Questions & Quantification

A study of first order inquisitive logic

Gianluca Grilletti

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Part of the content of this dissertation has been produced in collaboration with other authors and part of the content has been already published or submitted for publication. The publications used as source material are listed below.

- Chapter 3 is based on independent work currently unpublished.
- Chapters 4 and 5 are based on a joint work with Dr. I. Ciardelli. In particular, part of the material in Chapter 4 (Sections 4.1-4.4) has been published in [Grilletti and Ciardelli, 2019]; and part of the material in Chapter 4 (Sections 4.1-4.4) and the content of Chapter 5 have been submitted for publication in a joint paper with Dr. I. Ciardelli.
- Chapter 6 is based on independent work. The content of the chapter has been published in [Grilletti, 2019].
- Chapter 7 is based on independent work currently submitted for publication.
- Chapter 8 is based on independent work currently unpublished.
- Chapter 9 is based on a joint work with Dr. N. Bezhanishvili and Dr. W.H. Holliday. The content of the chapter has been published in [Bezhanishvili et al., 2019].

Contents

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people, after a whole day of interesting talks and discussions. I am grateful I was given the occasion to be part of this and to give my small contribution to such marvelous events. And I really hope I will be able to share this experience with others, in the same way Nick did with me.

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Amsterdam, October 2020 — Gianluca Grilletti

Chapter 1

Introduction

Inquisitive logic

Inquisitive logic provides a framework to encompass *questions* in order to employ them in formal inferences and study their logical properties. This also prepares the ground to investigate these properties from the perspective of natural language.

The main obstacle to tackle in order to reason about questions in a formal framework is that questions do not have an associated *truth value*. It is natural to judge a statement either true or false depending on the context in which it is evaluated (e.g., "it is raining" is true when water falls from the sky); and we can easily make inferences between different statements based on their truth values in different contexts (e.g., that "Either Thom or Tom has an umbrella" implies that "Someone has an umbrella", since the latter is true whenever the former is). However this is not the case for questions, since questions do not have an associated truth value in a given context (e.g., we cannot really judge the question "who has an umbrella?" to be true or false, independently from the context): questions *raise issues* which require some specific *information* to be *resolved*.

Semantics of logical formalisms are usually based on the concept of *truth* relative to a given *state of affair*. Examples of this are Tarskian semantics (where a single state of affairs is modeled at once), possible world semantics (where several states of affairs are modeled at once) and epistemic semantics (where the description of the state of affairs includes also the knowledge of one or more ideal agents). Thus these approaches do not seem suitable to reason about questions and logical inferences involving them.

One of the first formalisms to address this issue is the predecessor of inquisitive logic: *partition semantics* proposed by Groenendijk and Stokhof [1984]. This semantics interprets a question as a *partition* of a set of possible worlds, in which every cell of the partition corresponds to a distinct *answer* to the question true in every world of the cell. Inquisitive logic inherits many traits from this system: statements and questions are interpreted as formulas of a suitable logical language; a new semantics is developed to formalize the concepts of *asserting a statement* and *resolving a question*; and the corresponding *entailment relation* allows to study logical relations between statements and questions.

However, the partition approach has some inherent limitations which do not allow to define formal counterparts of some natural language concepts (e.g., determinacy between questions), and so this precludes expressing complex sentences like conditional questions and mention-some questions (e.g., "What is an instance of...?"). To overcome this hurdle, inquisitive logic proposes a semantics based on *information* instead of *truth* or *answerhood*. So rather than characterize whether a statement is *satisfied* in a given *state of affairs* or represent a *question* as the set of its *possible answers*, inquisitive logics focuses on whether a sentence is *supported* by a given *piece of information*.

More concretely, the semantics of inquisitive logic represents a certain *piece of information* I as a *set of possible worlds*, that is, the set of those worlds s_I in which I is *true*. This allows to give a uniform semantic account to both *statements and questions*: a statement is *implied* by a certain body of information I iff it is *true* in every world of s_I; and a question is *resolved* by a certain information I iff the issue it raises is *settled* in the same way in every world of s_I. We will refer to this as the *support semantics* of inquisitive logic and say that a sentence, being a statement or a question, is *supported* by a piece of information in case the right condition above is met.

This approach also allows to study *logical relations* involving both statements and questions in terms of the information supporting them (see [Ciardelli, 2018] for an in-depth discussion). For example, we say that a statement *resolves* a question in case every piece of information *implying* the former also *resolves* the issue raised by the latter (e.g., that "Ilaria has an umbrella" resolves the question "whether someone has an umbrella").

Since formulas are interpreted relative to information states, that is, sets of possible worlds, inquisitive logic qualifies as an example of *team semantics*, a family of formalisms firstly considered by Hodges [1997b] in which formulas are not interpreted relative to a single point of evaluation (e.g., a *world*, a *state*) but are interpreted at *sets of* points of evaluation (e.g., a *set of worlds*, a *set of assignments*). Several logics fall under this family, for example independence friendly logic [Hintikka and Sandu, 1989], dependence logic [Väänänen, 2007] and the set-theoretic multiverse [Hamkins, 2012]. Even though these formalisms differ on what is regarded as a *team* (e.g., in dependence logic a team is a set of variable assignments), there are strong connections thoroughly explored in the literature between inquisitive logic and the other members of the family of team logics (see for example [Yang and Väänänen, 2016] and [Ciardelli, 2016, Chapter 5]).

The first formalism implementing the team-based approach to questions is *inquisitive propositional logic*, introduced by Ciardelli and Roelofsen [2009] as a logical system extending classical propositional logic with questions (see also [Ciardelli and Roelofsen, 2011] and [Ciardelli, 2016, Chapter 3]). In this system, the role of possible worlds is played by propositional valuations (i.e., evaluations of atomic propositions) and formulas from classical logic are treated as *statements*. So a *piece of information* I in this context amounts to a property of propositional models, and a statement α is *supported* by I iff α is true in all the models having property I.

To include questions into the picture, a novel question-forming operator is added to the semantics: inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$. This operator allows to express several kinds of questions, as for example the *polar question* "whether p holds", represented by the formula $?p := p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg p$. In this particular case, a piece of information I *resolves* the formula $?p$ iff I *implies* either p or $\neg p$, that is, if every model with property I satisfies p *or* every model with property I satisfies $\neg p$.

This approach allows to infer in a formal system natural logical relations between statements and questions: for example, we can infer that the sentence "p holds" (represented by the formula p) *determines* the question "whether p holds" (represented by the formula $?p$). Moreover, introducing questions through additional question-forming operators allows to represent and study more complex sentences: for example the *conditional question* "if p is the case, is q also the case?", represented by the formula $p \to ?q$.

Inquisitive propositional logic is a concrete incarnation of the support semantics approach to questions. However, the approach itself is much more general and can be implemented in many different ways: the concepts of *possible world*, *piece of information* and *resolution* are deliberately underspecified so that, depending on how they are instantiated, we obtain a different *inquisitive logic*. Some examples are *inquisitive first order logic* [Ciardelli, 2016, Chapter 4], *inquisitive modal logic* [Ciardelli, 2016, Chapters 6-7], *inquisitive intuitionistic logic* [Ciardelli et al., 2020, Holliday, 2020, Punčochář, 2016], *inquisitive epistemic logic* [Ciardelli, 2014, Ciardelli and Roelofsen, 2015], *substructural inquisitive logics* [Punčochář, 2019], *relevant logics of questions* [Punčochář, 2020].

The support semantics has been studied in detail in many of these instances. For example, *inquisitive propositional logic* has been investigated from several different points of view: there are several proofs of completeness for the logic (e.g., [Ciardelli and Roelofsen, 2009, 2011] and [Ciardelli, 2016, Theorem 3.3.2]) and different proof systems (e.g., [Ciardelli, 2016, Section 3.1] and [Sano, 2009]); a constructive interpretation of proofs was given in some of these systems, in the style of the Curry-Howard interpretation [Ciardelli, 2018]; connections with intermediate logics [Ciardelli, 2009] and with logics of dependence [Yang and Väänänen, 2016] have been investigated in great detail; many extensions and generalizations of the logic have been proposed and studied [Punčochář, 2015, 2016, 2020, Ciardelli et al., 2020]; several algebraic approaches have been proposed to study the logic and variations of it [Punčochář, 2015, 2019]. Another significant example is *modal inquisitive logic*, for which interesting technical results have been proved recently: the axiomatization of the logic [Ciardelli, 2016, Theorem 7.3.30]; bisimulation results for the logic and extensions [Ciardelli and Otto, 2017, 2018]; a faithful translation of the system in first order logic [Ciardelli and Otto, 2018, Meißner and Otto, 2019]. These examples show that a considerable body of results and techniques have been proposed to study inquisitive logics. However, there is at least one aspect which has yet to be analyzed at this level of detail, and which is exactly the topic of this dissertation: quantification.

Inquisitive first order logic

The logic we are interested to study in this dissertation is *Inquisitive first order logic*, which extends *classical first order logic* with questions. Similarly to the propositional case, the role of *possible worlds* is played by classical first order models and a *piece of information* amounts to a property I of first order models. For example I_1: "the interpretation of c is in the extension of the relation P" and I_2: "the cardinality of the domain is finite and even" are considered pieces of information.

This semantics account allows to interpret the standard quantifiers from first and second order logic: for example "for all x it holds that…", "there exists an x such that…" and "there are finitely many x such that…". Formally, these quantifiers are combinators that, given a property, produce a statement on the set of elements respecting said property. So the statement "there exists an x such that P holds at x" is *supported* by a piece of information I iff I ensures that the extension of P is not empty.

In the more general semantic framework of inquisitive logic we can also naturally interpret question-forming expressions as quantifiers: for example "what is an x such that…?", "are there any x such that…?", "which x are such that…?" and "how many x are such that…?". All these expressions acts as combinators that, given a property, produce a *question* on the set of elements respecting said property. For example, "how many x are in the extension of P?" is *supported* by a piece of information I iff I allows to pinpoint the cardinality of the set of elements having property P. This novel approach generalizes the concept of quantifier and enriches significantly the expressive power of classical first order logic.

To represent questions in this context, we add to the syntax of first order classical logic two question-forming operators: the inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ (that we already encountered in the propositional case) and the inquisitive existential quantifier $\overline{\exists}$. The latter introduces questions of the form "what is an x such that…", as for example "what is an x in the extension of P?", represented by the formula $\overline{\exists}x.P(x)$ and resolved by the pieces of information I that implies for some particular element d that it has property P.

In this extended language we can capture and study several classes of natural language questions involving quantifiers. Among these classes we have *mention-some questions* [Ciardelli, 2016, Section 4.7], asking for an instance of an element with a given property. "What is an x in the extension of P?" and "if q is the case, what is an x in the extension of P?" (represented by the formula $q \to \overline{\exists} x.P(x)$) are both examples of mention-some questions. Another important class captured by inquisitive first order logic is comprised by *mention-all questions* [Ciardelli, 2016, Section 4.8], asking for the extension of a certain property or relation. For example, "what is the extension of P?" is a mention-all question represented by the formula $\forall x.(P(x) \vee\!\!\!\vee \neg P(x))$ (literally, "for every element x, is x in the extension of P?"). This last expression effectively acts as a second order quantifier—for instance in the formula $\forall x.(P(x) \vee\!\!\!\vee \neg P(x)) \to \forall x.(Q(x) \vee\!\!\!\vee \neg Q(x))$, which expresses that the extension of Q is determined by the extension of P—showing the enhanced expressive power of the logic.

We are also able to perform inferences involving *questions* and *quantifiers*. For a simple example, the statement that "c is in the extension of P" (represented by $P(c)$) *resolves* the question "what is an x in the extension of P" (represented by $\overline{\exists} x.P(x)$). In fact, every piece of information ensuring that c is in the extension of P also allows to pinpoint an element in said extension: the element c. The following (less trivial) example shows that we can also formalize inferences involving complex forms of quantification: the question "what is the extension of P?" (represented by $\forall x.(P(x) \vee\!\!\!\vee \neg P(x))$) *resolves* the conditional question "if the extension of P coincides with the extension of Q, what is the extension of Q?" (represented by the formula $\forall x.(P(x) \leftrightarrow Q(x)) \to \forall x.(Q(x) \vee\!\!\!\vee \neg Q(x))$).

Studying support semantics in the presence of quantification brings forth interesting theoretical issues, first and foremost on the *expressive power* of the logic. We mentioned that the semantics allows to interpret all quantifiers from classical logic, and even new quantifiers proper to questions. We even saw that we have access to some form of second order quantification in the logical language. So a natural question is how far we can go, what quantifiers can we express in first order inquisitive logic? Can we characterize them or find some properties they must meet to be expressible? And, more generally, what questions are expressible in this enhanced version of first order logic?

These issues on the expressive power also kindle several questions about the *entailment* of the logic. The most clear example is whether the entailment is axiomatizable, a non-trivial question since the logic allows for some forms of second order quantification. This problem was already tackled in the context of fragments (e.g., the *mention-some* and *mention-all* fragments [Ciardelli, 2016, Sections 4.7, 4.8]), but as of now it remains open for the full language. Other natural questions stem from the relation with its classical first order counterpart, as for example whether the entailment satisfies certain properties like compactness or the Löwenheim-Skolem theorem, and if not in which fragments these remain valid.

To tackle all these issues we need new tools and techniques, and providing them is the challenge this dissertation addresses. This work is meant to be a first step in the direction of understanding the interactions between questions and quantifiers and, more importantly, studying them in a systematic way. The focus was first and foremost on developing tools and techniques that could be used to better understand inquisitive first order logic and other logics of questions in the presence of quantification. As we will show with several examples, these tools have proved to do the job they were meant to: finding interesting properties of the logic and proving them.

Content of the dissertation

The dissertation can be divided into four parts, each considering a different approach to study the logic.

Games and expressive power

In the first part, consisting of Chapters 4 and 5, we adapt a tool from the field of model theory to inquisitive first order logic and use it to show different properties of the logic: *Ehrenfeucht-Fraïssé games.*

Ehrenfeucht-Fraïssé games (also known as EF games or back-and-forth games), introduced in 1967 by Ehrenfeucht [1967] developing model-theoretic results presented by Fraïssé [1954], have proved to be a powerful tool to study the expressiveness of classical first order logic. These games provide a particularly perspicuous way of understanding what differences between models can be detected by means of first order formulas of a certain quantifier rank.

One of the main merits of EF games is that they allow for relatively easy proofs that certain properties of first order structures are not first order expressible. A classical application of this kind is the characterization of the cardinality quantifiers definable in classical first order logic. This characterization yields a range of interesting undefinability results: for instance, it implies that the quantifiers *an even number of individuals* and *infinitely many individuals* are not first order definable.

In this part of the dissertation we are going to develop the game-theoretic approach to study inquisitive first order logic. We will show that the technique of EF games adapts to this context and allows to detect when two inquisitive models are indistinguishable by formulas of a given complexity. This requires to develop a more general version of the game, due to the expressive power of support semantics.

The game developed is quite flexible and can be modified to capture properties other than logical equivalence. For example, variations of the game allow to characterize in game-theoretic terms *structural relations* between inquisitive models (e.g., the *submodel* relation). Moreover, the general approach can be applied to study also fragments of inquisitive first order logic.

Regarding applications, the EF game can be employed to study the expressive limitations of the logic, similarly to the case of classical logic. For example, we can achieve a characterization of the cardinality quantifiers definable in inquisitive first order logic, generalizing the result for classical logic to this more expressive setting.

Chapter 4 introduces a variation of the Ehrenfeucht-Fraïssé game for inquisitive first order logic and shows that this game provides a characterization of the expressive power of the logic. Moreover, it presents further variations of the game (i.e., a *symmetric version* and a *transfinite version*), showing the flexibility of the game-theoretic approach.

Chapter 5 showcases an application of the Ehrenfeucht-Fraïssé game introduced in the previous chapter. We introduce the notion of cardinality quantifier in inquisitive first order logic, and we use the game to characterize which among them are definable using the game.

Manipulating models

The second part, consisting of Chapter 6, takes another step in the model-theoretic direction and presents several ways to manipulate and combine models of first order inquisitive logic. The theory developed allows to prove two interesting properties of the logic: the *Disjunction and Existence properties*.

There is a close connection between intuitionistic logic and inquisitive logic, as shown by several results. For example: there is an interpretation of proofs as programs for both logics [Ciardelli, 2016, Proposition 2.4.8]; there is a translation of inquisitive logic into intuitionistic logic [Ciardelli and Roelofsen, 2011, Section 6]; several intermediate logics have inquisitive logic as their negative translation [Ciardelli, 2009, Theorem 3.4.9].

Two hallmarks of constructive logics are the *disjunction property* and the *existence property*: the former states that, if a disjunction of the form $\varphi \vee\!\!\!\vee \psi$ is valid, then at least one of the disjuncts φ and ψ is valid too; and the latter states that, if an existential formula $\overline{\exists} x.\varphi$ is valid, then for a term t the formula $\varphi[t/x]$ is valid too. Both properties are famously true for intuitionistic logic, and the disjunction property has been proven to hold also for inquisitive propositional logic [Ciardelli, 2016, Corollary 2.5.6]. In this chapter we address whether these properties hold for inquisitive first order logic, as already conjectured by Ciardelli [2016].

The proof we give is semantical in nature: we develop several constructions to combine and transform inquisitive models, and use them to prove the disjunction and existence properties. Some of these constructions are inspired by operations on intuitionistic Kripke-frames (e.g., disjoint union, direct sum) while others are based on constructions typical of classical predicate logic (e.g., models of terms, permutation models).

This approach allows us to prove also more general results: we define several classes of theories for which the corresponding consequence relations have the disjunction and/or the existence property. Most notably, theories containing only statements have this property.

Axiomatization

In the third part, consisting of Chapters 7 and 8, we shift our attention on the axiomatization problem. As of now it is not known whether first order inquisitive logic is axiomatizable—even though there is a candidate deduction system proposed by Ciardelli [2016, Section 4.6]. However, we tackle a restricted version of the axiomatization problem, that is, we *axiomatize fragments and variations of the logic.*

This is not the first work focusing on this problem: Ciardelli [2016, Ch. 4] showed that two fragments, the *mention-some fragment* and the *mention-all fragment*, can be recursively axiomatized. This leads to the questions whether there are other interesting fragments or variations of inquisitive first order logic which are axiomatizable, and whether we can find novel techniques to axiomatize them. In this part of the dissertation we give positive answers to these questions: we introduce and axiomatize a new fragment—the *classical antecedent fragment*—and we study a family of inquisitive logics—the *finite-width inquisitive logics*.

Chapter 7 focuses on the classical antecedent fragment, which extends the mention-all and the mention-some fragments. It can be intuitively characterized as the fragment where questions are not allowed in the antecedent of a conditional. This fragment is particularly interesting since it contains—modulo logical equivalence—all formulas corresponding to natural language sentences. We prove that the natural deduction system proposed in [Ciardelli, 2016, Section 4.6], restricted to the classical antecedent fragment, provides a *sound and strongly complete* axiomatization.

Chapter 8 focuses on the finite-width inquisitive logics and on the bounded-width fragment. Finite-width inquisitive logics were already introduced by Sano [2011] as a hierarchy closely related to inquisitive first order logic (among which we also find pair semantics by Groenendijk and Stokhof [1984]). In the same paper, Sano axiomatized pair semantics by adapting the canonical model completeness technique for first order intuitionistic logic with constant domain [Gabbay et al., 2009, Section 7.2]. Two questions were left open in Sano's paper, that is, whether the other elements of the hierarchy are axiomatizable, and whether first order inquisitive logic is the limit of this hierarchy: we give a positive answer to the former and a negative answer to the latter.

The chapter also treats the bounded-width fragment, characterized by the following property similar to the finite model property of modal logic and the *coherence* property of dependence logic [Kontinen, 2010]: if a formula of the fragment is not supported by an information state s, then there exists a *finite* subset of s which still does not support the formula. This rather peculiar property allows to derive several interesting results on the fragment (e.g., validities in the fragment are recursively enumerable, the restricted entailment is compact), building on the completeness result for the finite-width inquisitive logics.

Alternative semantics

The fourth and last part, consisting of Chapter 9, is an exploratory work not yet developed for the first order case, but only for the propositional case: we present *an algebraic and a topological semantics for inquisitive propositional logics*. This line of research strengthens the bonds between inquisitive logic and intermediate logics, and opens new venues of research in the direction of universal algebra. Generalizing these semantic accounts to the first order case could prove to be a precious tool to study first order inquisitive logic from new perspectives, for example using the methods employed by Rasiowa and Sikorski [1950] or Görnemann [1971].

On the algebraic side, we introduce a new semantics based on Heyting algebras by restricting the valuations of propositional atoms only over *regular elements*. We also show that the possible-world semantics for inquisitive logic can be seen as a particular instance of this algebraic semantics. From this we obtain an algebraic semantics for inquisitive logic by imposing additional conditions—stemming from the linguistic interpretation of the logic—and thus obtaining the class of *inquisitive Heyting algebras*. We also prove rather interesting properties of these special algebras: for example, an inquisitive Heyting algebra is univocally determined by its set of regular elements (modulo isomorphism).

On the topological side, we apply a duality result developed by Bezhanishvili and Holliday [2020] to characterize inquisitive algebras in terms of their dual *topological UV-spaces*. This allows to define a topological semantics for inquisitive logic which, as far as we know, is the first attempt to study inquisitive logic from a topological perspective.

Chapter 2

Background

In this chapter we recall the main definitions and properties of inquisitive first order logic InqBQ. We also introduce some tools that will be used in later chapters.

Before going to the main content of the chapter, let us recall some basic notation and definitions from *first order classical logic* CQC. Henceforth we will assume to have fixed a first order logic signature Σ consisting of *relation symbols* (usually indicated with the letters p, P, R) and *function symbols* (usually indicated with the letters c, f). 0-ary relation symbols will be referred to as *propositional atoms* and 0-ary relation symbols will be referred to as *constants*. We will also assume to have fixed an infinite set of variables $\mathtt{Var}$.

The set of *terms of* Σ is produced by the following grammar:

$$t \;:=\; x \mid c \mid f(t_1, \ldots, t_{\mathrm{Ar}(f)})$$

where $x \in \mathtt{Var}$ is a variable, $c \in \Sigma$ is a constant, $f \in \Sigma$ is a non-0-ary function symbol and $t_1, \ldots, t_{\mathrm{Ar}(f)}$ are other terms.

To make the connection between classical logic and inquisitive logic more clear, we will present the syntax and semantics of CQC in a slightly unconventional way: we will consider three different syntaxes and distinct associated semantics.

2.0.1. Definition (Syntaxes for CQC). The syntaxes $\mathcal{L}^{\mathsf{CQC}}_{\neq}$, $\mathcal{L}^{\mathsf{CQC}}_{=}$ and $\mathcal{L}^{\mathsf{CQC}}_{\asymp}$ are defined by the following grammars:

$$\begin{array}{ll}
\mathcal{L}^{\mathsf{CQC}}_{\neq}: & \varphi ::= \bot \mid R(\bar{t}) \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \forall x.\varphi \\
\mathcal{L}^{\mathsf{CQC}}_{=}: & \varphi ::= \bot \mid R(\bar{t}) \mid t_1 = t_2 \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \forall x.\varphi \\
\mathcal{L}^{\mathsf{CQC}}_{\asymp}: & \varphi ::= \bot \mid R(\bar{t}) \mid t_1 \asymp t_2 \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \forall x.\varphi
\end{array}$$

where t_1 and t_2 are terms of Σ and $\bar{t}$ is a sequence of terms of arity $\mathrm{Ar}(R)$.

The difference between the three languages is the equality symbol and its interpretation: $\mathcal{L}^{\mathsf{CQC}}_{\neq}$ does not have an equality; $\mathcal{L}^{\mathsf{CQC}}_{=}$ contains the equality symbol $=$, which will be interpreted as the usual *identity between elements of a model*; $\mathcal{L}^{\mathsf{CQC}}_{\asymp}$

contains the equality symbol $\asymp$, which will be interpreted as a congruence relation explicitly encoded in the models. To introduce a terminological distinction, we will call $=$ the *rigid equality* and $\asymp$ the *non-rigid equality.*

Depending on which syntax we are working with, the models for CQC are slightly different. For brevity, we give a compact but exhaustive definition covering the three cases.

2.0.2. Definition (Models of CQC). A model of CQC (or CQC-model) is a sequence

$$M = \langle D^M, \mathcal{I} \rangle \qquad \text{(if we are working with } \mathcal{L}^{\mathsf{CQC}}_{\not=} \text{ or } \mathcal{L}^{\mathsf{CQC}}_{=}\text{)}$$
$$M = \langle D^M, \mathcal{I}, \asymp^M \rangle \qquad \text{(if we are working with } \mathcal{L}^{\mathsf{CQC}}_{\asymp}\text{)}$$

where:

- D^M is a non-empty set, called the *domain of the model*;
- $\mathcal{I}$ is an interpretation map that associates to every symbol in the signature an appropriate interpretation in the model, that is:[1]
  - For R an n-ary relation symbol, $\mathcal{I}(R) \subseteq D^n$;
  - For f an n-ary function symbol, $\mathcal{I}(f) : D^n \to D$;
- $\asymp^M \subseteq D^2$ (if present) is an equivalence relation over D, that is also a congruence with respect to the interpretation of relation and function symbols.[2]

When M is clear from the context, we will indicate D^M with D, omitting the reference to the model. As a notational convention, we will indicate $\mathcal{I}(R)$ and $\mathcal{I}(f)$ with the symbols R^M and f^M respectively.

Before moving to the semantics, we introduce assignments and the interpretation of terms. Given a model M, an *assignment over* M is a function $g : \mathtt{Var} \to D$. We will indicate with $g[x \mapsto d]$ the assignment that maps x to d and that coincides with g on every other variable. Given a model M and an *assignment* $g : \mathtt{Var} \to D$ we can define t^g_M the *interpretation of a term t in M relative to g* by the following clauses:

$$x^g_M := g(x) \qquad c^g_M := c^M \qquad f(t_1, \dots, t_{\mathrm{Ar}(f)})^g_M := f^M(\ (t_1)^g_M, \dots, (t_{\mathrm{Ar}(f)})^g_M\ )$$

If M is clear from the context, we will simply write t^g instead of t^g_M.

Now we give a compact presentation of the semantics containing the clauses of all the syntaxes.

[1]We assume the following notational convention: D^0 indicates a fixed singleton set $\{*\}$ not dependent on D. This way the interpretations of the symbol of the signature encompass naturally propositional atoms ($\mathcal{I}(p) \subseteq \{*\}$) and constant symbols ($\mathcal{I}(c) : \{*\} \to D$).

[2]An equivalence relation $\asymp \subseteq D^2$ is called a *congruence* with respect to an n-ary relation R (resp., function symbol f) if $d_1 \asymp d'_1, \dots, d_n \asymp d'_n$ implies that $R(d_1, \dots, d_n) \Leftrightarrow R(d'_1, \dots, d'_n)$ (resp., $f(d_1, \dots, d_n) \asymp f(d'_1, \dots, d'_n)$).

2.0.3. DEFINITION (Semantics of CQC). Let M be a CQC-model and $g : \texttt{Var} \to D$ an assignment. We define the satisfaction relation $\vDash^{\mathsf{CQC}}$ over formulas of CQC by the following inductive clauses:

$$
\begin{array}{ll}
M \nvDash^{\mathsf{CQC}}_g \bot & \\
M \vDash^{\mathsf{CQC}}_g t_1 = t_2 & \iff t_1^g = t_2^g \\
M \vDash^{\mathsf{CQC}}_g t_1 \asymp t_2 & \iff t_1^g \asymp^M t_2^g \\
M \vDash^{\mathsf{CQC}}_g R(t_1, \dots, t_n) & \iff R^M(t_1^g, \dots, t_n^g) \\
M \vDash^{\mathsf{CQC}}_g \psi_1 \wedge \psi_2 & \iff M \vDash^{\mathsf{CQC}}_g \psi_1 \text{ and } M \vDash^{\mathsf{CQC}}_g \psi_2 \\
M \vDash^{\mathsf{CQC}}_g \psi_1 \to \psi_2 & \iff M \nvDash^{\mathsf{CQC}}_g \psi_1 \text{ or } M \vDash_g \psi_2 \\
M \vDash^{\mathsf{CQC}}_g \forall x.\psi & \iff \text{For all } d \in D \text{ we have } M \vDash^{\mathsf{CQC}}_{g[x \mapsto d]} \psi
\end{array}
$$

As in the propositional case, we indicate with $\Gamma \vDash^{\mathsf{CQC}} \varphi$ that every model which satisfies every formula in Γ satisfies φ too. For Γ empty we simply write $\vDash^{\mathsf{CQC}} \varphi$ and, in case this holds, we say that φ is valid. We will indicate with $\mathsf{CQC}^{\neq}$, $\mathsf{CQC}^{=}$ and $\mathsf{CQC}^{\asymp}$ the sets of valid formulas in the syntaxes $\mathcal{L}^{\mathsf{CQC}}_{\neq}$, $\mathcal{L}^{\mathsf{CQC}}_{=}$ and $\mathcal{L}^{\mathsf{CQC}}_{\asymp}$ respectively.

It is easy to show that $\mathsf{CQC}^{=}$ and $\mathsf{CQC}^{\asymp}$ contain the same formulas, modulo changing the equality symbol used. As we will see, this is not the case for inquisitive logic: the different models used in combination to the syntaxes $\mathcal{L}_{=}$ and $\mathcal{L}_{\asymp}$ lead to differences between the interpretations of $=$ and $\asymp$ which are reflected at the level of validity.

2.1 First Order Inquisitive Logic InqBQ

We now proceed to present the main definitions for the first order version of inquisitive logic InqBQ. As anticipated in Chapter 1, this logic can be thought of as an extension of CQC with question-forming operators.

As for CQC, we will work with different syntaxes depending on whether and which equality we are using. We consider the syntaxes $\mathcal{L}_{\neq}$ (without equality), $\mathcal{L}_{=}$ (with rigid equality) and $\mathcal{L}_{\asymp}$ (with non-rigid equality).

2.1.1. DEFINITION (Syntaxes for the logic InqBQ). The syntaxes $\mathcal{L}_{\neq}$, $\mathcal{L}_{=}$ and $\mathcal{L}_{\asymp}$ are defined by the following grammars:

$$
\begin{array}{ll}
\mathcal{L}_{\neq}: & \varphi ::= \bot \mid R(\bar{t}) \mid \varphi \wedge \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid \varphi \to \varphi \mid \forall x.\varphi \mid \overline{\exists} x.\varphi \\
\mathcal{L}_{=}: & \varphi ::= \bot \mid R(\bar{t}) \mid t_1 = t_2 \mid \varphi \wedge \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid \varphi \to \varphi \mid \forall x.\varphi \mid \overline{\exists} x.\varphi \\
\mathcal{L}_{\asymp}: & \varphi ::= \bot \mid R(\bar{t}) \mid t_1 \asymp t_2 \mid \varphi \wedge \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid \varphi \to \varphi \mid \forall x.\varphi \mid \overline{\exists} x.\varphi
\end{array}
$$

where t_1 and t_2 are terms of Σ (defined as for CQC) and $\bar{t}$ is a sequence of terms of arity $\mathrm{Ar}(R)$.

Notice that the syntaxes introduced are that of CQC with in addition the two new operators $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ (*inquisitive disjunction*) and $\overline{\exists}$ (*inquisitive existential quantifier*). The role of these two operators is to introduce *questions* into the picture: $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ is used to introduce *alternative questions* (e.g., "Does c have property P, or property Q?" translates to $P(c) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} Q(c)$) and $\overline{\exists}$ to introduce *existential questions* (e.g., "What is an element with property P?" translates to $\overline{\exists}x.P(x)$).

Another important difference between CQC and InqBQ is the role of identity, that is, the ways in which we interpret the equality symbol. We introduced the rigid and non-rigid equalities for CQC-models, but the semantics of classical logic cannot distinguish between the interpretations of the two: every classically valid inference involving the rigid equality remains classically valid if we consider the non-rigid equality instead, and vice versa.

However, in the context of inquisitive logic we can give a different interpretation to the two symbols, compatible with the view of InqBQ as a generalization of CQC: the rigid equality $=$ is interpreted as the *real identity* between elements, that is, the one relating two formal objects only if they are *the same object*; while the non-rigid equality $\asymp$ is interpreted as an intensional equality, for which different formal objects may or may not represent the same individual and this is an information encoded in the semantics itself.

We will discuss the technical consequences of this distinction after we introduce the semantics of InqBQ, but for now let us point out that the two identities can model different scenarios and have different logical properties. An emblematic example is the following puzzle by Frege [1892]. The names *Hesperus* and *Phosphorus* both refer to the same entity, that is, the planet Venus. However the sentences "Hesperus is Hesperus" and "Hesperus is Phosphorus" seem to have a different *meaning*, even though they are both true. The point to make here is that the former is *tautological*, while the latter can be inferred only when we have the *information* that the two names actually refer to the same entity.[3] This is a situation which *can* be modeled using the non-rigid equality since the *piece of information* that the two names refer to the same object *can be encoded in the semantics itself*; but it cannot be captured by the rigid equality. For a more extensive treatment of Frege's puzzle from the point of view of inquisitive logic we refer to [Ciardelli, 2016, Chapter 4].

To enrich the language we introduce the following shorthands.

$$\neg\varphi :\equiv \varphi \to \bot \qquad \varphi \vee \psi :\equiv \neg(\neg\varphi \wedge \neg\psi) \qquad \exists x.\varphi :\equiv \neg\forall x.\neg\varphi$$

$$?\varphi :\equiv \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg\varphi \qquad t_1 \neq t_2 :\equiv \neg(t_1 = t_2) \qquad t_1 \not\asymp t_2 :\equiv \neg(t_1 \asymp t_2)$$

We will call the symbols $\vee$ and $\exists$ *classical disjunction* and *classical existential quantifier* respectively. $\vee$ and $\exists$ play the role of statement-forming operators, as

[3] I found myself very confused when I heard this example for the first time during a seminar, without knowing the premise that Hesperus and Phosphorus are both names for Venus: I was thinking "Of course these two sentences have a different meaning, what is everybody talking about?".

in their interpretation in CQC: "c has property P or property Q" translates to $P(c) \vee Q(c)$; "There exists an element with property P" translates to $\exists x.P(x)$.

We will call a formula *classical* if it does not contain the symbols $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\overline{\exists}$, that is, if it is generated by the grammar of CQC. Notice that, given two classical formulas α and β, also $\neg\alpha$, $\alpha \vee \beta$ and $\exists x.\alpha$ are classical formulas. In the rest of this dissertation, we will assume the following notational convention: with the symbols α, β, γ, ... we indicate classical formulas; while with the symbols φ, χ, ψ, ... we indicate generic formulas, possibly classical.

2.1.1 Models of InqBQ

As previously mentioned, the logic InqBQ aims to capture the logical relations involving statements and questions through an information based approach. To formalize this concept, we need an appropriate mathematical structure to represent *information.*

2.1.2. Definition (Information model). An *information model* $\mathcal{M}$ of the syntax $\mathcal{L}_{\neq}$ (resp. $\mathcal{L}_{=}$, $\mathcal{L}_{\asymp}$) is a multiset $\{M_w \mid w \in W^{\mathcal{M}}\}$ where $W^{\mathcal{M}}$ is a set (called the *set of worlds* of the model) and the M_w are CQC-models of the syntax $\mathcal{L}_{\neq}^{\mathsf{CQC}}$ (resp. $\mathcal{L}_{=}^{\mathsf{CQC}}$, $\mathcal{L}_{\asymp}^{\mathsf{CQC}}$) with the same domain $D^{\mathcal{M}}$ (called the *domain* of the model) and the same interpretation of function symbols over $D^{\mathcal{M}}$.[4]

As a notational convention, we will always indicate the CQC-models constituting an information model with the same name and we will use different fonts to distinguish them (e.g.: N_w is a structure in $\mathcal{N}$). Regarding the interpretation of the symbols from Σ, we will use the following notational convention: for R a relation symbol, we indicate with $R_w^{\mathcal{M}}$ the interpretation of R in the model M_w (we assume the same convention for $\asymp$, whose interpretation is world-dependent); for f a function symbol, we indicate with $f^{\mathcal{M}}$ the interpretation of f at any of the models $M_w \in \mathcal{M}$ (recall that the interpretation of f does not depend on the world w; we assume the same convention for $=$, whose interpretation is not world-dependent). When $\mathcal{M}$ is clear from the context, we will indicate $W^{\mathcal{M}}$, $D^{\mathcal{M}}$ and $R_w^{\mathcal{M}}$ with W, D and R_w respectively, omitting the reference to the model (while we maintain the notation $f^{\mathcal{M}}$ to distinguish the interpretation of f from the symbol).

Notice that we require function symbols to have a *rigid* (i.e., non world-dependent) interpretation. Symbols with a rigid interpretation further enhance the expressive power of the logic, and as we will see some results proved in this dissertation depend on their presence in the language—for example, the *existence property* (Chapter 6). Alternatively, we could have introduced also *rigid*

[4]With the term *multiset* we refer to a family of elements $X := \{ x_w \mid w \in W \}$ indexed by another set W. Using multisets allows to have multiple copies of the same element (distinguished by the index) without implicitly imposing an order on X (as we have with *sequences*).

relation symbols and *non-rigid function symbols* in the signature, following the presentation of [Ciardelli, 2016, Chapter 4]. However, this choice does not restrict the expressive power of the logic since we can simulate them at the level of logical entailment (both *functionality* and *rigidity* are definable properties). And in addition, this allows to simplify the axiomatic systems proposed to capture the support semantics. For example, the *introduction rule* for the existential quantifier is not valid for terms containing *non-rigid function symbols*.

Information models are used to represent *pieces of information*. In this context, we use the term *piece of information* to refer to any property I of first order models in the given signature. For example I_1: "the interpretation of c is in the extension of P" and I_2: "the extensions of P and Q are disjoint" are considered pieces of information. Given an information model $\mathcal{M}$ we can encode a piece of information I with the set of worlds $s_I := \{w \in W \mid M_w \text{ has property } I\}$; that is, by selecting the worlds corresponding to the first order models having property I. So in the examples given above, s_{I_1} consists of all the worlds $w \in W$ for which M_w satisfies $P(c)$; and s_{I_2} consists of all the worlds $w \in W$ for which $P_w \cap Q_w = \emptyset$. This intuition motivates the following definition.

2.1.3. DEFINITION (Information state). Given a model $\mathcal{M}$, we will refer to a subset $s \subseteq W$ as an *information state* or *info state*. We will say that $t \subseteq s$ is an *enhancement* of s. Given an information state $s \subseteq W$, we define the *restriction of* $\mathcal{M}$ *to* s as the model

$$\mathcal{M}|_s := \{M_w \mid w \in s\}$$

We now give some examples of information models and we introduce the graphical representation we will use to visualize them.

2.1.4. EXAMPLE. A graphical representation of an information model $\mathcal{M}$ of the language $\mathcal{L}_{\asymp}$ is depicted in Figure 2.1. The signature considered is $\Sigma = \{c, P\}$, consisting of a constant symbol c and a unary predicate P. $\mathcal{M}$ is defined by the following clauses.

- The set of worlds of $\mathcal{M}$ is $W = \{w_0, w_1\}$. In Figure 2.1, the column with label w_i represents the classical structure M_{w_i}.
- The domain of $\mathcal{M}$ is $D = \{a, b\}$. In Figure 2.1, we depict one copy of the domain for every classical structure.
- The interpretation of c is $c^{\mathcal{M}} = a$. In Figure 2.1, it is indicated by placing the symbol c near a. Notice that by definition the interpretation of c has to be the same for every world.
- For every column, the elements in the extension of P are represented by star-shaped nodes, while the others are represented by square-shaped nodes. In Figure 2.1 we have $P_{w_0}(a)$, $P_{w_1}(b)$ but *not* $P_{w_0}(b)$, $P_{w_1}(a)$.

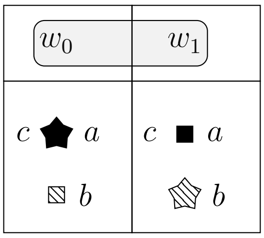


Figure 2.1: The model $\mathcal{M}$ described in Example 2.1.4.

- For every column, the identity $\asymp$ is represented by the pattern of the nodes (filled or striped). Two elements in the column labelled w are related by the identity $\asymp_w$ of M_w if and only if they have the same pattern. In Figure 2.1 we have $a \not\asymp_{w_0} b$ and $a \not\asymp_{w_1} b$.

Finally, the grey rectangle represents the information state $s = \{w_0, w_1\}$. s represents the following piece of information: it is known that there are two distinct individuals and that exactly one of these individuals has property P. It is not known, however, which of the two individuals has property P.

2.1.5. Example (A model representing all finite cyclic groups). Consider the signature $\{0, S, +\}$ and the set of worlds $W = \{ w_n \,|\, n \geq 1 \}$. Consider the model $\mathbb{Z}_* = \{\mathbb{Z}/n\mathbb{Z} \mid w_n \in W\}$ in the syntax $\mathcal{L}_{\asymp}$ where $\mathbb{Z}/n\mathbb{Z}$ indicates the group $\mathbb{Z}$ where $\asymp$ is interpreted by the following clause: $a \asymp_{w_n} b$ iff $a - b \equiv 0 \mod n$. A graphical representation of this model is given in Figure 2.2.

This model is suitable to represent informational scenarios in which we are working with a finite cyclic group, but we have only partial information on which one. For example, the information that we are working with a cyclic group of even order is encoded by the info state $s := \{ w_2, w_4, w_6, \ldots \}$.

As previously pointed out, some components of the model are not world-dependent, that is, their interpretation does not depend on the particular structure M_w we consider: the domain, the interpretation of the function symbols and the interpretation of $=$ (if present in the language). This suggests the following definition, which we will extensively make use of in Chapter 6.

2.1.6. Definition (Skeleton). Given an information model $\mathcal{M}$, we call the *skeleton of* $\mathcal{M}$ the pair[5]

$$\text{Sk}(\mathcal{M}) = \left\langle D^{\mathcal{M}}, \{ f^{\mathcal{M}} \}_{f \in \Sigma} \right\rangle$$

We will refer to a pair $S = \left\langle D^{\mathcal{M}}, \{ f^{\mathcal{M}} \}_{f \in \Sigma} \right\rangle$ simply as a *skeleton*, and we call an information model $\mathcal{M}$ such that $\text{Sk}(\mathcal{M}) = S$ a *model over the skeleton* S.

[5]The interpretation of $=$ is not indicated since, as we will see later, $=$ is always interpreted as the extensional equality between objects in D.

w_1	w_2	w_3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	
▧ -2	▧ -2	▧ -2	
▧ -1	■ -1	■ -1	
▧ 0	▧ 0	⚀ 0	$\cdots$
▧ 1	■ 1	▧ 1	
▧ 2	▧ 2	■ 2	
$\vdots$	$\vdots$	$\vdots$	

Figure 2.2: A graphical representation of the model $\mathbb{Z}_*$ from Example 2.1.5. The same conventions as Figure 2.1 are adopted. Notice that the classical models associated with two distinct worlds are distinguished only by the identity relation. For example, at world w_2 it holds that $-2 \asymp_{w_2} 2$ because $-2 \equiv 2 \mod 2$, while at world w_3 we have $-2 \not\asymp_{w_3} 2$ because $-2 \not\equiv 2 \mod 3$.

Notice that to define an information model $\mathcal{M}$ it suffices to specify separately W, $\mathrm{Sk}(\mathcal{M})$, R_w for every $w \in W$ and relation symbol $R \in \Sigma$ and $\asymp_w$ (if we are working with the language $\mathcal{L}_\asymp$). In case we are working with the language $\mathcal{L}_\asymp$, to ensure that the structure so defined is an information model it is necessary and sufficient to check that the relation $\asymp_w$ is a congruence with respect to the interpretation of the symbols for every choice of the world $w \in W$.

2.1.2 Semantics of InqBQ

Now we have all the ingredients needed to introduce the semantics for the logic InqBQ. For brevity, we do not distinguish between models for the different languages, but note that the clause for $\asymp$ can be interpreted only in models of the syntax $\mathcal{L}_\asymp$.

2.1.7. Definition (Semantics of InqBQ). Let $\mathcal{M} = \{M_w \mid w \in W\}$ be an information model, $s \subseteq W$ an info state and $g : \mathtt{Var} \to D$ an assignment. We define the support relation $\vDash$ over formulas of InqBQ by the following inductive clauses:

$$
\begin{array}{ll}
\mathcal{M}, s \vDash_g \bot & \iff s = \emptyset \\
\mathcal{M}, s \vDash_g t_1 = t_2 & \iff t_1^g = t_2^g \\
\mathcal{M}, s \vDash_g t_1 \asymp t_2 & \iff \text{For all } w \in s \text{ we have } t_1^g \asymp_w^{\mathcal{M}} t_2^g \\
\mathcal{M}, s \vDash_g R(t_1, \dots, t_n) & \iff \text{For all } w \in s \text{ we have } R_w^{\mathcal{M}}(t_1^g, \dots, t_n^g) \\
\mathcal{M}, s \vDash_g \psi_1 \wedge \psi_2 & \iff \mathcal{M}, s \vDash_g \psi_1 \text{ and } \mathcal{M}, s \vDash_g \psi_2 \\
\mathcal{M}, s \vDash_g \psi_1 \vee\!\!\!\vee \psi_2 & \iff \mathcal{M}, s \vDash_g \psi_1 \text{ or } \mathcal{M}, s \vDash_g \psi_2 \\
\mathcal{M}, s \vDash_g \psi_1 \to \psi_2 & \iff \text{For all } t \subseteq s, \text{ if } \mathcal{M}, t \vDash_g \psi_1 \text{ then } \mathcal{M}, t \vDash_g \psi_2 \\
\mathcal{M}, s \vDash_g \forall x.\psi & \iff \text{For all } d \in D \text{ we have } \mathcal{M}, s \vDash_{g[x \mapsto d]} \psi \\
\mathcal{M}, s \vDash_g \exists\!\!\!\exists x.\psi & \iff \text{There exists } d \in D \text{ such that } \mathcal{M}, s \vDash_{g[x \mapsto d]} \psi
\end{array}
$$

In case $s = W$ we simply write $\mathcal{M} \vDash_g \varphi$, omitting the info state. In case φ is a sentence (i.e., a formula without free variables) we simply write $\mathcal{M}, s \vDash \varphi$, omitting the assignment. We use the notation $\mathcal{M}, s \vDash \varphi(a_1, \dots, a_n)$ as a shorthand for $\mathcal{M}, s \vDash_g \varphi(x_1, \dots, x_n)$ for an arbitrary assignment g such that $g(x_i) = a_i$ for every $i \in \{1, \dots, n\}$.

Before showcasing some example, let us present the support conditions that can be derived for some of the shorthands previously introduced.

$$
\begin{array}{ll}
\mathcal{M}, s \vDash_g \neg\psi & \iff \text{For all } t \subseteq s, \text{ if } t \neq \emptyset \text{ then } \mathcal{M}, t \nvDash_g \psi \\
\mathcal{M}, s \vDash_g ?\psi & \iff \mathcal{M}, s \vDash_g \psi \text{ or } \mathcal{M}, s \vDash_g \neg\psi \\
\mathcal{M}, s \vDash_g t_1 \neq t_2 & \iff \text{For all } w \in s, t_1^g \neq t_2^g
\end{array}
$$

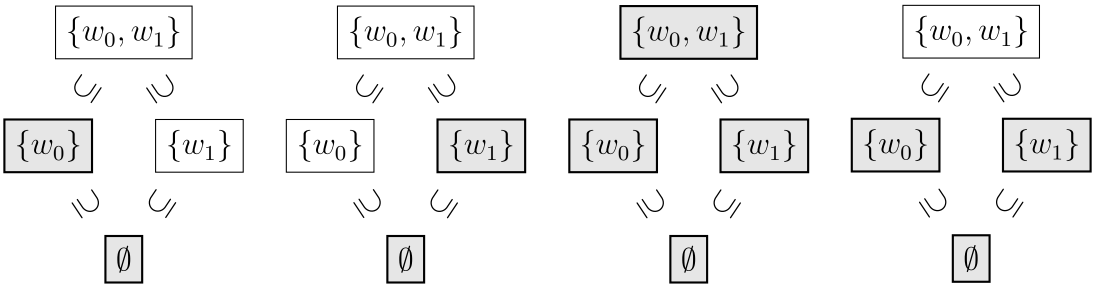
2.1.8. Example. Consider the model $\mathcal{M}$ depicted in Figure 2.1. Following the semantics clauses defined above we have:

- $\mathcal{M} \nvDash P(c)$ since for the world w_1 the element $c^{\mathcal{M}} = a$ is not in the extension of P_{w_1}. However if we restrict to the information state $\{w_0\}$ we have $\mathcal{M}, \{w_0\} \vDash P(c)$ since for every world in the info state—namely only w_0—we have that $c^{\mathcal{M}}$ is in the extension of P.

- $\mathcal{M} \nvDash \neg P(c)$, since for the state $\{w_0\} \subseteq W$ we showed that $\mathcal{M}, \{w_0\} \vDash P(c)$ holds. For a similar reason, $\mathcal{M}, \{w_1\} \vDash \neg P(c)$ since for every non-empty substate of $\{w_1\}$ (namely $\{w_1\}$ itself) the formula $P(c)$ is not supported.

- $\mathcal{M} \vDash P(a) \to \neg P(b)$, since every substate of W supporting $P(a)$—namely $\emptyset$ and $\{w_0\}$—supports also $\neg P(b)$.

- $\mathcal{M} \nvDash \exists\!\!\!\exists x.P(x)$, since for every choice of $d \in D$ there is a substate of W not supporting $P(d)$: if we pick the element a we have $\mathcal{M}, \{w_1\} \nvDash P(a)$; if we



(a) $P(c)$ (b) $\neg P(c)$ (c) $P(a) \to \neg P(b)$ (d) $\overline{\exists} x.P(x)$

Figure 2.3: A graphical description of Example 2.1.8. In gray the states that support the formula.

pick the element b we have $\mathcal{M}, \{w_0\} \nvDash P(b)$. Notice that if we restrict to the singleton information states $\{w_0\}$ and $\{w_1\}$ then the formula is supported, since $\mathcal{M}, \{w_0\} \vDash P(a)$ and $\mathcal{M}, \{w_1\} \vDash P(b)$.

A graphical recap of the example is given in Figure 2.3.

Intuitively an information state s, which encodes a certain piece of information, *supports a statement (classical formula)* α if all the worlds of W (the current context) compatible with the information carried by s satisfy α; or in other terms, if by knowing that s, we know that α is true. So for example, the formula $P(c)$ representing the statement "c has property P" is supported by the info state s if and only if every classical structure M_w associated with a world $w \in s$ satisfies $P(c)$.

When it comes to questions, an information state s *supports a question* φ if the issue raised by the question is resolved by the information carried by s. For example, the formula $\overline{\exists} x.P(x)$ representing the question "What is an element with property P?" is supported by s if and only if we can identify an element a such that $P(a)$ is satisfied for every M_w with $w \in s$. Notice that, with the information carried by s, we need to be able to *identify* an element a, and not just know that there is one.

There are two characteristic properties of inquisitive semantics, which stem from the information-based interpretation: the *empty state property* and the *persistency property.*

2.1.9. Lemma. *For every formula* φ *of the logic*

Empty state $\mathcal{M}, \emptyset \vDash_g \varphi$.

Persistency *If* $\mathcal{M}, s \vDash_g \varphi$ *and* $u \subseteq s$, *then* $\mathcal{M}, u \vDash_g \varphi$.

These two properties have a quite intuitive interpretation: the incoherent information state ($\emptyset$) supports every sentence—this can be thought as an *ex falso quod* *libet* principle; and when an information state (s) supports a sentence, then every enhancement ($t \subseteq s$) supports that sentence too.

These properties allow to simplify the semantic clause for negation and to obtain simple clauses also for the remaining shorthands.

$$\begin{aligned}
\mathcal{M}, s \vDash_g \neg\psi &\iff \text{For all } w \in s \text{ we have } \mathcal{M}, \{w\} \nvDash_g \psi \\
\mathcal{M}, s \vDash_g \psi_1 \vee\!\!\!\vee\, \psi_2 &\iff \text{For all } w \in s \text{ we have } \mathcal{M}, \{w\} \vDash_g \psi_1 \text{ or } \mathcal{M}, \{w\} \vDash_g \psi_2 \\
\mathcal{M}, s \vDash_g \exists x.\psi &\iff \text{For all } w \in s \text{ there exists } d_w \in D \\
&\qquad\qquad \text{such that } \mathcal{M}, \{w\} \vDash_{g[x \mapsto d_w]} \psi \\
\mathcal{M}, s \vDash_g t_1 \not\asymp t_2 &\iff \text{For all } w \in s \text{ we have } t_1^g \not\asymp_w^{\mathcal{M}} t_2^g
\end{aligned}$$

We give a couple more examples to familiarize the reader with the semantics and showcase properties of some interesting formulas.

2.1.10. Example. Consider the formula $?P(c)$ and the model from Example 2.1.8. Expanding the semantic clause for ? we obtain $\mathcal{M} \nvDash ?P(c)$, since we have $\mathcal{M} \nvDash P(c)$ and $\mathcal{M} \nvDash \neg P(c)$. Intuitively, the information encoded in the info state $\{w_0, w_1\}$ is not enough to determine whether c is in the extension of P, since there are possible worlds for which this is the case and possible worlds for which this is not the case.

It is interesting to point out that the formula $?P(c)$ is always satisfied at singleton states. In fact, expanding the definition we have:

$$\mathcal{M}, \{w\} \vDash ?P(c) \qquad \text{iff} \qquad c^{\mathcal{M}} \in P_w \;\text{ or }\; c^{\mathcal{M}} \notin P_w$$

and the right-hand side is trivially satisfied.

2.1.11. Example. Consider the formula $\forall x.?P(x)$. This formula is particularly interesting, since it requires the extension of P to be *determined*. In fact, expanding the semantic clause for $\forall$ we have

$$\begin{aligned}
\mathcal{M}, s \vDash \forall x.?P(x) \quad &\text{iff} \quad \text{For all } d \in D \text{ we have } \mathcal{M}, s \vDash P(d) \text{ or } \mathcal{M}, s \vDash \neg P(d) \\
&\text{iff} \quad \text{For all } w, w' \in s \text{ we have } P_w = P_{w'}
\end{aligned}$$

In other terms, for every element d of the domain, all the worlds of s must agree on *whether* the element has property P.

A similar formula, but with quite a different interpretation is $?\forall x.P(x)$. In this case we have

$$\mathcal{M}, s \vDash ?\forall x.P(x) \qquad \text{iff} \qquad \begin{array}{r} \text{either for every } w \in s,\; P_w = D \\ \text{or for every } w \in s,\; P_w \neq D \end{array}$$

So there are two cases in which the formula is supported: either all the worlds of s agree *that* all the elements are in the extension of P; or the formula $\forall x.P(x)$ can be refuted in every world of s, that is, for every world w there exists an element d_w which is not in the extension of P. Notice that the element d_w could depend on the the world w, so in the latter case we do not have definitive information on the extension of P.

2.1.12. Example. Notice the difference between the semantics of the quantifiers $\overline{\exists}$ and $\exists$:

$$\mathcal{M}, s \vDash \overline{\exists} x.P(x) \iff \text{There exists } d \in D \text{ such that for every } w \in s,\ d \in P_w$$

$$\mathcal{M}, s \vDash \exists x.P(x) \iff \text{For all } w \in s \text{ there exists } d_w \in D \text{ such that } d_w \in P_w$$

The former requires to exhibit a single element d that is in the extension of P for every world in s; while the latter requires to find for every world w of s an element d_w—possibly depending on the world w—which is in the extension of P in M_w.

This difference in the semantics is in line with the intuitive interpretation of the two formulas: $\overline{\exists} x.P(x)$ represents the question "what is an x in the extension of P", and in fact it requires to exhibit an element in the extension; while $\exists x.P(x)$ represents the statement "there is an x in the extension of P", which does not require to *exhibit* such an element, but just to assert its existence.

As we did for CQC, we can define an entailment relation corresponding to the semantics introduced.

2.1.13. Definition. We define the *entailment relation* of InqBQ as follows: given a theory Φ and a formula ψ, we indicate with $\Phi \vDash \psi$ that for every $\mathcal{M}$, s and g such that $\mathcal{M}, s \vDash_g \Phi$, it holds that $\mathcal{M}, s \vDash_g \psi$ too. In particular, we indicate with $\varphi \equiv \psi$ that $\varphi \vDash \psi$ and $\psi \vDash \varphi$.

Since we are dealing with different syntaxes we introduce different sets of valid formulas, one for each syntax.

2.1.14. Definition. We indicate with $\mathsf{InqBQ}^{\neq}$, $\mathsf{InqBQ}^{=}$ and $\mathsf{InqBQ}^{\asymp}$ the sets of valid formulas for the semantics introduced for the syntaxes $\mathcal{L}_{\neq}$, $\mathcal{L}_{=}$ and $\mathcal{L}_{\asymp}$ respectively.

As a notational convention, when we do not want to refer to a particular language or the language is clear from the context we will simply use—with a slight abuse of notation—the symbol InqBQ to indicate the set of valid formulas considered. As usual, we will refer to elements of these sets as *tautologies* or *validities*.

2.1.3 Properties of the Logic

In this subsection we review some of the main properties characterizing InqBQ. For an in-depth treatment of these properties we refer to [Ciardelli, 2016, Section 4.4]. We start with a simple lemma showing that the semantics introduced is indeed a generalization of the one for CQC.

2.1.15. Lemma. *For $\mathcal{M}$ an information model, s an information state, g an assignment and α a classical formula, it holds that*

$$\mathcal{M}, s \vDash_g \alpha \iff \forall w \in s.\ M_w \vDash_g^{\mathsf{CQC}} \alpha$$

Unsurprisingly, this means that we can recover the semantics of CQC by simply restricting the syntax to classical formulas and the semantics to single-world information models (i.e., models for which W is a singleton). This result is also in line with the interpretation of classical formulas as statements: a classical formula (statement) α is supported by an information state (piece of information) s if the statement is *true* in every world of s. This leads to the following definition.

2.1.16. Definition. A formula φ is called *truth-conditional* if for every model $\mathcal{M}$, info state s and assignment g it holds that

$$\mathcal{M}, s \vDash_g \varphi \iff \forall w \in s.\ \mathcal{M}, \{w\} \vDash_g \varphi$$

So Lemma 2.1.15 states exactly that classical formulas are truth-conditional. Notice that not every formula is truth-conditional: for example $P(c) \vee\!\!\!\vee \neg P(c)$ is not as we saw in Example 2.1.10.

As a consequence of Lemma 2.1.15 we obtain the following result, which shows that the entailment relation of InqBQ is a generalization of its classical counterpart.

2.1.17. Theorem. *For every set of classical formulas $\Gamma \cup \{\alpha\}$ it holds that*

$$\Gamma \vDash \alpha \iff \Gamma \vDash^{\mathsf{CQC}} \alpha$$

We will refer to this result by saying that $\vDash$ is a *conservative extension* of $\vDash^{\mathsf{CQC}}$.

As we recalled in Chapter 1 and in line with the result just presented, CQC is a logic meant to represent *statements*, that is, sentences whose semantics is completely determined by their truth-conditions. The definition of truth-conditional formulas given above expresses the same property: for these formulas the support conditions boil down to truth at every world. In fact we can prove that (modulo logical equivalence) classical formulas are all *and only* the formulas with this property.

2.1.18. Theorem (Truth-conditionality and classical formulas). *For every formula φ, φ is truth-conditional if and only if there exists a classical formula such that $\varphi \equiv \alpha$.*

There is another interesting class of formulas which are truth-conditionals: negated formulas. As previously pointed out, the semantic clause for negation is

$$\begin{aligned}\mathcal{M}, s \vDash_g \neg\psi \quad &\Longleftrightarrow \text{For all } w \in s \text{ we have } \mathcal{M}, \{w\} \nvDash_g \psi \\ &\Longleftrightarrow \text{For all } w \in s \text{ we have } \mathcal{M}, \{w\} \vDash_g \neg\psi\end{aligned}$$

which amounts exactly to $\neg\psi$ being truth-conditional. Moreover by Theorem 2.1.18 we can also infer that every truth-conditional formula is equivalent to a negation, since $\varphi \equiv \alpha \equiv \neg\neg\alpha \equiv \neg\neg\varphi$. Consequently, we obtain yet another characterization of truth-conditional formulas.

2.1.19. Corollary. *For φ a formula, the following are equivalent:*

- *φ is truth-conditional;*
- *φ is equivalent to a classical formula α;*
- *φ is equivalent to $\neg\psi$ for some ψ;*
- *$\varphi \equiv \neg\neg\varphi$.*

2.2 Relations with the Intuitionistic Logic CD

If we change perspective on how to look at information models, we can see a clear connection between support semantics and *intuitionistic forcing*. For an introduction to intuitionistic forcing see [Dalen, 2002, Section 3], [Gabbay, 1981, Section 2.2] and [Gabbay et al., 2009, Chapter 3]. So far we presented info states as *sets of worlds*—highlighting that they are a *collection* of objects—and we derived the interpretation of the logical symbols on info states starting from their interpretation on worlds. But we can also take a different approach. We can think about info states as *points of the ordered structure* $\langle \mathcal{P}_0(W), \supseteq \rangle$[6] and define a suitable interpretation of the logical symbols over this structure: de facto, we can think of an information model as a particular instance of a *Kripke model*.[7]

This intuition and Lemma 2.1.9 hint at a connection between InqBQ and *intuitionistic first order logic with constant domain* CD: the semantical clauses defining the operators are nearly identical (when we treat $\mathbin{\rotatebox[origin=c]{-90}{\guilsinglright}}$ and $\overline{\exists}$ as the corresponding operators in intuitionistic logic); both support and Kripke semantics (for intuitionistic logic) satisfy similar principles like *persistency*; and the treatment of equality is similar in the two formalisms (see e.g. [Gabbay et al., 2009,

[6] $\mathcal{P}_0(W)$ indicates the set of all *non-empty* subsets of W.

[7] Notice that by considering the ordered structure $\langle \mathcal{P}_0(W), \supseteq \rangle$ we are cutting out the empty information state. By Lemma 2.1.9, the empty state satisfies *every formula* of the logic and so we are not loosing any information on the original model by excluding it. Moreover, the empty state cannot be interpreted as a point of a Kripke model if we want to preserve the semantics, since the set of formulas it supports is not consistent.

Section 3.5]). As we will show in this section, this is not a coincidence: there is a one-to-one correspondence between information models and a special class of intuitionistic Kripke models. We focus on the case of the syntax $\mathcal{L}_{\asymp}$, but all the results here presented can be easily adapted to the syntaxes $\mathcal{L}_{\neq}$ and $\mathcal{L}_{=}$.

To introduce this class, we need some notions from intuitionistic Kripke semantics and some definitions. We start by recalling the definition of a first order intuitionistic Kripke frame with constant domain.[8]

2.2.1. Definition (Constant domain Kripke models). An intuitionistic constant-domain Kripke model, henceforth referred to as CD-model, is a tuple

$$\mathcal{K} = \langle S, \leq, D, \mathcal{I}, \asymp \rangle$$

where:

- S is a non-empty set called the *set of states* of $\mathcal{K}$;
- $\leq$ is a partial order on S;
- D is a non-empty set called the *domain* of $\mathcal{K}$;
- $\mathcal{I}$ is an interpretation map that associates to every symbol in the signature an appropriate interpretation in the model, that is:
  - For R an n-ary relation symbol, $\mathcal{I}(R) : S \to D^n$;
  - For f an n-ary function symbol, $\mathcal{I}(f) : D^n \to D$;
- $\asymp: S \to \mathcal{P}(D^2)$ is a map that associates to each state an equivalence relation.

Additionally we require the following properties to hold:

Persistency of relations For every pair of states $s \leq t$ and every relation symbol R, it holds that $\mathcal{I}(R)(s) \subseteq \mathcal{I}(R)(t)$;

Persistency of equality For every pair of states $s \leq t$, it holds that $\asymp(s) \subseteq \asymp(t)$;

Congruence condition For every state s, $\asymp(s)$ is a congruence with respect to $\mathcal{I}(R)(s)$ and $\mathcal{I}(f)$ for every relation symbol $R \in \Sigma$ and every function symbol $f \in \Sigma$.

[8] The definition of constant-domain intuitionistic Kripke model that we present is not the most general one (see for example [Gabbay et al., 2009, Section 3.4]) since the interpretation of function symbols is not state-dependent. However this restriction allows to simplify the formulation of some of the lemmas that follow and the theory presented is general enough for the purposes of the current work.

As we did for information models, we introduce the shorthands $R_s^{\mathcal{K}} := \mathcal{I}(R)(s)$, $f^{\mathcal{K}} := \mathcal{I}(f)$ and $\asymp_s^{\mathcal{K}} := \asymp(s)$. The partial order $\langle S, \leq \rangle$ is commonly referred to as the *frame* of the model.

The models introduced can be used to interpret intuitionistic formulas through the so-called *forcing relation*.[9]

2.2.2. Definition (Forcing). Given a CD-model $\mathcal{K}$, a state $s \in S$ and a valuation $g : \texttt{Var} \to D$, we define recursively the forcing relation by the following clauses:

$$
\begin{array}{ll}
\mathcal{K}, s \Vdash_g \bot & \iff s = \emptyset \\
\mathcal{K}, s \Vdash_g t_1 \asymp t_2 & \iff t_1^g \asymp_s^{\mathcal{K}} t_2^g \\
\mathcal{K}, s \Vdash_g R(t_1, \ldots, t_n) & \iff R_s^{\mathcal{K}}(t_1^g, \ldots, t_n^g) \\
\mathcal{K}, s \Vdash_g \psi_1 \wedge \psi_2 & \iff \mathcal{K}, s \Vdash_g \psi_1 \text{ and } \mathcal{K}, s \Vdash_g \psi_2 \\
\mathcal{K}, s \Vdash_g \psi_1 \vee\!\!\!\vee \psi_2 & \iff \mathcal{K}, s \Vdash_g \psi_1 \text{ or } \mathcal{K}, s \Vdash_g \psi_2 \\
\mathcal{K}, s \Vdash_g \psi_1 \to \psi_2 & \iff \text{For all } t \geq s, \text{ if } \mathcal{K}, t \Vdash_g \psi_1 \text{ then } \mathcal{K}, t \Vdash_g \psi_2 \\
\mathcal{K}, s \Vdash_g \forall x.\psi & \iff \text{For all } d \in D \text{ we have } \mathcal{K}, s \Vdash_{g[x \mapsto d]} \psi \\
\mathcal{K}, s \Vdash_g \overline{\exists} x.\psi & \iff \text{There exists } d \in D \text{ such that } \mathcal{K}, s \Vdash_{g[x \mapsto d]} \psi
\end{array}
$$

We indicate with $\mathcal{K} \Vdash_g \varphi$ that for every state $s \in S$ it holds that $\mathcal{K}, s \Vdash_g \varphi$. We indicate with $[\![\varphi]\!]_g^{\mathcal{K}} \subseteq S$ the set of states s such that $\mathcal{K}, s \Vdash_g \varphi$.

