Polyatomic Logics and Generalised Blok-Esakia Theory with Applications to Orthologic and KTB

MSc Thesis *(Afstudeerscriptie)*

written by

Rodrigo Nicolau Almeida
(born November 5 1995 in Tomar, Portugal)

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Date of the public defense:	**Members of the Thesis Committee:**
September 20 2022	prof. dr. Yde Venema (chair)
	dr. Nick Bezhanishvili (co-supervisor)
	dr. Tommaso Moraschini (co-supervisor)
	prof. dr. Wesley Holliday
	dr. Johannes Marti

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Abstract

This thesis presents a study of translations, special "hybrid" logical systems developed on the basis of these translations, and general Blok-Esakia theory. This is done on two levels: the development of a theoretical framework for analysing such questions, as well as an analysis of the special case of orthologic.

On the first front, inspired by the Kolmogorov Double Negation Translation, and DNA-Logics, we develop a general notion of "Polyatomic Logics", which can be studied for a wide class of translations. These logics serve as "hybrids", between the two logical systems being translated, which can assist in applications, as well as in the study of the interrelations between the translated systems. We develop the basic theory of Polyatomic Logics, and prove algebraic completeness, and Birkhoff-style definability theorems of such systems.

We then develop a theory connecting such logics with "generalised modal companions" – abstracting from the classic Blok-Esakia isomorphism, and taken to mean a strong and property-preserving connection between the extensions of two logical systems. This is contrasted with the famous Gödel-McKinsey-Tarski situation, where we show that many of the motivating results of that theory can be recovered for a class of translations we call "sober translations". Our main contribution in this respect is the introduction of the notion of a "Polyatomic Blok-Esakia isomorphism", which is shown to hold for any sober translation, and which provides a new natural correspondence between logical systems.

As a case study, we provide an analysis of the logic of ortholattices, and the Goldblatt translation of Orthologic into KTB modal logic. Our results show that many natural invariance conditons, including the Polyatomic Blok-Esakia introduced, fail for this setting. We undertake a study of the reasons for this failure, and analyse whether restricted versions of it might hold. With this goal in mind, we introduce a new duality between a subcategory of the category of ortholattices, and a subcategory of the category of orthospaces. This representation is shown to have desirable category-theoretic properties, which we use to identify appropriate expansions of orthologic and KTB. With these tools, we prove the existence of a Polyatomic Blok-Esakia isomorphism between "Orthoimplicative Logic" and "Sober KTB".

Keywords: Polyatomic Logics, Blok-Esakia theory, Orthologic

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Contents

Chapter 1

Introduction

In the 20th century, classical logic, intuitionistic logic, modal logic, provability logic, and quantum logic, were some of many logical calculi introduced to solve questions such as what counts as a valid proof, or a valid inference in an experimental setting. Rather than wholly independent formalisms, it was noted soon after their creation that many of these systems were intimately related. Early examples included:

- *Double Negation Translations*, introduced by Kolmogorov [45], and developed by Gödel and Gentzen [31], which we refer to as the Kolmogorov-Gödel-Gentzen translation (KGG) which translated classical logic, CPC into intuitionistic logic, IPC.
- The *Gödel-McKinsey-Tarski translation* (GMT) [33], which translated intuitionistic logic into $\mathsf{S4}$ modal logic.
- The *Goldblatt translation* [35], which translated orthologic into KTB modal logic.

All of these share the fact that they are sound and faithful translations, and that they contain a syntactic part - the actual translation - coupled with a semantic part - a transformation of the underlying models. For instance, in the KGG translation, given a Heyting algebra $\mathbf{H}$, one considers those elements $a = \neg\neg a$, called *regular elements*. Then we can look at the set

$$\mathbf{H}^{\neg} = \{a \in H : a = \neg\neg a\}.$$

When equipped with the meet, implication and bounds of H, as well as the operation $x \dot{\vee} y := \neg\neg(x \vee y)$, yields a Boolean algebra. We call Heyting algebras $\mathbf{H}$ such that $\mathbf{H}$ is generated by its regular elements *regularly generated*.

Such translations have found applications in logic, mathematics, philosophy and linguistics, through their capacity to hybridise different domains. In this setting, the KGG translation appears as a paradigmatic example. The key idea in applications (see e.g. [43] or [16]) is that one has a set of "constructive worlds", which should be run with intuitionistic principles, together with a set of "ideal worlds", which should be run with classical principles. Thus, the double negation serves as a way to go from constructive to ideal worlds.

Motivated by this interpretation, Ciardelli et al. [16] introduced a specific logic to model questions, where the prototypical model was a regularly generated Heyting algebra. Its success later lead to the development of a general analysis of such models [5], and *DNA-logics* (named for "Double

Negation on Atoms") in [5, 6], which vastly abstract this situation: given any superintuitionistic logic L, one can take the Heyting algebras validating L and consider:

$$\mathsf{RegVar}(L) = \{H : H \text{ is a regularly generated Heyting algebra and } H \vDash L\}.$$

Then consider the formulas, φ, such that: whenever $H \in \mathsf{RegVar}$, and $v : \mathsf{Prop} \to H$ is a valuation taking values in regular elements, $H, v \vDash \varphi$. Such a collection of formulas is called the *DNA-variant* of L. Such "logics" have the feature that they are not closed under uniform substitution, and can be axiomatised over L by closing under Modus Ponens the logic L together with the axiom on propositional variables:

$$\neg\neg p \to p.$$

The intuition behind this is worthwhile to spell out. Intuitionistic models have the property that each world sees a classical world. If $\neg\neg p$ holds in a world, this means that each classical world it sees is p – even if at the present world, p cannot be asserted. To say that one takes values in regular models, means that whenever $\neg\neg p$ holds, then p holds already. However, more complex statements such as $p \vee q$ might not be settled yet, as we can still conceive of worlds where p and not q holds and worlds where q and not p holds.

Such logics provide new domains for analysing the relationship between classical and intuitionistic logic, as well as influence new potential applications. However, through consideration of this discussion, one can be lead to conclude that this structure does not depend entirely on properties of Heyting algebras and the KGG translation. Indeed, the work was further extended to other translations by Grilletti and Quadrellaro [38], and similar subjects, aiming towards more general classes of logics, have been explored in unpublished work by Quadrellaro.

To see the breadth of these ideas, let us quickly turn to the GMT translation. We recall the semantic transformation in the GMT translation: here, given $\mathsf{S4}$ algebra $(B, \Box)$, an element a is called *open* if $a = \Box a$. Then we consider:

$$B_{\Box} := \{a : a = \Box a\}.$$

Equipping this set with the induced operations from the distributive lattice as well as $a \Rightarrow b := \Box(\neg a \vee b)$, we obtain a Heyting algebra. In analogy with the situation above, given a normal extension of $\mathsf{S4}$, M, we consider the set of formulas which are valid on all openly generated M-algebras, when the atomic propositions are sent to open elements. We could call this logic $M^{\Box}$ the $\Box$-*variant* of M. Such logics might be naturally considered under a provability interpretation: one could think of hybridising contextual facts, as the domain of $\mathsf{S4}$, with provable facts, as the domain of IPC. The fact that $p \to \Box p$ should be added as an axiom on atomic propositions would mean that our basic sentences (e.g., axioms), should all be provable, whereas their interrelations might not be immediately provable. For example, the sentence $p \to q$ might be true, meaning that whenever we observe p we obtain q, without $\Box(p \to q)$ being so.

As we will have opportunity to see, similar arguments can be made for the Goldblatt translation, since it works semantically by taking the $\Box\Diamond$-fixed points. Translations with a similar selector term are ubiquitous: we have translations from orthomodular lattices to residuated ortholattices [27]; reflexivisation translations [14, Chapter 4] in modal logic; as well as numerous substructural logics into modal systems [40]. For a great number of these, as we will see, an appropriate notion of a "hybrid" logic could be developed, but it certainly does not seem reasonable to do it one-by-one.

Hence, a natural question arises as to what conditions are needed on a translation to ensure that one can construct such a logic, and how this can be developed in general.

One of the contributions of this thesis lies in developing a theory of "Polyatomic Logics". This provides a broad generalisation of DNA-logics, to the setting of many naturally ocurring translations. This is done in Chapter 3, inspired by the work of Moraschini [53], which identifies the concept of a translation with categorical adjunction. In it, we identify the class of *selective translations* as the appropriate setting for Polyatomic logics (PAt-logics). In this sense, and in analogy with [6], we prove algebraic completeness and definability theorems for these logics with respect to appropriate (quasi-)equational classes and (quasi-)varieties. In analogy with the above situation, we also obtain completeness of these logics with respect to regularly generated algebras. We then turn to one of the interesting applications of such a theory, which takes up the rest of the thesis – the development of a general Blok-Esakia theory.

To understand this, let us return to the GMT translation. Gödel established that this translation was sound and faithful. Later work proved that the same translation also translated IPC into extensions of S4, namely the system S4.Grz. This lead to the idea of a "modal companion": given a normal extension of S4, $M \in \mathbf{NExt}(\mathsf{S4})$, and L a superintuitionistic logic, we say that M is a *modal companion* of L iff:

$$\varphi \in L \iff GMT(\varphi) \in M$$

where $GMT(\varphi)$ is the translation of φ under the GMT translation. Developments of these ideas eventually lead to the celebrated Blok-Esakia theorem, proved independently by Blok ([10]) and Esakia ([24], see also [23]). This result establishes an *isomorphism* between the lattice of superintuitionistic logics, and the lattice $\mathbf{NExt}(\mathsf{S4.Grz})$ of normal extensions of S4.Grz. Relevantly, this isomorphism carries with it explicit maps transforming the semantic models of the respective systems. This allows one to transfer a number of properties - such as FMP, tabularity, Kripke completeness, decidability, canonicity, amongst many others - from axiomatic extensions of S4.Grz to axiomatic extensions of IPC, and for some of these to be transfered back (see [14] for a survey of these). This further allows a number of methods to be developed uniformly between the two systems, and has highlighted, through what *does not* transfer (e.g., local tabularity, Craig interpolation), the key differences of intuitionistic versus modalised classical logic. This area of research concerning the interplay between S4.Grz and IPC can be generally described as "Blok-Esakia theory".

Given the success and acclaim of this theory, it would be desirable to understand whether, and when, one can expect similar results in the interplay between two systems. In Chapter 4, we explore this question, armed with the tools developed before. In this sense, we outline the natural generalisations of the concepts present in Blok-Esakia theory, and prove some basic properties of these. We then make use of our PAt-logics to give meaning to these concepts, and show that the study of Blok-Esakia theory can be appropriately conducted through looking at Polyatomic logics. This is done by showing that the general concepts of Blok-Esakia theory – such as greatest or least companions – correspond exactly with natural concepts from Polyatomic logic. Additionally, we introduce the concept of a PAt-Blok Esakia isomorphism, which holds when the lattice of logics of the translated system is isomorphic to the lattice of Polyatomic logics in the other system.

This approach thus bypasses the famous "Blok's lemma", which establishes that every S4.Grz logic is sound and complete with respect to its openly generated algebras, and which establishes the isomorphism in full. In fact, via these tools, all Polyatomic logics are sound and complete with respect to their regularly generated algebras; hence, from the point of view of two systems for which a PAt-Blok Esakia theorem holds, the analogue of Blok's Lemma becomes a question of axiomatising the greatest logic having the same Polyatomic variant. This explains both its importance and its general difficulty: for instance, the greatest logic having the same DNA-variant as IPC is the well-known Medvedev Logic, a logic which is not known to have a recursive axiomatisation.

As an illustration of these methods, in Chapter 5 we take up orthologic and the Goldblatt translation. This seems like a natural use case, since it fits pretty closely to the other translations in its style, and relates systems which have long been known, but not extensively studied. A recent revival of interest in questions related to ortholattices as related to residuated structures [27] as well as modal logics [41] makes this a timely addition to studies of the topic. And importantly, this is a case which has a long history of having been signalled as potentially having some analogue of the Blok-Esakia theory [35, 18, 52, 17, 49]. Making use of the tools developed in Chapter 4, we thus proceed to investigate this case, and prove that all currently considered types of Blok-Esakia isomorphisms fail for this setting. Not only is an isomorphism impossible between "orthologics" and KTB logics, but the Polyatomic Blok-Esakia isomorphism also fails.

This raises further questions about the ways in which systems can fail to be structurally similar, even when their semantics are superficially very similar. This theme is taken up in Chapter 6. Whereas in the IPC and S4 case, the key difficulties have solely to do with regular generation of algebras, and the underlying axiomatisation, the Goldblatt translation faces other issues which we deem *sobriety problems*. In an abstract way, this can be seen as a mismatch between a more restrictive semantics of orthologic, and a broader semantics of KTB, which requires the models to be violently transformed in order to witness the translation. To fix this situation, we propose two new classes, to feature in an adapted Goldblatt translation:

- A new class of algebraic structure of ortholattices, called orthoimplicative systems, and a new logic system, called *Orthoimplicative Logic*, which is a conservative extension of Orthologic.
- A suitable fragment of KTB, called *sober* KTB, which includes an additional non-standard Π_2-rule [2, 3] which is admissible for many interesting cases.

These developments allow us to prove a PAt Blok-Esakia isomorphism between these logical systems, which vindicate the stated intuitions that orthologics and KTB-logics should be related in a deeper manner than simply through a translation. The latter theorem requires the development of a number of topological, logical and algebraic tools, which might have independent interest.

In Chapter 7 we conclude the work, providing a brief summary of the findings as well outlining some further work. We also include some some remarks regarding work that, for reasons of space and coherence, we could not include here.

Our main contributions in this thesis can be summarised as follows:

- We introduce the notion of "selective translation", "strongly selective translation", and "sober translations", and study their basic properties;
- We introduce the concept of PAt-logics and prove algebraic completeness for them;
- We develop a generalised Blok-Esakia theory for strongly selective and sober translations, and show that PAt-logics are adequate structures for the development of such a theory;
- We introduce the concept of a "Polyatomic Blok-Esakia isomorphism";
- We prove that no isomorphism can hold between the lattices of Orthologics and KTB modal logic can hold, and the Polyatomic Blok-Esakia isomorphism fails;

- We introduce the notion of "slim orthospace", and provide a duality between a subcategory of the category of ortholattices and the category of slim orthospaces;
- We show that the above duality realises in the ortholattice case the "Distributivisation" functor;
- We introduce the concept of "Orthoimplicative System", and "Sober KTB algebra", providing a non-standard axiomatisation of them;
- We translate these results into logic, obtaining Orthoimplicative Logics and Sober KTB logics, and showing that these are logically conservative over orthologics and KTB logics for a variety of cases;
- We prove that a Polyatomic Blok-Esakia isomorphism holds between the lattices of extensions of Orthoimplicative Logic and Sober KTB logic.

Chapter 2

Preliminaries

In this chapter we fix notation, and discuss some essential concepts that will be needed throughout the thesis. Basic mathematical terminology, as well as familiarity with first order logic is assumed throughout.

2.1 Lattices and Ordered Sets

In this chapter we present a number of different lattice-based algebras.

Definition 2.1.1. Let (X, R) be a set with a relation $R \subseteq X \times X$. We say that (X, R) is a *quasi-ordered set* if R is:

- *Reflexive*: for all $a \in X$, aRa
- *Transitive*: for all $a, b, c \in X$, aRb and bRc implies aRc.

We say that (X, R) is *partially ordered* (a *poset*) if in addition, R is antisymmetric:

- *Antisymmetric*: for all $a, b \in X$ if aRb and bRa then $a = b$.

When (X, R) is a partially ordered set, we often write $\leqslant$ instead of R. We also use $<$ and $\geqslant$ as abbreviations with their usual meaning.

Definition 2.1.2. Let $(X, \leqslant)$ be a poset. Given a subset $\{a, b\} \subseteq X$, we say that $c \in X$ is a *lower bound* of $\{a, b\}$ if $c \leqslant a$ and $c \leqslant b$. We say that c is a *greatest lower bound* if it is a lower bound of $\{a, b\}$, and whenever $d \leqslant a$ and $d \leqslant b$ then $d \leqslant c$. We define upper bounds and least upper bounds dually.

Definition 2.1.3. Let $(X, \leqslant)$ be a poset. We say that $(X, \leqslant)$ is a *lattice* if for every pair $a, b \in X$, $\{a, b\} \subseteq X$ has a greatest lower bound and a least upper bound. In this case we denote:

$$\begin{aligned} a \wedge b &:= \text{ greatest lower bound of } \{a, b\} \text{ and} \\ a \vee b &:= \text{ least upper bound of } \{a, b\} \ . \end{aligned}$$

We say that furthermore $(X, \leqslant)$ is *bounded* if it has a least element (which we denote by 0) and a greatest element (which we denote by 1). We say that $(X, \leqslant)$ is a *complete lattice* if for each subset $A \subseteq X$, A has a greatest lower bound in X, denoted by $\bigwedge A$.

It is a general fact that lattices can be given an equational presentation (see for instance [19, Chapter 2]). In this case, a lattice $\mathbf{L}$ is understood as an algebraic structure $\mathbf{L} = (L, \wedge, \vee)$, in the language with $\wedge, \vee$, which satisfies the following for each $a, b, c \in L$:

- (Idempotence) $a \wedge a = a$ and $a \vee a = a$;
- (Commutativity) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$;
- (Associativity) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$;
- (Absorption) $a \wedge (b \vee a) = a$ and $a \vee (b \wedge a) = a$.

