certainty iff P is *closed* in the topology $\tau(\mathcal{O})$;
- finally, P is decidable with certainty iff P is *clopen* in the topology $\tau(\mathcal{O})$.

The corresponding inductive notions can be similarly characterised (see Kelly, 1996), when the smallest topology generated by \mathcal{O} , $\tau(\mathcal{O})$, satisfies the separation condition $T1^6$: in this case,

- P is inductively verifiable iff it is Σ_2 in the Borel hierarchy for this topology (i.e., a countable union of closed sets); in the same conditions,
- P is inductively falsifiable iff it is Π_2 (a countable intersection of open sets), and
- P is inductively decidable iff it is Δ_2 (i.e., Σ_2 and Π_2).

For a detailed discussion about the Borel Hierarchy, we refer to (Kechris, 1995, p. 167).

More recently, in (Baltag et al., 2015), these characterisations were generalised to arbitrary topologies satisfying the weaker separation condition $T0^7$; in particular, P is inductively verifiable iff it is a countable union of locally closed sets. Obviously, $T0$ is a minimally necessary condition for any kind of learnability of the real world from observations. For further explanations of the above topological separation conditions, we refer to (Engelking, 1989; Kelley, 1991).

3.3 A Dynamic Logic for Learning Theory

In this section we present our sound and complete Dynamic Logic for Learning Theory for an unrestricted learner who produces conjectures from fully-determined observations.

⁶A topology \mathcal{O} is $T1$ iff for every two distinct points $x \neq y$ there exist an open $O \in \mathcal{O}$ with $x \in O$ and $y \notin O$.

⁷A topology \mathcal{O} is $T0$ iff points can be distinguished by opens in \mathcal{O} ; i.e., if x and y satisfy the same observable properties in \mathcal{O} , then $x = y$.

3.3.1 Syntax and Semantics

Let $\text{Prop} = \{p, q, \dots\}$ be a countable set of *propositional variables*, denoting arbitrary “ontic” (i.e., non-epistemic) facts and $\text{Prop}_\mathcal{O} = \{o, u, v, \dots\}$ a countable set of *observational variables*, denoting “observable facts”.

Observational Events. We consider *observational events* e (or, in short, *observations*) by which the learner acquires some evidence about the world. We denote the set of all observational events by Π and define it by the following recursive clauses:

$$e := !\top \mid !o \mid (e; e)$$

where $o \in \text{Prop}_\mathcal{O}$. We will follow the usual rules for the elimination of the parentheses. Intuitively: for every observational variable o , we have a primitive observational event, denoted by $!o$, corresponding to *the event of observing variable* o . We also denote by $!\top$ the *null event* (in which *no new observation has taken place yet*). Observational events are naturally closed under regular operations on programs, of which we consider only one, $e; e'$, that represents *sequential composition* of fully determined observations (first observation e is made then observation e' is made). By a “fully determined observation e ” we mean that the observation is fully determined by the event e . With this condition, we make sure that it is clear which event determines which observation for the learner.

The Language of DLLT. The dynamic language \mathcal{L}_Π of the Dynamic Logic for Learning Theory is defined recursively as

$$\varphi := p \mid o \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid L(e) \mid K\varphi \mid [e]\varphi \mid \Box\varphi$$

where $p \in \text{Prop}$, $o \in \text{Prop}_\mathcal{O}$, and $e \in \Pi$. We employ the usual abbreviations for propositional connectives $\top, \perp, \vee, \rightarrow, \leftrightarrow$, and $\langle K \rangle\varphi, \langle e \rangle\varphi$ and $\diamond\varphi$ denote $\neg K\neg\varphi, \neg[e]\neg\varphi$, and $\neg\Box\neg\varphi$, respectively. Given a formula $\varphi \in \mathcal{L}_\Pi$ and $e \in \Pi$, we denote by O_φ and O_e the set of all observational variables occurring in φ and e , respectively.

Intuitively, $L(e)$ denotes the learner’s *conjecture* given observation e ; i.e., her “strongest belief” after having performed observation e . We read $K\varphi$ as “the learner *knows* φ (with absolute certainty)”. The operator $[e]\varphi$ is similar to the update operator in Public Announcement Logic: we read $[e]\varphi$ as “after event e is observed, φ holds”. Finally, \Box is the so-called effort modality from Subset Space Logic (Moss and Parikh, 1992; Dabrowski et al., 1996); we read $\Box\varphi$ as “ φ is *stably true*” (i.e. it is true and will stay true under any further observations).

Now we define the structures that we use to interpret our language \mathcal{L}_Π .

3.3.1. DEFINITION. [Learning Frame/Model] A *learning frame* is a triplet $(X, \mathcal{O}, \mathbb{L})$, where (X, \mathcal{O}) is an intersection frame (as given in Definition 2.2.1 in Section 2.2) and $\mathbb{L} : \mathcal{O} \rightarrow \mathcal{P}(X)$ is a *learner*, i.e., a map associating to every information state $O \in \mathcal{O}$ some conjecture $\mathbb{L}(O) \subseteq X$, and satisfying two properties:

1. $\mathbb{L}(O) \subseteq O$ (conjectures fit the evidence), and
2. if $O \neq \emptyset$ then $\mathbb{L}(O) \neq \emptyset$ (consistency of conjectures based on consistent evidence).

We can extend \mathbb{L} to range over *strings* of information states $\vec{O} = (O_1, \dots, O_n)$ in a natural way, by putting $\mathbb{L}(\vec{O}) := \mathbb{L}(\bigcap \vec{O})$, where $\bigcap \vec{O} := O_1 \cap \dots \cap O_n$. A *learning model* $M = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ is a learning frame $(X, \mathcal{O}, \mathbb{L})$ together with a valuation map $\|\cdot\| : \text{Prop} \cup \text{Prop}_{\mathcal{O}} \rightarrow \mathcal{P}(X)$ as above; equivalently, it consists of an intersection model $(X, \mathcal{O}, \|\cdot\|)$ together with a learner \mathbb{L} , as defined above.

Intuitively, the states in X represent possible worlds. The tautological evidence $X = \bigcap \emptyset$ represents the state of “no information” (before anything is observed), while the contradictory evidence \emptyset represents inconsistent information. Finally, $\mathbb{L}(O)$ represents the learner’s conjecture after observing O , while $\mathbb{L}(O_1, \dots, O_n) = \mathbb{L}(O_1 \cap \dots \cap O_n)$ represents the conjecture after observing a finite sequence of observations O_1, \dots, O_n . The fact that \mathcal{O} is closed under finite intersections is important here for identifying any finite sequence O_1, \dots, O_n with a single observation $O = O_1 \cap \dots \cap O_n \in \mathcal{O}$. This will become more clear later when we define the observational updates.

Epistemic Scenarios. As in Subset Space Semantics, the formulas of our logic are interpreted *not* at possible worlds, but at so-called *epistemic scenarios*: pairs (x, U) of an ontic state $x \in X$ and an information state $U \in \mathcal{O}$ such that $x \in U$. Therefore, only the *truthful* observations about the actual state play a role in the evaluation of formulas. Intuitively, x represents the *actual state* of the world, while U represents the *learner’s current evidence* (based on her previous observations). We denote by $ES(M) := \{(x, U) \mid x \in U \in \mathcal{O}\}$ the set of all epistemic scenarios of model M .

Each observational event $e \in \Pi$ induces a *dynamic update* of the learner’s information state. This is encoded in an *update function* e .

3.3.2. DEFINITION. [Observational updates, update function] An *update function* (also denoted by) $e : \mathcal{O} \rightarrow \mathcal{O}$ maps any information state $U \in \mathcal{O}$ to an updated information state $e(U) \in \mathcal{O}$. The map is given by recursion:

$$!\top(U) = U, \quad !o(U) = U \cap \|o\|, \quad (e; e')(U) = e'(e(U)).$$

The meaning of these clauses should be obvious: the null event $!\top$ does not change the learner’s information state; the single observation of variable o simply adds $\|o\|$ to the current evidence U (so that the learner will know the world is in $U \cap \|o\|$) and the information state after a sequential composition $e; e'$ is the same as the one obtained by updating first with e then with e' .

By the following lemmas, it is easy to see that the update map is appropriately defined:

3.3.3. LEMMA. *Let $M = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ be a learning model and $U \in \mathcal{O}$ be an information state. Then, for all $e \in \Pi$ we have $e(U) \in \mathcal{O}$.*

Proof:

The proof follows easily by induction on the structure of e . For the base cases $!\top$ and $!o$, we have $!\top(U) = U \in \mathcal{O}$ and $!o(U) = \|o\| \cap U \in \mathcal{O}$ by the closure of \mathcal{O} under finite intersections. In the inductive case $e; e'$, we apply the inductive hypothesis to e and U , yielding that $e(U) \in \mathcal{O}$, then we obtain that $(e; e') = e'(e(U))$ (by applying again the inductive hypothesis to e' and $e(U)$). \square

3.3.4. LEMMA. *Let $M = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ be a learning model and $U \in \mathcal{O}$ be an information state. Then, for all $e \in \Pi$, we have $e(U) \subseteq U$.*

Proof:

The proof follows easily by induction on the structure of e . Base cases follows easily by the definitions of the dynamic updates $!\top$ and $!o$. We only prove the following inductive case:

Case $e := f; f'$: $(f; f')(U) = f'(f(U)) \subseteq f(U)$ (by IH on f') $\subseteq U$ (by IH on f). \square

3.3.5. DEFINITION. [Size of events in Π] The *size* $s(e)$ of an event $e \in \Pi$ is a natural number recursively defined as:

$$\begin{aligned} s(!\top) = s(!o) &= 1, \\ s(e; e') &= s(e) + s(e') + 1. \end{aligned}$$

3.3.6. LEMMA. *Let $M = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ be a learning model and $U \in \mathcal{O}$ be an information state. Then, for all $e, e' \in \Pi$ we have: $(e; e')(U) = e(U) \cap e'(U)$.*

Proof:

The proof follows by induction on the size of $(e; e')$, with the induction hypothesis: (IH): for all $(f; f') \in \Pi$ such that $s(f; f') < s(e; e')$, $(f; f')(U) = f(U) \cap f'(U)$.

Base case $e' := !\top$

$(e; !\top)(U) = !\top(e(U)) = e(U)$ (by the definition of $!\top(U)$) $= e(U) \cap U$ (by Lemma 3.3.4) $= e(U) \cap !\top(U)$ (by the definition of $!\top(U)$).

Base case $e' := !o$

$(e; !o)(U) = !o(e(U)) = e(U) \cap \|o\|$ (by the definition of $!o(U)$) $= e(U) \cap (\|o\| \cap U)$ (by Lemma 3.3.4) $= e(U) \cap !o(U)$ (by the definition of $!o(U)$).

Case $e' := f; f'$

$$\begin{aligned} (e; e')(U) &= e'(e(U)) = (f; f')(e(U)) = f(e(U)) \cap f'(e(U)) \text{ (by IH, since } \\ &s(f; f') < s(e; e')) = (e; f)(U) \cap (e; f')(U) = (f(U) \cap e(U)) \cap (f'(U) \cap e(U)) \\ &\text{(by IH, since } s(e; f) < s(e; e') \text{ and also } s(e; f') < s(e; e')) = e(U) \cap (f(U) \cap \\ &f'(U)) = e(U) \cap (f; f')(U) \text{ (by IH, since } s(f; f') < s(e; e')) = e(U) \cap e'(U). \end{aligned}$$

□

We now proceed with the semantic definition.

3.3.7. DEFINITION. [Semantics] Given a learning model $M = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ and an epistemic scenario (x, U) , the semantics of the language \mathcal{L}_Π is given by a binary relation $(x, U) \models_M \varphi$ between epistemic scenario and formulas, called the *satisfaction relation*, as well as a *truth set* (interpretation) $\llbracket \varphi \rrbracket_M^U := \{x \in U \mid (x, U) \models_M \varphi\}$, for all formulas $\varphi \in \mathcal{L}_\Pi$. We typically omit the subscript, simply writing $(x, U) \models \varphi$ and $\llbracket \varphi \rrbracket^U$, whenever the model M is understood. The satisfaction relation is defined by the following recursive clauses:

$$\begin{array}{lll} (x, U) \models p & \text{iff} & x \in \llbracket p \rrbracket \\ (x, U) \models o & \text{iff} & x \in \llbracket o \rrbracket \\ (x, U) \models \neg\varphi & \text{iff} & (x, U) \not\models \varphi \\ (x, U) \models \varphi \wedge \psi & \text{iff} & (x, U) \models \varphi \text{ and } (x, U) \models \psi \\ (x, U) \models L(e) & \text{iff} & x \in \mathbb{L}(e(U)) \\ (x, U) \models K\varphi & \text{iff} & (\forall y \in U) ((y, U) \models \varphi) \\ (x, U) \models [e]\varphi & \text{iff} & x \in e(U) \text{ implies } (x, e(U)) \models \varphi \\ (x, U) \models \Box\varphi & \text{iff} & (\forall O \in \mathcal{O}) (x \in O \subseteq U \text{ implies } (x, O) \models \varphi) \\ & \text{i.e.} & (\forall O \in \mathcal{O}) (x \in O \text{ implies } (x, U \cap O) \models \varphi) \end{array}$$

where $p \in \text{Prop}$, $o \in \text{Prop}_\mathcal{O}$, and $e \in \Pi$.

We say that a formula φ is *valid in a learning model* M , and write $M \models \varphi$, if $(x, U) \models_M \varphi$ for all epistemic scenarios $(x, U) \in ES(M)$. We say φ is *valid*, and write $\models \varphi$, if it is valid in *all* learning models.

In our formal language, belief is not a primitive notion, but can be defined as an abbreviation:

$$B\varphi := K(L(!\top) \rightarrow \varphi),$$

that reads “the learner *believes* φ ”. Similarly we can define conditional belief as

$$B^e\varphi := K(L(e) \rightarrow \varphi),$$

that reads “the learner *believes* φ conditional on observational event e ”.

3.3.8. DEFINITION. [Precondition (informational content)] To each observational event $e \in \Pi$, we associate a formula $pre(e) \in \mathcal{L}_\Pi$, called the *precondition* of event e . The definition is by recursion: $pre(!\top) = \top$, $pre(!o) = o$ and $pre(e; e') = pre(e) \wedge pre(e')$.

The precondition formula $pre(e)$ captures the “condition of possibility” of the event e (i.e. e can happen in a world x iff $pre(e)$ is true at (x, U) , for any $U \in \mathcal{O}$ with $x \in U$), as well as its *informational content* (the learner’s new information after e). This leads to the following result:

3.3.9. LEMMA. *Let $M = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ be a learning model and $U \in \mathcal{O}$ be an information state. Then, for all $e \in \Pi$ we have:*

$$\llbracket pre(e) \rrbracket^U = e(U) = \llbracket \langle e \rangle \top \rrbracket^U.$$

Proof:

Equivalence $e(U) = \llbracket \langle e \rangle \top \rrbracket^U$ follows directly from the semantic clause of $[e]\varphi$ given in Definition 3.3.7. We prove $\llbracket pre(e) \rrbracket^U = e(U)$ by induction on the structure of e , using Lemma 3.3.6.

Base case $e := !\top$

$$\llbracket pre(!\top) \rrbracket^U = \llbracket \top \rrbracket^U = U = !\top(U) \text{ (by the definitions of } pre(e) \text{ and } !\top(U)).$$

Base case $e := !o$

$$\llbracket pre(!o) \rrbracket^U = \llbracket o \rrbracket^U = U \cap \|o\| = !o(U) \text{ (by the definitions of } pre(e) \text{ and } !o(U)).$$

Case $e := f; f'$

$$\begin{aligned} \llbracket pre(f; f') \rrbracket^U &= \llbracket pre(f) \wedge pre(f') \rrbracket^U \text{ (by the definitions of } pre(e)) = \llbracket pre(f) \rrbracket^U \\ &\cap \llbracket pre(f') \rrbracket^U = f(U) \cap f'(U) \text{ (by IH)} = (f; f')(U) \text{ (by Lemma 3.3.6)}. \end{aligned}$$

□

3.3.2 Axiomatization

Table 3.1 presents the axioms and inference rules for the Dynamic Logic for Learning Theory (DLLT). Further on, in Sections 3.3.5 and 4.6, we will show that DLLT is sound and complete with respect to the learning models.

Intuitive reading of the axioms and rules. Group Basic Axioms and rules are quite standard: **S5** axioms and rules for K says that the notion of knowledge with absolute certainty we study in this paper is factive and fully (both positively and negatively) introspective. $(K_{[e]})$ and $(Nec_{[e]})$ together show that dynamic modalities $[e]\varphi$ behave like normal modal operators. The reduction axioms

Table 3.1: The axiom schemas for the Dynamic Logic for Learning Theory (DLLT)

Basic axioms and rules:		
(CPL)	all instantiations of propositional tautologies	
(K_K)	$K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$	
(T_K)	$K\varphi \rightarrow \varphi$	
(4_K)	$K\varphi \rightarrow KK\varphi$	
(5_K)	$\neg K\varphi \rightarrow K\neg K\varphi$	
($K_{[e]}$)	$[e](\psi \rightarrow \chi) \rightarrow ([e]\psi \rightarrow [e]\chi)$	
Learning axioms:		
(CC)	$pre(e) \rightarrow \langle K \rangle L(e)$	Consistency of Conjecture
(EC)	$K(pre(e) \leftrightarrow pre(e')) \rightarrow (L(e) \leftrightarrow L(e'))$	Extensionality of Conjecture
(SP)	$L(e) \rightarrow pre(e)$	Success Postulate
Reduction axioms:		
(R_p)	$[e]p \leftrightarrow (pre(e) \rightarrow p)$	
(R_o)	$[e]o \leftrightarrow (pre(e) \rightarrow o)$	
(R_L)	$[e]L(e') \leftrightarrow (pre(e) \rightarrow L(e; e'))$	
(R_{\neg})	$[e]\neg\psi \leftrightarrow (pre(e) \rightarrow \neg[e]\psi)$	
(R_K)	$[e]K\psi \leftrightarrow (pre(e) \rightarrow K[e]\psi)$	
(R_e)	$[e][e']\psi \leftrightarrow [e; e']\psi$	
(R_{\square})	$[e]\square\psi \leftrightarrow \square[e]\psi$	
Effort axiom and rule:		
($\square Ax$)	$\square\varphi \rightarrow [e]\varphi$, for $e \in \Pi$	
($\square Ru$)	From $\vdash \psi \rightarrow [e; !o]\varphi$, infer $\vdash \psi \rightarrow [e]\square\varphi$, where $o \notin O_\psi \cup O_e \cup O_\varphi$	

are as in Epistemic Action Logic (EAL) (Baltag et al., 1998; Baltag and Renne, 2016) (a.k.a., Action Model Logic from van Ditmarsch et al., 2007), where the precondition of an observational event e is captured by $pre(e)$, that is, the informational content of the event e being true. The three learning axioms state the following: (CC) states that the learner conjectures consistent propositions upon having received truthful information; (EC) says that the form of the observational

event (primitive or sequential) is irrelevant for learning, what is important is the informational content of the observation: observing informationally equivalent events gives rise to equivalent conjectures. Moreover, (SP) states that what the learner conjectures fits what is observed, that is, the learner conjectures propositions that support what she has observed. Finally, we have the Effort rule ($\Box\text{Ru}$) and axiom ($\Box\text{Ax}$) which together explain the dynamic behaviour of the effort modality. While the latter expresses that if φ is stably true than it holds after any observational event has taken place, the former states that if φ holds after any more informative event has taken place ($[e; o]$), φ is stably true after e has taken place.

3.3.10. PROPOSITION. *The following formulas are derivable in **DLLT** for all $\varphi \in \mathcal{L}_\Pi$ and $e \in \Pi$:*

1. $[e](\varphi \wedge \psi) \leftrightarrow ([e]\varphi \wedge [e]\psi)$
2. $\langle e \rangle \psi \leftrightarrow (\text{pre}(e) \wedge [e]\psi)$
3. *from $\vdash \varphi \leftrightarrow \psi$, infer $\vdash [e]\varphi \leftrightarrow [e]\psi$*
4. $\langle e \rangle \text{pre}(e') \leftrightarrow \text{pre}(e; e')$
5. *from $\vdash \text{pre}(e) \leftrightarrow \text{pre}(e')$, infer $\vdash [e]\varphi \leftrightarrow [e']\varphi$*
6. $[!\top]\varphi \leftrightarrow \varphi$ (we denote it $R[\top]$)
7. *from $\vdash \psi \rightarrow [!o]\varphi$, infer $\vdash \psi \rightarrow \Box\varphi$ (where $o \notin O_\psi \cup O_\varphi$).*

Proof:

1. Follows from $K_{[e]}$ and $\text{Nec}_{[e]}$ standardly.
2. From $\langle e \rangle \psi := \neg[e]\neg\psi$ and the reduction axiom R_{\neg} .
3. From $K_{[e]}$ and $\text{Nec}_{[e]}$.
4. The proof goes by induction on the structure of e' . For the derivation of the base case $e' := !o$: use (2.) in Proposition 3.3.10, the reduction axiom R_o , CPL and the definition of precondition (Definition 3.3.8) for $e; !o$, namely $\text{pre}(e; !o) = \text{pre}(e) \wedge \text{pre}(!o)$.

For the base case $e' := !\top$:

1. $\vdash \langle e \rangle \text{pre}(!\top) \leftrightarrow \langle e \rangle \top$ (Definition 3.3.8 for $!\top$)
2. $\vdash \langle e \rangle \top \leftrightarrow \text{pre}(e)$ (by (2.) in Proposition 3.3.10, R_o and CPL)
3. $\vdash \text{pre}(e) \leftrightarrow (\text{pre}(e) \wedge \text{pre}(!\top))$ (CPL, Definition 3.3.8 for $!\top$)
4. $\vdash (\text{pre}(e) \wedge \text{pre}(!\top)) \leftrightarrow \text{pre}(e; !\top)$ (Definition 3.3.8 for $e; !\top$)
5. $\vdash \langle e \rangle \text{pre}(!\top) \leftrightarrow \text{pre}(e; !\top)$ (CPL, 1, 4)

We now prove the inductive case $e' := f; f'$. First note that at this point of the proof we have that $\vdash \langle e \rangle \text{pre}(f) \leftrightarrow \text{pre}(e; f)$ and $\vdash \langle e \rangle \text{pre}(f') \leftrightarrow \text{pre}(e; f')$. The derivation goes as follows:

1. $\vdash \langle e \rangle \text{pre}(e') \leftrightarrow \text{pre}(e) \wedge [e] \text{pre}(e')$ (by (2) in Proposition 3.3.10)
2. $\vdash \langle e \rangle \text{pre}(f; f') \leftrightarrow \text{pre}(e) \wedge [e] \text{pre}(f; f')$ (since $e' := f; f'$)
3. $\vdash \text{pre}(e) \wedge [e] \text{pre}(f; f') \leftrightarrow \text{pre}(e) \wedge [e](\text{pre}(f) \wedge \text{pre}(f'))$
(Definition 3.3.8 for $f; f'$)
4. $\vdash (\text{pre}(e) \wedge [e](\text{pre}(f) \wedge \text{pre}(f'))) \leftrightarrow (\text{pre}(e) \wedge ([e] \text{pre}(f) \wedge [e] \text{pre}(f')))$
($K_{[e]}$)
5. $\vdash (\text{pre}(e) \wedge ([e] \text{pre}(f) \wedge [e] \text{pre}(f'))) \leftrightarrow (\langle e \rangle \text{pre}(f) \wedge \langle e \rangle \text{pre}(f'))$
(CPL and (2) in Proposition 3.3.10)
6. $\vdash (\langle e \rangle \text{pre}(f) \wedge \langle e \rangle \text{pre}(f')) \leftrightarrow (\text{pre}(e; f) \wedge \text{pre}(e; f'))$ (IH on f and f')
7. $\vdash (\text{pre}(e; f) \wedge \text{pre}(e; f')) \leftrightarrow (\text{pre}(e) \wedge \text{pre}(f) \wedge \text{pre}(e) \wedge \text{pre}(f'))$
(Definition 3.3.8 for $e; f$ and $e; f'$)
8. $\vdash (\text{pre}(e) \wedge \text{pre}(f) \wedge \text{pre}(e) \wedge \text{pre}(f')) \leftrightarrow (\text{pre}(e) \wedge \text{pre}(f; f'))$
(CPL and Definition 3.3.8 for $f; f'$)
9. $\vdash (\text{pre}(e) \wedge \text{pre}(f; f')) \leftrightarrow \text{pre}(e; e')$
(since $e' := f; f'$ and Definition 3.3.8 for $e; e'$)
10. $\vdash \langle e \rangle \text{pre}(e') \leftrightarrow \text{pre}(e; e')$ (CPL, 1, 9)

5. Follows by induction on the structure of φ using the corresponding reduction axiom, CPL, and replacement of provable equivalences for the formulas outside of the dynamic observation operators.
6. Follows by induction on the structure of φ and the corresponding reduction axiom.
7. Suppose $\vdash \psi \rightarrow [!o]\varphi$ such that $o \notin O_\psi \cup O_\varphi$, then we have the following derivation for $\psi \rightarrow \Box\varphi$:

1. $\vdash \psi \rightarrow [!o]\varphi$ (by assumption, $o \notin O_\psi \cup O_\varphi$)
2. $\vdash [!o]\varphi \rightarrow [!\top; !o]\varphi$
(by (5) in Proposition 3.3.10 since $\vdash \text{pre}(!o) \leftrightarrow \text{pre}(!\top; !o)$)
3. $\vdash \psi \rightarrow [!\top; !o]\varphi$ (CPL, 1 and 2)
4. $\vdash \psi \rightarrow [!\top]\Box\varphi$ (by $\Box\text{Ru}$)
5. $\vdash \psi \rightarrow \Box\varphi$ (by $R[\top]$ and CPL)

□

In our framework, belief and conditional beliefs (B and $B^e\varphi$) are defined in terms of the operators K and L . The axiomatic system **DLT** given in Table 3.1 over the language \mathcal{L}_Π can therefore derive the properties describing the type of belief and conditional belief modalities we intend to formalize here. More precisely, as stated in Proposition 3.3.11, the system **DLT** yields the standard belief system KD45 for B . More generally, if we replace the (D) axiom for a “weaker” version $(D') := \langle K \rangle pre(e) \rightarrow \neg B^e \perp$ then we have a weak version of the system KD45, denoted by wKD45, for conditional beliefs B^e .

3.3.11. PROPOSITION (WKD45 SYSTEM FOR CONDITIONAL BELIEF). *The standard axioms and rules of the doxastic logic KD45 (see Section 2.1) are derivable for our belief operator B in the system **DLT**. More generally, the following axioms and rules of the weaker system wKD45 are derivable for our conditional belief operator B^e in the system **DLT**:*

- (Nec $_{B^e}$) From $\vdash \varphi$, infer $\vdash B^e\varphi$.
- (K $_{B^e}$) $B^e(\varphi \rightarrow \psi) \rightarrow (B^e\varphi \rightarrow B^e\psi)$
- (D' $_{B^e}$) $\langle K \rangle pre(e) \rightarrow \neg B^e \perp$
- (4 $_{B^e}$) $B^e\varphi \rightarrow B^e(B^e\varphi)$
- (5 $_{B^e}$) $\neg B^e\varphi \rightarrow B^e(\neg B^e\varphi)$

Proof:

We first prove (Nec $_{B^e}$), (K $_{B^e}$), (D' $_{B^e}$), (4 $_{B^e}$) and (5 $_{B^e}$) for B^e . Then we prove the system KD45 for B . Recall that $B^e\varphi := K(L(e) \rightarrow \varphi)$ and let $e \in \Pi$.

(Nec $_{B^e}$): From $\vdash \varphi$, infer $\vdash B^e\varphi$.

1. $\vdash \varphi$ (assumption)
2. $\vdash L(e) \rightarrow \varphi$ (1, CPL)
3. $\vdash K(L(e) \rightarrow \varphi)$ (Nec $_K$)

(K $_{B^e}$): we need to show that

$$\vdash K(L(e) \rightarrow (\varphi \rightarrow \psi)) \rightarrow (K(L(e) \rightarrow \varphi) \rightarrow K(L(e) \rightarrow \psi)).$$

1. $\vdash (L(e) \rightarrow (\varphi \rightarrow \psi)) \rightarrow ((L(e) \rightarrow \varphi) \rightarrow (L(e) \rightarrow \psi))$ (CPL)
2. $\vdash K((L(e) \rightarrow (\varphi \rightarrow \psi)) \rightarrow ((L(e) \rightarrow \varphi) \rightarrow (L(e) \rightarrow \psi)))$ (Nec $_K$)
3. $\vdash K(L(e) \rightarrow (\varphi \rightarrow \psi)) \rightarrow K((L(e) \rightarrow \varphi) \rightarrow (L(e) \rightarrow \psi))$ (K $_K$)
4. $\vdash K(L(e) \rightarrow (\varphi \rightarrow \psi)) \rightarrow (K(L(e) \rightarrow \varphi) \rightarrow K(L(e) \rightarrow \psi))$ (K $_K$, 3, CPL)

(D'_{B^e}): we need to show that

$$\vdash \langle K \rangle pre(e) \rightarrow \langle K \rangle L(e).$$

1. $\vdash pre(e) \rightarrow \langle K \rangle L(e)$ (CC)
2. $\vdash K \neg L(e) \rightarrow \neg pre(e)$ (contraposition of CC)
3. $\vdash K(K \neg L(e) \rightarrow \neg pre(e))$ (Nec_K)
4. $\vdash K K \neg L(e) \rightarrow K \neg pre(e)$ (K_K, 3, MP)
5. $\vdash K \neg L(e) \rightarrow K \neg pre(e)$ (4_K)
6. $\vdash \langle K \rangle pre(e) \rightarrow \langle K \rangle L(e)$ (contraposition of 5)

(4_{B^e}): we need to show that

$$\vdash K(L(e) \rightarrow \varphi) \rightarrow K(L(e) \rightarrow K(L(e) \rightarrow \varphi)).$$

1. $\vdash K(L(e) \rightarrow \varphi) \rightarrow (L(e) \rightarrow K(L(e) \rightarrow \varphi))$ (CPL)
2. $\vdash K K(L(e) \rightarrow \varphi) \rightarrow K(L(e) \rightarrow K(L(e) \rightarrow \varphi))$ (Nec_K, K_K, 1, MP)
3. $\vdash K(L(e) \rightarrow \varphi) \rightarrow K(L(e) \rightarrow K(L(e) \rightarrow \varphi))$ (4_K)

(5_{B^e}): we need to show that

$$\vdash \neg K(L(e) \rightarrow \varphi) \rightarrow K(L(e) \rightarrow \neg K(L(e) \rightarrow \varphi)).$$

1. $\vdash \neg K(L(e) \rightarrow \varphi) \rightarrow (L(e) \rightarrow \neg K(L(e) \rightarrow \varphi))$ (CPL)
2. $\vdash K \neg K(L(e) \rightarrow \varphi) \rightarrow K(L(e) \rightarrow \neg K(L(e) \rightarrow \varphi))$ (Nec_K, K_K, 1, MP)
3. $\vdash \neg K(L(e) \rightarrow \varphi) \rightarrow K(L(e) \rightarrow \neg K(L(e) \rightarrow \varphi))$ (5_K)

The KD45 axioms and rules for B follows from the system wKD45 as a special case when $e := !\top$. \square

3.3.12. PROPOSITION (S4 SYSTEM FOR EFFORT). *The S₄ axioms and rules for the effort modality \square are derivable in DLLT.*

Proof:

The derivation of (Nec_□) easily follows from (Nec_[e]) and (□Ru). The T-axiom for \square follows from (□Ax) for $e := !\top$.

For the K-axiom:

1. $\vdash (\square(\varphi \rightarrow \psi) \wedge \square\varphi) \rightarrow ([!o](\varphi \rightarrow \psi) \wedge ![o]\varphi)$ (□Ax, for some $o \notin O_\varphi \cup O_\psi$)
2. $\vdash ([!o](\varphi \rightarrow \psi) \wedge ![o]\varphi) \rightarrow ![o]\psi$ (K_[e])
3. $\vdash (\square(\varphi \rightarrow \psi) \wedge \square\varphi) \rightarrow ![o]\psi$ (CPL, 1 and 2)
4. $\vdash (\square(\varphi \rightarrow \psi) \wedge \square\varphi) \rightarrow \square\psi$ ((7) in Proposition 3.3.10, $o \notin O_\varphi \cup O_\psi$)

For the 4-axiom:

1. $\vdash \Box \varphi \rightarrow [!o; !u]\varphi$ (\Box Ax, for some $o, u \notin O_\varphi$)
2. $\vdash \Box \varphi \rightarrow [!o]\Box \varphi$ (\Box Ru)
3. $\vdash \Box \varphi \rightarrow \Box \Box \varphi$ ((7) in Proposition 3.3.10)

□

3.3.3 Expressivity of \mathcal{L}_Π

In this section we compare the expressive power of \mathcal{L}_Π to those of its fragments of interest. Let $\mathcal{L}_\Pi^{-!}$ denote the fragment of \mathcal{L}_Π obtained by removing only the update operators $[e]\varphi$. The fragment obtained by further removing the effort modality $\Box \varphi$ is called the *static fragment* and denoted by $\mathcal{L}_\Pi^{-!\Box}$. Finally, we denote the epistemic fragment having only K as its modality by \mathcal{L}_Π^{epis} .

As usual in Dynamic Epistemic Logic, the dynamic observational event modalities $[e]\varphi$ are only a convenient way to express complex properties in a succinct manner, but they can in principle be eliminated.

In some of our inductive proofs, we need a complexity measure on formulas different from the standard one which is based on subformula complexity. The standard notion requires only that formulas are more complex than their subformulas, while we also need that $[e]\Box \varphi$ is more complex than $\Box[e]\varphi$ for all $e \in \Pi$. We provide such suitable complexity measure and a strict partial order \prec_1 on \mathcal{L}_Π in the Technical Appendix A.

3.3.13. PROPOSITION (EXPRESSIVITY). *The language \mathcal{L}_Π is co-expressive with its fragment $\mathcal{L}_\Pi^{-!}$. Moreover, this can be proved in **DLLT**: for every formula φ there exists some formula φ' free of any dynamic modalities, such that $\vdash \varphi \leftrightarrow \varphi'$. Furthermore, if φ contains dynamic modalities then φ' can be chosen such that $\varphi' \prec_1 \varphi$.*

Proof:

Suppose, towards contradiction, that φ is not in $\mathcal{L}_\Pi^{-!}$, and that φ is not provably equivalent to any formula of lower complexity (in the sense of \prec_1 from Lemma A.1.2) that is in $\mathcal{L}_\Pi^{-!}$. We construct an infinite descending sequence

$$\varphi_0 \succ_1 \varphi_1 \succ_1 \dots \succ_1 \varphi_n \succ_1 \dots$$

of provably equivalent formulas, none of which is in $\mathcal{L}_\Pi^{-!}$. The construction goes as follows. We first put $\varphi_0 := \varphi$. Then, at step n , assuming given φ_n not in $\mathcal{L}_\Pi^{-!}$, and provably equivalent to all the previous formulas, we chose the first dynamic modality occurring in φ_n , and apply once to it the relevant Reduction Axiom (from left to right), obtaining a provably equivalent formula φ_{n+1} , which

by Lemma [A.1.2](#) has the property that $\varphi_{n+1} \prec_1 \varphi_n$. By transitivity of provable equivalence, φ_{n+1} is provably equivalent to $\varphi_0 = \varphi$, and (by transitivity of \prec_1) it is of lower complexity than $\varphi_0 = \varphi$; so, by our assumption above, φ_{n+1} is still not in \mathcal{L}_Π^{-1} . But the existence of this infinite descending sequence contradicts the well-foundedness of \prec_1 . \square

In the following theorem, we compare the expressive power of the languages \mathcal{L}_Π , $\mathcal{L}_\Pi^{-! \square}$ and \mathcal{L}_Π^{epis} .

3.3.14. THEOREM (EXPRESIVITY OF FRAGMENTS OF \mathcal{L}_Π). *The full language \mathcal{L}_Π (and thus \mathcal{L}_Π^{-1}) is strictly more expressive than the static fragment $\mathcal{L}_\Pi^{-! \square}$ with respect to learning models. Moreover, $\mathcal{L}_\Pi^{-! \square}$ is strictly more expressive than the epistemic fragment \mathcal{L}_Π^{epis} .*

Proof:

For the first claim, consider the following two-state models $M_1 = (X, \mathcal{O}_1, \mathbb{L}, \|\cdot\|)$ and $M_2 = (X, \mathcal{O}_2, \mathbb{L}, \|\cdot\|)$ where $X = \{x, y\}$, $\mathbb{L} := \mathcal{O} \rightarrow \mathcal{P}(X)$ is such that $\mathbb{L}(O) := O$ for every $O \in \mathcal{O}$ and the valuation $\|p\| = \{y\}$. And, take $\mathcal{O}_1 = \{X, \emptyset\}$ (the trivial topology on X) and $\mathcal{O}_2 = \mathcal{P}(X)$ (the discrete topology on X). It is then easy to see that $M_1, (x, \{x, y\})$ and $M_2, (x, \{x, y\})$ are modally equivalent with respect to the language $\mathcal{L}_\Pi^{-! \square}$. However, $M_2, (x, \{x, y\}) \models \diamond K \neg p$ (since $\{x\}$ is an open set of M_2 and $x \notin \|p\| = \{y\}$) whereas $M_1, (x, \{x, y\}) \not\models \diamond K \neg p$, since the only open including x is $\{x, y\}$ and $y \in \|p\| = \{y\}$. To prove that $\mathcal{L}_\Pi^{-! \square}$ is strictly more expressive than the epistemic fragment \mathcal{L}_Π^{epis} , consider the models $M'_1 = (X, \mathcal{O}_1, \mathbb{L}_1, \|\cdot\|)$ and $M'_2 = (X, \mathcal{O}_2, \mathbb{L}_2, \|\cdot\|)$, where X, \mathcal{O}_1 , and \mathcal{O}_2 are as above but $\mathbb{L}_1 = \mathbb{L}$ and \mathbb{L}_2 is such that $\mathbb{L}_2(\emptyset) = \emptyset$, $\mathbb{L}_2(\{x\}) = \{x\}$ and $\mathbb{L}_2(\{y\}) = \mathbb{L}_2(\{x, y\}) = \{y\}$. It is then easy to see that $M'_1, (x, \{x, y\})$ and $M'_2, (x, \{x, y\})$ are modally equivalent with respect to the language \mathcal{L}_Π^{epis} whereas $M'_2, (x, \{x, y\}) \not\models L(!\top)$ (since $x \notin \mathbb{L}_2(!\top(\{x, y\})) = \{y\}$) but $M'_1, (x, \{x, y\}) \models L(!\top)$ (since $x \in \mathbb{L}_1(!\top(\{x, y\})) = \{x, y\}$). \square

The expressivity diagram in Figure [3.2](#) summarizes Proposition [3.3.13](#) and Theorem [3.3.14](#).

3.3.4 Expressing belief, inductive learning and learning in the limit

We investigate how various notions of learnability, already mentioned in Sections [3.2](#) and [3.2.1](#), can be expressed in the framework of DLLT. We start with the notions that were already captured in ([Moss and Parikh, 1992](#)). Then we will see how to define in our language *inductive knowledge* and *inductive learnability*, which cannot be expressed in the language of SSL.

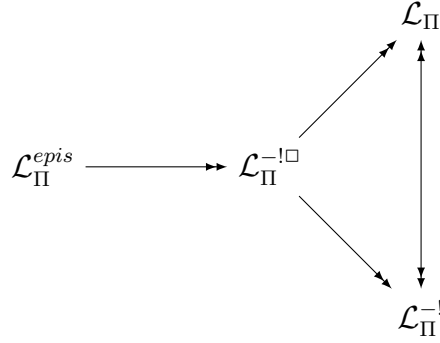


Figure 3.2: Expressivity diagram (Arrows point to the more expressive languages, and reflexive and transitive arrows are omitted.)

3.3.15. PROPOSITION. $\diamond Kp$ is true at (x, U) in a model M iff $\|p\|$ is learnable with certainty at state x . Similarly, $p \rightarrow \diamond Kp$ is valid (i.e., true at all epistemic scenarios) in a model M iff $\|p\|$ is verifiable with certainty. A similar statement holds for falsifiability with certainty.

Proof:

As we know from Section 3.2.1, $\|p\|$ is learnable with certainty in x iff it exists $O \in \mathcal{O}$ such that $x \in O \subseteq \|p\|$. We also know that $\|p\|$ is verifiable with certainty iff $\|p\|$ is learnable with certainty at all worlds $x \in \|p\|$ with respect to all information states $U \in \mathcal{O}$ that contain x . Note that the latter, verifiability with certainty, is equivalent to the following statement: $\|p\|$ is open in the smallest topology generated by \mathcal{O} . It is well-known that these properties are expressible in SSL via the formulas above (Moss and Parikh, 1992), namely $\diamond Kp$ and $p \rightarrow \diamond Kp$. \square

In particular, the following validity of our logic expresses the fact that *all observable properties are verifiable with certainty* (contrary to the language in (Moss and Parikh, 1992) with no observational variables):

$$o \rightarrow \diamond Ko.$$

By adding the learning operator to subset space logic, DLLT can capture, not only belief, but also the various inductive notions of knowledge and learnability we presented in Definitions 3.2.4 and 3.2.5. The notion of infallible knowledge is obviously very strong: we know very few things with such certainty (maybe some logical or mathematical truths that require only hard thinking and no empirical evidence). One needs weaker notions of knowledge if one desires to model the type of knowledge we can acquire from experimental evidence that is typically partial and incomplete. This type of knowledge is taken to be fallible, yet resistant to

truthful evidence gain and stronger than plain belief. In this learning theoretical context, it is captured by an evidence-based notion of *inductive knowledge* defined in terms of *undefeated belief* and *tracking the truth* (see Definition 3.2.5). Recall that we can define belief in our language as $B\varphi := K(L(!\top) \rightarrow \varphi)$.

3.3.16. PROPOSITION (INDUCTIVE LEARNING THEORETIC NOTIONS). *Given a learning model $M = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ and $(x, U) \in ES(M)$,*

1. $(x, U) \models \Box Bp$ iff the learner \mathbb{L} has undefeated belief in $\|p\|$ (at world x in information state U).
2. $(x, U) \models K(\Diamond \Box Bp \leftrightarrow p)$ iff the learner \mathbb{L} tracks the truth of $\|p\|$ (at world x in information state U). Similarly, $(x, U) \models K(\Diamond \Box B\neg p \leftrightarrow \neg p)$ iff the learner \mathbb{L} tracks the falsehood of $\|p\|$ (at world x in information state U).
3. $(x, U) \models \Box Bp \wedge K(\Diamond \Box Bp \leftrightarrow p)$ iff the learner \mathbb{L} inductively knows $\|p\|$ (at world x in information state U).

We denote $K^{ind}p := \Box Bp \wedge K(\Diamond \Box Bp \leftrightarrow p)$.

4. $(x, U) \models \Diamond K^{ind}p$ iff $\|p\|$ is inductively learnable by \mathbb{L} (at world x in information state U).
5. $(x, X) \models \Diamond L(!\top)$ iff (given enough observations) the learner \mathbb{L} will eventually reach a true conjecture (though she might later fall again into false ones); and similarly, $(x, X) \models \Diamond \Box L(!\top)$ iff (given enough observations) the learner will eventually produce only true conjectures thereafter.
6. The formula $p \rightarrow \Diamond K^{ind}p$ is valid in M iff $\|p\|$ is inductively verifiable by \mathbb{L} . For the corresponding generic notion: $\|p\|$ is inductively verifiable (by some learner) iff $p \rightarrow \Diamond K^{ind}p$ is valid in the intersection space (X, \mathcal{O}) .
7. Similarly, $\neg p \rightarrow \Diamond K^{ind}\neg p$ is valid in M iff $\|p\|$ is inductively falsifiable by \mathbb{L} . In other words, $\|p\|$ is inductively falsifiable by \mathbb{L} iff $X - \|p\|$ is inductively verifiable by \mathbb{L} . For the corresponding generic notion: $\|p\|$ is inductively falsifiable (by some learner) iff $\neg p \rightarrow \Diamond K^{ind}\neg p$ is valid in the intersection space (X, \mathcal{O}) .

Proof:

Let $M = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ be a learning model and $(x, U) \in ES(M)$.

1. For undefeated belief:

$(x, U) \models \Box Bp$ iff $\forall V \in \mathcal{O}$ such that $x \in V \subseteq U$, $(x, V) \models Bp$ (Definition 3.3.7 for \Box) iff $\forall V \in \mathcal{O}$ such that $x \in V$, $(x, V) \models K(L(!\top) \rightarrow p)$ (abbreviation for B) iff $\forall V \in \mathcal{O}$ such that $x \in V$, $\mathbb{L}(V) \subseteq \llbracket p \rrbracket^V$ (Definition 3.3.7 for K , L and p , and $!\top(V) = V$) iff \mathbb{L} has undefeated belief in $\llbracket p \rrbracket^U$ at world x (by Definition 3.2.4).

2. For tracking the truth:

$(x, U) \models K(\diamond \Box Bp \leftrightarrow p)$ iff $\forall y \in U, (y, U) \models \diamond \Box Bp \leftrightarrow p$ (Definition 3.3.7 for K) iff $\forall y \in U, ((y, U) \models \diamond \Box Bp$ iff $(y, U) \models p)$ iff $\forall y \in U ((\exists V \in \mathcal{O}$ with $y \in V \subseteq U$ such that $(y, V) \models \Box Bp$) iff $y \in \llbracket p \rrbracket^U$) (Definition 3.3.7 for \diamond) iff $\forall y \in U ((\exists V \in \mathcal{O}$ with $y \in V \subseteq U$ such that \mathbb{L} has undefeated belief in $\llbracket p \rrbracket^V$ at y) iff $y \in \llbracket p \rrbracket^U$) (by the proof above for undefeated belief) iff \mathbb{L} tracks the truth of $\llbracket p \rrbracket^U$ at world x (by Definition 3.2.5).

Tracking the falsehood follows by a similar reasoning.

3. For inductive knowledge: it follows straightforwardly from the proofs above that $(x, U) \models \Box Bp \wedge K(\diamond \Box Bp \leftrightarrow p)$ iff \mathbb{L} has undefeated belief in $\llbracket p \rrbracket^U$ and \mathbb{L} tracks the truth of $\llbracket p \rrbracket^U$ at world x iff $\llbracket p \rrbracket^U$ is inductively known at world x (by Definition 3.2.5).

4. For inductive learnability:

$(x, U) \models \diamond K^{ind} p$ iff $\exists V \in \mathcal{O}$ such that $x \in V \subseteq U$ and $(x, V) \models K^{ind} p$ (Definition 3.3.7 for \Box) iff there exists some observable $V \in \mathcal{O}$ of x such that $\llbracket p \rrbracket^V$ is inductively known by \mathbb{L} at world x (by the proof above for inductive knowledge) iff $\llbracket p \rrbracket^U$ is inductively learnable by \mathbb{L} at world x (by Definition 3.2.5).

5. $(x, X) \models \diamond L(!\top)$ iff $\exists V \in \mathcal{O}$ such that $x \in V$ and $(x, V) \models L(!\top)$ (Definition 3.3.7 for \diamond) iff $\exists V \in \mathcal{O}$ such that $x \in V$ and $x \in \mathbb{L}(V)$ (Definition 3.3.7 for L and since $!\top(V) = V$) iff (this already expresses what we wanted) the learner \mathbb{L} will eventually reach a true conjecture at world x . The remaining case follows standardly.

6. For inductive verifiability:

$p \rightarrow \diamond K^{ind} p$ is valid in a learning model M iff for every $(x, U) \in ES(M)$, $(x, U) \models p \rightarrow \diamond K^{ind} p$ iff for every $(x, U) \in ES(M)$, $x \in \llbracket p \rrbracket^U$ implies $(x, U) \models \diamond K^{ind} p$ (CPL, Definition 3.3.7 for p) iff for every $(x, U) \in ES(M)$, if $\llbracket p \rrbracket^U$ is true at x then $\llbracket p \rrbracket^U$ is inductively learnable by \mathbb{L} at world x (by the proof above for inductive learnability) iff $\llbracket p \rrbracket^U$ is inductively verifiable by \mathbb{L} (by Definition 3.2.5). The generic case follows standardly.

7. For inductive falsifiability: follows as in the previous case with respect to the validity of $\neg p \rightarrow \diamond K^{ind} \neg p$.

□

3.3.5 Soundness of DLLT

In this section we prove soundness of **DLLT**. This is not a trivial matter due to the non-standard rule ($\Box\text{Ru}$), thus we first need the following lemma.

3.3.17. LEMMA. *Let $M = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ and $M' = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|')$ be two learning models and $\varphi \in \mathcal{L}_\Pi$ such that M and M' differ only in the valuation of some $o \notin O_\varphi$. Then, for all $U \in \mathcal{O}$, we have $\llbracket \varphi \rrbracket_M^U = \llbracket \varphi \rrbracket_{M'}^U$.*

Proof:

Follows by subformula induction on φ . Let $M = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ and $M' = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|')$ be two learning models such that M and M' differ only in the valuation of some $o \notin O_\varphi$, let $U \in \mathcal{O}$ and $e \in \Pi$.

Base case $\varphi := q \in \text{Prop}$. Follows by the fact that q is not an observational variable, thus, for all $q \in \text{Prop}$ we have that $\|q\| = \|q\|'$. Since M and M' have the same set of opens \mathcal{O} , for all $U \in \mathcal{O}$ we have that $\llbracket q \rrbracket_M^U = U \cap \|q\| = U \cap \|q\|' = \llbracket q \rrbracket_{M'}^U$.

Base case $\varphi := o \in \text{Prop}_\mathcal{O}$. Since $o \in O_\varphi$, we have that $\|o\| = \|o\|'$. By the same reasoning as above, $\llbracket o \rrbracket_M^U = \llbracket o \rrbracket_{M'}^U$.

Case $\varphi := L(e)$

Note that $O_{L(e)} = O_e$ and for every $o \in O_e$, $\|o\| = \|o\|'$. Therefore, since \mathcal{O} is the same in both models, $e(U) \in \mathcal{O}$ is the same in both models. Also, \mathbb{L} is the same in both models, thus $\llbracket L(e) \rrbracket_M^U = \mathbb{L}(e(U)) = \llbracket L(e) \rrbracket_{M'}^U$.

The cases for Booleans $\varphi := \neg\psi$ and $\varphi := \psi \wedge \chi$ are straightforward.

Case $\varphi := K\psi$

Note that $O_{K\psi} = O_\psi$. Then, by induction hypothesis (IH), we have that $\llbracket \psi \rrbracket_M^U = \llbracket \psi \rrbracket_{M'}^U$. We have two case (1) if $U = \llbracket \psi \rrbracket_M^U = \llbracket \psi \rrbracket_{M'}^U$, then $\llbracket K\psi \rrbracket_M^U = \llbracket K\psi \rrbracket_{M'}^U = U$, and (2) if $\llbracket \psi \rrbracket_M^U = \llbracket \psi \rrbracket_{M'}^U \neq U$, then $\llbracket K\psi \rrbracket_M^U = \llbracket K\psi \rrbracket_{M'}^U = \emptyset$.

Case $\varphi := [e]\psi$

Note that $O_{[e]\psi} = O_e \cup O_\psi$. Suppose $x \in \llbracket [e]\psi \rrbracket_M^U$. For every $o \in O_e$ we have that $\|o\| = \|o\|'$, thus $e(U) \in \mathcal{O}$ is the same in both models. We must show that, $x \in e(U)$ implies $(x, e(U)) \models_{M'} \psi$. So suppose $x \in e(U)$, since $x \in \llbracket [e]\psi \rrbracket_M^U$ we have that $(x, e(U)) \models_M \psi$. By IH we then have $(x, e(U)) \models_{M'} \psi$. The other direction follows similarly.

Case $\varphi := \Box\psi$

Suppose $x \in \llbracket \Box\psi \rrbracket_M^U$. This means, by the semantics of \Box , that for all $V \in \mathcal{O}$ with $x \in V \subseteq U$ we have that $(x, V) \models_M \psi$, i.e., that $x \in \llbracket \psi \rrbracket_M^V$. Therefore, by IH and the fact that $O_{\Box\psi} = O_\psi$, we obtain $x \in \llbracket \psi \rrbracket_{M'}^V$ for all $V \subseteq U$. Since M and M' carry exactly the same collection \mathcal{O} , we conclude that $x \in \llbracket \Box\psi \rrbracket_{M'}^U$. The opposite direction follows similarly.

□

The following lemma will be useful for the soundness proof of the reduction axiom (R_\Box).

3.3.18. LEMMA. *Let $M = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ be a learning model and $U, O \in \mathcal{O}$ be information states. Then, for all $e \in \Pi$, we have: $e(U) \cap O = e(U \cap O)$.*

Proof:

The proof follows by induction on the structure of e for all $V \in \mathcal{O}$. Base cases follow easily by the definitions of the dynamic updates $!T$ and $!o$. We only prove the inductive case.

Case $e := f; f'$

$(f; f')(U) \cap O = f'(f(U)) \cap O = f'(f(U) \cap O)$ (by IH on f' and $f(U) \in \mathcal{O}$ by Lemma 3.3.3) $= f'(f(U \cap O))$ (by IH on f) $= (f; f')(U \cap O)$. □

3.3.19. THEOREM (SOUNDNESS OF DLLT). *The system DLLT in Table 3.1 is sound wrt the class of learning models.*

Proof:

The soundness proof follows via standard validity check. We here only present the validity proofs for the three learning axioms, (CC), (EC), (SP), the reduction axioms R_L , R_\Box , and for the Effort-axiom ($\Box Ax$) and the Effort-rule ($\Box Ru$). The validity proofs for the other axioms and rules are as usual. Let $M = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ be a learning model and $(x, U) \in ES(M)$.

(CC): Suppose $(x, U) \models pre(e)$ with $e \in \Pi$. We want to show that $(x, U) \models \langle K \rangle L(e)$, i.e., that there is a $y \in U$ s.t. $(y, U) \models L(e)$, i.e., by the semantic definition of L , that there is a $y \in U$ s.t. $y \in \mathbb{L}(e(U))$. Recall that $\mathbb{L}(O) \subseteq O$ for every $O \in \mathcal{O}$ (by clause (1) in Definition 3.3.1), so $\mathbb{L}(e(U)) \subseteq e(U)$. Also, $\mathbb{L}(O) \neq \emptyset$ if $O \neq \emptyset$ (by (2) in Definition 3.3.1). Since $x \in \llbracket pre(e) \rrbracket^U = e(U)$ by Lemma 3.3.9, we have that $\mathbb{L}(e(U)) \neq \emptyset$, i.e., there is $y \in \mathbb{L}(e(U))$. Since $y \in \mathbb{L}(e(U)) \subseteq U$, by the semantics of L , we obtain that $(y, U) \models L(e)$. Thus $(x, U) \models \langle K \rangle L(e)$.

(EC): Suppose $(x, U) \models K(\text{pre}(e) \leftrightarrow \text{pre}(e'))$. This means that $x \in \llbracket K(\text{pre}(e) \leftrightarrow \text{pre}(e')) \rrbracket^U$. Therefore, by the semantics of K , we obtain that $\llbracket \text{pre}(e) \rrbracket^U = \llbracket \text{pre}(e') \rrbracket^U$. This means that $\llbracket \text{pre}(e) \rrbracket^U = e(U) = \llbracket \text{pre}(e') \rrbracket^U = e'(U)$. Hence, since \mathbb{L} is a function, we obtain $\mathbb{L}(e(U)) = \mathbb{L}(e'(U))$. Therefore, by the semantics of L , we conclude that $\llbracket L(e) \rrbracket^U = \llbracket L(e') \rrbracket^U$, i.e., that $(x, U) \models L(e) \leftrightarrow L(e')$.

(SP): Suppose $(x, U) \models L(e)$. By the semantics, we have $x \in \mathbb{L}(e(U))$. Since we have that $\mathbb{L}(e(U)) \subseteq e(U) = \llbracket \text{pre}(e) \rrbracket^U$ by clause (1) in Definition 3.3.1 and Lemma 3.3.9, we obtain that $x \in \llbracket \text{pre}(e) \rrbracket^U$, i.e., that $(x, U) \models \text{pre}(e)$.

(R_L): From left-to-right: Suppose $(x, U) \models [e]L(e')$. This means that $x \in e(U)$ implies $(x, e(U)) \models L(e')$, i.e., that $x \in e(U) = \llbracket \text{pre}(e) \rrbracket^U$ implies $x \in \mathbb{L}(e'(e(U)))$. Then, by the semantics of L , we obtain that $(x, U) \models \text{pre}(e) \rightarrow L(e; e')$. The opposite direction follows similarly.

(R_□):

$$\begin{aligned} (x, U) \models [e]\Box\varphi & \\ \text{iff } x \in e(U) \text{ implies } (x, e(U)) \models \Box\varphi & \quad (\text{by the semantics of } [e]) \\ \text{iff } x \in e(U) \text{ implies } (\forall O \in \mathcal{O})(x \in O \text{ implies } (x, e(U) \cap O) \models \varphi) & \\ & \quad (\text{by the semantics of } \Box) \\ \text{iff } (\forall O \in \mathcal{O})((x \in e(U) \text{ and } x \in O) \text{ implies } (x, e(U) \cap O) \models \varphi) & \\ \text{iff } (\forall O \in \mathcal{O})(x \in O \text{ implies } (x \in e(U) \cap O \text{ implies } (x, e(U) \cap O) \models \varphi)) & \\ \text{iff } (\forall O \in \mathcal{O})(x \in O \text{ implies } (x \in e(U \cap O) \text{ implies } (x, e(U \cap O)) \models \varphi)) & \\ & \quad (\text{Lemma 3.3.18}) \\ \text{iff } (\forall O \in \mathcal{O})(x \in O \text{ implies } (x, U \cap O) \models [e]\varphi) & \quad (\text{by the semantics of } [e]) \\ \text{iff } (x, U) \models \Box[e]\varphi. & \quad (\text{by the semantics of } \Box) \end{aligned}$$

(□Ax): Suppose $(x, U) \models \Box\varphi$. This means, by the semantics of \Box , that for all $O \in \mathcal{O}$ with $x \in O$ we have $(x, U \cap O) \models \varphi$. By Lemmas 3.3.3 and 3.3.9, we have that $\llbracket \text{pre}(e) \rrbracket^U \in \mathcal{O}$ for all $e \in \Pi$ and $U \in \mathcal{O}$, so in particular $x \in \llbracket \text{pre}(e) \rrbracket^U$ implies $(x, U \cap \llbracket \text{pre}(e) \rrbracket^U) \models \varphi$ for every $e \in \Pi$. By the semantics of $[e]$, we obtain that $(x, U) \models [e]\varphi$ for every observational event $e \in \Pi$.

(□Ru): Suppose $\models \chi \rightarrow [e;!o]\varphi$ and $\not\models \chi \rightarrow [e]\Box\varphi$, where $o \notin O_\chi \cup O_e \cup O_\varphi$. The latter means that there exists a learning model $M = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ such that for some $U \in \mathcal{O}$ and some $w \in U$, we have $w \notin \llbracket \chi \rightarrow [e]\Box\varphi \rrbracket_M^U$. Therefore $w \in \llbracket \chi \wedge \neg[e]\Box\varphi \rrbracket_M^U$. Thus we have (1): $w \in \llbracket \chi \rrbracket_M^U$ and (2): $w \in \llbracket \neg[e]\Box\varphi \rrbracket_M^U$. Because of (2), $w \in \llbracket \langle e \rangle \diamond \neg\varphi \rrbracket_M^U$, and, by the semantics of $\langle e \rangle$, $w \in \llbracket \diamond \neg\varphi \rrbracket_M^{\llbracket \text{pre}(e) \rrbracket^U}$. Therefore, applying the semantics of

□, we obtain (3): there exists $V \in \mathcal{O}$ s.t. $w \in V \subseteq \llbracket pre(e) \rrbracket_M^U \subseteq U$ and $w \in \llbracket \neg\varphi \rrbracket_M^V$.

Now consider the learning model $M' = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|')$ such that $\|o\|' = V$ and $\|u\|' = \|u\|$ for all $u \in \text{Prop}_{\mathcal{O}}$ such that $u \neq o$. In order to use Lemma 3.3.17 we must show that M' is a learning model. Since X , \mathbb{L} , and \mathcal{O} are as in the learning model M , we only need to verify that $\|u\|' \in \mathcal{O}$ for all $u \in \text{Prop}_{\mathcal{O}}$. As $\|o\|' = V \in \mathcal{O}$, the condition is satisfied for o . For every $u \neq o$, since $\|u\|' = \|u\|$ and $\|u\| \in \mathcal{O}$ we have $\|u\|' \in \mathcal{O}$. Therefore, M' is a learning model. Now continuing with our soundness proof, since $o \notin O_X \cup O_e \cup O_\varphi$, by Lemma 3.3.17, we obtain $\llbracket \chi \rrbracket_{M'}^U = \llbracket \chi \rrbracket_M^U$, $\llbracket pre(e) \rrbracket_{M'}^U = \llbracket pre(e) \rrbracket_M^U$ and $\llbracket \neg\varphi \rrbracket_{M'}^U = \llbracket \neg\varphi \rrbracket_M^U$. Since $\|o\|' = V \subseteq \llbracket pre(e) \rrbracket_{M'}^U \subseteq U$, we have $\|o\|' = \llbracket o \rrbracket_{M'}^U$. As $pre(!o) = o$ and $\|o\|' \subseteq U$, we obtain $\|o\|' = \llbracket pre(!o) \rrbracket_{M'}^U$. Thus $\llbracket o \rrbracket_{M'}^U = \llbracket pre(!o) \rrbracket_{M'}^U = V$. Because of (3) we have that $w \in \llbracket pre(e) \rrbracket_{M'}^U$ and $w \in \llbracket \neg\varphi \rrbracket_{M'}^V = \llbracket \neg\varphi \rrbracket_{M'}^{\llbracket pre(!o) \rrbracket_{M'}^U} = \llbracket \langle !o \rangle \neg\varphi \rrbracket_{M'}^U$. Thus, $w \in \llbracket pre(!o) \rrbracket_{M'}^U$, so $w \in \llbracket pre(e; !o) \rrbracket_{M'}^U = \llbracket pre(e) \rrbracket_{M'}^U \cap \llbracket pre(!o) \rrbracket_{M'}^U = \llbracket pre(!o) \rrbracket_{M'}^U$ simply because $\llbracket pre(!o) \rrbracket_{M'}^U = V \subseteq \llbracket pre(e) \rrbracket_{M'}^U$. Since $w \in \llbracket \neg\varphi \rrbracket_{M'}^{\llbracket pre(!o) \rrbracket_{M'}^U}$ we obtain $w \in \llbracket \neg\varphi \rrbracket_{M'}^{\llbracket pre(e; !o) \rrbracket_{M'}^U}$. Putting everything together, $w \in \llbracket pre(e; !o) \rrbracket_{M'}^U$ and $w \in \llbracket \neg\varphi \rrbracket_{M'}^{\llbracket pre(e; !o) \rrbracket_{M'}^U}$, we obtain that $w \in \llbracket \langle e; !o \rangle \neg\varphi \rrbracket_{M'}^U$ and $w \in \llbracket \chi \rrbracket_{M'}^U$. Therefore $w \in \llbracket \chi \wedge \langle e; !o \rangle \neg\varphi \rrbracket_{M'}^U$, which contradicts the validity of $\chi \rightarrow [e; !o]\varphi$.

□

3.3.6 Completeness of DLLT

We now move to the completeness proof of our axiomatization **DLLT**. Although **DLLT** is more expressive than Subset Space Logic (interpreted on intersection spaces), our completeness proof is *much simpler*, and follows via a *canonical model construction*: this is one of the advantages of having the (expressively redundant) dynamic observation modalities. There are two main technical differences between our construction and the standard canonical model from Basic Modal Logic. First, this is *not* a relational (Kripke) model, but a *neighborhood model*; so the closest analogue is the type of canonical construction used in Topological Modal Logic or Neighborhood Semantics (Aiello et al., 2007). Second, the standard notion of maximally consistent theory⁸ is *not* very useful for our logic, since such theories do not “internalize” the rule $\square\text{Ru}$. To do this, we need instead to consider “witnessed” (*maximally consistent*) theories, in which every occurrence of a $\diamond\varphi$ in any “existential context” is “witnessed” by some $\langle !o \rangle\varphi$ (with o observational variable). The appropriate notion of “existential contexts” is represented by *possibility forms*, in the following sense:

⁸Recall from Section 2.1 that a set of formulas $\Gamma \subseteq \mathcal{L}_\Pi$ is consistent if Γ does not derive a contradiction, and it is maximally consistent if any consistent theory $\Gamma' \supseteq \Gamma$, $\Gamma' = \Gamma$.

3.3.20. DEFINITION. [Pseudo-modalities: necessity and possibility forms] The set of *necessity-form expressions* of our language is given by $NF_{\mathcal{L}_{\Pi}} := (\{\varphi \rightarrow \mid \varphi \in \mathcal{L}_{\Pi}\} \cup \{K\} \cup \{e : e \in \Pi\})^*$. For any finite string $s \in NF_{\mathcal{L}_{\Pi}}$, we define pseudo-modalities $[s]$ (called *necessity form*) and $\langle s \rangle$ (called *possibility form*) that generalize our dynamic modalities $[e]$ and $\langle e \rangle$. These pseudo-modalities are functions mapping any formula $\varphi \in \mathcal{L}_{\Pi}$ to another formula $[s]\varphi \in \mathcal{L}_{\Pi}$, and respectively $\langle s \rangle\varphi \in \mathcal{L}_{\Pi}$. Necessity forms are defined recursively, by putting: $[\epsilon]\varphi := \varphi$ (where ϵ is the empty string), $[s, \varphi \rightarrow]\varphi := [s](\varphi \rightarrow \varphi)$, $[s, K]\varphi := [s]K\varphi$, $[s, e]\varphi := [s][e]\varphi$. As for possibility forms, we put $\langle s \rangle\varphi := \neg[s]\neg\varphi$.

To illustrate, expression $[K, !o, \diamond p \rightarrow, e]$ constitutes a necessity form such that $[K, !o, \diamond p \rightarrow, e]\varphi = K[!o](\diamond p \rightarrow [e]\varphi)$.

The following lemma expresses that any necessity form s is characterized by an observational event $e \in \Pi$ and a formula $\psi \in \mathcal{L}_{\Pi}$ independently of which formula $\varphi \in \mathcal{L}_{\Pi}$ it is applied to.

3.3.21. LEMMA. *For every necessity form $[s]$, there exist an observational event $e \in \Pi$ and a formula $\psi \in \mathcal{L}_{\Pi}$, with $O_{\psi} \cup O_e \subseteq O_s$, such that for all $\varphi \in \mathcal{L}_{\Pi}$, we have*

$$\vdash [s]\varphi \text{ iff } \vdash \psi \rightarrow [e]\varphi.$$

Proof:

We proceed by induction on the structure of necessity forms. For the empty string $s := \epsilon$, take $\psi := \top$ and $e := !\top$, then it follows from classical propositional logic.

$$s := s', \eta \rightarrow$$

$$\begin{aligned} \vdash [s', \eta \rightarrow]\varphi &\text{ iff } \vdash [s'](\eta \rightarrow \varphi) && \text{(Definition \textcolor{red}{3.3.20})} \\ &\text{ iff } \vdash \psi' \rightarrow [e](\eta \rightarrow \varphi) \\ &\quad \text{(for some } \psi' \in \mathcal{L}_{\Pi} \text{ and } e \in \Pi \text{ with } O_{\psi'} \cup O_e \subseteq O_{s'}, \text{ by IH)} \\ &\text{ iff } \vdash \psi' \rightarrow ([e]\eta \rightarrow [e]\varphi) && \text{(K}_{[e]}\text{)} \\ &\text{ iff } \vdash (\psi' \wedge [e]\eta) \rightarrow [e]\varphi && \text{(CPL)} \\ &\text{ iff } \vdash \psi \rightarrow [e]\varphi && (\psi := \psi' \wedge [e]\eta, \text{ thus, } O_{\psi} \cup O_e \subseteq O_s) \end{aligned}$$

$$s := s', K$$

$$\begin{aligned} \vdash [s', K]\varphi &\text{ iff } \vdash [s']K\varphi && \text{(Definition \textcolor{red}{3.3.20})} \\ &\text{ iff } \vdash \psi' \rightarrow [e]K\varphi \\ &\quad \text{(for some } \psi' \in \mathcal{L}_{\Pi} \text{ and } e \in \Pi \text{ with } O_{\psi'} \cup O_e \subseteq O_{s'}, \text{ by IH)} \\ &\text{ iff } \vdash \psi' \rightarrow (pre(e) \rightarrow K[e]\varphi) && \text{(R}_K\text{)} \\ &\text{ iff } \vdash (\psi' \wedge pre(e)) \rightarrow K[e]\varphi && \text{(CPL)} \\ &\text{ iff } \vdash \langle K \rangle(\psi' \wedge pre(e)) \rightarrow [e]\varphi \\ &\quad \text{(pushing } K \text{ back with its dual } \langle K \rangle, \text{ since } K \text{ is an S5 modality)} \\ &\text{ iff } \vdash \psi \rightarrow [e]\varphi && (\psi := \langle K \rangle(\psi' \wedge pre(e)) \in \mathcal{L}_{\Pi}, \text{ thus, } O_{\psi} \cup O_e \subseteq O_s) \end{aligned}$$

$$s := s', e'$$

$$\begin{aligned} \vdash [s', e']\varphi &\text{ iff } \vdash [s'][e']\varphi && \text{(Definition \textcolor{red}{3.3.20})} \\ &\text{ iff } \vdash \psi' \rightarrow [e][e']\varphi \\ &\text{ (for some } \psi' \in \mathcal{L}_\Pi \text{ and } e \in \Pi \text{ with } O_{\psi'} \cup O_e \subseteq O_{s'}, \text{ by IH)} \\ &\text{ iff } \vdash \psi' \rightarrow [e; e']\varphi && \text{(R}_e\text{)} \end{aligned}$$

□

3.3.22. LEMMA. *The following rule is admissible in **DLIT**:*

$$\text{if } \vdash [s][!o]\varphi \text{ then } \vdash [s]\Box\varphi, \text{ where } o \notin O_s \cup O_\varphi.$$

Proof:

Suppose $\vdash [s][!o]\varphi$ where $o \notin O_s \cup O_\varphi$. Then, by Lemma [3.3.21](#), there exist $e \in \Pi$ and $\psi \in \mathcal{L}_\Pi$ with $O_\psi \cup O_e \subseteq O_s$ such that $\vdash \psi \rightarrow [e][!o]\varphi$. Thus we get $\vdash \psi \rightarrow [e; !o]\varphi$ by an instance of R_e . Therefore, by the Effort rule ($\Box Ru$) we have $\vdash \psi \rightarrow [e]\Box\varphi$. Then, again by Lemma [3.3.21](#), we obtain $\vdash [s]\Box\varphi$. □

3.3.23. DEFINITION. [Maximal O-witnessed theories] For every countable set of observational variables O , let \mathcal{L}_Π^O be the language of the logic **DLIT**^O based only on the observational variables in O . Let $NF_{\mathcal{L}_\Pi^O}^O$ denote the set of necessity-form expressions of \mathcal{L}_Π^O (i.e., necessity forms involving only observational variables in O).

- An *O-theory* is a *consistent* set of formulas in \mathcal{L}_Π^O . Here, “consistent” means consistent with respect to the axiomatization **DLIT** formulated for \mathcal{L}_Π^O .
- A *maximal O-theory* is an O-theory Γ that is maximal with respect to \subseteq among all O-theories; in other words, Γ cannot be extended to another O-theory.
- An *O-witnessed theory* is an O-theory Γ such that, for every $s \in NF_{\mathcal{L}_\Pi^O}^O$ and $\varphi \in \mathcal{L}_\Pi^O$, if $\langle s \rangle \Diamond \varphi$ is consistent with Γ then there is $o \in O$ such that $\langle s \rangle \langle !o \rangle \varphi$ is consistent with Γ . A *maximal O-witnessed theory* Γ is an O-witnessed theory that is not a proper subset of any O-witnessed theory.

The following lemmas will be useful in the proof of Lindenbaum’s Lemma.

3.3.24. LEMMA. *For every maximal O-witnessed theory Γ , and any $\varphi, \psi \in \mathcal{L}_\Pi^O$,*

1. *either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$,*
2. *$\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$,*

3. $\varphi \in \Gamma$ and $\varphi \rightarrow \psi \in \Gamma$ implies $\psi \in \Gamma$.

Proof:

The proofs follow in a standard way as Γ is maximal. \square

3.3.25. LEMMA. *For every $\Gamma \subseteq \mathcal{L}_{\Pi}^{\text{O}}$, if Γ is an O-theory and $\Gamma \not\vdash \neg\varphi$ for some $\varphi \in \mathcal{L}_{\Pi}^{\text{O}}$, then $\Gamma \cup \{\varphi\}$ is an O-theory. Moreover, if Γ is O-witnessed, then $\Gamma \cup \{\varphi\}$ is also O-witnessed.*

Proof:

Let $\Gamma \subseteq \mathcal{L}_{\Pi}^{\text{O}}$ be an O-theory and $\varphi \in \mathcal{L}_{\Pi}^{\text{O}}$ such that $\Gamma \not\vdash \neg\varphi$. We first show that $\Gamma \cup \{\varphi\}$ is an O-theory. Suppose, toward a contradiction, that $\Gamma \cup \{\varphi\}$ is not an O-theory, i.e., that $\Gamma \cup \{\varphi\} \vdash \perp$. Thus, there is a finite set $\Delta \subseteq \Gamma$ such that $\Delta \vdash \neg\varphi$ and therefore $\Gamma \vdash \neg\varphi$, which contradicts the assumption that $\Gamma \not\vdash \neg\varphi$.

Now suppose that Γ is O-witnessed but $\Gamma \cup \{\varphi\}$ is not O-witnessed. By the previous statement, we know that $\Gamma \cup \{\varphi\}$ is consistent. Therefore, the latter means that there is $s \in NF_{\mathcal{L}_{\Pi}^{\text{O}}}^{\text{O}}$ and $\psi \in \mathcal{L}_{\Pi}^{\text{O}}$ such that $\Gamma \cup \{\varphi\}$ is consistent with $\langle s \rangle \diamond \psi$ but $\Gamma \cup \{\varphi\} \vdash \neg \langle s \rangle \langle !o \rangle \psi$ for all $o \in \text{O}$. This implies that $\Gamma \cup \{\varphi\} \vdash [s][!o]\neg\psi$ for all $o \in \text{O}$. Therefore, $\Gamma \vdash \varphi \rightarrow [s][!o]\neg\psi$ for all $o \in \text{O}$. Note that $\varphi \rightarrow [s][!o]\neg\psi := [\varphi \rightarrow, s][!o]\neg\psi$, and $[\varphi \rightarrow, s] \in NF_{\mathcal{L}_{\Pi}^{\text{O}}}^{\text{O}}$. We thus have $\Gamma \vdash [\varphi \rightarrow, s][!o]\neg\psi$ for all $o \in \text{O}$. Since Γ is O-witnessed, we obtain $\Gamma \vdash [\varphi \rightarrow, s]\Box\neg\psi$. By unraveling the necessity form $[\varphi \rightarrow, s]$, we get $\Gamma \vdash \varphi \rightarrow [s]\Box\neg\psi$, thus, $\Gamma \cup \{\varphi\} \vdash [s]\Box\neg\psi$, i.e., $\Gamma \cup \{\varphi\} \vdash \neg \langle s \rangle \diamond \psi$, contradicting the assumption that $\Gamma \cup \{\varphi\}$ is consistent with $\langle s \rangle \diamond \psi$. \square

3.3.26. LEMMA. *If $\{\Gamma_i\}_{i \in \mathbb{N}}$ is an increasing chain of O-theories such that $\Gamma_i \subseteq \Gamma_{i+1}$, then $\bigcup_{n \in \mathbb{N}} \Gamma_n$ is an O-theory.*

Proof:

Let $\Gamma_0 \subseteq \Gamma_1 \subseteq \dots \subseteq \Gamma_n \subseteq \dots$ be an increasing chain of O-theories and suppose, toward contradiction, that $\bigcup_{n \in \mathbb{N}} \Gamma_n$ is not an O-theory, i.e., suppose that $\bigcup_{n \in \mathbb{N}} \Gamma_n \vdash \perp$. This means that there exists a finite $\Delta \subseteq \bigcup_{n \in \mathbb{N}} \Gamma_n$ such that $\Delta \vdash \perp$. Then, since $\bigcup_{n \in \mathbb{N}} \Gamma_n$ is a union of an increasing chain of O-theories, there is some $m \in \mathbb{N}$ such that $\Delta \subseteq \Gamma_m$. Therefore, $\Gamma_m \vdash \perp$ contradicting the fact that Γ_m is an O-theory. \square

3.3.27. LEMMA (LINDENBAUM'S LEMMA). *Every O-witnessed theory Γ can be extended to a maximal O-witnessed theory T_{Γ} .*

Proof:

The proof follows by constructing an increasing chain

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \dots \subseteq \Gamma_n \subseteq \dots,$$

of O-witnessed theories where $\Gamma_0 := \Gamma$, and each Γ_i will be recursively defined. We have to guarantee that each Γ_i is O-witnessed and, in order to do so, we follow a two-fold construction, where $\Gamma_0 = \Gamma'_0 := \Gamma$. Let $\gamma_n := (s_n, \varphi_n)$ be the n th-pair in the enumeration \mathcal{A} of all pairs of the form (s, φ) consisting of a necessity form expression $s \in NF_{\mathcal{L}_\Pi}^O$ and a formula $\varphi \in \mathcal{L}_\Pi^O$. Note that the empty string ϵ is in $NF_{\mathcal{L}_\Pi}^O$, so for every formula $\varphi \in \mathcal{L}_\Pi^O$ the pair (ϵ, φ) is in \mathcal{A} , and $\langle \epsilon \rangle \varphi := \varphi$ by the definition of possibility forms. We then set

$$\Gamma'_n = \begin{cases} \Gamma_n \cup \{ \langle s_n \rangle \varphi_n \} & \text{if } \Gamma \not\vdash \neg \langle s_n \rangle \varphi_n, \\ \Gamma_n & \text{otherwise.} \end{cases}$$

By Lemma 3.3.25, each Γ'_n is O-witnessed. Therefore, if φ_n is of the form $\varphi_n := \diamond \theta$ for some $\theta \in \mathcal{L}_\Pi^O$, there must exist an $o \in O$ such that Γ'_n is consistent with $\langle s \rangle \langle !o \rangle \theta$ (since Γ'_n is O-witnessed). We then define

$$\Gamma_{n+1} = \begin{cases} \Gamma'_n & \text{if } \Gamma_n \not\vdash \neg \langle s_n \rangle \varphi_n \text{ and } \varphi_n \text{ is not of the form } \diamond \theta, \\ \Gamma'_n \cup \{ \langle s_n \rangle \langle !o \rangle \theta \} & \text{if } \Gamma_n \not\vdash \neg \langle s_n \rangle \varphi_n \text{ and } \varphi_n := \diamond \theta \text{ for some } \theta \in \mathcal{L}_\Pi^O, \\ \Gamma_n & \text{otherwise,} \end{cases}$$

where $o \in O$ is such that Γ'_n is consistent with $\langle s \rangle \langle !o \rangle \theta$. Again by Lemma 3.3.25, it is guaranteed that each Γ_n is O-witnessed. Now consider the union $T_\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$. By Lemma 3.3.26, we know that T_Γ is an O-theory. To show that T_Γ is O-witnessed, let $s \in NF_{\mathcal{L}_\Pi}^O$ and $\theta \in \mathcal{L}_\Pi^O$ and suppose $\langle s \rangle \diamond \theta$ is consistent with T_Γ . The pair $(s, \diamond \theta)$ appears in the enumeration \mathcal{A} , thus $\gamma_m := (s_m, \varphi_m) = (s, \diamond \theta)$ with $s_m := s$ and $\varphi_m := \diamond \theta$, for some $\gamma_m \in \mathcal{A}$. Since $\langle s_m \rangle \varphi_m$ is consistent with T_Γ and $\Gamma_m \subseteq T_\Gamma$, we know that $\langle s_m \rangle \diamond \theta$ is in particular consistent with Γ_m . Therefore, by the above construction, $\langle s \rangle \langle !o \rangle \theta \in \Gamma_{m+1}$ for some $o \in O$ such that Γ'_m is consistent with $\langle s \rangle \langle !o \rangle \theta$. Thus, as T_Γ is consistent and $\Gamma_{m+1} \subseteq T_\Gamma$, we have that $\langle s \rangle \langle !o \rangle \theta$ is also consistent with T_Γ , moreover $\langle s \rangle \langle !o \rangle \theta \in T_\Gamma$. Hence, we conclude that T_Γ is O-witnessed. Finally, T_Γ is also maximal by construction: otherwise there would be an O-witnessed theory T such that $T_\Gamma \subset T$. This implies that there exists $\varphi \in \mathcal{L}_\Pi^O$ with $\varphi \in T$ but $\varphi \notin T_\Gamma$. Then, by the construction of T_Γ , we obtain $\Gamma_i \vdash \neg \varphi$ for all $i \in \mathbb{N}$. Therefore, since $T_\Gamma \subseteq T$, we have $T \vdash \neg \varphi$. Hence, since $\varphi \in T$, we obtain $T \vdash \perp$ (contradicting T being consistent). \square

3.3.28. LEMMA (EXTENSION LEMMA). *Let O be a set of observational variables and O' be a countable set of fresh observational variables, i.e., $O \cap O' = \emptyset$. Let*

$\tilde{O} = O \cup O'$. Then, every O -theory Γ can be extended to an \tilde{O} -witnessed theory $\tilde{\Gamma} \supseteq \Gamma$, and hence to a maximal \tilde{O} -witnessed theory $T_\Gamma \supseteq \Gamma$.

Proof:

Let $\mathcal{A} = \{\gamma_0, \gamma_1, \dots, \gamma_n, \dots\}$ be an enumeration of all pairs of the form $\gamma_i := (s_i, \varphi_i)$ consisting of any necessity form $s_i \in NF_{\mathcal{L}_\Pi}^{\tilde{O}}$ and every formula $\varphi_i \in \mathcal{L}_\Pi^{\tilde{O}}$. We will recursively construct a chain of \tilde{O} -theories $\Gamma_0 \subseteq \dots \subseteq \Gamma_n \subseteq \dots$ such that:

1. $\Gamma_0 = \Gamma$,
2. $O'_n = \{o \in O' : o \text{ occurs in } \Gamma_n\}$ is finite for every $n \in \mathbb{N}$ (we will verify this later in the proof). Finally,
3. for every $\gamma_n := (s_n, \varphi_n)$ with $s_n \in NF_{\mathcal{L}_\Pi}^{\tilde{O}}$ and $\varphi_n \in \mathcal{L}_\Pi^{\tilde{O}}$:
If $\Gamma_n \not\vdash \neg\langle s_n \rangle \diamond \varphi_n$ then there is o_m “fresh” such that $\langle s_n \rangle \langle !o_m \rangle \varphi_n \in \Gamma_{n+1}$. Otherwise we will define $\Gamma_{n+1} = \Gamma_n$.

For every $\gamma_n \in \mathcal{A}$, let $O'(n) = \{o \in O' : o \text{ occurs either in } s_n \text{ or } \varphi_n\}$. Clearly every $O'(n)$ is finite. We now construct an increasing chain of \tilde{O} -theories recursively. We fix $\Gamma_0 := \Gamma$ and let

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\langle s_n \rangle \langle !o_m \rangle \varphi_n\} & \text{if } \Gamma_n \not\vdash \neg\langle s_n \rangle \diamond \varphi_n, \\ \Gamma_n & \text{otherwise,} \end{cases}$$

where m is the least natural number bigger than the indices in $O'_n \cup O'(n)$, i.e., o_m is fresh. To see that $O'_n \cup O'(n)$ is finite for every $n \in \mathbb{N}$, we just need to check that O'_n is finite. First note that since $\Gamma := \Gamma_0$ is an O -theory, no observational variables in O' occur in Γ_0 . For each pair $\gamma_i := (s_i, \varphi_i)$ with $i \in \{0, \dots, n-1\}$ the set $O'(n)$ is finite. At each step n of the construction, we can only add finitely many fresh variables to Γ_n . Thus, finitely many observational variables in O' occur in Γ_n and so O'_n is finite.

We now need to show that $\tilde{\Gamma} := \bigcup_{n \in \mathbb{N}} \Gamma_n$ is an \tilde{O} -witnessed theory. We first show that $\tilde{\Gamma}$ is an \tilde{O} -theory. By Lemma 3.3.26, it suffices to show by induction that every Γ_n is an \tilde{O} -theory. Clearly Γ_0 is an \tilde{O} -theory. For the inductive step suppose that Γ_n is consistent but Γ_{n+1} is not. Hence $\Gamma_{n+1} \neq \Gamma_n$, and moreover $\Gamma_{n+1} \vdash \perp$. Then, since $\Gamma_{n+1} = \Gamma_n \cup \{\langle s_n \rangle \langle !o_m \rangle \varphi_n\}$, we have $\Gamma_n \vdash [s_n][!o_m] \neg \varphi_n$. Therefore there exists $\{\theta_1, \dots, \theta_k\} \subseteq \Gamma_n$ such that $\{\theta_1, \dots, \theta_k\} \vdash [s_n][!o_m] \neg \varphi_n$. Let $\theta = \bigwedge_{1 \leq i \leq k} \theta_i$. Then $\vdash \theta \rightarrow [s_n][!o_m] \neg \varphi_n$, so $\vdash [\theta \rightarrow, s_n][!o_m] \neg \varphi_n$ with $o_m \notin O_{\Gamma_n} \cup O_{s_n} \cup O_{\varphi_n}$.

By the admissible rule in Lemma 3.3.22 we obtain that $\vdash [\theta \rightarrow, s_n] \Box \neg \varphi_n$, thus $\vdash \theta \rightarrow [s_n] \Box \neg \varphi_n$. It follows $\theta \vdash [s_n] \Box \neg \varphi_n$ and therefore $\theta \vdash \neg \langle s_n \rangle \diamond \varphi_n$. Since $\{\theta_1, \dots, \theta_k\} \subseteq \Gamma_n$, we obtain $\Gamma_n \vdash \neg \langle s_n \rangle \diamond \varphi_n$. But this would mean $\Gamma_n =$

Γ_{n+1} , contradicting our assumption. Therefore Γ_{n+1} is consistent and thus an $\tilde{\text{O}}$ -theory. Therefore, by Lemma 3.3.26, $\tilde{\Gamma}$ is an $\tilde{\text{O}}$ -theory. Condition (3) above implies that $\tilde{\Gamma}$ is also $\tilde{\text{O}}$ -witnessed. Then, by Lindenbaum's Lemma, there is a maximal $\tilde{\text{O}}$ -witnessed theory $T_{\tilde{\Gamma}}$ such that $T_{\tilde{\Gamma}} \supseteq \tilde{\Gamma} \supseteq \Gamma$. \square

We are now ready to build the canonical model.

Canonical Model for T_0 . For any consistent set of formulas Φ , consider a maximally consistent O -witnessed extension $\Phi \subseteq T_0$. As our canonical set of worlds, we take the set $X^c := \{T : T \text{ maximally consistent } \text{O}\text{-witnessed theory with } T \sim_K T_0\}$, where we put

$$T \sim_K T' \text{ iff } \forall \varphi \in \mathcal{L}_{\Pi}^{\text{O}} (K\varphi \in T \rightarrow \varphi \in T').$$

As usual, it is easy to see (given the **S5** axioms for K) that \sim_K is an equivalence relation. For any formula φ , we use the notation $\hat{\varphi} := \{T \in X^c : \varphi \in T\}$. In particular, for any observational variable $o \in \text{O}$, we have $\hat{o} = \{T \in X^c : o \in T\}$. As the canonical set of information states, we take $\mathcal{O}^c := \{\widehat{\text{pre}(e)} : e \in \Pi^{\text{O}}\}$. Finally, our canonical learner is given by $\mathbb{L}^c(\widehat{\text{pre}(e)}) := \widehat{L(e)}$, and the canonical valuation $\|\cdot\|_c$ is given as $\|p\|_c = \hat{p}$ and $\|o\|_c = \hat{o}$. The learning model $M^c = (X^c, \mathcal{O}^c, \mathbb{L}^c, \|\cdot\|_c)$ is called the *canonical model*. Note that we use c as a subscript instead of a superscript for the canonical valuation $\|\cdot\|_c$, this is in order to avoid confusion with our ‘‘open-restriction’’ notation for the truth set of a formula $\llbracket \varphi \rrbracket^U$.

Before proving that the canonical model is well-defined, we need the following lemmas.