As originally shown by Görnemann [1971], the logic of this class of models is axiomatized by the axioms and rules of *intuitionistic first order logic* IQC [Dalen, 2002] with in addition the *constant domain* axiom schema:

$$\mathsf{CD}: \quad \forall x.(\varphi \vee\!\!\!\vee \psi) \to \varphi \vee\!\!\!\vee \forall x.\psi \qquad \text{for } x \text{ not free in } \varphi$$

This schema captures exactly the condition that all the points of the model share the same domain of individuals.

We want to pinpoint a particular class of CD-models which are in one-to-one correspondence with information models. We characterize this class as an intersection of two other classes: *negative models* and $\mathcal{P}_0$-CD-*models*. Let us start by introducing negative models.

2.2.3. Definition (Negative models). We call a CD-model $\mathcal{K}$ a *negative model* if, for every relation symbol $R \in \Sigma$ and every state $s \in S$, it holds that

$$
\begin{array}{c}
\forall \overline{d} \in D^{\mathrm{Ar}(R)}. \left[\, R_s(\overline{d}) \iff \forall t \geq s.\ \exists u \geq t.\ R_u(\overline{d}) \,\right] \\
\forall d_1, d_2 \in D. \left[\, d_1 \asymp_s d_2 \iff \forall t \geq s.\ \exists u \geq t.\ d_1 \asymp_u d_2 \,\right]
\end{array}
$$

[9]Notice that the inquisitive symbols $\vee\!\!\!\vee$ and $\overline{\exists}$ are playing the role of the intuitionistic disjunction and the intuitionistic existential quantifier in the clauses. This is to highlight the close relationship between the operators and to greatly simplify the presentation of the results in this dissertation.

The definition above, which seems quite arbitrary, stems from a natural condition: we are requiring the sets of states for which $R_\bullet(\overline{d})$ hold and for which $d_1 \asymp_\bullet d_2$ hold to be *stable under* $\neg\neg$, as the following lemma shows.

2.2.4. LEMMA. *Let $\mathcal{K}$ be a* CD*-model. $\mathcal{K}$ is a negative model iff for every relation symbol R it holds that*

$$\mathcal{K} \Vdash \forall \overline{x}.\, (R(\overline{x}) \leftrightarrow \neg\neg R(\overline{x}))$$
$$\mathcal{K} \Vdash \forall x.\, \forall y.\, (x = y \leftrightarrow \neg\neg x = y)$$

The second class we introduce is the class of $\mathcal{P}_0$ - CD-models.

2.2.5. DEFINITION ($\mathcal{P}_0$ - CD-models). We say that a CD-model is a $\mathcal{P}_0$ - CD-model if $\langle S, \leq \rangle = \langle \mathcal{P}_0(W), \supseteq \rangle$ for some non-empty set W.

Being a $\mathcal{P}_0$ - CD-model is a particularly strong restriction on the frame, and this is reflected also at the level of the semantics. For example, the following schemata are forced in every $\mathcal{P}_0$ - CD-model.

$$\begin{array}{ll} \mathsf{KF}: & \neg\neg\forall x.\varphi \to \forall x.\neg\neg\varphi \\ \mathsf{KP}: & (\neg\varphi \to \psi \vee\!\!\!\vee \chi) \to (\neg\varphi \to \psi) \vee\!\!\!\vee (\neg\varphi \to \chi) \\ \mathsf{UP}: & (\neg\varphi \to \overline{\exists} x.\psi) \to \overline{\exists} x.(\neg\varphi \to \chi) \end{array}$$

The schema KF is forced since $\mathcal{P}_0$ - CD-models satisfy the *McKinsey property*, that is, every state of a $\mathcal{P}_0$ - CD-model has a successor which is an endpoint. As for the schemata KP and UP, the property follows from the following folklore results.

2.2.6. LEMMA. *Given W a non-empty set, the upsets of $\langle \mathcal{P}_0(W), \supseteq \rangle$ stable under $\neg\neg$ are exactly the rooted ones.*[10]

This lemma, in combination with the intuitionistically valid principle $\neg\neg\neg\varphi \equiv \neg\varphi$, implies the following.

2.2.7. COROLLARY. *Let $\mathcal{K}$ be a $\mathcal{P}_0$ - CD-model, g : `Var` $\to D$ be an assignment and φ be a formula. Then the set $[\![\neg\varphi]\!]_g^{\mathcal{K}} := \{ s \in S \mid \mathcal{K}, s \vDash_g \neg\varphi \}$ is a rooted upset.*

Now that we introduced negative and $\mathcal{P}_0$ - CD-models we are finally ready to relate the semantics of InqBQ and of CD.

[10] An upset is called *rooted* if it admits a minimum (usually called the *root* of the upset).

2.2.8. Lemma (From information models to negative $\mathcal{P}_0$ - CD-models). *Let $\mathcal{M}$ be an information model. Define the* CD*-model*

$$\mathcal{K} := \langle \mathcal{P}_0(W^{\mathcal{M}}), \supseteq, D^{\mathcal{M}}, \mathcal{I}, \asymp \rangle$$

where:[11]

- *For every relation symbol R, every state $s \in \mathcal{P}_0(W^{\mathcal{M}})$ and every sequence of elements $\overline{d} \in (D^{\mathcal{M}})^{\mathrm{Ar}(R)}$ we let $R^{\mathcal{K}}_s(\overline{d})$ iff $\forall w \in s.\ R^{\mathcal{M}}_w(\overline{d})$.*
- *For every function symbol f, every state $s \in \mathcal{P}_0(W^{\mathcal{M}})$ and every sequence of elements $\overline{d} \in (D^{\mathcal{M}})^{\mathrm{Ar}(f)}$ we let $f^{\mathcal{K}}(\overline{d}) = f^{\mathcal{M}}(\overline{d})$.*
- *For every $d_1, d_2 \in D^{\mathcal{M}}$ and every state $s \in \mathcal{P}_0(W^{\mathcal{M}})$ we let $d_1 \asymp^{\mathcal{K}}_s d_2$ iff $\forall w \in s.\ d_1 \asymp^{\mathcal{M}}_w d_2$.*

Then $\mathcal{K}$ is a negative $\mathcal{P}_0$ - CD*-model and for every $s \in \mathcal{P}_0(W)$, for every $g : \mathtt{Var} \to D^{\mathcal{M}}$ and for every formula φ it holds that*

$$\mathcal{K}, s \Vdash_g \varphi \qquad \textit{iff} \qquad \mathcal{M}, s \vDash_g \varphi$$

2.2.9. Lemma (From negative $\mathcal{P}_0$ - CD-models to information models). *Let $\mathcal{K} = \langle \mathcal{P}_0(W), \supseteq, D, \mathcal{I}, \asymp \rangle$ be a negative $\mathcal{P}_0$ -* CD*-model. Define the information model*

$$\mathcal{M} := \langle M_w \mid w \in W \rangle$$

where:

- *The common domain of the structures M_w is D (the domain of $\mathcal{K}$).*
- *For every relation symbol R, every world $w \in W$ and every sequence of elements $\overline{d} \in D^{\mathrm{Ar}(R)}$ it holds that $R^{\mathcal{M}}_w(\overline{d})$ iff $R^{\mathcal{K}}_{\{w\}}(\overline{d})$.*
- *For every function symbol f, every world $w \in W$ and every sequence of elements $\overline{d} \in D^{\mathrm{Ar}(f)}$ it holds that $f^{\mathcal{M}}(\overline{d}) = f^{\mathcal{K}}(\overline{d})$.*
- *For every $d_1, d_2 \in D$, every world $w \in W$ it holds that $d_1 \asymp^{\mathcal{M}}_w d_2$ iff $d_1 \asymp^{\mathcal{K}}_{\{w\}} d_2$.*

Then for every $s \in \mathcal{P}_0(W)$, for every $g : \mathtt{Var} \to D^{\mathcal{M}}$ and for every formula φ it holds that

$$\mathcal{M}, s \vDash_g \varphi \qquad \textit{iff} \qquad \mathcal{K}, s \Vdash_g \varphi$$

[11] Recall that we indicate the interpretation of the relation symbol R in the information model $\mathcal{M}$ at world w with the notation $R^{\mathcal{M}}_w$; and we indicate the interpretation of R in the model $\mathcal{K}$ at state s with the notation $R^{\mathcal{K}}_s$. The same notational convention also applies to function and equality symbols.

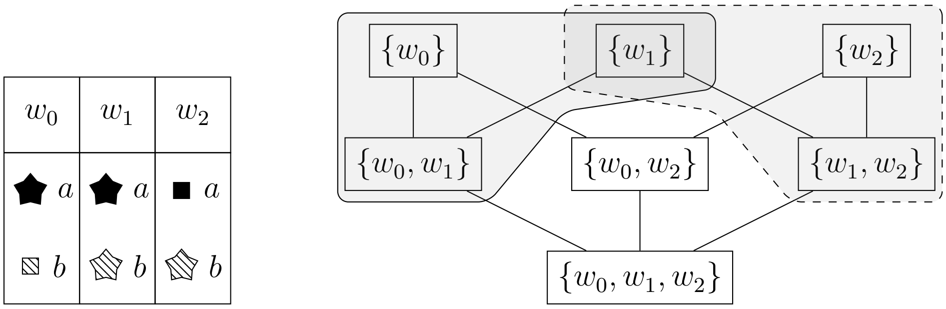


Figure 2.4: An example of an information model in the signature $\Sigma = \langle P \rangle$ for P a unary predicate (on the left) and the frame of the corresponding $\mathcal{P}_0$ - CD-model (on the right). The set of points satisfying $P(a)$ are contained in the area delimited with a continuous line; and the set of points satisfying $P(b)$ are contained in the area delimited with a dashed line. Notice in particular that the interpretations of atomic formulas respect the persistency conditions of CD-models.

An example of this correspondence is showcased in Figure 2.4. This correspondence shows the close connection between the logics CD and InqBQ, and opens up new approaches to study meta-theoretical properties of InqBQ. A clear example of this is the axiomatization of the ClAnt fragment presented in Chapter 7, inspired by completeness results for the logic CD as presented by Gabbay [1981, Section 3.3].

The relation between information models and $\mathcal{P}_0$ - CD-models showed in Lemmas 2.2.8 and 2.2.9 is one of the main motivations behind the natural deduction system to axiomatize InqBQ proposed by Ciardelli [2016, Chapter 4]. We present a slightly modified version of this system in Figure 2.5: we substitute the *classical negation* rule with the KF − rule; we add rules to account for the equality symbols and the rigidity conditions. We also propose an equivalent Hilbert-style presentation of the system in Figure 2.6. Notice that the rules $\vee\!\!\!\vee$-split and $\overline{\exists}$-split are substituted with the schemata KP and UP.[12]

As already discussed, the schemata KP, UP and KF are valid on $\mathcal{P}_0$ - CD-models. Moreover, the DNC formulas (where DNC stands for *Double Negation for Classical formulas*) is a direct consequence of Theorem 2.1.17. Consequently, all the axioms in the table are sound for InqBQ.

As of now, it is not known whether this axiomatization is complete for the logic InqBQ. It was shown by Ciardelli [2016, Sections 4.7,4.8] that *fragments* of InqBQ are axiomatized by restrictions of this system. Moreover, in Chapters

[12] The equivalence of the two systems follows easily from the following observation: in the natural deduction system without the rules $\vee\!\!\!\vee$-split and $\overline{\exists}$-split (resp., in the Hilbert-style system without the axioms KP and UP), every classical formula is provably equivalent to a negated formula and viceversa.

$$\wedge\text{i}\ \ \frac{\varphi \qquad \psi}{\varphi \wedge \psi} \qquad\qquad \wedge\text{e}\ \ \frac{\varphi \wedge \psi}{\varphi} \quad \frac{\varphi \wedge \psi}{\psi}$$

$$\mathbin{\vee\!\!\!\vee}\text{i}\ \ \frac{\varphi}{\varphi \mathbin{\vee\!\!\!\vee} \psi} \quad \frac{\psi}{\varphi \mathbin{\vee\!\!\!\vee} \psi} \qquad\qquad \mathbin{\vee\!\!\!\vee}\text{e}\ \ \frac{\varphi \mathbin{\vee\!\!\!\vee} \psi \qquad \begin{array}{c}[\varphi]\\ \vdots \\ \chi\end{array} \qquad \begin{array}{c}[\psi]\\ \vdots \\ \chi\end{array}}{\chi}$$

$$\to\text{i}\ \ \frac{\begin{array}{c}[\varphi]\\ \vdots \\ \psi\end{array}}{\varphi \to \psi} \qquad\qquad \to\text{e}\ \ \frac{\varphi \qquad \varphi \to \psi}{\psi}$$

$$\forall\text{i}\ \ \frac{\varphi[y/x]}{\forall x.\varphi} \qquad\qquad \forall\text{e}\ \ \frac{\forall x.\varphi}{\varphi[t/x]}$$

$$\overline{\exists}\text{i}\ \ \frac{\varphi[t/x]}{\overline{\exists} x.\varphi} \qquad\qquad \overline{\exists}\text{e}\ \ \frac{\overline{\exists} x.\varphi \qquad \begin{array}{c}[\varphi[y/x]]\\ \vdots \\ \psi\end{array}}{\psi}$$

$$\text{Ex Falsum}\ \ \frac{\bot}{\varphi} \qquad\qquad \text{DNC-rule}\ \ \frac{\neg\neg\alpha}{\alpha}$$

$$\mathbin{\vee\!\!\!\vee}\text{-split}\ \ \frac{\alpha \to \psi \mathbin{\vee\!\!\!\vee} \chi}{(\alpha \to \psi) \mathbin{\vee\!\!\!\vee} (\alpha \to \chi)} \qquad\qquad \text{CD-rule}\ \ \frac{\forall x.(\varphi \mathbin{\vee\!\!\!\vee} \psi)}{\forall x.\varphi \mathbin{\vee\!\!\!\vee} \psi}$$

$$\overline{\exists}\text{-split}\ \ \frac{\alpha \to \overline{\exists} x.\psi}{\overline{\exists} x.(\alpha \to \psi)} \qquad\qquad \text{KF-rule}\ \ \frac{\neg\neg\forall x.\,\varphi}{\forall x.\,\neg\neg\varphi}$$

$$=\text{i}\ \ \frac{}{t = t} \qquad\qquad =\text{e}\ \ \frac{\varphi[t/x] \qquad t = t'}{\varphi[t'/x]}$$

$$\asymp\text{i}\ \ \frac{}{t \asymp t} \qquad\qquad \asymp\text{e}\ \ \frac{\varphi[t/x] \qquad t \asymp t'}{\varphi[t'/x]}$$

$$\text{Rig} =\ \ \frac{}{\forall x, y.\, ?x = y} \qquad\qquad \text{Rig } f^{=}\ \ \frac{}{\forall \overline{x}.\, \overline{\exists} y.\, f(\overline{x}) = y}$$

$$\text{Rig } f^{\asymp}\ \ \frac{}{\forall \overline{x}.\, \overline{\exists} y.\, f(\overline{x}) \asymp y}$$

Figure 2.5: A slight variation of the natural deduction system for InqBQ proposed by Ciardelli [2016, Chapter 4]. The rules containing the equality symbols $=$ and $\asymp$ are present only when working with syntaxes containing said symbols. In ($\forall e$) and ($\overline{\exists} i$), t must be free for x in φ; in ($\forall i$), y must not occur free in any undischarged assumption; in ($\overline{\exists} e$), y must not occur free in ψ or any undischarged assumption; in (DNC-rule), α ranges over classical formulas; in ($\mathbin{\vee\!\!\!\vee}$-split), α ranges over classical formulas; in ($\overline{\exists}$-split), α ranges over classical formulas and x is not free in α; in (CD), x must not occur free in ψ.

Schemata of IQC

$\varphi \to (\psi \to \varphi)$
$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\psi \to \chi))$
$\varphi \wedge \psi \to \varphi$
$\varphi \wedge \psi \to \psi$
$\varphi \to (\psi \to \varphi \wedge \psi)$
$\varphi \to \varphi \mathbin{\vee\!\!\!\vee} \psi$
$\psi \to \varphi \mathbin{\vee\!\!\!\vee} \psi$
$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \mathbin{\vee\!\!\!\vee} \psi \to \chi))$
$\bot \to \varphi$

$\forall x.\varphi \to \varphi[y/x]$
$\varphi(y) \to \overline{\exists} x.\varphi(x)$

Rules of IQC

Modus Ponens: $\varphi,\ \varphi \to \psi \ / \ \psi$
$\varphi \to \psi \ / \ \overline{\exists} x.\varphi \to \psi$ for x not free in ψ
$\varphi \to \psi \ / \ \varphi \to \forall x.\psi$ for x not free in φ

Additional schemata

CD schema:	$\forall x.(\varphi \mathbin{\vee\!\!\!\vee} \psi) \to \varphi \mathbin{\vee\!\!\!\vee} \forall x.\psi$ for x not free in φ	
KP schema:	$(\neg\varphi \to \psi \mathbin{\vee\!\!\!\vee} \chi) \to (\neg\varphi \to \psi) \mathbin{\vee\!\!\!\vee} (\neg\varphi \to \chi)$	
UP schema:	$(\neg\varphi \to \overline{\exists} x.\psi) \to \overline{\exists} x.(\neg\varphi \to \psi)$ for x not free in φ	
KF schema:	$\neg\neg\forall x.\varphi(x) \to \forall x.\neg\neg\varphi(x)$	
DNC formulas:	$\neg\neg\alpha \to \alpha$ for α classical	
Rigid functions:	$\forall \overline{x}.\, \overline{\exists} y.\, f(\overline{x}) = y$	
	$\forall \overline{x}.\, \overline{\exists} y.\, f(\overline{x}) \asymp y$	
Rigid identity:	$\forall x.\, \forall y.\, ?x = y$	
Identity axioms:	$\forall x.\, \forall y.\, x = y$	$\forall x.\, \forall y.(\ (x = y) \wedge \varphi(x) \to \varphi(y)\ )$
	$\forall x.\, \forall y.\, x \asymp y$	$\forall x.\, \forall y.(\ (x \asymp y) \wedge \varphi(x) \to \varphi(y)\ )$

Figure 2.6: An Hilbert-style axiomatization for InqBQ equivalent to the natural deduction system in Figure 2.5. The axiomatic system for IQC is presented and discussed in [Gabbay, 1981, Section 2.2].

7 and 8 we will present two more fragments axiomatized by slight variations of this systems. But the question remains open: are these principles enough to axiomatize the whole logic?

Chapter 3

Preliminaries

In this chapter we collect some technical results which will be used at various points of the dissertation. In particular, we are concerned with when two worlds or individuals can be regarded as the same from the perspective of the logic—in different terms, when two worlds or elements are *indistinguishable*. Studying this issue, we will be able to define natural relations between information models preserving or reflecting the formulas supported: the *strong equivalence* and the *submodel relation*. These constructions are particularly useful to manipulate the structure of information models while preserving their logic.

3.1 Essential Equivalence

We start by defining a relation telling us when two worlds *instantiate exactly the same state of affairs*, that is, when they have the same associated first order model.

3.1.1. Definition (Essential equivalence between worlds). Let $\mathcal{M}$ be an information model and w, w' two worlds of $\mathcal{M}$. We say that w is *essentially equivalent* to w' (and indicate it with $w \approx^e w'$) iff $M_w = M_{w'}$. Given W the set of worlds of $\mathcal{M}$, we define the *essential set of worlds* (and indicate it with W^e) the quotient $W/{\approx^e}$.

Notice that $w \approx^e w'$ does not only imply that the two models M_w and $M_{w'}$ are isomorphic, but it requires them to be *exactly the same structure*. In the former case, we could still be able to tell the worlds w and w' apart using the information encoded by info states of the model, as shown in Figure 3.1.

So two worlds of a model $\mathcal{M}$ are considered essentially equivalent if they describe *exactly the same* state of affairs. Essential equivalence can be thought of as a *bisimilarity relation* between worlds: two worlds are considered essentially equivalent if they are indistinguishable from one another.

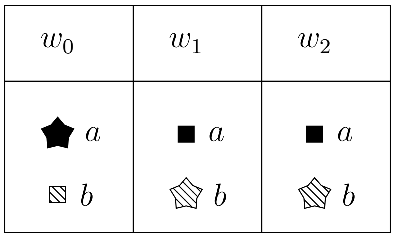


Figure 3.1: In the information model depicted we have $w_0 \not\approx^e w_1 \approx^e w_2$. Notice that, even though M_{w_0} is isomorphic to M_{w_1}, we do not consider the worlds w_0 and w_1 essentially equivalent. In fact we have that the info states $\{w_0, w_2\}$ and $\{w_1, w_2\}$ do not support the same formulas (e.g., $\overline{\exists} x.P(x)$ is supported only by the latter), showing that the worlds w_0 and w_1 are not indistinguishable.

As for worlds, we can define also an essential equivalence between elements: two elements are considered essentially equivalent if their role can be swapped without altering the structural properties of the model.

3.1.2. Definition (Essential equivalence between elements). Let $\mathcal{M}$ be an information model. We define recursively a chain of binary relations $\sim^0$, $\sim^1$, ... by the following clauses:

- Given d, d' two elements of $\mathcal{M}$, $d \sim^0 d'$ iff for every relation symbol $R(\overline{x})$, for every world $w \in \mathcal{M}$, for every tuple of elements $\langle d_1, \dots, d_{\mathrm{Ar}(R)} \rangle$ and for every index $i \leq \mathrm{Ar}(R)$ it holds that

$$R_w(d_1, \dots, d_{i-1}, d, d_{i+1}, \dots, d_{\mathrm{Ar}(R)}) \Leftrightarrow R_w(d_1, \dots, d_{i-1}, d', d_{i+1}, \dots, d_{\mathrm{Ar}(R)})$$
$$d = d_0 \Leftrightarrow d' = d_0 \qquad \text{(If working with } \mathcal{L}_=\text{)}$$
$$d \asymp_w d_0 \Leftrightarrow d' \asymp_w d_0 \qquad \text{(If working with } \mathcal{L}_\asymp\text{)}$$

- Given d, d' two elements of $\mathcal{M}$, $d \sim^{n+1} d'$ iff $d \sim^n d'$ and for every function symbol $f \in \Sigma$, for every tuple of elements $\langle d_1, \dots, d_{\mathrm{Ar}(f)} \rangle$, and for every $i \leq \mathrm{Ar}(f)$ it holds that

$$f(\, d_1, \dots, d_{i-1}, d, d_{i+1}, \dots, d_{\mathrm{Ar}(f)} \,) \sim^n f(\, d_1, \dots, d_{i-1}, d', d_{i+1}, \dots, d_{\mathrm{Ar}(f)} \,)$$

We say that two elements d, d' are *essentially equivalent* (and indicate it with $d \sim^e d'$) iff $d \sim^n d'$ for every $n \in \mathbb{N}$. Given D the set of worlds of $\mathcal{M}$, we define the *essential domain* of $\mathcal{M}$ (and indicate with D^e) the quotient $D/{\sim^e}$.

Notice that $\sim^e$ is not only an equivalence relation, but also a congruence with respect to the interpretation of the symbols of the signature. Intuitively, two elements are considered essentially equivalent iff they are indistinguishable.

We also introduce relativized versions of the relation $\sim^e$: given an info state s we write $d \sim^e_s d'$ if d, d' are essentially equivalent for the model $\mathcal{M}|_s$; and given a world w we write $d \sim^e_w d'$ for $d \sim^e_{\{w\}} d'$. Notice that if we are working with the languages $\mathcal{L}_=$ or $\mathcal{L}_\asymp$ the definition of essential equivalence can be simplified: $d \sim^e d'$ iff $d = d'$ in the former case; and $d \sim^e d'$ iff $d \asymp^{\mathcal{M}}_W d'$ in the latter case.

The relations $\approx^e$ and $\sim^e$ allow us to obtain a *compressed* version of a model, carrying the same information as the original one.

3.1.3. Definition (Essential quotient). Let $\mathcal{M}$ be an information model. We define its essential quotient as the information model

$$\mathcal{M}^e = \left\langle\, M^e_{[w]} \;\middle|\; [w] \in W^e \,\right\rangle$$

where $M^e_{[w]} := M_w/{\sim^e}$.

Notice that, by definition of $\approx^e$, $M^e_{[w]}$ does not depend on the choice of the representative w and so $\mathcal{M}^e$ is well-defined.

3.1.4. Lemma. *Let $\mathcal{M}$ be an information model, s an information state of $\mathcal{M}$ and $g : \mathtt{Var} \to D$ an assignment over the domain of $\mathcal{M}$. Consider the information state $s^e := \{\, [w] \in W^e \mid w \in s\}$ and the assignment $g^e : \mathtt{Var} \to D^e$ defined as $g^e(x) = [g(x)]$. Then for every formula φ it holds that*

$$\mathcal{M}, s \vDash_g \varphi \qquad \Longleftrightarrow \qquad \mathcal{M}^e, s^e \vDash_{g^e} \varphi$$

Proof:

In the rest of this proof, we will indicate with $\approx^e$ and $\sim^e$ the essential equivalence relations relative to $\mathcal{M}$. The proof proceeds by induction on the structure of the formula φ. We will spell out only the non-trivial cases.

- If φ is an atomic formula $A(\bar{t})$, then we have

$$\begin{aligned} \mathcal{M}, s \vDash_g A(\bar{t}) &\iff \forall w \in s.\ M_w \vDash_g A(\bar{t}) \\ &\iff \forall [w] \in s^e.\ M_w/{\sim^e} \vDash_{g^e} A(\bar{t}) \\ &\iff \mathcal{M}^e, s^e \vDash_{g^e} A(\bar{t}) \end{aligned}$$

- If $\varphi \equiv \psi \to \chi$, firstly suppose $\mathcal{M}, s \nvDash_g \psi \to \chi$. Then for some information state t it holds that $\mathcal{M}, t \vDash_g \psi$ and $\mathcal{M}, t \nvDash_g \chi$. Then, by inductive hypothesis, for $t^e := \{[w] \in W^e \mid w \in t\,\}$ we have $\mathcal{M}^e, t^e \vDash_{g^e} \psi$ and $\mathcal{M}^e, t^e \nvDash_{g^e} \chi$, which in turn entails $\mathcal{M}^e, s^e \nvDash_{g^e} \psi \to \chi$.

  Secondly suppose $\mathcal{M}^e, s^e \nvDash_{g^e} \psi \to \chi$. We proceed with a similar argument: let $u \subseteq s^e$ such that $\mathcal{M}^e, u \vDash_{g^e} \psi$ and $\mathcal{M}^e, u \nvDash_{g^e} \chi$, and consider the state $t \subseteq s$ defined as $t := \{\, w \in s \mid [w] \in u\,\}$. In particular, we have that $t^e = u$. Then, by inductive hypothesis, it holds that $\mathcal{M}, t \vDash_g \psi$ and $\mathcal{M}, t \nvDash_g \chi$, which in turn implies $\mathcal{M}, s \nvDash_g \psi \to \chi$.

- If $\varphi \equiv \overline{\exists} x.\psi$, firstly suppose $\mathcal{M}, s \vDash_g \overline{\exists} x.\psi$. This means that for an element $d \in D$ it holds that $\mathcal{M}, s \vDash_{g[x \mapsto d]} \psi$. By inductive hypothesis it follows $\mathcal{M}^e, s^e \vDash_{g^e[x \mapsto [d]]} \psi$, which in turn entails $\mathcal{M}^e, s^e \vDash_{g^e} \overline{\exists} x.\psi$.

  Secondly suppose $\mathcal{M}^e, s^e \vDash_{g^e} \overline{\exists} x.\psi$. This means that for some element $[d] \in D^e$ it holds that $\mathcal{M}^e, s^e \vDash_{g^e[x \mapsto [d]]} \psi$. By inductive hypothesis, it follows that $\mathcal{M}, s \vDash_{g[x \mapsto d]} \psi$, and consequently $\mathcal{M}, s \vDash_g \overline{\exists} x.\psi$.

- If $\varphi \equiv \forall x.\psi$, the proof can be carried out as in the case for $\varphi \equiv \overline{\exists} x.\psi$ with minimal modifications.

□

3.2 Strong Equivalence and Submodel Relation

With the relations $\approx^e$ and $\sim^e$, we considered ways to characterize "behavioral equivalence" internal to a model. Now we are also going to present an *external* concept of equivalence, basically generalizing the concept of *isomorphism* between CQC-models.

3.2.1. Definition (Strong equivalence). Let $\mathcal{M} = \langle M_w | w \in W^{\mathcal{M}} \rangle$ and $\mathcal{N} = \langle N_w | w \in W^{\mathcal{N}} \rangle$ be two information models. We say that $\mathcal{M}$ is a *strongly equivalent* to $\mathcal{N}$ (and we indicate it with $\mathcal{M} \cong \mathcal{N}$) if there exist two maps $F : W^{\mathcal{M}} \to W^{\mathcal{N}}$ and $G : D^{\mathcal{M}} \to D^{\mathcal{N}}$ such that

SEq1 F and G are bijections;

SEq2 G commutes with the interpretation of the logical symbols:

$$\begin{array}{lcl} G(f^{\mathcal{M}}(d_1, \ldots, d_n)) & = & f^{\mathcal{N}}(G(d_1), \ldots, G(d_n)) \\ R^{\mathcal{M}}_w(d_1, \ldots, d_n) & \iff & R^{\mathcal{N}}_{F(w)}(G(d_1), \ldots, G(d_n)) \\ d_1 \asymp^{\mathcal{M}}_w d_2 & \iff & G(d_1) \asymp^{\mathcal{N}}_{F(w)} G(d_2) \end{array}$$

The intended interpretation of this relation is quite simple: $\mathcal{M}$ is strongly equivalent to $\mathcal{N}$ iff $\mathcal{M}$ and $\mathcal{N}$ are the same model modulo renaming of the worlds and of the elements. The following lemma follows immediately from the definition of strong equivalence.

3.2.2. Lemma. *Let $\mathcal{M}$ and $\mathcal{N}$ be two strongly equivalent models, and let $F : W^{\mathcal{M}} \to W^{\mathcal{N}}$ and $G : D^{\mathcal{M}} \to D^{\mathcal{N}}$ be a pair of functions witnessing the equivalence. Let s be a state of $\mathcal{M}$ and $g :$ `Var` $\to D^{\mathcal{M}}$ be an assignment over $\mathcal{M}$. Then for every formula φ it holds that*

$$\mathcal{M}, s \vDash_g \varphi \quad \iff \quad \mathcal{N}, F[s] \vDash_{G \circ g} \varphi$$

We introduce one last relation between information models which will be particularly useful to study the properties of InqBQ (especially in Chapters 4 and 6): the submodel relation. Recall the definition of $\mathcal{M}|_s := \{M_w \mid w \in s\}$, the restriction of $\mathcal{M}$ to an information state s.

3.2.3. DEFINITION (Submodel). Let $\mathcal{M}$ and $\mathcal{N}$ be information models. We say that $\mathcal{M}$ is a *submodel* of $\mathcal{N}$ (and we indicate it with $\mathcal{M} \hookrightarrow \mathcal{N}$) if there exists two maps $F : W^{\mathcal{M}} \to W^{\mathcal{N}}$ and $G : D^{\mathcal{M}} \to D^{\mathcal{N}}$ such that:

Sub1 For every $b \in D^{\mathcal{N}}$, there exists $a \in D^{\mathcal{M}}$ such that $G(a) \sim^e_{F[W^{\mathcal{M}}]} b$;

Sub2 G commutes with the interpretation of the logical symbols (modulo essential equivalence):

$$\begin{array}{lcl} G(f^{\mathcal{M}}(d_1, \dots, d_n)) & \sim^e_{F[W^{\mathcal{M}}]} & f^{\mathcal{N}}(G(d_1), \dots, G(d_n)) \\ R^{\mathcal{M}}_w(d_1, \dots, d_n) & \iff & R^{\mathcal{N}}_{F(w)}(G(d_1), \dots, G(d_n)) \\ d_1 \asymp^{\mathcal{M}}_w d_2 & \iff & G(d_1) \asymp^{\mathcal{N}}_{F(w)} G(d_2) \end{array}$$

It is easy to verify that $\mathcal{M}|_s \hookrightarrow \mathcal{M}$, with $F : s \to W$ the inclusion map and $G : D \to D$ the identity map. So the submodel relation can be seen as a generalization of the restriction between models taking into account also the essential equivalence between worlds and elements. The following lemma formalizes this intuition.

3.2.4. LEMMA. *Let $\mathcal{M}$ and $\mathcal{N}$ be two information models. Then $\mathcal{M} \hookrightarrow \mathcal{N}$ iff there exists an information state s of $\mathcal{N}$ such that $\mathcal{M}^e \cong (\mathcal{N}|_s)^e$.*

Proof:
We prove separately the two implications: (1) firstly that $\mathcal{M} \hookrightarrow \mathcal{N}$ implies $\mathcal{M}^e \cong (\mathcal{N}|_s)^e$ for some s; (2) secondly that $\mathcal{M}^e \cong (\mathcal{N}|_s)^e$ for some s implies $\mathcal{M} \hookrightarrow \mathcal{N}$.

Part (1): Firstly suppose that $\mathcal{M} \hookrightarrow \mathcal{N}$, and let $F : W^{\mathcal{M}} \to W^{\mathcal{N}}$ and $G : D^{\mathcal{M}} \to D^{\mathcal{N}}$ be a pair of functions witnessing it. Consider the information state $s := F\left[W^{\mathcal{M}}\right]$. We want to prove that $\mathcal{M}^e \cong (\mathcal{N}|_s)^e$.

Consider the functions $F' : W^{\mathcal{M}^e} \to W^{(\mathcal{N}|_s)^e}$ and $G' : D^{\mathcal{M}^e} \to D^{(\mathcal{N}|_s)^e}$ defined as

$$F'([w]) = [F(w)] \qquad\qquad G'([d]) = [G(d)]$$

Firstly, we can show that F' is well-defined: let $w, w' \in W^{\mathcal{M}}$ such that $w \approx^e w'$; we have to show that $F(w) \approx^e F(w')$. By definition of $\approx^e$, we have $M_w = M_{w'}$. Consider now a relation symbol R and a sequence $b_1, \dots, b_{\mathrm{Ar}(R)} \in D^{\mathcal{N}}$. By Sub1, there exists a sequence $a_1, \dots, a_{\mathrm{Ar}(R)} \in D^{\mathcal{M}}$ such that $G(a_i) \sim^e_{F[W^{\mathcal{M}}]} b_i$; fix one sequence with this property. We have

$$\begin{aligned}
& \langle b_1, \ldots, b_{\mathrm{Ar}(R)} \rangle \in R^{\mathcal{N}}_{F(w)} & \\
\iff & \langle G(a_1), \ldots, G(a_{\mathrm{Ar}(R)}) \rangle \in R^{\mathcal{N}}_{F(w)} & \text{(by Condition Sub1)} \\
\iff & \langle a_1, \ldots, a_{\mathrm{Ar}(R)} \rangle \in R^{\mathcal{M}}_{w} & \text{(by Condition Sub2)} \\
\iff & \langle a_1, \ldots, a_{\mathrm{Ar}(R)} \rangle \in R^{\mathcal{M}}_{w'} & \text{(by hypothesis)} \\
\iff & \langle G(a_1), \ldots, G(a_{\mathrm{Ar}(R)}) \rangle \in R^{\mathcal{N}}_{F(w')} & \text{(by Condition Sub2)} \\
\iff & \langle b_1, \ldots, b_{\mathrm{Ar}(R)} \rangle \in R^{\mathcal{N}}_{F(w')} & \text{(by Condition Sub1)}
\end{aligned}$$

So we have that $R^{\mathcal{N}}_{F(w)}$ and $R^{\mathcal{N}}_{F(w')}$ coincide; in the same way we can show that also $\asymp^{\mathcal{N}}_{F(w)}$ and $\asymp^{\mathcal{N}}_{F(w')}$ coincide. We conclude that $N_{F(w)} = N_{F(w')}$, and consequently that $F(w) \approx^e F(w')$.

Secondly, we can show that also G' is well-defined: let $d, d' \in D^{\mathcal{M}}$ such that $d \sim^e d'$; we have to show that $G(d) \sim^e G(d')$, that is, that for every $n \in \mathbb{N}$ it holds that $G(d) \sim^n G(d')$ (compare with Definition 3.1.2). By definition of $d \sim^e d'$ for every relation symbol R, for every world $w \in W^{\mathcal{M}}$, for every tuple $\langle d_1, \ldots, d_{\mathrm{Ar}(R)} \rangle$ and for every index $i \leq \mathrm{Ar}(R)$ it holds that

$$\begin{gathered}
\langle d_1, \ldots, d_{i-1}, d, d_{i+1}, \ldots, d_{\mathrm{Ar}(R)} \rangle \in R^{\mathcal{M}}_{w} \\
\Updownarrow \\
\langle d_1, \ldots, d_{i-1}, d', d_{i+1}, \ldots, d_{\mathrm{Ar}(R)} \rangle \in R^{\mathcal{M}}_{w}
\end{gathered}$$

By Condition *Sub*2 this entails

$$\begin{gathered}
\langle G(d_1), \ldots, G(d_{i-1}), G(d), G(d_{i+1}), \ldots, G(d_{\mathrm{Ar}(R))} \rangle \in R^{\mathcal{N}|_s}_{F(w)} \\
\Updownarrow \\
\langle G(d_1), \ldots, G(d_{i-1}), G(d'), G(d_{i+1}), \ldots, G(d_{\mathrm{Ar}(R))} \rangle \in R^{\mathcal{N}|_s}_{F(w)}
\end{gathered}$$

Since F is surjective over $s = F[W^{\mathcal{M}}]$ and, by Condition Sub1, every element in $D^{\mathcal{N}|_s}$ is $\sim^e$-equivalent to some element in $G[D^{\mathcal{M}}]$, it follows that: for every relation symbol R, for every world $v \in W^{\mathcal{N}|_s}$, for every tuple $\langle c_1, \ldots, c_{\mathrm{Ar}(R)} \rangle$ of elements of $D^{\mathcal{N}|_s}$ and for every index $i \leq \mathrm{Ar}(R)$ it holds that

$$\begin{gathered}
\langle c_1, \ldots, c_{i-1}, G(d), c_{i+1}, \ldots, c_{\mathrm{Ar}(R)} \rangle \in R^{\mathcal{N}|_s}_{v} \\
\Updownarrow \\
\langle c_1, \ldots, c_{i-1}, G(d'), c_{i+1}, \ldots, c_{\mathrm{Ar}(R)} \rangle \in R^{\mathcal{N}|_s}_{v}
\end{gathered}$$

that is, $G(d) \sim^0 G(d')$.

With a simple induction we can now show that $G(d) \sim^n G(d')$ for every $n \in \mathbb{N}$, and thus conclude $G(d) \sim^e G(d')$. The basic case ($G(d) \sim^0 G(d')$) has been shown above. Suppose $G(d) \sim^n G(d')$ for a certain value $n \in \mathbb{N}$. Consider now a function symbol f and a sequence $\langle c_1, \ldots, c_{\mathrm{Ar}(f)} \rangle$ of elements of $D^{\mathcal{N}|_s}$. By Condition Sub1, there exists a sequence $\langle d_1, \ldots, d_{\mathrm{Ar}(f)} \rangle$ of elements of $D^{\mathcal{M}}$ such that $G(d_i) \sim^e_s c_i$ for every $i \leq \mathrm{Ar}(f)$. Since $d \sim^e d'$, it follows that $d \sim^{n+1} d'$ and consequently, for every $i \leq \mathrm{Ar}(f)$, it holds that

$$\begin{array}{lll} & f^{\mathcal{N}}(c_1, \ldots, c_{i-1}, G(d), c_{i+1}, \ldots, c_{\mathrm{Ar}(f)}) & \\ \sim^e & G(f^{\mathcal{M}}(d_1, \ldots, d_{i-1}, d, d_{i+1}, \ldots, d_{\mathrm{Ar}(f)})) & \text{(by Condition Sub2)} \\ \sim^n & G(f^{\mathcal{M}}(d_1, \ldots, d_{i-1}, d', d_{i+1}, \ldots, d_{\mathrm{Ar}(f)})) & \text{(by Definition of } \sim^{n+1}) \\ \sim^e & f^{\mathcal{N}}(c_1, \ldots, c_{i-1}, G(d'), c_{i+1}, \ldots, c_{\mathrm{Ar}(f)}) & \text{(by Condition Sub2)} \end{array}$$

and consequently $G(d) \sim^{n+1} G(d')$. Notice that all the steps in this part of the proof are reversible, thus showing that $d \sim^e d'$ iff $G(d) \sim^e G(d')$.

To conclude this part of the proof we need to show that Conditions SEq1 and SEq2 hold for F' and G'. Injectivity of G' follows from previous considerations: $G'([d]) = G'([d'])$ iff $G(d) \sim^e G(d')$ iff $d \sim^e d'$ iff $[d] = [d']$. Surjectivity of F' follows by definition of $s = F[W']$ and surjectivity of G' follows from Condition Sub1. As for Condition SEq2, it follows directly from Condition Sub2.

We conclude by showing that F' is injective: suppose now that $F'([w]) = F'([w'])$ for some worlds $w, w' \in W^{\mathcal{M}}$, that is, $N_{F(w)} = N_{F(w')}$. By condition Sub2 it follows that also $M_w = M_{w'}$, that is, $w \approx^e w'$. Thus $[w] = [w']$, showing that F' is injective.

Part (2): Suppose now that $\mathcal{M}^e \cong (\mathcal{N}|_s)^e$ for some information state s, and let $F' : W^{\mathcal{M}^e} \to W^{(\mathcal{N}|_s)^e}$ and $G' : D^{\mathcal{M}^e} \to D^{(\mathcal{N}|_s)^e}$ a pair of functions witnessing it, that is, respecting Conditions SEq1 and SEq2. We need to define a pair of functions $F : W^{\mathcal{M}} \to W^{\mathcal{N}}$ and $G : D^{\mathcal{M}} \to D^{\mathcal{N}}$ respecting Conditions Sub1 and Sub2.

Consider any *sections* F and G of the functions F' and G' respectively, that is, maps respecting the following conditions:

$$F(w) \in F'([w]) \qquad\qquad G(d) \in G'([d])$$

We want to show that F and G respect all the conditions. For Condition Sub1: since G' is a bijection, it follows that every element of $[c] \in D^{(\mathcal{N}|_s)^e}$ is of the form $G'([d])$ for some $d \in \mathcal{M}$. By definition of G, this entails that $c \sim^e G(d)$, and since c was arbitrary Condition Sub1 follows.

For Condition Sub2: let us start with the condition on relation symbols (the case for $\asymp$ is completely analogous). For $R(\overline{x})$ a relation symbol, $w \in W^{\mathcal{M}}$ and $d_1, \ldots, d_{\mathrm{Ar}(R)} \in D^{\mathcal{M}}$ it holds that

$$\begin{array}{llll} & \langle d_1, \ldots, d_{\mathrm{Ar}(R)} \rangle & \in R^{\mathcal{M}}_w & \\ \iff & \langle [d_1], \ldots, [d_{\mathrm{Ar}(R)}] \rangle & \in R^{\mathcal{M}^e}_{[w]} & \text{(by Lemma 3.1.4)} \\ \iff & \langle G'([d_1]), \ldots, G'([d_{\mathrm{Ar}(R)}]) \rangle & \in R^{(\mathcal{N}|_s)^e}_{F'([w])} & \text{(by Condition SEq2)} \\ \iff & \langle G(d_1), \ldots, G(d_{\mathrm{Ar}(R)}) \rangle & \in R^{\mathcal{N}}_{F(w)} & \text{(by Lemma 3.1.4)} \end{array}$$

We now show the condition on function symbols. For f a function symbol and $d_1, \ldots, d_{\mathrm{Ar}(f)} \in D^{\mathcal{M}}$ it holds that

$$\begin{array}{lll} & [G(f^{\mathcal{M}}(d_1, \ldots, d_{\mathrm{Ar}(f)}))] & \\ = & G'([f^{\mathcal{M}}(d_1, \ldots, d_{\mathrm{Ar}(f)})]) & \text{(by definition of } G) \\ = & G'(f^{\mathcal{M}^e}([d_1], \ldots, [d_{\mathrm{Ar}(f)}])) & \text{(by definition of } \mathcal{M}^e) \\ = & f^{(\mathcal{N}|_s)^e}(G'([d_1]), \ldots, G'([d_{\mathrm{Ar}(f)}])) & \text{(by Condition SEq2)} \\ = & f^{(\mathcal{N}|_s)^e}([G(d_1)], \ldots, [G(d_{\mathrm{Ar}(f)})]) & \text{(by definition of } G) \\ = & [f^{\mathcal{N}}(G(d_1), \ldots, G(d_{\mathrm{Ar}(f)}))] & \text{(by definition of } (\mathcal{N}|_s)^e) \end{array}$$

So it follows that $G(f^{\mathcal{M}}(d_1, \ldots, d_{\mathrm{Ar}(f)})) \sim^e f^{\mathcal{N}}(G(d_1), \ldots, G(d_{\mathrm{Ar}(f)}))$; and since f and the elements $d_1, \ldots, d_{\mathrm{Ar}(f)}$ were arbitrary, Condition Sub2 follows.

□

Combining this result with Lemmas 3.1.4 and 3.2.2, we obtain the following property of the submodel relation.

3.2.5. Corollary. *Let $\mathcal{M}$ and $\mathcal{N}$ be information models such that $\mathcal{M} \hookrightarrow \mathcal{N}$ and let $F : W^{\mathcal{M}} \to W^{\mathcal{N}}$ and $G : D^{\mathcal{M}} \to D^{\mathcal{N}}$ be a pair of functions witnessing it, that is, respecting Conditions Sub1 and Sub2. Consider an information state $s \subseteq W^{\mathcal{M}}$ and an assignment $g : \mathtt{Var} \to D^{\mathcal{M}}$. Then for every formula φ it holds that*

$$\mathcal{M}, s \vDash_g \varphi \quad \Longleftrightarrow \quad \mathcal{N}, F[s] \vDash_{G \circ g} \varphi$$

In particular, for every sentence ψ it holds that

$$\mathcal{N} \vDash \psi \quad \Longrightarrow \quad \mathcal{M} \vDash \psi$$

Proof:
By Lemma 3.2.4 $\mathcal{M}^e \cong (\mathcal{N}|_{F[W^{\mathcal{M}}]})^e$, and the functions witnessing this are F' and G' defined as

$$F'([w]) = [F(w)] \qquad\qquad G'([d]) = [G(d)]$$

Combining this with Lemmas 3.1.4 and 3.2.2 we obtain

$$\begin{array}{llll} & \mathcal{M}, s & \vDash_g \varphi & \\ \Longleftrightarrow & \mathcal{M}^e, s^e & \vDash_{g^e} \varphi & \text{(by Lemma 3.1.4)} \\ \Longleftrightarrow & (\mathcal{N}|_{F[W^{\mathcal{M}}]})^e, F'[s^e] & \vDash_{G' \circ g^e} \varphi & \text{(by Lemma 3.2.2)} \\ \Longleftrightarrow & \mathcal{N}|_{F[W^{\mathcal{M}}]}, F[s] & \vDash_{G \circ g} \varphi & \text{(by Lemma 3.1.4)} \\ \Longleftrightarrow & \mathcal{N}, F[s] & \vDash_{G \circ g} \varphi & \text{(since } F[s] \subseteq F[W^{\mathcal{M}}]) \end{array}$$

The second claim follows by persistency of the semantics.

□

We conclude this section by pointing out a particularly interesting application of Corollary 3.2.5: when two models are one a submodel of the other, then they satisfy the same formulas. This result is used extensively in Chapter 6.

3.2.6. Corollary. *Let $\mathcal{M}$ and $\mathcal{N}$ be two information models. Suppose that*

- $\mathcal{M} \hookrightarrow \mathcal{N}$ *and F, G is a pair of functions witnessing this.*
- $\mathcal{N} \hookrightarrow \mathcal{M}$ *and F', G' is a pair of functions witnessing this.*
- $F' \circ F = \mathsf{id}$ *(the identity function).*

Then for every information state $s \subseteq W^{\mathcal{M}}$, for every assignment $g : \mathtt{Var} \to D^{\mathcal{M}}$ and for every formula φ it holds that

$$\mathcal{M}, s \vDash_g \varphi \qquad \Longleftrightarrow \qquad \mathcal{N}, F[s] \vDash_{G \circ g} \varphi$$

Proof:
Firstly, by Corollary 3.2.5 we have the left-to-right implication. Moreover by the same corollary (since $F'[F[s]] = s$) we also obtain that

$$\mathcal{N}, F[s] \vDash_{G \circ g} \varphi \qquad \Longrightarrow \qquad \mathcal{M}, s \vDash_{G' \circ G \circ g} \varphi$$

By Condition Sub2 we have that $G' \circ G$ preserve (modulo essential equivalence) the interpretation of function, relation and equality symbols. It follows by a simple structural induction on φ that:

$$\mathcal{M}, s \vDash_{G' \circ G \circ g} \varphi \qquad \Longleftrightarrow \qquad \mathcal{M}, s \vDash_g \varphi$$

from which we conclude.

□

Chapter 4

Ehrenfeucht-Fraïssé Games

A powerful tool to study the expressiveness of several logical systems is given by Ehrenfeucht-Fraïssé games (also known as EF games or back-and-forth games), introduced in 1967 by Ehrenfeucht [1967], developing model-theoretic results presented by Fraïssé [1954]. These games provide a particularly perspicuous way of understanding what differences between models can be detected by means of formulas of a given logic. Reasoning about winning strategies in this game, one can prove that two first order structures are elementarily equivalent, or one can find a formula telling them apart.

One of the main merits of Ehrenfeucht-Fraïssé games is that they allow for relatively easy proofs that certain properties are not expressible by formulas. A classical application of this kind is the characterization of the cardinality quantifiers definable in classical first order logic. This characterization says that the only cardinality quantifiers definable in classical first order logic are those which, for some natural number m, are insensitive to the difference between any cardinals larger than m. This characterization yields a range of interesting undefinability results: for instance, it implies that the quantifiers *an even number of individuals* and *infinitely many individuals* are not first order definable.

The basic idea of EF games has proven to be very flexible and adaptable to a wide range of logical settings, including fragments of first order logic with finitely many variables [Immerman, 1982]; extensions of first order logic with generalized quantifiers [Kolaitis and Väänänen, 1995]; monadic second order logic [Fagin, 1975]; modal logic [Benthem, 1976]; and intuitionistic logic [Visser, 2001, Połacik, 2008]; logics based on team semantics such as dependence logic [Väänänen, 2007, Sec. 6.6] and inclusion logic [Grädel, 2016]. In each case, the game provides an insightful characterization of the distinctions that can and cannot be made by means of formulas in the logic.

In this chapter we introduce a variation of the Ehrenfeucht-Fraïssé game for InqBQ and show that this game provides a characterization of the expressive power of the logic. We then present further variations of the game, showing that this game-theoretic approach can be adapted to study other aspects of the logic.

4.1 The Ehrenfeucht-Fraïssé Game

The Ehrenfeucht-Fraïssé game for InqBQ is played by two players, Spoiler and Duplicator (abbreviated as S and D respectively), using two inquisitive models $\mathcal{M}, \mathcal{N}$ as a board. The intuition behind the game is similar to the one for classical logic: Spoiler wants to exhibit that the information encoded in $\mathcal{M}$ is *not subsumed* by the information encoded in $\mathcal{N}$; while Duplicator wants to show otherwise.

As in the classical case, the game proceeds in turns: each turn, Spoiler picks an object from one of the two models and Duplicator must respond by picking a corresponding object from the other model. At the end of the game, a winner is decided by comparing the atomic formulae supported by the sub-structures built during the game.

However, there are two crucial differences with the game for classical logic. Firstly, the objects that are picked during the game are not just individuals $d \in D$, but also information states $s \subseteq W$. The reason is that the implication of InqBQ behaves like a *restrictor*, that is, like a quantifier ranging over enhancements of the current information state (this is apparent from the semantic clauses for implication presented in Definition 2.1.7). Secondly, the roles of the two models in the game are not symmetric: it could be that the information encoded by $\mathcal{M}$ is subsumed by the information encoded by $\mathcal{N}$, but not vice versa; this is the case, for example, when $\mathcal{M}$ is a submodel of $\mathcal{N}$ (Definition 3.2.3). This asymmetry is connected to the absence of a classical negation in the language of InqBQ.

Now we describe the game and introduce a standard notation for the objects chosen during a run of the game.

Consider the following *ingredients*:

- $\mathcal{M}$ and $\mathcal{N}$ are information models;
- s and t are information states of $\mathcal{M}$ and $\mathcal{N}$ respectively;
- $\overline{a}$ and $\overline{b}$ are tuples (of the same length) of elements from $\mathcal{M}$ and $\mathcal{N}$ respectively.

A position in an EF game for InqBQ is a tuple $\langle \mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b} \rangle$ where:

If not otherwise specified, a game between the models $\mathcal{M}$ and $\mathcal{N}$ starts from position $\langle \mathcal{M}, W_{\mathcal{M}}, \varepsilon; \mathcal{N}, W_{\mathcal{N}}, \varepsilon; \rangle$, where ε indicates the empty tuple.

Starting a round from a position $\langle \mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b} \rangle$, S performs a move choosing it from the following list:[1]

[1] In the following, the notation $\overline{a}a'$ indicates the sequence obtained by adding the element a' at the end of the sequence $\overline{a}$.

$\overline{\exists}$-move: S picks an element $a' \in D_{\mathcal{M}}$; D responds with an element $b' \in D_{\mathcal{N}}$; the game continues from position $\langle \mathcal{M}, s, \overline{a}a'; \mathcal{N}, t, \overline{b}b' \rangle$;

$\forall$-move: S picks an element $b' \in D_{\mathcal{N}}$; D responds with an element $a' \in D_{\mathcal{M}}$; the game continues from position $\langle \mathcal{M}, s, \overline{a}a'; \mathcal{N}, t, \overline{b}b' \rangle$;

$\rightarrow$-move: S picks a sub-state $t' \subseteq t$; D responds with a sub-state $s' \subseteq s$. After that, S decides if the game continues from position $\langle \mathcal{M}, s', \overline{a}; \mathcal{N}, t', \overline{b} \rangle$ or from position $\langle \mathcal{N}, t', \overline{b}; \mathcal{M}, s', \overline{a} \rangle$.

Notice that the game has a *chirality*, given by which of $\mathcal{M}$ and $\mathcal{N}$ plays the role of the "first model", which S partially controls: by performing a quantifier-move—that is, an $\overline{\exists}$-move or an $\forall$-move—S picks an element from a model of his choice, but then the game continues with the same chirality; on the contrary, by performing an $\rightarrow$-move, S can pick an information state only from the second model, and then the game proceeds with a chirality of his choice.

With respect to the termination condition, a pair of numbers $\langle i, q \rangle \in \mathbb{N}^2$ is fixed in advance and it is known to the two players; we call i, q the *indexes* of the game. These numbers constrain the development of the game: in total, S can play only i implication moves and only q quantifier moves. When there are no more moves available, the game ends. Assuming $\langle \mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b} \rangle$ is the final position, the game is won by Player D if the following condition is satisfied, and by player S otherwise:

Winning condition for D: for all atomic formulas $A(x_1, \ldots, x_n)$ where n is the size of the tuples $\overline{a}$ and $\overline{b}$, we have:

$$\mathcal{M}, s \models A(\overline{a}) \implies \mathcal{N}, t \models A(\overline{b}) \tag{4.1}$$

We indicate with $\mathrm{EF}_{i,q}(\mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b})$ the game with indexes i, q starting from position $\langle \mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b} \rangle$. For brevity, we indicate with $\mathrm{EF}_{i,q}(\mathcal{M}, \mathcal{N})$ the game $\mathrm{EF}_{i,q}(\mathcal{M}, W_{\mathcal{M}}, \varepsilon; \mathcal{N}, W_{\mathcal{N}}, \varepsilon)$.

4.1.1. EXAMPLE. Consider the signature $\Sigma = \{P^{(1)}\}$ in the language $\mathcal{L}^{\neq}$. Given the models $\mathcal{M}$ and $\mathcal{N}$ in Figure 4.1, we simulate a run of the game $\mathrm{EF}_{0,2}(\mathcal{M}, \mathcal{N})$:

- S starts by performing an $\overline{\exists}$-move and choses $a_1 = d_1$ from $\mathcal{M}$; D responds by choosing $b_1 = e_2$ from $\mathcal{N}$. Current position: $\langle \mathcal{M}, \{w_0, w_1\}, \langle d_1 \rangle; \mathcal{N}, \{v_0, v_1\}, \langle e_2 \rangle \rangle$; moves left: 0 $\rightarrow$-moves, 1 quantifier-move.

- S performs an $\forall$-move and choses $b_2 = e_1$ from $\mathcal{N}$; D responds by choosing $a_2 = d_2$ from $\mathcal{M}$. Current position: $\langle \mathcal{M}, \{w_0, w_1\}, \langle d_1, d_2 \rangle; \mathcal{N}, \{v_0, v_1\}, \langle e_2, e_1 \rangle \rangle$; moves left: 0 $\rightarrow$-moves, 0 quantifier-moves.

$\mathcal{M}$

	w_0	w_1
d_1	•	×
d_2	×	•

$\mathcal{N}$

	v_0	v_1
e_1	•	•
e_2	×	•

Figure 4.1: The information models used in Example 4.1.1.

	$\mathcal{M}$	$\mathcal{N}$
S	$a_1 := \mathbf{d_1}$	
D		$b_1 := \mathbf{e_2}$
S		$b_2 := \mathbf{e_1}$
D	$a_2 := \mathbf{d_2}$	

(a) Example of run of $\mathrm{EF}_{0,2}(\mathcal{M},\mathcal{N})$.

	$\mathcal{M}$	$\mathcal{N}$
S	$a_1 := \mathbf{d_1}$	
D		$b_1 := \mathbf{e_2}$
S		$t_1 := \{\mathbf{v_1}\}$
D	$s_1 := \{\mathbf{w_0}\}$	
S	$i := \mathbf{1}$	
S		$b_2 := \mathbf{e_1}$
D	$a_2 := \mathbf{d_2}$	

(b) Example of run of $\mathrm{EF}_{1,2}(\mathcal{M},\mathcal{N})$.

- Since there are no moves left, the game is over. **Duplicator is the winner**, since Condition 4.1 is met:

$$\mathcal{M}, \{w_0, w_1\} \nvDash P(d_1) \implies [\mathcal{M}, \{w_0, w_1\} \vDash P(d_1) \implies \mathcal{N}, \{v_0, v_1\} \vDash P(e_2)]$$
$$\mathcal{M}, \{w_0, w_1\} \nvDash P(d_2) \implies [\mathcal{M}, \{w_0, w_1\} \vDash P(d_2) \implies \mathcal{N}, \{v_0, v_1\} \vDash P(e_1)]$$

The run is represented in Table 4.2a.

4.1.2. Example. Referring to the same models of Example 4.1.1, we simulate a run of $\mathrm{EF}_{1,2}(\mathcal{M},\mathcal{N})$:

- S starts by performing an $\overline{\exists}$-move and choses $a_1 = d_1$ from $\mathcal{M}$; D responds by choosing $b_1 = e_2$ from $\mathcal{N}$. Current position: $\langle \mathcal{M}, \{w_0, w_1\}, \langle d_1 \rangle ; \mathcal{N}, \{v_0, v_1\}, \langle e_2 \rangle \rangle$; moves left: 1 $\rightarrow$-move, 1 quantifier-move.

- S performs a $\rightarrow$-move and choses $t_1 = \{v_1\}$ an info state of $\mathcal{N}$; D responds by choosing $s_1 = \{w_0\}$ an info state of $\mathcal{M}$; finally S chooses to change the chirality of the system. Current position: $\langle \mathcal{N}, \{v_1\}, \langle e_2 \rangle ; \mathcal{M}, \{w_0\}, \langle d_1 \rangle \rangle$; moves left: 0 $\rightarrow$-moves, 1 quantifier-move.

- S performs an $\overline{\exists}$-move and choses $b_2 = e_1$ from $\mathcal{N}$; D responds by choosing $a_2 = d_2$ from $\mathcal{M}$. Current position: $\langle \mathcal{N}, \{v_0\}, \langle e_2, e_1 \rangle ; \mathcal{M}, \{w_0\}, \langle d_1, d_2 \rangle \rangle$; moves left: 0 $\rightarrow$-moves, 0 quantifier-moves.

- Since there are no moves left, the game is over. **Spoiler is the winner**, since Condition 4.1 is not met:

$$\mathcal{N}, \{v_1\} \vDash P(\underbrace{e_1}_{b_2}) \quad \text{and} \quad \mathcal{M}, \{w_0\} \nvDash P(\underbrace{d_2}_{a_2})$$

The run is represented in Table 4.2b. Notice that, since we are working in the language $\mathcal{L}^{\neq}$, Duplicator could have won the run by choosing $a_2 = d_1$ in the last exchange.

As usual, a winning strategy for a player is a strategy which guarantees victory to them, no matter which moves the opponent performs. If Duplicator has a winning strategy in the game $\mathrm{EF}_{i,q}(\mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b})$ we write:

$$(\mathcal{M}, s, \overline{a}) \preceq_{i,q} (\mathcal{N}, t, \overline{b})$$

We indicate with $\approx_{i,q}$ the relation $\preceq_{i,q} \cap \succeq_{i,q}$. Notice that the game is finite (since the number of turns is bounded by $i + q$), zero-sum and has perfect information. Therefore, if $(\mathcal{M}, s, \overline{a}) \preceq_{i,q} (\mathcal{N}, t, \overline{b})$ does not hold, then it follows from the Gale-Stewart Theorem that Spoiler has a winning strategy in the game $\mathrm{EF}_{i,q}(\mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b})$. We simply write $\mathcal{M} \preceq_{i,q} \mathcal{N}$ as a shorthand for $(\mathcal{M}, W_{\mathcal{M}}, \varepsilon) \preceq_{i,q} (\mathcal{N}, W_{\mathcal{N}}, \varepsilon)$.

The following two propositions follow easily from the definition of the game, and exhibit the recursive nature of the game.

4.1.3. PROPOSITION. *If* $(\mathcal{M}, s, \overline{a}) \preceq_{i,q} (\mathcal{N}, t, \overline{b})$ *then* $(\mathcal{M}, s, \overline{a}) \preceq_{i',q'} (\mathcal{N}, t, \overline{b})$ *for all* $i' \leq i$ *and* $q' \leq q$.

4.1.4. PROPOSITION. *Suppose* $\langle i, q \rangle \neq \langle 0, 0 \rangle$. $(\mathcal{M}, s, \overline{a}) \preceq_{i,q} (\mathcal{N}, t, \overline{b})$ *iff the following three conditions are satisfied:*

- *If* $i > 0$, *then* $\forall t' \subseteq t.\ \exists s' \subseteq s.\ (\mathcal{M}, s', \overline{a}) \approx_{i-1,q} (\mathcal{N}, t', \overline{b})$;
- *If* $q > 0$, *then* $\forall a' \in D_{\mathcal{M}}.\ \exists b' \in D_{\mathcal{N}}.\ (\mathcal{M}, s, \overline{a}a') \preceq_{i,q-1} (\mathcal{N}, t, \overline{b}b')$;
- *If* $q > 0$, *then* $\forall b' \in D_{\mathcal{N}}.\ \exists a' \in D_{\mathcal{M}}.\ (\mathcal{M}, s, \overline{a}a') \preceq_{i,q-1} (\mathcal{N}, t, \overline{b}b')$.

4.2 IQ Degree and Types

We already mentioned the intuition behind the game: Spoiler wants to exhibit that the information encoded in $\mathcal{M}$ is not subsumed by the information encoded in $\mathcal{N}$; while Duplicator wants to show otherwise. We want to formalize this intuition, and to do so we need to specify the notion of *information encoded by a model*. More precisely, what Spoiler tries to show is that the *definable* *information*—that is, the properties definable by formulas of InqB—encoded by $\mathcal{M}$ is not subsumed by the information encoded in $\mathcal{N}$. This Subsection introduces the tools to formalize this intuition.

