Arbitrary Public Announcement Logic with Memory*

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Abstract. We introduce Arbitrary Public Announcement Logic with Memory (APALM), obtained by adding to the models a 'memory' of the initial states, representing the information before any communication took place ("the prior"), and adding to the syntax operators that can access this memory. We show that APALM is recursively axiomatizable (in contrast to the original Arbitrary Public Announcement Logic, for which the corresponding question is still open). We present a complete recursive axiomatization, that includes a natural finitary rule, and study this logic's expressivity and the appropriate notion of bisimulation. We then examine Group Announcement Logic with Memory (GALM), the extension of APALM obtained by adding to its syntax group announcement operators, and provide a complete finitary axiomatization (again in contrast to the original Group Announcement Logic, for which the only known axiomatization is infinitary). We also show that, in the memory-enhanced context, there is a 'natural' reduction of the so-called coalition announcement modality to group announcements (in contrast to the memory-free case, where this natural translation was shown to be invalid).

Keywords. arbitrary public announcement logic, group announcement logic, coalition announcement logic, arbitrary announcement modality, coalition announcement modality, dynamic epistemic logic, modal logic, recursive axiomatization, subset space semantics

1 Introduction

Arbitrary Public Announcement Logic (APAL) and its relatives are natural extensions of Public Announcements Logic (PAL), involving the addition of operators $\Box\varphi$ and $\Diamond\varphi$, quantifying over public announcements $[\theta]\varphi$ of some given type. These logics are of great interest both philosophically and from the point of view of applications. Motivations range from supporting an analysis of Fitch's paradox [30] by modeling notions of 'knowability' (expressible as $\Diamond K\varphi$), to determining the existence of communication protocols that achieve certain goals (cf. the famous Russian Card problem, given at a mathematical Olympiad [31]), and more generally to epistemic planning [16], and to inductive learnability in empirical science [10]. Many such extensions have been investigated, starting with the original APAL [6], and continuing with its variants CAL (Coalition Announcement Logic) [26], GAL (Group Announcement Logic) [1], Future Event Logic [35], FAPAL (Fully Arbitrary Public Announcement Logic) [39], $APAL^+$ (Positive Arbitrary Announcement Logic) [34], BAPAL (Boolean Arbitrary Public Announcement Logic) [33] etc. Similar ideas for arbitrary-quantification modalities have been adopted in other contexts, see [22] for a modality quantifying over action models in Arbitrary Action Model Logic and [17] for a quantifier over the set of all refinements of a given model (a variation on a bisimulation quantifier) in Refinement Modal Logic.

One problem with the above formalisms, with the exception of BAPAL[1], is that they all have *infinitary* axiomatizations. It is known that APAL and GAL are undecidable [20,4]. It is therefore not guaranteed that the validities of these logics are recursively enumerable.[2] The seminal paper on APAL [6] proved completeness using an infinitary rule, and

*This work is an extended version of our paper [12].

[1]BAPAL is a very weak version, allowing $\Box\varphi$ to quantify over only purely propositional announcements - no epistemic formulas.

[2]$APAL^+$ is known to be decidable, hence its validities must be r.e., but no recursive axiomatization is known. Also, note that $APAL^+$ is still very weak, in that it quantifies only over positive epistemic announcements, thus not allowing public announcements of ignorance, which are precisely the ones driving the solution process in puzzles such as the Muddy Children.

then went on to claim that in theorem-proving[3] this rule can be replaced by the following finitary rule: from $\psi \to [\theta][p]\varphi$, infer $\psi \to [\theta]\Box\varphi$, as long as the propositional variable p is "fresh". A similar method is adopted in the completeness proof of GAL in [1] and it was claimed that the infinitary rule used in the completeness proof could be replaced by the finitary rule 'from $\chi \to [\theta][\bigwedge_{i\in G} K_i p_i]\psi$ infer $\chi \to [\theta][G]\psi$', where p_i's are "fresh" and $[G]$ is the group announcement operator, quantifying over updates with formulas of the form $\bigwedge_{i\in G} K_i\varphi_i$. These are natural $\Box$ and $[G]$-introduction rules, similar to the introduction rule for universal quantifier in First Order Logic (FOL), and they are based on the intuition that variables that do not occur in a formula are irrelevant for its truth value, and thus can be taken to stand for any "arbitrary" formula (via some appropriate change of valuation). However, the soundness of the $\Box$-introduction rule was later disproved via a counterexample by Kuijer [23]. Moreover, a slightly modified version of Kuijer's counterexample also proves that the aforementioned $[G]$-introduction rule for GAL is unsound. Thus, a long-standing open question concerns *finding 'strong' versions of APAL and GAL for which there exists a recursive axiomatization*. Here, by 'strong' version we mean one that allows quantification over a sufficiently wide range of announcements (sufficiently wide to avoid Liar-like circles) as intended by a similar restriction in the original APAL.

In this paper, we solve these open questions for both APAL and GAL, starting with a diagnosis of the APAL counterexample, that leads naturally to our proposed solution. The framework for the 'strong' version of GAL will be developed analogously, as an extension of the one for APAL. Due to the similar syntactic and semantic behaviours of the group announcement ($[G]$) and arbitrary announcement ($\Box$) operators, most of our analysis of the latter also applies to the former.

Our diagnosis of Kuijer's counterexample is that it makes an essential use of a known undesirable feature of PAL and APAL, namely their *lack of memory*: the updated models "forget" the initial states. As a consequence, the expressivity of the APAL $\Box$-modality reduces after any update. This is what invalidates the above rule. We fix this problem by adding to the models a memory of the *initial* epistemic situation W^0, representing the information *before any non-trivial communication took place* ("the prior"). Since communication – gaining more information – deletes possibilities, the set W of currently possible states is a (possibly proper) subset of the set W^0 of initial states. On the syntactic side, we add an operator φ^0 saying that "φ was initially the case" (before all communication). To mark the initial states, we also need a constant 0, stating that "no non-trivial communication has taken place yet". Therefore, 0 will be true only in the initial epistemic situation. It is convenient, though maybe not absolutely necessary, to add a universal modality $U\varphi$ that quantifies over all currently possible states.[4] In the resulting Arbitrary Public Announcement Logic with Memory (APALM), the arbitrary announcement operator $\Box$ quantifies over updates (not only of epistemic formulas but) of arbitrary formulas that do not contain the operator $\Box$ itself.[5] As a result, the range of $\Box$ is wider than in standard APAL, covering announcements that may refer to the initial situation (by the use of the operators 0 and φ^0) or to all currently possible states (by the use of $U\varphi$).

Although APALM is inspired from APAL and designed specifically to avoid the aforementioned flaws that affect the soundness of the 'natural' axiomatization of APAL, the exact relationship between APAL and APALM is not easy to elucidate. This is because of the extreme sensitivity to context of the "arbitrary public announcement" operator: since its semantics quantifies over the remaining syntax, any addition to the syntax changes its meaning. One way to look at APALM is to say that we simply added new operators to the syntax, while essentially keeping "the same" semantics for the arbitrary announcement operator $\Box$, namely in terms of quantifying over the announcement of all sentences in the language that do not contain $\Box$ itself. However, this does change the semantics of $\Box$ by enlarging the scope of the quantifier. This is a very unusual situation in Logic. Although it may look like a stronger language (and from a conceptual and intuitive perspective it is indeed a form of strengthening), APALM is not necessarily stronger than APAL from a logical point of view. In fact, we conjecture that its expressivity is incomparable to the one of APAL. Proving this claim is a highly non-trivial task, which remains an open question for now.

[3]This means that from any proof of a theorem from the axioms that uses the infinitary rule we can obtain a finitary proof of the same theorem, by using the finitary rule instead.

[4]From an epistemic point of view, it would be more natural to replace U by an operator Ck that describes current common knowledge and quantifies only over currently possible states that are accessible by epistemic chains from the actual state. We chose to stick with U for simplicity and leave the addition of Ck to APAL for future work.

[5]This restriction is necessary to produce a well-defined semantics that avoids Liar-like vicious circles. In standard APAL, the quantification restriction of $\Diamond$ is present in its semantics and with respect to formulas of the form $\langle\theta\rangle\varphi$ and $\Diamond\varphi$. Formulas of the form $\langle\Diamond p\rangle\varphi$ are allowed in the syntax of APAL. APAL and APALM expressivities seem to be incomparable, and that would still be the case if we dropped the above restriction.

We show that the original finitary rule proposed in [6] *is* sound for APALM and, moreover, it forms the basis of a *complete recursive axiomatization*.[6] It can therefore be applied to all the puzzles and examples that motivated APAL, with the difference that one can now also use the complete proof system to reason axiomatically about them. Besides its technical advantages, APALM is valuable in its own respect. Maintaining a record of the initial situation in our models helps us to formalize updates that refer to the 'epistemic past' such as "what you said, I knew already" [29]. This may be useful in treating certain epistemic puzzles involving reference to past information states, e.g. "What you said did not surprise me" [24]. The more recent *Cheryl's Birthday* problem also contains an announcement of the form "At first I didn't know when Cheryl's birthday is, but now I know" (although in this particular puzzle the past-tense announcement is redundant and plays no role in the solution).[7] See [29] for more examples.

Note though that the 'memory' of APALM is *very limited*: our models do not remember the whole history of communication, but only the initial epistemic situation (before any communication). Correspondingly, in the syntax we do *not* have a 'yesterday' operator $Y\varphi$, referring to the previous state just before the last announcement as in [27], but only the operator φ^0 referring to the initial state. We think of this 'economy' of memory as *a (positive) "feature, not a bug"* of our logic: a detailed record of *all* history is simply not necessary for solving the problem at hand. In fact, keeping all the history and adding a $Y\varphi$ operator would greatly complicate our task by invalidating some of the standard nice properties of PAL and APAL.[8] We will briefly return to this connection with the yesterday operator in Section 6.

So we opt for simplicity, enriching the models and language with just enough memory to recover the full expressivity of $\Box$ after updates, and thus establish the soundness of the $\Box$-introduction rule. Such a limited-memory semantics is sufficient for our purposes, but it also has an intrinsic naturality and simplicity, similar to the one encountered in some Bayesian models, with their distinction between 'prior' and 'posterior' (aka current) probabilities.[9]

Having established the desired results for APALM, we also study a version of GAL with the same memory mechanism – Group Announcement Logic with Memory (GALM) – obtained by extending APALM with group announcement operators. In this logic, the group announcement operators $[G]\varphi$ quantify over updates with formulas of the form $\bigwedge_{i\in G} K_i\varphi_i$, thus, represents *what a group of agents can bring about via simultaneous public announcements*. In order to avoid Liar-like vicious circles, these updates can have occurrences of every component of the language but $\Box$ and $[G]$ (see footnote 5). A related modality is the *coalition announcement* operator $\langle[G]\rangle\varphi$, which says that the members group G can bring about φ by simultaneous public announcements *no matter what other public announcements are simultaneously made by the 'outsiders'* (i.e. the agents *not* in G). The resulting logic was called Coalition Announcement Logic (CAL) [2]. In the memory-free context, it was shown in [19] that the natural and apparently 'obvious' translation of $\langle[G]\rangle\varphi$ into GAL is in fact *invalid*.[10] Moreover, no recursive axiomatization is known for CAL, or for any of its extensions. In contrast, in this paper we show that, in our memory-enhanced framework, the obvious translation *is* valid: the analogue of the coalition announcement modality is definable in GALM. Further, we provide a complete *finitary* axiomatization for GALM, thus proving that validities of both GALM and CALM (the memory-enhanced variant of CAL) are recursively enumerable.

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[6]We use a slightly different version of this rule, which is easily seen to be equivalent to the original version in the presence of the usual PAL reduction axioms.

[7]Cheryl's Birthday problem was part of the 2015 Singapore and Asian Schools Math Olympiad, and became viral after it was posted on Facebook by Singapore TV presenter Kenneth Kong.

[8]E.g. the standard Composition Axiom (stating that any sequence of announcements is equivalent to a single announcement) fails in the presence of the Y operator. As a consequence, a logic with full memory of all history would lose some of the appealing features of the APAL operator (e.g. its $S4$ character: $\Box\varphi \to \Box\Box\varphi$). Moreover, this would force us to distinguish between "knowability via one communication step" $\Diamond K$ versus "knowability via a finite communication sequence" $\Diamond^* K$, leading to an unnecessarily complex logic.

[9]In such models, only the 'prior' and the 'posterior' information states are taken to be relevant, while all the intermediary steps are forgotten. As a consequence, all the evidence gathered in between the initial and the current state can be compressed into one set E, called "the evidence" (rather than keeping a growing tail-sequence of all past evidence sets). Similarly, in our logic, all the past communication is compressed in its end-result, namely in the set W of current possibilities, which plays the same role as the evidence set E in Bayesian models.

[10]But it is not known if another, valid translation exists.

This is done by following the same steps as for APALM, and showing that the finitary $[G]$-introduction rule that was originally proposed for GAL in [1] is in fact sound for GALM.[11]

On the technical side, our completeness proof involves an essential detour into an alternative semantics for APALM and GALM ('pseudo-models'), in the style of Subset Space Logics (SSL) [25,18]. This reveals deep connections between apparently very different formalisms. Moreover, this alternative semantics is of independent interest, giving us a more general setting for modeling knowability and learnability (see, e.g., [13,14,10]). Various combinations of PAL or APAL with subset space semantics have been investigated in the literature [9,41,40,36,37,13,11,10], including a version of SSL with backward looking public announcement operators that refer to what was true *before* a public announcement [8]. Following the SSL-style, our pseudo-models come with a given family of *admissible sets* of worlds, which in our context represent "publicly announceable" (or communicable) propositions.[12] We interpret $\Box$ in pseudo-models as the so-called 'effort' modality of SSL, which quantifies over updates with announceable propositions (regardless of whether they are syntactically definable or not). The modality $[G]$ on the other hand quantifies over updates with those announceable propositions that are known by some agents in G. The operator $[G]$ is thus modelled as a restricted version of the effort modality. The finitary $\Box$-introduction rule is obviously *sound for the effort modality*, because of its more 'semantic' character. Similarly, the finitary $[G]$-introduction rule is also sound for this effort-like group announcement operator. These observations, together with the important fact that our models for APALM and GALM (unlike original APAL models) can be seen as a special case of pseudo-models, lie at the core of our soundness and completeness proofs.

The paper is organized as follows. In Section 2, we introduce the syntax and semantics of APALM (Section 2.1); then discuss Kuijer's counterexample for the soundness of the finitary $\Box$-introduction rule of the original APAL (Section 2.2); present a complete finitary axiomatization for APALM (Section 2.3); and define a notion of bisimulation appropriate for the language of APALM and prove expressivity results comparing fragments of this language (Section 2.4). Section 3 presents first the syntax and semantics of GAL and CAL, as well as their problems; then proceeds to introduce our group announcement logic with memory GALM, providing a complete finitary axiomatization, and showing that the coalition announcement modality is definable in this memory-enhanced framework. In Section 4 we prove soundness, and in Section 5 we prove completeness, for both APALM and GALM. Section 6 contains some concluding comments and ideas for future work.

For readability, most of the rather technical proofs are omitted from the main text and presented in the appendix.

2 Arbitrary Public Announcement Logic with Memory

We start by introducing APALM, obtained by enriching the models of APAL with a record of the initial information states (representing the informational situation before any communication took place) and the language of APAL[13] with operators that can refer to this memory.

2.1 Syntax and Semantics of APALM

Let *Prop* be a countable set of propositional variables and $\mathcal{AG} = \{1, \ldots, n\}$ be a finite set of agents. The language $\mathcal{L}$ of APALM (Arbitrary Public Announcement Logic with Memory) is defined by the grammar:

$$\varphi \ ::= \ p \mid \top \mid 0 \mid \varphi^0 \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid U\varphi \mid \langle\theta\rangle\varphi \mid \Diamond\varphi,$$

where $p \in Prop$, $i \in \mathcal{AG}$, and $\theta \in \mathcal{L}_{-\Diamond}$ is a formula in the sublanguage $\mathcal{L}_{-\Diamond}$ obtained from $\mathcal{L}$ by removing the $\Diamond$ operator. Given a formula $\varphi \in \mathcal{L}$, we denote by P_φ the set of all propositional variables occurring in φ. We follow the standard rules for omission of the parentheses. We define $\bot$ as $\neg\top$. The propositional connectives $\vee, \rightarrow$, and $\leftrightarrow$ are defined as

[11] We again use a slightly different version of this rule, which can easily be proven to be equivalent to the original version in the presence of the PAL reduction axioms. This choice is clearly cosmetic and made in order to simplify the soundness and completeness proofs.

[12] In SSL, the set of admissible sets is sometimes, but not always, taken to be a topology. Here, it will be a Boolean algebra with epistemic operators.

[13] The language of APAL is defined recursively as $\varphi ::= \ p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid \langle\varphi\rangle\varphi \mid \Diamond\varphi$, where $p \in Prop$.

$\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi := \neg(\varphi \wedge \neg\psi)$, and $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$. The dual modalities are defined as $\hat{K}_i\varphi := \neg K_i\neg\varphi$, $E\varphi := \neg U\neg\varphi$, $\Box\varphi := \neg\Diamond\neg\varphi$, and $[\theta]\varphi := \neg\langle\theta\rangle\neg\varphi$.[14]

We read $K_i\varphi$ as "φ is *known* by agent i"; $\langle\theta\rangle\varphi$ as "θ can be *truthfully announced*, and *after* this φ is true". U and E are the universal and existential modalities quantifying over all *current* possibilities: $U\varphi$ says that "φ is true in *all current alternatives* of the actual state". $\Diamond\varphi$ and $\Box\varphi$ are the (existential and universal) arbitrary announcement operators, quantifying over updates with formulas in $\mathcal{L}_{-\Diamond}$. We can read $\Box\varphi$ as "φ is *stably true* (under public announcements)", i.e., φ stays true no matter what (true) announcements are made. The constant 0, meaning that "*no (non-trivial) announcements* took place yet", holds only at the *initial time*. Similarly, the formula φ^0 means that "*initially* (prior to all communication), φ *was* true".

In some of our inductive proofs pertaining to APALM, we need a complexity measure on formulas that is different from the standard one based on subformula complexity. The standard notion requires only that formulas are more complex than their subformulas, while we also need that $\Diamond\varphi$ is more complex than $\langle\theta\rangle\varphi$ for all $\theta \in \mathcal{L}_{-\Diamond}$. A similar complexity measure is also required for the language of Group Announcement Logic, denoted by $\mathcal{L}_G$, that we define in Section 3. In order to avoid repetitions, we prefer to present the relevant syntactic definition for both $\mathcal{L}$ and $\mathcal{L}_G$ together in Appendix A.1, where the reader can find the definition of a *(proper) subformula in* $\mathcal{L}$, $\Diamond$*-depth of a formula*, and a *well-founded strict order* $<$ *on formulas of* $\mathcal{L}$ (similarly for the language of $\mathcal{L}_G$). Here we only state the core lemma pertaining to this complexity measure that will be useful in our expressivity and completeness proofs.

Lemma 1. *There exists a well-founded strict partial order* $<$ *on* $\mathcal{L}$ *such that:*

1. *if* φ *is a proper subformula of* ψ*, then* $\varphi < \psi$,
2. $(\theta \to p) < [\theta]p$,
3. $(\theta \to \neg[\theta]\psi) < [\theta]\neg\psi$,
4. $(\theta \to K_i[\theta]\psi) < [\theta]K_i\psi$,
5. $[\langle\theta\rangle\rho]\chi < [\theta][\rho]\chi$,
6. $(\theta \to \varphi^0) < [\theta]\varphi^0$,
7. $(\theta \to U[\theta]\varphi) < [\theta]U\varphi$,
8. $(\theta \to (U\theta \wedge 0)) < [\theta]0$,
9. $\langle\theta\rangle\varphi < \Diamond\varphi$*, for all* $\theta \in \mathcal{L}_{-\Diamond}$.

Proof. The proof is via easy arithmetic calculations following the definitions in Appendix A.1, restricted to language $\mathcal{L}$. Note that, Definition 15 is redundant for the cases restricted to language $\mathcal{L}_{-\Diamond}$.

Definition 1 (Model, Initial Model, and Relativized Model).

- *A* model *is a tuple* $\mathcal{M} = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$*, where* $W \subseteq W^0$ *are non-empty sets of states,* $\sim_i \subseteq W^0 \times W^0$ *are equivalence relations labeled by 'agents'* $i \in \mathcal{AG}$*, and* $\|\cdot\| : Prop \to \mathcal{P}(W^0)$ *is a valuation function that maps every propositional variable* $p \in Prop$ *to a set of states* $\|p\| \subseteq W^0$. W^0 *is the* initial domain*, representing the initial informational situation before any communication took place; its elements are called* initial states. *In contrast,* W *is the* current domain*, and its elements are called* current states.
- *For every model* $\mathcal{M} = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$*, we define its* initial model $\mathcal{M}^0 = (W^0, W^0, \sim_1, \dots, \sim_n, \|\cdot\|)$*, whose both current and initial domains are the initial domain of the original model* $\mathcal{M}$.
- *Given a model* $\mathcal{M} = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$ *and a non-empty set* $A \subseteq W$*, we define the* relativized model $\mathcal{M}|A = (W^0, A, \sim_1, \dots, \sim_n, \|\cdot\|)$.

For states $w \in W$ and agents i, we will use the notation $w_i := \{s \in W : w \sim_i s\}$ to denote the equivalence class of w modulo $\sim_i$ restricted to W.

Definition 2 (Semantics). *Given a model* $\mathcal{M} = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$*, we recursively define a* truth set $[\![\varphi]\!]_{\mathcal{M}}$ *for every formula* $\varphi \in \mathcal{L}$ *as follows:*

$$[\![0]\!] = \begin{cases} W^0 & \text{if } W = W^0 \\ \emptyset & \text{otherwise} \end{cases}$$

$$[\![p]\!] = \|p\| \cap W \qquad [\![\varphi^0]\!] = [\![\varphi]\!]_{\mathcal{M}^0} \cap W$$

$$[\![\top]\!] = W$$

[14]The update operator $\langle\theta\rangle\varphi$ is often denoted by $\langle!\theta\rangle\varphi$ in Public Announcement Logic literature. We skip the exclamation sign, but we will use the notation $\langle!\rangle$ for this modality and $[!]$ for its dual when we do not want to specify the announcement formula θ (so that ! functions as a placeholder for the content of the announcement).

$$[\![\neg\varphi]\!] = W - [\![\varphi]\!] \qquad [\![U\varphi]\!] = \begin{cases} W & \textit{if } [\![\varphi]\!] = W \\ \emptyset & \textit{otherwise} \end{cases}$$

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!] \qquad [\![\langle\theta\rangle\varphi]\!] = \begin{cases} [\![\varphi]\!]_{\mathcal{M}|[\![\theta]\!]} & \textit{if } [\![\theta]\!] \neq \emptyset \\ \emptyset & \textit{otherwise} \end{cases}$$

$$[\![K_i\varphi]\!] = \{w \in W : w_i \subseteq [\![\varphi]\!]\} \qquad [\![\Diamond\varphi]\!] = \bigcup\{[\![\langle\theta\rangle\varphi]\!] \; : \; \theta \in \mathcal{L}_{-\Diamond}\}$$

In this definition and elsewhere, we skip the subscript and simply write $[\![\varphi]\!]$ when the current model $\mathcal{M}$ is understood. In particular, $\mathcal{M}|[\![\theta]\!]$ is here an abbreviation for $\mathcal{M}|[\![\theta]\!]_{\mathcal{M}}$. More generally, we use this notation also in iterated contexts, with the formula being evaluated in the last model that is being relativized e.g. $(\mathcal{M}|[\![\theta]\!])|[\![\rho]\!]$ stands for the further relativization $\mathcal{M}'|[\![\rho]\!]_{\mathcal{M}'}$ of the relativised model $\mathcal{M}' := \mathcal{M}|[\![\theta]\!]_{\mathcal{M}}$.

Observation 1 *Note that we have*

$$w \in [\![[\theta]\varphi]\!] \;\; \textit{iff} \;\; w \in [\![\theta]\!] \textit{ implies } w \in [\![\varphi]\!]_{\mathcal{M}|[\![\theta]\!]}, \textit{ and}$$

$$w \in [\![\Box\varphi]\!] \quad \textit{iff} \quad w \in [\![[\theta]\varphi]\!] \textit{ for every } \theta \in \mathcal{L}_{-\Diamond}.$$

A straightforward consequence of this fact and of the semantics of $\langle\theta\rangle\varphi$ is the following:

Observation 2 *Given a model $\mathcal{M} = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$ and formulas $\theta, \rho, \varphi \in \mathcal{L}$, if $[\![\theta]\!]_{\mathcal{M}} = [\![\rho]\!]_{\mathcal{M}}$, then $[\![\langle\theta\rangle\varphi]\!]_{\mathcal{M}} = [\![\langle\rho\rangle\varphi]\!]_{\mathcal{M}}$ and $[\![[\theta]\varphi]\!]_{\mathcal{M}} = [\![[\rho]\varphi]\!]_{\mathcal{M}}$.*

Proposition 1. *We have $[\![\varphi]\!] \subseteq W$, for all formulas $\varphi \in \mathcal{L}$.*

Proof. See the proof of Proposition 8 in Appendix A.3.

Lemma 2. *For all models $\mathcal{M} = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$ and formulas $\theta, \rho \in \mathcal{L}_{-\Diamond}$: $(\mathcal{M}|[\![\theta]\!])|[\![\rho]\!] = \mathcal{M}|[\![\langle\theta\rangle\rho]\!]$.*

Proof. The model $(\mathcal{M}|[\![\theta]\!])|[\![\rho]\!]$ is obtained by first relativizing $\mathcal{M}$ to the set $[\![\theta]\!]_{\mathcal{M}}$, then relativizing the resulting model again to $[\![\rho]\!]_{\mathcal{M}|[\![\theta]\!]} \subseteq [\![\theta]\!]_{\mathcal{M}}$. But overall this is the same as the model obtained by directly relativizing $\mathcal{M}$ to the set $[\![\rho]\!]_{\mathcal{M}|[\![\theta]\!]} = [\![\langle\theta\rangle\rho]\!]_{\mathcal{M}}$, i.e. $\mathcal{M}|[\![\langle\theta\rangle\rho]\!]$.

What we study in this paper is information update via public announcements. But the models given in Definition 1 are too general for this purpose: their current domain W can be *any* subset of the initial domain W^0. Our intended models (which we call "announcement models") will thus be a subclass of these models, in which the current domain comes from updating the initial domain with some public announcement.

Definition 3 (Announcement Models and Validity). *An* announcement model *(or a-*model*, for short) is a model $\mathcal{M} = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$ such that $W = [\![\theta]\!]_{\mathcal{M}^0}$ for some $\theta \in \mathcal{L}_{-\Diamond}$; equivalently, $\mathcal{M} = \mathcal{M}^0|[\![\theta]\!]$, i.e. $\mathcal{M}$ can be obtained by updating its initial model $\mathcal{M}^0$ with some formula in $\mathcal{L}_{-\Diamond}$. A formula φ is* APALM valid *(or* valid*, for short) if it is true in every current state $s \in W$ (i.e. $[\![\varphi]\!]_{\mathcal{M}} = W$) of every announcement model $\mathcal{M} = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$.*

Given this definition and the semantics, the following fact is obvious:

Lemma 3. *If $\mathcal{M} = (W^0, W, \sim_1, \dots, \sim_n, \|\cdot\|)$ is an a-model and $\theta \in \mathcal{L}_{-\Diamond}$ is a formula such that $\mathcal{M} = \mathcal{M}^0|[\![\theta]\!]$, then for all formulas $\varphi \in \mathcal{L}$, we have:*

$$[\![\varphi]\!]_{\mathcal{M}} = [\![(\langle\theta\rangle\varphi)^0]\!]_{\mathcal{M}}.$$

More generally, for all formulas $\rho \in \mathcal{L}_{-\Diamond}$, we have:

$$[\![\rho \wedge \varphi]\!]_{\mathcal{M}} = [\![\langle\rho\rangle(\langle\theta\rangle\varphi)^0]\!]_{\mathcal{M}}.$$

Proof. The first identity follows from the sequence of equalities: $[\![(\langle\theta\rangle\varphi)^0]\!]_{\mathcal{M}} = [\![\langle\theta\rangle\varphi]\!]_{\mathcal{M}^0} = [\![\varphi]\!]_{\mathcal{M}^0|[\![\theta]\!]} = [\![\varphi]\!]_{\mathcal{M}}$, where we used first the semantics of ψ^0 and then the semantics of $\langle\theta\rangle\varphi$.

For the second identity, we use the first one to get the equalities: $[\![\langle\rho\rangle(\langle\theta\rangle\varphi)^0]\!]_{\mathcal{M}} = [\![\langle\rho\rangle(\langle\theta\rangle\varphi)^0]\!]_{\mathcal{M}^0|[\![\theta]\!]} = [\![(\langle\theta\rangle\varphi)^0]\!]_{(\mathcal{M}^0|[\![\theta]\!])|[\![\rho]\!]} = [\![(\langle\theta\rangle\varphi)^0]\!]_{\mathcal{M}^0|[\![\langle\theta\rangle\rho]\!]} = [\![\langle\theta\rangle\varphi]\!]_{\mathcal{M}^0} \cap [\![\langle\theta\rangle\rho]\!]_{\mathcal{M}^0} = [\![\varphi]\!]_{\mathcal{M}^0|[\![\theta]\!]} \cap [\![\rho]\!]_{\mathcal{M}^0|[\![\theta]\!]} = [\![\varphi]\!]_{\mathcal{M}} \cap [\![\rho]\!]_{\mathcal{M}} = [\![\rho \wedge \varphi]\!]_{\mathcal{M}}$, where at key points we used the semantics of $\langle\rho\rangle\psi$, then Lemma 2, then the semantics of ψ^0 in the model $\mathcal{M}^0|[\![\langle\theta\rangle\rho]\!]$ (whose set of worlds is given by $[\![\langle\theta\rangle\rho]\!]_{\mathcal{M}^0}$), then the semantics of $\langle\theta\rangle\psi$.

