TRANSLATIONAL EMBEDDINGS VIA STABLE CANONICAL RULES

NICK BEZHANISHVILI AND ANTONIO M. CLEANI

Abstract. This paper presents a new uniform method for studying modal companions of superintuitionistic deductive systems and related notions, based on the machinery of stable canonical rules. Using our method, we obtain an alternative proof of the Blok-Esakia theorem both for logics and for rule systems, and prove an analogue of the Dummett-Lemmon conjecture for rule systems. Since stable canonical rules may be developed for any rule system admitting filtration, our method generalises smoothly to richer signatures. We illustrate this by applying our techniques to prove analogues of the Blok-Esakia theorem (for both logics and rule systems) and of the Dummett-Lemmon conjecture (for rule systems) in the setting of tense companions of bi-superintuitionistic deductive systems. We also use our techniques to prove that the lattice of rule systems (logics) extending the modal intuitionistic logic KM and the lattice of rule systems (logics) extending the provability logic GL are isomorphic.

1. Introduction

A modal companion of a superintuitionistic logic L is defined as any normal modal logic M extending S4 such that the *Gödel translation* fully and faithfully embeds L into M. The notion of a modal companion has sparked a remarkably prolific line of research, documented, e.g., in the surveys [16] and [60]. The jewel of this research line is the celebrated *Blok-Esakia theorem*, first proved independently by Blok [11] and Esakia [24]. The theorem states that the lattice of superintuitionistic logics is isomorphic to the lattice of normal extensions of Grzegorczyk's modal logic GRZ, via the mapping which sends each superintuitionistic logic L to the normal extension of GRZ with the set of all Gödel translations of formulae in L.

Zakharyashchev [61] developed a unified approach to the theory of modal companions, via his technique of *canonical formulae*. These formulae generalise the subframe formulae of Fine [27]. Like a subframe formula, a canonical formula syntactically encodes the structure of a finite *refutation pattern*, i.e., a finite transitive frame together with a (possibly empty) set of parameters. By applying a version of the *selective filtration* construction, every formula can be matched with a finite set of finite refutation patterns, in such a way that the conjunction of all the canonical formulae associated with the refutation patterns is equivalent to the original formula. By studying how the Gödel translation affects superintuitionistic canonical formulae, Zakharyashchev gave alternative proofs of classic theorems in the theory of modal companions, and extended this theory with several novel results. Among these, he confirmed the *Dummett-Lemmon conjecture*, formulated in [21], which states that a superintuitionistic logic is Kripke complete iff its weakest modal companion is. Jeřábek [34] generalized canonical formulae to *canonical rules*, and applied this notion to extend Zakharyaschev's approach to theory of modal companions to *rule systems* (also known as *multi-conclusion consequence relations.*)

In [4, 5, 2], *stable canonical formulae* and *rules* were introduced as an alternative to Zakharyaschev and Jeřábek-style canonical rules and formulae. The basic idea is the same: a stable canonical formula or rule syntactically encodes the semantic structure of a finite refutation pattern. The main difference lies in how such structure is encoded, which affects how refutation patterns are constructed in the process of rewriting a formula (or rule) into a conjunction of stable canonical formulae (or rules). Namely, in the case of stable canonical formulae and rules finite refutation patterns are constructed by taking *filtrations* rather than selective filtrations of countermodels. A survey of stable canonical formulae and rules can be found in [3].

This paper applies stable canonical rules to develop a novel, uniform approach to the study of modal companions and related notions. Our approach echoes the Zakharyaschev-Jeřábek approach in using rules encoding finite refutation patterns, but also bears circumscribed similarities with Blok's original algebraic approach in some proof strategies (see Remark 3.40). Our techniques deliver central results in the theory of modal companions in a notably uniform fashion, and with high potential for further generalisation. In particular, we obtain an alternative proof of the Blok-Esakia theorem for both logics and rule systems, and generalise the Dummett-Lemmon conjectures to rule systems. Moreover, due to the flexibility of filtration, our techniques easily generalise to rule systems in richer signatures. We illustrate this via two case studies. Firstly, we apply our methods to study the notion of *tense companions* of bi-superintuitionistic deductive systems, introduced by Wolter [57] for logics. Here we generalise [57, Theorem 23], an analogue of the Blok-Esakia theorem, from logics to rule systems. Moreover, we obtain an analogue of the Dummett-Lemmon conjecture. Notably, these results are obtained via minimal adaptations of our technique, whereas extending the Zakharyaschev-Jeřábek technique to this setting is far from straightforward, as we argue in Section 4.2.3. Secondly, we apply our methods to study a Gödel translation-like correspondence between normal extensions of the intuitionistic provability logic $\mathtt{KM}$ and the normal extensions of the Gödel-Löb provability logic $\mathtt{GL}$. Here we prove that the lattice of normal modal superintuitionistic rule systems extending $\mathtt{KM}$ is isomorphic to the lattice of normal modal rule systems extending $\mathtt{GL}$. The corresponding result for logics, known as the *Kuznetsov-Muravitsky theorem* [37, Proposition 3] follows as a corollary. In pursuing these two generalisations of our technique, we also develop new kinds of stable (or stable-like) canonical rules: for bi-superintuitionistic and tense logics on the one hand, and (more significantly) for modal superintuitionistic rule systems over $\mathtt{KM}$ and modal rule systems over $\mathtt{GL}$ on the other.

The techniques described in this paper can also be used to obtain axiomatic characterizations of the modal companion maps (and their counterparts in the richer signatures discussed here) in terms of stable canonical rules, as well as some results concerning the notion of stability [6]. These results can be found in the recent master's thesis [18], on which the present paper is based.

The paper is organised as follows. Section 2 reviews general preliminaries. Each subsequent section presents and applies our methods to deductive systems in a specific pair of signature. Section 3 studies modal companions of superintuitionistic deductive systems. Section 4 studies tense companions of bi-superintuitionistic deductive system. Section 5 studies the Kuznetsov-Muravitsky isomorphism between normal extensions of $\mathtt{KM}$ and normal extensions of $\mathtt{GL}$. We conclude in Section 6.

2. General Preliminaries

This section fixes notational conventions and reviews the background theory needed throughout the paper. We collect here all definitions and results which all subsequent sections rely on. Preliminary information specific to the topic of a particular section is instead presented therein. We use [14] as our standard reference for universal algebra, and [32] for rule systems.

2.1. **Relations.** We begin by fixing some notation concerning binary relations. Let X be a set, R a transitive binary relation on X, and $U \subseteq X$. We define:

$$qmax_R(U) := \{x \in U : \text{ for all } y \in U, \text{ if } Rxy \text{ then } Ryx\} \tag{1}$$

$$max_R(U) := \{x \in U : \text{ for all } y \in U, \text{ if } Rxy \text{ then } x = y\} \tag{2}$$

$$qmin_R(U) := \{x \in U : \text{ for all } y \in U, \text{ if } Ryx \text{ then } Rxy\} \tag{3}$$

$$min_R(U) := \{x \in U : \text{ for all } y \in U, \text{ if } Ryx \text{ then } x = y\}. \tag{4}$$

The elements of $qmax_R(U)$ and $max_R(U)$ are called *R-quasi-maximal* and *R-maximal* elements of U respectively, and similarly the elements of $qmin_R(U)$ and $min_R(U)$ are called *R-quasi-minimal* and *R-minimal* elements of U respectively. Note that if R is a partial order then both $qmax_R(U) = max_R(U)$ and $qmin_R(U) = min_R(U)$. Lastly, we say that an element $x \in U$ is *R-passive* in U if for all $y \in X \smallsetminus U$, if Rxy then there is no $z \in U$ such that Ryz. Intuitively, an R-passive element of U is an $x \in U$ such that one cannot "leave" and "re-enter" U starting from x and "moving through" R. The set of all R-passive elements of U is denoted by $pas_R(U)$.

2.2. **Deductive Systems.** We now review *deductive system*, which span both propositional logics and rule systems. The set $Frm_\nu(X)$ of *formulae* in signature ν over a set of variables X is the least set containing X and such that for every $f \in \nu$ and $\varphi_1, \dots, \varphi_n \in Frm_\nu(X)$ we have $f(\varphi_1, \dots, \varphi_n) \in Frm_\nu(X)$, where n is the arity of f. Henceforth we will take *Prop* to be a fixed arbitrary countably infinite set of variables and write simply Frm_ν for $Frm_\nu(Prop)$. We occasionally write formulae in the form $\varphi(p_1, \dots, p_n)$ to indicate that the variables occurring in φ are among $p_1, \dots, p_n$. A *substitution* is a map $s : Prop \to Frm_\nu(Prop)$. Every substitution may be extended to a map $\bar{s} : Frm_\nu(Prop) \to Frm_\nu(Prop)$ recursively, by setting $\bar{s}(p) = s(p)$ if $p \in Prop$, and $\bar{s}(f(\varphi_1, \dots, \varphi_n)) = f(\bar{s}(\varphi_1), \dots, \bar{s}(\varphi_n))$.

Definition 2.1. A *logic* over Frm_ν is a set $\mathtt{L} \subseteq Frm_\nu$, such that

(structurality) $\quad \varphi \in \mathtt{L} \Rightarrow \bar{s}(\varphi) \in \mathtt{L}$ for every substitution s.

Interesting examples of logics, including those discussed in this paper, are normally closed under conditions other than structurality. If Γ, Δ are sets of formulae and $\mathcal{S}$ is a set of logics, we write $\Gamma \oplus_{\mathcal{S}} \Delta$ for the least logic in $\mathcal{S}$ extending both Γ, Δ.

For any sets X, Y, write $X \subseteq_\omega Y$ to mean that $X \subseteq Y$ and $|X|$ is finite. A (*multi-conclusion*) *rule* in signature ν over a set of variables X is a pair (Γ, Δ) such that $\Gamma, \Delta \subseteq_\omega Frm_\nu(X)$. In case $\Delta = \{\varphi\}$ we write Γ/Δ simply as Γ/φ, and analogously if $\Gamma = \{\psi\}$. We use ; to denote union between finite sets of formulae, so that $\Gamma;\Delta = \Gamma\cup\Delta$ and $\Gamma;\varphi = \Gamma\cup\{\varphi\}$. We write $Rul_\nu(X)$ for the set of all rules in ν over X, and simply Rul_ν when $X = Prop$.

Definition 2.2. A *rule system* is a set $\mathtt{S} \subseteq Rul_\nu(X)$ satisfying the following conditions.

1. If $\Gamma/\Delta \in \mathtt{S}$ then $\bar{s}[\Gamma]/\bar{s}[\Delta] \in \mathtt{S}$ for all substitutions s (structurality).
2. $\varphi/\varphi \in \mathtt{S}$ for every formula φ (reflexivity).
3. If $\Gamma/\Delta \in \mathtt{S}$ then $\Gamma;\Gamma'/\Delta;\Delta' \in \mathtt{S}$ for any finite sets of formulae Γ', Δ' (monotonicity).
4. If $\Gamma/\Delta;\varphi \in \mathtt{S}$ and $\Gamma;\varphi/\Delta \in \mathtt{S}$ then $\Gamma/\Delta \in \mathtt{S}$ (cut).

Remark 2.3. Rule systems are also called *multiple-conclusion consequence relations* (e.g., in [4, 32]). We prefer the terminology of rule systems (used in [34]) for brevity.

If $\mathcal{S}$ is a set of rule systems and Σ, Ξ are sets of rules, we write $\Xi \oplus_{\mathcal{S}} \Sigma$ for the least rule system in $\mathcal{S}$ extending both Ξ and Σ. A set of rules Σ is said to *axiomatise* a rule system $\mathtt{S} \in \mathcal{S}$ *over* some rule system $\mathtt{S}' \in \mathcal{S}$ if $\mathtt{S}' \oplus_{\mathcal{S}} \Sigma = \mathtt{S}$.

If $\mathtt{S}$ is a rule system we let the set of *tautologies* of $\mathtt{S}$ be the set

$$\mathsf{Taut}(\mathtt{S}) := \{\varphi \in Frm_\nu : /\varphi \in \mathtt{S}\}.$$

By the structurality condition for rule systems, it follows that $\mathsf{Taut}(\mathtt{S})$ is a logic for every rule system $\mathtt{S}$.

We interpret deductive systems over algebras in the same signature. If $\mathfrak{A}$ is a ν-algebra we denote its carrier as A. Let $\mathfrak{A}$ be some ν-algebra. A *valuation* on $\mathfrak{A}$ is a map $V : Prop \to A$. Every valuation V on $\mathfrak{A}$ may be recursively extended to a map $\bar{V} : Frm_\nu \to A$, by setting

$$\bar{V}(p) := V(p)$$
$$\bar{V}(f(\varphi_1, \dots, \varphi_n)) := f^{\mathfrak{A}}(\bar{V}(\varphi_1), \dots, \bar{V}(\varphi_n)).$$

A pair $(\mathfrak{A}, V)$ where $\mathfrak{A}$ is a ν-algebra and V a valuation on $\mathfrak{A}$ is called a *model*. A rule Γ/Δ is *valid* on a ν-algebra $\mathfrak{A}$ if the following holds: for any valuation V on $\mathfrak{A}$, if $\bar{V}(\gamma) = 1$ for all $\gamma \in \Gamma$, then $\bar{V}(\delta) = 1$ for some $\delta \in \Delta$. When this holds we write $\mathfrak{A} \models \Gamma/\Delta$, otherwise we write $\mathfrak{A} \nvDash \Gamma/\Delta$ and say that $\mathfrak{A}$ *refutes* Γ/Δ. As a special case, a formula φ is valid on a ν-algebra $\mathfrak{A}$ if the rule $/\varphi$ is. We write $\mathfrak{A} \models \varphi$ when this holds, $\mathfrak{A} \nvDash \varphi$ otherwise. The notion of validity extends to classes of ν-algebras: $\mathcal{K} \models \Gamma/\Delta$ means that $\mathfrak{A} \models \Gamma/\Delta$ for every $\mathfrak{A} \in \mathcal{K}$, and $\mathcal{K} \nvDash \Gamma/\Delta$ means that $\mathfrak{A} \nvDash \Gamma/\Delta$ for some $\mathfrak{A} \in \mathcal{K}$. Analogous notation is used for formulae. Finally, if Ξ is a set of formulae or rules and $\mathfrak{A}$ a ν-algebra, $\mathfrak{A} \models \Xi$ means that every formula or rule in Ξ is valid on $\mathfrak{A}$, $\mathfrak{A} \nvDash \Xi$ means that some formula or rule in Ξ is not valid on $\mathfrak{A}$, and similarly for classes of ν-algebras.

Write $\mathcal{A}_\nu$ for the class of all ν-algebras. For every deductive system $\mathtt{S}$ we define

$$\mathsf{Alg}(\mathtt{S}) := \{\mathfrak{A} \in \mathcal{A}_\nu : \mathfrak{A} \models \mathtt{S}\}.$$

Conversely, if $\mathcal{K}$ is a class of ν-algebras we set

$$\mathsf{ThR}(\mathcal{K}) := \{\Gamma/\Delta \in Rul_\nu : \mathcal{K} \models \Gamma/\Delta\}$$
$$\mathsf{Th}(\mathcal{K}) := \{\varphi \in Frm_\nu : \mathcal{K} \models \varphi\}$$

We also interpret deductive systems over ν-formulae on expansions of Stone spaces dual to ν-algebras, which for the moment we refer to as *ν-spaces*. Precise definitions of these topological sturctures and of valuations over them are given in each subsequent section. If $\mathfrak{X}$ is a ν-space we denote its underlying domain as X, its family of open sets as $\mathcal{O}$, and its family of clopen sets as $\mathsf{Clop}(\mathfrak{X})$. Moreover, if $U \subseteq X$ we write $-U$ for $X \smallsetminus U$. Given a valuation V on a ν-space $\mathfrak{X}$ and a point $x \in X$, we call $(\mathfrak{X}, V)$ a (global) *model*. A formula φ is *satisfied* on a model $(\mathfrak{X}, V)$ at a point x if $x \in \bar{V}(\varphi)$. In this case we write $\mathfrak{X}, V, x \models \varphi$, otherwise we write $\mathfrak{X}, V, x \nvDash \varphi$ and say that the model $(\mathfrak{X}, V)$ *refutes* φ at a point x. A rule Γ/Δ is *valid* on a model $(\mathfrak{X}, V)$ if the following holds: if for every $x \in X$ we have $\mathfrak{X}, V, x \models \gamma$ for each $\gamma \in \Gamma$, then for every $x \in X$ we have $\mathfrak{X}, V, x \models \delta$ for some $\delta \in \Delta$. In this case we write $\mathfrak{X}, V \models \Gamma/\Delta$, otherwise we write $\mathfrak{X}, V \nvDash \Gamma/\Delta$ and say that the model $(\mathfrak{X}, V)$ *refutes* φ. A rule Γ/Δ is *valid* on a ν-space $\mathfrak{X}$ if it is valid on the model $(\mathfrak{X}, V)$ for every valuation V on $\mathfrak{X}$, otherwise $\mathfrak{X}$ *refutes* Γ/Δ. We write $\mathfrak{X} \models \Gamma/\Delta$ to mean that Γ/Δ is valid on $\mathfrak{X}$, and $\mathfrak{X} \nvDash \Gamma/\Delta$ to mean that $\mathfrak{X}$ refutes Γ/Δ. As in the case of algebras we define validity on models and ν-spaces for a formula φ as validity of the rule $/\varphi$, and write $\mathfrak{X} \models \varphi$ if φ is valid in $\mathfrak{X}$, otherwise $\mathfrak{X} \nvDash \varphi$. The notion of validity generalises to classes of ν-spaces, so that if $\mathcal{K}$ is a class of ν-space then $\mathcal{K} \models \Gamma/\Delta$ means $\mathfrak{X} \models \Gamma/\Delta$ for every $\mathfrak{X} \in \mathcal{K}$, and $\mathcal{K} \nvDash \Gamma/\Delta$ means $\mathfrak{X} \nvDash \Gamma/\Delta$ for some $\mathfrak{X} \in \mathcal{K}$. We extend the present notation for validity to sets of formulae or rules the same way as for algebras.

Write $\mathcal{S}_\nu$ for the class of all ν-spaces. For every deductive system $\mathtt{S}$ we define

$$\mathsf{Spa}(\mathtt{S}) := \{\mathfrak{X} \in \mathcal{S}_\nu : \mathfrak{X} \models \mathtt{S}\}.$$

Conversely, if $\mathcal{K}$ is a class of ν-spaces we set

$$\mathsf{ThR}(\mathcal{K}) := \{\Gamma/\Delta \in \mathit{Rul}_\nu : \mathcal{K} \models \Gamma/\Delta\}$$
$$\mathsf{Th}(\mathcal{K}) := \{\varphi \in \mathit{Frm}_\nu : \mathcal{K} \models \varphi\}$$

Throughout the paper we study the structure of lattices of deductive systems via semantic methods. This is made possible by the following fundamental result, connecting the syntactic types of deductive systems to closure conditions on the classes of algebras validating them. Item 1 is widely known as *Birkhoff's theorem*, after [9].

Theorem 2.4 ([14, Theorems II.11.9 and V.2.20])**.** *For every class $\mathcal{K}$ of ν-algebras, the following conditions hold:*

(1) $\mathcal{K}$ is a variety iff $\mathcal{K} = \mathsf{Alg}(\mathtt{S})$ for some set of ν-formulae $\mathtt{S}$.
(2) $\mathcal{K}$ is a universal class iff $\mathcal{K} = \mathsf{Alg}(\mathtt{S})$ for some set of ν-rules $\mathtt{S}$.

In this sense, ν-logics correspond to varieties of ν-algebras, whereas ν-rule systems correspond to universal classes of ν-algebras.

This concludes our general preliminaries. We now begin the study of modal companions via stable canonical rules.

3. Modal Companions of Superintuitionistic Deductive Systems

This section studies the theory of modal companions of superintuitionistic deductive systems via stable canonical rules. Its main purpose is to present our method in detail and show that it performs as expected. After some brief preliminaries (Section 3.1), we present superintuitionistic and modal stable canonical rules (Section 3.2). The main results of this section are included in Section 3.3.3 and Section 3.3.4. The former uses stable canonical rules to give a characterisation of the set of modal companions of a superintuitionistic deductive system, and proves the Blok-Esakia theorem for both logics and rule systems. The latter proves an extension of the Dummett-Lemmon conjecture to rule systems, again using stable canonical rules.

The techniques presented in this section can also be applied to obtain axiomatic characterisations of the modal companion maps via stable canonical rules, as well as some results concerning the preservation of stability by the modal companion maps. More details on these topics can be found in [18, Sections 2.3.3, 2.3.4].

3.1. **Modal and Superintuitionistic Deductive Systems.** We begin with a brief overview of the semantic and syntactic structures discussed throughout the present section.

3.1.1. *Superintuitionistic Deductive Systems, Heyting Algebras, and Esakia Spaces.* We work with the *superintuitionistic signature*,

$$si := \{\wedge, \vee, \to, \bot, \top\}.$$

The set Frm_{si} of superintuitionistic (si) formulae is defined recursively as follows.

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi.$$

We abbreviate $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$. We let $\mathtt{IPC}$ denote the *intuitionistic propositional calculus*, and point the reader to [17, Ch. 2] for an axiomatisation.

Definition 3.1. A *superintuitionistic logic*, or si-logic for short, is a logic $\mathtt{L}$ over Frm_{si} satisfying the following additional conditions:

(1) $\mathtt{IPC} \subseteq \mathtt{L}$;
(2) $\varphi \to \psi, \varphi \in \mathtt{L}$ implies $\psi \in \mathtt{L}$ (MP).

A *superintuitionistic rule system*, or si-rule system for short, is a rule system $\mathtt{L}$ over Frm_{si} satisfying the following additional requirements.

(1) $/\varphi \in \mathtt{L}$ whenever $\varphi \in \mathtt{IPC}$.
(2) $\varphi, \varphi \to \psi/\psi \in \mathtt{L}$ (MP-R).

For every si-logic $\mathtt{L}$ write $\mathbf{Ext}(\mathtt{L})$ for the set of si-logics extending $\mathtt{L}$, and similarly for si-rule systems. Then $\mathbf{Ext}(\mathtt{IPC})$ is the set of all si-logics. It is well known that $\mathbf{Ext}(\mathtt{IPC})$ admits the structure of a complete lattice, with $\oplus_{\mathbf{Ext}(\mathtt{IPC})}$ serving as join and intersection as meet. Clearly, for every $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC})$ there exists a least si-rule system $\mathtt{L_R}$ containing $/\varphi$ for each $\varphi \in \mathtt{L}$. Hence $\mathtt{IPC_R}$ is the least rule system. The set $\mathbf{Ext}(\mathtt{IPC_R})$ is also a lattice when endowed with $\oplus_{\mathbf{Ext}(\mathtt{IPC_R})}$ as join and intersection as meet. Slightly abusing notation, we refer to these lattices as we refer to their underlying sets, i.e., $\mathbf{Ext}(\mathtt{IPC})$ and $\mathbf{Ext}(\mathtt{IPC_R})$ respectively. Additionally, we make use of systematic ambiguity and write both $\oplus_{\mathbf{Ext}(\mathtt{IPC})}$ and $\oplus_{\mathbf{Ext}(\mathtt{IPC_R})}$ simply as $\oplus$, leaving context to clarify which operation is meant.

The following proposition is central for transferring results about si-rule systems to si-logics. Its proof is routine.

Proposition 3.2. *The mappings* $(\cdot)_\mathtt{R}$ *and* $\mathsf{Taut}(\cdot)$ *are mutually inverse complete lattice isomorphisms between* $\mathbf{Ext}(\mathtt{IPC})$ *and the sublattice of* $\mathbf{Ext}(\mathtt{IPC_R})$ *consisting of all si-rule systems* $\mathtt{L}$ *such that* $\mathsf{Taut}(\mathtt{L})_\mathtt{R} = \mathtt{L}$.

A *Heyting algebra* is a tuple $\mathfrak{H} = (H, \wedge, \vee, \to, 0, 1)$ such that $(H, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and for every $a, b, c \in A$ we have

$$c \leq a \to b \iff a \wedge c \leq b.$$

We let HA denote the class of all Heyting algebras. By Theorem 2.4, HA is a variety. If $\mathcal{V} \subseteq \mathsf{HA}$ is a variety (resp: universal class) we write $\mathbf{Var}(\mathcal{V})$ and $\mathbf{Uni}(\mathcal{V})$ respectively for the lattice of subvarieties (resp: of universal subclasses) of $\mathcal{V}$. The connections between $\mathbf{Ext}(\mathtt{IPC})$ and $\mathbf{Var}(\mathsf{HA})$ on the one hand, and between $\mathbf{Ext}(\mathtt{IPC_R})$ and $\mathbf{Uni}(\mathsf{HA})$ on the other, are as intimate as they come.

Theorem 3.3. *The following maps are pairs of mutually inverse dual isomorphisms:*

1. $\mathsf{Alg} : \mathbf{Ext}(\mathtt{IPC}) \to \mathbf{Var}(\mathsf{HA})$ *and* $\mathsf{Th} : \mathbf{Var}(\mathsf{HA}) \to \mathbf{Ext}(\mathtt{IPC})$*;*
2. $\mathsf{Alg} : \mathbf{Ext}(\mathtt{IPC_R}) \to \mathbf{Uni}(\mathsf{HA})$ *and* $\mathsf{ThR} : \mathbf{Uni}(\mathsf{HA}) \to \mathbf{Ext}(\mathtt{IPC_R})$*.*

Item 1 is proved in [17, Theorem 7.56], whereas Item 2 follows from [34, Theorem 2.2] by standard techniques.

Corollary 3.4. *Every si-logic* (*resp. si-rule system*) *is complete with respect to some variety* (*resp. universal class*) *of Heyting algebras.*

An *Esakia space* is a tuple $\mathfrak{X} = (X, \leq, \mathcal{O})$, such that $(X, \mathcal{O})$ is a Stone space, $\leq$ is a partial order on X, and

(1) ${\uparrow}x := \{y \in X : x \leq y\}$ is closed for every $x \in X$;
(2) ${\downarrow}U := \{x \in X : {\uparrow}x \cap U \neq \varnothing\} \in \mathsf{Clop}(\mathfrak{X})$ for every $U \in \mathsf{Clop}(\mathfrak{X})$.

We let Esa denote the class of all Esakia spaces. If $\mathfrak{X}, \mathfrak{Y}$ are Esakia spaces, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is called a *bounded morphism* if for all $x, y \in X$ we have that $x \leq y$ implies $f(x) \leq f(y)$, and $h(x) \leq y$ implies that there is $z \in X$ with $x \leq z$ and $h(z) = y$.

If $\mathfrak{X}$ is an Esakia space and $U \subseteq X$, we say that U is an *upset* if ${\uparrow}[U] = U$. We let $\mathsf{ClopUp}(\mathfrak{X})$ denote the set of clopen upsets in $\mathfrak{X}$. A *valuation* on an Esakia space $\mathfrak{X}$ is a map $V : Prop \to \mathsf{ClopUp}(\mathfrak{X})$. A valuation V on $\mathfrak{X}$ extends to a truth-set assignment $\bar{V} : Frm_{si} \to \mathsf{ClopUp}(\mathfrak{X})$ in the standard way, with

$$\bar{V}(\varphi \to \psi) := -{\downarrow}(\bar{V}(\varphi) \smallsetminus \bar{V}(\psi)).$$

The following result recalls some important properties of Esakia spaces, used throughout the paper. For proofs the reader may consult [26, Lemma 3.1.5, Theorem 3.2.1].

Proposition 3.5. *Let* $\mathfrak{X} \in \mathsf{Esa}$*. Then for all* $x, y \in X$ *we have:*

1. *If* $x \not\leq y$ *then there is* $U \in \mathsf{ClopUp}(\mathfrak{X})$ *such that* $x \in U$ *and* $y \notin U$*;*
2. *For all* $U \in \mathsf{Clop}(U)$ *and* $x \in U$*, there is* $y \in max_{\leq}(U)$ *such that* $x \leq y$*.*

Esakia [22] proved that the category of Heyting algebras with corresponding homomorphisms is dually equivalent to the category of Esakia spaces with continuous bounded morphisms. The reader may consult [26, §3.4] for a detailed proof of this result. We denote the Esakia space dual to a Heyting algebra $\mathfrak{H}$ as $\mathfrak{H}_*$, and the Heyting algebra dual to an Esakia space $\mathfrak{X}$ as $\mathfrak{X}^*$.

3.1.2. *Modal Deductive Systems, Modal Algebras, and Modal Spaces.* We shall now work in the *modal signature*,

$$md := \{\wedge, \vee, \neg, \Box, \bot, \top\}.$$

The set Frm_{md} of modal formulae is defined recursively as follows.

$$\varphi ::= p \,|\, \bot \,|\, \top \,|\, \varphi \wedge \varphi \,|\, \varphi \vee \varphi \,|\, \neg\varphi \,|\, \Box\varphi.$$

As usual we abbreviate $\Diamond\varphi := \neg\Box\neg\varphi$. Further, we let $\varphi \to \psi := \neg\varphi \vee \psi$ and $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Definition 3.6. A *normal modal logic*, henceforth simply *modal logic*, is a logic $\mathtt{M}$ over Frm_{md} satisfying the following conditions:

(1) $\mathtt{CPC} \subseteq \mathtt{M}$, where $\mathtt{CPC}$ is the classical propositional calculus;
(2) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \in \mathtt{M}$;
(3) $\varphi \to \psi, \varphi \in \mathtt{M}$ implies $\psi \in \mathtt{M}$ (MP);
(4) $\varphi \in \mathtt{M}$ implies $\Box\varphi \in \mathtt{M}$ (NEC).

We denote the least modal logic as $\mathtt{K}$. A *normal modal rule system*, henceforth simply *modal rule system*, is a rule system $\mathtt{M}$ over Frm_{md}, satisfying the following additional requirements:

(1) $/\varphi \in \mathtt{M}$ whenever $\varphi \in \mathtt{K}$;
(2) $\varphi \to \psi, \varphi/\psi \in \mathtt{M}$ (MP-R);
(3) $\varphi/\Box\varphi \in \mathtt{M}$ (NEC-R).

If $\mathtt{M}$ is a modal logic let $\mathbf{NExt}(\mathtt{M})$ be the set of modal logics extending $\mathtt{M}$, and similarly for modal rule systems. Obviously, the set of modal logics coincides with $\mathbf{NExt}(\mathtt{K})$. It is well known that $\mathbf{NExt}(\mathtt{K})$ forms a lattice under the operations $\oplus_{\mathbf{NExt}(\mathtt{K})}$ as join and intersection as meet. Clearly, for each $\mathtt{M} \in \mathbf{NExt}(\mathtt{K})$ there is always a least modal rule system $\mathtt{K_R}$ containing $/\varphi$ for each $\varphi \in \mathtt{M}$. Therefore, $\mathtt{K_R}$ is the least modal rule system. The set $\mathbf{NExt}(\mathtt{K_R})$ is also a lattice when endowed with $\oplus_{\mathbf{NExt}(\mathtt{K_R})}$ as join and intersection as meet. With slight abuse of notation, we refer to these lattices as we refer to their underlying sets, i.e., $\mathbf{NExt}(\mathtt{K})$ and $\mathbf{NExt}(\mathtt{K_R})$ respectively. Additionally, we make use of systematic ambiguity and write both $\oplus_{\mathbf{NExt}(\mathtt{K})}$ and $\oplus_{\mathbf{NExt}(\mathtt{K_R})}$ simply as $\oplus$, leaving context to clarify which operation is meant.

We have a modal counterpart of Proposition 3.2.

Proposition 3.7. *The mappings $(\cdot)_{\mathtt{R}}$ and $\mathsf{Taut}(\cdot)$ are mutually inverse complete lattice isomorphisms between $\mathbf{NExt}(\mathtt{K})$ and the sublattice of $\mathbf{NExt}(\mathtt{K_R})$ consisting of all si-rule systems $\mathtt{M}$ such that $\mathsf{Taut}(\mathtt{M})_{\mathtt{R}} = \mathtt{M}$.*

A *modal algebra* is a tuple $\mathfrak{A} = (A, \wedge, \vee, \neg, \Box, 0, 1)$ such that $(A, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra and the following equations hold:

$$\Box 1 = 1, \tag{5}$$

$$\Box(a \wedge b) = \Box a \wedge \Box b. \tag{6}$$

We let MA denote the class of all modal algebras. By Theorem 2.4, MA is a variety. We let $\mathbf{Var}(\mathsf{MA})$ and $\mathbf{Uni}(\mathsf{MA})$ be the lattice of subvarieties and the lattice of universal subclasses of MA respectively. We have the following analogue of Theorem 3.3.