We define the *implication degree* (Ideg) and *quantification degree* (Qdeg) of a formula by the following inductive clauses. Here, p stands for a generic atomic formula:

$$\begin{array}{ll}
\text{Ideg}(p) & = 0 \\
\text{Ideg}(\bot) & = 0 \\
\text{Ideg}(\varphi_1 \wedge \varphi_2) & = \max(\text{Ideg}(\varphi_1), \text{Ideg}(\varphi_2)) \\
\text{Ideg}(\varphi_1 \vee\!\!\!\vee \varphi_2) & = \max(\text{Ideg}(\varphi_1), \text{Ideg}(\varphi_2)) \\
\text{Ideg}(\varphi_1 \to \varphi_2) & = \max(\text{Ideg}(\varphi_1), \text{Ideg}(\varphi_2)) + 1 \\
\text{Ideg}(\forall x.\varphi) & = \text{Ideg}(\varphi) \\
\text{Ideg}(\exists\!\!\!\exists x.\varphi) & = \text{Ideg}(\varphi)
\end{array}
\qquad
\begin{array}{ll}
\text{Qdeg}(p) & = 0 \\
\text{Qdeg}(\bot) & = 0 \\
\text{Qdeg}(\varphi_1 \wedge \varphi_1) & = \max(\text{Qdeg}(\varphi_1), \text{Qdeg}(\varphi_2)) \\
\text{Qdeg}(\varphi_1 \vee\!\!\!\vee \varphi_1) & = \max(\text{Qdeg}(\varphi_1), \text{Qdeg}(\varphi_2)) \\
\text{Qdeg}(\varphi_1 \to \varphi_1) & = \max(\text{Qdeg}(\varphi_1), \text{Qdeg}(\varphi_2)) \\
\text{Qdeg}(\forall x.\varphi) & = \text{Qdeg}(\varphi) + 1 \\
\text{Qdeg}(\exists\!\!\!\exists x.\varphi) & = \text{Qdeg}(\varphi) + 1
\end{array}$$

To keep track of both these measures of complexity of a formula with a unique formal object, we define the combined degree of a formula φ as $\text{IQdeg}(\varphi) = \langle \text{Ideg}(\varphi), \text{Qdeg}(\varphi) \rangle$. The natural order $\leq$ on such degrees is defined by

$$\langle a, b \rangle \leq \langle a', b' \rangle \iff a \leq a' \text{ and } b \leq b'$$

So a formula φ is *(strictly) more complex* than a formula ψ if and only if $\text{IQdeg}(\varphi) \geq \text{IQdeg}(\psi)$ and at least one among $\text{Ideg}(\varphi) > \text{Ideg}(\psi)$ and $\text{Qdeg}(\varphi) > \text{Qdeg}(\psi)$ holds.
We denote by $\mathcal{L}^l$ the set of formulas φ with free variables included in $\{x_1, \ldots, x_l\}$; and with $\mathcal{L}^l_{i,q}$ the formulas $\varphi \in \mathcal{L}^l$ with $\text{IQdeg}(\varphi) \leq \langle i, q \rangle$.[2]

The notion of IQ degree allows us to adapt the notion of type from classical model theory to the setup of inquisitive logic.

4.2.1. Definition (types and $\langle i, q \rangle$-types). Let $\mathcal{M}$ be a model, s an information state of $\mathcal{M}$, and $\overline{a}$ a tuple of elements in $\mathcal{M}$ of length l. The *type* and $\langle i, q \rangle$*-type* of $\langle \mathcal{M}, s, \overline{a} \rangle$ are respectively

$$\begin{array}{ll}
\text{tp}(\mathcal{M}, s, \overline{a}) & := \left\{ \varphi \in \mathcal{L}^l \;\middle|\; \mathcal{M}, s \models \varphi(\overline{a}) \right\} \\
\text{tp}_{i,q}(\mathcal{M}, s, \overline{a}) & := \left\{ \varphi \in \mathcal{L}^l_{i,q} \;\middle|\; M, s \models \varphi(\overline{a}) \right\}
\end{array}$$

Additionally, we define the relations $\sqsubseteq$, $\equiv$, $\sqsubseteq_{i,q}$ and $\equiv_{i,q}$ as:

$$\begin{array}{lllllll}
\langle \mathcal{M}, s, \overline{a} \rangle & \sqsubseteq & \langle \mathcal{N}, t, \overline{b} \rangle & \text{iff} & \text{tp}(\mathcal{M}, s, \overline{a}) & \subseteq & \text{tp}(\mathcal{N}, t, \overline{b}) \\
\langle \mathcal{M}, s, \overline{a} \rangle & \equiv & \langle \mathcal{N}, t, \overline{b} \rangle & \text{iff} & \text{tp}(\mathcal{M}, s, \overline{a}) & = & \text{tp}(\mathcal{N}, t, \overline{b}) \\
\langle \mathcal{M}, s, \overline{a} \rangle & \sqsubseteq_{i,q} & \langle \mathcal{N}, t, \overline{b} \rangle & \text{iff} & \text{tp}_{i,q}(\mathcal{M}, s, \overline{a}) & \subseteq & \text{tp}_{i,q}(\mathcal{N}, t, \overline{b}) \\
\langle \mathcal{M}, s, \overline{a} \rangle & \equiv_{i,q} & \langle \mathcal{N}, t, \overline{b} \rangle & \text{iff} & \text{tp}_{i,q}(\mathcal{M}, s, \overline{a}) & = & \text{tp}_{i,q}(\mathcal{N}, t, \overline{b})
\end{array}$$

[2]The results of this chapter are independent from the presence and the interpretation of the equality symbol, so $\mathcal{L}$ will indicate a generic syntax among $\mathcal{L}_{\neq}$, $\mathcal{L}_{=}$ and $\mathcal{L}_{\asymp}$.

4.2.2. Example. Consider the models $\mathcal{M}$, $\mathcal{N}$ in Figure 4.1. We have

$$\begin{gathered}\mathcal{N}, \{v_0, v_1\} \vDash P(e_1) \qquad \text{and} \qquad \mathcal{M}, \{w_0, w_1\} \nvDash P(d_1) \\ \Downarrow \\ (\mathcal{N}, \{v_0, v_1\}, \langle e_1 \rangle) \not\sqsubseteq_{0,0} (\mathcal{M}, \{w_0, w_1\}, \langle d_1 \rangle)\end{gathered}$$

Notice that, if the signature is finite, there are only a finite number of non-equivalent formulas with a combined degree of at most $\langle i, q\rangle$, and consequently only a finite number of $\langle i, q\rangle$-types. This can be shown by well-founded induction on $\langle i, q\rangle$, relying on the fact that $(\mathcal{L}^l, \wedge, \vee\!\!\!\vee)$ is a distributive lattice:

- Up to logical equivalence, there are only finitely many formulas in $\mathcal{L}^l_{0,0}$—atoms and combinations of them obtained using the operators $\wedge$ and $\vee\!\!\!\vee$. Notice that the hypothesis of working with a finite signature is crucial to have this result.

- Formulas in $\mathcal{L}^l_{i,q}$ are equivalent to combinations of formulas in $I \cup A \cup E$ obtained using the operators $\wedge$ and $\vee\!\!\!\vee$, where $I = \{\varphi \to \psi \mid \varphi, \psi \in \mathcal{L}^l_{i-1,q}\}$, $A = \{\forall x.\varphi \mid \varphi \in \mathcal{L}^{l+1}_{i,q-1}\}$ and $E = \{\exists x.\varphi \mid \varphi \in \mathcal{L}^{l+1}_{i,q-1}\}$ (we impose the convention $\mathcal{L}^l_{i,q} = \emptyset$ if $i < 0$ or $q < 0$). By inductive hypothesis I, A and E contain only finitely many non-equivalent formulas, thus the same holds for $\mathcal{L}^l_{i,q}$.

4.3 Ehrenfeucht-Fraïssé Theorem

In the following we state and prove the main result connecting the notion of type and the Ehrenfeucht-Fraïssé game. In this section we will require the signature Σ to satisfy the additional condition of being *relational*.

4.3.1. Definition (Relational signature). We say that a signature Σ is *relational* if it contains only relation and constant symbols.

This restriction can be easily dispensed with when we consider the languages $\mathcal{L}_=$ and $\mathcal{L}_\asymp$, but not when we work with the language $\mathcal{L}_{\neq}$. We will discuss this issue further in Section 4.4.

We are now ready to state the main theorem of this chapter.

4.3.2. Theorem. *Suppose the signature Σ is finite and relational. Then*

$$(\mathcal{M}, s, \overline{a}) \preceq_{i,q} (\mathcal{N}, t, \overline{b}) \quad \Longleftrightarrow \quad (\mathcal{M}, s, \overline{a}) \sqsubseteq_{i,q} (\mathcal{N}, t, \overline{b})$$

Proof:
We will prove this by well-founded induction on $\langle i, q\rangle$. For the basic case, $\langle i, q\rangle =$

$\langle 0,0\rangle$, we just have to verify that Condition 4.1 holds for all atomic formulas iff it holds for all formulas $\varphi \in \mathcal{L}^l_{0,0}$: this is just a straightforward verification.

Next, we prove the right-to-left direction of the inductive step. Suppose $\langle i,q\rangle > \langle 0,0\rangle$ and suppose the claim holds for all $\langle i',q'\rangle < \langle i,q\rangle$. For the left-to-right direction, we proceed by contraposition. Suppose that some $\varphi \in \mathcal{L}^l_{i,q}$ witnesses that $(\mathcal{M}, s, \overline{a}) \not\sqsubseteq_{i,q} (\mathcal{N}, t, \overline{b})$, that is:

$$\mathcal{M}, s \vDash \varphi(\overline{a}) \qquad\qquad \mathcal{N}, t \nvDash \varphi\left(\overline{b}\right)$$

We proceed by induction on the structure of φ.[3]

We first treat the easy cases:

- If φ is an atomic formula, it follows $(\mathcal{M}, s, \overline{a}) \not\preceq_{0,0} (\mathcal{N}, t, \overline{b})$; so, by Proposition 4.1.3, we also have $(\mathcal{M}, s, \overline{a}) \not\preceq_{i,q} (\mathcal{N}, t, \overline{b})$.
- If φ is a conjunction $\psi \wedge \chi$ then we have:

$$\begin{cases} \mathcal{M}, s \ \vDash \psi(\overline{a}) \wedge \chi(\overline{a}) & \Longrightarrow \quad \mathcal{M}, s \ \vDash \psi(\overline{a}) \quad \text{and} \quad \mathcal{M}, s \ \vDash \chi(\overline{a}) \\ \mathcal{N}, t \ \nvDash \psi(\overline{b}) \wedge \chi(\overline{b}) & \Longrightarrow \quad \mathcal{N}, t \ \nvDash \psi(\overline{b}) \quad \text{or} \quad \mathcal{N}, t \ \nvDash \chi(\overline{b}) \end{cases}$$

  So, either ψ or χ is a less complex witness of $(\mathcal{M}, s, \overline{a}) \not\sqsubseteq_{i,q} (\mathcal{N}, t, \overline{b})$. By the nested inductive hypothesis, we conclude that $(\mathcal{M}, s, \overline{a}) \not\preceq_{i,q} (\mathcal{N}, t, \overline{b})$.
- If φ is a disjunction $\psi \mathbin{\vee\!\!\!\vee} \chi$, the reasoning is analogue to the case for conjunction.

So the cases left are: ($\Rightarrow^1$) φ of the form $\psi \to \chi$, ($\Rightarrow^2$) φ of the form $\forall x\psi$ and ($\Rightarrow^3$) φ of the form $\overline{\exists} x\psi$. Let us consider the three cases separately.

Case $\Rightarrow^1$: φ is an implication of the form $\psi \to \chi$. In this case we have

$$\begin{aligned} \mathcal{N}, t \nvDash \psi(\overline{b}) \to \chi(\overline{b}) \quad &\Longrightarrow \quad \exists t' \subseteq t. \begin{cases} \mathcal{N}, t' \vDash \psi(\overline{b}) \\ \mathcal{N}, t' \nvDash \chi(\overline{b}) \end{cases} \\ \mathcal{M}, s \vDash \psi(\overline{a}) \to \chi(\overline{a}) \quad &\Longrightarrow \quad \nexists s' \subseteq s. \begin{cases} \mathcal{M}, s' \vDash \psi(\overline{a}) \\ \mathcal{M}, s' \nvDash \chi(\overline{a}) \end{cases} \end{aligned}$$

Thus there exists a state $t' \subseteq t$ with a different $\langle i-1,q\rangle$-type than every $s' \subseteq s$—either because it supports ψ or because it does not support χ. So by main inductive hypothesis, if Spoiler performs a $\to$-move and chooses t', for every choice s' of Duplicator we have $(\mathcal{M}, s', \overline{a}) \not\approx_{i-1,q} (\mathcal{N}, t', \overline{b})$. It follows by Proposition 4.1.4 that $(\mathcal{M}, s, \overline{a}) \not\preceq_{i,q} (\mathcal{N}, t, \overline{b})$.

[3]Since we are performing a *nested induction*, we will refer to the first induction on the indexes $\langle i,q\rangle$ as the *main induction*, and to the second induction on the structure of φ as the *nested induction*.

Case $\Rightarrow^2$: φ is a universal formula of the form $\forall x.\psi$. In this case we have

$$\begin{array}{lcl} \mathcal{N}, t \nvDash \forall x.\psi(\bar{b}, x) & \Longrightarrow & \exists b' \in D^{\mathcal{N}}. \quad \mathcal{N}, t \nvDash \psi\left(\bar{b}, b'\right) \\ \mathcal{M}, s \vDash \forall x.\psi(\bar{a}, x) & \Longrightarrow & \forall a' \in D^{\mathcal{M}}.\ \mathcal{M}, s \vDash \psi\left(\bar{a}, a'\right) \end{array}$$

If Spoiler performs an $\forall$-move and chooses b', then by main inductive hypothesis, for every possible choice a' of Duplicator, we have

$$(\mathcal{M}, s, \bar{a}a') \not\preceq_{i,q-1} (\mathcal{N}, t, \bar{b}b')$$

It follows by Proposition 4.1.4 that $(\mathcal{M}, s, \bar{a}) \not\preceq_{i,q} (\mathcal{N}, t, \bar{b})$.

Case $\Rightarrow^3$: φ is an existential formula of the form $\overline{\exists} x\psi$. This case is similar to the previous one: Spoiler can perform an $\overline{\exists}$-move and pick an element $a' \in D^{\mathcal{M}}$ with no counterpart in $D^{\mathcal{N}}$, and by Proposition 4.1.4 we obtain $(\mathcal{M}, s, \bar{a}) \not\preceq_{i,q} (\mathcal{N}, t, \bar{b})$.

This completes the proof of the left-to-right direction of the inductive step. Now consider the converse direction. Again, we proceed by contraposition: Suppose that Spoiler has a winning strategy in the game $\mathrm{EF}_{i,q}(\mathcal{M}, s, \bar{a}; \mathcal{N}, t, \bar{b})$. We consider again three cases, depending on the first move of Spoiler's winning strategy (cases $\Leftarrow^1$, $\Leftarrow^2$, $\Leftarrow^3$ respectively).

Case $\Leftarrow^1$: the first move is a $\rightarrow$-move. Suppose that Spoiler starts by choosing $t' \subseteq t$. As this is a winning strategy for Spoiler, for every choice $s' \subseteq s$ of Duplicator we have

$$(\mathcal{M}, s', \bar{a}) \not\preceq_{i-1,q} (\mathcal{N}, t', \bar{b}) \quad \text{or} \quad (\mathcal{N}, t', \bar{b}) \not\preceq_{i-1,q} (\mathcal{M}, s', \bar{a})$$

By inductive hypothesis, this translates to

$$\exists \psi_{s'} \in \mathrm{tp}(s') \setminus \mathrm{tp}(t') \quad \text{or} \quad \exists \theta_{s'} \in \mathrm{tp}(t') \setminus \mathrm{tp}(s')$$

where $\mathrm{tp}(s') := \mathrm{tp}_{i-1,q}(\mathcal{M}, s', \bar{a})$ and $\mathrm{tp}(t') := \mathrm{tp}_{i-1,q}(\mathcal{N}, t', \bar{b})$.

So there exist two families $\{\psi_{s'} \mid s' \subseteq s\}$ and $\{\theta_{s'} \mid s' \subseteq s\}$ such that:

$$\begin{cases} \psi_{s'} \in \mathrm{tp}(s') \setminus \mathrm{tp}(t') & \text{if } \mathrm{tp}(s') \setminus \mathrm{tp}(t') \neq \emptyset \\ \psi_{s'} := \bot & \text{otherwise} \end{cases}$$

$$\begin{cases} \theta_{s'} \in \mathrm{tp}(t') \setminus \mathrm{tp}(s') & \text{if } \mathrm{tp}(t') \setminus \mathrm{tp}(s') \neq \emptyset \\ \theta_{s'} := \top & \text{otherwise} \end{cases}$$

Moreover we can suppose the two families to be finite, since there are only a finite number of formulas of degree $\langle i-1, q\rangle$ up to logical equivalence (see Section 4.2). Define now

$$\varphi := \bigwedge_{s' \subseteq s} \theta_{s'} \rightarrow \bigvee\!\!\!\!\bigvee_{s' \subseteq s} \psi_{s'}$$

We have:

1. $\text{IQdeg}(\varphi) \leq \langle i, q \rangle$;
2. $\varphi \notin \text{tp}_{i,q}(\mathcal{N}, t, \bar{b})$ (since by construction φ is falsified at $t' \subseteq t$);
3. $\varphi \in \text{tp}_{i,q}(\mathcal{M}, s, \bar{a})$ (since by construction φ holds at every state $s' \subseteq s$).

Thus we have $(\mathcal{M}, s, \bar{a}) \not\sqsubseteq_{i-1,q} (\mathcal{N}, t, \bar{b})$, as we wanted.

Case $\Leftarrow^2$: the first move is an $\forall$-move. Suppose Spoiler starts by choosing $b' \in D^{\mathcal{N}}$. As this is a winning strategy for Spoiler, for every choice $a' \in D^{\mathcal{M}}$ of Duplicator we have

$$(\mathcal{M}, s, \bar{a}a') \not\preceq_{i,q-1} (\mathcal{N}, t, \bar{b}b')$$

By inductive hypothesis, the above translates to

$$\exists \psi_{a'} \in \text{tp}(a') \setminus \text{tp}(b')$$

where $\text{tp}(a') := \text{tp}_{i,q-1}(\mathcal{M}, s, \bar{a}a')$ and $\text{tp}(b') := \text{tp}_{i,q-1}(\mathcal{N}, t, \bar{b}b')$.

Now the formula

$$\varphi := \forall x \bigvee_{a' \in D^{\mathcal{M}}} \psi_{a'}$$

has IQ-degree at most $\langle i, q \rangle$, and by construction we have $\varphi \in \text{tp}_{i,q}(\mathcal{M}, s, \bar{a})$ and $\varphi \notin \text{tp}_{i,q}(\mathcal{N}, t, \bar{b})$. Thus, we have $(\mathcal{M}, s, \bar{a}) \not\sqsubseteq_{i,q} (\mathcal{N}, t, \bar{b})$.

Case $\Leftarrow^3$: the first move is an $\overline{\exists}$-move. Reasoning as in the previous case, we find that there exists a $a' \in D^{\mathcal{M}}$—the element chosen by Spoiler following the winning strategy—such that for every $b' \in D^{\mathcal{N}}$

$$\exists \theta_{b'} \in \text{tp}(s') \setminus \text{tp}(t')$$

In particular, it follows that the formula

$$\varphi := \overline{\exists} x \bigwedge_{b' \in D^{\mathcal{N}}} \psi_{b'}$$

is a formula of complexity at most $\langle i, q \rangle$ such that $\varphi \in \text{tp}_{i,q}(\mathcal{M}, s, \bar{a})$ and $\varphi \notin \text{tp}_{i,q}(\mathcal{N}, t, \bar{b})$. Again, it follows that $(\mathcal{M}, s, \bar{a}) \not\sqsubseteq_{i,q} (\mathcal{N}, t, \bar{b})$.

This concludes the proof of the right-to-left direction.

□

As an immediate corollary, we also obtain a game-theoretic characterization of the relation $\equiv_{i,q}$.

4.3.3. Corollary. *For a finite signature Σ, we have:*

$$(\mathcal{M}, s, \bar{a}) \approx_{i,q} (\mathcal{N}, t, \bar{b}) \quad \Longleftrightarrow \quad (\mathcal{M}, s, \bar{a}) \equiv_{i,q} (\mathcal{N}, t, \bar{b})$$

4.4 Extending the Result to Function Symbols

The results we just obtained assume that the signature Σ is finite and relational. However, under opportune hypotheses we can extend them to finite non-relational signatures, that is, signatures containing also function symbols other than constants. Recall that in InqBQ, function symbols are interpreted rigidly, that is, their interpretation is independent from the world considered.

As in the case of classical logic [Hodges, 1997a, Section 3.3], the presence of function symbols requires some care in formulating the Ehrenfeucht-Fraïssé game. The reason is that allowing atomic formulas to contain arbitrary occurrences of function symbols allows us to generate, with a finite number of choices in the game, a sub-structure of the model of infinite size, which in turn spoils the crucial locality feature of the game.

A simple way to generalize the result to finite signatures containing function symbols is to follow the treatment by Hodges [1997a, Section 3.3] and define $\sqsubseteq$, $\sqsubseteq_{i,q}$ in terms of *unnested formulas*.

4.4.1. Definition (Unnested formula). An *unnested atomic formula* is a formula of one of the following forms:

$$x = y \qquad c = y \qquad f(\overline{x}) = y \qquad x \asymp y \qquad c \asymp y \qquad f(\overline{x}) \asymp y \qquad R(\overline{x})$$

A formula φ is said to be *unnested* if every atomic subformula of φ is an unnested atomic formula.

Examples of *nested formulas*—i.e., non-unnested formulas—are $f(x) = g(y)$, $R(f(x))$ and $f(c) \asymp x$.

We can now make the following amendments to the previous section to account for function symbols:

1. the winning conditions for the game are determined by looking at whether Condition 4.1 is satisfied for all *unnested* atomic formulas;

2. the $\langle i, q\rangle$-types are defined as sets of *unnested* formulas of degree at most $\langle i, q\rangle$, instead of generic formulas of said degree.

Except for these small tweaks, the statement of the result and the proof are completely analogous.

In case we are working with the languages $\mathcal{L}_=$ or $\mathcal{L}_\asymp$, that is, in case we have an equality symbol in the language, we can turn an arbitrary formula into an equivalent unnested one (e.g., replacing $P(f(x))$ with $\forall y.(y \asymp f(x) \to Py)$). So the restriction to unnested formulas is not a limitation to the generality of the game-theoretic characterization *as long as we have the equality in the language*; rather, it can be seen as an indirect way of assigning formulas containing function symbols with the appropriate $\langle i, q\rangle$-degree—making explicit the quantifications which are implicit in the presence of function symbols.

Unfortunately, in case we are working with the language $\mathcal{L}_{\neq}$, which does not contain an equality symbol, there are infinitely many atomic formulas not equivalent to any unnested formula, which in turns implies that we need to consider a non-finitary winning condition for Duplicator. Since this goes beyond the scope of this chapter, we will leave the study of this case for further work.

4.5 Variations of the Game

In this section we introduce some variations of the game presented in Section 4.3.

4.5.1 Symmetric Version

We introduce a *symmetric version* of the game, to address an issue of the original game: the game presented is too convoluted. The players need to choose and keep track of the elements picked, the states chosen and the chirality associated to the position. These choices could interact in non-trivial ways during a run of the game, so this makes the game quite complicated to play.

The symmetric version gets rid of the chirality from the game, paying a price for that: this version of the game corresponds to the relations $\equiv_{i,q}, \equiv$, and not to the relations $\sqsubseteq_{i,q}, \sqsubseteq$; and only one direction of Theorem 4.3.2 holds, namely the support-equivalence of two models given a winning strategy for Duplicator. In later sections we will use this symmetric version of the game to show certain expressive limitations of InqBQ.

The symmetric game is the same as the Ehrenfeucht-Fraïssé game presented in Section 4.1, except for two things: the $\rightarrow$-move and the winning condition for Duplicator. The following are the clauses for the symmetric version of the game.

Possible moves, $\rightarrow$-move: Starting a round from a position $\langle \mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b}\rangle$, S can perform a $\rightarrow$-move, which consists in the following: S picks a sub-state $t' \subseteq t$ or a sub-state $s' \subseteq s$; D responds with a sub-state from the other model. After that, the game continues from position $\langle \mathcal{M}, s', \overline{a}; \mathcal{N}, t', \overline{b}\rangle$.

Winning condition: The game is won by Player D if the following condition is satisfied, and by player S otherwise:

Winning condition for D: for all atomic formulas $A(x_1, \dots, x_n)$ where n is the size of the tuples $\overline{a}$ and $\overline{b}$, we have:

$$\mathcal{M}, s \models A(\overline{a}) \iff \mathcal{N}, t \models A(\overline{b}) \tag{4.2}$$

All the other components of the game—including the side conditions needed to perform an $\overline{\exists}$-move, $\forall$-move or $\rightarrow$-move—remain unchanged. Notice that at the end of the $\rightarrow$-move, S does not choose the chirality of the game as in the original game. Moreover, the winning condition for this version of the game is *symmetric* with respect to the two models.

We will indicate with $\mathrm{EF}^s_{i,q}(\mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b})$ the game just described. We will indicate with $\mathcal{M}, s, \overline{a} \approx^s_{i,q} \mathcal{N}, t, \overline{b}$ the existence of a winning strategy for Duplicator in the game $\mathrm{EF}^s_{i,q}(\mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b})$; and with $\mathcal{M}, s, \overline{a} \approx^s \mathcal{N}, t, \overline{b}$ the existence of a winning strategy for arbitrary $i, q \in \mathbb{N}$. We will also use notational conventions analogous to the ones introduced for the original game.

Notice that the roles of the two models in the game are interchangeable:

4.5.1. Lemma. *$\approx^s_{i,q}$ and $\approx^s$ are symmetric relations.*

Comparing this version with the original one, we clearly made Spoiler's life much easier: now he can perform $\rightarrow$-moves choosing states from both models; and the winning condition for Duplicator is more restrictive than the original one. So the following result should not come as a surprise:

4.5.2. Lemma. *If $\mathcal{M}, s, \overline{a} \approx^s_{i,q} \mathcal{N}, t, \overline{b}$, then $\mathcal{M}, s, \overline{a} \approx_{i,q} \mathcal{N}, t, \overline{b}$.*

Proof:
The idea of the proof is really simple: given a winning strategy for Duplicator in the game $\mathrm{EF}^s_{i,q}(\mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b})$, this is also a winning strategy in the game $\mathrm{EF}_{i,q}(\mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b})$. The details are left to the reader. $\square$

As an immediate corollary we obtain the following result, which we will use extensively in applications.

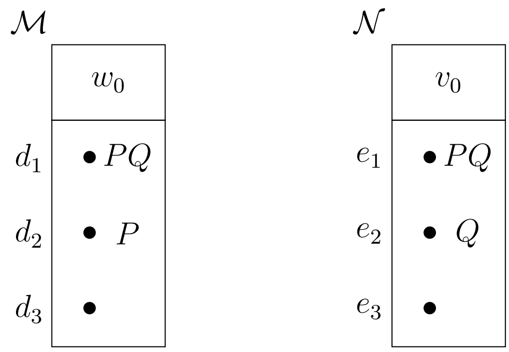
4.5.3. Corollary. *Suppose the signature Σ is finite. Then*

$$(\mathcal{M}, s, \overline{a}) \approx^s_{i,q} (\mathcal{N}, t, \overline{b}) \quad \Longrightarrow \quad (\mathcal{M}, s, \overline{a}) \equiv_{i,q} (\mathcal{N}, t, \overline{b})$$

We can also easily show that the converse of Corollary 4.5.3 does not hold.

4.5.4. Example. Consider the models $\mathcal{M}$ and $\mathcal{N}$ in Figure 4.3a. It is easy to show that:

- $\mathcal{M} \preceq_{0,1} \mathcal{N}$: the winning strategy for Duplicator is described in Table 4.3b;
- $\mathcal{M} \succeq_{0,1} \mathcal{N}$: the winning strategy for Duplicator is described in Table 4.3c;
- $\mathcal{M} \not\approx^s_{0,1} \mathcal{N}$: if Spoiler picks the element d_2 with its first (and only) move, every move of Duplicator leads to Spoiler's victory.



(a) The two models considered.

S plays...	D responds...
d_1	e_1
d_2	e_1
d_3	e_3
e_1	d_1
e_2	d_3
e_3	d_3

(b) Winning strategy for D in the game $\mathrm{EF}_{0,1}(\mathcal{M},\mathcal{N})$.

S plays...	D responds...
d_1	e_1
d_2	e_3
d_3	e_3
e_1	d_1
e_2	d_1
e_3	d_3

(c) Winning strategy for D in the game $\mathrm{EF}_{0,1}(\mathcal{N},\mathcal{M})$.

Figure 4.3: The information models and the winning strategy considered in Example 4.5.4.

4.5.2 Transfinite Version

As for the classical case, we can introduce a transfinite version of the game. The intuition to achieve this is to introduce a "timer" to keep track of the stage of the game reached during a run. After performing a move, the timer increases by one, and this way we keep track of how many moves we performed. In the Ehrenfeucht-Fraïssé game presented at the beginning of the Chapter, we can define the value of the timer to be the ordered pair $\langle h, k\rangle$, where h is the number of implication moves performed and k is the number of quantification moves performed. We will call this pair the *stage* of the game.

If we allow the values of k to range over ordinals, and not only natural numbers, then we easily obtain a transfinite version of the game. The position of the game when the timer assumes the value $\langle h, \alpha+1\rangle$—where $\alpha+1$ is a successor ordinal—is determined by the position and the actions performed at stage $\langle h, \alpha\rangle$; while the position of the game when the timer assumes the value $\langle h, \theta\rangle$ for θ a limit ordinal, is the *limit position* (a concept that will be clarified in what follows) of the stages of the form $\langle h, \alpha\rangle$ for $\alpha < \theta$. Notice that, while there is a natural way to define a limit position after performing a transfinite amount of quantification moves, there is no natural way to define a limit position after performing a transfinite number of implication moves: For example, what should be the chirality of the limit position?

The transfinite version of the game is defined as follows:

Position: A position in the game is a tuple $\langle \mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b}; h, \gamma, i \rangle$. h and γ represent the stage reached in the run of the game, that is, h is the number of implication moves performed and γ is the number of quantification moves performed; in particular we require $\overline{a} = \langle a_\alpha | \alpha < \gamma \rangle$ and $\overline{b} = \langle b_\alpha | \alpha < \gamma \rangle$ to be sequences of elements in the respective domains, indexed by the ordinal γ. Additionally, the index $i \in \{0, 1\}$ encodes the chirality of the game: if $i = 0$ then $\mathcal{M}$ is the first model, otherwise $\mathcal{N}$ is the first model.[4]

Possible moves and limit positions: We define the positions reached during the game by transfinite recursion on the ordinal representing the number of moves performed at that point of the run. The starting position is given at the start of the game.

Starting a round from a position $\langle \mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b}; h, \gamma, i \rangle$, the following are the possible moves:

$\exists$-move: S picks an element $a' \in D_\mathcal{M}$; D responds with an element $b' \in D_\mathcal{N}$; the game reaches position $\langle \mathcal{M}, s, \overline{a}a'; \mathcal{N}, t, \overline{b}b'; h, \gamma + 1, i \rangle$;

$\forall$-move: S picks an element $b' \in D_\mathcal{N}$; D responds with an element $a' \in D_\mathcal{M}$; the game reaches position $\langle \mathcal{M}, s, \overline{a}a'; \mathcal{N}, t, \overline{b}b'; h, \gamma + 1, i \rangle$;

$\to$-move: S picks a sub-state $t' \subseteq t$; D responds with a sub-state $s' \subseteq s$. After that, S decides a chirality, that is, which of the following positions the game reaches:

$$\langle \mathcal{M}, s', \overline{a};\ \mathcal{N}, t', \overline{b};\ h + 1, \gamma, i \rangle \qquad \langle \mathcal{M}, s', \overline{a};\ \mathcal{N}, t', \overline{b};\ h + 1, \gamma, 1 - i \rangle$$

Let θ be a limit ordinal. Suppose the positions encountered during the run of the game up to the ordinal θ are

$$\langle\ \langle \mathcal{M}, s_\alpha, \overline{a}_\alpha; \mathcal{N}, t_\alpha, \overline{b}_\alpha; h_\alpha, \gamma_\alpha, i_\alpha \rangle \mid \alpha < \theta\ \rangle$$

Then we say that the run has reached the *limit position*

$$\langle \mathcal{M}, s_\theta, \overline{a}_\theta; \mathcal{N}, t_\theta, \overline{b}_\theta; h_\theta, \theta, i_\theta \rangle$$

defined by the following clauses:

[4]Using an index to encode the chirality of the game is not as intuitive as using the position of the models in the tuple, but allows to simplify significantly the definition of limit position that follows.

- Since only a finite amount of $\rightarrow$-moves is allowed, the values of $s_\alpha, t_\alpha, i_\alpha$ are eventually constant in α; we define $s_\theta, t_\theta, i_\theta$ as these constant values.
- We define

$$\overline{a}_\theta = \bigcup_{\alpha<\theta} \overline{a}_\alpha \qquad\qquad \overline{b}_\theta = \bigcup_{\alpha<\theta} \overline{b}_\alpha$$

  Since $\langle \overline{a}_\alpha | \alpha < \theta \rangle$ and $\langle \overline{b}_\alpha | \alpha < \theta \rangle$ are pairwise coherent increasing sequences of sequences, then $\overline{a}_\theta$ and $\overline{b}_\theta$ are also sequences. Moreover, since only a finite amount of $\rightarrow$-moves is allowed, the sequences are indexed by θ.

Termination conditions: A number $i \in \mathbb{N}$ and an ordinal $\mathcal{Q} \in \mathrm{ORD}$ are fixed in advance and these are the highest values allowed for h and γ respectively during the run. In particular, a move is not allowed if performing it makes the game reach a position with $h > i$ or $\gamma > \mathcal{Q}$. When there are no more available moves, the game ends.

Winning condition: The game is won by Player D if the following condition is satisfied, and by player S otherwise:

Winning condition for D: for all atomic formulas $A(x_1, \ldots, x_n)$ and for every sequence of ordinals $\alpha_1, \ldots, \alpha_n < \mathcal{Q}$, we have:

$$\begin{cases} \mathcal{M}, s \vDash A\left(a_{\alpha_1}, \ldots, a_{\alpha_n}\right) \implies \mathcal{N}, t \vDash A\left(b_{\alpha_1}, \ldots, b_{\alpha_n}\right) & \text{if } i = 0 \\ \mathcal{N}, t \vDash A\left(b_{\alpha_1}, \ldots, b_{\alpha_n}\right) \implies \mathcal{M}, s \vDash A\left(a_{\alpha_1}, \ldots, a_{\alpha_n}\right) & \text{if } i = 1 \end{cases}$$

We will indicate with $\mathrm{EF}_{i,\mathcal{Q}}(\mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b})$ the game just described (where $i \in \mathbb{N}$, while $\mathcal{Q} \in \mathrm{ORD}$) starting from the initial position $\langle \mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b}; 0, 0, 0 \rangle$. We will indicate with $\mathcal{M}, s, \overline{a} \preceq_{i,\mathcal{Q}} \mathcal{N}, t, \overline{b}$ the existence of a winning strategy for Duplicator in the game $\mathrm{EF}_{i,\mathcal{Q}}(\mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b})$. We will also use notational conventions analogous to the ones introduced for the original game. Notice that these notations extend the ones presented for the original game.

As for the previous games, the existence of a winning strategy for Duplicator in the game $\mathrm{EF}_{i,\mathcal{Q}}(\mathcal{M}, \mathcal{N})$ gives us sensible information on the relation between the two models. For example, the submodel relation (Definition 3.2.3) is captured definable in terms of $\preceq_{i,\mathcal{Q}}$.

4.5.5. Theorem. *Let $\mathcal{M}$ and $\mathcal{N}$ be information models and define $\mathcal{Q} := |D^{\mathcal{M}}| + |D^{\mathcal{N}}| + 1$ (where $+$ denotes the usual cardinal sum). Then the following are equivalent:*

1. $\mathcal{M} \hookrightarrow \mathcal{N}$;
2. $\mathcal{N} \preceq_{2,\mathcal{Q}} \mathcal{M}$;

3. $\mathcal{N} \preceq_{h,\mathcal{Q}} \mathcal{M}$ *for every* $h \geq 1$.

Proof:
Firstly, suppose that Condition 1 holds, and let F, G be the functions that witness it. We describe here a winning strategy for Duplicator in the game $\mathrm{EF}_{h,\mathcal{Q}}(\mathcal{N}, \mathcal{M})$ for an arbitrary $h \in \mathbb{N} \setminus \{0\}$, thus showing that Conditions 2 and 3 follow:

- If Spoiler plays a quantifier move and picks an element a of $\mathcal{M}$, then Duplicator picks $G(a)$ of $\mathcal{N}$.
- If Spoiler plays a quantifier move and picks an element b of $\mathcal{N}$, then Duplicator picks a of $\mathcal{M}$ such that $G(a) \sim^{e}_{F[W^{\mathcal{M}}]} b$ (such an element exists by Condition Sub1 of Definition 3.2.3).
- The first time that Spoiler plays an implication move and picks $s \subseteq W^{\mathcal{M}}$, Duplicator responds by picking $t = F[s] \subseteq W^{\mathcal{N}}$.
- After the first implication move, whenever Spoiler performs an implication move and chooses $s \subseteq W^{\mathcal{M}}$ (resp. $t \subseteq W^{\mathcal{N}}$), then Duplicator picks $t \subseteq W^{\mathcal{N}}$ (resp. $s \subseteq W^{\mathcal{M}}$) so that the invariant $F(s) = t$ is maintained.

This is a winning strategy in the game since by definition of F, G:

$$\begin{array}{c} \forall w \in s.\ [\ \mathcal{M}, \{w\} \vDash A(d_1, \ldots, d_n) \iff \mathcal{N}, \{F(w)\} \vDash A(G(d_1), \ldots, G(d_n))\ ] \\ \Downarrow \\ \mathcal{M}, s \vDash A(d_1, \ldots, d_n) \iff \mathcal{M}, F(s) \vDash A(G(d_1), \ldots, G(d_n)) \end{array}$$

and since $h \geq 1$, we have that $F[s] = t$ and thus $\mathcal{N} \preceq_{h,\mathcal{Q}} \mathcal{M}$.

Since Condition 3 trivially implies Condition 2, then we just need to show that Condition 2 implies Condition 1. To show this, we will play the part of Spoiler in the game $\mathrm{EF}_{2,\mathcal{Q}}(\mathcal{N}, \mathcal{M})$ and use the winning strategy of Duplicator to build the functions f and g.

Consider enumeration $\langle\, a_\alpha \,|\, \alpha < |D^{\mathcal{M}}| \,\rangle$ of the elements of $\mathcal{M}$; and an enumeration $\langle\, b_\alpha \,|\, \alpha < |D^{\mathcal{N}}| \,\rangle$ of the elements of $\mathcal{N}$. Fix a winning strategy for Duplicator:

1. Firstly Spoiler performs $|D^{\mathcal{N}}|$ many $\exists$-moves: each time Spoiler chooses a distinct element b_α of $\mathcal{N}$, so that he covers the whole domain; define $H(b_\alpha) \in D^{\mathcal{M}}$ to be the element that Duplicator chooses in response.
2. Secondly Spoiler performs $|D^{\mathcal{M}}|$ many $\forall$-moves: each time Spoiler chooses a distinct element a_α of $\mathcal{M}$, so that he covers the whole domain; define $G(a_\alpha) \in D^{\mathcal{N}}$ to be the element that Duplicator chooses in response.

3. After the previous moves, Spoiler can perform an implication move and choose a singleton state $\{w\} \subseteq W^{\mathcal{M}}$. Following the winning strategy, Duplicator has to answer with a state $t \subseteq W^{\mathcal{N}}$ comprised only of worlds equivalent under the relation $\approx^{e}_{W^{\mathcal{N}}}$ (Definition 3.1.1).

   If this were not the case, then for a relation symbol R and some elements $d_1, \ldots, d_{\mathrm{Ar}(R)}$ we would have $\mathcal{N}, t \nvDash ?R(d_1, \ldots, d_{\mathrm{Ar}(R)})$, and consequently $\mathcal{N}, t \nvDash \forall\overline{x}.?R(\overline{x})$. However a direct verification yields $\mathcal{M}, \{w\} \vDash \forall\overline{x}.?R(\overline{x})$, and thus $\mathcal{M}, \{w\} \not\preceq_{1,1} \mathcal{N}, t$. This contradicts the assumption that Duplicator is employing a winning strategy.

   In the notation above, for every possible choice $w \in W^{\mathcal{M}}$ define $F(w)$ to be any element of $t \subseteq W^{\mathcal{N}}$, the state chosen by Duplicator following the strategy.

Now that we defined F and G, we need to show that they respect the properties of Definition 3.2.3. First of all, notice that for every world $w \in W^{\mathcal{M}}$, every sequence $\overline{a}$ of $D^{\mathcal{M}}$, every sequence $\overline{b}$ of $D^{\mathcal{N}}$ and every atomic formula $A(\overline{x})$ it holds that

$$\begin{array}{lcl} \mathcal{M}, \{w\} \vDash A(\overline{a}) & \iff & \mathcal{N}, \{F(w)\} \vDash A(G(\overline{a})) \\ \mathcal{M}, \{w\} \vDash A(H(\overline{b})) & \iff & \mathcal{N}, \{F(w)\} \vDash A(\overline{b}) \end{array}$$

In fact, after performing passage 3 of the run described above, we have by hypothesis that[5]

$$\begin{gathered} \mathcal{M}, \{w\}, \langle\, a_\alpha \mid \alpha < |D^{\mathcal{M}}|\,\rangle^\frown \langle\, H(b_\alpha) \mid \alpha < |D^{\mathcal{N}}|\,\rangle \\ \equiv_{1,1} \\ \mathcal{N}, \{F(w)\}, \langle\, G(a_\alpha) \mid \alpha < |D^{\mathcal{M}}|\,\rangle^\frown \langle\, b_\alpha \mid \alpha < |D^{\mathcal{N}}|\,\rangle \end{gathered}$$

So Condition Sub2 of Definition 3.2.3 is respected.

As for Condition Sub1, notice that by the equivalences above we have, for every $b \in D^{\mathcal{N}}$, $b \sim^{e}_{F[W^{\mathcal{M}}]} G(H(b))$. Thus the element $H(b) \in \mathcal{M}$ witnesses that Condition Sub1 holds.

□

4.6 Conclusions

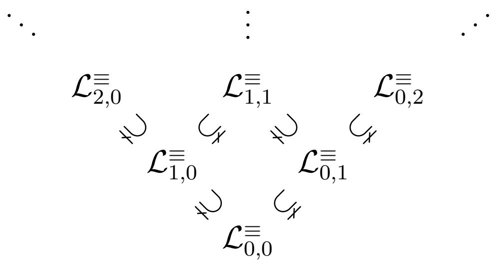
In this chapter we presented a generalization of the Ehrenfeucht-Fraïssé game in order to study the expressive power of InqBQ. Moreover, we proved that a suitable version of the *Ehrenfeucht-Fraïssé theorem* holds for this game (Theorem 4.3.2). One of the main differences between the new game and its classical counterpart is the novel notion of *chirality*: we are able to capture in game-theoretical terms

[5]With $\overline{x}^\frown\overline{y}$ we indicate the concatenation of the sequences $\overline{x}$ and $\overline{y}$.

not only when two models support the same formulas, but also when a model supports all the formulas supported by the other.

The game introduced is quite flexible and can be redesigned in several ways. One example is the *symmetric version* of the game, excluding chirality. Even though this version does not fully charactetrize support-equivalence between models (as shown in Example 4.5.4), it is a simpler game much easier to use in applications, as it will be showcased in Chapter 5. Another salient example is the *transfinite version* of the game, allowing for infinite runs indexed by an ordinal. This version can be used to capture other relations other than support-equivalence, as for example the submodel relation (Theorem 4.5.5).

The Ehrenfeucht-Fraïssé game proved to be a valuable tool to study the expressive power of the logic in several different contexts. One question that seems crucial to address is how the IQ degree captures the expressiveness of InqBQ. More concretely, if we call $\mathcal{L}^{\equiv}_{i,q}$ the set of formulas equivalent to formulas of IQdeg at most $\langle i, q\rangle$, a natural conjecture would be that $\mathcal{L}^{\equiv}_{i,q} \subseteq \mathcal{L}^{\equiv}_{i',q'}$ iff $\langle i, q\rangle \leq \langle i', q'\rangle$ and that the containments are all strict. In other terms, we are conjecturing that the IQdeg is a suitable indicator to distinguish the expressive power of InqBQ formulas. This issue seems natural to tackle using the Ehrenfeucht-Fraïssé games introduced in this chapter.



Another question which we leave for further work is whether the game can be modified to capture support-equivalence for interesting fragments of the logic or for other inquisitive logics. For example, in Chapter 7 we define and study a variation of the game that captures support-equivalence restricted to formulas of the *classical antecedent fragment*. It is still an open problem whether a suitable version of the game can be defined for the *mention-some* and *mention-all* fragments studied by Ciardelli [2016, Sections 4.7]; and the same question remains open for the *inquisitive logics of finite-width* and the *bounded-width fragment* introduced in this dissertation (Chapter 8).

Chapter 5

Cardinality Quantifiers

In this chapter we will use the EF-game for InqBQ to study in detail what InqBQ can express about the number of individuals satisfying a predicate P. The sentences we are concerned with include not only statements about the number of individuals satisfying P, like those in (1), but also questions about the number of individuals satisfying P, like those in (2).

(1) a. There is no P.
b. There are at least three P.
c. The number of P is even.
d. There are infinitely many P.

(2) a. Are there any P?
b. How many P are there?
c. Is the number of P even, or odd?
d. Are there infinitely many P?

Which among the statements in (1) and the questions in (2) can be expressed in InqBQ? Instead of pursuing a direct answer to this question, we will tackle the problem from a more general perspective. We will see that, in an inquisitive setting, all these sentences instantiate the form $Qx.P(x)$, where Q is a quantifier which is sensitive only to the cardinality of its argument. Thus—interestingly—in the inquisitive setting, not only *no* and *at least three*, but also *how many* can be viewed as generalized quantifiers. We can then ask which cardinality quantifers are expressible in InqBQ. In this chapter, we will establish a simple answer to this question. From this answer, a verdict about the definability of the examples above, as well as many other similar examples, can be easily reached.

We will first look at cardinality quantifiers in the setting of standard first order logic, CQC, and recall the characterization of cardinality quantifiers expressible in CQC; we will then present a generalization of the notion of a cardinality quantifier to InqBQ, which encompasses also inquisitive quantifiers like *how many*; finally, we will use the Ehrenfeucht-Fraïssé game introduced in Chapter 4 to provide a characterization of the cardinality quantifiers expressible in InqBQ, and use this characterization to show that, just like many interesting statements about cardinalities are not expressible in CQC, many interesting questions about cardinalities are not expressible in InqBQ.

5.1 Cardinality Quantifiers in Classical First Order Logic

In classical logic, a formula $\alpha(x)$, with at most the variable x free, determines, relative to a model M, a corresponding set of individuals:

$$\alpha_M := \{d \in D \mid M \vDash \alpha(d)\}$$

Recall that the elements of α_M are divided in equivalence classes under the relation $\sim^e$ (Definition 3.1.2).[1] Since $\sim^e$ acts as an intensional equality for the model, the natural way to define how many individuals satisfy α is to consider the *essential cadinality* of the set α_M.

$$|\alpha_M|_e^M := |\ \{[d]_{\sim^e} \mid d \in \alpha_M\}\ |$$

Notice that the values that $|\alpha_M|_e^M$ depend on the language and the signature we consider: for example, in the language $\mathcal{L}_{\neq}$ and in the empty signature $\Sigma = \{\emptyset\}$ all the formulas of InqBQ are equivalent to either $\top$ or $\bot$, and so we have either $|\alpha_M|_e^M = 0$ or $|\alpha_M| = |D^e|$. To avoid this issue we restrict our attention to the syntaxes $\mathcal{L}_=$ and $\mathcal{L}_{\asymp}$—for which there exists models of every cardinality in every signature—and leave the case of the syntax $\mathcal{L}_{\neq}$ for future work.

Let K be a class of cardinals. This is an operator that can be added to classical first order logic by stipulating that if $\alpha(x)$ is a classical formula with at most x free, then $Q_K x.\alpha(x)$ is a formula, with the following semantics:[2]

$$M \vDash Q_K x.\alpha(x) \iff |\alpha_M|_e^M \in K$$

By a cardinality quantifier we mean a quantifier which is of the form Q_K for some class of cardinals K. Notice that the existential quantifier $\exists$ is a cardinality quantifier, since $\exists = Q_{\mathrm{CARD}\setminus\{0\}}$, for CARD the class of all cardinals. However,

[1] Definition 3.1.2 considers essential equivalence between elements of information models and not CQC-models. However we can extend the definition to CQC-models under the identification of a CQC-model M with the singleton information model $\{M\}$.

[2] One can, more generally, allow the formation of the formula $Q_K x.\alpha$ for any formula α, even when α contains free variables besides x. Extending the semantic clause to this case is straightforward: we just have to relativize the clause to an assignment function g. However, we restrict to the case in which $Q_K x.\alpha$ is a sentence, since this does not lead to a loss of generality and it is convenient not to have assignments around all the time.

the universal quantifier $\forall$ is not a cardinality quantifier, since the condition $M \vDash \forall x.P(x)$, namely, $P_M = D$, cannot be formulated solely in terms of $|P_M|_e^M$.[3]

Let $\chi_K[P]$ be a CQC-formula (thus, not containing Q_K). We say that $\chi_K[P]$ defines Q_K if $Q_K x.P(x) \equiv \chi_K[P]$, that is, if the two formulas are satisfied by the same CQC-models. If this is the case, then for every formula $\alpha(x)$ we have $Q_K x.\alpha(x) \equiv \chi_K[\alpha]$. We say that the quantifier Q_K is definable in CQC if there is a CQC-formula which defines it.

The statements in (1) can all be seen as having the form $Qx.P(x)$, where Q is a cardinality quantifier. Indeed, we have the following characterizations, where $[3, \dots)$ is the class of cardinals ≥ 3; Even is the set of even natural numbers; Inf is the class of infinite cardinals.

(3) a. $M \vDash$ (1-a) $\iff P_M = \emptyset \iff |P_M|_e^M \in \{0\}$
b. $M \vDash$ (1-b) $\iff |P_M|_e^M \geq 3 \iff |P_M|_e^M \in [3, \dots)$
c. $M \vDash$ (1-c) $\iff |P_M|_e^M$ is even $\iff |P_M|_e^M \in \mathsf{Even}$
d. $M \vDash$ (1-d) $\iff |P_M|_e^M$ is infinite $\iff |P_M|_e^M \in \mathsf{Inf}$

What cardinality quantifiers are definable in classical first order logic? That is, for what classes K of cardinals is the quantifier Q_K definable? The answer is given by the following theorem, which can be proved using the Erenfeucht-Fraïssé games for CQC (and seems, to the best of our knowledge, to be folklore).

5.1.1. THEOREM. *Let K be a class of cardinals. The quantifier Q_K is definable in first order logic if and only if there exists a natural number n such that K contains either all or none of the cardinals $\kappa \geq n$.*

Consider again the statements in (1), repeated below for convenience with the corresponding classes of cardinals given on the right. It follows immediately from the characterization that statements (1-a) and (1-b) are expressible in classical first order logic, while statements (1-c) and (1-d) are not.

(4)	a.	There is no P.	$K = \{0\}$
	b.	There are at least three P.	$K = [3, \dots)$
	c.	The number of P is even.	$K = \mathsf{Even}$
	d.	There are infinitely many P.	$K = \mathsf{Inf}$

[3] In this chapter, we focus on cardinality quantifiers of type $\langle 1 \rangle$, which operate on a single unary predicate. More generally, one could consider cardinality quantifiers of type $\langle n_1, \dots, n_k \rangle$, which operate on k predicates of arities $n_1, \dots, n_k$ respectively. It seems quite possible that the characterization result given here can be extended to this general setting. However, we leave this extension for future work.

5.2 Cardinality Quantifiers in InqBQ

Let us now turn to the inquisitive case. An information model $\mathcal{M}$ represents a variety of states of affairs, one for each possible world w. At each world w, the state of affairs is represented by the first order structure M_w, having domain D.

Let $\alpha(x)$ be a classical formula with at most the variable x free. Relative to each world w, $\alpha(x)$ determines an extension α_w, which is a set of individuals from D:

$$\alpha_w := \{d \in D \mid M_w \vDash \alpha(d)\}$$

Each one of these extensions has an associated essential cardinality dependent on the set of individuals α_w and on the model M_w, which we will henceforth indicate with $|\alpha|_w^{\mathcal{M}} := |\alpha_w|_e^{M_w}$ (or simply $|\alpha|_w$ if the model is clear from the context).

Therefore, relative to an information state s, the formula $\alpha(x)$ determines a corresponding set of cardinals, $\{|\alpha|_w \mid w \in s\}$. We refer to this set of cardinals as the *cardinality trace* of $\alpha(x)$ in s.

5.2.1. Definition (Cardinality trace). Let $\mathcal{M}$ be an information model, s be an information state, and $\alpha(x)$ be a classical formula where at most the variable x occurs free. The cardinality trace of $\alpha(x)$ in s is the set of cardinals:

$$\mathrm{tr}_s(\alpha) = \{\ |\alpha|_w \mid w \in s\ \}$$

A cardinal κ is in $\mathrm{tr}_s(\alpha)$ if, according to the information available in s, κ might be the number of elements satisfying $\alpha(x)$; that is, if it might be the case that the extension of $\alpha(x)$ has cardinality κ. Thus, $\mathrm{tr}_s(\alpha)$ captures exactly the information available in s about the number of individuals satisfying $\alpha(x)$.

Now let $\mathcal{K}$ be a class of sets of cardinals. We associate with $\mathcal{K}$ a corresponding quantifier $Q_{\mathcal{K}}$. We can add this quantifier to InqBQ by stipulating that if $\alpha(x)$ is a classical formula with at most x free, then $Q_{\mathcal{K}}x.\alpha(x)$ is a formula, interpreted by the following clause:[4]

$$M, s \vDash Q_{\mathcal{K}}x.\alpha(x) \iff \mathrm{tr}_s(\alpha) \in \mathcal{K}$$

A cardinality quantifier is a quantifier which is of the form $Q_{\mathcal{K}}$, where $\mathcal{K}$ is a class of sets of cardinals.

Let $\chi_{\mathcal{K}}[P]$ be an InqBQ-formula (thus, without cardinality quantifiers). We say that $\chi_{\mathcal{K}}[P]$ defines the quantifier $Q_{\mathcal{K}}$ if $Q_{\mathcal{K}}x.P(x) \equiv \chi_{\mathcal{K}}[P]$, that is, the two formulas are supported by the same information models. Again, it is not hard to see that if this holds, then for every classical formula $\alpha(x)$ we have

[4]The reason for restricting the application of Q_K to classical formulas is that $Q_Kx.\alpha(x)$ only looks at the semantics of α with respect to worlds. Non-classical formulas only become significant when interpreted relative to information states; relative to single worlds, the operators $\vee\!\!\!\vee$ and $\overline{\exists}$ collapse on their classical counterparts $\vee$ and $\exists$. Therefore, while extending our quantifiers to operate on non-classical formulas is not problematic, it is also not interesting.

$Q_{\mathcal{K}}x.\alpha(x) \equiv \chi_{\mathcal{K}}[\alpha]$. We say that $Q_{\mathcal{K}}$ is definable in InqBQ if there is an InqBQ-formula that defines it.

In order to make the notion of a cardinality quantifier more concrete, let us see how the statements in (1) and the questions in (2) can be seen as instantiating the form $Qx.P(x)$ where Q is a cardinality quantifier in the sense of inquisitive logic.

Consider first the statements in (1). In general, in inquisitive semantics a statement α is supported by a state s iff the information available in s implies that α is true. This means that α is true at all worlds $w \in s$. Keeping this in mind, we can see that the statements in (1) have the following semantics:

(5) a. $M, s \models \text{(1-a)} \iff \forall w \in s : P_w = \emptyset \iff \text{tr}_s(P) \subseteq \{0\}$
b. $M, s \models \text{(1-b)} \iff \forall w \in s : |P|_w \geq 3 \iff \text{tr}_s(P) \subseteq [3, \dots)$
c. $M, s \models \text{(1-c)} \iff \forall w \in s : |P|_w \text{ is even} \iff \text{tr}_s(P) \subseteq \mathsf{Even}$
d. $M, s \models \text{(1-d)} \iff \forall w \in s : |P|_w \text{ is infinite} \iff \text{tr}_s(P) \subseteq \mathsf{Inf}$

Let us now check that all these statements correspond to statements of the form $Qx.P(x)$ for Q a cardinality quantifier. For this, we introduce a useful notation.

5.2.2. Definition (Downward closure of a class). Let K be a class. We denote by $K^{\downarrow}$ the class consisting of all sets X such that $X \subseteq K$.

Thus, if K is a set, then $K^{\downarrow} = \wp(K)$. However, if K is a proper class, then $K^{\downarrow}$ will not be a set either; moreover, $K^{\downarrow}$ will not contain K, since K is not a set.

Now consider the cardinality quantifiers Q_1–Q_4 determined by the following classes:

$$\mathcal{K}_1 = \{0\}^{\downarrow} \qquad \mathcal{K}_2 = [3, \dots)^{\downarrow} \qquad \mathcal{K}_3 = \mathsf{Even}^{\downarrow} \qquad \mathcal{K}_4 = \mathsf{Inf}^{\downarrow}$$

We have:

$$\begin{array}{llllll}
\mathcal{M}, s \models Q_1x.P(x) & \iff \text{tr}_s(P) \in \mathcal{K}_1 & \iff \text{tr}_s(P) \subseteq \{0\} & \iff \mathcal{M}, s \models \text{(1-a)} \\
\mathcal{M}, s \models Q_2x.P(x) & \iff \text{tr}_s(P) \in \mathcal{K}_2 & \iff \text{tr}_s(P) \subseteq [3, \dots) & \iff \mathcal{M}, s \models \text{(1-b)} \\
\mathcal{M}, s \models Q_3x.P(x) & \iff \text{tr}_s(P) \in \mathcal{K}_3 & \iff \text{tr}_s(P) \subseteq \mathsf{Even} & \iff \mathcal{M}, s \models \text{(1-c)} \\
\mathcal{M}, s \models Q_4x.P(x) & \iff \text{tr}_s(P) \in \mathcal{K}_4 & \iff \text{tr}_s(P) \subseteq \mathsf{Inf} & \iff \mathcal{M}, s \models \text{(1-d)}
\end{array}$$

Next, consider the questions in (2). Start with (2-a), the question whether there are any P. This question is settled in in an information state s in case the information in s implies that there are no P, or it implies that there are some P. The former is the case if the extension of P is empty in all worlds $w \in s$. The latter is the case if the extension of P is non-empty in all worlds $w \in s$. Thus, the semantics of (2-a) is as follows.

(6)
$$\begin{aligned}
\mathcal{M}, s \models \text{(2-a)} &\iff (\forall w \in W : P_w = \emptyset) \text{ or } (\forall w \in W : P_w \neq \emptyset) \\
&\iff (\forall w \in W : |P|_w = 0) \text{ or } (\forall w \in W : |P|_w \geq 1) \\
&\iff \text{tr}_s(P) = \{0\} \text{ or } \text{tr}_s(P) \subseteq [1, \dots)
\end{aligned}$$

Second, consider the question (2-b), how many individuals are P. This question is settled in an information state s if the information available in s determines exactly how many individuals are P. This is the case if there is a cardinal κ such that at every world $w \in s$, the extension P_w contains κ elements.[5]

(7)
$$\begin{aligned} \mathcal{M}, s \models \text{(2-b)} &\iff \exists\kappa.\,\forall w \in W.\ |P|_w = \kappa \\ &\iff \mathrm{tr}_s(P) \text{ contains at most one element} \\ &\iff \mathrm{tr}_s(P) \subseteq \{\kappa\} \text{ for some cardinal } \kappa \end{aligned}$$

Next, consider (2-c), the question whether the number of P is even or odd. This is settled in an information state s in case the information available in s implies that the number of P is even, or that the number of P is odd.[6] The former holds if the extension of P is even at every world in s. The latter holds if the extension of P is odd al every world in s.

(8)
$$\begin{aligned} \mathcal{M}, s \models \text{(2-c)} &\iff (\forall w \in W : |P|_w \text{ is even}) \text{ or } (\forall w \in W : |P|_w \text{ is odd}) \\ &\iff \mathrm{tr}_s(P) \subseteq \mathsf{Even} \text{ or } \mathrm{tr}_s(P) \subseteq \mathsf{Odd} \end{aligned}$$

Finally, consider (2-d), the question whether there are infinitely many P. This is settled in an information state s in case the information available in s implies that there are infinitely many P, or it implies that the aren't infinitely many P. The former is the case if the extension of P is infinite at every world $w \in s$, while the latter is the case if the extension of P is finite at every world $w \in s$.

(9)
$$\begin{aligned} \mathcal{M}, s \models \text{(2-d)} &\iff (\forall w \in s : |P|_w \text{ is finite}) \text{ or } (\forall w \in s : |P|_w \text{ is infinite}) \\ &\iff \mathrm{tr}_s(P) \subseteq \mathsf{Fin} \text{ or } \mathrm{tr}_s(P) \subseteq \mathsf{Inf} \end{aligned}$$

Now consider four cardinality quantifiers, Q_5–Q_8, determined by the following classes:

(10)
a. $\mathcal{K}_5 = \{0\}^{\downarrow} \cup [1, \dots)^{\downarrow}$
b. $\mathcal{K}_6 = \bigcup\{\{\kappa\}^{\downarrow} \mid \kappa \text{ a cardinal}\}$
c. $\mathcal{K}_7 = \mathsf{Even}^{\downarrow} \cup \mathsf{Odd}^{\downarrow}$
d. $\mathcal{K}_8 = \mathsf{Fin}^{\downarrow} \cup \mathsf{Inf}^{\downarrow}$

Then we have:

$$\begin{aligned} \mathcal{M}, s \models Q_5 x.P(x) \iff \mathrm{tr}_s(P) \in \mathcal{K}_5 &\iff \mathrm{tr}_s(P) \subseteq \{0\} \text{ or } \mathrm{tr}_s(P) \subseteq [1, \dots) \\ &\iff \mathcal{M}, s \models \text{(2-a)} \end{aligned}$$

$$\mathcal{M}, s \models Q_6 x.P(x) \iff \mathrm{tr}_s(P) \in \mathcal{K}_6 \iff \mathrm{tr}_s(P) \subseteq \{\kappa\} \text{ for some } \kappa$$

[5]An equivalent way of formulating the same condition is to say that (2-b) is settled in s iff the number of P is the same at all the worlds in s: $M, s \models \text{(2-b)} \iff \forall w, w' \in s : \#P_w = \#P_{w'}$.

[6]Notice that the question presupposes that the number of P is either even or odd. Since all and only the finite cardinals are even or odd, the question presupposes that the number of P is finite. About the way presuppositions of questions are interpreted in inquisitive logic, see [Ciardelli, 2016, Section 1.3].

$$\begin{aligned} & & & \iff \mathcal{M}, s \vDash \text{(2-b)} \\ \mathcal{M}, s \vDash Q_7 x.P(x) & \iff & \mathrm{tr}_s(P) \in \mathcal{K}_7 & \iff \mathrm{tr}_s(P) \subseteq \mathsf{Even} \text{ or } \mathrm{tr}_s(P) \subseteq \mathsf{Odd} \\ & & & \iff \mathcal{M}, s \vDash \text{(2-c)} \\ \mathcal{M}, s \vDash Q_8 x.P(x) & \iff & \mathrm{tr}_s(P) \in \mathcal{K}_8 & \iff \mathrm{tr}_s(P) \subseteq \mathsf{Fin} \text{ or } \mathrm{tr}_s(P) \subseteq \mathsf{Inf} \\ & & & \iff \mathcal{M}, s \vDash \text{(2-d)} \end{aligned}$$

So, in the inquisitive setting, a new range of "inquisitive" cardinality quantifiers come into play, which combine with a property to yield questions like those exemplified in (2). In addition to standard cardinality quantifiers like '*no*', '*at least three*', '*infinitely many*', we also have new, question-forming cardinality quantifiers like '*how many*' and '*whether finitely or infinitely many*'.

5.3 Characterization

What cardinality quantifiers can be expressed in InqBQ? Given that, in the inquisitive setting, cardinality quantifiers are in one-to-one correspondence with classes of sets of cardinals, this question can be made precise as follows.

5.3.1. QUESTION. For which classes of sets of cardinals $\mathcal{K}$ is the quantifier $Q_{\mathcal{K}}$ definable in InqBQ?

The next theorem provides an answer to this question. In essence, what the theorem says is that the cardinality quantifiers definable in InqBQ are all and only the inquisitive disjunctions of cardinality quantifiers definable in classical first order logic.[7] Before stating the Theorem, let us fix some useful notations. For any natural number n, we let:

- $[0, n] := \{m \in \mathrm{CARD} \mid m \leq n\}$
- $[n, \dots) := \{\kappa \in \mathrm{CARD} \mid \kappa \geq n\}$

Moreover, we introduce an equivalence relation $=_n$ that disregards differences between cardinals larger than n. More precisely, if κ and κ' are two cardinals:

- $\kappa =_n \kappa' \iff \kappa = \kappa'$ or $\kappa, \kappa' > n$

If A and B are sets of cardinals, we write $A =_n B$ if A and B are the same set, modulo identifying all cardinals larger than n:

[7]While we have not specified a general notion of inquisitive generalized quantifier here, a natural notion should allow as an instance the quantifier Q_0 whose semantics is given by: $\mathcal{M}, s \vDash Q_0 x.P(x) \iff \forall w, w' \in s : P_w = P_{w'}$. Informally, $Q_0 x.P(x)$ expresses the question "*which elements are P?*". Now this quantifier is definable in InqBQ by the formula $\forall x.?P(x)$, which is clearly not equivalent to an inquisitive disjunction of classical formulas. This shows that the characterization in Theorem 5.3.2 is really specific for cardinality quantifiers.

- $A =_n B \iff \forall\kappa \in A.\exists\kappa' \in B.$ s.t. $\kappa =_n \kappa'$ and
  $\forall\kappa' \in B.\exists\kappa \in A.$ s.t. $\kappa =_n \kappa'$

Moreover, we say that a class of sets of cardinals $\mathcal{K}$ is:

- $=_n$-invariant, if whenever $B \in \mathcal{K}$ and $A =_n B$ we have $A \in \mathcal{K}$;
- downward-closed, if whenever $B \in \mathcal{K}$ and $A \subseteq B$ we have $A \in \mathcal{K}$.

We can now state our main result.

5.3.2. Theorem (Characterization of cardinality quantifiers definable in InqBQ). *Let $\mathcal{K}$ be a class of sets of cardinals. The following are equivalent:*

1. *The cardinality quantifier $Q_{\mathcal{K}}$ is definable in* InqBQ.
2. $\mathcal{K} = K_1^{\downarrow} \cup \cdots \cup K_n^{\downarrow}$ *where for each* $K_i \subseteq$ CARD *there exists a natural number m such that K_i contains either all or none of the cardinals $\kappa \geq m$.*
3. *$\mathcal{K}$ is downward closed and $=_m$-invariant for some natural number m.*

In the proof we will assume to be working with the syntax $\mathcal{L}_{\asymp}$. However, since all the models employed satisfy for every world the condition $d \asymp_w d'$ iff $d = d'$, the same proof applies also to the syntax $\mathcal{L}_{=}$.