Example 1. *Consider the a-model $\mathcal{M} = (W^0, W, \sim_a, \sim_b, \|\cdot\|)$ given in Figure 1, where the initial states include all the nodes of the graph and the current states are the nodes in the shaded area. It is easy to see that the current domain W is obtained by updating the initial domain by $\hat{K}_b p$: the shaded area corresponds to $[\![\hat{K}_b p]\!]_{\mathcal{M}^0}$. The representation in Figure 1 makes it clear that the a-model does not lose the initial domain and specifies the current domain as a subset of the initial one. Since $W^0 \neq W$ (the shaded area does not cover the whole initial domain), 0 is false everywhere in the model, that is, $[\![0]\!] = \emptyset$. Moreover, while $\hat{K}_b\hat{K}_a K_b r$ was initially true at w, it currently is not: $w \in [\![(\hat{K}_b\hat{K}_a K_b r)^0]\!]$ but $w \notin [\![\hat{K}_b\hat{K}_a K_b r]\!]$ (as $[\![r]\!] = \emptyset$).*

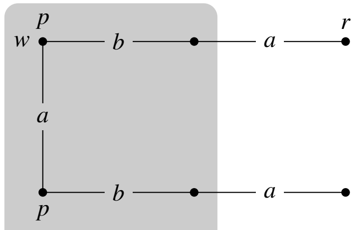


Fig. 1: An a-model $\mathcal{M}$. All nodes in the graph are initial states, while the current states are the nodes in shaded areas. Valuation is given by labeling the nodes, and epistemic relations by labeling arrows with agent names.

2.2 An Analysis of Kuijer's counterexample

The language of APAL is defined recursively as

$$\varphi ::= \ p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid \langle\varphi\rangle\varphi \mid \Diamond\varphi$$

where $p \in Prop$. In APAL, $\Box$ quantifies only over updates with *epistemic* formulas. More precisely, the APAL semantics of $\Box$ is given by

$$w \in [\![\Box\varphi]\!] \quad \text{iff} \quad w \in [\![[\theta]\varphi]\!] \text{ for every } \theta \in \mathcal{L}_{epi},$$

where $\mathcal{L}_{epi}$ is the sublanguage generated from propositional variables $p \in Prop$ using only the Boolean connectives $\neg$ and $\wedge$, and the epistemic operators K_i. The finitary $\Box$-introduction rule proposed for the original APAL: "from $\psi \to [\theta][p]\varphi$, infer $\psi \to [\theta]\Box\varphi$ (where $p \notin P_\psi \cup P_\theta \cup P_\varphi$)" [, p. 323].

Kuijer's counterexample [23] shows the unsoundness of this rule, by taking the formula $\gamma := p \wedge \hat{K}_b\neg p \wedge \hat{K}_a K_b p$, and showing that $[\hat{K}_b p]\Box\neg\gamma \to [q]\neg\gamma$ is valid in APAL models, i.e., in multi-agent epistemic models with equivalence relations. (In fact, it is also valid in our a-models.) By the above $\Box$-intro rule, the formula $[\hat{K}_b p]\Box\neg\gamma \to \Box\neg\gamma$ should also be valid. But this is contradicted by the model $\mathcal{M}$ in Figure 2:

The premise $[\hat{K}_b p]\Box\neg\gamma$ is true at w in $\mathcal{M}$, since $\Box\neg\gamma$ holds at w in the updated model $\mathcal{M}|[\![\hat{K}_b p]\!]$ in Figure 3(a): indeed, the only way to falsify $\Box\neg\gamma$ in Figure 3(a) would be by deleting the node u_2 while keeping (at least) node u_1. But in $\mathcal{M}|[\![\hat{K}_b p]\!]$, u_1 and u_2 cannot be separated by epistemic sentences: they are bisimilar!

In contrast, the conclusion $\Box\neg\gamma$ is *false* at w in the original model $\mathcal{M}$, since there u_1 and u_2 *could be separated*. Indeed, we could perform an alternative update with the formula $p \vee \hat{K}_a r$, yielding the epistemic model $\mathcal{M}|[\![p \vee \hat{K}_a r]\!]$ in Figure 3(b), where γ *is true* at w (so that $\Box\neg\gamma$ was *false* in $\mathcal{M}$).

To see that the counterexample does not apply to APALM, notice that a-models keep track of the initial states. The a-model corresponding to our initial model $\mathcal{M}$ is the one drawn in Figure 4 - where the initial states and current states are the same:

The updated a-model $\mathcal{M}|[\![\hat{K}_b p]\!]$ (consisting now of the initial structure together with current set of worlds W from Figure 3(a)) is now drawn in Figure 5a, where the nodes in the shaded area are the current states. But in this a-model,

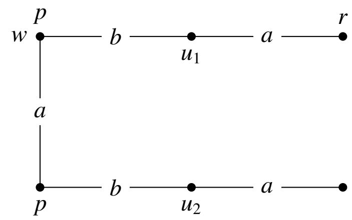
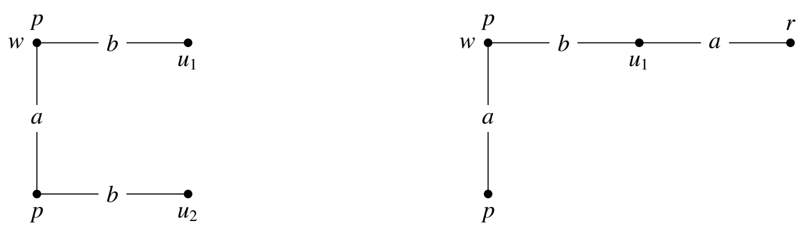


Fig. 2: An epistemic model $\mathcal{M}$. Worlds are nodes in the graph (e.g. w, u_1, u_2), valuation is given by labeling the nodes with the true atoms (e.g. p and r), and epistemic relations are given by labeled arrows.



(a) $\mathcal{M}|[\![\hat{K}_b p]\!]$ (b) $\mathcal{M}|[\![p \vee \hat{K}_a r]\!]$

Fig. 3: Two updates of $\mathcal{M}$.

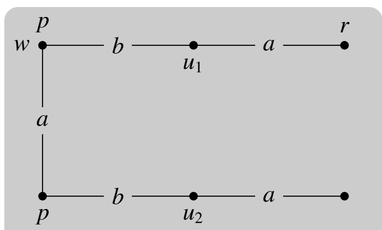


Fig. 4: $\mathcal{M}$ as an a-model. As before, initial states are all nodes in the graph and current states are represented by the nodes in the shaded area. So this is an initial model: all initial states are current states.

$\Box\neg\gamma$ is *no longer true* at w (and so the premise $[\hat{K}_b p]\Box\neg\gamma$ was *not* true in $\mathcal{M}$ when considered as an a-model!). Indeed, we *can* perform a new update of the a-model $\mathcal{M}|[\![\hat{K}_b p]\!]$ with the formula $(p \vee \hat{K}_a r)^0$, which yields the updated a-model given in Figure 5b.

(a) $\mathcal{M}|[\![\hat{K}_b p]\!]$, when $\mathcal{M}$ is an a-model

(b) $(\mathcal{M}|[\![\hat{K}_b p]\!])|[\![(p \vee \hat{K}_a r)^0)]\!]_{\mathcal{M}|[\![\hat{K}_b p]\!]}$

Fig. 5: Two updates of $\mathcal{M}$, when $\mathcal{M}$ is an a-model. Initial states are nodes in the graphs and current states are represented by the nodes in shaded areas.

Note that, in this new model, γ *is* the case at w (- thus showing that $\Box\neg\gamma$ was *not* true at w in $\mathcal{M}|[\![\hat{K}_b p]\!]$). So the counterexample simply does not work for APALM.

Moreover, we can see that the unsoundness of $[!]\Box$-intro rule for APAL has to do with its lack of memory, which leads to information loss after updates: while initially (in $\mathcal{M}$) there were epistemic sentences (e.g. $p \vee \hat{K}_a r$) that could separate u_1 and u_2 , there are no such sentences after the update.

APALM solves this by keeping track of the initial states, and referring back to them, as in $(p \vee \hat{K}_a r)^0$.

2.3 Axiomatization

Table 1 presents a complete proof system **APALM** for our logic APALM (where recall that P_φ is the set of propositional variables in φ).

The notion of derivation, denoted by $\vdash$, in **APALM** is defined as usual. Thus, $\vdash \varphi$ means φ is a theorem of **APALM**. For any set of formulas $\Gamma \subseteq \mathcal{L}$ and any $\varphi \in \mathcal{L}$, we write $\Gamma \vdash \varphi$ if there exists a finitely many formulas $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \varphi$. We say that Γ is **APALM**-*consistent* if $\Gamma \nvdash \bot$, and **APALM**-*inconsistent* otherwise.[15] We drop mention of the logic **APALM** when it is clear from the context.

Intuitive Reading of the Axioms Parts (I) and (II) should be obvious. The axiom R[$\top$] says that *updating with tautologies is redundant*. The reduction laws that do not contain ^{0}, U or 0 are well-known PAL axioms. R_U is the natural reduction law for the universal modality. The axiom R^0 says that the truth value of φ^0 formulas stays the same in time (because the superscript ^{0} serves as a time stamp), so they can be treated similarly to atoms. Ax_0 says that 0 was initially the case, and R_0 says that at any later stage (after any update) 0 can only be true if it was already true before the update and the update was trivial (universally true). Together, these two say that the constant 0 characterizes states where no non-trivial communication has occurred. Axiom 0-U is a synchronicity constraint: *if no non-trivial communication has taken place yet, then this is the case in all the currently possible states*. Axiom 0-*eq* says that *initially, φ is equivalent to its initial correspondent* φ^0. The Equivalences with ^{0} express that ^{0} distributes over negation and over conjunction. $\text{Imp}^0_\Box$ says that *if initially φ was stably true (under any further announcements), then φ is the case now*. Taken together, the elimination axiom $[!]\Box$-elim and introduction rule $[!]\Box$-intro say that *φ is a stable truth after an announcement θ iff φ stays true after any more informative announcement* (of the form $\theta \wedge \rho$). [16]

[15]Notions of derivation and (in)consistent sets for **GALM** studied in Section 3 are defined similarly.

[16]The "freshness" of the variable $p \in P$ in the rule $[!]\Box$-intro ensures that it represents any generic announcement.

	(I) Basic Axioms of system APALM:
(CPL)	all classical propositional tautologies and Modus Ponens
($S5_{K_i}$)	all $S5$ axioms and rules for knowledge operator K_i
($S5_U$)	all $S5$ axioms and rules for U operator
(U-K_i)	$U\varphi \to K_i\varphi$
	(II) Axioms and rules for dynamic modalities $[!]$:
($K_!$)	*Kripke's axiom for* $[!]$: $[\theta](\psi \to \varphi) \to ([\theta]\psi \to [\theta]\varphi)$
($Nec_!$)	*Necessitation for* $[!]$: from φ, infer $[\theta]\varphi$.
(RE)	*Replacement of Equivalents* $[!]$: from $\theta \leftrightarrow \rho$, infer $[\theta]\varphi \leftrightarrow [\rho]\varphi$.
Reduction laws:	
($R[\top]$)	$[\top]\varphi \leftrightarrow \varphi$
(R_p)	$[\theta]p \leftrightarrow (\theta \to p)$
($R_\neg$)	$[\theta]\neg\psi \leftrightarrow (\theta \to \neg[\theta]\psi)$
(R_{K_i})	$[\theta]K_i\psi \leftrightarrow (\theta \to K_i[\theta]\psi)$
($R_{[!]}$)	$[\theta][\rho]\chi \leftrightarrow [\langle\theta\rangle\rho]\chi$
(R^0)	$[\theta]\varphi^0 \leftrightarrow (\theta \to \varphi^0)$
(R_U)	$[\theta]U\varphi \leftrightarrow (\theta \to U[\theta]\varphi)$
(R_0)	$[\theta]0 \leftrightarrow (\theta \to (U\theta \wedge 0))$
	(III) Axioms and rules for $\top$, 0, and *initial* operator ^{0}:
(Ax_0)	0^0
(0-U)	$0 \to U0$
(0-*eq*)	$0 \to (\varphi \leftrightarrow \varphi^0)$
(Nec^0)	*Necessitation for* ^{0}: from φ, infer φ^0
Equivalences with ^{0}:	
(Eq^0_p)	$p^0 \leftrightarrow p$
($Eq^0_\neg$)	$(\neg\varphi)^0 \leftrightarrow \neg\varphi^0$
($Eq^0_\wedge$)	$(\varphi \wedge \psi)^0 \leftrightarrow (\varphi^0 \wedge \psi^0)$
Implications with ^{0}:	
(Imp^0_U)	$(U\varphi)^0 \to U\varphi^0$
(Imp^0_i)	$(K_i\varphi)^0 \to K_i\varphi^0$
($Imp^0_\Box$)	$(\Box\varphi)^0 \to \varphi$
	(III) Elim-axiom and Intro-rule for $\Box$:
($[!]\Box$-elim)	$[\theta]\Box\varphi \to [\theta \wedge \rho]\varphi$
($[!]\Box$-intro)	from $\chi \to [\theta \wedge p]\varphi$, infer $\chi \to [\theta]\Box\varphi$ (for $p \notin P_\chi \cup P_\theta \cup P_\varphi$).

Table 1: The axiomatization **APALM**. (Here, $\varphi, \psi, \chi \in \mathcal{L}$, while $\theta, \rho \in \mathcal{L}_{-\Diamond}$.)

Proposition 2. [17] *The following schemas and rules are derivable in* **APALM**, *for* $\varphi, \psi, \chi \in \mathcal{L}$ *and* $\theta \in \mathcal{L}_{-\Diamond}$*:*

1. *from* $\vdash \varphi \leftrightarrow \psi$*, infer* $\vdash [\theta]\varphi \leftrightarrow [\theta]\psi$
2. $\vdash \langle\theta\rangle 0 \leftrightarrow (0 \wedge U\theta)$
3. $\vdash \langle\theta\rangle\psi \leftrightarrow (\theta \wedge [\theta]\psi)$
4. *($\Box$-elim):* $\vdash \Box\varphi \to [\theta]\varphi$
5. *($\Box$-intro): from* $\vdash \chi \to [p]\varphi$*, infer* $\vdash \chi \to \Box\varphi$ $(p \notin P_\chi \cup P_\varphi)$
6. *all* $S4$ *axioms and rules for* $\Box$
7. $\vdash (\varphi \to \psi)^0 \leftrightarrow (\varphi^0 \to \psi^0)$
8. $\vdash \varphi^{00} \leftrightarrow \varphi^0$
9. $\vdash \Box\varphi^0 \leftrightarrow \varphi^0$*, and* $\vdash \Diamond\varphi^0 \leftrightarrow \varphi^0$
10. $\vdash (\Box\varphi)^0 \to \Box\varphi^0$
11. $\vdash (0 \wedge \Diamond\varphi^0) \to \varphi$
12. $\vdash \varphi \to (0 \wedge \Diamond\varphi)^0$
13. $\vdash \varphi \to \psi^0$ *if and only if* $\vdash (0 \wedge \Diamond\varphi) \to \psi$
14. $\vdash [\theta](\psi \wedge \varphi) \leftrightarrow ([\theta]\psi \wedge [\theta]\varphi)$
15. $\vdash [\theta][p]\psi \leftrightarrow [\theta \wedge p]\psi$
16. $\vdash [\theta]\bot \leftrightarrow \neg\theta$

[17] See Appendix A.2 for proof.

We arrive now at one of the main results of our paper.

Theorem 1 (Soundness and Completeness of APALM). *APALM validities are recursively enumerable. Indeed, the axiom system* **APALM** *in Table 1 is sound and complete wrt a-models.*

Soundness is proved in Section 4, and completeness in Section 4 and 5.

2.4 Expressivity: Comparisons and Bisimulation

To compare APALM and its fragments with basic epistemic logic (and its extension with the universal modality), consider the *static fragment* $\mathcal{L}_{-\Diamond,\langle!\rangle}$, obtained from $\mathcal{L}$ by removing both $\Diamond$ operator and the dynamic modality $\langle\varphi\rangle\psi$; as well as the *present-only fragment* $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$, obtained by removing operators 0 and φ^0 from $\mathcal{L}_{-\Diamond,\langle!\rangle}$; and finally the *epistemic fragment* $\mathcal{L}_{epi}$, obtained by removing U from $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$.

For every a-model $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$, consider its *initial epistemic model* $\mathcal{M}^{initial} = (W^0, \sim_1, \ldots, \sim_n, \|\cdot\|)$ and its *current epistemic model* $\mathcal{M}^{current} = (W, \sim_1 \cap (W \times W), \ldots, \sim_n \cap (W \times W), \|\cdot\| \cap W)$.

Proposition 3. *The fragment $\mathcal{L}_{-\Diamond}$ is co-expressive with the static fragment $\mathcal{L}_{-\Diamond,\langle!\rangle}$. In fact, every formula $\varphi \in \mathcal{L}_{-\Diamond}$ is provably equivalent to some formula $\psi \in \mathcal{L}_{-\Diamond,\langle!\rangle}$ (by using* **APALM** *reduction laws, given in Table 1) to eliminate dynamic modalities, as in standard PAL).*

Proof. Use step-by-step the reduction axioms (given in Table 1, Section 2.3), as a rewriting process, and prove termination by $<$-induction on φ by using Lemma 1.

Proposition 4. *The static fragment $\mathcal{L}_{-\Diamond,\langle!\rangle}$ (and hence, also $\mathcal{L}_{-\Diamond}$) is strictly more expressive than the present-only fragment $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$, which in turn is more expressive than the epistemic fragment $\mathcal{L}_{epi}$. In fact, each of the operators* 0 *and φ^0 independently increase the expressivity of $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$.*

Proof. Consider the a-model in Figure 5a: while u_1 and u_2 are indistinguishable for $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$, the sentence $(p \vee \hat{K}_a r)^0$ distinguishes the two. This shows that $\mathcal{L}_{-\Diamond,\langle!\rangle,0}$ is strictly more expressive than $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$. To see that $\mathcal{L}_{-\Diamond,\langle!\rangle,\varphi^0}$ is strictly more expressive than $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$, we just need to consider two a-models $\mathcal{M}_1 = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$ and $\mathcal{M}_2 = (W, W, \sim_1 \cap (W \times W), \ldots, \sim_n \cap (W \times W), \|\cdot\|)$ such that $W \subset W^0$. As both models have the same underlying current models, they make the same formulas of $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$ true at the same states in W. However, only the latter makes 0 true (at every state) since it is an initial model. Finally, note that the present-only fragment, $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$, is precisely the extension of $\mathcal{L}_{epi}$ with the universal modality and it is well-known that $\mathcal{L}_{epi}$ is strictly less expressive than its extension with the universal modality (see, e.g., [15, Chapter 7.1]). Thus, $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$ is more expressive than $\mathcal{L}_{epi}$.

Kuijer's counterexample presented in Section 2.2 shows that the standard epistemic bisimulation is not appropriate for APALM. In the following, we define an appropriate notion of bisimulation for APALM that leads to modal invariance and Hennessy-Milner property. We then also compare the expressive power of $\mathcal{L}_\Diamond$ and the *static fragment* $\mathcal{L}_{-\Diamond,\langle!\rangle}$ in Proposition 7, whose proof uses the notion of APALM bisimulation.

Definition 4 (Total/APALM Bisimulation).

- *A* total bisimulation *between epistemic models* $(W, \sim_1, \ldots, \sim_n, \|\cdot\|)$ *and* $(W', \sim'_1, \ldots, \sim'_n, \|\cdot\|')$ *is a non-empty binary relation* $B \subseteq W \times W'$ *such that*
  1. *if sBs', then $s \in \|p\|$ iff $s' \in \|p\|'$ for all $p \in Prop$,*
  2. *if sBs' and $s \sim_i t$, then there exists $t' \in W'$ such that tBt' and $s' \sim_i t'$ (the forth condition),*
  3. *if sBs' and $s' \sim'_i t'$, then there exists $t \in W$ such that tBt' and $s \sim_i t$ (the back condition), and*
  4. *for every $s \in W$ there exists some $s' \in W'$ with sBs'; and dually, for every $s' \in W'$ there exists some $s \in W$ with sBs'.*
- *An* APALM bisimulation *between a-models* $\mathcal{M}_1 = (W_1^0, W_1, \sim_1, \ldots, \sim_n, \|\cdot\|_1)$ *and* $\mathcal{M}_2 = (W_2^0, W_2, \sim'_1, \ldots, \sim'_n, \|\cdot\|_2)$ *is a total bisimulation B (as defined above) between the corresponding initial epistemic models* $\mathcal{M}_1^{initial}$ *and* $\mathcal{M}_2^{initial}$*, with the property that:* if $s_1 B s_2$, then $s_1 \in W_1$ iff $s_2 \in W_2$. *Two current states* $s_1 \in W_1$ *and* $s_2 \in W_2$ *are* APALM-bisimilar *if there exists an APALM bisimulation B between the underlying a-models such that* $s_1 B s_2$.

Since a-models are always of the form $\mathcal{M} = \mathcal{M}^0|[\![\theta]\!]$ for some $\theta \in \mathcal{L}_{-\diamond}$, we have a characterization of APALM-bisimulation only in terms of the initial models as stated in Proposition 5. To prove Propositions 5 and 6, we need the following auxiliary Lemmas 4 and 5.

Lemma 4. *Let B be a total epistemic bisimulation between initial epistemic models $\mathcal{M}_1^{initial}$ and $\mathcal{M}_2^{initial}$ (or equivalently, an APALM-bisimulation between initial a-models $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$), and let $s_1 \in W_1^0$, $s_2 \in W_2^0$ be two initial states such that $s_1 B s_2$. Then we have*

$$s_1 \in [\![\alpha]\!]_{\mathcal{M}_1^0} \textit{ iff } s_2 \in [\![\alpha]\!]_{\mathcal{M}_2^0}$$

for all formulas $\alpha \in \mathcal{L}_{-\diamond}$.

Proof. By Proposition 3, it is enough to prove the claim for all formulas $\alpha \in \mathcal{L}_{-\diamond,\langle!\rangle}$. Let B be an APALM bisimulation between initial a-models $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$. The proof is by subformula induction on α, using the following *induction hypothesis* (IH): for all $\beta \in Sub(\alpha)$, we have $s_1 \in [\![\beta]\!]_{\mathcal{M}_1^0}$ iff $s_2 \in [\![\beta]\!]_{\mathcal{M}_2^0}$ for all $s_1 \in W_1^0$, $s_2 \in W_2^0$ such that $s_1 B s_2$.

Base case $\alpha := p$: Since $s_1 B s_2$, $s_1 \in [\![p]\!]_{\mathcal{M}_1^0}$ iff $s_2 \in [\![p]\!]_{\mathcal{M}_2^0}$ follows by Definition 4, valuation condition.

Base case $\alpha := \top$: Since $s_1 \in W_1^0 = [\![\top]\!]_{\mathcal{M}_1^0}$ and $s_2 \in W_2^0 = [\![\top]\!]_{\mathcal{M}_2^0}$, we trivially obtain that $s_1 \in [\![\top]\!]_{\mathcal{M}_1^0}$ iff $s_2 \in [\![\top]\!]_{\mathcal{M}_2^0}$.

Base case $\alpha := 0$: Since $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$ are initial a-models, by the semantics, we have $s_1 \in W_1^0 = [\![0]\!]_{\mathcal{M}_1^0}$ and $s_2 \in W_2^0 = [\![0]\!]_{\mathcal{M}_2^0}$. We therefore trivially obtain that $s_1 \in [\![0]\!]_{\mathcal{M}_1^0}$ iff $s_2 \in [\![0]\!]_{\mathcal{M}_2^0}$.

Case $\alpha := \beta \wedge \gamma$ and $\alpha := \neg\beta$ follow straightforwardly by the semantics and IH.

In the following sequence of equivalencies, we make repeated use of the semantic clauses in Defn. 2.

Case $\alpha := \beta^0$
$s_1 \in [\![\beta^0]\!]_{\mathcal{M}_1^0}$ iff $s_1 \in [\![\beta]\!]_{\mathcal{M}_1^0} \cap W_1^0$ iff $s_2 \in [\![\beta]\!]_{\mathcal{M}_2^0} \cap W_2^0$ (by IH and $s_2 \in W_2^0$) iff $s_2 \in [\![\beta^0]\!]_{\mathcal{M}_2^0}$.

Case $\alpha := U\beta$
$s_1 \in [\![U\beta]\!]_{\mathcal{M}_1^0}$ iff $\forall s \in W_1^0, s \in [\![\beta]\!]_{\mathcal{M}_1^0}$ iff $\forall s' \in W_2^0, s' \in [\![\beta]\!]_{\mathcal{M}_2^0}$ (since B is total and IH) iff $s_2 \in [\![U\beta]\!]_{\mathcal{M}_2^0}$.

Case $\alpha := K_i\beta$
$s_1 \in [\![K_i\beta]\!]_{\mathcal{M}_1^0}$ iff $(\forall s \in W_1^0)(s \sim_i s_1$ implies $s \in [\![\beta]\!]_{\mathcal{M}_1^0})$ iff $(\forall s \in W_2^0)(s \sim_i' s_2$ implies $s \in [\![\beta]\!]_{\mathcal{M}_2^0})$ (back and forth condition, IH) iff $s_2 \in [\![K_i\beta]\!]_{\mathcal{M}_2^0}$.

Lemma 5. *Let B be a total epistemic bisimulation between initial epistemic models $\mathcal{M}_1^{initial}$ and $\mathcal{M}_2^{initial}$ (or equivalently, an APALM-bisimulation between initial a-models $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$), and let $s_1 \in W_1^0$, $s_2 \in W_2^0$ be two initial states such that $s_1 B s_2$. Then, for all $\varphi \in \mathcal{L}$, we have*

$$s_1 \in [\![\langle\alpha\rangle\varphi]\!]_{\mathcal{M}_1^0} \textit{ iff } s_2 \in [\![\langle\alpha\rangle\varphi]\!]_{\mathcal{M}_2^0}$$

for all formulas $\alpha \in \mathcal{L}_{-\diamond}$.

Proof. Let B be an APALM bisimulation between initial a-models $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$. The proof goes by $<$-induction on φ, using Lemma 1). We assume the following induction hypothesis: for all $\psi < \varphi$ in $\mathcal{L}$ and all states $s_1 \in W_1^0$, $s_2 \in W_2^0$ with $s_1 B s_2$, we have: $s_1 \in [\![\langle\alpha\rangle\psi]\!]_{\mathcal{M}_1^0}$ iff $s_2 \in [\![\langle\alpha\rangle\psi]\!]_{\mathcal{M}_2^0}$, for all $\alpha \in \mathcal{L}_{-\diamond}$.

Base cases $\varphi := p$, $\varphi := \top$, and $\varphi := 0$ follow directly from Lemma 4 and the fact that the formulas $\langle\alpha\rangle p$, $\langle\alpha\rangle\top$, and $\langle\alpha\rangle 0$ are in $\mathcal{L}_{-\diamond}$.

In the following sequence of equivalencies, we make repeated use of the semantic clauses in Defn. 2.

Case $\varphi := \psi^0$
$s_1 \in [\![\langle\alpha\rangle\psi^0]\!]_{\mathcal{M}_1^0}$ iff $s_1 \in [\![\psi^0]\!]_{\mathcal{M}_1^0|[\![\alpha]\!]_{\mathcal{M}_1^0}}$ iff $s_1 \in [\![\psi]\!]_{\mathcal{M}_1^0} \cap [\![\alpha]\!]_{\mathcal{M}_1^0}$ (since $\mathcal{M}_1^0|[\![\alpha]\!]_{\mathcal{M}_1^0} = (W_1^0, [\![\alpha]\!]_{\mathcal{M}_1^0}, \sim_1, \ldots, \sim_n, \|\cdot\|_1)$) iff $s_2 \in [\![\psi]\!]_{\mathcal{M}_2^0} \cap [\![\alpha]\!]_{\mathcal{M}_2^0}$ (by IH, Lemma 4: $\alpha \in \mathcal{L}_{-\diamond}$) iff $s_2 \in [\![\psi^0]\!]_{\mathcal{M}_2^0|[\![\alpha]\!]_{\mathcal{M}_2^0}}$ (since $\mathcal{M}_2^0|[\![\alpha]\!]_{\mathcal{M}_2^0} = (W_2^0, [\![\alpha]\!]_{\mathcal{M}_2^0}, \sim_1', \ldots, \sim_n', \|\cdot\|_2)$) iff $s_2 \in [\![\langle\alpha\rangle\psi^0]\!]_{\mathcal{M}_2^0}$.

Cases $\varphi := K_i\psi$ and $\varphi := U\psi$ follow similarly as in Lemma 4. We spell out here only the case $\varphi := U\psi$. We have two sub-cases:

Sub-case $[\![\alpha]\!]_{\mathcal{M}_1^0} = \emptyset$: This implies that $s_1 \notin [\![\alpha]\!]_{\mathcal{M}_1^0}$, thus, by Lemma 4, $s_2 \notin [\![\alpha]\!]_{\mathcal{M}_2^0}$. Therefore, $s_1 \notin [\![\langle\alpha\rangle U\psi]\!]_{\mathcal{M}_1^0}$ and $s_2 \notin [\![\langle\alpha\rangle U\psi]\!]_{\mathcal{M}_2^0}$. This implies that, if $s_1 \in [\![\langle\alpha\rangle U\psi]\!]_{\mathcal{M}_1^0}$ then $s_2 \in [\![\langle\alpha\rangle U\psi]\!]_{\mathcal{M}_2^0}$ (as the antecedent is false).

Sub-case $[\![\alpha]\!]_{\mathcal{M}_1^0} \neq \emptyset$: Observe that $s_1 \in [\![\langle\alpha\rangle U\psi]\!]_{\mathcal{M}_1^0}$ iff $s_1 \in [\![U\psi]\!]_{\mathcal{M}_1^0|[\![\alpha]\!]_{\mathcal{M}_1^0}}$ iff $\forall s \in [\![\alpha]\!]_{\mathcal{M}_1^0}, s \in [\![\psi]\!]_{\mathcal{M}_1^0|[\![\alpha]\!]_{\mathcal{M}_1^0}}$ iff $\forall s \in [\![\alpha]\!]_{\mathcal{M}_1^0}, s \in [\![\langle\alpha\rangle\psi]\!]_{\mathcal{M}_1^0}$. Suppose that $s_1 \in [\![\langle\alpha\rangle U\psi]\!]_{\mathcal{M}_1^0}$ and let $s' \in [\![\alpha]\!]_{\mathcal{M}_2^0}$. Since B is a total bisimulation, there is $s_1' \in W_1^0$ such that $s_1' B s'$. Since $\alpha \in \mathcal{L}_{-\diamond}$, by Lemma 4, we have $s_1' \in [\![\alpha]\!]_{\mathcal{M}_1^0}$. Then, by the above observation, we have $s_1' \in [\![\langle\alpha\rangle\psi]\!]_{\mathcal{M}_1^0}$. Thus, by IH, we obtain that $s' \in [\![\langle\alpha\rangle\psi]\!]_{\mathcal{M}_2^0}$. As $s_2 \in W_2^0$, we then conclude, via similar steps as in the above observation, that $s_2 \in [\![\langle\alpha\rangle U\psi]\!]_{\mathcal{M}_2^0}$. The other direction is similar. For the case $\varphi := K_i\psi$, we also use the back and forth conditions of B.