3.3.29. LEMMA. *For every maximal O -witnessed theory T , the set $\{\theta \in \mathcal{L}_{\Pi}^{\text{O}} : K\theta \in T\}$ is O -witnessed.*

Proof:

Observe that, by axiom (T_K) , $\{\theta : K\theta \in T\} \subseteq T$. Therefore, as T is consistent, the set $\{\theta : K\theta \in T\}$ is consistent. Let $s \in NF_{\mathcal{L}_{\Pi}^{\text{O}}}^{\text{O}}$ and $\varphi \in \mathcal{L}_{\Pi}^{\text{O}}$ such that $\{\theta : K\theta \in T\} \vdash [s][!o]\neg\varphi$ for all $o \in \text{O}$. We must show that $\{\theta : K\theta \in T\} \vdash [s]\Box\neg\varphi$. By normality of K , $T \vdash K[s][!o]\neg\varphi$ for all $o \in \text{O}$. Since $K[s][!o]\neg\varphi := [K, s][!o]\neg\varphi$ is a necessity form and T is O -witnessed, we obtain $T \vdash [K, s]\Box\neg\varphi$, i.e., $T \vdash K[s]\Box\neg\varphi$. As T is maximal, we have $K[s]\Box\neg\varphi \in T$, thus $[s]\Box\neg\varphi \in \{\theta : K\theta \in T\}$. \square

3.3.30. LEMMA. *Let $T \in X^c$. Then, $K\varphi \in T$ iff $\varphi \in T'$ for all $T' \in X^c$.*

Proof:

From left-to-right follows directly from the definition of X^c and \sim_K . For the right-to-left direction, we prove the contrapositive. Let $\varphi \in \mathcal{L}_\Pi$ such that $K\varphi \notin T$. Then, by Lemma 3.3.29 and Lemma 3.3.25 we obtain that $\{\psi : K\psi \in T\} \cup \{\neg\varphi\}$ is an O-witnessed theory. We can then apply Lindenbaum's Lemma (Lemma 3.3.27) and extend it to a maximal O-witnessed theory S such that $\varphi \notin S$. As $\{\psi : K\psi \in T\} \subseteq S$, we have $S \in X^c$. \square

3.3.31. COROLLARY. *Let $T \in X^c$. Then, $\langle K \rangle \varphi \in T$ iff there is $S \in X^c$ such that $\varphi \in S$.*

3.3.32. PROPOSITION. *The canonical model is well-defined, i.e., the canonical model is a learning model as in Definition 3.3.1.*

Proof:

We need to show that the following properties hold:

1. For all $o \in \text{Prop}_\mathcal{O}$, $\|o\|_c \in \mathcal{O}^c$: let $o \in \text{Prop}_\mathcal{O}$. By the definitions of $\|\cdot\|_c$ and \mathcal{O}^c , we have $\|o\|_c = \widehat{o}$. As $\text{pre}(!o) = o$, we obtain that $\widehat{o} = \widehat{\text{pre}(!o)} \in \mathcal{O}$.
2. If $F = \{\widehat{\text{pre}(e_1)}, \dots, \widehat{\text{pre}(e_m)}\} \subseteq \mathcal{O}^c$ is finite then $\bigcap F \in \mathcal{O}^c$: Let $F = \{\widehat{\text{pre}(e_1)}, \dots, \widehat{\text{pre}(e_m)}\} \subseteq \mathcal{O}^c$. It is easy to see that $\bigcap \{\widehat{\text{pre}(e_1)}, \dots, \widehat{\text{pre}(e_m)}\} = \widehat{\text{pre}(e_1) \wedge \dots \wedge \text{pre}(e_m)} = \widehat{\text{pre}(e_1; \dots; e_m)}$ by the definition of precondition. Since $(e_1; \dots; e_m) \in \Pi^0$, by the definition of \mathcal{O}^c in the canonical model we have $\widehat{\text{pre}(e_1; \dots; e_m)} \in \mathcal{O}^c$ and thus $\bigcap F \in \mathcal{O}^c$.
3. \mathbb{L}^c is a well-defined function and a learner: For this, note that $\mathbb{L}^c(\widehat{\text{pre}(e)}) := \widehat{L(e)} \subseteq X^c$. We will first prove that:
 (2a) if $\widehat{\text{pre}(e)} = \widehat{\text{pre}(e')}$ then $\mathbb{L}^c(\widehat{\text{pre}(e)}) = \mathbb{L}^c(\widehat{\text{pre}(e')})$: Suppose $\widehat{\text{pre}(e)} = \widehat{\text{pre}(e')}$. This means that: $\forall T \in X^c, \text{pre}(e) \in T$ iff $\text{pre}(e') \in T$. Therefore, we obtain $\vdash \text{pre}(e) \leftrightarrow \text{pre}(e')$. Then, by (Nec_K) , we have $\vdash K(\text{pre}(e) \leftrightarrow \text{pre}(e'))$. Since $\mathbb{L}^c(\widehat{\text{pre}(e)}) := \widehat{L(e)}$, showing $\mathbb{L}^c(\widehat{\text{pre}(e)}) = \mathbb{L}^c(\widehat{\text{pre}(e')})$ boils down to showing that $\widehat{L(e)} = \widehat{L(e')}$, i.e., that $\vdash L(e) \leftrightarrow L(e')$, which follows from axiom (EC) and that $\vdash K(\text{pre}(e) \leftrightarrow \text{pre}(e'))$.

Next, we must prove that,

- (2b) \mathbb{L}^c is a learner, i.e., \mathbb{L}^c satisfies the properties of a learner given in Definition 3.3.1. To show this, we first check that $\mathbb{L}^c(\widehat{\text{pre}(e)}) \subseteq \widehat{\text{pre}(e)}$ holds. Let $T \in \mathbb{L}^c(\widehat{\text{pre}(e)})$. This means, by the definition of $\mathbb{L}^c(\widehat{\text{pre}(e)})$, that $L(e) \in T$. Since $(L(e) \rightarrow \text{pre}(e)) \in T$ (by the axiom (SP)), we have that $\text{pre}(e) \in T$ as T is maximally consistent. Thus, $T \in \widehat{\text{pre}(e)}$. Finally we

show that if $\widehat{pre(e)} \neq \emptyset$ then $\mathbb{L}^c(\widehat{pre(e)}) \neq \emptyset$. Suppose $\widehat{pre(e)} \neq \emptyset$, i.e., there is $T \in X^c$ with $T \in \widehat{pre(e)}$. This means, by the definition of $\widehat{pre(e)}$, that $pre(e) \in T$. By the axiom (CC), we obtain that $\langle K \rangle L(e) \in T$. Then, by Corollary 3.3.31, there is $S \in X^c$ such that $L(e) \in S$. Thus, by the definition of $\widehat{L(e)}$, we have $S \in \widehat{L(e)}$ meaning that $\widehat{L(e)} = \mathbb{L}^c(\widehat{pre(e)}) \neq \emptyset$.

□

Our aim is to prove a Truth Lemma for the canonical model, which, as usual, will immediately imply completeness, as usual. For this we need the following result.

3.3.33. LEMMA. *Let $T \in X^c$. Then, $\Box\varphi \in T$ iff $[e]\varphi \in T$ for all $e \in \Pi^O$.*

Proof:

The direction from left-to-right follows by the axiom (\Box Ax). For the direction from right-to-left, suppose, toward a contradiction, that for all $e \in \Pi^O$, $[e]\varphi \in T$ and $\Box\varphi \notin T$. Then, since T is a maximally consistent theory, $\Diamond\neg\varphi \in T$. Since T is an O-witnessed theory, there is $o \in O$ such that $\langle !o \rangle \neg\varphi$ is consistent with T . Since T is also maximally consistent, we obtain that $\langle !o \rangle \neg\varphi \in T$, i.e., that $\neg[!o]\varphi \in T$, contradicting our initial assumption. □

3.3.34. LEMMA (TRUTH LEMMA). *Let $M^c = (X^c, \mathcal{O}^c, \mathbb{L}^c, \|\cdot\|_c)$ be the canonical model for some T_0 . For all formulas $\varphi \in \mathcal{L}_{\Pi}^O$, all $T \in X^c$ and all $e \in \Pi^O$, we have:*

$$\langle e \rangle \varphi \in T \quad \text{iff} \quad (T, \widehat{pre(e)}) \models_{M^c} \varphi.$$

Proof:

The proof is by induction on the structure of φ and uses the following *induction hypothesis*,

(IH): for all ψ subformula of φ , and $e \in \Pi^O$, $\langle e \rangle \psi \in T$ iff $(T, \widehat{pre(e)}) \models_{M^c} \psi$.

The base case for propositional and observational variables, as well as Boolean formulas are straightforward. We only verify the remaining inductive cases. Observe that at this point of the proof we have that: $\forall e, e' \in \Pi^O$, $\widehat{\langle e \rangle pre(e')} = \llbracket pre(e') \rrbracket_{M^c}^{\widehat{pre(e)}}$ since $pre(e)$ is a Boolean formula.

Case $\varphi := L(e')$.

$$\begin{aligned}
\langle e \rangle L(e') \in T &\text{ iff } (pre(e) \wedge [e]L(e')) \in T && \text{((2) in Proposition 3.3.10)} \\
&\text{ iff } (pre(e) \wedge L(e; e')) \in T && \text{(R}_L \text{ and CPL)} \\
&\text{ iff } pre(e) \in T \text{ and } L(e; e') \in T \\
&\text{ iff } T \in \widehat{pre(e)} \text{ and } T \in \widehat{L(e; e')} && \text{(Definition of } \widehat{\varphi}\text{)} \\
&\text{ iff } T \in \widehat{pre(e)} \text{ and } T \in \mathbb{L}^c(\widehat{pre(e; e')}) && \text{(Definition of } \mathbb{L}^c\text{)} \\
&\text{ iff } T \in \widehat{pre(e)} \text{ and } T \in \mathbb{L}^c(\langle e \rangle \widehat{pre(e')}) \\
&&& \text{((4) in Proposition 3.3.10)} \\
&\text{ iff } T \in \widehat{pre(e)} \text{ and } T \in \mathbb{L}^c(\llbracket pre(e') \rrbracket_{M^c}^{\widehat{pre(e')}}) \\
&&& \text{(by the above observation)} \\
&\text{ iff } (T, \widehat{pre(e)}) \models_{M^c} L(e') && \text{(semantics)}
\end{aligned}$$

Case $\varphi := K\psi$.

$$\begin{aligned}
\langle e \rangle K\psi \in T &\text{ iff } (pre(e) \wedge K[e]\psi) \in T && \text{((2) in Proposition 3.3.10 and R}_K\text{)} \\
&\text{ iff } pre(e) \in T \text{ and } K[e]\psi \in T \\
&\text{ iff } pre(e) \in T \text{ and } (\forall T' \sim_K T)([e]\psi \in T') && \text{(Lemma 3.3.30)} \\
&\text{ iff } pre(e) \in T \text{ and } (\forall T' \sim_K T \text{ s.t. } pre(e) \in T')(\langle e \rangle \psi \in T') \\
&&& \text{((2) in Proposition 3.3.10)} \\
&\text{ iff } T \in \widehat{pre(e)} \text{ and } (\forall T' \in \widehat{pre(e)})(T', \widehat{pre(e)}) \models_{M^c} \psi && \text{(IH)} \\
&\text{ iff } (T, \widehat{pre(e)}) \models_{M^c} K\psi && \text{(semantics)}
\end{aligned}$$

Case $\varphi := \langle e' \rangle \psi$.

$$\begin{aligned}
\langle e \rangle \langle e' \rangle \psi \in T &\text{ iff } \langle e; e' \rangle \psi \in T && \text{(R}_e\text{)} \\
&\text{ iff } pre(e; e') \wedge \langle e; e' \rangle \psi \in T && \text{((2) in Proposition 3.3.10)} \\
&\text{ iff } pre(e; e') \in T \text{ and } \langle e; e' \rangle \psi \in T \\
&\text{ iff } T \in \widehat{pre(e)} \cap \widehat{pre(e')} \text{ and } T \in \widehat{\langle e; e' \rangle \psi} \\
&\text{ iff } (T, \widehat{pre(e)} \cap \widehat{pre(e')}) \models_{M^c} \psi && \text{(IH)} \\
&\text{ iff } (T, \widehat{pre(e)}) \models_{M^c} \langle e' \rangle \psi && \text{(semantics)}
\end{aligned}$$

Case $\varphi := \Box\psi$. First observe the following,

$$\begin{aligned}
\langle e \rangle \Box\psi \in T &\text{ iff } e \wedge [e]\Box\psi \in T && \text{((2.) in Proposition 3.3.10)} \\
&\text{ iff } \text{pre}(e) \in T \text{ and } \Box[e]\psi \in T && \text{(R}_\Box\text{)} \\
&\text{ iff } \text{pre}(e) \in T \text{ and } [e']\psi \in T, \forall e' \in \Pi^0 && \text{(Lemma 3.3.33)} \\
&\text{ iff } \text{pre}(e) \in T \text{ and } [e'; e]\psi \in T, \forall e' \in \Pi^0 && \text{(R}_e\text{)} \\
&\text{ iff } \text{pre}(e) \in T \text{ and } [e; e']\psi \in T, \forall e' \in \Pi^0 && \text{((5.) in Proposition 3.3.10)}
\end{aligned}$$

For (\Rightarrow): suppose $\langle e \rangle \Box\psi \in T$. Now let $U \in \mathcal{O}^c$ such that $T \in U$. By the definition of \mathcal{O}^c we know that $U = \widehat{\text{pre}(e')}$ for some $e' \in \Pi^0$. Since $T \in \widehat{\text{pre}(e')}$ and by the observation above we obtain, $\text{pre}(e; e') \in T$ and $[e; e']\psi \in T$. Thus, $T \in \widehat{\text{pre}(e; e')}$ and $T \in \widehat{[e; e']\psi}$. By (2.) in Proposition 3.3.10 we have $T \in \widehat{\text{pre}(e; e')}$ and $T \in \widehat{\langle e; e' \rangle \psi}$. Since $\widehat{\text{pre}(e; e')} = \widehat{\text{pre}(e) \cap \text{pre}(e')}$ and $\langle e; e' \rangle \psi \in T$, by IH we get $(T, \widehat{\text{pre}(e) \cap \text{pre}(e')}) \models_{M^c} \psi$. Since $\widehat{\text{pre}(e')} = U$ was taken arbitrarily in \mathcal{O}^c , by the semantics, we obtain $(T, \widehat{\text{pre}(e)}) \models_{M^c} \Box\psi$.

For (\Leftarrow): suppose $(T, \widehat{\text{pre}(e)}) \models_{M^c} \Box\psi$. By the semantics of \Box and the definition of \mathcal{O}^c , we have that for all $e' \in \Pi^0$, if $T \in \widehat{\text{pre}(e) \cap \text{pre}(e')}$ then $(T, \widehat{\text{pre}(e) \cap \text{pre}(e')}) \models_{M^c} \psi$. Let $e'' \in \Pi^0$ such that $T \in \widehat{\text{pre}(e')}$, therefore $T \in \widehat{\text{pre}(e) \cap \text{pre}(e')}$. Since $\widehat{\text{pre}(e) \cap \text{pre}(e')} \in \mathcal{O}^c$, we obtain, by the assumption, that $(T, \widehat{\text{pre}(e) \cap \text{pre}(e')}) \models_{M^c} \psi$. Thus, by IH, we have $\text{pre}(e; e') \in T$ and $\langle e; e' \rangle \psi \in T$. By clause (2.) from Proposition 3.3.10 and clause (5.) from Proposition 3.3.10, $[e''; e]\psi \in T$. By (R_e) we have $[e'']\psi \in T$. Since e'' was taken arbitrarily, by Lemma 3.3.33, we have $\Box[e'']\psi \in T$. Then, by (R_□), we obtain $[e]\Box\psi \in T$. Since $\text{pre}(e) \in T$ and $[e]\Box\psi \in T$, we have $\langle e \rangle \Box\psi \in T$ by (2.) in Proposition 3.3.10. □

3.3.35. THEOREM (COMPLETENESS OF DLLT). *DLLT is complete with respect to the class of learning models.*

Proof:

Let φ be an **DLLT**-consistent formula, i.e., it is an O_φ -theory. Then, by Lemma 3.3.28, it can be extended to some maximal O -witnessed theory T . Then, we have $\langle !\top \rangle \varphi \in T$ i.e., $T \in \widehat{\langle !\top \rangle \varphi}$ (by (6.) in Proposition 3.3.10). Then, by Truth Lemma (Lemma 3.3.34), we obtain that $(T, \widehat{\text{pre}(\langle !\top \rangle)}) \models_{M^c} \varphi$, where $M^c = (X^c, \mathcal{O}^c, \mathbb{L}^c, \|\cdot\|_c)$ is the canonical model for T . This proves completeness. □

3.4 A logic for AGM learning from partial observations

In this section we extend the logic DLLT, from Section 3.3, to reason about an AGM learner that produces conjectures from partial observations, i.e., from partial information.

3.4.1 Syntax and Semantics

In order to capture partial observations, the set of observational events is extended with epistemic non-determinism, events of the form $e \sqcup e'$. Such events capture, e.g., situations when a learner obtains indirect evidence from other learners' reports (which are usually incomplete), or when she observes only some feature of the evidence, so her information is compatible with multiple fully-determined events.

Partial Observational Events. We consider *partial observational events* e (or, in short, *partial observations*) by which the learner acquires some evidence about the world. We denote the set of all partial observational events by Π_{\sqcup} which extends the recursive definition of observational events Π as follows:

$$e := !\top \mid !o \mid (e; e) \mid (e \sqcup e)$$

where $!\top$, $!o$, $e; e'$ are as in Π and $e \sqcup e'$ captures *epistemic non-determinism*: one of the two observational events e or e' happens, but the learner is uncertain which of the two. The reader must not confuse this notion with the one of *ontic non-determinism* where the action is non-deterministic. In our framework, the partial observational event $e \sqcup e'$ is a deterministic action: one of e or e' took place deterministically, however the learner cannot distinguish which action from e and e' took place.

Observe that Π_{\sqcup} is a proper extension of Π . Thus, the language of AGM learning given below is a proper extension of the DLLT language \mathcal{L}_{Π} from Section 3.3.1.

The Language of AGM Learning. The dynamic language $\mathcal{L}_{\Pi_{\sqcup}}$ of AGM learning from partial observations is defined recursively as

$$\varphi := p \mid o \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid L(e) \mid K\varphi \mid [e]\varphi \mid \Box\varphi$$

where $p \in \text{Prop}$, $o \in \text{Prop}_{\mathcal{O}}$ and $e \in \Pi_{\sqcup}$. We employ the usual abbreviations from Section 3.3.1. Given a formula $\varphi \in \mathcal{L}_{\Pi_{\sqcup}}$, we denote by O_{φ} and O_e the set of all observational variables occurring in φ and e , respectively.

As before, $L(e)$ denotes the learner's *conjecture* given observation e ; i.e., her "strongest belief" after having performed observational event e . Observe that $\mathcal{L}_{\Pi_{\sqcup}}$ is a proper extension of the DLLT language \mathcal{L}_{Π} since Π_{\sqcup} is a proper extension of Π .

We interpret $\mathcal{L}_{\Pi_{\cup}}$ on *plausibility learning models* in the style of subset space semantics, as given in turn. Contrary to the learning models defined for DLLT, here the set of information states \mathcal{O}_{\cup} will be also closed under finite unions, i.e. \mathcal{O}_{\cup} forms a lattice.

3.4.1. DEFINITION. [Plausibility Learning Frame/Model, AGM Learner]

- A *plausibility learning frame* is a triple $(X, \mathcal{O}_{\cup}, \leq)$, where (X, \mathcal{O}_{\cup}) is a lattice space⁹ such that X is a non-empty set of *possible worlds* (or “ontic states”) and $\emptyset \neq \mathcal{O}_{\cup} \subseteq \mathcal{P}(X)$ is called *partial information states* (or “partial observations”, or simply “evidence”); and \leq is a total preorder on X , called *plausibility order* such that it satisfies the observational version of Lewis’ “Limit Condition”: *every non-empty information state O has maximal elements*. More precisely, if for any evidence $O \in \mathcal{O}_{\cup}$, we put

$$Max_{\leq}(O) := \{x \in O : y \leq x \text{ for all } y \in O\}$$

for the set of maximal (“most plausible”) worlds compatible with the evidence, then the Limit Condition requires that $Max_{\leq}(O) \neq \emptyset$ whenever $O \neq \emptyset$. The plausibility relation $x \leq y$ reads as “world y is at least as plausible as world x ”.

- A *plausibility learning model* $M = (X, \mathcal{O}_{\cup}, \leq, \|\cdot\|)$ consists of a plausibility learning frame $(X, \mathcal{O}_{\cup}, \leq)$ together with a valuation map $\|\cdot\| : \text{Prop} \cup \text{Prop}_{\mathcal{O}_{\cup}} \rightarrow \mathcal{P}(X)$ that maps propositional variables p into arbitrary sets $\|p\| \subseteq X$ and observational variables o into information states $\|o\| \in \mathcal{O}_{\cup}$.
- A *learner* $\mathbb{L}_{\leq} : \mathcal{O}_{\cup} \rightarrow \mathcal{P}(X)$ on a plausibility learning frame $(X, \mathcal{O}_{\cup}, \leq)$ is a function that maps to every information state $O \in \mathcal{O}_{\cup}$ some conjecture $\mathbb{L}_{\leq}(O) \subseteq X$. An *AGM-learner* is a learner who, upon having observed $O \in \mathcal{O}_{\cup}$, always conjectures the set of most plausible O -states. That is, $\mathbb{L}_{\leq} : \mathcal{O}_{\cup} \rightarrow \mathcal{P}(X)$ is an AGM-learner on $(X, \mathcal{O}_{\cup}, \leq)$ if $\mathbb{L}_{\leq}(O) := Max_{\leq}(O)$ for all $O \in \mathcal{O}_{\cup}$.

By the observational Limit Condition given in Definition 3.4.1, it is then guaranteed that $\mathbb{L}_{\leq}(O) \neq \emptyset$ for all $O \in \mathcal{O}_{\cup}$ with $O \neq \emptyset$ (on DLLT, this corresponds to the condition 2 in Definition 3.3.1). This means that an AGM-learner makes consistent conjectures whenever the received information is consistent.

Similar to Definition 3.3.2, each observational event $e \in \Pi_{\cup}$ induces a *dynamic update* of the learner’s information state. We now extend such definition with the appropriate clause for epistemic non-determinism.

⁹Recall from Section 2.2 in Chapter 2: \mathcal{O}_{\cup} is assumed to be *closed under finite intersections and finite unions*, i.e., if $\mathcal{F} \subseteq \mathcal{O}_{\cup}$ is finite then $\bigcap \mathcal{F} \in \mathcal{O}_{\cup}$ and $\bigcup \mathcal{F} \in \mathcal{O}_{\cup}$

3.4.2. DEFINITION. [Partial Observational updates] A *partial observational update function* (in short, “update function”) $e : \mathcal{O}_U \rightarrow \mathcal{O}_U$ maps any information state $U \in \mathcal{O}_U$ to an updated information state $e(U) \in \mathcal{O}_U$. The map is given by recursion as in Definition 3.3.2 with the additional clause for epistemic non-determinism $e \sqcup e'$: $(e \sqcup e')(U) = e(U) \cup e'(U)$.

The definition above matches our intuition: $e \sqcup e'(U)$ is the disjunction of the information states produced by the two events (since the learner does not know which of the two happened).

The following lemmas will help us to prove that the update map is appropriately defined:

3.4.3. LEMMA. Let $M = (X, \mathcal{O}_U, \leq, \|\cdot\|)$ be a plausibility learning model and $U \in \mathcal{O}_U$ be a partial information state. Then, for all $e \in \Pi_{\sqcup}$ we have $e(U) \in \mathcal{O}_U$.

Proof:

The proof follows easily by induction on the structure of e as in the proof of Lemma 3.3.3. The proof of the remaining case, namely the inductive case $e \sqcup e'$ is as follows: $(e \sqcup e')(U) = e(U) \cup e'(U)$ (by Definition 3.4.2). By the induction hypothesis on $e(U)$ and $e'(U)$ we have that $e(U) \in \mathcal{O}_U$ and $e'(U) \in \mathcal{O}_U$. Since \mathcal{O}_U is closed under finite unions we obtain that $(e \sqcup e')(U) = e(U) \cup e'(U) \in \mathcal{O}_U$. \square

3.4.4. LEMMA. Let $M = (X, \mathcal{O}_U, \leq, \|\cdot\|)$ be a plausibility learning model and $U \in \mathcal{O}_U$ be a partial information state. Then, for all $e \in \Pi_{\sqcup}$, we have $e(U) \subseteq U$.

Proof:

The proof follows easily by induction on the structure of e as in the proof of Lemma 3.3.4. We prove the remaining inductive case $e := f \sqcup f'$:

$(f \sqcup f')(U) \subseteq f'(U) \cup f(U) \subseteq U$ (by Definition 3.4.2 and induction hypothesis on both f and f' for any $U \in \mathcal{O}_U$, we conclude $f'(U), f(U) \subseteq U$). \square

3.4.5. DEFINITION. [Size of events in Π_{\sqcup}] The *size* $s(e)$ of an event $e \in \Pi_{\sqcup}$ is a natural number recursively defined as in Definition A.2.1 with the additional clause for $s(e \sqcup e')$ as: $s(e \sqcup e') = s(e) + s(e') + 1$.

3.4.6. LEMMA. Let $M = (X, \mathcal{O}_U, \leq, \|\cdot\|)$ be a plausibility learning model and $U \in \mathcal{O}_U$ a partial information state. Then, for all $e, e' \in \Pi_{\sqcup}$ we have: $(e; e')(U) = e(U) \cap e'(U)$.

Proof:

The proof follows by induction on the size of $(e; e')$ as in Lemma 3.3.6 with respect to the size measure in Definition 3.4.5, with the following induction hypothesis:

(IH) for all $(f; f') \in \Pi_{\sqcup}$ such that $s(f; f') < s(e; e')$, $(f; f')(U) = f(U) \cap f'(U)$.

Here we only prove the remaining inductive case $e' := f \sqcup f'$:

$(e; e')(U) = e'(e(U)) = (f \sqcup f')(e(U)) = f(e(U)) \cup f'(e(U))$ (by Definition 3.4.2 for $f \sqcup f'$) $= (e; f)(U) \cup (e; f')(U) = (f(U) \cap e(U)) \cup (f'(U) \cap e(U))$ (by IH, since $s(e; f) < s(e; e')$ and $s(e; f') < s(e; e') = e(U) \cap (f(U) \cup f'(U)) = e(U) \cap (f \sqcup f')(U)$ (by Definition 3.4.2 for $f \sqcup f'$) $= e(U) \cap e'(U)$. \square

The semantic definition for the language $\mathcal{L}_{\Pi_{\sqcup}}$ is defined similarly to the semantic definition for \mathcal{L}_{Π} in DLLT but with the appropriate semantic clause for the AGM learning operator.

3.4.7. DEFINITION. [Semantics for an AGM-learner] Given a plausibility learning model $M = (X, \mathcal{O}_{\sqcup}, \leq, \|\cdot\|)$ and an epistemic scenario (x, U) , the semantics for the language $\mathcal{L}_{\Pi_{\sqcup}}$ is defined recursively as in Definition 3.3.7 with the following modification for the clause of the learning operator $L(e)$:

$$(x, U) \models L(e) \text{ iff } x \in \text{Max}_{\leq} e(U)$$

We extend our previous notion of precondition as follows:

3.4.8. DEFINITION. [Precondition of Partial Observational Events] To each partial observational event $e \in \Pi_{\sqcup}$, the *precondition* of event e is defined by recursion as in Definition 3.3.8 with the additional clause for $e \sqcup e'$: $\text{pre}(e \sqcup e') = \text{pre}(e) \vee \text{pre}(e')$.

As in Definition 3.3.8, here the precondition formula $\text{pre}(e)$ captures the “condition of possibility” of the event e . Thus, the event $e \sqcup e'$ can happen in a world x iff $\text{pre}(e) \vee \text{pre}(e')$ is true at (x, U) , for any $U \in \mathcal{O}$ with $x \in U$, as well as its *informational content* (the learner’s new information after $e \sqcup e'$).

The following lemma corresponds to Lemma 3.3.9 of DLLT that expresses what the formula $\text{pre}(e)$ captures.

3.4.9. LEMMA. *Let $M = (X, \mathcal{O}_{\sqcup}, \leq, \|\cdot\|)$ be a plausibility learning model and $U \in \mathcal{O}_{\sqcup}$ be a partial information state. Then, for all $e \in \Pi_{\sqcup}$ we have:*

$$\llbracket \text{pre}(e) \rrbracket^U = e(U) = \llbracket \langle e \rangle \top \rrbracket^U.$$

Proof:

The proof follows as in the proof of Lemma 3.3.9 by induction on the structure of e , now using Lemma 3.4.6. We prove the remaining inductive case $e := f \sqcup f'$:

$\llbracket \text{pre}(f \sqcup f') \rrbracket^U = \llbracket \text{pre}(f) \vee \text{pre}(f') \rrbracket^U$ (by Definition 3.4.8) $= \llbracket \text{pre}(f) \rrbracket^U \cup \llbracket \text{pre}(f') \rrbracket^U = f(U) \cup f'(U)$ (by IH) $= (f \sqcup f')(U)$ (by Definition 3.4.2). \square

3.4.2 Expressivity of language $\mathcal{L}_{\Pi_{\sqcup}}$

In this section we first compare the expressive power of the language $\mathcal{L}_{\Pi_{\sqcup}}$ to those of its fragments of interest. Then, we investigate how the learnability notions addressed in Section 3.2.1 can be expressed in $\mathcal{L}_{\Pi_{\sqcup}}$ for the corresponding plausibility learning models.

As it is expected, the full language $\mathcal{L}_{\Pi_{\sqcup}}$ and the one obtained by removing the update modalities $\mathcal{L}_{\Pi_{\sqcup}}^{-!}$ are equally expressive by the same argument as the one in the proof of Proposition 3.3.13 in Section 3.3.3. In the following theorem, we compare the languages $\mathcal{L}_{\Pi_{\sqcup}}^{-!}$ (and therefore the full language $\mathcal{L}_{\Pi_{\sqcup}}$), the static language $\mathcal{L}_{\Pi_{\sqcup}}^{-! \square}$ and the epistemic fragment $\mathcal{L}_{\Pi_{\sqcup}}^{epi}$, obtaining similar results as the ones in Theorem 3.3.14.

3.4.10. THEOREM (EXPRESSIVITY OF $\mathcal{L}_{\Pi_{\sqcup}}$). *$\mathcal{L}_{\Pi_{\sqcup}}$ is equally expressive as $\mathcal{L}_{\Pi_{\sqcup}}^{-!}$, and they are strictly more expressive than the static fragment $\mathcal{L}_{\Pi_{\sqcup}}^{-! \square}$ with respect to plausibility learning models. Moreover, $\mathcal{L}_{\Pi_{\sqcup}}^{-! \square}$ is strictly more expressive than the epistemic fragment $\mathcal{L}_{\Pi_{\sqcup}}^{epi}$.*

Proof:

$\mathcal{L}_{\Pi_{\sqcup}}$ is equally expressive as $\mathcal{L}_{\Pi_{\sqcup}}^{-!}$: as in the proof of Proposition 3.3.13, use step-by-step the reduction axioms in Table 3.1 as a rewriting process and prove termination by the strict partial order \prec_1 on $\mathcal{L}_{\Pi_{\sqcup}}$ defined in Lemma A.1.2 in the Technical Appendix A¹⁰. For the second claim, consider the following two-state models $M_1 = (X, \mathcal{O}_{\sqcup}^1, \leq, \|\cdot\|)$ and $M_2 = (X, \mathcal{O}_{\sqcup}^2, \leq, \|\cdot\|)$ where $X = \{x, y\}$, $\leq = \{(x, x), (y, y), (x, y)\}$ and the valuation $\|p\| = \{y\}$. And, take $\mathcal{O}_{\sqcup}^1 = \{X, \emptyset\}$ (the trivial topology on X) and $\mathcal{O}_{\sqcup}^2 = \mathcal{P}(X)$ (the discrete topology on X). It is then easy to see that $M_1, (x, \{x, y\})$ and $M_2, (x, \{x, y\})$ are modally equivalent with respect to the language $\mathcal{L}_{\Pi_{\sqcup}}^{-! \square}$. However, $M_2, (x, \{x, y\}) \models \diamond K \neg p$ (since $\{x\}$ is an open set of M_2) whereas $M_1, (x, \{x, y\}) \not\models \diamond K \neg p$, since the only open including x is $\{x, y\}$ and $y \in \|p\| = \{y\}$. To prove that $\mathcal{L}_{\Pi_{\sqcup}}^{-! \square}$ is strictly more expressive than the epistemic fragment $\mathcal{L}_{\Pi_{\sqcup}}^{epi}$, consider the models $M'_1 = (X, \mathcal{O}_{\sqcup}^1, \leq_1, \|\cdot\|)$ and $M'_2 = (X, \mathcal{O}_{\sqcup}^2, \leq_2, \|\cdot\|)$, where $X, \mathcal{O}_{\sqcup}^1$, and \mathcal{O}_{\sqcup}^2 are as above but $\leq_1 = \leq$ and $\leq_2 = \{(x, x), (y, y), (y, x)\}$. It is then easy to see that $M'_1, (x, \{x, y\})$ and $M'_2, (x, \{x, y\})$ are modally equivalent with respect to the language $\mathcal{L}_{\Pi_{\sqcup}}^{epi}$ whereas $M'_1, (x, \{x, y\}) \not\models L(!\top)$ (since $x \notin \text{Max}_{\leq_1}(!\top(\{x, y\})) = \{y\}$) but $M'_2, (x, \{x, y\}) \models L(!\top)$ (since $x \in \text{Max}_{\leq_2}(!\top(\{x, y\})) = \{x\}$). \square

The expressivity diagram in Figure 3.3 summarizes Theorem 3.4.10.

Belief, Inductive Knowledge and Inductive Learnability. We now explore how the notions of belief, inductive knowledge and inductive learnability can be expressed within our Dynamic Logic of AGM Learning.

¹⁰This is a standard method in Dynamic Epistemic Logic and we refer the reader to (van Ditmarsch et al., 2007, Chapter 7.4) for further details.

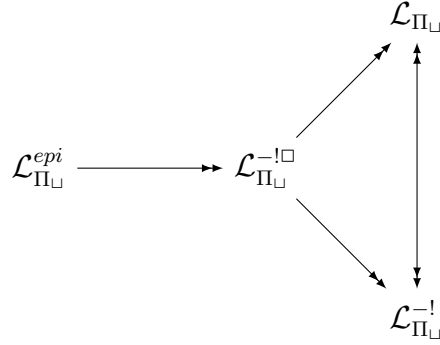


Figure 3.3: Expressivity diagram (Arrows point to the more expressive languages, and reflexive and transitive arrows are omitted.)

We first recall the definitions of the notions given in Section 3.2.1 but now with respect to an AGM learner: recall that the notion of infallible knowledge is in our logic directly represented by the modality K , whose semantic clause mimics the following definition. The AGM learner is said to *infallibly know* a proposition $P \subseteq X$ in a partial information state $U \in \mathcal{O}_U$ if her partial information state U entails P , i.e., $U \subseteq P$. The possibility of learning a proposition with such certainty in a possible world $x \in X$ by a learner \mathbb{L}_{\leq} if given enough evidence (true at x) is called *learnability with certainty*. In other words, P is learnable with certainty at world x if there exists some truthful partial information state $O \in \mathcal{O}_U$ (i.e., $x \in O$) such that the learner infallibly knows P in information state O .

As in \mathcal{L}_{Π} , the notion of learnability with certainty (anticipated by Parikh et al., 2007) is characterised in \mathcal{L}_{Π_U} by the formula $\Diamond Kp$ (following the same argument as in the proof of Proposition 3.3.15 from Section 3.2.1).

In a partial information state U , we say that the AGM learner *believes* a proposition $P \subseteq X$ if her conjecture given U entails P , that is, $\mathbb{L}_{\leq}(U) \subseteq P$. This gives us the standard interpretation of belief on plausibility models (see e.g., Board, 2004; van Benthem, 2007; Baltag and Smets, 2008):

$$(x, U) \models B\varphi \text{ iff } \text{Max}_{\leq} e(U) \subseteq \llbracket \varphi \rrbracket^U.$$

Recall that belief can be defined as an abbreviation in \mathcal{L}_{Π} . The same abbreviation holds in \mathcal{L}_{Π_U} :

$$B\varphi := K(L(!\top) \rightarrow \varphi).$$

Indeed, it is easy to check that this notion satisfies the semantic clause above.

We say that, in partial information state U and ontic state x , the AGM learner *has undefeated belief* in a proposition $P \subseteq X$ if she believes P and will continue to believe P no matter what new true observations will be made; i.e. iff $(x, O) \models BP$ for every $O \in \mathcal{O}_U$ with $x \in O$. We then say, in a partial information state U , the AGM learner *inductively knows* P at world x if the learner has undefeated belief

in P at x and the learner tracks the truth of P at x . Finally, P is *inductively learnable* by the AGM learner \mathbb{L}_{\leq} at world x in U if there exists some truthful partial information state $O \in \mathcal{O}_U$ (i.e., $x \in O \subseteq U$) such that \mathbb{L}_{\leq} inductively knows P in information state O at x . Note that these notions correspond to the learning theoretic notions presented in Section 3.3.4.

3.4.11. PROPOSITION. *Given a plausibility learning model $M = (X, \mathcal{O}_U, \leq, \|\cdot\|)$ and $(x, U) \in ES(M)$, the equivalences (1) to (7) hold as in Proposition 3.3.16 but with respect to the AGM-learner \mathbb{L}_{\leq} .*

Proof:

The proof follows easily as in the proof of Proposition 3.3.16 using the relevant definitions above with respect to an AGM learner and our semantic definition (Definition 3.4.7). \square

To illustrate these notions, consider the following example¹¹:

3.4.12. EXAMPLE. [The alcohol inspector] An alcohol inspector needs to randomly check cars that pass through a security point in a perimetrical highway of Munich during the October fest to check the driver’s alcohol levels. The maximum alcoholic-level allowed is 30 points (which corresponds to two small beers). His alcohol-measuring tool, known as *breathalyser*, has an accuracy of ± 20 .

At some point, a young woman gets the stop sign in order to get inspected. The breathalyser outputs a reading of 40 points. Given the accuracy of the tool, this first measurement can be represented by the interval $(20, 60) \subseteq \mathbb{R}$. At this point, the inspector cannot know for sure that the driver has drunk more beers than allowed. The inspector then borrows a more advanced and accurate breathalyser from one of his colleagues, with an accuracy of ± 5 . The more accurate breathalyser outputs a reading of 35 points. So the measurement of the second breathalyser can be represented by the interval $(30, 40)$. Therefore, after the reading of the second device, the inspector knows with certainty that the woman has exceeded legal alcohol limit, so she needs to wait for a couple of hours before driving again and pay a costly fine. Moreover, let us assume that inspector obeys the legal principle of “*believing in innocence until proven guilty beyond doubt*”: so, whenever he is in doubt (because his measurements do not prove either case), he believes the driver is *not drunk*.

This situation can be represented in a plausibility learning frame $(X, \mathcal{O}_U, \preceq)$ ¹², where (X, \mathcal{O}_U) is a lattice with $X = [0, \infty) \subseteq \mathbb{R}$ as the set of “possible worlds”

¹¹This example is very similar to Example 3.2.1 in Section 3.2. Note that the logical structure of the story and the mathematical formalization are analogous as the ones for Example 3.2.1. To avoid repetition (and for the fun of it), here we use a different story.

¹²We use \preceq to denote the plausibility order in this frame, to distinguish it from the natural order on $X \subseteq \mathbb{R}$.

(=possible alcohol levels)¹³, while the family of partial observations \mathcal{O}_U is the closure under finite intersections and finite unions of the family of breathalyser measurements (=single-step total observations) represented by

$$\mathcal{B} = \{[0, b] \subseteq \mathbb{R} : 0 < b \in \mathbb{Q}\} \cup \{(a, b) \subseteq \mathbb{R} : 0 < a, b \in \mathbb{Q}\}.$$

The sets in \mathcal{B} represent all possible readings of arbitrarily accurate breathalysers, while the sets in \mathcal{O}_U represent all possible information states of the inspector, based on iterated (and possibly) partial reports of such readings. Finally, the policy of believing in “innocence until proven guilty” is captured by assuming that (in the absence of any evidence) the inspector considers all non-drunk states to be a priori *more plausible* than all drunk states: i.e. $x \preceq y$ for $x > 30$ and $y \leq 30$. This policy is not enough to fully determine the plausibility relation. To make it precise, let us assume for now that the inspector has *no other strong belief on the matter*, i.e. he considers all the *drunk states to be equally plausible* (and similarly for the non-drunk states). So the relation is given by putting: $x \preceq y$ iff either $y \leq 30$ or else $30 < x, y$. It is easy to check that \preceq is indeed a total preorder.

Consider the propositions *drunk* $D = (30, \infty)$ and *not drunk* $ND = [0, 30]$ in the context of this example. We can then ask if the inspector knows with certainty that the woman is outside the permitted alcohol levels, namely if the inspector knows proposition D . After the second reading, the inspector knows with certainty that the woman has drunk more than allowed. Thus, given enough more accurate measurements, the inspector can infallibly know D (whenever D is actually the case); i.e. D is always learnable with certainty. However proposition ND is *not* always learnable with certainty: if the real level of alcohol happens to be exactly 30, then the driver is not drunk (ND) but the inspector will never come to infallibly know ND . This is simply because any interval containing 30 has non-empty intersection with D . Still, ND is falsifiable with certainty (since its negation is learnable with certainty whenever true). A property that is *neither learnable with certainty nor falsifiable with certainty* is having alcohol level *barely-above-permitted* $BAP = (30, 31]$. Note that we already talked about such notions in Example [3.2.1](#).

Inductive learnability is of course a weaker, more general form of knowledge: *both* properties *drunk* $D := (30, \infty)$ and *not-drunk* $ND := [0, 30]$ are inductively learnable by the inspector, if endowed with the above plausibility order \preceq . Indeed,

¹³We are aware of the fact that not having an upper bound for the possible alcohol levels is not very realistic. One can make a more realistic example by fixing a large enough upper bound, for instance $50 < m \in \mathbb{R}$, so that $X = [0, m)$. In the corresponding *bounded* lattice space (X, \mathcal{O}_U) , the analysis goes similarly as for $[0, \infty)$ but with respect to the base for the subset topology generated from the standard one in \mathbb{R} . We chose to keep the example wrt $X = [0, \infty)$, since this does not make a significant difference in our discussion and our analysis. Moreover, we want to keep the example as similar as Example [3.2.1](#) in order to make the analogy explicit between the relevant learning theoretic notions and the two types of learners.

if the true alcohol level is some $x \in ND = [0, 30]$, then the inspector (in the absence of any evidence), starts by believing ND (since $\mathbb{L}_{\preceq}(X) = \text{Max}_{\preceq}X = [0, 30]$); and, no matter what further direct evidence (a, b) , with $a < x < b$, the learner gets, she will still believe ND (since $\mathbb{L}_{\preceq}((a, b)) = \text{Max}_{\preceq}(a, b) \cap [0, 30] \subseteq [0, 30]$). So in this case the inspector inductively knows ND from the start. While if $x \in D = (30, \infty)$, then after taking an accurate enough measurement, the inspector will obtain some evidence (a, b) , with $30 < a < x < b$. For any further refinement $(a', b') \subseteq (a, b)$ of this evidence, we will have $x \in (a', b') \subseteq (a, b) \subseteq (30, \infty) = D$, hence $\mathbb{L}_{\preceq}((a', b')) = \text{Max}_{\preceq}(a', b') = (a', b') \subseteq D$. Which means that, after reading (a, b) , the inspector achieves inductive knowledge of D : he will believe D no matter what further observations might be made. The AGM learner with respect to \preceq is then defined as follows,

$$\mathbb{L}_{\preceq}((a, b)) = \begin{cases} [0, 30], & \text{if } (a, b) = X \\ \text{Max}_{\preceq}(a, b) \cap ND, & \text{if } (a, b) \cap ND \neq \emptyset \\ \text{Max}_{\preceq}(a, b), & \text{otherwise (i.e., if } (a, b) \subseteq D). \end{cases}$$

What about the property $BAP = (30, 31]$ of having a *barely-above-permitted* alcohol level? This property is in principle also inductively learnable (by *some* learners), but *not* by the above AGM learner! To design an AGM learner who can inductively learn it, we need to change the plausibility relation, using a different refinement of the general “innocent until proven guilty” policy. The inspector still believes *all the non-drunk states to be more plausible than all the drunk ones*; but now, within the drunk-world zone, he has a similarly generous attitude: “if guilty then barely guilty”. In other words, he considers *the barely-above-permitted levels in $BAP = (30, 31]$ to be more plausible than the way-above-permitted ones in $WAV = (31, \infty)$* ; and for the rest, he is indifferent, as before. This amounts to adopting a plausibility order \ll , given by putting $x \ll y$ iff: either we have $y \leq 30$, or else we have both $30 < x$ and $y \leq 31$, or otherwise we have $31 < x, y < \infty$. It is easy to check that \ll is a total pre-order, and moreover that properties D , ND and BAP are all inductively learnable by an inspector endowed with this plausibility order and the resulting learning function \mathbb{L}_{\ll} .

3.4.3 Axiomatization

In this section, we present a sound and complete axiomatization for the Dynamic Logic of AGM Learning from Partial Observations (**AGML**).

3.4.13. THEOREM (SOUNDNESS AND COMPLETENESS OF AGML). *The sound and complete axiomatization of **AGML** with respect to plausibility learning models is obtained by extending the axiomatic system in Table 3.1 (DLLT) with the two axioms in Table 3.2 all with respect to the language $\mathcal{L}_{\Pi_{\perp}}$. The axioms in Table 3.2 correspond to the AGM postulates Inclusion and Rational Monotonicity.*

AGM Learning axioms:	
(Inc)	$(pre(e) \wedge L(e')) \rightarrow L(e; e')$
(RMon)	$\langle K \rangle(L(e') \wedge pre(e)) \rightarrow (L(e; e') \rightarrow (pre(e) \wedge L(e')))$

Table 3.2: The two additional AGM axioms of (**AGML**)**Proof:**

The soundness proof will be presented in Section 3.4.4 and the completeness proof in Section 3.4.5. \square

Besides the theorems given in Proposition 3.3.10, we have the following:

3.4.14. PROPOSITION. *The following formula is derivable in **AGML** for all $\varphi \in \mathcal{L}_{\Pi_{\perp}}$ and $e \in \Pi_{\perp}$: $\langle K \rangle(L(e') \wedge pre(e)) \rightarrow (L(e; e') \leftrightarrow (pre(e) \wedge L(e'))$.*

Proof:

Follows straightforwardly from (RMon) and (Inc). \square

The two AGM learning axioms (Inc) and (RMon) are novel to the current system. They correspond to the Inclusion and Expansion (Subexpansion and Superexpansion) AGM postulates in (Alchourrón et al., 1985), respectively. These are better understood in terms of belief. (Inc) states that the learner believes a proposition P after having observed e only if she initially believes that e entails P . (RMon) on the other hand says that the learner revises her beliefs in a monotonic way as long as the newly observed event is consistent with her previous conjecture.

3.4.4 Soundness of **AGML**

For the soundness of the non-standard rule $\square Ru$ with respect to plausibility learning models, we follow a similar strategy as the one used for the soundness of **DLIT** in Section 3.3.5. For this, we need the following lemma which is the corresponding counterpart of Lemma 3.3.17.

3.4.15. LEMMA. *Let $M = (X, \mathcal{O}_{\cup}, \leq, \|\cdot\|)$ and $M' = (X, \mathcal{O}_{\cup}, \leq, \|\cdot\|')$ be two plausibility learning models and $\varphi \in \mathcal{L}_{\Pi_{\perp}}$ such that M and M' differ only in the valuation of some $o \notin O_{\varphi}$. Then, for all $U \in \mathcal{O}_{\cup}$, we have $\llbracket \varphi \rrbracket_M^U = \llbracket \varphi \rrbracket_{M'}^U$.*

Proof:

Follows by subformula induction on φ . Let $M = (X, \mathcal{O}_{\cup}, \leq, \|\cdot\|)$ and $M' = (X, \mathcal{O}_{\cup}, \leq, \|\cdot\|')$ be two learning models such that M and M' differ only in the valuation of some $o \notin O_{\varphi}$, and let $U \in \mathcal{O}_{\cup}$. Note that in the proof of Lemma

3.3.17 the structure of the observational events e does not play any role. Thus, the base cases and the inductive cases for Booleans, $K\varphi$, $\Box\varphi$, and $[e]\varphi$ follow as in the proof of Lemma **3.3.17** (but with respect to the language $\mathcal{L}_{\Pi_{\cup}}$). Here we only prove the case $\varphi := L(e)$. We need to show that $\llbracket L(e) \rrbracket_M^U = \llbracket L(e) \rrbracket_{M'}^U$. Note that $O_{L(e)} = O_e$ and for every $o \in O_e$, $\|o\| = \|o\|'$. Therefore, since \mathcal{O}_{\cup} is the same in both models, $e(U) \in \mathcal{O}_{\cup}$ is the same in both models. Also, \leq is the same in both models, thus $\llbracket L(e) \rrbracket_M^U = \text{Max}_{\leq} e(U) = \llbracket L(e) \rrbracket_{M'}^U$. \square

3.4.16. THEOREM. *The system **AGML** given in Tables **3.1** and **3.2** with language $\mathcal{L}_{\Pi_{\cup}}$ is sound with respect to the class of plausibility learning models.*

Proof:

The soundness proof follows via standard validity check as in the proof of Theorem **3.3.19** but with respect to plausibility learning models and language $\mathcal{L}_{\Pi_{\cup}}$. Note that the structure of the observational events that appear in the axioms is not relevant for the soundness proof of Theorem **3.3.19**. Thus, we can follow here the same reasoning steps as before. In particular, the soundness of the non-standard ($\Box\text{Ru}$) follows as in the proof of Theorem **3.3.19** but using Lemma **3.4.15**. Here we only present the proofs of the AGM learning axioms (Inc) and (RMon).

Let $M = (X, \mathcal{O}, \leq, \|\cdot\|)$ be a learning model, $(x, U) \in ES(M)$ and $e, e' \in \Pi_{\cup}$.

- (Inc) We need to show that $(x, U) \models \text{pre}(e) \wedge L(e') \rightarrow L(e, e')$. Suppose $(x, U) \models \text{pre}(e) \wedge L(e')$. We need to show that $x \in \llbracket L(e; e') \rrbracket$, i.e., that $x \in \text{Max}_{\leq}(e(U) \cap e'(U))$ (by Definition **3.4.7** and Lemma **3.4.6**). By the initial assumption and the semantic definition (Definition **3.4.7**), we have that $x \in e(U)$ and $x \in \text{Max}_{\leq}(e'(U))$. Let $y \in e(U) \cap e'(U)$. Since $y \in e'(U)$ and $x \in e(U) \cap \text{Max}_{\leq}(e'(U))$, by Definition **3.4.1** for $\text{Max}_{\leq}(e'(U))$ we have that $y \leq x$. Since y was arbitrary in $e(U) \cap e'(U)$ such that $y \leq x$, we have that $x \in \text{Max}_{\leq}(e(U) \cap e'(U))$. Therefore $(x, U) \models L(e; e')$.
- (RMon) We need to show that $(x, U) \models \langle K \rangle(L(e') \wedge \text{pre}(e)) \rightarrow (L(e; e') \rightarrow \text{pre}(e) \wedge L(e'))$. Suppose (a) $(x, U) \models \langle K \rangle(L(e') \wedge \text{pre}(e))$ and suppose (b) $(x, U) \models L(e; e')$. We need to show that $(x, U) \models \text{pre}(e) \wedge L(e')$, i.e., that $x \in e(U)$ and $x \in \text{Max}_{\leq}(e'(U))$ (by the semantic definition (Definition **3.4.7**) and Lemma **3.3.9**). By (a) we have that: there is $y \in U$ such that $(y, U) \models L(e') \wedge \text{pre}(e)$. By Definition **3.4.7** it follows that $y \in e(U)$ and $y \in \text{Max}_{\leq}(e'(U))$. By assumption (b), Definition **3.4.7** and Lemma **3.4.6** we have that $x \in \text{Max}_{\leq}(e(U) \cap e'(U))$. Since $y \in e(U)$ and $y \in \text{Max}_{\leq}(e'(U))$, we have $y \in e(U) \cap e'(U)$. By Definition **3.4.1** for $\text{Max}_{\leq}(e(U) \cap e'(U))$ it follows that $y \leq x$. Since $y \in \text{Max}_{\leq}(e'(U))$, $x \in e'(U)$ and $y \leq x$, it follows that $x \in \text{Max}_{\leq}(e'(U))$. Therefore $x \in e(U)$ and $x \in \text{Max}_{\leq}(e'(U))$, thus $(x, U) \models \text{pre}(e) \wedge L(e')$.

\square

3.4.5 Completeness of AGML

We prove completeness of **AGML** via a canonical model construction, although its construction is not a trivial matter because of the following reasons. First, we need to consider “witnessed” (*maximally consistent*) theories as in the canonical model for **DLT** in Section 3.3.6 (due to the presence of the Effort rule $\square\text{Ru}$). Recall that in witnessed theories, every occurrence of a $\diamond\varphi$ in any “existential context” is “witnessed” by some $\langle!o\rangle\varphi$ (with o observational variable). This is represented by *possibility forms*, exactly as in Definition 3.3.20 but with respect to $\mathcal{L}_{\Pi_{\perp}}$. Second, we need to define the plausibility order in the canonical model, \leq^c . The definition of \leq^c is inspired by the construction of the so-called *order models* from *spheres systems* and *selection models* presented in (Grahne, 1998). This will become clear once we start with the construction.

First note that Lemmas 3.3.21 and 3.3.22 hold for our current language $\mathcal{L}_{\Pi_{\perp}}$ (with respect to Π_{\perp}). This is because the structure of events e do not play any role throughout the proofs of these lemmas. Thus, in what follows, we will refer to these lemmas directly as stated in Section 3.3.6. The reader should keep in mind that here we consider them with respect to the language $\mathcal{L}_{\Pi_{\perp}}$. In a similar fashion, we use the Lindenbaum’s Lemma and the Extension Lemma (Lemmas 3.3.27 and 3.3.28) as presented in Section 3.3.6. Their proofs with respect to the language $\mathcal{L}_{\Pi_{\perp}}$ follow exactly as before. This is because witnessed theories are defined similarly as for **DLT** and the only events that play a role in the proofs are of the form $e :=!o$, which are contained in $\Pi \subseteq \Pi_{\perp}$.

Canonical Model for T_0 . For any consistent set of formulas Φ , consider a maximally consistent O-witnessed extension $T_0 \supseteq \Phi$. As our canonical set of worlds, we take the set $X^c := \{T : T \text{ maximally consistent O-witnessed theory with } T \sim_K T_0\}$, where we put

$$T \sim_K T' \text{ iff } \forall \varphi \in \mathcal{L}_{\Pi_{\perp}}^{\text{O}} (K\varphi \in T \text{ implies } \varphi \in T').$$

It is easy to see (given the **S5** axioms for K) that \sim_K is an equivalence relation. For any formula φ , we use the notation $\widehat{\varphi} := \{T \in X^c : \varphi \in T\}$. As the canonical set of information states, we take $\mathcal{O}_{\perp}^c := \{\widehat{\text{pre}(e)} : e \in \Pi_{\perp}^{\text{O}}\}$. Towards defining the canonical plausibility relation \leq^c , let

$$S_e = \bigcup \{\widehat{L(e')} : \widehat{\text{pre}(e)} \subseteq \widehat{\text{pre}(e')} \text{ and } e' \in \Pi_{\perp}^{\text{O}}\},$$

and $\$ = \{S_e : e \in \Pi_{\perp}^{\text{O}}\} \cup \{X^c\}$. The canonical plausibility order \leq^c on X^c is given by, for any $T, T' \in X^c$:

$$T \leq^c T' \text{ iff } \forall S \in \$ (T \in S \text{ implies } T' \in S).$$

As we mentioned before, the definition of \leq^c is inspired by the construction of a sphere system from a set selection model presented in (Grahne, 1998). Roughly

speaking, while $\widehat{L}(e')$ plays the role of a selection function that picks out a set of maximally consistent O-witnessed theories given e' (see e.g., [Chellas, 1975](#); [Grahne, 1998](#)), the collection of sets \mathcal{S} forms a sphere system (see e.g., [Lewis, 1973](#)).

The canonical valuation $\|\cdot\|_c$ is given as $\|p\|_c = \widehat{p}$ and $\|o\|_c = \widehat{o}$. The tuple $M^c = (X^c, \mathcal{O}_{\sqcup}^c, \leq^c, \|\cdot\|_c)$ is called the *canonical model*.

Before proving that $M^c = (X^c, \mathcal{O}_{\sqcup}^c, \leq^c, \|\cdot\|_c)$ is in fact a plausibility learning model, we first need the following lemmas that will guide us step-by-step through the properties of the sphere system:

3.4.17. LEMMA. *For all $e, e' \in \Pi_{\sqcup}^O$, if $\widehat{pre}(e) \cap S_{e'} \neq \emptyset$ then $\widehat{L}(e) \subseteq S_{e'}$.*

Proof:

Suppose $\widehat{pre}(e) \cap S_{e'} \neq \emptyset$. This means, since $S_{e'} = \bigcup \{\widehat{L}(e'') : \widehat{pre}(e') \subseteq \widehat{pre}(e'') \text{ and } e'' \in \Pi_{\sqcup}^O\}$, that there is a $T \in \widehat{pre}(e) \cap \widehat{L}(e'')$ for some $e'' \in \Pi_{\sqcup}^O$ such that $\widehat{pre}(e') \subseteq \widehat{pre}(e'')$. Let $T' \in \widehat{L}(e)$ and consider the observational event $e \sqcup e''$. As $\widehat{pre}(e') \subseteq \widehat{pre}(e'')$, we also have $\widehat{pre}(e') \subseteq \widehat{pre}(e \sqcup e'') = \widehat{pre}(e) \vee \widehat{pre}(e'')$. Hence, $\widehat{L}(e \sqcup e'') \subseteq S_{e'}$. Now suppose, toward contradiction, that $\widehat{pre}(e) \cap \widehat{L}(e \sqcup e'') = \emptyset$. This implies, since $\widehat{L}(e \sqcup e'') \subseteq \widehat{pre}(e \sqcup e'')$ (by (SP)) = $\widehat{pre}(e) \vee \widehat{pre}(e'')$, that $\widehat{L}(e \sqcup e'') \subseteq \widehat{pre}(e'')$. Moreover, as $\widehat{pre}(e \sqcup e'') \neq \emptyset$ (by assumption), we also have $\widehat{L}(e \sqcup e'') \neq \emptyset$ (by (CC)). Thus, as $\widehat{L}(e \sqcup e'') \cap \widehat{pre}(e'') \neq \emptyset$ and $\widehat{pre}(e'') \subseteq \widehat{pre}(e \sqcup e'')$, by [Proposition 3.4.14](#), we have $\widehat{L}(e'') = \widehat{pre}(e'') \cap \widehat{L}(e \sqcup e'')$. This implies that $\widehat{pre}(e) \cap \widehat{L}(e'') = \widehat{pre}(e) \cap (\widehat{pre}(e'') \cap \widehat{L}(e \sqcup e'')) \neq \emptyset$, contradicting $\widehat{pre}(e) \cap \widehat{L}(e \sqcup e'') = \emptyset$. Therefore, $\widehat{pre}(e) \cap \widehat{L}(e \sqcup e'') \neq \emptyset$. Then, by [Proposition 3.4.14](#) again, $\widehat{L}(e) = \widehat{pre}(e) \cap \widehat{L}(e \sqcup e'')$. As $T' \in \widehat{L}(e)$, we have $T' \in \widehat{L}(e \sqcup e'') \subseteq S_{e'}$, i.e., $T' \in S_{e'}$. \square

3.4.18. LEMMA. *The following holds for all $e \in \Pi_{\sqcup}^O$:*

1. $X^c \in \mathcal{S}$,
2. for all $S, S' \in \mathcal{S}$, either $S \subseteq S'$ and $S' \subseteq S$ (nestedness),
3. if $\widehat{pre}(e) \neq \emptyset$, then $\{S \in \mathcal{S} : S \cap \widehat{pre}(e) \neq \emptyset\}$ has a smallest¹⁴ member (restricted Limit assumption). More precisely, S_e is the smallest member of $\{S \in \mathcal{S} : S \cap \widehat{pre}(e) \neq \emptyset\}$.

Proof:

¹⁴Smallest with respect to the subset relation in $\mathcal{P}(X^c)$.

1. By the definition of $\$$.
2. Let $S, S' \in \$$. If $S = X^c$ or $S' = X^c$, it is trivially the case that either $S \subseteq S'$ or $S' \subseteq S$. Now consider the case $S = S_e$ and $S' = S_{e'}$ for some $e, e' \in \Pi_{\sqcup}^0$ and suppose that (a) $S_e \not\subseteq S_{e'}$ and (b) $S_{e'} \not\subseteq S_e$. This means that there are $T, T' \in X^c$ such that $T \in S_e$ but $T \notin S_{e'}$, and $T' \in S_{e'}$ but $T' \notin S_e$. By the definitions of S_e and $S_{e'}$, we then have $T \in \widehat{L(e_1)}$ and $T' \in \widehat{L(e_2)}$ for some $e_1, e_2 \in \Pi_{\sqcup}^0$ such that $\widehat{pre(e)} \subseteq \widehat{pre(e_1)}$ and $\widehat{pre(e')} \subseteq \widehat{pre(e_2)}$. Moreover, since $\widehat{pre(e)} \subseteq \widehat{pre(e_1 \sqcup e_2)}$ and $\widehat{pre(e')} \subseteq \widehat{pre(e_1 \sqcup e_2)}$, we also have $\widehat{L(e_1 \sqcup e_2)} \subseteq S_{e'} \cap S_e$. As $\widehat{pre(e)} \neq \emptyset$ (or, $\widehat{pre(e')} \neq \emptyset$, since otherwise we would have either $S_e \subseteq S_{e'}$ or $S_{e'} \subseteq S_e$), we have $\widehat{pre(e_1 \sqcup e_2)} \neq \emptyset$. Thus, by (CC), $\widehat{L(e_1 \sqcup e_2)} \neq \emptyset$. Finally, we also have $\widehat{pre(e_1)} \cap S_{e'} = \emptyset$ and $\widehat{pre(e_2)} \cap S_e = \emptyset$. Therefore, recalling $\widehat{L(e_1 \sqcup e_2)} \subseteq S_{e'} \cap S_e$, we have $\widehat{L(e_1 \sqcup e_2)} \not\subseteq \widehat{pre(e_1 \sqcup e_2)}$, contradicting (SP). Therefore, either $S_e \subseteq S_{e'}$ or $S_{e'} \subseteq S_e$.
3. Suppose $\widehat{pre(e)} \neq \emptyset$ and show that S_e is the smallest element of $\{S \in \$: S \cap \widehat{pre(e)} \neq \emptyset\}$.
 - (a) $S_e \in \{S \in \$: S \cap \widehat{pre(e)} \neq \emptyset\}$: Since $\widehat{pre(e)} \neq \emptyset$, we have, by (CC), that $\widehat{L(e)} \neq \emptyset$. Moreover, as $\widehat{pre(e)} \subseteq \widehat{pre(e)}$, we have $\widehat{L(e)} \subseteq S_e$. Then, by (SP), we obtain that $\widehat{L(e)} \subseteq \widehat{pre(e)} \cap S_e \neq \emptyset$, thus, $S_e \in \{S \in \$: S \cap \widehat{pre(e)} \neq \emptyset\}$.
 - (b) S_e is the smallest element in $\{S \in \$: S \cap \widehat{pre(e)} \neq \emptyset\}$: Let $S' \in \{S \in \$: S \cap \widehat{pre(e)} \neq \emptyset\}$ and assume that $T \in S_e$. The former means that $S' \cap \widehat{pre(e)} \neq \emptyset$. The latter means, by the definition of S_e , that $T \in \widehat{L(e')}$ for some $e' \in \Pi_{\sqcup}^0$ such that $\widehat{pre(e)} \subseteq \widehat{pre(e')}$. By the initial assumption, we have that $S' \cap \widehat{pre(e')} \neq \emptyset$. Then, by Lemma 3.4.17, we have $\widehat{L(e')} \subseteq S'$. Therefore, $T \in S'$. As T has been chosen arbitrarily from S_e , we conclude that $S_e \subseteq S'$.

□

3.4.19. LEMMA. For all $e \in \Pi_{\sqcup}^0$, we have $\widehat{L(e)} = S_e \cap \widehat{pre(e)}$.

Proof:

For (\subseteq) : let $T \in \widehat{L(e)}$. By (SP), we also have $T \in \widehat{pre(e)}$. Moreover, since $\widehat{pre(e)} \subseteq \widehat{pre(e)}$, we have $\widehat{L(e)} \subseteq S_e$, thus, $T \in S_e$. Therefore, $T \in S_e \cap \widehat{pre(e)}$.

For (\supseteq) : let $T \in S_e \cap \widehat{pre(e)}$. Then, by the definition of S_e , we have that $T \in \widehat{pre(e)} \cap \widehat{L(e')}$ for some $e' \in \Pi_{\sqcup}^0$ such that $\widehat{pre(e)} \subseteq \widehat{pre(e')}$. Thus, $\widehat{pre(e; e')} = \widehat{pre(e)}$. Then, by (EC), we have $\widehat{L(e)} = \widehat{L(e; e')}$. Hence, by (Inc), we obtain that

$\widehat{pre(e)} \cap \widehat{L(e')} \subseteq \widehat{L(e)}$, thus, $T \in \widehat{L(e)}$. \square

In the following theorem we prove that the canonical model is well-defined.

3.4.20. THEOREM. $M^c = (X^c, \mathcal{O}_{\sqcup}^c, \leq^c, \|\cdot\|_c)$ is a plausibility learning model.

Proof:

We need to prove:

1. For all $o \in \text{Prop}_{\mathcal{O}}$, $\|o\|_c \in \mathcal{O}_{\sqcup}^c$: follows as clause (1) in the proof of Proposition 3.3.32 with respect to \mathcal{O}_{\sqcup}^c .
2. $(X^c, \mathcal{O}_{\sqcup}^c)$ is a lattice frame: let \mathcal{F} be a finite subset of \mathcal{O}_{\sqcup}^c . We need to show that $\bigcap \mathcal{F} \in \mathcal{O}_{\sqcup}^c$ and $\bigcup \mathcal{F} \in \mathcal{O}_{\sqcup}^c$. The former follows as in clause (2) from the proof of Proposition 3.3.32 with respect to \mathcal{O}_{\sqcup}^c . The latter follows similarly by using \sqcup .
3. \leq^c is a total preorder on X^c such that $\text{Max}_{\leq^c}(\widehat{pre(e)}) \neq \emptyset$ for all $e \in \Pi_{\sqcup}$ with $\widehat{pre(e)} \neq \emptyset$:
 - (a) \leq^c is reflexive: obvious by the definition of \leq^c .
 - (b) \leq^c is transitive: let $T_1, T_2, T_3 \in X^c$ such that $T_1 \leq^c T_2$ and $T_2 \leq^c T_3$. Moreover, let $e \in \Pi_{\sqcup}^{\mathcal{O}}$ with $T_1 \in S_e$. Then, since $T_1 \leq^c T_2$, we have $T_2 \in S_e$. Similarly, this implies, by $T_2 \leq^c T_3$, that $T_3 \in S_e$. We then conclude that $T_1 \leq^c T_3$.
 - (c) \leq^c is total: let $T, T' \in X^c$ and assume, toward contradiction, that (c.1) $T' \not\leq^c T$ and (c.2) $T \not\leq^c T'$. (c.1) means that there is $S \in \mathcal{S}$ such that $T' \in S$ but $T \notin S$. And, similarly, (c.2) means that there is $S' \in \mathcal{S}$ such that $T \in S'$ but $T' \notin S'$. But, by (2) in Lemma 3.4.18, we know that either $S \subseteq S'$ or $S' \subseteq S$. W.l.o.g, suppose that $S \subseteq S'$. Then, $T' \in S$ implies that $T' \in S'$, which contradicts with (c.2). Therefore, we have either $T \leq^c T'$ or $T' \leq^c T$.
 - (d) for all $e \in \Pi_{\sqcup}^{\mathcal{O}}$ such that $\widehat{pre(e)} \neq \emptyset$, $\text{Max}_{\leq^c}(\widehat{pre(e)}) \neq \emptyset$:
We first show that $\text{Max}_{\leq^c}(\widehat{pre(e)}) = S_e \cap \widehat{pre(e)}$. Then, by (3) in Lemma 3.4.18, we conclude that $\text{Max}_{\leq^c}(\widehat{pre(e)}) \neq \emptyset$.
For (\subseteq): let $T \in \text{Max}_{\leq^c}(\widehat{pre(e)})$. This means that $T' \leq^c T$ for all $T' \in \widehat{pre(e)}$. We already know that $T \in \widehat{pre(e)}$. Suppose, toward contradiction, that $T \notin S_e$. As $S_e \cap \widehat{pre(e)} \neq \emptyset$ (see the proof of (3) in Lemma 3.4.18), there is $T' \in S_e \cap \widehat{pre(e)}$. Since $T \neq T'$ and \leq^c is a total order, we have either $T <^c T'$ or $T' <^c T$. But the latter cannot be the case, since $S_e \in \mathcal{S}$, $T' \in S_e$, and $T \notin S_e$ and the former contradicts with $T \in \text{Max}_{\leq^c}(\widehat{pre(e)})$. So, we obtain that $T \in S_e$.

For (\supseteq) : let $T \in S_e \cap \widehat{pre(e)}$, $T' \in \widehat{pre(e)}$ and $S \in \$$ such that $T' \in S$. Since S_e is the smallest element of $\$$, we have that $S_e \subseteq S$. Therefore, $T \in S$. Thus, by the definition of \leq^c , we obtain that $T' \leq^c T$. Since T' has been chosen arbitrarily from $\widehat{pre(e)}$, we conclude that $T \in \text{Max}_{\leq^c}(\widehat{pre(e)})$.

□

The Truth Lemma corresponding to **AGML** will mostly follow as the Truth Lemma with respect to **DLT** in Section 3.3.6 (Lemma 3.3.34). The only difference is with formulas of the form $L(e)$ since its semantic definition is not the same as the one for **DLT**. For proving that particular case, the following lemma will be useful.

3.4.21. LEMMA. For all $e \in \Pi_{\sqcup}^O$, $\text{Max}_{\leq^c}(\widehat{pre(e)}) = \widehat{L(e)}$.

Proof:

See the proofs of Theorem 3.4.20 item 3.d and Lemma 3.4.19. □

3.4.22. LEMMA (TRUTH LEMMA). Let $M^c = (X^c, \mathcal{O}_{\sqcup}^c, \leq^c, \|\cdot\|_c)$ be the canonical model for some T_0 . For all formulas $\varphi \in \mathcal{L}_{\Pi_{\sqcup}^O}^O$, all $T \in X^c$ and all $e \in \Pi_{\sqcup}^O$, we have:

$$\langle e \rangle \varphi \in T \text{ iff } (T, \widehat{pre(e)}) \models_{M^c} \varphi.$$

Proof:

The proof is by induction on the structure of φ and uses the following *induction hypothesis*:

(IH): for all ψ subformula of φ , and $e \in \Pi_{\sqcup}^O$, $\langle e \rangle \psi \in T$ iff $(T, \widehat{pre(e)}) \models_{M^c} \psi$.

The base case for propositional and observational variables, as well as Boolean formulas are straightforward. We only verify the inductive case $\varphi := L(e')$ since the rest of the inductive cases are as in the proof of Lemma 3.3.34.

Observe that at this point of the proof we have that: $\forall e, e' \in \Pi_{\sqcup}^O$, $\langle e \rangle \widehat{pre(e')} = \llbracket \widehat{pre(e')} \rrbracket_{M^c}^{\widehat{pre(e)}}$ since $pre(e)$ is a Boolean formula.

Case $\varphi := L(e')$. We have the following sequence of equivalences:

$$\begin{aligned}
\langle e \rangle L(e') \in T &\text{ iff } (pre(e) \wedge [e]L(e')) \in T && \text{((2) in Proposition 3.3.10)} \\
&\text{ iff } (pre(e) \wedge L(e; e')) \in T && \text{(R}_L \text{ and CPL)} \\
&\text{ iff } pre(e) \in T \text{ and } L(e; e') \in T \\
&\text{ iff } T \in \widehat{pre(e)} \text{ and } T \in \widehat{L(e; e')} && \text{(Definition of } \widehat{\varphi}\text{)} \\
&\text{ iff } T \in \widehat{pre(e)} \text{ and } T \in Max_{\leq^c}(\widehat{pre(e; e')}) && \text{(Lemma 3.4.21)} \\
&\text{ iff } T \in \widehat{pre(e)} \text{ and } T \in Max_{\leq^c}(\widehat{\langle e \rangle pre(e')}) \\
&&& \text{((4) in Proposition 3.3.10)} \\
&\text{ iff } T \in \widehat{pre(e)} \text{ and } T \in Max_{\leq^c}(\llbracket pre(e') \rrbracket_{M^c}^{\widehat{pre(e)}}) \\
&&& \text{(by IH with the above observation)} \\
&\text{ iff } (T, \widehat{pre(e)}) \models_{M^c} L(e') \\
&&& \text{(Lemma 3.4.9 and semantic definition (Definition 3.4.7))}
\end{aligned}$$

□

3.4.23. THEOREM (COMPLETENESS OF AGML). *AGML is complete with respect to the class of plausibility learning models.*

Proof:

Let φ be an **AGML**-consistent formula, i.e., it is an O_φ -theory. Then, by Lemma 3.3.28, it can be extended to some maximal O -witnessed theory T . Then, we have $\langle !T \rangle \varphi \in T$ i.e., $T \in \widehat{\langle !T \rangle \varphi}$ (by (6) in Proposition 3.3.10). Then, by Truth Lemma (Lemma 3.4.22), we obtain that $(T, \widehat{pre(!T)}) \models_{M^c} \varphi$, where $M^c = (X^c, \mathcal{O}_\cup^c, \leq^c, \|\cdot\|_c)$ is the canonical model for T . This proves completeness. □

3.5 Conclusions and Open Questions

In this chapter we proposed two dynamic logics for learning from observations that allow us to reason about inductive inference. The first one with an unrestricted learner, in the style of Formal Learning Theory, who produces conjectures from fully determined observations. The second one with an AGM learner (a fully rational learner) who produces conjectures from partial information.

Our first framework, Dynamic Logic for Learning Theory (DLLT) (Sections 3.3.1 - 3.3.5), is an extension of previously studied Subset Space Logics. It is a natural continuation of the work bridging Dynamic Epistemic Logic and Formal Learning Theory. The syntax, with a topological semantics, features an S5

knowledge operator, the effort modality, dynamic observation operators in a PDL-format as “observational events” and a learning operator. The dynamic observation operators encode the fully determined observational events taking place and the learning operator encodes the learner’s conjecture after an observational event occurs. We give a sound and complete axiomatisation for this logic. We showed how natural learnability properties, such as learnability with certainty and learnability in the limit, can be expressed in DLLT.

Our technical results with respect to our DLLT system (the expressivity results and the complete axiomatisation), as well as the methods used to prove them (the reduction laws and the canonical neighbourhood model), may look deceptively simple. But in fact, achieving this simplicity is one of the major contributions of the work in this chapter. The most well-known relative to DLLT is *Subset Space Logic (SSL) over intersection spaces*, completely axiomatized by Weiss and Parikh (2002) (and, indeed, our operator \Box originates in the effort modality of the SSL formalism introduced in (Moss and Parikh, 1992; Dabrowski et al., 1996)). Although less expressive than our logic (since it has no notion of belief B or conjecture L), the Weiss-Parikh axiomatisation of SSL over intersection spaces is in a sense more complex and less transparent (such as is their completeness proof, which is non-canonical). That axiomatisation consists of the following list:

S5_K	The S5 axioms and rules for K
S4_□	The S4 axioms and rules for \Box
Cross Axiom	$K\Box\varphi \rightarrow \Box K\varphi$
Weak Directedness	$\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$
M_n (for all n)	$(\Box\langle K \rangle\varphi \wedge \Diamond K\psi_1 \wedge \dots \wedge \Diamond K\psi_n) \rightarrow \langle K \rangle(\Diamond\varphi \wedge \Diamond K\psi_1 \wedge \dots \wedge \Diamond K\psi_n)$

Although this list looks shorter than our list in Table 3.1, each of our axioms is simple and readable and has a transparent intuitive interpretation. In contrast, note the complexity and opaqueness of the last axiom schemata M_n above (having one schema for each natural number n). Our completeness result implies that all these complex validities are provable in our simple system (and in fact in the even simpler system that omits all the axioms that refer to the learner L). This shows the usefulness of adding the (expressively redundant) dynamic observation modalities: they help to describe the behaviour of the effort modality \Box in a much simpler and natural manner, by combining $(\Box Ax)$ and $(\Box Ru)$ (which together capture the meaning of \Box as universally quantifying over observation modalities).

Moreover, our completeness proof is also much simpler (though with some technical twists). Traditionally, the use of canonical models has been considered

impossible for Subset Space Logics, and so authors had to use other, more *ad-hoc* methods (e.g., step-by-step constructions). The fact that in the work presented here we can get away with a canonical construction is again due to the addition of the dynamic modalities.

For our second framework, the Logic of AGM Learning from partial observations (AGML) (Section 3.4), we enriched the logic DLLT with additional structure in order to model learners whose conjectures satisfy standard rationality constraints (namely, the AGM postulates for belief revision). The standard model for such learners is provided by “AGM conditioning”: learners are endowed with a total preorder, describing their prior plausibility relation, and at each step they believe the set of most plausible states compatible with all the previous observations. To axiomatize our proposed logic of AGM Learning, we needed to assume that the learner has access to a wider range of potential information than in DLLT: not only sequences of fully determined observations, but also *partial* observations (representing imprecise information captured by finite unions of observations). Semantically, this required a technical shift from intersection spaces to lattice spaces. On the syntactic side, we needed to extend our dynamic modalities from observations to partial observations (still in the PDL-format), adding epistemic non-determinism. This led to a rich evidential setting, with a more interesting logic and an elegant axiomatization.

On the technical results concerning our logic AGML, note that our move to partial observations (and so to partial observational events) still requires *less* information than the classical axiomatizations of AGM conditioning in the literature (which assumed full Boolean closure of the set of “conditions”, i.e. the observable sets formed a Boolean algebra). Still, while this move to partial observations seems general enough, as well as natural and desirable in itself, it does require a much wider access to information than the setting in DLLT. So it is fair to ask the question: is there a way to axiomatize AGM learners *without* requiring them to access partial information? In other words, is AGM conditioning over intersection spaces axiomatizable in a simple, elegant way (similar to our axiomatization)? This problem is still open, though we conjecture that the answer is *no*. If we are right, this would be an argument for a deeper philosophical point: it may be that AGM postulates are best suited to “rich” evidential settings, in which both fully determined and partial observations are available.

The closest relative to AGML, in the subset space logics approach, is the so-called *Topologic* (Moss and Parikh, 1992). Topologic consists of $S5_K$, $S4_{\square}$, the Cross Axiom, Weak Directedness, and the Union Axiom,

$$\diamond\varphi \wedge \langle K \rangle \diamond\psi \rightarrow \diamond(\diamond\varphi \wedge \langle K \rangle \diamond\psi \wedge K \diamond \langle K \rangle (\varphi \wedge \psi)).$$

Dabrowski et al. (1996) showed that the Union Axiom is sound for lattice spaces. Independently, Dabrowski et al. (1996) and Georgatos (1993, 1994) proved that Topologic is complete for topological spaces and decidable. Moreover, it was

proved to be complete for complete lattice spaces (closed under infinitary intersections and unions). However, their completeness proofs involve somewhat complicated constructions. Once again, note the opaqueness in the Union Axiom compared to our simple and readable list of axioms for AGML. As for DLLT, the simplicity in our axiomatic system and in our canonical completeness proof is due to the presence of the dynamic modalities in our language. In fact, adding dynamic modalities $[\varphi]$ for arbitrary formulas to SSL (with respect to the semantics given in Bjorndahl (2018)) was studied in (Baltag et al., 2017), which results in a *Dynamic Topologic* with a canonical behaviour.

Other recent work involving SSL and closely related to our contribution, are (Bjorndahl, 2018) and (van Ditmarsch et al., 2014, 2015b). Bjorndahl (2018) introduces dynamic modalities $[\varphi]$ for arbitrary formulas (rather than restricting to observational variables $[o]$, as we do), though with a different semantics (according to which $[\varphi]$ restricts the space to the interior of φ , in contrast to our simpler semantics, that follows the standard definition of update or “public announcement”). His syntax does *not* contain the effort modality, or any other form of quantifying over observations.

The work of (van Ditmarsch et al., 2014, 2015b) uses Bjorndahl-style dynamic modalities in combination with a topological version of the so-called arbitrary public announcement operator, which is a more syntactic-driven relative of the effort modality. This syntactic nature comes with a price: the logic of arbitrary public announcements is much less well-behaved than SSL (or our logic), in particular it has non-compositional features (the meaning of a formula may depend on the meaning of *all* atomic variables, including the ones that do not occur in that formula). As a consequence, the soundness of (the arbitrary-public announcement analogue of) our $(\Box Ru)$ is not at all obvious for their logic, which instead relies on an infinitary inference rule. Since that rule makes use of infinitely many premisses, their complete axiomatization is truly infinitary, and impossible to automatise: indeed, it does not even necessarily imply that the set of their validities is recursively enumerable (in contrast with our finitary axiomatisation, which immediately implies such a result). The problems in (van Ditmarsch et al., 2014, 2015b) are solved in (Baltag et al., 2017) by replacing the arbitrary announcement modality with the effort modality (or equivalently, extending SSL with Bjorndahl-style dynamic modalities). In fact, similar problems with respect to the much older Kripke style *Arbitrary Public Announcement Logic* (APAL) in (Balbiani et al., 2008), are solved in the chapter that follows (Chapter 4).

Note that in contrast to the work presented here, *all the above papers are concerned mostly with axiomatisations over topological spaces* (rather than the wider class of intersection spaces or lattice spaces), and that *none of them has any belief B or conjecture operators L* . Hence, none of them can be used to capture any learning-theoretic notions going beyond learning with certainty.

Chapter 4

Arbitrary Public Announcement Logic with Memory

4.1 Introduction

Since very early in life, while growing up, and through our daily experiences, we all *learn by public announcements*. When our parents or the adults taking care of us become our first teachers, we learn basic facts about ourselves and about the world we are growing in. The way we learn from announcements went to the next level during our first years of schooling, by interacting and exchanging information with our lecturer and our classmates. As a matter of fact, a lot of the knowledge we obtain, comes from public truthful information. It would be very difficult if each of us could *have to rediscover* by ourselves that the Earth rotates around the sun or that we have billions of bacterias and micro-organisms living in our body. Unless we are a genius child, we should have obtain this information from a trustworthy source. Even simpler facts such as “Today is sunny in Amsterdam” can be learn by anyone living in Rotterdam by checking a trustworthy public weather-app.

Not surprisingly, scenarios of learning from (public) announcements that involve multiple learners are more complex than single-learner ones. These (multi-learners) situations happen everyday in every stage of human life. It is then interesting to question what and how agents can learn from trustworthy public announcements or public communication with other agents. In other words, we can question what are the changes brought about in the learners’ knowledge and beliefs from acquiring completely trustworthy, truthful information (from one or more agents). Public Announcement Logic (PAL) is a modal logic used to reason precisely about these questions¹, first introduced and studied by Plaza (1989, 2007). Various interesting epistemic puzzles and examples have been analysed

¹Recall from Chapter 2, Section 2.1.1 that the language of PAL extends the language of EL with formulas of the form $[\theta]\varphi$ that express “after θ is publicly announced, φ holds”.

(and solved) using PAL. One of the most famous ones, with a surprising solution, is the *Muddy Children Puzzle*. Its solution reveals that knowledge can be acquired also from simultaneous and repetitive public announcements of ignorance, e.g., “I don’t know if I am muddy”². The Muddy Children Puzzle is a clear example showcasing the “power” of truthful communication to acquire knowledge.

Arbitrary Public Announcement Logic (APAL) and its relatives are natural extensions of Public Announcement Logic, involving the addition of operators $\blacksquare\varphi$ and $\blacklozenge\varphi$, quantifying over public announcements, $[\theta]\varphi$, of some given type. APAL and PAL are of great interest both philosophically and from the point of view of applications. Motivations range from supporting an analysis of Fitch’s paradox by modelling notions of “knowability” (van Benthem, 2004), i.e., what a learner can come to know by getting new information (expressible as $\blacklozenge K\varphi$), to determining the existence of communication protocols that achieve certain goals (cf. the famous Russian Card problem given at a mathematical Olympiad, see van Ditmarsch, 2003), and more generally, to epistemic planning (Bolander and Andersen, 2011). These two frameworks also provide technical tools for a wide range of conceptual variations that involve quantification over public information (not necessarily announcements). Some of these variations involve *quantifying over observational updates* in the single-agent logics for inductive learnability in empirical sciences (as in Chapter 3, (Baltag et al., 2018a, 2020)).

Many extensions have been investigated, starting with the original APAL (Balbiani et al., 2008), and continuing with its variants GAL (Group Announcement Logic) (Ågotnes et al., 2010), Future Event Logic (van Ditmarsch et al., 2010), FAPAL (Fully Arbitrary Public Announcement Logic) (van Ditmarsch et al., 2016), APAL⁺ (Positive Arbitrary Announcement Logic) (van Ditmarsch et al., 2018), BAPAL (Boolean Arbitrary Public Announcement Logic) (van Ditmarsch and French, 2017), etc. One problem with the above formalisms, with the exception of BAPAL, is that they all use *infinitary* axiomatizations. In the seminal paper on APAL, Balbiani et al. (2008) proved completeness using an infinitary rule and claimed that the rule can be replaced by a proposed finitary rule in theorem proving³. Many of APAL variants adopted a similar strategy by replacing the infinitary rule with a similar version of the finitary one proposed for APAL. However, the soundness of the finitary rule was later disproved by Kuijjer (2015). It is therefore not guaranteed that the validities of these logics are recursively enumerable.

On the other hand, BAPAL is a very weak version of APAL, allowing $\blacksquare\varphi$ to quantify over only purely propositional announcements. APAL⁺ is known to be decidable, hence its validities must be r.e., but no recursive axiomatization is known. Even with a larger formula quantification range (over \blacksquare) than in BA-

²For a three-child version of the puzzle see e.g. Fagin et al., 1995.

³This means that from any proof of a theorem from the axioms that uses the infinitary rule we can obtain a finitary proof of the same theorem, by using the finitary rule instead.

PAL, APAL⁺ is still very weak, in that it quantifies only over positive epistemic announcements. Thus, in APAL⁺ public announcements of ignorance are not allowed, which are precisely the ones driving the solution process in puzzles such as the Muddy Children. Thus, a long-standing open question concerns *finding a “strong” version of APAL for which there exists a recursive axiomatization*. Here, by “strong” version we mean one that allows quantification over a sufficiently wide range of announcements (sufficiently wide to avoid Liar-like circles) as intended by a similar restriction in the original APAL. Such a question for APAL renders a similar open question for GAL, namely finding a “strong” version of GAL for which there exists a recursive axiomatization.

In this chapter, we solve these open questions and focus primarily on the problem concerning APAL. The framework for the “strong” version of GAL will be developed analogously, as an extension of the one for APAL. Due to the similar syntactic and semantic behaviours of the group announcement ($[G]$, representing *what a group of agents can bring about via simultaneous public announcements*) and arbitrary announcement (\blacksquare) operators, most of our analysis of the latter also applies to the former. We introduce Arbitrary Public Announcement Logic with Memory (APALM), obtained by adding to the epistemic models (which are the intended models in the original APAL) a “memory” of the initial states, representing the information before any communication took place, and adding to the syntax operators that can access this memory. We show that APALM is recursively axiomatizable, providing a sound and complete finitary Hilbert-style system (in contrast to the original Arbitrary Public Announcement Logic, for which the corresponding question is still open).

Outline

This chapter is organized as follows. In Section 4.2, we introduce the problem with the unsound finitary rule proposed for the original APAL (and for GAL). We describe our strategy for solving the *unsoundness issue* for a strong version of APAL. We start with the formalization of our framework APALM in Section 4.3, introducing the syntax and semantics in Section 4.3.1. In Section 4.3.2, we discuss in detail Kuijer’s counterexample (Kuijer, 2015) for the soundness of the aforementioned finitary rule. In Sections 4.3.3 and 4.3.4, we prove some expressivity results comparing fragments of the language of APALM and we define the appropriate notion of bisimulation for our logic. We present a sound and complete finitary axiomatization in Section 4.3.5. In Section 4.4 we present the syntax, semantics, and axiomatization of our Group Announcement Logic with Memory (GALM). In Section 4.5 we prove soundness and in Section 4.6 we prove completeness, for both GALM and APALM. Section 4.7 contains some concluding remarks and ideas for future work.

This chapter is based on (Baltag et al., 2018b).

4.2 The “issue” in APAL and our solution

The seminal paper on APAL (Balbiani et al., 2008) proved completeness using an infinitary rule formalized as follows,

$$\text{from } [s][\rho]\varphi \text{ for all } \rho \in \mathcal{L}_{\text{epis}}, \text{ infer } [s]\blacksquare\varphi,$$

where $[s]$ is a pseudo-modality, a function mapping any formula in the language of APAL into a new formula in the language.⁴

The authors went on to claim that in theorem-proving the infinitary rule can be replaced by the following finitary rule:

$$\text{from } \chi \rightarrow [\theta][p]\varphi, \text{ infer } \chi \rightarrow [\theta]\blacksquare\varphi,$$

where the propositional variable p is “fresh”. The “freshness” of the variable $p \in P$ in the rule ensures that it represents any generic announcement. A similar method is adopted in the completeness proof of GAL in (Ågotnes et al., 2010) and it was claimed that a similar infinitary rule used in the completeness proof could be replaced by the finitary rule:

$$\text{from } \chi \rightarrow [\theta][\bigwedge_{i \in G} K_i p_i]\psi, \text{ infer } \chi \rightarrow [\theta][G]\psi,$$

where p_i 's are “fresh” and $[G]$ is the group announcement operator. These are natural \blacksquare and $[G]$ -introduction rules, similar to the introduction rule for the universal quantifier in First Order Logic (FOL) (and similar to the introduction rule (\square Ru) for the effort modality presented in Chapter 3), and they are based on the intuition that variables that do not occur in a formula are irrelevant for its truth value, and thus can be taken to stand for any arbitrary formula (via some appropriate change of valuation). However, the soundness of the \blacksquare -introduction rule was later disproved via a counterexample by Kuijjer (2015). Moreover, as we will observe later on in this chapter, a slightly modified version of Kuijjer's counterexample also proves that the aforementioned $[G]$ -introduction rule is unsound.

Our diagnosis of Kuijjer's counterexample in short (we provide a full analysis later in Section 4.3.2) is that it makes an essential use of a known undesirable feature of PAL and APAL, namely their *lack of memory*: the updated models “forget” the initial states. As a consequence, the expressivity of the APAL \blacksquare -modality reduces after any update. This is what invalidates the above rule.

Our strategy for solving the issue. We fix the problem mentioned above by adding to the models a memory of the *initial* epistemic situation W^0 , representing the information *before any non-trivial communication took place* (“the prior”). Since communication – gaining more information – deletes possibilities, the set W of currently possible states is a (possibly proper) subset of the set

⁴Such pseudo-modalities are called *necessity forms* and they are defined in Definition 4.6.1

W^0 of initial states. On the syntactic side, we add an operator φ^0 saying that “ φ was initially the case” (before all communication). To mark the initial states, we also need a constant 0, stating that “no non-trivial communication has taken place yet”. Therefore, 0 will be true only in the initial epistemic situation. It is convenient, though maybe not absolutely necessary, to add a universal modality $U\varphi$ that quantifies over all currently possible states.⁵

In the resulting Arbitrary Public Announcement Logic with Memory (APALM), the arbitrary announcement operator \blacksquare quantifies over updates (not only of epistemic formulas but) of arbitrary formulas that do not contain the operator \blacksquare itself. This restriction is necessary to produce a well-defined semantics that avoids Liar-like vicious circles. In standard APAL, the restriction is with respect to inductive construct $\blacklozenge\varphi$. Thus, formulas of the form $\langle\blacklozenge p\rangle\varphi$ are allowed in original APAL. The expressive powers of APAL and APALM seem to be incomparable, and that would still be the case if we dropped the above restriction. As a result, the range of \blacksquare is wider than in standard APAL, covering announcements that may refer to the initial situation (by the use of the operators 0 and φ^0) or to all currently possible states (by the use of $U\varphi$).

We show that the original finitary rule proposed in (Balbiani et al., 2008) is sound for APALM and, moreover, it forms the basis of a *complete recursive axiomatization*.⁶ Besides its technical advantages, APALM is valuable in its own respect. Maintaining a record of the initial situation in our models helps us to formalize updates that refer to the “epistemic past” such as “what you said, I knew already” (van Benthem, 2002). This may be useful in treating certain epistemic puzzles involving reference to past information states, e.g. “What you said did not surprise me” (McCarthy, 1990). The more recent *Cheryl’s Birthday* problem also contains an announcement of the form “At first I didn’t know when Cheryl’s birthday is, but now I know” (although in this particular puzzle the past-tense announcement is redundant and plays no role in the solution).⁷ See (van Benthem, 2002) for more examples.