Theorem 3.8. *The following maps are pairs of mutually inverse dual isomorphisms:*

(1) $\mathsf{Alg} : \mathbf{NExt}(\mathtt{K}) \to \mathbf{Var}(\mathsf{MA})$ *and* $\mathsf{Th} : \mathbf{Var}(\mathsf{MA}) \to \mathbf{NExt}(\mathtt{K})$*;*
(2) $\mathsf{Alg} : \mathbf{NExt}(\mathtt{K_R}) \to \mathbf{Uni}(\mathsf{MA})$ *and* $\mathsf{ThR} : \mathbf{Uni}(\mathsf{MA}) \to \mathbf{NExt}(\mathtt{K_R})$*.*

Item 2 is proved in [17, Theorem 7.56], whereas Item 2 follows from [8, Theorem 2.5].

Corollary 3.9. *Every modal logic (resp. modal rule system) is complete with respect to some variety (resp. universal class) of modal algebras.*

A *modal space* is a tuple $\mathfrak{X} = (X, R, \mathcal{O})$, such that $(X, \mathcal{O})$ is a Stone space, $R \subseteq X \times X$ is a binary relation, and

(1) $R[x] := \{y \in X : Rxy\}$ is closed for every $x \in X$;
(2) $R^{-1}(U) := \{x \in X : R[x] \cap U \neq \varnothing\} \in \mathsf{Clop}(\mathfrak{X})$ for every $U \in \mathsf{Clop}(\mathfrak{X})$.

We let Mod denote the class of all modal spaces. If $\mathfrak{X}, \mathfrak{Y}$ are modal spaces, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is called a *bounded morphism* when for all $x, y \in X$, if Rxy then $Rf(x)f(y)$, and $Rf(x)y$ implies that there is $z \in X$ with Rxz and $f(z) = y$. A *valuation* on a modal space $\mathfrak{X}$ is a map $V : Prop \to \mathsf{Clop}(\mathfrak{X})$. A valuation extends to a full truth-set assignment $\bar{V} : Frm \to \mathsf{Clop}(\mathfrak{X})$ in the usual way.

By a generalisation of Stone duality, the category of modal algebras with corresponding homomorphisms is dually equivalent to the category of modal spaces with continuous bounded morphisms. A proof of this result can be found, e.g., in [49, Sections 3, 4]. We denote the modal space dual to a modal algebra $\mathfrak{A}$ as $\mathfrak{A}_*$, and the modal algebra dual to an modal space $\mathfrak{X}$ as $\mathfrak{X}^*$.

In this paper we are mostly concerned with modal algebras and modal spaces validating one of the following modal logics.

$$\mathtt{K4} := \mathtt{K} \oplus \Box p \to \Box\Box p$$
$$\mathtt{S4} := \mathtt{K4} \oplus \Box p \to p$$

We let $\mathsf{K4} := \mathsf{Alg}(\mathtt{K4})$ and $\mathsf{S4} := \mathsf{Alg}(\mathtt{S4})$. We call algebras in $\mathsf{K4}$ *transitive algebras*, and algebras in $\mathsf{S4}$ *closure algebras*. It is obvious that for every $\mathfrak{A} \in \mathsf{MA}$, $\mathfrak{A} \in \mathsf{K4}$ iff $\Box\Box a \leq \Box a$ for every $a \in A$, and $\mathfrak{A} \in \mathsf{S4}$ iff $\mathfrak{A} \in \mathsf{K4}$ and additionally $\Box a \leq a$ for every $a \in A$. Moreover, it is easy to see that a modal space validates $\mathtt{K4}$ iff it has a transitive relation, and that it validates $\mathtt{S4}$ iff it has a reflexive and transitive relation (see, e.g., Chagrov and Zakharyaschev 17, Section 3.8).

Let $\mathfrak{X} \in \mathsf{Spa}(\mathtt{K4})$. A subset $C \subseteq X$ is called a *cluster* if it is an equivalence class under the relation $\sim$ defined by $x \sim y$ iff Rxy and Ryx. A cluster is called *improper* if it is a singleton, otherwise we call it *proper*.

We recall some basic properties of $\mathtt{K4}$- and $\mathtt{S4}$-spaces.

Proposition 3.10. *Let $\mathfrak{X} \in \mathsf{Spa}(\mathtt{S4})$ and $U \in \mathsf{Clop}(\mathfrak{X})$. Then the following conditions hold:*

(1) *The set $qmax_R(U)$ is closed;*
(2) *If $x \in U$ then there is $y \in qmax_R(U)$ such that Rxy.*

Moreover, let $\mathfrak{X} \in \mathsf{Spa}(\mathtt{K4})$ and $U \in \mathsf{Clop}(\mathfrak{X})$. Then the following conditions hold:

(3) *The structure (X, R^+), with the same topology as $\mathfrak{X}$, is a $\mathtt{S4}$-space, where for all $x, y \in X$ we have R^+xy iff Rxy or $x = y$;*
(4) *The set $qmax_R(U)$ is closed;*
(5) *If $x \in U$ then there is $y \in qmax_R(U)$ such that Rxy.*

Properties 1, 2 are proved in [26, Theorems 3.2.1, 3.2.3]. Property 3 is straightforward to check, and properties 4, 5 are immediate consequences of 1, 2, and 3.

Among extensions of $\mathtt{S4}$, the modal logic $\mathtt{GRZ}$ plays a particularly central role in this paper.

$$\begin{aligned}\mathtt{GRZ} :&= \mathtt{K} \oplus \Box(\Box(p \to \Box p) \to p) \to p \\ &= \mathtt{S4} \oplus \Box(\Box(p \to \Box p) \to p) \to p\end{aligned}$$

We let $\mathsf{GRZ} := \mathsf{Alg}(\mathtt{GRZ})$. It is not difficult to see that GRZ coincides with the class of all closure algebras $\mathfrak{A}$ such that for every $a \in A$ we have

$$\Box(\Box(a \to \Box a) \to a) \leq a$$

or equivalently,

$$a \leq \Diamond(a \wedge \neg\Diamond(\Diamond a \wedge \neg a)).$$

A poset (X, R) is called *Noetherian* if it contains no infinite R-ascending chain of pairwise distinct points. It is well known that $\mathtt{GRZ}$ is complete with respect to the class of Noetherian partially ordered Kripke frames [17, Corollary 5.52]. In general, $\mathtt{GRZ}$-spaces may fail to be partially ordered. Still, clusters cannot occur just anywhere in a $\mathtt{GRZ}$-space, as the following result clarifies.

Proposition 3.11. *For every* $\mathtt{GRZ}$*-space* $\mathfrak{X}$ *and* $U \in \mathsf{Clop}(\mathfrak{X})$*, the following hold:*

(1) $qmax_R(U) \subseteq max_R(U)$*;*
(2) The set $max_R(U)$ *is closed;*
(3) For every $x \in U$ *there is* $y \in pas_R(U)$ *such that* Rxy*;*
(4) $max_R(U) \subseteq pas_R(U)$*.*

Item 1 is proved in [26, Theorem 3.5.6]. Item 2 follows from Item 1 and Proposition 3.10. Item 3 is immediate from the $\mathtt{GRZ}$-axiom. Item 4 then follows from Proposition 3.10, Item 1, and Item 3.

Let us say that $U \subseteq X$ *cuts* a cluster $C \subseteq X$ if both $U \cap C \neq \varnothing$ and $U \smallsetminus C \neq \varnothing$. As an immediate consequence of Item 4 in Proposition 3.11 we obtain that for any $U \in \mathsf{Clop}(\mathfrak{X})$, neither $max_R(U)$ or $pas_R(U)$ cut any clusters in $\mathfrak{X}$.

3.2. **Stable Canonical Rules for Superintuitionistic and Modal Rule Systems.** In both the si and the modal cases, the *filtration* technique can be used to construct finite countermodels to a non-valid rule Γ/Δ. Roughly, this construction consists of expanding finitely generated subreducts in a locally finite signature of arbitrary counter-models to Γ/Δ, in such a way that the new operation added to the subreduct agrees with the original one on selected elements. Si and modal *stable canonical rules* are essentially syntactic devices for encoding finite filtrations. The present section briefly reviews this method in both the si and modal case. We point the reader to [4, 5, 2, 3] and [33, Ch. 5] for more in-depth discussion.

3.2.1. *Supertintuitionistic Case.* We begin by defining si stable canonical rules.

Definition 3.12. Let $\mathfrak{H} \in \mathsf{HA}$ be finite and $D \subseteq A \times A$. For every $a \in H$ introduce a fresh propositional variable p_a. The *si stable canonical rule* of $(\mathfrak{H}, D)$, is defined as the rule $\eta(\mathfrak{H}, D) = \Gamma/\Delta$, where

$$\begin{aligned}\Gamma =& \{p_0 \leftrightarrow 0\} \cup \{p_1 \leftrightarrow 1\} \cup \\ & \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in H\} \cup \{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in H\} \cup \\ & \{p_{a\to b} \leftrightarrow p_a \to p_b : (a, b) \in D\} \\ \Delta =& \{p_a \leftrightarrow p_b : a, b \in H \text{ with } a \neq b\}.\end{aligned}$$

We write si stable canonical rules of the form $\eta(\mathfrak{H}, \varnothing)$ simply as $\eta(\mathfrak{H})$, and call them *stable rules*.

If $\mathfrak{H}, \mathfrak{K} \in \mathsf{HA}$, let us call a map $h : \mathfrak{H} \to \mathfrak{K}$ *stable* if h is a bounded lattice homomorphism, i.e., if it preserves $0, 1, \wedge$, and $\vee$. If $D \subseteq H \times H$, we say that h satisfies the *bounded domain condition* (BDC) for D if

$$h(a \to b) = h(a) \to h(b)$$

for every $(a, b) \in D$. It is not difficult to check that every stable map $h : \mathfrak{H} \to \mathfrak{K}$ satisfies $h(a \to b) \leq h(a) \to h(b)$ for every $(a, b) \in H$.

Remark 3.13. The BDC was originally called *closed domain condition* in, e.g., [4, 2], following Zakharyaschev's terminology for a similar notion in the theory of his canonical formulae. The name *stable domain condition* was later used in [3] to stress the difference with Zakharyaschev's notion. However, this choice may create confusion between the BDC and the property of being a stable map. The terminology used in this paper is meant to avoid this, while concurrently highlighting the similarity between the geometric version of the BDC, to be presented in a few paragraphs, and the definition of a bounded morphism.

The next two results characterise refutation conditions for si stable canonical rules. For detailed proofs the reader may consult [5, Proposition 3.2].

Proposition 3.14. *For every finite $\mathfrak{H} \in \mathsf{HA}$ and $D \subseteq H \times H$, we have $\mathfrak{H} \nvDash \eta(\mathfrak{H}, D)$.*

Proof sketch. Use the valuation $V(p_a) = a$. □

Proposition 3.15. *For every $\mathfrak{H}, \mathfrak{K} \in \mathsf{HA}$ with $\mathfrak{H}$ finite, and every $D \subseteq H \times H$, we have $\mathfrak{K} \nvDash \eta(\mathfrak{H}, D)$ iff there is a stable embedding $h : \mathfrak{H} \to \mathfrak{K}$ satisfying the BDC for D.*

Proof sketch. ($\Rightarrow$) Assume $\mathfrak{K} \nvDash \eta(\mathfrak{H}, D)$, and take a valuation V on $\mathfrak{K}$ such that $\mathfrak{K}, V \nvDash \eta(\mathfrak{H}, D)$. Define a map $h : \mathfrak{H} \to \mathfrak{K}$ by setting $h(a) = V(p_a)$. Then h is the desired stable embedding satisfying the BDC for D.

($\Leftarrow$) Assume we have a stable embedding $h : \mathfrak{H} \to \mathfrak{K}$ satisfying the BDC for D. By the proof of Proposition 3.14 we know that the valuation V with $V(p_a) = a$ witnesses $\mathfrak{H} \nvDash \eta(\mathfrak{H}, D)$. So put $V(p_a) = h(a)$. □

Si stable canonical rules also have uniform refutation conditions on Esakia spaces. If $\mathfrak{X}, \mathfrak{Y}$ are Esakia spaces, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is called *stable* if $x \leq y$ implies $f(x) \leq f(y)$, for all $x, y \in X$. If $\mathfrak{d} \subseteq Y$ we say that f satisfies the BDC for $\mathfrak{d}$ if for all $x \in X$,

$${\uparrow} f(x) \cap \mathfrak{d} \neq \varnothing \Rightarrow f[{\uparrow} x] \cap \mathfrak{d} \neq \varnothing.$$

If $\mathfrak{D} \subseteq \wp(Y)$ then we say that f satisfies the BDC for $\mathfrak{D}$ if it does for each $\mathfrak{d} \in \mathfrak{D}$. If $\mathfrak{H}$ is a finite Heyting algebra and $D \subseteq H$, for every $(a,b) \in D$ set $\mathfrak{d}_{(a,b)} := \beta(a) \smallsetminus \beta(b)$. Finally, put

$$\mathfrak{D} := \{\mathfrak{d}_{(a,b)} : (a,b) \in D\}.$$

The following result follows straightforwardly from [2, Lemma 4.3].

Proposition 3.16. *For every Esakia space $\mathfrak{X}$ and any si stable canonical rule $\eta(\mathfrak{H}, D)$, we have $\mathfrak{X} \nvDash \eta(\mathfrak{H}, D)$ iff there is a continuous stable surjection $f : \mathfrak{X} \to \mathfrak{H}_*$ satisfying the BDC for the family $\mathfrak{D} := \{\mathfrak{d}_{(a,b)} : (a,b) \in D\}$.*

In view of Proposition 3.16, when working with Esakia spaces we shall often write a si stable canonical rule $\eta(\mathfrak{H}, D)$ as $\eta(\mathfrak{H}_*, \mathfrak{D})$.

Stable maps and the BDC are closely related to the filtration construction. We recall its definition in an algebraic setting, and state the fundamental theorem used in most of its applications.

Definition 3.17. Let $\mathfrak{H}$ be a Heyting algebra, V a valuation on $\mathfrak{A}$, and Θ a finite, subformula closed set of formulae. A (finite) model $(\mathfrak{K}', V')$ is called a (*finite*) *filtration of* $(\mathfrak{H}, V)$ *through* Θ if the following hold:

(1) $\mathfrak{K}' = (\mathfrak{K}, \to)$, where $\mathfrak{K}$ is the bounded sublattice of $\mathfrak{H}$ generated by $\bar{V}[\Theta]$;
(2) $V(p) = V'(p)$ for every propositional variable $p \in \Theta$;
(3) The inclusion $\subseteq : \mathfrak{H} \to \mathfrak{K}$ is a stable embedding satisfying the BDC for the set
$$\{(\bar{V}'(\varphi), \bar{V}'(\psi)) : \varphi \to \psi \in \Theta\}.$$

Theorem 3.18 (Filtration theorem for Heyting algebras)**.** *Let $\mathfrak{H} \in \mathsf{HA}$ be a Heyting algebra, V a valuation on $\mathfrak{H}$, and Θ a a finite, subformula closed set of formulae. If $(\mathfrak{K}', V')$ is a filtration of $(\mathfrak{H}, V)$ through Θ then for every $\varphi \in \Theta$ we have*

$$\bar{V}(\varphi) = \bar{V}'(\varphi).$$

Consequently, for every rule Γ/Δ such that $\gamma, \delta \in \Theta$ for each $\gamma \in \Gamma$ and $\delta \in \Delta$ we have

$$\mathfrak{H}, V \models \Gamma/\Delta \iff \mathfrak{K}, V' \models \Gamma/\Delta.$$

A proof of the filtration theorem above follows from, e.g., the proof of [2, Lemma 3.6].

The next result establishes that every si rule is equivalent to finitely many si stable canonical rules. This lemma was proved in [5, Proposition 3.3], but we rehearse the proof here to illustrate the exact role of filtration in the machinery of stable canonical rules.

Lemma 3.19. *For every si rule Γ/Δ there is a finite set Ξ of si stable canonical rules such that for any $\mathfrak{K} \in \mathsf{HA}$ we have $\mathfrak{K} \nvDash \Gamma/\Delta$ iff there is $\eta(\mathfrak{H}, D) \in \Xi$ such that $\mathfrak{K} \nvDash \eta(\mathfrak{H}, D)$.*

Proof. Since bounded distributive lattices are locally finite there are, up to isomorphism, only finitely many pairs $(\mathfrak{H}, D)$ such that

- $\mathfrak{H}$ is at most k-generated as a bounded distributive lattice, where $k = |Sfor(\Gamma/\Delta)|$;
- $D = \{(\bar{V}(\varphi), \bar{V}(\psi)) : \varphi \to \psi \in Sfor(\Gamma/\Delta)\}$, where V is a valuation on $\mathfrak{H}$ refuting Γ/Δ.

Let Ξ be the set of all rules $\eta(\mathfrak{H}, D)$ for all such pairs $(\mathfrak{H}, D)$, identified up to isomorphism.

($\Rightarrow$) Assume $\mathfrak{K} \nvDash \Gamma/\Delta$ and take a valuation V on $\mathfrak{K}$ refuting Γ/Δ. Consider the bounded distributive sublattice $\mathfrak{J}$ of $\mathfrak{K}$ generated by $\bar{V}[Sfor(\Gamma/\Delta)]$. Since bounded distributive lattices are locally finite, $\mathfrak{J}$ is finite. Define a binary operation $\rightsquigarrow$ on $\mathfrak{J}$ by setting, for all $a, b \in J$,

$$a \rightsquigarrow b := \bigvee\{c \in J : a \wedge c \leq b\}.$$

Clearly, $\mathfrak{J}' := (\mathfrak{J}, \rightsquigarrow)$ is a Heyting algebra. Define a valuation V' on $\mathfrak{J}'$ with $V'(p) = V(p)$ if $p \in \Theta$, $V'(p)$ arbitrary otherwise. Since $\mathfrak{J}'$ is a sublattice of $\mathfrak{K}$, the inclusion $\subseteq$ is a stable embedding. Now let $\varphi \to \psi \in \Theta$. Then $\bar{V}'(\varphi) \to \bar{V}'(\psi) \in J$. From the fact that $\subseteq$ is a stable embedding it follows that $\bar{V}'(\varphi) \rightsquigarrow \bar{V}'(\psi) \leq \bar{V}'(\varphi) \to \bar{V}'(\psi)$. Conversely, by definition of $\rightsquigarrow$ we find $\bar{V}'(\varphi) \rightsquigarrow \bar{V}'(\psi) \wedge \bar{V}'(\varphi) \leq \bar{V}'(\psi)$. But then by the properties of Heyting algebras it follows that $\bar{V}'(\varphi) \to \bar{V}'(\psi) \leq \bar{V}'(\varphi) \rightsquigarrow \bar{V}'(\psi)$. Thus $\bar{V}'(\varphi) \rightsquigarrow \bar{V}'(\psi) = \bar{V}'(\varphi) \to \bar{V}'(\psi)$ as desired. We have shown that the model $(\mathfrak{J}', V')$ is a filtration of the model $(\mathfrak{K}, V)$ through $Sfor(\Gamma/\Delta)$, which implies $\mathfrak{J}', V' \nvDash \Gamma/\Delta$.

($\Leftarrow$) Assume that there is $\eta(\mathfrak{H}, D) \in \Xi$ such that $\mathfrak{K} \nvDash \eta(\mathfrak{H}, D)$. Let V be the valuation associated with D in the sense spelled out above. Then $\mathfrak{H}, V \nvDash \Gamma/\Delta$. Moreover $(\mathfrak{H}, V)$ is a filtration of the model $(\mathfrak{K}, V)$, so by the filtration theorem it follows that $\mathfrak{K}, V \nvDash \Gamma/\Delta$. ☐

As an immediate consequence we obtain a uniform axiomatisation of all si-rule systems by means of si stable canonical rules.

Theorem 3.20 ([5, Proposition 3.4])**.** *Any si-rule system* $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ *is axiomatisable over* $\mathtt{IPC_R}$ *by some set of si stable canonical rules.*

Proof. Let $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$, and take a set of rules Ξ such that $\mathtt{L} = \mathtt{IPC_R} \oplus \Xi$. By Lemma 3.19 and the completeness of $\mathtt{IPC_R}$ (Corollary 3.4), for every $\Gamma/\Delta \in \Xi$ there is a finite set $\Pi_{\Gamma/\Delta}$ of si stable canonical rules whose conjunction is equivalent to Γ/Δ. But then $\mathtt{L} = \mathtt{IPC_R} \oplus \bigcup_{\Delta/\Gamma \in \Xi} \Pi_{\Gamma/\Delta}$. ☐

3.2.2. *Modal Case.* We now turn to modal stable canonical rules.

Definition 3.21. Let $\mathfrak{A} \in \mathsf{MA}$ be finite and $D \subseteq A$. For every $a \in A$ introduce a fresh propositional variable p_a. The *modal stable canonical rule* of $(\mathfrak{A}, D)$ is defined as the rule $\mu(\mathfrak{A}, D) = \Gamma/\Delta$, where

$$\begin{aligned}\Gamma = \{&p_{\neg a} \leftrightarrow \neg p_a : a \in A\} \cup \\ &\{p_{a \wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \cup \{p_{a \vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \cup \\ &\{p_{\Box a} \to \Box p_a : a \in A\} \cup \{\Box p_a \to p_{\Box a} : a \in D\} \\ \Delta = \{&p_a : a \in A \smallsetminus \{1\}\}.\end{aligned}$$

As in the si case, a modal stable canonical rule of the form $\mu(\mathfrak{A}, \varnothing)$ is written simply as $\mu(\mathfrak{A})$ and called a *stable rule*.

If $\mathfrak{A}, \mathfrak{B} \in \mathsf{MA}$ are modal algebras, let us call a map $h : \mathfrak{A} \to \mathfrak{B}$ *stable* if for every $a \in A$ we have $h(\Box a) \leq \Box h(a)$. If $D \subseteq A$, we say that h satisfies the *bounded domain condition* (BDC) for D if $h(\Box a) = \Box h(a)$ for every $a \in D$.

The following two propositions are modal counterparts to Propositions 3.14 and 3.15. Their proofs are similar to the latter's, and can be found in [4, Lemma 5.3, Theorem 5.4].

Proposition 3.22. *For every finite $\mathfrak{A} \in \mathsf{MA}$ and $D \subseteq A$, we have $\mathfrak{A} \nvDash \mu(\mathfrak{A}, D)$.*

Proposition 3.23. *For every $\mathfrak{A}, \mathfrak{B} \in \mathsf{MA}$ with $\mathfrak{A}$ finite, and every $D \subseteq A$, we have $\mathfrak{B} \nvDash \mu(\mathfrak{A}, D)$ iff there is a stable embedding $h : \mathfrak{A} \to \mathfrak{B}$ satisfying the BDC for D.*

Refutation conditions for modal stable canonical rules on modal spaces are obtained in analogous fashion to the si case. If $\mathfrak{X}, \mathfrak{Y}$ are modal spaces, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is called *stable* if for all $x, y \in X$, we have that Rxy implies $Rf(x)f(y)$. If $\mathfrak{d} \subseteq Y$ we say that f satisfies the BDC for $\mathfrak{d}$ if for all $x \in X$,

$$R[f(x)] \cap \mathfrak{d} \neq \varnothing \Rightarrow f[R[x]] \cap \mathfrak{d} \neq \varnothing.$$

If $\mathfrak{D} \subseteq \wp(Y)$ then we say that f satisfies the BDC for $\mathfrak{D}$ if it does for each $\mathfrak{d} \in \mathfrak{D}$. If $\mathfrak{A}$ is a finite modal algebra and $D \subseteq H$, for every $a \in D$ set $\mathfrak{d}_a := -\beta(a)$. Finally, put $\mathfrak{D} := \{\mathfrak{d}_a : a \in D\}$. The following result is proved in [4, Theorem 3.6].

Proposition 3.24. *For every modal space $\mathfrak{X}$ and any modal stable canonical rule $\mu(\mathfrak{A}, D)$, $\mathfrak{X} \nvDash \mu(\mathfrak{A}, D)$ iff there is a continuous stable surjection $f : \mathfrak{X} \to \mathfrak{A}_*$ satisfying the BDC for $\mathfrak{D}$.*

In view of Proposition 3.24, when working with modal spaces we may write a modal stable canonical rule $\mu(\mathfrak{A}, D)$ as $\mu(\mathfrak{A}_*, \mathfrak{D})$.

As in the si case, stable maps and the BDC are closely related to the filtration technique.

Definition 3.25. Let $\mathfrak{A}$ be a modal algebra, V a valuation on $\mathfrak{A}$, and Θ a finite, subformula closed set of formulae. A (finite) model $(\mathfrak{B}, V')$ is called a (*finite*) *filtration of* $(\mathfrak{A}, V)$ *through* Θ if the following conditions hold:

(1) $\mathfrak{B} = (\mathfrak{B}', \Box)$, where $\mathfrak{B}'$ is the Boolean subalgebra of $\mathfrak{A}$ generated by $\bar{V}[\Theta]$;
(2) $V(p) = V'(p)$ for every propositional variable $p \in \Theta$;
(3) The inclusion $\subseteq : \mathfrak{B} \to \mathfrak{A}$ is a stable embedding satisfying the BDC for the set
$$\{\bar{V}(\varphi) : \Box\varphi \in \Theta\}$$

The following result is proved, e.g., in [4, Lemma 4.4].

Theorem 3.26 (Filtration theorem for modal algebras)**.** *Let $\mathfrak{A} \in \mathsf{MA}$ be a modal algebra, V a valuation on $\mathfrak{A}$, and Θ a finite, subformula closed set of formulae. If $(\mathfrak{B}', V')$ is a filtration of $(\mathfrak{A}, V)$ through Θ then for every $\varphi \in \Theta$ we have*

$$\bar{V}(\varphi) = \bar{V}'(\varphi).$$

Consequently, for every rule Γ/Δ such that $\gamma, \delta \in \Theta$ for each $\gamma \in \Gamma$ and $\delta \in \Delta$ we have

$$\mathfrak{A}, V \models \Gamma/\Delta \iff \mathfrak{B}, V' \models \Gamma/\Delta.$$

Unlike the si case, filtrations of a given model through a given set of formulae are not necessarily unique when they exist. Depending on which construction is preferred, different properties of the original model may or may not be preserved. In this section we mainly deal with closure algebras, whence we are particularly interested in filtrations preserving reflexivity and transitivity. It is easy to see that any filtration preserves reflexivity. Whilst, in general, the filtration of a transitive model may fail to be transitive, transitive filtrations of transitive models can be constructed in multiple ways. Here we restrict attention to one particular construction.

Definition 3.27. Let $\mathfrak{A} \in \mathsf{S4}$, V a valuation on $\mathfrak{A}$ and Θ a finite, subformula closed set of formula. The (*least*) *transitive filtration* of $(\mathfrak{A}, V)$ is a pair $(\mathfrak{B}', V')$ with $\mathfrak{B}' = (\mathfrak{B}, \blacksquare)$, where $\mathfrak{B}$ and V' are as per Definition 3.25, and for all $b \in B$ we have

$$\blacksquare b := \bigvee \{\Box a : \Box a \leq \Box b \text{ and } a, \Box a \in B\}$$

It is easy to see that transitive filtrations of transitive models are indeed based on closure algebras (cf., e.g., [4, Lemma 6.2]).

Transitive filtrations provide the necessary countermodels to rewrite modal rules into (conjunctions of) modal stable canonical rules. The following lemma, which is a modal counterpart to Lemma 3.19, explains how.

Lemma 3.28 ([4, Theorem 5.5])**.** *For every modal rule Γ/Δ there is a finite set Ξ of modal stable canonical rules of the form $\mu(\mathfrak{A}, D)$ with $\mathfrak{A} \in \mathsf{S4}$, such that for any $\mathfrak{B} \in \mathsf{S4}$ we have that $\mathfrak{B} \nvDash \Gamma/\Delta$ iff there is $\mu(\mathfrak{A}, D) \in \Xi$ such that $\mathfrak{B} \nvDash \mu(\mathfrak{A}, D)$.*

Proof. Since Boolean algebras are locally finite there are, up to isomorphism, only finitely many pairs $(\mathfrak{A}, D)$ such that

- $\mathfrak{A}$ is at most k-generated as a Boolean algebra, where $k = |Sfor(\Gamma/\Delta)|$;
- $D = \{\bar{V}(\varphi) : \Box\varphi \in Sfor(\Gamma/\Delta)\}$, where V is a valuation on $\mathfrak{H}$ refuting Γ/Δ.

Let Ξ be the set of all rules $\mu(\mathfrak{A}, D)$ for all such pairs $(\mathfrak{A}, D)$, identified up to isomorphism. Then we reason as in the proof of Lemma 3.19, using the well-known fact that every model $(\mathfrak{B}, V)$ with $\mathfrak{B} \in \mathsf{S4}$ has a transitive filtration through $Sfor(\Gamma/\Delta)$ to establish the $(\Rightarrow)$ direction. □

Exactly mirroring the si case we apply Lemma 3.28 to obtain the following uniform axiomatisation of modal rule systems extending $\mathtt{S4_R}$.

Theorem 3.29. *Every modal rule system $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$ is axiomatisable over $\mathtt{S4_R}$ by some set of modal stable canonical rules of the form $\mu(\mathfrak{A}, D)$, for $\mathfrak{A} \in \mathsf{S4}$.*

3.3. **Modal Companions of Superintuitionistic Deductive Systems via Stable Canonical Rules.** We now turn to the main topic of this section. Section 3.3.1 reviews the basic ingredients of the theory of modal companions. Section 3.3.3 shows how to apply stable canonical rules to give a novel proof of the Blok-Esakia theorem. Lastly, Section 3.3.4 applies our methods to obtain an analogue of the Dummett-Lemmon conjecture to rule systems.

3.3.1. *Semantic Mappings.* We begin by defining semantic transformation between Heyting and closure algebras. For more details, consult [26, Section 3.5].

Definition 3.30. The mapping $\sigma : \mathsf{HA} \to \mathsf{S4}$ assigns every $\mathfrak{H} \in \mathsf{HA}$ to the algebra $\sigma\mathfrak{H} := (B(\mathfrak{H}), \Box)$, where $B(\mathfrak{H})$ is the free Boolean extension of $\mathfrak{H}$ and

$$\Box a := \bigvee \{b \in H : b \leq a\}.$$

It can be shown that for each $\mathfrak{H} \in \mathsf{HA}$ we have that $\sigma\mathfrak{H}$ is in fact a $\mathtt{GRZ}$-algebra [26, Corollary 3.5.7].

Definition 3.31. The mapping $\rho : \mathsf{S4} \to \mathsf{HA}$ assigns every $\mathfrak{A} \in \mathsf{S4}$ to the algebra $\rho\mathfrak{A} := (O(A), \wedge, \vee, \to, 0, 1)$, where

$$O(A) := \{a \in A : \Box_F a = a\} = \{a \in A : \Diamond_P a = a\}$$
$$a \to b := \Box_F(\neg a \vee b)$$

The algebra $\rho(\mathfrak{A})$ is called the *Heyting algebra of open elements* associated with $\mathfrak{A}$. It is easy to verify that $\rho(\mathfrak{A})$ is indeed a Heyting algebra for any closure algebra $\mathfrak{A}$.

We now give a dual description of the maps σ, ρ on modal and Esakia spaces.

Definition 3.32. If $\mathfrak{X} = (X, \leq, \mathcal{O})$ is an Esakia space we set $\sigma\mathfrak{X} := (X, R, \mathcal{O})$ with $R := \leq$. Let $\mathfrak{Y} := (Y, R, \mathcal{O})$ be an $\mathsf{S4}$-space. For $x, y \in Y$ write $x \sim y$ iff Rxy and Ryx. Define a map $\rho : Y \to \wp(Y)$ by setting $\rho(x) = \{y \in Y : x \sim y\}$. We define $\rho\mathfrak{Y} := (\rho[Y], \leq, \rho[\mathcal{O}])$ where $\rho(x) \leq \rho(y)$ iff Rxy.

Note that σ here is effectively the identity map, though we find useful to distinguish an Esakia space $\mathfrak{X}$ from $\sigma\mathfrak{X}$ notationally in order to signal whether we are treating the space as a model for si or modal deductive systems. On the other hand, the map ρ affects a modal space $\mathfrak{Y}$ by collapsing its R-clusters and endowing the result with the quotient topology. We shall refer to $\rho\mathfrak{Y}$ as the *Esakia skeleton* of $\mathfrak{Y}$, and to $\sigma\rho\mathfrak{Y}$ as the *modal skeleton* of $\mathfrak{Y}$. It is easy to see that the map $\rho : \mathfrak{Y} \to \rho\mathfrak{Y}$ is a surjective bounded morphism which moreover reflects $\leq$.