Case $\varphi := \langle\theta\rangle\psi$ uses the validity of the formula $\langle\alpha\rangle\langle\theta\rangle\psi \leftrightarrow \langle\langle\alpha\rangle\theta\rangle\psi$ which can be easily verified.
$s_1 \in [\![\langle\alpha\rangle\langle\theta\rangle\psi]\!]_{\mathcal{M}_1^0}$ iff $s_1 \in [\![\langle\langle\alpha\rangle\theta\rangle\psi]\!]_{\mathcal{M}_1^0}$ (by $\models \langle\alpha\rangle\langle\theta\rangle\psi \leftrightarrow \langle\langle\alpha\rangle\theta\rangle\psi$) iff $s_2 \in [\![\langle\langle\alpha\rangle\theta\rangle\psi]\!]_{\mathcal{M}_2^0}$ (IH, using $\psi < \langle\theta\rangle\psi$) iff $s_2 \in [\![\langle\alpha\rangle\langle\theta\rangle\psi]\!]_{\mathcal{M}_2^0}$.

Case $\varphi := \diamond\psi$
$s_1 \in [\![\langle\alpha\rangle\diamond\psi]\!]_{\mathcal{M}_1^0}$ iff $s_1 \in [\![\diamond\psi]\!]_{\mathcal{M}_1^0|[\![\alpha]\!]_{\mathcal{M}_1^0}}$ iff $s_1 \in \bigcup_{\theta\in\mathcal{L}_{-\diamond}} [\![\langle\theta\rangle\psi]\!]_{\mathcal{M}_1^0|[\![\alpha]\!]_{\mathcal{M}_1^0}}$ iff $s_1 \in \bigcup_{\theta\in\mathcal{L}_{-\diamond}} [\![\langle\alpha\rangle\langle\theta\rangle\psi]\!]_{\mathcal{M}_1^0}$ iff $s_2 \in \bigcup_{\theta\in\mathcal{L}_{-\diamond}} [\![\langle\alpha\rangle\langle\theta\rangle\psi]\!]_{\mathcal{M}_2^0}$ (IH, $\langle\theta\rangle\psi < \diamond\psi$) iff $s_2 \in \bigcup_{\theta\in\mathcal{L}_{-\diamond}} [\![\langle\theta\rangle\psi]\!]_{\mathcal{M}_2^0|[\![\alpha]\!]_{\mathcal{M}_2^0}}$ iff $s_2 \in [\![\diamond\psi]\!]_{\mathcal{M}_2^0|[\![\alpha]\!]_{\mathcal{M}_2^0}}$ iff $s_2 \in [\![\langle\alpha\rangle\diamond\psi]\!]_{\mathcal{M}_2^0}$.

Proposition 5. *Let $\mathcal{M}_1 = (W_1^0, W_1, \sim_1, \ldots, \sim_n, \|\cdot\|_1)$ and $\mathcal{M}_2 = (W_2^0, W_2, \sim_1', \ldots, \sim_n', \|\cdot\|_2)$ be a-models, and let $B \subseteq W_1^0 \times W_2^0$. The following are equivalent:*

1. *B is an APALM bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$;*
2. *B is a total epistemic bisimulation between $\mathcal{M}_1^{initial}$ and $\mathcal{M}_2^{initial}$ (or equivalently, an APALM bisimulation between $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$), and $\mathcal{M}_1 = \mathcal{M}_1^0|[\![\theta]\!]_{\mathcal{M}_1^0}$, $\mathcal{M}_2 = \mathcal{M}_2^0|[\![\theta]\!]_{\mathcal{M}_2^0}$ for some common formula $\theta \in \mathcal{L}_{-\diamond}$.*

Proof. (1) → (2): Let B be an APALM bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then it is obvious (from the definition) that B is also a total bisimulation between $\mathcal{M}_1^{initial}$ and $\mathcal{M}_2^{initial}$. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are a-models, there must exist $\theta_1, \theta_2 \in \mathcal{L}_{-\diamond}$ s.t. $\mathcal{M}_1 = \mathcal{M}_1^0|[\![\theta_1]\!]_{\mathcal{M}_1^0}$, $\mathcal{M}_2 = \mathcal{M}_2^0|[\![\theta_2]\!]_{\mathcal{M}_2^0}$. Hence, $W_1 = [\![\theta_1]\!]_{\mathcal{M}_1^0}$ and $W_2 = [\![\theta_2]\!]_{\mathcal{M}_2^0}$. To show that $[\![\theta_1]\!]_{\mathcal{M}_1^0} = [\![\theta_2]\!]_{\mathcal{M}_1^0}$, let first $s_1 \in [\![\theta_1]\!]_{\mathcal{M}_1^0} = W_1$. By the definition of APALM bisimulation, there must exist $s_2 \in W_2^0$ such that $s_1 B s_2$. Again by the definition, $s_1 \in W_1$ implies that $s_2 \in W_2 = [\![\theta_2]\!]_{\mathcal{M}_2^0}$. This, together with $s_1 B s_2$, gives us by Lemma 4 that $s_1 \in [\![\theta_2]\!]_{\mathcal{M}_1^0}$. For the converse, let $s_1 \in [\![\theta_2]\!]_{\mathcal{M}_1^0}$; by the definition of APALM bisimulation, there must exist $s_2 \in W_2^0$ such that $s_1 B s_2$. By Lemma 4, we have $s_2 \in [\![\theta_2]\!]_{\mathcal{M}_2^0} = W_2$, and again by the definition of APALM bisimulation (and the fact that $s_1 B s_2$), this implies that $s_1 \in W_1 = [\![\theta_1]\!]_{\mathcal{M}_1^0}$. Given that $\mathcal{M}_1 = \mathcal{M}_1^0|[\![\theta_1]\!]_{\mathcal{M}_1^0}$ and $\mathcal{M}_2 = \mathcal{M}_2^0|[\![\theta_2]\!]_{\mathcal{M}_2^0}$ such that $[\![\theta_1]\!]_{\mathcal{M}_1^0} = [\![\theta_2]\!]_{\mathcal{M}_1^0}$, we can take $\theta := \theta_2$. Then $\mathcal{M}_1 = \mathcal{M}_1^0|[\![\theta_1]\!]_{\mathcal{M}_1^0} = \mathcal{M}_1^0|[\![\theta_2]\!]_{\mathcal{M}_1^0}$.

(2) → (1): Suppose that B is a total bisimulation between $\mathcal{M}_1^{initial}$ and $\mathcal{M}_2^{initial}$, and $\mathcal{M}_1 = \mathcal{M}_1^0|[\![\theta]\!]_{\mathcal{M}_1^0}$, $\mathcal{M}_2 = \mathcal{M}_2^0|[\![\theta]\!]_{\mathcal{M}_2^0}$ for some common formula $\theta \in \mathcal{L}_{-\diamond}$. Hence, $W_1 = [\![\theta]\!]_{\mathcal{M}_1^0}$ and $W_2 = [\![\theta]\!]_{\mathcal{M}_2^0}$. We need to verify that $\mathcal{M}_1$ and $\mathcal{M}_2$ are APALM-bisimilar. For this we just need to verify the property that if $s_1 B s_2$, then $s_1 \in W_1$ holds iff $s_2 \in W_2$ holds. Suppose $s_1 B s_2$ and let $s_1 \in W_1 = [\![\theta]\!]_{\mathcal{M}_1^0} \subseteq W_1^0$. By the totality of the bisimulation B, there must exist some $s_2 \in W_2^0$ with $s_1 B s_2$. By Lemma 4, $s_1 \in [\![\theta]\!]_{\mathcal{M}_1^0}$ implies that $s_2 \in [\![\theta]\!]_{\mathcal{M}_2^0} = W_2$. The converse is analogous.

So, to check for APALM-bisimilarity, it is enough to check for total bisimilarity between the initial models and for both models being updates with the same formula.

Next, we verify that this is indeed the appropriate notion of bisimulation.

Corollary 1. APALM formulas are invariant under APALM-bisimulation*: if $s_1 B s_2$ for some APALM-bisimulation relation B between a-models $\mathcal{M}_1 = (W_1^0, W_1, \sim_1, \ldots, \sim_n, \|\cdot\|_1)$ and $\mathcal{M}_2 = (W_2^0, W_2, \sim_1', \ldots, \sim_n', \|\cdot\|_2)$, then: $s_1 \in [\![\varphi]\!]_{\mathcal{M}_1}$ iff $s_2 \in [\![\varphi]\!]_{\mathcal{M}_2}$ for all $\varphi \in \mathcal{L}$.*

Proof. Let B be some APALM-bisimulation relation between a-models $\mathcal{M}_1 = (W_1^0, W_1, \sim_1, \ldots, \sim_n, \|\cdot\|_1)$ and $\mathcal{M}_2 = (W_2^0, W_2, \sim_1', \ldots, \sim_n', \|\cdot\|_2)$. By Proposition 5, there exists some formula $\theta \in \mathcal{L}_{-\diamond}$ such that $\mathcal{M}_1 = \mathcal{M}_1^0|[\![\theta]\!]_{\mathcal{M}_1^0}$, $\mathcal{M}_2 = \mathcal{M}_2^0|[\![\theta]\!]_{\mathcal{M}_2^0}$. By the same proposition, B is a total epistemic bisimulation between the initial epistemic models $\mathcal{M}_1^{initial}$ and $\mathcal{M}_2^{initial}$. Thus, for every formula φ, we have the sequence of equivalences: $s_1 \in [\![\varphi]\!]_{\mathcal{M}_1}$ iff $s_1 \in [\![\langle\theta\rangle\varphi]\!]_{\mathcal{M}_1^0}$ iff (by Lemma 5) $s_2 \in [\![\langle\theta\rangle\varphi]\!]_{\mathcal{M}_2^0}$ iff $s_2 \in [\![\varphi]\!]_{\mathcal{M}_2}$.

Proposition 6 (Hennessy-Milner). *Let* $\mathcal{M}_1 = (W_1^0, W_1, \sim_1, \ldots, \sim_n, \|\cdot\|_1)$ *and* $\mathcal{M}_2 = (W_2^0, W_2, \sim_1', \ldots, \sim_n', \|\cdot\|_2)$ *be a-models with* W_1^0 *and* W_2^0 *finite. Then,* $s_1 \in W_1$ *and* $s_2 \in W_2$ *satisfy the same APALM formulas iff they are APALM-bisimilar.*

Proof. We only need to prove the left-to-right direction. Let $s_1 \in W_1$ and $s_2 \in W_2$ such that for all $\varphi \in \mathcal{L}$, $s_1 \in [\![\varphi]\!]_{\mathcal{M}_1}$ iff $s_2 \in [\![\varphi]\!]_{\mathcal{M}_2}$. This implies that for all $\varphi \in \mathcal{L}$, $s_1 \in [\![\varphi]\!]_{\mathcal{M}_1^0}$ iff $s_2 \in [\![\varphi]\!]_{\mathcal{M}_2^0}$. To see this, let $\varphi \in \mathcal{L}$ such that $s_1 \in [\![\varphi]\!]_{\mathcal{M}_1^0}$. This means, by the semantics, that $s_1 \in [\![\varphi^0]\!]_{\mathcal{M}_1}$. As s_1 in $\mathcal{M}_1$ and s_2 in $\mathcal{M}_2$ satisfy the same APALM formulas, we obtain that $s_2 \in [\![\varphi^0]\!]_{\mathcal{M}_2}$, thus, $s_2 \in [\![\varphi]\!]_{\mathcal{M}_2^0}$. The opposite direction is analogous. We then show that the modal equivalence relation in $W_1^0 \times W_2^0$ between the models $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$ is an APALM bisimulation. We thus need to show the following:

- (Totality) For all $s \in W_1^0$, there exists $s' \in W_2^0$ such that, $s \in [\![\varphi]\!]_{\mathcal{M}_1^0}$ iff $s' \in [\![\varphi]\!]_{\mathcal{M}_2^0}$ for all $\varphi \in \mathcal{L}$, and for all $s' \in W_2^0$, there exists $s \in W_1^0$ such that $s \in [\![\varphi]\!]_{\mathcal{M}_1^0}$ iff $s' \in [\![\varphi]\!]_{\mathcal{M}_2^0}$ for all $\varphi \in \mathcal{L}$.
  Let $s \in W_1^0$ and suppose, toward contradiction, that for no element s' of W_2^0 we have that $s \in [\![\varphi]\!]_{\mathcal{M}_1^0}$ iff $s' \in [\![\varphi]\!]_{\mathcal{M}_2^0}$ for all $\varphi \in \mathcal{L}$. Since W_2^0 is finite, we can list its elements $W_2^0 = \{w_1, w_2, \ldots, w_n\}$. The first assumption then implies that for all $w_i \in W_2^0$, there exists $\psi_i \in \mathcal{L}$ such that $s \in [\![\psi_i]\!]_{\mathcal{M}_1^0}$ but $w_i \notin [\![\psi_i]\!]_{\mathcal{M}_2^0}$. Thus, $s_1 \in [\![E(\psi_1 \wedge \cdots \wedge \psi_n)]\!]_{\mathcal{M}_1^0}$ but $s_2 \notin [\![E(\psi_1 \wedge \cdots \wedge \psi_n)]\!]_{\mathcal{M}_2^0}$, contradicting the assumption that s_1 in $\mathcal{M}_1^0$ and s_2 in $\mathcal{M}_2^0$ satisfy the same APALM formulas. The second clause follows similarly.
- (Valuation) This follows immediately from modal equivalence.
- (Forth for $\sim_i$) Let $w_1, w_1' \in W_1^0$ and $w_2 \in W_2^0$ such that $w_1 \in [\![\varphi]\!]_{\mathcal{M}_1^0}$ iff $w_2 \in [\![\varphi]\!]_{\mathcal{M}_2^0}$ for all $\varphi \in \mathcal{L}$ and $w_1 \sim_i w_1'$. Suppose, toward contradiction, that for no element $w_2' \in W_2^0$ with $w_2 \sim_i' w_2'$, $\mathcal{M}_1^0, w_1'$ and $\mathcal{M}_2^0, w_2'$ satisfy the same APALM formulas. Since W_2^0 is finite, the set $\sim_i'(w_2) = \{t \in W_2^0 : w_2 \sim_i' t\}$ is finite, thus, we can write $\sim_i'(w_2) = \{t_1, \ldots, t_k\}$. As in the proof of (Totality), the assumption implies that for all t_j with $w_2 \sim_i' t_j$, there exists $\psi_j \in \mathcal{L}$ such that $w_1' \in [\![\psi_j]\!]_{\mathcal{M}_1^0}$ but $t_j \notin [\![\psi_j]\!]_{\mathcal{M}_2^0}$. Therefore, $w_1 \in [\![\hat{K}_i(\psi_1 \wedge \cdots \wedge \psi_k)]\!]_{\mathcal{M}_1^0}$ but $w_2 \notin [\![\hat{K}_i(\psi_1 \wedge \cdots \wedge \psi_k)]\!]_{\mathcal{M}_2^0}$, contradicting the assumption that $\mathcal{M}_1^0, w_1$ and $\mathcal{M}_2^0, w_2$ satisfy the same APALM formulas. Back condition for $\sim_i$ follows analogously.

We have therefore proven that the modal equivalence relation in $W_1^0 \times W_2^0$ between the models $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$ is an APALM bisimulation between $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$. By Proposition 5, it suffices to further prove that $\mathcal{M}_1 = \mathcal{M}_1^0|[\![\theta]\!]_{\mathcal{M}_1^0}$, $\mathcal{M}_2 = \mathcal{M}_2^0|[\![\theta]\!]_{\mathcal{M}_2^0}$ for some common formula $\theta \in \mathcal{L}_{-\diamond}$. It then suffices to show that $[\![\theta_1]\!]_{\mathcal{M}_2^0} = [\![\theta_2]\!]_{\mathcal{M}_2^0}$, where $W_1 = [\![\theta_1]\!]_{\mathcal{M}_1^0}$ and $W_2 = [\![\theta_2]\!]_{\mathcal{M}_2^0}$.

- $[\![\theta_2]\!]_{\mathcal{M}_2^0} \subseteq [\![\theta_1]\!]_{\mathcal{M}_2^0}$: Observe that $s_1 \in [\![U\theta_1^0]\!]_{\mathcal{M}_1}$, since $W_1 = [\![\theta_1]\!]_{\mathcal{M}_1^0}$. Moreover, as $\mathcal{M}_1, s_1$ and $\mathcal{M}_2, s_2$ satisfy the same APALM formulas, we obtain that $s_2 \in [\![U\theta_1^0]\!]_{\mathcal{M}_2}$. Therefore, for all $y \in [\![\theta_2]\!]_{\mathcal{M}_2^0}$, we have $y \in [\![\theta_1^0]\!]_{\mathcal{M}_2}$, implying that $y \in [\![\theta_1]\!]_{\mathcal{M}_2^0}$. Hence, $[\![\theta_2]\!]_{\mathcal{M}_2^0} \subseteq [\![\theta_1]\!]_{\mathcal{M}_2^0}$.

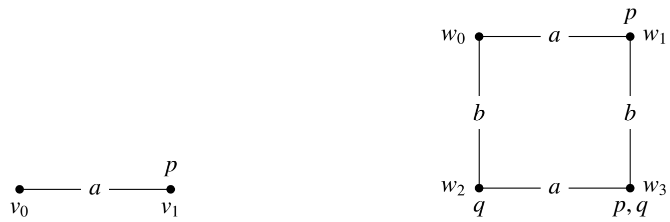
- $[\![\theta_1]\!]_{\mathcal{M}_2^0} \subseteq [\![\theta_2]\!]_{\mathcal{M}_2^0}$: Observe that $s_2 \in [\![U\theta_2^0]\!]_{\mathcal{M}_2}$, since $W_2 = [\![\theta_2]\!]_{\mathcal{M}_2^0}$. Moreover, as $\mathcal{M}_1, s_1$ and $\mathcal{M}_2, s_2$ satisfy the same APALM formulas, we obtain that $s_1 \in [\![U\theta_2^0]\!]_{\mathcal{M}_1}$. Now suppose, toward contradiction, that $[\![\theta_1]\!]_{\mathcal{M}_2^0} \not\subseteq [\![\theta_2]\!]_{\mathcal{M}_2^0}$, i.e., there is $y \in W_2^0$ such that $y \in [\![\theta_1]\!]_{\mathcal{M}_2^0}$ but $y \notin [\![\theta_2]\!]_{\mathcal{M}_2^0}$. By the totality of the modal equivalence relation, there exists $x \in W_1^0$ such that $x \in [\![\theta_1]\!]_{\mathcal{M}_1^0}$ but $x \notin [\![\theta_2]\!]_{\mathcal{M}_1^0}$. The former implies that $x \in W_1$. Therefore, by the latter, we have that $x \notin [\![\theta_2^0]\!]_{\mathcal{M}_1}$. This implies, since $s_1, x \in W_1$, that $s_1 \notin [\![U\theta_2^0]\!]_{\mathcal{M}_1}$, contradicting $s_1 \in [\![U\theta_2^0]\!]_{\mathcal{M}_1}$.

Therefore, we obtain that $[\![\theta_1]\!]_{\mathcal{M}_2^0} = [\![\theta_2]\!]_{\mathcal{M}_2^0}$. Given that $\mathcal{M}_1 = \mathcal{M}_1^0|[\![\theta_1]\!]_{\mathcal{M}_1^0}$ and $\mathcal{M}_2 = \mathcal{M}_2^0|[\![\theta_2]\!]_{\mathcal{M}_2^0}$ such that $[\![\theta_1]\!]_{\mathcal{M}_2^0} = [\![\theta_2]\!]_{\mathcal{M}_2^0}$, we can take $\theta := \theta_1$. Then $\mathcal{M}_2 = \mathcal{M}_2^0|[\![\theta_2]\!]_{\mathcal{M}_2^0} = \mathcal{M}_2^0|[\![\theta_1]\!]_{\mathcal{M}_2^0}$.

As a last result in this section, we compare the expressive powers of $\mathcal{L}$ and $\mathcal{L}_{-\diamond}$.

Proposition 7. $\mathcal{L}$ *is strictly more expressive than* $\mathcal{L}_{-\diamond}$ *and, therefore, than the static fragment* $\mathcal{L}_{-\diamond,\langle!\rangle}$.

Proof. By Proposition 3, it suffices to show that $\mathcal{L}$ is strictly more expressive than $\mathcal{L}_{-\diamond,\langle!\rangle}$. Wlog, we assume that $\mathcal{AG} = \{a, b\}$. The proof follows by a similar argument as in [6, Proposition 3.13] via contradiction: suppose that $\mathcal{L}$ and $\mathcal{L}_{-\diamond,\langle!\rangle}$ are equally expressive for a-models, i.e., for all $\varphi \in \mathcal{L}$ there exists $\psi \in \mathcal{L}_{-\diamond,\langle!\rangle}$ such that $\models \varphi \leftrightarrow \psi$. Consider the formula



(a) $\mathcal{M}$ (b) $\mathcal{M}'$

Fig. 6: $\mathcal{M} = (W, W, \sim_a, \sim_b, \|\cdot\|)$ and $\mathcal{M}' = (W', W', \sim'_a, \sim'_b, \|\cdot\|')$ in Proposition 7.

$\Diamond(K_a p \wedge \neg K_b K_a p)$ in $\mathcal{L}$. By the assumption, there must be $\psi \in \mathcal{L}_{-\Diamond,\langle!\rangle}$ such that $\models \Diamond(K_a p \wedge \neg K_b K_a p) \leftrightarrow \psi$. To reach the desired contradiction, we now construct two a-models $\mathcal{M}$ and $\mathcal{M}'$ (similar to the ones in the proof in [6]) which agree on ψ at the actual world but disagree on $\Diamond(K_a p \wedge \neg K_b K_a p)$. For this argument it is crucial to observe that any such ψ contains only finitely many propositional variables. As we have countably infinitely many propositional variables, there is a propositional variable q that does not occur in ψ (that is also different from p). Without loss of generality, suppose ψ is built using only one variable p. Consider the a-models $\mathcal{M} = (W, W, \sim_a, \sim_b, \|\cdot\|)$ and $\mathcal{M}' = (W', W', \sim'_a, \sim'_b, \|\cdot\|')$ given in Figure 6. It is easy to see that both $\mathcal{M}$ and $\mathcal{M}'$ are initial a-models. Moreover, they are also APALM-bisimilar with respect to the language of APALM constructed from using *only* the propositional variable p and agents a and b. In particular, the corresponding bisimulation relation is $B = \{(v_0, w_0), (v_0, w_2), (v_1, w_1), (v_1, w_3)\}$. Then, by Lemma 4, we have, e.g., that $v_1 \in [\![\psi]\!]_{\mathcal{M}}$ iff $w_1 \in [\![\psi]\!]_{\mathcal{M}'}$. However, while $v_1 \notin [\![\Diamond(K_a p \wedge \neg K_b K_a p)]\!]_{\mathcal{M}}$, we have $w_1 \in [\![\Diamond(K_a p \wedge \neg K_b K_a p)]\!]_{\mathcal{M}'}$, since $w_1 \in [\![\langle p \vee q\rangle(K_a p \wedge \neg K_b K_a p)]\!]_{\mathcal{M}'}$.

All the expressivity results of this section are summarized by the diagram in Figure 7.

$$\begin{tikzcd}
 & & \mathcal{L}_{-\Diamond,\langle!\rangle,0} \arrow[r, two heads] & \mathcal{L}_{-\Diamond} \arrow[d, two heads, shift left] \arrow[dr, two heads] & \\
\mathcal{L}_{epi} \arrow[r, two heads] & \mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0} \arrow[ur, two heads] \arrow[dr, two heads] & & & \mathcal{L} \\
 & & \mathcal{L}_{-\Diamond,\langle!\rangle,\varphi^0} \arrow[r, two heads] & \mathcal{L}_{-\Diamond,\langle!\rangle} \arrow[u, two heads, shift left] \arrow[ur, two heads] &
\end{tikzcd}$$

Fig. 7: Expressivity diagram. Arrows point to the more expressive languages.

3 Group Announcement Logic with Memory

In this section we turn our focus on the Group Announcement Logic (GAL), and we propose a memory-enhanced version GALM. As in the case of APALM, the addition (to both models and logic) of a memory of the initial states helps to provide a recursive axiomatization, by re-establishing the soundness of the natural GAL inference rule (already proposed in [1], but later shown to be unsound). Moreover, the same move makes possible the reduction of the related Coalition Announcement Logic CAL (or more precisely, its memory-enhanced version CALM) to a fragment of GALM, via an intuitively 'obvious' equivalence (-which was proved to be invalid on memory-free models, but becomes valid in the memory-enhanced version).

3.1 GAL, CAL and their problems

Like APAL, Group Announcement Logic GAL (first introduced in [1]) is also an extension of PAL, involving group announcement operators $[G]\varphi$ and $\langle G\rangle\varphi$ (instead of the arbitrary announcement operators $\Box\varphi$ and $\Diamond\psi$). More precisely, the language of GAL is defined recursively as

$$\varphi ::= \ p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid \langle\psi\rangle\varphi \mid \langle G\rangle\varphi$$

where $p \in Prop$, $i \in \mathcal{AG}$, and $G \subseteq \mathcal{AG}$.

The group announcement operator can be seen as a restricted version of the arbitrary public announcement operator in the sense that it quantifies only over updates with formulas of the form $\bigwedge_{i\in G} K_i\theta_i$, where $\theta_i \in \mathcal{L}_{epi}$ and $i \in G \subseteq \mathcal{AG}$. Ågotnes et al. [1] interpret the operator $[G]\varphi$ on epistemic models $M = (W, \sim_1, \dots, \sim_n, \|\cdot\|)$ as

$$w \in [\![[G]\varphi]\!] \quad \text{iff} \quad \text{for every set } \{\psi_i : i \in G\} \subseteq \mathcal{L}_{epi}, w \in [\![[\bigwedge_{i\in G} K_i\psi_i]\varphi]\!]. \tag{1}$$

This operator intends to capture communication among a group of agents and what a coalition can bring about via public announcements. While GAL seems to provide more adequate tools than APAL to treat puzzles involving epistemic dialogues, the axiomatization of GAL presented in [1] has a similar shape as the one for APAL in [6]. To recall, [1] proves completeness of GAL also by using an infinitary rule and claims that it is replaceable in theorem-proving by the finitary rule

$$\text{from } \varphi \to [\theta][\bigwedge_{i\in G} K_i p_i]\psi \text{ infer } \varphi \to [\theta][G]\psi, \tag{$R[G]$}$$

where $p_i \notin P_\varphi \cup P_\psi \cup P_\theta$. However, Kuijer's counterexample presented in Section 2.2 constitutes a counterexample also for the soundness of this rule. Consider again the formula $\gamma := p \wedge \hat{K}_b\neg p \wedge \hat{K}_a K_b p$ and let $G = \{a\}$. We first show that

$$[\hat{K}_b p][G]\neg\gamma \to [K_a q]\neg\gamma$$

is valid on epistemic models. For this, suppose that $[\hat{K}_b p][G]\neg\gamma \to [K_a q]\neg\gamma$ is not valid on epistemic models, i.e., that there is an epistemic model $\mathcal{N} = (W, \sim_1, \dots, \sim_n, \|\cdot\|)$ and $w \in W$ such that $w \in [\![[\hat{K}_b p][G]\neg\gamma]\!]$ but $w \notin [\![[K_a q]\neg\gamma]\!]$. The latter means that $w \in [\![\langle K_a q\rangle\gamma]\!]$. Therefore, $w \in [\![K_a q]\!]$ and $w \in [\![\gamma]\!]_{\mathcal{N}|[\![K_a q]\!]}$. The latter implies that $w \in \|p\|$ and there are two states w_1, w_2 in $\mathcal{N}|[\![K_a q]\!]$ such that (1) w_1 is $\sim_b$-connected to w and $w_1 \notin \|p\|$, and (2) w_2 is $\sim_a$-connected to w and $w_2 \in [\![K_b p]\!]_{\mathcal{N}|[\![K_a q]\!]}$. In other words, the model in Figure 8 is guaranteed to be a submodel of $\mathcal{N}|[\![K_a q]\!]$.

Moreover, since $w \in [\![[\hat{K}_b p][G]\neg\gamma]\!]$ and $w \in [\![\hat{K}_b p]\!]$, we also have that $w \in [\![[G]\neg\gamma]\!]_{\mathcal{N}|[\![\hat{K}_b p]\!]}$. Recall that $w \in [\![K_a q]\!]$. Therefore, neither w nor w_2 have $\sim_a$-access to a states in $\mathcal{N}$ that makes q false. Furthermore, since $K_a q$ is a positive knowledge formula, we have $w \in [\![K_a q]\!]_{\mathcal{N}|\theta}$ for any θ such that $w \in [\![\theta]\!]_{\mathcal{N}}$. Then, $w \in [\![[G]\neg\gamma]\!]_{\mathcal{N}|[\![\hat{K}_b p]\!]}$ implies that $w \in [\![\neg\gamma]\!]_{(\mathcal{N}|[\![\hat{K}_b p]\!])|[\![K_a q]\!]_{\mathcal{N}|[\![\hat{K}_b p]\!]}} = [\![\neg\gamma]\!]_{(\mathcal{N}|[\![\langle\hat{K}_b p\rangle K_a q]\!])}$. It is not difficult to see that the model in Figure 8 is also a submodel of $\mathcal{N}|[\![\langle\hat{K}_b p\rangle K_a q]\!]$ (recall that w_1 is in $\mathcal{N}|[\![K_a q]\!]$), thus, $w \in [\![\gamma]\!]_{(\mathcal{N}|[\![\langle\hat{K}_b p\rangle K_a q]\!])}$. This contradicts the assumption that $w \in [\![[\hat{K}_b p][G]\neg\gamma]\!]$. Therefore, $[\hat{K}_b p][G]\neg\gamma \to [K_a q]\neg\gamma$ is indeed valid on epistemic models.

If the $R[G]$-rule were sound, then by applying it we obtain that

$$[\hat{K}_b p][G]\neg\gamma \to [G]\neg\gamma$$

should be valid. But this is *not* the case: the model $\mathcal{M}$ in Figure 2 constitutes a counterexample, since $w \in [\![[\hat{K}_b p][G]\neg\gamma]\!]_{\mathcal{M}}$, but $w \in [\![\langle K_a\hat{K}_a(p \vee r)\rangle\gamma]\!]_{\mathcal{M}}$, and thus $w \notin [\![[G]\neg\gamma]\!]$.