Note though that the “memory” of APALM is *very limited*: our models do not remember the whole history of communication, but only the initial epistemic situation (before any communication). Correspondingly, in the syntax we do *not* have a “yesterday” operator $Y\varphi$, referring to the previous state just before the last announcement as in (Renne et al., 2009), but only the operator φ^0 refer-

⁵From an epistemic point of view, it would be more natural to replace U by an operator Ck that describes current common knowledge and quantifies only over currently possible states that are accessible by epistemic chains from the actual state. We chose to stick with U for simplicity and leave the addition of Ck to APAL for future work.

⁶We use a slightly different version of this rule, which is easily seen to be equivalent to the original version in the presence of the usual PAL reduction axioms.

⁷Cheryl’s Birthday problem was part of the 2015 Singapore and Asian Schools Math Olympiad, and became viral after it was posted on Facebook by Singapore TV presenter Kenneth Kong.

ring to the initial state. We think of this “economy” of memory as a (*positive*) “*feature, not a bug*” of our logic: a detailed record of *all* history is simply not necessary for solving the problem at hand. In fact, keeping all the history and adding a $Y\varphi$ operator would greatly complicate our task by invalidating some of the standard nice properties of PAL and APAL. For instance, the standard Composition Axiom (stating that any sequence of announcements is equivalent to a single announcement) fails in the presence of the Y operator. As a consequence, a logic with full memory of all history would lose some of the appealing features of the APAL operator (e.g. its **S4** character: $\blacksquare\varphi \rightarrow \blacksquare\blacksquare\varphi$). Moreover, this would force us to distinguish between “knowability via one communication step” $\blacklozenge\varphi$ versus “knowability via a finite communication sequence” $\blacklozenge^*\varphi$, leading to an unnecessarily complex logic.

So we opt for simplicity, enriching the models and language with just enough memory to recover the full expressivity of \blacksquare after updates, and thus establish the soundness of the \blacksquare -introduction rule. Such a limited-memory semantics is sufficient for our purposes, but it also has an intrinsic naturality and simplicity, similar to the one encountered in some Bayesian models, with their distinction between “prior” and “posterior” (aka current) probabilities.⁸

Having established the desired results for APALM, we also study a version of GAL with the same memory mechanism – Group Announcement Logic with Memory (GALM) – obtained by extending APALM with group announcement operators. In this logic, the group announcement operators $[G]\varphi$ quantify over updates with formulas of the form $\bigwedge_{i \in G} K_i\varphi_i$, thus, represents *what a group of agents can bring about via simultaneous public announcements*. These updates can have occurrences of every component of the language but \blacksquare and $[G]$ for the same reason explained before for APALM. We then show, following the same steps as for APALM, that the original finitary $[G]$ -introduction rule proposed in (Ågotnes et al., 2010) is sound for GALM.⁹ By using this rule, we provide a complete *finitary* axiomatization for GALM, thus, prove that it is recursively axiomatizable.

On the technical side, our completeness proof involves an essential detour into an alternative semantics for APALM and GALM (pseudo-models), in the style of Subset Space Logics (SSL) (Moss and Parikh, 1992; Dabrowski et al., 1996). This reveals deep connections between apparently very different formalisms. Moreover,

⁸In such models, only the “prior” and the “posterior” information states are taken to be relevant, while all the intermediary steps are forgotten. As a consequence, all the evidence gathered in between the initial and the current state can be compressed into one set E , called “the evidence” (rather than keeping a growing tail-sequence of all past evidence sets). Similarly, in our logic, all the past communication is compressed in its end-result, namely in the set W of current possibilities, which plays the same role as the evidence set E in Bayesian models.

⁹We again use a slightly different version of this rule, which can easily be proven to be equivalent to the original version in the presence of the PAL reduction axioms. This choice is clearly cosmetic and made in order to simplify the soundness and completeness proofs.

this alternative semantics is of independent interest, giving us a more general setting for modeling knowability and learnability as acknowledge in Chapter 3 (see also, e.g., [Bjorndahl, 2018; Bjorndahl and Özgün, 2017; Bjorndahl and Özgün, 2019]). Various combinations of PAL or APAL with subset space semantics have been investigated in the literature ([Balbiani et al., 2013; Wang and Ågotnes, 2013b,a; van Ditmarsch et al., 2014, 2015b; Bjorndahl, 2018; Baltag et al., 2017]), including a version of SSL with backward looking public announcement operators that refer to what was true *before* a public announcement ([Balbiani et al., 2016]).

Following the SSL-style, our pseudo-models come with a given family of *admissible sets* of worlds, which in our context represent “publicly announceable” (or communicable) propositions.¹⁰ We interpret \blacksquare in pseudo-models as the so-called “effort” modality of SSL, discussed in Chapter 3, denoted as \square . The effort modality quantifies over updates with announceable propositions regardless of whether they are syntactically definable or not. The modality $[G]$ on the other hand quantifies over updates with those announceable propositions that are known by some agents in G . The operator $[G]$ is thus modelled as a restricted version of the effort modality. The finitary \blacksquare -introduction rule is obviously *sound for the effort modality*, because of its more “semantic” character. Similarly, the finitary $[G]$ -introduction rule is also sound for this effort-like group announcement operator $[G]$. These observations, together with the important fact that our models for APALM and GALM (unlike original APAL models) can be seen as a special case of pseudo-models, lie at the core of our soundness and completeness proofs.

4.3 Arbitrary Public Announcement Logic with Memory

In this section we present our Arbitrary Public Announcement Logic with Memory.

4.3.1 Syntax and Semantics of APALM

Let Prop be a countable set of propositional variables and $\mathcal{AG} = \{1, \dots, n\}$ be a finite set of agents. The language \mathcal{L} of APALM (Arbitrary Public Announcement Logic with Memory) is recursively defined by the grammar:

$$\varphi ::= \top \mid p \mid 0 \mid \varphi^0 \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid U\varphi \mid \langle\theta\rangle\varphi \mid \blacklozenge\varphi,$$

where $p \in \text{Prop}$, $i \in \mathcal{AG}$, and $\theta \in \mathcal{L}_{-\blacklozenge}$ is a formula in the sublanguage $\mathcal{L}_{-\blacklozenge}$ obtained from \mathcal{L} by removing the \blacklozenge operator. Given a formula $\varphi \in \mathcal{L}$, we denote by P_φ the set of all propositional variables occurring in φ . We employ the

¹⁰In SSL, the set of admissible sets is sometimes, but not always, taken to be a topology. Here, it will be a Boolean algebra with epistemic operators.

usual abbreviations for \perp and the propositional connectives $\vee, \rightarrow, \leftrightarrow$. The dual modalities are defined as $\hat{K}_i\varphi := \neg K_i\neg\varphi$, $E\varphi := \neg U\neg\varphi$, $\blacksquare\varphi := \neg\blacklozenge\neg\varphi$, and $\langle\theta\rangle\varphi := \neg\langle\theta\rangle\neg\varphi$.^[11]

We read $K_i\varphi$ as “ φ is *known* by agent i ”; $\langle\theta\rangle\varphi$ as “ θ can be *truthfully announced*, and *after* this public announcement φ is true”. U and E are, respectively, the universal and existential modalities quantifying over all *current* possibilities: $U\varphi$ says that “ φ is true in *all current alternatives* of the actual state”. $\blacklozenge\varphi$ and $\blacksquare\varphi$ are the (existential and universal) arbitrary announcement operators, quantifying over updates with formulas in $\mathcal{L}_{-\blacklozenge}$. We can read $\blacksquare\varphi$ as “ φ is *stably true* (under public announcements)”, i.e., φ stays true no matter what (true) announcements are made. The constant 0 , meaning that “*no (non-trivial) announcements* took place yet”, holds only at the *initial time*. Similarly, the formula φ^0 means that “*initially* (prior to all communication), φ was true”.

4.3.1. DEFINITION. [Model, Initial Model, and Relativized Model]

- A *model* is a tuple $M = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$, where $W \subseteq W^0$ are non-empty sets of states, $\sim_i \subseteq W^0 \times W^0$ are equivalence relations labeled by “agents” $i \in \mathcal{AG}$, and $\|\cdot\| : \text{Prop} \rightarrow \mathcal{P}(W^0)$ is a valuation function that maps every propositional variable $p \in \text{Prop}$ to a set of states $\|p\| \subseteq W^0$. W^0 is the *initial domain*, representing the initial informational situation before any communication took place; its elements are called *initial states*. In contrast, W is the *current domain*, representing the current informational situation, and its elements are called *current states*.
- For every model $M = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$, we define its *initial model* $M^0 = (W^0, W^0, \sim_1, \dots, \sim_n, \|\cdot\|)$, whose both current and initial domains are the initial domain of the original model M .
- Given a model $M = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$ and a set $A \subseteq W$, we define the *relativized model* as $M|A = (W^0, A, \sim_1, \dots, \sim_n, \|\cdot\|)$.

For states $w \in W$ and agents i , we will use the notation $w_i := \{s \in W : w \sim_i s\}$ to denote the restriction to W of w ’s equivalence class modulo \sim_i .

4.3.2. DEFINITION. [Semantics] Given a model $M = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$, we recursively define a *truth set* $\llbracket\varphi\rrbracket_M$ for every formula $\varphi \in \mathcal{L}$ as follows (we skip the subscript and simply write $\llbracket\varphi\rrbracket$ when the current model M is understood):

¹¹The update operator $\langle\theta\rangle\varphi$ is often denoted by $\langle!\theta\rangle\varphi$ in Public Announcement Logic literature; we skip the exclamation sign, but we will use the notation $\langle!\rangle$ for this modality and $\llbracket!\rrbracket$ for its dual when we do not want to specify the announcement formula θ (so that $!$ functions as a placeholder for the content of the announcement).

$$\begin{aligned}
\llbracket \top \rrbracket &= W & \llbracket \varphi \wedge \psi \rrbracket &= \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket \\
\llbracket p \rrbracket &= \llbracket p \rrbracket \cap W & \llbracket K_i \varphi \rrbracket &= \{w \in W : w_i \subseteq \llbracket \varphi \rrbracket\} \\
\llbracket 0 \rrbracket &= \begin{cases} W^0 & \text{if } W = W^0 \\ \emptyset & \text{otherwise} \end{cases} & \llbracket U\varphi \rrbracket &= \begin{cases} W & \text{if } \llbracket \varphi \rrbracket = W \\ \emptyset & \text{otherwise} \end{cases} \\
\llbracket \varphi^0 \rrbracket &= \llbracket \varphi \rrbracket_{M^0} \cap W & \llbracket \langle \theta \rangle \varphi \rrbracket &= \llbracket \varphi \rrbracket_{M \llbracket \theta \rrbracket} \\
\llbracket \neg \varphi \rrbracket &= W - \llbracket \varphi \rrbracket & \llbracket \blacklozenge \varphi \rrbracket &= \bigcup \{ \llbracket \langle \theta \rangle \varphi \rrbracket : \theta \in \mathcal{L}_{-\blacklozenge} \}
\end{aligned}$$

1. OBSERVATION. Note that we have

$$w \in \llbracket \blacksquare \varphi \rrbracket \text{ iff } w \in \llbracket \langle \theta \rangle \varphi \rrbracket \text{ for every } \theta \in \mathcal{L}_{-\blacklozenge}.$$

What we study in this chapter is information update via public announcements. But the models given in Definition 4.3.1 are too general for this purpose: their current domain W can be *any* subset of the initial domain W^0 . Our intended models (which we call “announcement models”) will thus be a subclass of these models, in which the current domain comes from updating the initial domain with some public announcement.

4.3.3. DEFINITION. [Announcement Models and Validity] An *announcement model* (or *a-model*, for short) is a model $M = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$ such that $W = \llbracket \theta \rrbracket_{M^0}$ for some $\theta \in \mathcal{L}_{-\blacklozenge}$; i.e., M can be obtained by updating its initial model M^0 with some formula in $\mathcal{L}_{-\blacklozenge}$. A formula φ is *APALM valid* (or *valid*, for short) if it is true in every current state $s \in W$ (i.e. $\llbracket \varphi \rrbracket_M = W$) of every announcement model $M = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$. We sometimes write $M \models \varphi$ when $\llbracket \varphi \rrbracket_M = W$ and also $\models \varphi$ when the formula φ is valid.

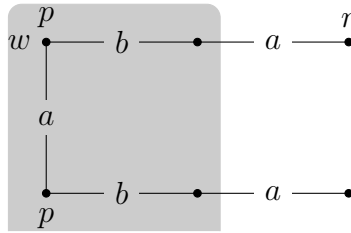


Figure 4.1: An *a-model* M . Initial states are represented by all nodes in the graph, current states are the nodes in shaded areas. Valuation is given by labeling each node with the true atoms, and epistemic relations are represented by arrows with agent names. Reflexive and transitive arrows are omitted.

4.3.4. EXAMPLE. Consider the a -model $M = (W^0, W, \sim_a, \sim_b, \|\cdot\|)$ given in Figure 4.1, where the initial states include all the nodes of the graph and the current states are the nodes in the shaded area. It is easy to see that the current domain W is obtained by updating the initial domain by $\hat{K}_b p$: the shaded area corresponds to $\llbracket \hat{K}_b p \rrbracket_{M^0}$. The representation in Figure 4.1 makes it clear that the a -model does not lose the initial domain and specifies the current domain as a subset of the initial one. Since $W^0 \neq W$ (the shaded area does not cover the whole initial domain), 0 is false everywhere in the model, that is, $\llbracket 0 \rrbracket = \emptyset$. Moreover, while $\hat{K}_b \hat{K}_a K_b r$ was initially true at w , it currently is not: $w \in \llbracket (\hat{K}_b \hat{K}_a K_b r)^0 \rrbracket$ but $w \notin \llbracket \hat{K}_b \hat{K}_a K_b r \rrbracket$ (as $\llbracket r \rrbracket = \emptyset$).

4.3.2 An Analysis of Kuijer's counterexample

To understand Kuijer's counterexample (Kuijer, 2015) to the soundness of the finitary \blacksquare -introduction rule for the original APAL, recall that, in APAL, \blacksquare quantifies only over updates with *epistemic* formulas. More precisely, the APAL semantics of \blacksquare in (Balbiani et al., 2008) is given by

$$w \in \llbracket \blacksquare \varphi \rrbracket \quad \text{iff} \quad w \in \llbracket [\theta] \varphi \rrbracket \text{ for every } \theta \in \mathcal{L}^{epi},$$

where \mathcal{L}^{epi} is the sublanguage generated from propositional atoms $p \in \text{Prop}$ using only the Boolean connectives \neg and \wedge , and the epistemic operators K_i . The APAL semantic clauses for the propositional variables, \neg , \wedge , and K_i are the same as the ones given in Definition 4.3.2 with respect to multi-agent epistemic models of the form $M := (W, \sim_i, \|\cdot\|)$.

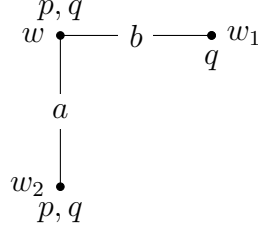
Kuijer takes the formula $\gamma := p \wedge \hat{K}_b \neg p \wedge \hat{K}_a K_b p$, and shows that

$$\llbracket \hat{K}_b p \rrbracket \blacksquare \neg \gamma \rightarrow \llbracket q \rrbracket \neg \gamma.$$

is valid in epistemic models. (In fact, it is also valid in our a -models.) But then, by the $[\!| \blacksquare$ -intro rule (or rather, by its weaker consequence (5)) in Proposition 4.3.14), the formula

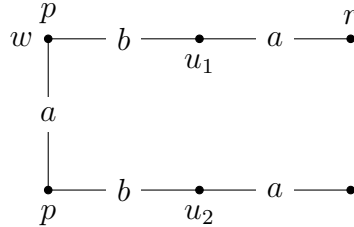
$$\llbracket \hat{K}_b p \rrbracket \blacksquare \neg \gamma \rightarrow \blacksquare \neg \gamma$$

should also be valid. We first present Kuijer's argument for the validity of the former and then we present the model that contradicts the validity of the latter. Suppose that $\llbracket \hat{K}_b p \rrbracket \blacksquare \neg \gamma \rightarrow \llbracket q \rrbracket \neg \gamma$ is not valid in epistemic models, i.e., that there is an epistemic model $N = (W, \sim_1, \dots, \sim_n, \|\cdot\|)$ and $w \in W$ such that $w \in \llbracket \llbracket \hat{K}_b p \rrbracket \blacksquare \neg \gamma \rrbracket$ but $w \notin \llbracket \llbracket q \rrbracket \neg \gamma \rrbracket$. The latter means that $w \in \llbracket \langle q \rangle \gamma \rrbracket$. Therefore, $w \in \llbracket q \rrbracket$ and $w \in \llbracket \gamma \rrbracket_{N|[\!|q]$. The latter implies that $w \in \llbracket p \rrbracket$ and there are two states w_1, w_2 in $N|[\!|q]$ such that (1) w_1 is \sim_b -connected to w and $w_1 \notin \llbracket p \rrbracket$, and (2) w_2 is \sim_a -connected to w and $w_2 \in \llbracket K_b p \rrbracket_{N|[\!|q]$. In other words, the model in Figure 4.8 is guaranteed to be a submodel of $N|[\!|q]$ (all three worlds in N are retained after the update with q).

Figure 4.2: Submodel of $N[[q]]$. Reflexive and transitive arrows are omitted.

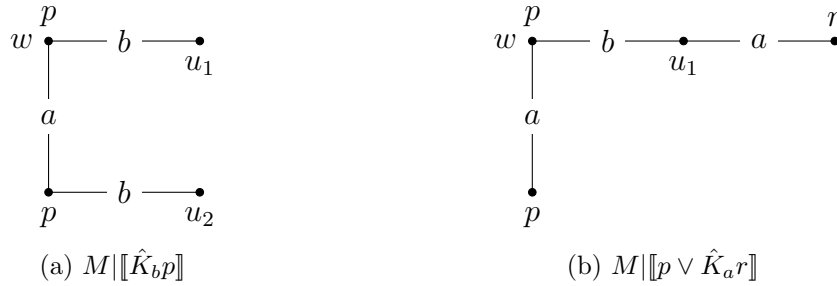
Moreover, since $w \in [[\hat{K}_b p] \blacksquare \neg \gamma]$ and $w \in [[\hat{K}_b p]]$, we also have that $w \in [[\blacksquare \neg \gamma]_{N[[\hat{K}_b p]]}]$. Note that whether a world satisfies an atomic formula does not change after any update, thus $w \in [[q]_{N|\theta}]$ for any θ . In particular, $w \in [[q]_{N[[\hat{K}_b p]]}]$. Then, $w \in [[\blacksquare \neg \gamma]_{N[[\hat{K}_b p]]}]$ implies that $w \in [[\neg \gamma]_{(N|[[\hat{K}_b p]])|[[q]_{N[[\hat{K}_b p]]}]} = [[\neg \gamma]_{(N|[[\langle \hat{K}_b p \rangle q])}]$ by the semantic definition for the original APAL. It is not difficult to see that the model in Figure 4.2 is also a submodel of $N|[[\langle \hat{K}_b p \rangle q]]$ (recall that w_1 is in $N|[[q]]$), thus, $w \in [[\gamma]_{(N|[[\langle \hat{K}_b p \rangle q])}]$. This contradicts the assumption that $w \in [[\hat{K}_b p] \blacksquare \neg \gamma]$.

We now see that the validity of $[\hat{K}_b p] \blacksquare \neg \gamma \rightarrow \blacksquare \neg \gamma$ is contradicted by the model M in Figure 4.3. The premise $[\hat{K}_b p] \blacksquare \neg \gamma$ is true at w in M , since $\blacksquare \neg \gamma$ holds at w in the updated model $M|[[\hat{K}_b p]]$ in Figure 4.4a: indeed, the only way to falsify $\blacksquare \neg \gamma$ would be by deleting the node u_2 from Figure 4.4a while keeping (all other nodes, and in particular) node u_1 . But in $M|[[\hat{K}_b p]]$, u_1 and u_2 can not be separated by epistemic sentences: they are bisimilar.

Figure 4.3: An epistemic model M . Worlds are nodes in the graph (for instance w , u_1 , u_2), valuation is given by labeling the nodes with the true atoms (for instance p and r), and epistemic relations are given by labeled arrows. Reflexive and transitive arrows are omitted.

In contrast, the conclusion $\blacksquare \neg \gamma$ is *false* at w in M , since in that original model u_1 and u_2 could be separated. Indeed, we could perform an alternative update with the formula $p \vee \hat{K}_a r$, yielding the epistemic model $M|[[p \vee \hat{K}_a r]]$ shown in Figure 4.4b, in which γ is true at w (contrary to the assertion that $\blacksquare \neg \gamma$ was true in M).

To see that the counterexample does not apply to APALM, notice that a -models keep track of the initial states. When we take M as an a -model as drawn

Figure 4.4: Two updates of M .

in Figure 4.5 (where the initial states and current states collapse) the updated model $M|[\hat{K}_b p]$ consists now of the initial structure together with current set of worlds W in Figure 4.4a. This structure is given in Figure 4.6a, where the nodes in the shaded area are the current states. But in this model, $\blacksquare \neg \gamma$ is *no longer true* at w (and so the premise $[\hat{K}_b p] \blacksquare \neg \gamma$ was *not* true in M when considered as an a -model!). Indeed, we *can* perform a new update of the a -model $M|[\hat{K}_b p]$ with the formula $(p \vee \hat{K}_a r)^0$, which yields the updated model given in Figure 4.6b:

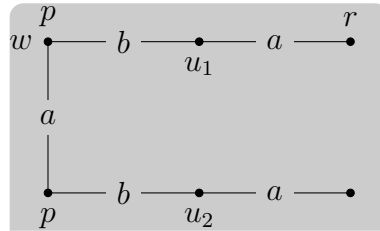


Figure 4.5: M as an a -model. Initial states are nodes in the graphs and current states are represented by the nodes in shaded areas. Reflexive and transitive arrows are omitted.

Note that, in this new model, γ is the case at w (- thus showing that $\blacksquare \neg \gamma$ was *not* true at w in $M|[\hat{K}_b p]$). So the counterexample simply does not work for APALM.

Moreover, we can see that the unsoundness of $[!] \blacksquare$ -intro rule for APAL has to do with its lack of memory, which leads to information loss after updates: while initially (in M) there were epistemic sentences (e.g. $p \vee \hat{K}_a r$) that could separate u_1 and u_2 , there are no such sentences after the update.

APALM solves this by keeping track of the initial states, and referring back to them, by means of formulas such as $(p \vee \hat{K}_a r)^0$.

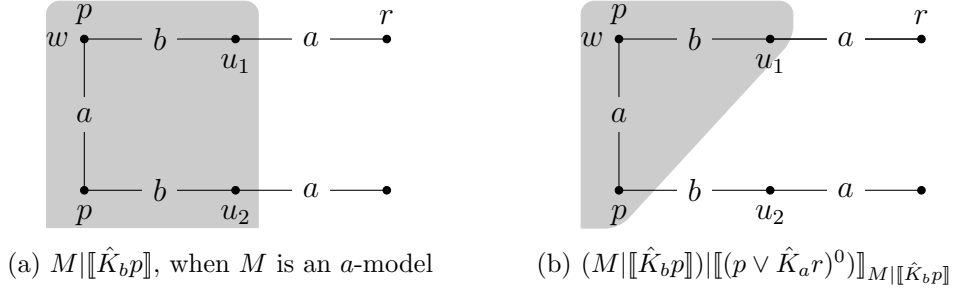


Figure 4.6: Two updates of M , when M is an a -model. Initial states are nodes in the graphs and current states are represented by the nodes in shaded areas. Reflexive and transitive arrows are omitted.

4.3.3 Expressivity of APALM

To compare APALM and its fragments with basic epistemic logic (and its extension with the universal modality), consider the *static fragment* $\mathcal{L}_{-\diamond, \langle \cdot \rangle}$, obtained from \mathcal{L} by removing both the \diamond operator and the dynamic modality $\langle \varphi \rangle \psi$; as well as the *present-only fragment* $\mathcal{L}_{-\diamond, \langle \cdot \rangle, 0, \varphi^0}$, obtained by removing the operators 0 and φ^0 from $\mathcal{L}_{-\diamond, \langle \cdot \rangle}$ (namely, taking out the operators whose interpretations refer to the initial model). Finally consider the *epistemic fragment* \mathcal{L}^{epi} , obtained by further removing the universal modality U from $\mathcal{L}_{-\diamond, \langle \cdot \rangle, 0, \varphi^0}$.

In some of our inductive proofs, we need a complexity order on formulas different from the standard one based on subformula complexity. The standard notion requires only that formulas are more complex than their subformulas, while we also need that $\diamond \varphi$ and $\langle G \rangle \varphi$ are more complex than $\langle \theta \rangle \varphi$ for all $\theta \in \mathcal{L}_{-\diamond}$ (a similar, but simpler, complexity order was defined for Chapter 3 with respect to the effort modality and observational updates). To the best of our knowledge, such a complexity order was first introduced in (Balbiani and van Ditmarsch, 2015) for the original APAL language from (Balbiani et al., 2008). Similar measures have later been introduced for topological versions of APAL in (van Ditmarsch et al., 2015b, 2019; Baltag et al., 2017). We define such appropriate complexity order (\prec_2) in the Technical Appendix A.2.

We now proceed with the following result.

4.3.5. PROPOSITION. *The fragment $\mathcal{L}_{-\diamond}$ is co-expressive with the static fragment $\mathcal{L}_{-\diamond, \langle \cdot \rangle}$. In fact, every formula $\varphi \in \mathcal{L}_{-\diamond}$ is provably equivalent to some formula $\psi \in \mathcal{L}_{-\diamond, \langle \cdot \rangle}$ (by using APALM reduction laws, given in Table 4.1 to eliminate dynamic modalities, as in standard PAL).*

Proof:

The strategy for the proof is to use step-by-step the reduction axioms (given in

Table 4.1, Section 4.3.5), as a rewriting process, and prove termination by \prec_2 -induction on φ by using Lemma A.2.5 from the Technical Appendix A.A.2. To see this, suppose towards contradiction that φ is a formula in $\mathcal{L}_{-\diamond}$, and moreover that φ is not provably equivalent to any formula of lower complexity (in the sense of \prec_2 from Lemma A.2.5) that is in $\mathcal{L}_{-\diamond(\uparrow)}$. We construct an infinite descending sequence

$$\varphi_0 \succ_2 \varphi_1 \succ_2 \dots \succ_2 \varphi_n \succ_2 \dots$$

of provably equivalent formulas, none of which is in $\mathcal{L}_{-\diamond(\uparrow)}$. The construction goes as follows. We first put $\varphi_0 := \varphi$. At step n , assuming that given $\varphi_n \in \mathcal{L}_{-\diamond}$ is not in $\mathcal{L}_{-\diamond(\uparrow)}$, and provably equivalent to all the previous formulas, we chose the first dynamic modality occurring in φ_n . We then apply once to this modality the relevant Reduction Axiom (from left to right), obtaining a provably equivalent formula φ_{n+1} . By Lemma A.2.5, φ_{n+1} has the property that $\varphi_{n+1} \prec_2 \varphi_n$. By transitivity of provable equivalence, φ_{n+1} is provably equivalent to $\varphi_0 = \varphi$, and (by transitivity of \prec_2) it is of lower complexity than $\varphi_0 = \varphi$. Thus, by our assumption above, φ_{n+1} is still not in $\mathcal{L}_{-\diamond(\uparrow)}$. But the existence of this infinite descending sequence contradicts the well-foundedness of \prec_2 . \square

4.3.6. DEFINITION. [Initial/Current epistemic model] For every a -model $M = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$, we call the epistemic model $M^{initial} = (W^0, \sim_1, \dots, \sim_n, \|\cdot\|)$ the *initial epistemic model* of M and $M^{current} = (W, \sim_1 \cap W \times W, \dots, \sim_n \cap W \times W, \|\cdot\| \cap W)$ its *current epistemic model*.

4.3.7. PROPOSITION. *The static fragment $\mathcal{L}_{-\diamond,(\uparrow)}$ (and hence, also $\mathcal{L}_{-\diamond}$) is strictly more expressive than the present-only fragment $\mathcal{L}_{-\diamond,(\uparrow),0,\varphi^0}$, which in turn is more expressive than the epistemic fragment \mathcal{L}^{epi} . In fact, each of the operators 0 and φ^0 independently increase the expressivity of $\mathcal{L}_{-\diamond,(\uparrow),0,\varphi^0}$.*

Proof:

Consider the a -model in Figure 4.6a: while u_1 and u_2 are indistinguishable for $\mathcal{L}_{-\diamond,(\uparrow),0,\varphi^0}$, the sentence $(p \vee \hat{K}_{ar})^0$ distinguishes the two. This shows that $\mathcal{L}_{-\diamond,(\uparrow),0}$ is strictly more expressive than $\mathcal{L}_{-\diamond,(\uparrow),0,\varphi^0}$. To see that $\mathcal{L}_{-\diamond,(\uparrow),\varphi^0}$ is strictly more expressive than $\mathcal{L}_{-\diamond,(\uparrow),0,\varphi^0}$, we just need to consider two a -models $M_1 = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$ and $M_2 = (W, W, \sim_1 \cap W \times W, \dots, \sim_n \cap W \times W, \|\cdot\|)$ such that $W \subset W^0$. As both models have the same underlying current models, they make the same formulas of $\mathcal{L}_{-\diamond,(\uparrow),0,\varphi^0}$ true at the same states in W . However, only M_2 makes 0 true (at every state) since it is an initial model. Moreover, it is well-known that \mathcal{L}^{epi} is strictly less expressive than its extension with the universal modality (see, e.g., Blackburn et al., 2001, Chapter 7.1). \square

The expressivity diagram in Figure 4.7 summarizes Propositions 4.3.5 and 4.3.7.

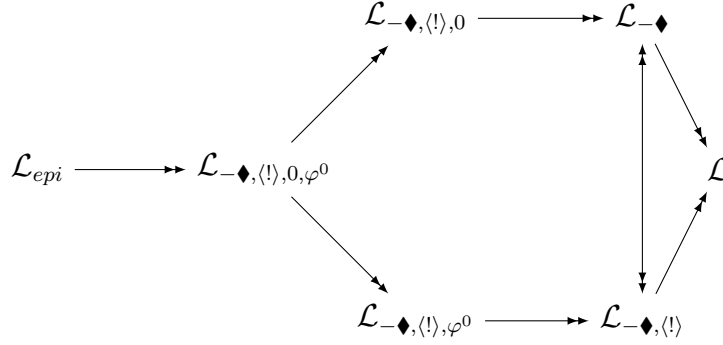


Figure 4.7: Expressivity diagram (Arrows point to the more expressive languages, and reflexive and transitive arrows are omitted.)

4.3.4 APALM Bisimulation

Kuijjer’s counterexample shows that the standard epistemic bisimulation is not appropriate for APALM, so in this section we define a new such notion.

4.3.8. DEFINITION. [APALM Bisimulation] An *APALM bisimulation* between a -models $M_1 = (W_1^0, W_1, \sim_1, \dots, \sim_n, \|\cdot\|_1)$ and $M_2 = (W_2^0, W_2, \sim'_1, \dots, \sim'_n, \|\cdot\|_2)$ is a total bisimulation B (in the usual sense)¹² between the corresponding initial epistemic models $M_1^{initial}$ and $M_2^{initial}$, with the property that: if $s_1 B s_2$, then $s_1 \in W_1$ iff $s_2 \in W_2$. Two current states $s_1 \in W_1$ and $s_2 \in W_2$ are *APALM-bisimilar* if there exists an APALM bisimulation B between the underlying a -models such that $s_1 B s_2$.

Since a -models are always of the form $M = M^0[[\theta]]$ for some $\theta \in \mathcal{L}_{-\diamond}$, we have a characterization of APALM-bisimulation only in terms of the initial models as stated in Proposition [4.3.11](#). First we need the following auxiliary Lemmas.

4.3.9. LEMMA. Let B be a total epistemic bisimulation between initial epistemic models $M_1^{initial}$ and $M_2^{initial}$ (or equivalently, an APALM-bisimulation between initial a -models M_1^0 and M_2^0), and let $s_1 \in W_1^0$, $s_2 \in W_2^0$ be two initial states such that $s_1 B s_2$. Then we have

$$s_1 \in [[\alpha]]_{M_1^0} \text{ iff } s_2 \in [[\alpha]]_{M_2^0}$$

for all formulas $\alpha \in \mathcal{L}_{-\diamond}$.

¹²A total bisimulation between epistemic models $(W, \sim_1, \dots, \sim_n, \|\cdot\|)$ and $(W', \sim'_1, \dots, \sim'_n, \|\cdot\|')$ is an epistemic bisimulation relation (satisfying the usual valuation and back-and-forth conditions from Modal Logic) $B \subseteq W \times W'$, such that: for every $s \in W$ there exists some $s' \in W'$ with $s B s'$; and dually, for every $s' \in W'$ there exists some $s \in W$ with $s B s'$.

Proof:

By Proposition 4.3.5, it is enough to prove the claim for all formulas $\alpha \in \mathcal{L}_{-\diamond, \langle \rangle}$. Let B be an APALM bisimulation between initial a -models M_1^0 and M_2^0 . The proof is by subformula induction on α , using the following *induction hypothesis* (IH): for all $\beta \in \text{Sub}(\alpha)$, we have $s_1 \in \llbracket \beta \rrbracket_{M_1^0}$ iff $s_2 \in \llbracket \beta \rrbracket_{M_2^0}$ for all $s_1 \in W_1^0$, $s_2 \in W_2^0$ such that $s_1 B s_2$.

Base case $\alpha := \top$: Since $s_1 \in W_1^0 = \llbracket \top \rrbracket_{M_1^0}$ and $s_2 \in W_2^0 = \llbracket \top \rrbracket_{M_2^0}$, we trivially obtain that $s_1 \in \llbracket \top \rrbracket_{M_1^0}$ iff $s_2 \in \llbracket \top \rrbracket_{M_2^0}$.

Base case $\alpha := p$: Since $s_1 B s_2$, $s_1 \in \llbracket p \rrbracket_{M_1^0}$ iff $s_2 \in \llbracket p \rrbracket_{M_2^0}$ follows by Definition 4.3.8, valuation condition.

Base case $\alpha := 0$: Since M_1^0 and M_2^0 are initial a -models, by the semantics, we have $s_1 \in W_1^0 = \llbracket 0 \rrbracket_{M_1^0}$ and $s_2 \in W_2^0 = \llbracket 0 \rrbracket_{M_2^0}$. We therefore trivially obtain that $s_1 \in \llbracket 0 \rrbracket_{M_1^0}$ iff $s_2 \in \llbracket 0 \rrbracket_{M_2^0}$.

Case $\alpha := \beta \wedge \gamma$ and $\alpha := \neg \beta$ follow straightforwardly by the semantics and IH.

In the following sequence of equivalencies, we make repeated use of the semantic clauses in Definition 4.3.2

Case $\alpha := \beta^0$
 $s_1 \in \llbracket \beta^0 \rrbracket_{M_1^0}$ iff $s_1 \in \llbracket \beta \rrbracket_{M_1^0} \cap W_1^0$ iff $s_2 \in \llbracket \beta \rrbracket_{M_2^0} \cap W_2^0$ (by IH and $s_2 \in W_2^0$) iff $s_2 \in \llbracket \beta^0 \rrbracket_{M_2^0}$.

Case $\alpha := U\beta$
 $s_1 \in \llbracket U\beta \rrbracket_{M_1^0}$ iff $\forall s \in W_1^0, s \in \llbracket \beta \rrbracket_{M_1^0}$ iff $\forall s' \in W_2^0, s' \in \llbracket \beta \rrbracket_{M_2^0}$ (since B is total and IH) iff $s_2 \in \llbracket U\beta \rrbracket_{M_2^0}$.

Case $\alpha := K_i\beta$
 $s_1 \in \llbracket K_i\beta \rrbracket_{M_1^0}$ iff $(\forall s \in W_1^0)(s \sim_i s_1 \text{ implies } s \in \llbracket \beta \rrbracket_{M_1^0})$ iff $(\forall s \in W_2^0)(s \sim'_i s_2 \text{ implies } s \in \llbracket \beta \rrbracket_{M_2^0})$ (back and forth condition, IH) iff $s_2 \in \llbracket K_i\beta \rrbracket_{M_2^0}$. \square

4.3.10. LEMMA. *Let B be a total epistemic bisimulation between initial epistemic models M_1^{initial} and M_2^{initial} (or equivalently, an APALM-bisimulation between initial a -models M_1^0 and M_2^0), and let $s_1 \in W_1^0$, $s_2 \in W_2^0$ be two initial states such that $s_1 B s_2$. Then, for all $\varphi \in \mathcal{L}$, we have*

$$s_1 \in \llbracket \langle \alpha \rangle \varphi \rrbracket_{M_1^0} \text{ iff } s_2 \in \llbracket \langle \alpha \rangle \varphi \rrbracket_{M_2^0}$$

for all formulas $\alpha \in \mathcal{L}_{-\diamond}$.

Proof:

Let B be an APALM bisimulation between initial a -models M_1^0 and M_2^0 . The proof goes by \prec_2 -induction on φ , using Lemma A.2.5 from the Technical Appendix

A.A.2. We assume the following induction hypothesis (IH): for all $\psi \in \mathcal{L}$ such that $\psi \prec_2 \varphi$ and all states $s_1 \in W_1^0$, $s_2 \in W_2^0$ with $s_1 B s_2$, we have: $s_1 \in \llbracket \langle \alpha \rangle \psi \rrbracket_{M_1^0}$ iff $s_2 \in \llbracket \langle \alpha \rangle \psi \rrbracket_{M_2^0}$, for all $\alpha \in \mathcal{L}_{-\diamond}$.

Base cases $\varphi := \top$, $\varphi := p$, and $\varphi := 0$ follow directly from Lemma 4.3.9 and the fact that the formulas $\langle \alpha \rangle \top$, $\langle \alpha \rangle p$, and $\langle \alpha \rangle 0$ are in $\mathcal{L}_{-\diamond}$.

In the following sequence of equivalencies, we make repeated use of the semantic clauses in Definition 4.3.2

Case $\varphi := \psi^0$
 $s_1 \in \llbracket \langle \alpha \rangle \psi^0 \rrbracket_{M_1^0}$ iff $s_1 \in \llbracket \psi^0 \rrbracket_{M_1^0 | \llbracket \alpha \rrbracket_{M_1^0}}$ iff $s_1 \in \llbracket \psi \rrbracket_{M_1^0} \cap \llbracket \alpha \rrbracket_{M_1^0}$ (since $M_1^0 | \llbracket \alpha \rrbracket_{M_1^0} = (W_1^0, \llbracket \alpha \rrbracket_{M_1^0}, \sim_1, \dots, \sim_n, \|\cdot\|_1)$) iff $s_2 \in \llbracket \psi \rrbracket_{M_2^0} \cap \llbracket \alpha \rrbracket_{M_2^0}$ (by IH, Lemma 4.3.9, $\alpha \in \mathcal{L}_{-\diamond}$) iff $s_2 \in \llbracket \psi^0 \rrbracket_{M_2^0 | \llbracket \alpha \rrbracket_{M_2^0}}$ (since $M_2^0 | \llbracket \alpha \rrbracket_{M_2^0} = (W_2^0, \llbracket \alpha \rrbracket_{M_2^0}, \sim'_1, \dots, \sim'_n, \|\cdot\|_2)$) iff $s_2 \in \llbracket \langle \alpha \rangle \psi^0 \rrbracket_{M_2^0}$.

Cases $\varphi := K_i \psi$ and $\varphi := U\psi$ follow similarly as in Lemma 4.3.9. We spell out here only the case $\varphi := U\psi$. First observe that,

$s_1 \in \llbracket \langle \alpha \rangle U\psi \rrbracket_{M_1^0}$ iff $s_1 \in \llbracket U\psi \rrbracket_{M_1^0 | \llbracket \alpha \rrbracket_{M_1^0}}$ iff $\forall s \in \llbracket \alpha \rrbracket_{M_1^0}, s \in \llbracket \psi \rrbracket_{M_1^0 | \llbracket \alpha \rrbracket_{M_1^0}}$ iff $\forall s \in \llbracket \alpha \rrbracket_{M_1^0}, s \in \llbracket \langle \alpha \rangle \psi \rrbracket_{M_1^0}$.

Suppose $s_1 \in \llbracket \langle \alpha \rangle U\psi \rrbracket_{M_1^0}$ and let $s' \in \llbracket \alpha \rrbracket_{M_2^0}$. Since B is a total bisimulation, there is $s'_1 \in W_1^0$ such that $s'_1 B s'$. Since $\alpha \in \mathcal{L}_{-\diamond}$, by Lemma 4.3.9, we have $s'_1 \in \llbracket \alpha \rrbracket_{M_1^0}$. Then, by the above observation, we have $s'_1 \in \llbracket \langle \alpha \rangle \psi \rrbracket_{M_1^0}$. Thus, by IH, we obtain that $s' \in \llbracket \langle \alpha \rangle \psi \rrbracket_{M_2^0}$. As $s_2 \in W_2^0$, we then conclude, via similar steps as in the above observation, that $s_2 \in \llbracket \langle \alpha \rangle U\psi \rrbracket_{M_2^0}$. The other direction is similar. For the case $\varphi := K_i \psi$, we also use the back and forth conditions of B .

Case $\varphi := \langle \theta \rangle \psi$ uses the validity of the formula $\langle \alpha \rangle \langle \theta \rangle \psi \leftrightarrow \langle \langle \alpha \rangle \theta \rangle \psi$ which can be easily verified.

$s_1 \in \llbracket \langle \alpha \rangle \langle \theta \rangle \psi \rrbracket_{M_1^0}$ iff $s_1 \in \llbracket \langle \langle \alpha \rangle \theta \rangle \psi \rrbracket_{M_1^0}$ (by $\models \langle \alpha \rangle \langle \theta \rangle \psi \leftrightarrow \langle \langle \alpha \rangle \theta \rangle \psi$) iff $s_2 \in \llbracket \langle \langle \alpha \rangle \theta \rangle \psi \rrbracket_{M_2^0}$ (IH, using $\psi \prec_2 \langle \theta \rangle \psi$) iff $s_2 \in \llbracket \langle \alpha \rangle \langle \theta \rangle \psi \rrbracket_{M_2^0}$.

Case $\varphi := \diamond \psi$
 $s_1 \in \llbracket \langle \alpha \rangle \diamond \psi \rrbracket_{M_1^0}$ iff $s_1 \in \llbracket \diamond \psi \rrbracket_{M_1^0 | \llbracket \alpha \rrbracket_{M_1^0}}$ iff $s_1 \in \bigcup_{\theta \in \mathcal{L}_{-\diamond}} \llbracket \langle \theta \rangle \psi \rrbracket_{M_1^0 | \llbracket \alpha \rrbracket_{M_1^0}}$ iff $s_1 \in \bigcup_{\theta \in \mathcal{L}_{-\diamond}} \llbracket \langle \alpha \rangle \langle \theta \rangle \psi \rrbracket_{M_1^0}$ iff $s_2 \in \bigcup_{\theta \in \mathcal{L}_{-\diamond}} \llbracket \langle \alpha \rangle \langle \theta \rangle \psi \rrbracket_{M_2^0}$ (IH, $\langle \theta \rangle \psi \prec_2 \diamond \psi$)
 iff $s_2 \in \bigcup_{\theta \in \mathcal{L}_{-\diamond}} \llbracket \langle \theta \rangle \psi \rrbracket_{M_2^0 | \llbracket \alpha \rrbracket_{M_2^0}}$ iff $s_2 \in \llbracket \diamond \psi \rrbracket_{M_2^0 | \llbracket \alpha \rrbracket_{M_2^0}}$ iff $s_2 \in \llbracket \langle \alpha \rangle \diamond \psi \rrbracket_{M_2^0}$. \square

4.3.11. PROPOSITION. Let $M_1 = (W_1^0, W_1, \sim_1, \dots, \sim_n, \|\cdot\|_1)$ and $M_2 = (W_2^0, W_2, \sim'_1, \dots, \sim'_n, \|\cdot\|_2)$ be a -models, and let $B \subseteq W_1^0 \times W_2^0$. The following are equivalent:

1. B is an APALM bisimulation between M_1 and M_2 ;
2. B is a total epistemic bisimulation between $M_1^{initial}$ and $M_2^{initial}$ (or equivalently, an APALM bisimulation between M_1^0 and M_2^0), and $M_1 = M_1^0 | \llbracket \theta \rrbracket_{M_1^0}$, $M_2 = M_2^0 | \llbracket \theta \rrbracket_{M_2^0}$ for some common formula $\theta \in \mathcal{L}_{-\diamond}$.

Proof:

(1) \Rightarrow (2): Let B be an APALM bisimulation between M_1 and M_2 . Then it is obvious (from the definition) that B is also a total bisimulation between $M_1^{initial}$ and $M_2^{initial}$. Since M_1 and M_2 are a -models, there must exist $\theta_1, \theta_2 \in \mathcal{L}_{-\blacklozenge}$ s.t. $M_1 = M_1^0 | [\theta_1]_{M_1^0}$, $M_2 = M_2^0 | [\theta_2]_{M_2^0}$. Hence, $W_1 = [\theta_1]_{M_1^0}$ and $W_2 = [\theta_2]_{M_2^0}$. To show that $[\theta_1]_{M_1^0} = [\theta_2]_{M_1^0}$, let first $s_1 \in [\theta_1]_{M_1^0} = W_1$. By the definition of APALM bisimulation, there must exist $s_2 \in W_2^0$ such that $s_1 B s_2$. Again by the definition, $s_1 \in W_1$ implies that $s_2 \in W_2 = [\theta_2]_{M_2^0}$. This, together with $s_1 B s_2$, gives us by Lemma 4.3.9 that $s_1 \in [\theta_2]_{M_1^0}$. For the converse, let $s_1 \in [\theta_2]_{M_1^0}$; by the definition of APALM bisimulation, there must exist $s_2 \in W_2^0$ such that $s_1 B s_2$. By Lemma 4.3.9, we have $s_2 \in [\theta_2]_{M_2^0} = W_2$, and again by the definition of APALM bisimulation (and the fact that $s_1 B s_2$), this implies that $s_1 \in W_1 = [\theta_1]_{M_1^0}$. Given that $M_1 = M_1^0 | [\theta_1]_{M_1^0}$ and $M_2 = M_2^0 | [\theta_2]_{M_2^0}$ such that $[\theta_1]_{M_1^0} = [\theta_2]_{M_1^0}$, we can take $\theta := \theta_2$. Then $M_1 = M_1^0 | [\theta_1]_{M_1^0} = M_1^0 | [\theta_2]_{M_1^0}$.

(2) \Rightarrow (1): Suppose that B is a total bisimulation between $M_1^{initial}$ and $M_2^{initial}$, and $M_1 = M_1^0 | [\theta]_{M_1^0}$, $M_2 = M_2^0 | [\theta]_{M_2^0}$ for some common formula $\theta \in \mathcal{L}_{-\blacklozenge}$. Hence, $W_1 = [\theta]_{M_1^0}$ and $W_2 = [\theta]_{M_2^0}$. We need to verify that M_1 and M_2 are APALM-bisimilar. For this we just need to verify the property that if $s_1 B s_2$, then $s_1 \in W_1$ holds iff $s_2 \in W_2$ holds. Suppose $s_1 B s_2$ and let $s_1 \in W_1 = [\theta]_{M_1^0} \subseteq W_1^0$. By the totality of the bisimulation B , there must exist some $s_2 \in W_2^0$ with $s_1 B s_2$. By Lemma 4.3.9, $s_1 \in [\theta]_{M_1^0}$ implies that $s_2 \in [\theta]_{M_2^0} = W_2$. The converse is analogous. \square

So, to check for APALM-bisimilarity, it is enough to check for total bisimilarity between the initial models and for both models being updates with the same formula.

Next, we verify that this is indeed the appropriate notion of bisimulation, namely that APALM formulas are invariant under APALM-bisimulation.

4.3.12. COROLLARY. *If $s_1 B s_2$ for some APALM-bisimulation relation B between a -models $M_1 = (W_1^0, W_1, \sim_1, \dots, \sim_n, \|\cdot\|_1)$ and $M_2 = (W_2^0, W_2, \sim'_1, \dots, \sim'_n, \|\cdot\|_2)$, then s_1 in M_1 and s_2 in M_2 satisfy the same APALM formulas, i.e., $s_1 \in [\varphi]_{M_1}$ iff $s_2 \in [\varphi]_{M_2}$ for all $\varphi \in \mathcal{L}$.*

Proof:

Let B be some APALM-bisimulation relation between a -models $M_1 = (W_1^0, W_1, \sim_1, \dots, \sim_n, \|\cdot\|_1)$ and $M_2 = (W_2^0, W_2, \sim'_1, \dots, \sim'_n, \|\cdot\|_2)$. By Proposition 4.3.11, there exists some formula $\theta \in \mathcal{L}_{-\blacklozenge}$ such that $M_1 = M_1^0 | [\theta]_{M_1^0}$, $M_2 = M_2^0 | [\theta]_{M_2^0}$. By the same proposition, B is a total epistemic bisimulation between the initial epistemic models $M_1^{initial}$ and $M_2^{initial}$. Thus, for every formula φ , we have the sequence of equivalences: $s_1 \in [\varphi]_{M_1}$ iff $s_1 \in [\langle \theta \rangle \varphi]_{M_1^0}$ iff (by Lemma 4.3.10) $s_2 \in [\langle \theta \rangle \varphi]_{M_2^0}$

iff $s_2 \in \llbracket \varphi \rrbracket_{M_2}$. □

4.3.13. PROPOSITION (HENNESSY-MILNER). *Let $M_1 = (W_1^0, W_1, \sim_1, \dots, \sim_n, \|\cdot\|_1)$ and $M_2 = (W_2^0, W_2, \sim'_1, \dots, \sim'_n, \|\cdot\|_2)$ be a -models with W_1^0 and W_2^0 finite. Then, $s_1 \in W_1$ and $s_2 \in W_2$ satisfy the same APALM formulas iff they are APALM-bisimilar.*

Proof:

We only need to prove the left-to-right direction. Let $s_1 \in W_1$ and $s_2 \in W_2$ such that for all $\varphi \in \mathcal{L}$, $s_1 \in \llbracket \varphi \rrbracket_{M_1}$ iff $s_2 \in \llbracket \varphi \rrbracket_{M_2}$. This implies that for all $\varphi \in \mathcal{L}$, $s_1 \in \llbracket \varphi \rrbracket_{M_1^0}$ iff $s_2 \in \llbracket \varphi \rrbracket_{M_2^0}$. To see this, let $\varphi \in \mathcal{L}$ such that $s_1 \in \llbracket \varphi \rrbracket_{M_1^0}$. This means, by the semantics, that $s_1 \in \llbracket \varphi^0 \rrbracket_{M_1}$. As s_1 in M_1 and s_2 in M_2 satisfy the same APALM formulas, we obtain that $s_2 \in \llbracket \varphi^0 \rrbracket_{M_2}$, thus, $s_2 \in \llbracket \varphi \rrbracket_{M_2^0}$. The opposite direction is analogous. We then show that the modal equivalence relation in $W_1^0 \times W_2^0$ between the models M_1^0 and M_2^0 is an APALM bisimulation. We thus need to show the following:

(Totality) For all $s \in W_1^0$, there exists $s' \in W_2^0$ such that, $s \in \llbracket \varphi \rrbracket_{M_1^0}$ iff $s' \in \llbracket \varphi \rrbracket_{M_2^0}$ for all $\varphi \in \mathcal{L}$, and for all $s' \in W_2^0$, there exists $s \in W_1^0$ such that $s \in \llbracket \varphi \rrbracket_{M_1^0}$ iff $s' \in \llbracket \varphi \rrbracket_{M_2^0}$ for all $\varphi \in \mathcal{L}$.

Let $s \in W_1^0$ and suppose, toward contradiction, that for no element s' of W_2^0 we have that $s \in \llbracket \varphi \rrbracket_{M_1^0}$ iff $s' \in \llbracket \varphi \rrbracket_{M_2^0}$ for all $\varphi \in \mathcal{L}$. Since W_2^0 is finite, we can list its elements $W_2^0 = \{w_1, w_2, \dots, w_n\}$. The first assumption then implies that for all $w_i \in W_2^0$, there exists $\psi_i \in \mathcal{L}$ such that $s \in \llbracket \psi_i \rrbracket_{M_1^0}$ but $w_i \notin \llbracket \psi_i \rrbracket_{M_2^0}$. Thus, $s_1 \in \llbracket E(\psi_1 \wedge \dots \wedge \psi_n) \rrbracket_{M_1^0}$ but $s_2 \notin \llbracket E(\psi_1 \wedge \dots \wedge \psi_n) \rrbracket_{M_2^0}$, contradicting the assumption that s_1 in M_1^0 and s_2 in M_2^0 satisfy the same APALM formulas. The second clause follows similarly.

(Valuation) This follows immediately from modal equivalence.

(Forth for \sim_i) Let $w_1, w'_1 \in W_1^0$ and $w_2 \in W_2^0$ such that $w_1 \in \llbracket \varphi \rrbracket_{M_1^0}$ iff $w_2 \in \llbracket \varphi \rrbracket_{M_2^0}$ for all $\varphi \in \mathcal{L}$ and $w_1 \sim_i w'_1$. Suppose, toward contradiction, that for no element $w'_2 \in W_2^0$ with $w_2 \sim'_i w'_2$, M_1^0, w'_1 and M_2^0, w'_2 satisfy the same APALM formulas. Since W_2^0 is finite, the set $\sim'_i(w_2) = \{t \in W_2^0 : w_2 \sim'_i t\}$ is finite, thus, we can write $\sim'_i(w_2) = \{t_1, \dots, t_k\}$. As in the proof of (Totality), the assumption implies that for all t_j with $w_2 \sim'_i t_j$, there exists $\psi_j \in \mathcal{L}$ such that $w'_1 \in \llbracket \psi_j \rrbracket_{M_1^0}$ but $t_j \notin \llbracket \psi_j \rrbracket_{M_2^0}$. Therefore, $w_1 \in \llbracket \hat{K}_i(\psi_1 \wedge \dots \wedge \psi_k) \rrbracket_{M_1^0}$ but $w_2 \notin \llbracket \hat{K}_i(\psi_1 \wedge \dots \wedge \psi_k) \rrbracket_{M_2^0}$, contradicting the assumption that M_1^0, w_1 and M_2^0, w_2 satisfy the same APALM formulas. Back condition for \sim_i follows analogously.

We have therefore proven that the modal equivalence relation in $W_1^0 \times W_2^0$ between the models M_1^0 and M_2^0 is an APALM bisimulation between M_1^0 and

M_2^0 . By Proposition 4.3.11, it suffices to further prove that $M_1 = M_1^0 \mid \llbracket \theta \rrbracket_{M_1^0}$, $M_2 = M_2^0 \mid \llbracket \theta \rrbracket_{M_2^0}$ for some common formula $\theta \in \mathcal{L}_{-\blacklozenge}$. It then suffices to show that $\llbracket \theta_1 \rrbracket_{M_2^0} = \llbracket \theta_2 \rrbracket_{M_2^0}$, where $W_1 = \llbracket \theta_1 \rrbracket_{M_1^0}$ and $W_2 = \llbracket \theta_2 \rrbracket_{M_2^0}$.

$\llbracket \theta_2 \rrbracket_{M_2^0} \subseteq \llbracket \theta_1 \rrbracket_{M_2^0}$: Observe that $s_1 \in \llbracket U\theta_1^0 \rrbracket_{M_1}$, since $W_1 = \llbracket \theta_1 \rrbracket_{M_1^0}$. Moreover, as M_1, s_1 and M_2, s_2 satisfy the same APALM formulas, we obtain that $s_2 \in \llbracket U\theta_1^0 \rrbracket_{M_2}$. Therefore, for all $y \in \llbracket \theta_2 \rrbracket_{M_2^0}$, we have $y \in \llbracket \theta_1^0 \rrbracket_{M_2}$, implying that $y \in \llbracket \theta_1 \rrbracket_{M_2^0}$. Hence, $\llbracket \theta_2 \rrbracket_{M_2^0} \subseteq \llbracket \theta_1 \rrbracket_{M_2^0}$.

$\llbracket \theta_1 \rrbracket_{M_2^0} \subseteq \llbracket \theta_2 \rrbracket_{M_2^0}$: Observe that $s_2 \in \llbracket U\theta_2^0 \rrbracket_{M_2}$, since $W_2 = \llbracket \theta_2 \rrbracket_{M_2^0}$. Moreover, as M_1, s_1 and M_2, s_2 satisfy the same APALM formulas, we obtain that $s_1 \in \llbracket U\theta_2^0 \rrbracket_{M_1}$. Now suppose, toward contradiction, that $\llbracket \theta_1 \rrbracket_{M_2^0} \not\subseteq \llbracket \theta_2 \rrbracket_{M_2^0}$, i.e., there is $y \in W_2^0$ such that $y \in \llbracket \theta_1 \rrbracket_{M_2^0}$ but $y \notin \llbracket \theta_2 \rrbracket_{M_2^0}$. By the totality of the modal equivalence relation, there exists $x \in W_1^0$ such that $x \in \llbracket \theta_1 \rrbracket_{M_1^0}$ but $x \notin \llbracket \theta_2 \rrbracket_{M_1^0}$. The former implies that $x \in W_1$. Therefore, by the latter, we have that $x \notin \llbracket \theta_2^0 \rrbracket_{M_1}$. This implies, since $s_1, x \in W_1$, that $s_1 \notin \llbracket U\theta_2^0 \rrbracket_{M_1}$, contradicting M_1, s_1 and M_2, s_2 satisfying the same APALM formulas.

Therefore, we obtain that $\llbracket \theta_1 \rrbracket_{M_2^0} = \llbracket \theta_2 \rrbracket_{M_2^0}$. Given that $M_1 = M_1^0 \mid \llbracket \theta_1 \rrbracket_{M_1^0}$ and $M_2 = M_2^0 \mid \llbracket \theta_2 \rrbracket_{M_2^0}$ such that $\llbracket \theta_1 \rrbracket_{M_2^0} = \llbracket \theta_2 \rrbracket_{M_2^0}$, we can take $\theta := \theta_1$. Then $M_2 = M_2^0 \mid \llbracket \theta_2 \rrbracket_{M_2^0} = M_2^0 \mid \llbracket \theta_1 \rrbracket_{M_2^0}$. \square

4.3.5 Axiomatization for APALM

In this section, we present in Table 4.1 a complete proof system **APALM** for our Arbitrary Public Announcement Logic with Memory (where recall that P_φ is the set of propositional variables in φ).

Intuitive Reading of the Axioms. Parts (I) and (II) should be obvious. The axiom $R[\top]$ says that *updating with tautologies is redundant*. The reduction laws that do not contain 0 , U or 0 are well-known PAL axioms. R_U is the natural reduction law for the universal modality. The axiom R^0 says that the truth value of φ^0 formulas stays the same in time (because the superscript 0 serves as a time stamp), so they can be treated similarly to atoms. Ax_0 says that 0 was initially the case, and R_0 says that at any later stage (after any update) 0 can only be true if it was already true before the update and the update was trivial (universally true). Together, these two say that the constant 0 characterizes states where no non-trivial communication has occurred. Axiom $0-U$ is a synchronicity constraint: *if no non-trivial communication has taken place yet, then this is the case in all the currently possible states*. Axiom $0-eq$ says that *initially, φ is equivalent to its initial correspondent φ^0* . The Equivalences with 0 express that 0 distributes over negation and over conjunction. Imp_{\blacksquare}^0 says that *if initially φ was stably true*

	(I) Basic Axioms of system APALM:
(CPL)	all classical propositional tautologies and Modus Ponens
(S5 K_i)	all S5 axioms and rules for knowledge operator K_i
(S5 U)	all S5 axioms and rules for U operator
(U- K_i)	$U\varphi \rightarrow K_i\varphi$
	(II) Axioms and rules for dynamic modalities $[!]$:
(K $_!$)	<i>Kripke's axiom for $[!]$:</i> $[\theta](\psi \rightarrow \varphi) \rightarrow ([\theta]\psi \rightarrow [\theta]\varphi)$
(Nec $_!$)	<i>Necessitation for $[!]$:</i> from $\vdash \varphi$, infer $\vdash [\theta]\varphi$.
(RE)	<i>Replacement of Equivalents $[!]$:</i> from $\vdash \theta \leftrightarrow \rho$, infer $\vdash [\theta]\varphi \leftrightarrow [\rho]\varphi$.
	Reduction laws:
(R $[\top]$)	$[\top]\varphi \leftrightarrow \varphi$
(R $_p$)	$[\theta]p \leftrightarrow (\theta \rightarrow p)$
(R $_{\neg}$)	$[\theta]\neg\psi \leftrightarrow (\theta \rightarrow \neg[\theta]\psi)$
(R $_{K_i}$)	$[\theta]K_i\psi \leftrightarrow (\theta \rightarrow K_i[\theta]\psi)$
(R $_{[!]}$)	$[\theta][\rho]\chi \leftrightarrow [\langle\theta\rangle\rho]\chi$
(R 0)	$[\theta]\varphi^0 \leftrightarrow (\theta \rightarrow \varphi^0)$
(R $_U$)	$[\theta]U\varphi \leftrightarrow (\theta \rightarrow U[\theta]\varphi)$
(R $_0$)	$[\theta]0 \leftrightarrow (\theta \rightarrow (U\theta \wedge 0))$
	(III) Axioms and rules for 0 and <i>initial</i> operator 0:
(Ax $_0$)	0^0
(0- U)	$0 \rightarrow U0$
(0- eq)	$0 \rightarrow (\varphi \leftrightarrow \varphi^0)$
(Nec 0)	<i>Necessitation for 0:</i> from $\vdash \varphi$, infer $\vdash \varphi^0$
	Equivalences with 0:
(Eq $_p^0$)	$p^0 \leftrightarrow p$
(Eq $_{\neg}^0$)	$(\neg\varphi)^0 \leftrightarrow \neg\varphi^0$
(Eq $_{\wedge}^0$)	$(\varphi \wedge \psi)^0 \leftrightarrow (\varphi^0 \wedge \psi^0)$
	Implications with 0:
(Imp $_U^0$)	$(U\varphi)^0 \rightarrow U\varphi^0$
(Imp $_i^0$)	$(K_i\varphi)^0 \rightarrow K_i\varphi^0$
(Imp $_{\blacksquare}^0$)	$(\blacksquare\varphi)^0 \rightarrow \varphi$
	(IV) Elim-axiom and Intro-rule for \blacksquare:
($[!]\blacksquare$ -elim)	$[\theta]\blacksquare\varphi \rightarrow [\theta \wedge \rho]\varphi$
($[!]\blacksquare$ -intro)	from $\vdash \chi \rightarrow [\theta \wedge p]\varphi$, infer $\vdash \chi \rightarrow [\theta]\blacksquare\varphi$ (for $p \notin P_\chi \cup P_\theta \cup P_\varphi$).

Table 4.1: The axiomatization APALM. (Here, $\varphi, \psi, \chi \in \mathcal{L}$, while $\theta, \rho \in \mathcal{L}_{-\diamond}$.)

(under any further announcements), then φ is the case now. Taken together, the elimination axiom $[!]\blacksquare$ -elim and introduction rule $[!]\blacksquare$ -intro say that φ is a stable truth after an announcement θ iff φ stays true after any more informative announcement (of the form $\theta \wedge \rho$).¹³

4.3.14. PROPOSITION. *The following schemas and inference rules are derivable in APALM, where $\varphi, \psi, \chi \in \mathcal{L}$ and $\theta \in \mathcal{L}_{-\blacklozenge}$:*

1. from $\vdash \varphi \leftrightarrow \psi$,
infer $\vdash [\theta]\varphi \leftrightarrow [\theta]\psi$
2. $\vdash \langle \theta \rangle 0 \leftrightarrow (0 \wedge U\theta)$
3. $\vdash \langle \theta \rangle \psi \leftrightarrow (\theta \wedge [\theta]\psi)$
4. $\vdash \blacksquare\varphi \rightarrow [\theta]\varphi$
5. from $\vdash \chi \rightarrow [p]\varphi$, infer $\vdash \chi \rightarrow \blacksquare\varphi$
($p \notin P_\chi \cup P_\varphi$)
6. **S4** system for \blacksquare
7. $\vdash (\varphi \rightarrow \psi)^0 \leftrightarrow (\varphi^0 \rightarrow \psi^0)$
8. $\vdash \varphi^{00} \leftrightarrow \varphi^0$
9. $\vdash \blacksquare\varphi^0 \leftrightarrow \varphi^0$, and $\vdash \blacklozenge\varphi^0 \leftrightarrow \varphi^0$
10. $\vdash (\blacksquare\varphi)^0 \rightarrow \blacksquare\varphi^0$
11. $\vdash (0 \wedge \blacklozenge\varphi^0) \rightarrow \varphi$
12. $\vdash \varphi \rightarrow (0 \wedge \blacklozenge\varphi)^0$
13. $\vdash \varphi \rightarrow \psi^0$ if and only if
 $\vdash (0 \wedge \blacklozenge\varphi) \rightarrow \psi$
14. $\vdash [\theta](\psi \wedge \varphi) \leftrightarrow ([\theta]\psi \wedge [\theta]\varphi)$
15. $\vdash [\theta][p]\psi \leftrightarrow [\theta \wedge p]\psi$
16. $\vdash [\theta]\perp \leftrightarrow \neg\theta$

Proof:

1. from $\vdash \varphi \leftrightarrow \psi$, infer $\vdash [\theta]\varphi \leftrightarrow [\theta]\psi$: Follows directly by (K_!) and (Nec_!).
2. $\langle \theta \rangle 0 \leftrightarrow (0 \wedge U\theta)$: Follows from the definition of $\langle \theta \rangle 0 := \neg[\theta]\neg 0$ and the axiom (R₋).
3. $\langle \theta \rangle \psi \leftrightarrow (\theta \wedge [\theta]\psi)$: Follows from the definition $\langle \theta \rangle \psi := \neg[\theta]\neg\psi$ and the axiom (R₋).
4. $\blacksquare\varphi \rightarrow [\rho]\varphi$ ($\rho \in \mathcal{L}_{-\blacklozenge}$ arbitrary):

1. $\vdash \blacksquare\varphi \leftrightarrow [\top]\blacksquare\varphi$ (R[\top])
2. $\vdash [\top]\blacksquare\varphi \rightarrow [\top \wedge \rho]\varphi$, (for arbitrary $\rho \in \mathcal{L}_{-\blacklozenge}$) ([!] \blacksquare -elim)
3. $\vdash [\top \wedge \rho]\varphi \rightarrow [\rho]\varphi$, (for arbitrary $\rho \in \mathcal{L}_{-\blacklozenge}$) ($\vdash (\top \wedge \rho) \leftrightarrow \rho$ and (RE))
4. $\vdash \blacksquare\varphi \rightarrow [\rho]\varphi$, (for arbitrary $\rho \in \mathcal{L}_{-\blacklozenge}$) (1, 3, CPL)

¹³The “freshness” of the variable $p \in P$ in the rule $[!]\blacksquare$ -intro ensures that it represents any generic announcement.

5. from $\vdash \chi \rightarrow [p]\varphi$, infer $\vdash \chi \rightarrow \blacksquare\varphi$ ($p \notin P_\chi \cup P_\varphi$): proof follows analogously to the above case by using (RE), ($[!]\blacksquare$ -intro) with $\theta := \top$, and (R $[\top]$).

6. S4 system for \blacksquare : the derivation of (Nec) rule for \blacksquare easily follows from (Nec $!$) and (5) in Proposition 4.3.14. The T-axiom for \blacksquare follows from (4) in Proposition 4.3.14, RE, and R $[\top]$.

For the K-axiom:

1. $\vdash (\blacksquare(\varphi \rightarrow \psi) \wedge \blacksquare\varphi) \rightarrow ([p](\varphi \rightarrow \psi) \wedge [p]\varphi)$
($p \notin P_\varphi \cup P_\psi$, (4) in Proposition 4.3.14)
2. $\vdash ([p](\varphi \rightarrow \psi) \wedge [p]\varphi) \rightarrow [p]\psi$ (K $!$)
3. $\vdash (\blacksquare(\varphi \rightarrow \psi) \wedge \blacksquare\varphi) \rightarrow [p]\psi$ (1, 2, CPL)
4. $\vdash (\blacksquare(\varphi \rightarrow \psi) \wedge \blacksquare\varphi) \rightarrow \blacksquare\psi$ ($p \notin P_\varphi \cup P_\psi$, (5) in Proposition 4.3.14)

For the 4-axiom:

1. $\vdash \blacksquare\varphi \rightarrow [p \wedge q]\varphi$ (for some $p, q \notin P_\varphi$, (4) in Proposition 4.3.14)
2. $\vdash \blacksquare\varphi \rightarrow [p]\blacksquare\varphi$ ($[!]\blacksquare$ -intro)
3. $\vdash \blacksquare\varphi \rightarrow \blacksquare\blacksquare\varphi$ ($p \notin P_\varphi$, (5) in Proposition 4.3.14)

7. $(\varphi \rightarrow \psi)^0 \leftrightarrow (\varphi^0 \rightarrow \psi^0)$: This is straightforward by the set of axioms called *Equivalences with* 0 .

8. $\vdash \varphi^{00} \leftrightarrow \varphi^0$:

1. $\vdash 0 \rightarrow (\varphi \leftrightarrow \varphi^0)$ (0-eq)
2. $\vdash (0 \rightarrow (\varphi \leftrightarrow \varphi^0))^0$ (Nec 0)
3. $\vdash 0^0 \rightarrow (\varphi \leftrightarrow \varphi^0)^0$ ((7) in Proposition 4.3.14)
4. $\vdash 0^0 \rightarrow (\varphi^0 \leftrightarrow \varphi^{00})$ ((7) in Proposition 4.3.14 and (Eq $_{\wedge}^0$))
5. $\vdash 0^0$ (Ax $_0$)
6. $\vdash \varphi^0 \leftrightarrow \varphi^{00}$ (4, 5, MP)

9. $\blacksquare\varphi^0 \leftrightarrow \varphi^0$ and $\varphi^0 \leftrightarrow \blacklozenge\varphi^0$: From left-to-right direction of both cases follow from the T-axiom for \blacksquare . From right-to-left direction we will only prove $\varphi^0 \rightarrow \blacksquare\varphi^0$ since the remaining implication follows simply by definition of the dual for \blacksquare . By an instance of the rule (5) in Proposition 4.3.14 (\blacksquare -intro), it is sufficient to show that $\vdash \varphi^0 \rightarrow [p]\varphi^0$ for $p \notin P_\varphi$:

1. $\vdash \varphi^0 \rightarrow (p \rightarrow \varphi^0)$ (for $p \notin P_\varphi$, CPL)
2. $\vdash \varphi^0 \rightarrow [p]\varphi^0$ (R 0)
3. $\vdash \varphi^0 \rightarrow \blacksquare\varphi^0$ ($(\blacksquare$ -intro) rule for $p \notin P_\varphi$)

$$\boxed{10.} \vdash (\blacksquare\varphi)^0 \rightarrow \blacksquare\varphi^0$$

1. $\vdash \blacksquare\varphi \rightarrow \varphi$ (S4 for \blacksquare)
2. $\vdash (\blacksquare\varphi \rightarrow \varphi)^0$ (Nec⁰)
3. $\vdash (\blacksquare\varphi)^0 \rightarrow \varphi^0$ (($\boxed{7}$) in Proposition $\boxed{4.3.14}$, 2, MP)
4. $\vdash (\blacksquare\varphi)^0 \rightarrow \blacksquare\varphi^0$ (($\boxed{9}$) in Proposition $\boxed{4.3.14}$)

$$\boxed{11.} \vdash (0 \wedge \blacklozenge\varphi^0) \rightarrow \varphi$$

1. $\vdash 0 \rightarrow (\varphi^0 \leftrightarrow \varphi)$ (0-*eq*)
2. $\vdash 0 \rightarrow (\varphi^0 \rightarrow \varphi)$ (CPL)
3. $\vdash 0 \rightarrow (\blacklozenge\varphi^0 \rightarrow \varphi)$ (($\boxed{9}$) in Proposition $\boxed{4.3.14}$)
4. $\vdash (0 \wedge \blacklozenge\varphi^0) \rightarrow \varphi$ (CPL)

$$\boxed{12.} \vdash \varphi \rightarrow (0 \wedge \blacklozenge\varphi)^0$$

1. $\vdash (\blacksquare\neg\varphi)^0 \rightarrow \neg\varphi$ (Imp \blacksquare^0)
2. $\vdash \neg\neg\varphi \rightarrow \neg(\blacksquare\neg\varphi)^0$ (contraposition of 1)
3. $\vdash \neg\neg\varphi \rightarrow (\neg\blacksquare\neg\varphi)^0$ (Eq \neg^0)
4. $\vdash \varphi \rightarrow (\blacklozenge\varphi)^0$ (the definition of \blacklozenge)
5. $\vdash \varphi \rightarrow (0^0 \wedge (\blacklozenge\varphi)^0)$ (Ax $_0$)
6. $\vdash \varphi \rightarrow (0 \wedge \blacklozenge\varphi)^0$ (Eq \wedge^0)

$$\boxed{13.} \vdash \varphi \rightarrow \psi^0 \text{ if and only if } \vdash (0 \wedge \blacklozenge\varphi) \rightarrow \psi$$

From left-to-right: Suppose $\vdash \varphi \rightarrow \psi^0$ and show: $\vdash (0 \wedge \blacklozenge\varphi) \rightarrow \psi$.

1. $\vdash (0 \wedge \blacklozenge\psi^0) \rightarrow \psi$ (($\boxed{11}$) in Proposition $\boxed{4.3.14}$)
2. $\vdash \blacklozenge\varphi \rightarrow \blacklozenge\psi^0$ (by assumption and Nec \blacksquare)
3. $\vdash (0 \wedge \blacklozenge\varphi) \rightarrow (0 \wedge \blacklozenge\psi^0)$ (2 and CPL)
4. $\vdash (0 \wedge \blacklozenge\varphi) \rightarrow \psi$ (3, 1, CPL)

From right-to-left: Suppose $\vdash (0 \wedge \blacklozenge\varphi) \rightarrow \psi$ and show $\vdash \varphi \rightarrow \psi^0$.

1. $\vdash \varphi \rightarrow (0 \wedge \blacklozenge\varphi)^0$ ((12) in Proposition 4.3.14)
2. $\vdash (0 \wedge \blacklozenge\varphi) \rightarrow \psi$ (assumption)
3. $\vdash ((0 \wedge \blacklozenge\varphi) \rightarrow \psi)^0$ (Nec⁰)
4. $\vdash (0 \wedge \blacklozenge\varphi)^0 \rightarrow \psi^0$ ((7) in Proposition 4.3.14)
5. $\vdash \varphi \rightarrow \psi^0$ (1, 4, CPL)

14. $[\theta](\varphi \wedge \psi) \leftrightarrow ([\theta]\varphi \wedge [\theta]\psi)$: Follows from (K_I) and (Nec_I).

15. $[\theta][p]\varphi \leftrightarrow [\theta \wedge p]\varphi$

1. $\vdash [\theta][p]\varphi \leftrightarrow [(\theta)p]\varphi$ (R_[])
2. $\vdash [(\theta)p]\varphi \leftrightarrow [\theta \wedge [p]]\varphi$ ((3) in Proposition 4.3.14, RE)
3. $\vdash [\theta \wedge [p]]\varphi \leftrightarrow [\theta \wedge (\theta \rightarrow p)]\varphi$ (R_p, RE)
4. $\vdash [\theta \wedge (\theta \rightarrow p)]\varphi \leftrightarrow [\theta \wedge p]\varphi$ (CPL, RE)
5. $\vdash [\theta][p]\varphi \leftrightarrow [\theta \wedge p]\varphi$ (1, 4, CPL)

16. $[\theta]\perp \leftrightarrow \neg\theta$: this is an easy consequence of (14) in Proposition 4.3.14, (R_p), and (R₋).

□

We arrive now at the main result of this section.

4.3.15. THEOREM (SOUNDNESS AND COMPLETENESS OF APALM). *APALM validities are recursively enumerable. Indeed, the axiom system APALM in Table 4.1 is sound and complete wrt a-models.*

Both soundness and completeness proofs are rather involved, thus, given in separate Sections 4.5 and 4.6, respectively.

4.4 Group Announcement Logic with Memory: GALM

In this section we turn our focus on the Group Announcement Logic (GAL) introduced in (Ågotnes et al., 2010). As briefly mentioned in the Introduction, GAL is also an extension of PAL, involving group announcement operators $[G]\varphi$

and $\langle G \rangle \varphi$ (instead of the arbitrary announcement operators $\blacksquare \varphi$ and $\blacklozenge \varphi$). The group announcement operator can be seen as a restricted version of the arbitrary public announcement operator in the sense that it quantifies only over updates with formulas of the form $\bigwedge_{i \in G} K_i \theta_i$, where $\theta_i \in \mathcal{L}^{epi}$ and $i \in G \subseteq \mathcal{AG}$. More precisely, [Ågotnes et al. \(2010\)](#) interpret the operator $[G]\varphi$ on epistemic models $M = (W, \sim_1, \dots, \sim_n, \|\cdot\|)$ as

$$w \in \llbracket [G]\varphi \rrbracket \quad \text{iff} \quad \text{for every set } \{\psi_i : i \in G\} \subseteq \mathcal{L}^{epi}, w \in \llbracket \bigwedge_{i \in G} K_i \psi_i \rrbracket \varphi.$$

This operator intends to capture communication among a group of agents and what a coalition can bring about via public announcements. While GAL seems to provide more adequate tools than APAL to treat puzzles involving epistemic dialogues, the axiomatization of GAL presented in [\(Ågotnes et al., 2010\)](#) has a similar shape as the one for APAL in [\(Balbiani et al., 2008\)](#). To recall, [Ågotnes et al. \(2010\)](#) prove completeness of GAL also by using an infinitary rule and claims that it is replaceable in theorem-proving by the finitary rule

$$\text{from } \varphi \rightarrow [\theta] \left[\bigwedge_{i \in G} K_i p_i \right] \psi \text{ infer } \varphi \rightarrow [\theta] [G] \psi, \quad (R[G])$$

where $p_i \notin P_\varphi \cup P_\psi \cup P_\theta$. However, Kuijjer's counterexample presented in Section [4.3.2](#) constitutes a counterexample also for the soundness of this rule. Consider again the formula $\gamma := p \wedge \hat{K}_b \neg p \wedge \hat{K}_a K_b p$ and let $G = \{a\}$. We show that while

$$[\hat{K}_b p] [G] \neg \gamma \rightarrow [K_a q] \neg \gamma$$

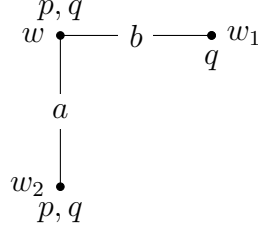
is valid in epistemic models, its $R[G]$ -conclusion

$$[\hat{K}_b p] [G] \neg \gamma \rightarrow [G] \neg \gamma$$

is not. For the former, suppose that $[\hat{K}_b p] [G] \neg \gamma \rightarrow [K_a q] \neg \gamma$ is not valid on epistemic models, i.e., that there is an epistemic model $N = (W, \sim_1, \dots, \sim_n, \|\cdot\|)$ and $w \in W$ such that $w \in \llbracket [\hat{K}_b p] [G] \neg \gamma \rrbracket$ but $w \notin \llbracket [K_a q] \neg \gamma \rrbracket$. The latter means that $w \in \llbracket [K_a q] \gamma \rrbracket$. Therefore, $w \in \llbracket [K_a q] \rrbracket$ and $w \in \llbracket \gamma \rrbracket_{N \upharpoonright \llbracket [K_a q] \rrbracket}$. The latter implies that $w \in \llbracket p \rrbracket$ and there are two states w_1, w_2 in $N \upharpoonright \llbracket [K_a q] \rrbracket$ such that (1) w_1 is \sim_b -connected to w and $w_1 \notin \llbracket p \rrbracket$, and (2) w_2 is \sim_a -connected to w and $w_2 \in \llbracket [K_b p] \rrbracket_{N \upharpoonright \llbracket [K_a q] \rrbracket}$. In other words, the model in [Figure 4.8](#) is guaranteed to be a submodel of $N \upharpoonright \llbracket [K_a q] \rrbracket$.

Moreover, since $w \in \llbracket [\hat{K}_b p] [G] \neg \gamma \rrbracket$ and $w \in \llbracket [\hat{K}_b p] \rrbracket$, we also have that $w \in \llbracket [G] \neg \gamma \rrbracket_{N \upharpoonright \llbracket [\hat{K}_b p] \rrbracket}$. Recall that $w \in \llbracket [K_a q] \rrbracket$. Therefore, neither w nor w_2 have \sim_a -access to a state in N that makes q false. Furthermore, since $K_a q$ is a positive knowledge formula¹⁴, we have $w \in \llbracket [K_a q]_{N \upharpoonright \theta} \rrbracket$ for any θ . Then, $w \in \llbracket [G] \neg \gamma \rrbracket_{N \upharpoonright \llbracket [\hat{K}_b p] \rrbracket}$

¹⁴The *positive formulas* are those that do not express ignorance, namely the formulas that do not contain subformulas of the form $\neg K_a \theta$ for any $a \in \mathcal{AG}$ and any formula θ in the corresponding language. [Balbiani et al. \(2008\)](#) proved that APAL positive formulas preserve truth under arbitrary epistemically definable model restriction (see Proposition 3.16, p. 320).

Figure 4.8: Submodel of $N[[K_a q]]$

implies that $w \in \llbracket \neg\gamma \rrbracket_{(N[[\hat{K}_b p]])[[K_a q]]_{N[[\hat{K}_b p]]}} = \llbracket \neg\gamma \rrbracket_{(N[[\langle \hat{K}_b p \rangle K_a q]])}$. It is not difficult to see that the model in Figure 4.8 is also a submodel of $N[[\langle \hat{K}_b p \rangle K_a q]]$ (recall that w_1 is in $N[[K_a q]]$), thus, $w \in \llbracket \gamma \rrbracket_{(N[[\langle \hat{K}_b p \rangle K_a q]])}$. This contradicts the assumption that $w \in \llbracket [\hat{K}_b p][G]\neg\gamma \rrbracket$. Therefore, $[\hat{K}_b p][G]\neg\gamma \rightarrow [K_a q]\neg\gamma$ is valid on epistemic models. However, model M in Figure 4.3 constitutes a counterexample for $[\hat{K}_b p][G]\neg\gamma \rightarrow [G]\neg\gamma$, as $w \in \llbracket [\hat{K}_b p][G]\neg\gamma \rrbracket_M$ and $w \in \llbracket \langle K_a \hat{K}_a(p \vee r) \rangle \gamma \rrbracket_M$, thus, $w \notin \llbracket [G]\neg\gamma \rrbracket$.

To the best of our knowledge, there had been no known recursive axiomatization for GAL or a stronger version of it. In this section, we provide a recursive axiomatization for Group Announcement Logic with Memory (GALM), obtained by extending the syntax of APALM with group announcement operators interpreted on a -models¹⁵

The language \mathcal{L}_G of GALM is defined recursively, for each group of agents $G \subseteq \mathcal{AG}$, as:

$$\varphi ::= \top \mid p \mid 0 \mid \varphi^0 \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid U\varphi \mid \langle \theta \rangle\varphi \mid \blacklozenge\varphi \mid \langle G \rangle\varphi,$$

where $p \in \text{Prop}$, $i \in \mathcal{AG}$, and $\theta \in \mathcal{L}_{-\blacklozenge}$. The dual modality for this new operator is defined as $[G]\varphi := \neg\langle G \rangle\neg\varphi$. $\langle G \rangle\varphi$ and $[G]\varphi$ are the (existential and universal) group announcement operators, quantifying over updates with formulas of the form $\bigwedge_{i \in G} K_i\theta_i$, where $\theta_i \in \mathcal{L}_{-\blacklozenge}$ and $i \in G$. This restricted quantification over $\mathcal{L}_{-\blacklozenge}$ captures the assumption that each agent can announce only the (\blacklozenge -free and $\langle G \rangle$ -free) propositions she knows and nothing else. Analogous to the reading of \blacksquare , we read $[G]\varphi$ as “ φ is *stably true* under group G ’s public announcements”, i.e., “ φ stays true no matter what group G truthfully announces”.

We introduce the following abbreviation of *relativized knowledge* for notational convenience:

$$K_i^\varphi\psi := K_i(\varphi \rightarrow \psi),$$

¹⁵We note that the language of the original GAL in (Ågotnes et al., 2010) does not include the arbitrary announcement operator \blacksquare . The fragment of GALM without the arbitrary announcement operators can be studied in a similar way. We prefer to work with this large language in order to be able to present the soundness and completeness proofs for APALM and GALM in a unified way.

where $\varphi, \psi \in \mathcal{L}_G$ and $i \in \mathcal{AG}$. The language \mathcal{L}_G is also interpreted on models introduced in Definition [4.3.1](#).

4.4.1. DEFINITION. Given a model $M = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$, the semantics for \mathcal{L}_G is defined recursively as in Definition [4.3.2](#) with the following additional clause for $\langle G \rangle$:

$$\llbracket \langle G \rangle \varphi \rrbracket = \bigcup_{i \in G} \{ \llbracket \langle \bigwedge_{i \in G} K_i \theta_i \rangle \varphi \rrbracket : \{ \theta_i : i \in G \} \subseteq \mathcal{L}_{-\diamond} \}.$$

2. OBSERVATION. Note that we have

$$w \in \llbracket [G] \varphi \rrbracket \quad \text{iff} \quad w \in \llbracket \langle \bigwedge_{i \in G} K_i \theta_i \rangle \varphi \rrbracket \quad \text{for every } \{ \theta_i : i \in G \} \subseteq \mathcal{L}_{-\diamond}.$$

4.4.2. PROPOSITION. We have $\llbracket \varphi \rrbracket \subseteq W$, for all formulas $\varphi \in \mathcal{L}_G$.

Proof:

The proof is by \prec_2 -induction on φ , using Lemma [A.2.5](#) from the Technical Appendix [A.A.2](#) and the following *induction hypothesis* (IH): for all $\psi \in \mathcal{L}_G$ such that $\psi \prec_2 \varphi$ and all models $M = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$, we have $\llbracket \psi \rrbracket \subseteq W$. The base cases $\varphi := \top$, $\varphi := p$, and $\varphi := 0$ are straightforward by the semantics given in Definition [4.3.2](#). The inductive cases for Booleans are immediate. Similarly, the following cases make use of the corresponding semantic clause in Definition [4.3.2](#).

Case $\varphi := \psi^0$: $\llbracket \psi^0 \rrbracket = \llbracket \psi \rrbracket_{M^0} \cap W \subseteq W$.

Case $\varphi := K_i \psi$: $\llbracket K_i \psi \rrbracket = \{ w \in W : w_i \subseteq \llbracket \psi \rrbracket \} \subseteq W$.

Case $\varphi := U\psi$: $\llbracket U\psi \rrbracket \in \{ \emptyset, W \}$, thus $\llbracket U\psi \rrbracket \subseteq W$.

Case $\varphi := \langle \theta \rangle \psi$: Since $\theta \prec_2 \langle \theta \rangle \psi$ (by [\(1.\)](#) in Lemma [A.2.5](#)), by the IH on θ , we have that $\llbracket \theta \rrbracket \subseteq W$. Moreover, since $\psi \prec_2 \langle \theta \rangle \psi$ (by [\(1.\)](#) in Lemma [A.2.5](#)), by the IH on ψ , we also have that $\llbracket \psi \rrbracket_{M \llbracket \theta \rrbracket} \subseteq \llbracket \theta \rrbracket$ (recall that $M \llbracket \theta \rrbracket = (W^0, \llbracket \theta \rrbracket, \sim_1, \dots, \sim_n, \|\cdot\|)$). Therefore, by Definition [4.3.2](#), we obtain that $\llbracket \langle \theta \rangle \psi \rrbracket = \llbracket \psi \rrbracket_{M \llbracket \theta \rrbracket} \subseteq \llbracket \theta \rrbracket \subseteq W$.

Case $\varphi := \diamond \psi$: By [\(9.\)](#) in Lemma [A.2.5](#), it follows that for each $\theta \in \mathcal{L}_{-\diamond}$, $\langle \theta \rangle \psi \prec_2 \diamond \psi$. Then, by the IH, we have that for all $\theta \in \mathcal{L}_{-\diamond}$, $\llbracket \langle \theta \rangle \psi \rrbracket \subseteq W$. Thus $\bigcup \{ \llbracket \langle \theta \rangle \psi \rrbracket : \theta \in \mathcal{L}_{-\diamond} \} \subseteq W$, i.e., $\llbracket \diamond \psi \rrbracket \subseteq W$.

Case $\varphi := \langle G \rangle \psi$: By [\(10.\)](#) in Lemma [A.2.5](#), it follows that for each $\theta \in \mathcal{L}_{-\diamond}$, $\langle \theta \rangle \psi \prec_2 \langle G \rangle \psi$. Then, by the IH, we have that for all $\theta_i \in \mathcal{L}_{-\diamond}$, $\llbracket \langle \bigwedge_{i \in G} K_i \theta_i \rangle \psi \rrbracket \subseteq W$. Thus $\bigcup \{ \llbracket \langle \bigwedge_{i \in G} K_i \theta_i \rangle \psi \rrbracket : \theta_i \in \mathcal{L}_{-\diamond} \} \subseteq W$, i.e., $\llbracket \langle G \rangle \psi \rrbracket \subseteq W$. \square

Our *a*-models given in Definition [4.3.3](#) are also the *intended* models for GALM, so GALM validities are defined with respect to *a*-models as in Definition [4.3.3](#). We can now state the main result of this section.

4.4.3. THEOREM (SOUNDNESS AND COMPLETENESS OF GALM). *GALM validities are recursively enumerable. In fact, the sound and complete axiomatization **GALM** wrt a -models is obtained by extending **APALM** with the axiom and rule given in Table 4.2.*

Elim-axiom and Intro-rule for $[G]$:	
($[!]$ $[G]$ -elim)	$[\theta][G]\varphi \rightarrow [\theta \wedge \bigwedge_{i \in G} K_i^\theta \rho_i]\varphi$
($[!]$ $[G]$ -intro)	from $\vdash \chi \rightarrow [\theta \wedge \bigwedge_{i \in G} K_i^\theta p_i]\varphi$, infer $\vdash \chi \rightarrow [\theta][G]\varphi$ (for $p_i \notin P_\chi \cup P_\theta \cup P_\varphi$).

Table 4.2: The additional axioms of **GALM**

The axiom and rule in Table 4.2 are very similar in spirit and in what they express to the $[!]\blacksquare$ -elim axiom and $[!]\blacksquare$ -intro rule, respectively. Together, the elimination axiom $[!][G]$ -elim and introduction rule $[!][G]$ -intro say that φ is a stable truth under group G 's announcements after an announcement θ iff φ stays true after any more informative announcement from the group G (of the form $\theta \wedge \bigwedge_{i \in G} K_i^\theta \rho_i$).

4.5 Soundness of GALM and APALM

As GALM is an extension of APALM, we present the soundness and completeness proofs directly for the former. The same results for APALM are obtained following similar steps. For the soundness and completeness proofs of only APALM, we refer the reader to (Baltag et al., 2018b).