Definition 2.1.4. Let $\mathbf{L} = (L, \wedge, \vee, 0, 1)$ be a bounded lattice. We say that L is *distributive* if for each $a, b, c \in L$:

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c).$$

A majority of lattices that appear in studies related to algebraic logic tend to be distributive. In this thesis, however, one of the main objects of study will be a kind of lattice which is not in general distributive:

Definition 2.1.5. An algebra $\mathbf{O} = (O, \wedge, \vee, ^{\perp}, 0, 1)$ is said to be an *ortholattice* when $(O, \wedge, \vee, 0, 1)$ is a bounded lattice, and $^{\perp}$ satisfies the following properties for every $a, b \in O$:

1. $(a \wedge b)^{\perp} = a^{\perp} \vee b^{\perp}$ and $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$;
2. $a \wedge a^{\perp} = 0$ and $a \vee a^{\perp} = 1$
3. $(a^{\perp})^{\perp} = a$.

Notice that distributive ortholattices coincide with *Boolean algebras* (see below). However, Figure 2.1, depicts an ortholattice which is not a Boolean algebra[1].

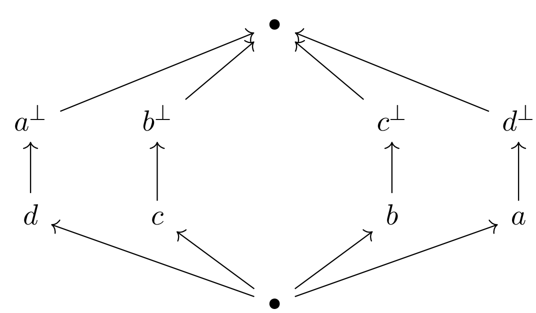


Figure 2.1: Example of an ortholattice

We also consider classes of lattices with various kinds of implications:

Definition 2.1.6. An algebra $\mathbf{H} = (H, \wedge, \vee, \to, 0, 1)$ is said to be a *Weak Heyting algebra* when $(H, \wedge, \vee, 0, 1)$ is a bounded distributive lattice, and $\to$ satisfies, for each $a, b \in H$:

1. $a \to a = 1$;

[1] Simply note that the Pentagon lattice N_5 embeds into the lattice, which means the lattice is not distributive.

2. $(a \vee b) \to c = a \to c \wedge b \to c$;

3. $a \to (b \wedge c) = a \to b \wedge a \to c$;

4. $a \to b \wedge b \to c \leqslant a \to c$.

A special class of Weak Heyting algebras are their namesake:

Definition 2.1.7. An algebra $\mathbf{H} = (H, \wedge, \vee, \to, 0, 1)$ is said to be a *Heyting algebra* if it is a Weak Heyting algebra, and additionally it satisfies, for each $a, b \in H$:

1. $a \wedge (a \to b) \leqslant b$;

2. $a \leqslant 1 \to a$.

The former are more usually presented through the so-called *residuation laws*: Heyting algebras are bounded distributive lattices with a binary implication operation $\to$ satisfying, for each $a, b, c \in H$:

$$c \wedge a \leqslant b \iff c \leqslant a \to b.$$

A special class of Heyting algebras which has a ubiquitous presence in logic is that of *Boolean algebras*:

Definition 2.1.8. An $\mathbf{B} = (B, \wedge, \vee, \to, 0, 1)$ is said to be a *Boolean algebra* if it is a Heyting algebra which additionally satisfies for every $a \in B$, $a \vee (a \to 0) = 1$.

The following is a well-known equivalence;

Proposition 2.1.9. Let $\mathbf{B} = (B, \wedge, \vee, \to, 0, 1)$ be a Heyting algebra. Then the following are equivalent:

- $\mathbf{B}$ is a Boolean algebra;
- $\mathbf{B} = (B, \wedge, \vee, \neg, 0, 1)$, where $\neg a := a \to 0$, is a distributive ortholattice.

Hence, we have that Heyting algebras and ortholattices constitute distinct generalisations of Boolean algebras, emphasising different phenomena. We usually present Boolean algebras in the language of ortholattices.

Intimately related to Weak Heyting algebras are specific classes of Boolean algebras with operators:

Definition 2.1.10. An algebra of type $\mathbf{B} = (B, \wedge, \vee, \neg, \Box, 0, 1)$ is said to be a *modal algebra* if $(B, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra, and $\Box$ satisfies, for each $a, b \in B$:

1. $\Box 1 = 1$;

2. $\Box(a \wedge b) = \Box a \wedge \Box b$.

We additionally say that $\mathbf{B}$ is a:

1. T-algebra, if it satisfies for $a \in B$, $\Box a \leqslant a$;

2. *Transitive modal algebra* if it satisfies for $a \in B$, $\Box a \leqslant \Box\Box a$;

3. *Symmetric modal algebra* if it satisfies for $a \in B$, $a \leqslant \Box\Diamond a$.

Where $\Diamond a = \neg\Box\neg a$. We say that **B** is an *interior algebra* or an *S4 algebra*, if it is both a T-algebra and a transitive modal algebra. We say that **B** is a *KTB*-algebra if it is both a T-algebra and a symmetric modal algebra.

We conclude this section by discussing some concepts that appear naturally when discussing lattices:

Definition 2.1.11. Let L be a lattice, and $F \subseteq L$ a subset. We say that F is:

- *Upwards closed* if whenever $x \in F$ and $x \leqslant y$ then $y \in F$;
- *Downwards closed* if whenever $y \in F$ and $x \leqslant y$ then $x \in F$;
- $\bigcirc$-*closed* for $\bigcirc \in \{\wedge, \vee\}$ if whenever $x, y \in F$ then $x \bigcirc y \in F$;
- A *filter* if it is non-empty, upwards closed and $\wedge$-closed;
- An *ideal* if it is non-empty, downwards closed and $\vee$-closed;
- A *prime filter* if it is a filter, and whenever $a \vee b \in F$, then either $a \in F$ or $b \in F$;
- A *prime ideal* if it is an ideal, and whenever $a \wedge b \in F$ then either $a \in F$ or $b \in F$.

We note that given a bounded lattice **L**, the set of downwards closed sets on **L**, the set of filters on **L**, and the set of ideals on **L** are all closed under arbitrary intersection. Hence, given $S \subseteq L$, we write:

- $\downarrow S \subseteq L$ for the smallest downwards closed set containing S. It is known that $\downarrow S = \{a \in L : \exists b \in S, a \leqslant b\}$;
- $\uparrow S \subseteq L$ for the smallest upwards closed set containing S. It is known that $\uparrow S = \{a \in L : \exists b \in S, b \leqslant a\}$;
- $\mathsf{Fil}(S) \subseteq L$ for the smallest filter containing S. It is known that $\mathsf{Fil}(S) = \{a \in L : \exists b_0, ..., b_n \in S, b_0 \wedge ... \wedge b_n \leqslant a\}$;
- $\mathsf{Id}(S) \subseteq L$ for the smallest ideal containing S. It is known that $\mathsf{Id}(S) = \{a \in L : \exists b_0, ..., b_n \in S, a \leqslant b_0 \vee ... \vee b_n\}$

In Boolean algebras, due to the presence of a negation, one can also consider the following:

Definition 2.1.12. Let B be a Boolean algebra. We say that $F \subseteq B$ is an *ultrafilter* if for all $a \in B$, either $a \in F$ or $\neg a \in F$.

As is well-known, in Boolean algebras, ultrafilters are exactly the maximal filters, and also the prime filters.

2.2 Duality Theory

In this section we briefly recall Stone, Priestley and Esakia dualities. We assume the reader is familiar with the notion of a topological space, and basic concepts of general topology. For references on general topology see [22]. For specific references on these dualities, see [19].

Definition 2.2.1. Let (X, τ) be a topological space. We say that (X, τ) is *zero-dimensional* if τ has a basis of clopen sets.

Definition 2.2.2. Given (X, τ) a topological space, we say that it is a *Stone space*, or a *Boolean space*, if it is a *Hausdorff, compact and zero-dimensional space.*

Denote by $\mathsf{Clop}(X)$ the collection of clopen sets. Then we have:

Proposition 2.2.3. If X is a topological space, then the structure $(\mathsf{Clop}(X), \cap, \cup, -, \varnothing, X)$ is a Boolean algebra.

Given a Boolean algebra, we can look at its *spectrum*, that is, the set of its ultrafilters, $\mathsf{Spec}(B)$. On the spectrum we can define a topology by declaring the following sets to be subbasic opens:

$$\varphi(a) = \{x \in \mathsf{Spec}(B) : a \in x\} \text{ for } a \in B$$

Call the resulting topology τ_B. Then:

Proposition 2.2.4. If B is a Boolean algebra, the structure $(\mathsf{Spec}(B), \tau_B)$ is a Boolean space.

These transformations are moreover inverse of one another. This is captured by the following duality result:

Theorem 2.2.5. *(Stone Duality) The category* **BA** *of Boolean algebras with Boolean homomorphisms, and the category* **BS** *of Boolean spaces with continuous maps, are dually equivalent.*

As a development of this, Priestley found a similar representation for the wider class of *bounded distributive lattices*. To see this, recall that we say that a structure $(X, \leqslant, \tau)$ is an *ordered topological space* if $(X, \leqslant)$ is a partially ordered set, and (X, τ) is a topological space. Denote by $\mathsf{ClopUp}(X)$ the class of clopen upwards closed sets.

Definition 2.2.6. Let $(X, \leqslant, \tau)$ be an ordered topological space. We say that X is *totally order-disconnected* if it satisfies the *Priestley Separation Axiom*: whenever $x \nleqslant y$, there is some $U \in \mathsf{ClopUp}(X)$ such that $x \in U$ and $y \notin U$.

We say that $(X, \leqslant, \tau)$ is a *Priestley space* if it is a compact totally order-disconnected space.

Similar to before, given such a space, we have that:

Proposition 2.2.7. If $(X, \leqslant, \tau)$ is a a Priestley Space, the structure $(\mathsf{ClopUp}(X), \cap, \cup, \varnothing, X)$ is a bounded distributive lattice.

Given a bounded distributive lattice L, we can likewise look at $\mathsf{Spec}(L)$ the space of *prime filters* of L, which come equipped with the natural inclusion order, and consider the subbasis induced by:

$$\{\varphi(a) : a \in L\} \cup \{\mathsf{Spec}(L) - \varphi(a) : a \in L\} \text{ for } a \in L$$

Denote again the resulting topology by τ_L:

Proposition 2.2.8. Given a bounded distributive lattice L, $(\mathsf{Spec}(L), \subseteq, \tau_L)$ is a Priestley space.

Hence we obtain the following (see [19, pp.262]):

Theorem 2.2.9. *(Priestley Duality) The categories* **DLat***, of bounded distributive lattices and bounded lattice homomorphisms, and* **Pries***, of Priestley spaces and order preserving continuous functions are dually equivalent.*

Note that Priestley duality specialises to Stone duality on Boolean algebras. There, the order becomes the identity, since as noted above, in Boolean algebras, prime filters are exactly the ultrafilters. A special case of Priestley duality is that which covers Heyting algebras:

Definition 2.2.10. Let $(X, \leqslant, \tau)$ be a Priestley space. We say that X is an *Esakia space* if whenever U is a clopen set, then $\downarrow U$ is also clopen.

Given an Esakia space, we define the Heyting implication using the downwards closure operator. Define the following for U a subset of $(X, \leqslant)$:

$$\Box_{\downarrow} U := \{x : \forall y \text{ if } x \leqslant y \text{ then } y \in U\}$$

Then define for two such subsets:

$$U \implies V := \Box_{\downarrow}(X - U \cup V)$$

Then we have:

Proposition 2.2.11. If $(X, \leqslant, \tau)$ is an Esakia space, then $(\mathsf{ClopUp}(X), \cap, \cup, \implies, \varnothing, X)$ is a Heyting algebra.

If H is a Heyting algebra, $(\mathsf{Spec}(H), \subseteq, \tau_H)$ is an Esakia space.

The final piece of structure concerns the morphisms of Esakia spaces:

Definition 2.2.12. Let $f : X \to Y$ be a map between two ordered sets. We say that f is a p-morphism if:

- Whenever $x \leqslant y$ then $f(x) \leqslant f(y)$;
- If $f(x) \leqslant y$ then there exists some z such that $x \leqslant z$ and $f(z) = y$.

Then we have (see [23, Corollary 3.4.8]):

Theorem 2.2.13. *(Esakia duality) The categories* **HA***, of Heyting algebras and Heyting algebra homomorphisms, and* **ES***, of Esakia spaces and continuous p-morphisms, are dually equivalent.*

A final natural duality we will have to discuss is that between modal algebras and so called *modal spaces*. This is a topological extension of Jonsson-Tarski duality, and is discussed extensively in [9, Chapter 5].

Definition 2.2.14. Let $X = (X, R, \tau)$ be a Stone space equipped with a relation R satisfying:

- (Point-Closedness) $R[x]$ is closed for each $x \in X$.
- (Clopen Closure) Whenever U is a clopen, then $R^{-1}(U)$ is clopen.

The key idea to obtain this is extending Stone duality with a relation $R_\Box$ induced on the ultrafilters as follows:

$$xRy \iff \Box a \in x \implies a \in y$$

In such a situation, recall that we denote by $\Box_R$ the following naturally induced operator:

$$\Box_R U = \{x : \forall y, \text{ if } xRy \text{ then } y \in U\}$$

Then we have:

Proposition 2.2.15. Given a modal algebra $(B, \Box)$, we have that $(X_B, R_\Box, \tau)$ is a modal space, and $(\mathsf{Clop}(X_B), \cap, \cup, \neg, \Box_{R_\Box}, \varnothing, X_B)$ is isomorphic to $(B, \Box)$.

2.3 Universal Algebra

For general references on universal algebra the reader can consult [12, 1].

Definition 2.3.1. We say that $\mathcal{L} = (F, ar)$ (also sometimes denoted as σ) is a *language* if F is a collection of symbols, and $ar : F \to \omega$ is a function, called the *arities* of F.

Given a language $\sigma = (F, ar)$, we say that a structure $\mathbf{A} = (A, \sigma)$ is an *algebra of type* σ if for each $g \in F$, such that $ar(g) = n$, there is a function $g : A^n \to A$.

We say that two structures $\mathbf{A}$ and $\mathbf{B}$ are *similar* if they have the same type.

Definition 2.3.2. Let $\mathbf{A} = (A, \sigma)$ and $\mathbf{B} = (B, \sigma)$ be two algebras. A map $f : \mathbf{A} \to \mathbf{B}$ is called a *homomorphism* if for each n-ary function symbol $g(x_0, ..., x_n) \in \sigma$, and $a_0, ..., a_n \in A$

$$f(g(a_0, ..., a_n)) = g(f(a_0), ..., f(a_n))$$

We say that an injective homomorphism is an *embedding*.

Definition 2.3.3. Given two algebras $\mathbf{A}$ and $\mathbf{B}$ of type σ, and a homomorphism $f : \mathbf{A} \to \mathbf{B}$ we say that:

- $\mathbf{A}$ is a *subalgebra* of $\mathbf{B}$ if $A \subseteq B$ and the inclusion map $i : A \to B$ is a homomorphism; we denote this by $\mathbf{A} \preceq \mathbf{B}$.
- $\mathbf{B}$ is a *homomorphic image* of $\mathbf{A}$ if f is surjective.
- $\mathbf{B}$ is *isomorphic* to $\mathbf{A}$ if f is injective and surjective.

Definition 2.3.4. Let $\mathbf{A}$ be an algebra of type σ. Let θ be an equivalence relation $\theta \subseteq A \times A$. We say that θ is a *congruence* if for all n-ary function symbols $g(x_0, ..., x_n) \in \sigma$, and $(a_0, b_0), ..., (a_n, b_n) \in \theta$:

$$(g(a_0, ..., a_n), g(b_0, ..., b_n)) \in \theta$$

We denote by $\mathsf{Con}_\mathbf{A}$ the set of congruences of an algebra $\mathbf{A}$.

We recall the following fact about congruences:

Definition 2.3.5. Given an algebra $\mathbf{A}$ in the language σ, $\mathsf{Con}_\mathbf{A}$ forms a complete lattice, where meets are given by intersection.

Given an algebra $\mathbf{A}$, and a collection $S \subseteq A \times A$, we denote by $\mathsf{Cg}(S)$ the smallest congruence containing S.

Definition 2.3.6. Given an algebra $\mathbf{A}$ and a congruence θ, we say that $(\mathbf{B}, \sigma)$ is the *quotient algebra of A under θ* if:

- $B = A/\theta$, i.e., the carrier set of B is the quotient set under θ.
- For each $g \in \sigma$, and $a_0, ..., a_n \in A$, $g^\theta(a_0/\theta, ..., a_n/\theta) = g(a_0, ..., a_n)/\theta$

Definition 2.3.7. Let $(\mathbf{A}_i)_{i \in I}$ be a family of algebras. We say that an algebra $\mathbf{B}$ is the *direct product* of the family $(A_i)_{i \in I}$ if:

- $B = \prod_{i \in I} A_i$, i.e., the carrier set of B is the cartesian product
- For each $g \in \sigma$, and each i:

$$g(a_0, ..., a_n)(i) = g(a_0(i), ..., a_n(i))$$

Definition 2.3.8. Let $(\mathbf{A}_i)_{i \in I}$ be a family of algebra of type σ, and $\mathbf{B}$ an algebra of type σ. We say that $\mathbf{B}$ is a *subdirect product* of the family $(\mathbf{A}_i)_{i \in I}$ if there is an embedding $f : B \to \prod_{i \in I} A_i$ which additionally satisfies: for each i, $\pi_i \circ f$ is surjective, i.e., f is surjective on factors.