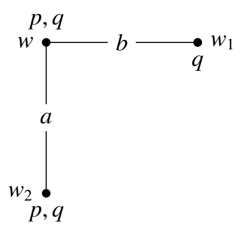


Fig. 8: Submodel of $\mathcal{N}|[\![K_a q]\!]$

Coalition modality. A related operator is the *coalition announcement* modality $\langle[G]\rangle\varphi$ that lies at the core of Coalition Announcement Logic (CAL), introduced in [2]: this is a coalition logic in the style of [26], but where the actions that agents can perform are restricted to public announcements. CAL is simply the extension of multi-agent epistemic logic with such coalition announcement modalities. Ågotnes et al. [2] interpret $\langle[G]\rangle\varphi$ on epistemic models $M = (W, \sim_1, \ldots, \sim_n, \|\cdot\|)$ as

$$w \in [\![\langle[G]\rangle\varphi]\!] \quad \text{iff} \quad \exists\{\psi_i : i \in G\} \subseteq \mathcal{L}_{epi}\, \forall\{\psi_j : j \in \mathcal{AG} - G\} \subseteq \mathcal{L}_{epi},\ w \in [\![\bigwedge_{i \in G} K_i\psi \wedge [\bigwedge_{m \in \mathcal{AG}} K_m\psi_m]\varphi]\!]. \tag{2}$$

The semantics of group and coalition announcement operators suggests at the first sight that the latter might be defined in terms of the former as $\langle[G]\rangle\varphi \leftrightarrow \langle G\rangle[\mathcal{AG} - G]\varphi$. However, this 'obvious' equivalence is proved to be *invalid* in [19].

Our diagnosis for the failure of this rather intuitive equivalence is the same as our explanation for the unsoundness of the finitary $\Box$-introduction rule for APAL or GAL: the models' *lack of memory* is the reason for the non-equivalence between the coalition announcement operator $\langle[G]\rangle\varphi$ (which expresses coalition G's ability to bring about φ by a joint announcement against any *simultaneous* joint announcement by the anti-coalition $\mathcal{AG} - G$) and the expression $\langle G\rangle[\mathcal{AG} - G]\varphi$ (which captures a similar ability of group G against any *subsequent* joint announcement by the anti-coalition).

As in the case of APAL, the same lack of memory leads also to difficulties in obtaining a recursive axiomatization for GAL, CAL and related logics. To the best of our knowledge, there are no known recursive axiomatizations for GAL, CAL etc.[18] In fact, the same state of affair applies to any logic that contains coalition announcement operators [2,32,3,4].

3.2 A principled solution: GALM

In this section, we develop a *Group Announcement Logic with Memory* (GALM), obtained by extending the syntax of APALM with group announcement operators interpreted on a-models.[19] Moreover, we show that the memory-enhanced version (CALM) of CAL is indeed embeddeded in GALM, via the natural analogue of the above-mentioned equivalence. Finally, we give a complete recursive axiomatization of GALM.

The language $\mathcal{L}_G$ of GALM is defined recursively as

$$\varphi \ ::= \ p \mid \top \mid 0 \mid \varphi^0 \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid U\varphi \mid \langle\theta\rangle\varphi \mid \Diamond\varphi \mid \langle G\rangle\varphi,$$

where $p \in Prop$, $i \in \mathcal{AG}$, $\theta \in \mathcal{L}_{-\Diamond}$, and $G \subseteq \mathcal{AG}$. Note that $\mathcal{L}_{-\Diamond}$ is the same as before, namely, it is the set of sentences in $\mathcal{L}_G$ that do not include $\Diamond$ or $\langle G\rangle$. In the context of GALM, the elements of $\mathcal{L}_{-\Diamond}$ are called $\Diamond, \langle G\rangle$*-free formulas*. The dual modality for this new operator is defined as $[G]\varphi := \neg\langle G\rangle\neg\varphi$. $\langle G\rangle\varphi$ and $[G]\varphi$ are the (existential and universal, respectively) group announcement operators, quantifying over updates with formulas of the form $\bigwedge_{i \in G} K_i\theta_i$,

[18]But in [21], the authors introduce a combination of an extension of GAL and CAL and present an *infinitary* axiomatization with two rules that resemble the infinitary rule in GAL.

[19]We note that the language of the original GAL in [1] does not include the arbitrary announcement operator $\Box$. The fragment of GALM without the arbitrary announcement operators can be studied in a similar way. We prefer to work with a larger language here, subsuming both APAL and GAL-type modalities, in order be able to present the soundness and completeness proofs for both APALM and GALM in a unified way.

where $\theta_i \in \mathcal{L}_{-\diamond}$ and $i \in G$. This restricted quantification over $\mathcal{L}_{-\diamond}$ captures the assumption that each agent can announce only the ($\diamond$ and G-free) propositions she knows and nothing else. Analogous to the reading of $\Box$, we read $[G]\varphi$ as "φ is *stably true* under group G's public announcements", i.e., "φ stays true no matter what group G truthfully announces".

We introduce the following abbreviation of *relativized knowledge* for notational convenience:

$$K_i^{\varphi}\psi := K_i(\varphi \rightarrow \psi),$$

where $\varphi, \psi \in \mathcal{L}_G$ and $i \in \mathcal{AG}$.

The following lemma will be useful in the completeness proof.

Lemma 6. *There exists a well-founded strict partial order $<$ on $\mathcal{L}_G$, such that:*

1. *if φ is a subformula of ψ, then $\varphi < \psi$,*
2. *$\langle\theta\rangle\varphi < \diamond\varphi$, for all $\theta \in \mathcal{L}_{-\diamond}$,*
3. *$\langle\theta\rangle\varphi < \langle G\rangle\varphi$, for all $\theta \in \mathcal{L}_{-\diamond}$.*

Proof. Similar to Lemma 1.

The language $\mathcal{L}_G$ is interpreted on the same models introduced in Definition 1.

Definition 5. *Given a model $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$, the semantics for $\mathcal{L}_G$ is defined recursively as in Definition 2 with the following additional clause for $\langle G\rangle$:*

$$[\![\langle G\rangle\varphi]\!] = \bigcup\{[\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\varphi]\!] \ : \ \{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\diamond}\}.$$

Observation 3 *Note that we have*

$$w \in [\![[G]\varphi]\!] \quad \textit{iff} \quad w \in [\![[\bigwedge_{i\in G} K_i\theta_i]\varphi]\!] \ \textit{ for every } \{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\diamond}.$$

The GALM analogue of Observation 2 is again a straightforward consequence of the semantics:

Observation 4 *Given a model $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$ and formulas $\theta, \rho, \varphi \in \mathcal{L}_G$, if $[\![\theta]\!]_{\mathcal{M}} = [\![\rho]\!]_{\mathcal{M}}$, then $[\![\langle\theta\rangle\varphi]\!]_{\mathcal{M}} = [\![\langle\rho\rangle\varphi]\!]_{\mathcal{M}}$ and $[\![[\theta]\varphi]\!]_{\mathcal{M}} = [\![[\rho]\varphi]\!]_{\mathcal{M}}$.*

Proposition 8. *We have $[\![\varphi]\!] \subseteq W$, for all formulas $\varphi \in \mathcal{L}_G$.*

Proof. See Appendix A.3.

As for APALM, the *intended* models for GALM are the *announcement models* (a-models, introduced in Definition 3). So GALM validities are defined with respect to a-models, as in Definition 3. It should also be obvious that the analogue of Lemma 3 still holds for GALM:

Lemma 7. *If $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$ is an a-model and $\theta \in \mathcal{L}_{-\diamond}$ is a formula such that $W = [\![\theta]\!]_{\mathcal{M}^0}$, then for all formulas $\varphi \in \mathcal{L}_G$ and all formulas $\rho \in \mathcal{L}_{-\diamond}$, we have:*

$$[\![\varphi]\!]_{\mathcal{M}} = [\![\langle\theta\rangle\varphi]\!]_{\mathcal{M}^0} = [\![(\langle\theta\rangle\varphi)^0]\!]_{\mathcal{M}},$$

$$[\![\rho \wedge \varphi]\!]_{\mathcal{M}} = [\![\langle\rho\rangle(\langle\theta\rangle\varphi)^0]\!]_{\mathcal{M}}.$$

The proof is exactly the same as the proof of Lemma 3. This result can be used to prove that in the memory-enhanced environment of GALM, the (memory-enhanced) coalition announcement modalities $\langle[G]\rangle\varphi$ *are* in fact definable using group announcement modalities (in contrast to the situation in memory-free GAL):

Proposition 9. *Let $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$ be an a-model, and $\theta \in \mathcal{L}_{-\diamond}$ be a formula such that $W = [\![\theta]\!]_{\mathcal{M}^0}$. For every group $G \subseteq \mathcal{AG}$ and every formula $\varphi \in \mathcal{L}_G$, we have:*

$$w \in [\![\langle G\rangle[\mathcal{AG} - G]\varphi]\!]_{\mathcal{M}} \quad \textit{iff} \quad \exists\{\psi_i : i \in G\} \subseteq \mathcal{L}_{-\diamond}\ \forall\{\psi_j : j \in \mathcal{AG} - G\} \subseteq \mathcal{L}_{-\diamond},\ w \in [\![\bigwedge_{i\in G} K_i\psi_i \wedge [\bigwedge_{m\in\mathcal{AG}} K_m\psi_m]\varphi]\!]_{\mathcal{M}}.$$

Proof. For the right-to-left implication: suppose there exists a set of formula $\{\psi_i : i \in G\} \subseteq \mathcal{L}_{-\Diamond}$ satisfying the property in the right-hand side of the above Proposition, i.e.:

$$\forall\{\psi_j : j \in \mathcal{AG} - G\} \subseteq \mathcal{L}_{-\Diamond},\ w \in [\![\bigwedge_{i\in G} K_i\psi_i \wedge [\bigwedge_{m\in\mathcal{AG}} K_m\psi_m]\varphi]\!]_{\mathcal{M}}.$$

Let $\{\psi_j : j \in \mathcal{AG} - G\} \subseteq \mathcal{L}_{-\Diamond}$ be some arbitrary set of formulas in $\mathcal{L}_{-\Diamond}$, indexed by agents in $\mathcal{AG} - G$. By applying the above claim (from the right-hand side of the Proposition) to the set $\{[\bigwedge_{i\in G} K_i\psi_i]\psi_j : j \in \mathcal{AG} - G\}$, we obtain that

$$w \in [\![\bigwedge_{i\in G} K_i\psi_i \wedge [\bigwedge_{i\in G} K_i\psi_i \wedge \bigwedge_{j\in\mathcal{AG}-G} K_j[\bigwedge_{i\in G} K_i\psi_i]\psi_j]\varphi]\!]_{\mathcal{M}}.$$

Applying to this Proposition 2.3, then Axiom (R_{K_i}), then Axiom $(R_!)$ and finally Proposition 2.3 again, we obtain that

$$w \in [\![\bigwedge_{i\in G} K_i\psi_i \wedge [\langle\bigwedge_{i\in G} K_i\psi_i\rangle \bigwedge_{j\in\mathcal{AG}-G} K_j\psi_j]\varphi]\!]_{\mathcal{M}} = [\![\bigwedge_{i\in G} K_i\psi_i \wedge [\bigwedge_{i\in G} K_i\psi_i][\bigwedge_{j\in\mathcal{AG}-G} K_j\psi_j]\varphi]\!]_{\mathcal{M}} = [\![\langle\bigwedge_{i\in G} K_i\psi_i\rangle[\bigwedge_{j\in\mathcal{AG}-G} K_j\psi_j]\varphi]\!]_{\mathcal{M}}.$$

So we established that

$$\exists\{\psi_i : i \in G\} \subseteq \mathcal{L}_{-\Diamond}\ \forall\{\psi_j : j \in \mathcal{AG} - G\} \subseteq \mathcal{L}_{-\Diamond},\ w \in [\![\langle\bigwedge_{i\in G} K_i\psi_i\rangle[\bigwedge_{j\in\mathcal{AG}-G} K_j\psi_j]\varphi]\!]_{\mathcal{M}},$$

which by the semantics of GALM is equivalent to $w \in [\![\langle G\rangle[\mathcal{AG} - G]\varphi]\!]_{\mathcal{M}}$, as desired.

For the left-to-right implication: suppose that we have $w \in [\![\langle G\rangle[\mathcal{AG} - G]\varphi]\!]_{\mathcal{M}}$, By the semantics, there exists a set of formula $\{\psi_i : i \in G\} \subseteq \mathcal{L}_{-\Diamond}$ s.t.

$$w \in [\![\langle\bigwedge_{i\in G} K_i\psi_i\rangle[\mathcal{AG} - G]\varphi]\!]_{\mathcal{M}} = [\![[\mathcal{AG} - G]\varphi]\!]_{\mathcal{M}|[\![\bigwedge_{i\in G} K_i\psi_i]\!]}.$$

Let $\{\psi_j : j \in \mathcal{AG} - G\} \subseteq \mathcal{L}_{-\Diamond}$ be any arbitrary set of formulas in $\mathcal{L}_{-\Diamond}$, indexed by agents in $\mathcal{AG} - G$. Applying the semantics of $[\mathcal{AG} - G]\varphi$ in $\mathcal{M}|[\![\bigwedge_{i\in G} K_i\psi_i]\!]$ to the set $\{(\langle\theta\rangle K_j\psi_j)^0 : j \in \mathcal{AG} - G\}$, the last displayed formula implies that:

$w \in [\![[\bigwedge_{j\in\mathcal{AG}-G}\left(\langle\theta\rangle K_j\psi_j\right)^0]\varphi]\!]_{\mathcal{M}|[\![\bigwedge_{i\in G} K_i\psi_i]\!]} = [\![\langle\bigwedge_{i\in G} K_i\psi_i\rangle[\left(\langle\theta\rangle\bigwedge_{j\in\mathcal{AG}-G} K_j\psi_j\right)^0]\varphi]\!]_{\mathcal{M}} = [\![\bigwedge_{i\in G} K_i\psi_i \wedge [\langle\bigwedge_{i\in G} K_i\psi_i\rangle\left(\langle\theta\rangle\bigwedge_{j\in\mathcal{AG}-G} K_j\psi_j\right)^0]\varphi]\!]_{\mathcal{M}}.$

But by Lemma 7, we have

$$[\![\langle\bigwedge_{i\in G} K_i\psi_i\rangle\left(\langle\theta\rangle\bigwedge_{j\in\mathcal{AG}-G} K_j\psi_j\right)^0]\!]_{\mathcal{M}} = [\![\bigwedge_{i\in G} K_i\psi_i \wedge \bigwedge_{j\in\mathcal{AG}-G} K_j\psi_j]\!]_{\mathcal{M}} = [\![\bigwedge_{m\in\mathcal{AG}} K_m\psi_m]\!]_{\mathcal{M}}.$$

By Observation 4, the above facts imply that $w \in [\![\bigwedge_{i\in G} K_i\psi_i \wedge [\bigwedge_{m\in\mathcal{AG}} K_m\psi_m]\varphi]\!]_{\mathcal{M}}$, as desired.

So, in contrast to the situation in the memory-free case of Coalition Announcement Logic, the memory-enhanced version is essentially a fragment of GALM.

We move now to the main result of this section.

Theorem 2 (Soundness and Completeness of GALM). *GALM validities are recursively enumerable. In fact, the sound and complete axiomatization* **GALM** *wrt a-models is obtained by extending* **APALM** *with the axiom and rule given in Table 2.*

	Elim-axiom and Intro-rule for $[G]$:
$([!][G]$-elim$)$	$[\theta][G]\varphi \to [\theta \wedge \bigwedge_{i\in G} K_i^\theta \rho_i]\varphi$
$([!][G]$-intro$)$	from $\chi \to [\theta \wedge \bigwedge_{i\in G} K_i^\theta p_i]\varphi$, infer $\chi \to [\theta][G]\varphi$ (for $p_i \notin P_\chi \cup P_\theta \cup P_\varphi$).

Table 2: The additional axioms of **GALM**

The axiom and rule in Table 2 are very similar in spirit (and in what they express) to the $[!]\Box$-elim axiom and $[!]\Box$-intro rule, respectively. Together, the elimination axiom $[!][G]$-elim and rule $[!][G]$-intro say that φ *is a stable truth under group G's announcements after an announcement* θ *iff* φ *stays true after any more informative announcement from the group* G (of the form $\theta \wedge \bigwedge_{i \in G} K_i^{\theta} \rho_i$).

4 Soundness via Pseudo-model Semantics

As GALM is an extension of APALM, we present the soundness and completeness proofs directly for the former. The same results for APALM are obtained following similar steps.

To start with, note that even the soundness of our axiomatic systems is not a trivial matter. As we saw from Kuijer's counterexample, the analogues of our finitary $\Box$ and $[G]$-introduction rules were *not* sound for APAL and GAL, respectively. To prove their soundness on a-models, we need a detour into an equivalent semantics, in the style of Subset Space Logics (SSL) [25,18]: *pseudo-models*.[20]

We first introduce an auxiliary notion: 'pre-models' are just SSL models, coming with a given family $\mathcal{A}$ of "admissible sets" of worlds (which can be thought of as the *communicable* propositions). We interpret $\Box$ in these structures as the so-called "effort modality" of SSL, which quantifies over updates with admissible propositions in $\mathcal{A}$. Analogously, $\langle G \rangle$ quantifies over updates with conjunctions of those admissible propositions in the scope of an epistemic operator labeled by an agent in G. Our 'pseudo-models' are pre-models with additional closure conditions (saying that the family of admissible sets includes the valuations and is closed under complement, intersection, and epistemic operators). These conditions imply that *every set definable by a* $\Diamond, \langle G \rangle$*-free formula*[21] *is admissible*, and this ensures the soundness of our $\Box$-elimination and $[G]$-elimination axioms on pseudo-models. As for the soundness of the long-problematic $\Box$ and $[G]$-introduction rules on (both pre- and) pseudo-models, this is due to the fact that both the effort modality and $[G]$ operator interpreted on pseudo-models have a more 'robust' range than the arbitrary announcement versions of them: they quantify over admissible sets, regardless of whether these sets are syntactically definable or not. Soundness with respect to our a-models then follows from the observation that they (in contrast to the original APAL models) are in fact equivalent to a *special case of pseudo-models*: the "standard" ones (in which the admissible sets in $\mathcal{A}$ are *exactly* the sets definable by $\Diamond, \langle G \rangle$-free formulas).

Definition 6 (Pre-model). *A* pre-model *is a tuple* $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$*, where* W^0 *is the initial domain,* $\sim_i$ *are equivalence relations on* W^0*,* $\|\cdot\| : Prop \to \mathcal{P}(W^0)$ *is a valuation map, and* $\mathcal{A} \subseteq \mathcal{P}(W^0)$ *is a family of subsets of the initial domain, called* admissible sets *(representing the propositions that can be publicly announced).*

Given a set $A \subseteq W^0$ and a state $w \in A$, we use the notation $w_i^A := \{s \in A : w \sim_i s\}$ to denote the restriction to A of w's equivalence class modulo $\sim_i$. We also introduce the following abbreviation for the semantic counterpart of relativized knowledge: $K_i^A B = \{w \in W^0 : w_i \cap A \subseteq B\}$.

Definition 7 (Pre-model Semantics for $\mathcal{L}_G$). *Given a pre-model* $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$*, we recursively define a* truth set $[\![\varphi]\!]_A$ *for every formula* φ *and* $A \in \mathcal{A}$*:*

$$[\![p]\!]_A = \|p\| \cap A \qquad [\![0]\!]_A = \begin{cases} A & \text{if } A = W^0 \\ \emptyset & \text{otherwise} \end{cases}$$

$$[\![\top]\!]_A = A \qquad [\![\varphi^0]\!]_A = [\![\varphi]\!]_{W^0} \cap A$$

$$[\![\neg\varphi]\!]_A = A - [\![\varphi]\!]_A$$

[20] A more direct soundness proof on a-models is in principle possible, but would require at least as much work as our detour. Unlike in standard EL, PAL or DEL, the meaning of an APALM formula (and therefore of a GALM formula) depends, not only on the valuation of the atoms occurring in it, but also on the family $\mathcal{A}$ of all sets definable by $\mathcal{L}_{-\Diamond}$-formulas. The move from models to pseudo-models makes explicit this dependence on the family $\mathcal{A}$, while also relaxing the demands on $\mathcal{A}$ (which is no longer required to be exactly the family of $\mathcal{L}_{-\Diamond}$-definable sets), and thus makes the soundness proof both simpler and more transparent. Since we will need pseudo-models for our completeness proof anyway, we see no added value in trying to give a more direct soundness proof.

[21] $\Diamond, \langle G \rangle$-free formulas are the sentences in $\mathcal{L}_{-\Diamond}$.

$$[\![\varphi \wedge \psi]\!]_A = [\![\varphi]\!]_A \cap [\![\psi]\!]_A$$

$$[\![K_i\varphi]\!]_A = \{w \in A : w_i^A \subseteq [\![\varphi]\!]_A\}$$

$$[\![U\varphi]\!]_A = \begin{cases} A & \text{if } [\![\varphi]\!]_A = A \\ \emptyset & \text{otherwise} \end{cases}$$

$$[\![\langle\theta\rangle\varphi]\!]_A = \begin{cases} [\![\varphi]\!]_{[\![\theta]\!]_A} & \text{if } [\![\theta]\!]_A \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

$$[\![\Diamond\varphi]\!]_A = \bigcup\{[\![\varphi]\!]_B : B \in \mathcal{A}, B \subseteq A\}$$

$$[\![\langle G\rangle\psi]\!]_A = \bigcup\{[\![\psi]\!]_{A\cap\bigcap_{i\in G} K_i^A B_i} : \{B_i : i \in G\} \subseteq \mathcal{A}\}$$

Observation 5 *Note that, for all $w \in A$, we have*

1. $w \in [\![\Box\varphi]\!]_A$ *iff* $\forall B \in \mathcal{A}\ (w \in B \subseteq A \Rightarrow w \in [\![\varphi]\!]_B)$*;*
2. $w \in [\![[G]\varphi]\!]_A$ *iff for every* $\{B_i : i \in G\} \subseteq \mathcal{A}$ $(w \in A \cap \bigcap_{i\in G} K_i^A B_i \Rightarrow w \in [\![\varphi]\!]_{A\cap\bigcap_{i\in G} K_i^A B_i})$*;*
3. $[\![\varphi]\!]_A \subseteq A$ *for all* $A \in \mathcal{A}$ *and* $\varphi \in \mathcal{L}_G$.

Observation 5.1 shows that our proposed semantics of $\Box$ on pre-models fits with the semantics of the effort modality in SSL [25,18]. The proof of Observation 5.3 is similar to that of Proposition 8.

Definition 8 (Pseudo-models and Validity). *A* pseudo-model *is a pre-model* $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$*, satisfying the following closure conditions:*

1. $\|p\| \in \mathcal{A}$*, for all* $p \in Prop$*,*
2. $W^0 \in \mathcal{A}$*,*
3. *if* $A \in \mathcal{A}$ *then* $(W^0 - A) \in \mathcal{A}$*,*
4. *if* $A, B \in \mathcal{A}$ *then* $(A \cap B) \in \mathcal{A}$*,*
5. *if* $A \in \mathcal{A}$ *then* $K_iA \in \mathcal{A}$*, where* $K_iA := \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow s \in A)\}$.

A formula $\varphi \in \mathcal{L}_G$ *is* valid in pseudo-models *if it is true in all admissible sets* $A \in \mathcal{A}$ *of every pseudo-model* $\mathcal{M}$*, i.e,* $[\![\varphi]\!]_A = A$ *for all* $A \in \mathcal{A}$ *and all* $\mathcal{M}$.

Lemma 8. *Given a pseudo-model* $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ *and* $A, B \in \mathcal{A}$*, we have* $K_i^A B \in \mathcal{A}$.

Proof. First note that by Definition 8.5 and Boolean operations of sets we have,
$K_i^A B = \{w \in W^0 : w_i \cap A \subseteq B)\} = \{w \in W^0 : \forall s \in W^0((s \in A \text{ and } w \sim_i s) \Rightarrow s \in B)\}$
$= \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow (s \in A \Rightarrow s \in B))\} = \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow (s \in (W^0 - A) \text{ or } s \in B))\}$
$= \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow s \in (W^0 - A) \cup B)\} = K_i((W^0 - A) \cup B)$.
Then, by Definition 8.(3-5) and $A, B \in \mathcal{A}$, we obtain $K_i^A B = K_i((W^0 - A) \cup B) \in \mathcal{A}$.

Proposition 10. *Given a pseudo-model* $(W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$*,* $A \in \mathcal{A}$*, and* $\theta \in \mathcal{L}_{-\Diamond}$*, we have* $[\![\theta]\!]_A \in \mathcal{A}$.

Proof. The proof is by subformula induction on θ. The base cases and the inductive cases for the Booleans are immediate (using the conditions in Definition 8).

Case $\theta := \psi^0$: By the semantics, $[\![\psi^0]\!]_A = [\![\psi]\!]_{W^0} \cap A \in \mathcal{A}$, since $[\![\psi]\!]_{W^0} \in \mathcal{A}$ (by the fact that $W^0 \in \mathcal{A}$ and IH), $A \in \mathcal{A}$ (by assumption), and $\mathcal{A}$ is closed under intersection.

Case $\theta := K_i\psi$: Note that $[\![K_i\psi]\!]_A = \{w \in A : w_i^A \subseteq [\![\psi]\!]_A\} = A \cap \{w \in W^0 : w_i^A \subseteq [\![\psi]\!]_A\}$ (by Definition 8) $= A \cap \{w \in W^0 : \forall s \in W^0((s \in A \text{ and } w \sim_i s) \Rightarrow s \in [\![\psi]\!]_A)\}$. We then obtain, by CPL and Boolean operations of sets that $[\![K_i\psi]\!]_A = A \cap \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow s \in ((W^0 - A) \cup [\![\psi]\!]_A)\}$. Moreover, $A \cap \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow s \in ((W^0 - A) \cup [\![\psi]\!]_A)\} = A \cap K_i((W^0 - A) \cup [\![\psi]\!]_A)$ by Definition 8.(3-5) (since $A \in \mathcal{A}$ and $[\![\psi]\!]_A \in \mathcal{A}$ by IH). Therefore, $[\![K_i\psi]\!]_A = A \cap K_i((W^0 - A) \cup [\![\psi]\!]_A)$ is in $\mathcal{A}$.

Case $\theta := U\psi$: By Definition 8, $[\![U\psi]\!]_A \in \{\emptyset, A\} \subseteq \mathcal{A}$.

Case $\theta := \langle\varphi\rangle\psi$: Since $A \in \mathcal{A}$, we have $[\![\varphi]\!]_A \in \mathcal{A}$ (by IH on φ), and hence $[\![\langle\varphi\rangle\psi]\!]_A = [\![\psi]\!]_{[\![\varphi]\!]_A} \in \mathcal{A}$ (by the semantics and IH on ψ).

To prove the soundness of our axioms, we need the following lemmas:

Lemma 9. *Given a pseudo-model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$, $A \in \mathcal{A}$ and $\theta \in \mathcal{L}_{-\Diamond}$ such that $w \in [\![\theta]\!]_A$, $w \in [\![K_i^\theta \rho]\!]_A$ iff $w \in K_i^{[\![\theta]\!]_A}[\![\rho]\!]_A$ for all $\rho \in \mathcal{L}_{-\Diamond}$.*

Proof. Observe that $K_i^{[\![\theta]\!]_A}[\![\rho]\!]_A = K_i((W^0 - [\![\theta]\!]_A) \cup [\![\rho]\!]_A)$ (as in Lemma 8). Moreover, it's easy to see that, $[\![K_i^\theta \rho]\!]_A = [\![K_i(\theta \to \rho)]\!]_A = \{w \in A : w_i^A \subseteq [\![\theta \to \rho]\!]_A\} = A \cap \{w \in W^0 : \forall s \in W^0 (w \sim_i s \Rightarrow s \in (W^0 \setminus [\![\theta]\!]_A \cup [\![\rho]\!]_A))\}$ (since $[\![\theta]\!]_A \subseteq A$). We therefore obtain that $[\![K_i^\theta \rho]\!]_A = A \cap K_i((W^0 - [\![\theta]\!]_A) \cup [\![\rho]\!]_A)$ (by Boolean operations of sets and the defn. of K_i). Thus, $[\![K_i^\theta \rho]\!]_A = A \cap K_i((W^0 - [\![\theta]\!]_A) \cup [\![\rho]\!]_A) = A \cap K_i^{[\![\theta]\!]_A}[\![\rho]\!]_A$. Therefore if $w \in [\![\theta]\!]_A \subseteq A$, $w \in [\![K_i^\theta \rho]\!]_A$ iff $w \in K_i^{[\![\theta]\!]_A}[\![\rho]\!]_A$.

Lemma 10. *Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ and $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|')$ be two pseudo-models and $\varphi \in \mathcal{L}_G$ such that $\mathcal{M}$ and $\mathcal{M}'$ differ only in the valuation of some finite number of propositional variables $p_1, \dots, p_n \notin P_\varphi$. Then, for all $A \in \mathcal{A}$, we have $[\![\varphi]\!]_A^{\mathcal{M}} = [\![\varphi]\!]_A^{\mathcal{M}'}$.*

Proof. The proof follows by subformula induction on φ. Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ and $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|')$ be two pseudo-models such that $\mathcal{M}$ and $\mathcal{M}'$ differ only in the valuation of some $p_1, \dots, p_n \notin P_\varphi$ and let $A \in \mathcal{A}$. We want to show that $[\![\varphi]\!]_A^{\mathcal{M}} = [\![\varphi]\!]_A^{\mathcal{M}'}$. The base cases $\varphi := q (\notin P_\varphi)$, $\varphi := \top$, $\varphi := 0$, and the inductive cases for Booleans are standard.

Case $\varphi := \psi^0$. Note that $P_{\psi^0} = P_\psi$. Then, by IH, we have that $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}}$ for every $A \in \mathcal{A}$, in particular for $W^0 \in \mathcal{A}$. Thus $[\![\psi]\!]_{W^0}^{\mathcal{M}'} = [\![\psi]\!]_{W^0}^{\mathcal{M}}$. Then, $[\![\psi]\!]_{W^0}^{\mathcal{M}'} \cap A = [\![\psi]\!]_{W^0}^{\mathcal{M}} \cap A$ for all $A \in \mathcal{A}$. By the semantics of the initial operator on pseudo-models, we obtain $[\![\psi^0]\!]_A^{\mathcal{M}'} = [\![\psi^0]\!]_A^{\mathcal{M}}$.