To start with, note that even the soundness of our axiomatic systems is not a trivial matter. As we saw from Kuijer's counterexample, the analogues of our finitary \blacksquare and $[G]$ -introduction rules were *not* sound for APAL and GAL, respectively. To prove their soundness on a -models, we need a detour into an equivalent semantics, in the style of Subset Space Logics (SSL) (Moss and Parikh, 1992; Dabrowski et al., 1996): *pseudo-models*. A more direct soundness proof on a -models is in principle possible, but would require at least as much work as our detour.

Unlike in standard EL, PAL or DEL, the meaning of an APALM formula (and therefore of a GALM formula) depends, not only on the valuation of the atoms occurring in it, but also on the family \mathcal{A} of all sets definable by $\mathcal{L}_{-\blacklozenge}$ -formulas. The move from a -models to pseudo-models makes explicit this dependence on the family \mathcal{A} , while also relaxing the demands on \mathcal{A} (which is no longer required to be exactly the family of $\mathcal{L}_{-\blacklozenge}$ -definable sets), and thus makes the soundness proof both simpler and more transparent. Since we will need pseudo-models for our

completeness proof anyway, we see no added value in trying to give a more direct soundness proof.

We first introduce an auxiliary notion: *pre-models* are just SSL models, coming with a given family \mathcal{A} of “admissible sets” of worlds (which can be thought of as the *communicable* propositions). We interpret \blacksquare in these structures as the so-called “effort modality” of SSL, which quantifies over updates with admissible propositions in \mathcal{A} . Analogously, $\langle G \rangle$ quantifies over updates with conjunctions of those admissible propositions in the scope of an epistemic operator labeled by an agent in G . Our *pseudo-models* are pre-models with additional closure conditions (saying that the family of admissible sets includes the valuations and is closed under complement, intersection, and epistemic operators). These conditions imply that *every set definable by a $\blacklozenge, \langle G \rangle$ -free formula¹⁶ is admissible*, and this ensures the soundness of our \blacksquare -elimination and $[G]$ -elimination axioms on pseudo-models.

As for the soundness of the long-problematic \blacksquare and $[G]$ -introduction rules on (both pre- and) pseudo-models, this is due to the fact that both the effort modality and the corresponding $[G]$ operator interpreted on pseudo-models have a more “robust” range than the arbitrary announcement versions of them: they quantify over admissible sets, regardless of whether these sets are syntactically definable or not. Soundness with respect to our *a*-models then follows from the observation that they (in contrast to the original APAL models) are in fact equivalent to a *special case of pseudo-models*: the “standard” ones, in which the admissible sets in \mathcal{A} are *exactly* the sets definable by $\blacklozenge, \langle G \rangle$ -free formulas.

4.5.1. DEFINITION. [Pre-model] A *pre-model* is a tuple $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$, where W^0 is the initial domain, \sim_i are equivalence relations on W^0 , $\|\cdot\| : \text{Prop} \rightarrow \mathcal{P}(W^0)$ is a valuation map, and $\mathcal{A} \subseteq \mathcal{P}(W^0)$ is a family of subsets of the initial domain, called *admissible sets* (representing the propositions that can be publicly announced).

Given a set $A \subseteq W^0$ and a state $w \in A$, we use the notation $w_i^A := \{s \in A : w \sim_i s\}$ to denote the restriction to A of w 's equivalence class modulo \sim_i . We also introduce the following abbreviation for the semantic counterpart of relativized knowledge: $K_i^A B = \{w \in W^0 : w_i \cap A \subseteq B\}$.

4.5.2. DEFINITION. [Pre-model Semantics for \mathcal{L}_G] Given a pre-model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$, we recursively define a *truth set* $\llbracket \varphi \rrbracket_A$ for every formula φ and subset $A \subseteq W^0$:

¹⁶ $\blacklozenge, \langle G \rangle$ -free formulas are the sentences in $\mathcal{L}_{-\blacklozenge}$.

$$\begin{array}{ll}
\llbracket \top \rrbracket_A = A & \llbracket K_i \varphi \rrbracket_A = \{w \in A : w_i^A \subseteq \llbracket \varphi \rrbracket_A\} \\
\llbracket p \rrbracket_A = \llbracket p \rrbracket \cap A & \llbracket U \varphi \rrbracket_A = \begin{cases} A & \text{if } \llbracket \varphi \rrbracket_A = A \\ \emptyset & \text{otherwise} \end{cases} \\
\llbracket 0 \rrbracket_A = \begin{cases} A & \text{if } A = W^0 \\ \emptyset & \text{otherwise} \end{cases} & \llbracket \langle \theta \rangle \varphi \rrbracket_A = \llbracket \varphi \rrbracket_{\llbracket \theta \rrbracket_A} \\
\llbracket \varphi^0 \rrbracket_A = \llbracket \varphi \rrbracket_{W^0} \cap A & \llbracket \blacklozenge \varphi \rrbracket_A = \bigcup \{ \llbracket \varphi \rrbracket_B : B \in \mathcal{A}, B \subseteq A \} \\
\llbracket \neg \varphi \rrbracket_A = A - \llbracket \varphi \rrbracket_A & \llbracket \langle G \rangle \psi \rrbracket_A = \bigcup \{ \llbracket \psi \rrbracket_{A \cap \bigcap_{i \in G} K_i^A B_i} : \{B_i : i \in G\} \subseteq \mathcal{A} \} \\
\llbracket \varphi \wedge \psi \rrbracket_A = \llbracket \varphi \rrbracket_A \cap \llbracket \psi \rrbracket_A &
\end{array}$$

3. OBSERVATION. Note that, for all $w \in A$, we have

1. $w \in \llbracket \blacksquare \varphi \rrbracket_A$ iff $\forall B \in \mathcal{A} (w \in B \subseteq A \Rightarrow w \in \llbracket \varphi \rrbracket_B)$;
2. $w \in \llbracket \langle G \rangle \varphi \rrbracket_A$ iff $w \in \llbracket \varphi \rrbracket_{A \cap \bigcap_{i \in G} K_i^A B_i}$ for every $\{B_i : i \in G\} \subseteq \mathcal{A}$;
3. $\llbracket \varphi \rrbracket_A \subseteq A$ for all $A \in \mathcal{A}$ and $\varphi \in \mathcal{L}_G$.

Observation 3.1 shows that our proposed semantics of \blacksquare on pre-models fits with the semantics of the effort modality \square in SSL (Moss and Parikh, 1992; Dabrowski et al., 1996). The proof of Observation 3.3 is similar to that of Proposition 4.4.2.

4.5.3. DEFINITION. [Pseudo-models and Validity] A *pseudo-model* is a pre-model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$, satisfying the following closure conditions:

1. $\llbracket p \rrbracket \in \mathcal{A}$, for all $p \in \text{Prop}$,
2. $W^0 \in \mathcal{A}$,
3. if $A \in \mathcal{A}$ then $(W^0 - A) \in \mathcal{A}$,
4. if $A, B \in \mathcal{A}$ then $(A \cap B) \in \mathcal{A}$,
5. if $A \in \mathcal{A}$ then $K_i A \in \mathcal{A}$, where $K_i A := \{w \in W^0 : \forall s \in W^0 (w \sim_i s \Rightarrow s \in A)\}$.

A formula $\varphi \in \mathcal{L}_G$ is *satisfied in a pseudo-model* \mathcal{M} if $\llbracket \varphi \rrbracket_A = A$ for all $A \in \mathcal{A}$ in \mathcal{M} . A formula $\varphi \in \mathcal{L}_G$ is *valid in pseudo-models* if it is true in all admissible sets $A \in \mathcal{A}$ of every pseudo-model \mathcal{M} , i.e., $\llbracket \varphi \rrbracket_A = A$ for all $A \in \mathcal{A}$ and all \mathcal{M} .

4.5.4. LEMMA. Given a pseudo-model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ and $A, B \in \mathcal{A}$, we have $K_i^A B \in \mathcal{A}$.

Proof:

First note that by clause (5) in Definition 4.5.3 and Boolean operations of sets we have,

$$K_i^A B = \{w \in W^0 : w_i \cap A \subseteq B\} = \{w \in W^0 : \forall s \in W^0 ((s \in A \text{ and } w \sim_i s) \Rightarrow$$

$s \in B\}$
 $= \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow (s \in A \Rightarrow s \in B))\} = \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow (s \in (W^0 - A) \text{ or } s \in B))\} = \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow s \in (W^0 - A) \cup B)\} = K_i((W^0 - A) \cup B).$

Then, by Definition 4.5.3 (3-5) and $A, B \in \mathcal{A}$, we obtain $K_i^A B = K_i((W^0 - A) \cup B) \in \mathcal{A}$. \square

The following result shows that a truth set of any $\blacklozenge, \langle G \rangle$ -free formula is an admissible set.

4.5.5. PROPOSITION. *Given a pseudo-model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$, $A \in \mathcal{A}$, and $\theta \in \mathcal{L}_{-\blacklozenge}$, we have $\llbracket \theta \rrbracket_A \in \mathcal{A}$.*

Proof:

The proof is by subformula induction on θ . The base cases and the inductive cases for the Booleans are immediate (using the conditions in Definition 4.5.3).

Case $\theta := \psi^0$: By the semantics, $\llbracket \psi^0 \rrbracket_A = \llbracket \psi \rrbracket_{W^0} \cap A \in \mathcal{A}$, since $\llbracket \psi \rrbracket_{W^0} \in \mathcal{A}$ (by the fact that $W^0 \in \mathcal{A}$ and IH), $A \in \mathcal{A}$ (by assumption), and \mathcal{A} is closed under intersection.

Case $\theta := K_i \psi$: Note that $\llbracket K_i \psi \rrbracket_A = \{w \in A : w_i^A \subseteq \llbracket \psi \rrbracket_A\} = A \cap \{w \in W^0 : w_i^A \subseteq \llbracket \psi \rrbracket_A\}$ (by Definition 4.5.3) $= A \cap \{w \in W^0 : \forall s \in W^0((s \in A \text{ and } w \sim_i s) \Rightarrow s \in \llbracket \psi \rrbracket_A)\}$. We then obtain, by CPL and Boolean operations of sets that $\llbracket K_i \psi \rrbracket_A = A \cap \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow s \in ((W^0 - A) \cup \llbracket \psi \rrbracket_A))\}$. Moreover, $A \cap \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow s \in ((W^0 - A) \cup \llbracket \psi \rrbracket_A))\} = A \cap K_i((W^0 - A) \cup \llbracket \psi \rrbracket_A)$ by (3 to 5) in Definition 4.5.3 (since $A \in \mathcal{A}$ and $\llbracket \psi \rrbracket_A \in \mathcal{A}$ by IH). Therefore, $\llbracket K_i \psi \rrbracket_A = A \cap K_i((W^0 - A) \cup \llbracket \psi \rrbracket_A)$ is in \mathcal{A} .

Case $\theta := U\psi$: By Definition 4.5.3, $\llbracket U\psi \rrbracket_A \in \{\emptyset, A\} \subseteq \mathcal{A}$.

Case $\theta := \langle \varphi \rangle \psi$: Since $A \in \mathcal{A}$, we have $\llbracket \varphi \rrbracket_A \in \mathcal{A}$ (by IH on φ), and hence $\llbracket \langle \varphi \rangle \psi \rrbracket_A = \llbracket \psi \rrbracket_{\llbracket \varphi \rrbracket_A} \in \mathcal{A}$ (by the semantics and IH on ψ). \square

To prove the soundness of our axioms, we need the following lemmas:

4.5.6. LEMMA. *Given a pseudo-model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$, $A \in \mathcal{A}$ and $\theta \in \mathcal{L}_{-\blacklozenge}$ such that $w \in \llbracket \theta \rrbracket_A$, $w \in \llbracket K_i^\theta \rho \rrbracket_A$ iff $w \in K_i^{\llbracket \theta \rrbracket_A} \llbracket \rho \rrbracket_A$ for all $\rho \in \mathcal{L}_{-\blacklozenge}$.*

Proof:

Observe that $K_i^{\llbracket \theta \rrbracket_A} \llbracket \rho \rrbracket_A = K_i((W^0 - \llbracket \theta \rrbracket_A) \cup \llbracket \rho \rrbracket_A)$ (as in Lemma 4.5.4). Moreover, it's easy to see that,

$\llbracket K_i^\theta \rho \rrbracket_A = \llbracket K_i(\theta \rightarrow \rho) \rrbracket_A = \{w \in A : w_i^A \subseteq \llbracket \theta \rightarrow \rho \rrbracket_A\} = A \cap \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow s \in ((W^0 - \llbracket \theta \rrbracket_A) \cup \llbracket \rho \rrbracket_A))\}$ (since $\llbracket \theta \rrbracket_A \subseteq A$). We therefore obtain that $\llbracket K_i^\theta \rho \rrbracket_A = A \cap K_i((W^0 - \llbracket \theta \rrbracket_A) \cup \llbracket \rho \rrbracket_A)$ (by Boolean operations

of sets and the defn. of K_i). Thus, $\llbracket K_i^\theta \rho \rrbracket_A = A \cap K_i((W^0 - \llbracket \theta \rrbracket_A) \cup \llbracket \rho \rrbracket_A) \subseteq K_i((W^0 - \llbracket \theta \rrbracket_A) \cup \llbracket \rho \rrbracket_A) = K_i^{\llbracket \theta \rrbracket_A} \llbracket \rho \rrbracket_A$. Therefore if $w \in \llbracket \theta \rrbracket_A \subseteq A$, $w \in \llbracket K_i^\theta \rho \rrbracket_A$ iff $w \in K_i^{\llbracket \theta \rrbracket_A} \llbracket \rho \rrbracket_A$. \square

4.5.7. LEMMA. *Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ and $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|')$ be two pseudo-models and $\varphi \in \mathcal{L}_G$ such that \mathcal{M} and \mathcal{M}' differ only in the valuation of some $p \notin P_\varphi$. Then, for all $A \in \mathcal{A}$, we have $\llbracket \varphi \rrbracket_A^{\mathcal{M}} = \llbracket \varphi \rrbracket_A^{\mathcal{M}'}$.*

Proof:

The proof follows by subformula induction on φ . Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ and $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|')$ be two pseudo-models such that \mathcal{M} and \mathcal{M}' differ only in the valuation of some $p \notin P_\varphi$ and let $A \in \mathcal{A}$. We want to show that $\llbracket \varphi \rrbracket_A^{\mathcal{M}} = \llbracket \varphi \rrbracket_A^{\mathcal{M}'}$. The base cases $\varphi := q (\neq p)$, $\varphi := \top$, $\varphi := 0$, and the inductive cases for Booleans are standard.

Case $\varphi := \psi^0$. Note that $P_{\psi^0} = P_\psi$. Then, by IH, we have that $\llbracket \psi \rrbracket_A^{\mathcal{M}'} = \llbracket \psi \rrbracket_A^{\mathcal{M}}$ for every $A \in \mathcal{A}$, in particular for $W^0 \in \mathcal{A}$. Thus $\llbracket \psi \rrbracket_{W^0}^{\mathcal{M}'} = \llbracket \psi \rrbracket_{W^0}^{\mathcal{M}}$. Then, $\llbracket \psi \rrbracket_{W^0}^{\mathcal{M}'} \cap A = \llbracket \psi \rrbracket_{W^0}^{\mathcal{M}} \cap A$ for all $A \in \mathcal{A}$. By the semantics of the initial operator on pseudo-models, we obtain $\llbracket \psi^0 \rrbracket_A^{\mathcal{M}'} = \llbracket \psi^0 \rrbracket_A^{\mathcal{M}}$.

Case $\varphi := K_i \psi$. Note that $P_{K_i \psi} = P_\psi$. Then, by IH, we have that $\llbracket \psi \rrbracket_A^{\mathcal{M}} = \llbracket \psi \rrbracket_A^{\mathcal{M}'}$. Observe that $\llbracket K_i \psi \rrbracket_A^{\mathcal{M}} = \{w \in A : w_i^A \subseteq \llbracket \psi \rrbracket_A^{\mathcal{M}}\}$ and, similarly, $\llbracket K_i \psi \rrbracket_A^{\mathcal{M}'} = \{w \in A : w_i^A \subseteq \llbracket \psi \rrbracket_A^{\mathcal{M}'}\}$. Then, since $\llbracket \psi \rrbracket_A^{\mathcal{M}} = \llbracket \psi \rrbracket_A^{\mathcal{M}'}$, we obtain $\llbracket K_i \psi \rrbracket_A^{\mathcal{M}} = \llbracket K_i \psi \rrbracket_A^{\mathcal{M}'}$.

Case $\varphi := U\psi$. Note that $P_{U\psi} = P_\psi$. Then, by IH, we have that $\llbracket \psi \rrbracket_A^{\mathcal{M}'} = \llbracket \psi \rrbracket_A^{\mathcal{M}}$ for every $A \in \mathcal{A}$. We have two case: (1) If $\llbracket \psi \rrbracket_A^{\mathcal{M}'} = \llbracket \psi \rrbracket_A^{\mathcal{M}} = A$, then $\llbracket U\psi \rrbracket_A^{\mathcal{M}'} = A = \llbracket U\psi \rrbracket_A^{\mathcal{M}}$. (2) If $\llbracket \psi \rrbracket_A^{\mathcal{M}'} = \llbracket \psi \rrbracket_A^{\mathcal{M}} \neq A$, then $\llbracket U\psi \rrbracket_A^{\mathcal{M}'} = \llbracket U\psi \rrbracket_A^{\mathcal{M}} = \emptyset$.

Case $\varphi := \langle \theta \rangle \psi$. Note that $P_{\langle \theta \rangle \psi} = P_\theta \cup P_\psi$. By IH, we have $\llbracket \theta \rrbracket_A^{\mathcal{M}'} = \llbracket \theta \rrbracket_A^{\mathcal{M}}$ and $\llbracket \psi \rrbracket_A^{\mathcal{M}'} = \llbracket \psi \rrbracket_A^{\mathcal{M}}$ for every $A \in \mathcal{A}$. By Proposition 4.5.5, we know that $\llbracket \theta \rrbracket_A^{\mathcal{M}} = \llbracket \theta \rrbracket_A^{\mathcal{M}'} \in \mathcal{A}$. Therefore, in particular, we have $\llbracket \psi \rrbracket_{\llbracket \theta \rrbracket_A^{\mathcal{M}'}}^{\mathcal{M}'} = \llbracket \psi \rrbracket_{\llbracket \theta \rrbracket_A^{\mathcal{M}}}^{\mathcal{M}}$. Therefore, by the semantics of $\langle ! \rangle$ on pseudo-models, we obtain $\llbracket \langle \theta \rangle \psi \rrbracket_A^{\mathcal{M}'} = \llbracket \langle \theta \rangle \psi \rrbracket_A^{\mathcal{M}}$.

Case $\varphi := \blacklozenge \psi$. Note that $P_{\blacklozenge \psi} = P_\psi$. Since the same family of sets \mathcal{A} is carried by both models \mathcal{M} and \mathcal{M}' and since (by IH) $\llbracket \psi \rrbracket_B^{\mathcal{M}'} = \llbracket \psi \rrbracket_B^{\mathcal{M}}$ for all $B \in \mathcal{A}$, we get:

$$\llbracket \blacklozenge \psi \rrbracket_A^{\mathcal{M}'} = \bigcup \{ \llbracket \psi \rrbracket_B^{\mathcal{M}'} : B \in \mathcal{A}, B \subseteq A \} = \bigcup \{ \llbracket \psi \rrbracket_B^{\mathcal{M}} : B \in \mathcal{A}, B \subseteq A \} = \llbracket \blacklozenge \psi \rrbracket_A^{\mathcal{M}}.$$

Case $\varphi := [G]\psi$. Note that $P_{[G]\psi} = P_\psi$. Then, by (IH), we have that $\llbracket \psi \rrbracket_B^{\mathcal{M}'} = \llbracket \psi \rrbracket_B^{\mathcal{M}}$ for every $B \in \mathcal{A}$. In particular, $\llbracket \psi \rrbracket_B^{\mathcal{M}'} = \llbracket \psi \rrbracket_B^{\mathcal{M}}$ for the B 's of the form $A \cap K_i^A C$ with $A, C \in \mathcal{A}$ (recall that pseudo-models are closed under K_i^A operation and conjunction, see Definition 4.5.3 and Lemma 4.5.4). Since the same family of sets \mathcal{A} is carried by both models \mathcal{M} and \mathcal{M}' , we obtain: $\llbracket [G]\psi \rrbracket_A^{\mathcal{M}'} = \bigcup \{ \llbracket \psi \rrbracket_{A \cap \bigcap_{i \in G} K_i^A B_i}^{\mathcal{M}'} : \{B_i : i \in G\} \subseteq \mathcal{A} \} = \bigcup \{ \llbracket \psi \rrbracket_{A \cap \bigcap_{i \in G} K_i^A B_i}^{\mathcal{M}} : \{B_i :$

$$i \in G\} \subseteq \mathcal{A}\} = \llbracket [G]\psi \rrbracket_A^{\mathcal{M}}. \quad \square$$

4.5.8. PROPOSITION (SOUNDNESS OF GALM/APALM ON PSEUDO-MODELS).
The system GALM is sound wrt pseudo-models. Therefore, the system APALM is also sound wrt pseudo-models.

Proof:

The soundness proof follows by validity check. For most of the axioms and rules, the proof follows simply by spelling out the semantics wrt Definition 4.5.2. We present here only the soundness of (Nec⁰), (Imp_i⁰), (Imp_■⁰), the axioms [!]■-elim, [!][G]-elim and rules [!]■-intro, [!][G]-intro:

Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ be a pseudo-model, $A \in \mathcal{A}$, and $w \in A$ arbitrarily chosen:

(Nec⁰): Suppose φ is valid in all pseudo-models, i.e., $\llbracket \varphi \rrbracket_A = A$ for all $A \in \mathcal{A}'$ and every pseudo-model \mathcal{M}' . Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ be a pseudo-model, by assumption we have that $\llbracket \varphi \rrbracket_A = A$ for all $A \in \mathcal{A}$ in \mathcal{M} . We need to show that \mathcal{M} satisfies φ^0 , i.e., we need to show that $w \in \llbracket \varphi^0 \rrbracket_A$ for $A \in \mathcal{A}$ and $w \in A$ arbitrarily chosen. By the semantic definition (Definition 4.5.2), we must show that $w \in \llbracket \varphi \rrbracket_{W^0} \cap A$. We already have that $w \in A$. Moreover, since φ is valid in all pseudo-models, we have that $w \in \llbracket \varphi \rrbracket_{W^0}$ (since \mathcal{M}^0 is a pseudo-model). Thus, $w \in \llbracket \varphi \rrbracket_{W^0} \cap A$.

(Imp_i⁰): Suppose $w \in \llbracket (K_i\varphi)^0 \rrbracket_A$. We need to show that $w \in \llbracket K_i\varphi^0 \rrbracket_A$, i.e., $w \in \{u \in A : u_i^A \subseteq \llbracket \varphi^0 \rrbracket_A\}$ by Definition 4.5.2. By the semantic definition (Definition 4.5.2) of $\llbracket \varphi^0 \rrbracket_A$, we need to show that $w \in \{u \in A : u_i^A \subseteq \llbracket \varphi \rrbracket_{W^0} \cap A\}$. Since, by the definition of w_i^A , we have $w_i^A \subseteq A$, we just need to show that $w_i^A \subseteq \llbracket \varphi \rrbracket_{W^0}$. Because of our assumption, $w \in \llbracket K_i\varphi \rrbracket_{W^0} \cap A = \{v \in W^0 : v_i \subseteq \llbracket \varphi \rrbracket_{W^0}\} \cap A$. Thus $w_i \subseteq \llbracket \varphi \rrbracket_{W^0}$. Since $w_i^A = w_i \cap A \subseteq \llbracket \varphi \rrbracket_{W^0} \cap A$, we have $w_i^A \subseteq \llbracket \varphi \rrbracket_{W^0}$.

(Imp_■⁰): Suppose $w \in \llbracket (\blacksquare\varphi)^0 \rrbracket_A$. We need to show that $w \in \llbracket \varphi \rrbracket_A$. Note that $\llbracket (\blacksquare\varphi)^0 \rrbracket_A = \llbracket \blacksquare\varphi \rrbracket_{W^0} \cap A$. Therefore, $w \in \{u \in W^0 : \text{for all } B \in \mathcal{A} (u \in B \subseteq W^0 \text{ implies } u \in \llbracket \varphi \rrbracket_B)\}$ and $w \in A$. Then, by the former and the fact that $w \in A \subseteq W^0$, we obtain $w \in \llbracket \varphi \rrbracket_A$.

([!]■-elim): Let $\rho \in \mathcal{L}_{-\blacklozenge}$ and suppose (1) $w \in \llbracket [\theta]\blacksquare\varphi \rrbracket_A$ and (2) $w \in \llbracket \theta \wedge \rho \rrbracket_A$. We need to show that $w \in \llbracket \varphi \rrbracket_{[\theta \wedge \rho]_A}$. Assumption (1) means that if $w \in \llbracket \theta \rrbracket_A$ then $w \in \llbracket \blacksquare\varphi \rrbracket_{[\theta]_A}$. By assumption (2) and since $w \in \llbracket \theta \wedge \rho \rrbracket_A \subseteq \llbracket \theta \rrbracket_A$, we have $w \in \llbracket \blacksquare\varphi \rrbracket_{[\theta]_A}$. Thus, by the semantics of \blacksquare , we have $w \in \{u \in \llbracket \theta \rrbracket_A : \text{for all } B \in \mathcal{A} (u \in B \subseteq \llbracket \theta \rrbracket_A \Rightarrow u \in \llbracket \varphi \rrbracket_B)\}$. Therefore, for

$B := \llbracket \theta \wedge \rho \rrbracket_A \subseteq \llbracket \theta \rrbracket_A$ (since by Proposition 4.5.5, $\llbracket \theta \wedge \rho \rrbracket_A \in \mathcal{A}$) we have $w \in \llbracket \varphi \rrbracket_{\llbracket \theta \wedge \rho \rrbracket_A}$.

($\llbracket ! \rrbracket [G]$ -elim): Let $\{\rho_i : i \in G\} \subseteq \mathcal{L}_{-\blacklozenge}$ and suppose (1) $w \in \llbracket [\theta][G]\varphi \rrbracket_A$ and (2) $w \in \llbracket \theta \wedge \bigwedge_{i \in G} K_i^\theta \rho_i \rrbracket_A$. We need to show that $w \in \llbracket [\theta \wedge \bigwedge_{i \in G} K_i^\theta \rho_i] \varphi \rrbracket_A$, i.e., $w \in \llbracket \varphi \rrbracket_{\llbracket \theta \wedge \bigwedge_{i \in G} K_i^\theta \rho_i \rrbracket_A}$. Assumption (1) means that if $w \in \llbracket \theta \rrbracket_A$ then $w \in \llbracket [G]\varphi \rrbracket_{\llbracket \theta \rrbracket_A}$. I.e., by the semantic clause for $[G]$, we have that if $w \in \llbracket \theta \rrbracket_A$ then for all $\{B_i : i \in G\} \subseteq \mathcal{A}$, $w \in \llbracket \varphi \rrbracket_{\llbracket \theta \rrbracket_A \cap \bigcap_{i \in G} K_i^{\llbracket \theta \rrbracket_A} B_i}$. By (2) we have that $w \in \llbracket \theta \rrbracket_A$ and $w \in \llbracket \bigwedge_{i \in G} K_i^\theta \rho_i \rrbracket_A$. Thus, by (1), we obtain $w \in \llbracket \varphi \rrbracket_{\llbracket \theta \rrbracket_A \cap \bigcap_{i \in G} K_i^{\llbracket \theta \rrbracket_A} \llbracket \rho_i \rrbracket_A} = \llbracket \varphi \rrbracket_{\llbracket \theta \rrbracket_A \cap \bigcap_{i \in G} \llbracket K_i^\theta \rho_i \rrbracket_A} = \llbracket \varphi \rrbracket_{\llbracket \theta \wedge \bigwedge_{i \in G} K_i^\theta \rho_i \rrbracket_A}$ (by Proposition 4.5.5 and Lemma 4.5.6). Thus if $w \in \llbracket \theta \wedge \bigwedge_{i \in G} K_i^\theta \rho_i \rrbracket_A$ then $w \in \llbracket \varphi \rrbracket_{\llbracket \theta \wedge \bigwedge_{i \in G} K_i^\theta \rho_i \rrbracket_A}$.

($\llbracket ! \rrbracket \blacksquare$ -intro): Suppose $\chi \rightarrow [\theta \wedge p]\varphi$ with $p \notin P_\chi \cup P_\theta \cup P_\varphi$ is valid and $\chi \rightarrow [\theta] \blacksquare \varphi$ where $p \notin P_\chi \cup P_\theta \cup P_\varphi$ is not valid. The latter means that there exists a pseudo-model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ such that for some $A \in \mathcal{A}$ and some $w \in A$, $w \notin \llbracket \chi \rightarrow [\theta] \blacksquare \varphi \rrbracket_A^{\mathcal{M}}$. Therefore $w \in \llbracket \chi \wedge \neg[\theta] \blacksquare \varphi \rrbracket_A^{\mathcal{M}}$. Thus we have (1) $w \in \llbracket \chi \rrbracket_A^{\mathcal{M}}$ and (2) $w \in \llbracket \neg[\theta] \blacksquare \varphi \rrbracket_A^{\mathcal{M}}$. Because of (2), $w \in \llbracket \langle \theta \rangle \diamond \neg \varphi \rrbracket_A^{\mathcal{M}}$, and, by the semantics, $w \in \llbracket \diamond \neg \varphi \rrbracket_{\llbracket \theta \rrbracket_A^{\mathcal{M}}}$. Therefore, applying the semantics of \blacksquare , we obtain (3) there exists $B \in \mathcal{A}$ s.t. $w \in B \subseteq \llbracket \theta \rrbracket_A^{\mathcal{M}} \subseteq A$ and $w \in \llbracket \neg \varphi \rrbracket_B^{\mathcal{M}}$.

Now consider the pre-model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|')$ such that $\|p\|' := B$ and $\|q\|' = \|q\|$ for any $q \neq p \in \text{Prop}$. In order to use Lemma 4.5.7 we must show that \mathcal{M}' is a pseudo-model. For this we only need to verify that \mathcal{M}' satisfies the closure conditions given in Definition 4.5.3. First note that $\|p\|' := B \in \mathcal{A}$ by the construction of \mathcal{M}' , so $\|p\|' \in \mathcal{A}$. For every $q \neq p$, since $\|q\|' = \|q\|$ and $\|q\| \in \mathcal{A}$ we have $\|q\|' \in \mathcal{A}$. Since \mathcal{A} is the same for both \mathcal{M} and \mathcal{M}' , and \mathcal{M} is a pseudo-model, the rest of the closure conditions are already satisfied for \mathcal{M}' . Therefore \mathcal{M}' is a pseudo-model. Now continuing with our soundness proof, since $p \notin P_\chi \cup P_\theta \cup P_\varphi$, by Lemma 4.5.7, we obtain $\llbracket \chi \rrbracket_A^{\mathcal{M}'} = \llbracket \chi \rrbracket_A^{\mathcal{M}}$, $\llbracket \theta \rrbracket_A^{\mathcal{M}'} = \llbracket \theta \rrbracket_A^{\mathcal{M}}$ and $\llbracket \neg \varphi \rrbracket_A^{\mathcal{M}'} = \llbracket \neg \varphi \rrbracket_A^{\mathcal{M}}$. Since $\|p\|' = B \subseteq \llbracket \theta \rrbracket_A^{\mathcal{M}'} \subseteq A$ we have $\|p\|' = \llbracket p \rrbracket_A^{\mathcal{M}'}$. Because of (3) we have that $w \in \llbracket \theta \rrbracket_A^{\mathcal{M}'}$ and $w \in \llbracket \neg \varphi \rrbracket_B^{\mathcal{M}'} = \llbracket \neg \varphi \rrbracket_{\llbracket p \rrbracket_A^{\mathcal{M}'}}^{\mathcal{M}'} = \llbracket \langle p \rangle \neg \varphi \rrbracket_A^{\mathcal{M}'}$. Thus, $w \in \llbracket p \rrbracket_A^{\mathcal{M}'}$, so $w \in \llbracket \theta \wedge p \rrbracket_A^{\mathcal{M}'} = \llbracket \theta \rrbracket_A^{\mathcal{M}'} \cap \llbracket p \rrbracket_A^{\mathcal{M}'} = \llbracket p \rrbracket_A^{\mathcal{M}'}$ simply because $\llbracket p \rrbracket_A^{\mathcal{M}'} \subseteq \llbracket \theta \rrbracket_A^{\mathcal{M}'}$. Since $w \in \llbracket \neg \varphi \rrbracket_{\llbracket p \rrbracket_A^{\mathcal{M}'}}^{\mathcal{M}'}$, we obtain $w \in \llbracket \neg \varphi \rrbracket_{\llbracket \theta \wedge p \rrbracket_A^{\mathcal{M}'}}^{\mathcal{M}'}$. Putting everything together, $w \in \llbracket \theta \wedge p \rrbracket_A^{\mathcal{M}'}$ and $w \in \llbracket \neg \varphi \rrbracket_{\llbracket \theta \wedge p \rrbracket_A^{\mathcal{M}'}}^{\mathcal{M}'}$. Thus we obtain that $w \in \llbracket \langle \theta \wedge p \rangle \neg \varphi \rrbracket_A^{\mathcal{M}'}$ and $w \in \llbracket \chi \rrbracket_A^{\mathcal{M}'}$ which contradicts the validity of $\chi \rightarrow [\theta \wedge p]\varphi$.

($\llbracket ! \rrbracket [G]$ -intro): Suppose $\chi \rightarrow [\theta \wedge \bigwedge_{i \in G} K_i^\theta p_i]\varphi$ with $p_i \notin P_\chi \cup P_\theta \cup P_\varphi$ is valid and $\chi \rightarrow [\theta][G]\varphi$ where $p_i \notin P_\chi \cup P_\theta \cup P_\varphi$ is not valid. The latter means

that there exists a pseudo-model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ such that for some $A \in \mathcal{A}$ and some $w \in A$, $w \notin \llbracket \chi \rightarrow [\theta][G]\varphi \rrbracket_A^{\mathcal{M}}$. Therefore $w \in \llbracket \chi \wedge \neg[\theta][G]\varphi \rrbracket_A^{\mathcal{M}}$. Thus we have (1) $w \in \llbracket \chi \rrbracket_A^{\mathcal{M}}$, and (2) $w \in \llbracket \neg[\theta][G]\varphi \rrbracket_A^{\mathcal{M}}$. Item (2) means $w \in \llbracket \langle \theta \rangle \langle G \rangle \neg\varphi \rrbracket_A^{\mathcal{M}}$. Then, by the semantics of $\langle ! \rangle$, we have $w \in \llbracket \langle G \rangle \neg\varphi \rrbracket_{[\theta]_A^{\mathcal{M}}}^{\mathcal{M}}$. Therefore by the semantics of $\langle G \rangle$ we obtain: (3) there exists $\{B_i : i \in G\} \subseteq \mathcal{A}$ s.t. $w \in \llbracket \neg\varphi \rrbracket_{[\theta]_A^{\mathcal{M}} \cap \bigcap_{i \in G} K_i^{[\theta]_A^{\mathcal{M}}} B_i}^{\mathcal{M}}$.

Now consider the pre-model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|')$ such that $\|p_i\|' = B_i$ and $\|q\|' = \|q\|$ for any $q \neq p_i \in \text{Prop}$ for all $i \in G$. Observe that since $[\theta]_A \subseteq A$, by Boolean operations of sets we obtain that $K_i^{[\theta]_A}(A \cap B_i) = K_i^{[\theta]_A} B_i$. In order to use Lemma 4.5.7 we must show that \mathcal{M}' is a pseudo-model as in the soundness proof of $[!]\blacksquare$ -intro. First note that for every $q \neq p_i$, since $\|q\|' = \|q\|$ and $\|q\| \in \mathcal{A}$, we have $\|q\|' \in \mathcal{A}$. Moreover, since for every $i \in G$, $\|p_i\|' = B_i \in \mathcal{A}$, we conclude that \mathcal{M}' satisfies the clause (1.) in Definition 4.5.3. Since \mathcal{A} is the same for both \mathcal{M} and \mathcal{M}' , and \mathcal{M} is a pseudo-model, the rest of the closure conditions are satisfied already. Therefore \mathcal{M}' is a pseudo-model. Now continuing with our soundness proof, given $p_i \notin P_\chi \cup P_\theta \cup P_\varphi$ for all $i \in G$, by Lemma 4.5.7, we obtain $\llbracket \chi \rrbracket_A^{\mathcal{M}'} = \llbracket \chi \rrbracket_A^{\mathcal{M}}$ and $\llbracket [\theta] \rrbracket_A^{\mathcal{M}'} = \llbracket [\theta] \rrbracket_A^{\mathcal{M}}$. We moreover have that

$$\begin{aligned} \llbracket \neg\varphi \rrbracket_{[\theta]_A^{\mathcal{M}} \cap \bigcap_{i \in G} K_i^{[\theta]_A^{\mathcal{M}}} B_i}^{\mathcal{M}} &= \llbracket \neg\varphi \rrbracket_{[\theta]_A^{\mathcal{M}'} \cap \bigcap_{i \in G} K_i^{[\theta]_A^{\mathcal{M}'}} B_i}^{\mathcal{M}'} = \llbracket \neg\varphi \rrbracket_{[\theta]_A^{\mathcal{M}'} \cap \bigcap_{i \in G} K_i^{[\theta]_A^{\mathcal{M}'}} \|p_i\|'}^{\mathcal{M}'} \\ &= \llbracket \neg\varphi \rrbracket_{[\theta]_A^{\mathcal{M}'} \cap \bigcap_{i \in G} K_i^{[\theta]_A^{\mathcal{M}'}} (A \cap \|p_i\|')}^{\mathcal{M}'} \end{aligned}$$

by the above observation. And by Lemma 4.5.6, we obtain

$$\llbracket \neg\varphi \rrbracket_{[\theta]_A^{\mathcal{M}'} \cap \bigcap_{i \in G} K_i^{[\theta]_A^{\mathcal{M}'}} (A \cap \|p_i\|')}^{\mathcal{M}'} = \llbracket \neg\varphi \rrbracket_{[\theta]_A^{\mathcal{M}'} \cap \bigcap_{i \in G} K_i^{[\theta]_A^{\mathcal{M}'}} (\llbracket p_i \rrbracket_A^{\mathcal{M}'})}^{\mathcal{M}'} = \llbracket \neg\varphi \rrbracket_{[\theta \wedge \bigwedge_{i \in G} K_i^\theta p_i]_A^{\mathcal{M}'}}^{\mathcal{M}'}$$

Therefore, $w \in \llbracket \neg\varphi \rrbracket_{[\theta \wedge \bigwedge_{i \in G} K_i^\theta p_i]_A^{\mathcal{M}'}}^{\mathcal{M}'}$. I.e., $w \in \llbracket \langle \theta \wedge \bigwedge_{i \in G} K_i^\theta p_i \rangle \neg\varphi \rrbracket_A^{\mathcal{M}'}$. Since we also have that $w \in \llbracket \chi \rrbracket_A^{\mathcal{M}'}$, we conclude that $w \in \llbracket \chi \wedge \langle \theta \wedge \bigwedge_{i \in G} K_i^\theta p_i \rangle \neg\varphi \rrbracket_A^{\mathcal{M}'}$, contradicting the validity of $\chi \rightarrow [\theta \wedge \bigwedge_{i \in G} K_i^\theta p_i]\varphi$.

□

4.5.9. DEFINITION. [Standard Pre-model] A pre-model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ is *standard* if and only if $\mathcal{A} = \{\llbracket \theta \rrbracket_{W^0} : \theta \in \mathcal{L}_{-\diamond}\}$.

4.5.10. PROPOSITION. *Every standard pre-model is a pseudo-model.*

Proof:

Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ be a standard pre-model. This implies that

$\mathcal{A} = \{\llbracket \theta \rrbracket_{W^0} : \theta \in \mathcal{L}_{-\diamond}\}$. We need to show that \mathcal{M} satisfies the closure conditions given in Definition 4.5.3. Conditions (1) and (2) are immediate.

For (3): let $A \in \mathcal{A}$. Since \mathcal{M} is a standard pre-model, we know that $A = \llbracket \theta \rrbracket_{W^0}$ for some $\theta \in \mathcal{L}_{-\diamond}$. Since $\theta \in \mathcal{L}_{-\diamond}$, we have $\neg\theta \in \mathcal{L}_{-\diamond}$, thus, $\llbracket \neg\theta \rrbracket_{W^0} \in \mathcal{A}$. Observe that $\llbracket \neg\theta \rrbracket_{W^0} = W^0 - \llbracket \theta \rrbracket_{W^0}$, thus, we obtain $W^0 - A \in \mathcal{A}$.

For (4): let $A, B \in \mathcal{A}$. Since \mathcal{M} is a standard pre-model, $A = \llbracket \theta_1 \rrbracket_{W^0}$ and $B = \llbracket \theta_2 \rrbracket_{W^0}$ for some $\theta_1, \theta_2 \in \mathcal{L}_{-\diamond}$. Since $\theta_1, \theta_2 \in \mathcal{L}_{-\diamond}$, we have $\theta_1 \wedge \theta_2 \in \mathcal{L}_{-\diamond}$, thus, $\llbracket \theta_1 \wedge \theta_2 \rrbracket_{W^0} \in \mathcal{A}$. Observe that $\llbracket \theta_1 \wedge \theta_2 \rrbracket_{W^0} = \llbracket \theta_1 \rrbracket_{W^0} \cap \llbracket \theta_2 \rrbracket_{W^0} = A \cap B$, thus, we obtain $A \cap B \in \mathcal{A}$.

For (5): let $A \in \mathcal{A}$. Since \mathcal{M} is a standard pre-model, $A = \llbracket \theta \rrbracket_{W^0}$ for some $\theta \in \mathcal{L}_{-\diamond}$. Since $\theta \in \mathcal{L}_{-\diamond}$, we have $K_i\theta \in \mathcal{L}_{-\diamond}$, thus, $\llbracket K_i\theta \rrbracket_{W^0} \in \mathcal{A}$. Observe that $\llbracket K_i\theta \rrbracket_{W^0} = \{w \in W^0 : \forall s \in W^0 (w \sim_i s \Rightarrow s \in \llbracket \theta \rrbracket_{W^0})\} = K_i\llbracket \theta \rrbracket_{W^0}$, thus, we obtain $K_iA \in \mathcal{A}$. \square

Equivalence between the standard pseudo-models and announcement models. For Proposition 4.5.13 only, we use the notation $\llbracket \varphi \rrbracket_A^{PS}$ to refer to pseudo-model semantics (as in Definition 4.5.2) and use $\llbracket \varphi \rrbracket_M$ to refer to the semantics on a -models (as in Definition 4.4.1).

First we need a couple of useful lemmas.

4.5.11. LEMMA. *The sentence $(K_i(\varphi \rightarrow \psi))^0 \leftrightarrow K_i(K_i(\varphi \rightarrow \psi))^0$ is valid on pseudo-models.*

Proof:

It is easy to see that the direction from left-to-right follows from the fact that the semantics for φ^0 is state-independent, and the direction from right-to-left is an instance of the T-axiom for K_i . \square

4.5.12. LEMMA. *Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ be a standard pseudo-model, $A \in \mathcal{A}$ and $\varphi \in \mathcal{L}_G$, then the following holds:*

1. $\llbracket \blacklozenge\varphi \rrbracket_A = \bigcup \{\llbracket \langle \theta \rangle \varphi \rrbracket_A : \theta \in \mathcal{L}_{-\diamond}\},$
2. $\llbracket \langle G \rangle \varphi \rrbracket_A = \bigcup \{\llbracket \langle \bigwedge_{i \in G} K_i\theta_i \rangle \varphi \rrbracket_A : \{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\diamond}\}.$

Proof:

1. For (\subseteq): Let $w \in \llbracket \blacklozenge\varphi \rrbracket_A$. Then, by the semantics of \blacklozenge in pseudo-models, there exists some $B \in \mathcal{A}$ such that $w \in B \subseteq A$ and $w \in \llbracket \varphi \rrbracket_B$. Since \mathcal{M} is standard, we know that $A = \llbracket \psi \rrbracket_{W^0}$ and $B = \llbracket \chi \rrbracket_{W^0}$ for some $\psi, \chi \in \mathcal{L}_{-\diamond}$. Moreover, since $B = \llbracket \chi \rrbracket_{W^0} \subseteq A = \llbracket \psi \rrbracket_{W^0}$, we have $B = \llbracket \chi \rrbracket_{W^0} \cap$

$\llbracket \psi \rrbracket_{W^0} = \llbracket \chi^0 \rrbracket_{\llbracket \psi \rrbracket_{W^0}} = \llbracket \chi^0 \rrbracket_A$, and so $w \in \llbracket \varphi \rrbracket_B = \llbracket \varphi \rrbracket_{\llbracket \chi^0 \rrbracket_A} = \llbracket \langle \chi^0 \rangle \varphi \rrbracket_A \subseteq \bigcup \{ \llbracket \langle \theta \rangle \varphi \rrbracket_A : \theta \in \mathcal{L}_{-\diamond} \}$.

For (\supseteq) : Let $w \in \bigcup \{ \llbracket \langle \theta \rangle \varphi \rrbracket_A : \theta \in \mathcal{L}_{-\diamond} \}$. Then we have $w \in \llbracket \langle \theta \rangle \varphi \rrbracket_A = \llbracket \varphi \rrbracket_{\llbracket \theta \rrbracket_A}$, for some $\theta \in \mathcal{L}_{-\diamond}$. Moreover, since $\llbracket \theta \rrbracket_A \in \mathcal{A}$ (by Proposition 4.5.5) and $\llbracket \theta \rrbracket_A \subseteq A$ (by Observation 3), it follows that $w \in \llbracket \diamond \varphi \rrbracket_A$ (by the semantics of \diamond in pseudo-models).

2. For (\subseteq) : Let $w \in \llbracket \langle G \rangle \varphi \rrbracket_A$. Then, by Definition 4.5.2, we have

$$w \in \llbracket \varphi \rrbracket_{A \cap \bigcap_{i \in G} K_i^A B_i} \text{ for some } \{B_i : i \in G\} \subseteq \mathcal{A}.$$

Since \mathcal{M} is a standard pseudo-model, we know that each $B_i = \llbracket \rho_i \rrbracket_{W^0}$ and $A = \llbracket \psi \rrbracket_{W^0}$ for some $\rho_i, \psi \in \mathcal{L}_{-\diamond}$.

Thus, $w \in \llbracket \varphi \rrbracket_{\llbracket \psi \rrbracket_{W^0} \cap \bigcap_{i \in G} K_i^{\llbracket \psi \rrbracket_{W^0}} \llbracket \rho_i \rrbracket_{W^0}} = \llbracket \varphi \rrbracket_{\llbracket \psi \rrbracket_{W^0} \cap \bigcap_{i \in G} \llbracket K_i(\psi \rightarrow \rho_i) \rrbracket_{W^0}} = \llbracket \varphi \rrbracket_{\llbracket \psi \rrbracket_{W^0} \cap \llbracket \bigwedge_{i \in G} K_i(\psi \rightarrow \rho_i) \rrbracket_{W^0}}$ by Lemma 4.5.6 and the semantics. By the semantics of 0 and Lemma 4.5.11, we obtain $\llbracket \varphi \rrbracket_{\llbracket \psi \rrbracket_{W^0} \cap \llbracket \bigwedge_{i \in G} K_i(\psi \rightarrow \rho_i) \rrbracket_{W^0}} = \llbracket \varphi \rrbracket_{\llbracket \bigwedge_{i \in G} (K_i(\psi \rightarrow \rho_i))^0 \rrbracket_A} = \llbracket \varphi \rrbracket_{\llbracket \bigwedge_{i \in G} K_i(K_i(\psi \rightarrow \rho_i))^0 \rrbracket_A}$. Thus, for $\theta_i := (K_i(\psi \rightarrow \rho_i))^0$, $w \in \llbracket \varphi \rrbracket_{\llbracket \bigwedge_{i \in G} K_i \theta_i \rrbracket_A} = \llbracket \langle \bigwedge_{i \in G} K_i \theta_i \rangle \varphi \rrbracket_A$.

For (\supseteq) : Let $\{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\diamond}$ such that $w \in \llbracket \varphi \rrbracket_{\llbracket \bigwedge_{i \in G} K_i \theta_i \rrbracket_A}$. Note that $\llbracket \bigwedge_{i \in G} K_i \theta_i \rrbracket_A = \bigcap_{i \in G} \llbracket K_i \theta_i \rrbracket_A = A \cap \bigcap_{i \in G} K_i^A \llbracket \theta_i \rrbracket_A$. Since \mathcal{M} is a standard pseudo-model, we know that $B_i := \llbracket \theta_i \rrbracket_A \in \mathcal{A}$ for every $i \in G$ and by our initial assumption $w \in \llbracket \varphi \rrbracket_{A \cap \bigcap_{i \in G} K_i^A \llbracket \theta_i \rrbracket_A}$, so we obtain $w \in \llbracket \langle G \rangle \varphi \rrbracket_A$.

□

The following proposition addresses the correspondence between standard pseudo-models and a -models which is crucial for the aforementioned equivalence.

4.5.13. PROPOSITION.

1. For every standard pseudo-model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ and every set $A \in \mathcal{A}$, we denote by M_A the model $M_A = (W^0, A, \sim_1, \dots, \sim_n, \|\cdot\|)$. Then:

- (a) For every $\varphi \in \mathcal{L}_G$, we have $\llbracket \varphi \rrbracket_{M_A} = \llbracket \varphi \rrbracket_A^{PS}$ for all $A \in \mathcal{A}$.
(b) M_A is an a -model, for all $A \in \mathcal{A}$.

2. For every a -model $M = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$, we denote by \mathcal{M}' the pre-model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$, where $\mathcal{A} = \{\llbracket \theta \rrbracket_{M^0} : \theta \in \mathcal{L}_{-\diamond}\}$. Then

- (a) \mathcal{M}' is a standard pseudo-model.
(b) For every $\varphi \in \mathcal{L}_G$, we have $\llbracket \varphi \rrbracket_M = \llbracket \varphi \rrbracket_W^{PS}$.

Proof:

1. Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ be a standard pseudo-model. Then, $A \in \mathcal{A}$ implies $A = \llbracket \theta \rrbracket_{W^0}^{PS} \subseteq W^0$ for some θ , hence $M_A = (W^0, A, \sim_1, \dots, \sim_n, \|\cdot\|)$ is a model.

(a) The proof is by \prec_2 -induction (see Lemma [A.2.5](#)). The base cases and the inductive cases for Booleans are straightforward.

Case $\varphi := \psi^0$. We have $\llbracket \psi^0 \rrbracket_A^{PS} = \llbracket \psi \rrbracket_{W^0}^{PS} \cap A = \llbracket \psi \rrbracket_{M_A^0} \cap A = \llbracket \psi^0 \rrbracket_{M_A}$ (by Definition [4.5.2](#), IH, and Definition [4.3.2](#)).

Case $\varphi := K_i \psi$. We have $\llbracket K_i \psi \rrbracket_A^{PS} = \{w \in A : w_i^A \subseteq \llbracket \psi \rrbracket_A^{PS}\} = \{w \in A : w_i \subseteq \llbracket \psi \rrbracket_{M_A}\} = \llbracket K_i \psi \rrbracket_{M_A}$ (by Definition [4.5.2](#), IH, and Definition [4.3.2](#)).

Case $\varphi := U\psi$. By Definitions [4.3.2](#) and [4.5.2](#), we have:

$$\llbracket U\psi \rrbracket_{M_A} = \begin{cases} A & \text{if } \llbracket \psi \rrbracket_{M_A} = A \\ \emptyset & \text{otherwise} \end{cases}$$

$$\llbracket U\psi \rrbracket_A^{PS} = \begin{cases} A & \text{if } \llbracket \psi \rrbracket_A^{PS} = A \\ \emptyset & \text{otherwise} \end{cases}$$

By IH, $\llbracket \psi \rrbracket_A^{PS} = \llbracket \psi \rrbracket_{M_A}$, therefore, $\llbracket U\psi \rrbracket_A^{PS} = \llbracket U\psi \rrbracket_{M_A}$.

Case $\varphi := \langle \psi \rangle \chi$. By Definition [4.3.2](#), we know that $\llbracket \langle \psi \rangle \chi \rrbracket_{M_A} = \llbracket \chi \rrbracket_{M_A | \llbracket \psi \rrbracket_{M_A}}$. Now consider $M_A | \llbracket \psi \rrbracket_{M_A} = (W^0, \llbracket \psi \rrbracket_{M_A}, \sim_1, \dots, \sim_n, \|\cdot\|)$. By Lemma [A.2.5.1](#) and IH, we have $\llbracket \psi \rrbracket_{M_A} = \llbracket \psi \rrbracket_A^{PS}$. Moreover, by the definition of standard pseudo-models, we know that $A = \llbracket \theta \rrbracket_{W^0}^{PS}$ for some $\theta \in \mathcal{L}_{-\diamond}$. Therefore, $\llbracket \psi \rrbracket_{M_A} = \llbracket \psi \rrbracket_A^{PS} = \llbracket \psi \rrbracket_{\llbracket \theta \rrbracket_{W^0}^{PS}}^{PS} = \llbracket \langle \theta \rangle \psi \rrbracket_{W^0}^{PS}$. Therefore, $\llbracket \psi \rrbracket_{M_A} \in \mathcal{A}$. We then have

$$\llbracket \langle \psi \rangle \chi \rrbracket_{M_A} = \llbracket \chi \rrbracket_{M_{\llbracket \psi \rrbracket_{M_A}}} = \llbracket \chi \rrbracket_{M_{\llbracket \psi \rrbracket_A^{PS}}} = \llbracket \chi \rrbracket_{\llbracket \psi \rrbracket_A^{PS}}^{PS} = \llbracket \langle \psi \rangle \chi \rrbracket_A^{PS},$$

by the semantics and IH on ψ and on χ (since $\llbracket \psi \rrbracket_A^{PS} \in \mathcal{A}$).

Case $\varphi := \diamond \psi$. By Definition [4.3.2](#), [\(9\)](#) in Lemma [A.2.5](#), IH, the fact that \mathcal{M} is a standard pseudo-model, and [\(1\)](#) in Lemma [4.5.12](#) - applied in this order - we obtain the following equivalences:

$$\llbracket \diamond \psi \rrbracket_{M_A} = \bigcup \{ \llbracket \langle \chi \rangle \psi \rrbracket_{M_A} : \chi \in \mathcal{L}_{-\diamond} \} = \bigcup \{ \llbracket \langle \chi \rangle \psi \rrbracket_A^{PS} : \chi \in \mathcal{L}_{-\diamond} \} = \llbracket \diamond \psi \rrbracket_A^{PS}.$$

Case $\varphi := \langle G \rangle \psi$. By Definition 4.4.1, (10.) in Lemma A.2.5, IH, the fact that \mathcal{M} is a standard pseudo-model, and (2.) in Lemma 4.5.12 - applied in this order - we obtain the following equivalences:

$$\begin{aligned} \llbracket \langle G \rangle \psi \rrbracket_{M_A} &= \bigcup \{ \llbracket \langle \bigwedge_{i \in G} K_i \theta_i \rangle \varphi \rrbracket_{M_A} : \{ \theta_i : i \in G \} \subseteq \mathcal{L}_{-\blacklozenge} \} \\ &= \bigcup \{ \llbracket \langle \bigwedge_{i \in G} K_i \theta_i \rangle \varphi \rrbracket_A^{PS} : \{ \theta_i : i \in G \} \subseteq \mathcal{L}_{-\blacklozenge} \} = \llbracket \langle G \rangle \psi \rrbracket_A^{PS}. \end{aligned}$$

(b) By part (a), $\llbracket \varphi \rrbracket_{M_A^0} = \llbracket \varphi \rrbracket_{M_{W^0}} = \llbracket \varphi \rrbracket_{W^0}^{PS}$ for all φ . Since \mathcal{M} is standard, we have $A = \llbracket \theta \rrbracket_{W^0}^{PS} = \llbracket \theta \rrbracket_{M_A^0}$ for some $\theta \in \mathcal{L}_{-\blacklozenge}$, so M_A is an a -model.

2. Let $M = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$ be an a -model. Since $\mathcal{A} = \{ \llbracket \theta \rrbracket_{M^0} : \theta \in \mathcal{L}_{-\blacklozenge} \} \subseteq \mathcal{P}(W^0)$, the model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ is a pre-model. Therefore, the semantics given in Definition 4.5.2 is defined on $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$.

(a) By Proposition 4.5.10, it suffices to prove that the pre-model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ is standard, i.e. that $\{ \llbracket \theta \rrbracket_{M^0} : \theta \in \mathcal{L}_{-\blacklozenge} \} = \{ \llbracket \theta \rrbracket_{W^0}^{PS} : \theta \in \mathcal{L}_{-\blacklozenge} \}$. For this, we need to show that for every a -model $M = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$, we have $\llbracket \theta \rrbracket_M = \llbracket \theta \rrbracket_W^{PS}$ for all $\theta \in \mathcal{L}_{-\blacklozenge}$. We prove this by subformula induction on θ . The base cases and the inductive cases for Booleans are straightforward.

Case $\theta := \psi^0$. Then $\llbracket \psi^0 \rrbracket_M = \llbracket \psi \rrbracket_{M^0} \cap W = \llbracket \psi \rrbracket_{W^0}^{PS} \cap W = \llbracket \psi^0 \rrbracket_W^{PS}$ (by Definition 4.3.2, IH, and Definition 4.5.2).

Case $\theta := K_i \psi$. We have $\llbracket K_i \psi \rrbracket_M = \{ w \in W : w_i \subseteq \llbracket \psi \rrbracket_M \} = \{ w \in W : w_i^W \subseteq \llbracket \psi \rrbracket_W^{PS} \} = \llbracket K_i \psi \rrbracket_W^{PS}$ (by Definition 4.3.2, IH, and Definition 4.5.2).

Case $\theta := U\psi$. By Definitions 4.3.2 and 4.5.2, we have:

$$\begin{aligned} \llbracket U\psi \rrbracket_M &= \begin{cases} W & \text{if } \llbracket \psi \rrbracket_M = W \\ \emptyset & \text{otherwise} \end{cases} \\ \llbracket U\psi \rrbracket_W^{PS} &= \begin{cases} W & \text{if } \llbracket \psi \rrbracket_W^{PS} = W \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

By IH, $\llbracket \psi \rrbracket_W^{PS} = \llbracket \psi \rrbracket_M$, therefore, $\llbracket U\psi \rrbracket_W^{PS} = \llbracket U\psi \rrbracket_M$.

Case $\theta := \langle \psi \rangle \chi$. By Definition 4.3.2, we know that $\llbracket \langle \psi \rangle \chi \rrbracket_M = \llbracket \chi \rrbracket_{M[\llbracket \psi \rrbracket_M]}$. Now consider the relativized model $M[\llbracket \psi \rrbracket_M] = (W^0, \llbracket \psi \rrbracket_M, \sim_1, \dots, \sim_n, \|\cdot\|)$. By (1.) in Lemma A.2.5 and IH on ψ , we have $\llbracket \psi \rrbracket_M = \llbracket \psi \rrbracket_W^{PS}$. Moreover, by the definition of a -models, we know that $W =$

$\llbracket \theta \rrbracket_{M^0}$ for some $\theta \in \mathcal{L}_{-\diamond}$. Therefore, $\llbracket \psi \rrbracket_M = \llbracket \psi \rrbracket_{M^0 \llbracket \theta \rrbracket_{M^0}} = \llbracket \langle \theta \rangle \psi \rrbracket_{M^0}$. Hence, since $\langle \theta \rangle \psi \in \mathcal{L}_{-\diamond}$, the model $M \llbracket \psi \rrbracket_M$ is also an a -model obtained by updating the initial model M^0 by $\langle \theta \rangle \psi$. We then have $\llbracket \langle \psi \rangle \chi \rrbracket_M = \llbracket \chi \rrbracket_{M \llbracket \psi \rrbracket_M}$ (by Definition 4.3.2) = $\llbracket \chi \rrbracket_{M \llbracket \psi \rrbracket_W^{PS}}$ (by IH on ψ) = $\llbracket \chi \rrbracket_{\llbracket \psi \rrbracket_W^{PS}}$ (by IH on χ , $M \llbracket \psi \rrbracket_M$ is an a -model) = $\llbracket \langle \psi \rangle \chi \rrbracket_W^{PS}$ (by Definition 4.5.2).

- (b) The proof of this part follows by \prec_2 -induction on φ (where \prec_2 is as in Lemma A.2.5 from the Technical Appendix A.A.2). All the inductive cases are similar to ones in the above proof, except for the cases $\varphi := \diamond\psi$ and $\varphi := \langle G \rangle \psi$, shown below.

Case $\varphi := \diamond\psi$. By Definition 4.3.2, (9.) in Lemma A.2.5, IH, the fact that \mathcal{M}' is a standard pseudo-model, and (2.) in Lemma 4.5.12 - applied in that order - we obtain the following equivalences:

$$\llbracket \diamond\psi \rrbracket_M = \bigcup \{ \llbracket \langle \chi \rangle \psi \rrbracket_M : \chi \in \mathcal{L}_{-\diamond} \} = \bigcup \{ \llbracket \langle \chi \rangle \psi \rrbracket_W^{PS} : \chi \in \mathcal{L}_{-\diamond} \} = \llbracket \diamond\psi \rrbracket_W^{PS}.$$

Case $\varphi := \langle G \rangle \psi$. By Definition 4.4.1, (10.) in Lemma A.2.5, IH, the fact that \mathcal{M}' is a standard pseudo-model and (2.) in Lemma 4.5.12 - applied in that order - we obtain the following: equivalences,

$$\begin{aligned} \llbracket \langle G \rangle \psi \rrbracket_M &= \bigcup \{ \llbracket \langle \bigwedge_{i \in G} K_i \theta_i \rangle \varphi \rrbracket_M : \{ \theta_i : i \in G \} \subseteq \mathcal{L}_{-\diamond} \} \\ &= \bigcup \{ \llbracket \langle \bigwedge_{i \in G} K_i \theta_i \rangle \varphi \rrbracket_W^{PS} : \{ \theta_i : i \in G \} \subseteq \mathcal{L}_{-\diamond} \} = \llbracket \langle G \rangle \psi \rrbracket_W^{PS} \end{aligned}$$

□

Due to the correspondence between the standard pseudo-models and a -models, we obtain the following.

4.5.14. COROLLARY. *Validity on standard pseudo-models coincides with validity on the a -models.*

Proof:

This is a straightforward consequence of Proposition 4.5.13. □

4.5.15. COROLLARY (SOUNDNESS OF GALM/APALM ON a -MODELS). *The system GALM is sound wrt a -models. Moreover, the system APALM is sound wrt a -models.*

Proof:

Follows immediately from Proposition 4.5.8 and Corollary 4.5.14. □

It is important to note that the equivalence between standard pseudo-models and a -models (given by Proposition 4.5.13 above, and underlying our soundness result) is *not* trivial. It relies in particular on the equivalence between the effort modality and the arbitrary announcement operator \blacksquare (see (1.) in Lemma 4.5.12) and on the equivalence between the *purely syntactic* and *purely semantic* descriptions of the group announcement operator $[G]$ on standard pseudo-models (see (2.) in Lemma 4.5.12). These equivalences hold *only because our models and language retain the memory of the initial situation*. Hence, a similar equivalence fails for the original APAL and GAL.

4.6 Completeness of GALM and APALM

In this section we prove the completeness of **GALM** and **APALM**. First, we show completeness with respect to pseudo-models, via an innovative modification of the standard canonical model construction. This is based on a method previously used in Chapters 3, that makes an essential use of the finitary \blacksquare and $[G]$ -introduction rules, by requiring our canonical theories T to be (not only maximally consistent, but also) “witnessed”. Roughly speaking, a theory T is witnessed if: every $\blacklozenge\varphi$ occurring in every “existential context” in T is witnessed by some atomic formula p , meaning that $\langle p \rangle\varphi$ occurs in the same existential context in T , and if for every $\langle G \rangle\varphi$ occurring in every “existential context” in T is witnessed by some formula $\bigwedge_{i \in G} K_i p_i$, meaning that $\langle \bigwedge_{i \in G} K_i p_i \rangle\varphi$ occurs in the same existential context in T . Our canonical pre-model will consist of all *initial, maximally consistent, witnessed theories* (where a theory is “initial” if it contains the formula 0). A Truth Lemma is proved, as usual. Completeness for both pseudo-models and a -models follows from the observation that our canonical pre-model *is standard*, hence it is a standard pseudo-model, and thus equivalent to a genuine a -model.

We now proceed with the details. As in Chapter 3, the appropriate notion of “existential context” is represented by *possibility forms*, in the following sense.

4.6.1. DEFINITION. [Necessity forms and Possibility forms] For any finite string $s \in (\{\bullet^0\} \cup \{\varphi \rightarrow \mid \varphi \in \mathcal{L}_G\} \cup \{K_i : i \in \mathcal{A}\} \cup \{U\} \cup \{\rho \mid \rho \in \mathcal{L}_{-\blacklozenge}\})^* = NF_{\mathcal{L}_G}$, we define pseudo-modalities $[s]$ and $\langle s \rangle$. These pseudo-modalities are functions mapping any formula $\varphi \in \mathcal{L}_G$ to another formula $[s]\varphi \in \mathcal{L}_G$ (*necessity form*), respectively $\langle s \rangle\varphi \in \mathcal{L}_G$ (*possibility form*). The necessity forms are defined recursively as $[\epsilon]\varphi = \varphi$, $[s, \bullet^0]\varphi = [s]\varphi^0$, $[s, \varphi \rightarrow]\varphi = [s](\varphi \rightarrow \varphi)$, $[s, K_i]\varphi = [s]K_i\varphi$, $[s, U]\varphi = [s]U\varphi$, $[s, \rho]\varphi = [s][\rho]\varphi$, where ϵ is the empty string. For possibility forms, we set $\langle s \rangle\varphi := \neg[s]\neg\varphi$.

Example: $[K_i, \bullet^0, \blacklozenge p \rightarrow, 0, U]$ is a necessity form such that $[K_i, \bullet^0, \blacklozenge p \rightarrow, 0, U]\varphi = K_i(\blacklozenge p \rightarrow [0]U\varphi)^0$.

4.6.2. DEFINITION. [Theories: witnessed, initial, maximal] Let \mathcal{L}_G^P be the language of GALM based only on some countable set P of propositional variables. Similarly, let $NF_{\mathcal{L}_G^P}^P$ denote the corresponding set of strings defined based on \mathcal{L}_G^P (necessity and possibility forms are as given in Definition [4.6.1](#)).

- A *P-theory* is a consistent set of formulas in \mathcal{L}_G^P (where “consistent” means consistent with respect to the axiomatization of **GALM** formulated for \mathcal{L}_G^P).
- A *maximal P-theory* is a *P-theory* Γ that is maximal with respect to \subseteq among all *P-theories*; in other words, Γ cannot be extended to another *P-theory*.
- A *P-witnessed theory* is a *P-theory* Γ such that, for every $s \in NF_{\mathcal{L}_G^P}^P$ and $\varphi \in \mathcal{L}_G^P$, (1) if $\langle s \rangle \blacklozenge \varphi$ is consistent with Γ then there is $p \in P$ such that $\langle s \rangle \langle p \rangle \varphi$ is consistent with Γ (or equivalently: if $\Gamma \vdash [s][p] \neg \varphi$ for all $p \in P$, then $\Gamma \vdash [s] \blacksquare \neg \varphi$), and (2) for every $G \subseteq \mathcal{AG}$, if $\langle s \rangle \langle G \rangle \varphi$ is consistent with Γ then there is $\{p_i : i \in G\} \subseteq P$ such that $\langle s \rangle \langle \bigwedge_{i \in G} K_i p_i \rangle \varphi$ is consistent with Γ (or equivalently: if $\Gamma \vdash [s][\bigwedge_{i \in G} K_i p_i] \neg \varphi$ for all $p_i \in P$, then $\Gamma \vdash [s][G] \neg \varphi$).
- A *P-theory* Γ is called *initial* if $0 \in \Gamma$.
- A *maximal P-witnessed theory* Γ is a *P-witnessed theory* that is not a proper subset of any *P-witnessed theory*. A *maximal P-witnessed initial theory* Γ is a maximal *P-witnessed theory* such that $0 \in \Gamma$.

4.6.3. LEMMA. For every necessity form $[s]$, there exist formulas $\theta \in \mathcal{L}_{-\blacklozenge}$ and $\psi \in \mathcal{L}_G$, with $P_\psi \cup P_\theta \subseteq P_s$, such that for all $\varphi \in \mathcal{L}_G$, we have

$$\vdash [s]\varphi \text{ iff } \vdash \psi \rightarrow [\theta]\varphi.$$

Proof:

We proceed by induction on the structure of necessity forms. For $s := \epsilon$, take $\psi := \top$ and $\theta := \top$, then it follows from the axiom $R[\top]$. For the inductive cases we will verify only $s := s', \bullet^0$; $s := s', \eta \rightarrow$; $s := s', U$; and $s := s', \rho$. The case $s := s', K_i$ is analogous to the case $s := s', U$.

Case $s := s', \bullet^0$

$\vdash [s', \bullet^0]\varphi$ iff $\vdash [s']\varphi^0$ (by Definition [4.6.1](#)) iff $\vdash \psi' \rightarrow [\theta']\varphi^0$ (for some $\psi' \in \mathcal{L}_G$ and $\theta' \in \mathcal{L}_{-\blacklozenge}$, by IH) iff $\vdash \psi' \rightarrow (\theta' \rightarrow \varphi^0)$ (by R^0) iff $\vdash (\psi' \wedge \theta') \rightarrow \varphi^0$ iff $\vdash (0 \wedge \diamond(\psi' \wedge \theta')) \rightarrow \varphi$ (by [\(13\)](#)) in Proposition [4.3.14](#)) iff $\vdash \psi \rightarrow [\theta]\varphi$ (since $\psi := 0 \wedge \diamond(\psi' \wedge \theta') \in \mathcal{L}_G$ and $\theta := \top \in \mathcal{L}_{-\blacklozenge}$).

Case $s := s', \eta \rightarrow$

$\vdash [s', \eta \rightarrow]\varphi$ iff $\vdash [s'](\eta \rightarrow \varphi)$ (by Definition 4.6.1) iff $\vdash \psi' \rightarrow [\theta'](\eta \rightarrow \varphi)$ (for some $\psi' \in \mathcal{L}_G$ and $\theta' \in \mathcal{L}_{-\diamond}$, by IH) iff $\vdash \psi' \rightarrow ([\theta']\eta \rightarrow [\theta']\varphi)$ (by $K_!$) iff $\vdash (\psi' \wedge [\theta']\eta) \rightarrow [\theta']\varphi$ iff $\vdash \psi \rightarrow [\theta]\varphi$ (since $\psi := \psi' \wedge [\theta']\eta \in \mathcal{L}_G$ and $\theta := \theta' \in \mathcal{L}_{-\diamond}$).

Case $s := s', U$

$\vdash [s', U]\varphi$ iff $\vdash [s']U\varphi$ (by Definition 4.6.1) iff $\vdash \psi' \rightarrow [\theta']U\varphi$ (for some $\psi' \in \mathcal{L}_G$ and $\theta' \in \mathcal{L}_{-\diamond}$, by IH) iff $\vdash \psi' \rightarrow (\theta' \rightarrow U[\theta']\varphi)$ (by R_U) iff $\vdash (\psi' \wedge \theta') \rightarrow U[\theta']\varphi$ iff $\vdash E(\psi' \wedge \theta') \rightarrow [\theta']\varphi$ (pushing U back with its dual E , since U is an S5 modality) iff $\vdash \psi \rightarrow [\theta]\varphi$ ($\psi := E(\psi' \wedge \theta') \in \mathcal{L}_G$ and $\theta := \theta' \in \mathcal{L}_{-\diamond}$).

Case $s := s', \rho$

$\vdash [s', \rho]\varphi$ iff $\vdash [s'][\rho]\varphi$ (by Definition 4.6.1) iff $\vdash \psi' \rightarrow [\theta'][\rho]\varphi$ (for some $\psi' \in \mathcal{L}_G$ and $\theta' \in \mathcal{L}_{-\diamond}$, by IH) iff $\vdash \psi' \rightarrow [\langle \theta' \rangle \rho]\varphi$ (by $R_{[\cdot]}$) iff $\vdash \psi \rightarrow [\theta]\varphi$ ($\psi := \psi' \in \mathcal{L}_G$ and $\theta := \langle \theta' \rangle \rho \in \mathcal{L}_{-\diamond}$)

In each case, it is easy to see that $P_\psi \cup P_\theta \subseteq P_s$. \square

4.6.4. LEMMA. *The following rules are admissible in GALM (the first one also in APALM):*

1. from $\vdash [s][p]\varphi$, infer $\vdash [s]\blacksquare\varphi$, where $p \notin P_s \cup P_\varphi$,
2. from $\vdash [s][\bigwedge_{i \in G} K_i p_i]\varphi$, infer $\vdash [s][G]\varphi$, where $p_i \notin P_s \cup P_\varphi$.

Proof:

For (1), suppose $\vdash [s][p]\varphi$. Then, by Lemma 4.6.3, there exist $\theta \in \mathcal{L}_{-\diamond}$ and $\psi \in \mathcal{L}_G$ such that $\vdash \psi \rightarrow [\theta][p]\varphi$. By the auxiliary reduction principle (15.) in Proposition 4.3.14, we get $\vdash \psi \rightarrow [\theta \wedge p]\varphi$. By the construction of the formulas ψ and θ , we know that $P_\psi \cup P_\theta \subseteq P_s$, and so $p \notin P_\psi \cup P_\theta \cup P_\varphi$. Therefore, by ($[\cdot]\blacksquare$ -intro), we have $\vdash \psi \rightarrow [\theta]\blacksquare\varphi$. Applying again Lemma 4.6.3, we obtain $\vdash [s]\blacksquare\varphi$. The proof of (2) goes similarly given that $(\star) [\theta][\bigwedge_{i \in G} K_i p_i]\varphi \leftrightarrow [\theta \wedge \bigwedge_{i \in G} K_i^\theta p_i]\varphi$ is derivable in GALM (by using the appropriate reduction axioms and RE). Let $s \in NF_{\mathcal{L}_G}^P$ such that $\vdash [s][\bigwedge_{i \in G} K_i p_i]\varphi$ where $p_i \notin P_s \cup P_\varphi$. Then, by Lemma 4.6.3, we obtain that $\vdash \chi \rightarrow [\theta][\bigwedge_{i \in G} K_i p_i]\varphi$. Therefore, by (\star) , we have that $\vdash \chi \rightarrow [\theta \wedge \bigwedge_{i \in G} K_i^\theta p_i]\varphi$. By the $[\cdot][G]$ -intro rule we then obtain $\vdash \chi \rightarrow [\theta][G]\varphi$. Again by Lemma 4.6.3, we get $\vdash [s][G]\varphi$. \square

4.6.5. LEMMA. *For every maximal P -witnessed theory Γ , and every formula $\varphi, \psi \in \mathcal{L}_G^P$,*

1. $\Gamma \vdash \varphi$ iff $\varphi \in \Gamma$
2. $\varphi \notin \Gamma$ iff $\neg\varphi \in \Gamma$,
3. $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$,
4. $\varphi \in \Gamma$ and $\varphi \rightarrow \psi \in \Gamma$ implies $\psi \in \Gamma$.
5. $\mathbf{GALM}_P \subseteq \Gamma$, where \mathbf{GALM}_P is \mathbf{GALM} formulated for \mathcal{L}_G^P .

Proof:

The proof is standard. We prove only (5): suppose $\mathbf{GALM}_P \not\subseteq \Gamma$. This means that there is a sentence $\psi \in \mathcal{L}_G^P$ such that $\psi \in \mathbf{GALM}_P$ but $\psi \notin \Gamma$. The former means that $\vdash \psi$, thus, $\Gamma \vdash \psi$. Items (2) and (1) implies that if $\psi \notin \Gamma$ then $\Gamma \vdash \neg\psi$, contradicting consistency of Γ . \square

4.6.6. LEMMA. *For every $\Gamma \subseteq \mathcal{L}_G^P$, if Γ is a P -theory and $\Gamma \not\vdash \neg\varphi$ for some $\varphi \in \mathcal{L}_G^P$, then $\Gamma \cup \{\varphi\}$ is a P -theory. Moreover, if Γ is P -witnessed, then $\Gamma \cup \{\varphi\}$ is also P -witnessed.*

Proof:

The proof of the first claim is standard. We only prove the second claim. Suppose that Γ is P -witnessed but $\Gamma \cup \{\varphi\}$ is not P -witnessed. By the previous statement, we know that $\Gamma \cup \{\varphi\}$ is consistent. Since $\Gamma \cup \{\varphi\}$ is not P -witnessed, it violates either (1) or (2) in Definition 4.6.2. First suppose $\Gamma \cup \{\varphi\}$ does not satisfy (1), that is, there is $s \in NF_{\mathcal{L}_G}^P$ and $\psi \in \mathcal{L}_G^P$ such that $\Gamma \cup \{\varphi\}$ is consistent with $\langle s \rangle \blacklozenge \psi$ but $\Gamma \cup \{\varphi\} \vdash \neg \langle s \rangle \langle p \rangle \psi$ for all $p \in P$. This implies that $\Gamma \cup \{\varphi\} \vdash [s][p]\neg\psi$ for all $p \in P$. Therefore, $\Gamma \vdash \varphi \rightarrow [s][p]\neg\psi$ for all $p \in P$. Note that $\varphi \rightarrow [s][p]\neg\psi := [\varphi \rightarrow, s][p]\neg\psi$, and $[\varphi \rightarrow, s] \in NF_{\mathcal{L}_G}^P$. We thus have $\Gamma \vdash [\varphi \rightarrow, s][p]\neg\psi$ for all $p \in P$. Since Γ is P -witnessed, we obtain $\Gamma \vdash [\varphi \rightarrow, s]\blacksquare\neg\psi$. By unraveling the necessity form $[\varphi \rightarrow, s]$, we get $\Gamma \vdash \varphi \rightarrow [s]\blacksquare\neg\psi$, thus, $\Gamma \cup \{\varphi\} \vdash [s]\blacksquare\neg\psi$, i.e., $\Gamma \cup \{\varphi\} \vdash \neg \langle s \rangle \blacklozenge \psi$, contradicting the assumption that $\Gamma \cup \{\varphi\}$ is consistent with $\langle s \rangle \blacklozenge \psi$.

Now suppose $\Gamma \cup \{\varphi\}$ does not satisfy (2). This means that there is $s \in NF_{\mathcal{L}_G}^P$ and $\psi \in \mathcal{L}_G^P$ such that for some group $G \subseteq \mathcal{A}$, the set $\Gamma \cup \{\varphi\}$ is consistent with $\langle s \rangle \langle G \rangle \psi$ but $\Gamma \cup \{\varphi\} \vdash \neg \langle s \rangle \langle \bigwedge_{i \in G} K_i p_i \rangle \psi$ for all $\{p_i : i \in G\} \subseteq P$. This implies that $\Gamma \cup \{\varphi\} \vdash [s][\bigwedge_{i \in G} K_i p_i]\neg\psi$ for all $\{p_i : i \in G\} \subseteq P$. Therefore, $\Gamma \vdash \varphi \rightarrow [s][\bigwedge_{i \in G} K_i p_i]\neg\psi$ for all $\{p_i : i \in G\} \subseteq P$. Note that $\varphi \rightarrow [s][\bigwedge_{i \in G} K_i p_i]\neg\psi := [\varphi \rightarrow, s][\bigwedge_{i \in G} K_i p_i]\neg\psi$, and $[\varphi \rightarrow, s] \in NF_{\mathcal{L}_G}^P$. We thus have $\Gamma \vdash [\varphi \rightarrow, s][\bigwedge_{i \in G} K_i p_i]\neg\psi$ for all $\{p_i : i \in G\} \subseteq P$. Since Γ is P -witnessed, we obtain $\Gamma \vdash [\varphi \rightarrow, s][G]\neg\psi$. By unraveling the necessity form $[\varphi \rightarrow, s]$, we get $\Gamma \vdash \varphi \rightarrow [s][G]\neg\psi$, thus, $\Gamma \cup \{\varphi\} \vdash [s][G]\neg\psi$, i.e., $\Gamma \cup \{\varphi\} \vdash \neg \langle s \rangle \langle G \rangle \psi$, contradicting the assumption that $\Gamma \cup \{\varphi\}$ is consistent with $\langle s \rangle \langle G \rangle \psi$. All together we obtain that $\Gamma \cup \{\varphi\}$ is P -witnessed. \square

4.6.7. LEMMA. *If $\{\Gamma_i\}_{i \in \mathbb{N}}$ is an increasing chain of P -theories such that $\Gamma_i \subseteq \Gamma_{i+1}$, then $\bigcup_{n \in \mathbb{N}} \Gamma_n$ is a P -theory.*

Proof:

Let $\{\Gamma_i\}_{i \in \mathbb{N}}$ be an increasing chain of P -theories with $\Gamma_i \subseteq \Gamma_{i+1}$ and suppose, toward contradiction, that $\bigcup_{n \in \mathbb{N}} \Gamma_n$ is not a P -theory, i.e., suppose that $\bigcup_{n \in \mathbb{N}} \Gamma_n \vdash \perp$. This means that there exists a finite $\Delta \subseteq \bigcup_{n \in \mathbb{N}} \Gamma_n$ such that $\Delta \vdash \perp$. Then, since $\bigcup_{n \in \mathbb{N}} \Gamma_n$ is a union of an increasing chain of P -theories, there is some $m \in \mathbb{N}$ such that $\Delta \subseteq \Gamma_m$. Therefore, $\Gamma_m \vdash \perp$ contradicting the fact that Γ_m is a P -theory. Hence, $\bigcup_{n \in \mathbb{N}} \Gamma_n$ is a P -theory. \square

4.6.8. LEMMA. *For every maximal P -witnessed theory T , both $\{\theta \in \mathcal{L}_G^P : K_i\theta \in T\}$ and $\{\theta \in \mathcal{L}_G^P : U\theta \in T\}$ are P -witnessed theories.*

Proof:

Observe that, by axiom (T_{K_i}) , $\{\theta \in \mathcal{L}_G^P : K_i\theta \in T\} \subseteq T$. Therefore, as T is consistent, the set $\{\theta \in \mathcal{L}_G^P : K_i\theta \in T\}$ is consistent. Let $s \in NF_{\mathcal{L}_G}^P$, $\psi \in \mathcal{L}_G^P$, and $G \subseteq \mathcal{AG}$ such that $\{\theta \in \mathcal{L}_G^P : K_i\theta \in T\} \vdash [s][p]\neg\varphi$ for all $p \in P$ and $\{\theta \in \mathcal{L}_G^P : K_i\theta \in T\} \vdash [s][\bigwedge_{i \in G} K_i p_i]\neg\psi$ for all $\{p_i : i \in G\} \subseteq P$. By normality of K_i , $T \vdash K_i[s][p]\neg\varphi$ for all $p \in P$ and $T \vdash K_i[s][\bigwedge_{i \in G} K_i p_i]\neg\psi$ for all $\{p_i : i \in G\} \subseteq P$. Since $K_i[s][p]\neg\varphi := [K_i, s][p]\neg\varphi$ and $K_i[s][\bigwedge_{i \in G} K_i p_i]\neg\psi := [K_i, s][\bigwedge_{i \in G} K_i p_i]\neg\psi$ are necessity forms and T is P -witnessed, we obtain $T \vdash [K_i, s]\blacksquare\neg\varphi$ and $T \vdash [K_i, s][G]\neg\varphi$, i.e., $T \vdash K_i[s]\blacksquare\neg\varphi$ and $T \vdash K_i[s][G]\neg\varphi$. As T is maximal, we have $K_i[s]\blacksquare\neg\varphi \in T$ and $K_i[s][G]\neg\varphi \in T$, thus $[s]\blacksquare\neg\varphi \in \{\theta \mid K_i\theta \in T\}$ and $[s][G]\neg\varphi \in \{\theta \mid K_i\theta \in T\}$. The proof for $\{\theta \in \mathcal{L}_G^P : U\theta \in T\}$ follows similarly. \square

4.6.9. LEMMA (LINDENBAUM'S LEMMA). *Every P -witnessed theory Γ can be extended to a maximal P -witnessed theory T_Γ .*

Proof:

The proof proceeds by constructing an increasing chain $\Gamma_0 \subseteq \Gamma_1 \subseteq \dots \subseteq \Gamma_n \subseteq \dots$ of P -witnessed theories, where $\Gamma_0 := \Gamma$, and each Γ_i is recursively defined. Since we have to guarantee that each Γ_i is P -witnessed, we follow a two-fold construction, where $\Gamma_0 = \Gamma_0^+ := \Gamma$. Let $\gamma_0, \gamma_1, \dots, \gamma_n, \dots$ be an enumeration of all pairs of the form $\gamma_i = (s_i, \varphi_i)$ consisting of any necessity form $s_i \in NF_{\mathcal{L}_G}^P$ and any formula $\varphi_i \in \mathcal{L}_G^P$. Let (s_n, φ_n) be the n th pair in the enumeration. We then set

$$\Gamma_n^+ = \begin{cases} \Gamma_n \cup \{\langle s_n \rangle \varphi_n\} & \text{if } \Gamma_n \not\vdash \neg \langle s_n \rangle \varphi_n, \\ \Gamma_n & \text{otherwise.} \end{cases}$$

Note that the empty string ϵ is in $NF_{\mathcal{L}_G}^P$, and for every $\psi \in \mathcal{L}_G^P$ we have $\langle \epsilon \rangle \psi := \psi$ by the definition of possibility forms. Therefore, the above enumeration of pairs includes every formula ψ of \mathcal{L}_G^P in the form of its corresponding pair (ϵ, ψ) . By Lemma 4.6.6, each Γ_n^+ is P -witnessed. Then, if φ_n is of the form $\varphi_n := \blacklozenge\theta$ for some $\theta \in \mathcal{L}_G^P$, there exists a $p \in P$ such that Γ_n^+ is consistent with $\langle s_n \rangle \langle p \rangle \theta$ (since Γ_n^+

is P -witnessed). Similarly, if φ_n is of the form $\varphi_n := \langle G \rangle \theta$ for some $\theta \in \mathcal{L}_G^P$, there exists $\{p_i : i \in G\} \subseteq P$ such that Γ_n^+ is consistent with $\langle s_n \rangle \langle \bigwedge_{i \in G} K_i p_i \rangle \theta$. We then define

$$\Gamma_{n+1} = \begin{cases} \Gamma_n^+ & \text{if } \Gamma_n \not\vdash \neg \langle s_n \rangle \varphi_n \text{ and} \\ & \varphi_n \text{ is not of the form } \blacklozenge \theta \text{ or } \langle G \rangle \theta, \\ \Gamma_n^+ \cup \{\langle s_n \rangle \langle p \rangle \theta\} & \text{if } \Gamma_n \not\vdash \neg \langle s_n \rangle \varphi_n \text{ and} \\ & \varphi_n := \blacklozenge \theta \text{ for some } \theta \in \mathcal{L}_G^P, \\ \Gamma_n^+ \cup \{\langle s_n \rangle \langle \bigwedge_{i \in G} K_i p_i \rangle \theta\} & \text{if } \Gamma_n \not\vdash \neg \langle s_n \rangle \varphi_n \text{ and} \\ & \varphi_n := \langle G \rangle \theta \text{ for some } \theta \in \mathcal{L}_G^P, \\ \Gamma_n & \text{otherwise,} \end{cases}$$

where $p \in P$, $\{p_i : i \in G\} \subseteq P$ such that Γ_n^+ is consistent with $\langle s_n \rangle \langle p \rangle \theta$ or consistent with $\langle s_n \rangle \langle \bigwedge_{i \in G} K_i p_i \rangle \theta$, respectively. Again by Lemma 4.6.6, it is guaranteed that each Γ_n is P -witnessed. Now consider the union $T_\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$. By Lemma 4.6.7, we know that T_Γ is a P -theory. To show that T_Γ is P -witnessed, first let $s \in NF_{\mathcal{L}_G}^P$ and $\psi \in \mathcal{L}_G^P$ and suppose $\langle s \rangle \blacklozenge \psi$ is consistent with T_Γ . The pair $(s, \blacklozenge \psi)$ appears in the above enumeration of all pairs, thus $(s, \blacklozenge \psi) := (s_m, \varphi_m)$ for some $m \in \mathbb{N}$. Hence, $\langle s \rangle \blacklozenge \psi := \langle s_m \rangle \varphi_m$. Then, since $\langle s \rangle \blacklozenge \psi$ is consistent with T_Γ and $\Gamma_m \subseteq T_\Gamma$, we know that $\langle s \rangle \blacklozenge \psi$ is in particular consistent with Γ_m . Therefore, by the above construction, $\langle s \rangle \langle p \rangle \psi \in \Gamma_{m+1}$ for some $p \in P$ such that Γ_m^+ is consistent with $\langle s \rangle \langle p \rangle \psi$. Thus, as T_Γ is consistent and $\Gamma_{m+1} \subseteq T_\Gamma$, we have that $\langle s \rangle \langle p \rangle \psi$ is also consistent with T_Γ . Now, let us check the witnessing condition for $\langle G \rangle$. Let $G \in \mathcal{A}$, $s \in NF_{\mathcal{L}_G}^P$ and $\psi \in \mathcal{L}_G^P$ and suppose $\langle s \rangle \langle G \rangle \psi$ is consistent with T_Γ . The pair $(s, \langle G \rangle \psi)$ appears in the above enumeration of all pairs, thus $(s, \langle G \rangle \psi) := (s_m, \varphi_m)$ for some $m \in \mathbb{N}$. Hence, $\langle s \rangle \langle G \rangle \psi := \langle s_m \rangle \varphi_m$. Then, since $\langle s \rangle \langle G \rangle \psi$ is consistent with T_Γ and $\Gamma_m \subseteq T_\Gamma$, we know that $\langle s \rangle \langle G \rangle \psi$ is in particular consistent with Γ_m . Therefore, by the above construction, $\langle s \rangle \langle \bigwedge_{i \in G} K_i p_i \rangle \psi \in \Gamma_{m+1}$ for some $\{p_i : i \in G\} \subseteq P$ such that Γ_m^+ is consistent with $\langle s \rangle \langle \bigwedge_{i \in G} K_i p_i \rangle \psi$. Thus, as T_Γ is consistent and $\Gamma_{m+1} \subseteq T_\Gamma$, we have that $\langle s \rangle \langle \bigwedge_{i \in G} K_i p_i \rangle \psi$ is also consistent with T_Γ . Hence, we conclude that T_Γ is P -witnessed. Finally, T_Γ is also maximal by construction: otherwise there would be a P -witness theory T such that $T_\Gamma \subsetneq T$. This implies that there exists $\varphi \in \mathcal{L}_G^P$ with $\varphi \in T$ but $\varphi \notin T_\Gamma$. Then, by the construction of T_Γ , we obtain $\Gamma_i \vdash \neg \varphi$ for all $i \in \mathbb{N}$. Therefore, since $T_\Gamma \subseteq T$, we have $T \vdash \neg \varphi$. Hence, since $\varphi \in T$, we conclude $T \vdash \perp$ (contradicting T being consistent). \square

4.6.10. LEMMA (EXTENSION LEMMA). *Let P be a countable set of propositional variables and P' be a countable set of fresh propositional variables, i.e., $P \cap P' = \emptyset$. Let $\tilde{P} = P \cup P'$. Then, every initial P -theory Γ can be extended to an initial \tilde{P} -witnessed theory $\tilde{\Gamma} \supseteq \Gamma$, and hence to a maximal \tilde{P} -witnessed initial theory $T_\Gamma \supseteq \Gamma$.*

Proof:

Let $\gamma_0, \gamma_1, \dots, \gamma_n, \dots$ an enumeration of all pairs of the form (s_n, φ_n) consisting of any $s_n \in NF_{\mathcal{L}_G}^{\tilde{P}}$, and every formula $\varphi_n \in \mathcal{L}_G^{\tilde{P}}$ of the form $\varphi_n := \blacklozenge\psi$ or $\varphi_n := \langle G \rangle\psi$ with $\psi \in \mathcal{L}_G^{\tilde{P}}$. We will recursively construct a chain of initial \tilde{P} -theories $\Gamma_0 \subseteq \dots \subseteq \Gamma_n \subseteq \dots$ such that

1. $\Gamma_0 = \Gamma$,
2. $P'_n := \{p \in P' : p \text{ occurs in } \Gamma_n\}$ is finite for every $n \in \mathbb{N}$ (we will see why this is finite later on the proof), and
3. for every $\gamma_n := (s_n, \varphi_n)$ with $s_n \in NF_{\mathcal{L}_G}^{\tilde{P}}$ and $\varphi_n \in \mathcal{L}_G^{\tilde{P}}$, if $\Gamma_n \not\vdash \neg\langle s_n \rangle\blacklozenge\psi$ where $\varphi_n := \blacklozenge\psi$ then there is p_m “fresh” such that $\langle s_n \rangle\langle p_m \rangle\psi \in \Gamma_{n+1}$, and, if $\Gamma_n \not\vdash \neg\langle s_n \rangle\langle G \rangle\psi$ where $\varphi_n := \langle G \rangle\psi$ for some $G \subseteq \mathcal{A}$ then there is $\{p_{m_i} : i \in G\}$ where p_{m_i} is “fresh” for every $i \in G$ such that $\langle s_n \rangle\langle \bigwedge_{i \in G} K_i p_{m_i} \rangle\psi \in \Gamma_{n+1}$. Otherwise we will define $\Gamma_{n+1} = \Gamma_n$.