Especially important in the theory of quasivarieties is the notion of reduced product:

Definition 2.3.9. Let $(A_i)_{i \in I}$ be a family of algebras. Let F be a filter on $\mathcal{P}(I)$, the power set algebra of I. Define an equivalence relation, $\sim_F$, by saying that given $a, b \in \prod_{i \in I} A_i$, $a \sim_F b$ if and only if:

$$[\![a = b]\!] = \{i \in I : a(i) = b(i)\} \in F$$

Then $\sim_F$ is a congruence on $\prod_{i \in I} A_i$. We denote by $\prod_{i \in I} A_i/F$ the quotient algebra $\prod_{i \in I} A_i/ \sim_F$. We call this algebra the *reduced product* via F of the $(A_i)_{i \in I}$.

Definition 2.3.10. Let $\mathbf{A}$ be an algebra. We say that $\mathbf{A}$ is *subdirectly irreducible* if for all families $(\mathbf{B}_i)_{i \in I}$ such that $\mathbf{A}$ is a subdirect embedding of this collection, there is some i such that $\mathbf{A} \cong \mathbf{B}_i$.

The following is known as Birkhoff's Subdirect Decomposition Theorem (see [12, Chapter 2, Theorem 8.6]):

Theorem 2.3.11. *Every algebra $\mathbf{A}$ is isomorphic to a subdirect product of subdirectly irreducible algebras.*

Definition 2.3.12. Let $\mathbf{A}$ be an algebra. We say that $\mathbf{A}$ is *trivial* if A is a singleton.

Definition 2.3.13. We say that an algebra $\mathbf{A}$ is *simple* if whenever $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$, then $\mathbf{B}$ is isomorphic to $\mathbf{A}$.

For a class of similar algebras $\mathbf{K}$, we recall the following monotone and idempotent operators:

1. $\mathbb{I}(\mathbf{K})$ - isomorphic copies of algebras in $\mathbf{K}$;
2. $\mathbb{H}(\mathbf{K})$ - homomorphic images of algebras in $\mathbf{K}$;

3. $\mathbb{S}(\mathbf{K})$ - subalgebras of algebras in $\mathbf{K}$;

4. $\mathbb{P}(\mathbf{K})$ - direct products of algebras in $\mathbf{K}$;

5. $\mathbb{P}_U(\mathbf{K})$ - ultraproducts of algebras in $\mathbf{K}$;

6. $\mathbb{P}_R(\mathbf{K})$ - reduced products of algebras in $\mathbf{K}$.

7. $\mathbb{P}_S(\mathbf{K})$ - subdirect products of algebras in $\mathbf{K}$.

Definition 2.3.14. Let $\mathbf{K}$ be a class of algebras. Then we say that $\mathbf{K}$ is:

1. A *variety* if it is closed under subalgebras, homomorphic images and direct products;

2. A *quasi-variety* if it is closed under isomorphisms, subalgebras, direct products and ultraproducts; equivalently, if it is closed under isomorphic images, subalgebras, and reduced products.

We denote by $\mathbb{V}$, respectively $\mathbb{Q}$, the variety/quasivariety generator operator. The following are sometimes referred to as respectively as *Tarski's HSP* and *Mal'tsev's ISP$_R$* theorems (see respectively [12, Chapter 2, Theorem 9.5], [12, Chapter 5, Theorem 2.25]):

Theorem 2.3.15. *For every class* $\mathbf{K}$ *of algebras we have:*

- $\mathbf{K}$ *is a variety if and only if* $\mathbf{K} = \mathbb{HSP}(\mathbf{K}')$ *for some class* $\mathbf{K}'$ *of similar algebras.*
- $\mathbf{K}$ *is a quasivariety if and only if* $\mathbf{K} = \mathbb{ISP}_R(\mathbf{K}')$

We also recall the following, sometimes called Hall's Theorem (see for instance [12, Theorem 11.12]):

Theorem 2.3.16. *A class* $\mathbf{K}$ *is a variety if and only if* $\mathbb{HP}_s(\mathbf{K}) = \mathbf{K}$.

One of the early achievements of universal algebra was to relate the above "mathematical" notions, with purely logical ones, finding a connection between syntactic presentations and semantic operations. These are captured in the ideas of *equations* and *quasi-equations*.

Definition 2.3.17. Let $\mathcal{L}$ be an algebraic language. An *equation* is a positive atomic formula of the form:

$$\lambda \approx \gamma$$

where λ, γ are terms in the language $\mathcal{L}$. A *quasi-equation* is an implication of the form:

$$\lambda_0 \approx \gamma_0 \ \& \ ... \ \& \ \lambda_n \approx \gamma_n \to \lambda \approx \gamma$$

where λ_i, γ_i are terms in the language.

Definition 2.3.18. We say that an algebra $\mathbf{A}$ *satisfies an equation* $\lambda(x_0, ..., x_n) \approx \gamma(x_0, ..., x_n)$, in symbols, $\mathbf{A} \vDash \lambda \approx \gamma$, if for each $a_0, ..., a_n \in A$, we have that $\lambda(a_0, ..., a_n) = \gamma(a_0, ..., a_n)$, i.e., interpreting the term in the algebra with those elements yields equality.

We say that an algebra $\mathbf{A}$ satisfies a quasi-equation $\lambda_0(\overline{x}) \approx \gamma_0(\overline{x}) \ \& \ ... \ \& \ \lambda_n(\overline{x}) \approx \gamma_n(\overline{x}) \to \lambda(\overline{x}) \approx \gamma(\overline{x})$, in symbols $\mathbf{A} \vdash \lambda_0 \approx \gamma_0 \ \& \ ... \ \& \ \lambda_n \approx \gamma_n \to \lambda \approx \gamma$ if for each $a_0, ..., a_m \in A$, if $\lambda_i(a_0, ..., a_m) = \gamma(a_0, ..., a_m)$ for each $i \leqslant n$, then $\lambda(a_0, ..., a_m) = \gamma(a_0, ..., a_m)$.

Definition 2.3.19. Let V be a class of algebras. We say that V is an *equational class* (resp. quasi-equational class) if there exists some set of equations S (resp., quasi-equations) such that for each algebra $\mathbf{A}$, $\mathbf{A} \in V$ if and only if $\mathbf{A} \models \varphi$ for each $\varphi \in S$.

Theorem 2.3.20. *Let* $\mathbf{K}$ *be a class of algebras. Then:*

- *(Birkhoff)* $\mathbf{K}$ *is an equational class if and only if* $\mathbf{K}$ *is a variety;*
- *(Mal'tsev)* $\mathbf{K}$ *is a quasi-equational class if and only if* $\mathbf{K}$ *is a quasivariety.*

Proof. See [12, Chapter 2, Theorem 11.9, Chapter 5, Theorem 2.25]. ■

The following is a particularly useful tool developed by Jónsson, which we will use occasionally (noting that virtually all varieties we will be dealing with are congruence distributive):

Lemma 2.3.21. (Jónssons' Lemma) Let $\mathbf{K}$ be a set of algebras such that $\mathbb{V}(\mathbf{K})$ is congruence distributive. Then the subdirectly irreducible elements of $\mathbf{K}$ are in:

$$\mathbb{HSP}_U(\mathbf{K})$$

Proof. See [12, Corollary 6.10]. ■

2.4 Logical Preliminaries

Throughout we denote some well-known logical systems as follows:

- CPC - Classical Propositional logic;
- IPC - Intuitionistic Propositional Logic;
- K - Minimal Normal modal logic;
- $\mathsf{S4}$ - S4 modal logic;

We assume the reader is familiar with classical and intuitionistic logic. We briefly recall the notion of a *normal modal logic* (for a detailed discussion on this see [9]):

Definition 2.4.1. Let $\mathcal{L}$ be the language of classical logic together with a unary operator $\Diamond$. A collection of formulas $L \subseteq \mathcal{L}$ is called a *normal modal logic* if:

- $\mathsf{CPC} \subseteq L$, i.e., L extends classical logic;
- L contains the following axioms (called the normality axioms):

$$\Box\top \leftrightarrow \top$$
$$\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi.$$

- L is closed under Modus Ponens, Unifrom Substitution and Necessitation: if $\varphi \in L$ then $\Box\varphi \in L$.

It follows straightforwardly from the definitions that the arbitrary intersection of normal modal logics is again a normal modal logic. This implies that the collection of normal modal logics forms a complete lattice. We denote by K the minimal normal modal logic. Moreover, given an axiom φ, we denote by $\mathsf{K} \oplus \varphi$ the smallest normal modal logic containing φ. Given a normal modal logic L, we denote by $\mathbf{NExt}(L)$ the complete lattice of normal extensions of L.

Important both in the theory of modal and intuitionistic logics is the concept of Kripke completeness:

Definition 2.4.2. Given a class $\mathbf{K}$ of Kripke frames, and φ a formula (in the language of intuitionistic or modal logic), we write $\mathbf{K} \models \varphi$ to mean that for each $\mathfrak{F} \in \mathbf{K}$ where $\mathfrak{F} = (X, R)$, and each model v over $\mathfrak{F}$, and each $w \in X$, $(\mathfrak{F}, v), w \Vdash \varphi$. A logic $L \in \Lambda(\mathsf{IPC})$ (resp. $L \in \Lambda(\mathsf{K})$) is said to be *Kripke complete* if there exists a class $\mathbf{K}$ of Kripke frames such that $\mathbf{K} \models L$.

Equally relevant are the concepts of *FMP* and *tabular* logics:

Definition 2.4.3. Let L be superintuitionistic or modal. We say that L has the *FMP* if L is Kripke complete with respect to a class $\mathbf{K}$ of finite frames. We say that L is *tabular* if L is Kripke complete with respect to a single finite Kripke frame.

We let HA and BA denote the classes of Heyting and Boolean algebras respectively. We assume familiarity with the algebraic completeness of the respective logical systems with respect to these classes of algebras. Moreover we also have the following well-known facts:

Definition 2.4.4. For each Heyting algebra $\mathbf{H} = (H, \wedge, \vee, \rightarrow, 0, 1)$, consider the collection $\mathbf{H}^{\neg} = \{a : a = \neg\neg a\}$. Then this forms a Boolean algebra with the induced operations:

- $\dot{0} = 0$ and $\dot{1} = 1$
- $a \dot{\wedge} b = a \wedge b$
- $a \dot{\rightarrow} b = a \rightarrow b$
- $a \dot{\vee} b = \neg\neg(a \vee b)$

Definition 2.4.5. Let $\mathbf{A}$ be an algebra of type σ. We say that a map $v : \mathcal{VAR} \rightarrow H$ is a *assignment of variables* on H. Given an assignment v, we extend this to a *valuation* $\overline{v}$ on H recursively, by defining:

- $\overline{v}(p) = v(p)$
- $\overline{v}(0) = 0$ and $\overline{v}(1) = 1$;
- $\overline{v}(\varphi \wedge \psi) = \overline{v}(\varphi) \wedge \overline{v}(\psi)$;
- $\overline{v}(\varphi \vee \psi) = \overline{v}(\varphi) \vee \overline{v}(\psi)$;
- $\overline{v}(\varphi \rightarrow \psi) = \overline{v}(\varphi) \rightarrow \overline{v}(\psi)$;

where the symbols on the right hand side are computed inside of H. We write $H \models \varphi$ to mean that whenever v is a valuation on H, $v(\varphi) = 1$.

We remark that the former is the understood meaning, whenever we take any algebra such as a Boolean algebra, Heyting algebra or a bounded distributive lattice.

Theorem 2.4.6. *(Glivenko Theorem) For each φ, a formula in the language of classical logic, we have that $\varphi \in \mathsf{CPC}$ if and only if $\neg\neg\varphi \in \mathsf{IPC}$.*

Proof. To establish this, assume that $\varphi \in \mathsf{CPC}$, and $\mathbf{H}$ is a Heyting algebra. We will show that $H \vDash \neg\neg\varphi$. To see this, note that by completeness of CPC with respect to Boolean algebras, we have that $H^{\neg} \vDash \varphi$. Now suppose that v is a valuation on H. Define a valuation $w : \mathsf{Prop} \to H^{\neg}$ by letting $w(p) = \neg\neg v(p)$. This is well-defined, since as is known, for all $a \in H$:

$$\neg\neg a \leftrightarrow \neg\neg\neg\neg a$$

Then by induction, we can see that for each formula ψ in the language of CPC:

$$w(\psi) = \neg\neg v(\psi).$$

Indeed, the base case is given by hypothesis, and the remaining cases follow by well-known properties of intuitionistic calculus. Hence, by hypothesis, $w(\varphi) = 1$, which means that $\neg\neg v(\varphi) = 1$, i.e., $v(\neg\neg\varphi) = 1$.

Conversely, if $\varphi \notin \mathsf{CPC}$, then $\neg\neg\varphi \notin \mathsf{CPC}$. Since $\mathsf{IPC} \subseteq \mathsf{CPC}$ (since all Boolean algebras are Heyting algebras, and the mentioned completeness theorem), so $\neg\neg\varphi \notin \mathsf{IPC}$, which gets us completeness. ■

The classes we just met of Heyting algebras and $\mathsf{S4}$ algebras are also intimately related. To see why note the following:

Proposition 2.4.7. Let $\mathbf{B}$ be an S4 algebra. Then the bounded sublattice $\mathbf{B}_{\Box} := \{a \in B : a = \Box a\}$ is a Heyting algebra when equipped with the induced meet and join and $a \Rightarrow b := \Box(a \to b)$.

Moreover, it is also possible to have a weak inverse to this operation. This latter transformation will play an important role in our investigations, and is frequently called *Booleanisation* (see for instance [30, pp.25]):

Definition 2.4.8. Let D be a bounded distributive lattice, B a Boolean algebra, and $e : D \to B$ a bounded lattice embedding. We say that (B, e) is the *Booleanisation* of D if for each Boolean algebra C, and bounded lattice homomorphism $f : D \to C$, there is a unique map $h : B \to C$ such that $f = h \circ e$.

Proposition 2.4.9. Let D be a bounded distributive lattice, and X_D the dual Priestley space. Then $\mathsf{Clop}(X_D)$, the class of all clopens of D is the Booleanisation of D.

Denote by $B(D)$ the Booleanisation of D.

Proposition 2.4.10. Let $f : D \to C$ be an injective bounded lattice homomorphism, where C is a Boolean algebra. Then the unique map $\overline{f} : B(D) \to C$ is also injective.

Proof. Suppose that $f(a) = f(b)$. Suppose that these are:

$$\bigvee_{i=1} f(a_i) - f(c_i) = \bigvee_{j=1} f(b_j) - f(d_j)$$

Using distributivity, we can rewrite the latter into $\bigwedge_{m=1}^{k} f(k_m) \vee \neg f(l_m)$. Now since these are equal we have that:

$$f(a_i) - f(c_i) \leqslant f(k_m) \vee \neg f(l_m)$$

Then $f(a_i) \leqslant f(k_m) \vee \neg f(l_m) \vee f(c_i)$, and in turn, then, $f(a_i) \wedge f(l_m) \leqslant f(k_m) \vee f(c_i)$. Now since f is a homomorphism, and injective, this means that:

$$a_i \wedge l_m \leqslant k_m \vee c_i$$

Operating the same transformations backwards, we have that $a_i - c_i \leqslant k_m \vee \neg l_m$. Since this is true of arbitrary such elements, then $a \leqslant b$. By a similar argument, then $b \leqslant a$. So $a = b$ as intended. ■

Moreover, the following holds:

Proposition 2.4.11. The Booleanisation $B(H)$ of a Heyting algebra H, admits an $\mathsf{S4}$ structure: for each $a \in B(H)$, we let:

$$\Box a := \bigvee \{c \in H : c \leqslant a\}$$

Proof. See [23, Construction 2.5.7]. ■

2.5 Algebraic Logic

Throughout, if $\mathbf{K}$ is a class of similar algebras, we denote by $\mathcal{L}_K$ the algebraic language of $\mathbf{K}$; when the subscript is understood we drop it. Given a set X, we denote by $T_{\mathcal{L}}(X)$ the set of terms constructed in the language $\mathcal{L}$ with variables from X, and by $\mathbf{Tm}_{\mathcal{L}}(X)$ the corresponding absolutely free algebra. Let $\mathsf{Eq}_{\mathcal{L}}(X)$ be the set of equations built from this language, with variables from X.

We begin by recalling the concept of a consequence relation and of a logic.

Definition 2.5.1. Let X be a set. A *consequence relation* over X is a relation $\vdash \subseteq \mathcal{P}(X) \times X$ such that:

- (Reflexivity) If $a \in X$ then $(X, a) \in \vdash$
- (Cut) If $(X, y) \in \vdash$ for every $y \in Y$, and $(Y, z) \in \vdash$, then $(X, z) \in \vdash$

Notation 2.5.1. Throughout, whenever $\Vdash \subseteq \mathcal{P}(X) \times X$ is a consequence relation, $Y \subseteq X$, $a \in X$ we write $Y \Vdash a$ to mean that $(Y, a) \in \Vdash$. We write $\Vdash a$ to mean $(\varnothing, a) \in \Vdash$. Given $Y, Z \subseteq X$, we write $Y \vdash Z$ to mean that $Y \vdash a$ for each $a \in Z$.

In this thesis, all consequence relations are assumed to be finitary, that is, if $X \vdash a$, then there is some finite $Y \subseteq X$ such that $Y \vdash a$. This is not necessary for most results, but it makes the presentation simpler, and avoids complications with respect to infinitary axiomatisations.

We note that consequence relations on X form a complete lattice:

Lemma 2.5.2. The collection $Con(X)$ of consequence relations on a set X forms a complete lattice, where arbitrary meets are given by intersections.

This entitles us to the following definition:

Definition 2.5.3. Let X be a set, and $S \subseteq \mathcal{P}(X) \times X$. We denote by $\vdash_S \subseteq \mathcal{P}(X) \times X$ the smallest consequence relation on X such that $S \subseteq \vdash_S$.