Proof:

We show that $2 \Rightarrow 1 \Rightarrow 3 \Rightarrow 2$.

[2 ⇒ 1] Suppose $\mathcal{K} = K_1^{\downarrow} \cup \cdots \cup K_n^{\downarrow}$ where for each K_i there exists a natural number m such that K_i contains either all or none of the cardinals $\kappa \geq m$. By Theorem 5.1.1, for each K_i we have a classical formula χ_i such that, in classical first order logic:

$$M \vDash \chi_i \iff |P_M|_e^M \in K_i$$

These formulas are also formulas of InqBQ, and it follows from Theorem 2.1.18 that we have:

$$\begin{aligned} \mathcal{M}, s \vDash \chi_i &\iff \forall w \in s : M_w \vDash \chi_i \\ &\iff \forall w \in s : |P|_w \in K_i \\ &\iff \mathrm{tr}_s(P) \subseteq K_i \\ &\iff \mathrm{tr}_s(P) \in K_i^{\downarrow} \end{aligned}$$

Now consider the inquisitive disjunction $\chi_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \chi_n$. We have:

$$\begin{aligned} \mathcal{M}, s \vDash \chi_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \chi_n &\iff \mathcal{M}, s \vDash \chi_1 \text{ or } \ldots \text{ or } \mathcal{M}, s \vDash \chi_n \\ &\iff \mathrm{tr}_s(P) \in K_1^{\downarrow} \text{ or } \ldots \text{ or } \mathrm{tr}_s(P) \in K_n^{\downarrow} \\ &\iff \mathrm{tr}_s(P) \in K_1^{\downarrow} \cup \cdots \cup K_n^{\downarrow} \\ &\iff \mathrm{tr}_s(P) \in \mathcal{K} \\ &\iff \mathcal{M}, s \vDash Q_{\mathcal{K}} x.P(x) \end{aligned}$$

This shows that the InqBQ formula $\chi_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \chi_n$ defines the quantifier $Q_\mathcal{K}$.

[1 ⇒ 3] Next, consider the implication from 1 to 3. Suppose $Q_\mathcal{K}$ is definable in InqBQ by a formula $\varphi_\mathcal{K}$. We need to show that $\mathcal{K}$ is downward closed and $=_m$-invariant for some natural number m.

We firstly show that $\mathcal{K}$ is downward closed. Suppose $A \subseteq B \in \mathcal{K}$. This means that there exists a model $\mathcal{M}$ and an info state s such that $\mathcal{M}, s \vDash \varphi_\mathcal{K}$ and $\mathrm{tr}_s(P) = B$. Consider now the state $t := \{w \in s \mid |P|_w \in A\} \subseteq s$. By definition we have $\mathrm{tr}_t(P) = A$; and by persistency $\mathcal{M}, t \vDash \varphi_\mathcal{K}$. Thus $A \in \mathcal{K}$.

Next, we show that $\mathcal{K}$ is closed under $=_m$ for some m. We want to show that the condition above holds for $m = q$, where q is the quantifier degree of the defining formula $\varphi_\mathcal{K}$. So, suppose $A \in \mathcal{K}$ and $A =_q B$. If we find two information models $\mathcal{M}, \mathcal{N}$ such that $\mathrm{tr}_{W^\mathcal{M}}(P) = A$, $\mathrm{tr}_{W^\mathcal{N}}(P) = B$ and $\mathcal{M} \approx_{i,q} \mathcal{N}$ then we are done, since in this case:

$$A \in \mathcal{K} \iff \mathcal{M} \vDash \varphi_\mathcal{K} \iff \mathcal{N} \vDash \varphi_\mathcal{K} \iff B \in \mathcal{K}$$

Consider enumerations of the sets A and B:[8] $A := \{\kappa_\alpha \mid \alpha < \lambda\}$ and $B := \{\kappa'_\alpha \mid \alpha < \lambda\}$ which both start with the same initial sequence $\langle \kappa_1, \ldots, \kappa_l \rangle = \langle \kappa'_1, \ldots, \kappa'_l \rangle$ enumerating $A \cap [0, q] = B \cap [0, q]$. Let $\mathcal{M}, \mathcal{N}$ be the models defined by the following clauses:

$W^\mathcal{M} := \{w_\alpha \mid \alpha < \lambda\}$	$W^\mathcal{N} := \{w_\alpha \mid \alpha < \lambda\} = W^\mathcal{M}$
$D^\mathcal{M} := \left\{d^\alpha_\beta \mid \alpha < \lambda \text{ and } \beta < \kappa_\alpha\right\}$	$D^\mathcal{N} := \left\{e^\alpha_\beta \mid \alpha < \lambda \text{ and } \beta < \kappa'_\alpha\right\}$
$P^\mathcal{M}_{w_\gamma}\left(d^\alpha_\beta\right) \iff \alpha = \gamma$	$P^\mathcal{N}_{w_\gamma}\left(e^\alpha_\beta\right) \iff \alpha = \gamma$
$d \asymp^\mathcal{M}_{w_\gamma} d'$ iff $d = d'$	$e \asymp^\mathcal{N}_{w_\gamma} e'$ iff $e = e'$

An example of these models is given in Figure 5.1. Notice that $|P|^\mathcal{M}_{w_\alpha} = \kappa_\alpha$ and $|P|^\mathcal{N}_{w_\alpha} = \kappa'_\alpha$. In particular, it follows that $\mathrm{tr}_{W^\mathcal{M}}(P) = A$ and $\mathrm{tr}_{W^\mathcal{N}}(P) = B$. So if we show that $\mathcal{M} \approx_{i,q} \mathcal{N}$, then we are done. In order to show this, we present here a winning strategy for Duplicator in the symmetric version of the EF-game between $\mathcal{M}$ and $\mathcal{N}$ (see Subsection 4.5.1):

- If Spoiler plays an implication move and chooses an information state s from either of the models, then Duplicator responds by choosing the same state s from the other model (this is possible since $W^\mathcal{M} = W^\mathcal{N}$);
- If Spoiler plays a quantifier move and chooses an element d^α_β from the model $\mathcal{M}$, we consider two separate cases:

[8] In the enumerations, we allow for repetitions of the same elements with different indices. This allows us to use the same cardinal λ as the set of indices for both sets A and B.

$\mathcal{M}$

	w_0	w_1	w_2
d_0^0	•	×	×
d_1^0	•	×	×
d_0^1	×	•	×
d_1^1	×	•	×
d_2^1	×	•	×
d_0^2	×	×	•
d_1^2	×	×	•
d_2^2	×	×	•
d_3^2	×	×	•
d_4^2	×	×	•

$\mathcal{N}$

	w_0	w_1	w_2
e_0^0	•	×	×
e_1^0	•	×	×
e_0^1	×	•	×
e_1^1	×	•	×
e_2^1	×	•	×
e_3^1	×	•	×
e_0^2	×	×	•
e_1^2	×	×	•
e_2^2	×	×	•
e_3^2	×	×	•

Figure 5.1: Suppose $q = 2$, and consider the sets $A = \{2, 3, 5\}$ and $B = \{2, 4\}$. Notice that $A =_2 B$. We enumerate these sets as $\langle 2, 3, 5\rangle$ and $\langle 2, 4, 4\rangle$. The figure shows the models $\mathcal{M}$ and $\mathcal{N}$ derived from this enumeration. These models are indistinguishable in the EF-game with only 2 quantifier moves, regardless of the number of implication moves.

- If $d_\beta^\alpha = a_i$ for some i, that is, it has already been picked during the run—by either Spoiler or Duplicator—then Duplicator responds by choosing b_i;
- If d_β^α has not been previously picked, then Duplicator chooses an element e_γ^α (notice that the elements have the same superscript and possibly different subscripts) which has not been previously picked during the run. The fact that duplicator can find such an element is guaranteed by $A =_q B$: this means that either $\kappa_\alpha = \kappa'_\alpha$, or else $\kappa_\alpha, \kappa'_\alpha > q$. In the former case the number of elements d_β^α and e_γ^α is exactly the same; in the latter case the number of elements e_γ^α is larger than the number of quantifier moves in the game.

• If Spoiler plays a quantifier move and chooses an element e_β^α from the model $\mathcal{N}$, then Duplicator applies the same strategy as in the previous case, swapping the roles of the models $\mathcal{M}$ and $\mathcal{N}$.

Notice that with this strategy Duplicator ensures that at the end of the run the final position:

1. has the same state s for both models;
2. $a_i = a_j$ if and only if $b_i = b_j$;

3. corresponding elements a_i, b_i in the two models have the same superscripts, that is, a_i and b_i are of the form d^α_β and e^α_β respectively.

This is indeed a winning strategy, since:

$$\mathcal{M}, s \models P(d^\alpha_\beta) \iff s \subseteq \{\alpha\} \iff \mathcal{N}, s \models P(e^\alpha_\gamma)$$
$$\mathcal{M}, s \models a_i = a_j \iff \mathcal{N}, s \models b_i = b_j$$

[3 ⇒ 2] Suppose $\mathcal{K}$ is downward closed and $=_m$-invariant for some number m. Let $A_1, \dots, A_n$ be the subsets of $[0, m+1]$ which are contained in $\mathcal{K}$. Now define:

$$K_i = \begin{cases} A_i & \text{if } m+1 \notin A_i \\ A_i \cup [m+1, \dots) & \text{if } m+1 \in A_i \end{cases}$$

We claim that $\mathcal{K} = K_1^\downarrow \cup \dots \cup K_n^\downarrow$. Start with the right-to-left inclusion. Let $B \in K_1^\downarrow \cup \dots \cup K_n^\downarrow$. This means that $B \subseteq K_i$ for some i. Now we distinguish two cases.

- Case 1: $K_i = A_i$. Then $A_i \in \mathcal{K}$ by definition, and since $\mathcal{K}$ is downward closed, also $B \in \mathcal{K}$.

- Case 2: $K_i = A_i \cup [m+1, \dots)$. We claim that in this case, $A_i =_m A_i \cup B$: if so, since $A_i \in \mathcal{K}$, and $\mathcal{K}$ is $=_m$-invariant, we have $A_i \cup B \in \mathcal{K}$, which in turn by downward closure yields $B \in \mathcal{K}$. To see that $A_i =_n A_i \cup B$, the only non-trivial step is to show that for all $\kappa \in B$ there exists some $\kappa' \in A_i$ such that $\kappa' =_m \kappa$. So, take $\kappa \in B$: if $\kappa \leq m$ then $\kappa \in A_i$ (since $B \subseteq K_i = A_i \cup [m+1, \dots)$), so we can take $\kappa' = \kappa$; if on the other hand $\kappa > m$, then $\kappa =_m m+1 \in A_i$.

Either way, we conclude $B \in \mathcal{K}$, which gives the right-to-left inclusion.

For the converse inclusion, suppose $B \in \mathcal{K}$. Again, we distinguish two cases.

- Case 1: $B \subseteq [0, m]$. In this case, $B = A_i$ for some $i \leq n$, and thus $B \in K_i^\downarrow$.

- Case 2: $B \ni \kappa$ for some $\kappa > m$. In this case, $B =_m B \cup \{m+1\}$, since $\kappa =_m m+1$. Since $B \in \mathcal{K}$ and $\mathcal{K}$ is $=_m$-invariant, also $B \cup \{m+1\} \in \mathcal{K}$. Now take $(B \cup \{m+1\}) \cap [0, m+1]$: by downward closure, this set is in $\mathcal{K}$, and since it is a subset of $[0, m+1]$, it coincides with A_i for some $i \leq n$. Notice that $m+1 \in A_i$, and thus, $K_i = A_i \cup [m+1, \dots)$. Therefore, $B \subseteq K_i$, which implies $B \in K_i^\downarrow$.

In either case, we conclude that $B \in K_i$ for some $i \leq n$, which gives the left-to-right inclusion.

□

Theorem 5.3.2 allows us to tell immediately which among the questions in (2) are expressible in InqBQ: (2-a), the question whether there is any P, is expressible, since it has the form $Q_{\mathcal{K}}x.P(x)$ for the class $\mathcal{K}_5 = \{0\}^{\downarrow} \cup [1,\dots)^{\downarrow}$, where both $\{0\}$ and $[1,\dots)$ are definable in classical first order logic. Indeed, the defining formula is simply $?\exists x.P(x)$, which abbreviates $\exists x.P(x) \vee\!\!\!\vee \neg\exists x.P(x)$.

The remaining questions, (2-b), (2-c), and (2-d) are not expressible, since they have the form $Q_{\mathcal{K}}x.P(x)$ for the following classes $\mathcal{K}$:

$$\mathcal{K}_6 = \bigcup\{\{\kappa\}^{\downarrow} \mid \kappa \text{ a cardinal}\} \qquad \mathcal{K}_7 = \mathsf{Even}^{\downarrow} \cup \mathsf{Odd}^{\downarrow} \qquad \mathcal{K}_8 = \mathsf{Fin}^{\downarrow} \cup \mathsf{Inf}^{\downarrow}$$

Clearly, these classes are not the form $K_1^{\downarrow} \cup \dots \cup K_n^{\downarrow}$ for $K_1, \dots, K_n$ definable in classical first order logic. In a similar way, we can see that none of the following questions about the cardinality of P is expressible in InqBQ.

(11) a. How many P are there, modulo k? for $k \geq 2$
b. Is the number of P even, odd, or infinite?
c. Is the number of P a prime number, or a composite one?
d. Are there uncountably many P?

While InqBQ can express the question "*what objects are P?*" (by means of the formula $\forall x.?P(x)$), it cannot express the corresponding cardinality question "*how many objects are P?*". From the perspective of logical modeling of questions, this means that analyzing *how many* questions—an important class of questions—requires a proper extension of the logic InqBQ. Developing and investigating such an extension is an interesting prospect for future work.

Since the proof of Theorem 5.3.2 is quite flexible, the characterization result can be seen to hold also when we restrict to certain salient classes of models, for instance the class of *finite models*.

5.3.3. COROLLARY. *Let $\mathcal{K}$ be a set of sets of* finite *cardinals. The following are equivalent:*

1. *The cardinality quantifier $Q_{\mathcal{K}}$ is definable in* InqBQ *with respect to the class of finite models. That is, there is a formula $\chi_{\mathcal{K}}$ of* InqBQ *such that $Q_{\mathcal{K}}x.P(x)$ is equivalent to $\chi_{\mathcal{K}}[P]$ in restriction to finite models.*
2. *$\mathcal{K} = K_1^{\downarrow} \cup \dots \cup K_n^{\downarrow}$ for some sets $K_1, \dots, K_n \subseteq \mathbb{N}$, where for each K_i there exists $m \in \mathbb{N}$ such that K_i contains all or none of the numbers $k \geq m$.*

5.4 Conclusions

In this chapter we introduced and studied cardinality quantifiers in the context of inquisitive logic. In particular, we showed how some natural questions (e.g., "How many P are there?", "Are there infinitely many P?") can be expressed by

InqBQ extended with suitable cardinality quantifiers. Moreover, we characterized which cardinality quantifiers are *definable* in the logic InqBQ (Theorem 5.3.2), extending a folklore result for classical first order logic.

As mentioned in the main text, the cardinality quantifiers we consider are only of type $\langle 1 \rangle$, that is, they operate on a single unary predicate. But we could consider cardinality quantifiers of arbitrary types, operating on sequences of predicates without restrictions on the arities. This more general framework would allow to represent and study the logic of more complex expressions, as for example “How many pairs are in the extension of R?” for R a binary predicate. We leave this extended framework and the study of its properties for further work.

Another direction for future research is a generalization of the concept of *cardinality trace* (Definition 5.2.1). We defined the trace $\mathrm{tr}_s(\alpha)$ to be the set of possible cardinalities of the family of elements satisfying the sentence α according to the information carried by the info state s. In turns, we defined cardinality quantifiers essentially as classes of cardinality traces. This approach has a major limitation: the argument of a cardinality quantifier can be only a statement, and not a question. So expressions with nested cardinality quantifiers (e.g., “How many boys read how many books?”) cannot be captured and studied in the current framework. By generalizing the definition of cardinality trace we would be able to introduce a more flexible concept of cardinality quantifier. Moreover, we would be able to study cardinality quantifiers in a fully compositional environment, allowing to freely combine statements, questions and quantifiers.

Chapter 6

Disjunction and Existence Properties

As pointed out in Section 2.2, there is a close connection between intuitionistic logic and inquisitive logic. This connection is not limited to the reinterpretation of inquisitive semantics in terms of intuitionistic Kripke frames (Lemmas 2.2.8 and 2.2.9). Ciardelli [2016] showed that some constructive results that hold for intuitionistic logic have a natural counterpart in inquisitive logic, as for example the Curry-Howard interpretation of proofs as programs [Ciardelli, 2016, Proposition 2.4.8].

One of the hallmarks of constructive logics is the *disjunction property*: if a disjunction of the form $\varphi \vee \psi$ is valid intuitionistically, then at least one of the disjuncts φ and ψ is valid too. This property is famously true for intuitionistic logic, and it has been proven to hold also for inquisitive propositional logic InqB w.r.t. inquisitive disjunction [Ciardelli, 2016, Corollary 2.5.6]. In this chapter we address whether this property and its first order analogue—the existence property—hold for InqBQ, as suggested by the fact they hold for intuitionistic first order logic—and as already conjectured by Ciardelli [2016].

The proof we give is semantical in nature: we develop several constructions to combine and transform information models, and use them to prove the disjunction and existence properties. Some of these constructions are inspired by operations on intuitionistic Kripke-frames (e.g., disjoint union, direct sum) while others are based on constructions typical of classical logic (e.g., models of terms, permutation models). So, on a side note, this work can also be also interpreted as a first step into developing a model theory for inquisitive models.

This approach allows us to prove also more general results: we define several classes of theories for which the corresponding consequence relations have the disjunction and/or the existence property. Most notably, classical theories (i.e., theories containing only classical formulas) have this property: Given Γ a classical theory, if $\Gamma \vDash \varphi \mathbin{\rlap{\vee}\vee} \psi$ then $\Gamma \vDash \varphi$ or $\Gamma \vDash \psi$ (disjunction property); and if $\Gamma \vDash \overline{\exists} x.\varphi(x)$ then there exists a term t of the language such that $\Gamma \vDash \varphi(t)$.

There is an important point to highlight: the disjunction and existence properties hold for InqBQ only when we consider the non-rigid equality ($\mathcal{L}_{\asymp}$) or no equality at all ($\mathcal{L}_{\neq}$). Indeed, the disjunction property fails even for really simple formulas in the language with a rigid equality ($\mathcal{L}_{=}$): for example, the formula $c = d \vee c \neq d$ for c, d constant symbols is valid, but neither disjunct is. So we will assume, for the rest of the chapter, to be working with the syntax $\mathcal{L}_{\asymp}$. All the results and techniques can be immediately adapted to the language without equality.

The chapter is divided as follows: In Section 6.1 we develop the toolkit of model-theoretic constructions used to prove the main results; In Section 6.2 we prove the general forms of the disjunction and existence properties. Section 6.3 presents further refinements of the disjunction property. Section 6.4 provides some concluding remarks and discusses open questions.

6.1 Model Constructions

In this section we will present several constructions involving information models and we will study their characterizing properties. The aim is to build a toolkit to study the properties of InqBQ through its models. Most of the proofs in this section are technical but not particularly hard and they rely on the results on submodels proved in Chapter 3 (in particular Corollary 3.2.5).

To make the constructions more transparent, we will accompany them with examples and graphical representations. Moreover, we divide the constructions by theme: each subsection contains constructions based on the same approach to modify a model or to combine different models.

6.1.1 Extending a Model in Size

In this subsection we consider ways to extend a model in size: adding copies of the elements, combining a model and a skeleton (Definition 2.1.6), building a model of terms. These operations allow us to start from a model and define equivalent models (meaning models supporting the same formulas) with a different structure. This will give us more freedom to combine models in later parts of the chapter.

Adding copies of the elements.

The first and most intuitive way to modify the skeleton is *to add copies of the elements*, that is, additional elements which are $\sim^e$ to elements already present in the model. We have a lot of freedom when performing this kind of operation: for each element of the model we can specify a number of copies of that specific element to add to the structure. For our purposes, adding ω copies of each element will suffice, so we define only this simpler operation.

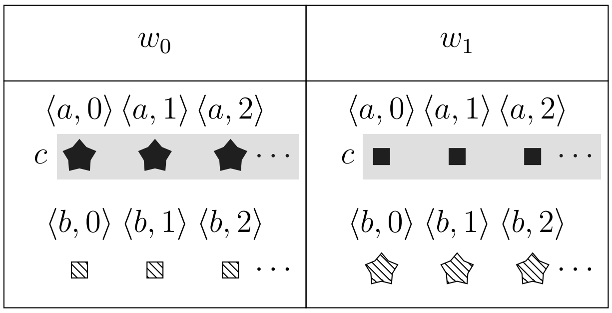


Figure 6.1: In the picture we depict the model $\mathcal{M}_\omega$, obtained from the model $\mathcal{M}$ of Figure 2.1. The new domain is obtained by adding a countable amount of copies of each element. For example, the elements in the gray box are the copies of the element a and are $\asymp_w^{\mathcal{M}_\omega}$-equivalent for every world w.

6.1.1. Definition. Given a model $\mathcal{M}$, we define the model $\mathcal{M}_\omega$ by the following clauses:

- $W^{\mathcal{M}_\omega} := W^{\mathcal{M}}$;
- $D^{\mathcal{M}_\omega} := D^{\mathcal{M}} \times \omega$;
- $f^{\mathcal{M}_\omega}(\langle d_1, k_1\rangle, \ldots, \langle d_n, k_n\rangle) := \langle f^{\mathcal{M}}(d_1, \ldots, d_n), 0\rangle$;
- $R_w^{\mathcal{M}_\omega}(\langle d_1, k_1\rangle, \ldots, \langle d_n, k_n\rangle)$ iff $R_w^{\mathcal{M}}(d_1, \ldots, d_n)$;
- $\langle d_1, k_1\rangle \asymp_w^{\mathcal{M}_\omega} \langle d_2, k_2\rangle$ iff $d_1 \asymp_w^{\mathcal{M}} d_2$.

The following lemma tells us that we only added redundant information by adding copies of the model.

6.1.2. Lemma. *Let $s \subseteq W^{\mathcal{M}}$ be an info state and $G : \mathtt{Var} \to D^{\mathcal{M}} \times \omega$ be an assignment. Define $\pi_1 : D^{\mathcal{M}} \times \omega \to D^{\mathcal{M}}$ as the projection on the first component (that is, $\pi_1(\langle d, k\rangle) = d$) and $g = \pi_1 \circ G$. Then for every formula φ*

$$\mathcal{M}_\omega, s \vDash_G \varphi \iff \mathcal{M}, s \vDash_g \varphi$$

Proof:
Firstly, notice that $\mathcal{M} \hookrightarrow \mathcal{M}_\omega$: in fact, the functions $\mathtt{id} : W^{\mathcal{M}} \to W^{\mathcal{M}_\omega}$ (the identity function) and $\iota_0 : D^{\mathcal{M}} \to D^{\mathcal{M}_\omega}$ defined as $\iota_0(d) := \langle d, 0\rangle$ respect Conditions Sub1 and Sub2. Secondly, we also have $\mathcal{M}_\omega \hookrightarrow \mathcal{M}$: the functions $\mathtt{id} : W^{\mathcal{M}_\omega} \to W^{\mathcal{M}}$ and $\pi_1 : D^{\mathcal{M}_\omega} \to D^{\mathcal{M}}$ respect Conditions Sub1 and Sub2. The conclusion follows by Corollary 3.2.6. □

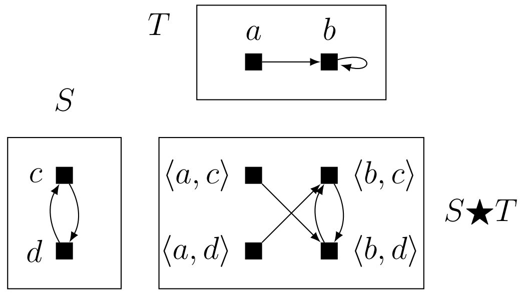


Figure 6.2: A simple example of the product of two skeletons in the signature $\Sigma = \{f\}$, for f a unary predicate function. In particular, the arrows represent the interpretation of f (the target is the image of the source).

Product of skeletons.

The previous construction can be thought as a *product* between a model and the set ω, and it is easy to generalize it to an arbitrary set instead of ω.

A slightly more challenging task is to define a product between a model $\mathcal{M}$ and a *skeleton* S (see Definition 2.1.6). S contains additional structure—the encoding of the functions—which needs to be reflected in the output of the construction, together with the structure of $\mathcal{M}$.

To do this we proceed in two steps: firstly we encode $\mathrm{Sk}(\mathcal{M})$ and S in a new skeleton having domain $D^{\mathcal{M}} \times S$; secondly we encode the relations of $\mathcal{M}$ by defining a structure of information model over the new skeleton. Let us start with the first step.

6.1.3. Definition (Product of skeletons). Given two skeletons S and T, we define the skeleton $S{\bigstar}T$ such that

- $D^{S\bigstar T} = D^S \times D^T$;
- $f^{S\bigstar T}(\langle d_1, e_1\rangle, \ldots, \langle d_n, e_n\rangle) = \left\langle f^S(d_1, \ldots, d_n), f^T(e_1, \ldots, e_n)\right\rangle$.

6.1.4. Definition. Given a model $\mathcal{M}$ and a skeleton S, we define the model $\mathcal{M}{\bigstar}S$ by the following clauses

- $\mathrm{Sk}(\mathcal{M}{\bigstar}S) = \mathrm{Sk}(\mathcal{M}){\bigstar}S$;
- $W^{\mathcal{M}\bigstar S} = W^{\mathcal{M}}$;
- $R_w^{\mathcal{M}\bigstar S}(\langle d_1, e_1\rangle, \ldots, \langle d_n, e_n\rangle)$ iff $R_w^{\mathcal{M}}(d_1, \ldots, d_n)$;
- $\langle d_1, e_1\rangle \asymp_w^{\mathcal{M}\bigstar S} \langle d_2, e_2\rangle$ iff $d_1 \asymp_w^{\mathcal{M}} d_2$.

Using the same technique used to prove Lemma 6.1.2 we can prove an analogous result for $\mathcal{M}{\bigstar}S$.

6.1.5. Lemma. *Let $s \subseteq W^{\mathcal{M}}$ an info state and $G : \texttt{Var} \to D^{\mathcal{M}} \times D^{S}$ an assignment. Define $g = \pi_1 \circ G$. Then for every formula φ*

$$\mathcal{M}\bigstar S, s \vDash_G \varphi \iff \mathcal{M}, s \vDash_g \varphi$$

We can generalize the construction above to a family of skeletons instead of a single one. We leave proofs and details to the reader.

6.1.6. Definition. Given a family of skeletons $\langle S^i | i \in I \rangle$, we define the skeleton $\bigstar_{i\in I} S^i$ such that

- $D^{\bigstar_{i\in I} S^i} = \prod_{i\in I} D^{S^i}$;
- $f^{\bigstar_{i\in I} S^i} \left(\langle d_1^i | i \in I \rangle, \dots, \langle d_1^i | i \in I \rangle \right) = \left\langle f^{S^i} \left(d_1^i, \dots, d_n^i \right) | i \in I \right\rangle$.

6.1.7. Definition. Given a model $\mathcal{M}$ with skeleton S^j and $\langle S^i | i \in I \rangle$ a family of skeletons such that $j \in I$, we define the model $\mathcal{M}\bigstar^j_{i\in I} S^i$ such that

- $\mathrm{Sk}(\mathcal{M}\bigstar^j_{i\in I} S^i) = \bigstar_{i\in I} S^i$;
- $W^{\mathcal{M}\bigstar^j_{i\in I} S^i} = W^{\mathcal{M}}$;
- $R_w^{\mathcal{M}\bigstar^j_{i\in I} S^i} \left(\langle d_1^i | i \in I \rangle, \dots, \langle d_n^i | i \in I \rangle \right)$ iff $R_w^{\mathcal{M}}(d_1^j, \dots, d_n^j)$;
- $\langle d_1^i | i \in I \rangle \preceq_w^{\mathcal{M}\bigstar^j_{i\in I} S^i} \langle d_2^i | i \in I \rangle$ iff $d_1^j \preceq_w^{\mathcal{M}} d_2^j$.

6.1.8. Lemma. *Let $s \subseteq W^{\mathcal{M}}$ an info state and $G : \texttt{Var} \to \prod_{i\in I} D^{S^i}$ an assignment. Define $g = \pi_j \circ G$. Then for every formula φ*

$$\mathcal{M}\bigstar^j_{i\in I} S^i, s \vDash_G \varphi \iff \mathcal{M}, s \vDash_g \varphi$$

Blowup model.

A different approach to modifying the skeleton is to extend the domain to an *algebra of terms* in a similar fashion as Hintikka's model of terms (as presented by Hodges [1997a], Section 2.3). This construction allows us to treat functions symbols as formal combinators and to encode the structure of the model using only relation symbols.

As for the product of a skeleton, we will define this construction in two steps: firstly we will define the blowup of a skeleton; and then we will give to the output the structure of an information model.

6.1.9. DEFINITION. Given a set A we define

$$\mathcal{T}\Sigma(A) = \{t\left(\underline{a_1}, \ldots, \underline{a_n}\right) | t(x_1, \ldots, x_n) \text{ term of } \Sigma \text{ and } a_1, \ldots, a_n \in A\}$$

where $\{\underline{a} | a \in A\}$ is a set of fresh constants not present in Σ.

We define the skeleton $\mathcal{B}A$ by the following clauses

- $D^{\mathcal{B}A} = \mathcal{T}\Sigma(A)$;
- $f^{\mathcal{B}A}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$ (notice that $f^{\mathcal{B}A}$ is a function while f is a formal symbol).

To study the relations between the structure of a skeleton and its blowup it is useful to introduce the following projection operator.

6.1.10. DEFINITION (Natural projection). Given a skeleton S, we define the *natural projection* $\Pi_{\mathcal{B}} : \mathcal{T}\Sigma(D^S) \to D^S$ as the only function such that

$$\Pi_{\mathcal{B}}(\underline{a}) = a \qquad \Pi_{\mathcal{B}}(f(t_1, \ldots, t_n)) = f^S(\Pi_{\mathcal{B}}(t_1), \ldots, \Pi_{\mathcal{B}}(t_n))$$

We are now ready to define the blowup of a model.

6.1.11. DEFINITION. Given a model $\mathcal{M}$, consider the natural projection $\Pi_{\mathcal{B}}$ from $\mathcal{B}(D^{\mathcal{M}})$ to $D^{\mathcal{M}}$. We define the *blowup* of $\mathcal{M}$ as the model $\mathcal{BM}$ defined by the following clauses

- $W^{\mathcal{BM}} = W^{\mathcal{M}}$;
- $\mathrm{Sk}(\mathcal{BM}) = \mathcal{B}\,\mathrm{Sk}(\mathcal{M})$;
- $R_w^{\mathcal{BM}}(t_1, \ldots, t_n)$ iff $R_w^{\mathcal{M}}(\Pi_{\mathcal{B}}(t_1), \ldots, \Pi_{\mathcal{B}}(t_n))$;
- $t_1 \asymp_w^{\mathcal{BM}} t_2$ iff $\Pi_{\mathcal{B}}(t_1) \asymp_w^{\mathcal{M}} \Pi_{\mathcal{B}}(t_2)$.

A graphical representation of the blowup of a simple model is given in Figure 6.3.

6.1.12. REMARK. Some observations to understand the structure of this new model:

- If the element e is the interpretation of the constant term c in $\mathcal{M}$, we have both $\underline{e}$ and c as distinct elements of $\mathcal{BM}$.
- Given $d, d' \in D^{\mathcal{M}}$ the congruence condition for $\underline{d}$ and $\underline{d'}$ reduces to $\underline{d} \asymp_w^{\mathcal{BM}} \underline{d'}$ iff $d \asymp_w^{\mathcal{M}} d'$, thus we can interpret $\asymp^{\mathcal{BM}}$ as an *extension* of $\asymp^{\mathcal{M}}$.

Using the natural projection, we can show that yet again we obtained a model carrying the same information as $\mathcal{M}$.

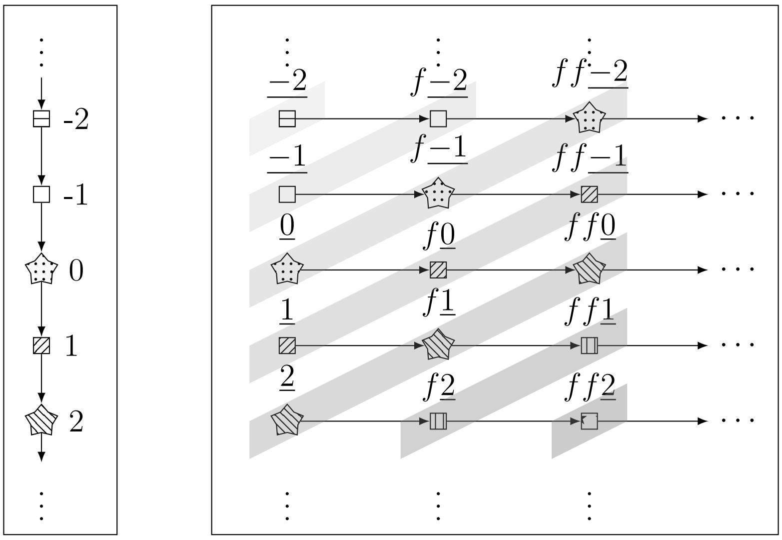


Figure 6.3: An example of blowup of a single-world model (i.e., a model such that $\left|W^{\mathcal{M}}\right| = 1$) in the signature $\Sigma = \{f^{(1)}\}$. The model $\mathcal{M}$ (on the left) represents the set $\mathbb{Z}$ where f (depicted by the arrows) is interpreted as the usual successor function and identity (represented by patterns) is interpreted as the real identity. The model $\mathcal{BM}$ (on the right) contains all the terms of the extended signature $\Sigma(\mathbb{Z})$ and interprets the function f as the formal term combinator (i.e., $f^{\mathcal{BM}}(t) = f(t)$). The equivalence classes of the relation $\asymp^{\mathcal{BM}}$ have been highlighted in different shades of gray.

6.1.13. THEOREM. *Let* $s \subseteq W^{\mathcal{M}}$ *and* $G : \mathtt{Var} \to \mathcal{T}\,\Sigma(D^{\mathcal{M}})$. *Define* $g = \Pi_{\mathcal{B}} \circ G$. *Then for every formula* φ

$$\mathcal{BM}, s \vDash_G \varphi \iff \mathcal{M}, s \vDash_g \varphi$$

Proof:
Firstly notice that $\mathcal{M} \hookrightarrow \mathcal{BM}$: the functions $\mathsf{id} : W^{\mathcal{M}} \to W^{\mathcal{BM}}$ and $G : D^{\mathcal{M}} \to D^{\mathcal{BM}}$ defined as $G(d) = \underline{d}$ respect Conditions Sub1 and Sub2. Secondly we also have $\mathcal{BM} \hookrightarrow \mathcal{M}$: the functions $\mathsf{id} : W^{\mathcal{BM}} \to W^{\mathcal{M}}$ and $\Pi_{\mathcal{B}} : D^{\mathcal{BM}} \to \mathcal{M}$ respect Conditions Sub1 and Sub2. The conclusion follows by Corollary 3.2.6. □

6.1.2 Combining Models

In the previous section we considered constructions that modified a single model at a time, possibly by combining it with other structures (sets, skeletons). Instead, in this section we focus on constructions involving different models and ways to combine them.

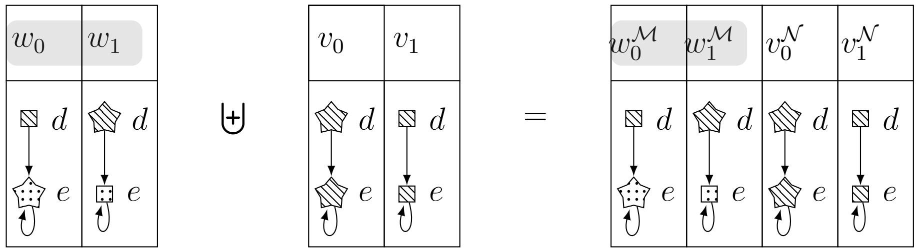


Figure 6.4: The disjoint union of the models $\mathcal{M}$ (first model) and $\mathcal{N}$ (second model). In the picture, the info state s of $\mathcal{M}$ (highlighted in gray in the first model) naturally corresponds to the state $s^{\mathcal{M}}$ of $\mathcal{M} \uplus \mathcal{N}$ (highlighted in gray in the third model).

Disjoint union.

The first construction allows us to combine two models $\mathcal{M}$ and $\mathcal{N}$ in a quite simple way, but the price to pay is that they must have the same skeleton S. Under this hypothesis, we can just define a structure of information model on S having one world w for each of the worlds of $\mathcal{M}$ and $\mathcal{N}$, with associated the corresponding structure of first order model.

6.1.14. Definition. Let $\mathcal{M}$ and $\mathcal{N}$ be two models with $\mathrm{Sk}(\mathcal{M}) = \mathrm{Sk}(\mathcal{N}) = S$. We define the *disjoint union of $\mathcal{M}$ and $\mathcal{N}$* as the model

$$\mathcal{M} \uplus \mathcal{N} = \left\{ M_w | w \in W^{\mathcal{M}} \right\} \cup \left\{ N_w | w \in W^{\mathcal{N}} \right\}$$

In particular we have

- $W^{\mathcal{M} \uplus \mathcal{N}} = W^{\mathcal{M}} \sqcup W^{\mathcal{N}}$;
- $D^{\mathcal{M} \uplus \mathcal{N}} = D^S$;
- For f a function symbol $f^{\mathcal{M} \uplus \mathcal{N}} = f^S$;
- For R a relation symbol and $w \in W^{\mathcal{M}}$ a world, $R^{\mathcal{M} \uplus \mathcal{N}}_{\langle w, \mathcal{M} \rangle} = R^{\mathcal{M}}_w$, and similarly for $w \in W^{\mathcal{N}}$;
- For $d_1, d_2 \in D^S$ and $w \in W^{\mathcal{M}}$ a world, $d_1 \asymp^{\mathcal{M} \uplus \mathcal{N}}_{\langle w, \mathcal{M} \rangle} d_2$ if and only if $d_1 \asymp^{\mathcal{M}}_w d_2$, and similarly for $w \in W^{\mathcal{N}}$.

As a notational convention we will write $w^{\mathcal{M}}$ when we want to indicate that w is a world in $W^{\mathcal{M}}$. We assume the same notation for information states: for $s \subseteq W^{\mathcal{M}}$ we will write $s^{\mathcal{M}}$. This notation is particularly useful when we consider the disjoint union of several models with the same set of worlds.
The following Theorem describes the relation between the support of $\mathcal{M} \uplus \mathcal{N}$ and its components.

6.1.15. Theorem (Disjoint union main property). *Given an assignment* $g : \texttt{Var} \to D^S$ *and an info state* $s \subseteq W^{\mathcal{M}}$ *of* $\mathcal{M}$*, then for every formula* φ

$$\mathcal{M} \uplus \mathcal{N}, s^{\mathcal{M}} \vDash_g \varphi \iff \mathcal{M}, s \vDash_g \varphi$$

Proof:
By definition we have $(\mathcal{M} \uplus \mathcal{N})_{s^{\mathcal{M}}} = \mathcal{M}_s$, and so the result follows trivially. □

This construction and the theorem above can be easily generalized to a family of models as follows.

6.1.16. Definition. Let $\langle \mathcal{M}^i | i \in I \rangle$ be a family of models where $\text{Sk}(\mathcal{M}^i) = S$ for all $i \in I$. We define the *disjoint union of the family* as the model

$$\biguplus_{i \in I} \mathcal{M}^i = \bigsqcup_{i \in I} \left\langle \; M^i_{\langle w, \mathcal{M}^i \rangle} \mid w \in W^{\mathcal{M}^i} \; \right\rangle$$

where $M^i_{\langle w, \mathcal{M}^i \rangle} = M^i_w$.

6.1.17. Theorem. *Given an assignment* $g : \texttt{Var} \to D^S$ *and* $s \subseteq W^{\mathcal{M}_j}$*, then for every formula* φ *it holds that*

$$\biguplus_{i \in I} \mathcal{M}_i, s^{\mathcal{M}_j} \vDash_g \varphi \iff \mathcal{M}_j, s \vDash_g \varphi$$

Direct sum.

The next construction presents a generalization of the disjoint union to models with different skeletons, using the same ideas of Definition 6.1.3. The construction is slightly more complex, but it does not require additional hypotheses on the models.

6.1.18. Definition. Let $\mathcal{M}$ and $\mathcal{N}$ be two models. We define the *direct sum of* $\mathcal{M}$ *and* $\mathcal{N}$ as the model $\mathcal{M} \oplus \mathcal{N}$ defined by the following clauses

- $W^{\mathcal{M} \oplus \mathcal{N}} = W^{\mathcal{M}} \sqcup W^{\mathcal{N}}$ (modulo renaming of the worlds);
- $D^{\mathcal{M} \oplus \mathcal{N}} = D^{\mathcal{M}} \times D^{\mathcal{N}}$;
- For f a function symbol $f^{\mathcal{M} \oplus \mathcal{N}} = \langle f^{\mathcal{M}}, f^{\mathcal{N}} \rangle$;
- For R a relation symbol and $w \in W^{\mathcal{M}}$ a world

  $$R^{\mathcal{M} \oplus \mathcal{N}}_{\langle w, \mathcal{M} \rangle} (\langle d_1, d_1' \rangle, \ldots, \langle d_n, d_n' \rangle) \iff R^{\mathcal{M}}_w (d_1, \ldots, d_n)$$

  and similarly for $\mathcal{N}$;

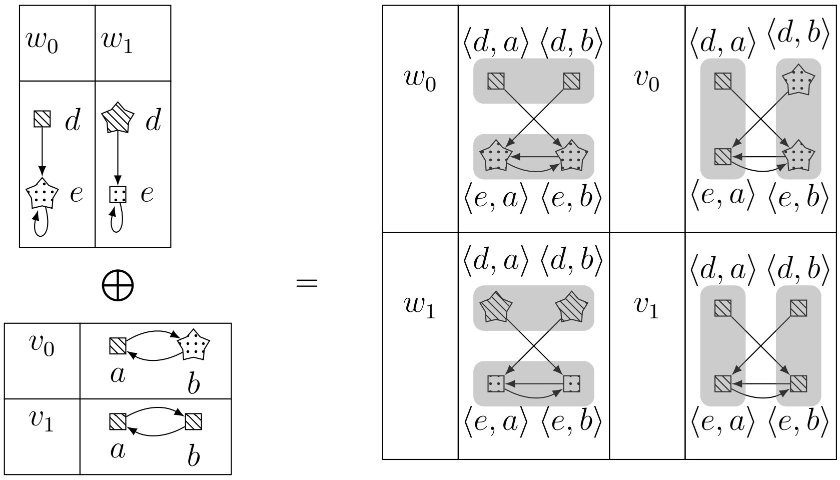


Figure 6.5: An example of direct sum: the model $\mathcal{M} \oplus \mathcal{N}$ on the right is obtained as the direct sum of the two models on the left, $\mathcal{M}$ (above) and $\mathcal{N}$ (below). Notice that for $w \in W^{\mathcal{M}}$ the projection map on the first component $\Pi_{\mathcal{M}} : D^{\mathcal{M}} \times D^{\mathcal{N}} \to D^{\mathcal{M}}$ respects the relation $\asymp_w^{\mathcal{M}\oplus\mathcal{N}}$ (i.e., the elements in the same gray box have the same image) and commutes with the interpretation of function symbols. The same holds for $w \in W^{\mathcal{N}}$ and the projection map on the second component $\Pi_{\mathcal{N}}$.

- For $d, d' \in D^S$ and $w \in W^{\mathcal{M}}$ a world

$$\langle d, d'\rangle \asymp_{\langle w,\mathcal{M}\rangle}^{\mathcal{M}\oplus\mathcal{N}} \langle e, e'\rangle \iff d \asymp_w^{\mathcal{M}} e$$

  and similarly for $\mathcal{N}$.

A graphical representation of a direct sum model is given in Figure 6.5.

6.1.19. Remark. Notice that Definition 6.1.18 amounts to the following expression, obtained by combining the constructions previously introduced.

$$\mathcal{M} \oplus \mathcal{N} = \left\{ M_w \bigstar \mathrm{Sk}(\mathcal{N}) \mid w \in W^{\mathcal{M}} \right\} \cup \left\{ \mathrm{Sk}(\mathcal{M}) \bigstar N_w \mid w \in W^{\mathcal{N}} \right\}$$

We assume the same notational convention for worlds and info states as for the disjoint union. In particular, $w^{\mathcal{M}}$ and $s^{\mathcal{M}}$ refer to a world and an info state of $\mathcal{M}$ respectively.

Notice that the disjoint union preserves the skeletons of the initial models, while the direct sum does not—not even when we start with models with the same skeleton. This property of the disjoint union is fundamental to carry out some of the constructions in this section.

Notice that there is a natural *projection operation* from the domain of this new model to the domain of its components.

6.1.20. Definition (Projection operation). We define the *projection operations*

$$\Pi_{\mathcal{M}} : D^{\mathcal{M}} \times D^{\mathcal{N}} \to D^{\mathcal{M}} \qquad \Pi_{\mathcal{N}} : D^{\mathcal{M}} \times D^{\mathcal{N}} \to D^{\mathcal{N}}$$
$$\langle d; e\rangle \mapsto d \qquad \langle d; e\rangle \mapsto e$$

The projection operations can also be extended to assignments in a natural way. Given an assignment $g : \texttt{Var} \to D^{\mathcal{M}} \times D^{\mathcal{N}}$, define $\Pi_{\mathcal{M}} g : \texttt{Var} \to D^{\mathcal{M}}$ such that $(\Pi_{\mathcal{M}} g)(x) = \Pi_{\mathcal{M}}(g(x))$. Moreover, given a term $t(x_1, \ldots, x_n)$ it follows easily that $\Pi_{\mathcal{M}}\left(t^{\mathcal{M}\oplus\mathcal{N}}(a_1, \ldots, a_n)\right) = t^{\mathcal{M}}(\Pi_{\mathcal{M}}(a_1), \ldots, \Pi_{\mathcal{M}}(a_n))$.

Reasoning as in the disjoint union case, we can obtain a strong connection between $\mathcal{M} \oplus \mathcal{N}$ and the models $\mathcal{M}$ and $\mathcal{N}$.

6.1.21. Theorem. *Let $g : \texttt{Var} \to D^{\mathcal{M}} \times D^{\mathcal{N}}$ be an assignment and let $s \subseteq W^{\mathcal{M}}$ be an info state of $\mathcal{M}$. Then for every formula φ*

$$\mathcal{M} \oplus \mathcal{N}, s^{\mathcal{M}} \vDash_g \varphi \iff \mathcal{M}, s \vDash_{\Pi_{\mathcal{M}} g} \varphi$$

Proof:
As noticed already, $(\mathcal{M} \oplus \mathcal{N})_{s^{\mathcal{M}}} = \mathcal{M}_s \bigstar \mathrm{Sk}(\mathcal{N})$. The result then follows by Lemma 6.1.5. □

6.1.22. Corollary. *Using the notations of Theorem 6.1.21, for $u \subseteq W^{\mathcal{M}\oplus\mathcal{N}}$*

$$\mathcal{M} \oplus \mathcal{N}, u \vDash_g \varphi \implies \mathcal{M}, u \cap W^{\mathcal{M}} \vDash_{\Pi_{\mathcal{M}} g} \varphi$$

Proof:
By persistency we obtain

$$\begin{aligned} \mathcal{M} \oplus \mathcal{N}, u \vDash_g \varphi &\implies \mathcal{M} \oplus \mathcal{N}, u \cap W^{\mathcal{M}} \vDash_g \varphi \\ &\implies \mathcal{M}, u \cap W^{\mathcal{M}} \vDash_{\Pi_{\mathcal{M}} g} \varphi \end{aligned}$$

□

As in the disjoint union case, the definition of direct sum can be extended to a family of models preserving the main property.

6.1.23. Definition. Let $\langle \mathcal{M}^i | i \in I\rangle$ a family of models. We define the *direct sum of the family* as the model

$$\bigoplus_{i\in I} \mathcal{M}^i = \bigsqcup_{j\in J} \left\langle \; M^j_w \bigstar^j_{i\in I} \mathrm{Sk}(\mathcal{M}^i) \mid w \in W^{\mathcal{M}^j} \; \right\rangle$$

6.1.24. Theorem. *Let $g : \texttt{Var} \to D^{\oplus \mathcal{M}^i}$ be an assignment and let $s \subseteq W^{\mathcal{M}^k}$ be an info state of $\mathcal{M}^k$. Then it holds that*

$$\bigoplus_{i\in I} \mathcal{M}^i, s^{\mathcal{M}_k} \vDash_g \varphi \iff \mathcal{M}^k, s \vDash_{\Pi_{\mathcal{M}^k} g} \varphi$$

Notice that from Theorems 6.1.17 and 6.1.24 we can derive an interesting result: if we apply one of the previous constructions—disjoint union or direct sum—to a family of models of a classical formula α, we obtain again a model of α.

6.1.25. Corollary.

$$\bigoplus_{i\in I}\mathcal{M}^i \vDash_G \alpha \iff \forall j \in I.\ \mathcal{M}^j \vDash_{\Pi^{\mathcal{M}^j}\circ G} \alpha$$

Proof:

$$\begin{aligned}\bigoplus_{i\in I}\mathcal{M}^i \vDash_G \alpha &\iff \forall w^j \in \bigsqcup_{j\in I} W^{\mathcal{M}^j}.\ \bigoplus_{i\in I}\mathcal{M}^i, \{w^j\} \vDash_G \alpha \\ &\iff \forall w^j \in \bigsqcup_{j\in I} W^{\mathcal{M}^j}.\ \mathcal{M}^j, \{w^j\} \vDash_{\Pi^{\mathcal{M}^j}\circ G} \alpha \\ &\iff \forall j \in I.\ \mathcal{M}^j \vDash_{\Pi^{\mathcal{M}^j}\circ G} \alpha\end{aligned}$$

□

6.1.3 Characteristic Model

The next construction we consider can be considered as a non-constructive canonical model for a classical theory. We start from a classical theory Γ and we combine several information models of Γ—one countermodel for each formula not following from Γ—to obtain an information model $\mathcal{M}_\Gamma$ with the canonical model property: $\mathcal{M}_\Gamma$ supports all and only the logical consequences of Γ.

Characteristic model of a classical theory.

Consider a classical theory Γ and define $\mathtt{C}(\Gamma) = \{\varphi \,|\, \Gamma \nvDash \varphi\}$ as the set of its *non-theorems*. By definition, for every $\varphi \in \mathtt{C}(\Gamma)$ we can find a pair $\langle \mathcal{M}_\varphi, g_\varphi \rangle$ that acts as a *witness* of the non-entailment $\Gamma \nvDash \varphi$, meaning that $\mathcal{M}_\varphi \vDash_{g_\varphi} \Gamma$ and $\mathcal{M}_\varphi \nvDash_{g_\varphi} \varphi$. Fixing now a family $\langle \langle \mathcal{M}_\varphi, g_\varphi \rangle \,|\varphi \in \mathtt{C}(\Gamma) \rangle$ of models as described, we are ready to define our next construction.

6.1.26. Definition (Characteristic model of Γ). Define $\mathcal{M}_\Gamma = \bigoplus_{\varphi\in\mathtt{C}(\Gamma)} \mathcal{M}_\varphi$ and $g_\Gamma : \mathtt{Var} \to \prod_{\varphi\in\mathtt{C}(\Gamma)} D^{\mathcal{M}_\varphi}$ as $g_\Gamma(x) = \langle g_\varphi(x) | \varphi \in \mathtt{C}(\Gamma) \rangle$.

The definition of the characteristic model of Γ strongly depends on the set $\langle \langle \mathcal{M}_\varphi, g_\varphi \rangle \,|\varphi \in \mathtt{C}(\Gamma) \rangle$. Choosing these models is the non-constructive part of the construction: notice that we need to make use of the Axiom of Choice.

As anticipated, this model supports exactly the logical consequences of Γ.

6.1.27. THEOREM. *For every formula ψ*

$$\Gamma \vDash \psi \iff \mathcal{M}_\Gamma \vDash_{g_\Gamma} \psi$$

Proof:
From Corollary 6.1.25 it follows that if $\Gamma \vDash \psi$, then $\mathcal{M}_\Gamma \vDash_{g_\Gamma} \psi$, and so the left-to-right implication of the theorem holds.

For the other direction, fix φ a non-theorem of Γ. We claim that the characteristic model does not support φ. By contradiction, if the model supports the formula then from Theorem 6.1.24 and the persistency of the logic we would obtain

$$\begin{aligned} \mathcal{M}_\Gamma \vDash_{g_\Gamma} \varphi &\implies \mathcal{M}_\Gamma, W^{\mathcal{M}_\varphi} \vDash_{g_\Gamma} \varphi \qquad \text{(by the equality } (\mathcal{M}_\Gamma)_{W^{\mathcal{M}_\varphi}} = \mathcal{M}_\varphi) \\ &\implies \mathcal{M}_\varphi \vDash_{g_\varphi} \varphi \end{aligned}$$

which gives a contradiction. □

Notice that this result gives us a non-trivial property of InqBQ with respect to CQC: given an arbitrary classical theory Γ, we can find a *single model* which entail *all and only* the theorems of Γ. It is well-known that the same property does not hold for CQC, as the set of formulas supported by a classical structure is necessarily a complete theory. We will see that this property is of fundamental importance for the proof of the generalized existence property presented in Section 6.2.

It is worth noticing that the same result does not hold for a generic theory. Consider for example the theory $\Phi = \{P(c) \vee\!\!\!\vee \neg P(c)\}$ and suppose toward a contradiction that there exists a pair $\langle \mathcal{M}, g \rangle$ such that

$$\Phi \vDash \psi \iff \mathcal{M} \vDash_g \psi$$

Then it is clear that $\mathcal{M} \vDash_g P(c)$ or $\mathcal{M} \vDash_g \neg P(c)$ by the semantic clause for $\vee\!\!\!\vee$. But clearly $\Phi \nvDash P(c)$ and $\Phi \nvDash \neg P(c)$, which leads to a contradiction.

6.1.4 Permutation Models

We introduce now some constructions with a combinatoric nature. These constructions are based on the idea that permuting the names of the elements of a model—unsurprisingly—does not change its logical properties. However this operation of permuting the names allows us to combine the models in new and more complex structures: this will be the key ingredient to prove the *existence property* in the following section.

Permutation model.

A simple way to obtain a new model is by simply swapping the *names* of the elements. As done before, we introduce the construction first by defining the skeleton of the resulting structure and then the interpretation of relations.

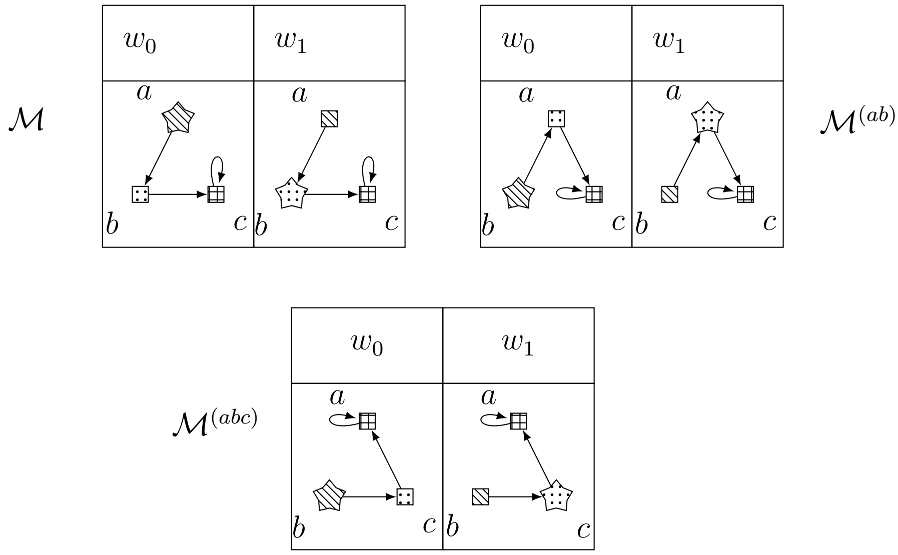


Figure 6.6: An example of a permutation model. In the Figure we depict the models $\mathcal{M}$ (upper left), $\mathcal{M}^{(ab)}$ (upper right) and $\mathcal{M}^{(abc)}$. Notice that the *roles* of the elements in the models are swapped according to the inverse of the permutation considered. For example, the element a in $\mathcal{M}^{(abc)}$ behaves like the element c in $\mathcal{M}$, as a is mapped onto c by the permutation $(abc)^{-1}$.

Henceforth, given a set X we will indicate with $\mathfrak{S}(X)$ the set of *permutations of* X, that is, the bijective functions $\sigma : X \to X$ having as domain and codomain X.

6.1.28. Definition. Let S be a skeleton and $\sigma \in \mathfrak{S}(D^S)$ a permutation. We define the skeleton S^σ by the following clauses

- $D^{S^\sigma} = D^S$;
- $f^{S^\sigma}(d_1, \ldots, d_n) = \sigma(f^S(\sigma^{-1}d_1, \ldots, \sigma^{-1}d_n))$.

6.1.29. Definition. Given a model $\mathcal{M}$ we define the model $\mathcal{M}^\sigma$ by the following clauses

- $W^{\mathcal{M}^\sigma} = W^{\mathcal{M}}$;
- $\mathrm{Sk}(\mathcal{M}^\sigma) = \mathrm{Sk}(\mathcal{M})^\sigma$;
- $R_w^{\mathcal{M}^\sigma}(d_1, \ldots, d_n)$ iff $R_w^{\mathcal{M}}(\sigma^{-1}d_1, \ldots, \sigma^{-1}d_n)$;
- $d_1 \asymp_w^{\mathcal{M}^\sigma} d_2$ iff $\sigma^{-1}d_1 \asymp_w^{\mathcal{M}} \sigma^{-1}d_2$.

6.1.30. THEOREM. *Let $s \subseteq W^{\mathcal{M}}$ be an info state and $g : \mathtt{Var} \to D^{\mathcal{M}}$ an assignment. Then for every formula φ*

$$\mathcal{M}, s \vDash_g \varphi \iff \mathcal{M}^\sigma, s \vDash_{\sigma g} \varphi$$

Proof:
Notice that $\mathcal{M} \hookrightarrow \mathcal{M}^\sigma$: the functions $\mathsf{id} : W^{\mathcal{M}} \to W^{\mathcal{M}^\sigma}$ and $\sigma : D^{\mathcal{M}} \to D^{\mathcal{M}^\sigma}$ respect Conditions Sub1 and Sub2. Moreover the inverse of id and σ also respect Conditions Sub1 and Sub2, thus showing that we also have $\mathcal{M}^\sigma \hookrightarrow \mathcal{M}$. The conclusion follows by Corollary 3.2.6.

□

Blowup and permutations.

There is a particularly interesting case of permutation models that we can consider, namely the one obtained by starting from a blowup model $\mathcal{BM}$ and considering a permutation induced by a certain $\sigma \in \mathfrak{S}(D^{\mathcal{M}})$.

6.1.31. DEFINITION. Given $\sigma \in \mathfrak{S}(D^{\mathcal{M}})$ we extend it inductively to a permutation $\tilde{\sigma} \in \mathfrak{S}(\mathcal{T}\,\Sigma(D^{\mathcal{M}}))$ by the following clauses:

- For $d \in D$, $\tilde{\sigma}(\underline{d}) = \underline{\sigma(d)}$;
- For f a function symbol, $\tilde{\sigma}\left(f(t_1, \ldots, t_n)\right) = f\left(\tilde{\sigma}(t_1), \ldots, \tilde{\sigma}(t_n)\right)$.

6.1.32. DEFINITION. Given $\sigma \in \mathfrak{S}(D^{\mathcal{M}})$, we define the model $\mathcal{B}^\sigma\mathcal{M} = (\mathcal{BM})^{\tilde{\sigma}}$. A graphical representation of a model of this kind is given in Figure 6.7.

Combining together the results we have on blowup models and permutation models, we obtain the following.

6.1.33. COROLLARY. *Let $s \subseteq W^{\mathcal{M}}$ an info state and $G : \mathtt{Var} \to \mathcal{T}\,\Sigma\left(D^{\mathcal{M}}\right)$ an assignment. Define $g = \Pi_{\mathcal{B}} \circ G$. Then for every formula φ*

$$\mathcal{B}^\sigma\mathcal{M}, s \vDash_{\tilde{\sigma} G} \varphi \iff \mathcal{M}, s \vDash_g \varphi$$

Proof:
This result follows by combining Theorem 6.1.13 and Theorem 6.1.30. □

Notice that $\mathrm{Sk}(\mathcal{B}^\sigma\mathcal{M}) = \mathrm{Sk}(\mathcal{BM})$ since we have

$$\begin{aligned}
\boldsymbol{f}^{\mathcal{B}^\sigma\mathcal{M}}(t_1, \ldots, t_n) &= \tilde{\sigma}\, \boldsymbol{f}^{\mathcal{BM}}(\tilde{\sigma}^{-1}t_1, \ldots, \tilde{\sigma}^{-1}t_n) \\
&= \tilde{\sigma} f(\tilde{\sigma}^{-1}t_1, \ldots, \tilde{\sigma}^{-1}t_n) \\
&= f(\tilde{\sigma}\tilde{\sigma}^{-1}t_1, \ldots, \tilde{\sigma}\tilde{\sigma}^{-1}t_n) \\
&= f(t_1, \ldots, t_n) \\
&= \boldsymbol{f}^{\mathcal{BM}}(t_1, \ldots, t_n)
\end{aligned}$$

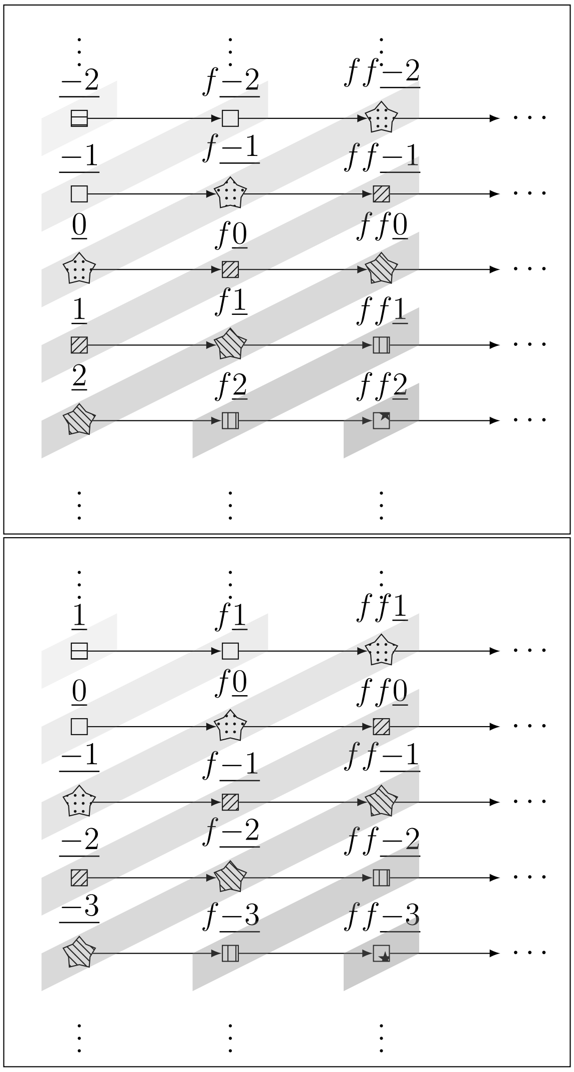


Figure 6.7: An example of a permutation model. Above, the model $\mathcal{BM}$ of Figure 6.3. Below, its corresponding permutation model $\mathcal{B}^\sigma\mathcal{M}$ for $\sigma(n) = -n - 1$. Notice that the skeleton is preserved by this operation, while the interpretation of predicate P and of identity changes in accordance with the permutation. For example, $\underline{0} \asymp^{\mathcal{BM}} f\underline{-1}$, but $\underline{0} \not\asymp^{\mathcal{B}^\sigma\mathcal{M}} f\underline{-1}$.

This remark allows to apply the disjoint union (Definition 6.1.16) to different permutation models of the kind $\mathcal{B}^\sigma\mathcal{M}$, as we will see with the following construction.

Full permutation model.

As we saw with the previous constructions, we can consider a permutation σ over the domain of a model $\mathcal{M}$, and this gives us a way to define a new model $\mathcal{M}^\sigma$. From here, it is not hard to define an action of the group $\mathfrak{S}(D^{\mathcal{M}})$ over the set $\{\mathcal{M}^\sigma | \sigma \in \mathfrak{S}(D^{\mathcal{M}})\}$. What is surprising, is that this action can be encoded by a single information model obtained by "gluing together" the blowups of the models $\mathcal{M}^\sigma$. This construction allows us to study which properties expressed by InqBQ are preserved under the action presented above and gives us the tools needed to prove the existence property.

6.1.34. Definition (Full Permutation Model). Let $\mathcal{M}$ be a model. We define its *full permutation model* as

$$\mathfrak{S}\mathcal{M} = \biguplus_{\sigma \in \mathfrak{S}(D^{\mathcal{M}})} \mathcal{B}^\sigma\mathcal{M}$$

Notice in particular that

- $W^{\mathfrak{S}\mathcal{M}} = W^{\mathcal{M}} \times \sigma(D^{\mathcal{M}})$;
- $\mathrm{Sk}(\mathfrak{S}\mathcal{M}) = \mathrm{Sk}(\mathcal{M})$;
- $\mathfrak{S}\mathcal{M}_{W^{\mathcal{B}^\sigma\mathcal{M}}} = \mathcal{B}^\sigma\mathcal{M}$.

To simplify the notation, we will write w^σ and s^σ instead of $w^{\mathcal{B}^\sigma\mathcal{M}}$ and $s^{\mathcal{B}^\sigma\mathcal{M}}$ to refer to worlds and info states in $\mathcal{B}^\sigma\mathcal{M}$.