For every γ_n , let $P'(n) := \{p \in P' \mid p \text{ occurs either in } s_n \text{ or } \varphi_n\}$. Clearly every $P'(n)$ is always finite. We now construct an increasing chain of initial \tilde{P} -theories recursively. We set $\Gamma_0 := \Gamma$, and let

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\langle s_n \rangle\langle p_m \rangle\psi\} & \text{if } \Gamma_n \not\vdash \neg\langle s_n \rangle\blacklozenge\psi \text{ and } \varphi_n := \blacklozenge\psi, \\ \Gamma_n \cup \{\langle s_n \rangle\langle \bigwedge_{i \in G} K_i p_{m_i} \rangle\psi\} & \text{if } \Gamma_n \not\vdash \neg\langle s_n \rangle\langle G \rangle\psi \text{ and } \varphi_n := \langle G \rangle\psi, \\ \Gamma_n & \text{otherwise,} \end{cases}$$

where m, m_i are, in each case, the least natural number greater than the indices in $P'_n \cup P'(n)$, i.e., p_m, p_{m_i} for all $i \in G$ are fresh in each case. To see that $P'_n \cup P'(n)$ is finite for every $n \in \mathbb{N}$, we just need to check that P'_n is finite. First note that since $\Gamma := \Gamma_0$ is a P -theory, no propositional variables in P' occur in Γ_0 . For each pair $\gamma_i := (s_i, \varphi_i)$ with $i \in \{0, \dots, n-1\}$ the set $P'(n)$ is finite. At each step n of the construction, we can only add finitely many fresh variables to Γ_n . Thus, finitely many propositional variables in P' occur in Γ_n and so P'_n is finite. We now show that $\tilde{\Gamma} := \bigcup_{n \in \mathbb{N}} \Gamma_n$ is an initial \tilde{P} -witnessed theory. First show that $\tilde{\Gamma}$ is a \tilde{P} -theory. By Lemma 4.6.7, it suffices to show by induction that every Γ_n is a \tilde{P} -theory. Clearly Γ_0 is a \tilde{P} -theory. For the inductive step suppose Γ_n is consistent but Γ_{n+1} is not. Hence, $\Gamma_n \neq \Gamma_{n+1}$ and moreover $\Gamma_{n+1} \vdash \perp$. Then, $\Gamma_{n+1} = \Gamma_n \cup \{\langle s_n \rangle\langle p_m \rangle\psi\}$ (when $\varphi_n := \blacklozenge\psi$) or $\Gamma_{n+1} = \Gamma_n \cup \{\langle s_n \rangle\langle \bigwedge_{i \in G} K_i p_{m_i} \rangle\psi\}$ (when $\varphi_n := \langle G \rangle\psi$). Here we will only check the latter case since the former case is analogous. Since $\Gamma_{n+1} = \Gamma_n \cup \{\langle s_n \rangle\langle \bigwedge_{i \in G} K_i p_{m_i} \rangle\psi\}$ we have $\Gamma_n \vdash [s_n][\bigwedge_{i \in G} K_i p_{m_i}]\neg\psi$. Therefore there exists $\{\theta_1, \dots, \theta_k\} \subseteq \Gamma_n$ such that $\{\theta_1, \dots, \theta_k\} \vdash [s_n][\bigwedge_{i \in G} K_i p_{m_i}]\neg\psi$. Let $\theta = \bigwedge_{1 \leq i \leq k} \theta_i$. Then $\vdash \theta \rightarrow [s_n][\bigwedge_{i \in G} K_i p_{m_i}]\neg\psi$, so $\vdash [\theta \rightarrow, s_n][\bigwedge_{i \in G} K_i p_{m_i}]\neg\psi$ with $p_{m_i} \notin P_\theta \cup P_{s_n} \cup P_{\varphi_n}$ for every $i \in G$. Thus, by the admissible rule (2) in

Lemma 4.6.4, we obtain $\vdash [\theta \rightarrow, s_n][G]\neg\psi$, i.e., $\vdash \theta \rightarrow [s_n][G]\neg\psi$. Therefore, $\theta \vdash \neg\langle s_n \rangle \langle G \rangle \psi$. Since $\{\theta_1, \dots, \theta_k\} \subseteq \Gamma_n$, we therefore have $\Gamma_n \vdash \neg\langle s_n \rangle \langle G \rangle \psi$. But, this would mean $\Gamma_n = \Gamma_{n+1}$, contradicting our assumption (that $\Gamma_{n+1} \neq \Gamma_n$). Therefore Γ_{n+1} is consistent and thus a \tilde{P} -theory. Hence, by Lemma 4.6.7, $\tilde{\Gamma}$ is a \tilde{P} -theory. Condition (3) above implies that $\tilde{\Gamma}$ is also \tilde{P} -witnessed. Then, by Lindenbaum's Lemma (Lemma 4.6.9), there is a maximal \tilde{P} -witnessed theory $T_{\tilde{\Gamma}}$ such that $T_{\tilde{\Gamma}} \supseteq \tilde{\Gamma} \supseteq \Gamma$. Moreover, since $0 \in \Gamma \subseteq \tilde{\Gamma} \subseteq T_{\tilde{\Gamma}}$, the set $T_{\tilde{\Gamma}}$ is in fact a maximal \tilde{P} -witnessed initial theory. \square

We are now ready to define our canonical pseudo-model. We first define, for all maximal P -witnessed theories T, S and for every $i \in \mathcal{AG}$:

$$T \sim_U S \text{ iff } \forall \varphi \in \mathcal{L}_G^P (U\varphi \in T \text{ implies } \varphi \in S), \text{ and}$$

$$T \sim_i S \text{ iff } \forall \varphi \in \mathcal{L}_G^P (K_i\varphi \in T \text{ implies } \varphi \in S).$$

4.6.11. LEMMA. *For every $i \in \mathcal{AG}$, $\sim_i \subseteq \sim_U$.*

Proof:

Let $i \in \mathcal{AG}$, let T and S be maximal P -witnessed theories such that $T \sim_i S$. Towards contradiction, suppose that $T \sim_U S$ is not the case. From the former we have that $\forall \varphi \in \mathcal{L}_G^P (K_i\varphi \in T \text{ implies } \varphi \in S)$. From the latter, we have that there is $\psi \in \mathcal{L}_G^P$ such that $U\psi \in T$ and $\psi \notin S$. Since $\vdash U\psi \rightarrow K_i\psi$ and T is a maximal P -witnessed theory, $U\psi \rightarrow K_i\psi \in T$. Therefore $K_i\psi \in T$ and $\psi \notin S$, contradicting that $T \sim_i S$. \square

4.6.12. DEFINITION. [Canonical Pre-Model] Given a maximal P -witnessed initial theory T_0 , the *canonical pre-model for T_0* is a tuple $\mathcal{M}^c = (W^c, \mathcal{A}^c, \sim_1^c, \dots, \sim_n^c, \|\cdot\|^c)$ such that:

- $W^c = \{T : T \text{ is a maximal } P\text{-witnessed theory such that } T_0 \sim_U T\}$,
- $\mathcal{A}^c = \{\hat{\theta} : \theta \in \mathcal{L}_{-\diamond}^P\}$ where $\hat{\varphi} = \{T \in W^c : \varphi \in T\}$ for any $\varphi \in \mathcal{L}_G^P$,
- for every $i \in \mathcal{AG}$ we define:

$$\sim_i^c = \sim_i \cap (W^c \times W^c).$$

- $\|p\|^c = \{T \in W^c : p \in T\} = \hat{p}$.

As usual, it is easy to see (given the $S5$ axioms for K_i and for U) that \sim_U and \sim_i^c are equivalence relations.

To prove that the canonical pre-model is indeed a pseudo-model, we first need to prove the truth lemma. For that we need the following lemmas.

4.6.13. LEMMA (EXISTENCE LEMMA FOR \sim_U). *Let T be a maximal P -witnessed theory, $\alpha \in \mathcal{L}_{\diamond}^P$, and $\varphi \in \mathcal{L}_G^P$ such that $\alpha \in T$ and $U[\alpha]\varphi \notin T$. Then, there is a maximal P -witnessed theory S such that $T \sim_U S$, $\alpha \in S$ and $[\alpha]\varphi \notin S$.*

Proof:

Let $\alpha \in \mathcal{L}_{\diamond}^P$ and $\varphi \in \mathcal{L}_G^P$ such that $\alpha \in T$ and $U[\alpha]\varphi \notin T$. The latter implies that $\{\psi \in \mathcal{L}_G^P : U\psi \in T\} \not\vdash [\alpha]\varphi$, hence, $\{\psi \in \mathcal{L}_G^P : U\psi \in T\} \not\vdash \neg\neg[\alpha]\varphi$. Then, by Lemmas 4.6.6 and 4.6.8, we obtain that $\{\psi \in \mathcal{L}_G^P : U\psi \in T\} \cup \{\neg[\alpha]\varphi\}$ is a P -witnessed theory. Note that $\vdash \neg[\alpha]\varphi \leftrightarrow (\alpha \wedge [\alpha]\neg\varphi)$ (see (3) in Proposition 4.3.14). We therefore obtain that $\{\psi \in \mathcal{L}_G^P : U\psi \in T\} \cup \{\neg[\alpha]\varphi\} \vdash \alpha$, thus, $\{\psi \in \mathcal{L}_G^P : U\psi \in T\} \cup \{\neg[\alpha]\varphi\} \not\vdash \neg\alpha$ (since $\{\psi \in \mathcal{L}_G^P : U\psi \in T\} \cup \{\neg[\alpha]\varphi\}$ is consistent). Therefore, by Lemma 4.6.6, $\{\psi \in \mathcal{L}_G^P : U\psi \in T\} \cup \{\neg[\alpha]\varphi\} \cup \{\alpha\}$ is also a P -witnessed theory. We can then apply Lindenbaum's Lemma (Lemma 4.6.9) and extend it to a maximal P -witnessed theory S such that $S \sim_U T$, $\alpha \in S$, and $[\alpha]\varphi \notin S$. \square

4.6.14. COROLLARY. *For $\varphi \in \mathcal{L}_G$, we have $\widehat{U}\varphi = W^c$ if $\widehat{\varphi} = W^c$, and $\widehat{U}\varphi = \emptyset$ otherwise.*

Proof:

If $\widehat{\varphi} = W^c$, suppose $\widehat{U}\varphi \neq W^c$. The latter means that there is a $T \in W^c$ such that $U\varphi \notin T$. Then, by Lemma 4.6.13 (when $\alpha := \top$), there is a maximal P -witnessed theory S such that $T \sim_U S$ and $\varphi \notin S$. Since $T_0 \sim_U T \sim_U S$ and \sim_U is transitive, we have $T_0 \sim_U S$, thus, $S \in W^c$. Therefore, $\widehat{\varphi} \neq W^c$, contradicting the initial assumption. If $\widehat{\varphi} \neq W^c$, then there is a $T \in W^c$ such that $\varphi \notin T$. Since $T \sim_U S$ for all $S \in W^c$, we obtain by the definition of \sim_U that $U\varphi \notin S$ for all $S \in W^c$. Therefore, $\widehat{U}\varphi = \emptyset$. \square

4.6.15. LEMMA (EXISTENCE LEMMA FOR \sim_i). *Let T be a maximal P -witnessed theory, let $\alpha \in \mathcal{L}_{\diamond}^P$, and $\varphi \in \mathcal{L}_G^P$ be such that $\alpha \in T$ and $K_i[\alpha]\varphi \notin T$. Then, there is a maximal P -witnessed theory S such that $T \sim_i S$, $\alpha \in S$ and $[\alpha]\varphi \notin S$.*

Proof:

Let $\alpha \in \mathcal{L}_{\diamond}^P$ and $\varphi \in \mathcal{L}_G^P$ such that $\alpha \in T$ and $K_i[\alpha]\varphi \notin T$. The latter implies that $\{\psi \in \mathcal{L}_G^P : K_i\psi \in T\} \not\vdash [\alpha]\varphi$, hence, $\{\psi \in \mathcal{L}_G^P : K_i\psi \in T\} \not\vdash \neg\neg[\alpha]\varphi$. Then, by Lemmas 4.6.6 and 4.6.8, we obtain that $\{\psi \in \mathcal{L}_G^P : K_i\psi \in T\} \cup \{\neg[\alpha]\varphi\}$ is a P -witnessed theory. Note that $\vdash \neg[\alpha]\varphi \leftrightarrow (\alpha \wedge [\alpha]\neg\varphi)$ (see (3) in Proposition 4.3.14). We therefore obtain that $\{\psi \in \mathcal{L}_G^P : K_i\psi \in T\} \cup \{\neg[\alpha]\varphi\} \vdash \alpha$, thus, $\{\psi \in \mathcal{L}_G^P : K_i\psi \in T\} \cup \{\neg[\alpha]\varphi\} \not\vdash \neg\alpha$ (since $\{\psi \in \mathcal{L}_G^P : K_i\psi \in T\} \cup \{\neg[\alpha]\varphi\}$ is consistent). Therefore, by Lemma 4.6.6, $\{\psi \in \mathcal{L}_G^P : K_i\psi \in T\} \cup \{\neg[\alpha]\varphi\} \cup \{\alpha\}$ is also a P -witnessed theory. We can then apply Lindenbaum's Lemma (Lemma 4.6.9) and extend it to a maximal P -witnessed theory S such that $S \sim_i T$, $\alpha \in S$,

and $[\alpha]\varphi \notin S$. □

4.6.16. COROLLARY. *Let T_0 be a maximal P -witnessed theory and $\mathcal{M}^c = (W^c, \mathcal{A}^c, \sim_1^c, \dots, \sim_n^c, \|\cdot\|^c)$ be the canonical pre-model for T_0 . For all $T \in \mathcal{M}^c$, $\alpha \in \mathcal{L}_{-\blacklozenge}^P$, and $\varphi \in \mathcal{L}_G^P$, if $\alpha \in T$ and $K_i[\alpha]\varphi \notin T$ then there is a maximal P -witnessed theory $S \in W^c$ such that $T \sim_i^c S$, $\alpha \in S$ and $[\alpha]\varphi \notin S$.*

Proof:

Let $T \in \mathcal{M}^c$, let $\alpha \in \mathcal{L}_{-\blacklozenge}^P$ and $\varphi \in \mathcal{L}_G^P$ be such that $\alpha \in T$ and $K_i[\alpha]\varphi \notin T$. By Lemma 4.6.15, there is a maximal P -witnessed theory S such that $T \sim_i S$, $\alpha \in S$ and $[\alpha]\varphi \notin S$. By Lemma 4.6.11, $T \sim_U S$. Since $T_0 \sim_U T$, by transitivity of \sim_U we have $T_0 \sim_U S$. Therefore $S \in W^c$ and so $T \sim_i^c S$. □

4.6.17. LEMMA. *Every element $T \in W^c$ is an initial theory (i.e. $0 \in T$).*

Proof:

Let $T \in W^c$. By the construction of W^c , we have $T_0 \sim_U T$. Since $0 \rightarrow U0$ is an axiom and T_0 is maximal, $(0 \rightarrow U0) \in T_0$. Thus, since $0 \in T_0$, we obtain $U0 \in T_0$ (by (4) in Lemma 4.6.5). Therefore, by the definition of \sim_U and since $T_0 \sim_U T$, we have that $0 \in T$. □

4.6.18. COROLLARY. *For all $\varphi \in \mathcal{L}_G^P$, we have $\widehat{\varphi} = \widehat{\varphi^0}$.*

Proof:

Since $0 \in T$ for all $T \in W^c$, we obtain by axiom (0-eq) that $\varphi \leftrightarrow \varphi^0 \in T$ for all $T \in W^c$. Therefore, $\widehat{\varphi} = \widehat{\varphi^0}$. □

4.6.19. LEMMA (TRUTH LEMMA). *Let $\mathcal{M}^c = (W^c, \mathcal{A}^c, \sim_1^c, \dots, \sim_n^c, V^c)$ be the canonical pre-model for some T_0 (in \mathcal{L}_G^P) and $\varphi \in \mathcal{L}_G^P$. Then, for all $\alpha \in \mathcal{L}_{-\blacklozenge}^P$, we have $\llbracket \varphi \rrbracket_{\widehat{\alpha}} = \widehat{\langle \alpha \rangle \varphi}$.*

Proof:

The proof is by \prec_2 -induction on φ , using the following *induction hypothesis* (IH): for all $\psi \in \mathcal{L}_G^P$ such that $\psi \prec_2 \varphi$, we have $\llbracket \psi \rrbracket_{\widehat{\alpha}} = \widehat{\langle \alpha \rangle \psi}$ for all $\alpha \in \mathcal{L}_{-\blacklozenge}^P$. The cases for the Boolean connectives are straightforward. The cases for K_i and U are standard, using $\vdash \langle \alpha \rangle K_i \psi \leftrightarrow \alpha \wedge K_i[\alpha]\psi$ and Corollary 4.6.16 for K_i , and $\vdash \langle \alpha \rangle U\psi \leftrightarrow \alpha \wedge U[\alpha]\psi$ and Lemma 4.6.13 for U .

Base case $\varphi := \top$. Then $\llbracket \top \rrbracket_{\widehat{\alpha}} = \widehat{\alpha} = \widehat{\langle \alpha \rangle \top}$, by Definition 4.5.2 and the fact that $\vdash \alpha \leftrightarrow \langle \alpha \rangle \top$.

Base case $\varphi := p$. Then $\llbracket p \rrbracket_{\hat{\alpha}} = \llbracket p \rrbracket^c \cap \hat{\alpha} = \widehat{p} \cap \hat{\alpha} = \widehat{p \wedge \alpha} = \langle \alpha \rangle p$, by Definition 4.5.2, the defn. of $\llbracket \cdot \rrbracket^c$, R_p , and (3.) in Proposition 4.3.14.

Base case $\varphi := 0$. Then $\llbracket 0 \rrbracket_{\hat{\alpha}} = W^c$ if $\hat{\alpha} = W^c$, and $\llbracket 0 \rrbracket_{\hat{\alpha}} = \emptyset$ otherwise. Also, $\langle \alpha \rangle 0 = \widehat{0 \wedge U\alpha}$ (by (2.) in Propositions 4.3.14) $= \{T \in W^c : 0 \wedge U\alpha \in T\} = \{T \in W^c : U\alpha \in T\} = \widehat{U\alpha}$ (by Lemma 4.6.17). By Corollary 4.6.14, $\widehat{U\alpha} = W^c$ if $\hat{\alpha} = W^c$, and $\widehat{U\alpha} = \emptyset$ otherwise. So $\llbracket 0 \rrbracket_{\hat{\alpha}} = \langle \alpha \rangle 0$.

Case $\varphi := \psi^0$. Follows easily from $\widehat{\top} = W^c$ and $R[\top]$, Corollary 4.6.18, and R^0 .

Case $\varphi := \langle \chi \rangle \psi$. Straightforward, using the fact that $\vdash \langle \alpha \rangle \langle \chi \rangle \psi \leftrightarrow \langle \langle \alpha \rangle \chi \rangle \psi$ (by $R[\cup]$)

Case $\varphi := \blacklozenge \psi$.

(\Rightarrow) Suppose $T \in \llbracket \blacklozenge \psi \rrbracket_{\hat{\alpha}}$. This means, by Definition 4.5.2, that $\alpha \in T$ and there exists $B \in \mathcal{A}^c$ such that $T \in B \subseteq \hat{\alpha}$ and $T \in \llbracket \psi \rrbracket_B$ (see Observation 3.1). By the construction of \mathcal{A}^c , we know that $B = \widehat{\theta}$ for some $\theta \in \mathcal{L}_{\blacklozenge}^P$. Therefore, $T \in \llbracket \psi \rrbracket_B$ means that $T \in \llbracket \psi \rrbracket_{\widehat{\theta}}$. Moreover, since $\widehat{\theta} \subseteq \hat{\alpha}$ and, thus, $\widehat{\theta} = \widehat{\alpha} \cap \widehat{\theta} = \widehat{\alpha \wedge \theta}$, we obtain $T \in \llbracket \psi \rrbracket_{\widehat{\alpha \wedge \theta}}$. By (1.) in Lemma A.2.5, we have $\psi \prec_2 \blacklozenge \psi$. Therefore, by IH, we obtain $T \in \langle \alpha \wedge \theta \rangle \psi$. Then, by axiom ($[\blacklozenge]$ -elim) and the fact that T is maximal, we conclude that $T \in \langle \alpha \rangle \blacklozenge \psi$.

(\Leftarrow) Suppose $T \in \langle \alpha \rangle \blacklozenge \psi$, i.e., $\langle \alpha \rangle \blacklozenge \psi \in T$. Then, since T is a maximal P -witnessed theory, there is $p \in P$ such that $\langle \alpha \rangle \langle p \rangle \psi \in T$. By (9.) in Lemma A.2.5, we know that $\langle p \rangle \psi \prec_2 \blacklozenge \psi$. Thus, by IH on $\langle p \rangle \psi$, we obtain that $T \in \llbracket \langle p \rangle \psi \rrbracket_{\hat{\alpha}}$. This means, by Definition 4.5.2 and Observation 3.3, that $T \in \llbracket \psi \rrbracket_{\llbracket p \rrbracket_{\hat{\alpha}}} \subseteq \llbracket \psi \rrbracket_{\hat{\alpha}}$. Since $p \prec_2 \blacklozenge \psi$, by IH on p , we obtain that $\llbracket p \rrbracket_{\hat{\alpha}} = \langle \alpha \rangle p \subseteq \hat{\alpha}$. By the construction of \mathcal{A}^c , we moreover have $\langle \alpha \rangle p \in \mathcal{A}^c$. Therefore, as $T \in \llbracket \psi \rrbracket_{\langle \alpha \rangle p}$ and $\langle \alpha \rangle p \subseteq \hat{\alpha}$, by Definition 4.5.2, we conclude that $T \in \llbracket \blacklozenge \psi \rrbracket_{\hat{\alpha}}$.

Case $\varphi := \langle G \rangle \psi$.

(\Rightarrow) Suppose $T \in \llbracket \langle G \rangle \psi \rrbracket_{\hat{\alpha}}$. This means by Definition 4.5.2 that $T \in \hat{\alpha}$ and there exists $\{B_i : i \in G\} \subseteq \mathcal{A}^c$ such that $T \in \llbracket \psi \rrbracket_{\widehat{\alpha} \cap \bigcap_{i \in G} K_i^{\hat{\alpha}} B_i}$. By the construction of \mathcal{A}^c we know that for all $i \in G$, $B_i = \widehat{\theta}_i$ for some $\theta_i \in \mathcal{L}_{\blacklozenge}^P$. Therefore $T \in \llbracket \psi \rrbracket_{\widehat{\alpha} \cap \bigcap_{i \in G} K_i^{\hat{\alpha}} \widehat{\theta}_i}$. It suffices to show that: $\widehat{\alpha} \cap \bigcap_{i \in G} K_i^{\hat{\alpha}} \widehat{\theta}_i = \widehat{\alpha \wedge \bigwedge_{i \in G} K_i^{\alpha} \theta_i}$. First we need to show that $K_i^{\hat{\alpha}} \widehat{\theta}_i = \widehat{K_i^{\alpha} \theta_i}$. Note that $K_i^{\hat{\alpha}} \widehat{\theta}_i = K_i(\widehat{\alpha} \rightarrow \widehat{\theta}_i) = K_i(\widehat{\alpha \rightarrow \theta_i})$ and $\widehat{K_i^{\alpha} \theta_i} = \widehat{K_i(\alpha \rightarrow \theta_i)}$.

For (\subseteq): Let $T \in K_i^{\hat{\alpha}} \widehat{\theta}_i$, then for all $S \sim_i^c T$, $S \in \widehat{\alpha \rightarrow \theta_i}$. Therefore $T \in \widehat{K_i(\alpha \rightarrow \theta_i)}$ and so $T \in \widehat{K_i^{\alpha} \theta_i}$.

For (\supseteq): Let $T \in \widehat{K_i^{\alpha} \theta_i}$, this means that $K_i(\alpha \rightarrow \theta_i) \in T$. Thus for all $S \sim_i^c T$, $\alpha \rightarrow \theta_i \in S$. Therefore $T \in K_i^{\hat{\alpha}} \widehat{\theta}_i$. Using this, it is easy to see that $\widehat{\alpha} \cap \bigcap_{i \in G} K_i^{\hat{\alpha}} \widehat{\theta}_i = \widehat{\alpha \wedge \bigwedge_{i \in G} K_i^{\alpha} \theta_i}$. We then obtain that $T \in \llbracket \psi \rrbracket_{\widehat{\alpha \wedge \bigwedge_{i \in G} K_i^{\alpha} \theta_i}}$. Since $\psi \prec_2 \langle G \rangle \psi$,

by IH we have that $T \in \overline{\langle \alpha \wedge \bigwedge_{i \in G} K_i^\alpha \theta_i \rangle \psi}$. Thus $\langle \alpha \wedge \bigwedge_{i \in G} K_i^\alpha \theta_i \rangle \psi \in T$. By ([!][G]-elim) we have $\langle \alpha \rangle \langle G \rangle \psi \in T$.

(\Leftarrow) Suppose $T \in \overline{\langle \alpha \rangle \langle G \rangle \psi}$, i.e., $\langle \alpha \rangle \langle G \rangle \psi \in T$. Since T is a maximal P -witnessed theory, there is $\{p_i : i \in G\} \subseteq P$ such that $\langle \alpha \rangle \langle \bigwedge_{i \in G} K_i p_i \rangle \psi \in T$. By (10.) in Lemma A.2.5, we know that $\langle \bigwedge_{i \in G} K_i p_i \rangle \psi \prec_2 \langle G \rangle \psi$. Thus, by IH on $\langle \bigwedge_{i \in G} K_i p_i \rangle \psi$, we obtain that $T \in \llbracket \langle \bigwedge_{i \in G} K_i p_i \rangle \psi \rrbracket_{\hat{\alpha}}$. This means, by Definition 4.5.2, that $T \in \llbracket \psi \rrbracket_{\llbracket \bigwedge_{i \in G} K_i p_i \rrbracket_{\hat{\alpha}}}$. By IH on $\bigwedge_{i \in G} K_i p_i$, we obtain that $T \in \llbracket \psi \rrbracket_{\overline{\langle \alpha \rangle \bigwedge_{i \in G} K_i p_i}}$. By (3.) in Proposition 4.3.14 and the reduction axioms (R_{K_i}) and (R_p), it is easy to see that the formula $\langle \alpha \rangle \bigwedge_{i \in G} K_i p_i \leftrightarrow \alpha \wedge \bigwedge_{i \in G} K_i^\alpha p_i$ is derivable in **GALM**. Therefore,

$$\llbracket \psi \rrbracket_{\overline{\langle \alpha \rangle \bigwedge_{i \in G} K_i p_i}} = \llbracket \psi \rrbracket_{\overline{\alpha \wedge \bigwedge_{i \in G} K_i^\alpha p_i}} = \llbracket \psi \rrbracket_{\hat{\alpha} \cap \bigcap_{i \in G} K_i^{\hat{\alpha}} \hat{p}_i}.$$

Thus $T \in \llbracket \psi \rrbracket_{\hat{\alpha} \cap \bigcap_{i \in G} K_i^{\hat{\alpha}} \hat{p}_i}$. Since $B_i := \hat{p}_i \in \mathcal{A}^c$ for every $i \in G$, we obtain that $T \in \llbracket \langle G \rangle \psi \rrbracket_{\hat{\alpha}}$. \square

4.6.20. COROLLARY. *The canonical pre-model \mathcal{M}^c is standard and hence a pseudo-model.*

Proof:

$$\mathcal{A}^c = \{\hat{\theta} : \theta \in \mathcal{L}_{-\diamond}^P\} = \{\widehat{\langle \top \rangle \theta} : \theta \in \mathcal{L}_{-\diamond}^P\} = \{\llbracket \theta \rrbracket_{\hat{\top}} : \theta \in \mathcal{L}_{-\diamond}^P\} = \{\llbracket \theta \rrbracket_{W^c} : \theta \in \mathcal{L}_{-\diamond}^P\}. \quad \square$$

4.6.21. LEMMA. *For every $\varphi \in \mathcal{L}_G^P$, if φ is consistent then $\{0, \blacklozenge \varphi\}$ is an initial P_φ -theory.*

Proof:

Let $\varphi \in \mathcal{L}_G^P$ s.t. $\varphi \not\vdash \perp$. By the *Equivalences with 0* in Table 4.1, we have $\vdash \perp^0 \leftrightarrow (p \wedge \neg p)^0 \leftrightarrow (p^0 \wedge \neg p^0) \leftrightarrow (p \wedge \neg p) \leftrightarrow \perp$. Therefore, $\vdash \psi \rightarrow \perp^0$ iff $\vdash \psi \rightarrow \perp$ for all $\psi \in \mathcal{L}_G^P$. Then, by (13.) in Proposition 4.3.14, we obtain $\vdash \varphi \rightarrow \perp$ iff $\vdash (0 \wedge \blacklozenge \varphi) \rightarrow \perp$. Since $\varphi \not\vdash \perp$, we have $0 \wedge \blacklozenge \varphi \not\vdash \perp$, i.e., $\{0, \blacklozenge \varphi\}$ is a P_φ -theory. By definition, it is an initial one. \square

4.6.22. COROLLARY. ***GALM** is complete with respect to standard pseudo-models.*

Proof:

Let φ be a consistent formula. By Lemma 4.6.21, $\{0, \blacklozenge \varphi\}$ is an initial P_φ -theory. By Extension and Lindenbaum Lemmas, respectively, we can extend P_φ to some $P \supseteq P_\varphi$ and extend $\{0, \blacklozenge \varphi\}$ to some maximal P -witnessed theory T_0 such that $(0 \wedge \blacklozenge \varphi) \in T_0$. So T_0 is initial and we can construct the canonical pseudo-model

\mathcal{M}^c for T_0 . Since $\blacklozenge\varphi \in T_0$ and T_0 is P -witnessed, there exists $p \in P$ such that $\langle p \rangle\varphi \in T_0$. By Truth Lemma (applied to $\alpha := p$), we get $T_0 \in \llbracket \varphi \rrbracket_{\hat{p}}$. Hence, φ is satisfied at T_0 in the set $\hat{p} \in \mathcal{A}^c$. \square

4.6.23. THEOREM. **APALM** is complete with respect to standard pseudo-models.

The completeness proof for **APALM** with respect to standard pseudo-models is obtained by following the same steps in the completeness proof of **GALM** without the parts required for the operator $\langle G \rangle$. This involves, for example, defining the witnessed theories only with respect to \blacklozenge and modifying the auxiliary lemmas accordingly. The reader can also see (Baltag et al., 2018b) for all the details in the completeness proof of **APALM**.

4.6.24. COROLLARY (COMPLETENESS ON a -MODELS). **GALM** and **APALM** are complete with respect to a -models.

Proof:

GALM completeness follows immediately from Corollaries 4.6.22 and 4.5.14. **APALM** completeness follows from Theorem 4.6.23 and Corollary 4.5.14. \square

4.7 Conclusions and Future Work

The work presented in this chapter solves the open question of finding a strong variant of APAL and GAL that is recursively axiomatizable. Our system APALM is inspired by our analysis of Kuijer’s counterexample (Kuijer, 2015), which leads us to add to APAL a “memory” of the initial situation. We then used similar methods to obtain a recursive axiomatization for the memory-enhanced variant GALM of GAL.

The soundness and completeness proofs crucially rely on a Subset Space-like semantics and on the equivalence between the effort modality and the arbitrary announcement modality (and on the equivalence between their $[G]$ counterparts), thus revealing the strong link between these two formalisms.

We just want to note again that the aforementioned issue persists in many of APAL and GAL variants with infinitary axiomatizations. As far as we know, there is no complete axiomatization for Coalition Announcement Logic (CAL): a coalition logic introduced by Pauly (2002) in the style of DEL where the actions that agents can perform are restricted to public announcements. In fact, the same affair renders to any logic that contains coalition announcement operators (Ågotnes and Van Ditmarsch, 2008; Van Ditmarsch, 2012; Ågotnes and van Ditmarsch, 2014; Ågotnes et al., 2016). The open question of finding a complete axiomatization for a strong version of CAL was resolved recently in (Galimullin

and Alechina, 2018). The authors introduce a combination of an extension of GAL and CAL and present a sound and complete infinitary axiomatization with two infinitary rules that resemble the infinitary rule in GAL. We believe that a memory-enhanced variant of the logic in (Galimullin and Alechina, 2018) is recursively axiomatizable. We leave these questions for future work.

We have a further comment on the connection with the yesterday operator. The limited form of memory provided by APALM is in fact enough to simulate the yesterday operator $Y\varphi$ on any given model, by using context-dependent formulas. For instance, the dialogue in Cheryl’s birthday puzzle (Albert: “I don’t know when Cheryl’s birthday is, but I know that Bernard doesn’t know it either”; Bernard: “At first I didn’t know when Cheryl’s birthday is, but I know now”; Albert: “Now I also know”), can be simulated by the following sequence of announcements¹⁷: first, the formula $0 \wedge \neg K_a c \wedge K_a \neg K_b c$ is announced (where 0 marks the fact that this is the first announcement), then $(\neg K_b c)^0 \wedge K_b c$ is announced, and finally $K_a c$ is announced.

For another example: if instead we change the story so that the third announcement (by Albert) is “I knew you knew it (just before you said so)”, then the last step of this alternative scenario corresponds to announcing the formula $([0 \wedge \neg K_a c \wedge K_a \neg K_b c] K_a K_b c)^0$ (saying that, just after the first announcement but before the second, Albert knew that Bernard knew the birthday). This shows how the logic can simulate the use of *any* (iterated) Y ’s in concrete examples, although at the cost of repeating the relevant part of history inside the announcement in order to mark the exact time when the announced formula was meant to be true.

A more systematic treatment of the yesterday operator on (a version of) our announcement models and its connection to arbitrary and group announcements are topics for future research. Yet another line of further work concerns other meta-logical properties, such as decidability and complexity, of APALM and GALM.

¹⁷Here, we use the abbreviation $K_a c = \bigvee \{K_a(d \wedge m) : d \in D, m \in M\}$, where D is the set of possible days and M is the set of possible months, to denote the fact that Albert knows Cheryl’s birthday and, similarly, use $K_b c$ for Bernard.

Part II

Finite Identification with positive
data (pfi) and
with complete data (cfi)

Chapter 5

Structural differences between pfi & cfi

5.1 Introduction

The groundbreaking work of Gold (1967) started a new era for developing mathematical and computational frameworks for studying the formal process of *learning*. The model in (Gold, 1967), *identification in the limit*, has been studied for learning recursive functions, recursively enumerable languages, and recursive languages with *positive data* and with *complete data*. The learning task consists of identifying a language amidst a family of languages on the basis of an infinite stream of inputs concerning the language. The stream consists of either positive information: an enumeration of all members of the language, or complete information labelling all sentences as belonging to the language or not.

The learning function will output infinitely many conjectures, and for a successful learning function these are required to stabilize into one permanent right one. In Gold's model, a huge difference in power between learning with positive data and with complete data is exposed. With positive data a family of languages containing all finite languages and at least one infinite one is not learnable. With complete data the learning task becomes almost trivial (for a general overview and further developments see Zeugmann and Lange (1995)).

Inspired by Gold's model and results, Angluin (1980) work focuses on indexed families of recursive languages, i.e., families of decidable languages with a uniform decision procedure for membership (for a summary of Angluin's results, see Angluin and Smith, 1983). Such families naturally occur as the sets of languages generated by many types of grammars. In particular, Angluin (1980) gave a characterization of cases in which Gold's learning task can be executed. Her work shows that many non-trivial families of recursive languages can be learned by means of positive data only. A few years later, Mukouchi (1992) and simultaneously Lange and Zeugmann (1992), introduced the framework of *finite identification* in the Angluin style for both positive and complete data. The learning task is as in Gold's model with the difference that the learning function can only output

a single conjecture. Mukouchi (1992) presents an Angluin style characterization theorem for positive and complete finite identification. As expected, finite identification with complete data is more powerful than with positive data only. However, the distinction is much less huge than in Gold's framework. The work in (Mukouchi, 1992) did not draw much attention until recently, when Gierasimeczuk and de Jongh (2013) further developed the theory of finite identification.

The difference between finite identification with positive and with complete data, if not as huge as in the limit case (for a detailed overview see, Zeugmann and Lange (1995)), is as we will show in this chapter, considerable not only in power but also in character.

In this chapter, we focus on a more fine-grained theoretical analysis of the distinction between finite identification with positive data and with complete data in the Angluin style. Our goal is to formally study the issue of the difference in learning power stemming from, on the one hand, positive and, on the other hand, positive and negative data. Here, we will focus only on the structural properties a family needs to have in order to be *pfi* or *cfi*. In the following chapter (Chapter 6), we will address this question with a different perspective, namely with respect to the computational features of a family that allow it to be *pfi* or *cfi*.

We start our analysis with finite identification of finite families. Here, the distinction between positive and complete data comes out very clearly: the difference is exactly described by the fact that with positive data families can only be identified if they are anti-chains with respect to the subset relation \subset .

Then, we investigate whether any finitely identifiable family is contained in a maximal finitely identifiable one. Maximal learnable families are of special interest because any learner which can learn a maximal learnable family can also learn any of its subfamilies. Moreover, it will turn out that we obtain more insight in the class of all learnable families if we know more about the class of the maximal ones. First, we address this in the setting of positive data. Simple examples of positively identifiable families are often maximal, like the set of all sets of exactly n elements for a fixed natural number n . If we widen the question to the existence of a *non-effectively finitely identifiable* maximal extensions of positively identifiable families, we get a positive answer for families containing only finite languages. We point out obstacles to generalising this result to arbitrary families containing also infinite languages, this wider question remains an open problem.

We then come to study the complete data setting. Surprisingly, we provide a strong negative result concerning maximal learnable families for effective or non-effective finite identification with complete data: any finitely identifiable family can be extended to a larger one which is also finitely identifiable, *ergo* maximal identifiable families do not exist in the case of complete data. The positive answer for positively identifiable families with finite languages brings about a natural follow-up question: how many maximal extensions does a positively identifiable family have? Addressing this question is not a trivial matter, therefore we conduct a case-by-case analysis. For instance, we prove that any family containing only

pairs of natural numbers has either only finitely many or uncountably many maximal non-efficient pfi extensions. We formulate this as a conjecture: any family of finite languages can only have finitely many maximal non-efficient pfi extensions or uncountably many. We are able to extend our result concerning families of pairs to families containing only n -tuples of natural numbers for a fixed n . We call these families equinumerous families. Moreover, we succeed in proving that this is also the case for families which languages have only two or three elements (i.e., families of pairs and triples). This is a first step in an attempt to prove our conjecture for families of finite languages with bounded cardinality. The general conjecture about families with finite sets of unbounded cardinality remains out of reach for the time being.

In all the sections that follow, we will refer as pfi and cfi to *finitely identifiable from positive data* and *finitely identifiable from complete data* respectively.

Outline

This chapter is structured as follows. In Section 5.2, we briefly discuss the relevance of negative data (and complete data) in empirical studies for learning, and the lack of a theoretical analysis for it in the literature. In Section 5.3 we study finite identification of finite families. Then, we investigate whether any finitely identifiable family is contained in a maximal finitely identifiable one in Section 5.4. We address this question first in the setting of positive data. Then in Section 5.4.1, we address the question in the setting of complete data. In Section 5.5, we study how many maximal extensions a positively identifiable family has. In Subsection 5.5.1 we address general conditions for a family to have uncountably many maximal nepfi extensions. In Subsections 5.5.3 and 5.5.4 we address the question for equinumerous families, first for subfamilies of only pairs and then for subfamilies with only n -tuples. In Subsection 5.5.5 we address a more complex case, for families containing pairs and triples. In Section 5.6 we conclude with final remarks and we give some directions for future research.

This Chapter is based on (de Jongh and Vargas-Sandoval, 2019).

5.2 The ignored value of negative data

The motivation for studying learning in this formal way is no longer predominantly first language learning by children as it was for Gold. His motivation for concentrating on positive data was because of indications that children do not use negative data when they learn their native language (Gold, 1967), but this is no longer believed in general. A large amount of theoretical and experimental work in computational linguistics (see e.g. Mitkov, 2005) has been conducted to analyze and test the intuition that there is a powerful contribution of “negative”

data for improving and speeding up children language acquisition (see e.g. [Saxton et al., 2005](#); [Hiller and Fernández, 2016](#)).

Formal learning theory goes beyond its linguistic purpose with more recent work that merges with Dynamic Epistemic Logic. Since the pioneering topological approach to Formal Learning Theory (FLT) initiated by [Kelly \(1996\)](#) (also studied and developed in [\(Baltag et al., 2015\)](#)), the general agenda of bridging DEL and FLT was deeply developed in [\(Gierasimczuk, 2010\)](#). Since then, the FLT-DEL merge gained some attention from the epistemic logicians. Substantial but scarce work combining these two fields of study, has been conducted in the last years for finite identification (learning with certainty) (see e.g., [Gierasimczuk \(2010\)](#)), and also for identifiability in the limit (learning in the limit) (for a framework using modal operators but not in the style of DEL, see e.g., [Kelly \(2014\)](#)).

Recall that in Chapter [3](#), inspired on identification in the limit (learning in the limit), we introduced two logics with subset space semantics to reason about inductive inference for two types of learners ([Baltag et al., 2018a, 2020](#)). For finite identification, [Dégremont and Gierasimczuk \(2011\)](#) show that finite identification of languages (sets) can be modelled in dynamic epistemic logic via a suitable translation of finite identification's basic concepts (data stream, class of languages, and languages) into the semantics of dynamic epistemic logic and alternatively of epistemic temporal logic. The purpose for this translation is to comprehend more deeply the semantics of learning, as in formal learning theory, and to analyse its multiple dimensions, including ways of formalisation (for more results on this, see e.g. [Bolander and Gierasimczuk, 2015, 2017](#)). Moreover, regarding the step by step information changes of dynamic epistemic logic, the study of fastest learning with respect to finite identification in [\(Gierasimczuk and de Jongh, 2013\)](#) gives the indication that the effectiveness of the procedure of retrieving specific information, needs to be studied seriously.

All these work, has been done with positive data only, leaving a gap for the complete data case. Filling this gap is particularly interesting for finite identification, since we already know by Gold's results that learning in the limit with complete data is not interesting enough (the learning task becomes almost trivial). Therefore, a step towards filling this gap, is to have a detailed analysis of the differences between finite identification with positive data and with complete data.

5.3 Finite families of languages

This section is dedicated to finite families of languages. A pair of simple but striking results already provides a good insight in a feature underlying the difference between finite identification on positive and on complete data. The feature of a finite family that allows it to be *pfi* is purely a structural one, namely being an anti-chain.

5.3.1. THEOREM. *A finite family of languages \mathcal{S} is pfi iff no language $S \in \mathcal{S}$ is a proper subset of another $S' \in \mathcal{S}$, i.e., \mathcal{S} is an anti-chain.*

Proof:

For (\Rightarrow): follows straightforwardly by contraposition from Corollary 2.4.10. For (\Leftarrow): take any language S_i in \mathcal{S} . Since $S_i \not\subseteq S_j$ for any $j \neq i$, choose $n_{ij} = \mu n \{n \in S_i - S_j\}$ where μ denotes the standard recursive *minimum function* in recursion theory.¹ and let $D_i = \{n_{ij} : j \neq i\}$. Let us verify that D_i is a DFTT for S_i : clearly it is finite because the family is finite, so $\{n_{ij} : j \neq i\}$ is finite and $D_i \subseteq S_i$. By construction, if $D_i \subseteq S_k \in \mathcal{S}$ then $i = k$. Since our construction involves a finite number of steps, it is effective and thus \mathcal{S} is pfi. \square

In contrast, we obtain the following result for complete data.

5.3.2. THEOREM. *Any finite collection of languages $\mathcal{S} = \{S_1, \dots, S_n\}$ is cfi.*

Proof:

Let \mathcal{S} be any finite family of languages and let S_i any language in \mathcal{S} . Take any $j \neq i$, then $S_i - S_j \neq \emptyset$ or $S_j - S_i \neq \emptyset$. If $S_i - S_j \neq \emptyset$, take the smallest $n_{ij} \in S_i - S_j$ to be in D_i . If $S_j - S_i \neq \emptyset$, take the smallest $m_{ij} \in S_j - S_i$ to be in \overline{D}_i . Repeat this for all $j \leq n$. The pair of sets obtained in that manner is consistent with S_i by construction, in fact they form a tell-tale pair for S_i . Note that this pair cannot be consistent with any other language $S_k \in \mathcal{S}$ such that $S_k \neq S_i$ simply by construction. Since i was arbitrary, by Mukouchi's characterization theorem for complete data we have that \mathcal{S} is cfi. \square

5.4 Looking for maximal learnable families

In this section we start by noting that being an anti-chain of finite languages is equivalent to being a nepfi family of finite languages. Then, we investigate whether any finitely identifiable family is contained in a finitely identifiable family which is maximal with respect to inclusion (\subseteq). We first address the question for nepfi families (recall that nepfi families are not always indexed), later on we will address it for pfi. We provide a positive result for maximal nepfi extensions of families with finite languages. For families containing infinite languages this question remains open. The case of necfi (and of cfi) is rather different, as we will show in Section 5.4.1 that maximal necfi extensions for cfi families never exist. First consider the following proposition.

¹ $\mu n(\dots n \dots)$ is the least integer n such that the expression $\dots n \dots$ is true (if this integer exists). For more details of the minimum function (or, *minimization operator*), the reader can look at any book of Recursion Theory, see e.g., (Rogers, 1967, p. xviii).

Not all anti-chains are nepfi, see the family in the proposition that follows just now. When all languages in the family are finite, being an anti-chain is equivalent to being a nepfi family, as we will see in Lemma [5.4.3](#).

5.4.1. PROPOSITION. *The family of all co-singletons, $\{\mathbb{N} - \{i\} : i \in \mathbb{N}\}$, is an anti-chain which is cfi but not nepfi.*

Proof:

Let $\mathcal{S} := \{\mathbb{N} - \{i\} : i \in \mathbb{N}\}$ and let $S_i = \mathbb{N} - \{i\}$ denote the language with index i . It is clearly cfi since $(\emptyset, \{i\})$ is a tell-tale pair for every $S_i \in \mathcal{S}$. It is not nepfi since there are no DFTTs for any $S_i \in \mathcal{S}$. To see this, towards contradiction suppose $S_i \in \mathcal{S}$ has a DFTT. Let D_i be a DFTT for $S_i \in \mathcal{S}$. Since D_i is finite, there is a $j \neq i$ such that $D_i \subseteq \mathbb{N} - \{j\}$ and $j \notin D_i$. Therefore, $D_i \subseteq S_j$ with $j \notin D_i$ and $j \neq i$, thus $S_j \neq S_i$. Thus, D_i is not a DFTT for S_i . Therefore, \mathcal{S} is not nepfi. \square

The following lemmas will be useful for the proof of Theorem [5.4.4](#).

5.4.2. LEMMA. *If \mathcal{C} is a chain with respect to (\subseteq) of anti-chains of languages \mathcal{S} then $\bigcup \mathcal{C}$ is an anti-chain.*

Proof:

Towards contradiction, suppose $\bigcup \mathcal{C}$ is not an anti-chain but every $\mathcal{S} \in \mathcal{C}$ is an anti-chain. Thus, there is $X, Y \in \bigcup \mathcal{C}$ such that $X \subseteq Y$ or $Y \subseteq X$. W.l.o.g. suppose $X \subseteq Y$. Since \mathcal{C} is a chain, we have that $X, Y \in \mathcal{S}$ for some $\mathcal{S} \in \mathcal{C}$. It follows that \mathcal{S} is not an anti-chain, contradicting that it is. \square

5.4.3. LEMMA. *A family of finite languages is nepfi if and only if is an anti-chain.*

Proof:

For (\Rightarrow) : follows directly by the contrapositive of Corollary [2.4.10](#). For (\Leftarrow) : take $D_i := S_i$ for every $S_i \in \mathcal{S}$. Clearly $D_i \subseteq S_i$ and it is finite. Also the following holds, if $D_i \subseteq S_j$ for some $S_j \in \mathcal{S}$, $S_i = S_j$. Thus D_i is a DFTT of $S_i \in \mathcal{S}$. \square

A fortiori this lemma implies that a cfi anti-chain of finite languages is nepfi. Nepfi cannot improve on nepfi regarding anti-chains of finite languages. Later we will see that this is different for cfi and pfi (see Proposition [6.4.7](#)).

In what follows we will not distinguish anti-chains of finite languages and nepfi families of finite languages.

We now proceed to the main result in this section.

5.4.4. THEOREM (EXISTENCE OF MAXIMAL NEPFI FAMILIES). *Every indexed family of finite languages which is pfi is contained in a maximal family of finite languages which is nepfi.*

Proof:

Let \mathcal{S} be an indexed pfi family of finite languages. If \mathcal{S} is not maximal, we can choose

$$S' \in \mathcal{P}(\mathbb{N})^{<\omega} \text{ such that } S_i \not\subseteq S' \wedge S' \not\subseteq S_i$$

for all $S_i \in \mathcal{S}$. Let $\mathcal{S}_1 = \mathcal{S} \cup \{S'\}$ be the obvious extension. We continue with the same procedure used for \mathcal{S} to extend \mathcal{S}_1 and so on so forth. We will have a chain of families of languages of the form,

$$\mathcal{S} \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \dots$$

By Lemmas 5.4.2 and 5.4.3 it follows that $\mathcal{S}_{max} := \bigcup_{n \in \mathbb{N}} \mathcal{S}_n$ is a nepfi family of finite languages and it is clearly a maximal nepfi extension of \mathcal{S} . \square

We want to remark a couple of things from the proof of Theorem 5.4.4. Recall from Chapter 2, an identification is *non-effective* if at least one of the two following situations holds: 1) the learning function is not recursive, and 2) the family is not effectively indexed. Nothing in the proof guarantees that the resulting maximal nepfi extension \mathcal{S}_{max} of a family \mathcal{S} is effectively indexable (even though at each step the family \mathcal{S}_n was effectively indexable). Another relevant observation is that if infinite languages are present in the family, such an argument cannot be applied since not every family which is an anti-chain is nepfi (see Proposition 5.4.1). Even if the starting family is a nepfi family, the resulting maximal extension, using Lemma 5.4.2, might not be nepfi. See the following example:

5.4.5. EXAMPLE. Consider the finite family of co-singletons $\mathcal{S} := \{S_0, S_1\}$ such that $S_0 := \mathbb{N} - \{0\}$ and $S_1 := \mathbb{N} - \{1\}$. This family is clearly nepfi, the DFTT for S_0 is $\{1\}$ and the DFTT for S_1 is $\{0\}$. Let $S_i := \mathbb{N} - \{i\}$. Now consider the following chain of anti-chains,

$$\mathcal{C} := \mathcal{S} \subseteq \mathcal{S} \cup \{S_2\} \subseteq (\mathcal{S} \cup \{S_2\}) \cup \{S_3\} \subseteq \dots$$

Clearly each family of the chain is an indexed nepfi family. Note that the family $\bigcup \mathcal{C}$ is precisely the family of all co-singletons, i.e., $\bigcup \mathcal{C} := \{\mathbb{N} - \{i\} : i \in \mathbb{N}\}$. By Proposition 5.4.1, we know that this family is not nepfi.

This example leads to the following.

1. OPEN QUESTION. *Do maximal nepfi families always exist for arbitrary (ne)pfi families?*

This question can be formulated as a purely combinatorial mathematical question. Even though it is not clear what happens when a family has an infinite language, we conjecture that any pfi family has a maximal nepfi extension. An answer to this conjecture, involves a more complex and deep structural analysis which we leave for future work.

We did answer negatively the question restricted to maximal pfi families in Theorem [6.3.14](#) from Chapter [6](#).

5.4.1 Do maximal necfi/cfi families exist?

In this section we address the question whether every necfi family is contained in a maximal one. Surprisingly, we show that the answer is negative, even for cfi families. All the results in this section will be stated for necfi families, but the same results follow for cfi families. Before addressing the main results, we first focus on a couple of useful propositions and we provide some notation.

Let $\overline{\mathcal{S}}$ be the complement family of any necfi family \mathcal{S} , i.e., $\overline{\mathcal{S}} = \{\overline{S} : S \in \mathcal{S}\}$ where $\overline{S} = \mathbb{N} - S$. Note that for every sequence σ of complete data for a family $\overline{\mathcal{S}}$ there is a mirror image of σ , say sequence $\overline{\sigma}$ (presented in exactly the same order), for the necfi family \mathcal{S} with inverted values of 0's and 1's. So $(k, 1)_j$ is in σ iff $(k, 0)_j$ is in $\overline{\sigma}$, for any $j \in \mathbb{N}$.

5.4.6. PROPOSITION. *If a family \mathcal{S} is necfi then $\overline{\mathcal{S}}$ is necfi as well. Similarly, if a family \mathcal{S} is cfi then $\overline{\mathcal{S}}$ is cfi as well.*

Proof:

Let λ be a (ne)cfi learner for \mathcal{S} . We can define a (ne)cfi learner $\overline{\lambda}$ for $\overline{\mathcal{S}}$ as follows:

$$\overline{\lambda}(\sigma[n]) = \uparrow \text{ iff } \lambda(\overline{\sigma}[n]) = \uparrow \quad \text{and} \quad \overline{\lambda}(\sigma[n]) = \overline{S} \text{ iff } \lambda(\overline{\sigma}[n]) = S,$$

for every $n \in \mathbb{N}$. Clearly $\overline{\lambda}$ is a recursive learner for $\overline{\mathcal{S}}$ if λ is recursive. □

5.4.7. COROLLARY. *If either \mathcal{S} or $\overline{\mathcal{S}}$ is (ne)cfi then \mathcal{S} and $\overline{\mathcal{S}}$ are (ne)cfi.*

This is not the case for nepfi families, since for instance the family of all singletons \mathcal{S}^s is nepfi (and pfi) but its complement family, namely the family of all co-singletons, is clearly not nepfi (see Proposition [5.4.1](#)). This is because no finite subset of a co-singleton can determine which co-singleton it is.

Consider any language S , then a *direct successor* of S is a language $S \cup \{n\}$ such that $n \notin S$. For every non-cofinite language $S_i \subseteq \mathbb{N}$ let $Suc(S_i)$ be *the set of all direct successors of S_i* and let $Suc_{\mathcal{S}}(S_i) := Suc(S_i) \cap \mathcal{S}$, i.e., the set of all distinct direct successors of S_i that are also languages in the family \mathcal{S} .

Consider the following example.

5.4.8. EXAMPLE. Take the family

$$\mathcal{S} = \{\{0\}, \{0, 1\}, \{0, 1, 2\}, \dots, \{0, 1, 2, 3, \dots, n\}, \dots\}.$$

This family is necfi (it is also cfi). To illustrate, consider the language $\{0\}$. Clearly, the pair $(\{0\}, \{1\})$ is a definite tell-tale pair for $\{0\}$.

Note that for each language $S_i \in \mathcal{S}$, there are infinitely many sets in \mathcal{S} that contain S_i . Moreover, there are infinitely many languages that properly contain a direct successor of S_i . Still, there are only finitely many direct successors that are languages in \mathcal{S} , in fact there is only one. Now, consider the extension $\mathcal{S} \cup \{\{0, n\} : n \in \mathbb{N}, n > 1\}$. This extension is not necfi, since the language $\{0\}$ no longer has a definite tell-tale pair. This is because any possible pair $(\{0\}, \overline{D}_{\{0\}})$ with a finite $\overline{D}_{\{0\}}$ will be consistent with infinitely many languages in \mathcal{S} .

In the proposition that follows, we show that a necfi family can only have finitely many direct successors of any language in the family. The purpose of this proposition is simply to ensure there is always room for a new language in a necfi family.

5.4.9. PROPOSITION. *If \mathcal{S} is necfi then $Suc_{\mathcal{S}}(S_i)$ is finite for every S_i in \mathcal{S} .*

Proof:

Let \mathcal{S} be a necfi family and $S_i \in \mathcal{S}$. Towards a contradiction, suppose there are infinitely many languages which contain a direct successor of S_i in \mathcal{S} . Thus, we assume that $Suc_{\mathcal{S}}(S_i)$ has infinitely many elements.

Since \mathcal{S} is necfi we have a definite tell-tale pair for S_i , namely (D_i, \overline{D}_i) . First note that $D_i \subseteq S_k$ for any $S_k \in Suc_{\mathcal{S}}(S_i)$. Since \overline{D}_i is finite, the contradiction will follow by showing that \overline{D}_i only serves to disambiguate between a finite number of languages in $Suc_{\mathcal{S}}(S_i) \subseteq \mathcal{S}$. We prove the following: $\overline{D}_i \cap S_k \neq \emptyset$ only for finitely many $S_k \in Suc_{\mathcal{S}}(S_i)$.

First note that \overline{D}_i is finite and for all distinct direct successors $S_l, S_k \in Suc_{\mathcal{S}}(S_i)$, $S_l = S_i \cup \{n_l\} \neq S_i \cup \{n_k\} = S_k$ for some $n_l, n_k \in \mathbb{N}$. Since \overline{D}_i is the negative member of the definite tell-tale pair for S_i , $\overline{D}_i \cap S_i = \emptyset$. Thus, if $S_k \in Suc_{\mathcal{S}}(S_i)$ and $\overline{D}_i \cap S_k \neq \emptyset$ then $n_k \in \overline{D}_i \cap S_k$. Since for each $S_k, S_l \in Suc_{\mathcal{S}}(S_i)$ we have some n_l, n_k such that $n_l \neq n_k$ and \overline{D}_i is finite, \overline{D}_i can only intersect finitely many languages in $S_k \in Suc_{\mathcal{S}}(S_i)$.

Continuing with the main proof, take $S_k \in Suc_{\mathcal{S}}(S_i)$ such that $S_k \cap \overline{D}_i = \emptyset$. We have such a language $S_k \in Suc_{\mathcal{S}}(S_i)$ by the previous claim and our initial assumption, that the set $Suc_{\mathcal{S}}(S_i)$ is infinite. Since also $S_k \supseteq S_i \supseteq D_i$, the existence of such S_k implies that (D_i, \overline{D}_i) is not a definite tell-tale pair for S_i . This contradicts our initial assumption on (D_i, \overline{D}_i) .

Since the choice of the definite tell-tale pair (D_i, \overline{D}_i) was arbitrary it follows that $Suc_{\mathcal{S}}(S_i)$ must be finite. \square

Next comes the main result of this section.

5.4.10. THEOREM (NON-EXISTENCE OF MAXIMAL NECFI/CFI FAMILIES). *Maximal necfi extensions do not exist for any necfi family \mathcal{S} .*

Proof:

Let \mathcal{S} be an arbitrary necfi family. It is sufficient to prove that a proper extension of \mathcal{S} exists which is necfi. We can assume that \mathcal{S} has a non-cofinite language, and such language we fix as S_i . This is simply because of the following: if all languages in \mathcal{S} are cofinite, then all languages in the complement family $\overline{\mathcal{S}}$ are finite. Thus we can find a necfi extension \mathcal{S}' for $\overline{\mathcal{S}}$ by proving the theorem for $\overline{\mathcal{S}}$. By proposition 5.4.6 we know that $\overline{\mathcal{S}'}$ is necfi. Note that $\overline{\mathcal{S}'}$ is an extension of our original family \mathcal{S} . Indeed, we can assume that $S_i \in \mathcal{S}$ is not cofinite. We proceed with the main proof. We start by proving the following claim.

Claim: For any definite tell-tale pair (D_i, \overline{D}_i) of S_i , there exists $n \in \mathbb{N}$ such that $n \notin \overline{D}_i \cup S_i$.

Let (D_i, \overline{D}_i) be any definite tell-tale pair of S_i and consider the negative member in the pair, \overline{D}_i . Since S_i is not cofinite we know that S_i has infinitely many direct successors, i.e., the set $Succ(S_i)$ is infinite. By Proposition 5.4.9 we have that \mathcal{S} contains only finitely many languages in $Succ(S_i)$, i.e., $Succ_{\mathcal{S}}(S_i)$ is finite. Since \overline{D}_i is also finite, there are infinitely many $m \in \mathbb{N}$ such that $m \notin \overline{D}_i$ and $m \notin S'$ for any $S' \in Succ_{\mathcal{S}}(S_i)$. Take an $n \in \mathbb{N}$ satisfying these characteristics and let $D_i \cup \{n\}$ the respective direct successor for D_i . The set $D_i \cup \{n\}$ is precisely the candidate for extending \mathcal{S} .

Now, if n satisfies the claim above, we show that the family $\mathcal{S}' := \mathcal{S} \cup \{D_i \cup \{n\}\}$ is necfi. To show that \mathcal{S}' is a proper extension of \mathcal{S} , towards contradiction suppose $D_i \cup \{n\} = S_k$ for some $S_k \in \mathcal{S}$. Since $D_i \subseteq S_k$ and \mathcal{S} is necfi, the pair (D_i, \overline{D}_i) cannot be consistent with S_k . Thus, there is $x \in S_k$ such that $x \in \overline{D}_i$. By our assumption, $x = n$ which contradicts our choice for n , namely that $n \notin \overline{D}_i$. Thus \mathcal{S}' is a proper extension of \mathcal{S} .

To show that \mathcal{S}' is necfi, we show that each $S_k \in \mathcal{S}'$ has a definite tell-tale pair (D_k, \overline{D}_k) . We first claim that the pairs

$$(D_i \cup \{n\}, \overline{D}_i) \text{ and } (D_i, \overline{D}_i \cup \{n\})$$

are the definite tell-tale pairs for the languages $D_i \cup \{n\}$ and S_i respectively in the extension \mathcal{S}' . Clearly these pairs allow disambiguation between S_i and $D_i \cup \{n\}$. Note that $(D_i, \overline{D}_i \cup \{n\})$ (for S_i) is not consistent with any other language $S_k \in \mathcal{S}$ such that $S_i \neq S_k$ since that would contradict that (D_i, \overline{D}_i) is a definite tell-tale pair for S_i in \mathcal{S} . Now we will check that the definite tell-tale pair for $D_i \cup \{n\}$, namely $(D_i \cup \{n\}, \overline{D}_i)$, is not consistent with any other language in \mathcal{S} . If $D_i \cup \{n\} \subseteq S_k$ for some S_k then $D_i \subseteq S_k$. Since \mathcal{S} is necfi and (D_i, \overline{D}_i) a definite tell-tale pair for S_i , there is $x \in \overline{D}_i$ such that $x \in S_k$. Analogously, if $\overline{D}_i \subseteq \mathbb{N} - S_k$, there is $x \in D_i$ such that $x \notin S_k$. It follows that $(D_i \cup \{n\}, \overline{D}_i)$ is a definite tell-tale pair for the new language $D_i \cup \{n\}$ in the extension \mathcal{S}' .

We now define the definite tell-tale pairs for the rest of the languages in \mathcal{S}' . Let $S_k \in \mathcal{S}' - (\{S_i\} \cup \{D_i \cup \{n\}\})$. If $D_k \not\subseteq D_i \cup \{n\}$ with (D_k, \overline{D}_k) the definite tell-tale pair for S_k in \mathcal{S} , then the definite tell-tale pair with respect to \mathcal{S}' will be exactly (D_k, \overline{D}_k) . Otherwise (if $D_k \subseteq D_i \cup \{n\}$), we consider two cases:

1. $D_k \subseteq S_k \subseteq D_i \cup \{n\}$.

First note that $S_k = D_i \cup \{n\}$ is not possible since \mathcal{S}' is a proper extension of \mathcal{S} . So $S_k \subset D_i \cup \{n\}$. Moreover $S_k \subseteq D_i$ since otherwise $n \in S_k$ and $S_k \not\subseteq D_i$ and this implies that there is $x \in S_k - D_i$ such that $x \neq n$, contradicting that $S_k \subset D_i \cup \{n\}$. Thus $S_k \subseteq D_i$. Now, note that the pair $(D_k, \overline{D}_k \cup \{n\})$ is not consistent with the language $D_i \cup \{n\}$, and thus, allows disambiguation between S_k and $D_i \cup \{n\}$ (and any other language in \mathcal{S}). Thus $(D_k, \overline{D}_k \cup \{n\})$ is the definite tell-tale pair for S_k in \mathcal{S}' .

2. $D_k \subseteq D_i \cup \{n\}$ with $n \in D_k$ and $D_k \not\subseteq D_i$.

Since $S_k = D_i \cup \{n\}$ is not possible, there is $x_k \in S_k - D_i$ such that $x_k \neq n$. Note that the pair $(D_k \cup \{x_k\}, \overline{D}_k)$ is not consistent with $D_i \cup \{n\}$ (and with any other language in \mathcal{S}'). Therefore, $(D_k \cup \{x_k\}, \overline{D}_k)$ is a definite tell-tale pair for S_k in \mathcal{S}' .

Note that there are only finitely many subsets of $D_i \cup \{n\}$, so we only need to take care of finitely many $S_k \in \mathcal{S}$ for which $D_k \subseteq D_i \cup \{n\}$ is the case.

Thus, there is a definite tell-tale pair for every language in \mathcal{S}' and, therefore, the family $\mathcal{S}' := \mathcal{S} \cup \{D_i \cup \{n\}\}$ is necfi. \square

There are other ways of extending a necfi family than the one described in Theorem 5.4.10 as in the case for the family in Example 5.4.8. The family $\mathcal{S} = \{\{0\}, \{0, 1\}, \{0, 1, 2\}, \dots, \{0, 1, 2, 3, \dots, n\}, \dots\}$ is necfi (it is also cfi). Note that for $S = \{0\}$ we can extend \mathcal{S} with $S \cup \{2\}$ and preserve necfi (also cfi) even though a tell-tale pair for S is $(\{0\}, \{1, 2\})$. Moreover we can extend it with $S \cup \{3\}$, $S \cup \{4\}$ and so on, and preserve necfi (and cfi).

The question concerning more possibilities (and the limitations) in which a cfi family can be extended, involves almost a purely combinatorial analysis. Even though we find this question intriguing, we think that it goes a bit further from the main purpose of this thesis, thus we leave this for future work.

5.5 Counting maximal extensions

Given that all anti-chains (nepfi families) of finite languages have a maximal nepfi extension (see Theorem 5.4.4), in this section we study their structure. Moreover, we investigate how many there are for nepfi families with finite languages. We conjecture the following.

1. CONJECTURE. *Every nepfi family of finite languages has only finitely many or uncountably many maximal nepfi extensions.*

By the above, this conjecture comes down to a purely combinatorial mathematical statement: each (\subseteq) anti-chain of finite sets has only finitely many or uncountably many extensions to a maximal such anti-chain. An answer to this conjecture can shed light to a more refined characterization for nepfi families (and pfi families), namely to distinguish the *finitely extendable families*, i.e., the ones that have only finitely many maximal nepfi extensions from the more sparse ones. In a way, maximal families give us the feeling of being “almost complete”, since we cannot add anything more to the family and preserve its structural properties. Thus, the finitely extendable families are the ones for which, in principle, we can keep track of its maximal extensions since there are only finitely many. We prove our conjecture for the special case of equinumerous families, families containing only n -tuples for a fixed $n \in \mathbb{N}$. We address this first for the simple cases of families of singletons and families of pairs. The analysis for those simple cases will be useful when we address the more general case. Then, we study the more complex case of families containing pairs and triples and prove Conjecture [1](#) for such families.

In this section our work is purely combinatorial. We are after structural properties only, so we ignore whether a family of languages is or can be represented as an indexed family. We also ignore whether the maximal extension is pfi or just nepfi. All the results here will be with respect to counting maximal nepfi extensions of a given family and their structure.

Consider the following example.

5.5.1. EXAMPLE. Let $\mathcal{S}^s = \{\{i\} : i \in \mathbb{N}\}$ be the family of all singletons. Clearly it is maximal with respect to nepfi. However, if we take out one of the singletons, say $\{0\}$, we obtain a nepfi subfamily \mathcal{S}_0^s which is no longer maximal and its only nepfi extension is \mathcal{S}^s . If we remove $\{1\}$ from \mathcal{S}_0^s , we can maximally extend this family in two different ways, either adding $\{0, 1\}$ or adding $\{0\}$ and $\{1\}$. Thus we have two independent maximal nepfi extensions for \mathcal{S}_1^s .

We can repeat the effective deletion-procedure described in Example [5.5.1](#) finitely many times and still obtain finitely many extensions. For regaining maximality, we are indeed “restricted” in the structural sense. The following lemma illustrates this.

5.5.2. LEMMA. *Let \mathcal{S} be a maximal nepfi family and let \mathcal{S}' be a maximal nepfi extension of $\mathcal{S} - \{\{x\}\}$ where $\{x\} \in \mathcal{S}$. Then for all $S \in \mathcal{S}'$ which are not in $\mathcal{S} - \{\{x\}\}$, S is of the form $\{x\} \cup A$, for some $A \subseteq S_i \in \mathcal{S} - \{\{x\}\}$.*

Proof:

Let \mathcal{S} and \mathcal{S}' as described in the lemma. Since \mathcal{S}' is an anti-chain that extends

$\mathcal{S} - \{\{x\}\}$ and \mathcal{S} is a maximal anti-chain, any new language $S \in \mathcal{S}'$ needs to have x as an element. Thus any $S \in \mathcal{S}' - (\mathcal{S} - \{\{x\}\})$ is such that $x \in S$. Let $A = S - \{\{x\}\}$. We will prove that $A \subset S_i$ for some $S_i \in \mathcal{S} - \{\{x\}\}$. By maximality of \mathcal{S} , A itself could not be added to \mathcal{S} and preserve nepfi. Thus, either $A \subset S_i$ or $S_i \subseteq A$ for some $S_i \in \mathcal{S} - \{\{x\}\}$. The latter cannot be since if $S_i \subseteq A$ then $S_i \subseteq A \cup \{x\}$ and \mathcal{S}' should be an anti-chain. Therefore $A \subset S_i$. \square

It is not always the case that we obtain only finitely many maximal extensions for a given nepfi/pfi family. In the following example we see that even when the languages are all finite, we may still obtain uncountably many maximal nepfi extensions.

5.5.3. EXAMPLE. Let $\mathcal{S} = \{\{0\} \cup \mathcal{S}'\}$ where $\mathcal{S}' = \{\{i, j, k\} : i, j, k \in \mathbb{N} - \{0\}\}$. Clearly \mathcal{S} is a maximal nepfi family. Consider \mathcal{S}' , by lemma [5.5.2](#) in order to regain maximality, the languages to add must be of the form $\{0\} \cup A$ for some $A \subseteq S_i$ and some $S_i \in \mathcal{S}'$. Therefore we have the following procedure for constructing uncountably many maximal nepfi extensions of \mathcal{S}' : For each $B \subseteq \mathbb{N} - \{0\}$ add the triples of the form $\{0, n, m\}$ with $n \neq m$ and $n, m \in B$ and all the pairs of the form $\{0, c\}$ with $c \notin B$. This construction is for all $B \subseteq \mathbb{N} - \{0\}$, thus \mathcal{S}' has uncountably many maximal nepfi extensions.

5.5.1 Uncountably many maximal nepfi extensions

We dedicate this subsection to study cases in which we can obtain uncountably many maximal nepfi extensions of a given family. We first address some cases of families with finite languages similar to example [5.5.3](#). After studying these cases, we exhibit some sufficient conditions for a family in order to have uncountably many maximal extensions.

In what follows, we denote as *pairs*, *triples* and *n-tuples* the unordered sets with 2, 3 and n elements respectively. Thus, an n -tuple (or, k -tuple) is a language with cardinality $n \in \mathbb{N}^+$ ($k \in \mathbb{N}^+$).²

Consider the following example that uses the fact (discussed in Section [5.5.3](#)) that the family of all pairs, \mathcal{S}^2 , is maximal nepfi.

5.5.4. EXAMPLE. Let \mathcal{S} be the family $\{\{0\}, \{1\}\} \cup \{\{i, n\} : i, n \in \mathbb{N} - \{0, 1\}\}$. This is clearly a nepfi family because every language is mutually incomparable with any other language in the family. Moreover it is maximal nepfi precisely because any other subset of \mathbb{N} is either a subset or a superset of $\{\{i, n\} : i, n \in \mathbb{N} - \{0, 1\}\}$, or a superset of $\{0\}, \{1\}$. Now consider the subfamily $\mathcal{S}' = \mathcal{S} - \{\{0\}, \{1\}\}$. Note that $\mathcal{S}' = \mathcal{S}^2 - \{\{0, a\}, \{1, b\} : a, b \in \mathbb{N}, a \neq 0, b \neq 1\}$. We will see that \mathcal{S}' has uncountably many maximal nepfi extensions. Clearly \mathcal{S} is one, and for every $B \subseteq \mathbb{N}$, the family $\mathcal{S}' \cup \{\{0, 1, b\} : b \in B\} \cup \{\{0, c\}, \{1, c\} : c \notin B\}$

²We use the standard notation for the set of all positive natural numbers \mathbb{N}^+ .

is a maximal nepfi extension of \mathcal{S}' . So we have uncountably many maximal nepfi extensions of \mathcal{S}' .

Consider now the similar maximal nepfi family $\{\{0\}\} \cup \{\{i, n\} : i, n \in \mathbb{N} - \{0\}\}$ and take the nepfi subfamily $\{\{i, n\} : i, n \in \mathbb{N} - \{0\}\}$. It turns out that $\{\{i, n\} : i, n \in \mathbb{N} - \{0\}\}$ has only two maximal nepfi extensions, namely \mathcal{S}^2 and $\{\{0\}\} \cup \{\{i, n\} : i, n \in \mathbb{N} - \{0\}\}$ itself. Although, note that $\{\{i, n\} : i, n \in \mathbb{N} - \{0\}\}$ has infinitely many non-maximal nepfi extensions. To see this, observe that we can add to $\{\{i, n\} : i, n \in \mathbb{N} - \{0\}\}$ one by one pairs of the form $\{0, n\}$ with $n \in \mathbb{N}$.

By a similar combinatorial argument as in Example 5.5.4, we straightforwardly obtain the following result.

5.5.5. PROPOSITION. *For every finite set $\{0, 1, \dots, m\}$ with $m > 0$, the subfamily $\mathcal{S}_{\mathbb{N}-\{0,1,\dots,m\}}^2 = \{\{i, n\} : n \in \mathbb{N}, i \in \{0, 1, \dots, m\}, n \neq i\}$, of the family of all pairs \mathcal{S}^2 , has uncountably many maximal nepfi extensions.*

Proof:

Simply because $\mathcal{S}_{\mathbb{N}-\{0,1,\dots,m\}}^2 \subset \mathcal{S}^2 - \{\{0, a\}, \{1, b\} : a, b \in \mathbb{N}, a \neq 0, b \neq 1\}$ and, by Example 5.5.4, the latter has uncountably many maximal nepfi extensions. \square

From Example 5.5.3 we know that the subfamily $\mathcal{S}^3 - \{\{0, a, b\} : a, b \in \mathbb{N}\}$ already has uncountably many maximal nepfi extensions. Therefore any subfamily $\mathcal{S}_{\mathbb{N}-\{0,1,\dots,m\}}^3$ obtained by removing all triples of the form $\{i, a, b\}$ with $a, b \in \mathbb{N}$, $i \in \{0, 1, \dots, m\}$ and $a, b \neq i$ has uncountably many maximal nepfi extensions for any $m \in \mathbb{N}$. Since a similar combinatorial argument works for any subfamily of quadruples, quintuples etc, we can generalize this result to all families of n -tuples, \mathcal{S}^n , with $n \in \mathbb{N}$ such that $n \geq 3$.

5.5.6. PROPOSITION. *Let $n \in \mathbb{N}$ such that $n \geq 3$ and \mathcal{S}^n be the class of all n -tuples. Any subfamily $\mathcal{S}_{\mathbb{N}-\{0,1,\dots,m\}}^n$ obtained by removing all n -tuples of the form $\{i, x_1, \dots, x_{n-1}\}$ with $x_j \in \mathbb{N}$, $i \in \{0, 1, \dots, m\}$ has uncountably many maximal nepfi extensions for any $m \in \mathbb{N}$.*

5.5.2 The class of families with only singletons

We dedicate this small section to study subfamilies of the family of all singletons, i.e., subfamilies $\mathcal{S} \subseteq \mathcal{S}^s$. We will present a simple argument that shows Conjecture 1 for such families. The structural simplicity of these families allows us to compute its maximal nepfi extensions in a simple generic manner. This will be clear in the proof of the following proposition.

5.5.7. PROPOSITION. *Let $\mathcal{S} \subseteq \mathcal{S}^s$ be a family of only singletons. The family \mathcal{S} has finitely many maximal nepfi extensions or uncountably many.*

Proof:

Let $\mathcal{S} \subseteq \mathcal{S}^s$ be a family of only singletons and let $X := \{n \in \mathbb{N} : \{n\} \in \mathcal{S}\}$. First note that, if $\mathcal{S} \cup \{Y\}$ is an anti-chain, $Y \subseteq \mathbb{N} - X$. Adding any new language Z to \mathcal{S} such that $Z \cap X \neq \emptyset$ and $Z \notin \mathcal{S}$, will impair the anti-chain condition, i.e., $\mathcal{S} \cup \{Z\}$ is not an anti-chain. To see this, consider $z \in Z \cap X$, then $\{z\} \in \mathcal{S}$ and so $\{z\} \subseteq Z$. We consider the following cases:

1. $\mathbb{N} - X$ is finite, (i.e., X is cofinite). Since $\mathbb{N} - X$ is finite, the number of anti-chains that we can construct with languages in $\mathcal{P}(\mathbb{N} - X)$ are finitely many. Moreover, there are only finitely many distinct maximal anti-chains \mathcal{C} with elements in $\mathcal{P}(\mathbb{N} - X)$. Such maximal anti-chains are precisely the ones that when added to \mathcal{S} we obtain a maximal nepfi family. It follows that \mathcal{S} has only finitely many maximal nepfi extensions.
2. $\mathbb{N} - X$ is countable, (i.e., X is not cofinite). Using a combinatorial argument, we will construct uncountably many maximal nepfi extensions of \mathcal{S} . For every $B \subseteq \mathbb{N} - X$, consider the family

$$\mathcal{S}' := \mathcal{S} \cup \{\{b\} : b \in B\} \cup \{\{c_1, c_2\} : c_1, c_2 \in (\mathbb{N} - X) - B\}.$$

Clearly \mathcal{S}' is an anti-chain of finite languages that extends \mathcal{S} i.e., it is a nepfi extension of \mathcal{S} . Moreover, it is easy to see that \mathcal{S}' is a maximal nepfi extension of \mathcal{S}' . Clearly, since adding any other language to \mathcal{S}' will impair the anti-chain condition.

□

5.5.3 The class of families with only pairs

In this subsection we study subfamilies of the family of all pairs, i.e., subfamilies $\mathcal{S} \subseteq \mathcal{S}^2$. We will prove our Conjecture [1](#) for such families. This will also bring some general insights into equinumerous families of languages with more than two elements. First we need some definitions.

5.5.8. DEFINITION. [Sets $NUM(\mathcal{Y})$, $PAIRS(\mathcal{Y})$, $nTUP(\mathcal{Y})$ and the family $\mathcal{S}^{\mathcal{Y}}$]

- Let $\mathcal{Y} = \{Y_1, \dots, Y_n\}$ be any set of pairs in \mathcal{S}^2 , let $NUM(\mathcal{Y})$ be the set of all numbers which appear in the pairs Y_1, \dots, Y_n , and let $PAIRS(\mathcal{Y})$ be the set of all pairs formed by elements in $NUM(\mathcal{Y})$. Finally, let $\mathcal{S}^{\mathcal{Y}}$ be the subfamily of all pairs which are not in $PAIRS(\mathcal{Y})$, i.e., $\mathcal{S}^{\mathcal{Y}} = \mathcal{S}^2 - PAIRS(\mathcal{Y})$.
- We can easily generalize the definition above to the family \mathcal{S}^n of all n -tuples for $n \geq 3$. We denote as $nTUP(\mathcal{Y})$ the set of all n -tuples formed by elements in $NUM(\mathcal{Y})$ and $\mathcal{S}^{\mathcal{Y}} = \mathcal{S}^n - nTUP(\mathcal{Y})$.

The combinatorial notion of Sperner family explains why, and in what way, for every finite set of pairs $\mathcal{Y} = \{Y_1, \dots, Y_n\}$, the subfamily $\mathcal{S}^{\mathcal{Y}}$ has finitely many maximal nepfi extensions.

5.5.9. DEFINITION. [Sperner family on a set] A *Sperner family* (or *Sperner system*) on a set $X \subseteq \mathbb{N}$ is a family of subsets of X in which none of the sets is contained in any other. Equivalently, a Sperner family on X is an *anti-chain* in the inclusion lattice over the power set of X , i.e., an anti-chain which elements are in $\mathcal{P}(X)$.

From here on we will refer to Sperner families as anti-chains. The number of different anti-chains on a set of n elements is counted by the so-called Dedekind numbers. Determining these numbers is known as the Dedekind problem. The number of anti-chains on sets of n elements for $n \in \mathbb{N}$ are 2, 3, 6, 20, 168, 7581, ... respectively. Concretely, we have for a set with 0 elements the anti-chains \emptyset and $\{\emptyset\}$, which is why the Dedekind number of a set with 0 elements is 2.

5.5.10. LEMMA. Let $\mathcal{S}^{\mathcal{Y}} \subseteq \mathcal{S}^2$ be the family corresponding to some finite set of pairs $\mathcal{Y} = \{Y_1, \dots, Y_n\}$. For every maximal nepfi extension \mathcal{S} of $\mathcal{S}^{\mathcal{Y}}$ and every $S \in (\mathcal{S} - \mathcal{S}^{\mathcal{Y}})$, $S \subseteq \text{NUM}(\mathcal{Y})$.

Proof:

Towards contradiction, suppose there is a maximal nepfi extension $\mathcal{S} \neq \mathcal{S}^2$ of $\mathcal{S}^{\mathcal{Y}}$ such that for some $S \in (\mathcal{S} - \mathcal{S}^{\mathcal{Y}})$, $S \not\subseteq \text{NUM}(\mathcal{Y})$. Thus, there is $z \in S$ such that $z \notin \text{NUM}(\mathcal{Y})$. Note that S cannot be a singleton simply because $\{z\}$ is contained in infinitely many $\{z, w\} \in \mathcal{S}^{\mathcal{Y}} \subseteq \mathcal{S}$. Thus a $w \neq z$ exists in S such that $\{w, z\} \notin \text{PAIRS}(\mathcal{Y})$. Therefore $\{w, z\} \in \mathcal{S}^{\mathcal{Y}}$ simply by definition of $\mathcal{S}^{\mathcal{Y}}$. But since $\{w, z\} \subseteq S \in \mathcal{S}$, \mathcal{S} cannot be nepfi extension of $\mathcal{S}^{\mathcal{Y}}$ contradicting our initial assumption. \square

5.5.11. PROPOSITION. For every finite set of pairs $\mathcal{Y} = \{Y_1, \dots, Y_n\}$, the number of maximal nepfi extensions of the subfamily $\mathcal{S}^{\mathcal{Y}} \subseteq \mathcal{S}^2$ is bounded by the Dedekind number of the set $\text{NUM}(\mathcal{Y}) = \{y_1, \dots, y_m\}$ or in other words, by the number of anti-chains in $\text{NUM}(\mathcal{Y}) = \{y_1, \dots, y_m\}$. Moreover, the maximal nepfi extensions of $\mathcal{S}^{\mathcal{Y}}$ correspond to the maximal singleton-free anti-chains on $\text{NUM}(\mathcal{Y})$. All such extensions are pfi.

Proof:

Let $\mathcal{Y} = \{Y_1, \dots, Y_n\}$ be any finite set of pairs and $\mathcal{S}^{\mathcal{Y}} \subseteq \mathcal{S}^2$ the corresponding family. By Lemma 5.5.10 we know that for every maximal nepfi extension \mathcal{S} of $\mathcal{S}^{\mathcal{Y}}$, if $S \in (\mathcal{S} - \mathcal{S}^{\mathcal{Y}})$ then $S \subseteq \text{NUM}(\mathcal{Y}) = \{y_1, \dots, y_m\}$. Therefore, for every maximal nepfi extension \mathcal{S} of $\mathcal{S}^{\mathcal{Y}}$ we have that $(\mathcal{S} - \mathcal{S}^{\mathcal{Y}}) \subseteq \mathcal{P}(\text{NUM}(\mathcal{Y}))$ which is finite. Clearly, $(\mathcal{S} - \mathcal{S}^{\mathcal{Y}})$ must be an anti-chain in $\mathcal{P}(\text{NUM}(\mathcal{Y}))$. By definition

of $\mathcal{S}^{\mathcal{Y}}$, every $x \in \mathbb{N}$ is contained in some pair in $\mathcal{S}^{\mathcal{Y}}$. Therefore, every maximal nepfi extension \mathcal{S} of $\mathcal{S}^{\mathcal{Y}}$ corresponds to some anti-chain in $\mathcal{P}(NUM(\mathcal{Y}))$ without singletons. Moreover, since $\mathcal{S} \supseteq \mathcal{S}^{\mathcal{Y}}$ is maximal nepfi, $(\mathcal{S} - \mathcal{S}^{\mathcal{Y}})$ is precisely a maximal singleton-free anti-chain in $\mathcal{P}(NUM(\mathcal{Y}))$. For the other direction, if we extend $\mathcal{S}^{\mathcal{Y}}$ with any maximal singleton-free anti-chain \mathcal{C} in $\mathcal{P}(NUM(\mathcal{Y}))$ then clearly the resulting family \mathcal{S} is a maximal nepfi extension. Simply because any $S \in (\mathcal{S} - \mathcal{S}^{\mathcal{Y}})$ has S itself as a DFTT set. Thus, there is a finite description of the anti-chain \mathcal{C} and their DFTTs which clearly makes the resulting family \mathcal{S} a pfi family. \square

So far we know the following about subfamilies of \mathcal{S}^2 : (1) By Proposition 5.5.11, any subfamily of \mathcal{S}^2 obtained by removing finitely many pairs from \mathcal{S}^2 has only finitely many maximal nepfi extensions; (2) by Example 5.5.4 and Proposition 5.5.5, we know that any subfamily of \mathcal{S}^2 obtained by removing all pairs of the form $\{i, n\}$ with $n \in \mathbb{N}$ and $i \in \{0, 1, \dots, m\}$ (of which there are infinitely many) has either 2 maximal nepfi extensions ($m = 0$) or uncountably many ($m > 0$). What happens when we consider subfamilies obtained by removing infinitely many arbitrary pairs? The answer to this question will also clarify what happens to subfamilies of all n -tuples \mathcal{S}^n for any $n \geq 3$.

We will first bring somewhat more structure in the removal of finitely many pairs. This will assist us later in the more complicated cases. In particular, we will study what happens when we remove from \mathcal{S}^2 a specific group of pairs called a *2-cluster*. A family that results from removing a finite 2-cluster can be extended by any language formed with numbers in the 2-cluster. This will become clear later on, when we prove Proposition 5.5.19. Then we will address the case of removing infinitely many pairs. We provide a complete overview of our investigations of the number of maximal nepfi extensions of subfamilies of \mathcal{S}^2 and will be able to conclude that every subfamily of \mathcal{S}^2 has either finitely or uncountably many maximal nepfi extensions.

5.5.12. DEFINITION. [2-cluster] We say that $\mathcal{G} \subseteq \mathcal{S}^2$ is a *2-cluster* in \mathcal{S}^2 if $PAIRS(\mathcal{G}) = \mathcal{G}$ (see Definition 5.5.8) and $|\mathcal{G}| > 1$.

Clearly for every $\mathcal{Y} \subseteq \mathcal{S}^2$, $PAIRS(\mathcal{Y})$ is a 2-cluster in \mathcal{S}^2 . The minimal-in-size 2-clusters of \mathcal{S}^2 are the ones that contain three pairs. The decision not to allow 2-clusters to have just a single pair is crucial for the cases considered in Proposition 5.5.16. In such cases, we construct uncountably many maximal nepfi extensions for a given family of pairs. The case for when we take out countably many single arbitrary pairs which do not form any 2-cluster, will be treated independently later on.

5.5.13. LEMMA. For any finite set $\mathcal{Y} \subseteq \mathcal{S}^2$, $\mathcal{Y} \subseteq PAIRS(\mathcal{Y})$ and this is the minimal 2-cluster that contains \mathcal{Y} .

Proof:

Straightforward from Definition [5.5.12](#) □

To illustrate the lemma above, let $\mathcal{Y} = \{\{1, 2\}, \{2, 3\}\}$. Then the minimal 2-cluster that contains \mathcal{Y} is $PAIRS(\mathcal{Y}) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. We have many finite 2-clusters that contain $PAIRS(\mathcal{Y})$ and therefore \mathcal{Y} . For instance the 2-cluster $\mathcal{G} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\}$.

The following proposition follows from Proposition [5.5.11](#), but proving it by itself clarifies matters about 2-clusters which will be useful in later proofs.