Definition 2.5.4. Given $S \subseteq \mathcal{P}(X) \times X$, define S^r as:

$$Ref = \{(A, a) : A \subseteq X, a \in A\}.$$

Also, define S° as:

$$S^\circ = \{(A, c) : \text{There exists some } (B, c) \in S, \text{ such that for all } b \in B \ (A, b) \in S\}.$$

We then define S°_n} as follows: $S^{\circ_0} = Ref$, $S^{\circ_{n+1}} = (S^{\circ_n})^\circ$.

Then it is not difficult to see that:

Lemma 2.5.5. For any $S \subseteq \mathcal{P}(X) \times X$:

$$\vdash_S = \bigcup_{n \in \omega} S^{\circ_n}.$$

A special kind of consequence relation we will be interested are *logics*. Throughout let $\mathcal{VAR}$ be a fixed but arbitrary set of variables.

Definition 2.5.6. Given an algebraic language $\mathcal{L}$ over $\mathcal{VAR}$, a given consequence relation $\vdash \subseteq \mathcal{P}(\mathbf{Tm}_{\mathcal{L}}(\mathcal{VAR})) \times \mathbf{Tm}_{\mathcal{L}}(\mathcal{VAR})$ is called a *logic* if $\vdash$ is substitution invariant: for every homomorphism $\sigma : \mathbf{Tm}_{\mathcal{L}}(\mathcal{VAR}) \to \mathbf{Tm}_{\mathcal{L}}(\mathcal{VAR})$, and every $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$:

$$\Gamma \vdash \varphi \text{ implies } \sigma[\Gamma] \vdash \sigma(\varphi)$$

In other words, logics are consequence relations where uniform substitutions are precisely the endomorphisms of the set $\mathbf{Tm}_{\mathcal{L}}(\mathcal{VAR})$. Similarly to above, we also have:

Lemma 2.5.7. Given a logic $\vdash$, the collection $\Lambda(\vdash)$ of finitary extensions of $\vdash$ forms a complete lattice with meet as intersection.

And similar to before, if $S \subseteq \mathcal{P}(\mathbf{Tm}_{\mathcal{L}}(\mathcal{VAR})) \times \mathbf{Tm}_{\mathcal{L}}(\mathcal{VAR})$, we denote by $Log(S)$ the smallest logic generated by S. Then it is clear that given $\vdash_i$ a collection of finitary logics, then $\bigvee_{i \in I} \vdash_i = Log(\bigcup_{i \in I} \vdash_i)$. Moreover, we get that the former description of the generated consequence relation in fact captures the generated logic in this particular case:

Lemma 2.5.8. If $(\vdash_i)_{i \in I}$ is a collection of finitary logics, then:

$$\bigvee_{i \in I} \vdash_i = \vdash_{\left(\bigcup_{i \in I} \vdash_i\right)}$$

Proof. By Lemma 2.5.5, if we show that $\vdash_{\left(\bigcup_{i \in I} \vdash_i\right)}$ is already a logic, it will surely be the smallest. Hence it suffices to check closure under substitution of that consequence relation. Suppose that $(X, \varphi) \in \vdash_{\bigcup_{i \in I} \vdash_i}$. By the same Lemma, we get that then $(X, \varphi) \in (\bigcup_{i \in I} \vdash_i)^{\circ_n}$. So by induction we can show that for any substitution σ, $(\sigma[X], \sigma(\varphi))$ also belongs there: for the base case, this follows by reflexivity and the fact that the logics are closed under substitution, whilst for the inductive case this follows by induction hypothesis and cut. ■

We now turn to a related concept, stemming from algebra:

Definition 2.5.9. Let $\mathbf{K}$ be a class of algebras and $\mathcal{L}$ its language, and $\mathsf{Eq}(\mathcal{VAR})$ the set of equations built over $\mathcal{VAR}$. We say that $\vDash_{\mathsf{K}} \subseteq \mathsf{Eq} \times \mathsf{Eq}$ is the *equational consequence relative to* $\mathbf{K}$ if it satisfies the following properties: for $\Theta \subseteq \mathsf{Eq}$ and $\gamma(\overline{x}), \delta(\overline{y}) \in Tm_{\mathcal{L}}(\mathcal{VAR})$:

$$\Theta \vDash_{\mathsf{K}} \gamma(\overline{x}) \approx \delta(\overline{y}) \iff \text{For every } \mathbf{A} \in \mathbf{K} \text{ and } h : \mathbf{Tm}_{\mathcal{L}}(X) \to \mathbf{A}$$
$$\text{if for all } \varphi \approx \psi \in \Theta \ \ h(\varphi) = h(\psi), \text{ then } h(\gamma) = h(\delta)$$

The equational consequence relative to a class of algebras is very often used as the semantics for specific logics. This is boiled down to the property of having a completeness theorem:

Definition 2.5.10. Let $\mathcal{L}$ be a language, and $\vdash$ be a logic in this language, and $\mu(x)$ a set of equations in one variable. We say that a class $\mathbf{K}$ of algebras is a μ-algebraic semantics for $\vdash$ if for all $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$:

$$\Gamma \vdash \varphi \iff \mu[\Gamma] \vDash_{\mathbf{K}} \mu(\varphi)$$

We say that $\mathbf{K}$ is an algebraic semantics if it is a μ-algebraic semantics for some μ.

Most of the logics we will deal with in this thesis will have an algebraic semantics. Examples are CPC, which has $\mathbf{BA}$ and the set $\mu(x) = \{x \approx 1\}$, and IPC, which has $\mathbf{HA}$ with the same set of equations. However, consider for instance the set:

$$\{\neg\neg x \approx 1\}$$

This set makes $\mathbf{HA}$ into an algebraic semantics for CPC by Theorem 2.4.6. This means that one logic can have multiple algebraic semantics. Nevertheless, for some logics, there is a way to determine a "canonical choice":

Definition 2.5.11. Let $\mathcal{L}$ be a language, and $\vdash$ a finitary logic for this language. We say that $\vdash$ is *algebraizable* if there are a quasivariety $\mathbf{K}$, $\mu(x)$ a set of equations in one variable, and $\Delta(x, y)$ a set of formulas in two variables, such that for all formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$ and $\Theta \cup \{\gamma \approx \delta\} \subseteq \mathsf{Eq}_{\mathcal{L}}$:

1. $\Gamma \vdash \varphi$ if and only if $\mu[\Gamma] \vDash_{\mathbf{K}} \mu(\varphi)$
2. $\Theta \vDash_{\mathbf{K}} \gamma \approx \delta$ if and only if $\Delta[\Theta] \vdash \Delta(\gamma, \delta)$
3. $x \dashv\vdash \Delta(\mu(x))$
4. $x \approx y \mathrel{=}\joinrel\mathrel{|}\joinrel\mathrel{\vDash}_{\mathbf{K}} \mu[\Delta(x, y)]$

In this case, $\mathbf{K}$ is said to be an *equivalent algebraic semantics.*

And indeed, we have:

Theorem 2.5.12. *[25, Theorem 3.17] Each algebraizable logic has a unique equivalent algebraic semantics.*

Chapter 3

Translations, Adjunctions and Polyatomic Logics

The purpose of the present chapter is to introduce the general aspects of the theory of Polyatomic Logics.

We begin by taking a close look at several translations which bear some resemblance to the KGG one. This is done to emphasise the similarities in them, and the way in which they allow for similar completeness theorems. We then embark on the general theory, abstracting from this situation. We recall the definition of a translation and its relation to right adjoints, as detailed in [53]. Building on that work, we introduce the notion of a *selective translation*, abstracting from some desirable properties present in the most salient examples. This allows us to provide an abstract framework to discuss semantically rich relationships between distinct logical contexts. Building on this, we introduce the idea of a PolyAtomic Logic (so named because atoms are sent to elements covered by a given term, in analogy with the naming convention of DNA-logics [6]), which blend properties of different logical settings into the same models.

In analogy with the setting of [6], we develop the theory of PAt-Logics and PAt-Quasivarieties, and establish a connection between these concepts, proving analogues of the Birkhoff theorem and algebraic completeness. This is then used to develop a basic abstract theory of "companionship" holding between logics under a selective translation[1].

3.1 Classical Translations and DNA-Logics

In the introduction and preliminaries we discussed several translations. Here we briefly recall them and discuss their similarities as respects the structure of their completeness theorems. We begin with the Double negation translation:

Definition 3.1.1. The *Kolmogorov-Gödel-Gentzen Double Negation Translation* (KGG)[2], denoted $K_{\neg\neg}$, maps $\mathbf{Tm}_{\mathcal{L}_{\mathsf{BA}}}(\mathcal{VAR})$ to $\mathbf{Tm}_{\mathcal{L}_{\mathsf{HA}}}(\mathcal{VAR})$ through the following assignment:

[1] A quick technical note: throughout we focus on the setting of *quasivarieties*, which is more natural from an algebraic-logical perspective. However, throughout, all results restrict to the setting of varieties unless otherwise specified.

[2] We note that this translation, as noted in the introduction, is originally due to Kolmogorov; however, his translation applies the $\neg\neg$-operation to every formula. The specific version outlined is due to Gödel, Gentzen and Glivenko.

1. $K_{\neg\neg}(p) = \neg\neg p$
2. $K_{\neg\neg}(\top) = \top$ and $K_{\neg\neg}(\bot) = \bot$
3. $K_{\neg\neg}(\varphi \wedge \psi) = K_{\neg\neg}(\varphi) \wedge K_{\neg\neg}(\psi)$
4. $K_{\neg\neg}(\varphi \to \psi) = K_{\neg\neg}(\varphi) \to K_{\neg\neg}(\psi)$
5. $K_{\neg\neg}(\varphi \vee \psi) = \neg\neg K_{\neg\neg}(\varphi \vee \psi)$

As we mentioned in the Introduction, and showed in the preliminaries, we have that given a Heyting algebra $\mathbf{H}$:

$$\mathbf{H}^{\neg} = \{a \in H : a = \neg\neg a\}$$

Is a Boolean algebra with the induced operations. Hence we have the following, which straightforwardly implies Theorem 2.4.6 once one accounts for the algebraizability of the systems at hand:

Theorem 3.1.2. *For each formula φ in the language of classical logic, and each Heyting algebra $\mathbf{H}$:*

$$\mathbf{H}^{\neg} \vDash \varphi \iff \mathbf{H} \vDash K_{\neg\neg}(\varphi)$$

Proof. First assume that $\mathbf{H} \nvDash K_{\neg\neg}(\varphi)$. Let $v : \mathsf{Prop} \to H$ be a valuation witnessing this. Construct a valuation $v' : \mathsf{Prop} \to H^{\neg}$ by defining $v'(p) = \neg\neg v(p)$. Then by induction on construction of terms $\varphi \in \mathbf{Tm}_{\mathcal{L}_{\mathsf{BA}}}(\mathcal{VAR})$ we can prove that:

$$v'(\varphi) = v(K_{\neg\neg}(\varphi))$$

Hence, $\mathbf{H}^{\neg} \nvDash \varphi$. The converse is wholly similar. ∎

As discussed in the introduction, one of the important applications of this translation was the development of inquisitive logic [16]. We recall here its development as DNA-logic, as presented in [6]:

Definition 3.1.3. Let $L \in \Lambda(\mathsf{IPC})$ be an intermediate logic. We consider:

$$L^{\neg} = \{\varphi : \varphi[\neg\neg p/p] \in L\}$$

where $\varphi[\neg p/p]$ is the result of substituting $\neg p$ for p in the term φ. Then we say that $L^{\neg}$ is the *DNA-variant* of L.

DNA-logics are thus defined as the DNA-variants of some intermediate logic. More explicitly, they are collections S of formulas such that:

- $\mathsf{IPC} \subseteq S$;
- $\neg\neg p \to p \in S$;
- S is closed under Modus Ponens.

This is the origin of the name DNA – short for "double negation on atoms". Note that such structures are not closed under uniform substitution. As algebraic models, one takes structures of the form:

$$(H, V^{\neg})$$

where H is a Heyting algebra, and $V^{\neg}$ is a valuation taking values in the regular elements of H, i.e, those $a \in H$ such that $a = \neg\neg a$. Similar to what we did above, given any valuation V one can produce a regular valuation by regularising the value of the atoms. Given this role of regular elements, given any Heyting algebra H, we denote by:

$$\langle H^{\neg} \rangle$$

The Heyting subalgebra of H generated by the regular elements. Those Heyting algebras H such that $H = \langle H^{\neg} \rangle$ are called *regularly generated*, and they are especially important due to carrying the essential properties of a logic. To correspond to this, a notion of *DNA-variety* was introduced in [6]:

Definition 3.1.4. Let H and H' be two Heyting algebras. We say that H' is a *core superalgebra* of H if $H \leqslant H'$ and $H^{\neg} = H'^{\neg}$.

One fact which can be noted is that core superalgebras are the operator which binds the least:

Proposition 3.1.5. For each variety of Heyting algebras K, the class of algebras obtained by taking core-superalgebras of $K^{\uparrow} := \{\langle B^{\neg} \rangle : B \in K\}$, is a DNA-variety.

Definition 3.1.6. Let K be a family of Heyting algebras. We say that K is a *DNA-variety* if it is closed under homomorphic images, subalgebras, products and core-superalgebras.

In [6] the following facts about these logics and varieties were established:

- DNA-logics form a complete lattice under inclusion;
- DNA-varieties form a complete lattice under inclusion;
- The lattices of DNA-logics and DNA-varieties are dually isomorphic.

These facts are enough to show soundness and completeness of DNA-logics with respect to Heyting algebra models as defined above, and moreover, completeness with respect to regularly generated Heyting algebras.

The reader will notice that in some respects it seems to be the main properties of the $\neg\neg$-operator, and the structure of the translation, which allow the above construction. However, many other translations exist which are of a very similar nature. Let us recall three of these:

Definition 3.1.7. The *Gödel-McKinsey-Tarski* (GMT) translation maps the set $\mathbf{Tm}_{\mathcal{L}_{\mathbf{HA}}}(\mathcal{VAR})$ to $\mathbf{Tm}_{\mathcal{L}_{\mathbf{S4}}}(\mathcal{VAR})$ through the following assignment:

1. $GMT(\top) = \top$
2. $GMT(\bot) = \bot$
3. $GMT(\varphi \wedge \psi) = GMT(\varphi) \wedge GMT(\psi)$

4. $GMT(\varphi \vee \psi) = GMT(\varphi) \vee GMT(\psi)$

5. $GMT(\varphi \to \psi) = \Box(\neg GMT(\varphi) \vee GMT(\psi))$

As mentioned also in the preliminaries and introduction, we have that given an S4-algebra $\mathbf{B} = (B, \Box)$:

$$\mathbf{B}_{\Box} = \{a \in B : a = \Box a\}$$

Is a Heyting algebra with the induced operations:

- $\hat{0} = 0$ and $\hat{1} = 1$
- $a \hat{\wedge} b = a \wedge b$
- $a \hat{\vee} b = a \vee b$
- $a \hat{\to} b = \Box(\neg a \vee b)$

Definition 3.1.8. The *Goldblatt translation* between the language of ortholattices and KTB logic is defined recursively as follows:

1. For each propositional variable p, $G(p) = \Box\Diamond p$;
2. $G(\psi \wedge \varphi) = G(\psi) \wedge G(\varphi)$
3. $G(\varphi^{\perp}) = \Box\neg G(\varphi)$

As in the other cases, given a KTB algebra $(B, \Box)$ we can consider the set:

$$\mathbf{B}_{\Box\Diamond} = \{a \in B : a = \Box\Diamond a\}$$

And equip this with the following operations: $a \dot{\wedge} b = a \wedge b$ and $a^{\perp} = \Box\neg a$. It can be straightforwardly verified that this yields an ortholattice.

For the last example, we need some definitions. These follow the terminology of [56].

Definition 3.1.9. Let $G = (G, \sqcap, \sqcup, \otimes, \multimap, 0, 1)$ be a structure where:

- $(G, \sqcap, \sqcup, 0, 1)$ is a bounded lattice.
- $(G, \otimes, 1)$ is a commutative monoid.
- For all $x, y, z \in P$:
$$x \otimes y \leqslant z \iff x \leqslant y \multimap z$$
- For all a, b, a', b': $a \leqslant a'$ and $b \leqslant b'$ implies $a \otimes a' \leqslant b \otimes b'$ and $a \multimap b \leqslant a' \multimap b'$.

We call this an *intuitionistic linear algebra* (ILA)[3]. Furthermore, we call an ILA a *ILS*-algebra if there is a modality $! : L \to L$ such that for all $a, b \in L$:

- The operation ! is a normal S4-modality: $!!a = !a \leqslant a$, $a \leqslant b$ implies $!a \leqslant !b$, $!\top = 1$, and furthermore, $!(a \sqcap b) = !a \otimes !b$.

[3]These are more frequently called today *commutative residuated lattices*, see for example [28]

The former structures are models of *Intuitionistic Linear Logic*. The specific details of this system are not important, but the reader is invited to consult [56] for a detailed exposition. Our concern will be with a translation of intuitionistic logic into this system:

Definition 3.1.10. The *Girard call-by-value* [32] C_v translation from intuitionistic logic IPC to Intuitionistic Linear Logic is defined as follows:

- $C_v(p) =!p$;
- $C_v(\varphi \wedge \psi) = C_v(\varphi) \otimes C_v(\psi)$;
- $C_v(\varphi \vee \psi) = C_v(\varphi) \sqcup C_v(\psi)$;
- $C_v(\varphi \to \psi) =!(C_v(\varphi) \multimap C_v(\psi))$.