Case $\varphi := K_i\psi$. Note that $P_{K_i\psi} = P_\psi$. Then, by IH, we have that $[\![\psi]\!]_A^{\mathcal{M}} = [\![\psi]\!]_A^{\mathcal{M}'}$. Observe that $[\![K_i\psi]\!]_A^{\mathcal{M}} = \{w \in A : w_i^A \subseteq [\![\psi]\!]_A^{\mathcal{M}}\}$ and, similarly, $[\![K_i\psi]\!]_A^{\mathcal{M}'} = \{w \in A : w_i^A \subseteq [\![\psi]\!]_A^{\mathcal{M}'}\}$. Then, since $[\![\psi]\!]_A^{\mathcal{M}} = [\![\psi]\!]_A^{\mathcal{M}'}$, we obtain $[\![K_i\psi]\!]_A^{\mathcal{M}} = [\![K_i\psi]\!]_A^{\mathcal{M}'}$.

Case $\varphi := U\psi$. Note that $P_{U\psi} = P_\psi$. Then, by IH, we have that $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}}$ for every $A \in \mathcal{A}$. We have two case: (1) If $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}} = A$, then $[\![U\psi]\!]_A^{\mathcal{M}'} = A = [\![U\psi]\!]_A^{\mathcal{M}}$. (2) If $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}} \neq A$, then $[\![U\psi]\!]_A^{\mathcal{M}'} = [\![U\psi]\!]_A^{\mathcal{M}} = \emptyset$.

Case $\varphi := \langle\theta\rangle\psi$. Note that $P_{\langle\theta\rangle\psi} = P_\theta \cup P_\psi$. By IH, we have $[\![\theta]\!]_A^{\mathcal{M}'} = [\![\theta]\!]_A^{\mathcal{M}}$ and $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}}$ for every $A \in \mathcal{A}$. By Proposition 10, we know that $[\![\theta]\!]_A^{\mathcal{M}} = [\![\theta]\!]_A^{\mathcal{M}'} \in \mathcal{A}$. Therefore, in particular, we have $[\![\psi]\!]_{[\![\theta]\!]_A^{\mathcal{M}'}}^{\mathcal{M}'} = [\![\psi]\!]_{[\![\theta]\!]_A^{\mathcal{M}}}^{\mathcal{M}}$. Therefore, by the semantics of $\langle ! \rangle$ on pseudo-models, we obtain $[\![\langle\theta\rangle\psi]\!]_A^{\mathcal{M}'} = [\![\langle\theta\rangle\psi]\!]_A^{\mathcal{M}}$.

Case $\varphi := \Diamond\psi$. Note that $P_{\Diamond\psi} = P_\psi$. Since *the same* family of sets $\mathcal{A}$ is carried by both models $\mathcal{M}$ and $\mathcal{M}'$ and since (by IH) $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}}$ for all $A \in \mathcal{A}$, we get:

$$[\![\Diamond\psi]\!]_A^{\mathcal{M}'} = \bigcup\{[\![\psi]\!]_B^{\mathcal{M}'} : B \in \mathcal{A}, B \subseteq A\} = \bigcup\{[\![\psi]\!]_B^{\mathcal{M}} : B \in \mathcal{A}, B \subseteq A\} = [\![\Diamond\psi]\!]_A^{\mathcal{M}}.$$

Case $\varphi := [G]\psi$. Note that $P_{[G]\psi} = P_\psi$. Then, by (IH), we have that $[\![\psi]\!]_B^{\mathcal{M}'} = [\![\psi]\!]_B^{\mathcal{M}}$ for every $B \in \mathcal{A}$. In particular, $[\![\psi]\!]_B^{\mathcal{M}'} = [\![\psi]\!]_B^{\mathcal{M}}$ for the B's of the form $A \cap K_i^A C$ with $A, C \in \mathcal{A}$ (recall that pseudo-models are closed under K_i^A operation and conjunction, see Definition 8 and Lemma 8). Since the same family of sets $\mathcal{A}$ is carried by both models $\mathcal{M}$ and $\mathcal{M}'$, we obtain:

$$[\![[G]\psi]\!]_A^{\mathcal{M}'} = \bigcup\{[\![\psi]\!]_{A \cap \bigcap_{i \in G} K_i^A B_i}^{\mathcal{M}'} : \{B_i : i \in G\} \subseteq \mathcal{A}\} = \bigcup\{[\![\psi]\!]_{A \cap \bigcap_{i \in G} K_i^A B_i}^{\mathcal{M}} : \{B_i : i \in G\} \subseteq \mathcal{A}\} = [\![[G]\psi]\!]_A^{\mathcal{M}}.$$

Proposition 11. *The system* **GALM** *is sound wrt pseudo-models. Therefore, the system* **APALM** *is also sound wrt pseudo-models.*

Proof. The soundness of most of the axioms follows simply by spelling out the semantics. We present here only the validity of the axioms $[!]\Box$-elim, $[!][G]$-elim, and that the rules $[!]\Box$-intro and $[!][G]$-intro preserve validity:

For the elimination axioms, let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ be a pseudo-model, $A \in \mathcal{A}$, and $w \in A$ arbitrarily chosen:

($[!]\Box$-elim): Let $\rho \in \mathcal{L}_{-\Diamond}$ and suppose (1) $w \in [\![[\theta]\Box\varphi]\!]_A$ and (2) $w \in [\![\theta \wedge \rho]\!]_A$. We need to show that $w \in [\![\varphi]\!]_{[\![\theta \wedge \rho]\!]_A}$. Assumption (1) means that if $w \in [\![\theta]\!]_A$ then $w \in [\![\Box\varphi]\!]_{[\![\theta]\!]_A}$. Then, by assumption (2) and since $w \in [\![\theta \wedge \rho]\!]_A \subseteq [\![\theta]\!]_A$, we have

$w \in [\![\Box\varphi]\!]_{[\![\theta]\!]_A}$. Thus, by the semantic clause for $\Box$, we have $w \in \{u \in [\![\theta]\!]_A :$ for all $B \in \mathcal{A}(u \in B \subseteq [\![\theta]\!]_A \Rightarrow u \in [\![\varphi]\!]_B\}$. Therefore, for $B := [\![\theta \wedge \rho]\!]_A \subseteq [\![\theta]\!]_A$ (since by Proposition 10, $[\![\theta \wedge \rho]\!]_A \in \mathcal{A}$) we have $w \in [\![\varphi]\!]_{[\![\theta\wedge\rho]\!]_A}$.

([!][G]-elim): Let $\{\rho_i : i \in G\} \subseteq \mathcal{L}_{-\Diamond}$ and suppose (1) $w \in [\![[\theta][G]\varphi]\!]_A$ and (2) $w \in [\![\theta \wedge \bigwedge_{i\in G} K_i^\theta \rho_i]\!]_A$. By assumption (1) and the semantic clause for $[G]$, we have that if $w \in [\![\theta]\!]_A$ then for all $\{B_i : i \in G\} \subseteq \mathcal{A}$, $w \in [\![\varphi]\!]_{[\![\theta]\!]_A \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A} B_i}$. By (2) we have that $w \in [\![\theta]\!]_A$ and $w \in [\![\bigwedge_{i\in G} K_i^\theta \rho_i]\!]_A$. Thus, by (1), we obtain $w \in [\![\varphi]\!]_{[\![\theta]\!]_A \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A} [\![\rho_i]\!]_A} = [\![\varphi]\!]_{[\![\theta]\!]_A \cap \bigcap_{i\in G} [\![K_i^\theta \rho_i]\!]_A} = [\![\varphi]\!]_{[\![\theta\wedge\bigwedge_{i\in G} K_i^\theta \rho_i]\!]_A}$ (by Proposition 10 and Lemma 9). Thus, by assumption (2), we obtain that $w \in [\![[\theta \wedge \bigwedge_{i\in G} K_i^\theta \rho_i]\varphi]\!]_A$.

([!]$\Box$-intro): Suppose $\models \chi \to [\theta \wedge p]\varphi$ and $\not\models \chi \to [\theta]\Box\varphi$, where $p \notin P_\chi \cup P_\theta \cup P_\varphi$. The latter means that there exists a pseudo model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ such that for some $A \in \mathcal{A}$ and some $w \in A$, $w \notin [\![\chi \to [\theta]\Box\varphi]\!]_A^{\mathcal{M}}$. Therefore $w \in [\![\chi \wedge \neg[\theta]\Box\varphi]\!]_A^{\mathcal{M}}$. Thus we have (1) $w \in [\![\chi]\!]_A^{\mathcal{M}}$ and (2) $w \in [\![\neg[\theta]\Box\varphi]\!]_A^{\mathcal{M}}$. Because of (2), $w \in [\![\langle\theta\rangle\Diamond\neg\varphi]\!]_A^{\mathcal{M}}$, and, by the semantics, $w \in [\![\Diamond\neg\varphi]\!]^{\mathcal{M}}_{[\![\theta]\!]_A^{\mathcal{M}}}$. Therefore, applying the semantics of $\Box$, we obtain (3) there exists $B \in \mathcal{A}$ s.t. $w \in B \subseteq [\![\theta]\!]_A^{\mathcal{M}} \subseteq A$ and $w \in [\![\neg\varphi]\!]_B^{\mathcal{M}}$.

Now consider the pre-model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|')$ such that $\|p\|' := B$ and $\|q\|' = \|q\|$ for any $q \neq p \in Prop$. In order to use Lemma 10 we must show that $\mathcal{M}'$ is a pseudo-model. For this we only need to verify that $\mathcal{M}'$ satisfies the closure conditions given in Definition 8. First note that $\|p\|' := B \in \mathcal{A}$ by the construction of $\mathcal{M}'$, so $\|p\|' \in \mathcal{A}$. For every $q \neq p$, since $\|q\|' = \|q\|$ and $\|q\| \in \mathcal{A}$ we have $\|q\|' \in \mathcal{A}$. Since $\mathcal{A}$ is the same for both $\mathcal{M}$ and $\mathcal{M}'$, and $\mathcal{M}$ is a pseudo-model, the rest of the closure conditions are already satisfied for $\mathcal{M}'$. Therefore $\mathcal{M}'$ is a pseudo-model. Now continuing with our soundness proof, since $p \notin P_\chi \cup P_\theta \cup P_\varphi$, by Lemma 10, we obtain $[\![\chi]\!]_A^{\mathcal{M}'} = [\![\chi]\!]_A^{\mathcal{M}}$, $[\![\theta]\!]_A^{\mathcal{M}'} = [\![\theta]\!]_A^{\mathcal{M}}$ and $[\![\neg\varphi]\!]_A^{\mathcal{M}'} = [\![\neg\varphi]\!]_A^{\mathcal{M}}$. Since $\|p\|' = B \subseteq [\![\theta]\!]_A^{\mathcal{M}'} \subseteq A$ we have $\|p\|' = [\![p]\!]_A^{\mathcal{M}'}$. Because of (3) we have that $w \in [\![\theta]\!]_A^{\mathcal{M}'}$ and $w \in [\![\neg\varphi]\!]_B^{\mathcal{M}'} = [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![p]\!]_A^{\mathcal{M}'}} = [\![\langle p\rangle\neg\varphi]\!]_A^{\mathcal{M}'}$. Thus, $w \in [\![p]\!]_A^{\mathcal{M}'}$, so $w \in [\![\theta \wedge p]\!]_A^{\mathcal{M}'} = [\![\theta]\!]_A^{\mathcal{M}'} \cap [\![p]\!]_A^{\mathcal{M}'} = [\![p]\!]_A^{\mathcal{M}'}$ simply because $[\![p]\!]_A^{\mathcal{M}'} \subseteq [\![\theta]\!]_A^{\mathcal{M}'}$. Since $w \in [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![p]\!]_A^{\mathcal{M}'}}$ we obtain $w \in [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta\wedge p]\!]_A^{\mathcal{M}'}}$. Putting everything together, $w \in [\![\theta \wedge p]\!]_A^{\mathcal{M}'}$ and $w \in [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta\wedge p]\!]_A^{\mathcal{M}'}}$, we obtain that $w \in [\![\langle\theta \wedge p\rangle\neg\varphi]\!]_A^{\mathcal{M}'}$ and $w \in [\![\chi]\!]_A^{\mathcal{M}'}$. Therefore $w \in [\![\chi \wedge \langle\theta \wedge p\rangle\neg\varphi]\!]_A^{\mathcal{M}'}$, which contradicts the validity of $\chi \to [\theta \wedge p]\varphi$.

([!][G]-intro): Suppose $\models \chi \to [\theta \wedge \bigwedge_{i\in G} K_i^\theta p_i]\varphi$ and $\not\models \chi \to [\theta][G]\varphi$ where $p_i \notin P_\chi \cup P_\theta \cup P_\varphi$. The latter means that there exists a pseudo model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ such that for some $A \in \mathcal{A}$ and some $w \in A$, $w \notin [\![\chi \to [\theta][G]\varphi]\!]_A^{\mathcal{M}}$. Therefore $w \in [\![\chi \wedge \neg[\theta][G]\varphi]\!]_A^{\mathcal{M}}$. Thus we have (1) $w \in [\![\chi]\!]_A^{\mathcal{M}}$, and (2) $w \in [\![\neg[\theta][G]\varphi]\!]_A^{\mathcal{M}}$. Item (2) means $w \in [\![\langle\theta\rangle\langle G\rangle\neg\varphi]\!]_A^{\mathcal{M}}$. Then, by the semantics of $\langle!\rangle$, we have $w \in [\![\langle G\rangle\neg\varphi]\!]^{\mathcal{M}}_{[\![\theta]\!]_A^{\mathcal{M}}}$. Therefore by the semantics of $\langle G\rangle$ we obtain: (3) there exists $\{B_i : i \in G\} \subseteq \mathcal{A}$ s.t. $w \in [\![\neg\varphi]\!]^{\mathcal{M}}_{[\![\theta]\!]_A^{\mathcal{M}} \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A^{\mathcal{M}}} B_i}$.

Now consider the pre-model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|')$ such that $\|p_i\|' = B_i$ and $\|q\|' = \|q\|$ for any $q \neq p_i \in Prop$ for all $i \in G$. Observe that since $[\![\theta]\!]_A \subseteq A$, by Boolean operations of sets we obtain that $K_i^{[\![\theta]\!]_A}(A \cap B_i) = K_i^{[\![\theta]\!]_A} B_i$. In order to use Lemma 10 we must show that $\mathcal{M}'$ is a pseudo-model as in the soundness proof of [!]$\Box$-intro. First note that for every $q \neq p_i$, since $\|q\|' = \|q\|$ and $\|q\| \in \mathcal{A}$, we have $\|q\|' \in \mathcal{A}$. Moreover, since for every $i \in G$, $\|p_i\|' = B_i \in \mathcal{A}$, we conclude that $\mathcal{M}'$ satisfies Definition 8.1. Since $\mathcal{A}$ is the same for both $\mathcal{M}$ and $\mathcal{M}'$, and $\mathcal{M}$ is a pseudo model, the rest of the closure conditions are satisfied already. Therefore $\mathcal{M}'$ is a pseudo model. Now continuing with our soundness proof, given that $p_i \notin P_\chi \cup P_\theta \cup P_\varphi$ for all $i \in G$, by Lemma 10, we obtain $[\![\chi]\!]_A^{\mathcal{M}'} = [\![\chi]\!]_A^{\mathcal{M}}$ and $[\![\theta]\!]_A^{\mathcal{M}'} = [\![\theta]\!]_A^{\mathcal{M}}$. We moreover have that

$$[\![\neg\varphi]\!]^{\mathcal{M}}_{[\![\theta]\!]_A^{\mathcal{M}} \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A^{\mathcal{M}}} B_i} = [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta]\!]_A^{\mathcal{M}'} \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A^{\mathcal{M}'}} B_i} = [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta]\!]_A^{\mathcal{M}'} \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A^{\mathcal{M}'}} \|p_i\|'} = [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta]\!]_A^{\mathcal{M}'} \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A^{\mathcal{M}'}} (A \cap \|p_i\|')}$$

by the above observation. And by Lemma 9, we obtain

$$[\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta]\!]_A^{\mathcal{M}'} \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A^{\mathcal{M}'}} (A \cap \|p_i\|')} = [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta]\!]_A^{\mathcal{M}'} \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A^{\mathcal{M}'}} ([\![p_i]\!]_A^{\mathcal{M}'})} = [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta\wedge\bigwedge_{i\in G} K_i^\theta p_i]\!]_A^{\mathcal{M}'}}.$$

Therefore, $w \in [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta\wedge\bigwedge_{i\in G} K_i^\theta p_i]\!]_A^{\mathcal{M}'}}$. I.e., $w \in [\![\langle\theta \wedge \bigwedge_{i\in G} K_i^\theta p_i\rangle\neg\varphi]\!]_A^{\mathcal{M}'}$. Since we also have that $w \in [\![\chi]\!]_A^{\mathcal{M}'}$, we conclude that $w \in [\![\chi \wedge \langle\theta \wedge \bigwedge_{i\in G} K_i^\theta p_i\rangle\neg\varphi]\!]_A^{\mathcal{M}'}$, contradicting the validity of $\chi \to [\theta \wedge \bigwedge_{i\in G} K_i^\theta p_i]\varphi$. Therefore, $\models \chi \to [\theta][G]\varphi$.

Definition 9 (Standard Pre-model). *A pre-model* $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ *is* standard *if and only if* $\mathcal{A} = \{[\![\theta]\!]_{W^0} : \theta \in \mathcal{L}_{-\Diamond}\}$.

Proposition 12. *Every standard pre-model is a pseudo-model.*

Proof. Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ be a standard pre-model. This implies that $\mathcal{A} = \{[\![\theta]\!]_{W^0} : \theta \in \mathcal{L}_{-\Diamond}\}$. We need to show that $\mathcal{M}$ satisfies the closure conditions given in Definition 8. Conditions (1) and (2) are immediate.

For (3): let $A \in \mathcal{A}$. Since $\mathcal{M}$ is a standard pre-model, we know that $A = [\![\theta]\!]_{W^0}$ for some $\theta \in \mathcal{L}_{-\Diamond}$. Since $\theta \in \mathcal{L}_{-\Diamond}$, we have $\neg\theta \in \mathcal{L}_{-\Diamond}$, thus, $[\![\neg\theta]\!]_{W^0} \in \mathcal{A}$. Observe that $[\![\neg\theta]\!]_{W^0} = W^0 - [\![\theta]\!]_{W^0}$, thus, we obtain $W^0 - A \in \mathcal{A}$.

For (4): let $A, B \in \mathcal{A}$. Since $\mathcal{M}$ is a standard pre-model, $A = [\![\theta_1]\!]_{W^0}$ and $B = [\![\theta_2]\!]_{W^0}$ for some $\theta_1, \theta_2 \in \mathcal{L}_{-\Diamond}$. Since $\theta_1, \theta_2 \in \mathcal{L}_{-\Diamond}$, we have $\theta_1 \wedge \theta_2 \in \mathcal{L}_{-\Diamond}$, thus, $[\![\theta_1 \wedge \theta_2]\!]_{W^0} \in \mathcal{A}$. Observe that $[\![\theta_1 \wedge \theta_2]\!]_{W^0} = [\![\theta_1]\!]_{W^0} \cap [\![\theta_2]\!]_{W^0} = A \cap B$, thus, we obtain $A \cap B \in \mathcal{A}$.

For (5): let $A \in \mathcal{A}$. Since $\mathcal{M}$ is a standard pre-model, $A = [\![\theta]\!]_{W^0}$ for some $\theta \in \mathcal{L}_{-\Diamond}$. Since $\theta \in \mathcal{L}_{-\Diamond}$, we have $K_i\theta \in \mathcal{L}_{-\Diamond}$, thus, $[\![K_i\theta]\!]_{W^0} \in \mathcal{A}$. Observe that $[\![K_i\theta]\!]_{W^0} = \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow s \in [\![\theta]\!]_{W^0})\} = K_i[\![\theta]\!]_{W^0}$, thus, we obtain $K_iA \in \mathcal{A}$.

Equivalence between the standard pseudo-models and announcement models. For Proposition 13 only, we use the notation $[\![\varphi]\!]^{PS}_A$ to refer to pseudo-model semantics (as in Definition 7) and use $[\![\varphi]\!]_{\mathcal{M}}$ to refer to the semantics on *a*-models (as in Definition 5).

The proof of Proposition 13 needs the following lemmas.

Lemma 11. *The sentence* $(K_i(\varphi \to \psi))^0 \leftrightarrow K_i(K_i(\varphi \to \psi))^0$ *is valid on pseudo-models.*

Proof. We only prove the direction from left-to-right since the direction from right-to-left is an instance of the T-axiom for K_i. Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ be a pseudo-model, $A \in \mathcal{A}$, and $w \in A$ such that $w \in [\![(K_i(\varphi \to \psi))^0]\!]_A$. This means, by the semantics of ^{0}, that $w \in [\![K_i(\varphi \to \psi)]\!]_{W^0} \cap A$. Let $v \in A$ such that $w \sim_i v$, that is, $v \in w_i^A$. Since $\sim_i$ is transitive and $w \in [\![K_i(\varphi \to \psi)]\!]_{W^0}$, we obtain that $v \in [\![K_i(\varphi \to \psi)]\!]_{W^0}$. Moreover, by the assumption, we have $v \in A$. Therefore, $v \in [\![K_i(\varphi \to \psi)]\!]_{W^0} \cap A$, that is, $v \in [\![(K_i(\varphi \to \psi))^0]\!]_A$. As v has been chosen arbitrarily from w_i^A, we obtain that $w_i^A \subseteq [\![(K_i(\varphi \to \psi))^0]\!]_A$, i.e., that $w \in [\![K(K_i(\varphi \to \psi))^0]\!]_A$.

Lemma 12. *Let* $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ *be a standard pseudo-model,* $A \in \mathcal{A}$ *and* $\varphi \in \mathcal{L}_G$. *Then we have the following:*

1. $[\![\Diamond\varphi]\!]_A = \bigcup\{[\![\langle\theta\rangle\varphi]\!]_A : \theta \in \mathcal{L}_{-\Diamond}\}$,
2. $[\![\langle G\rangle\varphi]\!]_A = \bigcup\{[\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\varphi]\!]_A : \{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\Diamond}\}$.

Proof.

1. For ($\subseteq$): Let $w \in [\![\Diamond\varphi]\!]_A$. Then, by the semantics of $\Diamond$ in pseudo-models, there exists some $B \in \mathcal{A}$ such that $w \in B \subseteq A$ and $w \in [\![\varphi]\!]_B$. Since $\mathcal{M}$ is standard, we know that $A = [\![\psi]\!]_{W^0}$ and $B = [\![\chi]\!]_{W^0}$ for some $\psi, \chi \in \mathcal{L}_{-\Diamond}$. Moreover, since $B = [\![\chi]\!]_{W^0} \subseteq A = [\![\psi]\!]_{W^0}$, we have $B = [\![\chi]\!]_{W^0} \cap [\![\psi]\!]_{W^0} = [\![\chi^0]\!]_{[\![\psi]\!]_{W^0}} = [\![\chi^0]\!]_A$, and so $w \in [\![\varphi]\!]_B = [\![\varphi]\!]_{[\![\chi^0]\!]_A} = [\![\langle\chi^0\rangle\varphi]\!]_A \subseteq \bigcup\{[\![\langle\theta\rangle\varphi]\!]_A : \theta \in \mathcal{L}_{-\Diamond}\}$.
For ($\supseteq$): Let $w \in \bigcup\{[\![\langle\theta\rangle\varphi]\!]_A : \theta \in \mathcal{L}_{-\Diamond}\}$. Then we have $w \in [\![\langle\theta\rangle\varphi]\!]_A = [\![\varphi]\!]_{[\![\theta]\!]_A}$, for some $\theta \in \mathcal{L}_{-\Diamond}$. Moreover, since $[\![\theta]\!]_A \in \mathcal{A}$ (by Proposition 10) and $[\![\theta]\!]_A \subseteq A$ (by Observation 5), it follows that $w \in [\![\Diamond\varphi]\!]_A$ (by the semantics of $\Diamond$ in pseudo-models).
2. For ($\subseteq$): Let $w \in [\![\langle G\rangle\varphi]\!]_A$. Then, by Definition 7, we have $w \in [\![\varphi]\!]_{A \cap \bigcap_{i\in G} K_i^A B_i}$ for some $\{B_i : i \in G\} \subseteq \mathcal{A}$. Since $\mathcal{M}$ is a standard pseudo-model, we know that each $B_i = [\![\rho_i]\!]_{W^0}$ and $A = [\![\psi]\!]_{W^0}$ for some $\rho_i, \psi \in \mathcal{L}_{-\Diamond}$. Thus,
$$w \in [\![\varphi]\!]_{[\![\psi]\!]_{W^0} \cap \bigcap_{i\in G} K_i^{[\![\psi]\!]_{W^0}} [\![\rho_i]\!]_{W^0}} = [\![\varphi]\!]_{[\![\psi]\!]_{W^0} \cap \bigcap_{i\in G} [\![K_i(\psi \to \rho_i)]\!]_{W^0}} = [\![\varphi]\!]_{[\![\psi]\!]_{W^0} \cap [\![\bigwedge_{i\in G} K_i(\psi\to\rho_i)]\!]_{W^0}}$$
by Lemma 9 and the semantics. By the semantics of 0 and Lemma 11, we obtain
$$[\![\varphi]\!]_{[\![\psi]\!]_{W^0} \cap [\![\bigwedge_{i\in G} K_i(\psi\to\rho_i)]\!]_{W^0}} = [\![\varphi]\!]_{[\![\bigwedge_{i\in G}(K_i(\psi\to\rho_i))^0]\!]_A} = [\![\varphi]\!]_{[\![\bigwedge_{i\in G} K_i(K_i(\psi\to\rho_i))^0]\!]_A}.$$
Thus, for $\theta_i := (K_i(\psi \to \rho_i))^0$, $w \in [\![\varphi]\!]_{[\![\bigwedge_{i\in G} K_i\theta_i]\!]_A} = [\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\varphi]\!]_A$. For ($\supseteq$): Let $\{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\Diamond}$ such that $w \in [\![\varphi]\!]_{[\![\bigwedge_{i\in G} K_i\theta_i]\!]_A}$. Note that $[\![\bigwedge_{i\in G} K_i\theta_i]\!]_A = \bigcap_{i\in G}[\![K_i\theta_i]\!]_A = A \cap \bigcap_{i\in G} K_i^A[\![\theta_i]\!]_A$. Since $\mathcal{M}$ is a standard pseudo-model, we know that $B_i := [\![\theta_i]\!]_A \in \mathcal{A}$ for every $i \in G$ and by our initial assumption $w \in [\![\varphi]\!]_{A \cap \bigcap_{i\in G} K_i^A [\![\theta_i]\!]_A}$, so we obtain $w \in [\![\langle G\rangle\varphi]\!]_A$.

Proposition 13.

1. *For every standard pseudo-model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ and every non-empty set $A \in \mathcal{A}$, we denote by $\mathcal{M}_A$ the model $\mathcal{M}_A = (W^0, A, \sim_1, \ldots, \sim_n, \|\cdot\|)$. Then:*
   (a) For every $\varphi \in \mathcal{L}_G$, we have $[\![\varphi]\!]_{\mathcal{M}_A} = [\![\varphi]\!]^{PS}_A$.
   (b) $\mathcal{M}_A$ is an a-model.

2. *For every a-model $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$, we denote by $\mathcal{M}'$ the pre-model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$, where $\mathcal{A} = \{[\![\theta]\!]_{\mathcal{M}^0} : \theta \in \mathcal{L}_{-\Diamond}\}$. Then*
   (a) $\mathcal{M}'$ is a standard pseudo-model.
   (b) For every $\varphi \in \mathcal{L}_G$, we have $[\![\varphi]\!]_{\mathcal{M}} = [\![\varphi]\!]^{PS}_W$.

Proof. See Appendix A.4.

Corollary 2. *Validity on standard pseudo-models coincides with validity on the a-models.*

Proof. This is a straightforward consequence of Proposition 13.

Corollary 3. *The system* **GALM** *is sound wrt a-models. Moreover, the system* **APALM** *is sound wrt a-models.*

Proof. Follows immediately from Proposition 11 and Corollary 2.

It is important to note that the equivalence between standard pseudo-models and *a*-models (given by Proposition 13 above, and underlying our soundness result) is *not* trivial (the proof is in Appendix A.4). It relies in particular on the equivalence between the effort modality and the arbitrary announcement operator $\Box$ (see Lemma 12.1), and on the equivalence between the *purely syntactic* and *purely semantic* descriptions of the group announcement operator $[G]$ on standard pseudo models (see Lemma 12.2). In turn, the equivalences between these operators hold *only because our models and language retain the memory of the initial situation*, i.e., having W^0 in *a*-models and in pseudo-models, and having the operators 0 and ^{0} in the language $\mathcal{L}_G$. Note that the most important steps in the proof of Lemma 12 make necessary use of the operator ^{0}. Hence, a similar equivalence of models fails for the original, memory lacking, APAL and GAL.

5 Completeness

In this section we prove the completeness of **GALM** and **APALM**. First, we show completeness with respect to pseudo-models, via an innovative modification of the standard canonical model construction. This is based on a method previously used in [11], that makes an essential use of the finitary $\Box$ and $[G]$-introduction rules, by requiring our canonical theories T to be (not only maximally consistent, but also) "witnessed". Roughly speaking, a theory T is witnessed if: every $\Diamond\varphi$ occurring in every "existential context" in T is witnessed by some atomic formula p, meaning that $\langle p\rangle\varphi$ occurs in the same existential context in T, and for every $\langle G\rangle\varphi$ occurring in every "existential context" in T is witnessed by some formula $\wedge_{i\in G}K_i p_i$, meaning that $\langle\wedge_{i\in G}K_i p_i\rangle\varphi$ occurs in the same existential context in T. Our canonical pre-model will consist of all *initial, maximally consistent, witnessed theories* (where a theory is 'initial' if it contains the formula 0). A Truth Lemma is proved, as usual. Completeness for (both pseudo-models and) *a*-models follows from the observation that our canonical pre-model *is standard*, hence it is (a standard pseudo-model, and thus) equivalent to a genuine *a*-model.

We now proceed with the details. The appropriate notion of "existential context" is represented by *possibility forms*, in the following sense.