Routine arguments show that that the algebraic and topological versions of the maps σ, ρ are indeed dual to each other, as stated in the following proposition.

Proposition 3.33. *The following hold.*

(1) Let $\mathfrak{H} \in \mathsf{HA}$. Then $(\sigma\mathfrak{H})_* \cong \sigma(\mathfrak{H}_*)$. Consequently, if $\mathfrak{X}$ is an Esakia space then $(\sigma\mathfrak{X})^* \cong \sigma(\mathfrak{X})^*$.

(2) Let $\mathfrak{X}$ be an $\mathsf{S4}$ modal space. Then $(\rho\mathfrak{X})^* \cong \rho(\mathfrak{X}^*)$. Consequently, if $\mathfrak{A} \in \mathsf{S4}$, then $(\rho\mathfrak{A})_* \cong \rho(\mathfrak{A}_*)$.

The dual description of ρ, σ makes the following result evident.

Proposition 3.34. *For every $\mathfrak{H} \in \mathsf{HA}$ we have $\mathfrak{H} \cong \rho\sigma\mathfrak{H}$. Moreover, for every $\mathfrak{A} \in \mathsf{S4}$ we have $\sigma\rho\mathfrak{A} \rightarrowtail \mathfrak{A}$.*

3.3.2. *The Gödel Translation.* The close connection between Heyting and closure algebras just outlined manifests syntactically as the existence of a well-behaved translation of si formulae into modal ones, called the *Gödel translation* after Gödel [31].

Definition 3.35 (Gödel translation)**.** The *Gödel translation* is a mapping $T : \mathit{Frm}_{si} \to \mathit{Frm}_{md}$ defined recursively as follows.

$$T(\bot) := \bot$$
$$T(\top) := \top$$
$$T(p) := \Box p$$
$$T(\varphi \wedge \psi) := T(\varphi) \wedge T(\psi)$$
$$T(\varphi \vee \psi) := T(\varphi) \vee T(\psi)$$
$$T(\varphi \to \psi) := \Box(\neg T(\varphi) \vee T(\psi))$$

We extend the Gödel translation from formulae to rules by setting

$$T(\Gamma/\Delta) := T[\Gamma]/T[\Delta].$$

We close this subsection by recalling the following key lemma due to Jeřábek [34].

Lemma 3.36 ([34, Lemma 3.13])**.** *For every $\mathfrak{A} \in \mathsf{S4}$ and si rule Γ/Δ,*

$$\mathfrak{A} \models T(\Gamma/\Delta) \iff \rho\mathfrak{A} \models \Gamma/\Delta$$

3.3.3. *Structure of Modal Companions.* We now have all the material needed to develop the theory of modal companions via the machinery of stable canonical rules.

Definition 3.37. Let $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ be a si-rule system and $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$ a modal rule system. We say that $\mathtt{M}$ is a *modal companion* of $\mathtt{L}$ (or that $\mathtt{L}$ is the si fragment of $\mathtt{M}$) whenever $\Gamma/\Delta \in \mathtt{L}$ iff $T(\Gamma/\Delta) \in \mathtt{M}$. Moreover, let $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC})$ be a si-logic and $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4})$ a modal logic. We say that $\mathtt{M}$ is a *modal companion* of $\mathtt{L}$ (or that $\mathtt{L}$ is the si fragment of $\mathtt{M}$) whenever $\varphi \in \mathtt{L}$ iff $T(\varphi) \in \mathtt{M}$.

Obviously, $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$ is a modal companion of $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ iff $\mathsf{Taut}(\mathtt{M})$ is a modal companion of $\mathsf{Taut}(\mathtt{L})$, and $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4})$ is a modal companion of $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC})$ iff $\mathtt{M_R}$ is a modal companion of $\mathtt{L_R}$.

Define the following three maps between the lattices $\mathbf{Ext}(\mathtt{IPC_R})$ and $\mathbf{NExt}(\mathtt{K_R})$.

$$\begin{aligned} \tau &: \mathbf{Ext}(\mathtt{IPC_R}) \to \mathbf{NExt}(\mathtt{S4_R}) & \sigma &: \mathbf{Ext}(\mathtt{IPC_R}) \to \mathbf{NExt}(\mathtt{S4_R}) \\ \mathtt{L} &\mapsto \mathtt{S4_R} \oplus \{T(\Gamma/\Delta) : \Gamma/\Delta \in \mathtt{L}\} & \mathtt{L} &\mapsto \mathtt{GRZ_R} \oplus \tau\mathtt{L} \end{aligned}$$

$$\begin{aligned} \rho &: \mathbf{NExt}(\mathtt{S4_R}) \to \mathbf{Ext}(\mathtt{IPC_R}) \\ \mathtt{M} &\mapsto \{\Gamma/\Delta : T(\Gamma/\Delta) \in \mathtt{M}\} \end{aligned}$$

These mappings are readily extended to lattices of logics.

$$\begin{aligned} \tau &: \mathbf{Ext}(\mathtt{IPC}) \to \mathbf{NExt}(\mathtt{S4}) & \sigma &: \mathbf{Ext}(\mathtt{IPC}) \to \mathbf{NExt}(\mathtt{S4}) \\ \mathtt{L} &\mapsto \mathsf{Taut}(\tau\mathtt{L_R}) = \mathtt{S4} \oplus \{T(\varphi) : \varphi \in \mathtt{L}\} & \mathtt{L} &\mapsto \mathsf{Taut}(\sigma\mathtt{L_R}) = \mathtt{GRZ} \oplus \{T(\varphi) : \varphi \in \mathtt{L}\} \end{aligned}$$

$$\begin{aligned} \rho &: \mathbf{NExt}(\mathtt{S4}) \to \mathbf{Ext}(\mathtt{IPC}) \\ \mathtt{M} &\mapsto \mathsf{Taut}(\rho\mathtt{M_R}) = \{\varphi : T(\varphi) \in \mathtt{M}\} \end{aligned}$$

Furthermore, extend the mappings $\sigma : \mathsf{HA} \to \mathsf{S4}$ and $\rho : \mathsf{S4} \to \mathsf{HA}$ to universal classes by setting

$$\begin{aligned} \sigma &: \mathbf{Uni}(\mathsf{HA}) \to \mathbf{Uni}(\mathsf{S4}) & \rho &: \mathbf{Uni}(\mathsf{S4}) \to \mathbf{Uni}(\mathsf{HA}) \\ \mathcal{U} &\mapsto \mathsf{Uni}\{\sigma\mathfrak{H} : \mathfrak{H} \in \mathcal{U}\} & \mathcal{W} &\mapsto \{\rho\mathfrak{A} : \mathfrak{A} \in \mathcal{W}\}. \end{aligned}$$

Finally, introduce a semantic counterpart to τ as follows.

$$\begin{aligned} \tau &: \mathbf{Uni}(\mathsf{HA}) \to \mathbf{Uni}(\mathsf{S4}) \\ \mathcal{U} &\mapsto \{\mathfrak{A} \in \mathsf{S4} : \rho\mathfrak{A} \in \mathcal{U}\} \end{aligned}$$

The goal of this subsection is to give alternative proofs of the following two classic results in the theory of modal companions. Firstly, that for every si-deductive system $\mathtt{L}$, the modal companions of $\mathtt{L}$ are exactly the elements of the interval $\rho^{-1}(\mathtt{L})$ (Theorem 3.43). Secondly, that the syntactic mappings σ, ρ are mutually inverse isomorphism (Theorem 3.44). This last result (restricted to logics) is widely known as the *Blok-Esakia theorem*.

The main problem one needs to deal with in order to prove the results just mentioned consists in showing that the mapping $\sigma : \mathbf{Ext}(\mathtt{IPC_R}) \to \mathbf{NExt}(\mathtt{GRZ_R})$ is surjective. We solve this problem by first applying stable canonical rules to show that the semantic mapping $\sigma : \mathbf{Uni}(\mathsf{HA}) \to \mathbf{Uni}(\mathsf{GRZ})$ is surjective, and subsequently establishing that the syntactic and semantic versions of σ capture essentially the same transformation. Our key tool is the following technical lemma.

Lemma 3.38. *Let $\mathfrak{A} \in \mathsf{GRZ}$. Then for every modal rule Γ/Δ, $\mathfrak{A} \models \Gamma/\Delta$ iff $\sigma\rho\mathfrak{A} \models \Gamma/\Delta$.*

Proof. ($\Rightarrow$) This direction follows from the fact that $\sigma\rho\mathfrak{A} \rightarrowtail \mathfrak{A}$ (Proposition 3.34).

($\Leftarrow$) We prove the dual statement that $\mathfrak{A}_* \not\models \Gamma/\Delta$ implies $\sigma\rho\mathfrak{A}_* \not\models \Gamma/\Delta$. Let $\mathfrak{X} := \mathfrak{A}_*$. By Theorem 3.29, Γ/Δ is equivalent to a conjunction of modal stable canonical rules of finite closure algebras, so without loss of generality we may assume $\Gamma/\Delta = \mu(\mathfrak{B}, D)$, for $\mathfrak{B} \in \mathsf{S4}$ finite. So suppose $\mathfrak{X} \not\models \mu(\mathfrak{B}, D)$ and let $\mathfrak{F} := \mathfrak{B}_*$. By Proposition 3.16, there is a stable map $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $\mathfrak{D} := \{\mathfrak{d}_a : a \in D\}$. We construct a stable map $g : \sigma\rho\mathfrak{X} \to \mathfrak{F}$ which also satisfies the BDC for $\mathfrak{D}$. By Proposition 3.16 again, this would show that $\sigma\rho\mathfrak{X} \not\models \mu(\mathfrak{B}, D)$, hence would conclude the proof.

Let $C \subseteq F$ be some cluster. Consider $Z_C := f^{-1}(C)$. As f is continuous, $Z_C \in \mathsf{Clop}(\mathfrak{X})$. Moreover, since f is stable Z_C does not cut any cluster. It follows that $\rho[Z_C]$ is clopen in $\sigma\rho\mathfrak{X}$, because $\sigma\rho\mathfrak{X}$ has the quotient topology. Enumerate $C := \{x_1, \dots, x_n\}$. Then $f^{-1}(x_i) \subseteq Z_C$ is clopen. By Proposition 3.11 we find that $M_i := max(f^{-1}(x_i))$ is closed. Furthermore, as $\mathfrak{X}$ is a $\mathtt{GRZ}$ space and every element of M_i is passive in M_i, by Proposition 3.11 again we have that M_i does not cut any cluster. Therefore $\rho[M_i]$ is closed, again because $\sigma\rho\mathfrak{X}$ has the quotient topology. Clearly, $\rho[M_i] \cap \rho[M_j] = \varnothing$ for each $i \neq j$.

We shall now separate the closed sets $\rho[M_1], \dots, \rho[M_n]$ by disjoint clopens. That is, we shall find disjoint clopens $U_1, \dots, U_n \in \mathsf{Clop}(\sigma\rho\mathfrak{X})$ with $\rho[M_i] \subseteq U_i$ and $\bigcup_i U_i = \rho[Z_C]$. Let $k \leq n$ and assume that U_i has been defined for all $i < k$. If $k = n$ put $U_n = \rho[Z_C] \smallsetminus \left(\bigcup_{i<k} U_i\right)$ and we are done. Otherwise set $V_k := \rho[Z_C] \smallsetminus \left(\bigcup_{i<k} U_i\right)$ and observe that it contains each $\rho[M_i]$ for $k \leq i \leq n$. By the separation properties of Stone spaces, for each i with $k < i \leq n$ there is some $U_{k_i} \in \mathsf{Clop}(\sigma\rho\mathfrak{X})$ with $\rho[M_k] \subseteq U_{k_i}$ and $\rho[M_i] \cap U_{k_i} = \varnothing$. Then set $U_k := \bigcap_{k<i\leq n} U_{k_i} \cap V_k$.

Now define a map

$$\begin{aligned} g_C &: \rho[Z_C] \to C \\ z &\mapsto x_i \iff z \in U_i. \end{aligned}$$

Note that g_C is relation preserving, evidently, and continuous because $g_C^{-1}(x_i) = U_i$. Finally, define $g : \sigma\rho\mathfrak{X} \to F$ by setting

$$g(\rho(z)) := \begin{cases} f(z) & \text{if } f(z) \text{ does not belong to any proper cluster} \\ g_C(\rho(z)) & \text{if } f(z) \in C \text{ for some proper cluster } C \subseteq F. \end{cases}$$

Now, g is evidently relation preserving. Moreover, it is continuous because both f and each g_C are. Suppose $Rg(\rho(x))y$ and $y \in \mathfrak{d}$ for some $\mathfrak{d} \in \mathfrak{D}$. By construction, $f(x)$ belongs to the same cluster as $g(\rho(x))$, so also $Rf(x)y$. Since f satisfies the BDC for $\mathfrak{D}$, there must be some $z \in X$ such that Rxz and $f(z) \in \mathfrak{d}$. Since $f^{-1}(f(z)) \in \mathsf{Clop}(\mathfrak{X})$, by Proposition 3.11 there is $z' \in max(f^{-1}(f(z)))$ with Rzz'. Then also Rxz' and $f(z') \in \mathfrak{d}$. But from $z' \in max(f^{-1}(f(z))$ it follows that $f(z') = g(\rho(z'))$ by construction, so we have $g(\rho(z')) \in \mathfrak{d}$. As clearly $R\rho(x)\rho(z')$, we have shown that g satisfies the BDC for $\mathfrak{D}$. By Proposition 3.16 this implies $\sigma\rho\mathfrak{X} \not\models \mu(\mathfrak{B}, D)$. □

Theorem 3.39. *Every* $\mathcal{U} \in \mathbf{Uni}(\mathsf{GRZ})$ *is generated by its skeletal elements, i.e.,* $\mathcal{U} = \sigma\rho\mathcal{U}$*.*

Proof. By $\sigma\rho\mathfrak{A} \rightarrowtail \mathfrak{A}$ (Proposition 3.34), surely $\sigma\rho\mathcal{U} \subseteq \mathcal{U}$. Conversely, suppose $\mathcal{U} \not\models \Gamma/\Delta$. Then there is $\mathfrak{A} \in \mathcal{U}$ with $\mathfrak{A} \not\models \Gamma/\Delta$. By Lemma 3.38 it follows that $\sigma\rho\mathfrak{A} \not\models \Gamma/\Delta$. This shows $\mathsf{ThR}(\sigma\rho\mathcal{U}) \subseteq \mathsf{ThR}(\mathcal{U})$, which is equivalent to $\mathcal{U} \subseteq \sigma\rho\mathcal{U}$. Hence indeed $\mathcal{U} = \sigma\rho\mathcal{U}$. □

Remark 3.40. The restriction of Theorem 3.39 to varieties plays an important role in the algebraic proof of the Blok-Esakia theorem given by Blok [11]. The unrestricted version is explicitly stated and proved in [52, Lemma 4.4] using a generalisation of Blok's approach, although it also follows from [34, Theorem 5.5]. Blok establishes the restricted version of Theorem 3.39 as a consequence of what is now known as the *Blok lemma*. The proof of the Blok lemma is notoriously involved. By contrast, our techniques afford a direct and, we believe, semantically transparent proof of Theorem 3.39.

Given Theorem 3.39, the main result of this section can be obtained via known routine arguments. First, we show that the syntactic modal companion maps τ, ρ, σ commute with $\mathsf{Alg}(\cdot)$.

Lemma 3.41 ([34, Theorem 5.9])**.** *For each* $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ *and* $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$*, the following hold:*

$$\mathsf{Alg}(\tau\mathtt{L}) = \tau\mathsf{Alg}(\mathtt{L}) \tag{7}$$

$$\mathsf{Alg}(\sigma\mathtt{L}) = \sigma\mathsf{Alg}(\mathtt{L}) \tag{8}$$

$$\mathsf{Alg}(\rho\mathtt{M}) = \rho\mathsf{Alg}(\mathtt{M}) \tag{9}$$

Proof. (7) For every $\mathfrak{A} \in \mathsf{S4}$ we have $\mathfrak{A} \in \mathsf{Alg}(\tau\mathtt{L})$ iff $\mathfrak{A} \models T(\Gamma/\Delta)$ for all $\Gamma/\Delta \in \mathtt{L}$ iff $\rho\mathfrak{A} \models \Gamma/\Delta$ for all $\Gamma/\Delta \in \mathtt{L}$ iff $\rho\mathfrak{A} \in \mathsf{Alg}(\mathtt{L})$ iff $\mathfrak{A} \in \tau\mathsf{Alg}(\mathtt{L})$.

(8) In view of Theorem 3.39 it suffices to show that $\mathsf{Alg}(\sigma\mathtt{L})$ and $\sigma\mathsf{Alg}(\mathtt{L})$ have the same skeletal elements. So let $\mathfrak{A} = \sigma\rho\mathfrak{A} \in \mathsf{GRZ}$. Assume $\mathfrak{A} \in \sigma\mathsf{Alg}(\mathtt{L})$. Since $\sigma\mathsf{Alg}(\mathtt{L})$ is generated by $\{\sigma\mathfrak{B} : \mathfrak{B} \in \mathsf{Alg}(\mathtt{L})\}$ as a universal class, by Proposition 3.34 and Lemma 3.36 we have $\mathfrak{A} \models T(\Gamma/\Delta)$ for every $\Gamma/\Delta \in \mathtt{L}$. But then $\mathfrak{A} \in \mathsf{Alg}(\sigma\mathtt{L})$. Conversely, assume $\mathfrak{A} \in \mathsf{Alg}(\sigma\mathtt{L})$. Then $\mathfrak{A} \models T(\Gamma/\Delta)$ for every $\Gamma/\Delta \in \mathtt{L}$. By Lemma 3.36 this is equivalent to $\rho\mathfrak{A} \in \mathsf{Alg}(\mathtt{L})$, therefore $\sigma\rho\mathfrak{A} = \mathfrak{A} \in \sigma\mathsf{Alg}(\mathtt{L})$.

(9) Let $\mathfrak{H} \in \mathsf{HA}$. If $\mathfrak{H} \in \rho\mathsf{Alg}(\mathtt{M})$ then $\mathfrak{H} = \rho\mathfrak{A}$ for some $\mathfrak{A} \in \mathsf{Alg}(\mathtt{M})$. It follows that for every si rule $T(\Gamma/\Delta) \in \mathtt{M}$ we have $\mathfrak{A} \models T(\Gamma/\Delta)$, and so by Lemma 3.36 in turn $\mathfrak{H} \models \Gamma/\Delta$. Therefore indeed $\mathfrak{H} \in \mathsf{Alg}(\rho\mathtt{M})$. Conversely, for all si rules Γ/Δ, if $\rho\mathsf{Alg}(\mathtt{M}) \models \Gamma/\Delta$ then by Lemma 3.36 $\mathsf{Alg}(\mathtt{M}) \models T(\Gamma/\Delta)$, hence $\Gamma/\Delta \in \rho\mathtt{M}$. Thus $\mathsf{ThR}(\rho\mathsf{Alg}(\mathtt{M})) \subseteq \rho\mathtt{M}$, and so $\mathsf{Alg}(\rho\mathtt{M}) \subseteq \rho\mathsf{Alg}(\mathtt{M})$. □

The result just proved leads straightforwardly to the following, purely semantic characterisation of modal companions.

Lemma 3.42. $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$ *is a modal companion of* $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ *iff* $\mathsf{Alg}(\mathtt{L}) = \rho\mathsf{Alg}(\mathtt{M})$.

Proof. ($\Rightarrow$) Assume $\mathtt{M}$ is a modal companion of $\mathtt{L}$. Then we have $\mathtt{L} = \rho\mathtt{M}$. By Lemma 3.41 $\mathsf{Alg}(\mathtt{L}) = \rho\mathsf{Alg}(\mathtt{M})$.

($\Leftarrow$) Assume that $\mathsf{Alg}(\mathtt{L}) = \rho\mathsf{Alg}(\mathtt{M})$. Therefore, by Proposition 3.34, $\mathfrak{H} \in \mathsf{Alg}(\mathtt{L})$ implies $\sigma\mathfrak{H} \in \mathsf{Alg}(\mathtt{M})$. This implies that for every si rule Γ/Δ, $\Gamma/\Delta \in \mathtt{L}$ iff $T(\Gamma/\Delta) \in \mathtt{M}$. □

We can now prove the main two results of this section.

Theorem 3.43 ([34, Theorem 5.5], [61, Theorem 3])**.** *The following conditions hold:*

1. *For every* $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$*, the modal companions of* $\mathtt{L}$ *form an interval* $\{\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R}) : \tau\mathtt{L} \leq \mathtt{M} \leq \sigma\mathtt{L}\}$.
2. *For every* $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC})$*, the modal companions of* $\mathtt{L}$ *form an interval* $\{\mathtt{M} \in \mathbf{NExt}(\mathtt{S4}) : \tau\mathtt{L} \leq \mathtt{M} \leq \sigma\mathtt{L}\}$.

Proof. (1) In view of Lemma 3.41 it suffices to prove that $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$ is a modal companion of $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ iff $\sigma\mathsf{Alg}(\mathtt{L}) \subseteq \mathsf{Alg}(\mathtt{M}) \subseteq \tau\mathsf{Alg}(\mathtt{L})$.

($\Rightarrow$) Assume $\mathtt{M}$ is a modal companion of $\mathtt{L}$. Then by Lemma 3.42 we have $\mathsf{Alg}(\mathtt{L}) = \rho\mathsf{Alg}(\mathtt{M})$, therefore it is clear that $\mathsf{Alg}(\mathtt{M}) \subseteq \tau\mathsf{Alg}(\mathtt{L})$. To see that $\sigma\mathsf{Alg}(\mathtt{L}) \subseteq \mathsf{Alg}(\mathtt{M})$ it suffices to show that every skeletal algebra in $\sigma\mathsf{Alg}(\mathtt{L})$ belongs to $\mathsf{Alg}(\mathtt{M})$. So let $\mathfrak{A} \cong \sigma\rho\mathfrak{A} \in \sigma\mathsf{Alg}(\mathtt{L})$. Then $\rho\mathfrak{A} \in \mathsf{Alg}(\mathtt{L})$ by Lemma 3.36, so there must be $\mathfrak{B} \in \mathsf{Alg}(\mathtt{M})$ such that $\rho\mathfrak{B} \cong \rho\mathfrak{A}$. But this implies $\sigma\rho\mathfrak{B} \cong \sigma\rho\mathfrak{A} \cong \mathfrak{A}$, and as universal classes are closed under subalgebras, by Proposition 3.34 we conclude $\mathfrak{A} \in \mathsf{Alg}(\mathtt{M})$.

($\Leftarrow$) Assume $\sigma\mathsf{Alg}(\mathtt{L}) \subseteq \mathsf{Alg}(\mathtt{M}) \subseteq \tau\mathsf{Alg}(\mathtt{L})$. It is an immediate consequence of Proposition 3.34 that $\rho\sigma\mathsf{Alg}(\mathtt{L}) = \mathsf{Alg}(\mathtt{L})$, which gives us $\rho\mathsf{Alg}(\mathtt{M}) \supseteq \mathsf{Alg}(\mathtt{L})$. But by construction $\rho\mathsf{Alg}(\mathtt{M}) = \rho\tau\mathsf{Alg}(\mathtt{L})$, hence $\rho\mathsf{Alg}(\mathtt{M}) \subseteq \mathsf{Alg}(\mathtt{L})$. Therefore indeed $\rho\mathsf{Alg}(\mathtt{M}) = \mathsf{Alg}(\mathtt{L})$, so by Lemma 3.42 we conclude that $\mathtt{M}$ is a modal companion of $\mathtt{L}$.

(2) Immediate from Item 1 and Propositions 3.2 and 3.7. □

Theorem 3.44 (Blok Esakia theorem)**.** *The following conditions hold:*

1. *The mappings* $\sigma : \mathbf{Ext}(\mathtt{IPC_R}) \to \mathbf{NExt}(\mathtt{GRZ_R})$ *and* $\rho : \mathbf{NExt}(\mathtt{GRZ_R}) \to \mathbf{Ext}(\mathtt{IPC_R})$ *are complete lattice isomorphisms and mutual inverses.*
2. *The mappings* $\sigma : \mathbf{Ext}(\mathtt{IPC}) \to \mathbf{NExt}(\mathtt{GRZ})$ *and* $\rho : \mathbf{NExt}(\mathtt{GRZ}) \to \mathbf{Ext}(\mathtt{IPC})$ *are complete lattice isomorphisms and mutual inverses.*

Proof. (1) It is enough to show that the mappings $\sigma : \mathbf{Uni}(\mathsf{HA}) \to \mathbf{Uni}(\mathsf{GRZ})$ and $\rho : \mathbf{NExt}(\mathsf{GRZ}) \to \mathbf{Ext}(\mathsf{HA})$ are complete lattice isomorphisms and mutual inverses. Both maps are evidently order preserving, and preservation of infinite joins is an easy consequence of Lemma 3.36. Let $\mathcal{U} \in \mathbf{Uni}(\mathsf{GRZ})$. Then $\mathcal{U} = \sigma\rho\mathcal{U}$ by Theorem 3.39, so σ is surjective and a left inverse of ρ. Now let $\mathcal{U} \in \mathbf{Uni}(\mathsf{HA})$. It is an immediate consequence of Proposition 3.34 that $\rho\sigma\mathcal{U} = \mathcal{U}$. Hence ρ is surjective and a left inverse of σ. Thus σ and ρ are mutual inverses, and therefore must both be bijections.

(2) Immediate from Item 1 and Propositions 3.2 and 3.7. □

As noted earlier, the arguments given in the proofs of Theorems 3.43 and 3.44 are standard. The novelty of our strategy consists in establishing the key fact on which these standard arguments depend on, namely Theorem 3.39, in a novel way using stable canonical rules.

3.3.4. *The Dummett-Lemmon Conjecture.* We call a modal or si-rule system *Kripke complete* if it is of the form $\mathtt{L} = \{\Gamma/\Delta : \mathcal{K} \models \Gamma/\Delta\}$ for some class of Kripke frames $\mathcal{K}$. Zakharyashchev [61, Corollary 2] applied his canonical formulae to prove the *Dummett-Lemmon conjecture* [21], which states that a si-logic is Kripke complete iff its weakest modal companion is. To our knowledge, a proof that the Dummett-Lemmon conjecture generalises to rule systems has not been published, although perhaps one could be given by applying Jeřábek-style canonical rule to adapt Zakharyaschev's argument. Here we give a proof that the Dummett-Lemmon conjecture does indeed generalise to rule systems using stable canonical rules.

It is easy to see that refutation conditions for stable canonical rules work essentially the same way for Kripke frames as they do for Esakia and modal spaces: for every Kripke frame $\mathfrak{X}$ and si stable canonical rule $\eta(\mathfrak{F}, \mathfrak{D})$, we have that $\mathfrak{X} \nvDash \eta(\mathfrak{F}, \mathfrak{D})$ iff there is a surjective stable homomorphism $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $\mathfrak{D}$, and analogously for the modal case. For details the reader may consult, e.g., [4]. The mappings σ, τ, ρ also extend to classes of Kripke frames in an obvious way. Finally Lemma 3.36 works for Kripke frames as well, the latter appropriately reformulated to incorporate the refutation conditions for stable canonical rules just stated.

We now introduce the notion of a *collapsed* stable canonical rule. We prefer to do so in a geometric setting, so to emphasize the main intuition behind this concept.

Definition 3.45. Let $\mu(\mathfrak{F}, \mathfrak{D})$ be some modal stable canonical rule, with $\mathfrak{F} \in \mathsf{Spa}(\mathtt{S4})$. The *collapsed stable canonical rule* $\eta(\rho\mathfrak{F}, \rho\mathfrak{D})$ is obtained by setting

$$\rho\mathfrak{D} := \{\rho[\mathfrak{d}] : \mathfrak{d} \in \mathfrak{D}\}.$$

Intuitively, $\eta(\rho\mathfrak{F}, \rho\mathfrak{D})$ is obtained from $\mu(\mathfrak{F}, \mathfrak{D})$ by collapsing all clusters in $\mathfrak{F}$ and in the set of domains $\mathfrak{D}$ as well.

Collapsed rules obey the following refutation condition.

Lemma 3.46 (Rule collapse lemma)**.** *For all $\mathfrak{X} \in \mathsf{Spa}(\mathtt{S4})$ and modal stable canonical rule $\mu(\mathfrak{F}, \mathfrak{D})$ with $\mathfrak{F} \in \mathsf{Spa}(\mathtt{S4})$, if $\mathfrak{X} \nvDash \mu(\mathfrak{F}, \mathfrak{D})$ then $\rho\mathfrak{X} \nvDash \eta(\rho\mathfrak{F}, \rho\mathfrak{D})$.*

Proof. Assume $\mathfrak{X} \nvDash \mu(\mathfrak{F}, \mathfrak{D})$. Then there is a continuous, relation preserving map $f : \mathfrak{X} \to \mathfrak{F}$ that satisfies the BDC for $\mathfrak{D}$. Consider the map $g : \rho\mathfrak{X} \to \rho\mathfrak{F}$ given by

$$g(\rho(x)) = \rho(f(x)).$$

Now $\rho(x) \leq \rho(y)$ implies Rxy, and since f is relation preserving also $Rf(x)f(y)$, which implies $\rho(f(x)) \leq \rho(f(y))$. So g is relation preserving. Furthermore, again because f is relation preserving we have that for any $U \subseteq F$, the set $f^{-1}(U)$ does not cut clusters, whence $g^{-1}(U) = \rho[f^{-1}(\rho^{-1}(U))]$ is clopen for any $U \subseteq \rho[F]$, as $\rho\mathfrak{X}$ has the quotient topology. Thus g is continuous. Let us check that g satisfies the BDC for $\rho\mathfrak{D}$. Assume that ${\uparrow}g(\rho(x)) \cap \rho[\mathfrak{d}] \neq \varnothing$ for $\mathfrak{d} \in \mathfrak{D}$. Then there is some

$\rho(y) \in \rho[F]$ with $\rho(f(x)) \leq \rho(y)$ and $\rho(y) \in \rho[\mathfrak{d}]$. By construction, wlog we may assume that $y \in \mathfrak{d}$. As ρ is relation reflecting it follows that $Rf(x)y$, and so we have that $R[f(x)] \cap \mathfrak{d} \neq \varnothing$. Since f satisfies the BDC for $\mathfrak{D}$ we conclude that $f[R[x]] \cap \mathfrak{d} \neq \varnothing$. So there is some $z \in X$ with Rxz and $f(z) \in \mathfrak{d}$. By definition, $\rho(f(z)) \in \rho[\mathfrak{d}]$. Hence we have shown that $\rho[f[R[x]]] \cap \rho[\mathfrak{d}] \neq \varnothing$, and so g indeed satisfies the BDC for $\mathfrak{D}$. □

We are now ready to prove the Dummett-Lemmon conjecture for rule systems.

Theorem 3.47 (Dummett-Lemmon conjecture for si-rule systems)**.** *For every si-rule system* $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$*, we have that* $\mathtt{L}$ *is Kripke complete iff* $\tau\mathtt{L}$ *is.*

Proof. ($\Rightarrow$) Let $\mathtt{L}$ be Kripke complete. Suppose that $\Gamma/\Delta \notin \tau\mathtt{L}$. Then there is $\mathfrak{X} \in \mathsf{Spa}(\tau\mathtt{L})$ such that $\mathfrak{X} \nvDash \Gamma/\Delta$. By Theorem 3.29, we may assume that $\Gamma/\Delta = \mu(\mathfrak{F}, \mathfrak{D})$ for $\mathfrak{F}$ a preorder. By the rule collapse lemma it follows that $\rho\mathfrak{X} \nvDash \eta(\rho\mathfrak{F}, \rho\mathfrak{D})$. Moreover, by Lemma 3.36 it follows that $\rho\mathfrak{X} \models \mathtt{L}$, and so we conclude $\eta(\rho\mathfrak{F}, \rho\mathfrak{D}) \notin \mathtt{L}$. Since $\mathtt{L}$ is Kripke complete, there is a si Kripke frame $\mathfrak{Y}$ such that $\mathfrak{Y} \nvDash \eta(\rho\mathfrak{F}, \rho\mathfrak{D})$. By Proposition 3.16, there is a stable map $f : \mathfrak{Y} \to \rho\mathfrak{F}$ satisfying the BDC for $\rho\mathfrak{D}$. Work in $\rho\mathfrak{F}$. For every $x \in \rho[F]$ look at $\rho^{-1}(x)$, let $k = |\rho^{-1}(x)|$ and enumerate $\rho^{-1}(x) = \{x_1, \ldots, x_k\}$. Now work in $\mathfrak{Y}$. For every $y \in f^{-1}(x)$ replace y with a k-cluster $y_1, \ldots, y_k$ and extend the relation R clusterwise: Ry_iz_j iff either $y = z$ or Ryz. Call the result $\mathfrak{Z}$. Clearly $\mathfrak{Z}$ is a Kripke frame, and moreover $\mathfrak{Z} \models \tau\mathtt{L}$, because $\rho\mathfrak{Z} \cong \mathfrak{Y}$. For convenience, identify $\rho\mathfrak{Z} = \mathfrak{Y}$. For every $x \in \rho[F]$ define a map $g_x : f^{-1}(x) \to \rho^{-1}(x)$ by setting $g_x(y_i) = x_i$ $(i \leq k)$. Finally, define $g : \mathfrak{Z} \to \mathfrak{F}$ by putting $g = \bigcup_{x \in \rho[F]} g_x$.