5.5.14. PROPOSITION. *Let $\mathcal{G}_1, \dots, \mathcal{G}_N$ with $N \in \mathbb{N}^+$ be a finite set of finite 2-clusters. Then the family*

$$\mathcal{S}^2 - (\mathcal{G}_1 \cup \dots \cup \mathcal{G}_N)$$

has finitely many maximal nepfi extensions.

Proof:

Take $\mathcal{G} \supseteq \mathcal{G}_1 \cup \dots \cup \mathcal{G}_N$ the minimal 2-cluster that contains all $\mathcal{G}_1, \dots, \mathcal{G}_N$, which exists by lemma [5.5.13](#). By Proposition [5.5.11](#), $\mathcal{S}^{\mathcal{G}}$ has finitely many maximal nepfi extensions, and therefore $\mathcal{S}^2 - (\mathcal{G}_1 \cup \dots \cup \mathcal{G}_N)$ as well, since $\mathcal{S}^2 - (\mathcal{G}_1 \cup \dots \cup \mathcal{G}_N) \supseteq \mathcal{S}^{\mathcal{G}}$. □

5.5.15. DEFINITION. [Maximal 2-cluster/Greatest set of 2-clusters outside \mathcal{S}]

- We say that a 2-cluster $\mathcal{G} \subset \mathcal{S}^2$ is a *maximal 2-cluster outside $\mathcal{S} \subset \mathcal{S}^2$* if $\mathcal{G} \cap \mathcal{S} = \emptyset$ and for any 2-cluster $\mathcal{G}' \supseteq \mathcal{G}$ outside \mathcal{S} it holds that $\mathcal{G}' = \mathcal{G}$.
- Let $N \in \mathbb{N}^+$, we say that $\mathcal{G} := \{\mathcal{G}_1, \dots, \mathcal{G}_N\}$ is *the greatest set of 2-clusters outside \mathcal{S}* , if it is the set of all maximal 2-clusters outside \mathcal{S} .

The result that follows addresses the cases when a subfamily of \mathcal{S}^2 has uncountably many maximal extensions.

5.5.16. PROPOSITION. *Let $\mathcal{S} \subseteq \mathcal{S}^2$.*

1. *If there is an infinite 2-cluster outside \mathcal{S} ; or*
2. *if $\{\mathcal{G}_1, \mathcal{G}_2, \dots\}$ is a countable sequence of disjoint finite 2-clusters such that $\bigcup_{i=1}^{\infty} \mathcal{G}_i \subseteq (\mathcal{S}^2 - \mathcal{S})$; or*
3. *if for more than one $k \in \mathbb{N}$ we have that $\{\{k, m\} : m \in \mathbb{N} - \{k\}\} \cap \mathcal{S} = \emptyset$;*

then \mathcal{S} has uncountably many maximal nepfi extensions.

Proof:

1. Let \mathcal{G} be an infinite 2-cluster outside \mathcal{S} , let $\{\{a_1, b_1\}, \{a_2, b_2\}, \dots\}$ be an enumeration of \mathcal{G} and let $\{a_1, b_1\}, \{a_2, b_2\}$ be the first two elements in the enumeration of \mathcal{G} . By definition of 2-cluster (defn. 5.5.12) we have that $\{a_1, a_2\}, \{a_1, b_2\}, \{b_1, a_2\}, \{b_1, b_2\}$ are also in \mathcal{G} . Thus they are outside \mathcal{S} . Then \mathcal{S} can be extended in at least two different ways where the extensions are nepfi; by adding $\{a_1, b_1\}, \{a_2, b_2\}$ or by adding $\{a_1, b_1, a_2\}, \{a_2, b_2, a_1\}$. These two ways of extending \mathcal{S} , namely \mathcal{S}_{1a} and \mathcal{S}_{1b} , are mutually exclusive. At step k , we repeat this procedure for both of the extensions resulting from the previous step, namely $\mathcal{S}_{(k-1)a}$ and $\mathcal{S}_{(k-1)b}$. We use the first couple of elements $\{a_{m_k}, b_{m_k}\}, \{a_{m_k+1}, b_{m_k+1}\}$ in \mathcal{G} that have not been used before. As before, we obtain two nepfi extensions for $\mathcal{S}_{(k-1)a}$ and two nepfi extensions for $\mathcal{S}_{(k-1)b}$.

By repeating this procedure, we are constructing a binary tree which branches correspond to chains of nepfi families that extend \mathcal{S} . By a standard combinatorial argument, we obtain uncountably many maximal nepfi extensions for \mathcal{S} .

2. Let $\mathcal{G}_1, \dots, \mathcal{G}_n, \dots$ be a countable sequence of finite 2-clusters such that $\bigcup_{i=1}^{\infty} \mathcal{G}_i \subseteq (\mathcal{S}^2 - \mathcal{S})$, and suppose the 2-clusters are pair-wise disjoint. For each \mathcal{G}_i consider $NUM(\mathcal{G}_i)$, the set of all numbers that appear in some pair of the 2-cluster \mathcal{G}_i . Then we can extend \mathcal{S} in two different ways: (1) by adding the 2-cluster $PAIRS(\mathcal{G}_i) = \mathcal{G}_i$, or (2) by adding the set $NUM(\mathcal{G}_i)$. Note that these two ways of extending \mathcal{S} are mutually exclusive. Therefore, and since we have countably many 2-clusters \mathcal{G}_i , by a well-known combinatorial argument, we have uncountably many maximal nepfi extensions for \mathcal{S} .
3. Suppose that for more than one $k \in \mathbb{N}$ we have that $\{\{k, m\} : m \in \mathbb{N}\} \cap \mathcal{S} = \emptyset$. Note that we already proved in Example 5.5.4 that the subfamily $\mathcal{S}' = \{\{i, n\} : i, n \in \mathbb{N} - \{0, 1\}\}$ of \mathcal{S}^2 has uncountably many maximal nepfi extensions. Therefore any subfamily of \mathcal{S}' has uncountably many as well. Clearly every \mathcal{S} satisfying the condition just mentioned will be isomorphic to a subfamily of \mathcal{S}' . Therefore \mathcal{S} has uncountably many maximal nepfi extensions.

□

In the following we address when a subfamily of \mathcal{S}^2 has finitely many maximal nepfi extensions. First we address the case when $\mathcal{X} \subseteq \mathcal{S}^2 - \mathcal{S}$ is an *unrelated* set of pairs outside of \mathcal{S} . The elements in such \mathcal{X} do not have a specific structure, namely they do not form 2-clusters nor a family of the form $\{\{k, x\} : x \in \mathbb{N} - \{k\}\}$.

5.5.17. DEFINITION. [Unrelated pairs outside a family] We say that $\mathcal{X} \subseteq \mathcal{S}^2 - \mathcal{S}$ is an *unrelated set of pairs outside of \mathcal{S}* if \mathcal{X} contains no 2-cluster and does not

contain a family of the form $\{\{k, m\} : m \in \mathbb{N} - \{k\}\}$ for any $k \in \mathbb{N}$, i.e., $\mathcal{S} \cap \{\{k, m\} : m \in \mathbb{N} - \{k\}\} \neq \emptyset$ for every $k \in \mathbb{N}$. Whenever $\mathcal{X} := \mathcal{S}^2 - \mathcal{S}$, \mathcal{X} is the *maximal unrelated set of pairs outside of \mathcal{S}* .

5.5.18. PROPOSITION. *If $\mathcal{X} = \mathcal{S}^2 - \mathcal{S}$ is the maximal unrelated set of pairs outside of \mathcal{S} , any maximal nepfi extension of \mathcal{S} contains \mathcal{X} . Moreover, \mathcal{S}^2 is the only maximal nepfi extension of \mathcal{S} .*

Proof:

It suffices to show that we can only add to \mathcal{S} pairs in \mathcal{X} if we want the extension to be an anti-chain. Take $Y \subseteq \mathbb{N}$ such that $Y \notin \mathcal{X}$. We will show that $\mathcal{S} \cup \{Y\}$ is not an anti-chain. Suppose $Y := \{y\}$, then since $\mathcal{S} \cap \{\{y, m\} : m \in \mathbb{N} - \{y\}\} \neq \emptyset$, $\mathcal{S} \cup \{Y\}$ will not be an anti-chain. Suppose $Y := \{x, y\}$. Since $Y \notin \mathcal{X}$ it must be that $Y \in \mathcal{S}$ already. Suppose $Y := \{x, y, z\}$ and consider the 2-cluster $\mathcal{G} := \{\{x, y\}, \{x, z\}, \{y, z\}\}$. Since \mathcal{X} contains no 2-cluster, there is a pair in \mathcal{G} that is already contained in \mathcal{S} . Again the extension is not an anti-chain. By a similar argument, we cannot add any larger set to \mathcal{S} . It follows that the only possible maximal nepfi extension of \mathcal{S} is $\mathcal{S} \cup \mathcal{X} = \mathcal{S}^2$. \square

5.5.19. PROPOSITION. *Let $\mathcal{S} \subseteq \mathcal{S}^2$ such that $\mathcal{S}^2 - \mathcal{S} = \bigcup \mathcal{G} \cup \mathcal{K} \cup \mathcal{X}$ where $\mathcal{G} = \{\mathcal{G}_1, \dots, \mathcal{G}_N\}$ with $N \in \mathbb{N}^+$ is the greatest set of finite 2-clusters outside \mathcal{S} , $\mathcal{K} := \{\{k, m\} : m \in \mathbb{N} - \{k\}\}$ for one fixed $k \in \mathbb{N}$, and $\mathcal{X} = \mathcal{S}^2 - (\mathcal{S} \cup \bigcup \mathcal{G} \cup \mathcal{K})$ is an unrelated set of pairs outside $\mathcal{S} \cup \bigcup \mathcal{G} \cup \mathcal{K}$. For any nepfi extension \mathcal{S}' of \mathcal{S} and any $A \in \mathcal{S}' - \mathcal{S}$, either $A \subseteq \text{NUM}(\bigcup \mathcal{G}) \cup \{k\}$, $A \in \mathcal{K}$ or $A \in \mathcal{X}$.*

Proof:

Towards contradiction, suppose there is $\mathcal{S}' \supseteq \mathcal{S}$ such that $A \in \mathcal{S}'$ is such that $A \not\subseteq \text{NUM}(\bigcup \mathcal{G}) \cup \{k\}$, $A \notin \mathcal{K}$ and $A \notin \mathcal{X}$. First note that since $A \not\subseteq \text{NUM}(\bigcup \mathcal{G})$, $A \notin \bigcup \mathcal{G}$. Since $A \not\subseteq \text{NUM}(\bigcup \mathcal{G}) \cup \{k\}$, there is $y \in A$ such that $y \notin \text{NUM}(\bigcup \mathcal{G}) \cup \{k\}$. Therefore $y \neq k$. Note that A cannot be a singleton, say $\{y\}$, simply because $\{\{y, n\} : n \in \mathbb{N} - \{y\}\} \cap \mathcal{S} \neq \emptyset$. So there is a language $S \in \mathcal{S}$ such that $\{y\} \subset S$. Thus, there is $z \neq y$ such that $\{z, y\} \subseteq A$. Note that the only pairs that are not in \mathcal{S} are in $\bigcup \mathcal{G} \cup \mathcal{K} \cup \mathcal{X}$, thus $A \neq \{z, y\}$ since we are supposing $A \notin \bigcup \mathcal{G}$, $A \notin \mathcal{K}$ and $A \notin \mathcal{X}$. Therefore, we have that $x \in A$ exists with $x \neq z, y$ and so $\{x, z, y\} \subseteq A$. Note that $\text{PAIRS}(\{x, z, y\}) \not\subseteq (\mathcal{S}^2 - \mathcal{S})$. Otherwise, since $\text{PAIRS}(\{x, z, y\})$ is a 2-cluster, \mathcal{G} is the greatest set of finite 2-clusters outside \mathcal{S} and the sets \mathcal{X} and \mathcal{K} contain no 2-cluster, $\text{PAIRS}(\{x, z, y\}) \subseteq \mathcal{G}_i$ for some $i \in \{1, \dots, N\}$ which cannot be since $y \notin \text{NUM}(\bigcup \mathcal{G}) \cup \{k\}$. Therefore $\text{PAIRS}(\{x, z, y\}) \cap \mathcal{S} \neq \emptyset$, i.e., there is a pair $\{a, b\} \subseteq \{x, z, y\} \subseteq A$ such that $\{a, b\} \in \mathcal{S}$. This contradicts that $A \in \mathcal{S}' - \mathcal{S}$ where \mathcal{S}' is a nepfi extension of \mathcal{S} . Thus, $A \subseteq \text{NUM}(\bigcup \mathcal{G}) \cup \{k\}$, $A \in \mathcal{K}$ or $A \in \mathcal{X}$. \square

5.5.20. COROLLARY. *Let $\mathcal{S} \subseteq \mathcal{S}^2$, \mathcal{G} , \mathcal{K} and \mathcal{X} as in Proposition 5.5.19. If \mathcal{S}' is a maximal nepfi extension of \mathcal{S} then $\mathcal{S}' - \mathcal{S}$ is of the form $\mathcal{C} \cup \mathcal{X}$ where \mathcal{C} is a maximal anti-chain in $\mathcal{P}(\text{NUM}(\bigcup \mathcal{G}) \cup \{k\})$ or of the form $\mathcal{D} \cup \mathcal{K} \cup \mathcal{X}$ where \mathcal{D} is a maximal anti-chain in $\mathcal{P}(\text{NUM}(\bigcup \mathcal{G}))$.*

Proof:

Note that since \mathcal{X} is a unrelated set outside $\mathcal{S} \cup \bigcup \mathcal{G} \cup \mathcal{K}$, $\mathcal{X} \cap (\bigcup \mathcal{G} \cup \mathcal{K}) = \emptyset$. Thus, it is easy to see that any singleton-free anti-chain $\mathcal{C} \in \mathcal{P}(\text{NUM}(\bigcup \mathcal{G}) \cup \{k\})$ together with \mathcal{X} form an anti-chain, i.e., $\mathcal{C} \cup \mathcal{X}$ is an anti-chain. By Proposition 5.5.19 and maximality of \mathcal{S}' , $\mathcal{S}' - \mathcal{S}$ is of the form $\mathcal{C} \cup \mathcal{X}$ where \mathcal{C} is a maximal anti-chain in $\mathcal{P}(\text{NUM}(\bigcup \mathcal{G}) \cup \{k\})$ or of the form $\mathcal{D} \cup \mathcal{K} \cup \mathcal{X}$ where \mathcal{D} is a maximal anti-chain in $\mathcal{P}(\text{NUM}(\bigcup \mathcal{G}))$. \square

The corollary above shows that any maximal nepfi family of such a family \mathcal{S} is characterized by some specific anti-chains of which there are only finitely many. Thus, we obtain the following result.

5.5.21. PROPOSITION. *Let $\mathcal{S} \subseteq \mathcal{S}^2$ such that $\mathcal{S}^2 - \mathcal{S} = \bigcup \mathcal{G} \cup \mathcal{K} \cup \mathcal{X}$ where $\mathcal{G} = \{\mathcal{G}_1, \dots, \mathcal{G}_N\}$ with $N \in \mathbb{N}^+$ is the greatest set of finite 2-clusters outside \mathcal{S} , $\mathcal{K} := \{\{k, m\} : m \in \mathbb{N} - \{k\}\}$ for some fixed $k \in \mathbb{N}$, and $\mathcal{X} = \mathcal{S}^2 - (\mathcal{S} \cup \bigcup \mathcal{G} \cup \mathcal{K})$ is the maximal unrelated set of pairs outside $\mathcal{S} \cup \bigcup \mathcal{G} \cup \mathcal{K}$. The family \mathcal{S} has finitely many maximal nepfi extensions.*

Proof:

Let the greatest set of 2-clusters outside \mathcal{S} be the finite set $\mathcal{G} = \{\mathcal{G}_1, \dots, \mathcal{G}_N\}$, $\mathcal{K} := \{\{k, m\} : m \in \mathbb{N} - \{k\}\} \subseteq \mathcal{S}^2 - \mathcal{S}$ and $\mathcal{X} = \mathcal{S}^2 - (\mathcal{S} \cup \bigcup \mathcal{G} \cup \mathcal{K})$ the maximal unrelated set of pairs outside $\mathcal{S} \cup \bigcup \mathcal{G} \cup \mathcal{K}$. First note that, $\mathcal{X} \cap (G_i \cup \mathcal{K}) = \emptyset$ for every $i \in \{1, \dots, N\}$. By Proposition 5.5.19, we can only extend \mathcal{S} with a set A such that $A \subseteq \text{NUM}(\bigcup \mathcal{G}) \cup \{k\}$, $A \in \mathcal{K}$ or $A \in \mathcal{X}$.

By Corollary 5.5.20, every maximal nepfi extension of \mathcal{S} will contain the set \mathcal{X} . Moreover, any maximal nepfi extension of \mathcal{S} is uniquely characterized by some maximal anti-chain in $\mathcal{P}(\text{NUM}(\bigcup \mathcal{G}) \cup \{k\})$ or by $\mathcal{D} \cup \mathcal{K}$ where \mathcal{D} is a maximal anti-chain in $\mathcal{P}(\text{NUM}(\bigcup \mathcal{G}))$. This is sufficient because the number of maximal anti-chains in $\mathcal{P}(\text{NUM}(\bigcup \mathcal{G}) \cup \{k\})$ and in $\mathcal{P}(\text{NUM}(\bigcup \mathcal{G}))$ is bounded by $d + d'$ where d and d' are the Dedekind numbers of $\text{NUM}(\bigcup \mathcal{G}) \cup \{k\}$ and $\text{NUM}(\bigcup \mathcal{G})$ respectively. Clearly both numbers d and d' are finite. Thus \mathcal{S} has only finitely many maximal nepfi extensions. \square

By Proposition 5.5.14, Proposition 5.5.16, Proposition 5.5.18 and Proposition 5.5.21 we have the following result.

5.5.22. THEOREM. *Any subfamily \mathcal{S} of \mathcal{S}^2 has either finitely many maximal nepfi extensions or uncountably many.*

Proof:

Follows by the fact that all the possible cases of subfamilies of \mathcal{S}^2 were covered in Proposition [5.5.14](#), Proposition [5.5.16](#), Proposition [5.5.18](#) and Proposition [5.5.21](#). \square

The proof of Theorem [5.5.22](#) allows us to obtain a similar general result for subfamilies whose languages have exactly cardinality n with $n \geq 3$, i.e., subfamilies of the family of all n -tuples \mathcal{S}^n for any $n \geq 3$. However, there are some subtle details so that we need to tread carefully. Therefore we dedicate the following section to the class of all subfamilies of \mathcal{S}^n .

5.5.4 The class of families with only n -tuples ($n \geq 3$)

Here we generalize all the notions and results obtained for subfamilies of \mathcal{S}^2 to subfamilies of \mathcal{S}^n with $n \geq 3$. In fact, we will mostly consider subfamilies of \mathcal{S}^3 and conduct our analysis on those. This analysis will clarify, straightforwardly, what happens in the general case of equinumerous families, namely families $\mathcal{S} \subseteq \mathcal{S}^n$ for a fixed (but arbitrary) $n \geq 3$.

Recall that an n -tuple (or, k -tuple) is a language with cardinality $n, k \in \mathbb{N}^+$.

To start with, we can straightforwardly generalize Proposition [5.5.11](#) with the following result.

5.5.23. PROPOSITION. *For every finite set of n -tuples $\mathcal{Y} = \{Y_1, \dots, Y_m\}$, the number of maximal *nepfi* extensions of the subfamily $\mathcal{S}^{\mathcal{Y}} \subseteq \mathcal{S}^n$ is bounded by the number of anti-chains in the finite set $NUM(\mathcal{Y})$. Moreover, the maximal *nepfi* extensions of $\mathcal{S}^{\mathcal{Y}}$ correspond to the maximal anti-chains in $NUM(\mathcal{Y})$ and such anti-chains contain no k -cardinality language for any $1 \leq k \leq n - 1$.*

Before continuing, we need some definitions.

5.5.24. DEFINITION. [n -cluster] We say that $\mathcal{G} \subseteq \mathcal{S}^n$ is an n -cluster in \mathcal{S}^n if the set $nTUP(\mathcal{G})$ of all n -tuples formed by numbers in $NUM(\mathcal{G})$ (see Definition [5.5.8](#)) is exactly \mathcal{G} and $|\mathcal{G}| > 1$, i.e., if $nTUP(\mathcal{G}) = \mathcal{G}$.

Clearly for every $\mathcal{Y} \subseteq \mathcal{S}^n$, $nTUP(\mathcal{Y})$ is an n -cluster in \mathcal{S}^n .

5.5.25. DEFINITION. [Maximal n -cluster/Greatest set of n -clusters outside \mathcal{S}]

- We say that an n -cluster $\mathcal{G} \subseteq \mathcal{S}^n$ is a *maximal n -cluster outside $\mathcal{S} \subseteq \mathcal{S}^n$* if $\mathcal{G} \cap \mathcal{S} = \emptyset$ and for any n -cluster $\mathcal{G}' \supseteq \mathcal{G}$ outside \mathcal{S} it holds that $\mathcal{G}' = \mathcal{G}$.
- We say that $\mathcal{G} := \{\mathcal{G}_1, \dots, \mathcal{G}_N\}$ with $N \in \mathbb{N}^+$ is the *greatest set of n -clusters outside \mathcal{S}* , if it is the set of all maximal n -clusters outside \mathcal{S} .

5.5.26. LEMMA. *For any finite set $\mathcal{Y} \subseteq \mathcal{S}^n$, $\mathcal{Y} \subseteq nTUP(\mathcal{Y})$, and this is the minimal n -cluster that contains \mathcal{Y} .*

Proof:

Straightforward from Definition [5.5.24](#). \square

The following proposition will be strengthened later on in Proposition [5.5.36](#), but what we want to emphasize here is that for this simple case things go as for the families of pairs.

5.5.27. PROPOSITION. *Let $N \in \mathbb{N}^+$ and $\mathcal{G}_1, \dots, \mathcal{G}_N$ be a finite set of n -clusters. Then the family $\mathcal{S}^n - (\mathcal{G}_1 \cup \dots \cup \mathcal{G}_N)$ has finitely many maximal nepfi extensions.*

Proof:

The proof goes as in the case for \mathcal{S}^2 , taking the minimal n -cluster that contains all $\mathcal{G}_1, \dots, \mathcal{G}_N$, which by lemma [5.5.26](#) we know exists. By Proposition [5.5.23](#) we know that \mathcal{S}^n has finitely many maximal nepfi extensions, and $\mathcal{S}^n - (\mathcal{G}_1 \cup \dots \cup \mathcal{G}_N) \supseteq \mathcal{S}^n$, so $\mathcal{S}^n - (\mathcal{G}_1 \cup \dots \cup \mathcal{G}_N)$ has finitely many as well. \square

For readability in the proofs that follow, we will denote $TRIP(\mathcal{Y})$ the 3-cluster $3TUP(\mathcal{Y})$.

5.5.28. EXAMPLE. Consider the family $\mathcal{S}^3 - \{\{0, 1, b\} : b \in \mathbb{N} - \{0, 1\}\}$. This family is very similar to the family studied in Example [5.5.3](#), namely the family $\mathcal{S}^3 - \{\{0, a, b\} : a, b \in \mathbb{N}\}$, however $\mathcal{S}^3 - \{\{0, 1, b\} : b \in \mathbb{N} - \{0, 1\}\}$ has only finitely many nepfi extensions. To see this, note that we cannot extend it with any singleton. Not even with $\{0\}$ or $\{1\}$ since there are triples S_i, S_j in $\mathcal{S}^3 - \{\{0, 1, b\} : b \in \mathbb{N} - \{0, 1\}\}$ that contain $\{0\}$ and $\{1\}$ respectively (note that S_i, S_j cannot contain both singletons). The only pair we can add to \mathcal{S} is $\{0, 1\}$, by the way the family $\mathcal{S}^3 - \{\{0, 1, b\} : b \in \mathbb{N} - \{0, 1\}\}$ is defined. Any other pair will be contained in a triple of the family. Analogously it is easy to see that for any set $Y \subseteq \mathbb{N}$ such that $|Y| \geq 4$, there is S_i in $\mathcal{S}^3 - \{\{0, 1, b\} : b \in \mathbb{N} - \{0, 1\}\}$ such that $S_i \subseteq Y$. Thus the only two maximal nepfi extensions are \mathcal{S}^3 and $\mathcal{S}^3 - \{\{0, 1, b\} : b \in \mathbb{N} - \{0, 1\}\} \cup \{0, 1\}$.

Recall from Example [5.5.3](#), that $\mathcal{S}^3 - \{\{0, a, b\} : a, b \in \mathbb{N}\}$ has uncountably many maximal nepfi extensions. What is different in $\mathcal{S}^3 - \{\{0, a, b\} : a, b \in \mathbb{N}\}$ that it allows for uncountably many maximal nepfi extensions, whereas $\mathcal{S}^3 - \{\{0, 1, b\} : b \in \mathbb{N}\}$ does not? The difference lies in the elements that are fixed and the ones that remain “free” in the triples that are discarded from the families. Whenever we fix two elements in the triples we are preventing the combinatorics to act out, since with only one “free” element in the triple, there is not much that combinatorics can do. However, with two non-fixed entries we can build

uncountably many nepfi extensions just as in Example [5.5.3](#). The following proposition generalizes what happens in Example [5.5.28](#), namely families obtained by removing from \mathcal{S}^3 finitely many families of the form $\{\{k, m, a\} : a \in \mathbb{N} - \{k, m\}\}$ for k and m fixed.

5.5.29. PROPOSITION. *The family $\mathcal{S}^3 - \bigcup_{i=1}^N \{\{k_i, m_i, a\} : a \in \mathbb{N} - \{k_i, m_i\}\}$ for some $N \in \mathbb{N}^+$ and some pairs $\{k_i, m_i\} \in \mathcal{S}^2$ has finitely many nepfi maximal extensions.*

Proof:

Let $\mathcal{S} := \mathcal{S}^3 - \bigcup_{i=1}^N \{\{k_i, m_i, a\} : a \in \mathbb{N} - \{k_i, m_i\}\}$. Consider N families of the form $\{\{k_i, m_i, a\} : a \in \mathbb{N} - \{k_i, m_i\}\}$ for some $N \in \mathbb{N}^+$ such that for any $i \in \{1, 2, \dots, N\}$, $\mathcal{S} \cap \{\{k_i, m_i, a\} : a \in \mathbb{N} - \{k_i, m_i\}\} = \emptyset$. First we will prove that we cannot add any other n -tuple with $n \geq 5$ to \mathcal{S} . For this it suffices to see that we can only add finitely many quadruples, namely $\{k_i, m_i, k_j, m_j\}$ for any $i, j \in \{1, 2, \dots, N\}$. This is because if we could add an n -tuple for $n \geq 5$, then we could add any quadruple of elements in the tuple. Let us prove it then by contradiction, suppose there is a quadruple $\{a, b, c, d\} \neq \{k_i, m_i, k_j, m_j\}$ for any $i, j \in \{1, 2, \dots, N\}$ and such quadruple can extend \mathcal{S} and preserve the anti-chain condition, i.e., the extension is nepfi. It is sufficient to verify the worst case scenario in which it differs in just one element from all the admissible quadruples $\{k_i, m_i, k_j, m_j\}$. Suppose $a \neq k_1, k_2, \dots, k_N$. It suffices to verify the case for $a \neq k_1$ because the others follows similarly. Note that $\{a, b, c, d\} = \{a, m_1, k_2, m_2\}$. Necessarily, the triple $\{a, m_1, k_2\} \in \mathcal{S}$. This is because otherwise the 3-cluster $TRIP(\{\{a, m_1, k_2\}, \{k_1, m_1, k_2\}\}) \subseteq \mathcal{S}^3 - \mathcal{S}$, which contradicts our assumption on \mathcal{S} . Thus, since $\{a, m_1, k_2\} \in \mathcal{S}$, we cannot use $\{a, m_1, k_2, m_2\}$ to extend \mathcal{S} and remain nepfi. \square

The result above does not apply when we consider infinitely many distinctive pairs $\{k_i, m_i\}$ such that $\mathcal{S} \cap \{\{k_i, m_i, a\} : a \in \mathbb{N} - \{k_i, m_i\}\} = \emptyset$. We will see later on in this section, that in the case of infinitely many pairs there are uncountably many maximal nepfi extensions. The following proposition generalizes what happens in examples [5.5.3](#) and [5.5.28](#).

5.5.30. PROPOSITION. *Let $n \geq 3$ and \mathcal{S}^n the corresponding family. Let $\{a_1, \dots, a_k\}$ be a fixed k -tuple of elements in \mathbb{N} for some $1 \leq k \leq n - 1$.*

1. *If $1 \leq k \leq n - 2$, the family*

$$\mathcal{S}^n - \{\{a_1, \dots, a_k, x_{k+1}, \dots, x_n\} \in \mathcal{S}^n : x_i \in \mathbb{N} - \{a_1, \dots, a_k\}\}$$

has uncountably many maximal nepfi extensions.

2. If $k = n - 1$ and $N \in \mathbb{N}^+$, the family

$$\mathcal{S}^n - \bigcup_{i=1}^N \{ \{a_{i,1}, \dots, a_{i,n-1}, b\} \in \mathcal{S}^n : b \in \mathbb{N} - \{a_{i,1}, \dots, a_{i,n-1}\} \},$$

has finitely many maximal nepfi extensions.

Proof:

For (1): it will be shown when we prove Proposition 5.5.31. For (2): it will be shown when we prove Proposition 5.5.33, 5.5.34. \square

As we mentioned before, the proof for the generalization of Theorem 5.5.22 needs to be treated carefully since there are cases that do not correspond exactly to the ones for the subfamilies of \mathcal{S}^2 . In the proof of Proposition 5.5.31 with respect to a subfamily of \mathcal{S}^3 (corresponding to the generalisation of Proposition 5.5.16), there are more cases of a similar kind in which the subfamily has uncountably many maximal extensions. The cases (1) to (3) in Proposition 5.5.31, correspond to the cases (1) to (3) in Proposition 5.5.16, case (4) only appears for subfamilies of \mathcal{S}^n when $n \geq 3$. The general proof for \mathcal{S}^n when $n \geq 3$ is basically the same as the proof for \mathcal{S}^3 . Therefore, here we present the proofs for subfamilies of \mathcal{S}^3 only.

5.5.31. PROPOSITION. *Let $\mathcal{S} \subseteq \mathcal{S}^n$. If \mathcal{S} satisfies one of the following cases:*

1. $\mathcal{S}^n - \mathcal{S}$ contains an infinite n -cluster, or
2. there is an infinite sequence of finite n -clusters $\{\mathcal{G}_1, \mathcal{G}_2, \dots\}$ such that $(\bigcup_{i=1}^{\infty} \mathcal{G}_i) \subseteq \mathcal{S}^n - \mathcal{S}$, or
3. for infinitely many $(n-1)$ -tuples $\{a_1, \dots, a_{n-1}\} \in \mathcal{S}^{n-1}$, $\{\{a_1, \dots, a_{n-1}, x\} \in \mathcal{S}^n : x \in \mathbb{N} - \{a_1, \dots, a_{n-1}\}\} \cap \mathcal{S} = \emptyset$, or
4. for some $k \in \mathbb{N}^+$ such that $k \leq n - 2$ and some k -tuple $\{a_1, \dots, a_k\} \in \mathcal{S}^k$, $\{\{a_1, \dots, a_k, x_{k+1}, \dots, x_n\} \in \mathcal{S}^n : x \in \mathbb{N} - \{a_1, \dots, a_k\}\} \cap \mathcal{S} = \emptyset$,

then \mathcal{S} has uncountably many maximal nepfi extensions.

Proof:

The proof is with respect to $\mathcal{S} \subseteq \mathcal{S}^3$, using a similar notation as the one used in the proof for subfamilies of \mathcal{S}^2 .

For (1) and (2), the proofs go exactly as their \mathcal{S}^2 counterparts, i.e., as in the proof of Proposition 5.5.16.

For (3). Suppose there are infinitely many families of the form $\{\{k_i, m_i, a\} : a \in \mathbb{N} - \{k_i, m_i\}\}$ such that $\bigcup_{i=1}^{\infty} \{\{k_i, m_i, a\} : a \in \mathbb{N} - \{k_i, m_i\}\} \subseteq \mathcal{S}^3 - \mathcal{S}$. Thus we have an infinite set of the form $\{k_1, m_1, k_2, m_2, \dots, k_n, m_n, \dots\}$. Note that

there is no difference between k_i and m_i , but we distinguish them since they are paired together and this will be relevant for our proof. For each quadruple $\{k_i, m_i, k_j, m_j\} \subseteq \{k_1, m_1, k_2, m_2, \dots, k_n, m_n, \dots\}$, we can maximally extend \mathcal{S} with either the pairs $\{k_i, m_i\}$, $\{k_j, m_j\}$ or with $\{k_i, m_i, k_j, m_j\}$ itself and with the rest of the pairs $\{k_l, m_l\} \not\subseteq \{k_i, m_i, k_j, m_j\}$ such that $\{k_l, m_l\} \notin \mathcal{S}$. Since there are countably many quadruples of this form, by a straightforward combinatorial argument we obtain uncountably many maximal extensions. Note that whether the families $\{\{k_i, m_i, a\} : a \in \mathbb{N} - \{k_i, m_i\}\}$ are disjoint or not does not matter for the argument of the proof. For the worst case scenario, suppose they all share the element $\{0\}$ i.e., $0 = k_i$ for every $i \in \mathbb{N}^+$. Then there will still be infinitely many m 's which are different from each other. Therefore we will have that for every triple $\{0, m_1, m_2\}$ we can add either $\{0, m_1, m_2\}$ or $\{0, m_1\}$, $\{0, m_2\}$ and we again obtain uncountably many maximal nepfi extensions.

For (4), the proof follows the same procedure as the one presented in Example 5.5.3 when there is only one $k \in \mathbb{N}$ such that $\{(k, a, b) : a, b \in \mathbb{N} - \{k\}\} \cap \mathcal{S} = \emptyset$. This is because, for every non-empty $B \subseteq \mathbb{N}$, we can add to \mathcal{S} the anti-chain

$$\{\{k, b\} : b \in B - \{k\}\} \cup \{\{k, c_1, c_2\} : c_i \in \mathbb{N} - (B \cup \{k\})\}$$

and the resulting extension is a maximal nepfi family. For the case that for more than one $k \in \mathbb{N}$, $\{(k, a, b) : a, b \in \mathbb{N} - \{k\}\} \cap \mathcal{S} = \emptyset$, the proof goes exactly as its \mathcal{S}^2 counterpart, i.e., as in the proof of Proposition 5.5.16

In all these cases, \mathcal{S} has uncountably many maximal extensions. \square

The following results, Proposition 5.5.35 and Proposition 5.5.36, are the suitable generalisations of Proposition 5.5.18 and Proposition 5.5.21 respectively. Lemma 5.5.32 and Propositions 5.5.32-5.5.33 will be useful for proving Proposition 5.5.34 which will be strengthened later on in Proposition 5.5.36. Then, we will address the appropriate notion of a *unrelated set of n -tuples outside of* a certain family $\mathcal{S} \subseteq \mathcal{S}^n$ concerning Proposition 5.5.35.

5.5.32. LEMMA. *Let a family $\mathcal{S} \subseteq \mathcal{S}^n$, \mathcal{S}' a nepfi extension of \mathcal{S} and $\mathcal{X} = \mathcal{S}^n - \mathcal{S}$ a set of n -tuples with the following properties:*

1. \mathcal{X} does not contain any n -cluster,
2. \mathcal{X} does not contain any family of the form $\{\{a_1, \dots, a_k, x_{k+1}, \dots, x_n\} \in \mathcal{S}^n : x \in \mathbb{N} - \{a_1, \dots, a_k\}\}$ for a k -tuple $\{a_1, \dots, a_k\}$ with $1 \leq k \leq n - 2$,
3. for just a finite set $\mathcal{P} := \{\{a_{i,1}, \dots, a_{i,n-1}\} : 1 \leq i \leq N\}$ with $(n - 1)$ -tuples $\{a_{i,1}, \dots, a_{i,n-1}\} \in \mathcal{S}^{n-1}$,
 $\mathcal{A}_i := \{\{a_{i,1}, \dots, a_{i,n-1}, x\} \in \mathcal{S}^n : x \in \mathbb{N} - \{a_{i,1}, \dots, a_{i,n-1}\}\} \subseteq \mathcal{X}$, i.e.,
 $\mathcal{A}_i \cap \mathcal{S} = \emptyset$,

then every $Y \in \mathcal{S}'$ is such that $Y \in \mathcal{X}$, $Y \in \mathcal{P}$ or Y is an m -tuple in $NUM(\mathcal{P})$ with $m \geq n - 1$.

Proof:

The proof is with respect to $\mathcal{S} \subseteq \mathcal{S}^3$, using a notation similar to the one in the proof for subfamilies of \mathcal{S}^2 . Let $N \in \mathbb{N}^+$ be the number of pairs $\{k, m\}$ such that the families are as described in condition (3). Let $\mathcal{P} = \{\{k_1, m_1\}, \dots, \{k_N, m_N\}\}$ be the set of those pairs and $\mathcal{KM}_i := \{\{k_i, m_i, z\} : z \in \mathbb{N} - \{k_i, m_i\}\}$ their respective associated families. Note that for every $i \in \{1, \dots, N\}$, $\mathcal{KM}_i \cap \mathcal{S} = \emptyset$.

The proof follows by a reasoning similar to the one in the proof of Proposition 5.5.18. Let $Y \subseteq \mathbb{N}$ be such that $\mathcal{S}' := \mathcal{S} \cup Y$ is an anti-chain, we will show that Y can only be a triple in \mathcal{X} (so it can be a triple in $\bigcup_{i=1}^N \mathcal{KM}_i$), a pair in \mathcal{P} or an m -tuple with $m \geq 2$ formed by elements in $NUM(\mathcal{P})$ (note that this excludes the singletons). We will show that adding any other set Y (that is not as described above) to \mathcal{S} will impair the anti-chain condition. In what follows, when we say that “we cannot add Y to \mathcal{S} ” we mean that *adding Y to \mathcal{S} will impair the anti-chain condition*. Suppose Y is the singleton $\{a\}$ for an arbitrary $a \in \mathbb{N}$. By the condition (2) on \mathcal{X} , we have $\{\{a, y, z\} : y, z \in \mathbb{N} - \{a\}\} \cap \mathcal{S} \neq \emptyset$. Thus there is a triple of the form $\{a, y, z\}$ contained in \mathcal{S} and therefore Y cannot be added to \mathcal{S} .

Suppose Y is a pair $\{a, b\} \notin \mathcal{P}$. Then $\{\{a, b, z\} : z \in \mathbb{N} - \{a, b\}\} \cap \mathcal{S} \neq \emptyset$. Thus there is a triple $\{a, b, z\} \in \mathcal{S}$ such that $\{a, b\} \subseteq \{a, b, z\}$, so we cannot add Y to \mathcal{S} .

Suppose Y is a pair such that $Y \not\subseteq NUM(\mathcal{P})$. W.l.o.g. suppose $Y = \{a, k_1\}$ with $a \notin NUM(\mathcal{P})$. Note that the triple $\{k_1, m_1, a\} \in \mathcal{KM}_1$ and $\{a, k_1\} \subseteq \{k_1, m_1, a\}$. Since $\{a, k_1\} \notin \mathcal{P}$, it follows that $\{\{a, k_1, z\} : z \in \mathbb{N} - \{a, k_1\}\} \cap \mathcal{S} \neq \emptyset$. Thus by the definition of \mathcal{S} there must be a triple of the form $\{a, k_1, x\}$ contained in \mathcal{S} for some $x \in \mathbb{N}$. Therefore, we cannot add Y to \mathcal{S} .

If Y is a triple not in \mathcal{X} then, by definition of \mathcal{S} , Y was already in \mathcal{S} . Suppose Y is a quadruple with at least one element that is not an element in $NUM(\mathcal{P})$. First consider $Y = \{a, k_1, k_2, k_3\}$ such that $a \notin NUM(\mathcal{P})$, $k_1, k_2, k_3 \in NUM(\mathcal{P})$ but the pairs formed by k_1, k_2, k_3 are not in \mathcal{P} , i.e., for instance $\{k_1, k_2\} \notin \mathcal{P}$. By the condition (1) on \mathcal{X} , the 3-cluster of elements in Y , i.e., $\mathcal{G}_3(Y)$, is not contained in \mathcal{X} . Thus, there is a triple $T \in \mathcal{G}_3(Y)$ such that $T \notin \mathcal{X}$. Since $\mathcal{S} = \mathcal{S}^3 - \mathcal{X}$, we have $T \in \mathcal{S}$. Since $T \subseteq Y$, we cannot add Y to \mathcal{S} . Now we consider the case $Y = \{a, k_1, m_1, k_2\}$ when $a \notin NUM(\mathcal{P})$ and $\{k_1, m_1\} \in \mathcal{P}$. By the condition (1) on \mathcal{X} , \mathcal{X} has no 3-clusters, and so the 3-cluster formed by elements in Y , $\mathcal{G}_3(Y)$, is such that $\mathcal{G}_3(Y) \cap \mathcal{S} \neq \emptyset$. So there must be a triple $T \subseteq Y$ such that $T \in \mathcal{S}$. Therefore, we cannot add Y to \mathcal{S} . By a similar argument, Y cannot be an m -tuple formed by some elements outside of $NUM(\mathcal{P})$, for any $m \geq 4$. Thus $Y \in \mathcal{X}$ or $Y \in \mathcal{P}$ or Y is an m -tuple with all elements in $NUM(\mathcal{P})$ and $m \geq 2$. \square

Some additional notation will be useful for the next result. Consider the set $\mathcal{P} := \{\{a_{i,1}, \dots, a_{i,n-1}\} : 1 \leq i \leq N\}$ defined in clause (3) from the proposition above. Given a subset $\mathcal{Q} \subseteq \mathcal{P}$, $I_{\mathcal{Q}} := \{1 \leq i \leq N : \{a_{i,1}, \dots, a_{i,n-1}\} \in \mathcal{Q}\}$. We can

generalize this notion for enumerations of languages of the form $\mathcal{V} := \{V_1, V_2, \dots\}$, i.e., given $\mathcal{U} \subseteq \mathcal{V}$, $I_{\mathcal{U}} := \{i : V_i \in \mathcal{U}\}$.

5.5.33. PROPOSITION. *If $\mathcal{S} \subseteq \mathcal{S}^n$, $\mathcal{X} = \mathcal{S}^n - \mathcal{S}$ are as in Lemma 5.5.32, and if \mathcal{S}_{max} is a maximal nepfi extension of \mathcal{S} , then $\mathcal{S}_{max} - \mathcal{S}$ has one of the following forms,*

1. \mathcal{X} , or
2. $\mathcal{X} - \bigcup_{j \in I_{\mathcal{Q}} \cup I_{\mathcal{R}}} \mathcal{A}_j \cup \mathcal{Q} \cup \mathcal{C}$
with $\mathcal{Q} \cup \mathcal{R} \subseteq \mathcal{P}$ such that $\mathcal{Q} \cap \mathcal{R} = \emptyset$ and $\mathcal{C} \in \mathcal{P}(NUM(\mathcal{R}))$ a maximal anti-chain without singletons.

Proof:

The proof is with respect to $\mathcal{S} \subseteq \mathcal{S}^3$. Let $\mathcal{P} = \{\{k_1, m_1\}, \dots, \{k_N, m_N\}\}$ be the set of pairs as described in Lemma 5.5.32 and $\mathcal{KM}_i := \{\{k_i, m_i, z\} : z \in \mathbb{N} - \{k_i, m_i\}\}$ their respective associated families. Let \mathcal{S}_{max} be any maximal nepfi extension of \mathcal{S} . First note that by reasoning similar to the one in the proof of Proposition 5.5.29, there are only finitely many m -tuples formed by elements in $NUM(\mathcal{P})$ and therefore there are only finitely many maximal anti-chains in $\mathcal{P}(NUM(\mathcal{P}))$. By Proposition 5.5.32, if $Y \in \mathcal{S}_{max} - \mathcal{S}$ then $Y \in \mathcal{X}$, $Y \in \mathcal{P}$ or Y is an m -tuple with all elements in $NUM(\mathcal{P})$ and $m \geq 2$. Thus, by a combinatorial argument and since \mathcal{S}_{max} is a maximal anti-chain, it is easy to see that $\mathcal{S}_{max} - \mathcal{S}$ can only have one of the following forms: \mathcal{X} , or

$$\mathcal{X} - \bigcup_{j \in I_{\mathcal{Q}}} \mathcal{KM}_j \cup \mathcal{Q}$$

with $\mathcal{Q} \subseteq \mathcal{P}$ and $\mathcal{R} = \emptyset$, or

$$\mathcal{X} - \bigcup_{j \in I_{\mathcal{R}}} \mathcal{KM}_j \cup \mathcal{C}$$

with $\mathcal{R} \subseteq \mathcal{P}$, $\mathcal{Q} = \emptyset$ and $\mathcal{C} \in \mathcal{P}(NUM(\mathcal{R}))$ a maximal anti-chain without singletons, or a combination of both, i.e., an anti-chain of the form

$$\mathcal{X} - \bigcup_{j \in I_{\mathcal{Q}} \cup I_{\mathcal{R}}} \mathcal{KM}_j \cup \mathcal{Q} \cup \mathcal{C}$$

with $\mathcal{Q} \cup \mathcal{R} \subseteq \mathcal{P}$ such that $\mathcal{Q} \cap \mathcal{R} = \emptyset$ and $\mathcal{C} \in \mathcal{P}(NUM(\mathcal{R}))$ a maximal anti-chain without singletons. \square

5.5.34. PROPOSITION. *If $\mathcal{S} \subseteq \mathcal{S}^n$ and $\mathcal{X} = \mathcal{S}^n - \mathcal{S}$ are as in Lemma 5.5.32, \mathcal{S} has only finitely many maximal nepfi extensions.*

Proof:

Follows straightforwardly by Proposition 5.5.33 since there are only finitely many such anti-chains and therefore finitely many ways to construct a maximal nepfi extension of \mathcal{S} . \square

Let $\mathcal{S} \subseteq \mathcal{S}^n$ and let \mathcal{X} be as in Lemma 5.5.32. Consider the set $\mathcal{X}' = \mathcal{X} - \bigcup_{j \in I_P} \mathcal{A}_j$. Note that such an \mathcal{X}' not only satisfies the conditions in Lemma 5.5.32 but also that for no $(n-1)$ -tuples $\{a_{i,1}, \dots, a_{i,n-1}\} \in \mathcal{S}^{n-1}$, it is the case that $\mathcal{A}_i := \{\{a_{i,1}, \dots, a_{i,n-1}, x\} \in \mathcal{S}^n : x \in \mathbb{N} - \{a_{i,1}, \dots, a_{i,n-1}\}\} \subseteq \mathcal{X}$. In such a case, we say that \mathcal{X}' is the *maximal unrelated set of n -tuples outside of \mathcal{S}* .

5.5.35. PROPOSITION. *Let $\mathcal{S} \subseteq \mathcal{S}^n$ and let $\mathcal{X} = \mathcal{S}^n - \mathcal{S}$ be the maximal unrelated set of n -tuples outside of \mathcal{S} . Any maximal nepfi extension of \mathcal{S} contains \mathcal{X} .*

Proof:

Follows by Proposition 5.5.33. \square

5.5.36. PROPOSITION. *Let $\mathcal{S} \subseteq \mathcal{S}^n$ and let there be no infinite n -cluster in $\mathcal{S}^n - \mathcal{S}$. If \mathcal{S} satisfies the following,*

1. *there are at most finitely many finite n -clusters $\mathcal{G} := \mathcal{G}_1, \dots, \mathcal{G}_N$ such that $(\bigcup_{i=1}^N \mathcal{G}_i) \subseteq \mathcal{S}^n - \mathcal{S}$,*
2. *for at most finitely many $(n-1)$ -tuples $\{a_{i,1}, \dots, a_{i,n-1}\} \in \mathcal{S}^{n-1}$ such that $\mathcal{A}_i := \{\{a_{i,1}, \dots, a_{i,n-1}, x\} \in \mathcal{S}^n : x \in \mathbb{N} - \{a_{i,1}, \dots, a_{i,n-1}\}\}$ is that $\mathcal{A}_i \cap \mathcal{S} = \emptyset$,*
3. *for all $k \in \mathbb{N}^+$ such that $k \leq n-2$ and all k -tuples $\{a_1, \dots, a_k\} \in \mathcal{S}^k$ it is the case that $\{\{a_1, \dots, a_k, x_{k+1}, \dots, x_n\} \in \mathcal{S}^n : x \in \mathbb{N} - \{a_1, \dots, a_k\}\} \cap \mathcal{S} \neq \emptyset$, and*
4. *$\mathcal{X} = \mathcal{S}^n - (\mathcal{S} \cup \bigcup \mathcal{G} \cup \bigcup \mathcal{A}_i)$ is the maximal unrelated set of pairs outside $\mathcal{S} \cup \bigcup \mathcal{G} \cup \bigcup \mathcal{A}_i$,*

then \mathcal{S} has finitely many maximal nepfi extensions.

Proof:

The proof is with respect to $\mathcal{S} \subseteq \mathcal{S}^3$, using a notation similar to the one in the proof for subfamilies of \mathcal{S}^2 . Let $\mathcal{S} \subseteq \mathcal{S}^3$ be such that there are only finitely many maximal 3-clusters $\mathcal{G}_1, \dots, \mathcal{G}_N$ with $(\bigcup_{i=1}^N \mathcal{G}_i) \subseteq (\mathcal{S}^3 - \mathcal{S})$. It is sufficient to show the proposition for the case that there are two pairs $\{k_1, m_1\}$ and $\{k_2, m_2\}$ such that $\mathcal{KM}_i := \{\{k_i, m_i, a\} : a \in \mathbb{N} - \{k, m\}\} \subseteq \mathcal{S}^3 - \mathcal{S}$ for $i \in \{1, 2\}$. W.l.o.g consider the family $\mathcal{S} = \mathcal{S}^3 - ((\bigcup_{i=1}^N \mathcal{G}_i) \cup \mathcal{KM}_1 \cup \mathcal{KM}_2 \cup \mathcal{X})$ where $\mathcal{X} =$

$\mathcal{S}^3 - (\mathcal{S} \cup \bigcup_{i=1}^N \mathcal{G}_i \cup \mathcal{KM}_1 \cup \mathcal{KM}_2)$ is the maximal unrelated set of triples outside of $\mathcal{S} \cup \bigcup_{i=1}^N \mathcal{G}_i \cup \mathcal{KM}_1 \cup \mathcal{KM}_2$. We want to show that there are only finitely many maximal nepfi extensions of \mathcal{S} . Note that by condition (3), for all $k \in \mathbb{N}$, $\{\{k, a, b\} : a, b \in \mathbb{N} - \{k\}\} \cap \mathcal{S} \neq \emptyset$.

In what follows, when we say “we can add Y to \mathcal{S} ” we mean that *we can add Y to \mathcal{S} without impairing the anti-chain condition (nepfi condition)*. Similarly, when we say “we cannot add Y to \mathcal{S} ” we mean that *adding Y to \mathcal{S} will impair the anti-chain condition*.

First we will prove that we can only add finitely many n -tuples with $n \geq 4$ to \mathcal{S} . For this, it suffices to see that we can only add finitely many quadruples different from $\{k_1, m_1, k_2, m_2\}$. This is because if we can add an n -tuple for $n \geq 4$, then we can add any quadruple of elements in the tuple. Let us prove it then by contradiction. Suppose there are infinitely many distinct quadruples $\{a_i, b_i, c_i, d_i\} \neq \{k_1, m_1, k_2, m_2\}$ for $i \in \mathbb{N}$ which can be added to \mathcal{S} . It is sufficient to check the case in which they differ from $\{k_1, m_1, k_2, m_2\}$ in one element only.

W.l.o.g. suppose $a_i \neq k_1$ for every $i \in \mathbb{N}$ and so $\{a_i, b_i, c_i, d_i\} = \{a_i, m_1, k_2, m_2\}$. Since we are considering them to be distinct from one another, we have that for every $i, j \in \mathbb{N}$, $\{a_i, m_1, k_2, m_2\} \neq \{a_j, m_1, k_2, m_2\}$. Note that either $\{a_i, m_1, k_2\} \in \mathcal{S}$ or $\{a_i, m_1, k_2\} \notin \mathcal{S}$. The former cannot be the case since then we cannot add $\{a_i, m_1, k_2, m_2\}$ to \mathcal{S} . So it must be the case that $\{a_i, m_1, k_2\} \notin \mathcal{S}$. Since also $\{k_1, m_1, k_2, \} \notin \mathcal{S}$, there is a 3-cluster $TRIP(\{\{a_i, m_1, k_2\}, \{k_1, m_1, k_2, \}\}) = TRIP(NUM(\{a_i, m_1, k_2, k_1\})) \subseteq \mathcal{S}^3 - \mathcal{S}$. Thus for every $i \in \mathbb{N}$,

$$TRIP(\{\{a_i, m_1, k_2\}, \{k_1, m_1, k_2, \}\}) \subseteq \mathcal{G}_j$$

for some $j \in \{1, \dots, N\}$. It follows that there are infinitely many triples of the form $\{a_i, m_1, k_2\}$ in $NUM(\bigcup_{i=1}^N \mathcal{G}_i)$ which cannot be since $NUM(\bigcup_{i=1}^N \mathcal{G}_i)$ contains only finitely many triples. Thus we can only add finitely many quadruples to \mathcal{S} , i.e., finitely many n -tuples.

Now note that we can only add triples in $\mathcal{X} \cup \mathcal{KM}_1 \cup \mathcal{KM}_2$ (this includes the triples formed by elements in $\{k_1, m_1, k_2, m_2\}$), and the ones formed by elements in $NUM(\bigcup_{i=1}^N \mathcal{G}_i)$. Note also that we can only add the pairs $\{k_1, m_1\}$ and $\{k_2, m_2\}$ since any other pair $\{a, b\}$ will be such that the family $\mathcal{AB} = \{\{a, b, x\} : x \in \mathbb{N} - \{a, b\}\} \cap \mathcal{S} \neq \emptyset$. Thus, adding $\{a, b\}$ impairs the anti-chain condition for the extension. Similarly, we cannot add any singleton because of our initial assumption, namely that for any $k \in \mathbb{N}$, $\{k, x, y : x, y \in \mathbb{N} - \{k\}\} \cap \mathcal{S} \neq \emptyset$. As in the case of \mathcal{S}^2 , for any maximal nepfi extension \mathcal{S}_m of \mathcal{S} and any $A \in \mathcal{S}_m - \mathcal{S}'$ we have that either $A \subseteq NUM(\bigcup_{i=1}^N \mathcal{G}_i)$, $A \subseteq \{k_1, m_1, k_2, m_2\}$, $A \in \mathcal{KM}_1$, $A \in \mathcal{KM}_2$ or $A \in \mathcal{X}$.

By an argument similar to the one used in the proof of Proposition 5.5.33 (and also in Proposition 5.5.19 and in Corollary 5.5.20), any maximal nepfi extension of \mathcal{S} is characterized by a maximal antichain having one of the following forms

$$\mathcal{C} \cup \mathcal{X}$$

with \mathcal{C} a maximal anti-chain in $\mathcal{P}(NUM(\bigcup_{i=1}^N \mathcal{G}_i) \cup \mathcal{P}(\{k_1, m_1, k_2, m_2\}))$,

$$\mathcal{D} \cup \mathcal{KM}_1 \cup \mathcal{KM}_2 \cup \mathcal{X}$$

with \mathcal{D} a maximal anti-chain in $\mathcal{P}(NUM(\bigcup_{i=1}^N \mathcal{G}_i))$,

$$\mathcal{B} \cup \mathcal{KM}_1 \cup \mathcal{X} \text{ or } \mathcal{H} \cup \mathcal{KM}_2 \cup \mathcal{X}$$

with \mathcal{B} a maximal anti-chain in $\mathcal{P}(NUM(\bigcup_{i=1}^N \mathcal{G}_i) \cup \mathcal{P}(\{k_1, m_1\}))$ and \mathcal{H} a maximal anti-chain in $\mathcal{P}(NUM(\bigcup_{i=1}^N \mathcal{G}_i) \cup \mathcal{P}(\{k_2, m_2\}))$. There are only finitely many such anti-chains. Thus \mathcal{S} can only have finitely many nepfi extensions. \square

We then have the following generalisation of Theorem [5.5.22](#).

5.5.37. THEOREM. *Let $n \in \mathbb{N}^+$. Any subfamily \mathcal{S} of the family of all n -tuples \mathcal{S}^n has either finitely many maximal nepfi extensions or uncountably many.*

Proof:

For $n = 1$ the proof follows from Proposition [5.5.7](#). For $n = 2$ it follows from Theorem [5.5.22](#). For $n \geq 3$ the proof follows by the fact that we cover all the possible cases for subfamilies of n -tuples with $n \geq 3$ in Proposition [5.5.23](#), Proposition [5.5.27](#), Proposition [5.5.30](#), Proposition [5.5.31](#), Proposition [5.5.34](#), and Proposition [5.5.36](#). \square

5.5.5 Families with pairs and triples

In this section we study how many maximal nepfi extensions a family $\mathcal{S} \subseteq \mathcal{S}^2 \cup \mathcal{S}^3$ has. In agreement with our general conjecture (but not trivially so), such a family has either finitely many or uncountably many maximal nepfi extensions. The theorems we have for subfamilies of \mathcal{S}^n do not apply straightforwardly to subfamilies of $\mathcal{S}^2 \cup \mathcal{S}^3$ since the combinatorics are not as simple. Thus we need to study these families more carefully.

In what follows we will always consider families of pairs and triples that have non-empty intersection with \mathcal{S}^2 and also with \mathcal{S}^3 , i.e., $\mathcal{S} \cap \mathcal{S}^2 \neq \emptyset$ and $\mathcal{S} \cap \mathcal{S}^3 \neq \emptyset$. This is because the cases when $\mathcal{S} \cap \mathcal{S}^2 = \emptyset$ are reduced to cases of families of only triples. Similarly for when $\mathcal{S} \cap \mathcal{S}^3 = \emptyset$, these families are subfamilies of \mathcal{S}^2 (already analyzed in Section [5.5.4](#)).

Our general strategy for studying such families, is to start by taking a maximal family $\mathcal{S} \subseteq \mathcal{S}^2 \cup \mathcal{S}^3$ and to consider different subfamilies of \mathcal{S} that result from taking out sets of languages (either pairs or triples). Then we analyze how many maximal extensions we can obtain from this resulting subfamily. As an illustration, consider the following example.

5.5.38. EXAMPLE. Take the family $\mathcal{S} = E_2 \cup O_3 \cup E_1 O_2$ where $E_2 = \{\{e, f\} : e \neq f \in \text{EVEN}\}$, $O_3 = \{\{o, p, q\} : o \neq p \neq q \in \text{ODD}\}$ and $E_1 O_2 = \{\{e, o, p\} : e \in \text{EVEN} \text{ and } o \neq p \in \text{ODD}\}$. This family is clearly maximal nepfi. Now consider the subfamily $\mathcal{S}' = \mathcal{S} - \{\{2, 4\}\}$ and the subfamily $\mathcal{S}'' = \mathcal{S} - \{\{2, 4\}, \{6, 8\}\}$. It is easy to see that \mathcal{S}' has only two maximal nepfi extensions, namely \mathcal{S} and $\mathcal{S}' \cup \{\{2, 4, o\} : o \in \text{ODD}\}$. And \mathcal{S}'' has four maximal nepfi extensions: $\mathcal{S}'' \cup \{\{2, 4, o\} : o \in \text{ODD}\} \cup \{\{6, 8\}\}$, $\mathcal{S}'' \cup \{\{2, 4\}\} \cup \{\{6, 8, o\} : o \in \text{ODD}\}$, $\mathcal{S}'' \cup \{\{2, 4, o\} : o \in \text{ODD}\} \cup \{\{6, 8, o\} : o \in \text{ODD}\}$ and \mathcal{S} itself. Thus, if we take out a finitely number of pairs from \mathcal{S} , we regain finitely many maximal nepfi extensions.

An important question comes up concerning the strategy described above, namely if this strategy is enough to treat all the cases for families $\mathcal{S}' \subseteq \mathcal{S}^2 \cup \mathcal{S}^3$. The answer to this question is positive if we find a way to characterize maximal families and then use these to compute any family $\mathcal{S}' \subseteq \mathcal{S}^2 \cup \mathcal{S}^3$. More concretely, to compute any family \mathcal{S}' from a maximal nepfi family $\mathcal{S} \subseteq \mathcal{S}^2 \cup \mathcal{S}^3$ by taking out certain languages. Surprisingly, we manage to do such thing, which was extremely useful for obtaining a simple analysis for such families. Before presenting the full description of the aforementioned characterization, we fix some notation and present an example.

Given a family \mathcal{S} , we denote \mathcal{S}_p and \mathcal{S}_t to be the sets of pairs and triples respectively of \mathcal{S} .

Let \mathcal{S}_p be a set of pairs. A first try to obtain a maximal family of pairs and triples, is to extend \mathcal{S}_p with the following anti-chain of triples $\mathcal{S}_t = \{\{a, b, c\} : \{a, b\}, \{b, c\}, \{a, c\} \notin \mathcal{S}_p\}$, i.e., $\{a, b, c\} \in \mathcal{S}_t$ iff $\{a, b\}, \{b, c\}, \{a, c\} \notin \mathcal{S}_p$. The resulting family \mathcal{S} seems like a good candidate for a *generic* maximal nepfi family of $\mathcal{S}^2 \cup \mathcal{S}^3$. Actually, such a family $\mathcal{S} = \mathcal{S}_p \cup \mathcal{S}_t$ is characteristic for a *generic description* of a nepfi family of $\mathcal{S}^2 \cup \mathcal{S}^3$, but to obtain maximality, we need more. Example [5.5.39](#) illustrates that a family with \mathcal{S}_p and \mathcal{S}_t as we just described is not always maximal. It also illustrates how we can extend the family $\mathcal{S} = \mathcal{S}_p \cup \mathcal{S}_t$ with pairs so that the resulting family is indeed maximal nepfi.

5.5.39. EXAMPLE. Let \mathcal{S}_p be the following set of pairs, $\{0, 1\} \notin \mathcal{S}_p$, $\{0, n\} \in \mathcal{S}_p$ for all $n > 1$ and $\{1, m\} \in \mathcal{S}_p$ for all $m > 1$. Consider the family $\mathcal{S} = \mathcal{S}_p \cup \mathcal{S}_t$ such that $\mathcal{S}_t = \{\{a, b, c\} : \{a, b\}, \{b, c\}, \{a, c\} \notin \mathcal{S}_p\}$. Note that any triple of the form $\{0, 1, n\}$ is not in \mathcal{S}_t , since $\{0, n\}$ or $\{1, n\}$ is in \mathcal{S}_p and this will impair the anti-chain condition. But \mathcal{S} is not maximal nepfi because we can add the pair $\{0, 1\}$ and remain nepfi, i.e., $\mathcal{S} \cup \{\{0, 1\}\}$ is nepfi. Moreover, $\mathcal{S} \cup \{\{0, 1\}\}$ is maximal nepfi.

We can characterize all maximal nepfi families of triples and pairs as described in Proposition [5.5.40](#). An important fact to remember is that, in this section, we only consider families that contain both pairs and triples, i.e., $\mathcal{S}_p \neq \emptyset$ and $\mathcal{S}_t \neq \emptyset$ (otherwise this characterization does not hold).

5.5.40. PROPOSITION. *Let \mathcal{S}'_p be any set of pairs. A family of the form $\mathcal{S}' = \mathcal{S}'_p \cup \mathcal{S}'_t \subseteq \mathcal{S}^2 \cup \mathcal{S}^3$ is nepfi iff $\mathcal{S}'_t \subseteq \mathcal{S}_t := \{\{a, b, c\} : \{a, b\}, \{b, c\}, \{a, c\} \notin \mathcal{S}'_p\}$. Moreover, there is a maximal nepfi extension $\mathcal{S} = \mathcal{S}_p \cup \mathcal{S}_t \supseteq \mathcal{S}'$ such that $\mathcal{S}_t := \{\{a, b, c\} : \{a, b\}, \{b, c\}, \{a, c\} \notin \mathcal{S}'_p\}$ and $\mathcal{S}_p := \{\{x, y\} : \{x, y, z\} \notin \mathcal{S}_t \text{ for any } z \in \mathbb{N}\}$.*

Proof:

The proof for the first part of the proposition is straightforward since \mathcal{S}'_p and $\mathcal{S}'_t \subseteq \{\{a, b, c\} : \{a, b\}, \{b, c\}, \{a, c\} \notin \mathcal{S}'_p\}$ form clearly an anti-chain and so $\mathcal{S}'_p \cup \mathcal{S}'_t$ is nepfi. We will now see that \mathcal{S} is maximal nepfi when $\mathcal{S}_t := \{\{a, b, c\} : \{a, b\}, \{b, c\}, \{a, c\} \notin \mathcal{S}'_p\}$ and $\mathcal{S}_p := \{\{x, y\} : \{x, y, z\} \notin \mathcal{S}_t \text{ for any } z \in \mathbb{N}\}$. First note that we can write \mathcal{S}_p also in terms of \mathcal{S}'_p . For this, note that we can also describe the set $\mathcal{S}_t := \{\{a, b, c\} : \{a, b\}, \{b, c\}, \{a, c\} \notin \mathcal{S}'_p\}$ as

$$\mathcal{S}_t = \{\{a, b, c\} : \forall \{x, y\} \in \mathcal{S}'_p, \{x, y\} \not\subseteq \{a, b, c\}\}.$$

So,

$$\{a, b, c\} \notin \mathcal{S}_t \text{ iff } \neg(\forall \{x, y\} \in \mathcal{S}'_p, \{x, y\} \not\subseteq \{a, b, c\}).$$

Therefore, we can also write the set $\mathcal{S}_p := \{\{x, y\} : \{x, y, z\} \notin \mathcal{S}_t \text{ for any } z \in \mathbb{N}\}$ as

$$\mathcal{S}_p = \{\{x, y\} : \forall c \in \mathbb{N} \neg(\forall \{x, y\} \in \mathcal{S}'_p, \{x, y\} \not\subseteq \{a, b, c\})\}.$$

This way, both sets \mathcal{S}_p and \mathcal{S}_t are described in terms of \mathcal{S}'_p . Clearly \mathcal{S} is an anti-chain. Since we exhausted in \mathcal{S}_t all the triples that we can add to \mathcal{S}'_p without impairing the anti-chain condition, we cannot add any other triple to $\mathcal{S}'_p \cup \mathcal{S}_t$. Similarly, we cannot add any more pairs to \mathcal{S}'_p than the ones in $\mathcal{S}_p \supseteq \mathcal{S}'_p$ since, by construction, we exhausted them in \mathcal{S}_p . Adding another pair not in \mathcal{S}_p will impair the anti-chain condition for the resulting extension $\mathcal{S} = \mathcal{S}_p \cup \mathcal{S}_t$. It is easy to see that adding singletons or any larger set than a triple will also impair the anti-chain condition. Therefore we cannot add anything more to the family $\mathcal{S}_p \cup \mathcal{S}_t$ and thus it is maximal nepfi. \square

Note that by Theorem 5.4.4 we know that any family $\mathcal{S}' := \mathcal{S}'_p \cup \mathcal{S}'_t$ has a maximal nepfi extension. It follows from Proposition 5.5.40 that \mathcal{S}' has a maximal nepfi extension that is also of only pairs and triples.

5.5.41. COROLLARY. *Any nepfi family of only pairs and triples has a maximal nepfi extension of pairs and tripels, i.e., has a maximal nepfi extension of the form $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$.*

Proof:

Let $\mathcal{S}' := \mathcal{S}'_p \cup \mathcal{S}'_t$ be any family of pairs and triples. By Proposition 5.5.40, $\mathcal{S}'_t \subseteq \{\{a, b, c\} : \{a, b\}, \{b, c\}, \{a, c\} \notin \mathcal{S}'_p\}$. Take $\mathcal{S}_t := \{\{a, b, c\} : \{a, b\}, \{b, c\}, \{a, c\} \notin \mathcal{S}'_p\}$ and let \mathcal{S}_p be such that $\mathcal{S}'_p \subseteq \mathcal{S}_p$ and satisfies

$$\{x, y\} \in \mathcal{S}_p \text{ iff } \{x, y, z\} \notin \mathcal{S}_t \text{ for any } z \in \mathbb{N}.$$

By Proposition [5.5.40](#), the family $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t \supseteq \mathcal{S}'$ is maximal *nepfi*. \square

We now proceed with our case-by-case analysis to show that every family of pairs and triples has either finitely many or uncountably many maximal *nepfi* extensions. The procedure we follow simulates the one we did for subfamilies of \mathcal{S}^2 (and subfamilies of \mathcal{S}^n). We will first see what happens to families that result from taking out finitely many pairs (or finitely many triples) from a maximal *nepfi* family $\mathcal{S} = \mathcal{S}_p \cup \mathcal{S}_t$. Then we will address, simultaneously, families that result when we take out infinitely many languages and the rest of families that result from taking out finitely many ones from $\mathcal{S} = \mathcal{S}_p \cup \mathcal{S}_t$. First consider the following useful lemmas.

Let \mathcal{Y}_p be the set of pairs obtained from some triple in \mathcal{S}_t , i.e., $\mathcal{Y}_p := \{\{b, c\} : \{y, b, c\} \in \mathcal{S}_t \text{ for some } y \in \mathbb{N}\}$.

5.5.42. LEMMA. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a *nepfi* family.*

1. *If \mathcal{S}_p has finitely many pairs, there are infinitely many triples that are not contained in \mathcal{S}_t .*
2. *If \mathcal{S}_p has infinitely many pairs, there are infinitely many triples that are not contained in \mathcal{S}_t .*
3. *If \mathcal{S}_t has infinitely many triples, there are infinitely many pairs that are not contained in \mathcal{S}_p .*
4. *If $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ is a maximal *nepfi* and \mathcal{S}_t has finitely many triples, there are only finitely many pairs that are not contained in \mathcal{S}_p .*

Proof:

1. Take for instance $\mathcal{S}_p := \{\{a, b\}\}$, the set of triples $\{\{a, b, y\} : y \in \mathbb{N} - \{a, b\}\}$ is infinite and $\{\{a, b, y\} : y \in \mathbb{N} - \{a, b\}\} \cap \mathcal{S}_t = \emptyset$.
2. Follows straightforwardly from case [\(1\)](#).
3. For every triple $\{x, y, z\} \in \mathcal{S}_t$, the pairs $\{x, y\}, \{x, z\}, \{y, z\}$ are not in \mathcal{S}_p . Since \mathcal{S}_t is infinite, \mathcal{Y}_p is also infinite. Thus we obtain infinitely many pairs that are not in \mathcal{S}_p .
4. Since there are only finitely many triples in \mathcal{S}_t , we obtain only finitely many pairs in \mathcal{Y}_p and these pairs that are not in \mathcal{S}_p . By maximality of \mathcal{S} those are the only pairs that are not in \mathcal{S}_p .

\square

5.5.43. LEMMA. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a maximal nepfi family. For every $x \in \mathbb{N}$, there is an infinite subfamily $\mathcal{S}_x \subseteq \mathcal{S}$ such that all languages in \mathcal{S}_x contain x .*

Proof:

Let $x \in \mathbb{N}$. Using maximality of \mathcal{S} with respect to nepfi, we will construct an infinite subfamily $\mathcal{S}_x \subseteq \mathcal{S}$ such that any language in \mathcal{S}_x contains x . For this, first note that either $\{x, y_1\} \in \mathcal{S}_p$ or $\{x, y_1, z_1\} \in \mathcal{S}_t$ (but not both since \mathcal{S} is an antichain) for some $y_1 \neq z_1 \in \mathbb{N}$ such that $x \neq y_1$ and $x \neq z_1$. Otherwise, we could add the singleton $\{x\}$ to \mathcal{S} , contradicting that \mathcal{S} is maximal nepfi. W.l.o.g suppose $\{x, y_1\} \in \mathcal{S}_p$. Now let $y_2 \in \mathbb{N}$ such that $y_2 \notin \{x, y_1\}$, by maximality of \mathcal{S} we have that either $\{x, y_2\} \in \mathcal{S}_p$ or $\{x, y_2, z_2\} \in \mathcal{S}_t$ (but not both) such that $y_2 \neq z_2 \in \mathbb{N}$ and $z_2 \notin \{x, y_1\}$. Otherwise, we could add either $\{x, y_2\}$ or $\{x, y_2, z_2\}$ to \mathcal{S} without impairing the anti-chain condition, contradicting that \mathcal{S} is maximal nepfi. W.l.o.g suppose $\{x, y_2, z_2\} \in \mathcal{S}_t$.

By an inductive argument, we can implement the previous procedure for any $y_m \in \mathbb{N}$ with $m \geq n$ and $y_m \notin \{x, y_1, y_2, z_2, \dots, y_m\}$, such that either $\{x, y_{m+1}\} \in \mathcal{S}_p$ or $\{x, y_{m+1}, z_{m+1}\} \in \mathcal{S}_t$ with $y_{m+1} \neq z_{m+1}$ and $z_{m+1} \notin \{x, y_1, y_2, z_2, \dots, y_m\}$. W.l.o.g suppose $\{x, y_{m+1}\} \in \mathcal{S}_p$.

Therefore, $\mathcal{S}_x := \{\{x, y_1\}, \{x, y_2, z_2\}, \dots, \{x, y_{m+1}\}, \dots\}$ is the desired subfamily. \square

5.5.44. PROPOSITION. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a maximal nepfi family and $\{a, b\} \in \mathcal{S}_p$. The family $\mathcal{S}' := \mathcal{S} - \{\{a, b\}\}$ has two maximal nepfi extensions.*

Proof:

Consider the family of triples that contains the pair $\{a, b\}$, namely $\mathcal{AB} := \{\{a, b, y\} : \{a, b\}, \{b, y\}, \{a, y\} \notin \mathcal{S}_p\}$. By maximality and Lemma [5.5.43](#), it is easy to see that adding any set that is not in $\{\{a, b\}\} \cup \mathcal{AB}$ to \mathcal{S}' will impair the anti-chain condition. Clearly the only two maximal extensions of $\mathcal{S}' := \mathcal{S} - \{\{a, b\}\}$ are \mathcal{S} and $\mathcal{S}' \cup \mathcal{AB}$. \square

5.5.45. PROPOSITION. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a maximal nepfi family and $\{a, b, c\} \in \mathcal{S}_t$. The family $\mathcal{S}' := \mathcal{S} - \{\{a, b, c\}\}$ has two maximal nepfi extensions.*

Proof:

By maximality and Lemma [5.5.43](#), it is easy to see that adding any set that is not in $\{\{a, b, c\}\} \cup \text{PAIRS}(\{a, b, c\})$ to \mathcal{S}' will violate the anti-chain condition. Thus, the only two maximal extensions of $\mathcal{S}' := \mathcal{S} - \{\{a, b, c\}\}$ are \mathcal{S} and $\mathcal{S}' \cup \text{PAIRS}(\{a, b, c\})$. \square

5.5.46. LEMMA. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a maximal nepfi family and let $\{a, b\} \in \mathcal{S}_p$. For any $y \in \mathbb{N}$, either $\{a, y\} \in \mathcal{S}_p$ or $\{a, y, z\} \in \mathcal{S}_t$ for some $z \in \mathbb{N}$. Similarly, either $\{b, y\} \in \mathcal{S}_p$ or $\{b, y, z\} \in \mathcal{S}_t$ for some $z \in \mathbb{N}$.*

Proof:

Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a maximal nepfi family and let $\{a, b\} \in \mathcal{S}_p$. Let $y \in \mathbb{N}$ and towards a contradiction, suppose $\{a, y\} \notin \mathcal{S}_p$ and $\{a, y, z\} \notin \mathcal{S}_t$ for all $z \in \mathbb{N}$. Then we can extend \mathcal{S} with the pair $\{a, y\}$ or with triples of the form $\{a, y, z\}$ with $z \in \mathbb{N}$ contradicting nepfi maximality of \mathcal{S} . Thus, either $\{a, y\} \in \mathcal{S}_p$ or $\{a, y, z\} \in \mathcal{S}_t$ for some $z \in \mathbb{N}$. Follows similarly for $\{b, y\}$, $\{b, y, z\}$. \square

5.5.47. PROPOSITION. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a maximal nepfi family. Let $\mathcal{F} := \{\{a_1, b_1\}, \dots, \{a_n, b_n\}\} \subseteq \mathcal{S}_p$ be a finite set of pairs (analogously, \mathcal{F} be a finite set of triples contained in \mathcal{S}_t). The family $\mathcal{S}' = \mathcal{S} - \mathcal{F}$ has finitely many maximal nepfi extensions.*

Proof:

Let $\mathcal{F} := \{\{a_1, b_1\}, \dots, \{a_n, b_n\}\} \subseteq \mathcal{S}_p$ be a finite set of pairs. We investigate which languages we can add to $\mathcal{S}' = \mathcal{S} - \mathcal{F}$ without impairing the anti-chain condition. First note that by Lemma 5.5.46, we can add to \mathcal{S}' a triple of the form $\{a_i, b_i, y\}$ only if $\{a_i, y\}, \{b_i, y\}$ are not in \mathcal{S}_p . We know by Proposition 5.5.44, that the only way to extend $\mathcal{S} - \{\{a, b\}\}$ for any $\{a, b\} \in \mathcal{F}$ and obtain a maximal anti-chain is by adding $\{a, b\}$ or adding the whole family $\mathcal{AB} := \{\{a, b, y\} : \{a, b\}, \{b, y\}, \{a, y\} \notin \mathcal{S}_p\}$. Thus, by a similar argument, it is easy to see that we can only add to $\mathcal{S} - \mathcal{F}$ combinations of pairs in \mathcal{F} and families of the form $\mathcal{AB}_i := \{\{a_i, b_i, y\} : \{a_i, b_i\} \in \mathcal{F} \text{ and } \{b, y\}, \{a, y\} \notin \mathcal{S}_p\}$ without impairing the anti-chain condition. Thus by a combinatorial argument, $\mathcal{S} - \mathcal{F}$ has only finitely many maximal nepfi extensions.

To illustrate the strategy of the proof for the case when $\mathcal{F} \subseteq \mathcal{S}_p$, here we just show it for a set of two distinctive pairs $\mathcal{F} = \{\{a, b\}, \{c, d\}\}$. Since \mathcal{S} is an anti-chain, we have that $\mathcal{AB} := \{\{a, b, y\} : \{a, b\}, \{b, y\}, \{a, y\} \notin \mathcal{S}_p\} \cap \mathcal{S}_t = \emptyset$ and $\mathcal{CD} := \{\{c, d, y\} : \{c, d\}, \{c, y\}, \{d, y\} \notin \mathcal{S}_p\} \cap \mathcal{S}_t = \emptyset$. The only maximal anti-chain extensions of $\mathcal{S} - \mathcal{F}$ are the following four families: \mathcal{S} ,

$$(\mathcal{S} - \mathcal{F}) \cup \{\{a, b\}\} \cup \mathcal{CD},$$

$$(\mathcal{S} - \mathcal{F}) \cup \{\{c, d\}\} \cup \mathcal{AB},$$

and $(\mathcal{S} - \mathcal{F}) \cup \mathcal{AB} \cup \mathcal{CD}$. To see this, we prove the following claim:

Claim: let $Z \subseteq \mathbb{N}$ be not a language in \mathcal{S}' such that $Z \notin \mathcal{F}$, $Z \notin \mathcal{AB}$ and $Z \notin \mathcal{CD}$, then $(\mathcal{S} - \mathcal{F}) \cup \{Z\}$ is not an anti-chain.

First note that if Z is a singleton, by maximality of \mathcal{S} and Lemma 5.5.43, there must be a pair or a triple in $(\mathcal{S} - \mathcal{F})$ that contains Z . Thus, the anti-chain

condition would be impaired if we add Z . If Z is a pair not contained in \mathcal{F} , by maximality we have that either $Z \in \mathcal{S}_p - \mathcal{F}$ or there is a triple $T \in \mathcal{S}_t$ such that $Z \subseteq T$. In both cases the anti-chain condition would be impaired if we add Z . If Z is a triple so that $Z \notin \mathcal{AB} \cup \mathcal{CD}$, by maximality of \mathcal{S} there is a pair P in $\mathcal{S}_p - \mathcal{F}$ such that $P \subseteq Z$. Thus, the anti-chain condition does not hold or Z is already in \mathcal{S}_t contradicting our assumption about Z .

Suppose $Z = \{z_1, z_2, z_3, z_4\}$ and let $\mathcal{G}_3(Z)$ be the smallest 3-cluster with elements in Z . We have two cases: 1) there is $x \in Z$ such that $x \notin \{a, b, c, d\}$, and 2) $Z = \{a, b, c, d\}$. For case 1), suppose w.l.o.g. that $Z = \{a, b, c, x\}$ with $x \neq d$. Thus by maximality of \mathcal{S} there is a triple $\{z_1, z_2, x\} \in \mathcal{G}_3(Z)$ such that either $\{z_1, z_2, x\} \in \mathcal{S}_t$ or $\text{PAIRS}(\{z_1, z_2, x\}) \cap \mathcal{S}_p \neq \emptyset$. In both cases there is a language of $\mathcal{S} - \mathcal{F}$ contained in Z , thus $(\mathcal{S} - \mathcal{F}) \cup \{Z\}$ is not an anti-chain. For case 2), we have by maximality of \mathcal{S} that, for instance, the pair $\{a, c\} \in (\mathcal{S} - \mathcal{F})$ or a triple of the form $\{a, c, y\}$ for some $y \in \mathbb{N}$ is in $(\mathcal{S} - \mathcal{F})$ (and similarly for the pairs $\{b, c\}$, $\{a, d\}$ and $\{b, d\}$). Thus $(\mathcal{S} - \mathcal{F}) \cup \{Z\}$ is not an anti-chain.

The argument for when $\mathcal{F} \subseteq \mathcal{S}_t$ will follow in a similar way (noting the argument in Proposition 5.5.45) and it also suffices to showcase the argument for when \mathcal{F} has only two triples. \square

5.5.48. COROLLARY. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a maximal nepfi family and $\mathcal{G} := \{\mathcal{G}_1, \dots, \mathcal{G}_n\}$ a finite set of finite 2-clusters in \mathcal{S}_p (analogously, a finite set of finite 3-clusters in \mathcal{S}_t). The family $\mathcal{S}' := \mathcal{S} - \mathcal{G}$ has finitely many maximal extensions.*

Proof:

Follows by Proposition 5.5.47 with $\mathcal{F} := \bigcup_{i=1}^n \mathcal{G}_i$. \square

Observe that we still need to analyse the case when \mathcal{F} is a finite set of both pairs and triples. Before addressing that, we need to study the case when we take out families of the form $\mathcal{KM}_i := \{\{k_i, m_i, a\} : a \in \mathbb{N} - \{k_i, m_i\}\} \subseteq \mathcal{S}_t$ for some pair $\{k_i, m_i\} \in \mathcal{S}^2$. We will discuss these cases further on, in Proposition 5.5.55 and Proposition 5.5.56.

First we will see, in the following propositions, the case when we take out an infinite family of finite 2-clusters or an infinite 2-cluster (or an infinite family of finite 3-clusters or an infinite 3-cluster respectively) from a maximal family $\mathcal{S} = \mathcal{S}_p \cup \mathcal{S}_t$.

5.5.49. PROPOSITION. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a maximal nepfi family and $\mathcal{G} := \{\mathcal{G}_1, \mathcal{G}_2, \dots\}$ a countable family of finite disjoint 2-clusters contained in \mathcal{S}_p (analogously, countably many finite 3-clusters in \mathcal{S}_t). The family $\mathcal{S}' := \mathcal{S} - \mathcal{G}$ has uncountably many maximal extensions.*

Proof:

Follows by the same reasoning as in Proposition 5.5.16 and as in case (2.) from Proposition 5.5.31. \square

5.5.50. PROPOSITION. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a maximal nepfi family and \mathcal{G} an infinite 2-cluster contained in \mathcal{S}_p (analogously, an infinite 3-cluster in \mathcal{S}_t). The family $\mathcal{S}' := \mathcal{S} - \mathcal{G}$ has uncountably many maximal extensions.*

Proof:

The proof uses the same argument as in the proof for the case (1.) in Proposition 5.5.16. \square

Recall from Section 5.5.2 when we study families of pairs that result by removing from \mathcal{S}^2 families of the form $\{\{k, x\} : x \in \mathbb{N} - \{k\}\}$ for a fixed $k \in \mathbb{N}$. We will now analyze similar cases for families containing pairs and triples. Given a maximal nepfi family $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$, we want to analyze how many maximal nepfi extensions we can recover for a family of the form $\mathcal{S}' := \mathcal{S} - \{\{k, x\} : x \in \mathbb{N} - \{k\}\}$ for a fixed $k \in \mathbb{N}$. First we need the following lemma.

5.5.51. LEMMA. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a maximal nepfi family, let $\mathcal{Y}_p := \{\{b, c\} : \{y, b, c\} \in \mathcal{S}_t \text{ for some } y \in \mathbb{N}\}$ and for some fixed $k \in \mathbb{N}$ the family $\mathcal{K}_p := \{\{k, x\} : x \in \mathbb{N} - \{k\}\} \subseteq \mathcal{S}_p$. Then for every $\mathcal{B} \in \mathcal{P}(\mathcal{Y}_p)$, the family $\mathcal{K}_t^{\mathcal{B}} := \{\{k, b, c\} : \{b, c\} \in \mathcal{B} \text{ and } b, c \neq k\}$ is such that $\mathcal{K}_t^{\mathcal{B}} \cap \mathcal{S}_t = \emptyset$.*

Proof:

First note that \mathcal{Y}_p is precisely the set of all pairs that are not in \mathcal{S} . This is because if $P \in \mathcal{Y}_p$ then there is a triple $T \in \mathcal{S}$ such that $P \subseteq T$, i.e., $\mathcal{Y}_p = \mathcal{S}^2 - \mathcal{S} = \mathcal{S}^2 - \mathcal{S}_p$. Let $\mathcal{B} \in \mathcal{P}(\mathcal{Y}_p)$ be arbitrary. Note that for every pair $\{b, c\} \in \mathcal{B}$ such that $b, c \neq k$, the pairs $\{k, b\}, \{k, c\} \in \mathcal{K}_p \subseteq \mathcal{S}_p$. Therefore any triple of the form $\{k, b, c\}$ with $b, c \in \mathcal{B}$ is not in \mathcal{S}_t since otherwise the anti-chain condition will be impaired. Therefore $\mathcal{K}_t^{\mathcal{B}} \cap \mathcal{S}_t = \emptyset$. \square

5.5.52. PROPOSITION. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a maximal nepfi family and for some fixed $k \in \mathbb{N}$, $\mathcal{K}_p := \{\{k, x\} : x \in \mathbb{N} - \{k\}\} \subseteq \mathcal{S}_p$. We obtain the following:*

1. *If \mathcal{S}_t is infinite, the subfamily $\mathcal{S}' := \mathcal{S} - \mathcal{K}_p$ has uncountably many maximal extensions.*
2. *If \mathcal{S}_t is finite, the subfamily $\mathcal{S}' := \mathcal{S} - \mathcal{K}_p$ has finitely many maximal extensions.*

Proof:

1. Since \mathcal{S}_t is infinite, by Lemma 5.5.42 we have infinitely many pairs in $\mathcal{Y}_p := \{\{b, c\} : \{y, b, c\} \in \mathcal{S}_t \text{ for some } y \in \mathbb{N}\}$ that are not in \mathcal{S}_p . Thus, we have uncountably many sets $\mathcal{B} \in \mathcal{P}(\mathcal{Y}_p)$ such that $\mathcal{B} \cap \mathcal{S}_p = \emptyset$. Thus, for any $\mathcal{B} \subseteq \mathcal{Y}_p$, the family $\mathcal{S}' \cup \mathcal{K}_t^{\mathcal{B}} \cup \{\{k, x\} : x \in \mathbb{N} - NUM(\mathcal{B})\}$ is a maximal nepfi extension of \mathcal{S} . Therefore, we obtain uncountably many maximal extensions.
2. First note that we can only add to \mathcal{S}' a set of the form $\{k\}$, $\{k, x\}$ or $\{k, x, y\}$ without impairing the anti-chain condition. By maximality of \mathcal{S} , any other triple will be in \mathcal{S}_t or will contain a pair that is in \mathcal{S}'_p . It follows that we cannot add any language larger than a triple since adding it will impair the anti-chain condition. Now let us proceed by counting the maximal nepfi extensions. Clearly \mathcal{S} is a maximal nepfi extension of \mathcal{S}' . By lemma 5.5.51, the structure of \mathcal{S}' and our initial observation, the family $\mathcal{S}' \cup \{k\}$ is also a maximal nepfi extension. By our initial observation and since we have only finitely many $\mathcal{B} \subseteq \mathcal{Y}_p$, we have only finitely many maximal nepfi extensions of the form $\mathcal{S}' \cup \mathcal{K}_t^{\mathcal{B}} \cup \{\{k, x\} : x \in \mathbb{N} - NUM(\mathcal{B})\}$. Altogether we obtain only finitely many maximal nepfi extensions for \mathcal{S}' .

□

The following result is the corresponding counterpart of Proposition 5.5.5.

5.5.53. PROPOSITION. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a maximal nepfi family and for at least two $k_1, k_2 \in \mathbb{N}$, the family $\mathcal{K}_{ip} := \{\{k_i, x\} : x \in \mathbb{N} - \{k_i\}\} \subseteq \mathcal{S}_p$ for $i \in \{1, 2\}$. The family $\mathcal{S}' := \mathcal{S} - \bigcup_{i \in \{1, 2\}} \mathcal{K}_{ip}$ has uncountably many maximal extensions.*

Proof:

The proof follows by a similar argument as in the proof of Proposition 5.5.5 (and Example 5.5.4). Note that any triple of the form $\{k_1, k_2, x\}$ with $x \in \mathbb{N}$ is not in \mathcal{S}' . Moreover, the pairs $\{k_1, k_2\}$, $\{k_1, x\}$, $\{k_2, x\}$ are not in \mathcal{S}' . Thus, for every $B \subseteq \mathbb{N}$, the family $\mathcal{S}' \cup \{\{k_1, b\} : b \in B - \{k_1\}\} \cup \{\{k_1, k_2, c\} : c \in \mathbb{N} - (B \cup \{k_1, k_2\})\}$ is a maximal nepfi extension of \mathcal{S}' . We obtain uncountably many maximal nepfi extensions of \mathcal{S}' . □

Recall from Example 5.5.3 and the discussion that followed, that we saw a big difference between families of the form

$$\mathcal{S}^3 - \{\{k, x, y\} : x, y \in \mathbb{N} - \{k\}\} \text{ for a fixed } k \in \mathbb{N}$$

and families of the form

$$\mathcal{S}^3 - \{\{k, m, y\} : y \in \mathbb{N} - \{k, m\}\} \text{ for a fixed pair } \{k, m\} \in \mathcal{S}^2,$$

when counting their maximal nepfi extensions. We now address these cases for families of pairs and triples. Given a maximal nepfi family $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$, we will see how many maximal nepfi extensions we can obtain for families of the form $\mathcal{S} - \{\{k, x, y\} : x, y \in \mathbb{N} - \{k\}\}$ for a fixed $k \in \mathbb{N}$ and for families of the form $\mathcal{S} - \{\{k, m, y\} : y \in \mathbb{N} - \{k, m\}\}$ for a fixed pair $\{k, m\} \in \mathcal{S}^2$.