Definition 10 (Necessity forms and possibility forms). *For any finite string $s \in (\{\bullet^0\} \cup \{\varphi\!\to \mid \varphi \in \mathcal{L}_G\} \cup \{K_i : i \in \mathcal{A}\} \cup \{U\} \cup \{\rho \mid \rho \in \mathcal{L}_{-\Diamond}\})^* = NF$, we define pseudo-modalities $[s]$ and $\langle s\rangle$. These pseudo-modalities are functions mapping any formula $\varphi \in \mathcal{L}_G$ to another formula $[s]\varphi \in \mathcal{L}_G$ (*necessity form*), respectively $\langle s\rangle\varphi \in \mathcal{L}_G$ (*possibility form*). The necessity forms are defined recursively as $[\epsilon]\varphi = \varphi$, $[s, \bullet^0]\varphi = [s]\varphi^0$, $[s, \psi\!\to]\varphi = [s](\psi \to \varphi)$, $[s, K_i]\varphi = [s]K_i\varphi$, $[s, U]\varphi = [s]U\varphi$, $[s, \rho]\varphi = [s][\rho]\varphi$, where ϵ is the empty string. For possibility forms, we set $\langle s\rangle\varphi := \neg[s]\neg\varphi$.*

Example: $[K_i, \bullet^0, \Diamond p{\to}, 0, U]$ is a pseudo-modality such that $[K_i, \bullet^0, \Diamond p{\to}, 0, U]\varphi = K_i(\Diamond p \to [0]U\varphi)^0$.

Definition 11 (Theories: witnessed, initial, maximal). *A* theory *Γ is a consistent set of formulas in $\mathcal{L}_G$ wrt the axiomatization of* **GALM**, *that is, $\Gamma \subseteq \mathcal{L}_G$ such that $\Gamma \nvdash \bot$. A formula φ is consistent with Γ if $\Gamma \cup \{\varphi\}$ is consistent (or, equivalently, if $\Gamma \nvdash \neg\varphi$). A* maximal theory *is a theory Γ that is maximal with respect to $\subseteq$ among all theories; in other words, Γ cannot be extended to another theory. A* witnessed theory *is a theory Γ such that, for every $s \in NF$ and $\varphi \in \mathcal{L}_G$, (1) if $\langle s\rangle\Diamond\varphi$ is consistent with Γ then there is $p \in Prop$ such that $\langle s\rangle\langle p\rangle\varphi$ is consistent with Γ (or equivalently: if $\Gamma \vdash [s][p]\neg\varphi$ for all $p \in Prop$, then $\Gamma \vdash [s]\Box\neg\varphi$) and (2) for every $G \subseteq \mathcal{AG}$, if $\langle s\rangle\langle G\rangle\varphi$ is consistent with Γ then there is $\{p_i : i \in G\} \subseteq Prop$ such that $\langle s\rangle\langle\wedge_{i\in G}K_ip_i\rangle\varphi$ is consistent with Γ. A theory Γ is called* initial *if $0 \in \Gamma$. A* maximal witnessed theory *Γ is a witnessed theory that is not a proper subset of any witnessed theory. A* maximal witnessed initial theory *Γ is a maximal witnessed theory such that $0 \in \Gamma$.*

Lemma 13. *For every $s \in NF$, there exist formulas $\theta \in \mathcal{L}_{-\Diamond}$ and $\psi \in \mathcal{L}_G$, with $P_\psi \cup P_\theta \subseteq P_s$, such that for all $\varphi \in \mathcal{L}_G$, we have*

$$\vdash [s]\varphi \textit{ iff } \vdash \psi \to [\theta]\varphi.$$

Proof. See Appendix A.5.

Lemma 14. *The following rules are derivable in* **GALM***:*

1. *if $\vdash [s][p]\varphi$ then $\vdash [s]\Box\varphi$, where $p \notin P_s \cup P_\varphi$,*
2. *if $\vdash [s][\bigwedge_{i\in G} K_ip_i]\varphi$ then $\vdash [s][G]\varphi$, where $p_i \notin P_s \cup P_\varphi$.*

Proof. For (1), suppose $\vdash [s][p]\varphi$. Then, by Lemma 13, there exist $\theta \in \mathcal{L}_{-\Diamond}$ and $\psi \in \mathcal{L}_G$ such that $\vdash \psi \to [\theta][p]\varphi$. By the auxiliary reduction in Proposition 2.15, we get $\vdash \psi \to [\theta \wedge p]\varphi$. By the construction of the formulas ψ and θ, we know that $P_\psi \cup P_\theta \subseteq P_s$, and so $p \notin P_\psi \cup P_\theta \cup P_\varphi$. Therefore, by ([!]$\Box$-intro), we have $\vdash \psi \to [\theta]\Box\varphi$. Applying again Lemma 13, we obtain $\vdash [s]\Box\varphi$. The proof of (2) goes in a similar way as the one before given that ($\star$) $[\theta][\bigwedge_{i\in G} K_ip_i]\varphi \leftrightarrow [\theta \wedge \bigwedge_{i\in G} K_i^\theta p_i]\varphi$ is derivable in **GALM** (by using the appropriate reduction axioms and RE). Let $s \in NF^P$ such that $\vdash [s][\bigwedge_{i\in G} K_ip_i]\varphi$ where $p_i \notin P_s \cup P_\varphi$. Then, by Lemma 13, we obtain that $\vdash \chi \to [\theta][\bigwedge_{i\in G} K_ip_i]\varphi$. Therefore, by ($\star$), we have that $\vdash \chi \to [\theta \wedge \bigwedge_{i\in G} K_i^\theta p_i]\varphi$. By the [!][G]-intro rule we then obtain $\vdash \chi \to [\theta][G]\varphi$. Again by Lemma 13, we get $\vdash [s][G]\varphi$.

Lemma 15. *For every maximal witnessed theory Γ, and every formula $\varphi, \psi \in \mathcal{L}_G$,*

1. *$\Gamma \vdash \varphi$ iff $\varphi \in \Gamma$*
2. *$\varphi \notin \Gamma$ iff $\neg\varphi \in \Gamma$,*
3. *$\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$,*
4. *$\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$ implies $\psi \in \Gamma$.*
5. **GALM** *$\subseteq \Gamma$.*

Proof. The proof is standard. We prove only item (5): suppose **GALM** $\nsubseteq \Gamma$. This means that there is a sentence $\psi \in \mathcal{L}_G$ such that $\psi \in$ **GALM** but $\psi \notin \Gamma$. The former means that $\vdash \psi$, thus, $\Gamma \vdash \psi$. Items (2) and (1) implies that if $\psi \notin \Gamma$ then $\Gamma \vdash \neg\psi$, contradicting consistency of Γ.

Lemma 16. *For every $\Gamma \subseteq \mathcal{L}_G$, if Γ is a theory and $\Gamma \nvdash \neg\varphi$ for some $\varphi \in \mathcal{L}_G$, then $\Gamma \cup \{\varphi\}$ is a theory. Moreover, if Γ is witnessed, then $\Gamma \cup \{\varphi\}$ is also witnessed.*

Proof. The proof of the first claim is standard. We only prove the second claim. Suppose that Γ is witnessed but $\Gamma \cup \{\varphi\}$ is not witnessed. By the previous statement, we know that $\Gamma \cup \{\varphi\}$ is consistent. Since $\Gamma \cup \{\varphi\}$ is not witnessed, it violates either (1) or (2) in Definition 11. First suppose $\Gamma \cup \{\varphi\}$ does not satisfy (1), that is, there is $s \in NF$ and $\psi \in \mathcal{L}_G$ such that $\Gamma \cup \{\varphi\}$ is consistent with $\langle s\rangle\Diamond\psi$ but $\Gamma \cup \{\varphi\} \vdash \neg\langle s\rangle\langle p\rangle\psi$ for all $p \in Prop$. This implies that $\Gamma \cup \{\varphi\} \vdash [s][p]\neg\psi$ for all $p \in Prop$. Therefore, $\Gamma \vdash \varphi \to [s][p]\neg\psi$ for all $p \in Prop$. Note that $\varphi \to [s][p]\neg\psi = [\varphi \to, s][p]\neg\psi$, and $[\varphi{\to}, s] \in NF$. We thus have $\Gamma \vdash [\varphi \to, s][p]\neg\psi$ for all $p \in Prop$. Since Γ is witnessed, we obtain $\Gamma \vdash [\varphi \to, s]\Box\neg\psi$. By unraveling the necessity form $[\varphi{\to}, s]$, we get $\Gamma \vdash \varphi \to [s]\Box\neg\psi$, thus, $\Gamma \cup \{\varphi\} \vdash [s]\Box\neg\psi$, i.e., $\Gamma \cup \{\varphi\} \vdash \neg\langle s\rangle\Diamond\psi$, contradicting the assumption that $\Gamma \cup \{\varphi\}$ is consistent with $\langle s\rangle\Diamond\psi$. Now suppose $\Gamma \cup \{\varphi\}$ does not satisfy (2). This means that there is $s \in NF$ and $\psi \in \mathcal{L}_G$ such that for some group $G \subseteq \mathcal{A}$, the set $\Gamma \cup \{\varphi\}$ is consistent with $\langle s\rangle\langle G\rangle\psi$ but $\Gamma \cup \{\varphi\} \vdash \neg\langle s\rangle\langle\bigwedge_{i\in G} K_ip_i\rangle\psi$ for all $\{p_i : i \in G\} \subseteq Prop$. This implies that $\Gamma \cup \{\varphi\} \vdash [s][\bigwedge_{i\in G} K_ip_i]\neg\psi$ for all $\{p_i : i \in G\} \subseteq Prop$. Therefore,

$\Gamma \vdash \varphi \to [s][\bigwedge_{i \in G} K_i p_i]\neg\psi$ for all $\{p_i : i \in G\} \subseteq Prop$. Note that $\varphi \to [s][\bigwedge_{i \in G} K_i p_i]\neg\psi = [\varphi \to, s][\bigwedge_{i \in G} K_i p_i]\neg\psi$, and $[\varphi \to, s] \in NF$. We thus have $\Gamma \vdash [\varphi \to, s][\bigwedge_{i \in G} K_i p_i]\neg\psi$ for all $\{p_i : i \in G\} \subseteq Prop$. Since Γ is witnessed, we obtain $\Gamma \vdash [\varphi \to, s][G]\neg\psi$. By unraveling the necessity form $[\varphi \to, s]$, we get $\Gamma \vdash \varphi \to [s][G]\neg\psi$, thus, $\Gamma \cup \{\varphi\} \vdash [s][G]\neg\psi$, i.e., $\Gamma \cup \{\varphi\} \vdash \neg\langle s\rangle\langle G\rangle\psi$, contradicting the assumption that $\Gamma \cup \{\varphi\}$ is consistent with $\langle s\rangle\langle G\rangle\psi$. Altogether we obtain that $\Gamma \cup \{\varphi\}$ is a witnessed theory.

Lemma 17. *If $\{\Gamma_i\}_{i \in \mathbb{N}}$ is an increasing chain of theories such that $\Gamma_i \subseteq \Gamma_{i+1}$, then $\bigcup_{n \in \mathbb{N}} \Gamma_n$ is a theory.*

Proof. Let $\{\Gamma_i\}_{i \in \mathbb{N}}$ be an increasing chain of theories with $\Gamma_i \subseteq \Gamma_{i+1}$ and suppose, toward contradiction, that $\bigcup_{n \in \mathbb{N}} \Gamma_n$ is not a theory, i.e., suppose that $\bigcup_{n \in \mathbb{N}} \Gamma_n \vdash \bot$. This means that there exists a finite $\Delta \subseteq \bigcup_{n \in \mathbb{N}} \Gamma_n$ such that $\Delta \vdash \bot$. Then, since $\bigcup_{n \in \mathbb{N}} \Gamma_n$ is a union of an increasing chain of theories, there is some $m \in \mathbb{N}$ such that $\Delta \subseteq \Gamma_m$. Therefore, $\Gamma_m \vdash \bot$ contradicting the fact that Γ_m is a theory. Hence, $\bigcup_{n \in \mathbb{N}} \Gamma_n$ is a theory.

Lemma 18. *For every maximal witnessed theory T, both $\{\theta \in \mathcal{L}_G : K_i\theta \in T\}$ and $\{\theta \in \mathcal{L}_G : U\theta \in T\}$ are witnessed theories.*

Proof. Observe that, by axiom (T_{K_i}), $\{\theta \in \mathcal{L}_G : K_i\theta \in T\} \subseteq T$. Therefore, as T is consistent, the set $\{\theta \in \mathcal{L}_G : K_i\theta \in T\}$ is consistent. Let $s \in NF$, $\psi \in \mathcal{L}$, and $G \subseteq \mathcal{AG}$ such that $\{\theta \in \mathcal{L}_G : K_i\theta \in T\} \vdash [s][p]\neg\varphi$ for all $p \in Prop$ and $\{\theta \in \mathcal{L}_G : K_i\theta \in T\} \vdash [s][\bigwedge_{i \in G} K_i p_i]\neg\psi$ for all $\{p_i : i \in G\} \subseteq Prop$. By normality of K_i, $T \vdash K_i[s][p]\neg\varphi$ for all $p \in Prop$ and $T \vdash K_i[s][\bigwedge_{i \in G} K_i p_i]\neg\psi$ for all $\{p_i : i \in G\} \subseteq Prop$. Since $K_i[s][p]\neg\varphi := [K_i, s][p]\neg\varphi$ and $K_i[s][\bigwedge_{i \in G} K_i p_i]\neg\psi := [K_i, s][[\bigwedge_{i \in G} K_i p_i]\neg\psi$ are necessity forms and T is witnessed, we obtain $T \vdash [K_i, s]\Box\neg\varphi$ and $T \vdash [K_i, s][G]\neg\varphi$, i.e., $T \vdash K_i[s]\Box\neg\varphi$ and $T \vdash K_i[s][G]\neg\varphi$. As T is maximal, we have $K_i[s]\Box\neg\varphi \in T$ and $K_i[s][G]\neg\varphi \in T$, thus $[s]\Box\neg\varphi \in \{\theta \in \mathcal{L}_G \mid K_i\theta \in T\}$ and $[s][G]\neg\varphi \in \{\theta \in \mathcal{L}_G \mid K_i\theta \in T\}$. The proof for $\{\theta \in \mathcal{L}_G : U\theta \in T\}$ follows similarly.

Lemma 19 (Lindenbaum's Lemma). *Every witnessed theory Γ can be extended to a maximal witnessed theory T_Γ.*

Proof. See Appendix A.5.

Lemma 20 (Extension Lemma). *For any $\theta \in \mathcal{L}_G$, if $\{0, \theta\}$ is a theory then there is a maximal witnessed initial theory Γ such that $\{0, \theta\} \subseteq \Gamma$.*

Proof. See Appendix A.5.

To define our canonical pseudo-model, we first put, for all maximal witnessed theories T, S and for every $i \in \mathcal{AG}$:

$$T \sim_U S \;\text{ iff }\; \forall\varphi \in \mathcal{L}_G\,(U\varphi \in T \;\text{ implies }\; \varphi \in S)\,,\;\text{and}$$

$$T \sim_i S \;\text{ iff }\; \forall\varphi \in \mathcal{L}_G\,(K_i\varphi \in T \;\text{ implies }\; \varphi \in S)\,.$$

Lemma 21. *For every $i \in \mathcal{AG}$, $\sim_i \subseteq \sim_U$.*

Proof. Let $i \in \mathcal{AG}$, let T and S be maximal witnessed theories such that $T \sim_i S$. Towards contradiction, suppose that $T \sim_U S$ is not the case. From the former we have that $\forall\varphi \in \mathcal{L}_G\,(K_i\varphi \in T$ implies $\varphi \in S)$. From the latter, we have that there is $\psi \in \mathcal{L}_G$ such that $U\psi \in T$ and $\psi \notin S$. Since $\vdash U\psi \to K_i\psi$ and T is a maximal witnessed theory, $U\psi \to K_i\psi \in T$. Therefore $K_i\psi \in T$ and $\psi \notin S$, contradicting that $T \sim_i S$.

Definition 12 (Canonical Pre-Model). *Given a maximal witnessed initial theory T_0, the* canonical pre-model for T_0 *is a tuple $\mathcal{M}^c = (W^c, \mathcal{A}^c, \sim_1^c, \ldots, \sim_n^c, \|\cdot\|^c)$ such that:*

- $W^c = \{T :$ *T is a maximal witnessed theory such that* $T_0 \sim_U T\}$,
- $\mathcal{A}^c = \{\widehat{\theta} : \theta \in \mathcal{L}_{-\Diamond}\}$ *where* $\widehat{\varphi} = \{T \in W^c : \varphi \in T\}$ *for any* $\varphi \in \mathcal{L}_G$,
- *for every $i \in \mathcal{AG}$ we define:*

$$\sim_i^c \;=\; \sim_i \,\cap\, (W^c \times W^c).$$

- $\|p\|^c = \{T \in W^c : p \in T\} = \widehat{p}$.

As usual, it is easy to see (given the $S5$ axioms for K_i and for U) that $\sim_U$ and $\sim_i^c$ are equivalence relations.

To prove that the canonical pre-model is indeed a pseudo-model, we first need to prove the truth lemma. For that we need the following lemmas.

Lemma 22 (Existence Lemma for $\sim_U$). *Let T be a maximal witnessed theory, $\alpha \in \mathcal{L}_{-\Diamond}$, and $\varphi \in \mathcal{L}_G$ such that $\alpha \in T$ and $U[\alpha]\varphi \notin T$. Then, there is a maximal witnessed theory S such that $T \sim_U S$, $\alpha \in S$ and $[\alpha]\varphi \notin S$.*

Proof. Let $\alpha \in \mathcal{L}_{-\Diamond}$ and $\varphi \in \mathcal{L}_G$ such that $\alpha \in T$ and $U[\alpha]\varphi \notin T$. The latter implies that $\{\psi \in \mathcal{L}_G : U\psi \in T\} \nvdash [\alpha]\varphi$, hence, $\{\psi \in \mathcal{L}_G : U\psi \in T\} \nvdash \neg\neg[\alpha]\varphi$. Then, by Lemmas 16 and 18, we obtain that $\{\psi \in \mathcal{L}_G : U\psi \in T\} \cup \{\neg[\alpha]\varphi\}$ is a witnessed theory. Note that $\vdash \neg[\alpha]\varphi \leftrightarrow (\alpha \wedge [\alpha]\neg\varphi)$ (see Proposition 2.3). We therefore obtain that $\{\psi \in \mathcal{L}_G : U\psi \in T\} \cup \{\neg[\alpha]\varphi\} \vdash \alpha$, thus, $\{\psi \in \mathcal{L}_G : U\psi \in T\} \cup \{\neg[\alpha]\varphi\} \nvdash \neg\alpha$ (since $\{\psi \in \mathcal{L}_G : U\psi \in T\} \cup \{\neg[\alpha]\varphi\}$ is consistent). Therefore, by Lemma 16, $\{\psi \in \mathcal{L}_G : U\psi \in T\} \cup \{\neg[\alpha]\varphi\} \cup \{\alpha\}$ is also a witnessed theory. We can then apply Lindenbaum's Lemma (Lemma 19) and extend it to a maximal witnessed theory S such that $S \sim_U T$, $\alpha \in S$, and $[\alpha]\varphi \notin S$.

Corollary 4. *For $\varphi \in \mathcal{L}_G$, we have $\widehat{U\varphi} = W^c$ if $\widehat{\varphi} = W^c$, and $\widehat{U\varphi} = \emptyset$ otherwise.*

Proof. If $\widehat{\varphi} = W^c$, suppose $\widehat{U\varphi} \neq W^c$. The latter means that there is a $T \in W^c$ such that $U\varphi \notin T$. Then, by Lemma 22 (when $\alpha := \top$), there is a maximal witnessed theory S such that $T \sim_U S$ and $\varphi \notin S$. Since $T_0 \sim_U T \sim_U S$ and $\sim_U$ is transitive, we have $T_0 \sim_U S$, thus, $S \in W^c$. Therefore, $\widehat{\varphi} \neq W^c$, contradicting the initial assumption. If $\widehat{\varphi} \neq W^c$, then there is a $T \in W^c$ such that $\varphi \notin T$. Since $T \sim_U S$ for all $S \in W^c$, we obtain by the definition of $\sim_U$ that $U\varphi \notin S$ for all $S \in W^c$. Therefore, $\widehat{U\varphi} = \emptyset$.

Lemma 23 (Existence Lemma for $\sim_i$). *Let T be a maximal witnessed theory, $\alpha \in \mathcal{L}_{-\Diamond}$, and $\varphi \in \mathcal{L}_G$ such that $\alpha \in T$ and $K_i[\alpha]\varphi \notin T$. Then, there is a maximal witnessed theory S such that $T \sim_i S$, $\alpha \in S$, and $[\alpha]\varphi \notin S$.*

Proof. Let $\alpha \in \mathcal{L}_{-\Diamond}$ and $\varphi \in \mathcal{L}_G$ such that $\alpha \in T$ and $K_i[\alpha]\varphi \notin T$. The latter implies that $\{\psi \in \mathcal{L}_G : K_i\psi \in T\} \nvdash [\alpha]\varphi$, hence, $\{\psi \in \mathcal{L}_G : K_i\psi \in T\} \nvdash \neg\neg[\alpha]\varphi$. Then, by Lemmas 16 and 18, we obtain that $\{\psi \in \mathcal{L}_G : K_i\psi \in T\} \cup \{\neg[\alpha]\varphi\}$ is a witnessed theory. Note that $\vdash \neg[\alpha]\varphi \leftrightarrow (\alpha \wedge [\alpha]\neg\varphi)$ (see Proposition 2.3). We therefore obtain that $\{\psi \in \mathcal{L}_G : K_i\psi \in T\} \cup \{\neg[\alpha]\varphi\} \vdash \alpha$, thus, $\{\psi \in \mathcal{L}_G : K_i\psi \in T\} \cup \{\neg[\alpha]\varphi\} \nvdash \neg\alpha$ (since $\{\psi \in \mathcal{L}_G : K_i\psi \in T\} \cup \{\neg[\alpha]\varphi\}$ is consistent). Therefore, by Lemma 16, $\{\psi \in \mathcal{L}_G : K_i\psi \in T\} \cup \{\neg[\alpha]\varphi\} \cup \{\alpha\}$ is also a witnessed theory. We can then apply Lindenbaum's Lemma (Lemma 19) and extend it to a maximal witnessed theory S such that $S \sim_i T$, $\alpha \in S$, and $[\alpha]\varphi \notin S$.

Corollary 5. *Let T_0 be a maximal witnessed initial theory and $\mathcal{M}^c = (W^c, \mathcal{A}^c, \sim_1^c, \ldots, \sim_n^c, \|\cdot\|^c)$ be the canonical pre-model for T_0. For all $T \in \mathcal{M}^c$, $\alpha \in \mathcal{L}_{-\Diamond}$ and $\varphi \in \mathcal{L}_G$, if $\alpha \in T$ and $K_i[\alpha]\varphi \notin T$ then there is a maximal witnessed theory $S \in W^c$ such that $T \sim_i^c S$, $\alpha \in S$ and $[\alpha]\varphi \notin S$.*

Proof. Let $T \in \mathcal{M}^c$, let $\alpha \in \mathcal{L}_{-\Diamond}$ and $\varphi \in \mathcal{L}_G$ be such that $\alpha \in T$ and $K_i[\alpha]\varphi \notin T$. By Lemma 23, there is a maximal witnessed theory S such that $T \sim_i S$, $\alpha \in S$ and $[\alpha]\varphi \notin S$. By Lemma 21, $T \sim_U S$. Since $T_0 \sim_U T$, by transitivity of $\sim_U$ we have $T_0 \sim_U S$. Therefore $S \in W^c$ and so $T \sim_i^c S$.

Lemma 24. *Every element $T \in W^c$ is an initial theory (i.e. $0 \in T$).*

Proof. Let $T \in W^c$. By the construction of W^c, we have $T_0 \sim_U T$. Since $0 \to U0$ is an axiom and T_0 is maximal, $(0 \to U0) \in T_0$. Thus, since $0 \in T_0$, we obtain $U0 \in T_0$ (by Lemma 15.4). Therefore, by the definition of $\sim_U$ and since $T_0 \sim_U T$, we have that $0 \in T$.

Corollary 6. *For all $\varphi \in \mathcal{L}_G^P$, we have $\widehat{\varphi} = \widehat{\varphi^0}$.*

Proof. Since $0 \in T$ for all $T \in W^c$, we obtain by axiom (0-*eq*) that $\varphi \leftrightarrow \varphi^0 \in T$ for all $T \in W^c$. Therefore, $\widehat{\varphi} = \widehat{\varphi^0}$.

Lemma 25 (Truth Lemma). *Let $\mathcal{N}^c = (W^c, \mathcal{A}^c, \sim_1^c, \ldots, \sim_n^c, V^c)$ be the canonical pre-model for some T_0 and $\varphi \in \mathcal{L}_G$. Then, for all $\alpha \in \mathcal{L}_{-\Diamond}$ we have $[\![\varphi]\!]_{\widehat{\alpha}} = \widehat{\langle\alpha\rangle\varphi}$.*

Proof. The proof is by <-induction on φ, using the following *induction hypothesis* (IH): for all $\psi < \varphi$, we have $[\![\psi]\!]_{\widehat{\alpha}} = \widehat{\langle\alpha\rangle\psi}$ for all $\alpha \in \mathcal{L}_{-\Diamond}$.

Base case $\varphi := p$. Then $[\![p]\!]_{\widehat{\alpha}} = \|p\|^c \cap \widehat{\alpha} = \widehat{p} \cap \widehat{\alpha} = \widehat{p \wedge \alpha} = \widehat{\langle\alpha\rangle p}$, by Defn.7, the defn. of $\|\cdot\|^c$, R_p, and Proposition 2.3.

Base case $\varphi := \top$. Then $[\![\top]\!]_{\widehat{\alpha}} = \widehat{\alpha} = \widehat{\langle\alpha\rangle\top}$, by Defn.7 and the fact that $\vdash \alpha \leftrightarrow \langle\alpha\rangle\top$.

Base case $\varphi := 0$. Then $[\![0]\!]_{\widehat{\alpha}} = W^c$ if $\widehat{\alpha} = W^c$, and $[\![0]\!]_{\widehat{\alpha}} = \emptyset$ otherwise. Also, $\widehat{\langle\alpha\rangle 0} = \widehat{0 \wedge U\alpha} = \widehat{0} \cap \widehat{U\alpha} = \widehat{U\alpha}$ (by Propositions 2.2 and Lemma 24). By Corollary 4, $\widehat{U\alpha} = W^c$ if $\widehat{\alpha} = W^c$, and $\widehat{U\alpha} = \emptyset$ otherwise. So $[\![0]\!]_{\widehat{\alpha}} = \widehat{\langle\alpha\rangle 0}$.

Case $\varphi := \neg\psi$. Then we have: $T \in [\![\neg\psi]\!]_{\widehat{\alpha}}$ iff $T \in \widehat{\alpha}$ and $T \notin [\![\psi]\!]_{\widehat{\alpha}}$ iff $T \in \widehat{\alpha}$ and $T \notin \widehat{\langle\alpha\rangle\psi}$ (by IH) iff $T \in \widehat{\alpha}$ and $T \in \widehat{\neg\langle\alpha\rangle\psi}$ iff $T \in \widehat{\alpha \wedge \neg\langle\alpha\rangle\psi}$ iff $T \in \widehat{\langle\alpha\rangle\neg\psi}$ (by the definition of $[\alpha]\psi$ and Prop. 2.3).

Case $\varphi := \psi \wedge \chi$. First observe that $\vdash \langle\alpha\rangle(\psi \wedge \chi) \leftrightarrow (\langle\alpha\rangle\psi \wedge \langle\alpha\rangle\chi)$, which can easily be derived from Propositions 2.3 and 2 .14. We then have:

$T \in [\![\psi \wedge \chi]\!]_{\widehat{\alpha}}$ iff $T \in [\![\psi]\!]_{\widehat{\alpha}}$ and $T \in [\![\chi]\!]_{\widehat{\alpha}}$ iff $T \in \widehat{\langle\alpha\rangle\psi}$ and $T \in \widehat{\langle\alpha\rangle\chi}$ (by IH) iff $T \in \widehat{\langle\alpha\rangle\psi \wedge \langle\alpha\rangle\chi}$ iff $T \in \widehat{\langle\alpha\rangle(\psi \wedge \chi)}$ (by the above theorem).

Case $\varphi := \psi^0$. First note that $[\![\psi^0]\!]_{\widehat{\alpha}} = \widehat{\alpha} \cap [\![\psi]\!]_{W^c}$ (by Definition 7) $= \widehat{\alpha} \cap [\![\psi]\!]_{\widehat{\top}}$ (since $\widehat{\top} = W^c$) $= \widehat{\alpha} \cap \widehat{\psi}$ (by IH on $[\![\psi]\!]_{\widehat{\top}}$ and by R[$\top$]) $= \widehat{\alpha} \cap \widehat{\psi}^0$ (by Corollary 6). By (3.) in Proposition 2 and R^0 is easy to see that $\widehat{\langle\alpha\rangle\psi^0} = \widehat{\alpha} \cap \widehat{\psi}^0$. Therefore, $[\![\psi^0]\!]_{\widehat{\alpha}} = \widehat{\langle\alpha\rangle\psi^0}$.

Case $\varphi := K_i\psi$.

($\Rightarrow$) Suppose $T \in [\![K_i\psi]\!]_{\widehat{\alpha}}$. This means, by Definition 7, that $T \in \widehat{\alpha}$ and $T_i^{\widehat{\alpha}} \subseteq [\![\psi]\!]_{\widehat{\alpha}}$. We need to show that $T \in \widehat{\langle\alpha\rangle K_i\psi}$. Since $\vdash \langle\alpha\rangle K_i\psi \leftrightarrow \alpha \wedge K_i[\alpha]\psi$, we therefore only need to show that $T \in \widehat{\alpha}$ and $T \in \widehat{K_i[\alpha]\psi}$. We already know the former, so we just need to prove the latter. Towards contradiction, suppose $T \notin \widehat{K_i[\alpha]\psi}$. By Corollary 5, there is $S \in W^c$ such that $T \sim_i^c S$, $\alpha \in S$, and $[\alpha]\psi \notin S$. Since $\vdash \langle\alpha\rangle\psi \to [\alpha]\psi$, we have that $\langle\alpha\rangle\psi \notin S$. Thus, $S \notin \widehat{\langle\alpha\rangle\psi}$. By IH, $S \notin [\![\psi]\!]_{\widehat{\alpha}}$ which is a contradiction since $T_i^{\widehat{\alpha}} \subseteq [\![\psi]\!]_{\widehat{\alpha}}$. Therefore, $T \in \widehat{K_i[\alpha]\psi}$.