The map g is evidently well defined, surjective, and relation preserving. We claim that moreover, it satisfies the BDC for $\mathfrak{D}$. To see this, suppose that $R[g(y_i)] \cap \mathfrak{d} \neq \varnothing$ for some $\mathfrak{d} \in \mathfrak{D}$. Then there is $x_j \in F$ with $x_j \in \mathfrak{d}$ and $Rg(y_i)x_j$. By construction also $\rho(x_j) \in \rho[\mathfrak{d}]$ and $Rf(\rho(y_i))\rho(x_j)$. As f satisfies the BDC for $\rho\mathfrak{D}$ it follows that there is some $z \in Y$ such that $R\rho(y_i)z$ and $f(z) \in \rho[\mathfrak{d}]$. We may view z as $\rho(z_n)$ where $\rho^{-1}(f(z))$ has cardinality $k \geq n$. Surely Ry_iz_n. Furtheromre, since $f(z) \in \rho[\mathfrak{d}]$ there must be some $m \leq k$ such that $f(z)_m = g(z_m) \in \mathfrak{d}$. By construction Rz_nz_m and so in turn Ry_iz_m. This establishes that g indeed satisfies the BDC for $\mathfrak{D}$. Thus we have shown $\mathfrak{Z} \nvDash \mu(\mathfrak{F}, \mathfrak{D})$. It follows that $\tau\mathtt{L}$ is Kripke complete.

($\Leftarrow$) Assume that $\tau(\mathtt{L})$ is Kripke complete. Suppose that $\Gamma/\Delta \notin \mathtt{L}$. Then there is an Esakia space $\mathfrak{X}$ such that $\mathfrak{X} \nvDash \Gamma/\Delta$. Therefore $\sigma\mathfrak{X} \nvDash T(\Gamma/\Delta)$. Surely $\sigma\mathfrak{X} \models \tau\mathtt{L}$, so $T(\Gamma/\Delta) \notin \tau\mathtt{L}$ and thus there is a Kripke frame $\mathfrak{Y}$ such that $\mathfrak{Y} \models \tau\mathtt{L}$ and $\mathfrak{Y} \nvDash T(\Gamma/\Delta)$. But then $\rho\mathfrak{Y} \nvDash \Gamma/\Delta$. $\rho\mathfrak{Y}$ is a Kripke frame, and validates $\mathtt{L}$ by Lemma 3.36. Therefore we have shown that $\mathtt{L}$ is indeed Kripke complete. □

4. Tense Companions of Super Bi-intuitionistic Deductive Systems

We now apply the techniques presented in Section 3 to the study of tense companions of bi-superintuitionistic deductive systems. We begin by reviewing some preliminaries in Section 4.1. In Section 4.2 we develop tense and bi-superintuitionistic stable canonical rules, which generalise the modal and si stable canonical rules seen in Section 3.2. We then apply such rules to extend the results of Section 3.3 to the bi-superintuitionistic and tense setting in Section 4.3. Here we obtain a characterisation of the set of tense companions of a bi-superintuitionistic deductive system, and extensions of the Blok-Esakia theorem and of the Gödel-Dummett conjecture to the bi-superintuitionistic and tense setting (Section 4.3.3). These results were known for logics (cf. [57]), but are new for rule systems.

Besides the original results just mentioned, the main contribution of this section is showcasing the uniformity of our method across signatures. The majority of results in this section are obtained via straightforward generalisations of arguments already seen in Section 3. This is a major virtue of our approach, which Zakharyaschev and Jeřábek's canonical formulae and rules-based approach does not seem to share to the same extent (Section 4.2.3).

As in the case of modal companions, our techniques also yield axiomatic characterisations of the tense companion maps via stable canonical rules, as well as some results concerning the preservation of stability by the tense companion maps. These topics are discussed in [18, Sections 3.3.3, 3.3.4, 3.3.5].

4.1. **Bi-superintuitionistic and Tense Deductive Systems.** We begin by reviewing definitions and basic facts concerning the structures dealt with in this section.

4.1.1. *Bi-superintuitionistic Deductive Systems, bi-Heyting Algebras, and bi-Esakia Spaces.* We work in the *bi-superintuitionistic signature*,

$$bsi := \{\wedge, \vee, \to, \leftarrow, \bot, \top\}.$$

The set Frm_{bsi} of bi-superintuitionistic (bsi) formulae is defined recursively as follows.

$$\varphi ::= p \,|\, \bot \,|\, \top \,|\, \varphi \wedge \varphi \,|\, \varphi \vee \varphi \,|\, \varphi \to \varphi \,|\, \varphi \leftarrow \varphi$$

We let $\llcorner\varphi := \varphi \leftarrow \top$ and $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$. The *bi-intuitionistic propositional calculus* $\mathtt{bi\text{-}IPC}$ is defined as the least logic over Frm_{bsi} containing $\mathtt{IPC}$, containing the axioms

$$p \to (q \vee (q \leftarrow p)) \qquad (q \leftarrow p) \to \llcorner(p \to q)$$
$$(r \leftarrow (q \leftarrow p)) \to ((p \vee q) \leftarrow p) \qquad \neg(p \leftarrow q) \to (p \to q)$$
$$\neg\llcorner(p \leftarrow p)$$

and such that if $\varphi, \varphi \to \psi \in \mathtt{bi\text{-}IPC}$ then $\psi \in \mathtt{bi\text{-}IPC}$, and if $\varphi \in \mathtt{bi\text{-}IPC}$ then $\neg\llcorner\varphi \in \mathtt{bi\text{-}IPC}$. The logic $\mathtt{bi\text{-}IPC}$ was introduced and extensively studied by Rauszer [44, 45, 46]. It was also investigated by Esakia [23], and more recently by Goré [30].

Definition 4.1. A *bsi-logic* is a logic $\mathtt{L}$ over Frm_{bsi} containing $\mathtt{bi\text{-}IPC}$ and satisfying the following conditions:

- If $\varphi, \varphi \to \psi \in \mathtt{L}$ then $\psi \in \mathtt{L}$ (MP);
- If $\varphi \in \mathtt{L}$ then $\neg\llcorner\varphi \in \mathtt{L}$ (DN).

A *bsi-rule system* is a rule system $\mathtt{L}$ over Frm_{bsi} satisfying the following conditions:

- $\varphi, \varphi \to \psi/\psi \in \mathtt{L}$ (MP-R);
- $\varphi/\neg\llcorner\varphi \in \mathtt{L}$ (DN-R);
- $/\varphi \in \mathtt{L}$ for every $\varphi \in \mathtt{bi\text{-}IPC}$.

If $\mathtt{L}$ is a bsi-logic let $\mathbf{Ext}(\mathtt{L})$ be the set of bsi-logics containing $\mathtt{L}$, and similarly for bsi-rule systems. Then $\mathbf{Ext}(\mathtt{bi\text{-}IPC})$ is the set of all bsi-logics. It is easy to see that $\mathbf{Ext}(\mathtt{bi\text{-}IPC})$ carries a complete lattice, with $\oplus_{\mathbf{Ext}(\mathtt{bi\text{-}IPC})}$ as join and intersection as meet. Observe that for every $\mathtt{L} \in \mathbf{Ext}(\mathtt{bi\text{-}IPC})$ there is a least bsi-rule system containing $/\varphi$ for each $\varphi \in \mathtt{L}$, which we denote by $\mathtt{L_R}$. Then $\mathtt{bi\text{-}IPC_R}$ is the least bsi-rule system and $\mathbf{Ext}(\mathtt{bi\text{-}IPC_R})$ is the set of all bsi-rule systems. Again, it is not hard to verify that $\mathbf{Ext}(\mathtt{bi\text{-}IPC_R})$ forms a complete lattice with $\oplus_{\mathbf{Ext}(\mathtt{bi\text{-}IPC_R})}$ as join and intersection as meet. Henceforth we write both $\oplus_{\mathbf{Ext}(\mathtt{bi\text{-}IPC})}$ and $\oplus_{\mathbf{Ext}(\mathtt{bi\text{-}IPC_R})}$ simply as $\oplus$, leaving context to clarify any ambiguity.

We generalise Proposition 3.2 to the bsi setting.

Proposition 4.2. *The mappings* $(\cdot)_{\mathtt{R}}$ *and* $\mathsf{Taut}(\cdot)$ *are mutually inverse complete lattice isomorphisms between* $\mathbf{Ext}(\mathtt{bi\text{-}IPC})$ *and the sublattice of* $\mathbf{Ext}(\mathtt{bi\text{-}IPC_R})$ *consisting of all bsi-rule systems* $\mathtt{L}$ *such that* $\mathsf{Taut}(\mathtt{L})_{\mathtt{R}} = \mathtt{L}$.

A *bi-Heyting algebra* is a tuple $\mathfrak{H} = (H, \wedge, \vee, \to, \leftarrow, 0, 1)$ such that the $\leftarrow$-free reduct of $\mathfrak{H}$ is a Heyting algebras, and such that for all $a, b, c \in H$ we have

$$a \leftarrow b \leq c \iff a \leq b \vee c.$$

Bi-Heyting algebras are discussed at length in [44, 45, 46] and more recently in [53, 43]. Let $\mathsf{bi\text{-}HA}$ denote the class of all bi-Heyting algebras. By Theorem 2.4, $\mathsf{bi\text{-}HA}$ is a variety.

Let $\mathfrak{L} = (L, \wedge, \vee, 0, 1)$ be a bounded lattice. The *order dual* of $\mathfrak{L}$ is the lattice $\bar{\mathfrak{L}} = (L, \vee, \wedge, 1, 0)$, where $\vee$ is viewed as the meet operation and $\wedge$ as the join operation. We have the following elementary but important fact.

Proposition 4.3 (Order duality principle for bi-Heyting algebras)**.** *For every bi-Heyting algebra* $\mathfrak{H}$*, the order dual* $\bar{\mathfrak{H}}$ *of* $\mathfrak{H}$ *is a Heyting algebra, where implication is defined, for all* $a, b \in H$*, by*

$$a \leftarrow b := \bigwedge \{c \in H : a \leq b \vee c\}.$$

This observation can be leveraged to establish a number of properties about bi-Heyting algebras via straightforward adaptations of the theory of Heyting algebras. We shall see numerous examples of this strategy in this section.

We write $\mathbf{Var}(\mathsf{bi\text{-}HA})$ and $\mathbf{Uni}(\mathsf{bi\text{-}HA})$ respectively for the lattice of subvarieties and of universal subclasses of $\mathsf{bi\text{-}HA}$. The following result may be proved via the same techniques used to prove Theorem 3.3. A recent self-contained proof of Item 1 may be found in [43, Theorem 2.8.3].

Theorem 4.4. *The following maps are pairs of mutually inverse dual isomorphisms:*

(1) $\mathsf{Alg} : \mathbf{Ext}(\mathtt{bi\text{-}IPC}) \to \mathbf{Var}(\mathsf{bi\text{-}HA})$ *and* $\mathsf{Th} : \mathbf{Var}(\mathsf{bi\text{-}HA}) \to \mathbf{Ext}(\mathtt{bi\text{-}IPC})$*;*
(2) $\mathsf{Alg} : \mathbf{Ext}(\mathtt{bi\text{-}IPC_R}) \to \mathbf{Uni}(\mathsf{bi\text{-}HA})$ *and* $\mathsf{ThR} : \mathbf{Uni}(\mathsf{bi\text{-}HA}) \to \mathbf{Ext}(\mathtt{bi\text{-}IPC_R})$.

Corollary 4.5. *Every bsi-logic* (*resp. bsi-rule system*) *is complete with respect to some variety* (*resp. universal class*) *of bi-Heyting algebras.*

A *bi-Esakia space* is an Esakia space $\mathfrak{X} = (X, \leq, \mathcal{O})$, satisfying the following additional conditions:

- ${\downarrow}x$ is closed for every $x \in X$;
- ${\uparrow}[U] \in \mathsf{Clop}(\mathfrak{X})$ whenever $U \in \mathsf{Clop}(\mathfrak{X})$.

Bi-Esakia spaces were introduced by Esakia [23]. We let $\mathsf{bi\text{-}Esa}$ denote the class of all bi-Esakia spaces. For $\mathfrak{X} \in \mathsf{bi\text{-}Esa}$, we write $\mathsf{ClopDown}(\mathfrak{X})$ for the set of clopen downsets in $\mathfrak{X}$. If $\mathfrak{X}, \mathfrak{Y} \in \mathsf{bi\text{-}Esa}$, a map $h : \mathfrak{X} \to \mathfrak{Y}$ is called a *bounded morphism* if for all $x, y \in X$, we have that $x \leq y$ implies that $f(x) \leq f(y)$, and moreover:

- $h(x) \leq y$ implies that there is $z \in X$ with $x \leq z$ and $h(z) = y$;

- $h(x) \geq y$ implies that there is $z \in X$ with $x \geq z$ and $h(z) = y$.

If $\mathfrak{X} = (X, \leq, \mathcal{O})$ is an Esakia space, the *order dual* $\bar{\mathfrak{X}}$ of $\mathfrak{X}$ is the structure $\mathfrak{X} = (X, \geq, \mathcal{O})$, where $\geq$ is the converse of $\leq$. The algebraic order duality principle of Proposition 4.3 has the following geometric counterpart.

Proposition 4.6. *For every bi-Esakia space $\mathfrak{X}$, the order dual $\bar{\mathfrak{X}}$ of $\mathfrak{X}$ is an Esakia space.*

As in the case of algebras, a number of results from the theory of Esakia spaces can be transferred smoothly to bi-Esakia spaces in virtue of this fact. For example, we may generalise Proposition 3.5 to the following result.

Proposition 4.7. *Let $\mathfrak{X} \in \mathsf{bi\text{-}Esa}$. Then for all $x, y \in X$ we have:*

(1) If $x \not\leq y$ then there is $U \in \mathsf{ClopUp}(\mathfrak{X})$ such that $x \in U$ and $y \notin U$;
(2) If $y \not\leq x$ then there is $U \in \mathsf{ClopDown}(\mathfrak{X})$ such that $x \in U$ and $y \notin U$.

Proof. (1) is just Proposition 3.5, whereas (2) follows from (1) and the order-duality principle. ☐

A *valuation* on a bi-Esakia space space $\mathfrak{X}$ is a map $V : Prop \to \mathsf{ClopUp}(\mathfrak{X}) \cup \mathsf{ClopDown}(\mathfrak{X})$. Bsi formulae are interpreted over bi-Esakia spaces the same way si formulae are interpreted over Esakia space, except for the following additional clause for co-implication (here $\mathfrak{X} \in \mathsf{bi\text{-}Esa}$, $x \in X$ and V is a valuation on $\mathfrak{X}$).

$$\mathfrak{X}, V, x \models \varphi \leftarrow \psi \iff \text{there is } y \in {\downarrow}x : \mathfrak{X}, V, x \models \varphi \text{ and } \mathfrak{X}, V, x \nvDash \psi$$

It is known that the category of bi-Heyting algebras with corresponding homomorphisms is dually equivalent to the category of bi-Esakia spaces with continuous bounded morphisms. This result generalizes Esakia duality, and is proved in [23]. We denote the bi-Esakia space dual to a bi-Heyting algebra $\mathfrak{H}$ as $\mathfrak{H}_*$, and the bi-Heyting algebra dual to a bi-Esakia space $\mathfrak{X}$ as $\mathfrak{X}^*$.

4.1.2. *Tense Deductive Systems, Tense Algebras, and Tense Spaces.* We now work in the *tense signature*,

$$ten := \{\wedge, \vee, \neg, \Box_F, \Diamond_P, \bot, \top\}.$$

We prefer this signature to one with two primitive boxes to strengthen the connection between bi-Heyting coimplication and backwards looking modalities. As usual, we write $\Diamond_F = \neg\Box_F\neg$ and $\Box_P = \neg\Diamond_P\neg$. The set Frm_{ten} of *tense formulae* is defined recursively as follows:

$$\varphi ::= p \,|\, \bot \,|\, \top \,|\, \varphi \wedge \varphi \,|\, \varphi \vee \varphi \,|\, \Box_F\varphi \,|\, \Diamond_P\varphi.$$

We introduce *tense deductive systems.* Good references on tense logics include [10, Ch. 1, Ch. 4] and [28]. Tense rule systems have not received much attention in the literature.

Definition 4.8. A *(normal) tense logic* is a logic $\mathtt{M}$ over Frm_{ten} satisfying the following conditions:

(1) $\mathtt{S4}_{\Box_F}, \mathtt{S4}_{\Diamond_P} \subseteq \mathtt{M}$, where $\mathtt{S4}_{\heartsuit}$ is the normal modal logic $\mathtt{S4}$ formulated in the modal signature with modal operator $\heartsuit \in \{\Box_F, \Diamond_P\}$;
(2) $\varphi \to \Box_F\Diamond_P\varphi \in \mathtt{M}$;
(3) $\varphi \to \psi, \varphi \in \mathtt{M}$ implies $\psi \in \mathtt{M}$ (MP);
(4) $\varphi \in \mathtt{M}$ implies $\Box_F\varphi \in \mathtt{M}$ (NEC$_F$);

(5) $\varphi \in \mathtt{M}$ implies $\Box_P\varphi \in \mathtt{M}$ (NEC$_P$);

We let $\mathtt{S4.t}$ denote the least normal tense logic. A *(normal) tense rule system* is a rule system $\mathtt{M}$ over Frm_{ten} satisfying the following requirements:

(1) $\varphi, \varphi \to \psi/\psi \in \mathtt{M}$ (MP-R);
(2) $\varphi/\Box_F\varphi \in \mathtt{M}$ (NEC$_F$-R);
(3) $\varphi/\Box_P\varphi \in \mathtt{M}$ (NEC$_P$-R);
(4) $/\varphi \in \mathtt{M}$ whenever $\varphi \in \mathtt{S4.t}$.

We note that, for convenience, we are using a somewhat non-standard notion of a tense deductive system by requiring that tense deductive system contain $\mathtt{S4}$. It is more customary to require only that tense deductive system contain $\mathtt{K}$.

If $\mathtt{M}$ is a tense logic let $\mathbf{NExt}(\mathtt{M})$ be the set of normal tense logics containing $\mathtt{M}$, and similarly for tense rule systems. Then $\mathbf{NExt}(\mathtt{S4.t})$ is the set of all tense logics. It is easily checked that $\mathbf{NExt}(\mathtt{S4.t})$ is a complete lattice, with $\oplus_{\mathbf{NExt}(\mathtt{S4.t})}$ as join and intersection as meet. Note that for every $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t})$ there is always a least tense rule system containing $/\varphi$ for each $\varphi \in \mathtt{M}$, which we denote by $\mathtt{M_R}$. Then $\mathtt{S4.t_R}$ is the least tense rule system and $\mathbf{NExt}(\mathtt{S4.t_R})$ is the set of all tense rule systems. Again, one can easily verify that $\mathbf{NExt}(\mathtt{S4.t_R})$ forms a complete lattice with $\oplus_{\mathbf{NExt}(\mathtt{S4.t_R})}$ as join and intersection as meet. As usual, we write both $\oplus_{\mathbf{NExt}(\mathtt{S4.t})}$ and $\oplus_{\mathbf{NExt}(\mathtt{S4.t_R})}$ simply as $\oplus$.

We have the following tense counterpart of Proposition 4.2.

Proposition 4.9. *The mappings* $(\cdot)_\mathtt{R}$ *and* $\mathsf{Taut}(\cdot)$ *are mutually inverse complete lattice isomorphisms between* $\mathbf{NExt}(\mathtt{S4.t})$ *and the sublattice of* $\mathbf{NExt}(\mathtt{S4.t_R})$ *consisting of all si-rule systems* $\mathtt{L}$ *such that* $\mathsf{Taut}(\mathtt{L})_\mathtt{R} = \mathtt{L}$.

A *tense algebra* is a structure $\mathfrak{A} = (A, \wedge, \vee, \neg, \Box_F, \Diamond_P, 0, 1)$, such that both the $\Box_F$-free and the $\Diamond_P$-free reducts of $\mathfrak{A}$ are closure algebras, and $\Box_F, \Diamond_P$ form a residual pair, that is, for all $a, b \in A$ we have the following identity:

$$\Diamond_P a \leq b \iff a \leq \Box_F y.$$

Tense algebras are extensively discussed in, e.g., [35] and [55, Section 8.1]. We let Ten denote the class of tense algebras. It is well known that Ten is equationally definable (see, e.g., [55, Proposition 8.5]), and hence is a variety by Theorem 2.4. We let $\mathbf{Var}(\mathsf{Ten})$ and $\mathbf{Uni}(\mathsf{Ten})$ be the lattice of subvarieties and of universal subclasses of Ten respectively. The following result can be obtained by similar techniques as Theorem 3.8.

Theorem 4.10. *The following maps are pairs of mutually inverse dual isomorphisms:*

(1) $\mathsf{Alg} : \mathbf{NExt}(\mathtt{S4.t}) \to \mathbf{Var}(\mathsf{Ten})$ *and* $\mathsf{Th} : \mathbf{Var}(\mathsf{Ten}) \to \mathbf{NExt}(\mathtt{S4.t})$*;*
(2) $\mathsf{Alg} : \mathbf{NExt}(\mathtt{S4.t_R}) \to \mathbf{Uni}(\mathsf{Ten})$ *and* $\mathsf{ThR} : \mathbf{Uni}(\mathsf{Ten}) \to \mathbf{NExt}(\mathtt{S4.t_R})$.

Corollary 4.11. *Every tense logic* (*resp. tense rule system*) *is complete with respect to some variety* (*resp. universal class*) *of tense algebras.*

A *tense space* is an $\mathtt{S4}$-modal space $\mathfrak{X} = (X, R, \mathcal{O})$, satisfying the following additional conditions:

- $R^{-1}(x)$ is closed for every $x \in X$;
- $R[U] \in \mathsf{Clop}(\mathfrak{X})$ whenever $U \in \mathsf{Clop}(\mathfrak{X})$.

It should be clear from the above definition that tense spaces, like bi-Esakia spaces, also satisfy an order-duality principle.

Proposition 4.12. *For every tense space* $\mathfrak{X} = (X, R, \mathcal{O})$*, its* order dual $\bar{\mathfrak{X}} = (X, \breve{R}, \mathcal{O})$*, where* $\breve{R}$ *is the converse of* R*, is an* $\mathtt{S4}$*-modal space.*

If $\mathfrak{X}, \mathfrak{Y}$ are tense spaces, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is called a *bounded morphism* if for all $x, y \in X$, if Rxy then $Rf(x)f(y)$, and moreover for all $x \in X$ and $y \in Y$ the following conditions hold:

- If $Rf(x)y$ then there is $z \in X$ such that Rxz and $f(z) = y$;
- If $Ryf(x)$ then there is $z \in X$ such that Rzx and $f(z) = y$.

A *valuation* on a tense space $\mathfrak{X}$ is a map $V : Prop \to \mathsf{Clop}(\mathfrak{X})$. The geometrical semantics of tense logics and rule systems over tense spaces is a routine generalisation of the semantics of modal logics and rule systems on modal spaces, using R to interpret $\Box_F$ and $\breve{R}$ to interpret $\Box_P$. We list some important properties of tense spaces, which are obtained straightforwardly from Proposition 3.10 and the order-duality principle.

Proposition 4.13. *Let* $\mathfrak{X} \in \mathsf{Spa}(\mathtt{S4.t})$ *and* $U \in \mathsf{Clop}(\mathfrak{X})$*. Then the following conditions hold:*

1. *The sets* $max_R(U)$*,* $min_R(U)$ *are closed;*
2. *If* $x \in U$ *then there is* $y \in qmax_R(U)$ *such that* Rxy*, and there is* $z \in qmin_R(U)$ *such that* Rzx

As a straightforward extension of the duality between modal algebras and modal spaces, one can prove that the category of tense algebras with homomorphisms is dually equivalent to the category of tense spaces with continuous bounded morphisms. We denote the tense space dual to a tense algebra $\mathfrak{A}$ as $\mathfrak{A}_*$, and the tense algebra dual to an tense space $\mathfrak{X}$ as $\mathfrak{X}^*$.

We will pay particular attention to tense algebras and spaces validating the tense logic $\mathtt{GRZ.T}$ below.

$$\begin{aligned}\mathtt{GRZ.T} := \mathtt{S4.t} &\oplus \Box_F(\Box_F(p \to \Box_F p) \to p) \to p \\ &\oplus p \to \Diamond_P(p \wedge \neg\Diamond_P(\Diamond_P p \wedge \neg p)).\end{aligned}$$

We name this logic $\mathtt{GRZ.T}$ rather than $\mathtt{GRZ.t}$ to emphasize that the $\mathtt{GRZ}$-axiom is required for both operators rather than just for $\Box_F$. We let $\mathsf{GRZ.T} := \mathsf{Alg}(\mathtt{GRZ.T})$. Clearly, for any $\mathfrak{A} \in \mathsf{Ten}$ we have $\mathfrak{A} \in \mathsf{GRZ.T}$ iff every $a \in A$ satisfies both the inequalties

$$\Box_F(\Box_F(a \to \Box_F a) \to a) \leq a,$$
$$a \leq \Diamond_P(a \wedge \neg\Diamond_P(\Diamond_P a \wedge \neg a)).$$

The following proposition is a counterpart to Proposition 3.11, and is proved straightforwardly using the latter and the order-duality principle.

Proposition 4.14. *For every* $\mathtt{GRZ}$*-space* $\mathfrak{X}$ *and* $U \in \mathsf{Clop}(\mathfrak{X})$*, the following hold:*

1. $qmax_R(U) \subseteq max_R(U)$*, and* $qmin_R(U) \subseteq min_R(U)$*;*
2. *The sets* $max_R(U)$ *and* $min_R(U)$ *is closed;*
3. *For every* $x \in U$ *there are* $y \in pas_R(U)$ *such that* Rxy*, and* $z \in pas_{\breve{R}}(U)$ *such that* Rzx*;*
4. $max_R(U) \subseteq pas_R(U)$ *and* $min_R(U) \subseteq pas_{\breve{R}}(U)$*.*

Recall that for $\mathfrak{X}$ a GRZ.T-space, a set $U \subseteq X$ is said to *cut* a cluster $C \subseteq X$ when both $U \cap C \neq \varnothing$ and $U \smallsetminus C \neq \varnothing$. As a consequence of Item 4 in Proposition 4.14 above, we obtain in particular that in any GRZ.T-space $\mathfrak{X}$, no cluster $C \subseteq X$ can be cut by either of $max_R(U), pas_R(U), min_R(U), pas_{\breve{R}}(U)$ for any $U \in \mathsf{Clop}(\mathfrak{X})$.

4.2. **Stable Canonical Rules for Bi-superintuitionistic and Tense Rule Systems.** In this section we generalise the si and modal stable canonical rules from Section 3.2 to the bsi and tense setting respectively. While bsi and tense stable canonical rules are not discussed in existing literature, the differences between their theory and that of si and modal stable canonical rules are few and inessential. In particular, all proofs of results in this sections are straightforward adaptations of corresponding results in Section 3.2, which is why we omit most of them.

4.2.1. *Bi-superintuitionistic Case.* We begin by defining bsi stable canonical rules.

Definition 4.15. Let $\mathfrak{H} \in \mathsf{bi\text{-}HA}$ be finite and $D^{\to}, D^{\leftarrow} \subseteq A \times A$. For every $a \in H$ introduce a fresh propositional variable p_a. The *bsi stable canonical rule* of $(\mathfrak{H}, D^{\to}, D^{\leftarrow})$, is defined as the rule $\eta_B(\mathfrak{H}, D^{\to}, D^{\leftarrow}) = \Gamma/\Delta$, where

$$\begin{aligned}\Gamma =&\{p_0 \leftrightarrow 0\} \cup \{p_1 \leftrightarrow 1\} \cup \\ &\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in H\} \cup \{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in H\} \cup \\ &\{p_{a\to b} \leftrightarrow p_a \to p_b : (a,b) \in D^{\to}\} \cup \{p_{a\leftarrow b} \leftrightarrow p_a \leftarrow p_b : (a,b) \in D^{\leftarrow}\} \\ \Delta =& \{p_a \leftrightarrow p_b : a, b \in H \text{ with } a \neq b\}.\end{aligned}$$

The notion of a stable map between bi-Heyting algebras is defined exactly as in the Heyting case, i.e., stable maps are simply bounded lattice homomorphisms. We note that for any stable map $h : \mathfrak{H} \to \mathfrak{K}$ with $\mathfrak{H}, \mathfrak{K} \in \mathsf{bi\text{-}HA}$, for any $a \in H$ we also have

$$h(a \leftarrow b) \geq h(a) \leftarrow h(b).$$

Indeed, this is obvious in view of the order-duality principle. If $D \subseteq H \times H$ and $\heartsuit \in \{\to, \leftarrow\}$, we say that h satisfies the $\heartsuit$-*bounded domain condition* (BDC$^{\heartsuit}$) for D if $h(a\heartsuit b) = h(a)\heartsuit h(b)$ for every $(a, b) \in D$. If $D^{\to}, D^{\leftarrow} \subseteq H \times H$, for brevity we say that h satisfies the BDC for $(D^{\to}, D^{\leftarrow})$ to mean that h satisfies the BDC$^{\to}$ for $D^{\to}$ and the BDC$^{\leftarrow}$ for $D^{\leftarrow}$.

The next two results characterise algebraic refutation conditions for bsi stable canonical rules.

Proposition 4.16. *For all finite $\mathfrak{H} \in \mathsf{bi\text{-}HA}$ and $D^{\to}, D^{\leftarrow} \subseteq H \times H$, we have $\mathfrak{H} \nvDash \eta_B(\mathfrak{H}, D^{\to}, D^{\leftarrow})$.*

Proposition 4.17. *For every bsi stable canonical rule $\eta_B(\mathfrak{H}, D^{\to}, D^{\leftarrow})$ and every $\mathfrak{K} \in \mathsf{bi\text{-}HA}$, we have $\mathfrak{K} \nvDash \eta_B(\mathfrak{H}, D^{\to}, D^{\leftarrow})$ iff there is a stable embedding $h : \mathfrak{H} \to \mathfrak{K}$ satisfying the BDC for $(D^{\to}, D^{\leftarrow})$.*

We now characterise geometric refutation conditions of bsi stable canonical rules on bi-Esakia spaces. Since bi-Esakia spaces are Esakia spaces, the notion of a stable map applies. Let $\mathfrak{X}, \mathfrak{Y} \in \mathsf{bi\text{-}Esa}$ and $\mathfrak{d} \subseteq Y$. A stable map $f : \mathfrak{X} \to \mathfrak{Y}$ is said to satisfy

- The BDC$^{\to}$ for $\mathfrak{d}$ if for all $x \in X$ we have
$$\uparrow f(x) \cap \mathfrak{d} \neq \varnothing \Rightarrow f[\uparrow x] \cap \mathfrak{d} \neq \varnothing;$$

- The BDC$^{\leftarrow}$ for $\mathfrak{d}$ if for all $x \in X$ we have

$$\downarrow f(x) \cap \mathfrak{d} \neq \varnothing \Rightarrow f[\downarrow x] \cap \mathfrak{d} \neq \varnothing.$$

If $\mathfrak{D} \subseteq \wp(Y)$, we say that f satisfies the BDC$^{\heartsuit}$ for $\mathfrak{D}$ when it does for each $\mathfrak{d} \in \mathfrak{D}$, where $\heartsuit \in \{\rightarrow, \leftarrow\}$. Given $\mathfrak{D}^{\rightarrow}, \mathfrak{D}^{\leftarrow} \in \wp(Y)$ we write that f satisfies the BDC for $(\mathfrak{D}^{\rightarrow}, \mathfrak{D}^{\leftarrow})$ if f satisfies the BDC$^{\rightarrow}$ for $\mathfrak{D}^{\rightarrow}$ and the BDC$^{\leftarrow}$ for $\mathfrak{D}^{\leftarrow}$. Finally, if $\eta_B(\mathfrak{H}, D^{\rightarrow}, D^{\leftarrow})$ is a bsi stable canonical rule consider $\mathfrak{X} := \mathfrak{H}_*$ and let

$$\mathfrak{D}^{\heartsuit} := \{\mathfrak{d}^{\heartsuit}_{(a,b)} : (a, b) \in D^{\heartsuit}\}$$

where

$$\mathfrak{d}^{\heartsuit}_{(a,b)} := \beta(a) \smallsetminus \beta(b)$$

for $\heartsuit \in \{\rightarrow, \leftarrow\}$.