6.1.35. Corollary. *Let $s \subseteq W^{\mathcal{M}}$ be an info state, $\sigma \in \mathfrak{S}(D^{\mathcal{M}})$ a permutation and $G : \mathtt{Var} \to \mathcal{T}\Sigma\left(D^{\mathcal{M}}\right)$ an assignment. Define $g = \Pi_{\mathcal{B}} \circ G$. Then for every formula φ*

$$\mathfrak{S}\mathcal{M}, s^\sigma \vDash_{\tilde{\sigma}G} \varphi \iff \mathcal{M}, s \vDash_g \varphi$$

Proof:
Using Theorem 6.1.17 and Corollary 6.1.33 we obtain

$$\begin{aligned} \mathfrak{S}\mathcal{M}, s^\sigma \vDash_{\tilde{\sigma}G} \varphi &\iff \mathcal{B}^\sigma\mathcal{M}, s \vDash_{\tilde{\sigma}G} \varphi \\ &\iff \mathcal{M}, s \vDash_g \varphi \end{aligned}$$

□

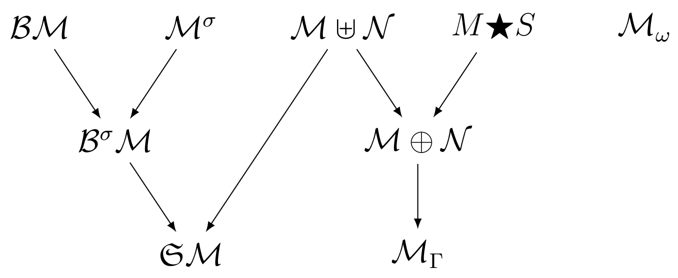


Figure 6.8: Summary of the constructions introduced. An arrow connects two constructions if the source was used to define the target.

6.1.36. COROLLARY. *Let $u \subseteq W^{\mathfrak{S}\mathcal{M}}$ be an info state, $G : \texttt{Var} \to \mathcal{T}\Sigma(D^{\mathcal{M}})$ an assignment and $\tau \in \mathfrak{S}(D^{\mathcal{M}})$ a permutation. Define $\tau u = \{w^{(\tau\sigma)} | w^\sigma \in u\}$. Then for every formula φ*

$$\mathfrak{S}\mathcal{M}, u \vDash_G \varphi \iff \mathfrak{S}\mathcal{M}, \tau u \vDash_{\tau G} \varphi$$

Proof:
As silly as it seems to spell it out, we firstly notice that $\mathfrak{S}\mathcal{M} \hookrightarrow \mathfrak{S}\mathcal{M}$. The interesting part is that we can find functions different from the identity that witness this: consider the functions

$$\begin{array}{rcl} F_\tau : W^{\mathfrak{S}\mathcal{M}} & \to & W^{\mathfrak{S}\mathcal{M}} \\ w^\sigma & \mapsto & w^{(\tau\sigma)} \end{array} \qquad \tilde{\tau} : D^{\mathfrak{S}\mathcal{M}} \to D^{\mathfrak{S}\mathcal{M}}$$

By definition of $\mathfrak{S}\mathcal{M}$, these functions respect Conditions Sub1 and Sub2. Moreover, their inverse also respects Conditions Sub1 and Sub2. The conclusion then follows by Corollary 3.2.6. ☐

In Figure 6.8 we give a graphical summary of the constructions introduced.

6.2 Disjunction and Existence Properties

6.2.1 Disjunction Property

We make use of the constructions introduced to prove that the logic InqBQ satisfies a strong form of the *disjunction property*, namely the *disjunction property over classical theories.*

6.2.1. THEOREM. *Given a classical theory Γ (i.e., $\Gamma \subseteq$ CQC), for every formulas φ and ψ*

$$\Gamma \vDash \varphi \vee\!\!\!\vee \psi \implies \Gamma \vDash \varphi \textit{ or } \Gamma \vDash \psi$$

Proof:
Suppose $\Gamma \not\models \varphi$ and $\Gamma \not\models \psi$. Let $\mathcal{M}_\varphi$, $\mathcal{M}_\psi$ two models and g_φ, g_ψ two assignments such that

$$\begin{cases} \mathcal{M}_\varphi \models_{g_\varphi} \Gamma \\ \mathcal{M}_\varphi \not\models_{g_\varphi} \varphi \end{cases} \qquad \begin{cases} \mathcal{M}_\psi \models_{g_\psi} \Gamma \\ \mathcal{M}_\psi \not\models_{g_\psi} \psi \end{cases}$$

Then consider the model $\mathcal{M}_\varphi \oplus \mathcal{M}_\psi$ and the assignment $G = \langle g_\varphi, g_\psi \rangle$. By Corollary 6.1.25 it follows that $\mathcal{M}_\varphi \oplus \mathcal{M}_\psi \models_G \Gamma$. Moreover by Theorem 6.1.21 and persistency it follows

$$\begin{aligned} \mathcal{M}_\varphi \not\models_{g_\varphi} \varphi &\implies \mathcal{M}_\varphi \oplus \mathcal{M}_\psi, W^{\mathcal{M}_\varphi} \not\models_G \varphi \\ &\implies \mathcal{M}_\varphi \oplus \mathcal{M}_\psi \not\models_G \varphi \end{aligned}$$

and with a similar argument we obtain $\mathcal{M}_\varphi \oplus \mathcal{M}_\psi \not\models_G \psi$ too. From this it follows that $\Gamma \not\models \varphi \vee\!\!\!\vee \psi$, thus the thesis. □

6.2.2. Remark. Another possible proof of the theorem could be carried out considering the characteristic model of Γ instead of the models $\mathcal{M}_\varphi$ and $\mathcal{M}_\psi$. The advantage of the proof presented above is that it does not make use of the axiom of choice.

6.2.2 Existence Property

In a similar fashion we can prove that InqBQ satisfies a strong form of the *existence property*, namely the *existence property for classical theories.*

6.2.3. Theorem. *Given a classical theory* Γ*, then for every formula* $\overline{\exists} x.\varphi(x)$

$$\Gamma \models \overline{\exists} x.\varphi(x) \quad \implies \quad \Gamma \models \varphi(t) \text{ for some } t \text{ term}$$

Proof:
Without loss of generality we can suppose Γ to be closed.[1]

Fix $\varphi(x, y_1, \ldots, y_m)$ (where $x, y_1, \ldots, y_m$ is a complete list of the distinct free variables in φ) and suppose that $\Gamma \not\models \varphi(t, y_1, \ldots, y_m)$ for every term t. Consider the characteristic model $\mathcal{M}_\Gamma$ of the theory Γ and the assignment $g_\Gamma : \texttt{Var} \to D^{\mathcal{M}_\Gamma}$ such that $\Gamma \models \psi$ iff $\mathcal{M}_\Gamma \models_{g_\Gamma} \psi$. Our aim is now, manipulating the model $\mathcal{M}_\Gamma$, to build a model of Γ but *not* of the formula $\overline{\exists} x.\varphi(x, \overline{y})$, thus proving that $\Gamma \not\models \overline{\exists} x.\varphi(x, \overline{y})$.

[1] It is trivial to prove that for InqBQ, as for CQC, entailment is invariant under substitutions of variables with fresh constants. Formally, consider a set of formulas $\Phi \cup \{\psi\}$ and a partial variable substitution $f : \texttt{Var} \to C$ for C a set of constants not appearing in $\Phi \cup \{\psi\}$. Define $\Phi[f]$ and $\psi[f]$ as the result of applying the substitution to the set Φ and ψ in the usual way. Then it holds that $\Phi \models \psi$ iff $\Phi[f] \models \psi[f]$.

As such a model consider $\mathfrak{S}((\mathcal{M}_\Gamma)_\omega)$. This is a model of Γ as we have

$$
\begin{aligned}
& \mathcal{M}_\Gamma \vDash \Gamma \\
\iff & \forall w \in W^{\mathcal{M}_\Gamma}.\ \mathcal{M}_\Gamma, \{w\} \vDash \Gamma && \text{by Lemma 2.1.15} \\
\iff & \forall w \in W^{(\mathcal{M}_\Gamma)_\omega}.\ (\mathcal{M}_\Gamma)_\omega, \{w\} \vDash \Gamma && \text{by Lemma 6.1.2} \\
\iff & \forall \sigma \in \mathfrak{S}(D^{(\mathcal{M}_\Gamma)_\omega}). \forall w^\sigma \in W^{(\mathcal{M}_\Gamma)_\omega}.\ \mathfrak{S}((\mathcal{M}_\Gamma)_\omega), \{w^\sigma\} \vDash \Gamma \\
& && \text{by Corollary 6.1.35} \\
\iff & \forall v \in W^{\mathfrak{S}((\mathcal{M}_\Gamma)_\omega)}.\ (\mathcal{M}_\Gamma)_\omega, \{v\} \vDash \Gamma \\
\iff & \mathfrak{S}((\mathcal{M}_\Gamma)_\omega) \vDash \Gamma && \text{by Lemma 2.1.15}
\end{aligned}
$$

Towards a contradiction suppose that $\Gamma \vDash \overline{\exists} x.\varphi(x, \overline{y})$. So for a certain element $t\left(\underline{\langle d_1, k_1\rangle}, \ldots, \underline{\langle d_n, k_n\rangle}\right) \in D^{\mathfrak{S}((\mathcal{M}_\Gamma)_\omega)} = \mathcal{T}\Sigma\left(D^{(\mathcal{M}_\Gamma)_\omega}\right)$ (where we suppose the elements $\langle d_i, k_i\rangle$ to be distinct) we have

$$\mathfrak{S}((\mathcal{M}_\Gamma)_\omega) \vDash_h \varphi(x, y_1, \ldots, y_m) \quad \text{if } h(x) = t\left(\underline{\langle d_1, k_1\rangle}, \ldots, \underline{\langle d_n, k_n\rangle}\right)$$

from which it follows that

$$\mathfrak{S}((\mathcal{M}_\Gamma)_\omega) \vDash_h \varphi(t(z_1, \ldots, z_n), y_1, \ldots, y_m) \quad \text{if } h(z_i) = \underline{\langle d_i, k_i\rangle} \tag{6.1}$$

where $z_1, \ldots, z_n$ are fresh variables distinct from $y_1, \ldots, y_m$.

Fix now the assignment H such that $H(z_i) = \underline{\langle g_\Gamma(z_i), i\rangle}$ for $1 \leq i \leq n$ and $H(y_j) = \underline{\langle g_\Gamma(y_j), n+j\rangle}$ for $1 \leq j \leq m$. As H is injective over the set $\{z_1, \ldots, z_n\}$, it is possible to find a permutation $\sigma \in (\mathcal{M}_\Gamma)_\omega$ such that $\sigma\left(\underline{\langle g_\Gamma(z_i), i\rangle}\right) = \underline{\langle d_i, k_i\rangle}$ for $1 \leq i \leq n$.

6.2.4. Remark. This passage justifies the fact that we are considering the model $\mathfrak{S}((\mathcal{M}_\Gamma)_\omega)$ instead of the model $\mathfrak{S}(\mathcal{M}_\Gamma)$. Indeed, if $g_\Gamma(z_i) = g_\Gamma(z_j)$ for $i \neq j$ then it would not be possible to find a permutation σ as above. A natural question (currently open) is if $\mathfrak{S}(\mathcal{M}_\Gamma) \nvDash \overline{\exists} x.\varphi(x)$ generally holds.

To conclude the proof, consider now the following steps

$$
\begin{aligned}
& \mathfrak{S}((\mathcal{M}_\Gamma)_\omega) \vDash_{\sigma H} \varphi(t(\overline{z}), \overline{y}) && \text{by Equation 6.1} \\
\Longrightarrow\ & \mathfrak{S}((\mathcal{M}_\Gamma)_\omega) \vDash_H \varphi(t(\overline{z}), \overline{y}) && \text{by Corollary 6.1.36} \\
\Longrightarrow\ & \mathfrak{S}((\mathcal{M}_\Gamma)_\omega), W^{(\mathcal{M}_\Gamma)_\omega} \vDash_H \varphi(t(\overline{z}), \overline{y}) && \text{by persistency} \\
\Longrightarrow\ & (\mathcal{M}_\Gamma)_\omega \vDash_{\Pi_{\mathcal{B}} H} \varphi(t(\overline{z}), \overline{y}) && \text{by Corollary 6.1.35} \\
\Longrightarrow\ & \mathcal{M}_\Gamma \vDash_{g_\Gamma} \varphi(t(\overline{z}), \overline{y}) && \text{by Lemma 6.1.2} \\
\Longrightarrow\ & \Gamma \vDash \varphi(t(\overline{z}), \overline{y}) && \text{by Theorem 6.1.27}
\end{aligned}
$$

and this is a contradiction, since it goes against the initial hypothesis that $\Gamma \not\models \varphi(t, \overline{y})$ for any term t. Thus we have $\Gamma \not\models \overline{\exists} x.\varphi(x, \overline{y})$, concluding the proof.

□

The following result is a simple corollary of the general form of the existence property, and shows the strongly constructive character of the logic InqBQ.

6.2.5. THEOREM. *Let $\varphi(x_1, \ldots, x_n, y)$ be a formula and Γ a classical theory. Suppose that*

$$\Gamma \models \forall \overline{x}.\overline{\exists} y.\varphi(\overline{x}, y)$$

Then there exists a term $t(x_1, \ldots, x_n)$ such that

$$\Gamma \models \forall \overline{x}.\varphi(\overline{x}, t(\overline{x}))$$

Proof:
In this proof we will adopt a slightly different notation to make explicit reference to the signature adopted: with $\models_\Sigma$ we indicate the entailment relation of InqBQ relative to the signature Σ.

Consider $c_1, \ldots, c_n$ fresh constant symbols not appearing in $\Gamma \cup \varphi$ and $\Sigma_{\overline{c}}$ the signature obtained by adding the new constants to the signature Σ. Then we have

$$\begin{aligned} \Gamma \models_\Sigma \forall \overline{x}.\overline{\exists} y.\varphi(\overline{x}, y) &\iff \Gamma \models_{\Sigma_{\overline{c}}} \overline{\exists} y.\varphi(\overline{c}, y) \\ &\iff \Gamma \models_{\Sigma_{\overline{c}}} \varphi(\overline{c}, t(\overline{c})) \text{ for some } t \qquad \text{(by Theorem 6.2.3)} \\ &\iff \Gamma \models_\Sigma \forall \overline{x}.\varphi(\overline{x}, t(\overline{x})) \text{ for some } t \end{aligned}$$

□

This theorem has an interesting interpretation connected to the notion of function definability. In the CQC case, we say that a formula $\varphi(x_1, \ldots, x_n, y)$ *defines a function under a classical theory* Γ if the following entailment holds

$$\Gamma \models \forall \overline{x}.\exists! y.\varphi(\overline{x}, y)$$

that is if $\varphi(\overline{x}, y)$ identifies a function in every classical model of Γ.

In InqBQ we can consider a stronger notion of function definability associated to the inquisitive quantifier. We say that a formula $\varphi(\overline{x}, y)$ *strongly defines a function under a classical theory* Γ if the following entailment holds

$$\Gamma \models \forall \overline{x}.\overline{\exists}! y.\varphi(\overline{x}, y)$$

where we have substituted the symbol $\exists$ with $\overline{\exists}$. In particular, the condition implies that in every model $\mathcal{M}$ of the theory Γ, $\varphi(\overline{x}, y)$ identifies the same function in every world of $\mathcal{M}$. Theorem 6.2.5 gives us then a complete characterization of which formulas strongly define a function, that is, only the ones that identify the interpretation of a fixed term of the language.

6.3 Further Refinements

It is worth noticing that the proof of the disjunction property given in Theorem 6.2.1 can be split into two passages:

1. Given two models $\mathcal{M}$ and $\mathcal{N}$ of a classical theory Γ, then $\mathcal{M} \oplus \mathcal{N}$ is a model of Γ too;

2. Given two models $\mathcal{M}_\varphi$ and $\mathcal{M}_\psi$ of Γ such that $\mathcal{M}_\varphi \not\models \varphi$ and $\mathcal{M}_\psi \not\models \psi$, then $\mathcal{M}_\varphi \oplus \mathcal{M}_\psi \not\models \varphi \vee\!\!\vee \psi$ by *persistency*.

This leads naturally to the following definition.

6.3.1. Definition ($\oplus$ property). Given a theory Φ, we say that it has the $\oplus$ property if

$$\text{If } \mathcal{M} \models_g \Phi \text{ and } \mathcal{N} \models_h \Phi \text{ then } \mathcal{M} \oplus \mathcal{N} \models_{\langle g,h \rangle} \Phi$$

Given a formula φ we say it has the $\oplus$ property if and only if the theory $\{\varphi\}$ has the $\oplus$ property.

6.3.2. Corollary. *Every classical theory has the $\oplus$ property.*

Proof:
This is a direct consequence of Corollary 6.1.25. □

And with the same proof as Theorem 6.2.1 we obtain the following corollary.

6.3.3. Corollary. *If a theory Φ has the $\oplus$ property, then it has the disjunction property.*

This also leads to the question whether the $\oplus$ property and the disjunction property actually coincide. The answer is no, as we will show with the next results.

6.3.4. Lemma. *There exists a theory with the disjunction property and without the $\oplus$ property.*

Proof:
Consider the signature $\{c, d\}$ with c and d constant symbols, the model $\mathcal{M}$ depicted in Figure 6.9 on the left and the set $\text{Th}(\mathcal{M}) = \{\varphi \mid \mathcal{M} \models \varphi\}$. This theory clearly has the disjunction property as

$$\begin{aligned} \text{Th}(\mathcal{M}) \models \varphi \vee\!\!\vee \psi &\iff \mathcal{M} \models \varphi \vee\!\!\vee \psi \\ &\iff \mathcal{M} \models \varphi \text{ or } \mathcal{M} \models \psi \\ &\iff \text{Th}(\mathcal{M}) \models \varphi \text{ or } \text{Th}(\mathcal{M}) \models \psi \end{aligned}$$

w_0
■ $\boldsymbol{c}$
▧ $\boldsymbol{d}$

$\langle w_0, 0\rangle$	$\langle w_0, 1\rangle$
■ $\langle \boldsymbol{c}, \boldsymbol{c}\rangle$ ▧ $\langle \boldsymbol{d}, \boldsymbol{c}\rangle$	■ $\langle \boldsymbol{c}, \boldsymbol{c}\rangle$ ■ $\langle \boldsymbol{d}, \boldsymbol{c}\rangle$
■ $\langle \boldsymbol{c}, \boldsymbol{d}\rangle$ ▧ $\langle \boldsymbol{d}, \boldsymbol{d}\rangle$	▧ $\langle \boldsymbol{c}, \boldsymbol{d}\rangle$ ▧ $\langle \boldsymbol{d}, \boldsymbol{d}\rangle$

Figure 6.9

Moreover it does not have the $\oplus$ property as the formula

$$\forall x.\forall y.(x = y \vee\!\!\!\vee x \neq y)$$

is supported at $\mathcal{M}$ but not at $\mathcal{M} \oplus \mathcal{M}$ (Figure 6.9 on the right).

□

Using Corollary 6.3.3 we can extend Theorem 6.2.1 to a larger class of theories, namely the *quasi* $\vee\!\!\!\vee$ *theories*.

6.3.5. Definition (q$\vee\!\!\!\vee$-free formulas). A formula of InqBQ is called *quasi* $\vee\!\!\!\vee$ *free* (q$\vee\!\!\!\vee$-free) if it is generated by the following grammar.

$$\psi \ ::= \ \alpha \mid \psi \wedge \psi \mid \varphi \to \psi \mid \overline{\exists} x.\psi \mid \forall x.\psi$$

where α ranges over classical formulas and φ ranges over formulas of InqBQ. A theory is called q$\vee\!\!\!\vee$-free if it contains only q$\vee\!\!\!\vee$-free formulas.

Basically a formula is q$\vee\!\!\!\vee$-free if and only if every occurrence of the symbol $\vee\!\!\!\vee$ (if there are any) appears in the antecedent of an implication. Thus for example the formula $(R(x) \to P(x) \vee\!\!\!\vee Q(x)) \to P(x)$ is q$\vee\!\!\!\vee$-free, while the formula $P(x) \to (R(x) \to P(x) \vee\!\!\!\vee Q(x))$ is *not* q$\vee\!\!\!\vee$-free.

Notice that this fragment is strictly more expressive than the classical fragment. For example the formula $\overline{\exists} x.P(x)$ is not truth-conditional, and so it is not equivalent to any classical formula (Theorem 2.1.18).

6.3.6. Lemma. *A q$\vee\!\!\!\vee$-free theory has the $\oplus$ property.*

Proof:
We will show by induction on the structure of φ a q$\vee\!\!\!\vee$-free formula that φ has the $\oplus$ property.In the rest of the proof $\mathcal{M}$ and $\mathcal{N}$ will indicate two generic information structures; g and h will indicate two generic assignments on $\mathcal{M}$ and $\mathcal{N}$ respectively. Moreover we indicate with the abbreviation IH the inductive hypothesis.

Case $\varphi \equiv \alpha$ this case coincides with Corollary 6.3.2.

Case $\varphi \equiv \psi \wedge \chi$

$$\begin{aligned}
& \mathcal{M} \vDash_g \psi \wedge \chi \text{ and } \mathcal{N} \vDash_g \psi \wedge \chi \\
\iff & \mathcal{M} \vDash_g \psi \text{ and } \mathcal{M} \vDash_g \chi \text{ and } \mathcal{N} \vDash_h \psi \text{ and } \mathcal{N} \vDash_h \chi \qquad \text{(by IH)} \\
\iff & \mathcal{M} \oplus \mathcal{N} \vDash_{\langle g,h\rangle} \psi \text{ and } \mathcal{M} \oplus \mathcal{N} \vDash_{\langle g,h\rangle} \chi \\
\iff & \mathcal{M} \oplus \mathcal{N} \vDash_{\langle g,h\rangle} \psi \wedge \chi
\end{aligned}$$

Case $\varphi \equiv \psi \to \chi$

$$\begin{aligned}
& \mathcal{M} \vDash_g \psi \to \chi \text{ and } \mathcal{N} \vDash_h \psi \to \chi \\
\iff & \begin{cases} \forall s \subseteq W^{\mathcal{M}}.\ \big[\mathcal{M}_s \vDash_g \psi \implies \mathcal{M}_s \vDash_g \chi\big] \\ \forall t \subseteq W^{\mathcal{N}}.\ [\mathcal{N}_t \vDash_h \psi \implies \mathcal{N}_t \vDash_h \chi] \end{cases} \qquad \left(\begin{array}{c}\text{by Corollary 6.1.22} \\ \text{and IH}\end{array}\right) \\
\implies & \forall s \subseteq W^{\mathcal{M}}. \forall t \subseteq W^{\mathcal{N}}.\ \big[\mathcal{M}_s \oplus \mathcal{N}_t \vDash_{\langle g,h\rangle} \psi \implies \mathcal{M}_s \oplus \mathcal{N}_t \vDash_{\langle g,h\rangle} \chi\big] \\
\iff & \forall u \subseteq W^{\mathcal{M}\oplus\mathcal{N}}.\ \big[(\mathcal{M} \oplus \mathcal{N})_u \vDash_{\langle g,h\rangle} \psi \implies (\mathcal{M} \oplus \mathcal{N})_u \vDash_{\langle g,h\rangle} \chi\big] \\
\iff & \mathcal{M} \oplus \mathcal{N} \vDash_{\langle g,h\rangle} \psi \to \chi
\end{aligned}$$

Case $\varphi \equiv \overline{\exists} x.\psi$

$$\begin{aligned}
& \mathcal{M} \vDash_g \overline{\exists} x.\psi \text{ and } \mathcal{N} \vDash_h \overline{\exists} x.\psi \\
\iff & \exists d \in D^{\mathcal{M}}.\ \mathcal{M} \vDash_{g[x\mapsto d]} \psi \text{ and } \exists e \in D^{\mathcal{N}}.\ \mathcal{N} \vDash_{h[x\mapsto e]} \psi \\
\iff & \exists d \in D^{\mathcal{M}}.\ \exists e \in D^{\mathcal{N}}.\ \big[\mathcal{M} \vDash_{g[x\mapsto d]} \psi \text{ and } \mathcal{N} \vDash_{h[x\mapsto e]} \psi\big] \qquad \text{(by IH)} \\
\iff & \exists \langle d, e\rangle \in D^{\mathcal{M}\oplus\mathcal{N}}.\ \mathcal{M} \oplus \mathcal{N} \vDash_{\langle g,h\rangle[x\mapsto\langle d,e\rangle]} \psi \\
\iff & \mathcal{M} \oplus \mathcal{N} \vDash_{\langle g,h\rangle} \overline{\exists} x.\psi
\end{aligned}$$

Case $\varphi \equiv \forall x.\psi$ the same passages as case $\varphi \equiv \overline{\exists} x.\psi$ apply here.

□

6.4 Conclusions

In this chapter we gave a proof of a conjecture formulated by Ciardelli [2016], namely that the disjunction and existence properties hold in InqBQ for every classical theory. Moreover we found a generalization of the disjunction property and determined two classes of theories with interesting features connected to said property, namely theories with the $\oplus$ property and q$\mathbin{\rotatebox[origin=c]{180}{$\wedge$}}$-free theories.

To do this, a toolkit of model-theoretic constructions was developed in Section 6.1. These constructions proved to be effective instruments to study the semantics and entailment of InqBQ, and gave more insight into the mechanisms governing this semantics. In view of this, we hope to have laid out the foundations of a model-theoretic approach to the study of InqBQ that could potentially lead to discover other meta-logical properties of the logic.

One additional remark: throughout the chapter, several definitions and results made a fundamental use of the varying identity of information models: to define $\mathcal{BM}$; to define the product of a model and a set/a skeleton; to define $\mathcal{M}_\Gamma$; to combine models by means of the operator $\oplus$. One may wonder if we obtain the same logic or a logic with similar properties if we restrict our attention to the class of models that interpret identity rigidly. The result of this chapter gives a negative answer to both these questions, as it can be easily proven that the disjunction and existence properties *do not hold* when restricting to this particular class of models.

We conclude the chapter with some questions of interest which, as of now, remain open. To summarize the results of Section 6.3, we have the following classes of theories

TC: the class of truth-conditional theories.

Cl: the class of theories equivalent to a classical theory.

q⩔f: the class of theories logically equivalent to a q⩔-free theory.

$\oplus$P: the class of theories with the $\oplus$ property.

DP: the class of theories with the disjunction property.

and the following hierarchy

$$\texttt{TC} \underset{\text{Theorem 2.1.18}}{=} \texttt{Cl} \subseteq \texttt{q}⩔\texttt{f} \underset{\text{Lemma 6.3.6}}{\subseteq} \oplus\texttt{P} \underset{\text{Lemma 6.3.4}}{\subsetneq} \texttt{DP}$$

Some questions on this hierarchy remain open, the first being the following: do the classes q⩔f and $\oplus$P coincide, as in the case of TC and Cl?

Another more conceptual question arises from this hierarchy: notice that we can divide the descriptions of the classes presented here into two kinds, *semantic descriptions* and *syntactic descriptions*. For example, the class TC is described semantically, as truth-conditionality of a theory Φ is a condition on the class of models of Φ (i.e., that it is closed under the operation $\biguplus$ for arbitrary families). On the other hand the class Cl, although it coincides with TC, is described syntacticly as it is the class of theories of a certain fragment of the language (modulo equivalence).

In this regard DP is described neither purely syntactically, as we explicitly refer to the entailment relation, nor purely semantically, as the condition strongly focuses on inquisitive disjunctions. Are there syntactic and semantic conditions characterizing the class DP? The same question can be asked for the classes q⩔f and $\oplus \mathtt{P}$. Is there a semantic condition characterizing the class $\mathtt{q} \mathbin{\vee\!\!\vee} \mathtt{f}$? Is there a syntactic condition characterizing the class $\oplus \mathtt{P}$?

If we move to the existence property, the same questions arise. Can we find a similar hierarchy for the existence property? And can we find syntactic and semantic characterizations for the existence property?

Chapter 7

Classical Antecedent Fragment

Although a recursive axiomatization has been found for several inquisitive logics [Ciardelli, 2014, Ciardelli et al., 2020, Ciardelli and Roelofsen, 2011, Punčochář, 2015, 2019, Sano, 2011], it is still not known whether InqBQ admits one. A sound natural deduction system for InqBQ has been proposed by Ciardelli [2016, Ch. 4], together with a conjecture of its completeness, which as of now remains open. In [Ciardelli, 2016, Chapter 4] it is also shown that two fragments of the logic—the *mention-some fragment* $\mathcal{L}_{\exists}$ and the *mention-all fragment* $\mathcal{L}_{\forall}$—can be recursively axiomatized. This leads to the questions whether there are other interesting fragments or variations of InqBQ which are axiomatizable, and whether we can find novel techniques to axiomatize them. In the following two chapters we give positive answers to these questions: we introduce two new fragments—the *classical antecedent fragment* ClAnt and the *bounded-width fragment* BW—and a class of logics "approximating" InqBQ—the *finite-width inquisitive logics* InqBQ_n—and we present some variations of the canonical model technique to axiomatize them.

This chapter focuses on the classical antecedent fragment ClAnt, which extends $\mathcal{L}_{\forall}$ and $\mathcal{L}_{\exists}$. It can be intuitively characterized as the fragment in which questions are not allowed in the antecedent of an implication. This fragment is particularly interesting since it contains—modulo logical equivalence—all formulas corresponding to natural language statements and several classes of formulas corresponding to natural language questions: for example polar questions ("Will Joey come to the party?"), alternative questions ("Is Joey coming to the party or is Chandler coming?"), mention-some and mention-all questions ("Who, for example, is coming to the party?", "Who exactly is coming to the party?"), and their conditional versions ("If Chandler comes to the party, will Joey come to the party too?"). We prove that the natural deduction system proposed for InqBQ by Ciardelli [2016], restricted to ClAnt, provides a *sound and strongly complete* axiomatization of InqBQ validities in the fragment.

7.1 ClAnt Fragment

In this section we present the main protagonist of this chapter, the ClAnt fragment.

7.1.1. Definition (ClAnt fragment). The ClAnt fragment is generated by the following grammar:

$$\varphi ::= \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid \alpha \to \varphi \mid \forall x.\varphi \mid \overline{\exists} x.\varphi$$

where p ranges over atomic formulas and α ranges over classical formulas.

In [Ciardelli, 2016, Ch. 4], two other fragments were presented and studied, namely the *mention-some* ($\mathcal{L}_{\overline{\exists}}$) and the *mention-all* fragment ($\mathcal{L}_{\forall}$). These two fragments are generated by the following grammars respectively:

$\mathcal{L}_{\overline{\exists}}$	Mention-some:	$\varphi ::= \alpha \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid \overline{\exists} x.\varphi \mid \varphi \wedge \varphi \mid \alpha \to \varphi$
$\mathcal{L}_{\forall}$	Mention-all:	$\varphi ::= \alpha \mid ?\alpha \mid \forall x.\varphi$

where α ranges over classical formulas.

It was proven by Ciardelli [2016] that the support relation for inquisitive logic restricted to both these fragments is finitely axiomatizable.[1] Interestingly, the proofs presented are quite different and cannot, *prima facie*, be adapted to the other fragment. For the mention-some fragment, the completeness proof uses a canonical model construction similar to the one proposed for propositional inquisitive logic by Ciardelli [2016, Ch. 3], heavily relying on the existence of a disjunctive normal form for formulas. For the mention-all fragment, the completeness proof passes through a translation to the *Logic of Interrogation* [Groenendijk, 1999, ten Cate and Shan, 2007], a logic with a partition-based semantics.

Notice that ClAnt subsumes both these fragments. So the axiomatization for ClAnt and the corresponding completeness proof, presented in Section 7.3 and 7.4 respectively, introduce a novel approach to axiomatize both $\mathcal{L}_{\overline{\exists}}$ and $\mathcal{L}_{\forall}$. Moreover, ClAnt is strictly more expressive than both these fragments, as shown by the following result.

7.1.2. Proposition. *The sentence* $\forall x.\overline{\exists} y.(P(x) \leftrightarrow \neg P(y))$ *is in* ClAnt *and it is not logically equivalent to any formula in* $\mathcal{L}_{\overline{\exists}} \cup \mathcal{L}_{\forall}$*.*

This formula holds if for every element x there is an associated element y such that *exactly one of them* has property P. This condition is particularly interesting in contexts where epistemic identity does not correspond to ontological identity, such as inquisitive logic (see [Ciardelli, 2016, Sec. 4.3.4] for a small discussion on the topic).

[1] In the scope of this dissertation, a semantics relation is considered finitely axiomatizable if it can be described in terms of finitely many axioms, schemata (ranging over all the formulas of the language or over formulas in given syntactic fragments) and rules of an opportune axiomatic system. For example, the systems presented in Figures 2.5 and 2.6 are regarded as finite axiomatizations.

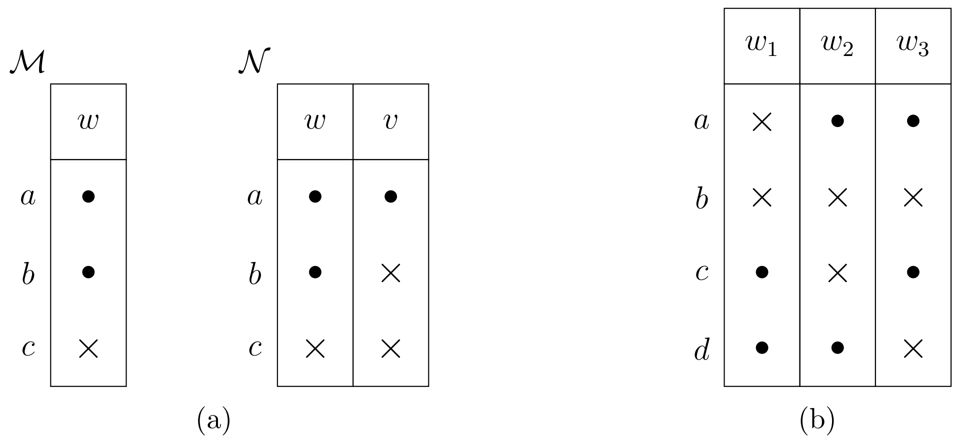


Figure 7.1: Three models in the syntax $\mathcal{L}_{\neq}$ and in the signature $\Sigma = \{P\}$ for P a unary predicate symbol, used in the proof of Proposition 7.1.2.

Proof:
In the scope of this proof, we will use the notation $\theta := \forall x.\overline{\exists} y.(P(x) \leftrightarrow \neg P(y))$. θ is clearly in ClAnt—notice that $P(x) \leftrightarrow \neg P(y)$ is a classical formula.

To show that θ is not equivalent to a formula in $\mathcal{L}_{\overline{\exists}}$, consider the models depicted in Figure 7.1a. It is straightforward to verify that $\mathcal{M} \vDash \theta$, while $\mathcal{N} \nvDash \theta$. Assume towards a contradiction that θ is equivalent to a formula in $\mathcal{L}_{\overline{\exists}}$. By the normal form described in [Ciardelli, 2016, Proposition 4.7.2], this means that

$$\theta \equiv \overline{\exists}\overline{x}_1.\alpha_1 \;\vee\!\!\!\vee\; \dots \;\vee\!\!\!\vee\; \overline{\exists}\overline{x}_n.\alpha_n$$

for some classical formulas $\alpha_1, \dots, \alpha_n$. In particular, this means that for some $i \in \{1, \dots, n\}$ we have $\mathcal{M} \vDash \overline{\exists}\overline{x}_i.\alpha_i$, that is, $\mathcal{M} \vDash_g \alpha_i$ for some assignment g. Since the extension of P in $\mathcal{M}$ consists of a and b, a straightforward induction over the the structure of α_i shows that $\mathcal{M} \vDash_h \alpha_i$ for h defined as follows:

$$h(x) = \begin{cases} g(x) & \text{if } g(x) \in \{a, c\} \\ a & \text{if } g(x) = b \end{cases}$$

Similarly, since the image of h is contained in $\{a, c\}$, another straightforward induction over the structure of α_i shows that $\mathcal{N} \vDash_h \alpha_i$, and consequently $\mathcal{N} \vDash \theta$. And this is a contradiction, as desired.

To show that θ is not equivalent to a formula in $\mathcal{L}_{\forall}$ we use [Ciardelli, 2016, Proposition 4.8.4], which states that every formula $\varphi \in \mathcal{L}_{\forall}$ is *pair-distributive*, that is:

$$\mathcal{M}, s \vDash_g \varphi \qquad \text{iff} \qquad \forall t \subseteq s. \left[\, |t| < 2 \implies \mathcal{M}, t \vDash_g \varphi \,\right]$$

So we just need to show that θ is not pair-distributive: given the model in Figure 7.1b, every state t with at most two worlds satisfies θ, but the whole model does not. $\square$

We conclude this section with an alternative presentation of the ClAnt fragment.

7.1.3. LEMMA. *Every formula in* ClAnt *is equivalent to a formula generated by the following grammar:*

$$\varphi ::= \alpha \mid \varphi \wedge \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid \forall x.\varphi \mid \overline{\exists} x.\varphi$$

where α ranges over classical formulas.

To distinguish between the two syntaxes, we will indicate with $\mathcal{F}^{\mathsf{ClAnt}}$ the set of formulas generated by the grammar in Definition 7.1.1 and with $\mathcal{F}^{\mathsf{ClAnt2}}$ the set of formulas generated by the grammar just introduced in Lemma 7.1.3.

Proof sketch:
The main idea of the proof is to "massage" the implications toward the classical formulas using the following equivalences, taking care of renaming the bounded variables when necessary.

$$\begin{array}{lcll}
\alpha \to \varphi \vee\!\!\!\vee \psi & \equiv & (\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi) & \\
\alpha \to \varphi \wedge \psi & \equiv & (\alpha \to \varphi) \wedge (\alpha \to \psi) & \\
\alpha \to (\beta \to \varphi) & \equiv & (\alpha \wedge \beta) \to \varphi & \\
\alpha \to \forall x.\varphi & \equiv & \forall x.(\alpha \to \varphi) & \text{(For } x \text{ not free in } \alpha) \\
\alpha \to \overline{\exists} x.\varphi & \equiv & \overline{\exists} x.(\alpha \to \varphi) & \text{(For } x \text{ not free in } \alpha)
\end{array}$$

Notice that the equivalences read from left to right reduce the complexity of the consequent of the implication. □

This result tells us that we can dispense with implications outside of classical formulas. At the level of expressive power this is a significant limitation, since → is the only logical operator acting as a second-order quantifier for the semantics—compare with Definition 2.1.7. It is not clear yet whether the completeness proof presented in the following sections relies on this limitation or it can be generalized to more expressive fragments, or even the whole logic. What is known, is that ClAnt is strictly less expressive than InqBQ, as the following result shows.

7.1.4. PROPOSITION. *The formula $\theta := \forall x.?P(x) \to ?r$ (for P a unary relation symbol and r a 0-ary relation symbol) is not logically equivalent to any formula in* ClAnt.

The proof of this result relies on yet another variation of the Ehrenfeucht-Fraïssé game we presented in Chapter 4, which captures the expressive power of the ClAnt fragment. We are going to present the game and its properties in the following section.

7.2 Ehrenfeucht-Fraïssé Game for ClAnt

In Chapter 4 we introduce a model-theoretic game that succesfully captures the notion of support-equivalence between models ($\equiv$), and more fine-grained relations meaningful to describe and study the expressive power of the logic ($\equiv_{i,q}$). We can achieve the same kind of result also for *fragments* of the logic, by modifying the game in a suitable way. We will now define a variation of the game which captures the expressive power of the ClAnt fragment.

Position: A position in the game is a tuple $\langle \mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b} \rangle$—the same as in the original version of the game.

Possible moves: Starting a round from a position $\langle \mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b} \rangle$, the following are the possible moves:

$\overline{\exists}$**-move:** S picks an element $a' \in D_{\mathcal{M}}$; D responds with an element $b' \in D_{\mathcal{N}}$; the game continues from position $\langle \mathcal{M}, s, \overline{a}a'; \mathcal{N}, t, \overline{b}b' \rangle$;

$\forall$**-move:** S picks an element $b' \in D_{\mathcal{N}}$; D responds with an element $a' \in D_{\mathcal{M}}$; the game continues from position $\langle \mathcal{M}, s, \overline{a}a'; \mathcal{N}, t, \overline{b}b' \rangle$;

w**-move:** S picks a world $v \in t$; D responds with a world $w \in s$. After that, the game continues from position $\langle \mathcal{M}, \{w\}, \overline{a}; \mathcal{N}, \{v\}, \overline{b} \rangle$. This move must be performed exactly only once during the game (notice that performing it a second time would not change the position).

Notice that the $\rightarrow$-move is replaced by the w-move (world-move). In particular, the chirality cannot be changed during the run.

Termination conditions: A number $q \in \mathbb{N}$ is fixed in advance, which is the number of quantifier-moves that Spoiler is allowed to perform during the run. As stated above, the w-move can be performed only once. When there are no more available moves, the game ends.

Winning condition: Suppose the final position of the game is $\langle \mathcal{M}, \{w\}, \overline{a}; \mathcal{N}, \{v\}, \overline{b} \rangle$. The game is won by Player D if the following condition is satisfied, and by player S otherwise:

Winning condition for D: for all atomic formulas $A(x_1, \dots, x_n)$ where n is the size of the tuples $\overline{a}$ and $\overline{b}$, we have:

$$\mathcal{M}, \{w\} \models A(\overline{a}) \iff \mathcal{N}, \{v\} \models A(\overline{b})$$

As in the symmetric version of the game presented in Subsection 4.5.1, the winning condition is symmetric with respect to the two models.

We will indicate with $\mathrm{EF}_q^{\mathsf{ClAnt}}(\mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b})$ the game just described. We will indicate with $\mathcal{M}, s, \overline{a} \approx_q^{\mathsf{ClAnt}} \mathcal{N}, t, \overline{b}$ the existence of a winning strategy for Duplicator in the game $\mathrm{EF}_q^{\mathsf{ClAnt}}(\mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b})$; and with $\mathcal{M}, s, \overline{a} \approx^{\mathsf{ClAnt}} \mathcal{N}, t, \overline{b}$ the existence of a winning strategy for arbitrary $q \in \mathbb{N}$. We will also use notational conventions analogous to the ones introduced for the original game.

Notice that after performing the w-move the game follows the same rules as the Ehrenfeucht-Fraïssé game for classical logic between the models M_w and N_v. So a run of the game can be divided in two phases: a first phase, before the w-move, in which the game is played as the inquisitive version of the game with no implication-moves; and a second phase, after the w-move, played as the classical version of the game.

To study the relations between this game and the expressive power of ClAnt, we need to introduce an alternative measure of complexity for formulas in the fragment. We define the ClAnt-degree (in symbols, CAdeg) of a formula $\varphi \in \mathcal{F}^{\mathsf{ClAnt}}$ by the following inductive clauses:

$$\begin{aligned}
&\mathrm{CAdeg}(p) &&:= 0\\
&\mathrm{CAdeg}(\bot) &&:= 0\\
&\mathrm{CAdeg}(\varphi \wedge \psi) &&:= \max\{\mathrm{CAdeg}(\varphi), \mathrm{CAdeg}(\psi)\}\\
&\mathrm{CAdeg}(\varphi \mathbin{\vee\!\!\!\vee} \psi) &&:= \max\{\mathrm{CAdeg}(\varphi), \mathrm{CAdeg}(\psi)\}\\
&\mathrm{CAdeg}(\alpha \to \varphi) &&:= \mathrm{Qdeg}(\alpha) + \mathrm{CAdeg}(\varphi)\\
&\mathrm{CAdeg}(\forall x.\varphi) &&:= \mathrm{CAdeg}(\varphi) + 1\\
&\mathrm{CAdeg}(\overline{\exists} x.\varphi) &&:= \mathrm{CAdeg}(\varphi) + 1
\end{aligned}$$

Notice in particular that in the clause for the formula $\alpha \to \varphi$, we consider the *quantifier degree* of α and we sum it to the ClAnt-*degree* of φ. As for the full signature, we indicate with $\mathcal{F}_q^{\mathsf{ClAnt}}$ the ClAnt formulas of CAdeg bounded by q. This allows us to introduce the relations $\sqsubseteq^{\mathsf{ClAnt}}$, $\sqsubseteq_{i,q}^{\mathsf{ClAnt}}$, $\equiv^{\mathsf{ClAnt}}$ and $\equiv_{i,q}^{\mathsf{ClAnt}}$, where we restrict our focus to formulas in ClAnt.

This notion of degree behaves similarly to the quantification degree when we focus on classical formulas. To make this more precise, we need an additional definition.

7.2.1. DEFINITION. Let φ be a ClAnt formula. We say that φ is classically-reduced if every classical subformula α of φ contains implications only in the form of negation, that is, if $\alpha \equiv \beta \to \gamma$ then $\gamma \equiv \bot$.

Notice that every ClAnt formula φ is equivalent to a ClAnt formula meeting the restriction: we just need to substitute the subformulas of the form $\beta \to \gamma$ with the equivalent formula $\neg\beta \vee \gamma$.

The notion of classically-reduced formula will play a crucial role in the proofs that follow. The reason is that the CAdeg and Qdeg coincide for classically-reduced classical formulas.

7.2.2. LEMMA. *Let* α *be a classical formula. Them* $\mathrm{CAdeg}(\alpha) \geq \mathrm{Qdeg}(\alpha)$*. Moreover, if* α *is classically-reduced then* $\mathrm{CAdeg}(\alpha) = \mathrm{Qdeg}(\alpha)$*.*

Proof:
The first statement can be proven by induction on the structure of the formula α. The only non-trivial case is that of implication: suppose α is of the form $\beta \to \gamma$. By inductive hypothesis we have:

$$\mathrm{CAdeg}(\alpha) \;=\; \mathrm{Qdeg}(\beta) + \mathrm{CAdeg}(\gamma) \;\geq\; \mathrm{Qdeg}(\beta) + \mathrm{Qdeg}(\gamma) \;\geq\; \mathrm{Qdeg}(\alpha)$$

Assume now that α is in disjunctive normal form. Then the second statement of the Theorem follows easily from the following identities: for arbitrary classical formulas β and γ we have

$$\begin{aligned}
\mathrm{CAdeg}(\neg\beta) &= \mathrm{Qdeg}(\beta) + \mathrm{CAdeg}(\bot)\\
&= \mathrm{Qdeg}(\neg\beta)\\[1ex]
\mathrm{CAdeg}(\beta \vee \gamma) &= \mathrm{CAdeg}(\neg(\neg\beta \wedge \neg\gamma))\\
&= \mathrm{Qdeg}(\neg\beta \wedge \neg\gamma) + \mathrm{CAdeg}(\bot)\\
&= \mathrm{Qdeg}(\neg(\neg\beta \wedge \neg\gamma))\\
&= \mathrm{Qdeg}(\beta \vee \gamma)\\[1ex]
\mathrm{CAdeg}(\exists x.\beta) &= \mathrm{CAdeg}(\neg\forall x.\neg\beta)\\
&= \mathrm{Qdeg}(\forall x.\neg\beta) + \mathrm{CAdeg}(\bot)\\
&= \mathrm{Qdeg}(\neg\forall x.\neg\beta)\\
&= \mathrm{Qdeg}(\exists x.\beta)
\end{aligned}$$

□

7.2.3. COROLLARY. *Every formula* φ *in* ClAnt *is logically equivalent to a classically-reduced formula* ψ *with* $\mathrm{CAdeg}(\varphi) \geq \mathrm{CAdeg}(\psi)$ *and* $\mathrm{Qdeg}(\varphi) = \mathrm{Qdeg}(\psi)$*.*

Proof:
As noticed above, to obtain a classically-reduced formula ψ starting from φ we can just recursively substitute classical subformulas of the form $\beta \to \gamma$ (for $\gamma \not\equiv \bot$) with $\neg\beta \vee \gamma$. We claim that each of these reduction steps preserves the Qdeg of the formula and does not increase the CAdeg of the formula. If this is the case, we can apply the reduction until we obtain a classically-reduced formula equivalent to the original one.

Suppose at a certain point of the reduction procedure we are starting from the formula $\theta[\beta \to \gamma]$ and we substitute $\beta \to \gamma$ with $\neg\beta \vee \gamma$, obtaining $\theta[\neg\beta \vee \gamma]$.[2] Since

$$\mathrm{Qdeg}(\beta \to \gamma) = \max\{\mathrm{Qdeg}(\beta), \mathrm{Qdeg}(\gamma)\} = \mathrm{Qdeg}(\neg\beta \vee \gamma)$$

[2]In this context, the square bracket notation $\theta[\chi]$ indicates a single occurrence of the subformula χ in the formula φ.

we have $\mathrm{Qdeg}(\theta[\beta \to \gamma]) = \mathrm{Qdeg}(\theta[\neg\beta \vee \gamma])$. Moreover, since

$$\begin{aligned}\mathrm{CAdeg}(\beta \to \gamma) &= \mathrm{Qdeg}(\beta) + \mathrm{CAdeg}(\gamma)\\ &\geq \max\{\mathrm{Qdeg}(\beta), \mathrm{Qdeg}(\gamma)\}\\ &= \mathrm{Qdeg}(\neg\beta \vee \gamma)\\ &= \mathrm{CAdeg}(\neg\beta \vee \gamma) \qquad \text{(by Lemma 7.2.2)}\end{aligned}$$

it follows that $\mathrm{CAdeg}(\theta[\beta \to \gamma]) \geq \mathrm{CAdeg}(\theta[\neg\beta \vee \gamma])$.

□

Before moving to the Ehrenfeucht-Fraïssé theorem we need one last result that will allow us to show the connection between the CAdeg and the game.

Indicate with $t\varphi$ the formula obtained starting from φ using the reduction steps described in the proof of Lemma 7.1.3. Notice that $\mathcal{F}^{\mathsf{ClAnt2}} \subseteq \mathcal{F}^{\mathsf{ClAnt}}$, and so CAdeg is defined also for formulas in $\mathcal{F}^{\mathsf{ClAnt2}}$. In particular, since the reduction steps in the proof above do not increase the CAdeg of the formulas considered, we obtain that $\mathrm{CAdeg}(t\varphi) \leq \mathrm{CAdeg}(\varphi)$.

Moreover, we can combine the results on the CAdeg with Lemma 7.1.3 to obtain the following corollaries.

7.2.4. Corollary. *Let $\varphi \in \mathcal{F}^{\mathsf{ClAnt2}}$ be classically-reduced. Then* $\mathrm{CAdeg}(\varphi) = \mathrm{Qdeg}(\varphi)$.

Proof:
By induction on the structure of the formula φ. The basic case follows from Lemma 7.2.2; the inductive steps follow trivially from the definition of CAdeg. □

7.2.5. Corollary. *Every $\varphi \in \mathcal{F}^{\mathsf{ClAnt}}$ is logically equivalent to a classically-reduced formula $\psi \in \mathcal{F}^{\mathsf{ClAnt2}}$ with* $\mathrm{Qdeg}(\psi) \leq \mathrm{CAdeg}(\varphi)$.

Proof:
Consider the formula ψ obtained by applying first Lemma 7.1.3 and then Corollary 7.2.3 starting from φ. As noticed, both these results do not increase the CAdeg of the formula considered, and so we obtain $\mathrm{CAdeg}(\psi) \leq \mathrm{CAdeg}(\varphi)$. The inequality then follows from Corollary 7.2.4. □

We are finally ready to prove the main result of this section, the Ehrenfeucht-Fraïssé theorem for the game.

7.2.6. Theorem. *Suppose the signature Σ is relational and finite. Then Duplicator has a winning strategy in the game $\mathrm{EF}_q^{\mathsf{ClAnt}}(\mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b})$ iff for every formula $\varphi(\overline{x}) \in \mathcal{F}^{\mathsf{ClAnt2}}$ with $\mathrm{Qdeg}(\varphi(\overline{x})) \leq q$ and every sequence $\overline{x}$ of the same length as $\overline{a}$ and $\overline{b}$ it holds that*

$$\mathcal{M}, s \vDash \varphi(\overline{a}) \implies \mathcal{N}, t \vDash \varphi(\overline{b})$$

Proof:
We prove the result by induction on q. Firstly, suppose $q = 0$. In this case, the game consists of only one w-move. So in particular, the condition of Duplicator having a winning strategy in the game is equivalent to the following condition:

$$\forall v \in t.\ \exists w \in s.\ \forall A(\overline{x}) \text{ atomic formula.} \left[\ \mathcal{M}, \{w\} \vDash A(\overline{a}) \iff \mathcal{N}, \{v\} \vDash A(\overline{b})\ \right]$$

or equivalently

$$\forall v \in t.\ \exists w \in s.\ \left[\ M_w, \overline{a} \equiv_0^{\mathsf{CQC}} N_v, \overline{b}\ \right] \tag{7.1}$$

where $\equiv_q^{\mathsf{CQC}}$ indicates satisfying the same classical formulas of quantifier degree up to q. We are going to see that this last condition is equivalent to

$$\mathcal{M}, s, \overline{a} \preceq_0^{\mathsf{ClAnt}} \mathcal{N}, t, \overline{b} \tag{7.2}$$

In fact:

- Suppose Condition 7.1 holds and assume that for $\varphi(\overline{x})$ a formula of quantifier degree 0 we have $\mathcal{N}, t \nvDash \varphi(\overline{b})$. By the normal form theorem [Ciardelli, 2016, Proposition 2.4.4] we can assume that $\varphi \equiv \alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \alpha_n$ for $\alpha_1, \ldots, \alpha_n$ classical formulas. So we have the following:

$$\begin{aligned} \mathcal{N}, t \nvDash \varphi \quad &\Longrightarrow\ \forall i \in \{1, \ldots, n\}\,.\ \exists v_i \in t.\ N_{v_i} \nvDash \alpha_i \\ &\Longrightarrow\ \forall i \in \{1, \ldots, n\}\,.\ \exists w_i \in s.\ M_{w_i} \nvDash \alpha_i \qquad \text{(by Condition 7.1)} \\ &\Longrightarrow\ \mathcal{M}, s \nvDash \varphi \end{aligned}$$

- Suppose Condition 7.2 holds and suppose towards a contradiction that Condition 7.1 does not hold. Spelling out the latter, we have[3]

  $$\exists v \in t.\ \forall w \in s.\ \exists \alpha_w \text{ classical.} \left[\ \mathrm{Qdeg}(\alpha_w) = 0 \text{ and } M_w \vDash \alpha_w(\overline{a}) \text{ and } N_v \nvDash \alpha_w(\overline{b})\ \right]$$

  This in particular implies that $\mathcal{M}, s \vDash \bigvee\!\!\!\bigvee_{w \in s} \alpha_w(\overline{a})$ and $\mathcal{N}, t \nvDash \bigvee\!\!\!\bigvee_{w \in s} \alpha_w(\overline{b})$, and this contradicts the hypothesis that Condition 7.2 holds.

This concludes the basic inductive step.

As for the inductive step, we follow the same structure of the proof of Theorem 4.3.2. We proceed by contradiction: firstly ($\Rightarrow$ direction) we show that we can use a formula φ with $\mathrm{CAdeg}(\varphi) = q$ supported by $\langle \mathcal{M}, s, \overline{a} \rangle$ and not by $\langle \mathcal{N}, t, \overline{b} \rangle$ to define a winning strategy for Duplicator in the game $\mathrm{EF}_q^{\mathsf{ClAnt}}(\mathcal{M}, s\overline{a}; \mathcal{N}, t, \overline{b})$; secondly ($\Leftarrow$ direction) we start from a winning strategy to obtain a formula supported by $\langle \mathcal{M}, s, \overline{a} \rangle$ and not by $\langle \mathcal{N}, t, \overline{b} \rangle$.

[3]Notice that Condition 7.1 entails the existence of a formula α_w distinguishing the two models, but we do not know a priori which model supports α_w and which does not. However, since α_w is classical, we can use either α_w or its negation, thus obtaining a classical formula supported by M_w and not by N_v.

Since this proof has the same structure of the proof of Theorem 4.3.2, we limit ourselves to show the key steps. In particular, the cases for the quantifiers (cases $\Rightarrow^2$, $\Rightarrow^3$, $\Leftarrow^2$ and $\Leftarrow^3$ in the proof of Theorem 4.3.2) are completely analogous. So we only consider the cases for the w-move (cases $\Rightarrow^1$ and $\Leftarrow^1$ in the proof of Theorem 4.3.2).

Case $\Rightarrow^1$: φ is an implication of the form $\beta \to \gamma$ with $\text{CAdeg}(\varphi) = q$ and it holds that

$$\mathcal{M}, s \vDash \varphi(\overline{a}) \qquad\qquad \mathcal{N}, t \nvDash \varphi(\overline{b})$$

Notice that, since $\varphi \in \mathcal{F}^{\mathsf{ClAnt2}}$, β and γ are necessarily classical formulas. In this case we have:

$$\mathcal{N}, t \nvDash \beta \to \gamma \quad \Longrightarrow \quad \exists v \in t. \begin{cases} \mathcal{N}, \{v\} \vDash \beta \\ \mathcal{N}, \{v\} \nvDash \gamma \end{cases}$$
$$\mathcal{M}, s \vDash \beta \to \gamma \quad \Longrightarrow \quad \nexists w \in s. \begin{cases} \mathcal{M}, \{w\} \vDash \beta \\ \mathcal{M}, \{w\} \nvDash \gamma \end{cases}$$

So by main inductive hypothesis, if Spoiler performs a w-move and chooses $v \in t$ as above, for every choice w of Duplicator we have $(M_w, \overline{a}) \not\approx_q^{\mathsf{CQC}} (N_v, \overline{b})$. Since after performing the w-move the game follows the rules of the EF-game for classical first order logic, it follows that Spoiler has a winning strategy from this point of the game.

Case $\Leftarrow^1$: Duplicator has a winning strategy in the game $\text{EF}_q^{\mathsf{ClAnt}}(\mathcal{M}, s, \overline{a}; \mathcal{N}, t, \overline{b})$ and the first move is a w-move.

Suppose that Spoiler starts by choosing $v \in t$. As this is a winning strategy for Spoiler, for every choice $w \in s$ of Duplicator we have

$$(\mathcal{M}, \{w\}, \overline{a}) \not\approx_q^{\mathsf{ClAnt}} (\mathcal{N}, \{v\}, \overline{b})$$

or stated in terms of the corresponding classical structures

$$(M_w, \overline{a}) \not\approx_q^{\mathsf{CQC}} (N_v, \overline{b})$$

So, by properties of the EF-game for classical first order logic, there exists a classical formula β_w with $\text{Qdeg}(\beta_w) \leq q$ such that

$$M_w \vDash \beta_w(\overline{a}) \qquad\qquad N_v \nvDash \beta_w(\overline{b})$$

By persistency of the support semantics (Lemma 2.1.9), it follows that

$$\mathcal{M}, s \vDash \bigvee_{w \in s} \beta_w(\overline{a}) \qquad\qquad \mathcal{N}, t \nvDash \bigvee_{w \in s} \beta_w(\overline{b})$$

If we show that $\mathrm{CAdeg}(\mathbb{V}_{w\in s}\beta_w) \leq q$, then we can conclude that there exists a formula in $\mathcal{F}^{\mathsf{ClAnt2}}$ which is supported by $\langle \mathcal{M}, s, \overline{a}\rangle$ and not by $\langle \mathcal{N}, t, \overline{b}\rangle$.

Notice that by Corollary 7.2.3, we can assume that β_w is classically-reduced. By Lemma 7.2.2 this means that $\mathrm{CAdeg}(\beta_w) = \mathrm{Qdeg}(\beta_w) \leq q$ for every $w \in s$, and consequently that $\mathrm{CAdeg}(\mathbb{V}_{w\in s}\beta_w) \leq q$. This concludes the proof.

□

7.2.7. Corollary.

$$(\mathcal{M}, s, \overline{a}) \preceq_q^{\mathsf{ClAnt}} (\mathcal{N}, t, \overline{b}) \quad \Longleftrightarrow \quad (\mathcal{M}, s, \overline{a}) \sqsubseteq_q^{\mathsf{ClAnt}} (\mathcal{N}, t, \overline{b})$$

Proof:
For the left-to-right direction, suppose $(\mathcal{M}, s, \overline{a}) \preceq_q^{\mathsf{ClAnt}} (\mathcal{N}, t, \overline{b})$. Consider a formula $\varphi \in \mathcal{F}^{\mathsf{ClAnt}}$ with $\mathrm{CAdeg}(\varphi) \leq q$. By Corollary 7.2.5, φ is logically equivalent to a classically-reduced formula $\psi \in \mathcal{F}^{\mathsf{ClAnt2}}$ with $\mathrm{Qdeg}(\psi) \leq \mathrm{CAdeg}(\varphi)$. So, by Theorem 7.2.6 we have:

$$\mathcal{M}, s \vDash \varphi(\overline{a}) \iff \mathcal{M}, s \vDash \psi(\overline{a}) \implies \mathcal{N}, s \vDash \psi(\overline{b}) \iff \mathcal{N}, t \vDash \varphi(\overline{b})$$

Since φ was an arbitrary formula in $\mathcal{F}^{\mathsf{ClAnt}}$, we have $(\mathcal{M}, s, \overline{a}) \sqsubseteq_q^{\mathsf{ClAnt}} (\mathcal{N}, t, \overline{b})$.

For the right-to-left direction, suppose that $(\mathcal{M}, s, \overline{a}) \sqsubseteq_q^{\mathsf{ClAnt}} (\mathcal{N}, t, \overline{b})$. Let $\varphi \in \mathcal{F}^{\mathsf{ClAnt2}}$ with $\mathrm{Qdeg}(\varphi) \leq q$. Applying Corollary 7.2.3 to φ, we obtain a classically-reduced formula $\psi \in \mathcal{F}^{\mathsf{ClAnt2}}$ equivalent to φ with $\mathrm{Qdeg}(\varphi) = \mathrm{Qdeg}(\psi)$.[4] Moreover, by Corollary 7.2.4, $\mathrm{CAdeg}(\psi) = \mathrm{Qdeg}(\psi)$

By hypothesis it follows that

$$\mathcal{M}, s \vDash \varphi(\overline{a}) \iff \mathcal{M}, s \vDash \psi(\overline{a}) \implies \mathcal{N}, s \vDash \psi(\overline{b}) \iff \mathcal{N}, t \vDash \varphi(\overline{b})$$

Since φ was an arbitrary formula in $\mathcal{F}^{\mathsf{ClAnt2}}$with $\mathrm{Qdeg}(\varphi) \leq q$, by Theorem 7.2.6 if follows $(\mathcal{M}, s, \overline{a}) \preceq_q^{\mathsf{ClAnt}} (\mathcal{N}, t, \overline{b})$.

□

This result allows us to study the expressive power of the ClAnt fragment. An example of this is given by the following theorem.

7.2.8. Theorem. *Let $\mathcal{M}$ and $\mathcal{N}$ be information models and suppose that $q := |D^{\mathcal{M}}| + |D^{\mathcal{N}}|$ is finite. Then the following are equivalent:*

1. $\mathcal{M} \hookrightarrow \mathcal{N}$;

[4]Notice that the reduction procedure described in the proof of Corollary 7.2.3 preserves the property of being a formula in $\mathcal{F}^{\mathsf{ClAnt2}}$.

2. $\mathcal{N} \sqsubseteq_q^{\mathsf{ClAnt}} \mathcal{M}$.

Proof:
By Corollary 7.2.7, Condition 2 is equivalent to $\mathcal{N} \preceq_q^{\mathsf{ClAnt}} \mathcal{M}$. So to prove the implication from Condition 1 to Condition 2 we just need to exhibit a winning strategy for Duplicator in the game $\mathrm{EF}_q^{\mathsf{ClAnt}}(\mathcal{N}, \mathcal{M})$. We can reason as in the proof of Theorem 4.5.5. Given f and g witnessing $\mathcal{M} \hookrightarrow \mathcal{N}$, the winning strategy for Duplicator is as follows:

- If Spoiler plays a quantifier move and picks an element $a \in \mathcal{M}$, then Duplicator picks $g(a) \in \mathcal{N}$.
- If Spoiler plays a quantifier move and picks an element $b \in \mathcal{N}$, then Duplicator picks $a \in \mathcal{M}$ such that $f(a) \sim^e b$ (such an element exists by definition of submodel 3.2.3).
- When Spoiler plays the w-move and picks $w \in W^{\mathcal{M}}$, Duplicator responds by picking $v = f(w) \in W^{\mathcal{N}}$.

This is a winning strategy in the game since by definition of f, g:

$$\mathcal{M}, \{w\} \vDash A(d_1, \ldots, d_n) \iff \mathcal{N}, \{f(w)\} \vDash A(g(d_1), \ldots, g(d_n))$$

To show the implication from Condition 2 to Condition 1, we will again reason as in the proof of Theorem 4.5.5: we will play the part of Spoiler in the game $\mathrm{EF}_q(\mathcal{N}, \mathcal{M})$ and use the winning strategy of Duplicator to build the functions f and g.

Consider enumeration $\langle a_1, \ldots, a_{|D^{\mathcal{M}}|} \rangle$ of the elements of $\mathcal{M}$; and an enumeration $\langle b_1, \ldots, b_{|D^{\mathcal{N}}|} \rangle$ of the elements of $\mathcal{N}$. Fix a winning strategy for Duplicator:

1. Firstly Spoiler performs $|D^{\mathcal{N}}|$ many $\overline{\exists}$-moves: each time Spoiler chooses a distinct element b_j of $\mathcal{N}$, so that he covers the whole domain; let $h(b_j) \in D^{\mathcal{M}}$ to be the element that Duplicator chooses in response.
2. Secondly Spoiler performs $|D^{\mathcal{M}}|$ many $\forall$-moves: each time Spoiler chooses a distinct element a_i of $\mathcal{M}$, so that he covers the whole domain; define $g(a_i) \in D^{\mathcal{N}}$ to be the element that Duplicator chooses in response.
3. After the previous moves, Spoiler can perform a w-move and choose a world $w \in W^{\mathcal{M}}$. Following the winning strategy, Duplicator has to answer with a world $v \in W^{\mathcal{N}}$. For every possible choice $w \in W^{\mathcal{M}}$ define $f(w) := v \in W^{\mathcal{N}}$ the world chosen by Duplicator following the strategy.