($\Leftarrow$) Suppose $T \in \widehat{\langle\alpha\rangle K_i\psi}$, i.e., $\langle\alpha\rangle K_i\psi \in T$. Since $\vdash \langle\alpha\rangle K_i\psi \leftrightarrow \alpha \wedge K_i[\alpha]\psi$, we therefore know that $\alpha \in T$ and $K_i[\alpha]\psi \in T$. We need to show that $T \in [\![K_i\psi]\!]_{\widehat{\alpha}}$. Let $S \in \widehat{\alpha}$ such that $T \sim_i^c S$. Since $T \in \widehat{K_i[\alpha]\psi}$, $[\alpha]\psi \in S$. Since $\alpha \in S$, $\langle\alpha\rangle\psi \in S$. This implies, by IH, that $S \in [\![\psi]\!]_{\widehat{\alpha}}$. Since this holds for all $S \in \widehat{\alpha}$ such that $T \sim_i^c S$, we have that $T_i^{\widehat{\alpha}} \subseteq [\![\psi]\!]_{\widehat{\alpha}}$ and, thus, $T \in [\![K_i\psi]\!]_{\widehat{\alpha}}$.

Case $\varphi := U\psi$. Follows similarly to the case for K_i using $\vdash \langle\alpha\rangle U\psi \leftrightarrow \alpha \wedge U[\alpha]\psi$ and Lemma 22 for U.

Case $\varphi := \langle\chi\rangle\psi$. We will use the fact that $\vdash \langle\alpha\rangle\langle\chi\rangle\psi \leftrightarrow \langle\langle\alpha\rangle\chi\rangle\psi$ (by R$_{[!]}$).

$T \in [\![\langle\chi\rangle\psi]\!]_{\widehat{\alpha}}$ iff $T \in [\![\chi]\!]_{\widehat{\alpha}}$ and $T \in [\![\psi]\!]_{[\![\chi]\!]_{\widehat{\alpha}}}$ (by Definition 7) iff $T \in \widehat{\langle\alpha\rangle\chi}$ and $T \in [\![\psi]\!]_{\widehat{\langle\alpha\rangle\chi}}$ (by IH on $[\![\chi]\!]_{\widehat{\alpha}}$) iff $T \in \widehat{\langle\langle\alpha\rangle\chi\rangle\psi}$ (by IH on $[\![\psi]\!]_{\widehat{\langle\alpha\rangle\chi}}$).

Case $\varphi := \Diamond\psi$.

($\Rightarrow$) Suppose $T \in [\![\Diamond\psi]\!]_{\widehat{\alpha}}$. This means, by Definition 7, that $\alpha \in T$ and there exists $B \in \mathcal{A}^c$ such that $T \in B \subseteq \widehat{\alpha}$ and $T \in [\![\psi]\!]_B$ (see Observation 5.1). By the construction of $\mathcal{A}^c$, we know that $B = \widehat{\theta}$ for some $\theta \in \mathcal{L}_{-\Diamond}$. Therefore, $T \in [\![\psi]\!]_B$ means that $T \in [\![\psi]\!]_{\widehat{\theta}}$. Moreover, since $\widehat{\theta} \subseteq \widehat{\alpha}$ and, thus, $\widehat{\theta} = \widehat{\alpha} \cap \widehat{\theta} = \widehat{\alpha \wedge \theta}$, we obtain $T \in [\![\psi]\!]_{\widehat{\alpha \wedge \theta}}$. By Lemma 6.1, we have $\psi < \Diamond\psi$. Therefore, by IH, we obtain $T \in \widehat{\langle\alpha \wedge \theta\rangle\psi}$. Then, by axiom ([!]$\Box$-elim) and the fact that T is maximal, we conclude that $T \in \widehat{\langle\alpha\rangle\Diamond\psi}$.

($\Leftarrow$) Suppose $T \in \widehat{\langle\alpha\rangle\Diamond\psi}$, i.e., $\langle\alpha\rangle\Diamond\psi \in T$. Then, since T is a maximal witnessed theory, there is $p \in Prop$ such that $\langle\alpha\rangle\langle p\rangle\psi \in T$. By Lemma 6.2, we know that $\langle p\rangle\psi < \Diamond\psi$. Thus, by IH on $\langle p\rangle\psi$, we obtain that $T \in [\![\langle p\rangle\psi]\!]_{\widehat{\alpha}}$. This means, by Definition 7 and Observation 5.3, that $T \in [\![\psi]\!]_{[\![p]\!]_{\widehat{\alpha}}} \subseteq [\![p]\!]_{\widehat{\alpha}}$. Since $p < \Diamond\psi$, by IH on p, we obtain that $[\![p]\!]_{\widehat{\alpha}} = \widehat{\langle\alpha\rangle p} \subseteq \widehat{\alpha}$. By the construction of $\mathcal{A}^c$, we moreover have $\widehat{\langle\alpha\rangle p} \in \mathcal{A}^c$. Therefore, as $T \in [\![\psi]\!]_{\widehat{\langle\alpha\rangle p}}$ and $\widehat{\langle\alpha\rangle p} \subseteq \widehat{\alpha}$, by Definition 7, we conclude that $T \in [\![\Diamond\psi]\!]_{\widehat{\alpha}}$.

Case $\varphi := \langle G\rangle\psi$.

($\Rightarrow$) Suppose $T \in [\![\langle G\rangle\psi]\!]_{\widehat{\alpha}}$. This means by Definition 7 that $T \in \widehat{\alpha}$ and there exists $\{B_i : i \in G\} \subseteq \mathcal{A}^c$ such that $T \in [\![\psi]\!]_{\widehat{\alpha} \cap \bigcap_{i \in G} K_i^{\widehat{\alpha}} B_i}$. By the construction of $\mathcal{A}^c$ we know that for all $i \in G$, $B_i = \widehat{\theta_i}$ for some $\theta_i \in \mathcal{L}_{-\Diamond}$. Therefore $T \in [\![\psi]\!]_{\widehat{\alpha} \cap \bigcap_{i \in G} K_i^{\widehat{\alpha}} \widehat{\theta_i}}$. It suffices to show that: $\widehat{\alpha} \cap \bigcap_{i \in G} K_i^{\widehat{\alpha}} \widehat{\theta_i} = \widehat{\alpha \wedge \bigwedge_{i \in G} K_i^{\alpha}\theta_i}$. First we need to show that $K_i^{\widehat{\alpha}}\widehat{\theta_i} = \widehat{K_i^{\alpha}\theta_i}$. Note that $\widehat{K_i^\alpha\theta_i} = K_i(\widehat{\alpha} \to \widehat{\theta_i}) = K_i(\widehat{\alpha \to \theta_i})$ and $\widehat{K_i^\alpha\theta_i} = \widehat{K_i(\alpha \to \theta_i)}$. For ($\subseteq$): Let $T \in \widehat{K_i^\alpha\theta_i}$, then for all $S \sim_i T$, $S \in \widehat{\alpha \to \theta_i}$. Therefore $T \in \widehat{K_i(\alpha \to \theta_i)}$ and so $T \in \widehat{K_i^\alpha\theta_i}$. For ($\supseteq$): Let $T \in \widehat{K_i^\alpha\theta_i}$, this means that $K_i(\alpha \to \theta_i) \in T$. Thus for all $S \sim_i T$, $\alpha \to \theta_i \in S$. Therefore $T \in \widehat{K_i^\alpha\theta_i}$. Using this, it is easy to see that $\widehat{\alpha} \cap \bigcap_{i\in G} \widehat{K_i^\alpha\theta_i} = \widehat{\alpha \wedge \bigwedge_{i\in G} K_i^\alpha\theta_i}$. We then obtain that $T \in [\![\psi]\!]_{\widehat{\alpha \wedge \bigwedge_{i\in G} K_i^\alpha\theta_i}}$. Since $\psi < \langle G\rangle\psi$, by I.H. we have that $T \in \widehat{\langle\alpha \wedge \bigwedge_{i\in G} K_i^\alpha\theta_i\rangle\psi}$. Thus $\langle\alpha \wedge \bigwedge_{i\in G} K_i^\alpha\theta_i\rangle\psi \in T$. By ([!][$G$]-elim) we have $\langle\alpha\rangle\langle G\rangle\psi \in T$.

($\Leftarrow$) Suppose $T \in \widehat{\langle\alpha\rangle\langle G\rangle\psi}$, i.e., $\langle\alpha\rangle\langle G\rangle\psi \in T$. Since T is a maximal witnessed theory, there is $\{p_i : i \in G\} \subseteq Prop$ such that $\langle\alpha\rangle\langle\bigwedge_{i\in G} K_i p_i\rangle\psi \in T$. By Lemma 6.3, we know that $\langle\bigwedge_{i\in G} K_i p_i\rangle\psi < \langle G\rangle\psi$. Thus, by IH on $\langle\bigwedge_{i\in G} K_i p_i\rangle\psi$, we obtain that $T \in [\![\langle\bigwedge_{i\in G} K_i p_i\rangle\psi]\!]_{\widehat{\alpha}}$. This means, by Definition 7, that $T \in [\![\psi]\!]_{[\![\bigwedge_{i\in G} K_i p_i]\!]_{\widehat{\alpha}}}$. By IH on $\bigwedge_{i\in G} K_i p_i$, we obtain that $T \in [\![\psi]\!]_{\widehat{\langle\alpha\rangle \bigwedge_{i\in G} K_i p_i}}$. By Proposition 3.2 and the reduction axioms (R_{K_i}) and (R_p), it is easy to see that the formula $\langle\alpha\rangle \bigwedge_{i\in G} K_i p_i \leftrightarrow \alpha \wedge \bigwedge_{i\in G} K_i^\alpha p_i$ is derivable in **GALM**. Therefore,

$$[\![\psi]\!]_{\widehat{\langle\alpha\rangle \bigwedge_{i\in G} K_i p_i}} = [\![\psi]\!]_{\widehat{\alpha \wedge \bigwedge_{i\in G} K_i^\alpha p_i}} = [\![\psi]\!]_{\widehat{\alpha}\cap\bigcap_{i\in G} K_i^{\widehat{\alpha}}\widehat{p_i}}.$$

Thus $T \in [\![\psi]\!]_{\widehat{\alpha}\cap\bigcap_{i\in G} K_i^{\widehat{\alpha}}\widehat{p_i}}$. Since $B_i := \widehat{p_i} \in \mathcal{A}^c$ for every $i \in G$, we obtain that $T \in [\![\langle G\rangle\psi]\!]_{\widehat{\alpha}}$.

Corollary 7. *The canonical pre-model $\mathcal{M}^c$ is standard (and hence a pseudo-model).*

Proof. $\mathcal{A}^c = \{\widehat{\theta} : \theta \in \mathcal{L}_{-\Diamond}\} = \{\widehat{\langle\top\rangle\theta} : \theta \in \mathcal{L}_{-\Diamond}\} = \{[\![\theta]\!]_{\widehat{\top}} : \theta \in \mathcal{L}_{-\Diamond}\} = \{[\![\theta]\!]_{W^c} : \theta \in \mathcal{L}_{-\Diamond}\}$.

Lemma 26. *For every $\varphi \in \mathcal{L}_G$, if φ is consistent then $\{0, \Diamond\varphi\}$ is an initial theory.*

Proof. Let $\varphi \in \mathcal{L}_G$ s.t. $\varphi \nvdash \bot$. By the *Equivalences with* ^{0} in Table 1, we have $\vdash \bot^0 \leftrightarrow (p \wedge \neg p)^0 \leftrightarrow (p^0 \wedge \neg p^0) \leftrightarrow (p \wedge \neg p) \leftrightarrow \bot$. Therefore, $\vdash \psi \to \bot^0$ iff $\vdash \psi \to \bot$ for all $\psi \in \mathcal{L}_G$. Then, by Proposition 2.13, we obtain $\vdash \varphi \to \bot$ iff $\vdash (0 \wedge \Diamond\varphi) \to \bot$. Since $\varphi \nvdash \bot$, we have $0 \wedge \Diamond\varphi \nvdash \bot$, i.e., $\{0, \Diamond\varphi\}$ is a theory. Since $0 \in \{0, \Diamond\varphi\}$, it is an initial one.

Corollary 8. GALM *is complete with respect to standard pseudo models.*

Proof. Let φ be a consistent formula. By Lemma 26, $\{0, \Diamond\varphi\}$ is an initial theory. Then, by Extension Lemma (Lemma 20), there is a maximal witnessed initial theory T_0 such that $\{0, \Diamond\varphi\} \subseteq T_0$. We can then construct the canonical pseudo-model $\mathcal{M}^c$ for T_0. Since $\Diamond\varphi \in T_0$ and T_0 is witnessed, there exists $p \in Prop$ such that $\langle p\rangle\varphi \in T_0$. By Truth Lemma (applied to $\alpha := p$), we get $T_0 \in [\![\varphi]\!]_{\widehat{p}}$. Hence, φ is satisfied at T_0 in the set $\widehat{p} \in \mathcal{A}^c$.

Theorem 3. APALM *is complete with respect to standard pseudo models.*

The completeness proof for **APALM** with respect to standard pseudo models is obtained by following the same steps in the completeness proof of **GALM** without the parts required for the operator $\langle G\rangle$. This involves, for example, defining the witnessed theories only with respect to $\Diamond$ and modifying the auxiliary lemmas accordingly. A reference to the completeness proof of **APALM** is omitted for the purpose of blind reviewing.

Corollary 9. GALM *is complete with respect to a-models. Moreover,* **APALM** *is complete with respect to a-models.*

Proof. **GALM** completeness follows immediately from Corollaries 8 and 2. **APALM** completeness follows from Theorem 3 and Corollary 2

6 Conclusions and Future Work

This paper solves the open question of finding a strong variant of APAL and GAL that is recursively axiomatizable. Our system APALM is inspired by our analysis of Kuijer's counterexample [23], which lead us to add to APAL a 'memory' of the initial situation. We then used similar methods to obtain a recursive axiomatization for the memory-enhanced variant GALM of GAL. The soundness and completeness proofs crucially rely on a Subset Space-like semantics and on the equivalence between the effort modality and the arbitrary announcement modality (and on the equivalence between their $[G]$ counterparts), thus revealing the strong link between these two formalisms.

We just want to note again that the problems with finding a recursive axiomatization apply to many other variants of APAL and GAL. As far as we know, there is no complete axiomatization for Coalition Announcement Logic (CAL) introduced in [2], though an extension of it (subsuming both GAL and CAL) was completely axiomatized in [21] using infinitary rules. But no recursive axiomatization is known for GAL, CAL, or any of their extensions [2,32,3,4]. In contrast, in this paper we showed that the memory-enhanced version of CAL is embeddable in the (recursively axiomatized) GALM. We believe that our methods could also be used to provide a direct recursive axiomatization of CALM (the memory-enhanced variant of CAL), but we leave this for future work. Another open question is to elucidate whether GALM and CALM are equally expressive. In the memory-free case, [19] provided a counterexample: there exists a property expressible in GAL that is not expressible in CAL. Is that still the case for the memory-enhanced versions? We leave this problem for future research as well.

A further comment is on the connection of our logic with the yesterday operator. The limited form of memory provided by APALM is in fact enough to 'simulate' the yesterday operator $Y\varphi$ on any given model, by using context-dependent formulas. For instance, the dialogue in Cheryl's birthday puzzle (Albert: "I don't know when Cheryl's birthday is, but I know that Bernard doesn't know it either"; Bernard: "At first I didn't know when Cheryl's birthday is, but I know now"; Albert: "Now I also know"), can be simulated by the following sequence of announcements[22]: first, the formula $0 \wedge \neg K_a c \wedge K_a \neg K_b c$ is announced (where 0 marks the fact that this is the first announcement), then $(\neg K_b c)^0 \wedge K_b c$ is announced, and finally $K_a c$ is announced.

For another example: if instead we change the story so that the third announcement (by Albert) is "I knew you knew it (just before you said so)", then the last step of this alternative scenario corresponds to announcing the formula $([0 \wedge \neg K_a c \wedge K_a \neg K_b c] K_a K_b c)^0$ (saying that, just after the first announcement but before the second, Albert knew that Bernard knew the birthday). This shows how the logic can simulate the use of *any* (iterated) Y's in concrete examples, although at the cost of repeating the relevant part of history inside the announcement in order to mark the exact time when the announced formula was meant to be true.

Therefore, APALM combines in a way the expressivity and advantages of APAL with *some* of the expressivity of TPAL (the extension of PAL with the yesterday operator, introduced in [28,27]), without any of "defects" of either of them: unlike APAL, it has a natural, straightforward recursive axiomatization, with intuitive axioms and rules; unlike TPAL, its semantics is not computationally much more demanding than the one of basic epistemic logic: instead of keeping track of a growing and unbounded number of past Kripke models, APALM keeps only the initial model and the current one. Nevertheless, a more systematic treatment of the yesterday operator on (a version of) our announcement models and its connection to arbitrary and group announcements deserves a closer look. Yet another line of further work concerns other meta-logical properties, such as decidability and complexity, of APALM and GALM.

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A Appendix

A.1 Definition of the complexity measure for the Proofs of Lemmas 1 and 6

In some of our inductive proofs, we need a complexity measure on formulas that is different from the standard one based on subformula complexity. The standard notion requires only that formulas are more complex than their subformulas, while we also need that $\Diamond\varphi$ and $\langle G\rangle\varphi$ are more complex than $\langle\theta\rangle\varphi$ for all $\theta \in \mathcal{L}_{-\Diamond}$. To the best of our knowledge, such a complexity measure was first introduced in [5] for the original APAL language from [6]. A similar measure is also used in [7] and has later been introduced for topological versions of APAL in [37,38,11]. Definitions below are given for our largest language $\mathcal{L}_G$. Their counterparts for $\mathcal{L}$ are obtained simply by eliminating the clauses for $\langle G\rangle\varphi$.

Definition 13 (Subformula). *Given a formula $\varphi \in \mathcal{L}_G$, the set $Sub(\varphi)$ of subformulas of φ is recursively defined as*

$$\begin{aligned} Sub(\varphi) &= \{\varphi\} \quad \text{if } \varphi \text{ is } p, \top \text{ or } 0\,, \\ Sub(\neg\varphi) &= Sub(\varphi) \cup \{\neg\varphi\} & Sub(\varphi \wedge \psi) &= Sub(\varphi) \cup Sub(\psi) \cup \{\varphi \wedge \psi\} \\ Sub(\varphi^0) &= Sub(\varphi) \cup \{\varphi^0\} & Sub(\langle\varphi\rangle\psi) &= Sub(\varphi) \cup Sub(\psi) \cup \{\langle\varphi\rangle\psi\} \\ Sub(K_i\varphi) &= Sub(\varphi) \cup \{K_i\varphi\} & Sub(\Diamond\varphi) &= Sub(\varphi) \cup \{\Diamond\varphi\} \\ Sub(U\varphi) &= Sub(\varphi) \cup \{U\varphi\} & Sub(\langle G\rangle\varphi) &= Sub(\varphi) \cup \{\langle G\rangle\varphi\}. \end{aligned}$$

Any formula in $Sub(\varphi) - \{\varphi\}$ is called a proper subformula *of φ.*

Definition 14 (Size of formulas in $\mathcal{L}_G$). *The* size *$s(\varphi)$ of formula $\varphi \in \mathcal{L}_G$ is a natural number recursively defined as:*

$$\begin{aligned} s(p) = s(\top) = s(0) &= 1, \\ s(\neg\varphi) = s(\varphi^0) = s(K_i\varphi) = s(U\varphi) = s(\Diamond\varphi) = s(\langle G\rangle\psi) &= s(\varphi) + 1, \\ s(\varphi \wedge \psi) &= s(\varphi) + s(\psi) + 1, \\ s(\langle\varphi\rangle\psi) &= (5 + s(\varphi)) \cdot s(\psi). \end{aligned}$$

Definition 15 ($\Diamond, G$-Depth of formulas in $\mathcal{L}_G$). *The $\Diamond, G$*-depth *$d(\varphi)$ of formula $\varphi \in \mathcal{L}_G$ is a natural number recursively defined as:*

$$\begin{aligned} d(p) = d(\top) = d(0) &= 0, \\ d(\neg\varphi) = d(\varphi^0) = d(K_i\varphi) = d(U\varphi) &= d(\varphi) \\ d(\varphi \wedge \psi) = d(\langle\varphi\rangle\psi) &= max\{d(\varphi), d(\psi)\}, \\ d(\Diamond\varphi) = d(\langle G\rangle\varphi) &= d(\varphi) + 1. \end{aligned}$$

Finally, we define our intended complexity relation $<$ as lexicographic merge of $\Diamond, G$-depth and size, exactly as in [5]:

Definition 16. *For any $\varphi, \psi \in \mathcal{L}_G$, we put*

$$\varphi < \psi \text{ iff either } d(\varphi) < d(\psi), \text{ or } d(\varphi) = d(\psi) \text{ and } s(\varphi) < s(\psi).$$

A.2 Proofs of results in Section 2

Proof of Proposition 2

1. from $\vdash \varphi \leftrightarrow \psi$, infer $\vdash [\theta]\varphi \leftrightarrow [\theta]\psi$: Follows directly by ($K_!$) and ($Nec_!$).

2. $\langle\theta\rangle 0 \leftrightarrow (0 \wedge U\theta)$: Follows from the definition of $\langle\theta\rangle 0 := \neg[\theta]\neg 0$ and the axiom ($R_\neg$.)

3. $\langle\theta\rangle\psi \leftrightarrow (\theta \wedge [\theta]\psi)$: Follows from the definition $\langle\theta\rangle\psi := \neg[\theta]\neg\psi$ and the axiom ($R_\neg$.)

4. $\Box\varphi \to [\rho]\varphi$ ($\rho \in \mathcal{L}_{-\Diamond}$ arbitrary):

1. $\vdash \Box\varphi \leftrightarrow [\top]\Box\varphi$	(R[$\top$])
2. $\vdash [\top]\Box\varphi \to [\top \wedge \rho]\varphi$, (for arbitrary $\rho \in \mathcal{L}_{-\Diamond}$)	([!]$\Box$-elim)
3. $\vdash [\top \wedge \rho]\varphi \to [\rho]\varphi$, (for arbitrary $\rho \in \mathcal{L}_{-\Diamond}$)	($\vdash (\top \wedge \rho) \leftrightarrow \rho$ and (RE))
4. $\vdash \Box\varphi \to [\rho]\varphi$, (for arbitrary $\rho \in \mathcal{L}_{-\Diamond}$)	(1-3, CPL)

5. from $\vdash \chi \to [p]\varphi$, infer $\vdash \chi \to \Box\varphi$ ($p \notin P_\chi \cup P_\varphi$):

1. $\vdash \chi \to [p]\varphi$	(assumption)
2. $\vdash \chi \to [\top][p]\varphi$	(R[$\top$])
3. $\vdash \chi \to [\langle\top\rangle p]\varphi$	($R_{[!]}$)
4. $\vdash \chi \to [\top \wedge p]\varphi$	(Prop.2.3, R_p, RE)
5. $\vdash \chi \to [\top]\Box\varphi$	($p \notin P_\chi \cup P_\varphi$ and [!]$\Box$-intro)
6. $\vdash \chi \to \Box\varphi$	(R[$\top$])

6. all $S4$ axioms and rules for $\Box$: The derivation of the necessitation rule for $\Box$, ($Nec_\Box$), easily follows from ($Nec_!$) and Prop. 2.5. The T-axiom for $\Box$ follows from Prop. 2.4 and R[$\top$].
For the K-axiom:

1. $\vdash (\Box(\varphi \to \psi) \wedge \Box\varphi) \to ([p](\varphi \to \psi) \wedge [p]\varphi)$	($p \notin P_\varphi \cup P_\psi$, Prop. 2.4)
2. $\vdash ([p](\varphi \to \psi) \wedge [p]\varphi) \to [p]\psi$	($K_!$)
3. $\vdash (\Box(\varphi \to \psi) \wedge \Box\varphi) \to [p]\psi$	(1, 2, CPL)
4. $\vdash (\Box(\varphi \to \psi) \wedge \Box\varphi) \to \Box\psi$	($p \notin P_\varphi \cup P_\psi$, Prop. 2.5)

For the 4-axiom:

1. $\vdash \Box\varphi \to [p \wedge q]\varphi$	(for some $p, q \notin P_\varphi$, Prop. 2.4)
2. $\vdash \Box\varphi \to [p]\Box\varphi$	([!]$\Box$-intro)
3. $\vdash \Box\varphi \to \Box\Box\varphi$	($p \notin P_\varphi$, Prop. 2.5)

7. $(\varphi \to \psi)^0 \leftrightarrow (\varphi^0 \to \psi^0)$: This is straightforward by the set of axioms called *Equivalences with* ^{0}.

8. $\vdash \varphi^{00} \leftrightarrow \varphi^0$:

1. $\vdash 0 \to (\varphi \leftrightarrow \varphi^0)$	(0-eq)
2. $\vdash (0 \to (\varphi \leftrightarrow \varphi^0))^0$	(Nec^0)
3. $\vdash 0^0 \to (\varphi \leftrightarrow \varphi^0)^0$	(Prop.2.7)
4. $\vdash 0^0 \to (\varphi^0 \leftrightarrow \varphi^{00})$	(Prop.2.7)
5. $\vdash 0^0$	(Ax_0)
6. $\vdash \varphi^0 \leftrightarrow \varphi^{00}$	(4, 5, MP)

9. $\Box\varphi^0 \leftrightarrow \varphi^0$ and $\varphi^0 \leftrightarrow \Diamond\varphi^0$: From left-to-right direction of both cases follow from the T-axiom for $\Box$. From right-to-left direction we will first prove that $\vdash \varphi^0 \to \Box\varphi^0$:

1. $\vdash \varphi^0 \to (p \to \varphi^0)$	(for $p \notin P_{\varphi^0}$, CPL)
2. $\vdash \varphi^0 \to [p]\varphi^0$	(R^0)
3. $\vdash \varphi^0 \to \Box\varphi^0$	(Prop. 2.5)

For $\vdash \Diamond\varphi^0 \to \varphi^0$, we have:

1. $\vdash (\neg\varphi)^0 \to \Box(\neg\varphi)^0$	(by the above result)
2. $\vdash \neg\varphi^0 \to \Box\neg\varphi^0$	($\mathrm{Eq}^0_\neg$)
3. $\vdash \Diamond\varphi^0 \to \varphi^0$	(contraposition of 2)

10. $\vdash (\Box\varphi)^0 \to \Box\varphi^0$

1. $\vdash \Box\varphi \to \varphi$	(S4 for $\Box$)
2. $\vdash (\Box\varphi \to \varphi)^0$	(Nec^0)
3. $\vdash (\Box\varphi)^0 \to \varphi^0$	(Prop.2.7, 2, MP)
4. $\vdash (\Box\varphi)^0 \to \Box\varphi^0$	(Prop.2.9)

11. $\vdash (0 \wedge \Diamond\varphi^0) \to \varphi$

1. $\vdash 0 \to (\varphi^0 \leftrightarrow \varphi)$	(0-*eq*)
2. $\vdash 0 \to (\varphi^0 \to \varphi)$	(CPL)
3. $\vdash 0 \to (\Diamond\varphi^0 \to \varphi)$	(Prop.2.9)
4. $\vdash (0 \wedge \Diamond\varphi^0) \to \varphi$	(CPL)

12. $\vdash \varphi \to (0 \wedge \Diamond\varphi)^0$

1. $\vdash (\Box\neg\varphi)^0 \to \neg\varphi$	($\mathrm{Imp}^0_\Box$)
2. $\vdash \neg\neg\varphi \to \neg(\Box\neg\varphi)^0$	(contraposition of 1)
3. $\vdash \neg\neg\varphi \to (\neg\Box\neg\varphi)^0$	($\mathrm{Eq}^0_\neg$)
4. $\vdash \varphi \to (\Diamond\varphi)^0$	(the defn. of $\Box$, CPL)
5. $\vdash \varphi \to (0^0 \wedge (\Diamond\varphi)^0)$	(Ax_0)
6. $\vdash \varphi \to (0 \wedge \Diamond\varphi)^0$	($\mathrm{Eq}^0_\wedge$)

13. $\vdash \varphi \to \psi^0$ if and only if $\vdash (0 \wedge \Diamond\varphi) \to \psi$
From left-to-right: Suppose $\vdash \varphi \to \psi^0$ and show: $\vdash (0 \wedge \Diamond\varphi) \to \psi$.

1. $\vdash (0 \wedge \Diamond\psi^0) \to \psi$	(Prop.2.11)
2. $\vdash \Diamond\varphi \to \Diamond\psi^0$	(by assumption and $\mathrm{Nec}_\Box$)
3. $\vdash (0 \wedge \Diamond\varphi) \to (0 \wedge \Diamond\psi^0)$	(2 and CPL)
4. $\vdash (0 \wedge \Diamond\varphi) \to \psi$	(1-3, CPL)

From right-to-left: Suppose $\vdash (0 \wedge \Diamond\varphi) \to \psi$ and show $\vdash \varphi \to \psi^0$.

1. $\vdash \varphi \to (0 \wedge \Diamond\varphi)^0$	(Prop.2.12)
2. $\vdash (0 \wedge \Diamond\varphi) \to \psi$	(assumption)
3. $\vdash ((0 \wedge \Diamond\varphi) \to \psi)^0$	(Nec^0)
4. $\vdash (0 \wedge \Diamond\varphi)^0 \to \psi^0$	(Prop.2.7)
5. $\vdash \varphi \to \psi^0$	(1-4, CPL)

14. $[\theta](\varphi \wedge \psi) \leftrightarrow ([\theta]\varphi \wedge [\theta]\psi)$: Follows from ($\mathrm{K}_!$) and ($\mathrm{Nec}_!$).

15. $[\theta][p]\varphi \leftrightarrow [\theta \wedge p]\varphi$

1. $\vdash [\theta][p]\varphi \leftrightarrow [\langle\theta\rangle p]\varphi$	($\mathrm{R}_{[!]}$)
2. $\vdash [\langle\theta\rangle p]\varphi \leftrightarrow [\theta \wedge [\theta]p]\varphi$	(Prop.2.3, RE)
3. $\vdash [\theta \wedge [\theta]p]\varphi \leftrightarrow [\theta \wedge (\theta \to p)]\varphi$	(R_p, RE)
4. $\vdash [\theta \wedge (\theta \to p)]\varphi \leftrightarrow [\theta \wedge p]\varphi$	(CPL, RE)
5. $\vdash [\theta][p]\varphi \leftrightarrow [\theta \wedge p]\varphi$	(1-4, CPL)

16. $[\theta]\bot \leftrightarrow \neg\theta$

1. $\vdash [\theta]\bot \leftrightarrow [\theta](p \wedge \neg p)$	(the defn. of $\bot$)
2. $\vdash [\theta](p \wedge \neg p) \leftrightarrow [\theta]p \wedge [\theta]\neg p$	(Prop.2.14)
3. $\vdash [\theta]p \wedge [\theta]\neg p \leftrightarrow ((\theta \to p) \wedge (\theta \to \neg(\theta \to p)))$	(R_p, $\mathrm{R}_\neg$)
4. $\vdash ((\theta \to p) \wedge (\theta \to \neg(\theta \to p))) \leftrightarrow \neg\theta$	(CPL)
5. $\vdash [\theta]\bot \leftrightarrow \neg\theta$	(1-4, CPL)

A.3 Proofs of results in Section 3

Proof of Proposition 8 The proof is by $<$-induction on φ, using Lemma 6 and the following *induction hypothesis* (IH): for all $\psi < \varphi$ and all models $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$, we have $[\![\psi]\!] \subseteq W$. The base cases $\varphi := p$, $\varphi := \top$, and $\varphi := 0$ are straightforward by the semantics given in Defn.2. The inductive cases for Booleans are immediate. Similarly, the following cases make use of the corresponding semantic clause in Defn.2.