Proposition 4.18. *For any bi-Esakia space $\mathfrak{X}$ and any bsi stable canonical rule $\eta_B(\mathfrak{H}, D^{\rightarrow}, D^{\leftarrow})$, we have $\mathfrak{X} \not\models \eta_B(\mathfrak{H}, D^{\rightarrow}, D^{\leftarrow})$ iff there is a continuous stable surjection $f : \mathfrak{X} \to \mathfrak{H}_*$ satisfying the BDC for $(\mathfrak{D}^{\rightarrow}, \mathfrak{D}^{\leftarrow})$ defined as above.*

In view of Proposition 4.18, in geometric settings we prefer to write a bsi stable canonical rule $\eta_B(\mathfrak{H}, D^{\rightarrow}, D^{\leftarrow})$ as $\eta_B(\mathfrak{H}_*, \mathfrak{D}^{\rightarrow}, \mathfrak{D}^{\leftarrow})$.

We now elucidate the notion of filtration for bi-Heyting algebras presupposed by our bsi stable canonical rules.

Definition 4.19. Let $\mathfrak{H}$ be a bi-Heyting algebra, V a valuation on $\mathfrak{H}$, and Θ a finite, subformula closed set of formulae. A (finite) model $(\mathfrak{K}', V')$ is called a *(finite) filtration of* $(\mathfrak{H}, V)$ *through* Θ if the following hold:

(1) $\mathfrak{K}' = (\mathfrak{K}, \rightarrow, \leftarrow)$, where $\mathfrak{K}$ is the bounded sublattice of $\mathfrak{H}$ generated by $\bar{V}[\Theta]$;
(2) $V(p) = V'(p)$ for every propositional variable $p \in \Theta$;
(3) The inclusion $\subseteq : \mathfrak{K}' \to \mathfrak{H}$ is a stable embedding satisfying the BDC for $(D^{\rightarrow}, D^{\leftarrow})$, where

$$D^{\heartsuit} := \{(\bar{V}(\varphi), \bar{V}(\psi)) : \varphi \heartsuit \psi \in \Theta\}$$

for $\heartsuit \in \{\rightarrow, \leftarrow\}$.

Theorem 4.20 (Filtration theorem for bi-Heyting algebras)**.** *Let $\mathfrak{H} \in \mathsf{bi\text{-}HA}$ be a bi-Heyting algebra, V a valuation on $\mathfrak{H}$, and Θ a a finite, subformula closed set of formulae. If $(\mathfrak{K}', V')$ is a filtration of $(\mathfrak{H}, V)$ through Θ then for every $\varphi \in \Theta$ we have*

$$\bar{V}(\varphi) = \bar{V}'(\varphi).$$

Consequently, for every bsi rule Γ/Δ such that $\gamma, \delta \in \Theta$ for each $\gamma \in \Gamma$ and $\delta \in \Delta$ we have

$$\mathfrak{H}, V \models \Gamma/\Delta \iff \mathfrak{K}', V' \models \Gamma/\Delta.$$

The next lemma is a counterpart to Lemma 3.19.

Lemma 4.21. *For every bsi rule Γ/Δ there is a finite set Ξ of bsi stable canonical rules such that for any $\mathfrak{K} \in \mathsf{bi\text{-}HA}$ we have that $\mathfrak{K} \not\models \Gamma/\Delta$ iff there is $\eta_B(\mathfrak{H}, D^{\rightarrow}, D^{\leftarrow}) \in \Xi$ such that $\mathfrak{K} \not\models \eta_B(\mathfrak{H}, D^{\rightarrow}, D^{\leftarrow})$.*

Proof. The proof is a straightforward generalisation of the proof of Lemma 3.19, using the fact that every finite bounded distributive lattice $\mathfrak{J}$ may be expanded to a bi-Heyting algebra $\mathfrak{J}' = (\mathfrak{J}, \rightsquigarrow, \leftsquigarrow)$ by setting:

$$a \rightsquigarrow b := \bigvee\{c \in J : a \wedge b \leq c\}$$
$$a \leftsquigarrow b := \bigwedge\{c \in J : a \leq b \vee c\}.$$

□

Reasoning as in the proof of Theorem 3.20 we obtain the following axiomatisation result.

Theorem 4.22. *Every bsi-rule system* $\mathtt{L} \in \mathbf{Ext}(\mathtt{bi\text{-}IPC_R})$ *is axiomatisable over* $\mathtt{bi\text{-}IPC_R}$ *by some set of bsi stable canonical rules.*

4.2.2. *Tense Case.* We now turn to tense stable canonical rules.

Definition 4.23. Let $\mathfrak{A} \in \mathsf{Ten}$ be finite and $D^{\Box_F}, D^{\Diamond_P} \subseteq A$. For every $a \in A$ introduce a fresh propositional variable p_a. The *tense stable canonical rule* of $(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$, is defined as the rule $\mu_T(\mathfrak{H}, D^{\Box_F}, D^{\Diamond_P}) = \Gamma/\Delta$, where

$$\begin{aligned}\Gamma =&\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \cup \{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\}\cup\\ &\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\}\cup\\ &\{\Box_F p_a \to p_{\Box_F a} : a \in A\} \cup \{p_{\Diamond_P a} \to \Diamond_P p_a : a \in A\}\cup\\ &\{p_{\Box_F a} \to \Box_F p_a : a \in D^{\Box_F}\} \cup \{\Diamond_P p_a \to p_{\Diamond_P a} : a \in D^{\Diamond_P}\}\\ \Delta =&\{p_a : a \in A \smallsetminus \{1\}\}.\end{aligned}$$

If $\mathfrak{A}, \mathfrak{B} \in \mathsf{MA}$ are tense algebras, a map $h : \mathfrak{A} \to \mathfrak{B}$ is called *stable* if for every $a \in A$ the following conditions hold:

$$h(\Box_F a) \leq \Box_F h(a) \qquad \Diamond_P h(a) \leq h(\Diamond_P a).$$

If $D \subseteq A$ and $\heartsuit \in \{\Box_F, \Diamond_P\}$, we say that h satisfies the $\heartsuit$*-bounded domain condition* (BDC$^\heartsuit$) for D if $h(\heartsuit a) = \heartsuit h(a)$ for every $a \in D$. If $D^{\Box_F}, D^{\Diamond_P} \subseteq A$, for brevity we say that h satisfies the BDC for $(D^{\Box_F}, D^{\Diamond_P})$ to mean that h satisfies the BDC$^{\Box_F}$ for $D^{\Box_F}$ and the BDC$^{\Diamond_P}$ for $D^{\Diamond_P}$.

We outline algebraic refutation conditions for tense stable canonical rules.

Proposition 4.24. *For all finite* $\mathfrak{A} \in \mathsf{Ten}$ *and* $D^{\Box_F}, D^{\Diamond_P} \subseteq A$*, we have* $\mathfrak{A} \nvDash \mu_T(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$.

Proposition 4.25. *For every tense stable canonical rule* $\mu_T(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$ *and any* $\mathfrak{B} \in \mathsf{Ten}$*, we have* $\mathfrak{B} \nvDash \mu_T(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$ *iff there is a stable embedding* $h : \mathfrak{A} \to \mathfrak{B}$ *satisfying the BDC for* $(D^{\Box_F}, D^{\Diamond_P})$.

Tense spaces are modal spaces, therefore the notion of a stable map applies. Let $\mathfrak{X}, \mathfrak{Y}$ be tense spaces. and $\mathfrak{d} \subseteq Y$. A stable map $f : \mathfrak{X} \to \mathfrak{Y}$ is said to satisfy

- The BDC$^{\Box_F}$ for $\mathfrak{d}$ if for all $x \in X$ we have
$$R[f(x)] \cap \mathfrak{d} \neq \varnothing \Rightarrow f[R[x]] \cap \mathfrak{d} \neq \varnothing;$$
- The BDC$^{\Diamond_P}$ for $\mathfrak{d}$ if for all $x \in X$ we have
$$\breve{R}[f(x)] \cap \mathfrak{d} \neq \varnothing \Rightarrow f[\breve{R}[x]] \cap \mathfrak{d} \neq \varnothing.$$

If $\mathfrak{D} \subseteq \wp(Y)$, we say that f satisfies the BDC$^\heartsuit$ for $\mathfrak{D}$ when it does for each $\mathfrak{d} \in \mathfrak{D}$, where $\heartsuit \in \{\Box_F, \Diamond_P\}$. Given $\mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P} \in \wp(Y)$ we write that f satisfies the BDC for $(\mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$ if f satisfies the BDC$^{\Box_F}$ for $\mathfrak{D}^{\Box_F}$ and the BDC$^{\Diamond_P}$ for $\mathfrak{D}^{\Diamond_P}$. Finally, if $\mu_T(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$ is a tense stable canonical rule consider $\mathfrak{X} := \mathfrak{A}_*$ and for $\heartsuit \in \{\Box_F, \Diamond_P\}$ let

$$\mathfrak{D}^\heartsuit := \{\mathfrak{d}^\heartsuit_a : a \in D^\heartsuit\}$$

where for each $a \in A$ we have

$$\mathfrak{d}^{\Box_F}_a := -\beta(a)$$
$$\mathfrak{d}^{\Diamond_P}_a := \beta(a)$$

Proposition 4.26. *For any tense space $\mathfrak{X}$ and any tense stable canonical rule $\mu_T(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$, we have $\mathfrak{X} \not\models \mu_T(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$ iff there is a continuous stable surjection $f : \mathfrak{X} \to \mathfrak{A}_*$ satisfying the BDC for $(\mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$ defined as above.*

In view of Proposition 4.26, in geometric settings we prefer to write a tense stable canonical rule $\mu_T(\mathfrak{A}, D^{\Box_F}, D^{\Diamond_P})$ as $\mu_T(\mathfrak{A}_*, \mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$.

We now introduce the notion of filtration implicit in tense stable canonical rules. Filtration for tense logics was considered, e.g., in [56] from a frame-theoretic perspective. Here we prefer an algebraic approach in line with Section 3.

Definition 4.27. Let $\mathfrak{A}$ be a tense algebra, V a valuation on $\mathfrak{A}$, and Θ a finite, subformula closed set of formulae. A (finite) model $(\mathfrak{B}', V')$ is called a *(finite) filtration of* $(\mathfrak{A}, V)$ *through* Θ if the following hold:

1. $\mathfrak{B}' = (\mathfrak{B}, \Box_F, \Diamond_P)$, where $\mathfrak{B}$ is the Boolean subalgebra of $\mathfrak{A}$ generated by $\bar{V}[\Theta]$;
2. $V(p) = V'(p)$ for every propositional variable $p \in \Theta$;
3. The inclusion $\subseteq: \mathfrak{B}' \to \mathfrak{A}$ is a stable embedding satisfying the BDC for $(D^{\Box_F}, D^{\Diamond_P})$, where
$$D^\heartsuit := \{\bar{V}(\varphi) : \heartsuit\varphi \in \Theta\}$$
for $\heartsuit \in \{\Box_F, \Diamond_P\}$.

Theorem 4.28 (Filtration theorem for tense algebras)**.** *Let $\mathfrak{A} \in \mathsf{Ten}$ be a tense algebra, V a valuation on $\mathfrak{A}$, and Θ a finite, subformula closed set of formulae. If $(\mathfrak{B}', V')$ is a filtration of $(\mathfrak{A}, V)$ through Θ then for every $\varphi \in \Theta$ we have*

$$\bar{V}(\varphi) = \bar{V}'(\varphi).$$

Consequently, for every tense rule Γ/Δ such that $\gamma, \delta \in \Theta$ for each $\gamma \in \Gamma$ and $\delta \in \Delta$ we have

$$\mathfrak{A}, V \models \Gamma/\Delta \iff \mathfrak{B}', V' \models \Gamma/\Delta.$$

Just like in the $\mathtt{S4}$ case, not every filtration of some model based on a tense algebra is itself based on a tense algebra, because the $\mathtt{S4}$-axiom for either $\Box_F$ or $\Diamond_P$ may not be preserved. However, given any model based on a tense algebra, there is always a method for filtrating it through any finite set of formulae which yields a model based on a tense algebra.

Definition 4.29. Let $\mathfrak{A} \in \mathsf{Ten}$, V a valuation on $\mathfrak{A}$ and Θ a finite, subformula closed set of formula. The (least) *transitive filtration* of $(\mathfrak{A}, V)$ is the pair $(\mathfrak{B}', V')$

with $\mathfrak{B} = (\mathfrak{B}', \blacksquare_F, \blacklozenge_P)$ where $\mathfrak{B}'$ and V' are as per Definition 3.25, and for all $b \in B$ we have

$$\blacksquare_F b := \bigvee \{\Box_F a : \Box_F a \leq \Box_F b \text{ and } a, \Box_F a \in B\}$$

$$\blacklozenge_P b := \bigwedge \{\Diamond_P a : \ \Diamond_P b \leq \Diamond_P a \text{ and } a, \Diamond_P a \in B\}$$

Via duality, it is not difficult to see that the least transitive filtration of any model based on a tense algebra is again a tense algebra.

At this stage, reasoning as in the proof of Lemma 3.28 using transitive filtrations we obtain the following results.

Lemma 4.30. *For every tense rule Γ/Δ there is a finite set Ξ of tense stable canonical rules such that for any $\mathfrak{K} \in \mathsf{Ten}$ we have that $\mathfrak{K} \nvDash \Gamma/\Delta$ iff there is $\eta_B(\mathfrak{H}, D^{\to}, D^{\leftarrow}) \in \Xi$ such that $\mathfrak{K} \nvDash \eta_B(\mathfrak{H}, D^{\to}, D^{\leftarrow})$.*

Theorem 4.31. *Every tense rule system is axiomatisable over $\mathtt{S4.t_R}$ by some set of tense stable canonical rules.*

4.2.3. *Comparison with Jeřábek-style Canonical Rules.* Our bsi and tense stable canonical rules generalise si and modal stable canonical rules in a way that mirrors the simple and intimate connection existing between Heyting and bi-Heyting algebras on the one hand, and modal and tense algebras on the other, explicated by the order-duality principles. Just like a bi-Heyting algebra is just a Heyting algebra whose order-dual is also a Heyting algebra, so every bsi stable canonical rule is a sort of "independent fusion" between two si stable canonical rules, whose associated Heyting algebras are order-dual to each other. Similarly for the tense case.

Jeřábek-style si and modal canonical rules (like Zakharyaschev-style si and modal canonical formulae), by contrast, do not generalise as smoothly to the bsi and tense case. Algebraically, a Jeřábek-style si canonical rule may be defined as follows (cf. [1, 5]).

Definition 4.32. Let $\mathfrak{H} \in \mathsf{HA}$ be finite and let $D \subseteq H$. The *si canonical rule* of $(\mathfrak{H}, D)$ is the rule $\zeta(\mathfrak{H}, D) = \Gamma/\Delta$, where

$$\begin{aligned}\Gamma := &\{p_0 \leftrightarrow \bot\} \cup \\ &\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in H\} \cup \{p_{a\to b} \leftrightarrow p_a \to p_b : a, b \in H\} \cup \\ &\{p_{a\vee b} \leftrightarrow p_a \vee p_b : (a, b) \in D\} \\ \Delta := &\{p_a \leftrightarrow p_b : a, b \in H \text{ with } a \neq b\}.\end{aligned}$$

Generalising the proof of [5, Corollary 5.10], one can show that every si rule is equivalent to finitely many si canonical rules. The key ingredient in this proof is a characterisation of the refutation conditions for si canonical rules: $\zeta(\mathfrak{H}, D)$ is refuted by a Heyting algebra $\mathfrak{K}$ iff there is a $(\wedge, \to, 0)$-embedding $h : \mathfrak{H} \to \mathfrak{K}$ preserving $\vee$ on elements from D. Because $(\wedge, \to, 0)$-algebras are locally finite, a result known as *Diego's theorem*, one can then reason as in the proof of, e.g., Lemma 3.19 to reach the desired result.

It should be clear that if one defined the bsi canonical rule $\zeta_B(\mathfrak{H}, D, D')$ by combining the rules $\zeta(\mathfrak{H}, D)$ and $\zeta(\bar{\mathfrak{H}}, D')$ the same way bsi stable canonical rule combine si stable canonical rules, then $\zeta_B(\mathfrak{H}, D, D')$ would be refuted by a bi-Heyting algebra $\mathfrak{K}$ iff there is a bi-Heyting algebra embedding $h : \mathfrak{H} \to \mathfrak{K}$. Since the variety of bi-Heyting algebras is not locally finite, this refutation condition is clearly too strong to deliver a result to the effect that every bsi rule is equivalent to a set of bsi canonical rules. Without such a result, in turn there is no hope of axiomatising every rule system over `bi-IPC` by means of bsi canonical rules.

Similar remarks hold in the tense case, although in this case the details are too complex to do them justice in the limited space we have at our disposal. We limit ourselves to a rough sketch of the tense case. Bezhanishvili et al. [7] show that the proof of the fact that every modal formula is equivalent, over `S4`, to finitely many modal Zakharyaschev-style canonical formulae of closure algebras rests on an application of Diego's theorem [cf. 7, Main Lemma]. This has to do with how selective filtrations of closure algebras are constructed. Given a closure algebra $\mathfrak{B}$ refuting a rule Γ/Δ, a key step in constructing a finite selective filtration of $\mathfrak{B}$ through $Sfor(\Gamma/\Delta)$ consists in generating a $(\wedge, \to, 0)$-subalgebra of $\rho\mathfrak{A}$ from a finite subset of $O(A)$. This structure is guaranteed to be finite by Diego's theorem. On the most obvious ways of generalising this construction to tense algebras, we would need to replace this step with one of the following:

(1) Generate both a $(\wedge, \to, 0)$-subalgebra of $\rho\mathfrak{A}$ and a $(\vee, \leftarrow, 1)$-subalgebra of $\rho\mathfrak{A}$ from a finite subset of $O(A)$;
(2) Generate a bi-Heyting subalgebra of $\rho\mathfrak{A}$ from a finite subset of $O(A)$.

On option 1, Diego's theorem and its order dual would guarantee that both the $(\wedge, \to, 0)$-subalgebra of $\rho\mathfrak{A}$ and the $(\vee, \leftarrow, 1)$-subalgebra of $\rho\mathfrak{A}$ are finite. However, it is not clear how one could then combine the two subalgebras into a bi-Heyting algebra, which is required to obtain a selective filtration based on a tense algebra. On option 2, on the other hand, we would indeed obtain a bi-Heyting subalgebra of $\rho\mathfrak{A}$, but not necessarily a finite one, since bi-Heyting algebras are not locally finite.

We realise that the argument sketches just presented are far from conclusive, so we do not go as far as ruling out the possibility that Jeřábek-style bsi and tense canonical rules could somehow be developed in such a way as to be a suitable tools for developing the theory of tense companions of bsi-rule systems. What such rules would look like, and in what sense they would constitute genuine generalisations of Jeřábek's canonical rules and Zakharyaschev's canonical formulae are interesting questions, but this paper is not the appropriate space to pursue them. At this stage we merely wish to stress that answering this sort of questions is a non-trivial matter, whereas generalising stable canonical rules to the bsi and tense setting and applying them to develop the theory of tense companions is a completely routine task. On our approach, exactly the same methods used in the si and modal case work equally well in the bsi-tense case.

4.3. **Tense Companions of Bi-superintuitionistic Rule Systems.** We turn to the main topic of this section. This section generalises the results of Section 3.3 to the bsi-tense setting. As anticipated, this is done using exactly the same techniques seen in the si and modal case, which is one of the main advantages of our method.

4.3.1. *Semantic Mappings.* We begin by generalising the semantic transformations for turning Heyting algebras into corresponding closure algebras and vice versa, seen in Section 3.3, to transformations between bi-Heyting and tense algebras. The results in this section are well known, and the reader may consult [57, Section 7] for a more detailed overview.

Definition 4.33. The mapping $\sigma : \mathsf{bi\text{-}HA} \to \mathsf{Ten}$ assigns every $\mathfrak{H} \in \mathsf{bi\text{-}HA}$ to the algebra $\sigma\mathfrak{H} := (B(\mathfrak{H}), \Box_F, \Diamond_P)$, where $B(\mathfrak{H})$ is the free Boolean extension of $\mathfrak{H}$ and

$$\Box_F a := \bigvee\{b \in H : b \leq a\}$$
$$\Diamond_P a := \bigwedge\{b \in H : a \leq b\}$$

That $\Box_F, \Diamond_P$ are well-defined operations on $B(\mathfrak{H})$ follows from the order-duality principle and the results in the previous section. It is easy to verify that $\sigma\mathfrak{H}$ validates the $\mathtt{S4}$ axioms for both $\Box_F$ and $\Diamond_P$. Moreover, for any $a \in B(H)$ clearly $\Diamond_P a \in H$, so $\Box_F \Diamond_P a = \Diamond_P a$. This implies $a \leq \Box_F \Diamond_P a$. Therefore indeed $\sigma\mathfrak{H} \in \mathsf{Ten}$.

Definition 4.34. The mapping $\rho : \mathsf{Ten} \to \mathsf{bi\text{-}HA}$ assigns every $\mathfrak{A} \in \mathsf{Ten}$ to the algebra $\rho\mathfrak{A} := (O(A), \wedge, \vee, \to, \leftarrow, 0, 1)$, where

$$O(A) := \{a \in A : \Box_F a = a\} = \{a \in A : \Diamond_P a = a\}$$
$$a \to b := \Box_F(\neg a \vee b)$$
$$a \leftarrow b := \Diamond_P(a \wedge \neg b).$$

Using the order-duality principle, it is easy to verify that for every $\mathfrak{A} \in \mathsf{Ten}$, the algebra $\rho\mathfrak{A}$ is indeed a bi-Heyting algebra.

Recall the geometric mappings $\sigma : \mathsf{Esa} \to \mathsf{Spa}(\mathtt{GRZ})$ and $\rho : \mathsf{Spa}(\mathtt{S4}) \to \mathsf{Esa}$. Since bi-Esakia spaces are Esakia spaces, and tense spaces are $\mathtt{S4}$-spaces, we may restrict these mappings to $\sigma : \mathsf{bi\text{-}Esa} \to \mathsf{Alg}(\mathtt{GRZ.T})$ and $\rho : \mathsf{Spa}(\mathtt{GRZ.T}) \to \mathsf{bi\text{-}Esa}$ and obtain geometric counterparts to the algebraic mappings between bi-Heyting and tense algebras defined in the present subsection. Reasoning as in the proof of Proposition 3.33 we find that the algebraic and geometric versions of the maps σ, ρ are indeed dual to each other.

Proposition 4.35. *The following hold.*

1. *Let $\mathfrak{H} \in \mathsf{bi\text{-}HA}$. Then $(\sigma\mathfrak{H})_* \cong \sigma(\mathfrak{H}_*)$. Consequently, if $\mathfrak{X}$ is a bi-Esakia space then $(\sigma\mathfrak{X})^* \cong \sigma(\mathfrak{X})^*$.*
2. *Let $\mathfrak{X}$ be a tense space. Then $(\rho\mathfrak{X})^* \cong \rho(\mathfrak{X}^*)$. Consequently, if $\mathfrak{A} \in \mathsf{Alg}(\mathtt{S4.t})$, then $(\rho\mathfrak{A})_* \cong \rho(\mathfrak{A}_*)$.*

As an easy corollary, we obtain the following analogue of Proposition 3.34.

Proposition 4.36. *For every $\mathfrak{H} \in \mathsf{bi\text{-}HA}$ we have $\mathfrak{H} \cong \rho\sigma\mathfrak{H}$. Moreover, for every $\mathfrak{A} \in \mathsf{Ten}$ we have $\sigma\rho\mathfrak{A} \rightarrowtail \mathfrak{A}$.*

4.3.2. *A Gödelian Translation.* We extend the Gödel translation of the previous section to a translation from bsi formulae to tense ones.

Definition 4.37 (Gödelian translation - bsi to tense)**.** The *Gödelian translation* is a mapping $T : Tm_{bsi} \to Tm_{ten}$ defined recursively as follows.

$$\begin{aligned} T(\bot) &:= \bot \\ T(\top) &:= \top \\ T(p) &:= \Box p \\ T(\varphi \wedge \psi) &:= T(\varphi) \wedge T(\psi) \\ T(\varphi \vee \psi) &:= T(\varphi) \vee T(\psi) \\ T(\varphi \to \psi) &:= \Box_F(\neg T(\varphi) \vee T(\psi)) \\ T(\varphi \leftarrow \psi) &:= \Diamond_P(T(\varphi) \wedge \neg T(\psi)) \end{aligned}$$

An essentially equivalent translation was considered in [57], though using $\Box_P$ instead of $\Diamond_P$ to interpret $\leftarrow$.

The following analogue of Lemma 3.36 is proved the same way as the latter.

Lemma 4.38. *For every* $\mathfrak{A} \in \mathsf{Ten}$ *and bsi rule* Γ/Δ*,*

$$\mathfrak{A} \models T(\Gamma/\Delta) \iff \rho\mathfrak{A} \models \Gamma/\Delta$$

We note that Lemma 4.38 does not appear the literature, which only mentions a similar results concerning formulae rather than rules.

4.3.3. *Structure of Tense Companions.* We are now ready to generalise Theorem 3.43 and Theorem 3.44 to the bsi-tense setting. We do so in this section. All the results of this section are new inasmuch as they involve rule systems. Their restrictions to logics were established by Wolter [57], although our proofs differ from Wolter's Blok-style algebraic approach.

We begin by formally defining the notion of a *tense companion*.

Definition 4.39. Let $\mathtt{L} \in \mathbf{Ext}(\mathtt{bi\text{-}IPC_R})$ be a bsi-rule system and $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t_R})$ a tense rule system. We say that $\mathtt{M}$ is a *tense companion* of $\mathtt{L}$ (or that $\mathtt{L}$ is the bsi fragment of $\mathtt{M}$) whenever $\Gamma/\Delta \in \mathtt{L}$ iff $T(\Gamma/\Delta) \in \mathtt{M}$ for every bsi rule Γ/Δ. Moreover, let $\mathtt{L} \in \mathbf{Ext}(\mathtt{bi\text{-}IPC})$ be a bsi-logic and $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t})$ a tense logic. We say that $\mathtt{M}$ is a *tense companion* of $\mathtt{L}$ (or that $\mathtt{L}$ is the bsi fragment of $\mathtt{M}$) whenever $\varphi \in \mathtt{L}$ iff $T(\varphi) \in \mathtt{M}$.

Clearly, $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t_R})$ is a modal companion of $\mathtt{L} \in \mathbf{Ext}(\mathtt{bi\text{-}IPC_R})$ iff $\mathsf{Taut}(\mathtt{M})$ is a modal companion of $\mathsf{Taut}(\mathtt{L})$, and $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t})$ is a modal companion of $\mathtt{L} \in \mathbf{Ext}(\mathtt{bi\text{-}IPC})$ iff $\mathtt{M_R}$ is a modal companion of $\mathtt{L_R}$.

Define the following three maps between $\mathbf{Ext}(\mathtt{bi\text{-}IPC_R})$ and $\mathbf{NExt}(\mathtt{S4.t_R})$.

$$\begin{aligned} \tau &: \mathbf{Ext}(\mathtt{bi\text{-}IPC_R}) \to \mathbf{NExt}(\mathtt{S4.t_R}) & \sigma &: \mathbf{Ext}(\mathtt{bi\text{-}IPC_R}) \to \mathbf{NExt}(\mathtt{S4.t_R}) \\ \mathtt{L} &\mapsto \mathtt{S4.t_R} \oplus \{T(\Gamma/\Delta) : \Gamma/\Delta \in \mathtt{L}\} & \mathtt{L} &\mapsto \mathtt{GRZ.T_R} \oplus \tau\mathtt{L} \end{aligned}$$

$$\begin{aligned} \rho &: \mathbf{NExt}(\mathtt{S4.t_R}) \to \mathbf{Ext}(\mathtt{bi\text{-}IPC_R}) \\ \mathtt{M} &\mapsto \{\Gamma/\Delta : T(\Gamma/\Delta) \in \mathtt{M}\} \end{aligned}$$

These mappings are readily extended to lattices of logics.

$$\begin{aligned} \tau &: \mathbf{Ext}(\mathtt{bi\text{-}IPC}) \to \mathbf{NExt}(\mathtt{S4.t}) & \sigma &: \mathbf{Ext}(\mathtt{bi\text{-}IPC}) \to \mathbf{NExt}(\mathtt{S4.t}) \\ \mathtt{L} &\mapsto \mathsf{Taut}(\tau\mathtt{L_R}) = \mathtt{S4.t} \oplus \{T(\varphi) : \varphi \in \mathtt{L}\} & \mathtt{L} &\mapsto \mathsf{Taut}(\sigma\mathtt{L_R}) = \mathtt{GRZ.T} \oplus \{T(\varphi) : \varphi \in \mathtt{L}\} \end{aligned}$$

$$\rho : \mathbf{NExt}(\texttt{S4.t}) \to \mathbf{Ext}(\texttt{bi-IPC})$$
$$\texttt{M} \mapsto \mathsf{Taut}(\rho\texttt{M}_{\texttt{R}}) = \{\varphi : T(\varphi) \in \texttt{M}\}$$

Furthermore, extend the mappings $\sigma : \mathsf{bi\text{-}HA} \to \mathsf{Ten}$ and $\rho : \mathsf{Ten} \to \mathsf{bi\text{-}HA}$ to universal classes by setting

$$\sigma : \mathbf{Uni}(\mathsf{bi\text{-}HA}) \to \mathbf{Uni}(\mathsf{Ten}) \qquad \rho : \mathbf{Uni}(\mathsf{Ten}) \to \mathbf{Uni}(\mathsf{bi\text{-}HA})$$
$$\mathcal{U} \mapsto \mathsf{Uni}\{\sigma\mathfrak{H} : \mathfrak{H} \in \mathcal{U}\} \qquad \mathcal{W} \mapsto \{\rho\mathfrak{A} : \mathfrak{A} \in \mathcal{W}\}.$$

Finally, introduce a semantic counterpart to τ as follows.

$$\tau : \mathbf{Uni}(\mathsf{bi\text{-}HA}) \to \mathbf{Uni}(\mathsf{Ten})$$
$$\mathcal{U} \mapsto \{\mathfrak{A} \in \mathsf{Ten} : \rho\mathfrak{A} \in \mathcal{U}\}$$

The following lemma is a counterpart to Lemma 3.38. It is proved via essentially the same argument which establishes the latter, though some adaptations are necessary which may be less than completely obvious. For this reason, as well as for the central place this lemma occupies in our strategy, we spell out the proof in some detail.

Lemma 4.40. *Let $\mathfrak{A} \in \mathsf{GRZ.T}$. Then for every modal rule Γ/Δ, we have $\mathfrak{A} \models \Gamma/\Delta$ iff $\sigma\rho\mathfrak{A} \models \Gamma/\Delta$.*

Proof. ($\Rightarrow$) This direction follows from the fact that $\sigma\rho\mathfrak{A} \rightarrowtail \mathfrak{A}$ (Proposition 4.36).

($\Leftarrow$) We prove the dual statement that $\mathfrak{A}_* \not\models \Gamma/\Delta$ implies $\sigma\rho\mathfrak{A}_* \not\models \Gamma/\Delta$. Let $\mathfrak{X} := \mathfrak{A}_*$. In view of Theorem 4.31 it is enough to consider the case $\Gamma/\Delta = \mu_T(\mathfrak{B}, D^{\Box_F}, D^{\Diamond_P})$, for $\mathfrak{B} \in \mathsf{Ten}$ finite. So suppose $\mathfrak{X} \not\models \mu(\mathfrak{B}, D)$ and let $\mathfrak{F} := \mathfrak{B}_*$. Then there is a stable map $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $(\mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$. We construct a stable map $g : \sigma\rho\mathfrak{X} \to \mathfrak{F}$ which satisfies the BDC for $(\mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$.