Now that we defined f and g, showing that they respect the properties of Definition 3.2.3 can be done exactly as in the proof of Theorem 4.5.5.

□

Using Theorem 7.2.6 we can finally show that ClAnt is strictly less expressive than InqBQ, that is, we can give a proof of Proposition 7.1.4.

Proof of Proposition 7.1.4:
To prove this result, for a given natural number q we will present two models $\mathcal{M}$ and $\mathcal{N}$ such that

1. $\mathcal{M} \not\models \theta$;
2. $\mathcal{N} \models \theta$;
3. $\mathcal{M} \equiv_q^{\mathsf{ClAnt}} \mathcal{N}$.

This implies that the formula θ is not equivalent to any ClAnt formula of CAdeg at most q; and since q is arbitrary, it follows that θ is not equivalent to any ClAnt formula.

Consider the model $\mathcal{G}$ defined by the following clauses:

- $W^{\mathcal{G}} = \{ \langle A, e\rangle \mid A \in \mathcal{P}_{fin}(\mathbb{N}) \text{ and } e \in \{+,-\} \}$;
- $D^{\mathcal{G}} = \mathbb{N}$;
- $P_{\langle A,e\rangle}(n)$ iff $n \in A$;
- $r_{\langle A,e\rangle}$ iff $e = +$.

We define $\mathcal{M}$ and $\mathcal{N}$ to be restrictions of $\mathcal{G}$ to certain information states. In particular:

$$\mathcal{M} = \mathcal{G}|_{s_\mathcal{M}} \text{ for } s_\mathcal{M} := \left\{ \langle A, e\rangle \in W^{\mathcal{G}} \;\middle|\; \begin{array}{ll} & (\ |A| \leq q \text{ and } e = +\ ) \\ \text{or} & (\ |A| > q \qquad\qquad\quad\ ) \end{array} \right\}$$

$$\mathcal{N} = \mathcal{G}|_{s_\mathcal{N}} \text{ for } s_\mathcal{N} := \left\{ \langle A, e\rangle \in W^{\mathcal{G}} \;\middle|\; \begin{array}{ll} & (\ |A| \leq q \text{ and } e = + \qquad\qquad\qquad\quad\ ) \\ \text{or} & (\ |A| > q \text{ and } |A| \text{ even and } e = +\ ) \\ \text{or} & (\ |A| > q \text{ and } |A| \text{ odd and } e = -\ \ ) \end{array} \right\}$$

The formula θ is supported by an information state s iff s does not contain two worlds with the same extension of P and that assign two different truth-values to r. In particular, an information state s of $\mathcal{G}$ supports θ iff s contains at most one between $\langle A,+\rangle$ and $\langle A,-\rangle$, for every $A \in \mathcal{P}_{fin}(\mathbb{N})$. It readily follows that $\mathcal{M} \not\models \theta$ and $\mathcal{N} \models \theta$.

It remains to show that $\mathcal{M} \equiv_q^{\mathsf{ClAnt}} \mathcal{N}$. Notice that $s_\mathcal{N} \subseteq s_\mathcal{M}$, so we have $\mathcal{M} \sqsubseteq \mathcal{N}$, and consequently $\mathcal{M} \sqsubseteq_q^{\mathsf{ClAnt}} \mathcal{N}$. To prove the converse, that is, $\mathcal{N} \sqsubseteq_q^{\mathsf{ClAnt}} \mathcal{M}$, we will use the variation of the Ehrenfeucht-Fraïssé game for ClAnt. By Corollary 7.2.7, we just need to prove $\mathcal{N} \preceq_q^{\mathsf{ClAnt}} \mathcal{M}$. We describe a strategy for Duplicator in the game $\mathrm{EF}_q^{\mathsf{ClAnt}}(\mathcal{N}, \mathcal{M})$, and prove it is indeed a winning strategy:

- If Spoiler plays a quantifier move *before the w-move* and picks an element n in one of the models, then Duplicator picks the same element in the other model.
- If Spoiler plays the w-move and picks a world $\langle A, e\rangle$ of $\mathcal{M}$, then Duplicator has to pick a world $\langle B, e'\rangle$ of $\mathcal{N}$. We consider two cases:
  - Either $|A| \leq q$, in which case Duplicator picks $\langle B, e'\rangle = \langle A, e\rangle$ in $\mathcal{N}$;
  - Or $|A| > q$. Suppose that the elements chosen in the previous turns are $a_1, \ldots, a_k \in \mathbb{N}$ (the same in both models). Then Duplicator picks $\langle B, e'\rangle$ such that

    $$a_i \in A \iff a_i \in B \qquad |B| > q \qquad |B| \text{ even iff } e = +$$

    Notice that, by definition of $\mathcal{N}$, the clauses on the cardinality of B entail that $e' = e$.
- If Spoiler plays a quantifier move *after the w-move* and picks an element already picked in previous rounds of the game, then Duplicator picks the corresponding element in the other model. On the other hand, if the element picked is fresh, say the element n of $\mathcal{M}$, then Duplicator picks an element n' such that $n \in A$ iff $n' \in B$. This is always possible since either $A = B$; or $|A|, |A^c|, |B|, |B^c| > q$ and at most q quantifier moves can be performed during a run of the game.

Call $a_1, \ldots, a_k$ the elements picked before the w-move (the same in both models); $d_1, \ldots, d_{q-k}$ the elements picked in $\mathcal{M}$ after the w-move; and $d'_1, \ldots, d'_{q-k}$ the elements picked in $\mathcal{N}$ after the w-move. It remains to show that this is a winning strategy for Duplicator.

- Since we choose $e' = e$ when performing the w-move, we have

  $$\mathcal{M}, \{\langle A, e\rangle\} \vDash r \iff e = e' = + \iff \mathcal{N}, \{\langle B, e'\rangle\} \vDash r$$

- By our choice of $\langle B, e'\rangle$ in the w-move, we have

  $$\mathcal{M}, \{\langle A, e\rangle\} \vDash P(a_i) \iff a_i \in A \iff a_i \in B \iff \mathcal{N}, \{\langle B, e'\rangle\} \vDash P(a_i)$$

- Since we imposed this condition in the moves after the w-move, we have

  $$\mathcal{M}, \{\langle A, e\rangle\} \vDash P(d_i) \iff d_i \in A \iff d'_i \in B \iff \mathcal{N}, \{\langle B, e'\rangle\} \vDash P(d'_i)$$

Thus the winning condition for Duplicator is met, showing that this is indeed a winning strategy. □

7.3 Deductive System

In this section we present the natural deduction system for the ClAnt fragment and we study the properties of some special classes of deductively closed theories.

We will focus on languages not containing rigid symbols, that is, on signatures containing only relational symbols and on either the syntax $\mathcal{L}_{\neq}$ or $\mathcal{L}_{\asymp}$. The reason why we restrict our scope to this case is that the rigidity of function symbols imposes an additional layer of complexity to the proof that the system is complete for the fragment. We leave the generalization of the results of this chapter to arbitrary signatures for future work.

7.3.1. Definition (Natural deduction system for ClAnt). For $\Phi \cup \{\psi\}$ ClAnt formulas we say that Φ derives ψ in ClAnt (in symbols $\Phi \rhd \psi$) if there is a derivation of ψ from Φ *containing only* ClAnt *formulas.*

Indicate with $\vdash$ the consequence relation of the system in Figure 2.5. Clearly if $\Phi \rhd \psi$ then $\Phi \vdash \psi$, but the converse is not true *a priori*; for example, it could be necessary to assume and discharge hypotheses not in ClAnt in every derivation of ψ from Φ.[5]

Notice that if we apply a rule in Figure 2.5—with the exception of $(\rightarrow i)$—to ClAnt formulas, the conclusion produced is again a ClAnt formula. A weaker version of the introduction rule with this property is the following:
So the relation $\rhd$ can be characterized in terms of the deductive system in Figure 7.2, obtained by replacing the rule ($\rightarrow$i) with the rule (ClAnt $\rightarrow$i).

To study the properties of this system and the relations with the system presented in Figure 2.5, we focus on *theories* of InqBQ and of ClAnt, that is, sets of formulas in InqBQ and in ClAnt respectively. Since we need to be particularly careful when handling free variables, we distinguish between *open* and *closed* theories.

7.3.2. Definition. Let Σ be a fixed signature.

- An *open* Σ*-theory* is any set Φ of InqBQ formulas in the signature Σ.
- *A closed* Σ*-theory* is any set Φ of InqBQ *sentences* in the signature Σ.

So closed Σ-theories do not contain formulas with free variables, while open Σ-theories may. It is easy to transform an open theory into a corresponding closed one, at the cost of adding new constant symbols in the signature. For A a set of parameters—that we assume disjoint from the set Σ—we define $\Sigma(A)$ as the signature extending Σ with the elements of A as fresh constant symbols.

[5]We will show in Theorem 7.4.3 that $\Phi \rhd \psi$ iff $\Phi \vdash \psi$, and so this kind of scenarios do not occur.

$\wedge$i	$\dfrac{\varphi \quad \psi}{\varphi \wedge \psi}$	$\wedge$e	$\dfrac{\varphi \wedge \psi}{\varphi} \quad \dfrac{\varphi \wedge \psi}{\psi}$
$\Vee$i	$\dfrac{\varphi}{\varphi \Vee \psi} \quad \dfrac{\psi}{\varphi \Vee \psi}$	$\Vee$e	$\dfrac{\varphi \Vee \psi \quad \begin{array}{c}[\varphi]\\ \vdots \\ \chi\end{array} \quad \begin{array}{c}[\psi]\\ \vdots \\ \chi\end{array}}{\chi}$
$\to$i	$\dfrac{\begin{array}{c}[\alpha]\\ \vdots \\ \psi\end{array}}{\alpha \to \psi}$	$\to$e	$\dfrac{\alpha \quad \alpha \to \psi}{\psi}$
$\forall$i	$\dfrac{\varphi[y/x]}{\forall x.\varphi}$	$\forall$e	$\dfrac{\forall x.\varphi}{\varphi[t/x]}$
$\overline{\exists}$i	$\dfrac{\varphi[t/x]}{\overline{\exists} x.\varphi}$	$\overline{\exists}$e	$\dfrac{\overline{\exists} x.\varphi \quad \begin{array}{c}[\varphi[y/x]]\\ \vdots \\ \psi\end{array}}{\psi}$
Ex Falsum	$\dfrac{\bot}{\varphi}$	DNC-rule	$\dfrac{\neg\neg\alpha}{\alpha}$
$\Vee$-split	$\dfrac{\alpha \to \psi \Vee \chi}{(\alpha \to \psi) \Vee (\alpha \to \chi)}$	CD-rule	$\dfrac{\forall x.(\varphi \Vee \psi)}{\forall x.\varphi \Vee \psi}$
$\overline{\exists}$-split	$\dfrac{\alpha \to \overline{\exists} x.\psi}{\overline{\exists} x.(\alpha \to \psi)}$		
$\asymp$i	$\dfrac{}{t \asymp t}$	$\asymp$e	$\dfrac{\varphi[t/x] \quad t \asymp t'}{\varphi[t'/x]}$

Figure 7.2: Natural deduction system for ClAnt. The rules containing the equality symbol $\asymp$ are present only when working with the syntax $\mathcal{L}_{\asymp}$. In (ClAnt $\to i$), α ranges over classical formulas; In ($\to e$), α ranges over classical formulas; In ($\forall e$) and ($\overline{\exists} i$), t must be free for x in φ; in ($\forall i$), y must not occur free in any undischarged assumption; in ($\overline{\exists} e$), y must not occur free in ψ or any undischarged assumption; in (DNC-rule), α ranges over classical formulas; in ($\Vee$-split), α ranges over classical formulas; in ($\overline{\exists}$-split), α ranges over classical formulas and x is not free in α; in (CD), x must not occur free in ψ.

$$\text{ClAnt} \rightarrow\text{i} \qquad \dfrac{\begin{matrix}[\alpha] \\ \vdots \\ \psi\end{matrix}}{\alpha \rightarrow \psi} \qquad \text{For } \alpha \text{ classical.}$$

7.3.3. Definition. Let Φ be an open Σ-theory and let V be the set of open variables appearing in Φ. Consider a set $\tilde{V} := \{\, \tilde{x} \,|\, x \in V \,\}$ of distinct formal parameters. We define the *closure of* Φ as the closed $\Sigma(\tilde{V})$-theory $\tilde{\Phi}$ obtained by substituting every free occurrence of the variable x in Φ with $\tilde{x}$, for every $x \in V$.

7.3.4. Proposition. *Let* $\Phi \cup \{\psi\}$ *be an open* Σ*-theory. Then*[6]

$$\Phi \vDash \psi \qquad \Longleftrightarrow \qquad \tilde{\Phi} \vDash \tilde{\psi}$$

The proof consists only in comparing the semantic clauses of the two entailments, and it is therefore omitted. This proposition allows us to focus our attention on closed theories and to highlight the role of the parameters in the proofs that follow. To lighten the notation, from now on we will simply write Σ*-theory* instead of *closed* Σ*-theory*.

To prove the completeness of the system introduced, we need to study more in detail three special classes of theories: *saturated theories*, *classically saturated theories* and ClAnt-*saturated theories*. In what follows, we will indicate with A a set of constant symbols not appearing in the signature Σ.

7.3.5. Definition (Saturated theory). A $\Sigma(A)$-theory Φ is called *saturated* (w.r.t. A) if for every pair of sentences φ, ψ of $\Sigma(A)$ it satisfies:

- Coherence: $\Phi \nvdash \bot$;
- Deductive closure: if $\Phi \vdash \varphi$ then $\varphi \in \Phi$;
- Disjunction property: If $\Phi \vdash \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi$ then $\Phi \vdash \varphi$ or $\psi \vdash \varphi$;
- Existence property: If $\Phi \vdash \overline{\exists} x.\varphi$ then $\Phi \vdash \varphi[a/x]$ for some $a \in A$;
- Normality condition: If $\Phi \nvdash \forall x.\varphi$ then $\Phi \nvdash \varphi[a/x]$ for some $a \in A$.

It is fairly easy to produce examples of saturated theories: consider an inquisitive model $\mathcal{M}$ on the signature $\Sigma(A)$ for which the interpretations of the symbols in A cover the whole domain, that is, $\{a^{\mathcal{M}} \mid a \in A\} = D$—henceforth we will call these models A-*covered*. Given an information state s, define the *theory of* $\langle \mathcal{M}, s\rangle$ as $\text{Th}(\mathcal{M}, s) := \{\, \varphi \text{ closed formula of } \Sigma(A) \,|\, \mathcal{M}, s \vDash \varphi \,\}$. It is immediate to show that $\text{Th}(\mathcal{M}, s)$ is a saturated $\Sigma(A)$-theory. In particular, the existence property and the normality condition rely on the fact that the model is A-covered.

If we restrict our attention only to classical formulas or ClAnt-formulas, we can define the corresponding concepts of *classically* and ClAnt-*saturated theories.*

[6]If the set Φ and ψ have a common free variable, let us say x, $\tilde{\Phi}$ and $\tilde{\psi}$ are obtained by substituting the free occurrences of x with the *same* formal parameter $\tilde{x}$.

7.3.6. Definition (Classical theories and classically saturated theories). A *classical* Σ*-theory* is a Σ-theory containing only classical formulas.

We say that a classical $\Sigma(A)$-theory Γ is classically-saturated (w.r.t. A) if for every pair of classical sentences α, β of $\Sigma(A)$ it satisfies:

- Coherence: $\Gamma \nvdash \bot$;
- Deductive closure: If $\Gamma \vdash \alpha$, then $\alpha \in \Gamma$;
- Classical disjunction property: If $\Gamma \vdash \alpha \vee \beta$ then $\Gamma \vdash \alpha$ or $\Gamma \vdash \beta$;
- Classical existence property: If $\Gamma \vdash \exists x.\alpha$ then $\Gamma \vdash \alpha[a/x]$, for some $a \in A$.

A simple induction shows that, given $\Gamma \cup \{\alpha\}$ classical formulas, $\Gamma \triangleright \alpha$ if and only if α is a consequence of Γ in classical first order logic. From this it easily follows that the condition corresponding to normality is also satisfied.

Classically saturated theories are examples of *Hintikka sets* for classical first order logic (see for example [Hodges, 1993, Section 2.3]). The following is a direct consequence of [Hodges, 1993, Theorem 2.3.3].

7.3.7. Fact. Every Hintikka set Γ in the signature $\Sigma(A)$ admits a (first order) model M_Γ with domain A such that each constant $a \in A$ is interpreted as itself in M_Γ.

As mentioned, a classically-saturated $\Sigma(A)$-theory Γ is a Hintikka set; moreover, since Γ is also deductively closed, it follows that Γ is complete, that is, for every sentence φ of $\Sigma(A)$ we either have $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.[7] This allows us to give a concise description of the model M_Γ in this particular case.

7.3.8. Lemma. *Let Γ be a classical $\Sigma(A)$-theory classically-saturated w.r.t. A, and let M_Γ be the unique first order model with domain A such that, for every atomic sentence p of $\Sigma(A)$ it holds that $M_\Gamma \models p$ iff $p \in \Gamma$. Then M_Γ is an A-covered model of Γ.*

These observations can be restated in terms of classically saturated theories.

7.3.9. Definition (Classical part and ClAnt part of Φ). Let Φ be a theory. The *classical part of* Φ ($\Phi\restriction_{cl}$) and the ClAnt *part of* Φ ($\Phi\restriction_{\mathsf{ClAnt}}$), are defined as the set of classical formulas contained in Φ and the set of ClAnt formulas contained in Φ respectively.

7.3.10. Corollary.

- *Let $\mathcal{M}$ be an A-covered model on the signature $\Sigma(A)$ and w a world of $\mathcal{M}$. Then* $\mathrm{Th}(\mathcal{M}, \{w\})\restriction_{cl}$ *is a classically-saturated* $\Sigma(A)$*-theory.*

[7] The proof is straightforward: for every sentence φ, $\varphi \vee \neg\varphi \in \Gamma$ (by deductive closure), and so $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ (by classical disjunction property).

- *Let Γ be a classically-saturated $\Sigma(A)$-theory. Then there exists a model $\mathcal{M}$ on the signature $\Sigma(A)$ and a world w of $\mathcal{M}$ such that $\Gamma = \mathrm{Th}(\mathcal{M}, \{w\})\upharpoonright_{cl}$.*

If we restrict our attention to A-covered models in the signature $\Sigma(A)$, Corollary 7.3.10 tells us that classically saturated theories are exactly the classical parts of theories of *singleton information states*, that is, states of the form $\{w\}$. It is worth noticing that for an arbitrary information state s, the set $\mathrm{Th}(\mathcal{M}, s)\upharpoonright_{cl}$ is not necessarily classically-saturated.

7.3.11. Definition (ClAnt-theories and ClAnt-saturated theories). A ClAnt Σ-*theory* is a Σ-theory containing only ClAnt formulas.

We say that a ClAnt $\Sigma(A)$-theory Φ is ClAnt-saturated (w.r.t. A) if for every pair of ClAnt sentences φ, ψ of $\Sigma(A)$ it satisfies:

- Coherence: $\Phi \not\rhd \bot$;
- Deductive closure: If $\Phi \rhd \varphi$, then $\varphi \in \Phi$;
- Disjunction property: If $\Phi \rhd \varphi \mathbin{\vee\!\!\!\vee} \psi$ then $\Phi \rhd \varphi$ or $\Phi \rhd \psi$;
- Existence property: If $\Phi \rhd \overline{\exists} x.\varphi$ then $\Phi \rhd \varphi[a/x]$ for some $a \in A$;
- Normality condition: If $\Phi \not\rhd \forall x.\varphi$ then $\Phi \not\rhd \varphi[a/x]$ for some $a \in A$.

From Definition 7.3.1 it follows readily that, given Φ a saturated $\Sigma(A)$-theory, the subset of its ClAnt formulas is a ClAnt-saturated $\Sigma(A)$-theory. The converse—that every ClAnt-saturated theory can be obtained by restricting a saturated theory in the whole language—is not as obvious, but surprisingly it is the case.

7.3.12. Theorem. *Let Φ be a* ClAnt-*saturated $\Sigma(A)$-theory. Then there exists a saturated $\Sigma(A)$-theory Ψ such that $\Phi = \Psi\upharpoonright_{\mathsf{ClAnt}}$.*

The rest of this section is devoted to proving this result. Henceforth, given a ClAnt-theory Φ, we will indicate with $\overline{\Phi}$ its deductive closure w.r.t. $\rhd$—which is again a ClAnt-theory.

7.3.13. Lemma. *Given Φ a* ClAnt-*saturated $\Sigma(A)$-theory and α a classical sentence such that $\Phi \not\rhd \neg\alpha$, then $\overline{\Phi \cup \{\alpha\}}$ is* ClAnt-*saturated $\Sigma(A)$-theory.*

Proof:
Let us call $\Theta := \overline{\Phi \cup \{\alpha\}}$. Clearly Θ is deductively closed. Moreover it is coherent, since $\Phi \not\rhd \neg\alpha$ iff $\Phi, \alpha \not\rhd \bot$: the left-to-right implication can be deduced using the rule (ClAnt $\to i$); the right-to-left implication can be deduced using the rule ($\to e$).

So we just need to show that Θ satisfies the disjunction property, the existence property and the normality condition.

Disjunction property

$$
\begin{array}{ll}
\Theta \triangleright \varphi \mathbin{\vee\!\!\!\vee} \psi & \\
\Longrightarrow \Phi \triangleright \alpha \to \varphi \mathbin{\vee\!\!\!\vee} \psi & (\mathsf{ClAnt} \to i) \\
\Longrightarrow \Phi \triangleright (\alpha \to \varphi) \mathbin{\vee\!\!\!\vee} (\alpha \to \psi) & (\mathbin{\vee\!\!\!\vee} - split) \\
\Longrightarrow \Phi \triangleright \alpha \to \varphi \text{ or } \Phi \triangleright \alpha \to \psi & \text{Disjunction property of } \Phi \\
\Longrightarrow \Theta \triangleright \varphi \text{ or } \Theta \triangleright \psi & (\to e)
\end{array}
$$

Existence property
Notice that since α is a sentence, x does not appear free in α.

$$
\begin{array}{ll}
\Theta \triangleright \overline{\exists} x.\varphi & \\
\Longrightarrow \Phi \triangleright \alpha \to \overline{\exists} x.\varphi & (\to i) \\
\Longrightarrow \Phi \triangleright \overline{\exists} x.(\alpha \to \varphi) & (\overline{\exists} - split) \\
\Longrightarrow \Phi \triangleright \alpha \to \varphi[a/x] \quad \text{for some } a & \text{Existence property of } \Phi \\
\Longrightarrow \Theta \triangleright \varphi[a/x] \quad \text{for some } a & (\to e)
\end{array}
$$

Normality condition
Notice that since α is a sentence, x does not appear free in α.

$$
\begin{array}{ll}
\Theta \triangleright \varphi[a/x] \quad \text{for all } a & \\
\Longrightarrow \Phi \triangleright \alpha \to \varphi[a/x] \quad \text{for all } a & (\mathsf{ClAnt} \to i) \\
\Longrightarrow \Phi \triangleright \forall x.(\alpha \to \varphi) & \text{Normality condition of } \Phi \\
\Longrightarrow \Phi \triangleright \alpha \to \varphi & (\forall e) \\
\Longrightarrow \Theta \triangleright \varphi & (\to e) \\
\Longrightarrow \Theta \triangleright \forall x.\varphi & (\forall i)
\end{array}
$$

Notice that the last passage is justified since Θ is a set of sentences, and consequently x does not appear free in Θ. $\square$

7.3.14. LEMMA. *Let Φ, Ψ be two ClAnt-saturated $\Sigma(A)$-theories such that $\Phi\lceil_{cl} = \Psi\lceil_{cl}$. Then $\Phi = \Psi$.*

Proof:
We start with a preliminary result: given any Φ and Ψ as in the hypothesis, then for every classical formula α it holds that $(\overline{\Phi \cup \{\alpha\}})\lceil_{cl} = (\overline{\Psi \cup \{\alpha\}})\lceil_{cl}$. The proof is straightforward:

$$
\begin{array}{ll}
\beta \in \overline{\Phi \cup \{\alpha\}} & \\
\Longrightarrow \alpha \to \beta \in \Phi & (\to i) \\
\Longrightarrow \alpha \to \beta \in \Psi & \text{Inductive hypothesis} \\
\Longrightarrow \beta \in \overline{\Psi \cup \{\alpha\}} & (\to e)
\end{array}
$$

Using this technical result, we can show by induction on the length of the ClAnt sentence θ—intended as the number of symbols appearing in θ—that $\theta \in \Phi \iff \theta \in \Psi$.

- If θ is an atom, the result follows by hypothesis—atoms are classical formulas.
- If $\theta \equiv \psi \wedge \chi$, then

$$\begin{aligned} & \theta \in \Phi \\ \iff & \psi \in \Phi \text{ and } \chi \in \Phi && \text{Deductive closure of } \Phi \\ \iff & \psi \in \Psi \text{ and } \chi \in \Psi && \text{Inductive hypothesis} \\ \iff & \theta \in \Psi && \text{Deductive closure of } \Psi \end{aligned}$$

- If $\theta \equiv \psi \mathbin{\underline{\vee}} \chi$, then

$$\begin{aligned} & \theta \in \Phi \\ \iff & \psi \in \Phi \text{ or } \chi \in \Phi && \text{Deductive closure and Disjunction property of } \Phi \\ \iff & \psi \in \Psi \text{ or } \chi \in \Psi && \text{Inductive hypothesis} \\ \iff & \theta \in \Psi && \text{Deductive closure and Disjunction property of } \Psi \end{aligned}$$

- If $\theta \equiv \alpha \to \psi$, then

$$\begin{aligned} & \theta \in \Phi \\ \iff & \psi \in \overline{\Phi \cup \{\alpha\}} && (\mathsf{ClAnt} \to i) \text{ and } (\to e) \\ \iff & \psi \in \overline{\Psi \cup \{\alpha\}} && \text{Inductive hypothesis applied to } \overline{\Phi \cup \{\alpha\}} \text{ and } \overline{\Psi \cup \{\alpha\}} \\ \iff & \theta \in \Psi && (\mathsf{ClAnt} \to i) \text{ and } (\to e) \end{aligned}$$

Notice that we can apply the inductive hypothesis to the theories $\overline{\Phi \cup \{\alpha\}}$ and $\overline{\Psi \cup \{\alpha\}}$ by Lemma 7.3.13.

- If $\theta \equiv \overline{\exists} x.\psi$, then

$$\begin{aligned} & \theta \in \Phi \\ \iff & \psi[a/x] \in \Phi \quad \text{for some } a && \text{Existence property for } \Phi \\ \iff & \psi[a/x] \in \Psi \quad \text{for some } a && \text{Inductive hypothesis} \\ \iff & \theta \in \Psi && \text{Existence property for } \Psi \end{aligned}$$

- If $\theta \equiv \forall x.\psi$, then

$$\begin{aligned} & \theta \in \Phi \\ \iff & \varphi[a/x] \in \Phi \quad \text{for all } a && \text{Normality condition for } \Phi \\ \iff & \psi[a/x] \in \Psi \quad \text{for all } a && \text{Inductive hypothesis} \\ \iff & \theta \in \Psi && \text{Normality condition for } \Psi \end{aligned}$$

□

To obtain a result analogous to Corollary 7.3.10, we need to introduce a construction resembling the canonical models for intuitionistic logic.

7.3.15. Definition (Canonical model). We define the *canonical model of* $\Sigma(A)$ as the model $\mathcal{M}^c$ defined by the following clauses:

- The set of worlds is W^c, the set of classically-saturated $\Sigma(A)$-theories;
- The common domain of the structures is $D^c := A$;
- The model corresponding to world Γ is M_Γ—introduced in Lemma 7.3.8.

A straightforward induction shows that $\mathrm{Th}(\mathcal{M}^c, \{\Gamma\})\restriction_{cl} = \Gamma$ for every $\Gamma \in W^c$. From this observation we obtain the following Lemma.

7.3.16. Lemma. *Let $s \subseteq W^c$ and Γ a classical theory. Then*

$$\mathcal{M}^c, s \vDash \Gamma \iff \forall \Theta \in s.\, \Gamma \subseteq \Theta$$

Proof:
For every $\alpha \in \Gamma$ we have

$$\begin{aligned} \mathcal{M}^c, s \vDash \alpha &\iff \forall \Theta \in s.\ \alpha \in \mathrm{Th}(\mathcal{M}^c, \{\Theta\}) \qquad \text{(By Theorem 2.1.18)} \\ &\iff \forall \Theta \in s.\ \alpha \in \Theta \end{aligned}$$

□

Given a coherent classical $\Sigma(A)$-theory Θ it is not generally true that there exists a world of the canonical model satisfying Θ. A simple example, for P a unary predicate symbol, is $\Theta := \{\, \neg\forall x.P(x) \,\} \cup \{\, P(a) \mid a \in A \,\}$. The problem in this case is that for every theory Γ of a world of the canonical model—that is, by Lemma 7.3.16, for every classically-saturated $\Sigma(A)$-theory Γ—if $\Gamma \nvdash \forall x.P(x)$ then there must be a witness $a \in A$ for which $\Gamma \nvdash P(a)$.

So the normality condition is necessary for such a world to exist. The following lemma shows that it is also a sufficient condition.[8]

7.3.17. Lemma (Classical saturation lemma). *Let Θ be a coherent classical $\Sigma(A)$-theory such that for every sentence α in the signature $\Sigma(A)$ it holds that*

$$\textit{Normality condition:} \qquad \Theta \nvdash \forall x.\alpha \implies \Theta \nvdash \alpha, \textit{ for some } a \in A.$$

Then there exists a classically-saturated $\Sigma(A)$-theory Γ such that $\Theta \subseteq \Gamma$.

[8]Recall that, by Theorem 2.1.17, $\vdash$ amounts to classical entailment when restricted to classical formulas.

Proof:
This proof is an adaptation of the proof by Gabbay [1981, Section 3.3, Theorem 2] for the intuitionistic case.

We start by showing a useful property, that we will later refer to as $(*)$-property: given β a classical sentence of $\Sigma(A)$, if Δ is a classical theory that satisfies the normality condition above then $\Delta \cup \{\beta\}$ also satisfies the condition. In fact for any classical formula α we have

$$\begin{aligned} \Delta \cup \{\beta\} \vdash \forall x.\alpha &\implies \Delta \vdash \beta \to \forall x.\alpha \\ &\implies \Delta \vdash \forall x.(\beta \to \alpha) \\ &\implies \Delta \vdash \beta \to \alpha[a/x] \text{ for some } a \in A \\ &\implies \Delta \cup \{\beta\} \vdash \alpha[a/x] \text{ for some } a \in A \end{aligned}$$

Now we go back to the main proof. Fix an enumeration $B_0, B_1, \ldots$ of the classical sentences in the signature $\Sigma(A)$. We will define inductively a chain of classical $\Sigma(A)$-theories Γ_i indexed by $\mathbb{N}$ such that:

1. Γ_i is coherent, that is, $\Gamma_i \nvdash \bot$.

2. For every index i, $\Gamma_i \subseteq \Gamma_{i+1}$.

3. For every index i, Γ_i respects the normality condition.

The plan is to take $\Gamma := \cup_{i\in\mathbb{N}}\Gamma_i$. During the construction we will impose some additional conditions to ensure Γ to be a classically-saturated $\Sigma(A)$-theory. We start the construction by defining $\Gamma_0 := \Theta$. By hypothesis Conditions 1 and 3 are satisfied; Condition 2 is trivially satisfied.

Suppose we already defined Γ_n with the properties above. We proceed by cases.

- *Case* $\Gamma_n \nvdash \neg B_n$ *and* $B_n \neq \exists x.\alpha(x)$. Define $\Gamma_{n+1} := \Gamma_n \cup \{B_n\}$. Condition 1 follows from $\Gamma_n \nvdash \neg B_n$; Condition 2 is trivially satisfied; Condition 3 follows from the $(*)$-property.

- *Case* $\Gamma_n \nvdash \neg B_n$ *and* $B_n = \exists x.\alpha(x)$. Notice that $\Gamma_n \cup \{\exists x.\alpha\} \nvdash \forall x.\neg\alpha$. So by Condition 3 and the $(*)$-property, there exists a parameter $a \in A$ such that $\Gamma_n \cup \{\exists x.\alpha\} \nvdash \neg\alpha[a/x]$. Define $\Gamma_{n+1} := \Gamma_n \cup \{B_n, \alpha[a/x]\}$. Condition 1 follows from $\Gamma_n \nvdash \neg B_n$ and $\Gamma_n \cup \{B_n\} \nvdash \neg\alpha[a/x]$; Condition 2 trivially holds; Condition 3 follows from the $(*)$-property.

- *Case* $\Gamma_n \vdash \neg B_n$. Define $\Gamma_{n+1} := \Gamma_n$. Conditions 1,2 and 3 trivially hold.

Define $\Gamma := \cup_{i\in\mathbb{N}}\Gamma_i$. By Condition 2, $\Theta \subseteq \Gamma$. So it remains to show that Γ is classically-saturated. First of all, Γ is coherent since $\Gamma \vdash \bot$ iff there exists an index $i \in \mathbb{N}$ such that $\Gamma_i \vdash \bot$, but the latter would contradict Condition 1.

Moreover Γ is deductively closed, since if $\Gamma \vdash B_n$ for some $n \in \mathbb{N}$, then $\Gamma_n \nvdash \neg B_n$ and so $B_n \in \Gamma_{n+1} \subseteq \Gamma$.

To show the classical disjunction property, suppose that $\Gamma \vdash B_m \vee B_n$. This implies that $\Gamma \nvdash \neg B_m$ or $\Gamma \nvdash \neg B_n$; without loss of generality, suppose the former is the case. Then $\Gamma_m \nvdash \neg B_m$, and by construction $B_{m+1} \in \Gamma_{m+1} \subseteq \Gamma$. Finally, to show the classical existence property suppose that $\Gamma \vdash \exists x.\alpha$ and let B_n be the enumeration of $\exists x.\alpha$. Then $\Gamma \nvdash \neg\exists x.\alpha$, and consequently $\Gamma_n \nvdash \neg\exists x.\alpha$. By construction, there exists an $a \in A$ such that $\alpha[a/x] \in \Gamma_{n+1} \subseteq \Gamma$.

This shows that Γ is a classically-saturated $\Sigma(A)$-theory as desired. □

Combining the results above, we can show the connection between ClAnt-saturated theories and the semantics of the logic.

7.3.18. Theorem. *Given Φ a* ClAnt*-saturated $\Sigma(A)$-theory, there exists a state E_Φ of $\mathcal{M}^c$ such that for every* ClAnt *formula ψ of $\Sigma(A)$:*

$$\psi \in \Phi \iff \mathcal{M}^c, E_\Phi \vDash \psi$$

Proof:
Define the state $E_\Phi := \{\, \Gamma \in W^c \mid \Phi\lceil_{cl} \subseteq \Gamma \,\}$ and consider the theory of this state $\Psi := \mathrm{Th}(\mathcal{M}^c, E_\Phi)$. By Lemma 7.3.16, $\Phi\lceil_{cl} \subseteq \Psi\lceil_{cl}$. We would like to show that also the other inclusion holds.

Fix a classical formula $\alpha \in \Psi$ and suppose toward a contradiction that $\alpha \notin \Phi\lceil_{cl}$. In particular, α is not a consequence of $\Phi\lceil_{cl}$ in classical first order logic—since $\Phi\lceil_{cl}$ is classically-saturated. By Lemma 7.3.17, there exists a classically-saturated $\Sigma(A)$-theory Θ such that $\Phi\lceil_{cl} \subseteq \Theta$ and $\alpha \notin \Theta$. But this leads to a contradiction, since we have:

$$\begin{aligned} \alpha \in \Psi \quad &\Longrightarrow \forall \Gamma \in E_\Phi.\, \alpha \in \Gamma \qquad \text{(by Lemma 7.3.16)} \\ &\Longrightarrow \forall \Gamma \supseteq \Phi\lceil_{cl}.\, \alpha \in \Gamma \end{aligned}$$

So we established that $\alpha \in \Phi\lceil_{cl}$, and since α was an arbitrary classical formula in Ψ, we also established that $\Psi\lceil_{cl} \subseteq \Psi\lceil_{cl}$.

By Lemma 7.3.14, since $\Phi\lceil_{cl} = \Psi\lceil_{cl}$, we obtain that $\Phi = \Psi\lceil_{\mathsf{ClAnt}}$, from which the result follows. □

From the previous result, Theorem 7.3.12 follows trivially.

Proof of Theorem 7.3.12:
Consider $\Psi := \mathrm{Th}(\mathcal{M}^c, E_\Phi)$. Since $\mathcal{M}^c$ is A-covered, Ψ is a saturated $\Sigma(A)$-theory. Moreover, by Theorem 7.3.18, $\Phi = \Psi\lceil_{\mathsf{ClAnt}}$, as desired. □

This result, together with the saturation lemma presented in Section 7.4, leads to the completeness of the natural deduction system introduced for ClAnt.

7.4 Completeness

This section is completely devoted to the proof of completeness for the ClAnt fragment. To lighten the proofs of the following lemmas, we introduce the following notation for inferences with multiple conclusions: let Φ and Ψ be sets of ClAnt formulas; we write $\Phi \triangleright \Psi$ to indicate that there exists $\psi_1, \ldots, \psi_n \in \Psi$ such that $\Phi \triangleright \psi_1 \vee\!\!\vee \ldots \vee\!\!\vee \psi_n$ or, in case Ψ is empty, that $\Phi \triangleright \bot$.

7.4.1. Lemma. *Let $\Phi \cup \Psi \cup \{\chi\}$ be a set of* ClAnt *formulas. If $\Phi \cup \{\chi\} \triangleright \Psi$ and $\Phi \triangleright \Psi \cup \{\chi\}$, then $\Phi \triangleright \Psi$.*

Proof:
By hypothesis, for some $\varphi_i, \varphi'_{i'} \in \Phi$ and $\psi_j, \psi'_{j'} \in \Psi$, we have[9]

$$\dfrac{\varphi_1 \quad \cdots \quad \varphi_h \quad \chi}{\psi_1 \vee\!\!\vee \ldots \vee\!\!\vee \psi_k}\, T_1 \qquad \text{and} \qquad \dfrac{\varphi'_1 \quad \cdots \quad \varphi'_{h'}}{\psi'_1 \vee\!\!\vee \ldots \vee\!\!\vee \psi'_{k'} \vee\!\!\vee \chi}\, T_2$$

Combining the two proofs together we get

$$\dfrac{\dfrac{\varphi'_1 \quad \cdots \quad \varphi'_{h'}}{\psi'_1 \vee\!\!\vee \ldots \vee\!\!\vee \psi'_{k'} \vee\!\!\vee \chi}\, T_2 \qquad \dfrac{[\psi'_1 \vee\!\!\vee \ldots \vee\!\!\vee \psi'_{k'}]}{\psi_1 \vee\!\!\vee \ldots \vee\!\!\vee \psi_k \vee\!\!\vee \psi'_1 \vee\!\!\vee \ldots \vee\!\!\vee \psi'_{k'}} \qquad \dfrac{\dfrac{\varphi_1 \quad \cdots \quad \varphi_h \quad [\chi]}{\psi_1 \vee\!\!\vee \ldots \vee\!\!\vee \psi_k}\, T_1}{\psi_1 \vee\!\!\vee \ldots \vee\!\!\vee \psi_k \vee\!\!\vee \psi'_1 \vee\!\!\vee \ldots \vee\!\!\vee \psi'_{k'}}\, \vee\!\!\vee\mathrm{i}}{\psi_1 \vee\!\!\vee \ldots \vee\!\!\vee \psi_k \vee\!\!\vee \psi'_1 \vee\!\!\vee \ldots \vee\!\!\vee \psi'_{k'}}\, \vee\!\!\vee\mathrm{e}$$

and thus $\Phi \triangleright \Psi$. □

7.4.2. Lemma (Saturation lemma). *Consider $\Phi \cup \{\psi\}$ a set of* ClAnt *formulas in the signature Σ such that $\Phi \not\triangleright \psi$. Consider the objects $\tilde{V}$, $\tilde{\Phi}$ and $\tilde{\psi}$ as defined in Proposition 7.3.4. Then given A a countable set of parameters disjoint from $\Sigma(\tilde{V})$, there exists a* ClAnt*-saturated $\Sigma(A \cup \tilde{V})$-theory Δ such that $\tilde{\Phi} \subseteq \Delta$ and $\tilde{\psi} \notin \Delta$.*

Proof:
This proof is an adaptation of the proof by Gabbay [1981, Section 3.3, Theorem 2] for the intuitionistic case.

First of all, by Proposition 7.3.4 we can assume that $\Phi \cup \{\psi\}$ contains only sentences, and that we just need to find a ClAnt-saturated $\Sigma(A)$-theory Δ such that $\Phi \subseteq \Delta$ and $\psi \notin \Delta$. Fix an enumeration $B_1, B_2, \ldots$ of the ClAnt sentences in the signature $\Sigma(A)$.[10] We will define inductively a chain of pairs of $\Sigma(A)$-theories $\langle \Delta_i, \Theta_i \rangle$ indexed by $i \in \mathbb{N}$ such that:

[9] In the natural deduction proofs that follows we will use a single line for the application of an instance of a rule; while we will use a double line for a subproof. On the right of a single line we will write the name of the rule applied; on the right of a double line we will write a label naming the subproof.

[10] Notice that this can be done without the use of the Axiom of Choice since we are considering a countable signature Σ and a countable set A.

1. $\Delta_i \not\triangleright \Theta_i$.

2. For every index i, $\Delta_i \subseteq \Delta_{i+1}$ and $\Theta_i \subseteq \Theta_{i+1}$.

3. $B_n \in \Delta_{n+1} \cup \Theta_{n+1}$.

The plan is to take $\Delta := \cup_{i\in\mathbb{N}}\Delta_i$. During the construction we will impose some additional conditions to ensure Δ to be a ClAnt-saturated $\Sigma(A)$-theory.

We start the construction by defining $\langle \Delta_0, \Theta_0 \rangle := \langle \Phi, \{\psi\}\rangle$. By hypothesis $\Phi \not\triangleright \psi$, and so Condition 1 is satisfied; Conditions 2 and 3 are trivially satisfied.

Suppose we already defined $\langle \Delta_n, \Theta_n \rangle$ with the properties above. We consider several cases—notice that Lemma 7.4.1 ensures that this is an exhaustive list, since we cannot have that $\Delta_n \triangleright \Theta_n \cup \{B_n\}$ and $\Delta_n \cup \{B_n\} \triangleright \Theta_n$.

- *Case* $\Delta_n \not\triangleright \Theta_n \cup \{B_n\}$ *and* $B_n \neq \forall x.\varphi(x)$. In this case we simply define $\Delta_{n+1} := \Delta_n$ and $\Theta_{n+1} := \Theta_n \cup \{B_n\}$. Conditions 1, 2 and 3 follow by construction.

- *Case* $\Delta_n \not\triangleright \Theta_n \cup \{B_n\}$ *and* $B_n = \forall x.\varphi(x)$. Consider a fresh parameter $a \in A$ (that is, an element not appearing in $\Delta_n \cup \Theta_n \cup \{B_n\}$) and define $\Delta_{n+1} := \Delta_n$ and $\Theta_{n+1} := \Delta_n \cup \{B_n, \varphi(a)\}$.

  Clearly Conditions 2 and 3 are satisfied. Moreover also Condition 1 holds, i.e. $\Delta_{n+1} \not\triangleright \Theta_{n+1}$; for otherwise, for some $\delta_1, \dots, \delta_h \in \Delta_n$ and some $\theta_1, \dots, \theta_k \in \Theta_n$, we would have:

$$\dfrac{\dfrac{\dfrac{\delta_1 \qquad \cdots \qquad \delta_h}{\theta_1 \Vee \dots \Vee \theta_k \Vee \forall x.\varphi(x) \Vee \varphi(a)}}{\forall x.(\theta_1 \Vee \dots \Vee \theta_k \Vee \forall x.\varphi(x) \Vee \varphi(x))}\ \forall i}{\theta_1 \Vee \dots \Vee \theta_k \Vee \forall x.\varphi(x) \Vee \forall x.\varphi(x)}\ CD$$

  So in particular $\Delta_n \triangleright \Theta_n \cup \{B_n\}$, which is a contradiction.

- *Case* $\Delta_n \cup \{B_n\} \not\triangleright \Theta_n$ *and* $B_n \neq \bar{\exists} x.\varphi(x)$. Define then $\Delta_{n+1} := \Delta_n \cup \{B_n\}$ and $\Theta_{n+1} := \Theta_n$; clearly Conditions 1, 2 and 3 are satisfied.

- *Case* $\Delta_n \cup \{B_n\} \not\triangleright \Theta_n$ *and* $B_n = \bar{\exists} x.\varphi(x)$. Consider a fresh parameter $a \in A$ (that is, an element not appearing in $\Delta_n \cup \Theta_n \cup \{B_n\}$) and define $\Delta_{n+1} := \Delta_n \cup \{B_n, \varphi(a)\}$ and $\Theta_{n+1} := \Theta_n$.

  Clearly Conditions 2 and 3 are satisfied. Also Condition 1 holds, for otherwise:

$$\dfrac{\delta_1 \qquad \cdots \qquad \delta_h \qquad \bar{\exists} x.\varphi(x) \qquad \varphi(a)}{\theta_1 \Vee \dots \Vee \theta_k}\ T$$

and so

$$\dfrac{\overline{\exists}x.\varphi(x) \qquad \dfrac{\delta_1 \quad \cdots \quad \delta_h \quad [\overline{\exists}x.\varphi(x)] \quad \dfrac{[\overline{\exists}x.\varphi(x)]}{\varphi(a)}\,\overline{\exists}\text{i}}{\theta_1 \mathbin{\vee\mkern-12mu\vee} \dots \mathbin{\vee\mkern-12mu\vee} \theta_k}\,T}{\theta_1 \mathbin{\vee\mkern-12mu\vee} \dots \mathbin{\vee\mkern-12mu\vee} \theta_k}\,\overline{\exists}\text{e}$$

Thus $\Delta_n \cup \{B_n\} \triangleright \Theta_n$, which is a contradiction.

Thus we built the sequence $\langle \Delta_i, \Theta_i \rangle$ for $i \in \mathbb{N}$. Define $\Delta := \cup_{i \in \mathbb{N}} \Delta_i$ and $\Theta := \cup_{i \in \mathbb{N}} \Theta_i$. We would like to show that Δ is ClAnt-saturated, that $\Phi \subseteq \Delta$ and that $\psi \notin \Delta$.

First of all, notice that $\Phi \subseteq \Delta_0 \subseteq \Delta$. Moreover by Condition 1 and 2 we have $\Delta \ntriangleright \Theta$, for otherwise there would be a finite n such that $\Delta_n \triangleright \Theta_n$. Consequently $\psi \in \Theta_0 \subseteq \Theta$, and so $\psi \notin \Delta$. What is left to show is that Δ is ClAnt-saturated.

By Condition 3, we have that every sentence in ClAnt is an element of $\Delta \cup \Theta$; by Condition 1 and 2, $\Delta \ntriangleright \Theta$. These two conditions (together with $\Theta \neq \emptyset$) ensure that Δ is deductively closed and $\bot \notin \Delta$.

As for the Disjunction property, suppose $\Delta \triangleright \varphi \mathbin{\vee\mkern-12mu\vee} \psi$. By contradiction, assume $\varphi, \psi \notin \Delta$, which in turn implies $\varphi, \psi \in \Theta$. In particular we would have $\Delta \triangleright \Theta$, which is a contradiction; thus at least one among φ and ψ has to be in Δ.

For the Existence property, suppose $\overline{\exists}x.\varphi(x) \in \Delta$. Let $B_n = \overline{\exists}x.\varphi(x)$ be the enumeration given to this sentence. Since $B_n \in \Delta_{n+1} \cup \Theta_{n+1}$ (Condition 3) and $\Theta_{n+1} \cap \Delta \subseteq \Theta \cap \Delta = \emptyset$, it follows that $B_n \in \Delta_{n+1}$. In particular, following the inductive construction presented above, we have that $\Delta_{n+1} := \Delta_n \cup \{B_n, \varphi(a)\}$ for some $a \in A$. So in particular we have $\varphi(a) \in \Delta_{n+1} \subseteq \Delta$. Since $\overline{\exists}x.\varphi(x)$ is an arbitrary existential sentence, it follows that Δ has the existence property.

The normality condition follows from considerations completely analogous to the one in the previous paragraph. $\square$

Using the results we collected so far, we can completely characterize the connection between the relation $\triangleright$ and the relations $\vdash$ and $\vDash$.

7.4.3. Theorem. *Let* $\Phi \cup \{\psi\}$ *be* ClAnt *formulas. Then it holds that*

$$\Phi \triangleright \psi \qquad \Longleftrightarrow \qquad \Phi \vdash \psi$$

Proof:
The left-to-right implication follows trivially, since every derivation in the deductive system for ClAnt is also a derivation for the deductive system for InqBQ.

For the right-to-left implication, we show the contrapositive. Suppose that $\Phi \ntriangleright \psi$. Then by Lemma 7.4.2 there exists a ClAnt-saturated theory Δ (in an extended signature) such that $\Phi \subseteq \Delta$ and $\psi \notin \Delta$. By Theorem 7.3.12, there exists a saturated theory Ψ such that $\Delta = \Psi\restriction_{\mathsf{ClAnt}}$, and so in particular $\Phi \subseteq \Psi$

and $\psi \notin \Psi$. Since Ψ is deductively closed w.r.t. $\vdash$, it follows that $\Phi \nvdash \psi$. □

7.4.4. Theorem (Completeness). *Let* $\Phi \cup \{\psi\}$ *be* ClAnt *formulas. Then it holds that*

$$\Phi \vDash \psi \qquad \Longrightarrow \qquad \Phi \rhd \psi$$

Proof:
We prove the result by contraposition: suppose that $\Phi \not\rhd \psi$. By Lemma 7.4.2, given A a countable set of fresh parameters there exists a ClAnt-saturated $\Sigma(\tilde{V} \cup A)$-theory Δ such that $\tilde{\Phi} \subseteq \Delta$ and $\tilde{\psi} \notin \Delta$. By Theorem 7.3.18, given $\mathcal{M}^c$ the canonical model for the signature $\Sigma(\tilde{V} \cup A)$, we have $\mathrm{Th}(\mathcal{M}^c, E_\Delta)\restriction_{\mathsf{ClAnt}} = \Delta$. Thus in particular $\mathcal{M}^c, E_\Delta \vDash \tilde{\Phi}$ and $\mathcal{M}^c, E_\Delta \nvDash \tilde{\psi}$.

Define the assignment $g : V \to D$ such that $g(x) = (\tilde{x})^{\mathcal{M}}$. An easy induction shows that, for every formula χ in the signature Σ with free variables in V we have $\mathcal{M}^c, E_\Delta \vDash_g \chi$ iff $\mathcal{M}^c, E_\Delta \vDash \tilde{\chi}$. In particular, it follows that $\mathcal{M}^c, E_\Delta \vDash_g \Phi$ and $\mathcal{M}^c, E_\Delta \nvDash_g \psi$. Thus $\Phi \nvDash \psi$. □

7.5 Conclusions

In this chapter we defined and studied the *classical antecedent fragment* ClAnt of inquisitive first order logic. In particular, we considered two problems regarding the fragment: how to capture its expressive power and how to axiomatize it. To address the former, we developed a variation of the Ehrenfeucht-Fraïssé game introduced in Chapter 4 and showed that we can characterize in game-theoretic terms support-equivalence restricted to ClAnt formulas. And to address the latter, we defined a natural deduction system based on the one proposed by Ciardelli [2016, Section 4.6] and proved its strong completeness with respect to support semantics restricted to ClAnt. The completeness proof proposed strongly relies on the connections of InqBQ with classical and intuitionistic first order logics: for example, to carry on the proof we needed to strengthen the folklore saturation lemma for CQC (Lemma 7.3.17); and we needed to adapt the saturation lemma for the logic CD to the inquisitive setting (Lemma 7.4.2).

A natural question which we did not address in this chapter is whether a similar proof can be used to study the entailment relation of InqBQ. For example, it is hard to say whether Lemma 7.3.14—one of the key results—generalizes to the full language, that is, whether if two saturated theories containing the same classical formulas coincide. If that were the case we would obtain a completeness proof for the whole language simply by adapting the one presented in this chapter.

Another interesting observation is that we can define a hierarchy of fragments of InqBQ in the same spirit as ClAnt. Define recursively ClAnt_n as follows: ClAnt_0

is the fragment comprised by all and only classical formulas (i.e., ClAnt_0 is simply CQC); and ClAnt_n is defined by restricting formulas in the antecedent of an implication to range only over formulas of ClAnt_{n-1}.

$$\begin{array}{ll} \mathsf{ClAnt}_0 & \varphi_0 ::= \bot \mid p \mid \varphi_0 \wedge \varphi_0 \mid \varphi_0 \to \varphi_0 \mid \forall x.\varphi_0 \\ \mathsf{ClAnt}_n & \varphi_n ::= \varphi_{n-1} \mid \varphi_n \wedge \varphi_n \mid \varphi_n \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \varphi_n \mid \varphi_{n-1} \to \varphi_n \mid \forall x.\varphi_n \mid \overline{\exists} x.\varphi_n \end{array}$$

In particular, under this definition ClAnt coincides with ClAnt_1. Adapting the completeness proof presented in this chapter to the fragments ClAnt_n would be a first step towards a completeness result for the whole language. Moreover, since every formula of InqBQ belongs to some of these fragments, axiomatizing the entailment relation restricted to every ClAnt_n would give a weak completeness result for InqBQ.

Chapter 8

Finite-Width Inquisitive Logics and Bounded-Width Fragment

As anticipated in Chapter 7, we are now going to study the axiomatization problem for the family of *finite-width inquisitive logics* and the *bounded-width fragment.*

Finite-width inquisitive logics were already introduced by Sano [2011] as a hierarchy approximating inquisitive first order logic. The idea to define these logics is rather simple: instead of considering arbitrary information models, we consider only models with at most n worlds for $n \in \mathbb{N}$ a fixed natural number. Sano noticed that this hierarchy is the first order version of the *inquisitive hierarchy* by Ciardelli [2009, Chapter 4], a chain of propositional logics $\langle \mathsf{InqB}_n \rangle_{n\in\mathbb{N}}$ which approximates inquisitive propositional logic: $\mathsf{InqB} = \bigcap_{n\in\mathbb{N}} \mathsf{InqB}_n$. An open question left by Sano [2011] is whether the first order hierarchy he presents approximates inquisitive first order logic, in analogy with the propositional case: as we will show, this is not the case. As a direct consequence of this fact, we obtain that the logic InqBQ cannot be characterized only in terms of information models with finitely many worlds.

Among the finite-width inquisitive logics we find a first order version of the pair semantics by Groenendijk and Stokhof [1984], obtained for $n = 2$. Sano axiomatized the latter by adapting the canonical model completeness technique for first order intuitionistic logic with constant domain outlined in [Gabbay et al., 2009, Section 7.2]. However he did not attempt to tackle the axiomatization problem for the other logics in the hierarchy, leaving it as an open question. In this chapter we will present an alternative completeness proof which adapts to any logic in the hierarchy, thus giving a positive answer to Sano's problem.

Finally, the chapter also treats the *bounded-width fragment* of InqBQ, characterized by a rather interesting property, analogous to the *finite model property* from modal logic and *coherence* from Dependence logic [Kontinen, 2010]: if a formula of the fragment is not supported by an information state s, then there exists a *finite* subset of s which still does not support the formula. This rather peculiar property allows us to derive interesting properties of the fragment, building on the novel completeness result for the finite-width inquisitive logics.

8.1 A Hierarchy of Inquisitive Logics

As mentioned above, we are interested in studying the hierarchy proposed by Sano [2011], comprised of the logics of the classes of information models with a finite and bounded number of worlds. Let us start by introducing these classes of models and the hierarchy.

8.1.1. Definition. Given a cardinal λ, define the classes of information models:

$$\mathbb{M}_{<\lambda} := \left\{ \mathcal{M} \;\middle|\; |W^{\mathcal{M}}| < \lambda \right\} \qquad\qquad \mathbb{M}_{\lambda} := \left\{ \mathcal{M} \;\middle|\; |W^{\mathcal{M}}| \leq \lambda \right\}$$

We indicate with $\mathsf{InqBQ}_{<\lambda}$ the logic of the class $\mathbb{M}_{<\lambda}$ and with InqBQ_{λ} the logic of the class $\mathbb{M}_{\lambda}$.[1]

So $\mathsf{InqBQ}_{<\lambda}$ and InqBQ_{λ} are the logics of inquisitive models with strictly less than λ worlds and with at most λ worlds respectively. Notice that $\mathsf{InqBQ}_{\lambda} = \mathsf{InqBQ}_{<\lambda+1}$ and that whenever $\lambda < \kappa$ we have $\mathbb{M}_{\lambda} \subseteq \mathbb{M}_{\kappa}$ and consequently $\mathsf{InqBQ}_{<\lambda} \supseteq \mathsf{InqBQ}_{<\kappa}$.

The propositional counterpart of this hierarchy has been defined and studied by Ciardelli [2009, Chapter 4]: the logic InqB_n for $n \in \mathbb{N}$ is defined as the logic of propositional information models with at most n worlds. The propositional hierarchy has been studied quite thoroughly in the literature: an explicit axiomatization has been given for all its elements [Ciardelli, 2009, Theorem 4.2.2]; they form a *strict chain*, that is, $\mathsf{InqB}_n \supsetneq \mathsf{InqB}_{n+1}$ [Ciardelli, 2009, Proposition 4.1.8]; and their intersection coincides with InqB [Ciardelli, 2009, Corollary 4.1.6].

$$\mathsf{InqB}_1 \supsetneq \mathsf{InqB}_2 \supsetneq \ldots \supsetneq \bigcap_{n\in\mathbb{N}} \mathsf{InqB}_n = \mathsf{InqB}$$

In analogy with the propositional case, Sano [2011] introduces only the logics InqBQ_n for n a natural number: we will call these logics the *finite-width inquisitive logics*. A natural question to ask is whether the chain $\langle \mathsf{InqBQ}_n \rangle_{n\in\mathbb{N}}$ respects the same properties as its propositional counterpart. The rest of this section is dedicated to study this problem, and we start by proving a useful technical result that will accompany us for the rest of the chapter.

[1] In what follows we use the classes $\mathbb{M}_{<\lambda}$ (resp., $\mathbb{M}_{\lambda}$) of models whose *essential cardinality* is strictly smaller than λ (resp., at most λ). This class is defined in terms of *cardinality*, and not in terms of *essential cardinality*. However the logic of the class does not depend no this choice: in fact the *essential quotients* of the elements of the former and the *essential quotients* of the elements of the latter are the same, and so by Lemma 3.1.4 they generate the same logics.

8.1.2. Definition. Given an information model $\mathcal{M}$ and an info state s, the *essential cardinality* of s relative to $\mathcal{M}$ is[2]

$$|s|_{\mathrm{e}}^{\mathcal{M}} \;:=\; \left|\, \{\, [w]_{\approx^e} \mid w \in s \,\} \,\right|$$

When $\mathcal{M}$ is clear from the context, we will simply write $|s|_{\mathrm{e}}$ omitting the model.

8.1.3. Lemma. *Let Σ be a signature containing only finitely many relation symbols. Let $R_1, \dots, R_l$ be all the non-rigid symbols of the syntax.*[3] *Consider the formulas recursively defined as follows:*[4]

$$\mathrm{C}_1 := \forall\overline{x}.\ \bigwedge_{j=1}^{l} ?R_j(\overline{x}) \qquad \mathrm{C}_{n+1} := \overline{\exists}\overline{x}.\ \bigvee_{i=1}^{n} \bigvee_{j=1}^{l} \Big[(R_j(\overline{x}) \to \mathrm{C}_i) \wedge (\neg R_j(\overline{x}) \to \mathrm{C}_{n+1-i})\Big]$$

Then for every model $\mathcal{M}$ and information state s it holds that

$$\mathcal{M}, s \models \mathrm{C}_n \qquad \textit{iff} \qquad |s|_{\mathrm{e}} \leq n$$

Notice that the hypothesis that Σ only finitely many relational symbols is necessary for the formulas C_n to be well defined.

Proof:
We prove this result by strong induction on n. For the base case for $n = 1$, by expanding the definition of C_1 we obtain

$$\mathcal{M}, s \models \mathrm{C}_1$$
$$\Updownarrow$$
$$\text{For all } j \in \{1, \dots, l\}\,,\ \text{for all } \overline{d} \in D^{\mathrm{Ar}(R_j)},\ \text{for all } w \in W,\ \overline{d} \in (R_j)_w \text{ or } \overline{d} \notin (R_j)_w$$

which amounts to saying that the extension of all the non-rigid symbols of the syntax is the same in every world; or in other terms, that all the models M_w for $w \in s$ coincide. By definition of $\approx^e$, it follows there is a unique $\approx^e$-equivalence class, that is, $|s|_{\mathrm{e}} = 1$.

For the inductive step, fix a model $\mathcal{M}$ and a state s. Firstly, suppose the model satisfies C_n. Then there exists an index $j \in \{1, \dots, l\}$, a sequence $\overline{d} \in D^{\mathrm{Ar}(R_j)}$ and a value $k \in [1, n-1]$ such that $\mathcal{M}, s \models R_j(\overline{d}) \to \mathrm{C}_k$ and $\mathcal{M}, s \models \neg R_j(\overline{d}) \to \mathrm{C}_{n-k}$. If we define $s^+ := \big\{\, w \in s \mid d \in (R_j)_w \,\big\}$ and $s^- := \big\{\, w \in s \mid d \notin (R_j)_w \,\big\}$,

[2] The relation $\approx^e$ was introduced in Definition 3.1.1.

[3] That is, either all the relation symbols in Σ, if we are working in the syntax $\mathcal{L}_{\neq}$ or $\mathcal{L}_{=}$; or all the relation symbols in Σ and the equality symbol $\asymp$, if we are working in the syntax $\mathcal{L}_{\asymp}$.

[4] To lighten the notation, in the following formulas we indicate with $\forall\overline{x}$ (resp., $\overline{\exists}\overline{x}$) the string of quantifiers $\forall x_1. \dots \forall x_k$ (resp., $\overline{\exists} x_1. \dots \overline{\exists} x_k$) where $k = \max\{\mathrm{Ar}(R_1), \dots, \mathrm{Ar}(R_l)\}$; and with $R_j(\overline{x})$ the formula $R_j\left(x_1, \dots, x_{\mathrm{Ar}(R_j)}\right)$.

by the inductive hypothesis the two conditions are equivalent to $|s^+|_e \leq k$ and $|s^-|_e \leq n - k$. And since $s = s^+ \cup s^-$, it follows that $|s|_e \leq n$ as wanted.

Secondly, suppose $|s|_e \leq n$. If $|s|_e = 1$, notice that by inductive hypothesis $\mathcal{M}, s \vDash \mathrm{C}_j$ for $j \leq n-1$, and so the formula C_n is trivially supported. So suppose that $|s|_e > 1$, that is, that there exists an inde $j \in \{1, \dots, l\}$, a sequence of elements $\overline{d} \in D^{\mathrm{Ar}(R_j)}$ and two worlds $w, w' \in s$ such that $\overline{d} \in (R_j)_w \setminus (R_j)_{w'}$. Defining s^+ and s^- as above, it follows that $w \in s^+ \setminus s^-$ and $w' \in s^- \setminus s^+$, and so $|s^+|_e \geq 1$ and $|s^-|_e \geq 1$.

Since all the worlds of s^+ support $R_j(\overline{d})$ and all the worlds of s^- do not, worlds from s^+ and worlds from s^- belong to different $\approx^e$-equivalence classes. It follows that $n = |s|_e = |s^+|_e + |s^-|_e$, and together with the previous results we have $|s^+|_e, |s^-| \in \{1, \dots, n-1\}$. If we define $k := |s^+|$, by inductive hypothesis we obtain $\mathcal{M}, s \vDash R_j(\overline{d}) \to \mathrm{C}_k$ and $\mathcal{M}, s \vDash \neg R_j(\overline{d}) \to \mathrm{C}_{n-k}$, from which $\mathcal{M}, s \vDash \mathrm{C}_n$ easily follows.

□

Using Lemma 8.1.3 we can already show that the logics InqBQ_n are pairwise distinct.[5]

8.1.4. Proposition. $\mathsf{InqBQ}_n \supsetneq \mathsf{InqBQ}_{n+1}$.

Proof:
Since $\mathbb{M}_n \subseteq \mathbb{M}_{n+1}$, we have $\mathsf{InqBQ}_n \supseteq \mathsf{InqBQ}_{n+1}$. To show that the containment is strict notice that, by Lemma 8.1.3, C_n is satisfied by all the models in $\mathbb{M}_n$ and not by all the models in $\mathbb{M}_{n+1}$. Consequently we have $\mathrm{C}_n \in \mathsf{InqBQ}_n \setminus \mathsf{InqBQ}_{n+1}$. □

The next question we tackle is whether the intersection of this chain is the logic InqBQ—this was left as an open problem by Sano [2011]. We show that the answer is negative, that is, there are formulas not valid in InqBQ that admit only counterexamples with infinitely many worlds.

8.1.5. Proposition. $\bigcap_{n \in \mathbb{N}} \mathsf{InqBQ}_n = \mathsf{InqBQ}_{\leq \aleph_0} \subsetneq \mathsf{InqBQ}$.

To prove this proposition, we need another technical result.

8.1.6. Definition (P-chains). Let $\mathcal{M}$ be an information model in a signature Σ containing a unary predicate symbol P. Consider the preorder $\preceq$ defined over the worlds of $\mathcal{M}$ as follows:

$$w \preceq w' \qquad \text{iff} \qquad P_w \subseteq P_{w'}$$

We call $\mathcal{M}$ a *P-chain* if the relation $\preceq$ is a *total preorder*.

	w_0	w_1	w_2	$\cdots$
a_0	■	■	■	
a_1	□	■	■	$\cdots$
a_2	□	□	■	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Figure 8.1: An example of P-chain. In this case we have $w_i \preceq w_j$ iff $i \leq j$ and so it is a total (pre)order.