5.5.54. PROPOSITION. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a maximal nepfi family and for some $k \in \mathbb{N}$ the family $\mathcal{K}_t := \{\{k, x, y\} : x, y \in \mathbb{N} - \{k\}\} \subseteq \mathcal{S}_t$ for a fixed $k \in \mathbb{N}$. The family $\mathcal{S}' := \mathcal{S} - \mathcal{K}_t$ has uncountably many maximal extensions.*

Proof:

First note that if $\{k, x, y\} \in \mathcal{S}_t$, the pairs $\{k, x\}, \{k, y\}, \{x, y\} \notin \mathcal{S}_p$. Therefore we can add these pairs to \mathcal{S}' . Moreover, by maximality of \mathcal{S} , we can only add to \mathcal{S}' a set of the form $\{k\}, \{k, x\}$ or $\{k, x, y\}$. Let $B \subseteq \mathbb{N}$, by a similar argument as in the proof of case (4) from Proposition 5.5.31 (and the proof of Proposition 5.5.5), the family $\mathcal{S}' \cup \{\{k, x, y\} : x, y \in B - \{k\}\} \cup \{\{k, z\} : z \in \mathbb{N} - (B \cup \{k\})\}$ is a maximal nepfi extension for \mathcal{S}' . Thus, we obtain uncountably many maximal nepfi extensions of \mathcal{S}' . \square

5.5.55. PROPOSITION. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a maximal nepfi family and $\mathcal{KM}_i := \{\{k_i, m_i, a\} : a \in \mathbb{N} - \{k_i, m_i\}\} \subseteq \mathcal{S}_t$ for some pair $\{k_i, m_i\} \in \mathcal{S}^2$. The following follows:*

1. *The family $\mathcal{S}' := \mathcal{S} - \bigcup_{i=1}^N \{\{k_i, m_i, a\} : a \in \mathbb{N} - \{k_i, m_i\}\}$ for a finite set of pairs $\{\{k_i, m_i\} : N \in \mathbb{N}, 1 \leq i \leq N\}$ has finitely many maximal nepfi extensions.*
2. *The family $\mathcal{S}' := \mathcal{S} - \bigcup_{i=1}^{\infty} \{\{k_i, m_i, a\} : a \in \mathbb{N} - \{k_i, m_i\}\}$ for infinitely many pairs $\{k_i, m_i\} \in \mathcal{S}^2$ has uncountably many maximal nepfi extensions.*

Proof:

First note that if $\{k, x, y\} \in \mathcal{S}_t$, the pairs $\{k, x\}, \{k, y\}, \{x, y\} \notin \mathcal{S}_p$. So we can add these pairs to \mathcal{S}' .

1. Follows by the same reasoning as in the proof of Proposition 5.5.29.
2. Follows by the same reasoning as in the proof of case (3) in Proposition 5.5.31.

\square

Now we address the case when we take out finitely many of both pairs and triples from a maximal nepfi family.

5.5.56. PROPOSITION. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a maximal nepfi family. Let $\mathcal{F} := \mathcal{F}_p \cup \mathcal{F}_t \subseteq \mathcal{S}_p \cup \mathcal{S}_t$ be a finite set of pairs and triples. The family $\mathcal{S} - \mathcal{F}$ has finitely many maximal nepfi extensions.*

Proof:

The proof follows by a similar argument as in the proof of Proposition 5.5.47 but using case (1.) from Proposition 5.5.55. We do this by simplifying the problem, namely we will check how many maximal nepfi extensions a certain “simpler” family has. The simpler family results from taking out only triples from a maximal nepfi family that is very similar to our original family \mathcal{S} .

Let $\mathcal{F}_p := \{\{a_1, b_1\}, \dots, \{a_n, b_n\}\}$. Recall from Proposition 5.5.47 that for every $\{a_i, b_i\} \in \mathcal{F}_p$, there is a family $\mathcal{AB}_i := \{\{a_i, b_i, y\} : \{a_i, b_i\}, \{b_i, y\}, \{a_i, y\} \notin \mathcal{S}_p\}$ such that $\mathcal{S}_t \cap \mathcal{AB}_i = \emptyset$. Now consider the maximal nepfi family $\mathcal{S}' = (\mathcal{S} - \mathcal{F}_p) \cup \bigcup_{i=1}^n \mathcal{AB}_i$ and let $\mathcal{F}_t^+ := \mathcal{F}_t \cup \bigcup_{i=1}^n \mathcal{AB}_i$. Note that $\mathcal{S}' - \mathcal{F}_t^+ = \mathcal{S} - (\mathcal{F}_p \cup \mathcal{F}_t) = \mathcal{S} - \mathcal{F}$. By Proposition 5.5.47 and case (1.) from Proposition 5.5.55 we have that $\mathcal{S}' - \mathcal{F}_t^+$ has finitely many maximal nepfi extensions. Thus $\mathcal{S} - \mathcal{F}$ has only finitely many as well. \square

To finish with our analysis of subfamilies of pairs and triples, we need to analyse families that result from taking out infinitely many pairs and triples from a maximal nepfi family that we did not cover in previous propositions. In particular, we rule out all the cases in which the resulting family will have uncountably many maximal nepfi extensions. For our purpose, we prove two propositions, the first when the set of triples \mathcal{S}_t is finite and the second when \mathcal{S}_t is infinite.

The following Lemma will be useful for the case when \mathcal{S}_t is finite.

5.5.57. LEMMA. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ be a maximal nepfi family such that \mathcal{S}_t is finite and let $\mathcal{X} := \mathcal{X}_p \cup \mathcal{X}_t$ be an infinite family such that $\mathcal{X} \subseteq \mathcal{S}$ and the following conditions are satisfied:*

1. \mathcal{X}_p contains only finitely many 2-clusters, the family of such 2-clusters is denoted by \mathcal{G}_2 ;
2. $\mathcal{S}_p - \mathcal{X}_p$ has no 2-clusters;
3. \mathcal{X}_t contains only finitely many 3-clusters, the family of such 3-clusters is denoted by \mathcal{G}_3 ;
4. $\mathcal{S}_t - \mathcal{X}_t$ has no 3-clusters;
5. \mathcal{X}_p contains at most one family of the form $\mathcal{K}_p := \{\{k, x\} : x \in \mathbb{N} - \{k\}\}$ for some $k \in \mathbb{N}$; and
6. \mathcal{X}_t contains no family of the form $\mathcal{K}_t := \{\{k, x, y\} : x, y \in \mathbb{N} - \{k\}\}$ for any $k \in \mathbb{N}$.

We can construct only finitely many maximal anti-chains that extend $\mathcal{S} - \mathcal{X}$ without impairing the anti-chain condition using specifically the following languages: languages in $\mathcal{X} - (\mathcal{K}_p \cup \mathcal{G}_2 \cup \mathcal{G}_3)$, pairs in \mathcal{K}_p , the singleton $\{k\}$, languages in $NUM(\mathcal{G}_2^i)$ for some $\mathcal{G}_2^i \in \mathcal{G}_2$ and languages in $NUM(\mathcal{G}_3^i)$ for some $\mathcal{G}_3^i \in \mathcal{G}_3$.

Proof:

It suffices to prove the case when \mathcal{X}_p contains one family of the form $\mathcal{K}_p := \{\{k, x\} : x \in \mathbb{N} - \{k\}\}$ for some $k \in \mathbb{N}$. First we will analyse why we can add the aforementioned languages to the family $\mathcal{S} - \mathcal{X}$ without impairing the anti-chain condition. Then we will see that there are only finitely many maximal anti-chains formed with such sets that can extend $\mathcal{S} - \mathcal{X}$. In what follows, when we say that “we can add a language Z to $\mathcal{S} - \mathcal{X}$ ” we will always mean that *we can add the language Z to $\mathcal{S} - \mathcal{X}$ without impairing the anti-chain condition.*

Now we proceed with the proof. Let $\mathcal{G}_2 := \{\mathcal{G}_2^i : N \in \mathbb{N}^+, 1 \leq i \leq N\}$ and let $\mathcal{G}_3 := \{\mathcal{G}_3^i : M \in \mathbb{N}^+, 1 \leq i \leq M\}$. Note that for every 2-cluster $\mathcal{G}_2^i \in \mathcal{G}_2$, $\mathcal{G}_2^i \subseteq \mathcal{X} \subseteq \mathcal{S}$. Thus, any triple formed by elements in $NUM(\mathcal{G}_2^i)$ is not in \mathcal{S}_t . Otherwise, \mathcal{S} will not be an anti-chain. Note also that we can add such triples to $\mathcal{S} - \mathcal{X}$. Moreover, we can add any language in $\mathcal{P}(NUM(\mathcal{G}_2^i))$ that is not a singleton simply because any pair or triple obtained from elements in $NUM(\mathcal{G}_2^i)$ is not in $\mathcal{S} - \mathcal{X}$. This holds for every 2-cluster in \mathcal{G}_2 .

Analogously, note that for every 3-cluster $\mathcal{G}_3^i \in \mathcal{G}_3$, $PAIRS(\mathcal{G}_3^i) \cap \mathcal{S}_p = \emptyset$. So we can add the whole family $PAIRS(\mathcal{G}_3^i)$ to $\mathcal{S} - \mathcal{X}$. Then it follows by a similar argument as the one before, that we can add any language in $\mathcal{P}(NUM(\mathcal{G}_3^i))$ that is not a singleton. This holds for every 3-cluster in \mathcal{G}_3 .

We shift our attention to the singleton $\{k\}$. Note that since $\mathcal{K}_p := \{\{k, x\} : x \in \mathbb{N} - \{k\}\} \subseteq \mathcal{X} \subseteq \mathcal{S}$, any triple that contains k is not in \mathcal{S}_t . Therefore we can add the singleton $\{k\}$ to $\mathcal{S} - \mathcal{X}$. Clearly, we can add \mathcal{X} to the family $\mathcal{S} - \mathcal{X}$.

Finally, note that since $\mathcal{P}(NUM(\mathcal{G}_2^i))$ is finite, it contains at most finitely many anti-chains and the same holds for $\mathcal{P}(NUM(\mathcal{G}_3^i))$.

Recall that $\mathcal{G}_2 := \{\mathcal{G}_2^i : N \in \mathbb{N}^+, 1 \leq i \leq N\}$ and $\mathcal{G}_3 := \{\mathcal{G}_3^i : M \in \mathbb{N}^+, 1 \leq i \leq M\}$. For every $\mathcal{G}_2^i \in \mathcal{G}_2$, let $\mathcal{C}_i = \{\mathcal{C}_j^i : 1 \leq j \leq n_i\}$ be the set of all maximal anti-chains in $\mathcal{P}(NUM(\mathcal{G}_2^i))$ that do not contain singletons and for every $\mathcal{G}_3^i \in \mathcal{G}_3$ let $\mathcal{H}_i = \{\mathcal{H}_k^i : 1 \leq k \leq m_i\}$ be the set of all maximal anti-chains in $\mathcal{P}(NUM(\mathcal{G}_3^i))$ that do not contain singletons. With all the previous observations considered, it is easy to see that the maximal anti-chains constructed with the aforementioned languages that we can add to $\mathcal{S} - \mathcal{X}$ are: \mathcal{X} itself, anti-chains of the form

$$\mathcal{X} - (\mathcal{K}_p \cup \mathcal{G}_2 \cup \mathcal{G}_3) \cup \{k\} \cup \bigcup_{i=1}^N \mathcal{C}_j^i \cup \bigcup_{i=1}^M \mathcal{H}_k^i$$

for some $\mathcal{C}_j^i \in \mathcal{C}_i$ and some $\mathcal{H}_k^i \in \mathcal{H}_i$; and anti-chains of the form

$$\mathcal{X} - (\mathcal{K}_p \cup \mathcal{G}_2 \cup \mathcal{G}_3) \cup \mathcal{K}_p \cup \bigcup_{i=1}^N \mathcal{C}_j^i \cup \bigcup_{i=1}^M \mathcal{H}_k^i$$

for some $\mathcal{C}_j^i \in \mathcal{C}_i$ and some $\mathcal{H}_k^i \in \mathcal{H}_i$. Obviously, these are only finitely many. \square

5.5.58. PROPOSITION. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ and let $\mathcal{X} := \mathcal{X}_p \cup \mathcal{X}_t \subseteq \mathcal{S}_p \cup \mathcal{S}_t$ be as in Lemma 5.5.57. The family $\mathcal{S}' := \mathcal{S} - \mathcal{X}$ has only finitely many maximal nepfi extensions.*

Proof:

It suffices to prove the case when \mathcal{X}_p contains one family of the form $\mathcal{K}_p := \{\{k, x\} : x \in \mathbb{N} - \{k\}\}$ for some $k \in \mathbb{N}$. We will prove that we can only add to $\mathcal{S}' := \mathcal{S} - \mathcal{X}$ languages in $\mathcal{X} - (\mathcal{K}_p \cup \mathcal{G}_2 \cup \mathcal{G}_3)$ pairs in \mathcal{K}_p , the singleton $\{k\}$, subsets of $NUM(\mathcal{G}_2^i)$ for some $\mathcal{G}_2^i \in \mathcal{G}_2$ that are not singletons, and subsets of $NUM(\mathcal{G}_3^i)$ for some $\mathcal{G}_3^i \in \mathcal{G}_3$ that are not singletons, so that the extension remains an anti-chain. Then, as in the proof for Lemma 5.5.57, the maximal nepfi extensions of $\mathcal{S}' := \mathcal{S} - \mathcal{X}$ are obtained by adding to it maximal anti-chains formed with the aforementioned languages. By Lemma 5.5.57, $\mathcal{S}' := \mathcal{S} - \mathcal{X}$ has only finitely many maximal extensions.

Let ADD be the family of all the languages described as in Lemma 5.5.57. In what follows, we will show that we can only extend $\mathcal{S}' := \mathcal{S} - \mathcal{X}$ with elements in ADD without impairing the anti-chain condition.

Let $Z \subseteq \mathbb{N}$ such that $Z \notin ADD$. We will show that $(\mathcal{S} - \mathcal{X}) \cup \{Z\}$ is not an anti-chain. Suppose Z is a singleton $\{z\} \neq \{k\}$. Note that by our assumption on \mathcal{X}_p , for all $k' \in \mathbb{N}$ such that $k' \neq k$, we have that $K'_p \cap (\mathcal{S}_p - \mathcal{X}_p) \neq \emptyset$ holds or, by maximality of \mathcal{S} , $K'_t \cap (\mathcal{S}_t - \mathcal{X}_t) \neq \emptyset$ holds. Thus there is a language in \mathcal{S} that contains Z , so $(\mathcal{S} - \mathcal{X}) \cup \{Z\}$ is not an anti-chain. Note that since \mathcal{K}_p is originally in \mathcal{S} (since it is in \mathcal{X}_p), the family $\mathcal{K}_t := \{\{k, x, y\} : x, y \in \mathbb{N} - \{k\}\}$ has empty intersection with \mathcal{X}_t , i.e., $\mathcal{K}_t \cap \mathcal{X}_t = \emptyset$. Suppose Z is a pair not in \mathcal{X}_p . By maximality of \mathcal{S} and Proposition 5.5.40, either $Z \in \mathcal{S}_p$ already or there is a triple $T \in \mathcal{S}_t$ such that $Z \subseteq T$. Thus adding Z impairs the anti-chain condition. Analogously, Z cannot be a triple that is not in \mathcal{X}_t (or in \mathcal{S}_t) since $(\mathcal{S} - \mathcal{X}) \cup \{Z\}$ is not an anti-chain. Suppose Z is a quadruple $\{z_1, z_2, z_3, z_4\}$ such that $Z \not\subseteq NUM(\mathcal{G}_2^i)$ for any $\mathcal{G}_2^i \in \mathcal{G}_2$ and $Z \not\subseteq NUM(\mathcal{G}_3^i)$ for any $\mathcal{G}_3^i \in \mathcal{G}_3$. Note that \mathcal{S}_p nor \mathcal{S}_t contain no 2-clusters or 3-clusters besides from the ones in \mathcal{G}_2 and in \mathcal{G}_3 . It follows that the 2-cluster and the 3-cluster of elements in Z , namely $\mathcal{G}_2(Z)$ and $\mathcal{G}_3(Z)$, are such that $\mathcal{G}_2(Z) \cap \mathcal{S}_p \neq \emptyset$ and $\mathcal{G}_3(Z) \cap \mathcal{S}_t \neq \emptyset$. Therefore there is a language in $\mathcal{S} = \mathcal{S}_p \cup \mathcal{S}_t$ that is contained in Z . Thus $(\mathcal{S} - \mathcal{X}) \cup \{Z\}$ is not an anti-chain. The argument follows similarly for any larger n -tuple Z such that $Z \not\subseteq NUM(\mathcal{G}_2^i)$ for any $\mathcal{G}_2^i \in \mathcal{G}_2$ and $Z \not\subseteq NUM(\mathcal{G}_3^i)$ for any $\mathcal{G}_3^i \in \mathcal{G}_3$. This shows that we can only extend $\mathcal{S} - \mathcal{X}$ with elements in ADD without impairing the anti-chain condition. \square

Now the remaining case, namely when \mathcal{S}_t is infinite. First we need the following lemma.

5.5.59. LEMMA. *Let \mathcal{S} be a maximal nepfi family of pairs and triples such that \mathcal{S}_t is infinite and let $\mathcal{X} := \mathcal{X}_p \cup \mathcal{X}_t$ be an infinite family such that $\mathcal{X} \subseteq \mathcal{S}$ and the following conditions are satisfied:*

1. \mathcal{X}_p contains only finitely many 2-clusters, the family of such 2-clusters is denoted by \mathcal{G}_2 ;
2. $\mathcal{S}_p - \mathcal{X}_p$ has no 2-clusters;
3. \mathcal{X}_t contains only finitely many 3-clusters, the family of such 3-clusters is denoted by \mathcal{G}_3 ;
4. $\mathcal{S}_t - \mathcal{X}_t$ has no 3-clusters;
5. \mathcal{X}_p does not contain any family of the form $\mathcal{K}_p := \{\{k, x\} : x \in \mathbb{N} - \{k\}\}$ for any $k \in \mathbb{N}$;
6. \mathcal{X}_t does not contain any family of the form $\mathcal{K}_t := \{\{k, x, y\} : x, y \in \mathbb{N} - \{k\}\}$ for any $k \in \mathbb{N}$; and
7. \mathcal{X}_t has at most finitely many families of the form $\mathcal{KM}_i := \{\{k_i, m_i, a\} : a \in \mathbb{N} - \{k_i, m_i\}\}$.

Let $\mathcal{KM}_1, \dots, \mathcal{KM}_N$ for some $N \in \mathbb{N}^+$ be the families contained in \mathcal{X}_t . We can construct only finitely many maximal anti-chains that extend $\mathcal{S} - \mathcal{X}$ without impairing the anti-chain condition using specifically the following languages: languages in $\mathcal{X} - (\bigcup_{i=1}^N \mathcal{KM}_i \cup \mathcal{G}_2 \cup \mathcal{G}_3)$, pairs in $\{k_i, m_i\}$, complete families of the form \mathcal{KM}_i , languages in $NUM(\mathcal{G}_2^i)$ for some $\mathcal{G}_2^i \in \mathcal{G}_2$ and languages in $NUM(\mathcal{G}_3^i)$ for some $\mathcal{G}_3^i \in \mathcal{G}_3$.

Proof:

First we will analyse why we can add the aforementioned languages to the family $\mathcal{S} - \mathcal{X}$ without impairing the anti-chain condition. Then we will see that there are only finitely many maximal anti-chains formed with such sets that can extend $\mathcal{S} - \mathcal{X}$.

In what follows, when we say that “we can add a language Z to $\mathcal{S} - \mathcal{X}$ ” we will always mean that *we can add the language Z to $\mathcal{S} - \mathcal{X}$ without impairing the anti-chain condition.*

Now we proceed with the proof. Clearly, we can add \mathcal{X} to the family $\mathcal{S} - \mathcal{X}$. Let $\mathcal{G}_2 := \{\mathcal{G}_2^i : M_2 \in \mathbb{N}^+, 1 \leq i \leq M_2\}$ and let $\mathcal{G}_3 := \{\mathcal{G}_3^i : M_3 \in \mathbb{N}^+, 1 \leq i \leq M_3\}$. Note that for every 2-cluster $\mathcal{G}_2^i \in \mathcal{G}_2$, any triple formed by elements in $NUM(\mathcal{G}_2^i)$ is not in \mathcal{S}_t , otherwise \mathcal{S} will not be an anti-chain. But note that we can add such triples to $\mathcal{S} - \mathcal{X}$. Moreover, we can add any language in $\mathcal{P}(NUM(\mathcal{G}_2^i))$ that is not a singleton simply because any pair or triple obtained from elements in $NUM(\mathcal{G}_2^i)$ is not in $\mathcal{S} - \mathcal{X}$. This holds for every 2-cluster in \mathcal{G}_2 . Analogously, note that for every 3-cluster $\mathcal{G}_3^i \in \mathcal{G}_3$, $PAIRS(\mathcal{G}_3^i) \cap \mathcal{S}_p = \emptyset$. So we can add

the whole family $PAIRS(\mathcal{G}_3^i)$ to $\mathcal{S} - \mathcal{X}$. Then it follows by a similar argument as the one before, that we can add any language in $\mathcal{P}(NUM(\mathcal{G}_3^i))$ that is not a singleton. This holds for every 3-cluster in \mathcal{G}_3 .

We shift our attention towards the families \mathcal{KM}_i with $i \in \{1, \dots, N\}$ and the corresponding pairs $\{k_i, m_i\}$. Note that since $\mathcal{KM}_i := \{\{k_i, m_i, x\} : x \in \mathbb{N} - \{k_i, m_i\}\} \subseteq \mathcal{X} \subseteq \mathcal{S}$, the pair $\{k_i, m_i\}$ is not in \mathcal{S}_p . Therefore, for every $i \in \{1, \dots, N\}$, we can add the pair $\{k_i, m_i\}$ or the family \mathcal{KM}_i (but not both simultaneously) to $\mathcal{S} - \mathcal{X}$. Moreover, we can add combinations of pairs $\{k_i, m_i\}$ and families \mathcal{KM}_j whenever $i \neq j$. Note that such combinations are just finitely many.

Finally, note that since $\mathcal{P}(NUM(\mathcal{G}_2^i))$ is finite, it contains at most finitely many anti-chains and the same holds for $\mathcal{P}(NUM(\mathcal{G}_3^i))$. Recall that $\mathcal{G}_2 := \{\mathcal{G}_2^i : M_2 \in \mathbb{N}^+, 1 \leq i \leq M_2\}$ and $\mathcal{G}_3 := \{\mathcal{G}_3^i : M_3 \in \mathbb{N}^+, 1 \leq i \leq M_3\}$. For every $\mathcal{G}_2^i \in \mathcal{G}_2$, let $\mathcal{C}_i = \{\mathcal{C}_j^i : 1 \leq j \leq n_i\}$ be the set of all maximal anti-chains in $\mathcal{P}(NUM(\mathcal{G}_2^i))$ that do not contain singletons and for every $\mathcal{G}_3^i \in \mathcal{G}_3$ let $\mathcal{H}_i = \{\mathcal{H}_k^i : 1 \leq k \leq m_i\}$ be the set of all maximal anti-chains in $\mathcal{P}(NUM(\mathcal{G}_3^i))$ that do not contain singletons. With all the previous observations considered, it is easy to see that the maximal anti-chains constructed with the aforementioned languages and that we can add to $\mathcal{S} - \mathcal{X}$ are: \mathcal{X} itself, anti-chains of the form

$$\mathcal{X} - \left(\bigcup_{i=1}^N \mathcal{KM}_i \cup \mathcal{G}_2 \cup \mathcal{G}_3 \right) \cup \bigcup_{i=1}^N \mathcal{KM}_i \cup \bigcup_{i=1}^{M_2} \mathcal{C}_j^i \cup \bigcup_{i=1}^{M_3} \mathcal{H}_k^i$$

for some $\mathcal{C}_j^i \in \mathcal{C}_i$ and some $\mathcal{H}_k^i \in \mathcal{H}_i$; anti-chains of the form

$$\mathcal{X} - \left(\bigcup_{i=1}^N \mathcal{KM}_i \cup \mathcal{G}_2 \cup \mathcal{G}_3 \right) \cup \bigcup_{i=1}^N \{\{k_i, m_i\}\} \cup \bigcup_{i=1}^{M_2} \mathcal{C}_j^i \cup \bigcup_{i=1}^{M_3} \mathcal{H}_k^i,$$

for some $\mathcal{C}_j^i \in \mathcal{C}_i$ and some $\mathcal{H}_k^i \in \mathcal{H}_i$; anti-chains of the form

$$\mathcal{X} - \left(\bigcup_{i=1}^N \mathcal{KM}_i \cup \mathcal{G}_2 \cup \mathcal{G}_3 \right) \cup \{k_i, m_i\} \cup \bigcup_{j \neq i}^N \mathcal{KM}_j \cup \bigcup_{i=1}^{M_2} \mathcal{C}_j^i \cup \bigcup_{i=1}^{M_3} \mathcal{H}_k^i,$$

for some $\mathcal{C}_j^i \in \mathcal{C}_i$ and some $\mathcal{H}_k^i \in \mathcal{H}_i$; anti-chains of the form

$$\mathcal{X} - \left(\bigcup_{i=1}^N \mathcal{KM}_i \cup \mathcal{G}_2 \cup \mathcal{G}_3 \right) \cup \bigcup_{j \neq i}^N \{\{k_j, m_j\}\} \cup \mathcal{KM}_i \cup \bigcup_{i=1}^{M_2} \mathcal{C}_j^i \cup \bigcup_{i=1}^{M_3} \mathcal{H}_k^i,$$

for some $\mathcal{C}_j^i \in \mathcal{C}_i$ and some $\mathcal{H}_k^i \in \mathcal{H}_i$; and anti-chains of the form

$$\mathcal{X} - \left(\bigcup_{i=1}^N \mathcal{KM}_i \cup \mathcal{G}_2 \cup \mathcal{G}_3 \right) \cup \bigcup_{j \in I_A} \{\{k_j, m_j\}\} \cup \bigcup_{i \in N - I_A} \mathcal{KM}_i \cup \bigcup_{i=1}^{M_2} \mathcal{C}_j^i \cup \bigcup_{i=1}^{M_3} \mathcal{H}_k^i,$$

for some $\mathcal{C}_j^i \in \mathcal{C}_i$, some $\mathcal{H}_k^i \in \mathcal{H}_i$, some set $\mathcal{A} \subseteq \{\{k_1, m_1\}, \dots, \{k_N, m_N\}\}$ and $I_{\mathcal{A}} := \{1 \leq j \leq N : \{k_j, m_j\} \in \mathcal{A}\}$. Clearly, there are only finitely many maximal anti-chains constructed in this way. \square

5.5.60. PROPOSITION. *Let $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ and $\mathcal{X} := \mathcal{X}_p \cup \mathcal{X}_t \subseteq \mathcal{S}_p \cup \mathcal{S}_t$ be as in Lemma 5.5.59. The family $\mathcal{S}' := \mathcal{S} - \mathcal{X}$ has only finitely many maximal nepfi extensions.*

Proof:

The proof follows the same strategy as in the one in the proof for Proposition 5.5.58. Namely, one needs to show that we can only add to $\mathcal{S}' := \mathcal{S} - \mathcal{X}$ languages as the ones described in Lemma 5.5.59. We call such family of languages the admissible languages and we denote it as ADD . So any language $Z \subseteq \mathbb{N}$ that is not in ADD cannot be added to $\mathcal{S} - \mathcal{X}$ without impairing the anti-chain condition. To illustrate, note that we cannot add any singleton $Z = \{z\}$ to $\mathcal{S} - \mathcal{X}$. This is because, by our initial assumption, \mathcal{X}_p and \mathcal{X}_t do not contain any families of the form \mathcal{K}_p and \mathcal{K}_t respectively for any $k \in \mathbb{N}$. Then, by maximality of \mathcal{S} , for every $k \in \mathbb{N}$ we have a pair or a triple in \mathcal{S}_p or \mathcal{S}_t respectively that contains k . In particular we have a pair or a triple in \mathcal{S}_p or \mathcal{S}_t respectively that contains z . By a case-by-case reasoning we can see that we cannot add any Z which is not in ADD . Therefore, by Lemma 5.5.59 $\mathcal{S}' := \mathcal{S} - \mathcal{X}$ has only finitely many maximal nepfi extensions. \square

By Proposition 5.5.40 to Proposition 5.5.60 we obtain the following result.

5.5.61. THEOREM. *Any family $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t \subseteq \mathcal{S}^2 \cup \mathcal{S}^3$ (of only pairs and triples) has only finitely many maximal nepfi extensions or uncountably many.*

Proof:

The strategy of the proof is to show that any family of pairs and triples $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ can be computed with a suitable maximal nepfi family of the form $\mathcal{S}' := \mathcal{S}'_p \cup \mathcal{S}'_t$ and a family of pairs and triples \mathcal{X} where $\mathcal{S}' - \mathcal{X} = \mathcal{S}$ is as described in the cases shown in Proposition 5.5.44 to Proposition 5.5.60.

First note that by Corollary 5.5.41, every family $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ that is not maximal nepfi, has a maximal nepfi extension of the form $\mathcal{S}' := \mathcal{S}'_p \cup \mathcal{S}'_t$ (so also of only pairs and triples). Therefore, we can compute any family $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ using a maximal nepfi family of the form $\mathcal{S}' := \mathcal{S}'_p \cup \mathcal{S}'_t$ and a family of pairs and triples \mathcal{X} so that $\mathcal{S}' - \mathcal{X} = \mathcal{S}$. Thus it suffices to argue about $\mathcal{S}' := \mathcal{S}'_p \cup \mathcal{S}'_t$ and \mathcal{X} for our purposes.

Note that any maximal family $\mathcal{S}' := \mathcal{S}'_p \cup \mathcal{S}'_t$ and any \mathcal{X} either have the properties described in some proposition from Proposition 5.5.44 to Proposition 5.5.56 or they do not. If they do not (i.e., if $\mathcal{S}' := \mathcal{S}'_p \cup \mathcal{S}'_t$ and \mathcal{X} are not as described in one of these propositions) then they must be as in Proposition

[5.5.58](#) or as in Proposition [5.5.60](#) (since the disjunction of these two corresponds to the negation of the disjunction of the propositions from Proposition [5.5.44](#) to Proposition [5.5.56](#)). Thus we cover all the possible cases of families of pairs and triples. By Proposition [5.5.44](#) to Proposition [5.5.60](#) we have that any family $\mathcal{S} := \mathcal{S}_p \cup \mathcal{S}_t$ of only pairs and triples has only finitely many maximal nepfi extensions or uncountably many. \square

We believe that generalizing the main results from this section with respect to nepfi subfamilies of $\mathcal{S}^n \cup \mathcal{S}^{n+1}$ will follow a similar strategy, guided by the same intuition. Due to the more complicated combinatorics involved, treating such families is not a simple task. Due to time constraints, we leave the study of the aforementioned families for future work.

5.6 Conclusions and future work

This chapter provides a detailed analysis and novel results on the structural difference between finite identification with positive data and with complete data. In particular, we study the structural properties of families which are finitely identifiable with positive data, in comparison with families that are only finitely identifiable with complete data. Our work solves questions that concern maximality with respect to finite identification. On one hand, we prove that there always exist a maximal non-effectively finitely identifiable family of an identifiable one that contains only finite languages. On the other hand, we prove that neither maximal cfi families nor maximal non-effective cfi families exist.

We also study the question that concerns the number of maximal finitely identifiable extensions with positive data. Expressly, for the cases of equinumerous families and for families which contain only pairs or triples. We prove that all these families have either finitely many or uncountably many maximal nepfi extensions. Directions of future work involve the general open question (with a combinatorial flavour) of how many maximal nepfi extensions a pfi family of finite languages has. In the light of our results, we conjecture that there are either finitely many or uncountably many. The next step is to investigate subfamilies containing only n -tuples and m -tuples for fixed numbers n and m . The answer for families of only pairs and triples support our conjecture for those. Then, it should follow an analysis of the question for families of languages of bounded cardinality, although dealing with such families involves a much more complex combinatorial analysis. We believe the answer will bring interesting insights, not only for finite identification but for discrete mathematics and combinatorics as well.

Some of the results presented in this chapter, use the fact that being an anti-chain of finite languages is a necessary and sufficient condition for a family to be nepfi. Recall that this is not the case when infinite languages are present in the family, as we saw for the family of co-singletons which is cfi only. A similar ques-

tion but concerning pfi is intriguing, namely, are anti-chains of finite languages always pfi? or at least cfi? In the following chapter (Chapter 6) we provide a first step in investigating the case of non-canonical anti-chains. We will present a, non-canonical but still computable, anti-chain of finite languages which is not pfi but also not cfi. Whether such an anti-chain exists which is cfi but not pfi is still an intriguing open question. Certainly a possible example will be hard to construct.

Chapter 6

Computational differences between pfi and cfi

6.1 Introduction

In this chapter, we study the computational differences and links between finite identification with positive and with complete data. Most of the results in Chapter 5 were with respect to nepfi of necfi families. Thus, the computational features of the family and *effectiveness* of its identification were being ignored. Here, we are interested in the computational properties of a family of languages and whether such properties allow pfi or cfi for the family in question. In particular, we analyze infinite anti-chains of finite languages, since for such cases it is not yet clear what the connection (or difference) is between cfi and pfi. As it happens, in most of the obvious examples of such anti-chains, both cfi and pfi hold (for instance, all the anti-chains of finite languages presented in Chapter 5). Moreover, for many maximal anti-chains of finite languages pfi holds. What can we say in general about these issues? Is every cfi anti-chain of finite languages pfi? Is every maximal anti-chain of finite languages pfi (or cfi)? In the sections that follow, we will provide negative answers to these questions.

We start with a short Section 6.2, where we extend the characterization theorem for finite identification of families with recursive languages in (Mukouchi, 1992; Lange and Zeugmann, 1992). For this, we prove a characterization theorem in Mukouchi style for finite identification of families with recursively enumerable languages. We will see that the *indexedness property* of the families in question is crucial for our result. Then, we focus on anti-chains of only singletons and pairs (i.e., nepfi families of singletons and pairs). Such families will be used to construct examples of the following kind:

- a canonical pfi family with no maximal pfi extension,
- a non-canonical pfi family,

- a non-canonical maximal nepfi family which is not pfi and not cfi, and
- a non-canonical anti-chain of finite languages which is cfi but not pfi.

Surprisingly, we will see in Sections [6.3.1](#), [6.3.2](#) and [6.4](#) that all such families exist. The simple structure of these families makes it easy to study their computational properties. Still, the analysis of these families is not a trivial matter. First, we focus on the computational properties of a subfamily of an, arbitrarily given, indexed nepfi family. In particular, we want to analyse cases when certain subfamilies of a given (ne)pfi family are decidable or recursively enumerable. As it happens in Chapter [5](#), some of the results in our analysis here are in terms of *maximal pfi* families. We show that for maximal pfi families with more than one pair, the set of singletons in the family is decidable. Since the examples in the list above are families containing more than one pair, we will mainly focus on those. We prove that such maximal families can be made canonical. Also, we investigate the close connection between the property of indexability of a given nepfi family and the r.e. property of a certain subfamily. We show that if an indexed family is nepfi then the subfamily of all its pairs is recursively enumerable.

Considering our list above, we present in Section [6.3.1](#) a family of finite languages coded standardly by canonical codes which does not have an *effectively finitely identifiable* maximal extension. Recall that infinite anti-chains of finite languages when the indexing of the languages is by canonical indices are always positively identifiable. In Section [6.3.2](#) we study non canonical families. We construct a non-canonical anti-chain which structural properties allow it to be pfi. We also present a non-canonical cfi family which is not an anti-chain and therefore is not pfi. Then, in Section [6.4](#), we exhibit a couple of examples of an indexable anti-chain of singletons and pairs that cannot be given a canonical indexing and is not pfi (and not cfi). One of these examples is a maximal anti-chain. We then present our example of a non-canonical anti-chain which is cfi but not pfi. This shows that finite identification with complete data of infinite anti-chains of finite languages is more powerful than with positive data only. This is a surprising result since families of finite languages which are anti-chains seem to be the domain of pfi par excellence. This result clearly shows the power of negative information in finite identification.

In Section [6.5](#) we study *fastest finite identification* which concerns a special kind of learner for finite identification, namely a *fastest learner*. Intuitively, a fastest learner identifies a language as soon as it is objectively certain which language it is. A family is positively identified in the fastest way iff all the DFTTs for all the languages in the family are *uniformly available and recognizable* by some recursive learning function. [Gierasimczuk and de Jongh \(2013\)](#) proved, by presenting a rather witty but complicated example, that fastest learning with positive data is more restrictive than pfi. Here, we present a much simpler example, that makes the distinction between these two ways of identification more transparent. Moreover, it provides a rather general way of constructing similar

examples. Then, we extend the definition in (Gierasimczuk and de Jongh, 2013), to reason about fastest learning with complete data. We show that every cfi anti-chain for which a fastest cfi learner exists, is also pfi. This means that in fastest learning there is no difference between cfi and pfi with respect to anti-chains of finite languages.

In Section 6.6 we were inspired by the work of Angluin (1987), where the problem of identifying languages from its members and non members by a learner asking queries is studied. In such a framework, the learner is allowed to produce queries and more than one conjecture, whereas the data presented by the teacher is meant to be, in a way, useful for the identification process. More precisely, the teacher presents the language to the learner by answering membership queries and testing its conjectures. We entertain a similar idea to study finite identification with a learner that can produce queries, *finite identification from queries* (in short, *learning from queries*). This learning notion seems to be closer to a more realistic learning scenario between a teacher and an *active learner* than the standard one in finite identification. By an active learner, we mean a learner that can ask questions to the teacher about the concept being learnt. The standard learner in cfi (and pfi) is, in some sense, a passive learner since it only receives the default information that nature (or teacher) fixed beforehand.

In our learning model, the learner receives answers from a teacher (or from nature) about her queries, and can produce either another query or a conjecture that concerns the target language. The identification process stops after one conjecture is produced. The learner can also abstain, in which case she will receive from the teacher a new element from a complete sequence chosen initially by the teacher from the target language. The *query learner* is a composition of two functions that act one after the other. One function produces the queries, and after the teacher gives an answer to her query, the second function produces a conjecture. This conjecture is based on the sequence of all the previous data from the teacher. We will see that a cfi learner and a query learner are very similar. In fact, we show that a family is cfi learnable if and only if it is learnable by queries. We also study a *strict* query learner, one that never abstains and always produces a query. We see that the class of families for which a strict query learner exists is also equivalent to cfi.

Outline

This chapter is structured as follows. In Section 6.2 we prove a characterization theorem in Mukouchi style for finite identification of families with recursive languages. In Section 6.3.1 we study families of singletons and pairs. Then we present in Section 6.3.1 a pfi anti-chain of singletons and pairs for which a maximal pfi extension never exists. In Section 6.3.2 we study non canonical families and prove that non-canonical pfi anti-chains of finite languages exist. In Section 6.4, we give

an example of an indexable non-canonical anti-chain of singletons and pairs that is not pfi and not cfi. Then, we construct an example of a non-canonical anti-chain of singletons and pairs that is cfi but not pfi. In Section 6.5 we study fastest learning with positive data and with complete data. We introduce and study learning by queries in Section 6.6. In Section 6.7 we present our final remarks and conclusions.

This Chapter is based on an unpublished manuscript (de Jongh and Vargas-Sandoval, 2020).

6.2 Extended Characterization Theorem

We dedicate this small section to a simple generalization of the Characterization Theorem from Mukouchi (1992), proved simultaneously by Lange and Zeugmann (1992), with respect to families which languages are recursively enumerable.

Recall that the aforementioned theorem, provides a characterization of finitely identifiable families with positive data and with complete data in terms of *definite tell-tale sets*. Before proving the theorem for families which languages are recursively enumerable, we recall the standard Characterization theorem in the style of Mukouchi (1992).

6.2.1. THEOREM (CHARACTERIZATION THEOREM).

- A family \mathcal{S} of languages is finitely identifiable with positive data (pfi) iff for every $S_i \in \mathcal{S}$ there is a DFTT set D_i obtainable in a uniformly computable way. That is, there exists an effective procedure Φ that on input i (index of S_i) produces the canonical index $\Phi(i)$ of some definite finite tell-tale set of S_i .
- A family \mathcal{S} of languages is finitely identifiable with complete data (cfi) iff for every $S_i \in \mathcal{S}$ there is a tell-tale pair (D_i, \overline{D}_i) in a uniformly computable way.

Proof:

See (Mukouchi, 1992, Theorem 7, p. 262) or (Lange and Zeugmann, 1992, Theorem 3, p. 382) for pfi and (Mukouchi, 1992, Theorem 10, p. 264) for cfi. \square

Clearly if a family is pfi then it is cfi.

6.2.2. DEFINITION. [Canonical sequence] Let $\mathcal{S} := \{S_i : i \in \mathbb{N}\}$ be an indexed family of languages. The *canonical positive sequence* (in short, *canonical sequence*) for $S_i \in \mathcal{S}$, σ^{can} , is defined in the following way:

$$\sigma^{can}(0) = \mu n(n \in S_i)$$

and

$$\sigma^{can}(n) = \begin{cases} n & \text{if } n \in S_i \\ \sigma^{can}(n-1) & \text{otherwise.} \end{cases}$$

where μ denotes the standard recursive *minimum function* in recursion theory. \square

We now present the main result from this section. A characterization theorem for families of recursively enumerable languages. In this case the indexing will be a recursive function $F(i, j)$ that enumerates S_i , so that the first element in the enumeration is $F(i, 0)$, the second element is $F(i, 1)$ and so on, i.e., $S_i = \{F(i, 0), F(i, 1), \dots, F(i, n), \dots\}$. Note that $F(i, j)$ produces σ^{can} , the canonical positive sequence for S_i . The argument of the proof follows a similar to the one for recursive languages but with the appropriate changes.

6.2.3. THEOREM. *An indexed family \mathcal{S} where all languages are recursively enumerable is pfi if and only if there is a uniform recursive procedure Φ such that for each i , $\Phi(i)$ is a DFTT for S_i .*

Proof:

For (\Rightarrow): suppose \mathcal{S} is pfi and let $S_i \in \mathcal{S}$ a r.e. language in the family. Since \mathcal{S} is pfi, there is a recursive learner λ such that for some $k \in \mathbb{N}$, $\lambda(F(i, k-1))$ halts with output j such that $S_j = S_i$. Thus the learner recognises language S_i on the k -th element in the sequence obtained by $F(i, j)$. The set $D_i = \{F(i, 0), F(i, 1), \dots, F(i, k-1)\}$ will be a DFTT for S_i . Clearly, since λ identifies S_i on initial segment $\sigma^{can}[k]$ from σ^{can} that results from $F(i, 0), F(i, 1), \dots, F(i, k-1)$. Note that we can repeat this process for any other language in the family.

For (\Leftarrow): follows as in the proof of the original theorem for recursive languages. Suppose there is a uniform recursive procedure that produces a definite tell-tale set D_i for any $S_i \in \mathcal{S}$. We will construct a pfi learner λ such that for any $S_i \in \mathcal{S}$ and any sequence of S_i , σ^+ , λ outputs i (see Definition 2.4.7). Let $S_i \in \mathcal{S}$ and σ^+ be any positive sequence of S_i . Consider the following recursive function,

$$\lambda(\sigma^+[n-1]) = \begin{cases} \mu(\text{set}(\sigma^+[n-1]) \subseteq S_i) & \text{if } i < n, D_i \subseteq \text{set}(\sigma^+[n-1]) \text{ and} \\ & \forall k < n-1, \lambda(\sigma^+[k]) = \uparrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

¹ $\mu n(\dots n \dots)$ is the least integer n such that the expression $\dots n \dots$ is true (if this integer exists). For more details of the minimum function (or, *minimization operator*), the reader can look at any book of Recursion Theory, see e.g., (Rogers, 1967, p. xviii).

where μ is the standard primitive recursive minimum function. For an $n \in \mathbb{N}$ large enough, the elements of D_i will occur in the positive sequence σ^+ for S_i (or even before the elements in some D_j for some $S_j \in \mathcal{S}$ such that $S_j = S_i$) Thus, after having observed D_i in the initial segment $\sigma^+[n-1]$ of σ^+ , the learner λ recognizes that this DFTT corresponds to S_i . Since S_i was taken arbitrarily, we have that λ is a pfi learner for the family \mathcal{S} . \square

The result in Theorem 6.2.3 is a sharp contrast with the case of identification in the limit. The characterization theorem for indexed families of recursively enumerable languages in (de Jongh and Kanazawa, 1996) is considerably more complex than Angluin's Characterization Theorem for recursive languages (Angluin, 1980). We will not exploit this result in this thesis to investigate which of our results will extend to this more general case, but is conformly something to explore in the future.

6.3 Families of singletons and pairs

In this section we study families of only singletons and pairs $\mathcal{S} \subseteq \mathcal{S}^1 \cup \mathcal{S}^2$. We first study some of the computational properties of such families. Because of their structural simplicity, we can characterize some of their computational properties with respect to pfi and cfi. First, we will focus on the computational properties of a subfamily of an, arbitrarily given, indexed nepfi family. In particular, we want to analyse cases when certain subfamilies of a given (ne)pfi family are decidable or recursively enumerable. Some of the most interesting examples of anti-chains that we will study in Sections 6.3.1, 6.3.2 and 6.4 are families of only singletons and pairs containing more than one pair, so we will focus on those.

We start with a simple lemma that characterizes maximal nepfi families of singletons and pairs.

6.3.1. LEMMA. *If $\mathcal{S} \subseteq \mathcal{S}^1 \cup \mathcal{S}^2$ is maximal nepfi then $\mathcal{S} \cap \mathcal{S}^2 = \{\{a, b\} : \{a\}, \{b\} \notin \mathcal{S} \cap \mathcal{S}^1\}$.*

Proof:

Follows straightforwardly by the definition of maximal anti-chain. \square

Note that for families of only singletons and pairs, the DFTTs have a very limited shape. In particular, for the singletons there is no other possibility for a DFTT than to be the singleton itself. On the contrary, for the pairs it will not always be necessary that the DFTTs are pairs. It will depend on the other elements of the family and their relation with the pair in question. We will see in Lemma 6.3.3 that for a maximal nepfi family of singletons and pairs with more than one pair, the DFTTs of the pairs are precisely the pairs.

To illustrate, consider the following example.

6.3.2. EXAMPLE. Take the family $\mathcal{S} := \{\{0, 1\}\} \cup \{\{i\} : i \notin \mathbb{N} - \{0, 1\}\}$. Clearly it is maximal pfi, the DFTTs for the singletons are the singletons themselves but for the pair $\{0, 1\}$ it could be either the singleton $\{0\}, \{1\}$ or the pair itself. Thus, the pair has more DFTTs than just the *trivial* one, so more DFTTs than only the language itself. We observed that something else happens when we consider maximal families with more than one pair. As it turns out, for such maximal families, the DFTTs can only be the languages themselves. To illustrate our intuition, consider now the family $\mathcal{S} = \{\{0, 1\}\} \cup \{\{2, 3\}\} \cup \{\{i\} : i \notin \mathbb{N} - \{0, 1, 2, 3\}\}$. First note that this family is not maximal pfi. The only way to make it maximal is to add the languages $\{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 3\}$. The resulting maximal family is then $\mathcal{S} \cup \{\{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 3\}\}$ and the only possible DFTTs are the languages themselves.

6.3.3. LEMMA. *If $\mathcal{S} \subseteq \mathcal{S}^1 \cup \mathcal{S}^2$ is maximal nepfi and has more than one pair then the only DFTT for each $S_i \in \mathcal{S}$ is S_i itself, i.e., if $D_i \subseteq S_i$ is a DFTT for S_i then $D_i = S_i$.*

Proof:

If S_i is a singleton then clearly the only possible DFTT is S_i itself. If $S_i = \{a, b\}$, by maximality of \mathcal{S} and since \mathcal{S} contains more than one pair, there must be $\{c\} \notin \mathcal{S}$ such that the pairs $\{a, c\}, \{b, c\} \in \mathcal{S}$, otherwise $\mathcal{S} \cup \{\{a, c\}, \{b, c\}\}$ would be a nepfi extension of \mathcal{S} contradicting maximality. So the only possible DFTT for S_i is the pair itself. Thus the only possible DFTT for $S_i \in \mathcal{S}$ is $D_i = S_i$. \square

For the case of families with only pairs and triples the situation is not so clear. The combinatorics make the analysis of these families more complex. For instance, the family $\mathcal{S} = \{\{0, 1, 2\}\} \cup \{\{3, 4, 5\}\} \cup (\mathcal{S}^2 - (PAIRS(\{0, 1, 2\}) \cup PAIRS(\{3, 4, 5\})))$ is a counterexample to the analogue of Lemma 6.3.3. This is because it is maximal pfi and, for instance, the pairs $\{0, 1\}, \{3, 4\}$ are DFTTs of the corresponding triples. Thus, neither an analogue of Lemma 6.3.1 (as we saw in Proposition 5.5.40) nor of Lemma 6.3.3 hold in this case. A natural question is, can we have some kind of counterpart of Lemma 6.3.3 that holds for families $\mathcal{S} \subseteq \mathcal{S}^2 \cup \mathcal{S}^3$ and for families with larger languages? We do not have an answer to that question.

Let us move on to a couple of nice results that follow from the lemmas above.

6.3.4. PROPOSITION. *If a family $\mathcal{S} \subseteq \mathcal{S}^1 \cup \mathcal{S}^2$ is maximal pfi and has more than one pair then $X \in \mathcal{S} \cap \mathcal{S}^1$ is decidable, i.e., the set $S(\mathcal{S}) := \{i : \{i\} \in \mathcal{S}\}$ is decidable.*

Proof:

We need to show that $S(\mathcal{S}) := \{i : \{i\} \in \mathcal{S}\}$ is decidable. Since \mathcal{S} is pfi, there is an effective procedure that outputs a DFTT for every language in \mathcal{S} . Thus we

can run through all the DFTTs of the languages in \mathcal{S} and see whether for $i \in \mathbb{N}$, $\{i\}$ occurs as DFTT or a pair $\{i, j\}$ occurs as DFTT for some $j \neq i$. If $D_i := \{i\}$ is the case, we know that $i \in S(\mathcal{S})$ and $i \notin S(\mathcal{S})$ otherwise. \square

This proposition characterizes the maximal pfi families of singletons and pairs completely: they consist of a recursive set of singletons plus all pairs of natural numbers in its complement.

6.3.5. PROPOSITION. *If $\mathcal{S} \subseteq \mathcal{S}^1 \cup \mathcal{S}^2$ is a maximal pfi family with more than one pair, \mathcal{S} can be represented canonically.*

Proof:

Let $\mathcal{S} \subseteq \mathcal{S}^1 \cup \mathcal{S}^2$ be a maximal pfi family. We can run a positive sequence σ_j^+ for some $S_j \in \mathcal{S}$. Let m_0 be the first element that appears in the sequence σ_j^+ . Since $\mathcal{S} \cap \mathcal{S}^1$ is decidable (by Proposition 6.3.4), we know whether $\{m\} \in \mathcal{S} \cap \mathcal{S}^1$. If $\{m\} \in \mathcal{S} \cap \mathcal{S}^1$, since \mathcal{S} is an anti-chain, it follows that $S_j = \{m\} \in \mathcal{S}$, which produces a canonical index for S_j . If $\{m\} \notin \mathcal{S} \cap \mathcal{S}^1$, we wait until the next element different than m_0 appears in σ_j^+ , namely m_1 . Since $\mathcal{S} \subseteq \mathcal{S}^1 \cup \mathcal{S}^2$ and it is a maximal anti-chain, it follows that $S_j = \{m_0, m_1\} \in \mathcal{S}$, which produces a canonical index for S_j . Since S_j and σ_j^+ were chosen arbitrarily, \mathcal{S} is canonical. \square

We will see in Proposition 6.3.20 that the pfi family $\mathcal{S}_2^T = \{\{2i\} \cup \{2y + 1 : Tiiy\} : i \in \mathbb{N}\}$ where T is Kleene's T -predicate cannot be made canonical. Thus, a generalization of Proposition 6.3.5 to all pfi families of singletons and pairs does not hold.

We saw in Theorem 5.4.10 that maximal cfi (and necfi) families do not exist. Still, what can we say about cfi families of singletons and pairs that contain a maximal pfi one? The following result gives us the possibility of characterizing all possible cfi extensions of maximal pfi families of singletons and pairs. In fact, as we saw in Proposition 5.4.9, Proposition 6.3.6 can be generalized straightforwardly for any family of finite languages. This is important when one is interested in seeing how far one can go with positive information only and only then add negative information to get more results.

To illustrate, consider the family of all singletons $\mathcal{S}^s := \{\{i\} : i \in \mathbb{N}\}$. We know that \mathcal{S}^s is pfi (and cfi) since every language is its own DFTT. Moreover, we know that it is maximal pfi since adding any other language will impair the anti-chain condition (necessary for pfi). Still, we can add languages to \mathcal{S}^s and check for cfi. In fact, any extension of \mathcal{S}^s that results from adding only finitely many supersets of some $\{i\}$ is a cfi family.

6.3.6. PROPOSITION. *If $\mathcal{S} \subseteq \mathcal{S}^1 \cup \mathcal{S}^2$ is cfi and there is $\{i\} \in \mathcal{S}$, there are only finitely many pairs of the form $\{i, j\}$ in \mathcal{S} .*

Proof:

Follows from Proposition 5.4.9 in Chapter 5. \square

6.3.7. PROPOSITION. *If an indexed family $\mathcal{S} \subseteq \mathcal{S}^1 \cup \mathcal{S}^2$ is nepfi then the set of languages which are pairs is r.e., i.e., the set $P(\mathcal{S}) := \{i : |S_i| = 2\}$ is r.e.*

Proof:

We provide a recursion theoretic argument for this proof. Since \mathcal{S} is indexable, the predicate $x \in S_i$ is recursive. Note that:

$$j \in \{i : |S_i| = 2\} \text{ iff } \exists k \exists m, k \neq m (k, m \in S_j).$$

The predicate $(\exists k \exists m, k \neq m (k, m \in S_j))$ is r.e.² Therefore the set $\{i : |S_i| = 2\}$ is r.e. \square

By a similar argument, the predicate $\exists k(i \in S_k \text{ and } \exists j(j \neq i \text{ and } j \in S_k))$ is r.e. Therefore the set $\{i : \exists j (\{i, j\} \in \mathcal{S})\}$ is r.e. Note that the predicate in $\{i : |S_i| = 2\}$ talks about the languages whereas the predicate in $\{i : \exists j (\{i, j\} \in \mathcal{S})\}$ talks about the elements in the languages.

6.3.1 The non-existence of maximal pfi families

In this section we investigate whether any finitely identifiable family is contained in a pfi family which is maximal with respect to inclusion. For the more common pfi families, maximal pfi extensions do exist. Nevertheless, this is not always the case. We will give an example of a canonical family (and thus a pfi family) which does not have a maximal pfi extension.

Before presenting our main result, first we prove the following useful lemmas.

6.3.8. LEMMA. *Let \mathcal{S} be a maximal canonical nepfi family. For every finite set $Y \in \mathbb{N}$, we can decide whether $Y \in \mathcal{S}$, i.e., \mathcal{S} is decidable.*

Proof:

Let $Y \in \mathbb{N}$ be arbitrary. First note that if $Y \notin \mathcal{S}$ then $Y \subset S_i$ or $S_i \subset Y$ for some $S_i \in \mathcal{S}$, otherwise Y can be added to \mathcal{S} as a new element without impairing the anti-chain condition (nepfi condition), which would make \mathcal{S} non-maximal nepfi. To decide whether $Y \in \mathcal{S}$, by canonicity, we simply run through S_0, S_1, S_2, \dots until we meet S_i which is Y itself, or a sub- or superset of Y . Thus \mathcal{S} is decidable. \square

Recall that a canonical anti-chain is pfi. Thus, a canonical nepfi family is pfi.

²This follows from the so-called Existential Quantifier Theorems, see e.g. Rogers [1967], Theorem X and Corollary XI, p. 66.

6.3.9. LEMMA. *Let \mathcal{S} be a maximal decidable nepfi family of finite languages. The family \mathcal{S} is canonical, i.e., there is a recursive function f such that for each i , $f(i)$ is a canonical index for S_i , and so, $S_i = F_{f(i)}$.*

Proof:

We run the canonical sequence (see Definition [6.2.2](#)), σ^{can} , for some $S_j \in \mathcal{S}$. The construction of $f(i)$ is in stages. At stage n , $set(\sigma^{can}[n])$ is computed and we ask whether $set(\sigma^{can}[n]) \in \mathcal{S}$. Note that the latter is possible because \mathcal{S} is decidable. To illustrate, suppose $m_0 \in \mathbb{N}$ is the first element that appears in σ^{can} . Since \mathcal{S} is decidable, there is a decision procedure which ensures whether $\{m_0\} \in \mathcal{S}$ or $\{m_0\} \notin \mathcal{S}$. If $\{m_0\} \in \mathcal{S}$, the procedure ends and produces a canonical index for S_j . Otherwise (if $\{m_0\} \notin \mathcal{S}$), more elements from σ^{can} will be given. Thus, wait until the next element different from m_0 appears in σ^{can} (which has to occur since $\{m_0\} \notin \mathcal{S}$), namely m_1 , and check whether the language $\{m_0, m_1\} \in \mathcal{S}$. Continue until the decision procedure confirms that a language S formed by elements that appear in the corresponding initial segment of σ_j^+ is in \mathcal{S} . Note that this must happen exactly when $S = S_j$ since \mathcal{S} is an anti-chain. Since S_j was taken arbitrarily, this procedure produces a canonical index for $S_j \in \mathcal{S}$. We then continue with σ^{can} for S_{j+1} and in the same manner we get a canonical index for S_{j+1} . Thus, \mathcal{S} is canonical. \square

6.3.10. PROPOSITION. *Let \mathcal{S} be a maximal decidable nepfi family of finite languages. The family \mathcal{S} is decidable iff \mathcal{S} is canonical.*

Proof:

Follows straightforwardly from Lemmas [6.3.8](#) and [6.3.9](#). \square

Consider $\pi(m, n) : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ the standard recursive pairing function (Cantor's pairing function).³ Let π_1 and π_2 be the standard first and second inverse functions of π respectively, i.e., $\pi_1(\pi(m, n)) = m$ and $\pi_2(\pi(m, n)) = n$.

6.3.11. LEMMA. *Let $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ and $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ two indexed families. The family $\mathcal{A} \cup \mathcal{B}$ is indexable.*

Proof:

Consider the indexing, $\mathcal{A} \cup \mathcal{B} := \{C_i : i \in \mathbb{N}\}$ where $C_{2i} = A_i$ and $C_{2i+1} = B_i$ for all $i \in \mathbb{N}$. Since \mathcal{A} and \mathcal{B} are indexed, the predicates $(x \in A_i)$ and $(x \in B_i)$ are both recursive and, thus, $x \in C_{2i}$ and $x \in C_{2i+1}$ are recursive. \square

³In simple words, a *recursive pairing function* is a process to uniquely encode two natural numbers into a single natural number. Cantor's pairing function defined as $\pi(m, n) = (m + n)(m + n + 1)/2 + n$ is a primitive recursive pairing function frequently used in the recursion theoretic literature, see e.g., [\(Rogers, 1967, p. 64\)](#).

In the special case of canonical families, the lemma above gets simplified with respect to r.e. sets of natural numbers in the following manner. If $A = \{a_i : i \in \mathbb{N}\}$ and $B = \{b_i : i \in \mathbb{N}\}$ are two indexed families. The family $A \cup B$ is indexable ($A \cup B := \{c_i : i \in \mathbb{N}\}$ where $c_{2i} = a_i$ and $c_{2i+1} = b_i$ for all $i \in \mathbb{N}$). Since every r.e. set is indexable, this version of the lemma can be linked directly to the standard theorem in recursion theory stating that the union of two r.e. sets is r.e. (see e.g., [Rogers, 1967](#), Theorem XIII, p.68).

We are now ready to prove the main result from this section.

6.3.12. THEOREM. *Let $A \subseteq \mathbb{N}$ and $B \subset \mathbb{N} - A$ be two recursively inseparable r.e. sets which are not recursive. Let $\mathcal{S} := \{\{a\} : a \in A\} \cup \{\{b, c\} : b, c \in B \text{ with } b \neq c\}$. The family \mathcal{S} can be given as an indexed family which is canonical, pfi and there is no canonical maximal pfi extension of \mathcal{S} .*

Proof:

First note that since both A and B are r.e. we have a recursive enumeration of $A = \{a_i : i \in \mathbb{N}\}$ and $B = \{b_i : i \in \mathbb{N}\}$. Let $B_p := \{c_i : i \in \mathbb{N}\}$ be a recursive enumeration of all pairs $b_n, b_m \in B$ with $b_n \neq b_m$, i.e., $c_i = \pi(b_m, b_n)$ with $b_n, b_m \in B$ and $b_n \neq b_m$. Clearly B_p is also indexed. Let $\mathcal{A} := \{\{a_i\} : a_i \in A\}$ and $\mathcal{B} := \{\{b, c\} : b, c \in B \text{ with } b \neq c\}$. Since A and B_p are indexed, the families \mathcal{A} and \mathcal{B} are also indexed. By [Lemma 6.3.11](#), the union of two indexed families is an indexable family, thus $\mathcal{S} := \mathcal{A} \cup \mathcal{B}$ can be given as a canonical indexed family. To see this explicitly, let $S_{2\pi(i,k)+1} = \{\pi_1(c_i), \pi_2(c_i)\}$ and $S_{2i} = \{a_i\}$, clearly the predicates $x \in S_{2\pi(i,k)+1}$ and $x \in S_{2i}$ are recursive. Note that it is pfi since any language serves as its own DFTT (as in the proof of [Theorem 2.4.11](#)). Now, towards a contradiction, suppose there is a maximal canonical pfi family extending \mathcal{S} , say \mathcal{S}' . Because of maximality and canonicity of \mathcal{S}' , we can decide by [Lemma 6.3.8](#), whether $Y \in \mathcal{S}'$ or $Y \notin \mathcal{S}'$ for each finite set $Y \subseteq \mathbb{N}$. Since \mathcal{S}' is decidable, the set $\mathcal{A}' \supseteq \mathcal{A}$ of singletons in \mathcal{S}' is recursive. Since \mathcal{S}' is an anti-chain, $\mathcal{A}' \cap \mathcal{B} = \emptyset$. Thus $A' := \{a_i : \{a\}_i \in \mathcal{A}'\}$ separates A from B and this contradicts the inseparability of A and B . \square

We can strengthen [Theorem 6.3.12](#) to conclude that \mathcal{S} has no maximal pfi extension at all.

Note that the family $\mathcal{B}_p := \{\{\pi_1(c_i), \pi_2(c_i)\} : c_i \in B_p\}$ (as in [Theorem 6.3.12](#)) shows that [Lemma 6.3.8](#) does not apply to non-maximal canonical families since \mathcal{B}_p is canonical.

6.3.13. LEMMA. *There is a canonical non-maximal pfi family which is not decidable.*

Proof:

The family \mathcal{B}_p above is clearly pfi and canonical. To see that \mathcal{B}_p is not decidable, towards contradiction suppose \mathcal{B}_p is decidable. It follows that B_p and thus B are

also decidable. Therefore B is recursive, which cannot be. \square

Observe that in the theorem above, we are talking about the languages of the family whereas in the theorem that follows we talk about the elements in the languages of the family.

6.3.14. THEOREM. *The family \mathcal{S} of Theorem [6.3.12](#) has no maximal pfi extension at all.*

Proof:

Let $\mathcal{S}' \supseteq \mathcal{S}$ be a maximal pfi family. On this supposition, we will define recursive A', B' , such that $A \subseteq A'$ and $B \subseteq B'$, $A' \cap B' = \emptyset$, $A' \cup B' = \mathbb{N}$ which leads to a contradiction (since A and B are recursively inseparable). Let $\{S_n : n \in \mathbb{N}\}$ be an indexing of \mathcal{S}' . For each i determine whether $i \in A'$ or $i \in B'$ as follows. Since \mathcal{S}' is indexed and maximal, we can find the first n such that $i \in S_n$. Now consider D_n the DFTT of S_n . We distinguish two possibilities: (1) $D_n = \{i\}$, and (2) $D_n \neq \{i\}$. In case (1) put $i \in A'$. Note that $A \subseteq A'$, because if $i \in A$ then $\{i\} \in \mathcal{S}$ and thus $\{i\} \in \mathcal{S}'$. Note also that $\{i\} \subset S_n$ is, in this case, impossible since \mathcal{S}' is pfi and thus an anti-chain. In case (2) put $i \in B'$, this means $\{i\}$ is not a DFTT. Note that $B \subseteq B'$, because if $i \in B$ then $\{i, k\}, \{i, j\} \in \mathcal{S} \subseteq \mathcal{S}'$ for some $j \neq i, j \neq k, i \neq k$ because of the definition of B . So $D_n \neq \{i\}$ because \mathcal{S}' is pfi. Therefore A', B' have been constructed as required, which is a contradiction. \square

This theorem applies not only to extensions to families with infinite members but also to non-canonical indexed families of only finite languages. We will see later on, in Section [6.3](#), that the latter kind of family exists.

6.3.2 Non-canonical families

In this section we will study non-canonical families. Here we are interested in analyzing some of the features that prevent a family to be canonical. In particular, we want to explore the connection between canonicity and pfi (and cfi). We prove that there is a non-canonical cfi family (not an anti-chain) which is not pfi. Then, we present a non-canonical anti-chain which is pfi and cannot be made canonical. Such anti-chain is pfi mainly because of its structural properties, pointing out again the importance of the structure of a family for being pfi.

We start by noting that not every family of finite languages is a canonical family. To see this, consider the following examples presented in ([Gierasimczuk and de Jongh, 2013](#)).

Recall that in the standard version of Kleene's predicate with respect to $Tiyy$, we have the following: if $Tiyy$ exists then i can be computed from the y and $i \leq y$. In what follows, we always consider the standard version of Kleene's predicate.

6.3.15. PROPOSITION. *The indexed family $\mathcal{S}_0^T := \{\{0\} \cup \{y : Tiiy\} : i \in \mathbb{N}\}$ where T is Kleene's predicate is not canonical and not pfi.*

Proof:

It holds that, given a y , the predicate $Tiiy$ is uniformly decidable (i.e., recursive) since i has to be smaller than y . However, the existence of such y is undecidable (the predicate $\exists y Tiiy$ is r.e.). Then, \mathcal{S}_0^T is an indexed but not canonical family, as is clear from the fact that we cannot decide whether $S \in \mathcal{S}_0^T$ has one or two elements (we will prove this in the proof of Theorem 6.3.17 below). Moreover, it is not pfi because contains languages which are proper subsets of other languages in the family. \square

6.3.16. EXAMPLE. Let us slightly modify the family in Proposition 6.3.15, to be $\mathcal{S}_1^T := \{\{i\} \cup \{y : Tiiy\} : i \in \mathbb{N}\}$. We might wrongly think that this family is pfi. The confusion arises from the idea that once we had seen i occur in the data we will be able to decide S_i . However, i could be precisely the computation y for some other j . In fact, whenever y is such that $Tjjy$ exists and corresponds to i the learner cannot decide whether she sees S_i or S_j . Note that if $Tjjy$ exists (appearing in the data), the index j can be computed from the y (since $j \leq y$). The problematic feature of \mathcal{S}_1^T preventing pfi is that, in general, \mathcal{S}_1^T is not an anti-chain. For instance, in the case i is precisely the computation y such that $Tjjy$ exist for j but $\neg(\exists z Tyyz)$ (for instance when i a code for the empty set). In this case, we will have that $S_y = \{y\}$, $\{y\} \subset \{j\} \cup \{y : Tjjy\}$, and so $S_y \subset S_j$. However, this family is cfi. To see this, we define the definite tale-tell pair (D_i, \overline{D}_i) for each S_i such that $D_i = \{i\}$ and $\overline{D}_i = \{j : Tjjj\}$. Note that this works precisely because $j \leq i$.

We now prove that \mathcal{S}_1^T is not canonical.

6.3.17. THEOREM. *The indexed family $\mathcal{S}_1^T := \{\{i\} \cup \{y : Tiiy\} : i \in \mathbb{N}\}$ where T is Kleene's predicate, cannot be given as a canonical family.*

Proof:

We provide a recursion-theoretic argument for this proof. Towards a contradiction, suppose that the computable function f enumerates \mathcal{S} as $\{F_{f(i)} : i \in \mathbb{N}\}$. Note that if $Tiiy$ exists then the index i can be computed from the y and $i \leq y$. Thus, by how the S_i 's are defined and our initial assumption, we know the following,

$$\exists z(Tiiz) \text{ iff } \forall x(x \in F_{f(i)} \text{ and } \forall y < x(y \notin F_{f(i)})) \text{ implies } F_{f(i)} \neq \{x\}$$

Note that the left part of the biconditional, namely $\exists z(Tiiz)$, is a r.e. predicate.⁴ However, the right part of the biconditional is clearly a Π_1 predicate simply

⁴This is the so-called *Halting Problem*, see Section 2.3. For a detailed discussion, see e.g., (Rogers, 1967, Section 1.9, p. 24)

because f is computable and no existential-quantifier occurs in the predicate. It follows that both predicates are Δ_1 , and so they are recursive. However, it is known that the predicate $\exists z(Tiiz)$ is not recursive. Therefore, a family \mathcal{S} whose languages are of the form $S_i = \{i\} \cup \{y : Ti iy\}$ cannot be represented as a canonical family. \square

6.3.18. COROLLARY. *There is a non-canonical family of finite languages which is cfi and not pfi.*

Proof:

Follows from Theorem 6.3.17 and the argument in Example 6.3.16 that the family \mathcal{S}_1^T is a non-canonical cfi family which is not pfi. \square

As we mentioned in Example 6.3.16, the family \mathcal{S}_1^T is not an anti-chain. Later on in Section 6.4, we will present a non-canonical cfi anti-chain of singletons and pairs which is not pfi.

6.3.19. COROLLARY. *The indexed family \mathcal{S}_0^T is not canonical and cannot be given as a canonical family.*

A natural question then arises, is there a non-canonical anti-chain which is pfi? We answer this question positively in the proposition that follows.

6.3.20. PROPOSITION. *The indexed family $\mathcal{S}_2^T := \{\{2i\} \cup \{2y+1 : Ti iy\} : i \in \mathbb{N}\}$ where T is Kleene's predicate is pfi but is not canonical and cannot be given as a canonical family.*

Proof:

First observe that the $2i$ element in each S_i is an even number whereas the other element $2(\mu y Ti iy) + 1$ (if it exists, where μ is the standard recursive minimum function) is always an odd number. It is easy to see that, for each $S_i \in \mathcal{S}_2^T$ the set $\{2i\}$ is a DFTT of S_i . Moreover, $\{2y + 1\}$ is also a DFTT of S_i . To see the latter, note that after $2y + 1$ appears in the data, the learner can compute i since $i < y$. In other words, if a learner encounters an odd number in the stream of data, she will know that this number does not correspond to the index of the language but it can be computed.

To see that \mathcal{S}_2^T cannot be given as a canonical family, note that by a similar argument as in the proof of Theorem 6.3.17, \mathcal{S}_2^T is not canonical. Moreover, \mathcal{S}_2^T has the following property 1): $\{2i\} \in \mathcal{S}_2^T$ is co-r.e. and not r.e. because $\neg \exists y(Ti iy)$. Now towards contradiction, suppose $\mathcal{S}' \supseteq \mathcal{S}_2^T$ is a canonical anti-chain. Let $\{S'_j : j \in \mathbb{N}\}$ be an enumeration of \mathcal{S}' . Then, $\{2i\} \in \mathcal{S}'$ iff $\{2i\} \in \mathcal{S}_2^T$ simply because \mathcal{S}_2^T and \mathcal{S}' are anti-chains such that $\mathcal{S}_2^T \subseteq \mathcal{S}'$. Thus the property 1) no longer holds for \mathcal{S}_2^T . But $\{2i\} \in \mathcal{S}' \Leftrightarrow \exists j(\{2i\} = S'_j)$ and

$\exists j(\{2i\} = S'_j)$ is a r.e. predicate, which is impossible since \mathcal{S}_2^T is not r.e. Therefore \mathcal{S}_2^T cannot be indexed canonically. \square

The fact that \mathcal{S}_2^T is pfi relies on the structural properties of the family and has nothing to do with whether the family is canonical or not. The families presented above (in Proposition [6.3.15](#), Example [6.3.16](#) and Proposition [6.3.20](#)), make clear the fact that canonicity is not a necessary condition for a family to be pfi or cfi.

6.4 Non-pfi anti-chains of singletons and pairs

We saw in Lemma [5.4.3](#) (in Chapter [5](#)) that nepfi families of finite languages and such anti-chains correspond to the same class of families. We will see that this is not the case for pfi families since *effectiveness* plays a crucial role in pfi. In this section, we present a maximal non-canonical (computable) anti-chain of finite languages which is not pfi (and not cfi). This clarifies the fact that, even for families of finite languages with “structurally nice properties”, like being a maximal anti-chain, pfi and cfi are not guaranteed. Surprisingly, we then prove that for infinite anti-chains of finite languages, cfi is more powerful than pfi. For this, we present a non-canonical cfi anti-chain of singletons and pairs which is not pfi.

We start this section by listing what we know so far concerning canonical and non-canonical anti-chains of finite languages:

- Canonical families which are anti-chains are always pfi.
- Cfi families of finite sets which are not canonical and not pfi do exist, for instance the family $\mathcal{S}_1^T = \{S_i : S_i = \{i\} \cup \{y : Tiiy\}\}$ where T is Kleene’s T -predicate.
- Pfi families which are not canonical do exist, for instance $\mathcal{S}_2^T := \{\{2i\} \cup \{2y + 1 : Tiiy\} : i \in \mathbb{N}\}$ where T is Kleene’s predicate (in Proposition [6.3.20](#)).

The following proposition presents the strategy and prepares the stage for the main results in this section.

6.4.1. PROPOSITION. *Let $A \subseteq \mathbb{N}$ r.e. and $B = \mathbb{N} - A$ co-r.e. There is an indexed nepfi family $\mathcal{S} = \{S_i : i \in \mathbb{N}\} \subseteq \mathcal{S}^1 \cup \mathcal{S}^2$ such that for any $S_i \in \mathcal{S}$ with $i \in A$, $S_i = \{i, a\}$ with $a \in A$ and $S_i = \{i\}$ otherwise.*

Proof:

We enumerate the set $A = \{a_0, a_1, a_2, \dots\}$. For each S_i we construct S_i in stages. At stage 0, let $i \in S_i$. At stage k , we check whether k is a computation showing that $i \in A$. If not, do nothing. If k is a computation for $i \in A$ we put the first $a \in A$

such that $a > k$, $a \neq i$ and $a \notin S_{i'j'}$ for $i', j' < i, j$ (the first a that has not been assigned before in S_i). Let \mathcal{S} be the family of all such S_i . By construction it is an anti-chain so it is nepfi, and for any $S_i \in \mathcal{S}$ such that $i \in A$, $S_i = \{i, a\}$ with $a \in A$, $S_i = \{i\}$ otherwise. Clearly, \mathcal{S} is indexed. The resulting family \mathcal{S} is not maximal because of the following: let $i, j \in A$ such that $S_i = \{i, a_k\}$ and $S_j = \{j, a_m\}$ have been constructed for some a_k, a_m in A . Note that the pair $\{a_k, a_m\}$ is not in \mathcal{S} . This is because S_{a_k} must be a pair, such that a_k is paired with the smallest element in A with certain properties that has not been used before, namely a_n . Thus, a_n cannot be a_m since a_m has been paired with j . Thus, the family $\mathcal{S} \cup \{a_k, a_m\}$ is an anti-chain that extends \mathcal{S} , thus it is nepfi. \square

We are now ready to present a non-canonical anti-chain which is not pfi and a maximal anti-chain which is not pfi.

6.4.2. PROPOSITION. *Consider the r.e. set $K := \{x : \exists y (Txy)\}$ where T is Kleene's predicate. There is a family \mathcal{S} such that \mathcal{S} contains all $\{i\}$ for $i \notin K$ and some $\{c, d\}$ with $c, d \in K$ which is an indexable anti-chain and not pfi.*

Proof:

First note that K is not recursive. Let $B \subset K$ be an infinite recursive. Let $C := \{y : \exists x (Txy)\}$, noting that $C = \{y : \exists x < y (Txy)\}$ and therefore C is recursive. Let $C := \{c_0, c_1, \dots\}$ be an enumeration of C . We will construct an indexed family $\mathcal{S} = \{S_0, S_1, S_2, \dots\}$ as follows,

$$S_{2i} = \begin{cases} \{i, b\} & \text{if } Tiiy \text{ for } y \in C \text{ and } b = \min\{z \in B : z > y\}, \\ \{i\} & \text{otherwise.} \end{cases}$$

Furthermore, $S_{2j+1} = \{i, b'\}$ computed from c_j where $Tiic_j$, $b = \min\{z \in B : z > c_j\}$, $Tbby$ for some $y \in C$ and $b' = \min\{z \in B : z > y\}$. Both b and b' always exist because B is an infinite subset of K . Note that $S_{2b} = \{b, b'\}$ and thus $b \neq b'$. Clearly, i will occur in more than one pair.

Altogether, if $i \notin K$ then $S_{2i} = \{i\}$. If $i \in K$ then S_{2i} will have two elements, i and some $b \in B \subseteq K$ such that $b > y > i$; and, S_{2j+1} will have two elements, i and some $b' \in B \subseteq K$ such that $b' > y > b > c_j > i$.

We will show that \mathcal{S} is indexed. Clearly, $i \in S_{2i}$ and $i \in S_{2j+1}$. Consider $x \neq i$, then

$$x \in S_{2i} \text{ iff } \exists y < x (Tiiy \text{ and } x = \min\{z \in B : y < z\}),$$

$$x \in S_{2j+1} \text{ iff } \exists y < x (Tbby, b = \min\{z \in B : c_j < z\}, \\ Tiic_j \text{ and } x = \min\{z \in B : y < z\}).$$

These two predicates are recursive. Thus, the family \mathcal{S} is indexed.

We will now show that \mathcal{S} is not pfi. Towards a contradiction, suppose λ is a pfi learner for \mathcal{S} . Take $i > 0$ and $m \in \mathbb{N}$ and suppose λ has received the segment $\sigma^+[m] = i, i, \dots, i$ for some $m \in \mathbb{N}$. Note that any such $\sigma^+[m]$ is an initial segment of some positive data sequence for some language in \mathcal{S} . If $i \in K$, by our construction above, there are at least two languages that contain i so $\lambda(\sigma^+[m]) \neq \uparrow$. If $i \notin K$, the segment $\sigma^+[m] = i, i, \dots, i$ corresponds to a positive data sequence σ^+ for $S_{2i} = \{i\}$. Thus, λ needs to give value $2i$ at some point and so

$$\exists m, \lambda(\sigma^+[m]) \neq \uparrow.$$

Altogether, and since $(i \in \mathbb{N} - K \text{ iff } \exists m, \lambda(\sigma^+[m]) \neq \uparrow)$, we obtain that $\mathbb{N} - K$ is r.e. which is a contradiction (since K is not recursive). Therefore, \mathcal{S} is not pfi. \square

We want to emphasize a few things about the proof of Proposition 6.4.2. First, observe that the languages in \mathcal{S} are all distinct from one to another, i.e., there are no $S_i, S_j \in \mathcal{S}$ with $S_i = S_j$ and $i \neq j$. This family will play a crucial role later on, when we prove Theorem 6.5.8 of Section 6.5. Second, that it can be generalized to a family constructed in this matter using any r.e. set W which is not recursive. This is because any r.e. set contains an infinite recursive set B (see e.g., Rogers, 1967), so the proof will follow as for K . Finally, note that \mathcal{S} is not a maximal anti-chain. This is simply because for every $i, k \in K - B$ the family $\mathcal{S} \cup \{i, k\}$ is a nepfi extension of \mathcal{S} . In fact, the non-maximality of \mathcal{S} follows from the next proposition.

6.4.3. PROPOSITION. *Consider the r.e. set $K := \{x : \exists y (Txy)\}$ where T is Kleene's predicate. The family \mathcal{S} such that \mathcal{S} contains all $\{i\}$ for $i \notin K$ and all $\{c, d\}$ for $c, d \in K$ is an indexable maximal anti-chain i.e., a maximal nepfi family.*

Proof:

Let $B \subset K$ be an infinite recursive set and let $C := \{y : \exists x (Txy)\}$, noting that $C = \{y : \exists x < y (Txy)\}$. We will construct an indexed family $\mathcal{S} = \{S_0, S_1, S_2, \dots\}$ as follows. First, let $S_{2n+1} = \{c, d\}$ with $c \neq d$ if $n \in \mathbb{N}$ codes a pair of computations of c and d being elements of K , i.e., if $n = \pi(y, z)$ such that π is Cantor's pairing function, $Tccy$ and $Tddz$. Now if i does not code such a pair then S_{2i} will be enumerated as follows:

$$S_{2i} = \begin{cases} \{i, b\} & \text{if } Tiiy \text{ for } y \in C \text{ and } b = \min\{z \in B : z > y\}, \\ \{i\} & \text{otherwise.} \end{cases}$$

In other words, whenever $Tiiy$ holds for some $y \in C$, we pair i with the smallest element in B greater than y such that $Tiiy$.

We will show that \mathcal{S} is indexed. Observe that by construction, for every $n \in \mathbb{N}$, if $n = \pi(c, d)$ then $c, d \in S_{2n+1}$. Thus, $(x \in S_{2n+1})$ is recursive, moreover S_{2n+1} can be computed from n with its canonical index. Regarding S_{2i} , $i \in S_{2i}$, and consider $x \neq i$. Then,

$$x \in S_{2i} \text{ iff } \exists y < x (T_{ii}y \text{ and } x = \min\{z \in B : y < z\}),$$

which is recursive. Therefore, \mathcal{S} is indexed.

By construction, the resulting family \mathcal{S} is an anti-chain and therefore nepfi. It is a maximal anti-chain simply because of the following, for any set $X \subseteq \mathbb{N}$ which is not a singleton nor a doubleton clearly $\mathcal{S} \cup X$ is not an anti-chain. If X is a singleton, then either $X \in \mathcal{S}$, or $X \subset S_{2n+1}$ for some n that codes a pair of computations or $X \subset S_{2i}$ for some $i \in K$. If X is a doubleton, then either $X = S_{2n+1}$ for some n , $X = S_{2i}$ for some $i \in K$ or $S_i = \{i\} \subset X$. In both cases $\mathcal{S} \cup X$ is not an anti-chain. Thus, \mathcal{S} is a maximal anti-chain. \square

We want to remark a couple of things from Proposition 6.4.3. First, as it happens for Proposition 6.4.2, Proposition 6.4.3 can be generalized to a family constructed in this matter using any r.e. set W which is not recursive. Second the following obvious fact, consider any set of singletons and extend it with the family of pairs formed by the elements in its complement. The resulting family is always nepfi. This gives us the following result.

6.4.4. COROLLARY. *If an (indexable) family of singletons and pairs \mathcal{S} is maximal nepfi and not pfi then $\mathcal{S}^1 \cap \mathcal{S}$ is non-r.e. and $\mathcal{S}^2 \cap \mathcal{S}$ is r.e.*

Proof:

Follows straightforwardly from Proposition 6.3.4, Proposition 6.3.7 and the general version of Proposition 6.4.3. \square

Given the result above, the family in Proposition 6.4.3 is a *generic* example of a maximal indexed nepfi extension for any pfi family of singletons and pairs. This gives us a partial answer for the Open Question 1 posed in Chapter 5 with respect to maximal *indexed* nepfi families of only singletons and pairs.

6.4.5. PROPOSITION. *The family in Proposition 6.4.3 is a non-canonical anti-chain which is not pfi.*

Proof:

Towards contradiction, suppose \mathcal{S} is canonical. By Proposition 6.3.10, \mathcal{S} is decidable. This implies that $\mathcal{S} \cap \mathcal{S}^1$ is also decidable, and therefore, $\mathbb{N} - K$ is decidable which cannot be since $\mathbb{N} - K$ is non-r.e. Towards contradiction, suppose \mathcal{S} is pfi. This implies, by Proposition 6.3.4 and a similar argument as before, that $\mathbb{N} - K$ is decidable, which cannot be since $\mathbb{N} - K$ is non-r.e. \square

6.4.6. PROPOSITION. *The family in Proposition [6.4.3](#) is not cfi.*

Proof:

Towards contradiction, suppose \mathcal{S} is cfi. If $b, c \in K$ then $\{b, c\} = L_j$ for some $L_j \in \mathcal{S}$. We will show that for any definite tell-tale pair (D_j, \overline{D}_j) for L_j , $D_j = \{b, c\}$. Towards contradiction and w.l.o.g., suppose $TTP := (\{b\}, \{x_0, \dots, x_{n-1}\})$ with $x_i \neq c$ is a definite tell-tale pair for L_j . Take $m = \min\{y \in K : y > b, c \text{ and } i \leq n-1, y > x_i\}$ (clearly such m exists). Since $m \in K$, $\{b, m\} \in \mathcal{S}$. Note that the proposed definite tell-tale pair TTP is consistent with the language $\{b, m\}$ but $\{b, m\} \neq \{b, c\}$, contradicting that TTP is a definite tell-tale pair for $\{b, c\}$. Thus, if (D_j, \overline{D}_j) is a definite tell-tale for L_j , $D_j = \{b, c\}$.

If $i \notin K$, $\{i\} = L_j$ for some $L_j \in \mathcal{S}$. Thus, by maximality of \mathcal{S} , $D_j = \{i\}$ is the positive element of any tell-tale pair (D_j, \overline{D}_j) for $\{i\}$. Clearly, since $D_j \neq \emptyset$ because $\mathbb{N} - K$ is infinite.

We now continue with the main proof. Take any $x \in \mathbb{N}$ and any $c \in K$. Then either $\{x\} \in \mathbb{N}$ or $\{x, c\} \in \mathbb{N}$. Moreover, either $\{x\}$ or $\{x, c\}$ corresponds to the set D_j of a definite tell-tale pair (D_j, \overline{D}_j) for some $L_j \in \mathcal{S}$. Thus, $(D_j := \{x\} \text{ iff } x \notin K)$ which gives a decision procedure for K and that cannot be. \square

Contrary to the results of Section [5.3](#), cfi identification is more powerful on infinite anti-chains of infinite languages than pfi identification (see Proposition [5.4.1](#), where we prove that the anti-chain of co-singletons is not even nepfi). The case of infinite anti-chains of finite languages is not so simple. Nevertheless, as the following proposition shows, cfi is more powerful than pfi in that case.

6.4.7. PROPOSITION. *There is a non-canonical cfi anti-chain of finite languages which is not pfi.*

Proof:

Consider the r.e. set $K := \{x : \exists y (Txy)\}$ where T is Kleene's predicate and $B \subseteq K \cup \{0\}$ a recursive subset. Let $C := \{y : \exists x (Txy)\}$, noting that $C = \{y : \exists x < y (Txy)\}$ and therefore C is recursive. Using these three sets we will construct a non-canonical cfi family which is not pfi.

Take an arbitrary $k_0 \in K$ and let $k_1 = \min\{k \in B : Tk_0k_0n \text{ for } n < k\}$. Let $\mathcal{S} := \{S_i : i \in \mathbb{N}\}$ such that $S_0 = S_1 = S_2 = \{k_0, k_1\}$. For any $n \geq 1$ and $i \geq 1$,

$$S_{3n} = \begin{cases} \{k_0, k_1\} & \text{if } n \notin C, \\ \{x_1, x_2\} & \text{if } n \in C, Tx_1x_1n \text{ and } x_2 = \min\{z \in B : z > n\}. \end{cases}$$

$$S_{3n+1} = \begin{cases} \{k_0, 0\} & \text{if } n \notin C, \\ \{x_1, 0\} & \text{if } n \in C \text{ such that } Tx_1x_1n. \end{cases}$$

$$S_{3i+2} = \begin{cases} \{i\} & \text{if } i \notin K, \\ \{i, x_2\} & \text{if } i \in K, T x_1 x_1 n \text{ for some } n \in C \text{ and } x_2 = \min\{z \in B : z > n\}. \end{cases}$$

Since C is recursive, the languages S_{3n} and S_{3n+1} can be computed from n with their canonical index. For the languages S_{3i+1} , we always include i and we also include x_2 whenever we find n showing that $i \in K$. Note that for all $x \in K - B$ there are exactly two $y, z \in \mathbb{N}$ such that $\{x, y\}, \{x, z\} \in \mathcal{S}$ and these languages contain x . To see this, note that the y and z are precisely 0 and $x_2 = \min\{x \in B : x > n\}$ with $n \in C$ from the description of the languages. Observe that $x_2 \in S_{3n}$ and $x_2 \in S_{3i+2}$, so S_{3n} and S_{3i+2} are identical if x_1 in S_{3n} is the same as i in S_{3i+2} .

For $x \in B$ there may be more than two elements $y \in \mathbb{N}$ pairing with x , but all such $y \in \mathbb{N}$ are smaller than x (this happens if x acts as x_2 in S_{3n} or in S_{3i+2}). Thus, there are only finitely many of such $y \in \mathbb{N}$. In fact, they can all be computed, but this does not seem relevant. The resulting family is clearly an anti-chain since i can occur in \mathcal{S} as the singleton $\{i\}$ only if $i \notin K$ and thus it will not pair with an element of K .

We now first prove that the resulting family is indexed. For this, we only have to check whether $x \in S_{3n}$, $x \in S_{3n+1}$ and $x \in S_{3i+2}$ are recursive (since $x \in S_0$, $x \in S_1$ and $x \in S_2$ clearly are recursive). As mentioned before, the languages S_{3n} and S_{3n+1} can be computed from n with their canonical index, thus $x \in S_{3n}$, $x \in S_{3n+1}$ are recursive. Now let us check it for S_{3i+2} . Clearly, $i \in S_{3i+2}$. Consider $x \neq i$, then

$$x \in S_{3i+2} \text{ iff } \exists y < x (T i i y \text{ and } x = \min\{z \in B : y < z\}),$$

which is recursive. Thus, the family \mathcal{S} is indexed.

Next we show that \mathcal{S} is not pfi. Towards a contradiction, suppose λ is a pfi learner for \mathcal{S} . Take $i > 0$ and $m \in \mathbb{N}$ and suppose λ has received the segment $\sigma^+[m] = i, i, \dots, i$ for some $m \in \mathbb{N}$. Note that any such $\sigma^+[m]$ is an initial segment of some positive data sequence for some language in \mathcal{S} . If $i \in K$, by our construction above, there are at least two languages that contain i so $\lambda(\sigma^+[m]) = \uparrow$. If $i \notin K$, the segment $\sigma^+[k] = i, i, \dots, i$ corresponds to a positive data sequence σ^+ for $S_{3i+2} = \{i\}$. Thus, λ needs to make a conjecture at some point and so

$$\exists m, \lambda(\sigma^+[m]) \neq \uparrow.$$

Altogether, and since $(i \in \mathbb{N} - K \text{ iff } \exists m, \lambda(\sigma^+[m]) \neq \uparrow)$, we obtain that $\mathbb{N} - K$ is r.e., which is a contradiction since K is not recursive. Therefore, \mathcal{S} is not pfi.