Let $C := \{x_1, \ldots, x_n\} \subseteq F$ be some cluster and let $Z_C := f^{-1}(C)$. Reasoning as in the proof of Lemma 3.38, we obtain that $\rho[Z_C]$ is clopen, and so is $f^{-1}(x_i)$ for each $x_i \in C$. Now for each $x_i \in C$ let

$$M_i := max_R(f^{-1}(x_i))$$
$$N_i := min_R(f^{-1}(x_i)).$$

By Proposition 4.14, both M_i, N_i are closed, and moreover neither cuts any cluster. Since $\sigma\rho\mathfrak{X}$ has the quotient topology, it follows that both $\rho[M_i], \rho[N_i]$ are closed as well.

For each $x_i \in C$ let $O_i := M_i \cup N_i$. Clearly, $O_i \cap O_j = \varnothing$ for each $i, j \leq n$. Therefore, using the separation properties of Stone spaces to reason as in the proof of Lemma 3.38, there are disjoint clopens $U_1, \ldots, U_n \in \mathsf{Clop}(\sigma\rho\mathfrak{X})$ with $\rho[O_i] \subseteq U_i$ and $\bigcup_{i\leq n} U_i = \rho[Z_C]$.

We can now define a map

$$g_C : \rho[Z_C] \to C$$
$$z \mapsto x_i \iff z \in U_i.$$

Clearly, g_C is relation preserving and continuous. Finally, define $g : \sigma\rho\mathfrak{X} \to F$ by setting

$$g(\rho(z)) := \begin{cases} f(z) & \text{if } f(z) \text{ does not belong to any proper cluster} \\ g_C(\rho(z)) & \text{if } f(z) \in C \text{ for some proper cluster } C \subseteq F. \end{cases}$$

Now, g is evidently relation preserving. Moreover, it is continuous because both f and each g_C are. Reasoning as in the proof of Lemma 3.38, we obtain that g satisfies the BDC$^{\Box_F}$ for $\mathfrak{D}^{\Box_F}$. The proof of the fact that g satisfies the BDC$^{\Diamond_P}$ for $\mathfrak{D}^{\Diamond_P}$ is a straightforward adaptation of the latter, using that for all $U \in \mathsf{Clop}(\mathfrak{X})$, if $x \in U$ there is $y \in min_R(U)$ such that Ryx (Proposition 4.14). □

Theorem 4.41. *Every $\mathcal{U} \in \mathbf{Uni}(\mathsf{GRZ.T})$ is generated by its skeletal elements, i.e. $\mathcal{U} = \sigma\rho\mathcal{U}$.*

Proof. Follows easily from Lemma 4.40, reasoning as in the proof of Theorem 3.39. □

As in the previous section, the next step is to apply Lemma 4.40 to prove that the syntactic tense companion maps τ, ρ, σ commute with $\mathsf{Alg}(\cdot)$, which leads to a purely semantic characterisation of tense companions.

Lemma 4.42. *For each $\mathtt{L} \in \mathbf{Ext}(\mathtt{bi\text{-}IPC_R})$ and $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t_R})$, the following hold:*

$$\mathsf{Alg}(\tau\mathtt{L}) = \tau\mathsf{Alg}(\mathtt{L}) \tag{10}$$

$$\mathsf{Alg}(\sigma\mathtt{L}) = \sigma\mathsf{Alg}(\mathtt{L}) \tag{11}$$

$$\mathsf{Alg}(\rho\mathtt{M}) = \rho\mathsf{Alg}(\mathtt{M}) \tag{12}$$

Proof. The proof of Equation (10) is trivial. To prove Equation (11), in view of Theorem 4.41 it is enough to show that $\mathsf{Alg}(\sigma\mathtt{L})$ and $\sigma\mathsf{Alg}(\mathtt{L})$ have the same skeletal elements. This is proved the same way as Equation (8) in Lemma 3.41. Finally, Equation (12) is proved analogously to Equation (9) in Lemma 3.41, applying Lemma 4.38 instead of Lemma 3.36. □

Lemma 4.43. *$\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t_R})$ is a tense companion of $\mathtt{L} \in \mathbf{Ext}(\mathtt{bi\text{-}IPC_R})$ iff $\mathsf{Alg}(\mathtt{L}) = \rho\mathsf{Alg}(\mathtt{M})$.*

Proof. Analogous to Lemma 3.42. □

The main results of this section can now be proved.

Theorem 4.44. *The following conditions hold:*

1. *For every $\mathtt{L} \in \mathbf{Ext}(\mathtt{bi\text{-}IPC_R})$, the modal companions of $\mathtt{L}$ form an interval*
$$\{\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t_R}) : \tau\mathtt{L} \leq \mathtt{M} \leq \sigma\mathtt{L}\};$$
2. *For every $\mathtt{L} \in \mathbf{Ext}(\mathtt{bi\text{-}IPC})$, the modal companions of $\mathtt{L}$ form an interval*
$$\{\mathtt{M} \in \mathbf{NExt}(\mathtt{S4.t}) : \tau\mathtt{L} \leq \mathtt{M} \leq \sigma\mathtt{L}\}.$$

Proof. Item 1 is proved the same way as Item 1 in Theorem 3.43. Item 2 is immediate from Item 1. □

Theorem 4.45 (Blok-Esakia theorem for bsi- and tense deductive systems)**.** *The following conditions hold:*

1. *The mappings $\sigma : \mathbf{Ext}(\mathtt{bi\text{-}IPC_R}) \to \mathbf{NExt}(\mathtt{GRZ.T_R})$ and $\rho : \mathbf{NExt}(\mathtt{GRZ.T_R}) \to \mathbf{Ext}(\mathtt{bi\text{-}IPC_R})$ are complete lattice isomorphisms and mutual inverses.*
2. *The mappings $\sigma : \mathbf{Ext}(\mathtt{bi\text{-}IPC}) \to \mathbf{NExt}(\mathtt{GRZ.T})$ and $\rho : \mathbf{NExt}(\mathtt{GRZ.T}) \to \mathbf{Ext}(\mathtt{bi\text{-}IPC})$ are complete lattice isomorphisms and mutual inverses.*

Proof. Item 1 is proved the same way as Item 1 in Theorem 3.44. Item 2 follows straightforwardly from Item 1 and Propositions 4.2 and 4.9. □

4.3.4. *The Dummett-Lemmon Conjecture for Bsi-Rule Systems.* The construction used to prove the Dummett-Lemmon conjecture for rule systems straightforwardly generalises to a proof of a variant of the conjecture applying to bsi-rule systems and their weakest tense companions. To establish this result, we first extend the notion of a collapsed rule and the rule collapse lemma to the bsi and tense setting.

Definition 4.46. Let $\mu_T(\mathfrak{F}, \mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$ be a tense stable canonical rule. The *collapsed tense stable canonical rule* $\eta_B(\rho\mathfrak{F}, \rho\mathfrak{D}^{\to}, \rho\mathfrak{D}^{\leftarrow})$ is defined by setting

$$\rho\mathfrak{D}^{\to} = \{\rho[\mathfrak{d}] : \mathfrak{d} \in \mathfrak{D}^{\Box_F}\}$$
$$\rho\mathfrak{D}^{\leftarrow} = \{\rho[\mathfrak{d}] : \mathfrak{d} \in \mathfrak{D}^{\Diamond_P}\}$$

Lemma 4.47 (Rule collapse lemma - bsi-tense)**.** *For every tense space $\mathfrak{X}$ and every tense stable canonical rule $\mu_T(\mathfrak{F}, D^{\Box_F}, D^{\Diamond_P})$, we have that $\mathfrak{X} \nvDash \mu_T(\mathfrak{F}, \mathfrak{D}^{\Box_F}, \mathfrak{D}^{\Diamond_P})$ implies $\rho\mathfrak{X} \nvDash \eta_B(\rho\mathfrak{F}, \rho\mathfrak{D}^{\to}, \rho\mathfrak{D}^{\leftarrow})$.*

Proof. Analogous to the proof of Lemma 3.46. □

At this point, we can establish the desired result via a straightforward adaptation of our proof of Theorem 3.47.

Theorem 4.48 (Dummett-Lemmon conjecture for bsi rule systems)**.** *For every bsi-rule system* $\mathtt{L} \in \mathbf{Ext}(\mathtt{bi\text{-}IPC_R})$*,* $\mathtt{L}$ *is Kripke complete iff* $\tau\mathtt{L}$ *is.*

Proof. ($\Rightarrow$) Let $\mathtt{L}$ be Kripke complete. Suppose that $\Gamma/\Delta \notin \tau\mathtt{L}$. Then there is $\mathfrak{X} \in \mathsf{Spa}(\tau\mathtt{L})$ such that $\mathfrak{X} \nvDash \Gamma/\Delta$. By Theorem 4.31, we may assume that $\Gamma/\Delta = \mu(\mathfrak{F}, \mathfrak{D})$ for $\mathfrak{F}$ a preorder. By Lemma 4.47 and Lemma 4.38 it follows that $\rho\mathfrak{X} \models \mathtt{L}$, and so $\eta(\rho\mathfrak{F}, \rho\mathfrak{D}) \notin \mathtt{L}$. Since $\mathtt{L}$ is Kripke complete, there is a bsi Kripke frame $\mathfrak{Y}$ such that $\mathfrak{Y} \nvDash \eta(\rho\mathfrak{F}, \rho\mathfrak{D})$. Take a stable map $f : \mathfrak{Y} \to \rho\mathfrak{F}$ satisfying the BDC for $\rho\mathfrak{D}$. Proceed as in the proof of Theorem 3.47 to construct a Kripke frame $\mathfrak{Z}$ with $\mathfrak{Z} \models \tau\mathtt{L}$ by expanding clusters in $\mathfrak{Y}$. We identify $\rho\mathfrak{Z} = \mathfrak{Y}$, and define a map $g : \mathfrak{Z} \to \mathfrak{F}$ via the same construction used in the proof of Theorem 3.47. Clearly, g is well defined, surjective, and relation preserving. We know that g satisfies the BDC for $\mathfrak{D}^{\to}$ from the proof of Theorem 3.47, and symmetric reasoning shows that g also satisfies the BDC for $\mathfrak{D}^{\leftarrow}$.

($\Leftarrow$) Analogous to the si and modal case. □

5. The Kuznetsov-Muravitsky Isomorphism for Logics and Rule Systems

In this section, we generalise our techniques further to study translational embeddings of (normal) *modal superintuitionistic* rule systems and logics into modal ones. We develop algebra-based rules for modal superintuitionistic rule systems over the intuitionistic provability logic KM, as well as a new kind of algebra-based rules for modal rule systems over the Gödel-Löb provability logic (Section 5.2). We call these *pre-stable canonical rules.* We apply pre-stable canonical rules to prove that the lattice of modal superintuitionistic rule systems (resp. logics) over KM is isomorphic to the lattice of modal rule systems (resp. logics) over $\mathtt{GL}$ via a Gödel-style translational embedding (Section 5.3). This result was proved for logics by Kuznetsov and Muravitsky [37], but appears to be new for rule systems.

For reasons of space, this section does not pursue the full theory of modal companions of superintuitionistic logics in the sense of either [25] or [59, 58], although we are confident that our techniques would work in that setting as well. Because of this the Dummett-Lemmon conjecture has no counterpart in the present section.

Besides supplying new results, this section further highlights the flexibility and uniformity of our techniques. Standard filtration does not work well for $\mathtt{KM}$ and $\mathtt{GL}$, suggesting a different, less standard notion of filtration should be used to generalise stable canonical rules to the present setting. The rest of our approach delivers the desired results despite this different design choice, which shows its flexibility. Moreover, it does so without needing any major changes and accommodations: the proofs of the main results in this section follow the basic blueprints of their counterparts from Section 3. This, once again, shows the uniformity of our approach.

5.1. **Deductive Systems for Provability.** We begin by briefly reviewing definitions and basic properties of the structures under discussion.

5.1.1. *Intuitionistic Provability, Frontons, and* $\mathtt{KM}$*-spaces.* In this subsection we shall work with the *modal superintuitionistic signature*,

$$msi := \{\wedge, \vee, \to, \boxtimes, \bot, \top\}.$$

The set Frm_{msi} of *modal superintuitionistic (msi) formulae* is defined recursively as follows.

$$\varphi ::= p \,|\, \bot \,|\, \top \,|\, \varphi \wedge \varphi \,|\, \varphi \vee \varphi \,|\, \varphi \to \varphi \,|\, \boxtimes \varphi$$

where $p \in Prop$.

The logic $\mathtt{IPCK}$ is obtained by extending $\mathtt{IPC}$ by the $\mathtt{K}$-axiom

$$\boxtimes(p \to q) \to (\boxtimes p \to \boxtimes q)$$

and closing under necessitation, that is, requiring that whenever $\varphi \in \mathtt{IPCK}$ then $\boxtimes\varphi \in \mathtt{IPCK}$ as well.

Definition 5.1. A *normal modal superintuitionistic logic*, or msi-logic for short, is a logic $\mathtt{L}$ over Frm_{msi} satisfying the following additional conditions:

(1) $\mathtt{IPCK} \subseteq \mathtt{L}$;
(2) If $\varphi \to \psi, \varphi \in \mathtt{L}$ then $\psi \in \mathtt{L}$ (MP);
(3) If $\varphi \in \mathtt{L}$ then $\boxtimes\varphi \in \mathtt{L}$ (NEC).

A *modal superintuitionistic rule system*, or msi-rule system for short, is a rule system $\mathtt{L}$ over Frm_{msi} satisfying the following additional requirements.

(1) $/\varphi \in \mathtt{L}$ whenever $\varphi \in \mathtt{IPCK}$;
(2) $\varphi, \varphi \to \psi/\psi \in \mathtt{L}$ (MP-R);
(3) $\varphi/\boxtimes\varphi \in \mathtt{L}$ (NEC-R).

If $\mathtt{L}$ is an msi-logic (resp. msi-rule system) we write $\mathbf{NExt}(\mathtt{L})$ for the set of msi-logics (resp. rule systems) extending $\mathtt{L}$. Surely, the set of msi-logics systems coincides with $\mathbf{NExt}(\mathtt{IPCK})$. It is easy to check that $\mathbf{NExt}(\mathtt{IPCK})$ forms a lattice under the operations $\oplus_{\mathbf{NExt}(\mathtt{K})}$ as join and intersection as meet. If $\mathtt{L} \in \mathbf{NExt}(\mathtt{IPCK})$, let $\mathtt{L_R}$ be the least msi-rule system containing $/\varphi$ for each $\varphi \in \mathtt{L_R}$. Then $\mathtt{IPCK_R}$ is the least msi-rule system. The set $\mathbf{NExt}(\mathtt{IPCK_R})$ of msi-rule systems is also a lattice when endowed with $\oplus_{\mathbf{NExt}(\mathtt{IPCK_R})}$ as join and intersection as meet. As usual, we refer to these lattices as we refer to their underlying sets, i.e. $\mathbf{NExt}(\mathtt{IPCK})$ and $\mathbf{NExt}(\mathtt{IPCK_R})$ respectively. We also write both $\oplus_{\mathbf{NExt}(\mathtt{IPCK})}$ and $\oplus_{\mathbf{NExt}(\mathtt{IPCK_R})}$ simply as $\oplus$, leaving context to resolve ambiguities. Clearly, for every $\mathtt{L} \in \mathbf{NExt}(\mathtt{IPCK})$ we have that $\mathsf{Taut}(\mathtt{L_R}) = \mathtt{L}$, which establishes the following result.

Proposition 5.2. *The mappings $(\cdot)_\mathtt{R}$ and $\mathsf{Taut}(\cdot)$ are mutually inverse complete lattice isomorphisms between $\mathbf{NExt}(\mathtt{IPCK})$ and the sublattice of $\mathbf{NExt}(\mathtt{IPCK_R})$ consisting of all msi-rule systems $\mathtt{L}$ such that $\mathsf{Taut}(\mathtt{L})_\mathtt{R} = \mathtt{L}$.*

Rather than studying $\mathbf{NExt}(\mathtt{IPCK_R})$ in its entirety, we shall focus on the sublattice of $\mathbf{NExt}(\mathtt{IPCK_R})$ consisting of all normal extensions of the rule system $\mathtt{KM_R}$, where $\mathtt{KM}$ is the msi-logic axiomatised as follows.

$$\mathtt{KM} := \mathtt{IPCK} \oplus p \to \boxtimes p \oplus (\boxtimes p \to p) \to p \oplus \boxtimes p \to (q \vee (q \to p)).$$

The logic $\mathtt{KM}$ was introduced by Kuznetsov [36] (see also [37]) and later studied by Esakia [25]. Its main motivation lies in its close connection with the Gödel-Löb provability logic, to be discussed in the next section. An extensive overview of both the history and theory of $\mathtt{KM}$ may be found in [42].

A *fronton* is a tuple $\mathfrak{H} = (H, \wedge, \vee, \to, \boxtimes, 0, 1)$ such that $(H, \wedge, \vee, \to, 0, 1)$ is a Heyting algebra and for every $a, b \in H$, $\boxtimes$ satisfies

$$\boxtimes 1 = 1 \tag{13}$$

$$\boxtimes(a \wedge b) = \boxtimes a \wedge \boxtimes b \tag{14}$$

$$a \leq \boxtimes a \tag{15}$$

$$\boxtimes a \to a = a \tag{16}$$

$$\boxtimes a \leq b \vee (b \to a) \tag{17}$$

Frontons are discussed in detail, e.g., in [25, 38]. We let Frt denote the class of all frontons. By Theorem 2.4, Frt is a variety. We write $\mathbf{Var}(\mathsf{Frt})$ and $\mathbf{Uni}(\mathsf{Frt})$ respectively for the lattice of subvarieties and of universal subclasses of Frt. Item 1 in the following result follows from, e.g., [42, Proposition 7], whereas Item 2 can be obtained via the techniques used in the proofs of Theorems 3.3 and 3.8.

Theorem 5.3. *The following maps are pairs of mutually inverse dual isomorphisms:*

(1) $\mathsf{Alg} : \mathbf{NExt}(\mathtt{KM}) \to \mathbf{Var}(\mathsf{Frt})$ *and* $\mathsf{Th} : \mathbf{Var}(\mathsf{Frt}) \to \mathbf{Ext}(\mathtt{KM})$;
(2) $\mathsf{Alg} : \mathbf{NExt}(\mathtt{KM_R}) \to \mathbf{Uni}(\mathsf{Frt})$ *and* $\mathsf{ThR} : \mathbf{Uni}(\mathsf{Frt}) \to \mathbf{NExt}(\mathtt{KM_R})$.

Corollary 5.4. *Every msi-logic (resp. si-rule system) extending $\mathtt{KM}$ is complete with respect to some variety (resp. universal class) of Frontons.*

We mention a simple yet important property of frontons, which plays a key role in the development of algebra-based rules for rule systems in $\mathbf{NExt}(\mathtt{KM_R})$.

Proposition 5.5 (cf. [25, Proposition 5]). *Every fronton $\mathfrak{H}$ satisfies the identity*

$$\boxtimes a = \bigwedge \{b \vee (b \to a) : b \in H\}.$$

for every $a \in H$.

It follows that for every Heyting algebra $\mathfrak{H}$, there is at most one way of expanding $\mathfrak{H}$ to a fronton, namely by setting

$$\boxtimes a := \bigwedge\{b \vee (b \to a) : b \in H\}$$

A $\mathtt{KM}$-*space* is a tuple $\mathfrak{X} = (X, \leq, \sqsubseteq, \mathcal{O})$, such that $(X, \leq, \mathcal{O})$ is an Esakia space, and $\sqsubseteq$ is a binary relation on X satisfying the following conditions, where $\Uparrow x := \{y \in X : x \sqsubseteq y\}$ and $\Downarrow x := \{y \in X : y \sqsubseteq x\}$, and $x < y$ iff $x \leq y$ and $x \neq y$:

(1) $x < y$ implies $x \sqsubseteq y$;
(2) $x \sqsubseteq y$ implies $x \leq y$;
(3) $\Uparrow x$ is closed for all $x \in X$;
(4) $\Downarrow[U] \in \mathsf{Clop}(\mathfrak{X})$ for every $U \in \mathsf{ClopUp}(\mathfrak{X})$;
(5) For every $U \in \mathsf{ClopUp}(\mathfrak{X})$ and $x \in X$, if $x \notin U$ then there is $y \in -U$ such that $x \leq y$ and $\Uparrow y \subseteq U$.

$\mathtt{KM}$-spaces are discussed in [25], and more at length in [15].

A *valuation* on a $\mathtt{KM}$ space space $\mathfrak{X}$ is a map $V : Prop \to \mathsf{ClopUp}(\mathfrak{X})$. The geometrical semantics for msi-rule systems extending $\mathtt{KM_R}$ over $\mathtt{KM}$-spaces is obtained straightforwardly by combining the geometrical semantics of si-rule systems and that of modal rule systems. The relation $\leq$ is used to interpret the implication connective $\to$, and the relation $\sqsubseteq$ is used to interpret the modal operator $\boxtimes$.

If $\mathfrak{X}, \mathfrak{Y}$ are $\mathtt{KM}$-spaces, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is called a *bounded morphism* if for all $x, y \in X$ we have:

- $x \leq y$ implies $f(x) \leq f(y)$;
- $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$;
- $f(x) \leq y$ implies that there is $z \in X$ with $x \leq z$ and $f(z) = y$;
- $f(x) \sqsubseteq y$ implies that there is $z \in X$ with $x \sqsubseteq z$ and $f(z) = y$

We recall some useful properties of $\mathtt{KM}$-spaces, which are proved in [15, Proposition 4.8].

Proposition 5.6. *For every* $\mathtt{KM}$*-space* $\mathfrak{X}$*, the following conditions hold:*

(1) For every $U \in \mathsf{ClopUp}(U)$ *we have* $\{x \in X : \Uparrow x \subseteq U\} = U \cup max_{\leq}(-U)$*;*
(2) If $\mathfrak{X}$ *is finite, then for all* $x, y \in X$ *we have* $x \sqsubseteq y$ *iff* $x < y$*.*

It is known that the category of frontons with corresponding homomorphisms is dually equivalent to the category of $\mathtt{KM}$-spaces with continuous bounded morphisms. This result was announced in [25, 354–5], and proved in detail in [15, Theorem 4.4]. We denote the $\mathtt{KM}$-space dual to a fronton $\mathfrak{H}$ as $\mathfrak{H}_*$, and the fronton dual to a $\mathtt{KM}$-space $\mathfrak{X}$ as $\mathfrak{X}^*$.

5.1.2. *Classical Provability, Magari Algebras, and* $\mathtt{GL}$*-spaces.* We now work in the modal signature *md* already discussed in Section 3. The modal logic $\mathtt{GL}$ is axiomatised by extending $\mathtt{K}$ with the well-known *Löb formula.*

$$\begin{aligned}\mathtt{GL} :=& \mathtt{K} \oplus \Box(\Box p \to p) \to \Box p\\ =& \mathtt{K4} \oplus \Box(\Box p \to p) \to \Box p\end{aligned}$$

The logic $\mathtt{GL}$ was independently discovered by Boolos and the Siena logic group led by Magari (cf. [47, 48, 40, 50, 12]) as a formalisation of the provability predicate of Peano arithmetic. The arithmetical completeness of $\mathtt{GL}$ was proved by Solovay [51] (see also [20]). The reader may consult [13] (as well as the more recent if less comprehensive [42]) for an overview of known results concerning $\mathtt{GL}$.

A modal algebra $\mathfrak{A}$ is called a *Magari algebra* (after [40]) if it satisfies the identity

$$\Box(\Box a \to a) = \Box a$$

for all $a \in A$. Magari algebras are also called $\mathtt{GL}$-algebras, e.g. in [38]. We let Mag denote the variety of all Magari algebras. Clearly, every Magari algebra is a transitive modal algebra, and moreover Mag coincides with the class of all modal algebras satisfying the equation

$$\Diamond a = \Diamond(\Box\neg a \wedge a).$$

The following result is a straightforward consequence of Theorem 3.8.

Theorem 5.7. *The following maps are pairs of mutually inverse dual isomorphisms:*

(1) $\mathsf{Alg} : \mathbf{NExt}(\mathtt{GL}) \to \mathbf{Var}(\mathsf{Mag})$ *and* $\mathsf{Th} : \mathbf{Var}(\mathsf{Mag}) \to \mathbf{Ext}(\mathtt{GL})$*;*
(2) $\mathsf{Alg} : \mathbf{NExt}(\mathtt{GL_R}) \to \mathbf{Uni}(\mathsf{Mag})$ *and* $\mathsf{ThR} : \mathbf{Uni}(\mathsf{Mag}) \to \mathbf{NExt}(\mathtt{GL_R})$.

Corollary 5.8. *Every modal logic* (*resp. modal rule system*) *extending* $\mathtt{GL}$ *is complete with respect to some variety* (*resp. universal class*) *of Magari algebras.*

Modal spaces dual to Magari algebras are called $\mathtt{GL}$*-spaces*. $\mathtt{GL}$-spaces display various similarities with $\mathtt{GRZ}$-spaces, as the reader can appreciate by comparing the following result with Proposition 3.11.

Proposition 5.9 (cf. [39])**.** *For every* $\mathtt{GL}$*-space* $\mathfrak{X}$ *and* $U \in \mathsf{Clop}(\mathfrak{X})$*, the following conditions hold:*

(1) *If* $x \in max_R(U)$ *then* $R[x] \cap U = \varnothing$*;*
(2) $max_R(U) \in \mathsf{Clop}(\mathfrak{X})$*;*
(3) *If* $x \in U$ *then either* $x \in max_R(U)$ *or there is* $y \in max_R(U)$ *such that* Rxy*;*
(4) *If* $\mathfrak{X}$ *is finite then* R *is irreflexive.*

$\mathtt{GL}$ is well-known to be complete with respect to the class of irreflexive and transitive Kripke frames containing no ascending chain. However, like $\mathtt{GRZ}$-spaces, $\mathtt{GL}$-spaces may contain clusters, and a fortiori reflexive points.

5.2. **Pre-stable Canonical Rules for Normal Extensions of $\mathtt{KM_R}$ and $\mathtt{GL_R}$.** In this section we develop a new kind of algebra-based rules, serving as analogues of stable canonical rules for rule systems in $\mathbf{NExt}(\mathtt{KM_R})$ and $\mathbf{NExt}(\mathtt{GL_R})$. These rules encode a notion of filtration weaker than standard filtration, and are better suited than the latter to the rule systems under discussion. We call them *pre-stable canonical rules.*

5.2.1. *The* $\mathtt{KM_R}$ *Case.* We have seen notions of filtration for both Heyting and modal algebras. One would hope that combining the latter would yield a suitable notion of filtration for frontons, which could then be used to develop stable canonical rules for rule systems in $\mathbf{NExt}(\mathtt{KM_R})$. This is in principle possible, but suboptimal. The reason is that with filtrations understood this way, rule systems in $\mathbf{NExt}(\mathtt{KM_R})$ would turn out to admit very few filtrations. To see this, recall (Proposition 5.6) that in every finite $\mathtt{KM}$-space $\mathfrak{X}$ we have that $x \sqsubseteq y$ iff $x < y$ for all $x, y \in X$. Now let $\mathfrak{X}$ be any $\mathtt{KM}$-space such that there are $x, y \in X$ with $x \neq y$ and $x \sqsubseteq y$. Then any finite image of $\mathfrak{X}$ under a $\sqsubseteq$-preserving map h with $h(x) = h(y)$ would contain a reflexive point, hence would fail to be a $\mathtt{KM}$-space.

We know that every finite distributive lattice has a unique Heyting algebra expansion, and moreover that every finite Heyting algebra has a unique fronton expansion. These constructions lead to a natural method for extracting finite countermodels based on frontons to non-valid msi rules, which we illustrate in the proof of Lemma 5.10. This result, in a somewhat different formulation, was first proved by Muravitsky [41] via frame-theoretic methods.

Lemma 5.10. *For any msi rule* Γ/Δ*, if* $\mathsf{Frt} \nvDash \Gamma/\Delta$ *then there is a finite fronton* $\mathfrak{H} \in \mathsf{Frt}$ *such that* $\mathfrak{H} \nvDash \Gamma/\Delta$*.*

Proof. Assume $\mathsf{Frt} \nvDash \Gamma/\Delta$ and let $\mathfrak{H} \in \mathsf{Frt}$ be a fronton with $\mathfrak{H} \nvDash \Gamma/\Delta$. Take a valuation V with $\mathfrak{H}, V \nvDash \Gamma/\Delta$. Put $\Theta = \mathit{Sfor}(\Gamma/\Delta)$ and set

$$D^{\to} := \{(\bar{V}(\varphi), \bar{V}(\psi)) \in H \times H : \varphi \to \psi \in \Theta\} \cup \{(\bar{V}(\varphi), a) : a \in D^{\boxtimes} \text{ and } \varphi \in \Theta\}$$
$$D^{\boxtimes} := \{\bar{V}(\varphi) \in H : \boxtimes\varphi \in \Theta\}$$

Let $\mathfrak{K}$ be the bounded distributive lattice generated by Θ. For all $a, b \in K$ define

$$a \rightsquigarrow b := \bigvee\{c \in H : a \wedge c \leq b\}$$
$$\boxtimes' a := \bigwedge_{b \in K} b \vee (b \rightsquigarrow a)$$

Obviously $(\mathfrak{K}, \rightsquigarrow)$ is a Heyting algebra, and by Proposition 5.5 it follows that $\mathfrak{K}' := (\mathfrak{K}, \rightsquigarrow, \boxtimes')$ is a fronton. Moreover, the inclusion $\subseteq : \mathfrak{K}' \to \mathfrak{A}$ is a bounded lattice embedding satisfying

$$a \rightsquigarrow b \leq a \to b \qquad \text{for all } (a, b) \in K \times K$$
$$a \rightsquigarrow b = a \to b \qquad \text{for all } (a, b) \in D^{\to}$$
$$\boxtimes' a = \boxtimes a \qquad \text{for all } a \in D^{\boxtimes}.$$

The first two claims are proved the same way as in the proof of Lemma 3.19. For the third claim we reason as follows. Suppose $a \in D^{\boxtimes}$. Then $(b, a) \in D^{\to}$ for every $b \in K$ by construction. Therefore,

$$\boxtimes' a = \bigwedge_{b \in K} b \vee (b \rightsquigarrow a) = \bigwedge_{b \in K} b \vee (b \to a).$$

By the axioms of frontons we have $\boxtimes a \leq b \vee (b \to a)$ for all $b \in H$, hence for all $b \in K$ in particular. Therefore $\boxtimes a \leq \boxtimes' a$. Conversely, for any $a \in K$ we have

$$\boxtimes' a \leq \boxtimes a \vee \boxtimes a \rightsquigarrow a$$
$$\text{(by } \boxtimes a \rightsquigarrow a \leq \boxtimes a \to a\text{)} \qquad \leq \boxtimes a \vee \boxtimes a \to a$$
$$\text{(by } \boxtimes a \to a = a \leq \boxtimes a\text{)} \qquad = \boxtimes a.$$

Let V' be an arbitrary valuation on $\mathfrak{K}'$ with $V'(p) = V(p)$ whenever $p \in \mathit{Sfor}(\Gamma/\Delta) \cap \mathit{Prop}$. Then for every $\varphi \in \Theta$ we have $V(\varphi) = V'(\varphi)$. This is shown easily by induction on the structure of φ. Therefore, $\mathfrak{K}', V' \nvDash \Gamma/\Delta$. ☐

The proof of Lemma 5.10 motivates an alternative notion of filtration for frontons. Let $\mathfrak{H}, \mathfrak{K} \in \mathsf{Frt}$. A map $h : \mathfrak{H} \to \mathfrak{K}$ is called *pre-stable* if for every $a, b \in H$ we have $h(a \to b) \leq h(a) \to h(b)$. For $a, b \in H$, we say that h satisfies the $\to$-*bounded domain condition* (BDC$^{\to}$) for (a, b) if $h(a \to b) = h(a) \to h(b)$. For $D \subseteq H$, we say that h satisfies the $\boxtimes$-*bounded domain condition* (BDC$^{\boxtimes}$) for D if $h(\boxtimes a) = \boxtimes h(a)$

for every $a \in D$. If $D \subseteq H \times H$, we say that h satisfies the BDC$^{\to}$ for D if it does for each $(a, b) \in D$, and analogously for the BDC$^{\boxtimes}$. Lastly, if $D^{\to} \subseteq H \times H$ and $D^{\boxtimes} \subseteq H$, we say that h satisfies the BDC for $(D^{\to}, D^{\boxtimes})$ if h satisfies the BDC$^{\to}$ for $D^{\to}$ and the BDC$^{\boxtimes}$ for $D^{\boxtimes}$.