An example of P-chain is presented in Figure 8.1. The proof of Proposition 8.1.5 that we are going to give relies on the fact that the class of P-chains is *definable* by inquisitive formulas, as we show in the following lemma.

8.1.7. LEMMA. *Consider the formula*

$$\mathrm{Pc} := \forall x, y. [\ (P(x) \to P(y)) \vee\!\!\!\vee (P(y) \to P(x))\ ] .$$

Let $\mathcal{M}$ be a model in a signature Σ containing the unary predicate P. Then the formula Pc *is satisfied by an information state s iff $\mathcal{M}|_s$ is a P-chain.*

Proof:
Firstly, suppose that $\mathcal{M}, s \nvDash \mathrm{Pc}$. Let a, b be elements such that $\mathcal{M}, s \nvDash P(a) \to P(b)$ and $\mathcal{M}, s \nvDash P(b) \to P(a)$. Since both are classical formulas and consequently truth-conditional formulas (Theorem 2.1.18), it follows that there exist two worlds w, w' such that

$$\mathcal{M}, \{w\} \vDash P(a) \quad \mathcal{M}, \{w\} \nvDash P(b) \qquad\qquad \mathcal{M}, \{w'\} \nvDash P(a) \quad \mathcal{M}, \{w'\} \vDash P(b)$$

or equivalently

$$a \in P_w \qquad b \notin P_w \qquad\qquad a \notin P_{w'} \qquad b \notin P_{w'} \tag{8.1}$$

In particular, w and w' are incomparable under $\preceq$ and so $\mathcal{M}|_s$ is not a P-chain.

Secondly, suppose that $\mathcal{M}|_s$ is not a P-chain. So there exist two incomparable worlds w, w' under $\preceq$; which in turns means there exist two elements a, b for which the relations in 8.1 hold. From this it easily follows that $\mathcal{M}, s \nvDash \mathrm{Pc}$. □

We are now ready to prove Proposition 8.1.5.

Proof of Proposition 8.1.5:
We prove this result only for the syntax $\mathcal{L}_{\neq}$ and for models in the signature

[5] This was already proved by Sano [2011] generalizing the approach for propositional logic used in [Ciardelli, 2009, Proposition 4.1.8].

$\Sigma = \{P\}$, but the proof can be easily generalized to and arbitrary syntax and an arbitrary signature containing at least one relation symbol.[6]

Consider the formula $\psi := \mathrm{Pc} \to \overline{\exists} x.[P(x) \to \mathrm{C}_1]$. We want to show that $\psi \in \mathsf{InqBQ}_{<\aleph_0} \setminus \mathsf{InqBQ}$. ψ is not a valid formula of InqBQ, since the model in Figure 8.1 is a counterexample. So it remains to show that $\psi \in \mathsf{InqBQ}_{<\aleph_0}$, that is, that ψ is valid on models with finitely many worlds.

Consider a model $\mathcal{M}$ and suppose that $|W| < \aleph_0$. Given a non-empty state $t \subseteq W$ such that $\mathcal{M}, t \vDash \mathrm{Pc}$, by Lemma 8.1.7 $\mathcal{M}|_t$ is a P-chain. To make the following passages easier to follow, we will assume that the relation $\preceq$ corresponding to $\mathcal{M}|_t$ is a *total order* (meaning that $\preceq$ is also anti-reflexive), but the same proof applies to the general case with minor changes.

We have two cases: either $|t| = 1$ or $|t| \geq 2$. In the former case we have $\mathcal{M}, t \vDash \mathrm{C}_1$, and consequently $\mathcal{M}, t \vDash \overline{\exists} x.[P(x) \to \mathrm{C}_1]$.

In the latter case, since $|t| \leq |W| < \aleph_0$, the total order $\preceq$ admits a maximum w and a second greatest element w'; pick an arbitrary $d \in P_w \setminus P_{w'}$. By definition of $\preceq$, it follows that w is the only world in t for which $\mathcal{M}, \{w\} \vDash P(d)$, and so in particular $\mathcal{M}, t \vDash \overline{\exists} x.[P(x) \to C_1]$. So in both cases we have $\mathcal{M}, t \vDash \overline{\exists} x.[P(x) \to C_1]$.

Since t was an arbitrary substate of W, it follows that $\mathcal{M} \vDash \psi$. □

This result shows that there is a difference between the propositional hierarchy and its first order version proposed by Sano [2011]. The propositional logics InqB_n *approximate* InqB, meaning that the formulas refuted by InqB are *exactly* the formulas refuted by some InqB_n; and as we saw in Proposition 8.1.5, this is not the case for the finite-width inquisitive logics, since there are formulas refuted by InqBQ and valid for every InqBQ_n. However, if we consider the generalized hierarchy from Definition 8.1.1 instead of the finite-width inquisitive logics, we regain this approximation result.

8.1.8. Proposition. *There exists a cardinal* $\lambda > \aleph_0$ *such that* $\mathsf{InqBQ} = \mathsf{InqBQ}_\lambda$.

Proof:
Let $\{\varphi_\alpha | \alpha < \rho\}$ be an enumeration of the non-valid formulas in the signature Σ, for ρ a suitable cardinal. For every φ_α we can find an information model $\mathcal{M}_\alpha$ such that $\mathcal{M}_\alpha \nvDash \varphi_\alpha$; let $\lambda_\alpha = |W^{\mathcal{M}_\alpha}|$. If we define $\lambda = \sup(\lambda_\alpha | \alpha < \rho)$, we have that all the models $\mathcal{M}_\alpha$ are in the class $\mathbb{M}_\lambda$ and consequently $\{\varphi_\alpha | \alpha < \rho\} \cap \mathsf{InqBQ}_\lambda = \emptyset$, that is, $\mathsf{InqBQ} = \mathsf{InqBQ}_\lambda$.[7] □

[6]Notice that if we work only with rigid symbols (i.e., either in the syntax $\mathcal{L}_{\neq}$ or $\mathcal{L}_{=}$, and without relation symbols in the signature) then information models are required to contain only instances of the same classical first order model. In this case the result trivially follows and, more interestingly, the support semantics boils down to the usual semantics of CQC.

[7]A priori this proof depends on the signature Σ considered. But since any formula contains only finitely-many symbols from Σ, the maximum value of λ is reached for any signature containing infinitely-many relation and function symbols of every arity.

Axioms and rules of IQC (Figure 2.6)

Additional schemata

CD schema:	$\forall x.(\varphi \vee\!\!\!\vee \psi) \to \varphi \vee\!\!\!\vee \forall x.\psi$ for x not free in φ
H2 schema:	$\varphi \vee\!\!\!\vee (\varphi \to \psi \vee\!\!\!\vee \neg\psi)$
W2 schema:	$(\varphi \to \psi) \vee\!\!\!\vee (\psi \to \varphi) \vee\!\!\!\vee ((\varphi \to \neg\psi) \wedge (\psi \to \neg\varphi))$
DNC formulas:	$\neg\neg\alpha \to \alpha$ for α classical

Figure 8.2: Hilbert-style axiomatization for InqBQ_2 proposed by Sano [2011]. In the original source Sano treats a syntax without equality, thus the axioms for identity are not required.

This leaves us with the question of what is the minimal λ such that $\mathsf{InqBQ} = \mathsf{InqBQ}_\lambda$. For example we could have $\mathsf{InqBQ} = \mathsf{InqBQ}_{\aleph_0}$, the logic of models with countably-many worlds (recall we only showed that $\mathsf{InqBQ} \neq \mathsf{InqBQ}_{<\aleph_0}$). Another natural question is what logics in this chain are axiomatizable. As of now there are no results regarding these logics. We leave these issues open, with the hope to address them in future research.

We briefly recap the results of the current section—and what is still not known—with the following expression.

$$\mathsf{CQC} = \mathsf{InqBQ}_1 \supsetneq \mathsf{InqBQ}_2 \ldots \supsetneq \mathsf{InqBQ}_{<\aleph_0} \supsetneq \mathsf{InqBQ}_{\aleph_0} \stackrel{?}{=} \mathsf{InqBQ}_{<\lambda} = \mathsf{InqBQ}$$

8.2 Axiomatizing the Finite-Width Inquisitive Logics

In this section we consider the axiomatization problem for the finite-width inquisitive logics. As we did for the completeness results of Chapter 7, we focus only on the languages not containing rigid symbols, that is, on signatures containing only relational symbols and on either the syntax $\mathcal{L}_{\neq}$ or $\mathcal{L}_{\asymp}$. We leave the generalization of the results of this chapter to arbitrary signatures for future work.

Sano [2011, Subsection 2.2] gave an axiomatization of InqBQ_2, which we report in Figure 8.2. This axiomatization is particularly simple and elegant and it highlights the essential features of the CD-models corresponding (through Lemma 2.2.8) to the elements in $\mathbb{M}_2$. However it does not seem easy to adapt this axiomatization to the other finite-width inquisitive logics, and in fact this is left as an open problem by Sano [2011].

To tackle this issue, we propose an alternative approach to the problem. We define a Hilbert-style system $\mathcal{H}\mathsf{InqBQ}_n$ *parametric in* n and show that this is strongly complete for the logic InqBQ_n. The system is derived from the variation

Axioms and rules of IQC (Figure 2.6)

Additional schemata

CD schema:	$\forall x.(\varphi \mathbin{\rlap{\vee}\,\vee} \psi) \to \varphi \mathbin{\rlap{\vee}\,\vee} \forall x.\psi$ for x not free in φ
KP schema:	$(\neg\varphi \to \psi \mathbin{\rlap{\vee}\,\vee} \chi) \to (\neg\varphi \to \psi) \mathbin{\rlap{\vee}\,\vee} (\neg\varphi \to \chi)$
UP schema:	$(\neg\varphi \to \overline{\exists} x.\psi) \to \overline{\exists} x.(\neg\varphi \to \psi)$ for x not free in φ
KF schema:	$\neg\neg\forall x.\varphi(x) \to \forall x.\neg\neg\varphi(x)$
DNC formulas:	$\neg\neg\alpha \to \alpha$ for α classical
Identity axioms:	$\forall x.\,\forall y.\,x \asymp y \qquad \forall x.\,\forall y.(\,(x \asymp y) \wedge \varphi(x) \to \varphi(y)\,)$

$$\mathrm{C}_n: \quad \begin{cases} \forall \overline{x}.\,\bigwedge_{j=1}^{l} ?R_j(\overline{x}) & \text{if } n = 1 \\ \overline{\exists} \overline{x}.\,\mathbin{\rlap{\vee}\,\vee}_{i=1}^{n-1} \mathbin{\rlap{\vee}\,\vee}_{j=1}^{l} \Big[(R_j(\overline{x}) \to \mathrm{C}_i) \wedge (\neg R_j(\overline{x}) \to \mathrm{C}_{n-i})\Big] & \text{otherwise} \end{cases}$$

Figure 8.3: Hilbert-style axiomatization for InqBQ_n. The formula C_n was introduced in Lemma 8.1.3.

of the one proposed by Ciardelli for InqBQ (Figure 2.6) and can be found in Figure 8.3. $\mathcal{H}\mathsf{InqBQ}_n$ is obtained simply by adding to the proposed axiomatization for InqBQ the formula C_n (so we are not adding a schema, only a single formula).

Let us point out some peculiarity of the system $\mathcal{H}\mathsf{InqBQ}_n$. Firstly, the system itself *depends* on the syntax considered. In fact the definition of the formula C_n (given in Lemma 8.1.3) *depends* on the non-rigid symbols present in the signature and on the presence of the non-rigid equality $\asymp$. Secondly, since the formulas C_n are not defined for a signature containing infinitely many non-rigid symbols, then the system itself *is not defined for arbitrary signatures*. Whether the axiomatization method presented in this chapter can be generalized to arbitrary signatures remains an open problem.

Let us indicate with $\vdash_n$ the consequence relation of the system $\mathcal{H}\mathsf{InqBQ}_n$ and with $\vDash_n$ the entailment of the logic InqBQ_n. In particular $\Phi \vDash_n \psi$ means that for every model $\mathcal{M} \in \mathbb{M}_n$, if $\mathcal{M} \vDash \varphi$ for every $\varphi \in \Phi$ then $\mathcal{M} \vDash \psi$. The rest of the section is dedicated to prove the following completeness theorem:

8.2.1. Theorem. *The system $\mathcal{H}\mathsf{InqBQ}_n$ is strongly complete for InqBQ_n, that is, for every set of formulas $\Phi \cup \{\psi\}$ it holds that*

$$\Phi \nvdash_n \psi \qquad \Longrightarrow \qquad \Phi \nvDash_n \psi$$

Our strategy to prove Theorem 8.2.1 consists of three steps. Firstly, in Subsection 8.2.1 we will show that the completeness problem for $\mathcal{H}\mathsf{InqBQ}_n$ can be restated in terms of superintuitionistic logics. In particular, the saturated theories of $\mathcal{H}\mathsf{InqBQ}_n$ coincide exactly with some saturated theories of the superintuitionistic logic $\mathsf{CD} + \mathsf{KF} + \mathsf{KP} + \mathsf{UP}$ meeting some additional requirements. Secondly, in Subsection 8.2.2 we will focus on the superintuitionistic logic mentioned above, and in particular on its *constant domain canonical model* $\mathcal{K}^c_A$ [Gabbay et al., 2009, Section 7.2]. To be more precise, we will study the *portion* of $\mathcal{K}^c_A$ comprised of theories of $\mathcal{H}\mathsf{InqBQ}_n$ and show that the point-generated submodels of this portion are actually $\mathcal{P}_0$ - CD-models, that is, CD-models corresponding to information models (recall Lemma 2.2.9). Finally, in Subsection 8.2.3 we will apply the results from the previous steps to show the completeness of $\mathcal{H}\mathsf{InqBQ}_n$.

8.2.1 Connection with $\mathsf{CD} + \mathsf{KF} + \mathsf{KP} + \mathsf{UP}$

For brevity, let us indicate the superintuitionistic logic $\mathsf{CD} + \mathsf{KF} + \mathsf{KP} + \mathsf{UP}$ simply with L. Notice that in the definition of $\mathcal{H}\mathsf{InqBQ}_n$ the axiomatic schemata required to be closed under arbitrary substitutions (so excluding DNC) are exactly CD, KF, KP and UP. So the choice of working with this logic to study the properties of $\mathcal{H}\mathsf{InqBQ}_n$ should not come as a surprise. In fact, the following result shows exactly the connection between this superintuitionistic logic and the axiomatic system in Figure 2.6.

Let $\vdash_L$ be the consequence relation of the logic L. Given a set of sentences Φ, let $\Phi^{\vdash_L}$ be its deductive closure w.r.t. $\vdash_L$ and let $\Phi^{\vdash}$ be its deductive closure w.r.t. $\vdash$.[8]

8.2.2. Lemma. *Given Φ a set of sentences in the signature $\Sigma(A)$, the following identity holds:*

$$\Phi^{\vdash} = (\Phi \cup \mathsf{DNC}_A)^{\vdash_L}$$

where DNC_A is the set of instances of the schema DNC in the signature $\Sigma(A)$.

The proof follows trivially from the definitions of $\vdash_L$ and $\vdash$; the details are left to the reader. This result taken on its own is not particularly relevant, but it has some really interesting consequences for the axiomatization problem, as the following corollary shows.

8.2.3. Corollary. *Theorem 8.2.1 is equivalent to the following condition:*
Suppose that Φ is a theory containing every instance of the schema DNC and the formula C_n. Moreover suppose that $\Phi \nvdash_L \psi$ for a certain formula ψ. Then $\Phi \nvDash_n \psi$.

Proof:
We restate the condition above in an equivalent form: for an arbitrary set of formulas Φ it holds that

$$\psi \notin (\Phi \cup \mathsf{DNC}_A \cup \{\mathrm{C}_n\})^{\vdash_L} \qquad \Rightarrow \qquad \Phi \nvDash_n \psi$$

[8] Recall that $\vdash$ indicates the consequence relation of the deductive system in Figure 2.6.

In turn, by Lemma 8.2.2 and the definition of $\mathcal{H}\mathsf{InqBQ}_n$ this condition is equivalent to

$$\Phi \nvdash_n \psi \qquad \Rightarrow \qquad \Phi \nvDash_n \psi$$

which is exactly the statement of Theorem 8.2.1. □

So the completeness problem for the system $\mathcal{H}\mathsf{InqBQ}_n$ boils down to showing a property of $\vdash_L$ in connection with the semantics of InqBQ_n. This does not seem a great step ahead in the proof, but it has a major consequence: it allows us to borrow techniques to show completeness of a system from the theory of superintuitionistic logic, and in particular the *canonical model technique*.

We will develop this aspect in the next Subsection, but for now let us show another auxiliary result using the correspondence in Lemma 8.2.2. In particular, we will now focus on *L-saturated theories*.

8.2.4. Definition (L-saturated theory). Let A be an infinite set of parameters. A $\Sigma(A)$-theory Φ is called *L-saturated* (w.r.t. A) if for every sentences φ, ψ of $\Sigma(A)$ it satisfies:

- Coherence: $\Phi \nvdash_L \bot$;
- Deductive closure: if $\Phi \vdash_L \varphi$ then $\varphi \in \Phi$;
- Disjunction property: If $\Phi \vdash_L \varphi \vee\!\!\!\vee \psi$ then $\Phi \vdash_L \varphi$ or $\psi \vdash_L \varphi$;
- Existence property: If $\Phi \vdash_L \overline{\exists} x.\varphi$ then $\Phi \vdash_L \varphi[a/x]$ for some $a \in A$;
- Normality condition: If $\Phi \nvdash_L \forall x.\varphi$ then $\Phi \nvdash_L \varphi[a/x]$ for some $a \in A$.

We will indicate with $\mathrm{ST}(A)$ the set of L-saturated theories.

As we will see in the next subsection, L-saturated theories are the points of the *canonical model* for the logic L. Connecting these theories with saturated theories (Definition 7.3.5) is another key to use the canonical model to study the system $\mathcal{H}\mathsf{InqBQ}_n$. This is exactly what we achieve with the following lemma.

8.2.5. Lemma. *Given Φ a set of formulas of $\Sigma(A)$, the following are equivalent:*

1. *Φ is a saturated theory (Definition 7.3.5);*
2. *Φ contains every instance of the schema* DNC *and it is a L-saturated theory.*

This result follows trivially from Lemma 8.2.2 and the definitions of saturated theory and of L-saturated theory; the details are left to the reader.

8.2.2 Constant Domain Canonical Model

Lemmas 8.2.2 and 8.2.5 allow us to study the consequence relation $\vdash$ of inquisitive logic using the consequence relation $\vdash_L$ of the logic L. In particular, we can employ the *constant domain canonical model* $\mathcal{K}^c_A$ for the logic L to study the properties of $\vdash$. Let us recall the definition and the main properties of $\mathcal{K}^c_A$.

8.2.6. Definition. Given an infinite set of parameters A, the *canonical model with constant domain for* L *over* A is the CD-model[9]

$$\mathcal{K}^c_A \;:=\; \langle\, \mathrm{ST}(A),\, \subseteq,\, A,\, \mathcal{I}^c_A,\, \asymp \,\rangle$$

where:

- $\mathrm{ST}(A)$ is the set of L-saturated theories (Definition 8.2.4);
- The interpretation of the relation and equality symbols is defined by the following clauses:

$$R^{\mathcal{K}^c_A}_{\Phi}(a_1,\dots,a_n) \quad \text{iff} \quad R(a_1,\dots,a_n) \in \Phi$$
$$a \asymp^{\mathcal{K}^c_A}_{\Phi} b \quad \text{iff} \quad a \asymp b \in \Phi$$

8.2.7. Theorem ([Gabbay et al., 2009, Theorem 7.2.6]). *For every* L*-saturated theory* Φ*, every sequence of elements* $a_1,\dots,a_n \in A$ *and every formula* $\psi(x_1,\dots,x_n)$*, it holds that*

$$\mathcal{K}^c_A, \Phi \Vdash \psi(a_1,\dots,a_n) \qquad \textit{iff} \qquad \psi(a_1,\dots,a_n) \in \Phi$$

8.2.8. Lemma (Saturation lemma [Gabbay et al., 2009, Lemma 7.2.3]). *Consider a theory* $\Phi \cup \{\psi\}$ *in the signature* Σ *such that* $\Phi \nvdash_L \psi$*. Then there exists an* A*-saturated theory* Δ *in the signature* $\Sigma(A)$ *such that* $\Phi \subseteq \Delta$ *and* $\psi \notin \Delta$.

The lemmas above are the cornerstone of the canonical model technique. Combining them, we can show that every non-derivation $\Phi \nvdash_L \psi$ is witnessed by a point of $\mathcal{K}^c_A$: we just need to apply Lemma 8.2.8 to obtain a theory $\Delta \in \mathrm{ST}(A)$ extending Φ and not containing ψ; and consequently Theorem 8.2.7 ensures that Δ (seen as a point of the canonical model) forces all the formulas in Φ and does not force ψ. So this results allow to show that the consequence relation $\vdash_L$ and the entailment relation $\Vdash$ restricted to models of the logic L actually coincide.

However our current aim is not to study the relation $\Vdash$, but the relation $\vDash_n$. Following the intuition given by Corollary 8.2.3, we will focus on some special theories belonging to $\mathcal{K}^c_A$, that is, we will focus on the theories containing all the instances of the schema DNC and the formula C_n.

As we will show in this subsection, the part of the canonical model comprised of these theories has a rather peculiar shape: all its point-generated submodels are negative $\mathcal{P}_0$-CD-models, which in turn correspond to inquisitive models by Lemma 2.2.9. This property allow us to relate the consequence relation $\vdash_L$ and the entailment relation $\vDash_n$.

[9]Recall that CD-models are defined in Chapter 3, Definition 2.2.1.

The strategy to prove this consists in collecting information on the *shape* of this portion of the canonical model: what are the theories corresponding to endpoints, how many endpoints has a theory among its successors, and so on. Once we have collected enough information, we can prove the key lemma, turning this cacophony of results into the right harmony to prove completeness.

Let us start with a folklore result, telling us what theories correspond to the endpoints of $\mathcal{K}^c_A$.

8.2.9. LEMMA. *Consider a theory $\Phi \in \mathrm{ST}(A)$. Then Φ is an endpoint of $\mathcal{K}^c_A$ iff Φ is classically saturated.*[10]

This is a well known result in the literature, which follows easily from Theorem 8.2.7; we omit the proof for brevity.

If we focus on the part of the canonical model which satisfies the schema DNC, we can infer more results about endpoints of the model. For example, this part of the canonical property satisfies the so called *McKinsey property*: every point of $\mathcal{K}^c_A$ that satisfies the schema DNC has an endpoint among its successors.

8.2.10. COROLLARY. *Let $\Phi \in \mathrm{ST}(A)$ be a theory satisfying every instance of the axiom DNC. Then one of the successors of Φ is an endpoint of $\mathcal{K}^c_A$.*

Proof:
Given a formula φ, define φ^{cl} as the formula obtained substituting every instance of the symbols $\mathbin{\rotatebox[origin=c]{180}{$\bigwedge$}}$ and $\overline{\exists}$ in φ with their classical analogue $\vee$ and $\exists$ respectively. Given a set of formulas Θ, define $\Theta^{cl} := \{ \varphi^{cl} \mid \varphi \in \Theta \}$. Notice that Θ^{cl} is always a classical theory (in the sense of Definition 7.3.6).

By Lemma 8.2.5, Φ is a saturated theory (in the sense of Definition 7.3.5). Notice that, since Φ is coherent by hypothesis, then as a direct consequence of [Ciardelli, 2016, Proposition 4.6.4] also Θ^{cl} is coherent.

By Lemma 7.3.17 there exists a classical theory $\Delta \supseteq \Phi^{cl}$ which is classically saturated; and by Lemma 8.2.9 Δ is an endpoint of $\mathcal{K}^c_A$. In particular, a direct verification of the semantics clauses for $\mathbin{\rotatebox[origin=c]{180}{$\bigwedge$}}$ and $\overline{\exists}$ shows that $\varphi \in \Delta$ iff $\varphi^{cl} \in \Delta$, and consequently $\Phi^{cl} \subseteq \Delta$ implies $\Phi \subseteq \Delta$. □

Let us point out that we basically proved this result in Chapter 7, under the guise of Lemma 7.3.17. Lemma 8.2.5 is the bridge that allows us to translate this result in terms of $\mathcal{K}^c_A$—and to skip a quite tedious proof at that.

The next few results do not focus on *whether* the points of the canonical model have endpoints among their successors, but they focus on *what* endpoints they have among their successors. Before implementing the strategy, let us start with a useful definition.

[10]Classically saturated theories first appear in Definition 7.3.6.

8.2.11. Definition (E-width). Given a CD-model $\mathcal{K}$ with underlying frame $\langle S, \leq \rangle$ and a state $s \in S$, we indicate with E_s the set of successors of s which are also endpoints.

$$E_s \;:=\; \{\, e \in S \mid s \leq e \text{ and } e \text{ is an endpoint } \}$$

The *E-width* (or endpoint-width) of s is $|E_s|$. The *E-width* of $\mathcal{K}$ is the sup of the E-widths of its states.

The definition of E-width arises naturally when we consider the correspondence between $\mathcal{P}_0$ - CD-models and information models (Lemmas 2.2.8 and 2.2.9): the E-width of a $\mathcal{P}_0$ - CD-model $\mathcal{K}$ is the cardinality of W, the set of worlds of its corresponding information model.

The following lemma tell us that, as long as we are interested in theories containing C_n, we can restrict our attention to points of $\mathcal{K}^c_A$ with E-width at most n.

8.2.12. Lemma. *Let $\Phi \in \mathrm{ST}(A)$ be a theory containing every instance of* DNC *and the formula C_n. Then Φ has E-width at most n.*

Proof:
We prove the result dividing the two cases $n = 1$ and $n > 1$. For $n = 1$, we have $\mathcal{K}^c_A, \Phi \Vdash \mathrm{C}_1$, that is, for every relation symbol R and elements $a_1, \ldots, a_{\mathrm{Ar}(R)} \in A$ we have $\mathcal{K}^c_A, \Phi \Vdash R(a_1, \ldots, a_{\mathrm{Ar}(R)})$ or $\mathcal{K}^c_A, \Phi \Vdash \neg R(a_1, \ldots, a_{\mathrm{Ar}(R)})$. This means that Φ is a classically saturated theory and so an endpoint of $\mathcal{K}^c_A$ by Lemma 8.2.9. It follows that the E-width of Φ is 1.

As for the case $n > 1$, suppose $\mathcal{K}^c_A, \Phi \Vdash \mathrm{C}_n$. We are going to find formulas $\alpha_1, \ldots, \alpha_n$ such that (1) every endpoint of $\mathcal{K}^c_A$ satisfies exactly one among them and (2) $\mathcal{K}^c_A, \Phi \Vdash \alpha_i \to \mathrm{C}_1$ for every $i \in \{1, \ldots, n\}$. From these two properties the conclusion follows easily.

Let us start by expanding the formula C_n in the assumption $\mathcal{K}^c_A, \Phi \Vdash \mathrm{C}_n$:

$$\mathcal{K}^c_A, \Phi \Vdash \overline{\exists} x. \mathbb{V}_{i=1}^{n-1}[(P(x) \to \mathrm{C}_i) \wedge (\neg P(x) \to \mathrm{C}_{n-i})]$$

$$\Longrightarrow \quad \text{for some } a_0 \in A, \text{ for some } 1 \leq k_0 \leq n-1 \; \begin{cases} \mathcal{K}^c_A, \Phi \Vdash P(a_0) \to \mathrm{C}_{k_0} \\ \mathcal{K}^c_A, \Phi \Vdash \neg P(a_0) \to \mathrm{C}_{n-k_0} \end{cases}$$

Notice that every endpoint of E_Φ satisfies exactly one among the formulas $P(a_0)$ and $\neg P(a_0)$: we define E^+_0 to be the set of endpoints that satisfy the former and E^-_0 to be the set of endpoints that satisfy the latter. So $\{E^+_0, E^-_0\}$ is a partition of E_Φ.

We found two interesting conditions, namely $\mathcal{K}^c_A, \Phi \Vdash P(a_0) \to \mathrm{C}_{k_0}$ and $\mathcal{K}^c_A, \Phi \Vdash \neg P(a_0) \to \mathrm{C}_{n-k_0}$. We need to work with both of them by expanding the definition of C_{k_0} and C_{n-k_0} in the respective formulas. To keep track of the expansion operations we perform, we associate the formula $P(a_0) \to \mathrm{C}_{k_0}$ to

E_0^+, the first element of the partition previously defined; and $\neg P(a_0) \to \mathrm{C}_{n-k_0}$ to E_0^-, the second element of the partition. Notice that three invariants hold: **(a)** each set collects exactly the endpoints satisfying the antecedent of the associated formula; **(b)** the sum of the indexes of the subformulas $\mathrm{C}_\bullet$ appearing in the consequents of the formulas is n; and **(c)** the formulas associated are all forced in Φ.

We now focus on the formula $P(a_0) \to \mathrm{C}_{k_0}$. If $k_0 = 1$, then by inductive hypothesis the formula $P(a_0)$ is supported by at most one endpoint, which is the only possible element of E_0^+. In this case we stop the procedure and shift our attention to the other formula. Otherwise we proceed by expanding the formula C_{k_0} in the condition $\mathcal{K}_A^c, \Phi \Vdash P(a_0) \to \mathrm{C}_{k_0}$.

$$\mathcal{K}_A^c, \Phi \Vdash P(a_0) \to \overline{\exists} x. \mathbb{V}_{i=1}^{k_0-1}[(P(x) \to C_i) \wedge (\neg P(x) \to C_{k_0-i})]$$

$$\Downarrow (\text{by } \mathsf{DNC})$$

$$\mathcal{K}_A^c, \Phi \Vdash \neg\neg P(a_0) \to \overline{\exists} x. \mathbb{V}_{i=1}^{k_0-1}[(P(x) \to C_i) \wedge (\neg P(x) \to C_{k_0-i})]$$

$$\Downarrow (\text{by } \mathsf{UP})$$

$$\mathcal{K}_A^c, \Phi \Vdash \overline{\exists} x. \left(\neg\neg P(a_0) \to \mathbb{V}_{i=1}^{k_0-1}[(P(x) \to C_i) \wedge (\neg P(x) \to C_{k_0-i})]\right)$$

$$\Downarrow (\text{by } \mathsf{KP})$$

$$\mathcal{K}_A^c, \Phi \Vdash \overline{\exists} x. \mathbb{V}_{i=1}^{k_0-1}[(\neg\neg P(a_0) \wedge P(x) \to C_i) \wedge (\neg\neg P(a_0) \wedge \neg P(x) \to C_{k_0-i})]$$

$$\Downarrow (\text{by } \mathsf{DNC})$$

$$\mathcal{K}_A^c, \Phi \Vdash \overline{\exists} x. \mathbb{V}_{i=1}^{k_0-1}[(P(a_0) \wedge P(x) \to C_i) \wedge (P(a_0) \wedge \neg P(x) \to C_{k_0-i})]$$

$$\Downarrow (\text{for some } a_{00} \in A,\ k_{00} \leq k_0)$$

$$\mathcal{K}_A^c, \Phi \Vdash P(a_0) \wedge P(a_{00}) \to C_{k_{00}} \quad \text{and} \quad \mathcal{K}_A^c, \Phi \Vdash P(a_0) \wedge \neg P(a_{00}) \to C_{k_0-k_{00}}$$

We are in a situation analogous to the previous one. The endpoints in E_0^+ satisfy exactly one among $P(a_0) \wedge P(a_{00})$ and $P(a_0) \wedge \neg P(a_{00})$: we define E_{00}^+ to be the set of endpoints that satisfy the former and E_{00}^- to be the set of endpoints that satisfy the latter. Notice that $\{E_{00}^+, E_{00}^-, E_0^-\}$ is a partition of E_Φ strictly finer than $\{E_0^+, E_0^-\}$.

To keep tabs of the expansion operations, we associate to each element of this partition a different formula: we associate $P(a_0) \wedge P(a_{00}) \to C_{k_{00}}$ to E_{00}^+; we associate $P(a_0) \wedge \neg P(a_{00}) \to C_{k_0-k_{00}}$ to E_{00}^+; and we already associated $\neg P(a_0) \to C_{n-k_0}$ to E_0^-. Notice that the three invariants still hold: **(a)** each set collects exactly the endpoints satisfying the antecedent of the associated formula; **(b)** the sum of the indexes of the subformulas $\mathrm{C}_\bullet$ appearing in the consequents of the formuals is n; and **(c)** the formulas associated are all forced in Φ.

Because this unpacking procedure maintains invariant **(b)**, we can keep performing it until we obtain a partition $\{E_1, \ldots, E_n\}$ of E_Φ such that the formula associated to every E_i is of the form $\alpha_i \to \mathrm{C}_1$. By invariant **(a)** each E_i collects all and only the endpoints satisfying α_i, and so by invariant **(c)** and inductive hypothesis each E_i contains at most one endpoint. Since this was a partition of E_Φ, it follows that $|E_\Phi| \leq n$.

□

So, given a theory Φ extending DNC and C_n, we gathered quite some information on the set E_Φ, in particular that it is non-empty and that it contains at most n points. But we still do not have any information on what happens between Φ and the points in E_Φ. The following lemma sheds some light on this issue.

8.2.13. Lemma (Interpolation). *Let $\Phi \in \mathrm{ST}(A)$ be a theory containing every instance of* DNC *and the formula C_n. Let $E \subseteq E_\Phi$ be a non-empty set of endpoints above Φ. Then there exists a state $\Theta \in \mathrm{ST}(A)$ such that $\Phi \subseteq \Theta$ and $E_\Theta = E$.*

We refer to Θ as an *interpolant* between Φ and E.

Proof:
We start by making some observations on the structure of the successors of Φ. If E consists of only one point the result follows trivially—just take said point as Θ; so suppose $|E| \leq 2$. Let $\Delta_1, \ldots, \Delta_m$ with $m < n$ be an enumeration of the elements of E_Φ and suppose (without loss of generality) that $E = \{\Delta_1, \ldots, \Delta_k\}$.

As shown in the proof of Lemma 8.2.12, we can find formulas $\alpha_1, \ldots, \alpha_m$ such that

$$\mathcal{K}^c_A, \Delta_i \Vdash \alpha_j \qquad \text{iff} \qquad i = j$$

By Corollary 8.2.10, Φ and all its successors have an endpoint as a successor. It follows that for a state $\Phi' \supseteq \Phi$ we have

$$\mathcal{K}^c_A, \Phi' \Vdash \neg\neg(\alpha_{i_1} \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} \alpha_{i_l}) \qquad \text{iff} \qquad E_{\Phi'} \subseteq \{\Delta_{i_1}, \ldots, \Delta_{i_l}\}$$

Define the formulas

$$\beta :\equiv \neg\neg(\alpha_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} \alpha_k) \qquad \beta_i :\equiv \neg\neg(\alpha_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} \alpha_{i-1} \mathbin{\vee\!\!\!\vee} \alpha_{i+1} \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} \alpha_n)$$

By the previous observations, $\mathcal{K}^c_A, \Phi' \Vdash \beta$ iff $E_{\Phi'} \subseteq E$; and $\mathcal{K}^c_A, \Phi' \Vdash \beta_i$ iff $\Delta_i \notin E_{\Phi'}$.

Now we go back to the proof of the main result. Suppose towards a contradiction that there is no state $\Theta \supseteq \Phi$ whose set of endpoints is E. It follows that for every point Φ' above Φ such that $E_{\Phi'} \subseteq E$, then there exists $i \in \{1, \ldots, k\}$ such that $E_{\Phi'} \subseteq E \setminus \{i\}$. Restating this in terms of the formulas introduced above, we have

$$\mathcal{K}^c_A, \Phi \Vdash \beta \to \beta_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} \beta_k$$

On the other hand, since $\mathcal{K}^c_A, \Delta_i \Vdash \beta$ and $\mathcal{K}^c_A, \Delta_i \not\Vdash \beta_i$ for every $i \in \{1, \dots, k\}$, we also have

$$\mathcal{K}^c_A, \Phi \not\Vdash (\beta \to \beta_1) \vee\!\!\!\vee \dots \vee\!\!\!\vee (\beta \to \beta_k)$$

But this leads to a contradiction, since Φ forces all the instances of the schema KP, and so it follows that

$$\mathcal{K}^c_A, \Phi \Vdash (\beta \to \beta_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee \beta_k) \to (\beta \to \beta_1) \vee\!\!\!\vee \dots \vee\!\!\!\vee (\beta \to \beta_k)$$

□

We gathered some interesting information about $\mathcal{K}^c_A$, that is, given a state Φ satisfying all the instances of DNC and C_n we have $1 \leq |E_\Phi| \leq n$ and for every non-empty subset of E_Φ there is an interpolant. So the submodel of $\mathcal{K}^c_A$ generated by Φ starts to look like a $\mathcal{P}_0$-CD-model. However we still miss an important piece of the puzzle: whether the interpolants we found are *unique*. The following proposition shows exactly this property.

8.2.14. PROPOSITION. *Given a finite non-empty set E of endpoints of $\mathcal{K}^c_A$, there exists a unique theory Φ satisfying every instance of DNC and such that $E_\Phi = E$. Moreover, this is the theory of an inquisitive model with $|E|$ worlds.*

Proof:
We prove first the existence of Φ and then the uniqueness.
Existence: Consider the inquisitive model $\mathcal{M}$ with domain A and set of worlds E, and such that for every relation $R \in \Sigma$, every $\Delta \in E$ and every a, a', $a_1, \dots, a_{\mathrm{Ar}(R)} \in A$ it holds that

$$\begin{array}{lll} \mathcal{M}, \{\Delta\} \vDash R(a_1, \dots, a_{\mathrm{Ar}(R)}) & \text{iff} & R(a_1, \dots, a_{\mathrm{Ar}(R)}) \in \Delta \\ \mathcal{M}, \{\Delta\} \vDash a \asymp a' & \text{iff} & a \asymp a' \in \Delta \end{array}$$

Since Δ is an endpoint of $\mathcal{K}^c_A$ and so a classically saturated theory (Lemma 8.2.9), it follows that for every formula φ it holds that

$$\mathcal{M}, \{\Delta\} \vDash \varphi \qquad \text{iff} \qquad \varphi \in \Delta$$

Since the domain of $\mathcal{M}$ is A (i.e., $\mathcal{M}$ is an A-covered model, in the notation of Chapter 7), it follows that[11]

$$\mathrm{Th}(\mathcal{M}) := \{\, \varphi \text{ closed formula of } \Sigma(A) \mid \mathcal{M} \vDash \varphi \,\}$$

is a state of the canonical model. Moreover $\mathcal{M}$ supports every instance of the schema DNC (Theorem 9.1.5) and, since $\mathcal{M}$ has exactly n worlds, it also supports C_n (Lemma 8.1.3). So by Lemma 8.2.12 $|E_{\mathrm{Th}(\mathcal{M})}| \leq n$, and by persistency

[11] $\mathrm{Th}(\mathcal{M})$ was already introduced in Section 7.3, after Definition 7.3.5. Recall also that $\mathrm{Th}(\mathcal{M})$ is a saturated theory whenever $\mathcal{M}$ is an A-covered model.

(Lemma 2.1.9) for every $\Delta \in E$we have $\mathrm{Th}(\mathcal{M}) \subseteq \mathrm{Th}(\mathcal{M}|_{\{\Delta\}}) = \Delta$. So we can conclude that $E = E_{\mathrm{Th}(\mathcal{M})}$, that is, $\mathrm{Th}(\mathcal{M})$ is the theory that we were looking for.

Uniqueness: We can show by structural induction on φ that, if $\Phi, \Phi' \in \mathrm{ST}(A)$ both satisfy every instance of DNC and $E_\Phi = E_{\Phi'} = E$, then $\varphi \in \Phi$ iff $\varphi \in \Phi'$.

The only non-trivial cases are the case of φ atomic and the case of $\varphi \equiv \psi \to \chi$.

If φ is an atomic formula, then we have:

$$\begin{aligned}
\mathcal{K}^c_A, \Phi \vDash \varphi &\iff \mathcal{K}^c_A, \Phi \vDash \neg\neg\varphi && \text{(by DNC)}\\
&\iff \text{For every } \Delta \in E,\ \mathcal{K}^c_A, \Delta \vDash \varphi && \text{(by Corollary 8.2.10)}\\
&\iff \mathcal{K}^c_A, \Phi' \vDash \neg\neg\varphi && \text{(by Corollary 8.2.10)}\\
&\iff \mathcal{K}^c_A, \Phi' \vDash \varphi && \text{(by DNC)}
\end{aligned}$$

If $\varphi \equiv \psi \to \chi$, we show that if $\psi \to \chi \notin \Phi$ then $\psi \to \chi \notin \Phi'$; the other implication follows with an analogous proof. Suppose that $\psi \to \chi \notin \Phi$, that is, that there exists a state $\Theta \supseteq \Gamma$ such that

$$\mathcal{K}^c_A, \Theta \Vdash \psi \qquad\qquad \mathcal{K}^c_A, \Theta \nVdash \chi$$

By Lemma 8.2.13, there exists a point Θ' above Φ' such that $E_\Theta = E_{\Theta'}$. By inductive hypothesis, it follows that

$$\mathcal{K}^c_A, \Theta' \Vdash \psi \qquad\qquad \mathcal{K}^c_A, \Theta' \nVdash \chi$$

and consequently that $\psi \to \chi \notin \Phi'$.

So we showed that for every φ it holds that $\varphi \in \Phi$ iff $\varphi \in \Phi'$, that is, $\Phi = \Phi'$. This concludes the proof of uniqueness.

□

Let us conclude this subsection by showing that the submodels of $\mathcal{K}^c_A$ generated by theories extending DNC and containing C_n are indeed negative $\mathcal{P}_0$ - CD-models. Given the previous results, this determines the shape of the submodel consisting of all the points of finite E-width.

8.2.15. Corollary. *Let $\Phi \in \mathrm{ST}(A)$ be a theory containing every instance of* DNC *and the formula C_n. Then the submodel generated by Φ is a negative $\mathcal{P}_0$ -* CD*-model with at most n endpoints.*

Proof:

Negativity follows from Lemma 2.2.4 and the hypothesis that Φ contains every instance of DNC, and so in particular every instance of $\neg\neg p \to p$ for p an atomic formula. Moreover by Lemma 8.2.12, $|E_\Phi| \leq n$.

It remains to show that the submodel generated by Φ is a $\mathcal{P}_0$ - CD-model. We already know that it is a CD-model by definition of $\mathcal{K}^c_A$. So, defining $\Phi^\uparrow := \{\, \Psi \in \mathrm{ST}(A) \mid \Psi \supseteq \Phi \,\}$, it suffices to show that $\langle \Phi^\uparrow, \subseteq \rangle \cong \langle \mathcal{P}_0(E_\Phi), \supseteq \rangle$.

By Proposition 8.2.14, given a set of endpoints $E \subseteq E_\Phi$ there exist a unique theory $\Psi \in \mathrm{ST}(A)$ such that $E_\Psi = E$; moreover by Lemma 8.2.13, this Ψ is a successor of Φ. Finally, every state in $\Phi^\uparrow$ has at least one endpoint for a successor by Lemma 8.2.10. This shows that $\Phi^\uparrow$ is in one-to-one correspondence with $\mathcal{P}_0(E_\Phi)$.

It remains to show that for every $\Psi_1, \Psi_2 \supseteq \Phi$ it holds that

$$\Psi_1 \subseteq \Psi_2 \qquad \text{iff} \qquad E_{\Psi_1} \supseteq E_{\Psi_2}$$

The left-to-right direction follows trivially by definition of endpoint. For the right-to-left direction, since $E_{\Psi_2} \subseteq E_{\Psi_1}$ by Lemma 8.2.13 there exists $\Theta \supseteq \Psi_1$ such that $E_\Theta = E_{\Psi_2}$. And by Lemma 8.2.14 it follows that $\Theta = \Psi_2$, showing that $\Psi_1 \subseteq \Psi_2$.

□

8.2.3 Completeness of $\mathcal{H}\mathsf{InqBQ}_n$

We are finally ready to collect the results from the previous subsections and to prove the completeness result for the system $\mathcal{H}\mathsf{InqBQ}_n$.

Proof of Theorem 8.2.1:

As we showed in Corollary 8.2.3, the statement of Theorem 8.2.1 is equivalent to the following:

Suppose that Φ is a theory containing every instance of the schema DNC *and the formula* C_n*. Moreover suppose that $\Phi \nvdash_L \psi$ for a certain formula ψ. Then $\Phi \nvDash_n \psi$.*

So we proceed to prove this condition. Consider Φ and ψ as in the hypothesis. By Lemma 8.2.8, there exists a theory $\Theta \in \mathrm{ST}(A)$ such that $\Phi \subseteq \Theta$ and $\psi \notin \Theta$. Moreover, since Φ contained every instance of the schema DNC and the formula C_n, so does Θ. By Lemmas 8.2.14 and 8.2.12, Θ is the theory of an information model $\mathcal{M}$ with at most $|E_\Theta| \leq n$ worlds. By definition of theory of a model, $\mathcal{M} \vDash \chi$ iff $\chi \in \Theta$, from which it follows $\mathcal{M} \vDash \varphi$ for every $\varphi \in \Phi$ and $\mathcal{M} \nvDash \psi$. Finally, since $\mathcal{M} \in \mathbb{M}_n$, we conclude that $\Phi \nvDash_n \psi$.

□

8.3 The BW Fragment

The results of the previous section suggest to study models with finitely many worlds in more detail. Unfortunately, as shown in Proposition 8.1.5, focusing our attention on these models does not allow us to capture the expressive power of InqBQ, since there are formulas which admit only countermodels with infinitely many worlds. However *there is* a fragment of the logic which can be studied using only models with finitely many worlds: the *bounded-width fragment* BW.

8.3.1. Definition (BW fragment). The *bounded-width fragment* (indicated with BW) is generated by the following grammar:

$$\psi ::= \bot \mid p \mid \psi \wedge \psi \mid \psi \vee\!\!\!\vee \psi \mid \varphi \to \psi \mid \forall x.\psi$$

where p ranges over atoms; φ ranges over generic formulas.

So the condition for a formula ψ to be part of the BW fragment is that every subformula of the form $\overline{\exists} x.\chi$ has to be on the left side of an implication. For example, the formulas $\overline{\exists} x.P(x) \to P(a)$ and $(\neg P(a) \to \overline{\exists} x.P(x)) \to \forall x.?P(x)$ are in the BW fragment; while the formulas $\overline{\exists} x.P(x)$ and $P(b) \to (\neg P(a) \to \overline{\exists} x.P(x))$ are not. Notice in particular that all the formulas not containing the symbol $\overline{\exists}$ are in the fragment.

This definition, which seems somewhat arbitrary, stems from a quite interesting property of the fragment, corresponding to *coherency* in Dependence logic [Kontinen, 2010]: for every formula φ of the BW fragment there exists a natural number $n(\varphi)$ such that, for every model $\mathcal{M}$, info state s and assignment g it holds that

$$\mathcal{M}, s \vDash_g \varphi \qquad \text{iff} \qquad \text{For all } t \subseteq s, \text{ if } |t| \leq n(\varphi) \text{ then } \mathcal{M}, t \vDash_g \varphi$$

So to verify that a formula in the fragment is supported by a model, we need to verify it is supported only on states of *bounded size*; moreover the bound is finite and depends only on the formula. Let us prove the property with the following lemma.

8.3.2. Lemma. *Let φ be a formula of the* BW *fragment. There exists a natural number $n = n(\varphi)$ such that, for every information model $\mathcal{M}$, every info state s and every assignment g :* `Var` $\to D$ *it holds that*

$$\mathcal{M}, s \vDash_g \varphi \qquad \textit{iff} \qquad \forall t \subseteq s. \left[\, |t| \leq n \implies \mathcal{M}, t \vDash_g \varphi \,\right]$$

Moreover we can give an explicit bound to $n(\varphi)$:

$$\begin{aligned} n(\bot) &= 1 & n(A) &= 1 \quad \textit{for } A \textit{ atomic} \\ n(\psi \wedge \chi) &= \max(n(\psi), n(\chi)) & n(\psi \vee\!\!\!\vee \chi) &= n(\psi) + n(\chi) \\ n(\psi \to \chi) &= n(\chi) & n(\forall x.\psi) &= n(\psi) \end{aligned}$$

Proof:
We prove the equivalent statement

$$\mathcal{M}, s \nvDash_g \varphi \qquad \text{iff} \qquad \exists t \subseteq s. \left[\, |t| \leq n \text{ and } \mathcal{M}, t \nvDash_g \varphi \,\right]$$

Notice that the right-to-left implication follows by persistency of the support semantics (Lemma 9.1.6). The proof of the left-to-right implication consists of a simple structural induction on φ. The cases of $\varphi \equiv \bot$, φ atomic, $\varphi \equiv \psi \wedge \chi$ are trivial and left to the reader. In what follows we abbreviate "inductive hypothesis" with IH.

- If $\varphi \equiv \psi \mathbin{\mathbb{V}} \chi$:

$$\begin{aligned}
& \mathcal{M}, s \not\models_g \psi \mathbin{\mathbb{V}} \chi \\
\iff & \mathcal{M}, s \not\models_g \psi \text{ and } \mathcal{M}, s \not\models_g \chi \\
\iff & \begin{cases} \exists t \subseteq s. \left[\, |t| \leq n(\psi) \text{ and } \mathcal{M}, t \not\models_g \psi \,\right] \\ \exists t' \subseteq s. \left[\, |t'| \leq n(\chi) \text{ and } \mathcal{M}, t' \not\models_g \chi \,\right] \end{cases} \\
\implies & \exists u \subseteq s. \left[\, |u| \leq n(\psi) + n(\chi) \text{ and } \mathcal{M}, u \not\models_g \psi \mathbin{\mathbb{V}} \chi \,\right] \quad (\text{for } u := t \cup t')
\end{aligned}$$

- If $\varphi \equiv \psi \to \chi$:

$$\begin{aligned}
& \mathcal{M}, s \not\models_g \psi \to \chi \\
\iff & \exists t \subseteq s. \begin{cases} \mathcal{M}, t \models_g \psi \\ \mathcal{M}, t \not\models_g \chi \end{cases} \\
\iff & \exists t \subseteq s. \begin{cases} \mathcal{M}, t \models_g \psi \\ \exists u \subseteq t.\ |u| \leq n(\chi) \text{ and } \mathcal{M}, u \not\models_g \chi \end{cases} && \text{(by IH)} \\
\implies & \exists u \subseteq s.\ |u| \leq n(\chi) \text{ and } \begin{cases} \mathcal{M}, u \models_g \psi \\ \mathcal{M}, u \not\models_g \chi \end{cases} && \text{(by persistency)} \\
\iff & \exists u \subseteq s.\ |u| \leq n(\chi) \text{ and } \mathcal{M}, u \not\models \psi \to \chi
\end{aligned}$$

- If $\varphi \equiv \forall x.\chi$:

$$\begin{aligned}
& \mathcal{M}, s \not\models_g \forall x.\psi \\
\iff & \exists a \in D.\ \mathcal{M}, s \not\models_{g[x \mapsto a]} \psi \\
\iff & \exists a \in D.\ \exists t \subseteq s. \left[\, |t| \leq n(\psi) \text{ and } \mathcal{M}, t \not\models_{g[x \mapsto a]} \psi \,\right] \quad \text{(by IH)} \\
\iff & \exists t \subseteq s. \left[\, |t| \leq n(\psi) \text{ and } \mathcal{M}, t \not\models_g \forall x.\psi \,\right]
\end{aligned}$$

□

Notice that the bound $n(\varphi)$ is not sharp: for example, the formula $P(a) \mathbin{\mathbb{V}} P(a)$ is truth-conditional since it is equivalent to the classical formula $P(a)$, but the bound previously defined is $n(P(a) \mathbin{\mathbb{V}} P(a)) = 2$. We leave as an open problem whether there exists a decision procedure to find the optimal bound.

Notice that this lemma entails the *finite model property* for the entailment restricted on BW consequences.

8.3.3. Corollary (Finite model property). *Let* $\Phi \cup \{\psi\}$ *a set of formulas and suppose that* $\psi \in \mathsf{BW}$*. Then it holds that*

$$\Phi \vDash \psi \qquad \textit{iff} \qquad \textit{For every } \mathcal{M} \in \mathbb{M}_{<\aleph_0},\ \textit{if } \mathcal{M} \vDash \varphi \textit{ for every } \varphi \in \Phi,\ \textit{then } \mathcal{M} \vDash \psi$$

Proof:
The left-to-right direction is trivial. For the right-to-left direction, we show the contrapositive of the statement. Suppose that $\Phi \nvDash \psi$, that is, there exists an information model $\mathcal{M}$ such that $\mathcal{M} \vDash \varphi$ for every $\varphi \in \Phi$ and $\mathcal{M} \nvDash \psi$. We want to find a model in $\mathbb{M}_{<\aleph_0}$ that supports all the formulas in Φ, but not ψ.

By Lemma 8.3.2, there exists a state s of $\mathcal{M}$ such that $|s| \leq n(\varphi)$ $\mathcal{M}, s \nvDash \psi$. Moreover by persistency (Lemma 2.1.9) we have $\mathcal{M}, s \vDash \varphi$ for every $\varphi \in \Phi$. It follows that $\mathcal{M}|_s$ is a model with the properties required.

□

So to study the entailment between BW formulas we only need models with finitely many worlds. Let us point out that this result *does not* imply that the set of BW validities is decidable: in fact even though we need only models with finitely many worlds, we do not have any restriction on the cardinality of the domain. For example, every classical first order model M belongs to the class $\mathbb{M}_{<\aleph_0}$, modulo identifying M with the singleton information model $\mathcal{M} = \{M\}$.

Combining the results from Section 8.2 and Lemma 8.3.2, we can show that the set of validities of the BW fragment is *recursively enumerable*. The key observation to obtain this result is the following.

8.3.4. Lemma. *Let* $\Phi \cup \{\psi\}$ *be a set of formulas and suppose that* $\psi \in \mathsf{BW}$*. Then it holds that*

$$\Phi \vDash \psi \qquad \textit{iff} \qquad \Phi \vdash_{n(\psi)} \psi$$

Proof:
Firstly we show that the condition $\Phi \vDash \psi$ is equivalent to $\Phi \vDash_{\mathbb{M}_{n(\psi)}} \psi$. It follows trivially by definition of $\vDash$ and $\vDash_{\mathbb{M}_{n(\psi)}}$ that $\Phi \vDash \psi$ implies $\Phi \vDash_{\mathbb{M}_{n(\psi)}} \psi$. As for the other implication, we prove the contrapositive. Suppose that $\Phi \nvDash \psi$. This means that there exists an information model $\mathcal{M}$ such that $\mathcal{M} \vDash \varphi$ for every $\varphi \in \Phi$ and $\mathcal{M} \nvDash \psi$. By Lemma 8.3.2, there exists an info state s of $\mathcal{M}$ such that $|s| \leq n(\psi)$ and $\mathcal{M}, s \nvDash \psi$. Moreover, by persistency of the semantics (Lemma 2.1.9) we have $\mathcal{M}, s \vDash \varphi$ for every $\varphi \in \Phi$. It follows that $\mathcal{M}|_s \in \mathbb{M}_{n(\psi)}$ supports all the formulas in Φ and does not support ψ, thus $\Phi \nvDash_{\mathbb{M}_{n(\psi)}} \psi$.

Since by Theorem 8.2.1 the condition $\Phi \vDash_{\mathbb{M}_{n(\psi)}} \psi$ amounts to $\Phi \vdash_{n(\psi)} \psi$, the result follows. □

8.3.5. Theorem. *The set of validities in the* BW *fragment is recursively enumerable.*

Proof:
For every natural number m, the set of formulas of InqBQ derivable in $\vdash_m$ is recursively enumerable. Since we have a decidable procedure to establish whether a formula is in the BW fragment and to compute $n(\varphi)$ given φ, then also the set $X_m := \{\varphi \in \mathsf{BW} \mid \vdash_m \varphi \text{ and } n(\varphi) = m\}$ is recursively enumerable. It follows that the set $X := \bigcup_{m\in\mathbb{N}} X_m$ is recursively enumerable too, and by Lemma 8.3.4 this is exactly the set of valid formulas in the BW fragment □

However, Theorem 8.3.5 is not the only consequence of Lemma 8.3.4. In fact there is another low-hanging fruit to pick: the compactness of $\vDash$ for consequences in BW.

8.3.6. Theorem (Compactness for BW consequences). *Let $\Phi \cup \{\psi\}$ be a set of formulas and suppose that $\psi \in \mathsf{BW}$. Then*

$$\Phi \vDash \psi \qquad \textit{iff} \qquad \textit{there exists } \Phi' \subseteq \Phi \textit{ finite such that } \Phi' \vDash \psi$$

Proof:
The right-to-left direction follows trivially by definition of the entailment relation $\vDash$. For the left-to-right direction, by Lemma 8.3.4 the condition $\Phi \vDash \psi$ is equivalent to $\Phi \vdash_{n(\psi)} \psi$, that is, there exists a derivation of ψ from Φ in the system $\mathcal{H}\mathsf{InqBQ}_n$. Since a derivation uses only finitely many premises, if we define Φ' to be the set of premises used in the derivation above we have $\Phi' \vdash_{n(\psi)} \psi$, and so by Lemma 8.3.4 we have $\Phi' \vDash \psi$. □

We still do not know whether the compactness property holds for the logic InqBQ, so this result is an interesting clue to tackle the problem. For example, since all $\overline{\exists}$-free formulas are in the BW fragment, if we want to find a derivation $\Phi \vDash \psi$ violating the compactness principle, ψ must contain the symbol $\overline{\exists}$.

As shown with the two theorems above, Lemma 8.3.4 gives us an effective way to study the BW fragment from a syntactic point of view. However this approach is somewhat unsatisfying: we are not gathering any new information about the relation $\vdash$, that is, the consequence relation of the system in Figure 2.6 for the full language. The most effective way to do so would be to find an *axiomatization* for BW consequences, not relying on the systems $\mathcal{H}\mathsf{InqBQ}_n$. Knowing whether additional axioms are required to capture logical consequences between BW formulas could lead us closer to finding whether InqBQ is finitely axiomatizable.

Unfortunately, as of now it is not known whether the system is finitely axiomatizable. We conclude this section with the following conjecture, with the hope that it will become a theorem in future work.

8.3.7. Conjecture (Completeness for BW). *The axiomatic system in Figure 2.6 is* complete *for the relation* $\vDash$ *restricted to the* BW *fragment: For every set of formulas* $\Phi \cup \{\psi\}$ *from the* BW *fragment it holds that*

$$\Phi \nvdash \psi \qquad \Longrightarrow \qquad \Phi \nvDash \psi$$

8.4 Conclusions

In this chapter we introduced the inquisitive logics $\mathsf{InqBQ}_{<\lambda}$ and InqBQ_{λ}, defined by restricting the support semantics to information models with less than λ worlds and with at most λ worlds respectively. These logics generalize the hierarchy introduced by Sano [2011] consisting of the *finite-width inquisitive logics*, that is, the logics InqBQ_n for $n \in \mathbb{N}$. We focused on tackling two open problems left by Sano: determining whether InqBQ is *approximated* by the logics InqBQ_n (i.e., $\mathsf{InqBQ} = \bigcap_{n\in\mathbb{N}} \mathsf{InqBQ}_n$); and finding an axiomatization for the finite-width inquisitive logics.

We gave a negative answer to the first problem by defining a formula valid in all the logics InqBQ_n, but not valid in InqBQ. In particular, this shows that the semantics of InqBQ cannot be characterized by a class of models with finitely many worlds. As for the second problem, we defined an Hilbert-style system for InqBQ_n and showed its strong completeness, thus axiomatizing the finite-width inquisitive logics. The completeness proof strongly relies on the close connection between InqBQ and the intuitionistic logic of constant domains CD. In particular, the proof strategy consists in taking the canonical model with constant domain $\mathcal{K}^c_A$ for the superintuitionistic logic $\mathsf{CD} + \mathsf{KF} + \mathsf{KP} + \mathsf{UP}$ and study the properties of the portion of $\mathcal{K}^c_A$ satisfying some additional axioms. So rather than *building* a canonical model starting from an axiomatization, we *borrow* the canonical model of a suitable superintuitionistic logic and use it to carry on the proof. It remains an open question whether we can generalize this strategy to axiomatize InqBQ or other members of the hierarchy introduced (e.g., the logic of information models with finitely many worlds $\mathsf{InqBQ}_{<\aleph_0}$).

In this chapter we also introduced and studied the *bounded-width fragment* BW, whose formulas have the following interesting property: if φ is in the fragment, then there is a finite number $n(\varphi)$ such that a model $\mathcal{M}$ supports φ iff for every info state s of size at most $n(\varphi)$ the restriction $\mathcal{M}|_s$ supports φ. In other terms, to verify that a formula in the fragment is supported by a model, it suffices to check the support condition for information states of a bounded finite size. In particular, to study these fragment we only need models with *finitely many worlds*. We proved that the entailment relation of InqBQ restricted to this fragment has several interesting properties: it can be characterized in terms of the axiomatic systems for the logics InqBQ_n; it is compact; and the set of its validities is recursively enumerable. However, we still lack a proper axiomatization for the fragment, as pointed out in Conjecture 8.3.7. Let us point out a possible strategy to prove the conjecture: the completeness result would follow directly by Lemma 8.3.4, if we were to prove that the conditions $\Phi \vdash \psi$ and $\Phi \vdash_{\mathrm{C}_{n(\psi)}} \psi$ are equivalent. We leave the study of this problem and possible implementations of this last proof step for further work.

Chapter 9

Algebraic and Topological Semantics

In this chapter we present *an algebraic and a topological semantics for inquisitive propositional logic* InqB. This line of research strengthens the bonds between inquisitive logic and intermediate logics, and opens new avenues of research in the direction of universal algebra. Generalizing these semantic accounts to the first order case could prove to be a precious tool for studying InqBQ from new perspectives, for example using the methods employed by Rasiowa and Sikorski [1950] or Görnemann [1971].

The starting point is the connection between InqB and several intermediate logics, including Medvedev's logic ML [Medvedev, 1966] and Kreisel-Putnam logic KP [Chagrov and Zakharyaschev, 1997, p. 148] (see [Ciardelli, 2009] for a thorough analysis of these connections). In particular InqB can be characterized as the logic of general intuitionistic Kripke models based on Medvedev's frames for which the valuations of atomic propositions are principal upsets [Ciardelli, 2009, Proposition 2.2.2]. So we can think of InqB as the logic corresponding to a certain *Kripke semantics*.

Even though the algebraic structures arising from this characterization have been already considered in the literature (e.g., by Frittella et al. [2016]), proper algebraic and topological semantics for inquisitive logic were introduced and developed only in the last few years ([Bezhanishvili et al., 2019]; see also [Quadrellaro, 2019, Bezhanishvili et al., 2020] for further developments). The aim of this chapter is to present these new approaches to the study of InqB.

After reviewing some topological preliminaries in Section 9.1, we start in Section 9.2 with an algebraic semantics for inquisitive logic based on Heyting algebras with propositional valuations ranging over only the $\neg\neg$-fixpoints. The Kripke semantics for inquistive logic can be seen as a particular instance of this algebraic semantics: for F a Medvedev frame, the algebra $\mathrm{Up}_p(F)$ of principal upsets of F is the algebra of $\neg\neg$-fixpoints of the Heyting algebra $\mathrm{Up}(F)$ of all upsets of F. For our algebraic semantics, we motivate restricting attention to only special Heyting algebras, which we call *inquisitive algebras*, of which $\mathrm{Up}(F)$ for a Medvedev frame

F is an example.

We also show how inquisitive algebras arise from Boolean algebras: for a given Boolean algebra B, we define in Section 9.3.1 its *inquisitive extension* $H(B)$ and prove in Section 9.3.2 that $H(B)$ is the unique inquisitive algebra having B as its algebra of $\neg\neg$-fixpoints. We also show that inquisitive algebras determine Medvedev's logic. In addition to the algebraic characterization of $H(B)$ in Section 9.3.2, we give a topological characterization of $H(B)$ in Section 9.3.3 in terms of the recently introduced choice-free duality for Boolean algebras employing upper Vietoris spaces (UV-spaces) [Bezhanishvili and Holliday, 2020], which we review in Section 9.1.3. In particular, while a Boolean algebra B is realized as the Boolean algebra of compact regular open elements of a UV-space dual to B, we show that $H(B)$ is realized as the algebra of compact open elements of this space.

The topological characterization of $H(B)$ leads in Section 9.4 to a new topological semantics for inquisitive logic based on UV-spaces. As an additional benefit, we obtain a new topological semantics for Medvedev's logic.

We conclude in Section 9.5 with some directions for future research.

9.1 Background

In what follows we will assume the reader to be familiar with the basic notions on Kripke semantics for intuitionistic logic (see, e.g., [Chagrov and Zakharyaschev, 1997, Sections 2.1-2.3]) and algebraic semantics for intuitionistic logic (see, e.g., [Chagrov and Zakharyaschev, 1997, Section 7.1-7.3]).

9.1.1 Propositional Inquisitive Logic InqB

In this chapter we will work with *inquisitive propositional logic* InqB (see, e.g., [Ciardelli and Roelofsen, 2011] and [Ciardelli, 2016, Chapters 2-3]), chronologically the first incarnation of the support semantics. InqB is an extension of *classical propositional logic* CPC with questions and has been by far the most studied inquisitive logic in the literature. We quickly recall in this section the definitions concerning InqB and the main properties of the logic. Henceforth, we will assume to have fixed an infinite set of atomic propositions AP.

9.1.1. Definition (Syntax of InqB). The syntax $\mathcal{L}_p$ of InqB is defined by the following grammar:

$$\mathcal{L}_p : \;\; \varphi ::= \bot \;\mid p \mid \varphi \wedge \varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid \varphi \to \varphi$$

where $p \in \mathrm{AP}$.

We will use the same shorthands for $\neg$, $\vee$ and ? as for InqBQ, that is:

$$\neg\varphi \;:\equiv\; \varphi \to \bot \qquad\qquad \varphi \vee \psi \;:\equiv\; \neg(\neg\varphi \mathbin{\vee\!\!\!\vee} \neg\psi) \qquad\qquad ?\varphi \;:\equiv\; \varphi \mathbin{\vee\!\!\!\vee} \neg\varphi$$

As in the first order case, $\mathcal{L}_p$ can be thought as the syntax of CPC with in addition the question-forming operator $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$. We maintain the terminology *classical formula* to indicate $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-free formulas and usually indicate them with lowercase first letters of the greek alphabet: $\alpha, \beta, \ldots$ The role of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ is, yet again, to introduce alternative questions: for example the formula $?p :\equiv p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg p$ stands for the question "whether p is the case".