Case $\varphi := \psi^0$: $[\![\psi^0]\!] = [\![\psi]\!]_{\mathcal{M}^0} \cap W \subseteq W$.

Case $\varphi := K_i\psi$: $[\![K_i\psi]\!] = \{w \in W : w_i \subseteq [\![\varphi]\!]\} \subseteq W$.

Case $\varphi := U\psi$: $[\![U\psi]\!] \in \{\emptyset, W\}$, thus $[\![U\psi]\!] \subseteq W$.

Case $\varphi := \langle\theta\rangle\psi$: Since $\theta < \langle\theta\rangle\psi$ (Lemma 6.1), by the IH on θ, we have that $[\![\theta]\!] \subseteq W$. Moreover, since $\psi < \langle\theta\rangle\psi$ (Lemma 6.1), by the IH on ψ, we also have that $[\![\psi]\!]_{\mathcal{M}|[\![\theta]\!]} \subseteq [\![\theta]\!]$ (recall that $\mathcal{M}|[\![\theta]\!] = (W^0, [\![\theta]\!], \sim_1, \ldots, \sim_n, \|\cdot\|)$). Therefore, by Defn.2, we obtain that $[\![\langle\theta\rangle\psi]\!] = [\![\psi]\!]_{\mathcal{M}|[\![\theta]\!]} \subseteq [\![\theta]\!] \subseteq W$.

Case $\varphi := \Diamond\psi$: By Lemma 6.2, it follows that for each $\theta \in \mathcal{L}_{-\Diamond}$, $\langle\theta\rangle\psi < \Diamond\psi$. Then, by the IH, we have that for all $\theta \in \mathcal{L}_{-\Diamond}$, $[\![\langle\theta\rangle\psi]\!] \subseteq W$. Thus $\bigcup\{[\![\langle\theta\rangle\psi]\!] : \theta \in \mathcal{L}_{-\Diamond}\} \subseteq W$, i.e., $[\![\Diamond\psi]\!] \subseteq W$.

Case $\varphi := \langle G\rangle\psi$: By Lemma 6.3, it follows that for each $\theta \in \mathcal{L}_{-\Diamond}$, $\langle\theta\rangle\psi < \langle G\rangle\psi$. Then, by the IH, we have that for all $\theta_i \in \mathcal{L}_{-\Diamond}$, $[\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\psi]\!] \subseteq W$. Thus $\bigcup\{[\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\psi]\!] : \theta_i \in \mathcal{L}_{-\Diamond}\} \subseteq W$, i.e., $[\![\langle G\rangle\psi]\!] \subseteq W$.

A.4 Proofs of results in Section 4

Proof of Proposition 13

1. Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ be a standard pseudo-model. Then, $A \in \mathcal{A}$ implies $A = [\![\theta]\!]^{PS}_{W^0} \subseteq W^0$ for some θ, hence $\mathcal{M}_A = (W^0, A, \sim_1, \ldots, \sim_n, \|\cdot\|)$ is a model whenever A in non-empty.
   (a) The proof is by $<$-induction from Lemma 6. The base cases and the inductive cases for Booleans are straightforward.

   Case $\varphi := \psi^0$. We have $[\![\psi^0]\!]^{PS}_A = [\![\psi]\!]^{PS}_{W^0} \cap A = [\![\psi]\!]_{\mathcal{M}^0_A} \cap A = [\![\psi^0]\!]_{\mathcal{M}_A}$ (by Defn.7, IH, and Defn. 2).

   Case $\varphi := K_i\psi$. We have $[\![K_i\psi]\!]^{PS}_A = \{w \in A : w^A_i \subseteq [\![\psi]\!]^{PS}_A\} = \{w \in A : w_i \subseteq [\![\psi]\!]_{\mathcal{M}_A}\} = [\![K_i\psi]\!]_{\mathcal{M}_A}$ (by Defn.7, IH, and Defn. 2).

   Case $\varphi := U\psi$. By Definitions 2 and 7, we have:
$$[\![U\psi]\!]_{\mathcal{M}_A} = \begin{cases} A & \text{if } [\![\psi]\!]_{\mathcal{M}_A} = A \\ \emptyset & \text{otherwise} \end{cases} \qquad [\![U\psi]\!]^{PS}_A = \begin{cases} A & \text{if } [\![\psi]\!]^{PS}_A = A \\ \emptyset & \text{otherwise} \end{cases}$$

   By IH, $[\![\psi]\!]^{PS}_A = [\![\psi]\!]_{\mathcal{M}_A}$, therefore, $[\![U\psi]\!]^{PS}_A = [\![U\psi]\!]_{\mathcal{M}_A}$.

   Case $\varphi := \langle\psi\rangle\chi$. By Defn. 2, we know that $[\![\langle\psi\rangle\chi]\!]_{\mathcal{M}_A} = [\![\chi]\!]_{\mathcal{M}_A|[\![\psi]\!]_{\mathcal{M}_A}}$. Now consider the relativized model $\mathcal{M}_A|[\![\psi]\!]_{\mathcal{M}_A} = (W^0, [\![\psi]\!]_{\mathcal{M}_A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$. By Lemma 6.1 and IH, we have $[\![\psi]\!]_{\mathcal{M}_A} = [\![\psi]\!]^{PS}_A$. Moreover, by the definition of standard pseudo-models, we know that $A = [\![\theta]\!]^{PS}_{W^0}$ for some $\theta \in \mathcal{L}_{-\Diamond}$. Therefore, $[\![\psi]\!]_{\mathcal{M}_A} = [\![\psi]\!]^{PS}_A = [\![\psi]\!]^{PS}_{[\![\theta]\!]^{PS}_{W^0}} = [\![\langle\theta\rangle\psi]\!]^{PS}_{W^0}$. Therefore, $[\![\psi]\!]_{\mathcal{M}_A} \in \mathcal{A}$. We then have
$$[\![\langle\psi\rangle\chi]\!]_{\mathcal{M}_A} = [\![\chi]\!]_{\mathcal{M}_{[\![\psi]\!]_{\mathcal{M}_A}}} = [\![\chi]\!]_{\mathcal{M}_{[\![\psi]\!]^{PS}_A}} = [\![\chi]\!]^{PS}_{[\![\psi]\!]^{PS}_A} = [\![\langle\psi\rangle\chi]\!]^{PS}_A,$$
   by the semantics and IH on ψ and on χ (since $[\![\psi]\!]^{PS}_A \in \mathcal{A}$).

   Case $\varphi := \Diamond\psi$. By Defn.2, Lemma 6.2, IH, the fact that $\mathcal{M}$ is a standard pseudo model, and Lemma 12.1 - applied in this order - we obtain the following equivalences:
$$[\![\Diamond\psi]\!]_{\mathcal{M}_A} = \bigcup\{[\![\langle\chi\rangle\psi]\!]_{\mathcal{M}_A} : \chi \in \mathcal{L}_{-\Diamond}\} = \bigcup\{[\![\langle\chi\rangle\psi]\!]^{PS}_A : \chi \in \mathcal{L}_{-\Diamond}\} = [\![\Diamond\psi]\!]^{PS}_A.$$

   Case $\varphi := \langle G\rangle\psi$. By Defn.5, Lemma 6.3, IH, the fact that $\mathcal{M}$ is a standard pseudo model, and Lemma 12.2 - applied in this order - we obtain the following equivalences:
$$[\![\langle G\rangle\psi]\!]_{\mathcal{M}_A} = \bigcup\{[\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\varphi]\!]_{\mathcal{M}_A} : \{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\Diamond}\} = \bigcup\{[\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\varphi]\!]^{PS}_A : \{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\Diamond}\}.$$

   Therefore $[\![\langle G\rangle\psi]\!]_{\mathcal{M}_A} = [\![\langle G\rangle\psi]\!]^{PS}_A$.

   (b) By part (a), $[\![\varphi]\!]_{\mathcal{M}^0_A} = [\![\varphi]\!]_{\mathcal{M}_{W^0}} = [\![\varphi]\!]^{PS}_{W^0}$ for all φ. Since $\mathcal{M}$ is standard, we have $A = [\![\theta]\!]^{PS}_{W^0} = [\![\theta]\!]_{\mathcal{M}^0_A}$ for some $\theta \in \mathcal{L}_{-\Diamond}$, so $\mathcal{M}_A$ is an a-model.

2. Let $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$ be an a-model. Since $\mathcal{A} = \{[\![\theta]\!]_{\mathcal{M}^0} : \theta \in \mathcal{L}_{-\diamond}\} \subseteq \mathcal{P}(W^0)$, the model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ is a pre-model. Therefore, the semantics given in Definition 7 is defined on $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$.

   (a) By Proposition 12, it suffices to prove that the pre-model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ is standard, i.e. that $\{[\![\theta]\!]_{\mathcal{M}^0} : \theta \in \mathcal{L}_{-\diamond}\} = \{[\![\theta]\!]^{PS}_{W^0} : \theta \in \mathcal{L}_{-\diamond}\}$. For this, we need to show that for every a-model $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$, we have $[\![\theta]\!]_{\mathcal{M}} = [\![\theta]\!]^{PS}_{W}$ for all $\theta \in \mathcal{L}_{-\diamond}$.
   We prove this by subformula induction on θ. The base cases and the inductive cases for Booleans are straightforward.

   Case $\theta := \psi^0$. Then $[\![\psi^0]\!]_{\mathcal{M}} = [\![\psi]\!]_{\mathcal{M}^0} \cap W = [\![\psi]\!]^{PS}_{W^0} \cap W = [\![\psi^0]\!]^{PS}_{W}$ (by Defn.2, IH, and Defn.7).

   Case $\theta := K_i\psi$. We have $[\![K_i\psi]\!]_{\mathcal{M}} = \{w \in W : w_i \subseteq [\![\psi]\!]_{\mathcal{M}}\} = \{w \in W : w_i^W \subseteq [\![\psi]\!]^{PS}_{W}\} = [\![K_i\psi]\!]^{PS}_{W}$ (by Defn.2, IH, and Defn.7).

   Case $\theta := U\psi$. By Definitions 2 and 7, we have:

   $$[\![U\psi]\!]_{\mathcal{M}} = \begin{cases} W & \text{if } [\![\psi]\!]_{\mathcal{M}} = W \\ \emptyset & \text{otherwise} \end{cases} \qquad [\![U\psi]\!]^{PS}_{W} = \begin{cases} W & \text{if } [\![\psi]\!]^{PS}_{W} = W \\ \emptyset & \text{otherwise} \end{cases}$$

   By IH, $[\![\psi]\!]^{PS}_{W} = [\![\psi]\!]_{\mathcal{M}}$, therefore, $[\![U\psi]\!]^{PS}_{W} = [\![U\psi]\!]_{\mathcal{M}}$.

   Case $\theta := \langle\psi\rangle\chi$. By Definition 2, we know that $[\![\langle\psi\rangle\chi]\!]_{\mathcal{M}} = [\![\chi]\!]_{\mathcal{M}|[\![\psi]\!]_{\mathcal{M}}}$. Now consider the relativized model $\mathcal{M}|[\![\psi]\!]_{\mathcal{M}} = (W^0, [\![\psi]\!]_{\mathcal{M}}, \sim_1, \ldots, \sim_n, \|\cdot\|)$. By Lemma 6.1 and IH on ψ, we have $[\![\psi]\!]_{\mathcal{M}} = [\![\psi]\!]^{PS}_{W}$. Moreover, by the definition of a-models, we know that $W = [\![\theta]\!]_{\mathcal{M}^0}$ for some $\theta \in \mathcal{L}_{-\diamond}$. Therefore, $[\![\psi]\!]_{\mathcal{M}} = [\![\psi]\!]_{\mathcal{M}^0|[\![\theta]\!]_{\mathcal{M}^0}} = [\![\langle\theta\rangle\psi]\!]_{\mathcal{M}^0}$. Hence, since $\langle\theta\rangle\psi \in \mathcal{L}_{-\diamond}$, the model $\mathcal{M}|[\![\psi]\!]_{\mathcal{M}}$ is also an a-model obtained by updating the initial model $\mathcal{M}^0$ by $\langle\theta\rangle\psi$. We then have $[\![\langle\psi\rangle\chi]\!]_{\mathcal{M}} = [\![\chi]\!]_{\mathcal{M}|[\![\psi]\!]_{\mathcal{M}}}$ (by Defn.2) $= [\![\chi]\!]_{\mathcal{M}|[\![\psi]\!]^{PS}_{W}}$ (by IH on ψ) $= [\![\chi]\!]^{PS}_{[\![\psi]\!]^{PS}_{W}}$ (by IH on χ, $\mathcal{M}|[\![\psi]\!]_{\mathcal{M}}$ is an a-model) $= [\![\langle\psi\rangle\chi]\!]^{PS}_{W}$ (by Defn.7).

   (b) The proof of this part follows by $<$-induction on φ (where $<$ is as in Lemma 6). All the inductive cases are similar to ones in the above proof, except for the cases $\varphi := \diamond\psi$ and $\varphi := \langle G\rangle\psi$, shown below.
   Case $\varphi := \diamond\psi$. By Defn.2, Lemma 6.2, IH, the fact that $\mathcal{M}'$ is a standard pseudo model, and Lemma 12.2 - applied in that order - we obtain the following equivalences:

   $$[\![\diamond\psi]\!]_{\mathcal{M}} = \bigcup\{[\![\langle\chi\rangle\psi]\!]_{\mathcal{M}} : \chi \in \mathcal{L}_{-\diamond}\} = \bigcup\{[\![\langle\chi\rangle\psi]\!]^{PS}_{W} : \chi \in \mathcal{L}_{-\diamond}\} = [\![\diamond\psi]\!]^{PS}_{W}.$$

   Case $\varphi := \langle G\rangle\psi$. By Defn.5, Lemma 6.3, IH, the fact that $\mathcal{M}'$ is a standard pseudo model and Lemma 12.2 - applied in that order - we obtain the following: equivalences,

   $$[\![\langle G\rangle\psi]\!]_{\mathcal{M}} = \bigcup\{[\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\varphi]\!]_{\mathcal{M}} : \{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\diamond}\} = \bigcup\{[\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\varphi]\!]^{PS}_{W} : \{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\diamond}\}.$$

   Therefore, $[\![\langle G\rangle\psi]\!]_{\mathcal{M}} = [\![\langle G\rangle\psi]\!]^{PS}_{W}$.

A.5 Proofs of results in Section 5

Proof of Lemma 13 We proceed by induction on the structure of $s \in NF$. For $s := \epsilon$, take $\psi := \top$ and $\theta := \top$, then it follows from the axiom R[$\top$]. For the inductive cases we will verify only $s := s', \bullet^0$; $s := s', \eta \rightarrow$; $s := s', U$; and $s := s', \rho$. The case $s := s', K_i$ is analogous to the case $s := s', U$.

Case $s := s', \bullet^0$

$\vdash [s', \bullet^0]\varphi$ iff $\vdash [s']\varphi^0$ (by Defn. 10) iff $\vdash \psi' \rightarrow [\theta']\varphi^0$ (for some $\psi' \in \mathcal{L}_G$ and $\theta' \in \mathcal{L}_{-\diamond}$, by IH) iff $\vdash \psi' \rightarrow (\theta' \rightarrow \varphi^0)$ (by R^0) iff $\vdash (\psi' \wedge \theta') \rightarrow \varphi^0$ iff $\vdash (0 \wedge \diamond(\psi' \wedge \theta')) \rightarrow \varphi$ (by Prop. 2.13) iff $\vdash \psi \rightarrow [\theta]\varphi$ (where $\psi := 0 \wedge \diamond(\psi' \wedge \theta') \in \mathcal{L}_G$ and $\theta := \top \in \mathcal{L}_{-\diamond}$).

Case $s := s', \eta \to$

$\vdash [s', \eta \to]\varphi$ iff $\vdash [s'](\eta \to \varphi)$ (by Defn. 10) iff $\vdash \psi' \to [\theta'](\eta \to \varphi)$ (for some $\psi' \in \mathcal{L}_G$ and $\theta' \in \mathcal{L}_{-\Diamond}$, by IH) iff $\vdash \psi' \to ([\theta']\eta \to [\theta']\varphi)$ (by K$_!$) iff $\vdash (\psi' \wedge [\theta']\eta) \to [\theta']\varphi$ iff $\vdash \psi \to [\theta]\varphi$ (where $\psi := \psi' \wedge [\theta']\eta \in \mathcal{L}_G$ and $\theta := \theta' \in \mathcal{L}_{-\Diamond}$).

Case $s := s', U$

$\vdash [s', U]\varphi$ iff $\vdash [s']U\varphi$ (by Defn. 10) iff $\vdash \psi' \to [\theta']U\varphi$ (for some $\psi' \in \mathcal{L}_G$ and $\theta' \in \mathcal{L}_{-\Diamond}$, by IH) iff $\vdash \psi' \to (\theta' \to U[\theta']\varphi)$ (by R$_U$) iff $\vdash (\psi' \wedge \theta') \to U[\theta']\varphi$ iff $\vdash E(\psi' \wedge \theta') \to [\theta']\varphi$ (pushing U back with its dual E, since U is an S5 modality) iff $\vdash \psi \to [\theta]\varphi$ ($\psi := E(\psi' \wedge \theta') \in \mathcal{L}_G$ and $\theta := \theta' \in \mathcal{L}_{-\Diamond}$).

Case $s := s', \rho$

$\vdash [s', \rho]\varphi$ iff $\vdash [s'][\rho]\varphi$ (by Defn. 10) iff $\vdash \psi' \to [\theta'][\rho]\varphi$ (by IH) iff $\vdash \psi' \to [\langle\theta'\rangle\rho]\varphi$ (by R$_{[!]}$) iff $\vdash \psi \to [\theta]\varphi$ (where $\psi := \psi' \in \mathcal{L}_G$ and $\theta := \langle\theta'\rangle\rho \in \mathcal{L}_{-\Diamond}$)

In each case, it is easy to see that $P_\psi \cup P_\theta \subseteq P_s$.

Proof of Lemma 19 (Lindenbaum's Lemma) The proof proceeds by constructing an increasing chain $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots \subseteq \Gamma_n \subseteq \ldots$ of witnessed theories, where $\Gamma_0 := \Gamma$, and each Γ_i is recursively defined. Since we have to guarantee that each Γ_i is witnessed, we follow a two-fold construction, where $\Gamma_0 = \Gamma_0^+ := \Gamma$. Let $\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots$ be an enumeration of all pairs of the form $\gamma_i = (s_i, \varphi_i)$ consisting of any $s_i \in NF$ and any formula $\varphi_i \in \mathcal{L}_G$. Let (s_n, φ_n) be the nth pair in the enumeration. We then set

$$\Gamma_n^+ = \begin{cases} \Gamma_n \cup \{\langle s_n\rangle\varphi_n\} & \text{if } \Gamma_n \nvdash \neg\, \langle s_n\rangle\varphi_n \\ \Gamma_n & \text{otherwise} \end{cases}$$

Note that the empty string ϵ is in NF and for every $\psi \in \mathcal{L}_G$ we have $\langle\epsilon\rangle\psi := \psi$ by the definition of possibility forms. Therefore, the above enumeration of pairs includes every formula ψ of $\mathcal{L}_G$ in the form of its corresponding pair (ϵ, ψ). By Lemma 16, each Γ_n^+ is witnessed. Then, if φ_n is of the form $\varphi_n := \Diamond\theta$ for some $\theta \in \mathcal{L}_G$, there exists a $p \in Prop$ such that Γ_n^+ is consistent with $\langle s_n\rangle\langle p\rangle\theta$ (since Γ_n^+ is witnessed). Similarly, if φ_n is of the form $\varphi_n := \langle G\rangle\theta$ for some $\theta \in \mathcal{L}_G$, there exists $\{p_i : i \in G\} \subseteq Prop$ such that Γ_n^+ is consistent with $\langle s_n\rangle\langle \bigwedge_{i\in G} K_i p_i\rangle\theta$. We then define

$$\Gamma_{n+1} = \begin{cases} \Gamma_n^+ & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\varphi_n \text{ and } \varphi_n \text{ is not of the form } \Diamond\theta \text{ or } \langle G\rangle\theta \\ \Gamma_n^+ \cup \{\langle s_n\rangle\langle p\rangle\theta\} & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\varphi_n \text{ and } \varphi_n := \Diamond\theta \text{ for some } \theta \in \mathcal{L}_G \\ \Gamma_n^+ \cup \{\langle s_n\rangle\langle \bigwedge_{i\in G} K_i p_i\rangle\theta\} & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\varphi_n \text{ and } \varphi_n := \langle G\rangle\theta \text{ for some } \theta \in \mathcal{L}_G \\ \Gamma_n & \text{otherwise} \end{cases}$$

where $p \in Prop$ and $\{p_i : i \in G\} \subseteq Prop$ such that Γ_n^+ is consistent with $\langle s_n\rangle\langle p\rangle\theta$ or consistent with $\langle s_n\rangle\langle \bigwedge_{i\in G} K_i p_i\rangle\theta$, respectively. Again by Lemma 16, it is guaranteed that each Γ_n is witnessed. Now consider the union $T_\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$. By Lemma 17, we know that T_Γ is a theory. To show that T_Γ is witnessed, first let $s \in NF$ and $\psi \in \mathcal{L}_G$ and suppose $\langle s\rangle\Diamond\psi$ is consistent with T_Γ. The pair $(s, \Diamond\psi)$ appears in the above enumeration of all pairs, thus $(s, \Diamond\psi) := (s_m, \varphi_m)$ for some $m \in \mathbb{N}$. Hence, $\langle s\rangle\Diamond\psi := \langle s_m\rangle\varphi_m$. Then, since $\langle s\rangle\Diamond\psi$ is consistent with T_Γ and $\Gamma_m \subseteq T_\Gamma$, we know that $\langle s\rangle\Diamond\psi$ is in particular consistent with Γ_m. Therefore, by the above construction, $\langle s\rangle\langle p\rangle\psi \in \Gamma_{m+1}$ for some $p \in Prop$ such that Γ_m^+ is consistent with $\langle s\rangle\langle p\rangle\psi$. Thus, as T_Γ is consistent and $\Gamma_{m+1} \subseteq T_\Gamma$, we have that $\langle s\rangle\langle p\rangle\psi$ is also consistent with T_Γ. Thus $\langle s\rangle\langle p\rangle\psi$ is also consistent with T_Γ for some $p \in Prop$. Now, let us check the witnessing condition for $\langle G\rangle$. Let $G \subseteq \mathcal{AG}$, $s \in NF$, and $\psi \in \mathcal{L}_G$ and suppose that $\langle s\rangle\langle G\rangle\psi$ is consistent with T_Γ. The pair $(s, \langle G\rangle\psi)$ appears in the above enumeration of all pairs, thus $(s, \langle G\rangle\psi) := (s_m, \varphi_m)$ for some $m \in \mathbb{N}$. Hence, $\langle s\rangle\langle G\rangle\psi := \langle s_m\rangle\varphi_m$. Then, since $\langle s\rangle\langle G\rangle\psi$ is consistent with T_Γ and $\Gamma_m \subseteq T_\Gamma$, we know that $\langle s\rangle\langle G\rangle\psi$ is in particular consistent with Γ_m. Therefore, by the above construction, $\langle s\rangle\langle \bigwedge_{i\in G} K_i p_i\rangle\psi \in \Gamma_{m+1}$ for some $\{p_i : i \in G\} \subseteq Prop$ such that Γ_m^+ is consistent with $\langle s\rangle\langle \bigwedge_{i\in G} K_i p_i\rangle\psi$. Thus, as T_Γ is consistent and $\Gamma_{m+1} \subseteq T_\Gamma$, we have that $\langle s\rangle\langle \bigwedge_{i\in G} K_i p_i\rangle\psi$ is also consistent with T_Γ. Hence, we conclude that T_Γ is witnessed. Finally, T_Γ is also maximal by construction: otherwise there would be a witness theory T such that $T_\Gamma \subsetneq T$. This implies that there exists $\varphi \in \mathcal{L}_G$ with $\varphi \in T$ but $\varphi \notin T_\Gamma$. Then, by the construction of T_Γ, we obtain $\Gamma_i \vdash \neg\varphi$ for all $i \in \mathbb{N}$. Therefore, since $T_\Gamma \subseteq T$, we have $T \vdash \neg\varphi$. Hence, since $\varphi \in T$, we conclude $T \vdash \bot$ (contradicting T being consistent).

Proof of Lemma 20 (Extension Lemma) Let $\theta \in \mathcal{L}_G$ and assume that $\{0, \theta\}$ is a theory. Moreover, let $\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots$ an enumeration of all pairs of the form (s_n, φ_n) consisting of any $s_n \in NF$, and every formula $\varphi_n \in \mathcal{L}_G$ of the form $\varphi_n := \Diamond\psi$ or $\varphi_n := \langle G\rangle\psi$ with $\psi \in \mathcal{L}_G$. We will recursively construct a chain of initial theories $\Gamma_0 \subseteq \ldots \subseteq \Gamma_n \subseteq \ldots$ such that

1. $\Gamma_0 = \{0, \theta\}$,
2. $P_n := \{p \in P : p \text{ occurs in } \Gamma_n\}$ is finite for every $n \in \mathbb{N}$, and
3. for every $\gamma_n := (s_n, \varphi_n)$ with $s_n \in NF$ and $\varphi_n \in \mathcal{L}_G$, if $\Gamma_n \nvdash \neg\langle s_n\rangle\varphi_n$ where $\varphi_n := \Diamond\psi$ then there is p_m "fresh" such that $\langle s_n\rangle\langle p_m\rangle\psi \in \Gamma_{n+1}$, and, if $\Gamma_n \nvdash \neg\langle s_n\rangle\varphi_n$ where $\varphi_n := \langle G\rangle\psi$ for some $G \subseteq \mathcal{A}$ then there is $\{p_{m_i} : i \in G\}$ where p_{m_i} is "fresh" for every $i \in G$ such that $\langle s_n\rangle\langle \bigwedge_{i\in G} K_i p_{m_i}\rangle\psi \in \Gamma_{n+1}$. Otherwise we will define $\Gamma_{n+1} = \Gamma_n$.

For every γ_n, let $P'(n) := \{p \in P' \mid p \text{ occurs either in } s_n \text{ or } \varphi_n\}$. Clearly every $P'(n)$ is always finite. We now construct an increasing chain of initial theories recursively. We set $\Gamma_0 := \{0, \theta\}$, and let

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\langle s_n\rangle\langle p_m\rangle\psi\} & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\varphi_n \text{ and } \varphi_n := \Diamond\psi \\ \Gamma_n \cup \{\langle s_n\rangle\langle \bigwedge_{i\in G} K_i p_{m_i}\rangle\psi\} & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\varphi_n \text{ and } \varphi_n := \langle G\rangle\psi \\ \Gamma_n & \text{otherwise,} \end{cases}$$

where m, m_i are, in each case, the least natural number greater than the indices in $P_n \cup P(n)$, i.e., p_m, p_{m_i} for all $i \in G$ are fresh in each case (since $P_n \cup P(n)$ is finite and $Prop$ is countably infinite, we always have enough fresh propositional variables). We now show that $\Gamma := \bigcup_{n\in\mathbb{N}} \Gamma_n$ is an initial witnessed theory. First show that Γ is a theory. By Lemma 17, it suffices to show by induction that every Γ_n is a theory. We are given that Γ_0 is a theory. For the inductive step suppose Γ_n is consistent but Γ_{n+1} is not. Hence, $\Gamma_n \neq \Gamma_{n+1}$ and moreover $\Gamma_{n+1} \vdash \bot$. Then, $\Gamma_{n+1} = \Gamma_n \cup \{\langle s_n\rangle\langle p_m\rangle\psi\}$ (when $\varphi_n := \Diamond\psi$) or $\Gamma_{n+1} = \Gamma_n \cup \{\langle s_n\rangle\langle \bigwedge_{i\in G} K_i p_{m_i}\rangle\psi\}$ (when $\varphi_n := \langle G\rangle\psi$). Here we will only check the latter case since the former case is analogous. Since $\Gamma_{n+1} = \Gamma_n \cup \{\langle s_n\rangle\langle \bigwedge_{i\in G} K_i p_{m_i}\rangle\psi\}$ we have $\Gamma_n \vdash [s_n][\bigwedge_{i\in G} K_i p_{m_i}]\neg\psi$. Therefore there exists $\{\theta_1, \ldots, \theta_k\} \subseteq \Gamma_n$ such that $\{\theta_1, \ldots, \theta_k\} \vdash [s_n][\bigwedge_{i\in G} K_i p_{m_i}]\neg\psi$. Let $\theta = \bigwedge_{1\leq i\leq k} \theta_i$. Then $\vdash \theta \to [s_n][\bigwedge_{i\in G} K_i p_{m_i}]\neg\psi$, so $\vdash [\theta\!\to, s_n][\bigwedge_{i\in G} K_i p_{m_i}]\neg\psi$ with $p_{m_i} \notin \mathrm{P}_\theta \cup \mathrm{P}_{s_n} \cup \mathrm{P}_{\varphi_n}$ for every $i \in G$. Thus, by the admissible rule in Lemma 14.2, we obtain $\vdash [\theta\!\to, s_n][G]\neg\psi$, i.e., $\vdash \theta \to [s_n][G]\neg\psi$. Therefore, $\theta \vdash \neg\langle s_n\rangle\langle G\rangle\psi$. Since $\{\theta_1, \ldots, \theta_k\} \subseteq \Gamma_n$, we therefore have $\Gamma_n \vdash \neg\langle s_n\rangle\langle G\rangle\psi$. But, this would mean $\Gamma_n = \Gamma_{n+1}$, contradicting our assumption (that $\Gamma_{n+1} \neq \Gamma_n$). Therefore Γ_{n+1} is consistent and thus a theory. Hence, by Lemma 17, Γ is a theory. Condition (3) above implies that Γ is also witnessed. Then, by Lindenbaum's Lemma (Lemma 19), there is a maximal witnessed theory T_Γ such that $\Gamma \subseteq T_\Gamma$. Moreover, since $0 \in \Gamma \subseteq T_\Gamma$, the set T_Γ is in fact a maximal witnessed initial theory.