Definition 5.11. Let $\mathfrak{H}$ be a fronton, V a valuation on $\mathfrak{H}$, and Θ a finite, subformula closed set of formulae. A (finite) model $(\mathfrak{K}', V')$, with $\mathfrak{K} \in \mathsf{Frt}$, is called a (*finite*) *weak filtration of* $(\mathfrak{H}, V)$ *through* Θ if the following hold:

(1) $\mathfrak{K}' = (\mathfrak{K}, \to, \boxtimes)$, where $\mathfrak{K}$ is the bounded sublattice of $\mathfrak{H}$ generated by $\bar{V}[\Theta]$;
(2) $V(p) = V'(p)$ for every propositional variable $p \in \Theta$;
(3) The inclusion $\subseteq: \mathfrak{B} \to \mathfrak{A}$ is a pre-stable embedding satisfying the BDC$^{\to}$ for the set $\{(\bar{V}(\varphi), \bar{V}(\psi)) : \varphi \to \psi \in \Theta\}$, and satisfying the BDC$^{\boxtimes}$ for the set $\{\bar{V}(\varphi) : \boxtimes\varphi \in \Theta\}$

A straightforward induction on structure establishes the following filtration theorem.

Theorem 5.12 (Filtration theorem for frontons)**.** *Let $\mathfrak{H}$ be a fronton, V a valuation on $\mathfrak{H}$, and Θ a a finite, subformula-closed set of formulae. If $(\mathfrak{K}', V')$ is a weak filtration of $(\mathfrak{H}, V)$ through Θ then for every $\varphi \in \Theta$ we have*

$$\bar{V}(\varphi) = \bar{V}'(\varphi).$$

Consequently, for every rule Γ/Δ such that $\gamma, \delta \in \Theta$ for each $\gamma \in \Gamma$ and $\delta \in \Delta$ we have

$$\mathfrak{H}, V \models \Gamma/\Delta \iff \mathfrak{K}', V' \models \Gamma/\Delta.$$

We now introduce algebra-based rules for rule systems in $\mathbf{NExt}(\mathtt{KM_R})$ by syntactically encoding weak filtrations as just defined. We call these *pre-stable canonical rules* to emphasize the role of pre-stable maps as opposed to stable maps in their refutation conditions.

Definition 5.13. Let $\mathfrak{H} \in \mathsf{Frt}$ be a finite fronton, and let $D^{\to} \subseteq H \times H$, $D^{\boxtimes} \subseteq H$ be such that $a \in D^{\boxtimes}$ implies $(b, a) \in D^{\to}$ for every $b \in H$. The *pre-stable canonical rule* of $(\mathfrak{H}, D^{\to}, D^{\boxtimes})$, is defined as $\eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes}) = \Gamma/\Delta$, where

$$\begin{aligned} \Gamma :=& \{p_0 \leftrightarrow 0\} \cup \{p_1 \leftrightarrow 1\} \cup \\ & \{p_{a \wedge b} \leftrightarrow p_a \wedge p_b : a \in H\} \cup \{p_{a \vee b} \leftrightarrow p_a \vee p_b : a \in H\} \cup \\ & \{p_{a \to b} \leftrightarrow p_a \to p_b : (a, b) \in D^{\to}\} \cup \{p_{\boxtimes a} \leftrightarrow \boxtimes p_a : a \in D^{\boxtimes}\} \\ \Delta :=& \{p_a \leftrightarrow p_b : a, b \in H \text{ with } a \neq b\}. \end{aligned}$$

The next two results outline algebraic refutation conditions for msi pre-stable canonical rules. They may be proved with straightforward adaptations of the proofs of similar results seen in earlier sections.

Proposition 5.14. *For every finite fronton $\mathfrak{H}$ and $D^{\to} \subseteq H \times H$, $D^{\boxtimes} \subseteq H$, we have $\mathfrak{H} \not\models \eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$.*

Proposition 5.15. *For every msi pre-stable canonical rule $\eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$ and any $\mathfrak{K} \in \mathsf{Frt}$, we have $\mathfrak{K} \not\models \eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$ iff there is a pre-stable embedding $h : \mathfrak{H} \to \mathfrak{K}$ satisfying the BDC for $(D^{\to}, D^{\boxtimes})$.*

We now give refutation conditions for msi pre-stable canonical rules on KM-spaces. If $\mathfrak{X}, \mathfrak{Y}$ are KM-spaces, a map $f : \mathfrak{X} \to \mathfrak{Y}$ is called *pre-stable* if for all $x, y \in X$, $x \leq y$ implies $f(x) \leq f(y)$. Clearly, if f is pre-stable then for all $x, y \in X$, $x \sqsubseteq y$ implies $f(x) \leq f(y)$. Now let $\mathfrak{d} \subseteq Y$. We say that f *satisfies the BDC$^{\to}$ for* $\mathfrak{d}$ if for all $x \in X$,

$$\uparrow[f(x)] \cap \mathfrak{d} \neq \varnothing \Rightarrow f[\uparrow x] \cap \mathfrak{d} \neq \varnothing.$$

We say that f *satisfies the BDC$^{\boxtimes}$ for* $\mathfrak{d}$ if for all $x \in X$ the following two conditions hold.

(BDC$^{\boxtimes}$-back) $\quad \Uparrow[h(x)] \cap \mathfrak{d} \neq \varnothing \Rightarrow h[\Uparrow x] \cap \mathfrak{d} \neq \varnothing$

(BDC$^{\boxtimes}$-forth) $\quad h[\Uparrow x] \cap \mathfrak{d} \neq \varnothing \Rightarrow \Uparrow[h(x)] \cap \mathfrak{d} \neq \varnothing$

If $\mathfrak{D} \subseteq \wp(Y)$, then we say that h satisfies the BDC$^{\to}$ for $\mathfrak{D}$ if it does for every $\mathfrak{d} \in \mathfrak{D}$, and similarly for the BDC$^{\boxtimes}$. Finally, if $\mathfrak{D}^{\to}, \mathfrak{D}^{\boxtimes} \subseteq \wp(Y)$, then we say that h satisfies the BDC for $(\mathfrak{D}^{\to}, \mathfrak{D}^{\boxtimes})$ if h satisfies the BDC$^{\to}$ for $\mathfrak{D}^{\to}$ and the BDC$^{\boxtimes}$ for $\mathfrak{D}^{\boxtimes}$.

Let $\mathfrak{H}$ be a finite fronton. If $D^{\to} \subseteq H \times H$, for every $(a, b) \in D^{\to}$ set $\mathfrak{d}^{\to}_{(a,b)} := \beta(a) \smallsetminus \beta(b)$. If $D^{\boxtimes} \subseteq H$, for every $a \in D^{\boxtimes}$ set $\mathfrak{d}^{\boxtimes}_a := -\beta(a)$. Finally, put $\mathfrak{D}^{\to} := \{\mathfrak{d}^{\to}_{(a,b)} : (a, b) \in D^{\to}\}$, $\mathfrak{D}^{\boxtimes} := \{\mathfrak{d}^{\boxtimes}_a : a \in D^{\boxtimes}\}$.

Proposition 5.16. *For every msi pre-stable canonical rule $\eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$ and any* KM*-space, we have $\mathfrak{X} \nvDash \eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$ iff there is a continuous pre-stable surjection $f : \mathfrak{X} \to \mathfrak{H}_*$ satisfying the BDC $(\mathfrak{D}^{\to}, \mathfrak{D}^{\boxtimes})$.*

Proof. $(\Rightarrow)$ Assume $\mathfrak{X} \nvDash \eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$. Then there is a pre-stable embedding $h : \mathfrak{H} \to \mathfrak{X}^*$ satisfying the BDC for $(D^{\to}, D^{\boxtimes})$. Reasoning as in the proofs of Proposition 3.16 and Proposition 3.24 it follows that there is a pre-stable map $f : \mathfrak{X} \to \mathfrak{H}_*$ satisfying the BDC$^{\to}$ for $\mathfrak{D}^{\to}$ and satisfying the BDC$^{\boxtimes}$-back for $\mathfrak{D}^{\boxtimes}$, namely the map $f = h^{-1}$. Let us check that f satisfies the BDC$^{\boxtimes}$-forth for $\mathfrak{D}^{\boxtimes}$. Let $\mathfrak{d}^{\boxtimes}_a \in \mathfrak{D}^{\boxtimes}$. Assume $f[\Uparrow x] \cap \mathfrak{d}^{\boxtimes}_a \neq \varnothing$, i.e., that there is $y \in \Uparrow x$ with $f(y) \in \mathfrak{d}^{\boxtimes}_a$. So $x \notin \boxtimes_{\sqsubseteq} h(U)$, where $U := -\mathfrak{d}^{\boxtimes}_a$. Since h satisfies the BDC$^{\boxtimes}$ for $\mathfrak{d}^{\boxtimes}_a$ we have $\boxtimes_{\sqsubseteq} h(U) = h(\boxtimes_{\sqsubseteq} U)$, and so $x \notin h(\boxtimes_{\sqsubseteq} U)$. This implies $f(x) \notin \boxtimes_{\sqsubseteq}(U)$, therefore there must be some $z \in \mathfrak{d}^{\boxtimes}_a$ such that $f(x) \sqsubseteq z$, i.e. $\Uparrow[f(x)] \cap \mathfrak{d}^{\boxtimes}_a \neq \varnothing$.

$(\Leftarrow)$ Assume that there is a continuous pre-stable surjection $f : \mathfrak{X} \to \mathfrak{H}_*$ satisfying the BDC for $(\mathfrak{D}^{\to}, \mathfrak{D}^{\boxtimes})$. By the proof of Proposition 3.16, $f^{-1} : \mathfrak{H} \to \mathfrak{X}^*$ is a pre-stable embedding satisfying the BDC$^{\to}$ for $D^{\to}$. Let us check that f^{-1} satisfies the BDC$^{\boxtimes}$ for $D^{\boxtimes}$. Let $U \subseteq X$ be such that $U = \beta(a)$ for some $a \in D^{\boxtimes}$, and reason as follows.

$$
\begin{aligned}
& x \notin f^{-1}(\boxtimes_{\sqsubseteq} U) && \iff \Uparrow x \cap f^{-1}(\mathfrak{d}^{\boxtimes}_a) \neq \varnothing \\
& (f \text{ satisfies the BDC}^{\boxtimes} \text{ for } \mathfrak{d}^{\boxtimes}_a) && \iff \Uparrow[f(x)] \cap \mathfrak{d}^{\boxtimes}_a \neq \varnothing \\
& && \iff x \notin \boxtimes_{\sqsubseteq} f^{-1}(U).
\end{aligned}
$$

□

In view of Proposition 5.16, when working with KM-spaces we may write an msi pre-stable canonical rule $\eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$ as $\eta_{\boxtimes}(\mathfrak{H}_*, \mathfrak{D}^{\to}, \mathfrak{D}^{\boxtimes})$.

We close this subsection by proving that our msi pre-stable canonical rules are expressive enough to axiomatise every rule system in $\mathbf{NExt}(\mathtt{KM_R})$.

Lemma 5.17. *For every msi rule Γ/Δ there is a finite set Ξ of msi pre-stable canonical rules such that for any $\mathfrak{K} \in \mathsf{Frt}$ we have $\mathfrak{K} \nvDash \Gamma/\Delta$ iff there is $\eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes}) \in \Xi$ such that $\mathfrak{K} \nvDash \eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$.*

Proof. Since bounded distributive lattices are locally finite there are, up to isomorphism, only finitely many triples $(\mathfrak{H}, D^{\to}, D^{\boxtimes})$ such that

- $\mathfrak{H} \in \mathsf{Frt}$ and $\mathfrak{H}$ is at most k-generated as a bounded distributive lattice, where $k = |Sfor(\Gamma/\Delta)|$;
- There is a valuation V on $\mathfrak{H}$ refuting Γ/Δ, such that

$$D^{\to} = \{(\bar{V}(\varphi), \bar{V}(\psi)) : \varphi \to \psi \in Sfor(\Gamma/\Delta)\} \cup \{(\bar{V}(\varphi), b) : \boxtimes\varphi \in Sfor(\Gamma/\Delta) \text{ and } b \in H\}$$
$$D^{\boxtimes} = \{\bar{V}(\varphi) : \boxtimes\varphi \in Sfor(\Gamma/\Delta)\}.$$

Let Ξ be the set of all msi pre-stable canonical rules $\eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$ for all such triples $(\mathfrak{H}, D^{\to}, D^{\boxtimes})$, identified up to isomorphism.

($\Rightarrow$) Let $\mathfrak{K} \in \mathsf{Frt}$ and suppose $\mathfrak{H} \nvDash \Gamma/\Delta$. Take a valuation V on $\mathfrak{H}$ such that $\mathfrak{K}, V \nvDash \Gamma/\Delta$. Then by the proof of Lemma 5.10 there is a weak filtration $(\mathfrak{H}', V')$ of $(\mathfrak{K}, V)$ through $Sfor(\Gamma/\Delta)$, which by the filtration theorem for frontons is such that $\mathfrak{H}', V' \nvDash \Gamma/\Delta$. This implies that there is a stable embedding $h : \mathfrak{H}' \to \mathfrak{K}$, which again by the proof of Lemma 5.10 satisfies the BDC for the pair $(\mathfrak{D}^{\to}, \mathfrak{D}^{\boxtimes})$ defined as above. Therefore $\eta_{\boxtimes}(\mathfrak{H}', D^{\to}, D^{\boxtimes}) \in \Xi$ and $\mathfrak{K} \nvDash \eta_{\boxtimes}(\mathfrak{H}', D^{\to}, D^{\boxtimes})$.

($\Leftarrow$) Analogous to the same direction in, e.g., Lemma 3.19. ◻

Theorem 5.18. *Every msi-rule system $\mathtt{L} \in \mathbf{NExt}(\mathtt{KM_R})$ is axiomatisable over $\mathtt{KM_R}$ by some set of msi pre-stable canonical rules of the form $\eta_{\boxtimes}(\mathfrak{H}, D^{\to}, D^{\boxtimes})$, where $\mathfrak{H} \in \mathsf{KM}$.*

Proof. Analogous to Theorem 3.20. ◻

5.2.2. *The $\mathtt{GL_R}$ Case.* Modal stable canonical rules as developed in Section 3.2.2 can axiomatise every rule system in $\mathbf{NExt}(\mathtt{GL_R})$ [4, Theorem 5.6]. However, modal stable canonical rules differ significantly from msi pre-stable canonical rules: they are based on a different notion of filtration, which is stated in terms of stable rather than pre-stable maps. Moreover, $\mathtt{GL_R}$ admits very few filtrations. The situation is similar to the case of $\mathbf{NExt}(\mathtt{KM_R})$. For recall (Proposition 5.9) that finite $\mathtt{GL}$-spaces are strict partial orders. If $\mathfrak{X}$ is a $\mathtt{GL}$-space and $f : \mathfrak{X} \to \mathfrak{Y}$ is a stable map from $\mathfrak{X}$ onto some finite modal space $\mathfrak{Y}$ such that $f(x) = f(y)$ for some $x, y \in X$ with Rxy, then $\mathfrak{Y}$ contains a reflexive point, hence cannot be a $\mathtt{GL}$-space.

In response to this problem, an alternative notion of filtration was introduced in [54], who note that the same technique was used already in [13]. We call it *weak filtration*. As usual, we prefer an algebraic definition. If $\mathfrak{A}, \mathfrak{B}$ are modal algebras and $D \subseteq A$, let us say that a map $h : \mathfrak{A} \to \mathfrak{B}$ satisfies the *$\Box$-bounded domain condition* (BDC$^{\Box}$) for D if $h(\Box a) = \Box(a)$ for every $a \in D$.

Definition 5.19. Let $\mathfrak{B} \in \mathsf{Mag}$ be a Magari algebra, V a valuation on $\mathfrak{B}$, and Θ a finite, subformula closed set of formulae. A (finite) model $(\mathfrak{A}', V')$, with $\mathfrak{A}' \in \mathsf{Mag}$, is called a *(finite) weak filtration of* $(\mathfrak{B}, V)$ *through* Θ if the following hold:

1. $\mathfrak{A}' = (\mathfrak{A}, \Box)$, where $\mathfrak{B}$ is the Boolean subalgebra of $\mathfrak{B}$ generated by $\bar{V}[\Theta]$;
2. $V(p) = V'(p)$ for every propositional variable $p \in \Theta$;

(3) The inclusion $\subseteq: \mathfrak{A}' \to \mathfrak{B}$ satisfies the BDC$^{\Box}$ for $D := \{\bar{V}'(\varphi) : \Box\varphi \in \Theta\}$.

Theorem 5.20. *Let $\mathfrak{B} \in \mathsf{Mag}$ be a Magari algebra, V a valuation on $\mathfrak{B}$, and Θ a finite, subformula closed set of formulae. Let $(\mathfrak{A}', V')$ be a weak filtration of $(\mathfrak{B}, V)$. Then for every $\varphi \in \Theta$ we have*

$$\bar{V}(\varphi) = \bar{V}'(\varphi).$$

Proof. Straightforward induction on the structure of φ. □

Unlike weak filtrations in the msi setting, modal weak filtrations are not in general unique. We will be particularly interested in weak filtrations satisfying an extra condition, which we will construe as a modal counterpart to pre-stability in the msi setting. For any modal algebra $\mathfrak{A}$ and $a \in A$ we write $\Box^+(a) := \Box a \wedge a$. Let $\mathfrak{A}, \mathfrak{B} \in \mathsf{Mag}$ be Magari algebras. A Boolean homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ is called *pre-stable* if for every $a \in A$ we have $h(\Box^+ a) \leq \Box^+ h(a)$. Clearly, every stable Boolean homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ is pre-stable, since $h(\Box a) \leq \Box h(a)$ implies $h(\Box a \wedge a) = h(\Box a) \wedge h(a) \leq \Box h(a) \wedge h(a)$. A weak filtration $(\mathfrak{A}', V')$ of some model $(\mathfrak{B}, V)$ through some finite, subformula closed set of formulae Θ is called *pre-stable* if the embedding $\subseteq: \mathfrak{A}' \to \mathfrak{B}$ is pre-stable.

If $\mathfrak{A}, \mathfrak{B}$ are modal algebras and $D \subseteq A$, a map $h : \mathfrak{A} \to \mathfrak{B}$ satisfies the $\Box^+$-*bounded domain condition* (BDC$^{\Box^+}$) for D if $h(\Box^+ a) = \Box^+ h(a)$ for every $a \in D$. Note that if $(\mathfrak{A}', V')$ is a filtration of $(\mathfrak{B}, V)$ through some Θ, then for every $D \subseteq A$ the inclusion $\subseteq: \mathfrak{A} \to \mathfrak{B}$ satisfies the BDC$^{\Box^+}$ for D iff it satisfies the BDC$^{\Box}$ for D. Indeed, since Θ is subformula-closed we have that $\Box^+\varphi \in \Theta$ implies $\Box\varphi \in \Theta$, which gives the "only if" direction, whereas the converse follows from the fact that $\subseteq$ is a Boolean embedding.

Our algebra-based rules encode pre-stable weak filtrations as defined above, and explicitly include a parameter $D^{\Box^+}$, linked to the BDC$^{\Box^+}$, intended as a counterpart to the parameter $D^{\to}$ of msi pre-stable canonical rules. We call these rules *modal pre-stable canonical rules.*

Definition 5.21. Let $\mathfrak{A} \in \mathsf{MA}$ be a finite modal algebra, and let $D^{\Box^+}, D^{\Box} \subseteq A$. Let $\Box^+\varphi := \Box\varphi \wedge \varphi$. The *pre-stable canonical rule* of $(\mathfrak{A}, D^{\Box^+}, D^{\Box})$, is defined as $\mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box}) = \Gamma/\Delta$, where

$$\begin{aligned}\Gamma :=& \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a \in H\} \cup \{p_{a\vee b} \leftrightarrow p_a \vee p_b : a \in H\} \cup \\ & \{p_{\neg a} \leftrightarrow \neg p_a : a \in A\} \cup \{p_{\Box^+ a} \to \Box^+ p_a : a \in a\} \cup \\ & \{\Box^+ p_a \to p_{\Box^+ a} : a \in D^{\Box^+}\} \cup \{p_{\Box a} \leftrightarrow \Box p_a : a \in D^{\Box}\} \\ \Delta :=& \{p_a : a \in A \smallsetminus \{1\}\}.\end{aligned}$$

It is helpful to conceptualise modal pre-stable canonical rules as algebra-based rules for bi-modal rule systems in the signature $\{\wedge, \vee, \neg, \Box, \Box^+, 0, 1\}$ (so that $\Box^+$ is an independent operator rather than defined from $\Box$) and containing $\Box^+ p \leftrightarrow \Box p \wedge p$ as an axiom.[1] From this perspective, modal pre-stable canonical rules are rather similar to msi pre-stable canonical rules.

[1]This view of GL as a bimodal logic is the main insight informing Litak's [38] strategy for deriving Item 2 of Theorem 5.35 from the theory of polymodal companions of msi-logics as developed by Wolter and Zakharyaschev [59, 58]. In that setting, msi formulae are translated into formulae in a bimodal signature, but the two modalities of the latter can be regarded as implicitly interdefinable in logics where one satisfies the Löb formula.

Using by now familiar reasoning, it is easy to verify that modal pre-stable canonical rules display the intended refutation conditions. For brevity, let us say that a pre-stable map h satisfies the BDC for $(D^{\Box+}, D^{\Box})$ if h satisfies the BDC$^{\Box+}$ for $D^{\Box+}$ and the BDC$^{\Box}$ for $D^{\Box}$.

Proposition 5.22. *For every finite modal algebra $\mathfrak{A} \in \mathsf{MA}$ and $D^{\Box+}, D^{\Box} \subseteq A$, we have $\mathfrak{H} \nvDash \mu_+(\mathfrak{A}, D^{\Box+}, D^{\Box})$.*

Proposition 5.23. *For every modal algebra $\mathfrak{B} \in \mathsf{MA}$ and any modal pre-stable canonical rule $\mu_+(\mathfrak{A}, D^{\Box+}, D^{\Box})$, we have $\mathfrak{B} \nvDash \mu_+(\mathfrak{A}, D^{\Box+}, D^{\Box})$ iff there is a pre-stable embedding $h : \mathfrak{B} \to \mathfrak{A}$ satisfying the BDC $(D^{\Box+} D^{\Box})$.*

If $\mathfrak{X}$ is any modal space, for any $x, y \in X$ define R^+xy iff Rxy or $x = y$. Let $\mathfrak{X}, \mathfrak{Y}$ be $\mathtt{GL}$-spaces. A map $f : \mathfrak{X} \to \mathfrak{Y}$ is called *pre-stable* if for all $x, y \in X$ we have that R^+xy implies $R^+f(x)f(y)$. If $\mathfrak{d} \subseteq Y$, we say that f satisfies the BDC$^{\Box+}$ for $\mathfrak{d}$ if for all $x \in X$,

$$R^+[f(x)] \cap \mathfrak{d} \neq \varnothing \Rightarrow f[R^+[x]] \cap \mathfrak{d} \neq \varnothing.$$

Furthermore, we say that f satisfies the BDC$^{\Box}$ for $\mathfrak{d}$ if for all $x \in X$ the following two conditions hold.

(BDC$^{\Box}$-back) $$R[f(x)] \cap \mathfrak{d} \neq \varnothing \Rightarrow f[R[x]] \cap \mathfrak{d} \neq \varnothing$$

(BDC$^{\Box}$-forth) $$f[R[x]] \cap \mathfrak{d} \neq \varnothing \Rightarrow R[f(x)] \cap \mathfrak{d} \neq \varnothing.$$

Finally, if $\mathfrak{D} \subseteq \wp(Y)$ we say that f satisfies the BDC$^{\Box+}$ (resp. BDC$^{\Box}$) for $\mathfrak{D}$ if it does for every $\mathfrak{d} \in \mathfrak{D}$, and if $\mathfrak{D}^{\Box+}, \mathfrak{D}^{\Box} \subseteq \wp(Y)$ we write that f satisfies the BDC for $(\mathfrak{D}^{\Box+}, \mathfrak{D}^{\Box})$ if f satisfies the BDC$^{\Box+}$ for $\mathfrak{D}^{\Box+}$ and the BDC$^{\Box}$ for $\mathfrak{D}^{\Box}$. Let $\mathfrak{A}$ be a finite Magari algebra. If $D^{\Box+} \subseteq A$, for every $a \in D^{\Box+}$ set $\mathfrak{d}^{\Box+}_a := -\beta(a)$. If $D^{\Box} \subseteq A$, for every $a \in D^{\Box}$ set $\mathfrak{d}^{\Box}_a := -\beta(a)$. Finally, put $\mathfrak{D}^{\Box+} := \{\mathfrak{d}^{\Box+}_a : a \in D^{\Box+}\}$, $\mathfrak{D}^{\Box} := \{\mathfrak{d}^{\Box}_a : a \in D^{\Box}\}$.

Proposition 5.24. *For all $\mathtt{GL}$-spaces $\mathfrak{X}$ and any modal pre-stable canonical rule $\mu_+(\mathfrak{A}, D^{\Box+}, D^{\Box})$, we have $\mathfrak{X} \nvDash \mu_+(\mathfrak{A}, D^{\Box+}, D^{\Box})$ iff there is a continuous pre-stable surjection $f : \mathfrak{X} \to \mathfrak{A}_*$ satisfying the BDC for $(\mathfrak{D}^{\Box+}, \mathfrak{D}^{\Box})$.*

As usual, in view of Proposition 5.24 we write a modal pre-stable canonical rule $\mu_+(\mathfrak{A}, D^{\Box+}, D^{\Box})$ as $\mu_+(\mathfrak{A}_*, \mathfrak{D}^{\Box+}, \mathfrak{D}^{\Box})$ in geometric settings.

We close this section by proving that pre-stable canonical rules axiomatise any rule system in $\mathbf{NExt}(\mathtt{GL_R})$.

Lemma 5.25. *For every modal rule Γ/Δ there is a finite set Ξ of modal pre-stable canonical rules of the form $\mu_+(\mathfrak{A}, D^{\Box+}, D^{\Box})$ with $\mathfrak{A} \in \mathsf{K4}$, such that for any $\mathfrak{B} \in \mathsf{Mag}$ we have $\mathfrak{B} \nvDash \Gamma/\Delta$ iff there is $\mu_+(\mathfrak{A}, D^{\Box+}, D^{\Box}) \in \Xi$ such that $\mathfrak{B} \nvDash \mu_+(\mathfrak{A}, D^{\Box+}, D^{\Box})$.*

Proof. Since Boolean algebras is locally finite there are, up to isomorphism, only finitely many triples $(\mathfrak{A}, D^{\Box+}, D^{\Box})$ such that

- $\mathfrak{A} \in \mathsf{K4}$ and $\mathfrak{A}$ is at most k-generated as a Boolean algebra, where $k = |Sfor(\Gamma/\Delta)|$;

- There is a valuation V on $\mathfrak{A}$ refuting Γ/Δ, such that

$$D^{\Box^+} = \{\bar{V}(\varphi) : \Box^+\varphi \in Sfor(\Gamma/\Delta)\}$$
$$D^{\Box} = \{\bar{V}(\varphi) : \Box\varphi \in Sfor(\Gamma/\Delta)\}$$

Let Ξ be the set of all modal pre-stable canonical rules $\mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box})$ for all such triples $(\mathfrak{A}, D^{\Box^+}, D^{\Box})$, identified up to isomorphism.

($\Rightarrow$) Let $\mathfrak{B} \in \mathsf{Mag}$ and suppose $\mathfrak{B} \nvDash \Gamma/\Delta$. Take a valuation V on $\mathfrak{B}$ such that $\mathfrak{B}, V \nvDash \Gamma/\Delta$. As is well-known, there is a transitive filtration $(\mathfrak{A}', V')$ of $(\mathfrak{B}, V)$ through $Sfor(\Gamma/\Delta)$. Then $\mathfrak{A}' \in \mathsf{K4}$. Moreover, clearly every filtration is a weak filtration, hence so is $(\mathfrak{A}', V')$. Therefore there is a Boolean embedding $h : \mathfrak{A}' \to \mathfrak{B}$ satisfying the BDC for $(D^{\Box^+}, D^{\Box})$, where $D^{\Box^+} := \{\bar{V}'(\varphi) : \Box^+\varphi \in Sfor(\Gamma/\Delta)\}$ and $D^{\Box} := \{\bar{V}'(\varphi) : \Box\varphi \in Sfor(\Gamma/\Delta)\}$. Indeed, it is obvious that h is a Boolean embedding which satisfies the BDC$^{\Box}$ for $D^{\Box}$. The fact that h satisfies the BDC$^{\Box^+}$ follows by noting that, additionally, $\Box\varphi \in Sfor(\Box^+\varphi)$ for every modal formula φ. Lastly, since $(\mathfrak{A}', V')$ is actually a filtration, f is stable, a fortiori pre-stable. Hence we have shown $\mathfrak{B} \nvDash \mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box})$.

($\Leftarrow$) Routine. $\Box$

Theorem 5.26. *Every modal rule system* $\mathtt{M} \in \mathbf{NExt}(\mathtt{GL_R})$ *is axiomatisable over* $\mathtt{GL_R}$ *by some set of modal pre-stable canonical rules of the form* $\mu_+(\mathfrak{A}, D^{\Box^+}, D^{\Box})$, *where* $\mathfrak{A} \in \mathsf{K4}$.

5.3. **The Kuznetsov-Muravitsky Isomorphism via Stable Canonical Rules.** We are ready for the main topic of this section, the Kuznetsov-Muravitsky isomorphism and its extension to rule systems. We apply pre-stable canonical rules to prove this and related results in the vicinity, using essentially the same techniques seen in Sections 3.3 and 4.3.

5.3.1. *Semantic Mappings.* We begin by reviewing the constructions for transforming frontons into corresponding Magari algebras and vice versa. The results in this paragraph are known, and recent proofs can be found in, e.g., [25].

Definition 5.27. The mapping $\sigma : \mathsf{Frt} \to \mathsf{Mag}$ assigns every $\mathfrak{H} \in \mathsf{Frt}$ to the algebra $\sigma\mathfrak{H} := (B(\mathfrak{H}), \Box)$, where $B(\mathfrak{H})$ is the free Boolean extension of $1\mathfrak{H}$ and for every $a \in B(H)$ we have

$$Ia := \bigvee\{b \in H : b \leq a\}$$
$$\Box a := \boxtimes Ia.$$

Observe that if $a \in H$ then $Ia = a$, and so $\Box a = \boxtimes a$. Consequently, if $a \in H$ also $\Box^+ a = \boxtimes^+ a$.

Definition 5.28. The mapping $\rho : \mathsf{Mag} \to \mathsf{Frt}$ assigns every Magari algebra $\mathfrak{A} \in \mathsf{Mag}$ to the algebra $\rho\mathfrak{A} := (O(A), \wedge, \vee, \to, \Box, 1, 0)$, where

$$O(A) := \{a \in A : \Box^+ a = a\}$$
$$a \to b := \Box^+(\neg a \vee b)$$
$$\boxtimes a := \Box a$$

By unpacking the definitions just presented it is not difficult to verify that the following Proposition holds.

Proposition 5.29. *For every* $\mathfrak{H} \in \mathsf{Frt}$ *we have* $\mathfrak{H} \cong \rho\sigma\mathfrak{H}$*. Moreover, for every* $\mathfrak{A} \in \mathsf{GRZ}$ *we have* $\sigma\rho\mathfrak{A} \rightarrowtail \mathfrak{A}$*.*

We call a Magari algebra $\mathfrak{A}$ *skeletal* if $\sigma\rho\mathfrak{A} \cong \mathfrak{A}$ holds.

We now give more suggestive dual descriptions of the maps σ, ρ on $\mathtt{KM}$- and $\mathtt{GL}$-spaces, which also make it easier to show that σ, ρ are the intended ranges.

Definition 5.30. If $\mathfrak{X} = (X, \leq, \sqsubseteq, \mathcal{O})$ is a $\mathtt{KM}$-space we set $\sigma\mathfrak{X} := (X, R, \mathcal{O})$, where $R = \sqsubseteq$. Let $\mathfrak{Y} := (Y, R, \mathcal{O})$ be a $\mathtt{GL}$-space. For $x, y \in Y$ write $x \sim y$ iff Rxy and Ryx. Define a map $\rho : Y \to \wp(Y)$ by setting $\rho(x) = \{y \in Y : x \sim y\}$. We define $\rho\mathfrak{Y} := (\rho[Y], \leq_\rho, \sqsubseteq_\rho \rho[\mathcal{O}])$ where $\rho(x) \sqsubseteq_\rho \rho(y)$ iff Rxy and $\rho(x) \leq_\rho \rho(y)$ iff $R^+_\rho \rho(x)\rho(y)$.