The following is discussed in Troelstra [56, Chapter 8]:

Definition 3.1.11. Let G be an ILS-algebra. Define:

$$!G := \{a : a =!a\}$$

And equip $!G$ with the following operations $0 := 0$, and $1 := 1$; $a \wedge b := a \otimes b$, $a \vee b := (a \sqcup b)$ and $a \to b :=!(a \multimap b)$.

Proposition 3.1.12. Given an ILS-algebra, the structure $!G = (!G, \wedge, \vee, \to, 0, 1)$ is a Heyting algebra.

In all of these cases, we are translating the algebraic languages, and transforming algebras of one similarity type into algebras of another. Hence, we may ask whether this transformation witnesses the translation, i.e., for instance, whether for an S4-algebras we have:

$$\mathbf{M}_{\Box} \vDash \varphi \iff \mathbf{M} \vDash GMT(\varphi).$$

This is indeed true, and for all of the translations we have discussed, the proof goes through similarly. Namely, we can additionally show for $\mathbf{G}$ an ILS-algebra:

$$\mathbf{M}_{\Box} \vDash \varphi \iff \mathbf{M} \vDash GMT(\varphi)$$
$$!\mathbf{G} \vDash \varphi \iff \mathbf{G} \vDash C_v(\varphi),$$

and for $\mathbf{B}$ a KTB algebra:

$$\mathbf{B}_{\Box\Diamond} \vDash \varphi \wedge \psi = \varphi \iff \mathbf{B} \vDash G(\varphi) \to G(\psi).$$

Let us sketch this in the case of ILS-algebras. Given a valuation $v : \mathcal{VAR} \to G$, we can construct a valuation $v' : \mathcal{VAR} \to !G$ such that $v'(p) =!v(p)$. Then this valuation will take values in the !-fixed points, and it can be shown by induction on the construction that:

$$v'(\varphi) = C_v(v(\varphi)).$$

Similarly, given such a valuation $v' : \mathsf{Prop} \to !G$, we can construct a valuation $v : \mathsf{Prop} \to G$ by letting $v(p) = v'(p)$. Then again by induction on construction we can show that:

$$v(\varphi) = C_v(v'(\varphi)).$$

Hence, assuming in turn that $G \not\models C_v(\varphi)$ and $!G \not\models \varphi$ will get us the result. It should be obvious that similar arguments will work for all other translations.

To obtain results similar to the Glivenko theorem, one also needs that, given any algebra from the system being translated, there exists some algebra of the target system of which it is the "skeleton" (e.g., the set of open elements, of $\Box\Diamond$-fixed points, of !-fixed points, etc). This can be less obvious to establish, but as we will see in the next sections, it is true in all the cases at hand.

Together, these two facts allow us a result analogous to Glivenko's theorem, with respect to the equational consequence relation, in total analogy with the KGG case.

To pursue this analogy further, let us now quickly sketch what a hypothetical $\Box$-logic could look like:

Definition 3.1.13. Let $L \in \mathbf{NExt}(\mathsf{S4})$ be a normal extension of $\mathsf{S4}$. Consider:

$$L^{\Box} = \{\varphi : \varphi[\Box p/p] \in L\}$$

where $\varphi[\Box p/p]$ is the resulting of replacing p with $\Box p$ in φ. Then we call $L^{\Box}$ the $\Box$-*variant* of L.

Just like in the DNA case, one can argue, analogously to Proposition 3.3 in [6], that adding:

$$p \to \Box p$$

and closing under Modus Ponens, Necessitation, but not uniform substitution, would yield precisely a $\Box$-variant of any given logic, and that all of them arise this way. Moreover, given an $\mathsf{S4}$-algebra B, we say that B is *openly generated* if $B = \langle B_{\Box} \rangle$. We can then take models of the form $(B, V^{\Box})$, where $V^{\Box}$ takes values only on open elements.

What about $\Box$-varieties? Just like in the DNA-case, one can show that core superalgebras bind the least, and hence, a class of algebras is a $\Box$-variety if and only if it is the closure of a variety of $\mathsf{S4}$ algebras.

The facts mentioned above can also be carried out for this setting:

- $\Box$-logics form a complete lattice under inclusion;
- $\Box$-varieties form a complete lattice under inclusion
- The lattices of $\Box$-logics and $\Box$-varieties are dually isomorphic.

However, the reader might now be persuaded that there is also nothing special about the GMT-translation. The Girard call-by-value translation or the Goldblatt translation would just as well yield hybridised logics. Hence two natural questions arise:

- What kinds of properties must a translation have in order for it to enjoy the sort of completeness theorem we found above;
- How can one develop a theory which generalises the DNA-logic structure to other such translations.

We will take on these tasks in the next sections.

3.2 Algebraic Translations and Adjunctions

In [53], the concept of a *contextual translation* was introduced with the goal of providing a characterisation of adjunctions between generalised quasivarieties. This is inspired by work of McKenzie [50] and Dukarm [21], and related to work by Freyd [26], which describes categorical equivalence through two "deformations" of the categories. The key idea of this approach is that right adjoints correspond to translations between the equational consequence relations of the categories of algebras. Below we present this concept, recalling its definition according to the presentation of [53], as well as some key results, which we provide without proof. We note that as previously noted, our discussion remains within a finitary context even though the results cited are proved in the broader setting of generalised quasivarieties. We also focus only on *unary translations*, though all of the results in this section could be proved for arbitrary κ-ary translations.

Definition 3.2.1. Let $\mathbf{X}$ and $\mathbf{Y}$ be classes of algebras. We say that a map $\zeta : \mathcal{L}_{\mathbf{X}} \to \mathbf{Tm}_{\mathcal{L}_{\mathbf{Y}}}(\mathcal{VAR})$ is a translation if for each n, ζ sends n-ary function symbols to n-ary terms.

Given a translation ζ we define a map $\zeta_* : \mathbf{Tm}_{\mathcal{L}_{\mathbf{X}}}(\mathcal{VAR}) \to \mathbf{Tm}_{\mathcal{L}_{\mathbf{Y}}}(\mathcal{VAR})$ by stipulating that:

1. $\zeta_*(x_i) = x_i$ for variables $x_i \in \mathcal{L}_{\mathbf{X}}$;

2. $\zeta_*(c) = \zeta(c)$ for constants $c \in \mathcal{L}$;

3. For complex terms $\varphi_0, ..., \varphi_n$, and n-ary operation ψ, such that $\zeta(\psi) = f$, where each f is a term of the form $f(x_0, ..., x_k)$. Then:

$$\zeta_*(\psi(\varphi_0, ..., \varphi_n)) = f(\zeta_*(\varphi_0), ..., \zeta_*(\varphi_n))$$

Moreover, we denote by ζ^* the natural lifting of ζ_* to sets of equations: let $\mathsf{Eq}_{\mathbf{Y}}(\mathcal{VAR})$ be the set of equations of the language $\mathcal{L}_{\mathbf{Y}}$. We define $\zeta^* : \mathcal{P}(\mathsf{Eq}_{\mathbf{X}}(\mathcal{VAR})) \to \mathcal{P}(\mathsf{Eq}_{\mathbf{Y}}(\mathcal{VAR}))$ by setting, for $\Phi \subseteq \mathsf{Eq}_{\mathbf{X}}(\mathcal{VAR})$:

$$\zeta^*(\Phi) = \{\zeta_*(\delta) \approx \zeta_*(\gamma) : \delta \approx \gamma \in \Phi\}$$

In other words, a translation for our purposes consists of an assignment of logical symbols, preserving arities, which recursively induces an assignment of formulas. We note that whilst this leaves out many notable translations - for instance, most instances of "standard translations" occurring in the modal logic literature do not conform in an obvious way to this shape, despite their relevance and uses - translations of this kind occur quite frequently in various natural logical frameworks.

More than a merely syntactic assignment, though, we want our semantics to reflect this translation in some sense. Hence we will need the notion of a *contextual translation*.

Definition 3.2.2. Let $\mathbf{X}$ and $\mathbf{Y}$ be two classes of algebras. We say that a pair (ζ, Θ), where ζ is a translation, and $\Theta \subseteq \mathsf{Eq}_{\mathbf{Y}}(\mathcal{VAR})$ a finite subset of equations written over a variable x, is a *contextual translation* if the following holds:

1. For every set $\Phi \cup \{\lambda \approx \gamma\} \subseteq \mathsf{Eq}_{\mathbf{X}}$, written in variables $\{x_i : i \in \omega\}$, we have:

$$\text{if } \Phi \vDash_{\mathsf{X}} \lambda \approx \gamma, \text{ then } \zeta^*(\Phi) \cup \bigcup_{i \in \omega} \Theta(x_i) \vDash_{\mathsf{Y}} \zeta^*(\lambda \approx \gamma)$$

2. For every k-ary operation $f(y_0, ..., y_n) \in \mathcal{L}_{\mathbf{X}}$, we have:

$$\Theta(x_0) \cup ... \cup \Theta(x_n) \vDash_{\mathsf{Y}} \Theta(\zeta^*(f(x_0, ..., x_n)))$$

In this case we refer to the set of equations Θ as the *context* of the translation.

Example 3.2.3. (KGG and GMT translations) The KGG and GMT translations we met in Definition 3.1.1 and Definition 3.1.7 are contextual translations: for the first, take a modification of the function $K_{\neg\neg}$ which maps proposition variables to themselves, and takes the context:

$$\Theta = \{x \approx \neg\neg x\}.$$

For the latter, take in addition to the function GMT, with the same modification, the context:

$$\Theta = \{x \approx \Box x\}.$$

In both cases, these translations were shown to be contextual in [53].

Looking at these examples, we see that the introduction of a context has the following effect: we take the equations we are interested in, when looking at a given algebra, and we interpret these equations in a smaller algebra, comprised only of the elements satisfying the equations in the context. We generically refer to these as *regular elements*. Algebraically, this corresponds to the following:

Definition 3.2.4. Let $\mathbf{Y}$ be a class of similar algebras, and $\mathcal{L}$ its language. Let θ be a set of equations of $\mathsf{Eq}_{\mathcal{L}'}$ in one variable, where $\mathcal{L}' \subseteq \mathbf{Tm}_{\mathcal{L}}(\mathcal{VAR})$ (i.e., n-ary terms are read as n-ary functions), and where θ is *compatible* with the operations of $\mathcal{L}'$, i.e, for each n-ary operation $t \in \mathcal{L}'$:

$$\theta(x_0) \cup ... \cup \theta(x_n) \vDash_{\mathsf{X}} \theta(t(x_0, ..., x_n)).$$

Let $\mathbf{A} \in \mathbf{Y}$ be some algebra. Then we let $\theta(\mathbf{A})$ be the following structure:

$$\theta(\mathbf{A}) := \{a : \mathbf{A} \vDash \theta(a)\}$$

equipped with the operations in $\mathcal{L}'$ (note that compatibility of θ allows this). We call this the *algebra of θ-regular elements of* $\mathbf{A}$.

If $\mathbf{Y}$ is a class of all similar algebras in the language $\mathcal{L}$, and $\mathbf{X}$ is the class of all algebras in the language $\mathcal{L}'$, this provides a map $\theta : \mathbf{Y} \to \mathbf{X}$, sending $\mathbf{A}$ to $\theta(\mathbf{A})$. Moreover, given a homomorphism of $\mathcal{L}$-algebras $f : \mathbf{A} \to \mathbf{B}$, we can define $\theta(f)$ as

$$\begin{aligned} \theta(f) : \theta(\mathbf{A}) &\to \theta(\mathbf{B}) \\ a &\mapsto f(a) \end{aligned}$$

i.e, as the restriction, and obtain that this is a well-defined homomorphism in the language $\mathcal{L}'$: it is well-defined since if $\Theta(a)$ holds in $\mathbf{A}$, then $\Theta(f(a)))$ holds in $\mathbf{B}$, since f is a homomorphism. Moreover, it is a $\mathcal{L}'$-homomorphism since the language is composed of terms in the language $\mathcal{L}$.

Thus, θ defines a functor. In other words, we have:

Proposition 3.2.5. Let $\mathbf{Y}$ be some class of algebras, and $\mathcal{L}$ its language, and let $\mathcal{L}' \subseteq \mathbf{Tm}_{\mathcal{L}}(\mathcal{VAR})$ be the language of $\mathbf{X}$. Let θ be a set of compatible equations. Then the assignment:

$$\theta : \mathbf{Y} \to \mathbf{X}$$
$$\mathbf{A} \mapsto \theta(\mathbf{A})$$

Is a functor, which acts on morphisms by restriction.

Given these definitions, one can see the connection between the notion of a contextual translation and the above functor: if $\mathbf{X}$ and $\mathbf{Y}$ are two classes of algebras, and $\langle \zeta, \Theta \rangle$ is a contextual translation between them, consider the following language:

$$\mathcal{L}' := \{\zeta(\psi) : \psi \in \mathcal{L}_{\mathbf{X}}\}.$$

This will be a language composed of terms from $\mathcal{L}_{\mathbf{Y}}$. By definition of being a contextual translation, Θ is a compatible set of equations. Thus, this yields a functor in the manner described above, which we denote by θ_ζ. More than that, we have the following characterisation (for a proof see [53], Theorem 5.1 and Lemma 5.4):

Theorem 3.2.6. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two quasivarieties. If $\langle \zeta, \Theta \rangle$ be a contextual translation between $\mathbf{X}$ and $\mathbf{Y}$. Then $\theta_\zeta : \mathbf{Y} \to \mathbf{X}$ is a right adjoint functor.*

The previous theorem establishes a deep connection between translations - an eminently syntactic domain - and adjunctions - which relate the semantic domains of interpretation we care about[4]. Through it we can investigate certain desirable properties of translations through a semantic lens, and seek to impose certain properties through syntactic restrictions.

Before that, we recall the explicit description given of the left adjoint of the functor described above:

Proposition 3.2.7. Let $\langle \zeta, \Theta \rangle$ be a contextual translation between $\mathcal{L}_{\mathbf{X}}$ and $\mathcal{L}_{\mathbf{Y}}$. Let θ_ζ be the induced right adjoint functor. Then the left adjoint functor, $\mathcal{F} : \mathbf{X} \to \mathbf{Y}$, acts on objects as follows: if $\mathbf{A} \in \mathbf{Y}$, then let Ψ be a congruence on $\mathbf{Tm}_{\mathcal{L}_{\mathbf{X}}}(\mathcal{VAR})$ such that $\mathbf{A} \cong \mathbf{Tm}_{\mathcal{L}_{\mathbf{X}}}(\mathcal{VAR})/\Psi$. Then define:

$$\mathcal{F}(A) = \mathbf{Tm}_{\mathcal{L}'}(X)/\mathsf{Cg}_{\mathbf{Y}}(\zeta^*(\Psi) \cup \bigcup_{i \in \omega} \Theta(x_i))$$

In many cases, more explicit descriptions of left adjoints are available. We also remark that, as we will see throughout the thesis, in the algebra of logic, such explicit descriptions tend to be facilitated by the existence of a topological duality.

Example 3.2.8. (Heyting algebras and S4, continued) As mentioned in the preliminaries, the functor $B(-) : \mathsf{HA} \to \mathsf{S4}$ which takes the Booleanisation of a Heyting algebra, and induces the modality $\Box$ as the relative complement in the Heyting algebra, is the left adjoint to the functor ρ which selects the open elements. We also note that, dually, σ corresponds to the forgetful functor from the category of Esakia spaces to the category of Boolean spaces which forgets the order.

[4] We remark that Moraschini in fact proved the converse of this Theorem also holds, except the resulting translation might not be unary, but instead, κ-ary.

3.3 Conditions on Adjunctions

For the rest of the chapter, let $\mathbf{X}$ and $\mathbf{Y}$ be two quasivarieties, and assume that $\langle\zeta, \Theta\rangle$ is a unary contextual translation, with associated functor θ_ζ. Also, for a given $\mathbf{A} \in \mathbf{X}$, let $\mathfrak{H}(\mathbf{A})$ denote $\theta_\zeta(\mathcal{F}(\mathbf{A}))$.