Finally, we will show that \mathcal{S} is cfi. For S_0, S_1 and S_2 , a definite tell-tale pair is (S_0, \emptyset) (recall that $S_0 = S_1 = S_2$). For S_{3n} and S_{3n+1} , the definite tell-tale pairs

are (S_{3n}, \emptyset) and (S_{3n+1}, \emptyset) respectively. This is simply because the languages S_{3n} and S_{3n+1} can be computed from n with their canonical index. For S_{3i+2} , first note the following: if $(D_{3i+2}, \overline{D}_{3i+2})$ is a definite tell-tale pair for S_{3i+2} , the positive set in $(D_{3i+2}, \overline{D}_{3i+2})$ cannot have two elements, i.e., $D_{3i+2} \neq S_{3i+2}$. This is because such a definite tell-tale exists only if $i \in K$ but we cannot decide this. It turns out that we can take the positive set of the definite tell-tale pair of S_{3i+2} to be the singleton $\{i\}$. Thus, if $i \notin B$ then $(\{i\}, \{0\})$ is a definite tell-tale pair for S_{3i+2} . To see this, consider the following. First, $i \in S_{3i+2}$ always. If $i \notin B$, by construction of \mathcal{S} we have the following cases:

$i \in K - B$: we have $S_{3i+2} = \{i, x_2\}$ with $Tiin$ for some n and $x_2 > n$ and thus $n \in C$. Therefore, the learner must be able to disambiguate S_{3i+2} from the language $S_{3n+1} = \{i, 0\}$. Clearly, $(\{i\}, \{0\})$ allows disambiguation and so it is a definite tell-tale pair for $S_{3i+2} = \{i, x_2\}$.

$i \notin K$: we have $S_{3i+2} = \{i\}$, thus the pair $(\{i\}, \{0\})$ works as a definite tell-tale pair for $S_{3i+2} = \{i\}$.

Now, if $i \in B$, $(\{i\}, \{0, \dots, i-1\})$ is a definite tell-tale pair for S_{3i+2} . To see this, note that i may appear also in finitely many other pairs in \mathcal{S} of the form $\{x, i\}$ where $x < i$ (when $i = x_2$ as in the construction above). Thus, it is good enough if we put the negative set in the definite tell-tale pair as $\overline{D}_{3i+2} = \{0, \dots, i-1\}$. Clearly, $(\{i\}, \{0, \dots, i-1\})$ will ensure disambiguation between S_{3i+2} and the other languages containing i . Altogether, \mathcal{S} is cfi. \square

In the section that follows, we will see that some “special” cfi anti-chains, are always pfi, namely the ones that can be learnt in the fastest way.

6.5 Fastest learning

In this section we focus on fastest learning with positive data, as defined by [Gierasimczuk and de Jongh \(2013\)](#), and formalize it for complete data. Intuitively, a fastest learner identifies a language as soon as it is objectively certain which language it is. Recall that a learner in DEL updates her information state as soon as new factual information has been made available. In those terms, a fastest learner is closer in spirit to a learner in DEL than the standard learner in FLT. Formally, a family is positively identified in the fastest way if all the DFTTs for all the languages in the family are *uniformly available and recognizable* by some recursive learning function. Such a learning function is called a *fastest learner*.

We present a much simpler example than the one in [\(Gierasimczuk and de Jongh, 2013\)](#) to show that not every pfi family is identifiable by a fastest learner. Then, we extend the definition in [\(Gierasimczuk and de Jongh, 2013\)](#), to reason about fastest learning with complete data. Finally, we show that every cfi anti-chain for which a fastest cfi learner exists, is also pfi.

Consider the following definition of fastest learning from [Gierasimczuk and de Jongh \(2013\)](#). Let \mathcal{S} be any indexed family, for every $S_i \in \mathcal{S}$ we denote \mathbb{DFTT}_i the set of all the DFTTs of S_i .

6.5.1. DEFINITION. [Fastest pfi] Let \mathcal{S} be an indexed family of languages. We say that \mathcal{S} is *finitely identifiable with positive data in the fastest way*, or *fastest pfi* in short, if and only if there is a learning function λ , such that, for each σ^+ and $i \in \mathbb{N}$,

$$\begin{aligned} \lambda(\sigma^+[n]) = j \text{ for some } S_j = S_i \quad \text{iff} \quad & \exists D_i^k \in \mathbb{DFTT}_i \text{ s.t. } D_i^k \subseteq \text{set}(\sigma^+[n]) \\ & \text{and } \nexists D_i^l \in \mathbb{DFTT}_i \text{ s.t. } D_i^l \subseteq \text{set}(\sigma^+[n-1]). \end{aligned}$$

We will call such λ a *fastest learning function* (or a *fastest learner*).

The definition above, is easily seen to be equivalent to the following more intuitive form:

$$\begin{aligned} \lambda(\sigma^+[n]) = j \text{ for some } S_j = S_i \quad \text{iff} \quad & \text{set}(\sigma^+[n]) \cap \mathcal{S} = \{S_i\} \\ & \text{and } \text{set}(\sigma^+[m]) \cap \mathcal{S} \neq \{S_i\} \text{ for } m < n. \end{aligned}$$

It is easy to see that the family of all pairs, \mathcal{S}^2 , is fastest pfi. This is simply because for any $S_i \in \mathcal{S}^2$, the only possible DFTT for S_i is the language itself. In fact, for every $n \in \mathbb{N}$, the family \mathcal{S}^n is fastest pfi.

We find the following theorem and definition by [Gierasimczuk and de Jongh \(2013\)](#) useful for getting more intuition about fastest learning. Theorem [6.5.3](#) states an equivalence between indexed families learnable in the fastest way and indexed families for which a *complete dftt-function* exists. In simple terms, a complete dftt-function for an indexed family \mathcal{S} is a recursive function that identifies all the DFTTs of every language S_i in the family.

6.5.2. DEFINITION. [Complete dftt-function] Let \mathcal{S} be an indexed family of languages. The *complete dftt-function* for \mathcal{S} is a recursive function $\nu_{\text{dftt}} : \mathcal{P}^{<\omega}(\mathbb{N}) \times \mathbb{N} \rightarrow \{0, 1\}$, such that:

1. $\nu_{\text{dftt}}(D, i) = 1$ if and only if D is a DFTT of S_i ,
2. for every $i \in \mathbb{N}$ there is a finite $D \subseteq \mathbb{N}$, such that $\nu_{\text{dftt}}(D, i) = 1$.

6.5.3. THEOREM ([GIERASIMCZUK AND DE JONGH \(2013\)](#)). *An indexed family \mathcal{S} is finitely identifiable in the fastest way iff there is a complete dftt-function for \mathcal{S} .*

Now let us move on to showing, in our way, that fastest learning is more restrictive than pfi learning. First, we need a definition and some useful propositions.

6.5.4. DEFINITION. Let \mathcal{S} and \mathcal{S}' be two indexed families of languages. We call *the sum of \mathcal{S} and \mathcal{S}'* the family $\mathcal{S} \oplus \mathcal{S}' := \{(S \oplus S')_i : i \in \mathbb{N}\}$ such that, $(S \oplus S')_i = \{2n : n \in S_i\} \cup \{2n + 1 : n \in S'_i\}$.

6.5.5. DEFINITION. [Singled valued families] We say that an indexed family of languages \mathcal{S} is *single valued* if each language occurs only once in the family, i.e., there are no $S_i, S_j \in \mathcal{S}$ such that $S_i = S_j$ and $i \neq j$.

6.5.6. PROPOSITION. *Let \mathcal{S} and \mathcal{S}' be two single valued indexed anti-chains. If \mathcal{S} or \mathcal{S}' is pfi then $\mathcal{S} \oplus \mathcal{S}'$ is pfi.*

Proof:

Suppose \mathcal{S} is pfi. We will prove that by using the learner ν for \mathcal{S} we can construct a pfi learner λ for $\mathcal{S} \oplus \mathcal{S}'$. In simple terms, λ will identify the part $\{2n : n \in S_i\}$ from every $(S \oplus S')_i$ using the learner ν . Let σ^+ be any positive sequence for a language $(S \oplus S')_i \in \mathcal{S} \oplus \mathcal{S}'$. For every $n \in \mathbb{N}$ such that $\sigma^+[n] := \langle \sigma^+(0), \dots, \sigma^+(n) \rangle$ we define $\lambda(\langle \sigma^+(0), \dots, \sigma^+(n) \rangle) = \nu(\langle \tau^+(0)/2, \dots, \tau^+(k)/2 \rangle)$ where $\langle \tau^+(0), \dots, \tau^+(k) \rangle$ is the subsequence of $\langle \sigma^+(0), \dots, \sigma^+(n) \rangle$ of its even members, and $\lambda(\epsilon) = \nu(\epsilon) = \uparrow$.

Now we will see that λ is a pfi learner for $\mathcal{S} \oplus \mathcal{S}'$. Let σ^+ be any positive sequence for a language $(S \oplus S')_i$. Note that the sequence $\tau^+(0)/2, \dots, \tau^+(k)/2, \dots$ obtained from σ^+ , is a positive sequence for the corresponding language $S_i \in \mathcal{S}$. Recall that ν is a pfi learner for \mathcal{S} , thus ν identifies the language $S_i \in \mathcal{S}$ with respect to the sequence $\tau^+(0)/2, \dots, \tau^+(k)/2, \dots$. Since \mathcal{S} is single valued, $S_i \in \mathcal{S}$ corresponds only to the language $(S \oplus S')_i$. Thus by construction of λ , $\lambda(\langle \sigma^+(0), \dots, \sigma^+(n) \rangle) = i$. For \mathcal{S}' the proof is analogous. Thus, $\mathcal{S} \oplus \mathcal{S}'$ is pfi. \square

Observe that in the proof of Proposition [6.5.7](#) the learner λ can be sure about her output i precisely because \mathcal{S} and \mathcal{S}' are single valued. To illustrate, suppose there is $S_j \in \mathcal{S}$ such that $S_j = S_i$ and $j \neq i$ but $S'_j \neq S'_i$ in \mathcal{S}' . Then, λ cannot distinguish on the basis of ν between the languages $(S \oplus S')_i$ and $(S \oplus S')_j$ which are normally not the same.

6.5.7. PROPOSITION. *Let \mathcal{S} and \mathcal{S}' be two single valued indexed nepfi families such that $\mathcal{S} \oplus \mathcal{S}'$ is fastest pfi. Then, both families \mathcal{S} and \mathcal{S}' are pfi.*

Proof:

Suppose there is a fastest pfi learner λ' for $\mathcal{S} \oplus \mathcal{S}'$. Thus, there is an effective uniform procedure that produces all the DFTTs of any language $(S \oplus S')_i \in \mathcal{S} \oplus \mathcal{S}'$. W.l.o.g. we will use λ' to construct a fastest learner ν' for \mathcal{S}' . Take any positive sequence σ^+ for some $S'_i \in \mathcal{S}'$ and assume $\langle \sigma^+(0), \dots, \sigma^+(n) \rangle$ is a sequence such that a DFTT, D , exists with $D \subseteq \{\sigma^+(0), \dots, \sigma^+(n)\}$ and no such DFTT exists for $\{\sigma^+(0), \dots, \sigma^+(n-1)\}$. Note that such n exists since \mathcal{S}' is

nepfi. Now note that $\{2\sigma^+(0) + 1, \dots, 2\sigma^+(n) + 1\}$ contains a DFTT for $(S \oplus S')_i$ namely $Y = \{2x + 1 : x \in D\}$ and $\{2\sigma^+(0) + 1, \dots, 2\sigma^+(n-1) + 1\}$ contains no DFTT. thus $\nu'(\langle \sigma^+(0), \dots, \sigma^+(n) \rangle) = \lambda'(\langle 2\sigma^+(0) + 1, \dots, 2\sigma^+(n) + 1 \rangle) = i$ because λ' is a fastest learner for $\mathcal{S} \oplus \mathcal{S}'$. It follows that, ν' is a fastest learner for \mathcal{S}' and therefore \mathcal{S}' is pfi. We can follow a similar reasoning to show that \mathcal{S} is pfi. \square

6.5.8. THEOREM. *Fastest learning with positive data is more restrictive than pfi learning on anti-chains of finite languages.*

Proof:

We will construct a pfi family which is not learnable by any fastest learner. Take for \mathcal{S} any single valued pfi family which is an anti-chain together with the non-pfi but nepfi single valued indexed family we constructed in Proposition 6.4.2 (in fact, any nepfi but not pfi single valued indexed family will do). We denote the latter family by $\mathcal{S}' = \{S'_i : i \in \mathbb{N}\}$. Take the sum $\mathcal{S} \oplus \mathcal{S}' = \{(S \oplus S')_i : i \in \mathbb{N}\}$. Since \mathcal{S} is pfi, by Proposition 6.5.7, $\mathcal{S} \oplus \mathcal{S}'$ is also pfi. Now, note that $\mathcal{S} \oplus \mathcal{S}'$ is not fastest learnable, because otherwise by Proposition 6.5.7, \mathcal{S}' would be pfi which is not the case. \square

The following proposition shows that every cfi anti-chain of finite languages for which a fastest cfi learner exists, is pfi. We first need to define what a fastest cfi learner is and give some notation.

Let us take a cfi family \mathcal{S} and a language $S_i \in \mathcal{S}$ and consider the family \mathbb{TTP}_i of all the tell-tale pairs of S_i .

For every complete data sequence σ of some $S \in \mathcal{S}$ and $n \in \mathbb{N}$, let $set^1(\sigma[n]) := \{m \in \mathbb{N} : (m, 1) \in \sigma[n]\}$ and $set^0(\sigma[n]) := \{m \in \mathbb{N} : (m, 0) \in \sigma[n]\}$. In simple words, $set^1(\sigma[n])$ is the set of all the elements in the finite segment $\sigma[n]$ that are in S and, analogously, $set^0(\sigma[n])$ is the set of the elements that are in the complement of S . We can now define the notion of a fastest cfi learner as follows.

6.5.9. DEFINITION. Let \mathcal{S} be an indexed family of languages. We say that \mathcal{S} is *finitely identifiable with complete data in the fastest way*, or *fastest cfi* in short, if and only if there is a learning function λ , such that, for each σ and $i \in \mathbb{N}$,

$$\begin{aligned} \lambda(\sigma[n]) = j \text{ for some } S_j = S_i \text{ iff } & \exists (D_i^k, \overline{D}_i^k) \in \mathbb{TTP}_i \\ & \left(D_i^k \subseteq set^1(\sigma[n]) \text{ and } \overline{D}_i^k \subseteq set^0(\sigma[n]) \right), \\ \text{and } & \nexists (D_i^l, \overline{D}_i^l) \in \mathbb{TTP}_i \\ & \left(D_i^l \subseteq set^1(\sigma[n-1]), \overline{D}_i^l \subseteq set^0(\sigma[n-1]) \right). \end{aligned}$$

We will call such λ a *fastest cfi learning function* (or a *fastest cfi learner*).

The definition above, is easily seen to be equivalent to:

$$\begin{aligned} \lambda(\sigma[n]) = j \text{ for some } S_j = S_i \text{ iff } \{S \in \mathcal{S} : S \text{ is consistent with the pair} \\ (set^1(\sigma[n]), set^0(\sigma[n]))\} = \{S_i\} \\ \text{and } \forall m < n, \\ \{S \in \mathcal{S} : S \text{ is consistent with the pair} \\ (set^1(\sigma[m]), set^0(\sigma[m]))\} \neq \{S_i\}. \end{aligned}$$

Similar to the fastest pfi learner, a fastest cfi learner needs access to all tell-tale pairs of all languages in the corresponding family in a uniform way. Then the learner makes a conjecture on the basis of the first tell-tale pair that appears in the sequence in question.

6.5.10. PROPOSITION. *If \mathcal{S} is an indexable antichain of finite languages for which a fastest cfi learner exists, then it is pfi.*

Proof:

Let \mathcal{S} be an indexable antichain of finite languages such that λ is a fastest cfi learner for \mathcal{S} . Using λ , we will show that there is a uniform effective procedure that outputs a DFTT for every $S_i \in \mathcal{S}$. Now let σ^+ be any positive sequence for any language $S_i \in \mathcal{S}$. Using σ^+ we can construct step-by-step a complete data sequence $\sigma := (\sigma^+(0), 1), (\sigma^+(1), 1), (\sigma^+(2), 1) \dots$, that will correspond to $S_i \in \mathcal{S}$. Note that since \mathcal{S} is an antichain and every language is finite, (S_i, \emptyset) is a tell-tale pair for $S_i \in \mathcal{S}$ and λ recognizes the tell-tale pair whenever it appears in a complete sequence. Clearly, after some $n \in \mathbb{N}$, $S_i \subseteq set(\sigma^1[n])$ and $\emptyset \subseteq set(\sigma^0[n])$ and there is no other pair $(D, \bar{D}) \in \mathbb{TTP}_i$ for which $D \subseteq set(\sigma^1[n-1])$ and $\bar{D} \subseteq set(\sigma^0[n-1])$ is the case. Thus $\lambda(\sigma[n]) = j$ for some $S_j = S_i$. Therefore $D_i = set(\sigma^1[n])$ is a DFTT for S_i . Since our choice of $S_i \in \mathcal{S}$ and of a sequence for S_i was arbitrary, it follows that \mathcal{S} is pfi. \square

This does not generalize to indexed families with infinite languages. The antichain of co-singletons is an obvious counterexample, because it is easy to see that its standard cfi learner is a fastest learner.

Of course, one can prove by a similar construction as in Theorem 6.5.8 that fastest cfi learning is restrictive with respect to cfi learning, but it follows already immediately from Proposition 6.5.7 and Proposition 6.5.10.

6.5.11. COROLLARY. *Fastest cfi learning is more restrictive than cfi learning.*

6.6 Finite identification from queries

In this section we study *finite identification from queries* or, in short, *learning from queries*. Our work is inspired by the work of [Angluin \(1987\)](#), where the problem of identifying languages from its elements and the elements in its complement

by asking queries is studied. The learner can produce queries and more than one conjecture, whereas the teacher presents the language to the learner by answering membership queries and testing the learner's conjectures. When a conjecture is not correct, the teacher then gives a "counter-example" to the learner, a string in the symmetric difference of the target language and the one conjectured.

Following our notation and methodology, we entertain a similar but simplified idea to study finite identification with a learner that can produce queries. This learning notion turns out to be close to cfi learning. In this model, the learner receives answers from nature (or a teacher) to her queries, and can produce either another query or a conjecture that aims at the target language. Clearly, after one conjecture is produced the identification procedure stops. The learner can also abstain, in which case she will receive from nature (or teacher) a new element from a complete sequence chosen initially by nature from the target language (as if the learner is a standard cfi learner).

The *query learner* will be a composition of two functions that act one after the other. One function produces the queries, and after nature (or a teacher) gives an answer to her query, the second function produces a conjecture. The conjecture is based on the sequence of all the previous data from nature. We will show that an indexed family is cfi learnable if and only if it is learnable by queries. We will also study a *strict* query learner, one that never abstains and always produces a query. We will see that the class of families for which a strict query learner exists is also equivalent to cfi.

Recall that we call *Seg* the set of all initial segments of all complete sequences. We say that an initial segment $\sigma[n] := \langle (x_0, t_0), \dots, (x_{n-1}, t_{n-1}) \rangle$ in *Seg* is consistent with a language S , if $\{x \in \mathbb{N} : (x, 1) \in \sigma[n]\} \subseteq S$ and $\{x \in \mathbb{N} : (x, 0) \in \sigma[n]\} \subseteq \mathbb{N} - S$.

6.6.1. DEFINITION. [Query learner]

- Given a complete sequence σ of some language S and an initial segment $\tau[n] := \langle (x_0, t_0), \dots, (x_{n-1}, t_{n-1}) \rangle$ in *Seg* consistent with S , a *question function* $\alpha : \text{Seg} \rightarrow \mathbb{N} \cup \{\#\}$ is a recursive function such that when $\alpha(\tau[n]) = x_n$, we say that α produces a query and when $\alpha(\tau[n]) = \#$ we say that α abstains. When $\alpha(\tau[n]) \cap S \in \mathbb{N}$ or $\alpha(\tau[n]) \cap S - \mathbb{N}$ has been decided, we say that an answer has been provided.
- A *query learner* is a pair of recursive functions $\Omega := (\alpha, \lambda)$ with a cfi learner $\lambda : \text{Seg} \rightarrow \mathbb{N} \cup \uparrow$ and a question function $\alpha : \text{Seg} \rightarrow \mathbb{N} \cup \{\#\}$. We define $\lambda(\sigma[0]) = \uparrow$ for the empty sequence $\sigma[0] = \epsilon$. To differentiate from cfi learners, we will call λ the *conjecture function* in Ω .
- Given a complete sequence σ of some language S and a query learner $\Omega := (\alpha, \lambda)$, we call σ_Ω the sequence produced by Ω from a complete sequence σ . As usual, $\sigma_\Omega(n)$ denotes the n -th element of the sequence and $\sigma_\Omega[n]$ the initial

segment of length n , i.e., $\sigma_\Omega[n] := \langle \sigma_\Omega(0), \dots, \sigma_\Omega(n-1) \rangle$. For every $n \in \mathbb{N}$, the sequence σ_Ω is defined recursively as follows: $\sigma[0] = \sigma_\Omega[0] = \epsilon$,

$$\sigma_\Omega(n) = \begin{cases} \emptyset & \text{if } \lambda(\sigma_\Omega[n]) = j \text{ for some } j \in \mathbb{N}, \\ (\alpha(\sigma_\Omega[n]), 1) & \text{if } \lambda(\sigma_\Omega[n]) = \uparrow \text{ and } \alpha(\sigma_\Omega[n]) \in \mathbb{N} \cap S, \\ (\alpha(\sigma_\Omega[n]), 0) & \text{if } \lambda(\sigma_\Omega[n]) = \uparrow \text{ and } \alpha(\sigma_\Omega[n]) \in \mathbb{N} - S, \\ \sigma(m) & \text{if } \lambda(\sigma_\Omega[n]) = \uparrow, \alpha(\sigma_\Omega[n]) = \# \\ & \text{and } \sigma(m) \text{ is the first element of } \sigma \text{ that} \\ & \text{does not appear in } \sigma_\Omega[n]. \end{cases}$$

Intuitively, after $\lambda(\sigma[0]) = \uparrow$, a query learner, at each step of the process, first uses the question function, α , to possibly output a query x_n (which will be answered by nature or by a teacher with $x_n \in \mathbb{N} \cap S$ or $x_n \in \mathbb{N} - S$ in each case, with S the target language). The query produced by α is based on an initial segment $\sigma_\Omega[n]$ (starting with the empty initial segment). Then the learner uses the conjecture function, λ , to produce a conjecture, where, as usual, \uparrow stands for undefined. Observe that given a language S and a complete sequence σ for S , for every $n \in \mathbb{N}$, $\sigma_\Omega(n)$ is consistent with σ . To illustrate, if nature answers $x \in \mathbb{N} \cap S$ for some query x produced by Ω , $(x, 1)$ must appear in σ because σ is a complete sequence for S . Thus, $set(\sigma_\Omega[n]) \subseteq set(\sigma)$.

In what follows, we will always consider that the answers come from nature (an analysis with respect to a teacher is essentially the same).

6.6.2. DEFINITION. [Finitely identifiable from queries] Let $\mathcal{S} = \{S_i : i \in \mathbb{N}\}$ be an indexed family of languages and let $\Omega := (\alpha, \lambda)$ be a query learner.

- Given a complete sequence σ of a language $S_i \in \mathcal{S}$. We say that Ω *identifies* S_i on σ , if for some $n \in \mathbb{N}$, $\lambda(\sigma_\Omega[n]) = j$ for some $j \in \mathbb{N}$ such that $S_i = S_j$ and stops.
- We say that Ω *identifies* S_i on every complete sequence σ of S_i , if for every σ of S_i , Ω identifies S_i on σ .
- A family \mathcal{S} is *finitely identifiable by the query learner* Ω iff for every $S_i \in \mathcal{S}$, Ω identifies S_i on every complete sequence σ of S_i .
- A family \mathcal{S} is *finitely identifiable from queries (qfi)* iff there is a recursive query learner $\Omega := (\alpha, \lambda)$ such that for every $S_i \in \mathcal{S}$, Ω identifies S_i on every complete sequence σ of S_i .

As in the case of nepfi and necfi, we occasionally relax the condition of indexability of the family in question and the condition of recursivity of both functions in the query learner Ω . In such cases, Ω is said to be a *non-effective query learner* and \mathcal{S} is said to be *non-effectively finitely identifiable from queries* (in short, neqfi). The next theorem is the corresponding counterpart of the Characterization Theorem by Mukouchi (1992), and Lange and Zeugmann (1992, 1996) with respect to learning by queries.

6.6.3. THEOREM (CHARACTERIZATION OF QFI FAMILIES). *An indexed family \mathcal{S} of languages is finitely identifiable with queries (qfi) iff for every $S_i \in \mathcal{S}$ there is a definite tell-tale pair (D_i, \overline{D}_i) in a uniformly computable way. That is, there exists an effective procedure Φ that on input i , index of S_i , produces the canonical index $\Phi(i)$ of some definite finite tell-tale pair of S_i .*

Proof:

Follows by a similar reasoning as in the proof of the Characterization Theorem for complete data in (Mukouchi, 1992).

For (\Rightarrow): Suppose there is a uniform computable procedure that produces a definite tell-tale pair (D_i, \overline{D}_i) for every $S_i \in \mathcal{S}$. We will construct a query learner $\Omega := (\alpha, \lambda)$ for \mathcal{S} using the definite tell-tale pairs. Take α a recursive function such that for any initial segment $\tau \in \text{Seg}$, $\alpha(\tau) = \#$ and take any cfi learner λ for \mathcal{S} . By the Characterization Theorem in (Mukouchi, 1992), we know λ exists. Clearly $\Omega = (\alpha, \lambda)$ is a query learner for \mathcal{S} , since obviously $\sigma_\Omega = \sigma$ where σ is the complete data sequence initially chosen.

For (\Leftarrow): Suppose \mathcal{S} is qfi. There is a query learner $\Omega := (\alpha, \lambda)$ that identifies every $S_i \in \mathcal{S}$ on any complete sequence σ for S_i . Let $S_i \in \mathcal{S}$, let σ be any complete sequence of S_i and consider any σ_Ω produced by Ω from σ . Then there is $n \in \mathbb{N}$ such that $\lambda(\sigma_\Omega[n]) = j$ with $S_j = S_i$ and for any $l < n$, $\lambda(\sigma[l]) = \uparrow$. Let (D_i, \overline{D}_i) be such that $D_i := \{(y_k, t_k) \in \sigma_\Omega[n] : t_k = 1\}$ and $\overline{D}_i := \{(y_k, t_k) \in \sigma_\Omega[n] : t_k = 0\}$. Clearly (D_i, \overline{D}_i) is a definite tell-tale pair for S_i . Obviously both D_i and \overline{D}_i are finite. Towards contradiction, suppose (D_i, \overline{D}_i) is consistent with another language $S_k \in \mathcal{S}$ such that $S_i \neq S_k$. Then, nature can extend the segment $\sigma_\Omega[n]$ with a complete sequence σ' consistent with S_k . Note that σ' will also be a complete sequence for S_k since the initial segment was consistent with S_k . But Ω will still identify σ' as a sequence for S_i . This contradicts that $\Omega := (\alpha, \lambda)$ is a query learner for \mathcal{S} and that it identifies S_k . Thus (D_i, \overline{D}_i) is a definite tell-tale pair for S_i . \square

6.6.4. PROPOSITION. *An indexed family \mathcal{S} is cfi learnable iff \mathcal{S} is finitely identifiable by queries.*

Proof:

Follows straightforwardly by the Characterization Theorem for complete data by

Mukouchi (1992) (also by Lange and Zeugmann (1992, 1996)) and from Theorem 6.6.3. \square

Consider a query learner who always produces a query and never abstains.

6.6.5. DEFINITION. [Strict query learner]

- A *strict query learner* is a query learner $\Omega := (\alpha, \lambda)$ with $\lambda : Seg \rightarrow \mathbb{N} \cup \uparrow$ and $\alpha : Seg \rightarrow \mathbb{N}$, such that $\alpha(\tau) = x_n \in \mathbb{N}$ and after an answer is produced, namely (x_n, t_n) , $\lambda(\tau, (x_n, t_n))$ produces a conjecture $i \in \mathbb{N}$ or \uparrow .
- Given a language S and a strict query learner $\Omega := (\alpha, \lambda)$, for every $n \in \mathbb{N}$, the sequence σ_Ω is defined recursively as follows: $\sigma_\Omega[0] = \epsilon$,

$$\sigma_\Omega(n) = \begin{cases} \emptyset & \text{if } \lambda(\sigma_\Omega[n]) = j \text{ for some } j \in \mathbb{N}, \\ (\alpha(\sigma_\Omega[n]), 1) & \text{if } \lambda(\sigma_\Omega[n]) = \uparrow \text{ and } \alpha(\sigma_\Omega[n]) \in \mathbb{N} \cap S, \\ (\alpha(\sigma_\Omega[n]), 0) & \text{if } \lambda(\sigma_\Omega[n]) = \uparrow \text{ and } \alpha(\sigma_\Omega[n]) \in \mathbb{N} - S. \end{cases}$$

6.6.6. DEFINITION. [Finitely identifiable by a strict query learner] Let $\mathcal{S} = \{S_i : i \in \mathbb{N}\}$ be an indexed family of languages. The learning notions defined in Definition 6.6.2 are defined similarly for a strict query learner $\Omega := (\alpha, \lambda)$.

One could think that the properties described in Definition 6.6.5 would restrict the capabilities of a query learner. We will show that this is not the case, a family \mathcal{S} is learned by a query learner if and only if it is learned by a strict query learner. In fact, the class of cfi families is equivalent to the class of strict-qfi families. First we need some definition and notation.

6.6.7. DEFINITION. [Finite canonical complete sequence] Let \mathcal{S} be an indexed family and Φ be an effective procedure such that $\Phi(i) = (D_i, \overline{D}_i)$ is a definite tell-tale pair for i . A sequence representation of (D_i, \overline{D}_i) is a finite sequence $\Phi_{seq}(i) := (x_0^i, t_0^i), (x_1^i, t_1^i), \dots, (x_k^i, t_k^i)$ where,

- $\{x_0^i, \dots, x_k^i\} = D_i \cup \overline{D}_i$,
- for each $j \leq k$, $t_j^i = 1$ iff $x_j^i \in D_i$ and $t_j^i = 0$ iff $x_j^i \in \overline{D}_i$, and
- for each $x \in D_i \cup \overline{D}_i$, the corresponding pair (x, t) occurs only once.

We will call such a sequence a *finite canonical complete sequence for i* (in short, *finite canonical sequence for i*).

In other words, we choose for each $i \in \mathbb{N}$ a finite canonical sequence $\Phi_{seq}(i)$ such that

$$(x, 1) \text{ occurs in } \Phi_{seq}(i) \text{ if and only if } x \in D_i$$

and

$$(x, 0) \text{ occurs in } \Phi_{seq}(i) \text{ if and only if } x \in \overline{D}_0.$$

Therefore, $\Phi_{seq}(i)$ is an initial segment of a complete sequence which pairs correspond exactly to the elements in the definite tell-tale pair produced by Φ on input i .

The notion presented in the following definition will be useful for the proof of the Characterization theorem for strict-qfi families in Mukouchi style.

6.6.8. DEFINITION. Let \mathcal{S} be an indexed family, let $S_j, S_i \in \mathcal{S}$, let $\Omega := (\alpha, \lambda)$ be a strict query learner and let $\Phi_{seq}(i) = (x_0^i, t_0^i), (x_1^i, t_1^i), \dots, (x_k^i, t_k^i)$ be a finite canonical sequence for $S_i \in \mathcal{S}$. Let τ be any initial segment of $\Phi_{seq}(i)$ of length $m < k$ and $\alpha(\tau) = x_{m+1}^i$ such that $(x_{m+1}^i, t_{m+1}^i) \in \Phi_{seq}(i)$ (i.e., α queries x_{m+1}^i). We say that (x_{m+1}^i, t_{m+1}^i) fits the answer to the query x_{m+1}^i with respect to S_j if $(x_{m+1}^i \in S_j \text{ and } t_{m+1}^i = 1)$ or $(x_{m+1}^i \in \mathbb{N} - S_j \text{ and } t_{m+1}^i = 0)$.

Equivalently, given a sequence σ of S_j , we can say that (x_{m+1}^i, t_{m+1}^i) fits the sequence σ of S_j if $\sigma(n) = (x_{m+1}^i, t_{m+1}^i)$ for some $n \in \mathbb{N}$.

6.6.9. THEOREM (CHARACTERIZATION OF STRICT-QFI FAMILIES). *An indexed family \mathcal{S} of languages is finitely identifiable with queries by a strict query learner (strict-qfi) iff for every $S_i \in \mathcal{S}$ there is a definite tell-tale pair (D_i, \overline{D}_i) in a uniformly computable way. That is, there exists an effective procedure Φ that on input i , index of S_i , produces the canonical index $\Phi(i)$ of some definite finite tell-tale pair of S_i .*

Proof:

For (\Rightarrow): Suppose there is a uniform computable procedure that produces a definite tell-tale pair (D_i, \overline{D}_i) for every $S_i \in \mathcal{S}$. We will construct a strict query learner $\Omega := (\alpha, \lambda)$ for \mathcal{S} using the definite tell-tale pairs. Suppose $S_j \in \mathcal{S}$ for some $j \in \mathbb{N}$ is the target language. For every $i \in \mathbb{N}$, consider a finite canonical sequence $\Phi_{seq}(i) = (x_0^i, t_0^i), (x_1^i, t_1^i), \dots, (x_k^i, t_k^i)$ as defined above. We run through all the sequences $\Phi_{seq}(i)$ in the natural order. At each stage $n \in \mathbb{N}$, the learner considers $\Phi_{seq}(i)$ at x , i.e., the learner will produce a query x corresponding to (x, t) of $\Phi_{seq}(i)$. We introduce the identification procedure in stages 0 and 1 then we describe it generally for any stage $k > 1$.

At stage 0, the learner issues \uparrow . At stage 1, the learner considers $\Phi_{seq}(0)$ and queries $\alpha((x_0^0, t_0^0)) = p_1((x_0^0, t_0^0)) = x_0^0$ where p_1 is the standard pair projection function that returns the first element of a pair. Thus, the learner queries the first element of $\Phi_{seq}(0)$ so that $\sigma_\Omega(0) = (x_0^0, t_0^0)$.

If (x_0^0, t_0^0) fits the answer to the query x_0^0 and (x_0^0, t_0^0) is the final element of the sequence $\Phi_{seq}(0)$ then the learner has confirmed (D_0, \overline{D}_0) . Thus, she issues 0

and stops (and thus $S_0 = S_j$). If (x_0^0, t_0^0) is not the final element of the sequence $\Phi_{seq}(0)$, at stage 2, the learner issues \uparrow and continues with $\Phi_{seq}(0)$ producing the query $\alpha((x_0^0, t_0^0), (x_1^0, t_1^0)) = p_1((x_1^0, t_1^0)) = x_1^0$ so that $\sigma_\Omega(1) = (x_1^0, t_1^0)$.

If (x_0^0, t_0^0) does not fit the answer to the query x_0^0 , the learner issues \uparrow on (x_0^0, t_0^0) , i.e., $\lambda((x_0^0, t_0^0)) = \uparrow$. Then the learner is in stage 2 and considers $\Phi_{seq}(1)$. The learner then queries $\alpha((x_0^0, t_0^0), (x_0^1, t_0^1)) = p_1((x_0^1, t_0^1)) = x_0^1$, i.e., queries the first element of $\Phi_{seq}(1)$, such that $\sigma_\Omega(1) = (x_0^1, t_0^1)$.

Generally, at stage k , if (x_m^i, t_m^i) is the $(m + 1)$ -th element in $\Phi_{seq}(i)$ and $p_1((x_m^i, t_m^i)) = x_m^i$ is queried, then,

- i) if (x_m^i, t_m^i) fits the answer to the query x_m^i , then, $\sigma_\Omega(k - 1) = (x_m^i, t_m^i)$ and
 - a) if (x_m^i, t_m^i) is not the last element of $\Phi_{seq}(i)$, the learner issues \uparrow on the sequence seen so far. Then, on stage $k + 1$, the learner queries the next element in $\Phi_{seq}(i)$, namely $p_1((x_{m+1}^i, t_{m+1}^i)) = x_{m+1}^i$.
 - b) if (x_m^i, t_m^i) is the last element of $\Phi_{seq}(i)$, the learner issues i and stops (then $S_i = S_j$).
- ii) if (x_m^i, t_m^i) does not fit the answer to the query x_m^i , the learner issues \uparrow on the sequence seen so far and $\sigma_\Omega(k - 1) = \sigma_\Omega(k - 2)$. Then, on stage $k + 1$, the learner considers (x_0^{i+1}, t_0^{i+1}) and queries $p_1((x_0^{i+1}, t_0^{i+1})) = x_0^{i+1}$, i.e., it queries the first element in $\Phi_{seq}(i + 1)$.

Note that in each step of the process k , $\sigma_\Omega(k - 1) = (x, t) \in D_j$ when $t = 1$ and $\sigma_\Omega(k - 1) = (x, t) \in \overline{D}_j$ when $t = 0$. Since S_j was arbitrarily chosen, the strict query learner described in this procedure finitely identifies any language from \mathcal{S} .

For (\Leftarrow) : The proof goes as the corresponding proof (this direction of the biconditional) of Theorem [6.6.3](#), but with respect to a strict query learner $\Omega := (\alpha, \lambda)$ and the corresponding σ_Ω produced by Ω for some $S_i \in \mathcal{S}$. \square

6.7 Conclusion and future research

In this chapter, we studied the computational differences and links between finite identification with positive and with complete data. In particular, we analyzed infinite indexed anti-chains of finite languages (i.e., nepfi indexed families of finite languages). For these cases it was not clear what the connection (or difference) between cfi and pfi is. Given our findings in Chapter [5](#) that nepfi and necfi differ strongly regarding the existence of maximal families, we concentrated first on the description and existence of maximal families in the pfi case. To do so, we restricted our analysis to a class of simple anti-chains, namely the ones with only singletons and pairs. Considering such anti-chains we were able to find examples of the following kind. First, we presented a family of finite languages coded

standardly by canonical codes (and hence pfi) which does not have an effectively finitely identifiable maximal extension. Next, we constructed a non-canonical anti-chain for which the structural properties allow it to be pfi. Then, we exhibited an example of an indexable anti-chain that cannot be given a canonical indexing, and that is not pfi and not cfi. Due to the structural properties of anti-chains of finite languages, cfi seems to be closer to pfi. Indeed, for many cases of cfi anti-chains of finite languages pfi holds (it holds in most of the obvious examples). We showed that this is not always the case. For this, we presented an example of a non-canonical anti-chain that is cfi but not pfi. This shows that finite identification with complete data of infinite anti-chains of finite languages is more powerful than with positive data only.

We then studied fastest learning (or fastest identification) for positive data as introduced in (Gierasimczuk and de Jongh, 2013). We provided an alternative example to the one in (Gierasimczuk and de Jongh, 2013) to show that not every pfi family can be fastest identified. We extended the definition in (Gierasimczuk and de Jongh, 2013), to reason about fastest learning with complete data. We proved that every cfi anti-chain for which a fastest cfi learner exists, is also pfi. This shows that in fastest learning there is no difference between cfi and pfi with respect to anti-chains of finite languages.

Then we investigated another variation of finite identification, learning from queries. For this, we consider an active learner that can ask to the teacher questions about the concept being learnt. The query learner is a composition of two functions that act one after the other. One function produces the queries, and after the teacher gives an answer to her query. The second function produces a conjecture. The query learner's conjecture is based on the sequence of all the previous data from the teacher. We showed that a family is cfi learnable if and only if it is learnable by queries. This result also holds for the case when the learner never abstains and always produces a query.

Directions of future work involve investigating how our results stand up in the context of other types of learning and their consequences for the study of learning in Dynamic Epistemic Logic (DEL). In particular, studying the connection between fastest learning and learning with certainty via updates in DEL and the relationship with learning in the limit with a restricted number of mind changes. Considering a learner that can make at most two conjectures in learning in the limit will be very close to a pfi learner. However, since all families of singletons and pairs are identified in this restricted version of learning in the limit, such a variation is still more powerful than finite identification.

Appendix A

Technical Specifications of Part I

A.1 Complexity order on formulas in DLLT and in AGML

The following lemma guarantees the necessary complexity order for the proofs of Lemmas 3.3.13 and 3.4.10 in Sections 3.3 and 3.4 respectively from Chapter 3.

For the definition that follows, note that $pre(e)$ is a boolean formula (see Definition 3.4.8).

A.1.1. DEFINITION. [Subformula] Given a formula $\varphi \in \mathcal{L}_{\Pi_{\perp}}$, the set $Sub(\varphi)$ of subformulas of φ is recursively defined as

$$\begin{aligned} Sub(\varphi) &= \{\varphi\} && \text{if } \varphi \text{ is } \top, o \text{ or } p, \\ Sub(\neg\varphi) &= Sub(\varphi) \cup \{\neg\varphi\}, \\ Sub(K\varphi) &= Sub(\varphi) \cup \{K\varphi\}, \\ Sub(\varphi \wedge \psi) &= Sub(\varphi) \cup Sub(\psi) \cup \{\varphi \wedge \psi\}, \\ Sub(L(e)) &= Sub(pre(e)) \cup \{L(e)\}, \\ Sub([e]\psi) &= Sub(pre(e)) \cup Sub(\psi) \cup \{[e]\psi\}, \\ Sub(\Box\varphi) &= Sub(\varphi) \cup \{\Box\varphi\}. \end{aligned}$$

Any formula in $Sub(\varphi) - \{\varphi\}$ is called a *proper subformula* of φ .

A.1.2. LEMMA. *There is a well-founded strict partial order \prec_1 on formulas in $\mathcal{L}_{\Pi_{\perp}}$ (and therefore in \mathcal{L}_{Π} since $\mathcal{L}_{\Pi_{\perp}}$ properly extends \mathcal{L}_{Π}) called ‘complexity order’, satisfying the following conditions:*

- if φ is a proper subformula of ψ then $\varphi \prec_1 \psi$,
- $(pre(e) \rightarrow p) \prec_1 [e]p$,

- $(pre(e) \rightarrow o) \prec_1 [e]o$,
- $(pre(e) \rightarrow L(e; e')) \prec_1 [e]L(e')$,
- $([e]\varphi \wedge [e]\psi) \prec_1 [e](\varphi \wedge \psi)$,
- $(pre(e) \rightarrow K[e]\varphi) \prec_1 [e]K\varphi$,
- $\Box[e]\varphi \prec_1 [e]\Box\varphi$.

Proof:

A complexity measure $c : \mathcal{L}_{\Pi_{\perp}} \rightarrow \mathbb{N}$ that gives such a strict partial order on $\mathcal{L}_{\Pi_{\perp}}$ can be defined similarly as the one in van Ditmarsch et al., 2007, Definition 7.21) as follows:

$$\begin{aligned}
c(\top) = c(o) = c(p) &= 1, \\
c(\varphi \wedge \psi) &= 1 + \max(c(\varphi), c(\psi)), \\
c(L(e)) &= 1 + c(pre(e)), \\
c(\neg\varphi) = c(K\varphi) = c(\Box\varphi) &= 1 + c(\varphi), \\
c([e]\varphi) &= (5 + c(pre(e))) \cdot c(\varphi).
\end{aligned}$$

We then define $\prec_1 \subseteq \mathcal{L}_{\Pi_{\perp}} \times \mathcal{L}_{\Pi_{\perp}}$ as $\varphi \prec_1 \psi$ iff $c(\varphi) < c(\psi)$. The lemma then follows via easy calculations using c . For this, it is useful to note that $c(pre(e)) < c([e]\varphi)$.

To illustrate, we will show that $(pre(e) \rightarrow L(e; e')) \prec_1 [e]L(e')$, i.e., we need to show that $c(pre(e) \rightarrow L(e; e')) < c([e]L(e'))$. First note that for every $\varphi, \psi \in \mathcal{L}_{\Pi_{\perp}}$, $c(\varphi \rightarrow \psi) = c(\neg(\neg\varphi \wedge \neg\psi)) = 2 + \max(2 + c(\varphi), 1 + c(\psi))$. Also note that for every $e \in \Pi_{\perp}$, $1 \leq c(pre(e))$. Clearly we also have that $c(pre(e)) < c(pre(e; e'))$. We obtain the following equivalences. On the one hand,

$$\begin{aligned}
c(pre(e) \rightarrow L(e; e')) &= 2 + \max(2 + c(pre(e)), 1 + c(L(e; e'))) \\
&= 2 + \max(2 + c(pre(e)), 2 + c(pre(e; e'))) \\
&\hspace{15em} \text{(calculating } c(L(e; e'))\text{)} \\
&= 4 + \max(c(pre(e)), c(pre(e; e'))) \\
&= 4 + c(pre(e; e')) \hspace{3em} \text{(by the observation above)} \\
&= 5 + \max(c(pre(e)), c(pre(e'))) \\
&\hspace{15em} \text{(calculating } c(pre(e) \wedge pre(e'))\text{)}
\end{aligned}$$

On the other, by easy calculations we obtain that,

$$c([e]L(e')) = 5 + c(pre(e)) + 5 \cdot c(pre(e')) + c(pre(e)) \cdot c(pre(e')).$$

Then, if $c(pre(e)) \leq c(pre(e'))$ we have that

$$\begin{aligned}
c(pre(e) \rightarrow L(e; e')) &= 5 + c(pre(e')) \\
&< 5 + 5 \cdot c(pre(e')) \hspace{3em} \text{(since } 1 \leq c(pre(e'))\text{)} \\
&< c([e]L(e')).
\end{aligned}$$

If $c(\text{pre}(e')) \leq c(\text{pre}(e))$ we have that

$$\begin{aligned} c(\text{pre}(e) \rightarrow L(e; e')) &= 5 + c(\text{pre}(e)) \\ &< 5 + c(\text{pre}(e)) + c(\text{pre}(e)) \cdot c(\text{pre}(e')) \\ &\hspace{15em} (\text{since } 1 \leq c(\text{pre}(e'))) \\ &< c([e]L(e')). \end{aligned}$$

In both cases we obtain that $c(\text{pre}(e) \rightarrow L(e; e')) < c([e]L(e'))$. \square

A.2 Complexity order on formulas in APALM and GALM

We need an appropriate complexity order on APALM formulas (and on GALM formulas) for some of our inductive proofs in Chapter 4. The language of GALM extends the language of APALM, therefore it is enough to define the complexity order for the formulas in the language of GALM, \mathcal{L}_G . First we need some definitions.

A.2.1. DEFINITION. [Size of formulas in \mathcal{L}_G] The *size* $s(\varphi)$ of a formula $\varphi \in \mathcal{L}_G$ is a natural number recursively defined as:

$$\begin{aligned} s(\top) = s(p) = s(0) &= 1, \\ s(\neg\varphi) = s(\varphi^0) = s(K_i\varphi) = s(U\varphi) = s(\blacklozenge\varphi) = s(\langle G \rangle\psi) &= s(\varphi) + 1, \\ s(\varphi \wedge \psi) = s(\langle \varphi \rangle\psi) &= s(\varphi) + s(\psi) + 1. \end{aligned}$$

A.2.2. DEFINITION. [\blacklozenge, G -Depth of formulas in \mathcal{L}_G] The \blacklozenge, G -*depth* $d(\varphi)$ of formula $\varphi \in \mathcal{L}_G$ is a natural number recursively defined as:

$$\begin{aligned} d(\top) = d(p) = d(0) &= 1, \\ d(\neg\varphi) = d(\varphi^0) = d(K_i\varphi) = d(U\varphi) &= 1 + d(\varphi), \\ d(\varphi \wedge \psi) = d(\langle \varphi \rangle\psi) &= 1 + \max\{d(\varphi), d(\psi)\}, \\ d(\blacklozenge\varphi) = d(\langle G \rangle\varphi) &= 2 + d(\varphi). \end{aligned}$$

Finally, we define our intended complexity relation \prec_2 as lexicographic merge of \blacklozenge, G -depth and size, exactly as in (Balbiani and van Ditmarsch (2015)):

A.2.3. DEFINITION. For any $\varphi, \psi \in \mathcal{L}_G$, we put

$$\varphi \prec_2 \psi \text{ iff either } d(\varphi) < d(\psi), \text{ or } d(\varphi) = d(\psi) \text{ and } s(\varphi) < s(\psi).$$

A.2.4. DEFINITION. [Subformula] Given a formula $\varphi \in \mathcal{L}_G$, the set $Sub(\varphi)$ of subformulas of φ is recursively defined as

$$\begin{aligned}
Sub(\varphi) &= \{\varphi\} && \text{if } \varphi \text{ is } \top, p \text{ or } 0, \\
Sub(\neg\varphi) &= Sub(\varphi) \cup \{\neg\varphi\}, \\
Sub(\varphi^0) &= Sub(\varphi) \cup \{\varphi^0\}, \\
Sub(K_i\varphi) &= Sub(\varphi) \cup \{K_i\varphi\}, \\
Sub(U\varphi) &= Sub(\varphi) \cup \{U\varphi\}, \\
Sub(\varphi \wedge \psi) &= Sub(\varphi) \cup Sub(\psi) \cup \{\varphi \wedge \psi\}, \\
Sub(\langle\varphi\rangle\psi) &= Sub(\varphi) \cup Sub(\psi) \cup \{\langle\varphi\rangle\psi\}, \\
Sub(\blacklozenge\varphi) &= Sub(\varphi) \cup \{\blacklozenge\varphi\}, \\
Sub(\langle G\rangle\varphi) &= Sub(\varphi) \cup \{\langle G\rangle\varphi\}.
\end{aligned}$$

A.2.5. LEMMA. *There exists a well-founded strict partial order \prec_2 on \mathcal{L}_G called ‘complexity order’, satisfying the following conditions:*

1. if φ is a subformula of ψ , then $\varphi \prec_2 \psi$,
2. $(\theta \rightarrow p) \prec_2 [\theta]p$,
3. $(\theta \rightarrow \neg[\theta]\psi) \prec_2 [\theta]\neg\psi$,
4. $(\theta \rightarrow K_i[\theta]\psi) \prec_2 [\theta]K_i\psi$,
5. $[\langle\theta\rangle\rho]\varphi \prec_2 [\theta][\rho]\varphi$,
6. $(\theta \rightarrow \varphi^0) \prec_2 [\theta]\varphi^0$,
7. $(\theta \rightarrow U[\theta]\varphi) \prec_2 [\theta]U\varphi$,
8. $(\theta \rightarrow (U\theta \wedge 0)) \prec_2 [\theta]0$,
9. $\langle\theta\rangle\varphi \prec_2 \blacklozenge\varphi$, for all $\theta \in \mathcal{L}_{-\blacklozenge}$.
10. $\langle\theta\rangle\varphi \prec_2 \langle G\rangle\varphi$, for all $\theta \in \mathcal{L}_{-\blacklozenge}$.

Proof:

The proof is via easy arithmetic calculations following the definitions above. Note that, Definition A.2.2 is redundant for the cases restricted to language $\mathcal{L}_{-\blacklozenge}$. \square

Bibliography

- Ågotnes, T., Balbiani, P., van Ditmarsch, H., and Seban, P. (2010). Group announcement logic. *Journal of Applied Logic*, 8(1):62–81.
- Ågotnes, T. and van Ditmarsch, H. (2014). Knowledge games and coalitional abilities. In *Johan van Benthem on Logic and Information Dynamics*, pages 451–485. Springer.
- Ågotnes, T., van Ditmarsch, H., and French, T. S. (2016). The undecidability of quantified announcements. *Studia Logica*, 104(4):597–640.
- Ågotnes, T. and Van Ditmarsch, H. (2008). Coalitions and announcements. In Padgham, L., Parkes, D. C., Müller, J. P., and Parsons, S., editors, *Proceedings of the 7th International Joint Conference on Autonomous Agents and Multiagent Systems (AAMAS 2008)*, volume 2, pages 673–680.
- Aiello, M., Pratt-Hartmann, I., van Benthem, J., et al. (2007). *Handbook of Spatial Logics*, volume 4. Springer-Verlag, Germany.
- Alchourrón, C. E., Gärdenfors, P., and Makinson, D. (1985). On the logic of theory change: Partial meet contraction and revision functions. *The journal of symbolic logic*, 50(2):510–530.
- Angluin, D. (1980). Inductive inference of formal languages from positive data. *Information and control*, 45(2):117–135.
- Angluin, D. (1987). Learning regular sets from queries and counterexamples. *Information and computation*, 75(2):87–106.
- Angluin, D. and Smith, C. H. (1983). Inductive inference: Theory and methods. *ACM Computing Surveys (CSUR)*, 15(3):237–269.
- Bacon, F. (1620). *Novum organum; with other parts of the great instauration*. La Salle: Open Court, 1994 edition.

- Balbiani, P., Baltag, A., van Ditmarsch, H., Herzig, A., Hoshi, T., and de Lima, T. (2008). ‘Knowable’ as ‘known after an announcement’. *The Review of Symbolic Logic*, 1(3):305–334.
- Balbiani, P. and van Ditmarsch, H. (2015). A simple proof of the completeness of APAL. *Studies in Logic*, 8 (1):65–78.
- Balbiani, P., van Ditmarsch, H., and Herzig, A. (2016). Before announcement. In *Advances in Modal Logic 11*, pages 58–77.
- Balbiani, P., van Ditmarsch, H., and Kudinov, A. (2013). Subset space logic with arbitrary announcements. In *Proceedings of the 5th Indian Conference on Logic and Its Applications (ICLA 2013)*, pages 233–244. Springer.
- Baltag, A., Ditmarsch, H. P. V., and Moss, L. S. (2008). Epistemic logic and information update. In Adriaans, P. and van Benthem, J., editors, *Handbook of the philosophy of information*. Elsevier Science Publishers.
- Baltag, A., Gierasimczuk, N., Özgün, A., Vargas Sandoval, A. L., and Smets, S. (2018a). A dynamic logic for learning theory. In Madeira, A. and Benevides, M., editors, *Dynamic Logic. New Trends and Applications*, pages 35–54, Cham. Springer International Publishing.
- Baltag, A., Gierasimczuk, N., and Smets, S. (2015). On the solvability of inductive problems: A study in epistemic topology. In Ramanujam, R., editor, *Proceedings of the 15th Conference on Theoretical Aspects of Rationality and Knowledge (TARK 2015)*. Also available as a technical report in ILLC Pre-publication Series PP-2015-13.
- Baltag, A., Moss, L. S., and Solecki, S. (1998). The logic of public announcements, common knowledge, and private suspicions. In *Proceedings of the 7th Conference on Theoretical Aspects of Rationality and Knowledge (TARK 1998)*, pages 43–56. Morgan Kaufmann Publishers Inc.
- Baltag, A., Özgün, A., and Vargas-Sandoval, A. L. (2017). Topo-logic as a dynamic-epistemic logic. In *Proceedings of the 6th International Workshop on Logic, Rationality and Interaction (LORI 2017)*, pages 330–346. Springer Berlin Heidelberg.
- Baltag, A., Özgün, A., and Vargas-Sandoval, A. L. (2018b). APAL with memory is better. In Moss, L. S., de Queiroz, R., and Martinez, M., editors, *Logic, Language, Information, and Computation*, pages 106–129, Berlin, Heidelberg. Springer Berlin Heidelberg.

- Baltag, A., Özgün, A., and Vargas Sandoval, A. L. (2020). The logic of AGM learning from partial observations. In Soares Barbosa, L. and Baltag, A., editors, *Dynamic Logic. New Trends and Applications*. Springer International Publishing.
- Baltag, A. and Renne, B. (2016). Dynamic epistemic logic. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2016 edition.
- Baltag, A. and Smets, S. (2008). A qualitative theory of dynamic interactive belief revision. *Texts in Logic and Games*, 3:9–58.
- van Benthem, J. (2002). *One is a lonely number: On the logic of communication*, pages 96–129. Lecture Notes in Logic. Cambridge University Press.
- van Benthem, J. (2004). What one may come to know. *Analysis*, 64(282):95–105.
- van Benthem, J. (2007). Dynamic logic for belief revision. *Journal of Applied Non-Classical Logics*, 17(2):129–155.
- van Benthem, J. (2011). *Logical Dynamics of Information and Interaction*. Cambridge University Press, New York, NY, USA.
- van Benthem, J. and Smets, S. (2015). Dynamic logics of belief change. In *Handbook of Logics for Knowledge and Belief*, pages 299–368. College Publications.
- Bezhanishvili, N., Marra, V., McNeill, D., and Pedrini, A. (2018). Tarski’s theorem on intuitionistic logic, for polyhedra. *Annals of Pure and Applied Logic*, 169(5):373–391.
- Bjorndahl, A. (2018). Topological subset space models for public announcements. In van Ditmarsch, H. and Sandu, G., editors, *Jaakko Hintikka on Knowledge and Game-Theoretical Semantics*, pages 165–186. Springer International Publishing, Cham.
- Bjorndahl, A. and Özgün, A. (2017). Logic and topology for knowledge, knowability and belief. In *Proceedings of the 16th Conference on Theoretical Aspects of Rationality and Knowledge (TARK 2017)*, volume 251 of *Electronic Proceedings in Theoretical Computer Science*, pages 88–101. Open Publishing Association.
- Bjorndahl, A. and Özgün, A. (2019). Logic and topology for knowledge, knowability, and belief. *The Review of Symbolic Logic*, pages 1–28.
- Blackburn, P., de Rijke, M., and Venema, Y. (2001). *Modal Logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge.

- Board, O. (2004). Dynamic interactive epistemology. *Games and Economic Behavior*, 49(1):49 – 80.
- Bogen, J. (2017). Theory and observation in science. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, summer 2017 edition.
- Bolander, T. and Andersen, M. B. (2011). Epistemic planning for single- and multi-agent systems. *Journal of Applied Non-Classical Logics*, 21(1):9–34.
- Bolander, T. and Gierasimczuk, N. (2015). Learning actions models: Qualitative approach. In van der Hoek, W., Holliday, W., and Wang, W., editors, *Proceedings of the 5th International Workshop on Logic, Rationality and Interaction (LORI 2015)*, volume 9394 of *Lecture Notes in Computer Science*, pages 40–52. Springer Berlin Heidelberg.
- Bolander, T. and Gierasimczuk, N. (2017). Learning to act: Qualitative learning of deterministic action models. *Journal of Logic and Computation*, 28(2):337–365.
- Bringsjord, S. and Govindarajulu, N. S. (2019). Artificial intelligence. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2019 edition.
- Chagrov, A. V. and Zakharyashev, M. (1997). *Modal Logic*, volume 35 of *Oxford logic guides*. Oxford University Press.
- Chellas, B. F. (1975). Basic conditional logic. *Journal of Philosophical Logic*, 4(2):133–153.
- Dabrowski, A., Moss, L. S., and Parikh, R. (1996). Topological reasoning and the logic of knowledge. *Annals of Pure and Applied Logic*, 78(1):73–110.
- Dégremont, C. and Gierasimczuk, N. (2011). Finite identification from the viewpoint of epistemic update. *Information and Computation*, 209(3):383–396.
- van Ditmarsch, H. (2003). The russian cards problem. *Studia logica*, 75(1):31–62.
- van Ditmarsch, H. and French, T. (2017). Quantifying over boolean announcements. Available online at <https://arxiv.org/abs/1712.05310>.
- van Ditmarsch, H., French, T., and Hales, J. (2018). Positive announcements. <https://arxiv.org/abs/1803.01696>.
- van Ditmarsch, H., French, T., and Pinchinat, S. (2010). Future event logic-axioms and complexity. *Advances in Modal Logic*, 8(8):77–99.

- van Ditmarsch, H., Halpern, J., van der Hoek, W., and Kooi, B. (2015a). *Handbook of Epistemic Logic*. College Publications.
- van Ditmarsch, H., van der Hoek, W., and Kooi, B. (2007). *Dynamic Epistemic Logic*. Springer Publishing Company, 1st edition.
- van Ditmarsch, H., van der Hoek, W., and Kuijer, L. B. (2016). Fully arbitrary public announcements. *Advances in Modal Logic*, 11:252–267.
- van Ditmarsch, H., Knight, S., and Özgün, A. (2014). Arbitrary announcements on topological subset spaces. In *Proceedings of the 12th European Conference on Multi-Agent Systems (EUMAS 2014)*, pages 252–266. Springer.
- van Ditmarsch, H., Knight, S., and Özgün, A. (2015b). Announcement as effort on topological spaces. In *Proceedings of the 15th Conference on Theoretical Aspects of Rationality and Knowledge (TARK 2015)*, volume 215 of *Electronic Proceedings in Theoretical Computer Science*, pages 283–297. Open Publishing Association.
- van Ditmarsch, H., Knight, S., and Özgün, A. (2019). Announcement as effort on topological spaces. *Synthese*, 196:2927–2969.
- Doyle, J. (1979). A truth maintenance system. *Artificial intelligence*, 12(3):231–272.
- Engelking, R. (1989). *General topology*, volume 6. Heldermann Verlag, Berlin, second edition.
- Fagin, R., Halpern, J. Y., Moses, Y., and Vardi, M. Y. (1995). *Reasoning About Knowledge*. MIT Press.
- Galimullin, R. and Alechina, N. (2018). Coalition and group announcement logic (corrected version). *arXiv preprint arXiv:1810.02769*.
- Georgatos, K. (1993). *Modal Logics for Topological Spaces*. PhD thesis, City University of New York.
- Georgatos, K. (1994). Knowledge theoretic properties of topological spaces. In Masuch, M. and Pólos, L., editors, *Knowledge Representation and Reasoning Under Uncertainty: Logic at Work*, volume 808 of *Lecture Notes in Artificial Intelligence*, pages 147–159. Springer.
- Georgatos, K. (1997). Knowledge on treelike spaces. *Studia Logica*, 59(2):271–301.
- Gerbrandy, J. and Groeneveld, W. (1997). Reasoning about information change. *Journal of Logic, Language and Information*, 6(2):147–169.

- Gierasimczuk, N. (2010). *Knowing one's limits: Logical analysis of inductive inference*. PhD thesis. Institute for Logic, Language and Computation.
- Gierasimczuk, N., Hendricks, V. F., and de Jongh, D. (2014). Logic and learning. In Baltag, A. and Smets, S., editors, *Johan van Benthem on Logic and Information Dynamics*, volume 5 of *Outstanding Contributions to Logic*, pages 267–288. Springer.
- Gierasimczuk, N. and de Jongh, D. (2013). On the complexity of conclusive update. *The Computer Journal*, 56(3):365–377.
- Gold, E. M. (1967). Language identification in the limit. *Information and control*, 10(5):447–474.
- Grahne, G. (1998). Updates and counterfactuals. *Journal of Logic and Computation*, 8(1):87–117.
- Hiller, S. and Fernández, R. (2016). A data-driven investigation of corrective feedback on subject omission errors in first language acquisition. *Proceedings of the 20th SIGNLL, Conference on Computational Natural Language Learning (CoNLL)*, pages 105–114.
- Hintikka, J. (1962). *Knowledge and Belief*. Cornell University Press.
- van der Hoek, W. (1993). Systems for knowledge and beliefs. *Journal of Logic and Computation*, 3(2):173–195.
- Jain, S., Osherson, D., Royer, J. S., and Sharma, A. (1999). *Systems that learn: An introduction to learning theory*. Learning, Development and Conceptual Change. MIT Press Cambridge, second edition edition.
- Jeřábek, E. (2005). Admissible rules of modal logics. *Journal of Logic and Computation*, 15(4):411–431.
- de Jongh, D. and Kanazawa, M. (1996). Angluin's theorem for indexed families of re sets and applications. In *Proceedings of the 9th Conference on Computational Learning Theory*, pages 193–204. ACM Press.
- de Jongh, D. and Vargas-Sandoval, A. L. (2019). Finite identification with positive and with complete data. In *International Tbilisi Symposium on Logic, Language, and Computation (TbiLLC 2017)*, pages 42–63. Springer.
- de Jongh, D. and Vargas-Sandoval, A. L. (2020). Finite identification of anti-chains of finite languages: Positive data vs complete data. Unpublished Manuscript.

- Kechris, A. S. (1995). *Classical Descriptive Set Theory*. Graduate Texts in Mathematics. Springer-Verlag.
- Kelley, J. L. (1991). *General Topology*, volume 27 of *Graduate Texts in Mathematics*. Springer Verlag.
- Kelly, K. T. (1996). *The Logic of Reliable Inquiry*. Oxford University Press.
- Kelly, K. T. (2014). A computational learning semantics for inductive empirical knowledge. In *Johan van Benthem on Logic and Information Dynamics*, pages 289–337. Springer.
- Kleene, S. C. (1943). Recursive predicates and quantifiers. *Transactions of the American Mathematical Society*, 53(1):41–73.
- Klein, J. (2016). Francis bacon. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2016 edition.
- Kuijjer, L. B. (2015). Unsoundness of $R(\Box)$. Unpublished Manuscript. Available online at http://personal.us.es/hvd/APAL_counterexample.pdf.
- Lange, S. and Zeugmann, T. (1992). Types of monotonic language learning and their characterization. In *Proceedings of the 5th Workshop on Computational Learning Theory*, pages 377–390. ACM Press.
- Lange, S. and Zeugmann, T. (1996). Set-driven and rearrangement-independent learning of recursive languages. *Theory of Computing Systems*, 29(6):599–634.
- Lange, S., Zeugmann, T., and Zilles, S. (2008). Learning indexed families of recursive languages from positive data: A survey. *Theoretical Computer Science*, 397(1):194–232.
- Lenzen, W. (1978). *Recent Work in Epistemic Logic*, volume 30. North Holland: Acta Philosophica Fennica.
- Lewis, D. K. (1973). *Counterfactuals*. Blackwell.
- Lorenzen, P. (1955). Einführung in die operative. *Logik und Mathematik, Berlin, Göttingen, Heidelberg*.
- McCarthy, J. (1990). Formalization of two puzzles involving knowledge. In *Formalizing Common Sense: Papers by John McCarthy*. Ablex Publishing Corporation. Original manuscript dated 1978-1981.
- McKinsey, J. C. C. (1941). A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology. *The Journal of Symbolic Logic*, 6(4):117–134.

- McKinsey, J. C. C. and Tarski, A. (1944). The algebra of topology. *Annals of Mathematics*, 45(1):141–191.
- Meyer, J.-J. C. and Hoek, W. V. D. (1995). *Epistemic Logic for AI and Computer Science*. Cambridge University Press, New York, NY, USA.
- Mill, J. S. (1843). *A System of Logic: Ratiocinative and inductive.*, volume 1. London: Jonh W. Parker.
- Mitkov, R. (2005). *The Oxford handbook of computational linguistics*. Oxford University Press.
- Moss, L. S. and Parikh, R. (1992). Topological reasoning and the logic of knowledge. In *Proceedings of the 4th Conference on Theoretical Aspects of Rationality and Knowledge (TARK 1992)*, pages 95–105. Morgan Kaufmann.
- Mukouchi, Y. (1992). Characterization of finite identification. In *Proceedings of the International Workshop on Analogical and Inductive Inference (AII 1992)*, pages 260–267. Springer-Verlag.
- Osherson, D. N., Stob, M., and Weinstein, S. (1986). *Systems that learn: An introduction to learning theory for cognitive and computer scientists*. Learning, Development and Conceptual Change. The MIT Press.
- Özgün, A. (2017). *Evidence in Epistemic Logic: A Topological Perspective*. PhD thesis, Université de Lorraine and University of Amsterdam.
- Parikh, R., Moss, L. S., and Steinsvold, C. (2007). Topology and epistemic logic. In *Handbook of Spatial Logics*, pages 299–341.
- Pauly, M. (2002). A modal logic for coalitional power in games. *Journal of logic and computation*, 12(1):149–166.
- Plaza, J. (1989). Logics of public communications. In Emrich, M., Pfeifer, M., Hadzikadic, M., and Ras, Z., editors, *Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems*, pages 201–216.
- Plaza, J. (2007). Logics of public communications. *Synthese*, 158(2):165–179.
- Putnam, H. (1963). Degree of confirmation and inductive logic. In Schilpp, P. A., editor, *The Philosophy of Rudolf Carnap*, pages 761–783. Open Court: La Salle.
- Rendsvig, R. and Symons, J. (2019). Epistemic logic. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, summer 2019 edition.

- Renne, B., Sack, J., and Yap, A. (2009). Dynamic epistemic temporal logic. In *Proceedings of the 2nd International Workshop on Logic, Rationality and Interaction (LORI 2009)*, pages 263–277. Springer.
- Rogers, H. (1967). *Theory of recursive functions and effective computability*. McGraw-Hill New York.
- Saxton, M., Backley, P., and Gallaway, C. (2005). Negative input for grammatical errors: Effects after a lag of 12 weeks. *Journal of child language*, 32(3):643–672.
- Schulte, O. (2018). Formal learning theory. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, spring 2018 edition.
- Smullyan, R. M. (1958). Undecidability and recursive inseparability. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 4(7-11):143–147.
- Soare, R. I. (1999). *Recursively enumerable sets and degrees: A study of computable functions and computably generated sets*. Springer-Verlag.
- Solomonoff, R. J. (1964a). A formal theory of inductive inference. Part I. *Information and control*, 7(1):1–22.
- Solomonoff, R. J. (1964b). A formal theory of inductive inference. Part II. *Information and control*, 7(2):224–254.
- Stalnaker, R. (2006). On logics of knowledge and belief. *Philosophical Studies*, 128(1):169–199.
- Talbott, W. (2016). Bayesian epistemology. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2016 edition.
- Tarski, A. (1938). Der aussagenkalkul und die topologie. *Fundam. Math*, 31:103–134.
- Toelstra, A. S. and van Dalen, D. (1988). Constructivism in mathematics: An introduction. *Studies in Logic and the foundation of Mathematics*, 1 and 2.
- Turing, A. M. (1937). On computable numbers, with an application to the entscheidungsproblem. *Proceedings of the London mathematical society, Series 2*, 42:230–265.
- Van Ditmarsch, H. (2012). Quantifying notes. In *Proceedings of the 19th International Workshop on Logic, Language, Information, and Computation (WoLLIC 2012)*, pages 89–109. Springer.

- Vickers, S. (1989). *Topology via logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press.
- Wáng, Y. N. and Ågotnes, T. (2013a). Multi-agent subset space logic. In *Proceedings of the 23rd International Joint Conference on Artificial Intelligence (IJCAI 2013)*, pages 1155–1161. IJCAI/AAAI.
- Wáng, Y. N. and Ågotnes, T. (2013b). Subset space public announcement logic. In *Proceedings of the 5th Indian Conference on Logic and Its Applications (ICLA 2013)*, pages 245–257. Springer.
- Weiss, M. A. and Parikh, R. (2002). Completeness of certain bimodal logics for subset spaces. *Studia Logica*, 71(1):1–30.
- Whewell, W. (1858a). *History of Scientific Ideas: Being the first part of the philosophy of the inductive sciences*, volume 1. John W. Parker and Son.
- Whewell, W. (1858b). *Novum organon renovatum*. In Butts, R., editor, *William Whewell Theory of Scientific Method*, volume 2. Hackett Publishing Company, 1989.
- von Wright, G. H. (1951). *An Essay in Modal Logic*, volume 5 of *Studies in logic and the foundations of mathematics*. North-Holland Publishing Company.
- Zeugmann, T. and Lange, S. (1995). A guided tour across the boundaries of learning recursive languages. In *Algorithmic learning for knowledge-based systems*, pages 190–258. Springer.

Samenvatting

Dit proefschrift is een studie vanuit verschillende gezichtspunten op leren en op het verband van leren met kennis en geloof (als mening, overtuiging) binnen een formele aanpak. Hierbij concentreren we ons grotendeels op inductief redeneren, inductief leren (*inductive inference*), het proces van het trekken van algemene conclusies uit binnenkomende informatie. Ons werk is geworteld in twee gebieden waarin, onafhankelijk van elkaar, de dynamiek van informatie wordt bestudeerd, Dynamische Epistemische Logica (DEL) en Formele Leertheorie (FLT).

In Deel [I](#) onderzoeken we de dynamiek van informatie die komt uit waarnemingen, of uit waarheidsgetrouwe mededelingen.

In Hoofdstuk [3](#) presenteren we twee dynamische modale logicas voor het redeneren over leren in de geest van FLT op grond van waarnemingen. Onze eerste logica gebruikt deelverzamelingsruimte-semantiek en het standaardbegrip van een leerfunctie om leren in de limiet te modelleren. Onze tweede logica breidt het eerste raamwerk uit om leren in de limiet te modelleren op grond van *partiële waarnemingen* met een *volledig rationele leerder* in de stijl van de AGM-theorie van geloofsherijking (*belief revision*). We presenteren resultaten over de uitdrukingskracht, correctheid en volledigheid voor beide logicas.

In Hoofdstuk [4](#), verschuiven we onze aandacht naar het verzamelen van informatie via openbare mededelingen en via willekeurige openbare mededelingen in scenarios met meer dan een leerder. We lossen problematische kwesties op die zich voordoen in het werk van [Balbiani et al. \(2008\)](#) aangaande de incorrecte finitaire regel die daar wordt voorgesteld voor de oorspronkelijke willekeurige openbare mededelingenlogica (*Arbitrary Public Announcement Logic*, APAL). Dit brengt ons ook tot een oplossing voor de lang openstaande vraag naar het vinden van een recursieve axiomatisering van een sterke versie van APAL (en de daarmee verbonden groepsmededelingslogica, *Group Announcement Logic*, GAL).

In Deel [III](#), richten we ons volledig op het leermodel in FLT van eindige identificatie. We verkrijgen een meer verfijnde theoretische analyse van het onderscheid tussen eindige identificatie met positieve informatie (pfi) en met volledige (posi-

tieve en negatieve) informatie (cfi). We laten zien dat het onderscheid tussen pfi en cfi, alhoewel niet zo enorm groot als bij het leren in de limiet, aanzienlijk is, niet alleen in kracht van leren maar ook in de aard ervan.

In Hoofdstuk 5 richten we ons uitsluitend op de structurele verschillen tussen pfi families en cfi families, voorbijgaande aan computationele aspecten. We onderzoeken of iedere eindig identificeerbare familie bevat is in een maximale eindig identificeerbare. Dit levert een positief antwoord op in het geval van positieve data voor families met alleen eindige talen, maar een sterk negatief resultaat in het geval van volledige data dat laat zien dat iedere eindig identificeerbare familie uitgebreid kan worden tot een grotere familie die ook eindig identificeerbaar is. We bestuderen ook hoeveel maximale uitbreidingen een positief identificeerbare familie heeft. We laten ons leiden door het vermoeden, gedeeltelijk bevestigd in dit proefschrift, dat iedere positief eindig identificeerbare familie van eindige talen, of slechts eindig veel maximale pfi uitbreidingen heeft, of niet-aftelbaar vele.

In Hoofdstuk 6 bestuderen we de computationele eigenschappen van een familie van talen. In het bijzonder bestuderen we oneindige antiketens van eindige talen. We verschaffen negatieve antwoorden op de vragen: Is iedere antiketen van eindige talen die cfi is ook pfi? Is iedere maximale antiketen van eindige talen pfi (of cfi)? We onderzoeken ook een variant van eindige identificatie met een leerder die een taal identificeert zodra het objectief zeker is welke taal het is en we exploreren de relatie tussen pfi en cfi in dit raamwerk. Tenslotte bestuderen we een variant van cfi met een leerder die vragen kan stellen aan de leraar.

Over het geheel genomen brengt dit proefschrift aan de ene kant Dynamische Epistemische Logica en Formele Leertheorie dichter bij elkaar, resulterend in nieuwe logicas voor de dynamiek van informatie die verschillende leertheoretische begrippen formaliseren. Aan de andere kant gebruikt het instrumenten uit de combinatoriek en recursietheorie om een gedetailleerde analyse te geven van de verschillen tussen eindige identificatie met positieve data en eindige identificatie met volledige data.

Abstract

In this dissertation we study various perspectives on learning and its relation to knowledge and belief within a formal approach. We mostly focus on *inductive inference* (or, *inductive learning*), the process of inferring general conclusions from incoming information. Our work is based in two areas that, independently, study dynamics of information, Dynamic Epistemic Logic (DEL) and Formal Learning Theory (FLT).

In Part [I](#) we investigate information dynamics from, on the one hand, incoming observations and, on the other, from incoming truthful announcements.

In Chapter [3](#) we present two dynamic modal logics to reason about learning from incoming observations in the spirit of FLT. Our first logic uses subset space semantics and the standard notion of a learning function to model learning in the limit. Our second logic extends the first framework in order to model learning in the limit from *partial observations* with a *fully rational learner* in the style of AGM belief revision theory. We present expressivity, soundness and completeness results for both logics.

In Chapter [4](#), we shift our focus to information gathering via public announcements and arbitrary public announcements in scenarios with multiple learners. We resolve problematic issues encountered in the work of [Balbiani et al. \(2008\)](#) concerning the unsound finitary rule proposed for the original Arbitrary Public Announcement Logic (APAL). This leads us also to solving the long standing open question of finding a recursive axiomatization for a strong version of APAL (and its variant Group Announcement Logic (GAL)).

In Part [II](#), we focus completely on the learning model of finite identification in FLT. We obtain a more fine-grained theoretical analysis of the distinction between finite identification with positive information (pfi) and with complete (positive and negative) information (cfi). We show that the difference between pfi and cfi, if not as huge as in learning in the limit, is considerable not only in power but also in character.

In Chapter [5](#), we focus purely on the structural differences between families

that are pfi and families that are cfi, ignoring computational aspects. We investigate whether any finitely identifiable family is contained in a maximal finitely identifiable one. We get a positive answer in the setting of positive data for families containing only finite languages. We provide a strong negative result in the setting of complete data showing that any finitely identifiable family can be extended to a larger one which is also finitely identifiable. We also study how many maximal extensions a positively identifiable family has. Our leading conjecture, which we partially resolve, is that any positively identifiable family of finite languages either has only finitely many maximal pfi extensions or uncountably many.

In Chapter 6 we study the computational properties of a family of languages. In particular, we analyze infinite anti-chains of finite languages. We provide negative answers to the questions: is every anti-chain of finite languages that is cfi also pfi? Is every maximal anti-chain of finite languages pfi (or cfi)? We also investigate a variation of finite identification that considers a learner who identifies a language as soon as it is objectively certain which language it is and explore the connection between pfi and cfi in this setting. We then study a variation of cfi which considers a learner that can ask queries to the teacher.

Overall, this dissertation, on the one hand brings closer together Dynamic Epistemic Logic and Formal Learning Theory, resulting in novel logics of information dynamics that formalize various learning theoretic notions. On the other hand, it uses tools in combinatorics and recursion theory to provide a detailed analysis of the differences between finite identification with positive data and finite identification with complete data.

Titles in the ILLC Dissertation Series:

- ILLC DS-2009-01: **Jakub Szymanik**
Quantifiers in TIME and SPACE. Computational Complexity of Generalized Quantifiers in Natural Language
- ILLC DS-2009-02: **Hartmut Fitz**
Neural Syntax
- ILLC DS-2009-03: **Brian Thomas Semmes**
A Game for the Borel Functions
- ILLC DS-2009-04: **Sara L. Uckelman**
Modalities in Medieval Logic
- ILLC DS-2009-05: **Andreas Witzel**
Knowledge and Games: Theory and Implementation
- ILLC DS-2009-06: **Chantal Bax**
Subjectivity after Wittgenstein. Wittgenstein's embodied and embedded subject and the debate about the death of man.
- ILLC DS-2009-07: **Kata Balogh**
Theme with Variations. A Context-based Analysis of Focus
- ILLC DS-2009-08: **Tomohiro Hoshi**
Epistemic Dynamics and Protocol Information
- ILLC DS-2009-09: **Olivia Ladinig**
Temporal expectations and their violations
- ILLC DS-2009-10: **Tikitu de Jager**
"Now that you mention it, I wonder...": Awareness, Attention, Assumption
- ILLC DS-2009-11: **Michael Franke**
Signal to Act: Game Theory in Pragmatics
- ILLC DS-2009-12: **Joel Uckelman**
More Than the Sum of Its Parts: Compact Preference Representation Over Combinatorial Domains
- ILLC DS-2009-13: **Stefan Bold**
Cardinals as Ultrapowers. A Canonical Measure Analysis under the Axiom of Determinacy.
- ILLC DS-2010-01: **Reut Tsarfaty**
Relational-Realizational Parsing

- ILLC DS-2010-02: **Jonathan Zvesper**
Playing with Information
- ILLC DS-2010-03: **Cédric Dégrement**
The Temporal Mind. Observations on the logic of belief change in interactive systems
- ILLC DS-2010-04: **Daisuke Ikegami**
Games in Set Theory and Logic
- ILLC DS-2010-05: **Jarmo Kontinen**
Coherence and Complexity in Fragments of Dependence Logic
- ILLC DS-2010-06: **Yanjing Wang**
Epistemic Modelling and Protocol Dynamics
- ILLC DS-2010-07: **Marc Staudacher**
Use theories of meaning between conventions and social norms
- ILLC DS-2010-08: **Amélie Gheerbrant**
Fixed-Point Logics on Trees
- ILLC DS-2010-09: **Gaëlle Fontaine**
Modal Fixpoint Logic: Some Model Theoretic Questions
- ILLC DS-2010-10: **Jacob Vosmaer**
Logic, Algebra and Topology. Investigations into canonical extensions, duality theory and point-free topology.
- ILLC DS-2010-11: **Nina Gierasimczuk**
Knowing One's Limits. Logical Analysis of Inductive Inference
- ILLC DS-2010-12: **Martin Mose Bentzen**
Stit, It, and Deontic Logic for Action Types
- ILLC DS-2011-01: **Wouter M. Koolen**
Combining Strategies Efficiently: High-Quality Decisions from Conflicting Advice
- ILLC DS-2011-02: **Fernando Raymundo Velazquez-Quesada**
Small steps in dynamics of information
- ILLC DS-2011-03: **Marijn Koolen**
The Meaning of Structure: the Value of Link Evidence for Information Retrieval
- ILLC DS-2011-04: **Junte Zhang**
System Evaluation of Archival Description and Access

- ILLC DS-2011-05: **Lauri Keskinen**
Characterizing All Models in Infinite Cardinalities
- ILLC DS-2011-06: **Rianne Kaptein**
Effective Focused Retrieval by Exploiting Query Context and Document Structure
- ILLC DS-2011-07: **Jop Briët**
Grothendieck Inequalities, Nonlocal Games and Optimization
- ILLC DS-2011-08: **Stefan Minica**
Dynamic Logic of Questions
- ILLC DS-2011-09: **Raul Andres Leal**
Modalities Through the Looking Glass: A study on coalgebraic modal logic and their applications
- ILLC DS-2011-10: **Lena Kurzen**
Complexity in Interaction
- ILLC DS-2011-11: **Gideon Borensztajn**
The neural basis of structure in language
- ILLC DS-2012-01: **Federico Sangati**
Decomposing and Regenerating Syntactic Trees
- ILLC DS-2012-02: **Markos Mylonakis**
Learning the Latent Structure of Translation
- ILLC DS-2012-03: **Edgar José Andrade Lotero**
Models of Language: Towards a practice-based account of information in natural language
- ILLC DS-2012-04: **Yurii Khomskii**
Regularity Properties and Definability in the Real Number Continuum: idealized forcing, polarized partitions, Hausdorff gaps and mad families in the projective hierarchy.
- ILLC DS-2012-05: **David García Soriano**
Query-Efficient Computation in Property Testing and Learning Theory
- ILLC DS-2012-06: **Dimitris Gakis**
Contextual Metaphilosophy - The Case of Wittgenstein
- ILLC DS-2012-07: **Pietro Galliani**
The Dynamics of Imperfect Information

- ILLC DS-2012-08: **Umberto Grandi**
Binary Aggregation with Integrity Constraints
- ILLC DS-2012-09: **Wesley Halcrow Holliday**
Knowing What Follows: Epistemic Closure and Epistemic Logic
- ILLC DS-2012-10: **Jeremy Meyers**
Locations, Bodies, and Sets: A model theoretic investigation into nominalistic mereologies
- ILLC DS-2012-11: **Floor Sietsma**
Logics of Communication and Knowledge
- ILLC DS-2012-12: **Joris Dormans**
Engineering emergence: applied theory for game design
- ILLC DS-2013-01: **Simon Pauw**
Size Matters: Grounding Quantifiers in Spatial Perception
- ILLC DS-2013-02: **Virginie Fiutek**
Playing with Knowledge and Belief
- ILLC DS-2013-03: **Giannicola Scarpa**
Quantum entanglement in non-local games, graph parameters and zero-error information theory
- ILLC DS-2014-01: **Machiel Keestra**
Sculpting the Space of Actions. Explaining Human Action by Integrating Intentions and Mechanisms
- ILLC DS-2014-02: **Thomas Icard**
The Algorithmic Mind: A Study of Inference in Action
- ILLC DS-2014-03: **Harald A. Bastiaanse**
Very, Many, Small, Penguins
- ILLC DS-2014-04: **Ben Rodenhäuser**
A Matter of Trust: Dynamic Attitudes in Epistemic Logic
- ILLC DS-2015-01: **María Inés Crespo**
Affecting Meaning. Subjectivity and evaluativity in gradable adjectives.
- ILLC DS-2015-02: **Mathias Winther Madsen**
The Kid, the Clerk, and the Gambler - Critical Studies in Statistics and Cognitive Science

- ILLC DS-2015-03: **Shengyang Zhong**
Orthogonality and Quantum Geometry: Towards a Relational Reconstruction of Quantum Theory
- ILLC DS-2015-04: **Sumit Sourabh**
Correspondence and Canonicity in Non-Classical Logic
- ILLC DS-2015-05: **Facundo Carreiro**
Fragments of Fixpoint Logics: Automata and Expressiveness
- ILLC DS-2016-01: **Ivano A. Ciardelli**
Questions in Logic
- ILLC DS-2016-02: **Zoé Christoff**
Dynamic Logics of Networks: Information Flow and the Spread of Opinion
- ILLC DS-2016-03: **Fleur Leonie Bouwer**
What do we need to hear a beat? The influence of attention, musical abilities, and accents on the perception of metrical rhythm
- ILLC DS-2016-04: **Johannes Marti**
Interpreting Linguistic Behavior with Possible World Models
- ILLC DS-2016-05: **Phong Lê**
Learning Vector Representations for Sentences - The Recursive Deep Learning Approach
- ILLC DS-2016-06: **Gideon Maillette de Buy Wenniger**
Aligning the Foundations of Hierarchical Statistical Machine Translation
- ILLC DS-2016-07: **Andreas van Cranenburgh**
Rich Statistical Parsing and Literary Language
- ILLC DS-2016-08: **Florian Speelman**
Position-based Quantum Cryptography and Catalytic Computation
- ILLC DS-2016-09: **Teresa Piovesan**
Quantum entanglement: insights via graph parameters and conic optimization
- ILLC DS-2016-10: **Paula Henk**
Nonstandard Provability for Peano Arithmetic. A Modal Perspective
- ILLC DS-2017-01: **Paolo Galeazzi**
Play Without Regret
- ILLC DS-2017-02: **Riccardo Pinosio**
The Logic of Kant's Temporal Continuum

- ILLC DS-2017-03: **Matthijs Westera**
Exhaustivity and intonation: a unified theory
- ILLC DS-2017-04: **Giovanni Cinà**
Categories for the working modal logician
- ILLC DS-2017-05: **Shane Noah Steinert-Threlkeld**
Communication and Computation: New Questions About Compositionality
- ILLC DS-2017-06: **Peter Hawke**
The Problem of Epistemic Relevance
- ILLC DS-2017-07: **Aybüke Özgün**
Evidence in Epistemic Logic: A Topological Perspective
- ILLC DS-2017-08: **Raquel Garrido Alhama**
Computational Modelling of Artificial Language Learning: Retention, Recognition & Recurrence
- ILLC DS-2017-09: **Miloš Stanojević**
Permutation Forests for Modeling Word Order in Machine Translation
- ILLC DS-2018-01: **Berit Janssen**
Retained or Lost in Transmission? Analyzing and Predicting Stability in Dutch Folk Songs
- ILLC DS-2018-02: **Hugo Huurdeman**
Supporting the Complex Dynamics of the Information Seeking Process
- ILLC DS-2018-03: **Corina Koolen**
Reading beyond the female: The relationship between perception of author gender and literary quality
- ILLC DS-2018-04: **Jelle Bruineberg**
Anticipating Affordances: Intentionality in self-organizing brain-body-environment systems
- ILLC DS-2018-05: **Joachim Daiber**
Typologically Robust Statistical Machine Translation: Understanding and Exploiting Differences and Similarities Between Languages in Machine Translation
- ILLC DS-2018-06: **Thomas Brochhagen**
Signaling under Uncertainty
- ILLC DS-2018-07: **Julian Schlöder**
Assertion and Rejection

- ILLC DS-2018-08: **Srinivasan Arunachalam**
Quantum Algorithms and Learning Theory
- ILLC DS-2018-09: **Hugo de Holanda Cunha Nobrega**
Games for functions: Baire classes, Weihrauch degrees, transfinite computations, and ranks
- ILLC DS-2018-10: **Chenwei Shi**
Reason to Believe
- ILLC DS-2018-11: **Malvin Gattinger**
New Directions in Model Checking Dynamic Epistemic Logic
- ILLC DS-2018-12: **Julia Ilin**
Filtration Revisited: Lattices of Stable Non-Classical Logics
- ILLC DS-2018-13: **Jeroen Zuiddam**
Algebraic complexity, asymptotic spectra and entanglement polytopes
- ILLC DS-2019-01: **Carlos Vaquero**
What Makes A Performer Unique? Idiosyncrasies and commonalities in expressive music performance
- ILLC DS-2019-02: **Jort Bergfeld**
Quantum logics for expressing and proving the correctness of quantum programs
- ILLC DS-2019-03: **Andras Gilyen**
Quantum Singular Value Transformation & Its Algorithmic Applications
- ILLC DS-2019-04: **Lorenzo Galeotti**
The theory of the generalised real numbers and other topics in logic
- ILLC DS-2019-05: **Nadine Theiler**
Taking a unified perspective: Resolutions and highlighting in the semantics of attitudes and particles
- ILLC DS-2019-06: **Peter T.S. van der Gulik**
Considerations in Evolutionary Biochemistry
- ILLC DS-2019-07: **Frederik Mollerstrom Lauridsen**
Cuts and Completions: Algebraic aspects of structural proof theory
- ILLC DS-2020-01: **Mostafa Dehghani**
Learning with Imperfect Supervision for Language Understanding
- ILLC DS-2020-02: **Koen Groenland**
Quantum protocols for few-qubit devices

ILLC DS-2020-03: **Jouke Witteveen**
Parameterized Analysis of Complexity

ILLC DS-2020-04: **Joran van Apeldoorn**
A Quantum View on Convex Optimization

ILLC DS-2020-05: **Tom Bannink**
Quantum and stochastic processes