Proposition 5.31. *The following conditions hold.*

1. *Let* $\mathfrak{H} \in \mathsf{Frt}$*. Then* $(\sigma\mathfrak{H})_* \cong \sigma(\mathfrak{H}_*)$*. Consequently, if* $\mathfrak{X}$ *is a* $\mathtt{KM}$*-space then* $(\sigma\mathfrak{X})^* \cong \sigma(\mathfrak{X})^*$*.*
2. *Let* $\mathfrak{X}$ *be a* $\mathtt{GL}$*-space. Then* $(\rho\mathfrak{X})^* \cong \rho(\mathfrak{X}^*)$*. Consequently, if* $\mathfrak{A} \in \mathsf{Mag}$*, then* $(\rho\mathfrak{A})_* \cong \rho(\mathfrak{A}_*)$*.*

Proposition 5.32. *For every fronton* $\mathfrak{H} \in \mathsf{Frt}$ *we have that* $\sigma\mathfrak{H}$ *is a Magari algebra, and for every Magari algebra* $\mathfrak{A} \in \mathsf{Mag}$ *we have that* $\rho\mathfrak{A}$ *is a fronton.*

5.3.2. *A Gödelian Translation.* We now show how to translate msi formulae into modal formulae in a way which suits our current goals. The main idea, already anticipated when developing msi stable canonical rules, is to conceptualise rule systems in $\mathbf{NExt}(\mathtt{GL_R})$ as stated in a signature containing two modal operators $\Box, \Box^+$, so to use $\Box$ to translate $\boxtimes$ and $\Box^+$ to translate $\to$. This leads to the following Gödelian translation function.

Definition 5.33. The Gödelian translation $T : Tm_{msi} \to Tm_{md}$ is defined recursively as follows.

$$\begin{aligned}
T(\bot) &:= \bot \\
T(\top) &:= \top \\
T(p) &:= \Box p \\
T(\varphi \wedge \psi) &:= T(\varphi) \wedge T(\psi) \\
T(\varphi \vee \psi) &:= T(\varphi) \vee T(\psi) \\
T(\varphi \to \psi) &:= \Box^+(\neg T(\varphi) \vee T(\psi)) \\
T(\boxtimes\varphi) &:= \Box T(\varphi)
\end{aligned}$$

The translation T above was originally proposed by Kuznetsov and Muravitsky [37], and is systematically studied in [59, 58]. Our presentation contains a revised clause for the case of $T(\boxtimes\varphi)$, which was originally defined as

$$T(\boxtimes\varphi) := \Box^+\Box T(\varphi).$$

However, it is not difficult to verify that $\mathsf{Mag} \models \Box p \leftrightarrow \Box^+\Box p$, which justifies our revised clause. As usual, we extend the translation T from terms to rules by setting

$$T(\Gamma/\Delta) := T[\Gamma]/T[\Delta].$$

The following key lemma describes the semantic behaviour of $T(\cdot)$ in terms of the map ρ.

Lemma 5.34. *For every* $\mathfrak{A} \in \mathsf{Mag}$ *and si rule* Γ/Δ,

$$\mathfrak{A} \models T(\Gamma/\Delta) \iff \rho\mathfrak{A} \models \Gamma/\Delta$$

Proof. A simple induction on structure shows that for every si term φ, every modal space $\mathfrak{X}$, every valuation V on $\mathfrak{X}$ and every point $x \in X$ we have

$$\mathfrak{X}, V, x \models T(\varphi) \iff \rho\mathfrak{X}, \rho[V], \rho(x) \models \varphi.$$

Using this equivalence and noting that every valuation V on some $\mathtt{KM}$-space $\rho\mathfrak{X}$ can be seen as of the form $\rho[V']$ for some valuation V' on $\mathfrak{X}$, the rest of the proof is easy. $\square$

5.3.3. *The Kuznetsov-Muravitsky Theorem.* We are now ready to state and prove the main result of the present section. Extend the mappings $\sigma : \mathsf{Frt} \to \mathsf{Mag}$ and $\rho : \mathsf{Mag} \to \mathsf{Frt}$ by setting

$$\begin{aligned} \sigma : \mathbf{Uni}(\mathsf{Frt}) &\to \mathbf{Uni}(\mathsf{Mag}) & \rho : \mathbf{Uni}(\mathsf{Mag}) &\to \mathbf{Uni}(\mathsf{Frt}) \\ \mathcal{U} &\mapsto \mathsf{Uni}\{\sigma\mathfrak{H} : \mathfrak{H} \in \mathcal{U}\} & \mathcal{W} &\mapsto \{\rho\mathfrak{A} : \mathfrak{A} \in \mathcal{W}\}. \end{aligned}$$

Now define the following two syntactic counterparts to σ, ρ between $\mathbf{NExt}(\mathtt{KM_R})$ and $\mathbf{NExt}(\mathtt{GL_R})$.

$$\begin{aligned} \sigma : \mathbf{NExt}(\mathtt{KM_R}) &\to \mathbf{NExt}(\mathtt{GL_R}) & \rho : \mathbf{NExt}(\mathtt{GL_R}) &\to \mathbf{NExt}(\mathtt{KM_R}) \\ \mathtt{L} &\mapsto \mathtt{GL_R} \oplus \{T(\Gamma/\Delta) : \Gamma/\Delta \in \mathtt{L}\} & \mathtt{M} &\mapsto \{\Gamma/\Delta : T(\Gamma/\Delta) \in \mathtt{M}\} \end{aligned}$$

These maps easily extend to lattices of logics, by setting:

$$\begin{aligned} \sigma : \mathbf{NExt}(\mathtt{KM}) &\to \mathbf{NExt}(\mathtt{GL}) & \rho : \mathbf{NExt}(\mathtt{GL}) &\to \mathbf{NExt}(\mathtt{KM}) \\ \mathtt{L} &\mapsto \mathsf{Taut}(\sigma\mathtt{L_R}) = \mathtt{GL} \oplus \{T(\varphi) : \varphi \in \mathtt{L}\} & \mathtt{M} &\mapsto \mathsf{Taut}(\rho\mathtt{M_R}) = \{\varphi : T(\varphi) \in \mathtt{M}\} \end{aligned}$$

The goal of this subsection is to establish the following result using pre-stable canonical rules.

Theorem 5.35 (Kuznetsov-Muravitsky theorem)**.** *The following conditions hold:*

(1) $\sigma : \mathbf{NExt}(\mathtt{KM_R}) \to \mathbf{NExt}(\mathtt{GL_R})$ *and* $\rho : \mathbf{NExt}(\mathtt{GL_R}) \to \mathbf{NExt}(\mathtt{KM_R})$ *are mutually inverse complete lattice isomorphisms.*
(2) $\sigma : \mathbf{NExt}(\mathtt{KM}) \to \mathbf{NExt}(\mathtt{GL})$ *and* $\rho : \mathbf{NExt}(\mathtt{GL}) \to \mathbf{NExt}(\mathtt{KM})$ *are mutually inverse complete lattice isomorphisms.*

Similarly to the previous sections, the main difficulty to overcome here consists in showing that $\sigma : \mathbf{NExt}(\mathtt{KM_R}) \to \mathbf{NExt}(\mathtt{GL_R})$ is surjective. We approach this problem by applying our pre-stable canonical rules, following a similar blueprint as that used in the previous sections. The following lemma is a counterpart of Lemma 3.38. Its proof is similar to the latter's, thanks to the similarities existing between $\mathtt{GRZ}$- and $\mathtt{GL}$-spaces.

Lemma 5.36. *Let* $\mathfrak{A} \in \mathsf{Mag}$*. Then for every modal rule* Γ/Δ *we have* $\mathfrak{A} \models \Gamma/\Delta$ *iff* $\sigma\rho\mathfrak{A} \models \Gamma/\Delta$*.*

Proof. ($\Rightarrow$) This direction follows from the fact that $\sigma\rho\mathfrak{A} \rightarrowtail \mathfrak{A}$ (Proposition 5.29).

($\Leftarrow$) We prove the dual statement that $\mathfrak{A}_* \nvDash \Gamma/\Delta$ implies $\sigma\rho\mathfrak{A}_* \nvDash \Gamma/\Delta$. Let $\mathfrak{X} := \mathfrak{A}_*$. In view of Theorem 5.26 it suffices to consider the case $\Gamma/\Delta = \mu_+(\mathfrak{B}, D^{\Box^+}, D^{\Box})$, for $\mathfrak{B} \in \mathsf{K4}$ finite. So suppose $\mathfrak{X} \nvDash \mu_+(\mathfrak{B}, D^{\Box^+}, D^{\Box})$ and let $\mathfrak{F} := \mathfrak{B}_*$. Then there is a pre-stable map $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $(\mathfrak{D}^{\square^+}, \mathfrak{D}^{\square})$. We construct a pre-stable map $g : \sigma\rho\mathfrak{X} \to \mathfrak{F}$ which also satisfies the BDC for $(\mathfrak{D}^{\square^+}, \mathfrak{D}^{\square})$.

Let C be a cluster in $\mathfrak{F}$. Consider $Z_C := f^{-1}(C)$. As f is continuous, Z_C is clopen. Moreover, since f is pre-stable Z_C does not cut any cluster. It follows that $\rho[Z_C]$ is clopen in $\rho\mathfrak{X}$, because $\rho\mathfrak{X}$ has the quotient topology.

Enumerate $C := \{x_1, \dots, x_n\}$. Then $f^{-1}(x_i) \subseteq Z_C$ is clopen. By Proposition 5.9, we have that $M_i := max_R(f^{-1}(x_i))$ is clopen. Furthermore, as every element of M_i is maximal in M_i, by Proposition 5.9 again we have that M_i does not cut any cluster. Therefore $\rho[M_i]$ is clopen, because $\rho\mathfrak{X}$ has the quotient topology. Clearly, $\rho[M_i] \cap \rho[M_j] = \varnothing$ for each $i \neq j$. Therefore there are disjoint clopens $U_1, \dots, U_n$ with $\rho[M_i] \subseteq U_i$ and $\bigcup_i U_i = \rho[Z_C]$. Just take $U_i := \rho[M_i]$ if $i \neq n$, and

$$U_n := \rho[Z_C] \smallsetminus \left(\bigcup_{i<n} U_i \right).$$

Now define

$$g_C : \rho[Z_C] \to C$$

$$g_C(z) = x_i \iff z \in U_i$$

Note that g_C is relation preserving, evidently, and continuous by construction. Finally, define $g : \sigma\rho\mathfrak{X} \to F$ by setting

$$g(\rho(z)) := \begin{cases} f(z) & \text{if } f(z) \text{ does not belong to any proper cluster} \\ g_C(\rho(z)) & \text{if } f(z) \in C \text{ for some proper cluster } C \subseteq F \end{cases}$$

Now, g is evidently pre-stable. Moreover, it is continuous because both f and each g_C are. Let us check that g satisfies the BDC for $(\mathfrak{D}^{\square^+}, \mathfrak{D}^{\square})$.

- (BDC$^{\square^+}$) This may be shown reasoning the same way as in the proof of Lemma 3.38.
- (BDC$^{\square}$-back) Let $\mathfrak{d} \in \mathfrak{D}^{\square}$ and $\rho(x) \in \rho[X]$. Suppose that $R[g(\rho(x))] \cap \mathfrak{d} \neq \varnothing$. Let $U := f^{-1}(f(x))$. Then $x \in U$, so by Proposition 5.9 either $x \in max_R(U)$ or there exists $x' \in max_R(U)$ such that Rxx'. We consider the former case only, the latter is analogous. Since $x \in max_R(U)$, by construction we have $g(\rho(x)) = f(x)$. Thus $R[f(x)] \cap \mathfrak{d} \neq \varnothing$. Since f satisfies the BDC for $\mathfrak{d}$, it follows that there is $y \in X$ such that Rxy and $f(y) \in \mathfrak{d}$. As $x \in max_R(U)$ we must have $f(x) \neq f(y)$. Now let $V := f^{-1}(f(y))$. As $y \in V$, by Proposition 5.9 either $y \in max_R(V)$ or there exists some $y' \in max_R(V)$ such that Ryy'. Wlog, suppose the former. Consequently, $f(y) = g(\rho(y))$. But then we have shown that $R\rho(x)\rho(y)$ and $g(\rho(y)) \in \mathfrak{d}$, i.e. $g[R[\rho(x)]] \cap \mathfrak{d} \neq \varnothing$.
- (BDC$^{\square}$-forth) Let $\mathfrak{d} \in \mathfrak{D}^{\square}$ and $\rho(x) \in \rho[X]$. Suppose that $g[R[\rho(x)]] \cap \mathfrak{d} \neq \varnothing$. Observe that $g[R[\rho(x)]] \cap \mathfrak{d} \neq \varnothing$ is equivalent to $R[\rho(x)] \cap g^{-1}(\mathfrak{d}) \neq \varnothing$. Therefore there is some $y \in \mathfrak{d}$ such that $R[\rho(x)] \cap g^{-1}(y) \neq \varnothing$. By Proposition 5.9 there is $z \in max_R(g^{-1}(y))$ with $R_\rho\rho(x)\rho(z)$. Observe that since g is pre-stable, $R^+g(\rho(x))g(\rho(z))$, whence if $g(\rho(x)) \neq g(\rho(z))$ in turn $Rg(\rho(x))g(\rho(z))$ and we are done. So suppose otherwise that $g(\rho(x)) = g(\rho(z))$. Distinguish two cases
  - *Case 1*: $y \notin R[y]$. Then y cannot belong to a proper cluster, so by construction $f(x) = g(\rho(x))$ and $f(z) = g(\rho(z))$. From $R\rho(x)\rho(z)$ it

follows that Rxz, whence $R[x]\cap f^{-1}(\mathfrak{d})\neq\varnothing$ Since f satisfies the BDC-forth for $\mathfrak{d}$, there must be some $u\in\mathfrak{d}$ with $Rf(x)u$ and $f(u)\in\mathfrak{d}$. Then also $Rg(\rho(x))u$, i.e. $R[g(\rho(x))]\cap\mathfrak{d}\neq\varnothing$ as desired.

– *Case 2*: $y\in R[y]$. But then $Rg(\rho(x))y$. This shows $R[g(\rho(x))]\cap\mathfrak{d}\neq\varnothing$ as desired.

□

Proposition 5.37. *Every universal class* $\mathcal{U}\in\mathbf{Uni}(\mathsf{Mag})$ *is generated by its skeletal elements, i.e.,* $\mathcal{U}=\sigma\rho\mathcal{U}$.

Proof. Analogous to Theorem 3.39, but applying Lemma 5.36 instead of Lemma 3.38. □

We now apply Lemma 5.36 to characterise the maps $\sigma:\mathbf{NExt}(\mathtt{KM_R})\to\mathbf{NExt}(\mathtt{GL_R})$ and $\rho:\mathbf{NExt}(\mathtt{KM_R})\to\mathbf{NExt}(\mathtt{GL_R})$ in terms of their semantic counterparts.

Lemma 5.38. *For each* $\mathtt{L}\in\mathbf{Ext}(\mathtt{KM_R})$ *and* $\mathtt{M}\in\mathbf{NExt}(\mathtt{GL_R})$, *the following hold:*

$$\mathsf{Alg}(\sigma\mathtt{L})=\sigma\mathsf{Alg}(\mathtt{L}) \tag{18}$$

$$\mathsf{Alg}(\rho\mathtt{M})=\rho\mathsf{Alg}(\mathtt{M}) \tag{19}$$

Proof. (18) By Theorem 3.39 it suffices to show that $\mathsf{Alg}(\sigma\mathtt{L})$ and $\sigma\mathsf{Alg}(\mathtt{L})$ have the same skeletal elements. So let $\mathfrak{A}=\sigma\rho\mathfrak{A}\in\mathsf{Mag}$. Assume $\mathfrak{A}\in\sigma\mathsf{Alg}(\mathtt{L})$. Since $\sigma\mathsf{Alg}(\mathtt{L})$ is generated by $\{\sigma\mathfrak{B}:\mathfrak{B}\in\mathsf{Alg}(\mathtt{L})\}$ as a universal class, by Proposition 5.29 and Lemma 5.34 we have $\mathfrak{A}\models T(\Gamma/\Delta)$ for every $\Gamma/\Delta\in\mathtt{L}$. But then $\mathfrak{A}\in\mathsf{Alg}(\sigma\mathtt{L})$. Conversely, assume $\mathfrak{A}\in\mathsf{Alg}(\sigma\mathtt{L})$. Then $\mathfrak{A}\models T(\Gamma/\Delta)$ for every $\Gamma/\Delta\in\mathtt{L}$. By Lemma 5.34 this is equivalent to $\rho\mathfrak{A}\in\mathsf{Alg}(\mathtt{L})$, therefore $\sigma\rho\mathfrak{A}=\mathfrak{A}\in\sigma\mathsf{Alg}(\mathtt{L})$.

(19) Let $\mathfrak{H}\in\mathsf{Frt}$. If $\mathfrak{H}\in\rho\mathsf{Alg}(\mathtt{M})$ then $\mathfrak{H}=\rho\mathfrak{A}$ for some $\mathfrak{A}\in\mathsf{Alg}(\mathtt{M})$. It follows that for every rule $T(\Gamma/\Delta)\in\mathtt{M}$ we have $\mathfrak{A}\models T(\Gamma/\Delta)$, and so by Lemma 5.34 in turn $\mathfrak{H}\models\Gamma/\Delta$. Therefore indeed $\mathfrak{H}\in\mathsf{Alg}(\rho\mathtt{M})$. Conversely, for all rules Γ/Δ, if $\rho\mathsf{Alg}(\mathtt{M})\models\Gamma/\Delta$ then by Lemma 5.34 $\mathsf{Alg}(\mathtt{M})\models T(\Gamma/\Delta)$, hence $\Gamma/\Delta\in\rho\mathtt{M}$. Thus $\mathsf{ThR}(\rho\mathsf{Alg}(\mathtt{M}))\subseteq\rho\mathtt{M}$, and so $\mathsf{Alg}(\rho\mathtt{M})\subseteq\rho\mathsf{Alg}(\mathtt{M})$. □

We are now ready to prove the main result of this section.

Theorem 5.39 (Kuznetsov-Muravitsky theorem)**.** *The following conditions hold:*

(1) $\sigma:\mathbf{NExt}(\mathtt{KM_R})\to\mathbf{NExt}(\mathtt{GL_R})$ *and* $\rho:\mathbf{NExt}(\mathtt{GL_R})\to\mathbf{NExt}(\mathtt{KM_R})$ *are mutually inverse complete lattice isomorphisms.*

(2) $\sigma:\mathbf{NExt}(\mathtt{KM})\to\mathbf{NExt}(\mathtt{GL})$ *and* $\rho:\mathbf{NExt}(\mathtt{GL})\to\mathbf{NExt}(\mathtt{KM})$ *are mutually inverse complete lattice isomorphisms.*

Proof. (1) It suffices to show that the two mappings $\sigma:\mathbf{Uni}(\mathsf{Frt})\to\mathbf{Uni}(\mathsf{Mag})$ and $\rho:\mathbf{Uni}(\mathsf{Mag})\to\mathbf{Uni}(\mathsf{Frt})$ are complete lattice isomorphisms and mutual inverses. Both maps are evidently order preserving, and preservation of infinite joins is an easy consequence of Lemma 5.34.

Let $\mathcal{U}\in\mathbf{Uni}(\mathsf{Mag})$. Then $\mathcal{U}=\sigma\rho\mathcal{U}$ by Proposition 5.37, so σ is surjective and a left inverse of ρ. Now let $\mathcal{U}\in\mathbf{Uni}(\mathsf{Frt})$. It follows immediately from Proposition 5.29 that $\rho\sigma\mathcal{U}=\mathcal{U}$. Therefore ρ is surjective and a left inverse of σ. But then σ and ρ are mutual inverses, whence both bijections.

(2) Follows immediately from Item 1 and Proposition 5.2. □

6. Conclusions and Future Work

This paper presented a novel approach to the study of modal companions and related notions based on stable canonical rules. We hope to have shown that our method is effective and quite uniform. With only minor adaptations to a fixed collection of techniques, we provided a unified treatment of the theories of modal and tense companions, and of the Kuznetsov-Muravitsky isomorphism. We both offered alternative proofs of classic theorems and established new results.

The techniques presented in this paper are based on a blueprint easily applicable across signatures. Stable canonical rules can be formulated for any class of algebras which admits a locally finite expandable reduct in the sense of [33, Ch. 5], and once stable canonical rules are available there is a clear recipe for adapting our strategy to the case at hand. We propose that further research be done in this direction, in particular addressing the following topics.

Firstly, for reasons of space we have not addressed the full theory of modal companions of msi deductive systems, as developed in [59, 58]. We conjecture that our techniques can recover several of the main known results in this area, and generalise them to rule systems. We hope that further work will confirm this.

Secondly, de Groot et al. [19] recently proved an analogue of the Blok-Esakia theorem for extensions of the *Heyting-Lemmon logic*, which expands superintuitionistic logic with a strict implication connective. Our techniques could be applied to generalise this result to rule systems, and more generally to develop a rich theory of modal companions of deductive systems over the Heyting-Lemmon logic.

Thirdly, Goldblatt [29] formulated a Gödel-style translation giving a full and faithful embedding of the propositional logic $\mathtt{Ort}$ of all *ortholattices* into the *Browerian modal logic* $\mathtt{B} = \mathtt{K} \oplus \Box p \to p \oplus p \to \Box\Diamond p$. To the best of our knowledge, the theory of modal companions of extensions of $\mathtt{Ort}$ (which include quantum logics) has not been developed, and in particular it is unknown whether Goldblatt's tranlsation gives rise to an analogue of the Blok-Esakia theorem. If a suitable expandable locally finite reduct of ortholattices can be found, stable canonical rules for rule systems over $\mathtt{Ort}$ can be developed, and a clear strategy for attacking the aforementioned problem becomes available.

References

[1] Bezhanishvili, G. and Bezhanishvili, N. [2009]. An Algebraic Approach to Canonical Formulas: Intuitionistic Case. *The Review of Symbolic Logic*, 2(3):517–549.

[2] ——— [2017]. Locally Finite Reducts of Heyting Algebras and Canonical Formulas. *Notre Dame Journal of Formal Logic*, 58(1):21–45.

[3] ——— [2020]. Jankov Formulas and Axiomatization Techniques for Intermediate Logics. *ILLC Prepublication (PP) Series*, PP-2020-12.

[4] Bezhanishvili, G., Bezhanishvili, N., and Iemhoff, R. [2016]. Stable canonical rules. *The Journal of Symbolic Logic*, 81(1):284–315.

[5] Bezhanishvili, G., Bezhanishvili, N., and Ilin, J. [2016]. Cofinal Stable Logics. *Studia Logica*, 104(6):1287–1317.

[6] ——— [2018]. Stable Modal Logics. *The Review of Symbolic Logic*, 11(3):436–469.

[7] Bezhanishvili, G., Ghilardi, S., and Jibladze, M. [2011]. An Algebraic Approach to Subframe Logics. Modal Case. *Notre Dame Journal of Formal Logic*, 52(2):187–202.

[8] Bezhanishvili, N. and Ghilardi, S. [2014]. Multiple-conclusion rules, hypersequents syntax and step frames. In R. Goré, B. Kooi, and A. Kurucz, eds., *Advances in Modal Logic*, vol. 10, pp. 54–73. CSLI Publications.

[9] Birkhoff, G. [1935]. On the Structure of Abstract Algebras. *Mathematical Proceedings of the Cambridge Philosophical Society*, 31(4):433–454.

[10] Blackburn, P., de Rijke, M., and Venema, Y. [2001]. *Modal logic*, *Cambridge Tracts in Theoretical Computer Science*, vol. 53. Cambridge: Cambridge University Press.

[11] Blok, W. [1976]. *Varieties of Interior Algebras*. Ph.D. thesis, Universiteit van Amsterdam.

[12] Boolos, G. [1980]. On Systems of Modal Logic with Provability Interpretations. *Theoria*, 46(1):7–18.

[13] Boolos, G. S. [1993]. *The Logic of Provability*. Cambdridge: Cambridge University Press.

[14] Burris, S. and Sankappanavar, H. P. [2012]. *A Course in Universal Algebra*. Graduate Texts in Mathematics. Berlin: Springer.

[15] Castiglioni, J. L., Sagastume, M. S., and San Martín, H. J. [2010]. On Frontal Heyting Algebras. *Reports on Mathematical Logic*, 45:201–224.

[16] Chagrov, A. and Zakharyaschev, M. [1992]. Modal Companions of Intermediate Propositional Logics. *Studia Logica: An International Journal for Symbolic Logic*, 51(1):49–82.

[17] ——— [1997]. *Modal Logic*. New York: Clarendon Press.

[18] Cleani, A. M. [2021]. *Translational Embeddings via Stable Canonical Rules*. Master's thesis, Institute for Logic, Language and Computation, University of Amsterdam, Amsterdam.

[19] de Groot, J., Litak, T., and Pattinson, D. [2021]. Gödel-McKinsey-Tarski and Blok-Esakia for Heyting-Lewis Implication. In *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pp. 1–15.

[20] de Jongh, D. and Montagna, F. [1988]. Provable Fixed Points. *Mathematical Logic Quarterly*, 34(3):229–250.

[21] Dummett, M. a. E. and Lemmon, E. J. [1959]. Modal Logics Between S4 and S5. *Mathematical Logic Quarterly*, 5(14-24):250–264.

[22] Esakia, L. [1974]. Topological kripke models. *Doklady Akademii Nauk SSSR*, 214(2):298–301.

[23] ——— [1975]. The Problem of Dualism in the Intuitionistic Logic and Browerian Lattices. In *V Inter. Congress of Logic, Methodology, and Philosophy of Science*, pp. 7–8. Canada.

[24] ——— [1976]. On Modal Companions of Superintuitionistic Logics. In *VII Soviet symposium on logic (Kiev, 1976)*, pp. 135–136.

[25] ——— [2006]. The modalized Heyting calculus: a Conservative Modal Extension of the Intuitionistic Logic. *Journal of Applied Non-Classical Logics*, 16(3-4):349–366.

[26] ——— [2019]. *Heyting Algebras: Duality Theory*, *Trends in Logic*, vol. 5. Amsterdam: Springer.

[27] Fine, K. [1985]. Logics Containing K4. Part II. *The Journal of Symbolic Logic*, 50(3):619–651.

[28] Gabbay, D. M., Hodkinson, I., and Reynolds, M. [1994]. *Temporal Logic: Mathematical Foundations and Computational Aspects*. Oxford Logic Guides. New York: Oxford University Press.

[29] Goldblatt, R. I. [1974]. Semantic Analysis of Orthologic. *Journal of Philosophical Logic*, 3(1/2):19–35.

[30] Goré, R. [2000]. Dual Intuitionistic Logic Revisited. In R. Dyckhoff, ed., *Automated Reasoning with Analytic Tableaux and Related Methods*, Lecture Notes in Computer Science, pp. 252–267. Berlin, Heidelberg: Springer.

[31] Gödel, K. [1933]. Eine Interpretation des Intuitionistischen Aussagenkalküls. *Ergebnisse einesmathematischen Kolloquiums*, 4:39–40.

[32] Iemhoff, R. [2016]. Consequence Relations and Admissible Rules. *Journal of Philosophical Logic*, 45(3):327–348.

[33] Ilin, J. [2018]. *Filtration revisited: Lattices of stable non-classical logics*. Ph.D. thesis, Universiteit van Amsterdam.

[34] Jeřábek, E. [2009]. Canonical rules. *The Journal of Symbolic Logic*, 74(4):1171–1205.

[35] Kowalski, T. [1998]. Varieties Of Tense Algebras. *Reports on Mathematical Logic*, pp. 53–95.

[36] Kuznetsov, A. V. [1978]. Proof-intuitionistic Logic. In *Modal and intensional logics, abstracts of the coordinating meeting, Moscow*, pp. 75–79.

[37] Kuznetsov, A. V. and Muravitsky, A. Y. [1986]. On Superintuitionistic Logics as Fragments of Proof Logic Extensions. *Studia Logica*, 45(1):77–99.

[38] Litak, T. [2014]. Constructive Modalities with Provability Smack. In G. Bezhanishvili, ed., *Leo Esakia on Duality in Modal and Intuitionistic Logic*, Outstanding Contributions to Logic, pp. 187–216. Dordrecht: Springer.

[39] Magari, R. [1975]. Representation and Duality Theory for Diagonalizable Algebras. *Studia Logica*, 34(4):305–313.

[40] ——— [1975]. The Diagonalizable Algebras. *Bollettino della Unione Matematica Italiana*, 4(12):117–125.

[41] Muravitsky, A. [1981]. Finite Approximability of the $I^{\triangle}$ Calculus and the Existence of an Extension Having no Model. *Mathematical notes of the Academy of Sciences of the USSR*, 29(6):463–468.

[42] ——— [2014]. Logic KM: A Biography. In G. Bezhanishvili, ed., *Leo Esakia on Duality in Modal and Intuitionistic Logic*, Outstanding Contributions to Logic, pp. 155–185. Dordrecht: Springer.

[43] Pedroso De Lima Martins, M. [2021]. *Bi-Gödel Algebras and Co-trees*. Master's thesis, Universiteit van Amsterdam.

[44] Rauszer, C. [1974]. A Formalization of the Propositional Calculus of H-B Logic. *Studia Logica: An International Journal for Symbolic Logic*, 33(1):23–34.

[45] ——— [1974]. Semi-Boolean Algebras and Their Applications to Intuitionistic Logic with Dual Operations. *Fundamenta Mathematicae*, 83:219–249.

[46] ——— [1977]. Applications of Kripke Models to Heyting-Brouwer Logic. *Studia Logica: An International Journal for Symbolic Logic*, 36(1/2):61–71.

[47] Sambin, G. [1974]. Un'estensione del teorema di Löb. *Rendiconti del Seminario Matematico della Università di Padova*, 52:193–199.

[48] ——— [1976]. An Effective Fixed-point Theorem in Intuitionistic Diagonalizable Algebras. *Studia Logica*, 35(4):345–361.

[49] Sambin, G. and Vaccaro, V. [1988]. Topology and Duality in Modal Logic. *Annals of pure and applied logic*, 37(3):249–296.

[50] Sambin, G. and Valentini, S. [1982]. The Modal Logic of Provability. The Sequential Approach. *Journal of Philosophical Logic*, 11(3):311–342.

[51] Solovay, R. M. [1976]. Provability interpretations of modal logic. *Israel Journal of Mathematics*, 25(3):287–304.

[52] Stronkowski, M. M. [2018]. On the Blok-Esakia Theorem for Universal Classes. *arXiv:1810.09286 [math]*. ArXiv: 1810.09286.

[53] Taylor, C. J. [2017]. *Double Heyting Algebras*. Ph.D. thesis, La Trobe University.

[54] van Benthem, J. and Bezhanishvili, N. [forthcoming]. Modern Faces of Filtration. In F. L. G. Faroldi and F. V. D. Putte, eds., *Outstanding Contributions to Logic: Kit Fine*. Springer.

[55] Venema, Y. [2007]. Algebras and Coalgebras. In P. Blackburn, J. van Benthem, and F. Wolter, eds., *Handbook of modal logic*, *Studies in logic and practical reasoning*, vol. 3, pp. 331–426. Amsterdam: Elsevier.

[56] Wolter, F. [1997]. Completeness and Decidability of Tense Logics Closely Related to Logics Above K4. *The Journal of Symbolic Logic*, 62(1):131–158.

[57] ——— [1998]. On Logics with Coimplication. *Journal of Philosophical Logic*, 27(4):353–387.

[58] Wolter, F. and Zakharyaschev, M. [1997]. On the Relation Between Intuitionistic and Classical Modal Logics. *Algebra and Logic*, 36(2):73–92.

[59] ——— [1998]. Intuitionistic Modal Logics as Fragments of Classical Bimodal Logics. In *Logic at Work: Essays in Honour of Helena Rasiowa*, pp. 168–186. Dodrecht: Springer.

[60] ——— [2014]. On the Blok-Esakia Theorem. In *Leo Esakia on Duality in Modal and Intuitionistic Logic*, pp. 99–118. Dodrecht: Springer.

[61] Zakharyashchev, M. V. [1991]. Modal Companions of Superintuitionistic Logics: Syntax, Semantics, and Preservation Theorems. *Mathematics of the USSR-Sbornik*, 68(1).

Institute for Logic, Language and Computation, University of Amsterdam

Department of Philosophy, University of Southern California