Notation 3.3.1. We write:

$$\mathbf{A}, \Theta \vDash \zeta^*(\lambda \approx \gamma)$$

to mean that $\mathbf{A}$ validates the equation $\zeta^*(\lambda \approx \gamma)$ in the context Θ, that is, in any valuation where all variables ocurring in λ, γ are assumed to satisfy Θ. ■

We begin by spelling out the structure of interpretations of elements from $\theta_\zeta(\mathbf{A})$. Note that for $a_0, ..., a_n \in \theta_\zeta(\mathbf{A})$, and $\psi(x_0, ..., x_n) \in \mathcal{L}_{\mathbf{X}}$, we have:

$$\psi^{\theta_\zeta(\mathbf{A})}(a_0, ..., a_n) = \zeta(\psi)(a_0, ..., a_n)$$

where the latter is calculated in $\mathbf{A}$. Having this in mind, we look at the following:

Proposition 3.3.1. For each $\mathbf{A} \in \mathbf{Y}$ and $\delta \approx \gamma \in \mathsf{Eq}_{\mathbf{X}}(\mathcal{VAR})$,

$$\mathbf{A}, \Theta \vDash \zeta^*(\delta \approx \gamma) \iff \theta_\zeta(\mathbf{A}) \vDash \delta \approx \gamma.$$

Proof. First suppose that $\mathbf{A}, \Theta \vDash \zeta^*(\delta \approx \lambda)$. Let $v : \mathbf{Tm}(X) \to \theta_\zeta(A)$ be an arbitrary valuation. Then define a new valuation:

$$\begin{aligned} v' : \mathbf{Tm}_{\mathbf{Y}}(X) &\mapsto A \\ x &\mapsto v(x) \end{aligned}$$

which is defined on variables, and lifted to all terms in X as expected. Note that since v was a valuation taking values in regular elements, then v' will also take values in regular elements. Moreover, we can see by induction on the construction of terms that:

$$v(p(\overline{x})) = v'(\zeta_*(p(\overline{x}))).$$

Indeed, for the base case this holds by assumption. Now assume that $t_0, ..., t_n$ are terms for which this holds. Let ψ be an n-ary operation in the language of $\mathbf{X}$. Then:

$$\begin{aligned} v(\psi(t_0, ..., t_n)) &= \psi(v(t_0), ..., v(t_n)) \\ &= \psi^{\theta_\zeta(\mathbf{A})}(\zeta_*(v'(t_0)), ..., \zeta_*(v'(t_n))) \\ &= \zeta_*(\psi)(v'(t_0), ..., v'(t_n)) \\ &= v'(\zeta_*(\psi(t_0, ..., t_n)) \end{aligned}$$

where these equalities follow by definition of the translation and our remarks about the explicit shape of operations in the algebra of regular elements. But then, since we have the equality, we obtain that $\theta_\zeta(A) \vDash \delta \approx \gamma$, as intended. The other direction follows by similar arguments. ■

The former is one of the key results we will need throughout this chapter. An easy consequence of it is a fact relating the structure of the adjunction and the correctness of the translation. First we recall a piece of category theory:

Definition 3.3.2. Let $F : \mathbb{C} \to \mathbb{D}$ be a functor. We say that F is *faithful* if for all $\mathbf{A}, \mathbf{B} \in \mathbb{C}$ and all maps $f, g : \mathbf{A} \to \mathbf{B}$: if $F(f) = F(g)$ then $f = g$. We say that it is *full* if whenever $f : F(\mathbf{A}) \to F(\mathbf{B})$ is a map on $\mathbb{D}$, then there is some map $g : \mathbf{A} \to \mathbf{B}$ such that $F(g) = f$. We say that F is *fully faithful* if it is both full and faithful. We say that F is *essentially surjective on objects* if for all $\mathbf{C} \in \mathbb{D}$ there is some $\mathbf{A} \in \mathbb{C}$ such that $F(\mathbf{A}) \cong \mathbf{C}$.

Definition 3.3.3. Let $\eta : \mathbf{1}_{\mathbb{C}} \implies G(F(-))$ and $\varepsilon : F(G(-)) \implies \mathbf{1}_{\mathbb{D}}$ be natural transformations. We say that these are respectively the *unit* and the *counit* if for all $\mathbf{B} \in \mathbb{C}$ we have that the composition,

$$F(\mathbf{B}) \xrightarrow{F(\eta_{\mathbf{B}})} F(G(F(\mathbf{B}))) \xrightarrow{\varepsilon_{F(\mathbf{B}}} F(\mathbf{B})$$

is equal to the identity on $\mathbf{B}$, and also, for all $\mathbf{A} \in \mathbb{D}$ we have that the composition,

$$G(\mathbf{A}) \xrightarrow{\eta_{G(\mathbf{A})}} G(F(G(\mathbf{A}))) \xrightarrow{G(\varepsilon_{\mathbf{A}}} G(\mathbf{A})$$

is equal to the identity on $\mathbf{A}$. We say that η is *pointwise injective* (resp. surjective) if for each $\mathbf{A} \in \mathbb{C}$, $\eta_{\mathbf{A}}$ is injective (surjecitve). Similarly we say that ε is pointwise injective/surjective.

Proposition 3.3.4. Let η be the unit of the adjunction $\mathcal{F} \vdash \theta_{\zeta}$. If η is pointwise injective then the translation is *faithful*: for every collection $\Phi \cup \{\lambda \approx \gamma\} \subseteq \mathsf{Eq}_{\mathbf{X}}$, written in variables $\{x_j : j \in \omega\}$, we have:

$$\Phi \vDash_{\mathsf{X}} \lambda \approx \gamma, \text{ iff } \zeta^*(\Phi) \cup \bigcup_{i \in \omega} \Theta(x_i) \vDash_{\mathsf{Y}} \zeta^*(\lambda \approx \gamma).$$

Proof. Assume that η is pointwise injective. One half of the above condition is true of every contextual translation, so we focus on the other one. Suppose that $\Phi \nvDash_{\mathbf{X}} \lambda \approx \gamma$. Let $\mathbf{A} \in \mathbf{X}$ be some algebra witnessing this. Then, let v be a valuation, such that $\mathbf{A}, v \vDash \Phi$ but $\mathbf{A}, v \nvDash \lambda \approx \gamma$. Since η is pointwise injective, we have that $\mathfrak{H}(\mathbf{A}), v \vDash \Phi$ but $\mathfrak{H}(\mathbf{A}), v \nvDash \lambda \approx \gamma$ as well. But then using the arguments from Proposition 3.3.1, we get that $\mathcal{F}(\mathbf{A}), v' \vDash \zeta^*(\Phi)$ whilst $\mathcal{F}(\mathbf{A}), v' \nvDash \zeta^*(\lambda \approx \gamma)$. This shows that:

$$\zeta^*(\Phi) \cup \bigcup_{i \in \omega} \Theta(x_i) \nvDash_{\mathbf{Y}} \zeta^*(\lambda \approx \gamma)$$

as desired. ■

We will be interested in translations where not only the unit is injective, but in fact an isomorphism. To work towards that, we need some definitions.

Definition 3.3.5. Let $\langle \zeta, \Theta, f \rangle$ be a contextual unary translation together with a unary term $f(x) \in \mathcal{L}_{\mathbf{Y}}$ called the *selector*. We say that the triple $\langle \zeta, \Theta, f \rangle$ is a *selective translation* if:

1. ζ is faithful;
2. $\vDash_{\mathsf{Y}} \Theta(f(x))$, i.e, selected elements are regular;
3. $\Theta(x) \vDash_{\mathsf{Y}} f(x) \approx x$.

We say that the translation is *strongly selective* if additionally we have:

4. The unit η of the adjunction is an isomorphism.

Example 3.3.6. The GMT and KGG translations we have encountered are both selective. Let the selector term be $f(x) = \Box x$. Then by the transitivity and reflexivity axioms, for any S4-modal algebra $\mathbf{A}$, and $a \in A$, $\Box\Box a = \Box\Box a$. By this fact and given the context is $\Box x$, we have the conditions. The arguments for the KGG translation are similar, using the selector term $\neg\neg x$, and using the fact that on Heyting algebras the equation $\neg\neg\neg\neg x \approx \neg\neg x$ holds.

Also note that they are faithful: we check this for the double negation translation, since the other has been mentioned. Indeed, if $\Phi \not\models \lambda \approx \gamma$, let $\mathbf{B}$ be a Boolean algebra witnessing this; since we know that $\mathbf{B}$ is such that $\theta(\mathbf{B}) = \mathbf{B}$, this is enough to prove faithfulness.

Finally, the GMT translation is strongly selective, whereas this is not clear for the double negation translation.

It is a general categorical fact that if the unit is an isomorphism, then the left adjoint is fully faithful and the right adjoint is essentially surjective on objects. We leave open the question of whether this and other sufficient conditions we give are actually necessary.

In addition to being well-behaved, these translations are quite abundant: all four translations we discussed in the previous section are examples of translations that are faithful, and as one can see by their definition, also selective[5].

We now turn to the selector term. Indeed, in the work of McKenzie [50], categorical equivalences were given by the presence of a unary idempotent and invertible term. The selector term is a similar device. Its role, for our purposes, will be to essentially internalise contexts, and allow the transformation of arbitrary valuations into regular valuations. However, it also has some categorical consequences. The following lemma follows by definition of being a selective translation:

Lemma 3.3.7. Let $\langle \zeta, \Theta, f \rangle$ be a selective translation, and $\mathbf{A}$ an algebra. Then whenever $a \in \theta_\zeta(\mathbf{A})$, then $f(a) = a$.

Proposition 3.3.8. Let $\langle \zeta, \Theta, f \rangle$ be a selective translation. Then the right adjoint functor, θ_ζ, preserves surjective homomorphisms.

Proof. Suppose that $h : A \to B$ is a surjective homomorphism. Let $a \in \theta_\zeta(\mathbf{B})$ be some regular element. By hypothesis let a' be an element such that $h(a') = a$, which is possible since h is surjective. Then note that:

$$\theta_\zeta(h)(f(a')) = f(h(a')) = f(a) = a,$$

where the second equality follows from Lemma 3.3.7. So $\theta_\zeta(h)$ is surjective as intended. ■

3.4 PAt-Logics and Quasivarieties

The previous section gave us a notion of a selective translation, which was shown to enjoy many of the nice properties of the double negation translation. In this and the following sections we will outline how such translations can be used to develop general "Polyatomic Logics" – so named since the atoms are 'covered' by the selector term.

Since we will be working with logics, we make a few assumptions on the objects at hand. For the following sections, unless specified assume that $\vdash$ is an algebraizable logic, $\mathbf{Y}$ is its equivalent

[5]In fact, most unary contextual translations that are considered tend to be of this kind. For a binary example, the reader may want to consider the example, also presented in [53], of the translation from Kleene algebras to distributive lattices, which the reader may find in that paper.

algebraic semantics, $\mu(x)$ and $\Delta(x,y)$ are the two sets witnessing algebraizability. Moreover, denote by $\Lambda(\vdash)$ the lattice of finitary extensions of this logic. We also fix a selective translation $\langle \zeta, \Theta, f\rangle$ between $\mathbf{X}$ and $\mathbf{Y}$, with all previous assumptions made on these classes.

We note that these assumptions are not necessary – indeed, throughout the thesis, namely through the Goldblatt translation, we will encounter examples which do not satisfy them, but are still amenable to the same treatment. The assumptions at hand make, however, for a more transparent read, and in any case, in concrete situations, it will be relatively clear how to overcome the lack of some assumptions.

We will be concerned here with "PAt-variants" of logics:

Definition 3.4.1. Let $\vdash_* \in \Lambda(\vdash)$ be a finitary extension of $\vdash$. We define the PAt-*variant* of $\vdash_*$, denoted $\vdash_*^f$, as follows: for each set $\Gamma \cup \varphi \subseteq \mathcal{L}_{\mathbf{Y}}$

$$\Gamma \vdash_*^f \varphi \iff \Gamma[f(p)/p] \vdash_* \varphi[f(p)/p].$$

where for each formula, $\chi[f(p)/p]$ is obtained by substituting p by $f(p)$ in χ, and $\Gamma[f(p)/p] = \{\chi[f(p)/p] : \chi \in \Gamma\}$.

The former abstracts from the idea of a *negative variant* [16].

Definition 3.4.2. Let $\vdash_s$ be an arbitrary subset of $\mathcal{P}(\mathbf{Tm}_{\mathcal{L}_Y}(\mathcal{VAR})) \times \mathbf{Tm}_{\mathcal{L}_Y}(\mathcal{VAR})$. We say that $\vdash_s$ is a PAt-logic if it is the PAt-variant of a logic $\vdash_* \in \Lambda(\vdash)$.

The main notable feature of PAt-logics is that, whilst they are closed under modus ponens, they need not be closed under uniform substitution. We collect some observations about these structures in the next proposition[6]:

Proposition 3.4.3. Let $\vdash_s$ be a PAt-logic, and $\vdash_* \in \Lambda(\vdash)$ some logic such that $\vdash_*^f = \vdash_s$. Then $\vdash_s$ is the least set of formulas such that:

1. $\vdash_s$ is a consequence relation in the language $\mathcal{L}_{\mathbf{Y}}$;
2. $\vdash_* \subseteq \vdash_s$;
3. For all atomic propositions p, we have $\vdash_s \Delta(f(p), p)$.

Proof. To see that $\vdash_s$ is a consequence relation, note this follows from the fact that $\vdash_*$ is one, and the definition of being a PAt variant. If $S \vdash_* \varphi$, then by closure under uniform substitution, $S[f(p)/p] \vdash_* \varphi[f(p)/p]$, so $S \vdash_s \varphi$. The third property follows from the fact that, $\vDash_{\mathsf{Y}} f^2(p) \approx f(p)$, and hence, $\vdash_* \Delta(f^2(p), f(p))$ by definition of algebraizability, so by definition of being a variant, $\vdash_s \Delta(f(p), p)$.
Now we note that this is least in these conditions: suppose that $\Vdash$ is another consequence relation satisfying the above properties. Suppose that $S \vdash_s \varphi$. By definition, then, $S[f(p)/p] \vdash_* \varphi[f(p)/p]$; hence $S[f(p)/p] \Vdash \varphi[f(p)/p])$ by the second property. Now by assumption, $\Vdash \Delta(f(p), p)$ for all atomic propositions; so by induction on complexity of formulas, and the properties of Δ, we get that:

$$\Vdash \Delta(\varphi[f(p)/p], \varphi)$$

And similar for any formulas in S. So since $S[f(p)/p] \Vdash \varphi[f(p)/p]$, by Δ-Modus Ponens (recalling that $\vdash_* \subseteq \Vdash$, and this is assumed to be algebraizable, hence satisfying $\Delta(x,y), x \vdash_* y$ for any x, y) and the Cut rule, $S \Vdash \varphi$. ■

[6] In fact, the properties outlined here could have been taken as the definition of PAt-logic.

As examples of the former, we recall from Section 1.1 the cases of DNA-Logics as well as the $\Box$-Logics which we sketched there. The requirement (3) can be seen as an abstract version of requiring that $p \to \Box p$ or $\neg\neg p \to p$ hold for atomic propositions. Indeed, in the case of DNA-logics, we have that $p \to \neg\neg p \approx 1$ is a theorem in IPC. Moreover, the set Δ in this case is given by:

$$\Delta(p,q) = \{p \to q, q \to p\}$$

Hence, $\Delta(\neg\neg p, p) = \{p \to \neg\neg p, \neg\neg p \to p\}$. Thus, requiring that $\neg\neg p \to p$ amounts to requiring that $\Delta(\neg\neg p, p) \in \mathsf{IPC}^{\neg}$, the negative variant of intuitionistic logic. The case for $\Box$-logics is wholly similar.

We now turn to showing that PAt-logics relative to a given translation form a complete lattice. Before that, recall from the preliminaries that given a collection $(\vdash_i)_{i\in I}$ of logics, we denote by $\vdash_{\left(\bigcup_{i\in I}\vdash_i\right)}$ the closure of the union under being a consequence relation.

Proposition 3.4.4. Let $(\vdash_i)_{i\in I}$ be a collection of PAt-logics. Then $\bigcap_{i\in I} \vdash_i$ is a PAt-logic. Moreover:

$$\bigvee_{i\in I} \vdash_i = \vdash_{\left(\bigcup_{i\in I}\vdash_i\right)}$$

where $\bigvee_{i\in I} \vdash_i$ denotes the supremum in the lattice of PAt-logics of $\vdash$.

Proof. Let $\vdash_i = \Vdash_i^f$. Let $\Vdash^f := \bigcap_{i\in I} \Vdash_i$ Then note that:

$$\begin{aligned}\bigcap_{i\in I} \vdash_i &:= \bigcap_{i\in I}\{(\Gamma,\varphi) : \Gamma[f(p)/p] \Vdash_i \varphi[f(p)/p]\} \\ &= \{(\Gamma,\varphi) : \Gamma[f(p)/p] \Vdash \varphi[f(p)/p]\} \\ &= \Vdash^f\end{aligned}$$

This follows by definition of being a PAt-variant.

To see that $\bigvee_{i\in I} \vdash_i = \vdash_{\left(\bigcup_{i\in I}\vdash_i\right)}$, we claim that if $\Vdash_i^f = \vdash_i$ for every i, then:

$$\left(\bigvee_{i\in I} \Vdash_i\right)^f = \vdash_{\left(\bigcup_{i\in I}\vdash_i\right)}$$

To see this, make some abbreviations: let $\Vdash = \bigvee_{i\in I} \Vdash_i$, and $\vdash_* = \vdash_{\left(\bigcup_{i\in I}\vdash_i\right)}$.

Now first, assume that $\Gamma \Vdash^f \varphi$. By assumption, then $\Gamma[f(p)/p] \Vdash \varphi[f(p)/p]$. By definition of being the supremum of a logic (see Proposition 2.5.8), we have that $(\Gamma[f(p)/p], \varphi[f(p)/p])$ belongs to the closure of $\bigcup_{i\in I} \Vdash_i$ under being a consequence relation. So first assume that it belongs to the union; then clearly it belongs to $\Vdash_i$ for some i, hence, $\Gamma \vdash_i \varphi$, as desired. Next, assume that for all $(\Gamma', \psi) \in \left(\bigcup_{i\in I} \Vdash_i\right)^{\circ n}$ this holds. If $(\Gamma[f(p)/p], \varphi[f(p)/p]) \in (\bigcup_{i\in I} \vdash_i)^{\circ n+1}$, then by assumption there is some $(\Gamma', \varphi[f(p)/p]) \in \bigcup_{i\in I} \Vdash_i^{\circ n}$, and $(\Gamma[f(p)/p], \chi)$ is in the same set for $\chi \in Y$. By closure under uniform substitution, we have that:

$$(\Gamma[f^2(p)/p], \chi[f(p)/p]) \text{ and } (\Gamma'[f(p)/p], \varphi[f^2(p)/p]) \in (\bigcup_{i\in I} \vdash_i)^{\circ n}$$

Hence, by induction hypothesis, we have that these will be in $\bigcup_{i\in I} \vdash_i$. Hence, in $\vdash_*$ we will have $(\Gamma[f(p)/p], \varphi[f(p)/p])$ which by assumption on some logics means that (Γ, φ) is there as well. The converse inclusion follows by similar arguments. ■

Hence, we denote by $\Lambda^f(\vdash)$ the complete lattice of PAt-variants of the logic $\vdash$. Moreover, we have the following:

Corollary 3.4.5. *The map $^f : \Lambda(\vdash) \to \Lambda^f(\vdash)$ that assigns to each logic $\vdash_*$ its PAt-variant is a complete lattice homomorphism.*

Proof. Preservation of bounds is obvious, and we have shown preservation of meets and joins in 3.4.4. ∎

The natural question then is what the appropriate semantics for these logics should be. Let $\mathbf{A} \in \mathbf{Y}$ be some algebra. Then define:

$$\mathbf{A}^f := \{a \in \mathbf{A} : \mathbf{A} \vDash f(a) = a\}.$$

Note that in light of Lemma 3.3.7, we have:

$$\mathbf{A}^f = \theta(\mathbf{A}).$$

We will use this fact freely in the sequel. We can then define the semantics of our models:

Definition 3.4.6. Let $\mathbf{A} \in \mathbf{Y}$. We denote by $\langle A^f \rangle$ the subalgebra of $\mathbf{A}$ generated by regular elements. We say that $\mathbf{A}$ is *regularly generated* if $\mathbf{A} = \langle A^f \rangle$.
We say that a pair $(\mathbf{A}, v^f)$ is a *polyatomic model* of the language $\mathcal{L}_{\mathbf{Y}}$ if v^f maps propositional variables to regular elements.