Models of InqB are analogous to their first order counterpart.

9.1.2. Definition (Propositional information model). A *(propositional) information model* $\mathcal{M}$ is a multiset $\{M_w \mid w \in W^{\mathcal{M}}\}$ where $W^{\mathcal{M}}$ is a set (called the set of worlds of the model) and the M_w are CPC-models, that is, propositional valuations over the set $\{\top, \bot\}$.

We will assume the same notational conventions as for InqBQ: we will indicate $W^{\mathcal{M}}$ with W when the model is clear from the context; we write p_w instead of $p^{\mathcal{M}_w}$ for the interpretation of the atomic formula p in the model M_w. We also maintain the terminology *information state* for the subsets of the set W.

9.1.3. Definition (Semantics of InqB). Let $\mathcal{M} = \{M_w \mid w \in W\}$ be a propositional information model and $s \subseteq W$ an info state. We define the support relation $\vDash$ over formulas of InqB by the following inductive clauses:

$$
\begin{aligned}
&\mathcal{M}, s \vDash \bot && \iff s = \emptyset \\
&\mathcal{M}, s \vDash p && \iff \text{For all } w \in s,\ p_w \text{ holds} \\
&\mathcal{M}, s \vDash \psi_1 \wedge \psi_2 && \iff \mathcal{M}, s \vDash \psi_1 \text{ and } \mathcal{M}, s \vDash \psi_2 \\
&\mathcal{M}, s \vDash \psi_1 \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi_2 && \iff \mathcal{M}, s \vDash \psi_1 \text{ or } \mathcal{M}, s \vDash \psi_2 \\
&\mathcal{M}, s \vDash \psi_1 \to \psi_2 && \iff \text{For all } t \subseteq s,\ \text{ if } \mathcal{M}, t \vDash \psi_1 \text{ then } \mathcal{M}, t \vDash \psi_2
\end{aligned}
$$

The intuition behind information models, information states and the semantics are completely analogous to the first order case: information models acts as *contexts* to represent *information*; information states encode *pieces of information*—in this case, properties of CPC-models; and the semantics describes whether a piece of information *supports* a given sentence (be it a statement or a question).

Also the other properties presented for the first order case are analogous for the propositional system:

9.1.4. Lemma. *Every classical formula is truth-conditional. That is, for $\mathcal{M}$ an information model, s an information state and α a classical formula, it holds that*

$$\mathcal{M}, s \vDash \alpha \iff \forall w \in s.\ M_w \vDash^{\mathsf{CPC}} \alpha$$

9.1.5. Theorem (Truth-conditionality and classical formulas). *For every formula φ, φ is truth-conditional if and only if there exists a classical formula such that $\varphi \equiv \alpha$.*

9.1.6. LEMMA. *For every formula φ of the logic*

Empty state $\mathcal{M}, \emptyset \vDash_g \varphi$.

Persistency *If* $\mathcal{M}, s \vDash_g \varphi$ *and* $u \subseteq s$, *then* $\mathcal{M}, u \vDash_g \varphi$.

9.1.7. THEOREM. *For every set of classical formulas* $\Gamma \cup \{\alpha\}$ *it holds that*

$$\Gamma \vDash \alpha \iff \Gamma \vDash^{\mathsf{CPC}} \alpha$$

Until now we focused on what InqB and InqBQ have in common—it does not come as a surprise that the two logics are so similar since they are defined through the same approach starting from CPC and CQC respectively. However, the following result distinguishes the two logics: the *disjunctive normal form lemma*.

9.1.8. LEMMA (Disjunctive normal form [Ciardelli, 2016, Proposition 2.4.4]). *Let* φ *be a formula in the language* $\mathcal{L}_p$. *Then there exist classical formulas* $\alpha_1, \dots, \alpha_n$ *such that*

$$\vDash \varphi \equiv \alpha_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee \alpha_n$$

Lemma 9.1.8 is particularly relevant for the study of propositional inquisitive logic and even goes beyond InqB. Most of the axiomatizations and the techniques presented in the literature to study InqB and other propositional inquisitive logics make use of some form of the disjunctive normal form lemma.

9.1.2 Kripke Semantics for InqB

As is the case for the semantics of InqBQ, also the semantics of InqB can be restated in terms of a special class of intuitionistic Kripke models based on *Medvedev frames*. Recall that *Medvedev frames* are Kripke frames of the form $\langle \mathcal{P}_0(W), \supseteq \rangle$ for W a finite non-empty set;[1] the logic of this class of models is called *Medvedev logic of finite problems* ML [Medvedev, 1966].

What follows is the propositional version of Lemma 2.2.8.

9.1.9. LEMMA ([Ciardelli, 2009, Proposition 2.2.2]). *Let* $\mathcal{M} := \{M_w \mid w \in W\}$ *be a (propositional) information model and* $s \subseteq W$ *a non-empty information state. Consider the (propositional) intuitionistic Kripke model* $\mathcal{K} := \langle \mathcal{P}_0(W), \supseteq, V \rangle$ *where*

$$V(p) := \mathcal{P}_0(\{w \in W \mid M_w \vDash^{\mathsf{CPC}} p\}).$$

Then for every formula $\varphi \in \mathcal{L}$, *we have*[2]

$$\mathcal{W}, s \vDash \varphi \iff \mathcal{K}, s \Vdash \varphi.$$

[1] Recall that $\mathcal{P}_0(W) := \{V \subseteq W \mid V \neq \emptyset\}$.

[2] Under the intuitionistic semantics, we interpret $\vee\!\!\!\vee$ as the intuitionistic disjunction.

If W is finite, then $\langle \mathcal{P}_0(W), \supseteq \rangle$ is a Medvedev frame, and $V(p)$ is a principal upset of this frame. InqB can also be characterized as the logic of *finite* information models [Ciardelli, 2009, Remark 3.1.11]; combining this and the previous result we obtain the following.

9.1.10. PROPOSITION. InqB *is the logic of the class of intuitionistic Kripke models*

$$\{ \langle \mathcal{P}_0(W), \supseteq, V \rangle \mid W \textit{ is finite and } V(p) \textit{ is principal for all } p \in \mathrm{AP} \}.$$

Based on this observation, Ciardelli [2016, Section 3.1] proposes a sound and complete natural deduction system for InqB, which is equivalent to the following Hilbert style system:

Axioms	**Rules**
Axioms of IPC	Modus Ponens
KP schema: $(\neg\varphi \to \psi \vee\!\!\!\vee \chi) \to (\neg\varphi \to \psi) \vee\!\!\!\vee (\neg\varphi \to \chi)$	
DNA formulas: $\neg\neg p \to p$ for p atomic	

Figure 9.1: Hilbert-style axiomatization for InqB.

9.1.3 UV-Spaces

In this section, we recall the basic constructions of the choice-free duality for Boolean algebras recently developed by Bezhanishvili and Holliday [2020]. They will be used in Sections 9.3.3 and 9.4, where we introduce a topological semantics for inquisitive logic. For an introduction the basic notions of topology used in this chapter, we refer to [Kelley, 1975].

Recall that for any poset $(X, \leq)$, we define

$$\texttt{Cl}_{\leq}(U) = \{x \in X \mid \exists y \geq x.\ y \in U\}, \tag{9.1}$$

$$\texttt{Int}_{\leq}(U) = X \setminus \texttt{Cl}_{\leq}(X \setminus U) = \{x \in X \mid \forall y \geq x.\ y \in U\}. \tag{9.2}$$

We call a set U *$\leq$-regular open* if $U = \texttt{Int}_{\leq}\texttt{Cl}_{\leq}(U)$. Let X be a topological space and $\leq$ its specialization order. Let $\mathcal{RO}(X)$ be the collection of $\leq$-regular open subsets of X. Let $\mathsf{CO}(X)$ denote the collection of compact open subsets of X. Finally, let $\mathsf{CO}\mathcal{RO}(X) = \mathsf{CO}(X) \cap \mathcal{RO}(X)$.

9.1.11. DEFINITION. An *upper Vietoris space* (UV-space) is a T_0 space X such that:

1. $\mathsf{CO}\mathcal{RO}(X)$ is closed under $\cap$ and $\texttt{Int}_{\leq}(X \setminus \cdot)$ and forms a basis for X;

2. every proper filter in $\mathsf{CO}\mathcal{RO}(X)$ is $\mathsf{CO}\mathcal{RO}(x) = \{U \in \mathsf{CO}\mathcal{RO}(X) \mid x \in U\}$ for some $x \in X$.

Given a UV-space X the set $\mathsf{CO}\mathcal{RO}(X)$ forms a Boolean algebra, where $\wedge$ is the intersection, $\vee$ is $\mathtt{Int}_{\leq}\mathtt{Cl}_{\leq}$ of the union, and $\neg$ is $\mathtt{Int}_{\leq}$ of the set-theoretic complement. It was observed by Bezhanishvili and Holliday [2020] that $\mathsf{CO}\mathcal{RO}(X)$ coincides with the set of compact regular open (in the topology of X) subsets of X. Conversely, for a Boolean algebra B we consider the set $UV(B)$ of all proper filters of B and define a topology generated by $\{\widehat{a} \mid a \in B\}$, where $\widehat{a} = \{x \in UV(B) \mid a \in x\}$. Then $UV(B)$ is a UV-space, where the specialization order is the inclusion order of filters, and B is isomorphic to the algebra $\mathsf{CO}\mathcal{RO}(UV(B))$. This correspondence can be extended to a full (choice-free) duality of the category of Boolean algebras and the category of UV-spaces [Bezhanishvili and Holliday, 2020]. The name "upper Vietoris" refers to the fact that, assuming the Axiom of Choice, the UV-dual of a Boolean algebra B is homeomorphic to the space of closed subsets of the Stone dual of B equipped with the upper Vietoris topology (for a choice-free version of this, see [Bezhanishvili and Holliday, 2020]).

9.2 Algebraic Semantics via Inquisitive Algebras

In this section, we define inquisitive algebras and a semantics for InqB via these algebras. We start with the following well known result (see, e.g., [Johnstone, 1982, p. 51]).

9.2.1. Proposition. *For any Heyting algebra H, let $H_{\neg\neg} = \{\neg\neg x \mid x \in H\}$. Then:*

1. *$H_{\neg\neg}$ forms a bounded $\{\wedge, \rightarrow\}$-subalgebra of H;*

2. *$H_{\neg\neg}$ forms a Boolean algebra with join given by $a \vee_{H_{\neg\neg}} b = \neg\neg(a \vee_H b)$.*

9.2.2. Example. Let B be a *complete* Boolean algebra and consider the Heyting algebras $\mathrm{Dw}_0(B)$ and $\mathrm{Dw}_p(B)$ of its non-empty and principal downsets, respectively. The latter is isomorphic to B, with the join in $\mathrm{Dw}_p(B)$ given by $\{a\}^{\downarrow} \vee \{b\}^{\downarrow} = \neg\neg(\{a\}^{\downarrow} \cup \{b\}^{\downarrow}) = \{a \vee_B b\}^{\downarrow}$, where $U^{\downarrow}$ is the downset generated by U. Then we can prove that the following identity holds:

$$\mathrm{Dw}_p(B) = (\mathrm{Dw}_0(B))_{\neg\neg}.$$

Firstly, if we consider a principal downset, we have

$$\neg\{b\}^{\downarrow} = \{a \in B \mid a \wedge b = \bot\} = \{\neg b\}^{\downarrow} \quad \Longrightarrow \quad \neg\neg\{b\}^{\downarrow} = \{b\}^{\downarrow}.$$

So $\mathrm{Dw}_p(B) \subseteq (\mathrm{Dw}_0(B))_{\neg\neg}$. For the other inclusion, it suffices to show that $\neg D$ is principal for every downset D. We have

$$\neg D = \{a \in B \mid \forall d \in D.\ a \wedge d \leq \bot\} \subseteq \left\{\bigvee \neg D\right\}^{\downarrow}.$$

On the other hand, $\bigvee \neg D \in \neg D$, since for every $e \in D$, we have

$$e \wedge \bigvee \neg D = \bigvee\{e \wedge a \mid \forall d \in D.\ a \wedge d \leq \bot\} = \bigvee\{\bot\} = \bot.$$

It follows that $\neg D = \{\bigvee \neg D\}^{\downarrow}$, thus $\neg D$ is principal.

9.2.3. Example. Let B be a Boolean algebra—not necessarily complete—and let $\mathrm{Dw}_{fg}(B)$ be the set of finitely generated downsets of B. Then we can show that:

$$\mathrm{Dw}_p(B) = (\mathrm{Dw}_{fg}(B))_{\neg\neg}.$$

The inclusion $\mathrm{Dw}_p(B) \subseteq (\mathrm{Dw}_{fg}(B))_{\neg\neg}$ is proved as in Example 9.2.2. For the other inclusion it suffices to show that for any $b_1, \dots, b_n \in B$, $\neg\{b_1, \dots, b_n\}^{\downarrow}$ is principal. This follows from the identities

$$\neg\{b_1, \dots, b_n\}^{\downarrow} = \{a \in B \mid \forall i \leq n.\ a \wedge b_i = \bot\} = \{\neg b_1 \wedge \cdots \wedge \neg b_n\}^{\downarrow}. \tag{9.3}$$

Elements of $\mathrm{Dw}_{fg}(B)$ can be represented in a special way that will be useful for later results. The proof of the next lemma consists of a simple induction on the number of generators of a finitely generated downset and is left to the reader.

9.2.4. Lemma. *Every downset $D \in \mathrm{Dw}_{fg}(B)$ can be represented in a unique way as $D = \{a_1, \dots, a_n\}^{\downarrow}$ with $a_i \not\leq a_j$ for $i \neq j$.*

We now want to define an algebraic semantics by restricting the interpretations of atoms to $H_{\neg\neg}$. This semantics was first introduced by Bezhanishvili et al. [2019] and was later studied from the perspective of universal algebra by Quadrellaro [2019] and Bezhanishvili et al. [2020]. Following Bezhanishvili et al. [2020] we will refer to this semantics as DNA-*semantics*, where DNA stands for "double negation for atoms". To highlight the connections with inquisitive logic, we will denote the meet, join, and implication in a Heyting algebra with the same symbols used for the connectives of our language, $\wedge$, $\vee\!\!\!\vee$, and $\rightarrow$.

We recall the standard definition of algebraic semantics for intuitionistic logic via Heyting algebras and we define DNA-semantics.

9.2.5. Definition (Intuitionistic and DNA-semantics). Let H be a Heyting algebra and let $V : \mathrm{AP} \to H$ be an atomic valuation over H. For each $\varphi \in \mathcal{L}_p$, we define $[\![\varphi]\!]^{H,V} \in H$ recursively as follows:

$$\begin{aligned}
[\![\bot]\!]^{H,V} &= \bot & [\![\varphi \wedge \psi]\!]^{H,V} &= [\![\varphi]\!]^{H,V} \wedge [\![\psi]\!]^{H,V} \\
[\![p]\!]^{H,V} &= V(p) & [\![\varphi \vee\!\!\!\vee \psi]\!]^{H,V} &= [\![\varphi]\!]^{H,V} \vee\!\!\!\vee [\![\psi]\!]^{H,V} \\
& & [\![\varphi \rightarrow \psi]\!]^{H,V} &= [\![\varphi]\!]^{H,V} \rightarrow [\![\psi]\!]^{H,V}.
\end{aligned}$$

A formula φ is *intuitionistically valid in* H iff for every $V : \mathrm{AP} \to H$ we have $[\![\varphi]\!]^{H,V} = \top$. Let $\mathrm{IntLog}(H)$ be the set of formulas intuitionistically valid in H. It is known that formulas valid in every Heyting algebra are exactly the formulas of intuitionistic propositional logic IPC [Chagrov and Zakharyaschev, 1997, Theorem 7.21].

A formula φ is DNA *valid in* H iff for every $V : \mathrm{AP} \to H_{\neg\neg}$, we have $[\![\varphi]\!]^{H,V} = \top$. Let $\mathsf{DNALog}(H)$ be the set of formulas inquisitively valid in H. A formula is DNA *valid* iff it is inquisitively valid in every Heyting algebra.

From now on we write $[\![\varphi]\!]$ instead of $[\![\varphi]\!]^{H,V}$ if H and V are clear from the context. Some properties of the semantics are straightforward to prove. For example:

9.2.6. Lemma. *If φ does not contain the symbol $\vee\!\!\!\vee$, then $[\![\varphi]\!] \in H_{\neg\neg}$.*

It follows from the definition above that every intuitionistic validity is also a DNA validity. And since the image of the valuations is restricted to $H_{\neg\neg}$—the regular elements of the algebra—also the formula $\neg\neg p \to p$ is DNA valid. However, for some formulas $\varphi \in \mathcal{L}_p$ we have that $\neg\neg\varphi \to \varphi$ is *not* a DNA validity, as shown in Example 9.2.7. So the set of DNA validities is *not* closed under uniform substitution.

9.2.7. Example. Consider $H = \mathrm{Dw}_{fg}(\mathcal{P}(W))$ for a finite set W with at least two elements. Notice that $H = \mathrm{Dw}_0(\mathcal{P}(W)) \cong \mathrm{Dw}(\mathcal{P}_0(W))$. Since the elements of $(\mathrm{Dw}_{fg(\mathcal{P}(W))})_{\neg\neg}$ are exactly the principal downsets—as shown in Example 9.2.3—in this case DNA semantics boils down to the support semantics for inquisitive logic (cf. Lemma 9.1.9).

For the same reason, given $A \subseteq W$ we have that $\neg\neg\{A\}^{\downarrow} = \{A\}^{\downarrow}$ and consequently $\neg\neg p \to p \in \mathsf{DNALog}(H)$. On the other hand, for $A, B \subseteq W$ we have $\neg\neg\{A, B\}^{\downarrow} = \{A \cup B\}^{\downarrow}$ (obtained by applying two times Equation 9.3), which is generally different from $\{A, B\}^{\downarrow}$. Thus $\neg\neg(p \vee\!\!\!\vee q) \to (p \vee\!\!\!\vee q) \notin \mathsf{DNALog}(H)$.

A natural question is whether we can characterize for which Heyting algebras H we have $\mathsf{InqB} \subseteq \mathsf{DNALog}(H)$, that is, every *inquisitive validity* is also a DNA valid formula of H. The following lemma gives a partial answer to this question. We call H a KP-*algebra* if H validates (intuitionistically) the axiom schema KP.

9.2.8. Lemma. *If H is a* KP-*algebra, then* $\mathsf{InqB} \subseteq \mathsf{DNALog}(H)$.

Combining Lemma 9.2.8 with the fact that the standard support semantics is a special case of our algebraic semantics (as shown in Example 9.2.7), we obtain the following:

9.2.9. Proposition. *The set of formulas* DNA *valid on* KP-*algebras is exactly the set of* InqB *validities.*

$$\{\ \varphi \in \mathcal{L}_p \mid \varphi \in \mathsf{DNALog}(H) \text{ for every } \mathsf{KP}\text{-algebra } H\ \} \ = \ \mathsf{InqB}$$

However, arbitrary KP-algebras are somewhat "too big" for our semantics. For example, if $H = \mathrm{Dw}_0(B)$ for a complete Boolean algebra B, then no matter what propositional valuation we consider, the semantic value $[\![\varphi]\!]$ of a formula φ has to be an element of the subalgebra generated by $\mathrm{Dw}_p(B)$, that is, $\mathrm{Dw}_{fg}(B)$. This observation is formalized by Point 3 of the following Lemma.

9.2.10. LEMMA. *Let H be a Heyting algebra and H' the subalgebra of H generated by $H_{\neg\neg}$. Then:*

1. $(H')_{\neg\neg} = H_{\neg\neg}$;
2. *for every valuation $V : \mathrm{AP} \to H_{\neg\neg}$ and formula φ we have $[\![\varphi]\!]^{H,V} = [\![\varphi]\!]^{H',V}$;*
3. *if H is a KP-algebra, so is H'.*

The proof follows directly from the definition of subalgebra, and it is thus omitted. As a consequence of this lemma, we can restrict our attention to algebras generated by $H_{\neg\neg}$.

9.2.11. DEFINITION. A Heyting algebra H is called *regularly generated* if it is generated by $H_{\neg\neg}$.

Notice that the algebras of the form $\mathrm{Dw}_0(B)$ for B a complete Boolean algebra (Example 9.2.2) are regularly generated iff B is finite. While the algebras of the form $\mathrm{Dw}_{fg}(B)$ for B a generic Boolean algebra (Example 9.2.3) are always regularly generated.

The choice of restricting to regularly generated algebras has a technical nature: since our semantics does not allow to leave the subalgebra generated by the regular elements, we restrict our attention to regularly generated algebras. There is also another restriction we can impose on the algebras we are studying, this time related to the intuitive interpretation we gave to formulas of inquisitive logic.

In Subsection 9.1.1 we recall this interpretation: formulas of InqB represent *statements and questions* and the support semantics is meant to represent *informational content* and model a certain *piece of information implying a statement/resolving a question*. For example, a question $p \vee\!\!\!\vee \neg p$ ("Does p hold?") is supported by an information state iff either p ("p holds") or $\neg p$ ("p does not hold") is supported by the state.

However, this is not necessarily the case in the algebraic setting just introduced. For example, a Boolean algebra B is trivially a regularly generated KP-algebra, since $B_{\neg\neg} = B$. And $[\![p \vee\!\!\!\vee \neg p]\!] = \top$ regardless of the value of $[\![p]\!]$ and $[\![\neg p]\!]$. This seems somewhat odd under the intuitive interpretation given: a question is considered *resolved*, but neither p nor $\neg p$ is considered *true*.

This leads to the following definition from [Chagrov and Zakharyaschev, 1997, p. 455].

9.2.12. Definition. A Heyting algebra H is *well connected* if for all $a, b \in H$, if $a \vee\!\!\!\vee\, b = 1$, then $a = 1$ or $b = 1$.

The algebras of the form $\mathrm{Dw}_0(B)$ for B a complete Boolean algebra (Example 9.2.2) and the algebras of the form $\mathrm{Dw}_{fg}(B)$ for B a generic Boolean algebra (Example 9.2.3) are both examples of well-connected algebras.

Working with a well-connected algebra H ensures that $[\![\varphi]\!] = 1$ iff $[\![\alpha]\!] = 1$ for at least one of the resolutions of φ (Lemma 9.1.8), in accordance with the linguistic interpretation.

We are finally ready to define the class of algebras which we will focus on to study InqB.

9.2.13. Definition (Inquisitive algebra). An *inquisitive algebra* is a regularly generated well-connected KP-algebra.

Summarizing the previous remarks, we have that algebras of the form $\mathrm{Dw}_0(B)$ for B a complete Boolean algebra (Example 9.2.2) are inquisitive algebras iff B is finite; while algebras of the form $\mathrm{Dw}_{fg}(B)$ for B a generic Boolean algebra (Example 9.2.3) are always inquisitive algebras. The latter is our standard example of inquisitive algebra.

9.3 Inquisitive Extension of a Boolean Algebra

In this section, we show an interesting property of inquisitive algebras highlighting their constructive character: given a Boolean algebra B, there is a unique (up to isomorphism) inquisitive algebra H such that $B = H_{\neg\neg}$; moreover there is a *constructive procedure* to obtain H starting from B. We call H the *inquisitive extension* of B.

9.3.1 Construction of the Inquisitive Extension

We construct the inquisitive extension of B as a quotient of the free Heyting algebra built using elements of B as constants. Firstly, consider the set

$$\mathcal{T} = \left\{ t(b_1, \ldots, b_n) \;\middle|\; t \text{ is a term in the signature } \left\{ \dot\wedge, \dot\vee, \dot\rightarrow, \dot\bot, \dot\top \right\} \right\}.$$

where $\dot\wedge, \dot\vee, \dot\rightarrow, \dot\bot, \dot\top$ are formal symbols. We also introduce the shorthand $\dot\neg t$ for $t \mathbin{\dot\rightarrow} \dot\bot$.

Define the binary relation $\approx$ on $\mathcal{T}$ as the smallest equivalence relation such that:

- $\approx$ respects all Heyting algebra equations (e.g., for commutativity of $\dot\wedge$ we require that $t_1 \mathbin{\dot\wedge} t_2 \approx t_2 \mathbin{\dot\wedge} t_1$);

- $\approx$ respects KP:

$$\dot\neg t_1 \dot\to (t_2 \dot\vee t_3) \approx (t_1 \dot\to t_2) \dot\vee (t_1 \dot\to t_3)$$

- $\approx$ agrees with the operations on B: for $a, b \in B$ it holds that

$$a \dot\wedge b \approx a \wedge b \qquad a \dot\to b \approx a \to b \qquad \dot\bot \approx \bot \qquad \dot\top \approx \top$$

$\mathcal{T}/\approx$ has a natural structure of KP-algebra, with operations defined as

$$[t_1] \wedge [t_2] = [t_1 \dot\wedge t_2] \qquad [t_1] \vee\!\!\!\vee [t_2] = [t_1 \dot\vee t_2] \qquad [t_1] \to [t_2] = [t_1 \dot\to t_2].$$

We call this algebra the *inquisitive extension of* B and denote it by $H(B)$. Notice that by construction it is a regularly generated KP-algebra. To simplify the notation, henceforth we drop the square brackets for the equivalence classes.

We can prove that the following universal property holds for inquisitive extensions.

9.3.1. Lemma. *Let B be a Boolean algebra and H a* KP*-algebra such that $B = H_{\neg\neg}$. Then there exists a unique homomorphism $h : H(B) \to H$ such that $h|_B = id_B$. Moreover, if H is regularly generated, then h is surjective.*

Proof:
Consider the map $f : \mathcal{T} \to H$ defined by the clauses

$$f(b) = b, \text{ for } b \in B \qquad f(t_1 \dot\wedge t_2) = f(t_1) \wedge f(t_2)$$
$$f(t_1 \dot\vee t_2) = f(t_1) \vee\!\!\!\vee f(t_2) \qquad f(t_1 \dot\to t_2) = f(t_1) \to f(t_2).$$

Since H is a KP-algebra and agrees with the operations on B, f factors through $H(B)$, and thus we obtain a quotient map $h : H(B) \to H$. Moreover, by construction, h is a Heyting algebra homomorphism.

The image of B is fixed and $H(B)$ is generated by B, so uniqueness follows. Moreover, if H is regularly generated, then h is surjective, since $B \subseteq h[H(B)]$ and B generates H. □

This result allows us to better understand the structure of the algebra $H(B)$. In particular, elements of $H(B)$ can be represented in a *disjunctive normal form* analogous to the normal form of InqB formulas (Lemma 9.1.8).

9.3.2. Proposition.

1. *Every $x \in H(B)$ can be represented in a unique way as $x = a_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee a_n$ with $a_1, \ldots, a_n \in B$ and $a_i \not\leq a_j$ for $i \neq j$.*
2. *$H(B) \cong \mathrm{Dw}_{fg}(B)$.*

We will call a representation of x as in item 1 *non-redundant*.

Proof:
For item 1, we divide the proof in two steps: proving that every element $x \in H(B)$ can be written in the form $x = b_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} b_m$ with $b_1, \ldots, b_m \in B$; and proving that from this form we can obtain a non-redundant representation.

We start by proving that every element $x \in H(B)$ can be written in the form $x = b_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} b_m$. Since $H(B)$ is the quotient of the set $\mathcal{T}$ of terms, we can proceed by induction on $t \in \mathcal{T}$.

- If $x \in B$, we are done.
- If $x = y \wedge z$, consider two representations $y = c_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} c_k$ and $z = d_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} d_l$. Then
$$x = y \wedge z = (c_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} c_k) \wedge (d_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} d_l) = \bigvee\!\!\!\!\bigvee \{c_i \wedge d_j \mid i \leq k, j \leq l\}.$$
- If $x = y \mathbin{\vee\!\!\!\vee} z$, then
$$x = y \mathbin{\vee\!\!\!\vee} z = c_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} c_k \mathbin{\vee\!\!\!\vee} d_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} d_l.$$
- If $x = y \to z$, then
$$\begin{aligned} x = y \to z &= (c_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} c_k) \to (d_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} d_l) \\ &= (c_1 \to d_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} d_l) \wedge \cdots \wedge (c_k \to d_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} d_l) \\ &= \wedge_{i=1}^{l} \left((c_i \to d_1) \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} (c_i \to d_l)\right) && \text{(by } KP\text{)} \\ &= \vee_{f:[n]\to[m]} \left(\wedge_{i=1}^{l} (c_i \to d_{f(i)})\right). \end{aligned}$$

Now we show that every element admits a non-redundant representation. Let $x = b_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} b_m$ be an arbitrary representation of x. If $\forall i, j.\ b_i \not\leq b_j$, then we are done. Otherwise, suppose (without loss of generality) that $b_1 \leq b_2$. Then

$$b_1 \mathbin{\vee\!\!\!\vee} b_2 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} b_n = b_2 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} b_n.$$

Repeating this procedure, we obtain a non-redundant representation of x.

For item 2, consider the map $h : H(B) \to \mathrm{Dw}_{fg}(B)$. Since

$$h(a_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} a_n) = h(a_1) \cup \cdots \cup h(a_n) = \{a_1, \ldots, a_n\}^{\downarrow},$$

h is injective. It is then easy to see that h is an isomorphism. □

A direct consequence of Proposition 9.3.2 is that $H(B)$ is well connected and thus an inquisitive algebra. We can also prove the following interesting property of $H(B)$, which will be useful for later applications.

9.3.3. Lemma. *Let H' be a finitely generated subalgebra of $H(B)$. Then H' is a subalgebra of a finite subalgebra of $H(B)$ of the form $H(B')$, where B' a Boolean subalgebra of B.*

Proof:
Let $a_1^1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee a_{k_1}^1, \ldots, a_1^n \vee\!\!\!\vee \ldots \vee\!\!\!\vee a_{k_n}^n$ be the non-redundant representations of the generators of H', and let A be the set $A = \{a_j^i \mid i \leq n, j \leq k_i\}$. Let B' be the Boolean subalgebra of B generated by A. Notice that this is a finite algebra. Clearly $H' \subseteq H(B') \subseteq H(B)$.

Finally, the isomorphism in Item 2 of Proposition 9.3.2 maps $H(B')$ onto $\mathrm{Dw}_{fg}(B')$—which is finite, since $|\mathrm{Dw}_{fg}(B')|$ is equal to the number of antichains in B'. Therefore, $H(B')$ is finite. □

The results of this section allow us to draw a strong connection between regularly generated KP-algebras and Medvedev's logic ML.

9.3.4. Theorem. *If H is a regularly generated KP-algebra, then H is an ML-algebra.*

Proof:
Let H be a regularly generated KP-algebra. Then, by Lemma 9.3.1, H is a homomorphic image of some algebra of the form $H(B)$. Thus, it suffices to show that $H(B)$ is an ML-algebra.

It is well known that for every Heyting algebra A and intermediate logic L we have that A is an L-algebra iff every finitely generated subalgebra of A is an L-algebra. Therefore, by Lemma 9.3.3, we obtain that $H(B)$ is an ML-algebra iff $H(B')$ is an ML-algebra for every finite Boolean subalgebra B' of B.

Thus, we only need to prove the result for algebras of the form $H(B')$ where B' is finite. Then $B' \cong \mathcal{P}(W)$ for some finite set W. By Proposition 9.3.2,

$$H(B') \cong \mathrm{Dw}_{fg}(B') \cong \mathrm{Dw}_{fg}(\mathcal{P}(W)) \cong \mathrm{Dw}_0(\mathcal{P}(W)) \cong \mathrm{Dw}(\mathcal{P}_0(W)),$$

which is exactly the algebra corresponding to the Medvedev frame $\langle \mathcal{P}_0(W), \supseteq \rangle$. We conclude that $H(B)$ is an ML-algebra and therefore H is also an ML-algebra.
□

9.3.5. Corollary.

$$\begin{aligned} & \mathrm{IntLog}(\{H \mid H \textit{ is a regularly generated } \mathsf{KP}\textit{-algebra}\}) \\ = \; & \mathrm{IntLog}(\{H(B) \mid B \textit{ is a finite Boolean algebra}\}) \\ = \; & \mathsf{ML}. \end{aligned}$$

Proof:
Let $\mathcal{C}_1$ be the class of regularly generated KP-algebras and $\mathcal{C}_2$ the class of $H(B)$'s for a finite Boolean algebra B. Firstly, notice that every $H(B)$ is a regularly generated KP-algebra, so $\mathcal{C}_2 \subseteq \mathcal{C}_1$. Consequently $\mathrm{IntLog}(\mathcal{C}_1) \subseteq \mathrm{IntLog}(\mathcal{C}_2)$. Therefore, we just need to prove that $\mathsf{ML} \subseteq \mathrm{IntLog}(\mathcal{C}_1)$ and $\mathrm{IntLog}(\mathcal{C}_2) \subseteq \mathsf{ML}$.

The first inclusion follows directly from Theorem 9.3.4. For the second inclusion, consider an arbitrary Medvedev frame $\langle \mathcal{P}_0(W), \supseteq \rangle$—recall that W is finite. As noticed in the proof of Theorem 9.3.4, the Heyting algebra corresponding to this frame is $\mathrm{Dw}(\mathcal{P}_0(W)) \cong H(\mathcal{P}(W))$. Hence it is isomorphic to an element of $\mathcal{C}_2$. It follows that $\mathrm{IntLog}(\mathcal{C}_2) \subseteq \mathsf{ML}$, as required. □

9.3.2 Algebraic Characterization of the Inquisitive Extension

We are now ready to provide our first characterization of $H(B)$.

9.3.6. THEOREM. *For a Boolean algebra B, its inquisitive extension $H(B)$ is the unique (up to isomorphism) inquisitive algebra such that $H(B)_{\neg\neg}$ is isomorphic to B.*

Proof:

Let H be an inquisitive algebra where $H_{\neg\neg} \cong B$, and fix an isomorphism $g : H_{\neg\neg} \to B$. By Lemma 9.3.1, there exists a unique morphism $h : H(B) \to H$ such that $h|_{H(B)} = g$, which is surjective since H is regularly generated.

It only remains to show that h is also injective, thus proving that h is an isomorphism. Let $x, y \in H(B)$ and suppose that $h(x) = h(y)$. Let $x = a_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee a_n$ and $y = b_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee b_m$ be their non-redundant representations. Then where $\sqcap, \sqcup, \Rightarrow$

are the operations of H, we have

$$
\begin{aligned}
& a_1 \sqcup \cdots \sqcup a_n = b_1 \sqcup \cdots \sqcup b_m \\
\Longrightarrow\ & (a_1 \sqcup \cdots \sqcup a_n) \Leftrightarrow (b_1 \sqcup \cdots \sqcup b_m) = \top \\
\Longrightarrow\ & \begin{cases} \bigsqcup_{f:[n]\to[m]} \bigsqcap_{i\leq n}(a_i \Rightarrow b_{f(i)}) = \top \\ \bigsqcup_{g:[m]\to[n]} \bigsqcap_{j\leq m}(b_j \Rightarrow a_{g(j)}) = \top \end{cases} \\
\Longrightarrow\ & \begin{cases} \exists f : [n] \to [m].\ \bigsqcap_{i\leq n}(a_i \Rightarrow b_{f(i)}) = \top \\ \exists g : [m] \to [n].\ \bigsqcap_{j\leq m}(b_j \Rightarrow a_{g(j)}) = \top \end{cases} && \text{(since } H \text{ is inquisitive)} \\
\Longrightarrow\ & \begin{cases} \forall i \leq n.\ \exists j \leq m.\ (a_i \Rightarrow b_j) = \top \\ \forall j \leq m.\ \exists i \leq n.\ (b_j \Rightarrow a_i) = \top \end{cases} \\
\Longrightarrow\ & \begin{cases} \forall i \leq n.\ \exists j \leq m.\ a_i \leq b_j \\ \forall j \leq m.\ \exists i \leq n.\ b_j \leq a_i \end{cases} && \text{(since } h|_B = id_B\text{)} \\
\Longrightarrow\ & \begin{cases} x \leq y \\ y \leq x \end{cases} \\
\Longrightarrow\ & x = y.
\end{aligned}
$$

So h is injective and thus an isomorphism, as required.

□

9.3.7. COROLLARY. *A Heyting algebra A is an inquisitive algebra iff A is isomorphic to $H(A_{\neg\neg})$.*

Proof:
The right-to-left implication is clear. For the left-to-right, consider an inquisitive algebra A. By Theorem 9.3.6, $H(A_{\neg\neg})$ is isomorphic to any inquisitive algebra with $A_{\neg\neg}$ as the set of $\neg\neg$-fixpoints. In particular, $A \cong H(A_{\neg\neg})$. □

We conclude this section with a result analogous to Corollary 9.3.5 but now for inquisitive logic.

9.3.8. COROLLARY.

$$
\begin{aligned}
& \mathsf{DNALog}(\{H \mid H \textit{ is a } \mathsf{KP}\textit{-algebra}\}) \\
= \ & \mathsf{DNALog}(\{H(B) \mid B \textit{ is a finite Boolean algebra}\}) \\
= \ & \mathsf{InqB}.
\end{aligned}
$$

Proof:
By Lemma 9.2.8, InqB is included in the inquisitive logic of the two classes of algebras. For the other inclusion: by Proposition 9.3.2, given a finite set W we have $H(\mathcal{P}(W)) \cong \mathrm{Dw}(\mathcal{P}_0(W))$. So by Proposition 9.1.10, the inquisitive logic of the second class of algebras is indeed InqB; and since the first class of algebras includes the second, we obtain both equalitites. □

9.3.3 Topological Characterization of the Inquisitive Extension

Using the UV-spaces of Section 9.1.3, we can give a topological realization of $H(B)$, which in the next section will lead to a topological semantics of inquisitive logic. By item 2 of the following theorem, $H(B)$ may be characterized as (isomorphic to) the Heyting algebra of compact open sets of the UV-space dual to B.

9.3.9. THEOREM. *Let B be a Boolean algebra and X its dual UV-space.*

1. $(\mathsf{O}(X), \subseteq) \cong \mathrm{Dw}_0(B)$.
2. $(\mathsf{CO}(X), \subseteq) \cong \mathrm{Dw}_{fg}(B) \cong H(B)$.

To prove Theorem 9.3.9, we will use the following lemma.

9.3.10. LEMMA. *Let $A = \bigcup_{i \in I} U_i$ and $B = \bigcup_{j \in J} V_j$ be open sets of a UV-space X, where U_i, V_j are $\mathsf{C}\mathcal{ORO}$-sets. Then $A \subseteq B$ iff $\forall i \in I.\ \exists j \in J.\ U_i \subseteq V_j$.*

Proof:
Firstly, we show that every $\mathsf{C}\mathcal{ORO}$-set U is the upset of a singleton: since $\{U\}^{\uparrow}$ is a filter in $\mathsf{C}\mathcal{ORO}(X)$, there exists a point x such that $\{U\}^{\uparrow} = \mathsf{C}\mathcal{ORO}(X)$. It follows that $U = \bigcap \mathsf{C}\mathcal{ORO}(x) = \{x\}^{\uparrow}$.

We can use this to prove the result. Call x_i the generator of U_i for each $i \in I$.

$$\begin{aligned} A \subseteq B &\iff \textstyle\bigcup_{i \in I} U_i \subseteq \bigcup_{j \in J} V_j &&\iff \forall i \in I.\ U_i \subseteq \textstyle\bigcup_{j \in J} V_j \\ &\iff \forall i \in I.\ U_i \subseteq \textstyle\bigcup_{j \in J} V_j &&\iff \forall i \in I.\ x_i \in \textstyle\bigcup_{j \in J} V_j \\ &\iff \forall i \in I.\ \exists j \in J.\ x_i \in V_j &&\iff \forall i \in I.\ \exists j \in J.\ U_i \subseteq V_j. \end{aligned}$$

□

Proof of Theorem 9.3.9:

For the first part: consider the map $f : O(X) \to \mathrm{Dw}_0(B)$ defined by[3]

$$f\left(\bigcup_{i \in I} \widehat{a_i}\right) = \{a_i \mid i \in I\}^{\downarrow}.$$

[3] Here we are adopting the convention $\{\}^{\downarrow} := \{\bot\}$, so that $f(\emptyset) = \{\bot\}$.

To show that f is well defined and order preserving and reflecting, we observe the following equivalences, using Lemma 9.3.10 for the first:

$$\begin{aligned}\bigcup_{i\in I}\widehat{a_i} \subseteq \bigcup_{j\in J}\widehat{b_j} &\iff \forall i \in I.\ \exists j \in J.\ \widehat{a_i} \subseteq \widehat{b_j} \\ &\iff \forall i \in I.\ \exists j \in J.\ a_i \leq b_j \\ &\iff \forall i \in I.\ \exists j \in J.\ \{a_i\}^{\downarrow} \subseteq \{b_j\}^{\downarrow} \\ &\iff \{a_i \mid i \in I\}^{\downarrow} \subseteq \{b_j \mid j \in J\}^{\downarrow}.\end{aligned}$$

Thus, f is also injective. Notice that surjectivity is trivially satisfied. Hence f is an isomorphism.

For the second part: since elements of $\mathsf{CO}(X)$ are exactly the sets of the form $\widehat{a_1} \cup \cdots \cup \widehat{a_n}$ for some $a_1, \ldots, a_n \in B$, we obtain that $f|_{\mathsf{CO}(X)}$ is an isomorphism with range $\mathrm{Dw}_{fg}(B)$, as required.

□

For the readers familiar with Esakia duality for Heyting algebras ([Esakia, 2019], in particular Section 3.4), we can further exploit Theorem 9.3.9 to obtain a connection between the choice-free duality for Boolean algebras and Esakia duality. This connection is based on the following proposition.

9.3.11. Proposition. *The following function defines an order isomorphism between the set* $\mathrm{Spec}(H(B))$ *of prime filters of* $H(B)$*, ordered by inclusion, and the set* $\mathrm{Filt}(B)$ *of filters of* B*, ordered by inclusion:*

$$\begin{array}{rccc} r: & (\mathrm{Spec}(H(B)), \subseteq) & \to & (\mathrm{Filt}(B), \subseteq) \\ & F & \mapsto & F \cap B \end{array}$$

Proof:
It is easy to verify that r is well defined and order preserving. For injectivity, notice that a prime filter $\mathfrak{p}$ of $H(B)$ is completely determined by the elements of B it contains, since for every non-redundant representation $a_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee a_n$, we have

$$a_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee a_n \in \mathfrak{p} \iff a_1 \in \mathfrak{p} \text{ or } \ldots \text{ or } a_n \in \mathfrak{p}. \tag{9.4}$$

Using this fact, we can also show surjectivity: let F be a filter of B and define $\mathfrak{p}_F$ as the smallest set including F and respecting (9.4). Then clearly $\mathfrak{p}_F$ is an upset and respects the primality condition ($a \vee\!\!\!\vee b \in \mathfrak{p}_F$ iff $a \in \mathfrak{p}_F$ or $b \in \mathfrak{p}_F$). Moreover,

algebras	**spaces**
$B \cong \mathsf{CO}\mathcal{RO}(UV(B))$	$UV(B)$
$H(B) \cong \mathsf{CO}(UV(B))$	$\wr\!\shortmid$
$H(B) \cong \mathsf{CO}(\mathrm{Spec}(H(B)))$	$\mathrm{Spec}(H(B))$

Figure 9.2: Summary of results of Section 9.3.3.

it is closed under meets, since

$$\begin{aligned}
& a_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee a_n \in \mathfrak{p}_F \text{ and } b_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee b_m \in \mathfrak{p}_F \\
\iff & \exists i.\ a_i \in \mathfrak{p}_F \text{ and } \exists j.\ b_j \in \mathfrak{p}_F \\
\iff & \exists i.\ a_i \in F \text{ and } \exists j.\ b_j \in F \\
\iff & \exists i.\ \exists j.\ a_i \wedge b_j \in F \\
\iff & \exists i.\ \exists j.\ a_i \wedge b_j \in \mathfrak{p}_F \\
\iff & (a_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee a_n) \wedge (b_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee b_m) = \bigvee\!\!\!\bigvee \{a_i \wedge b_j \mid i \leq n, j \leq m\} \in \mathfrak{p}_F.
\end{aligned}$$

Since $r(\mathfrak{p}_F) = F$, we also have surjectivity.

□

9.3.12. Proposition. *Given B a Boolean algebra, the Esakia space* $\mathrm{Spec}(H(B))$ *dual to $H(B)$ is homeomorphic to the UV-space $UV(B)$ dual to B.*

Proof:
The map r defined in Proposition 9.3.11 above is a homeomorphism; all the verifications are standard and left to the reader. □

We summarize the results of this section in Figure 9.2.

9.4 Topological Semantics for Inquisitive Logic

Theorem 9.3.9 and Lemma 9.4.2 allow us to define a topological semantics for InqB based on UV-spaces.

9.4.1. Definition (Topological semantics).

Let X be a UV-space and $V : \mathrm{AP} \to \mathsf{CO}\mathcal{RO}(X)$ an atomic valuation. For each inquisitive formula $\varphi \in \mathcal{L}$, we define its semantic valuation $[\![\varphi]\!]^{X,V} \in \mathsf{CO}(X)$ recursively as follows:[4]

$$\begin{aligned}
[\![\bot]\!]^{X,V} &= \emptyset & [\![\varphi \wedge \psi]\!]^{X,V} &= [\![\varphi]\!]^{X,V} \cap [\![\psi]\!]^{X,V} \\
[\![p]\!]^{X,V} &= V(p) & [\![\varphi \vee\!\!\!\vee \psi]\!]^{X,V} &= [\![\varphi]\!]^{X,V} \cup [\![\psi]\!]^{X,V} \\
& & [\![\varphi \to \psi]\!]^{X,V} &= \mathtt{Int}\left((X \setminus [\![\varphi]\!]^{X,V}) \cup [\![\psi]\!]^{X,V}\right).
\end{aligned}$$

We adopt the same notational conventions for validity as in Definition 9.2.5.

In the Boolean algebra $\mathsf{CO}\mathcal{RO}(X)$, implication is given by $U \to V = \neg U \vee V = \mathtt{Int}_{\leq}\mathtt{Cl}_{\leq}(\mathtt{Int}_{\leq}(X \setminus U) \cup V)$, and it is easy to verify that the right-hand side is equal to $\mathtt{Int}_{\leq}((X \setminus U) \cup V)$. Interestingly, in the semantic clause for $\to$ we can use either the operator Int or $\mathtt{Int}_{\leq}$, as shown in the following lemma.

9.4.2. Lemma. *Given* $A, B \in \mathsf{CO}(X)$, $\mathtt{Int}((X \setminus A) \cup B) = \mathtt{Int}_{\leq}((X \setminus A) \cup B)$.

To prove Lemma 9.4.2, we first need to establish some technical results. In the following we denote $X \setminus A$ by $\overline{A}$. For a UV space X and $x, y \in X$, let $x \sqcap y$ be the greatest lower bound of x and y in the specialization order of X [Bezhanishvili and Holliday, 2020, Corollary 5.5].

9.4.3. Lemma. *Let* $U \in \mathsf{CO}\mathcal{RO}(X)$ *and* $x_1, x_2 \in U$. *Then* $x_1 \sqcap x_2 \in U$.

Proof:
By [Bezhanishvili and Holliday, 2020, Corollary 5.5], $U = U \vee U = U \cup \{x \sqcap y \mid x, y \in U\}$.
□

9.4.4. Lemma. *Given* $U, V \in \mathsf{CO}\mathcal{RO}(X)$, $\mathtt{Int}_{\leq}\left(\overline{U} \cup V\right) = \neg U \vee V$.

Proof:
Left-to-right inclusion. Consider an element $x \in \mathtt{Int}_{\leq}\left(\overline{U} \cup V\right)$. If $x \in \neg U \cup V$, then there is nothing to prove; so suppose this is not the case. By [Bezhanishvili and Holliday, 2020, Corollary 5.5], there is a decomposition $x = x_1 \sqcap x_2$ such that $x_1 \in \neg U$ and $x_2 \in U$.

Since $x_2 \notin \overline{U}$ and $x_2 \geq x \in \mathtt{Int}_{\leq}\left(\overline{U} \cup V\right)$, it follows that $x_2 \in V$. So $x \in \{y \sqcap z \mid y \in \neg U, z \in V\} \subseteq \neg U \vee V$, as desired.

Right-to-left inclusion. Consider $x \in \neg U \vee V$ and take an arbitrary $w \geq x$. We want to show that $w \in \overline{U} \cup V$.

If $w \in \neg U \cup V \subseteq \overline{U} \cup V$, then there is nothing to prove; so suppose this is not the case. By [Bezhanishvili and Holliday, 2020, Corollary 5.5], we can write $w = w_1 \sqcap w_2$ with $w_1 \in \neg U$ and $w_2 \in V$. In particular, w_1 is a successor of w not in U, and since $\overline{U}$ is a $\leq$-downset, it follows that $w \in \overline{U} \subseteq \overline{U} \cup V$.

Since w was an arbitrary successor of x, it follows that $x \in \mathtt{Int}_{\leq}\left(\overline{U} \cup V\right)$.
□

[4]Notice that Theorem 9.3.9 ensures that $[\![\varphi \to \psi]\!]^{X,V} \in \mathsf{CO}(X)$.

9.4.5. Lemma. *Given $U_i, V_j \in \mathtt{CO}\mathcal{RO}(X)$, the following identity holds:*

$$\mathtt{Int}_{\leq}\left(\left(\bigcap_{i=1}^{m}\overline{U_i}\right) \cup \left(\bigcup_{j=1}^{n} V_j\right)\right) = \bigcup_{f:[m]\to[n]} \bigcap_{i=1}^{m} \left(\neg U_i \vee V_{f(i)}\right).$$

Proof:

By Lemma 9.4.4, the identity is equivalent to

$$\mathtt{Int}_{\leq}\left(\left(\bigcap_{i=1}^{m}\overline{U_i}\right) \cup \left(\bigcup_{j=1}^{n} V_j\right)\right) = \bigcup_{f:[m]\to[n]} \mathtt{Int}_{\leq}\left(\bigcap_{i=1}^{m} \left(\overline{U}_i \cup V_{f(i)}\right)\right).$$

Let L and R be the left-hand side and right-hand side, respectively.

Right-to-left inclusion. Consider $x \in R$. This means that:

$$\exists f : [m] \to [n].\ \forall y \geq x.\ y \in \bigcap_{i=1}^{m} \left(\overline{U}_i \cup V_{f(i)}\right).$$

So with fixed f as above, given $y \geq x$, we have:

$$y \in \bigcap_{i=1}^{m} \left(\overline{U}_i \cup V_{f(i)}\right) \subseteq \bigcap_{i=1}^{m}\left(\overline{U}_i \cup \left(\bigcup_{j=1}^{n} V_j\right)\right) = \left(\bigcap_{i=1}^{m}\overline{U_i}\right) \cup \left(\bigcup_{j=1}^{n} V_j\right).$$

As y was an arbitrary successor of x, it follows that $x \in L$.

Left-to-right inclusion. We will show this step by contradiction. Suppose that $x \notin R$. This means that:

$$\forall f : [m] \to [n].\ \exists y \geq x.\ \exists i \in [m].\ y \notin \overline{U}_i \cup V_{f(i)},$$

or equivalently

$$\exists i \in [m].\ \forall j \in [n].\ \{x\}^{\uparrow} \cap U_i \cap \overline{V}_j \neq \emptyset.$$

Fix an index k instantiating the first quantifier, and consider for each $j \in [n]$ an element $y_j \in \{x\}^{\uparrow} \cap U_k \cap \overline{V}_j$. Define $y = y_1 \sqcap \cdots \sqcap y_n$. We have:

- For every $j \in [n]$, $y_j \geq x$, and thus $y \geq x$.
- Since $y_j \in \overline{V}_j$ and V_j is open, it follows that $\mathtt{Cl}(y_j) \subseteq \overline{V}_j$; and consequently $y \in \overline{V}_j$, since $y \leq y_j$.
- Since $y_1, \ldots, y_n \in U_k$, we have $y \in U_k$ (see Lemma 9.4.3).

So it follows that $y \geq x$ and $y \in U_k \cap \overline{V_1} \cap \cdots \cap \overline{V_n}$. Thus in particular $y \notin \left(\bigcap_{i=1}^{m} \overline{U}_i\right) \cup \left(\bigcup_{j=1}^{n} V_j\right)$, from which we obtain $x \notin L$, as desired.

□

Proof of Lemma 9.4.2:
By Lemma 9.4.5, $\mathtt{Int}_{\leq}(\overline{A} \cup B) \in \mathsf{CO}(X)$. Since the order topology is finer than the main topology, we have

$$\mathtt{Int}(\overline{A} \cup B) = \mathtt{Int}\left(\mathtt{Int}_{\leq}(\overline{A} \cup B)\right) = \mathtt{Int}_{\leq}(\overline{A} \cup B).$$

□

9.4.6. THEOREM. *The set of formulas valid on UV-spaces under this semantics is exactly the set of theorems of* InqB.

Proof:
Let X be a UV-space. By Theorem 9.3.9, $\mathsf{CO}(X) \cong H(\mathsf{CO}\mathcal{RO}(X))$. Moreover, by [Bezhanishvili and Holliday, 2020], every Boolean algebra is isomorphic to one of the form $\mathsf{CO}\mathcal{RO}(X)$. Combining this result with Corollary 9.3.8, we obtain:

$$\begin{aligned} \mathsf{DNA}\mathrm{Log}(\{X \mid X \text{ a UV-space}\}) &= \mathsf{DNA}\mathrm{Log}(\{H(B) \mid B \text{ a Boolean algebra}\}) \\ &= \mathsf{InqB}. \end{aligned}$$

□

We conclude this section by pointing out a connection with Medvedev's logic ML. UV-spaces can be used to give a new topological semantics for ML in a way analogous to inquisitive logic, namely by allowing valuations to range over CO-sets in Definition 9.4.1—and not only $\mathsf{CO}\mathcal{RO}$-sets.

9.4.7. COROLLARY. ML *is sound and complete with respect to the topological semantics presented above.*

Proof:
As noticed in the proof of Theorem 9.4.6, $\mathsf{DNA}\mathrm{Log}(\{X \mid X \text{ a UV-space}\}) = \mathsf{DNA}\mathrm{Log}(\{H(B) \mid B \text{ a Boolean algebra}\})$. Combining this observation with Corollary 9.3.5 and Theorem 9.3.9, we obtain the desired result. □

9.5 Conclusions

In this chapter, we introduced algebraic and topological semantics for inquisitive logic and connected them via choice-free duality for Boolean algebras by Bezhanishvili and Holliday [2020]. This opens up new avenues for further research which we will briefly mention.

The main results of this chapter are concerned with KP-algebras, since the KP-axiom is essential for inquisitive logic. However, we could consider different classes of regularly generated Heyting algebras and study the corresponding generalized logics. This is the direction taken by Quadrellaro [2019], Bezhanishvili et al. [2020], where the DNA-semantics is further developed for equationally definable classes of Heyting algebras, leading to a Birkhoff-type result for varieties of regularly generated algebras [Bezhanishvili et al., 2020, Corollary 3.27].

We can also consider substituting the double negation nucleus $\neg\neg$ with another nucleus or interesting endomorphism of Heyting algebras. This direction was followed for example by Holliday [2020], which yields the nuclear semantics for "inquisitive intuitionistic logic". How to define and characterize inquisitive extensions in this setting and whether there is a corresponding topological semantics remain open problems. Another work on this topic is [Grilletti and Quadrellaro, forthcoming], where the endomorphism $\neg\neg$ is substituted with an arbitrary definable endomorphism. Interestingly, most of the theory developed in [Quadrellaro, 2019, Bezhanishvili et al., 2020] adapts to this case. How to develop a topological semantics using Esakia duality in this alternative setting is currently under investigation.

Finally, it is worth exploring whether these results can be generalized to the first order case. The major obstacle towards this generalization is that an axiomatization for InqBQ is still not known, and this prevents us from finding an equationally definable class of algebras whose logic is InqBQ. On the other hand, a generalization of the DNA-semantics to the first order case would provide new ways to tackle the axiomatization problem: for example trying to adapt the completeness proof for classical first order logic given by Rasiowa and Sikorski [1950], or the completeness proof for the logic CD by Görnemann [1971].

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Samenvatting

Deze dissertatie focust op de studie van inquisitieve eerste-ordelogica, een logisch formalisme dat vragen in de aanwezigheid van kwantificatie omvat, ontwikkeld met als doel het aanwenden van vragen in formele gevolgtrekkingen en het bestuderen van hun logische eigenschappen. In het bijzonder focussen we op het ontwikkelen van gereedschappen en technieken voor het bestuderen van de uitdrukkingskracht van inquisitieve eerste-ordelogica en de eigenschappen van haar logisch gevolg. De dissertatie bestaat uit vier delen, die elk een andere aanpak om de logica te bestuderen beschouwen.

In het eerste deel, bestaande uit hoofdstukken 4 en 5, maken we een gereedschap uit het gebied van modeltheorie geschikt voor inquisitieve eerste-ordelogica: Ehrenfeucht-Fraïssé-spellen. We laten zien dat de techniek van Ehrenfeucht-Fraïssé-spellen zich laat aanpassen aan deze context en dat het ons kan laten detecteren wanneer twee inquisitieve modellen niet te onderscheiden zijn door formules van een bepaalde complexiteit. Het ontwikkelde spel is vrij flexibel en kan aangepast worden om andere eigenschappen dan logische equivalentie te vatten, bijvoorbeeld de submodelrelatie. Door middel van het spel kunnen we kenschetsen welke kardinaliteitskwantoren definieerbaar zijn in inquisitieve eerste-ordelogica, waarmee we het resultaat voor klassieke logica generaliseren naar dit expressievere systeem.

Het tweede deel, bestaande uit hoofdstuk 6, neemt een andere stap in de modeltheoretische richting en presenteert verscheidene manieren om de modellen van inquisitieve eerste-ordelogica te manipuleren en combineren. De ontwikkelde theorie stelt ons in staat om te bewijzen dat twee belangrijke kenmerken van constructieve logica's ook van toepassing zijn op inquisitieve eerste-ordelogica: de disjunctie- en existentie-eigenschap. Het bewijs dat we geven is semantisch van aard: we ontwikkelen verscheidene constructies om inquisitieve modellen te combineren en transformeren, en gebruiken ze om de disjunctie- en existentie-eigenschap te bewijzen. Sommige van deze constructies zijn geinspireerd op bewerkingen van intuitionistische Kripke-frames (bijvoorbeeld disjuncte vereniging), terwijl andere gebaseerd zijn op constructies die typisch voor klassieke predikatenlogica zijn (bijvoorbeeld modellen van termen). Deze aanpak staat ons toe om ook algemenere resultaten te bewijzen: we definiëren verscheidene klassen van theorieën waarvan de overeenkomstige gevolgrelaties de disjunctie- en/of de existentieeigenschap hebben.

In het derde deel, bestaande uit hoofdstukken 7 en 8, verleggen we onze aandacht naar het axiomatiseringsprobleem. Op dit moment is het niet bekend of inquisitieve eerste-ordelogica axiomatiseerbaar is. We behandelen een beperkte versie van het axiomatiseringsprobleem, dat wil zeggen, we axiomatiseren fragmenten en variaties van de logica. Hoofdstuk 7 focust op het klassieke-antecedentfragment, dat intuitief gekenschetst kan worden als het fragment waarin vragen niet toegestaan zijn in het antecedent van een implicatie. Dit fragment is in het bijzonder interessant omdat het—modulo logische equivalentie—alle formules bevat die overeenkomen met zinnen uit de natuurlijke taal. We bewijzen dat het systeem van natuurlijke deductie zoals voorgesteld in [Ciardelli, 2016, Section 4.6], beperkt tot het klassieke-antecedentfragment, een correcte en sterk volledige axiomatisering biedt. Hoofdstuk 8 focust op het eindige-breedte inquisitieve logica's en op het begrensde-breedtefragment.

Eindige-breedte inquisitieve logica's zijn geïntroduceerd door Sano [2011] als een hiërarchie die nauw verwant is aan inquisitieve eerste-ordelogica. Sano axiomatiseerde één van deze logica's en liet twee vragen open: of de andere elementen van de hiërarchie axiomatiseerbaar zijn, en of inquisitieve eerste-ordelogica de limiet van deze hiërarchie is. We geven een positief antwoord op de eerste en een negatief antwoord op de laatste vraag. Hoofdstuk 8 behandelt ook het begrensde-breedtefragment, dat gekenmerkt wordt door de volgende eigenschap: als een formule van het fragment niet ondersteund wordt in een informatiestaat s, dan bestaat er een *eindige* deelverzameling van s die de formule ook niet ondersteunt. Deze nogal eigenaardige eigenschap staat ons toe om verscheidene interessante resultaten voor het fragment af te leiden (bijvoorbeeld dat validiteiten in het fragment recursief opsombaar zijn en dat de begrensde gevolgrelatie compact is), voortbouwend op het volledigheidsresultaat voor de eindige-breedte inquisitieve logica's.

Het vierde deel, bestaande uit hoofdstuk 9, is een verkennend werk dat nog niet ontwikkeld is voor eerste-ordelogica, maar alleen voor propositionele logica: we presenteren een algebraïsche en een topologische semantiek voor inquisitieve propositionele logica. Een generalisatie van deze aanpak naar eerste-ordelogica zou een kostbaar hulpmiddel kunnen blijken te zijn voor het bestuderen van inquisitieve eerste-ordelogica vanuit nieuwe perspectieven, bijvoorbeeld met behulp van de methoden die worden gebruikt door Rasiowa en Sikorski [1950] of Görnemann [1971]. Aan de algebraïsche kant introduceren we een nieuwe semantiek, gebaseerd op Heyting-algebra's, door de valuaties van propositionele atomen alleen te beperken tot reguliere elementen. Hieruit verkrijgen we een algebraïsche semantiek voor inquisitieve logica door de semantiek te beperken tot de klasse van inquisitieve Heyting-algebra's. Aan de topologische kant passen we een door Bezhanishvili en Holliday [2020] ontwikkeld dualiteitsresultaat toe om inquisitieve algebra's te karakteriseren in termen van hun duale topologische UV-ruimtes. Dit maakt het mogelijk om een topologische semantiek voor inquisitieve logica te definiëren die, voor zover de auteur weet, de eerste poging is om inquisitieve logica te bestuderen vanuit een topologisch perspectief.

Abstract

This dissertation focuses on the study of inquisitive first order logic, a logical formalism encompassing questions in the presence of quantification, developed in order to employ questions in formal inferences and study their logical properties. In particular, we focus on developing tools and techniques to study the expressive power of inquisitive first order logic and the properties of its entailment. The dissertation can be divided in four parts, each considering a different approach to study the logic.

In the first part, consisting of Chapters 4 and 5, we adapt a tool from the field of model theory to inquisitive first order logic: Ehrenfeucht-Fraïssé games. We show that the technique of Ehrenfeucht-Fraïssé games adapts to this context and allows to detect when two inquisitive models are indistinguishable by formulas of a given complexity. The game developed is quite flexible and can be modified to capture properties other than logical equivalence, as for example the submodel relation. Using the game, we achieve a characterization of the cardinality quantifiers definable in inquisitive first order logic, generalizing the result for classical logic to this more expressive setting.

The second part, consisting of Chapter 6, takes another step in the model-theoretic direction and presents several ways to manipulate and combine models of first order inquisitive logic. The theory developed allows us to prove that two hallmarks of constructive logics hold for inquisitive first order logic: the Disjunction and Existence properties. The proof we give is semantical in nature: we develop several constructions to combine and transform inquisitive models, and use them to prove the disjunction and existence properties. Some of these constructions are inspired by operations on intuitionistic Kripke-frames (e.g., disjoint union) while others are based on constructions typical of classical predicate logic (e.g., models of terms). This approach allows us to prove also more general results: we define several classes of theories for which the corresponding consequence relations have the disjunction and/or the existence property.

In the third part, consisting of Chapters 7 and 8, we shift our attention on the

axiomatization problem. As of now it is not known whether first order inquisitive logic is axiomatizable. We tackle a restricted version of the axiomatization problem, that is, we axiomatize fragments and variations of the logic. Chapter 7 focuses on the classical antecedent fragment, which can be intuitively characterized as the fragment where questions are not allowed in the antecedent of a conditional. This fragment is particularly interesting since it contains—modulo logical equivalence—all formulas corresponding to natural language sentences. We prove that the natural deduction system proposed in [Ciardelli, 2016, Section 4.6], restricted to the classical antecedent fragment, provides a sound and strongly complete axiomatization. Chapter 8 focuses on the finite-width inquisitive logics and on the bounded-width fragment. Finite-width inquisitive logics were introduced by Sano [2011] as a hierarchy closely related to inquisitive first order logic. Sano axiomatized one of these logics and left open two questions: whether the other elements of the hierarchy are axiomatizable, and whether first order inquisitive logic is the limit of this hierarchy. We give a positive answer to the former and a negative answer to the latter. Chapter 8 also treats the bounded-width fragment, characterized by the following property: if a formula of the fragment is not supported by an information state s, then there exists a *finite* subset of s which still does not support the formula. This rather peculiar property allows to derive several interesting results on the fragment (e.g., validities in the fragment are recursively enumerable, the restricted entailment is compact), building on the completeness result for the finite-width inquisitive logics.

The fourth part, consisting of Chapter 9, is an exploratory work not yet developed for the first order case, but only for the propositional case: we present an algebraic and a topological semantics for inquisitive propositional logics. Generalizing these semantic accounts to the first order case could prove to be a precious tool to study first order inquisitive logic from new perspectives, for example using the methods employed by Rasiowa and Sikorski [1950] or Görnemann [1971]. On the algebraic side, we introduce a new semantics based on Heyting algebras by restricting the valuations of propositional atoms only over regular elements. From this we obtain an algebraic semantics for inquisitive logic by restricting the semantics to the class of inquisitive Heyting algebras. On the topological side, we apply a duality result developed by Bezhanishvili and Holliday [2020] to characterize inquisitive algebras in terms of their dual topological UV-spaces. This allows to define a topological semantics for inquisitive logic which, as far as the author knows, is the first attempt to study inquisitive logic from a topological perspective.

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