We define the semantic clauses for interpreting formulas in a polyatomic model as follows:

- For x a propositional variable, we let $[\![x]\!] = v^f(x)$
- For any complex terms $\psi(t_0, ..., t_n)$ we have that $[\![\psi(t_0, ..., t_n)]\!] = \psi^{\mathbf{A}}([\![t_0]\!], ..., [\![t_n]\!])$

Given a model $(\mathbf{A}, v^f)$, and a set of equations λ, γ we write $(\mathbf{A}, v^f) \vDash \lambda \approx \gamma$ if for all $x_0, ..., x_n \in \mathcal{VAR}$, $\lambda(v^f(x_0), ..., v^f(x_n)) = \gamma(v^f(x_0), ..., v^f(x_n))$. Given an algebra $\mathbf{A}$, we write $\mathbf{A} \vDash_f \varphi \approx \psi$ to mean that for every PAt-model v^f over $\mathbf{A}$, $(\mathbf{A}, v^f) \vDash \varphi \approx \psi$; then we say that the equation is PAt-valid in $\mathbf{A}$. Given a class $\mathbb{C}$ of algebras, we write $\mathbb{C} \vDash_f \varphi \approx \psi$ to mean that each algebra has the equation as a PAt-validity.

Additionally, we abbreviate what it means to satisfy a sequent: given a pair of the form $(\Gamma, \varphi) \in \mathcal{P}(\mathbf{Tm}_{\mathcal{L}_{\mathbf{Y}}}(\mathcal{VAR})) \times \mathbf{Tm}_{\mathcal{L}_{\mathbf{Y}}}(\mathcal{VAR})$ we write $(\mathbf{A}, v) \vDash (\Gamma, \varphi)$ to mean that: for all $x_0, ..., x_n \in \mathcal{VAR}$, and each equation $\delta \approx \gamma \in \mu[\Gamma] = \bigcup\{\mu(\varphi) : \varphi \in \Gamma\}$, if $\delta(v(x_0), ..., v(x_n)) = \gamma(v(x_0), ..., v(x_n))$, then $\delta^*(v(x_0), ..., v(x_n)) = \gamma^*(v(x_0), ..., v(x_n))$ for each $\delta^* \approx \gamma^* \in \mu(\varphi)$. We write $\mathbf{A} \vDash (\Gamma, \varphi)$ to mean that for each valuation $v : \mathcal{VAR} \to \mathbf{A}$, $(\mathbf{A}, v) \vDash (\Gamma, \varphi)$. We write $\mathbf{A} \vDash_f (\Gamma, \varphi)$ to mean that for every polyatomic model over $\mathbf{A}$, v^f, $(\mathbf{A}, v^f) \vDash (\Gamma, \varphi)$.

Given a family $S \subseteq \mathcal{P}(\mathbf{Tm}_{\mathcal{L}_{\mathbf{Y}}}(\mathcal{VAR})) \times \mathbf{Tm}_{\mathcal{L}_{\mathbf{Y}}}(\mathcal{VAR})$ we write $\mathbf{A} \vDash S$ to mean that for each $(\Gamma, \varphi) \in S$, $\mathbf{A} \vDash (\Gamma, \varphi)$. Similarly we define $\mathbf{A} \vDash_f (\Gamma, \varphi)$.

Given an arbitrary v a valuation on $\mathbf{A}$, define v^f by letting $v^f(p) = v(f(p))$ and lifting to terms appropriately. Then note that in this situation, for any term χ:

$$v^f(\chi) = v(\chi[f(p)/p])$$

Lemma 3.4.7. Let $\mathbf{A} \in \mathbf{Y}$ and $\vdash_*$ a finitary extension of $\vdash$. Then we have:

1. $\mathbf{A} \vDash (\Gamma[f(p)/p], \varphi[f(p)/p])$ iff $\mathbf{A} \vDash_f (\Gamma, \varphi)$;

2. If $\mathbf{A} \vDash \vdash_*$ then $\mathbf{A} \vDash_f \ (\vdash_*)^f$

3. For any algebra $\mathbf{A}$, we have that $\mathbf{A} \vDash_f (\Gamma, \varphi)$ if and only if $\langle A^f \rangle \vDash_f (\Gamma, \varphi)$.

4. Let $\mathbf{A}$ be an algebra and $\vdash_*$ an extension of $\vdash$. Then we have that $\mathbf{A} \vDash_f (\vdash_*)^f$ implies $\langle A^f \rangle \vDash \vdash_*$

Proof. (1) Suppose that $\mathbf{A} \nvDash (\Gamma[f(p)/p], \varphi[f(p)/p])$; then take v a valuation witnessing this, and consider v^f; then since:

$$v^f(\chi) = v(\chi[f(p)/p])$$

We have that $(\mathbf{A}, v^f) \nvDash (\Gamma, \varphi)$. The converse is similar.

(2) Suppose that $\mathbf{A} \nvDash_f (\Gamma, \varphi)$ where $(\Gamma, \varphi) \in \vdash_*^f$. Then by the previous statement, $\mathbf{A} \nvDash (\Gamma[f(p)/p], \varphi[f(p)/p])$; but the latter is by definition in $\vdash_*$, as desired.

(3) Note that $\langle A^f \rangle^f = A^f$, since the former is a subalgebra of A. Hence a given v is a regular valuation on A if and only if it is a regular valuation on $\langle A^f \rangle$. This yields the result immediately.

(4) Suppose that $\langle A^f \rangle \nvDash \vdash_*$. Hence there is a sequent $(\Gamma, \varphi) \in \vdash_*$, and a valuation v, such that:

$$\langle A^f \rangle, v \vDash \lambda \approx \gamma, \text{ for } \lambda \approx \gamma \in \mu[\Gamma] \text{ and } \langle A^f \rangle, v \nvDash \lambda' \approx \gamma' \text{ for } \lambda' \approx \gamma' \in \mu(\varphi)$$

Since $\langle A^f \rangle$ is a generated subalgebra, we can express each element as a term over regular elements, say $a = \delta_a(\overline{y})$, for some trm $\delta_a(\overline{y})$. Then for $\lambda \approx \gamma \in \mu[\Gamma]$, we have:

$$[\![\lambda(x_0, ..., x_n)]\!]^{\langle A^f \rangle, v} = \lambda(\overline{\delta_x(\overline{y})})$$

and similar for γ. Hence consider a regular valuation, say w, on $\mathbf{A}$, such that $q_i \mapsto v(y_i)$, for each q_i ocurring in the above formulas. Then we have that for each λ, γ as noted, $[\![\lambda[\overline{\delta_x(\overline{y})}/\overline{x}]]\!]^{\langle A^f \rangle, w} = \lambda(\overline{\delta_x(\overline{y})})$ and similar for γ. Hence for each $\lambda \approx \gamma \in \mu[\Gamma]$

$$(\langle A^f \rangle, w) \vDash \lambda(\overline{\delta_x(\overline{y})}) \approx \gamma(\overline{\delta_x(\overline{y})})$$

And likewise, $(\langle A^f \rangle, w) \nvDash \lambda'(\overline{\delta_x(\overline{y})}) \approx \gamma'(\overline{\delta_x(\overline{y})})$. Now notice that:

$$\mu[\Gamma[\delta_x(\overline{y})/\overline{x}]] = \{\mu(\chi[\delta_x(\overline{y})]) : \chi \in \Gamma\}$$

and the latter is equal substituting every equation $\lambda \approx \gamma \in \mu[\Gamma]$ using the substitution given. Hence for each equation in $\mu[\Gamma[\delta_x(\overline{y})/\overline{x}]]$, $(\langle A^f \rangle, w)$ satisfies it, and it does not satisfy some equation in $\mu[\varphi[\delta_x(\overline{y})/\overline{x}]]$. Hence $(\langle A^f \rangle, w) \nvDash \Gamma[\overline{\delta_x(\overline{y})}], \varphi[\overline{\delta_x(\overline{y})}])$. Hence, since the valuation takes values in regular elements, also $(\langle A^f \rangle, w) \nvDash_f \Gamma[\overline{\delta_x(\overline{y})}], \varphi[\overline{\delta_x(\overline{y})}])$.

In light of all of this, by the previous statement, $\mathbf{A} \nvDash_f (\Gamma[\overline{\delta_x(\overline{y})}/\overline{p}], \varphi[\overline{\delta_x(\overline{y})}/\overline{p}])$. Since $\vdash_*$ is a genuine logic, and thus, is closed under substitution, and since $(\Gamma, \varphi) \in \vdash_*$:

$$(\Gamma[\overline{\delta_x(\overline{y})}/\overline{p}], \varphi[\overline{\delta_x(\overline{y})}/\overline{p}]) \in \vdash_* .$$

But then we have by Proposition 3.4.3 that $(\Gamma[\overline{\delta_x(\overline{y})}/\overline{p}], \varphi[\overline{\delta_x(\overline{y})}/\overline{p}]) \in (\vdash_*)^f$ as well. So we have that $\mathbf{A} \nvDash_f \ (\vdash_*)^f$, as intended. ■

We further isolate one fact which follows from the previous proof.

Corollary 3.4.8. *Suppose that* $\langle A^f \rangle \nvDash (\Gamma, \varphi)$*. Then there is a substitution* σ*, such that* $\mathbf{A} \nvDash_f (\Gamma[\sigma(p)/p], \varphi[\sigma(p)/p])$*.*

These lemmas are needed to establish the connection between PAt-logics and the PAt-*quasivarieties*, which we turn to now:

Definition 3.4.9. Let $\mathbf{A}, \mathbf{B} \in \mathbf{Y}$ be algebras. We say that $\mathbf{A}$ is a *core superalgebra* of $\mathbf{B}$ if $\mathbf{A} \preceq \mathbf{B}$ and $\mathbf{A}^f = \mathbf{B}^f$.

Definition 3.4.10. Let V be a subquasivariety of $\mathbf{Y}$. We define the PAt-*variant* of V as:

$$\mathsf{V}^{\uparrow} := \{\mathbf{A} : \exists \mathbf{B} \in \mathcal{V}, \mathbf{A}^f = \mathbf{B}^f, \text{ and } \mathbf{A} \preceq \mathbf{B}\}.$$

We say a quasivariety is a PAt-*quasivariety*, relative to the translation and the quasivariety $\mathbf{Y}$, if it is the PAt-variant of some subquasivariety of $\mathbf{Y}$.

Furthermore, we say that a PAt-quasivariety is a PAt-*variety*, relative to the translation and $\mathbf{Y}$, if it is the PAt-variant of a subvariety of $\mathbf{Y}$.

This definition parallels the one for DNA logics, and similar to what is found in that setting, we have:

Proposition 3.4.11. Let $\mathbf{K}$ be a family of algebras from $\mathbf{Y}$. Then:

1. $\mathbf{K}$ is a PAt-quasivariety if and only if it is closed under subalgebras, reduced products and core superalgebras.
2. $\mathbf{K}$ is a PAt-variety if and only if it is closed under subalgebras, homomorphic images, products and core superalgebras.

Proof. The proof is a generalisation of [6, Proposition 3.18], where the role of Heyting algebra regular elements is replaced by that of θ-regular elements. Indeed, we provide only the proof of the first statement, as the second follows by the arguments in that Proposition and the ideas sketched here.

For the right to left direction, we simply note that if V is closed under the specified operations, then it forms a quasivariety; hence if it is further closed under core superalgebras it will be its own PAt-variant.
For the other direction, let $\mathsf{V}^{\uparrow}$ be some PAt-variant. We check closure under the operations.

- (Subalgebras): Suppose that $\mathbf{A} \preceq \mathbf{B}$ and $\mathbf{B} \in \mathsf{V}^{\uparrow}$. By definition, then, there is some $\mathbf{C} \in \mathsf{V}$ such that $\mathbf{C} \preceq \mathbf{B}$, with the same set of regular elements. Then consider $C \cap A$. We have that this will be a subalgebra of $\mathbf{B}$ (since intersections of subalgebras are again subalgebras), so will be in V. Moreover, $(\mathbf{C} \cap \mathbf{A})^f = (\mathbf{A})^f$: whenever $a \in A \cap C$, then $a \in A$, so if $f(a) = a$ in $A \cap C$, then the same will hold in A; conversely, if $a \in A$, and $f(a) = a$, then already $f(a) = a$ in B, so by assumption, $f(a) \in A \cap C$. This then shows that $\mathbf{A} \in \mathsf{V}^{\uparrow}$.
- (Reduced Products): Suppose that $(\mathbf{A}_i)_{i \in I}$ are a family of algebras all in $\mathsf{V}^{\uparrow}$, and hence, that $(\mathbf{B}_i)_{i \in I}$ is a family of algebras in V such that $\mathbf{B}_i \preceq \mathbf{A}_i$ with the same regular elements. Now consider $\prod_{i \in I} \mathbf{A}_i / R$. By closure of V under reduced products, we obtain that $\prod_{i \in I} \mathbf{B}_i / R \in \mathsf{V}$. Moreover, by Proposition 3.3.8, we have that:

$$\Big(\prod_{i \in I} \mathbf{B}_i / R\Big)^f = \prod_{i \in I} \mathbf{B}_i^f / R.$$

Hence, we get the latter is equal to $\prod_{i\in I} \mathbf{A}_i^f/R$, which tells us by the same fact that $\prod_{i\in I} \mathbf{A}_i/R \in \mathsf{V}^{\uparrow}$.

■

We have that, as expected, PAt quasivarieties and PAt varieties form a complete lattice:

Lemma 3.4.12. Let $(\mathbf{K}_i)_{i\in I}$ be a non-empty collection of PAt-quasivarieties (resp. PAt-varieties). Then $\bigcap_{i\in I} \mathbf{K}_i$ is a PAt-quasivariety (resp. PAt-variety).

Proof. In light of Proposition 3.4.11, we simply note that the intersection will be closed under all the operations. ■

Indeed, it is not difficult to show that one needs only to look at the composed operator:

$$(\mathbb{ISP}_R)^{\uparrow} =$$

in the sense that whenever $\mathbf{K}$ is a class of algebras, this is a PAt-quasivariety iff $\mathbf{K} = (\mathbb{ISP}_R)^{\uparrow}(\mathbf{K})$. This follows by the same arguments as presented in [6, Theorem 3.21]. Similarly, a class $\mathbf{K}$ is a PAt-variety if and only if $\mathbf{K} = \mathbb{V}^{\uparrow}(\mathbf{K})$.

With this we have, denoting by $\mathsf{PAQ}(\mathbf{X})$ the class of PAt subquasivarieties of $\mathbf{X}$. For that we quickly recall the following simple fact:

Lemma 3.4.13. Let $(\mathbf{A}_i)_{i\in I})$ and $(\mathbf{B}_{i\in I})$ be two families of algebras, and assume that for each i, $\mathbf{A}_i \preceq \mathbf{B}_i$. Then:

- $\prod_{i\in I} \mathbf{A}_{i\in I} \preceq \prod_{i\in I} \mathbf{B}_{i\in I}$
- Given a filter F on I, $\prod_{i\in I} \mathbf{A}_{i\in I}/F \preceq \prod_{i\in I} \mathbf{B}_{i\in I}/F$

Proof. (1) Is straightforward by the definition: given $a \in \prod_{i\in I} A_i$, $a(i) \in A_i \preceq B_i$, hence, $a \in \prod_{i\in I} B_i$; moreover, the operations are computed pointwise.

To see (2), first note that for $a, b \in \prod_{i\in I} A_i$, $a \sim_F b$ implies that $a \sim_F b$ in $\prod_{i\in I} B_i$. Hence we can send the equivalence class $[a]_F \in \prod_{i\in I} A_i/F$ to $[a]_F \in \prod_{i\in I} B_i/F$. It is clear to see that this is then a homomorphism. ■

Proposition 3.4.14. The map $(-)^{\uparrow} : \Lambda(\mathbf{X}) \to \mathsf{PAQ}(\mathbf{X})$ is a complete lattice homomorphism.

Proof. First let $(\mathbf{K}_i)_{i\in I}$ be a collection of subquasivarieties of $\mathbf{X}$. Then we show that:

$$(\bigcap_{i\in I} \mathbf{K}_i)^{\uparrow} = \bigcap_{i\in I} \mathbf{K}_i^{\uparrow}.$$

Indeed, if $\mathbf{A} \in (\bigcap_{i\in I} \mathbf{K}_i)^{\uparrow}$ then $\mathbf{A}$ is the core superalgebra of some $\mathbf{B}$ which is in every subquasivariety; but then clearly $\mathbf{A}$ is in each such subquasivariety. For the converse, if $\mathbf{A}$ is a core superalgebra of $\mathbf{B}_i$ each respectively in $\mathbf{K}_i$, note that $\bigcap_{i\in I} \mathbf{B}_i$ will be a subalgebra of all $\mathbf{B}_i$, it will have the same regular elements, and will be in $\bigcap \mathbf{K}_i$, showing the result.

For the preservation of the join, we show that:

$$(\bigvee_{i\in I} \mathbf{K}_i)^{\uparrow} = \bigvee_{i\in I} \mathbf{K}_i^{\uparrow}$$

Where the left hand side join is taken in the class of quasivarieties, and the left hand side in PAt-quasivarieties. Indeed, by our previous remark, and general facts about closure operators, we have that:

$$\bigvee_{i \in I} \mathbf{K}_i^{\uparrow} = (\mathbb{ISP}_R)^{\uparrow}\big(\bigcup_{i \in I} \mathbf{K}_i^{\uparrow}\big)$$

Whereas in turn we have that $\bigvee_{i \in I} \mathbf{K}_i = \mathbb{ISP}_R\big(\bigcup_{i \in I} \mathbf{K}_i\big)$. So we show that:

$$\mathbb{ISP}_R^{\uparrow}\big(\bigcup_{i \in I} \mathbf{K}_i\big) = \mathbb{ISP}_R^{\uparrow}\big(\bigcup_{i \in I} \mathbf{K}_i^{\uparrow}\big).$$

One inclusion follows from the inflationarity and monotonicity of the $\uparrow$ operator. For the other, assume that $\mathbf{A}$ is a core superalgebra of $\mathbf{B}$, which is a subalgebra of $\mathbf{C} = \prod_{j \in J} \mathbf{D}_j/F$, where for each j, there is some $\mathbf{E}_j$ such that $\mathbf{E}_j \preceq \mathbf{D}_j$, and $\mathbf{E}_j \in \mathbf{K}_j$.

Now consider $\prod_{j \in J} \mathbf{E}_j/F$. Then by Lemma 3.4.13, we have that $\prod_{j \in J} \mathbf{E}_j/F \preceq \mathbf{C}$. Since $\mathbf{B} \preceq \mathbf{C}$, we let $\mathbf{G} = \mathbf{B} \cap \prod_{j \in J} \mathbf{E}_j/F$, which is a subalgebra of both of these algebras. Then we claim that $\mathbf{E}$ is a core subalgebra of $\mathbf{B}$: indeed, if $a \in B$, and $f(a) = a$, then surely $a \in \mathbf{C}$, and there $f(a) = a$. Hence, $a \in \prod_{j \in J} \mathbf{E}_j/F$ since this has the same regular elements as $\mathbf{C}$. So $a \in \mathbf{G}$ by definition. Thus, we have that $\mathbf{A}$ is a core superalgebra of $\mathbf{G}$, and $\mathbf{G}$ is in $\mathbb{SP}_R(\bigcup_{i \in I} \mathbf{K}_i)$. So $\mathbf{A} \in \mathbb{ISP}_R^{\uparrow}(\bigcup_{i \in I} \mathbf{K}_i)$, as desired. This shows the other inclusion, and concludes the proof. ∎

We conclude by mentioning some natural facts holding for PAt-Quasivarieties:

Corollary 3.4.15. *Every* PAt*-Quasivariety is generated by its regularly generated algebras.*

Proof. Note that if $\mathbf{K}$ is a PAt-Quasivariety, and $B \in K$ is not regularly generated, then $\langle B^f \rangle \leqslant B$, and so $\langle B^f \rangle \in \mathbf{K}$ is a regularly generated algebra which moreover has B as its core-superalgebra. ∎

This can be sharpened in the case of PAt-varieties into a Birkhoff-style theorem:

Definition 3.4.16. Let $\mathbf{A}$ be some algebra. We say that $\mathbf{A}$ is regular-subdirectly irreducible (RSI) if it is regularly generated and subdirectly irreducible.

Theorem 3.4.17. *Every* PAt*-Variety is generated by its regular-subdirectly irreducible elements.*

Proof. Let $\mathbf{K}$ be a PAt-Variety, and let $\mathbf{K}_{SI}$ be the class of subdirectly irreducible algebras in $\mathbf{K}$, and $\mathbf{K}_{RSI}$ the class of regular subdirectly irreducible algebras. Then clearly $\mathbb{V}^{\uparrow}(\mathbf{K}_{RSI}) \subseteq \mathbf{K}$. Conversely, we show that $\mathbf{K} \subseteq \mathbb{V}^{\uparrow}(\mathbf{K}_{RSI})$ by showing that every regularly generated algebra in $\mathbf{K}$ is in $\mathbb{V}^{\uparrow}(\mathbf{K}_{RSI})$; then the inclusion will follow from the last corollary.

So assume that $\mathbf{A}$ is a regularly generated algebra. By Birkhoff's Subdirect Decomposition Theorem (see Preliminaries), we have that $\mathbf{A}$ is a subdirect product of subdirectly irreducible algebras, i.e:

$$h : A \hookrightarrow \prod_{i \in I} B_i$$

Where the $\mathbf{B}_i \in \mathbf{K}_{SI}$. We will show that in fact $\mathbf{B}_i \in \mathbf{K}_{RSI}$. So let $\mathbf{B}_i$ and $c \in B_i$ be arbitrary. By assumption on the embedding, we have that for some $a \in A$:

$$h(a)(i) = c$$

Since a is regularly generated, there is some term $t(x_0, ..., x_n)$ written over regular elements from A, such that $t(e_0, ..., e_n) = a$ for some $e_0, ..., e_n \in A$. Hence:

$$h(a)(i) = h(t(e_0, ..., e_n))(i) = t(h(e_0)(i), ..., h(e_n)(i)) = c$$

But now notice that $h(e_j)(i)$ is regular: indeed, $f(h(x_j)(i) = h(f(e_j)(i) = h(e_j)(i)$, which follows from h being an embedding, and e_j being a regular element. So we obtain that c is regularly generated. Thus we obtain that A is a subdirect product of regular subdirectly irreducible algebras from $\mathbf{K}$, i.e, $A \in \mathbb{V}(\mathbf{K}_{RSI})$. ■

3.5 Dual equivalence between PAt-logics and quasivarieties

As before, let $\mathbf{X}$ and $\mathbf{Y}$ be given, and $\vdash$ a logic such that $\mathbf{Y}$ is its equivalent algebraic semantics. Let $\Lambda(\vdash)$ be the lattice of finitary extensions of $\vdash$, and $\Lambda(\mathbf{Y})$ the lattice of subquasivarieties of $\mathbf{Y}$. Consider the two following maps: for $\mathbf{K}$ a subquasivariety of $\mathbf{Y}$, and $\vdash_*$ a finitary extension. Then first consider the following operators:

$$\mathsf{QVar}(\vdash_*) := \{\mathbf{A} : \forall(\Gamma, \varphi) \in \vdash_*, \mathbf{A} \vDash (\Gamma, \varphi)\}$$
$$\mathsf{Log}(\mathbf{K}) := \{(\Gamma, \varphi) : K \vDash (\Gamma, \varphi)\}$$

Similarly, we define the polyatomic version of these maps:

$$\mathsf{QVar}^f(\vdash_*) := \{\mathbf{A} : \forall(\Gamma, \varphi) \in \vdash_*, \mathbf{A} \vDash_f (\Gamma, \varphi)\}$$
$$\mathsf{Log}^f(\mathbf{K}) := \{(\Gamma, \varphi) : K \vDash_f (\Gamma, \varphi)\}$$

We will now show that given any such K and $\vdash_*$, applying these operators yields a PAt-logic and a PAt-quasivariety respectively. First, we show the following, which has the exact same proof as [6, Lemma 3.22].

Proposition 3.5.1. The PAt-validity of a pair (Γ, φ) is preserved under the operations of subalgebras, homomorphic images, products and core superalgebras.

As a corollary, we note it is moreover preserved under reduced products.

Corollary 3.5.2. *Given a logic $\vdash_*$, the class $\mathsf{QVar}^f(\vdash_*)$ is a PAt-quasivariety.*

Proof. By Proposition 3.5.1, we have that $(\vdash_*)^f$ is closed under subalgebras, reduced products and core superalgebras. Hence by Proposition 3.4.11, we get the result. ■

Given a quasivariety $\mathbf{K} \in \Lambda(\mathbf{Y})$, we also denote by $\vdash_{\mathbf{K}}$ the logic $\mathsf{Log}(\mathbf{K})$, and say this is the logic *dually corresponding* to $\mathbf{K}$.

Proposition 3.5.3. Given a quasivariety $\mathbf{K}$, $\mathsf{Log}^f(\mathbf{K})$ is the PAt-variant of $\vdash_{\mathbf{K}}$, the finitary extension dually corresponding to $\mathbf{K}$.

Proof. To see this, note that $\Gamma \nvdash_{\mathbf{K}} \varphi$ if and only if there is some $\mathbf{A} \in \mathbf{K}$, and $\mathbf{A} \nvDash (\Gamma, \varphi)$. By Proposition 3.4.7, this holds iff $(\mathbf{A}, v^f) \nvDash (\Gamma[f(p)/p], \varphi[f(p)/p])$. But this holds if and only if $(\Gamma, \varphi) \notin \mathbf{K}^f$. ■

$$\begin{array}{ccc} \Lambda(\vdash) & \xrightarrow{\vdash^f} & \Lambda(\vdash)^f \\ \big\downarrow{\scriptstyle \cong^{op}} & & \big\downarrow{\scriptstyle \cong^{op}} \\ \Lambda(\mathsf{Y}) & \xrightarrow[\uparrow]{} & \Lambda(\mathsf{Y})^f \end{array}$$

Figure 3.1: Commuting Diagram of PAt-Logics and Quasivarieties

Denote by $\Lambda(\vdash)^f$ and $\Lambda(\mathbf{Y})^f$ respectively the lattices of PAt-variants of $\vdash$ and PAt-quasivarieties of $\mathbf{Y}$. Then we have that the diagram in Figure 3.1 commutes:

This follows, exactly as in [6] by observing:

1. $\mathsf{QVar}^f(\vdash^f) = \mathsf{QVar}(\vdash)^{\uparrow}$

2. $\mathsf{Log}^f(\mathbf{K}^{\uparrow}) = (\mathsf{Log}(\mathbf{K}))^f$

This follows by the same arguments as those found in the above cited paper, using in an essential fashion Lemma 3.4.7. We extract the consequences of this for our purposes:

Theorem 3.5.4. *(Definability Theorem) Every K, a* PAt*-quasivariety, is defined by its* PAt*-validities, i.e for every algebra $\mathbf{A}$:*

$$\mathbf{A} \in \mathbf{K} \iff \mathbf{A} \vDash_f \mathsf{Log}^f(\mathbf{K})$$

Proof. If $\mathbf{A} \in \mathbf{K}$, then certainly $\mathbf{A} \vDash_f \mathsf{Log}^f(\mathbf{K})$. Conversely, if $\mathbf{A} \notin \mathbf{K}$, by the above commutativity result, we have that $K = \mathsf{QVar}^f(\mathsf{Log}^f(\mathbf{K}))$, since $\mathsf{QVar}^f(\mathsf{Log}^f(\mathbf{K})) = \mathsf{QVar}^f(\mathsf{Log}(K)^f) = \mathsf{QVar}(\mathsf{Log}(\mathbf{K}))^{\uparrow} = K^{\uparrow} = K$. So $\mathbf{A} \notin \mathsf{QVar}^f(\mathsf{Log}^f(\mathbf{K}))$, hence $\mathbf{A} \nvDash \mathsf{Log}^f(\mathbf{K})$. ■

As a corollary we have the following Birkhoff theorem analogue:

Theorem 3.5.5. *(* PAt*-Birkhoff) A class $\mathbf{K}$ of algebras is a* PAt*-quasivariety if and only if it is* PAt*-definable by a collection of quasi-equations.*

Similarly, we have completeness:

Theorem 3.5.6. *(Algebraic completeness) Every* PAt*-logic $\vdash_s$ is complete with respect to its corresponding* PAt*-quasivariety, i.e, for every pair (Γ, φ):*

$$\Gamma \vdash_s \varphi \iff \mathsf{QVar}^f(\vdash_s) \vDash_f (\Gamma, \varphi)$$

Proof. If $\Gamma \vdash_s \varphi$, then certainly $\mathsf{QVar}^f(\vdash_s) \vDash_f (\Gamma, \varphi)$. Now assume that $\Gamma \nvdash_s \varphi$. Then note that:

$$\vdash_s \;=\; \mathsf{Log}^f(\mathsf{QVar}^f(\vdash_s))$$

Which follows since $\mathsf{Log}^f(\mathsf{QVar}^f(\vdash_s)) = \mathsf{Log}^f(\mathsf{QVar}(\vdash_s)^{\uparrow}) = \mathsf{Log}(\mathsf{QVar}(\vdash_s))^f = \vdash_s$. Hence we obtain the result. ■

Then in light of this, together with Corollary 3.4.5 and Proposition 3.4.14, we have:

Corollary 3.5.7. *The map* QVar^f *is a dual isomorphism between the complete lattices* $\Lambda^f(\vdash)$ *and* $\Lambda^f(\mathbf{Y})$.

Moreover, in light of Corollary 3.4.15 we have:

Corollary 3.5.8. *Every* PAt*-logic* $\vdash_f$ *is sound and complete with respect to regularly generated algebras, with respect to* PAt*-validity.*

We thus obtain a completeness theorem for all PAt-logics. We expect that these logics should be ubiquitous, both in light of our examples, and the relative ease of finding selective translations. Given the motivations provided in the introduction, we also hope that they could be useful in applications. For our purposes, we will show in the next chapter that they are intimately related to Blok-Esakia theory.

3.6 Chapter Summary

We summarise our contributions in this chapter as follows:

- We develop the framework proposed in [53], and introduce the concepts of *selective translations*, and prove some basic properties of these translations.

- We generalise the notions introduced in [6] of DNA-logics and varieties to the setting of *Polyatomic* logics and quasivarieties, and prove analogues of Birkhoff's theorem and algebraic completeness for this setting

Chapter 4

General Blok-Esakia Theory

In this chapter we provide a general theoretical application of PAt-logics: the development of a generalised Blok-Esakia theory. This is done by an analysis of the GMT translation, which we carry out in the first sections, in the setting of strongly selective translations. We identify a more specific class of translations called *sober translations* which allow for a Blok-Esakia theory mimicking the GMT case, and note the fact that not all translations we have so far considered are of this kind. We then outline a path to generalising this, making use of natural concepts from DNA-logics which we generalise to Polyatomic Logics. We conclude by showing that these are genuine generalisations of the concepts as they appear in the GMT case.

4.1 The GMT Translation and Classic Blok-Esakia Theory

In this section we recall some known facts from Blok-Esakia theory. For an in-depth reference, we refer the reader to [23], see also [17]. We will provide full proofs of some known facts, as these turn out to have all the relevant ideas which are needed for the general case, and we prefer to outline them in this more concrete setting.

Throughout this and following sections, let $\Lambda(\mathsf{IPC})$ denote the lattice of axiomatic extensions of IPC and $\mathbf{NExt}(\mathsf{S4})$ denote the class of normal extensions of $\mathsf{S4}$. Let $\Lambda(\mathbf{HA})$ and $\Lambda(\mathbf{S4})$ respectively denote the lattices of subvarieties of Heyting algebras and S4-algebras, respectively. Similar notation is used for other systems and varieties.

We begin with the notion of a modal companion which grounds the whole endeavour:

Definition 4.1.1. For $M \in \mathbf{NExt}(\mathsf{S4})$ and $L \in \Lambda(\mathsf{IPC})$, we say that M is a *modal companion* of L iff:

$$\varphi \in L \iff GMT(\varphi) \in M.$$

It follows by our discussion in Chapter 3 (see the discussion in page 30) and the known algebraic completeness theorems of $\mathsf{S4}$ with respect to $\mathbf{S4}$-algebras and IPC with respect to Heyting algebras, that $\mathsf{S4}$ is a modal companion of IPC. It is however, by far the only one. Another well-known companion is the logic known as Grz: this can be axiomatised as:

$$\mathsf{Grz} := \mathsf{K} \oplus \Box(\Box(p \to \Box p) \to p) \to p.$$

It was shown by Grzegorczyk [39] that this logic is a modal companion of IPC[1]. Given this diversity of companions, one generally makes use of three maps, carrying logics to logics: the first two are denoted by τ and ρ:

$$\begin{aligned} \tau : \Lambda(\mathsf{IPC}) &\to \mathbf{NExt}(\mathsf{S4}) \\ L &\mapsto \mathsf{S4} \oplus \{GMT(\varphi) : \varphi \in L\} \end{aligned}$$

And

$$\begin{aligned} \tau : \Lambda(\mathsf{IPC}) &\to \mathbf{NExt}(\mathsf{Grz}) \\ L &\mapsto \mathsf{Grz} \oplus \{GMT(\varphi) : \varphi \in L\} \end{aligned}$$

By definition, we have that given $L \in \Lambda(\mathsf{IPC})$, $\tau(L)$ and $\sigma(L)$ are logics, though we have no guarantee that they are – as desired – modal companions. In the opposite direction, one defines the following map, from logics $M \in \mathbf{NExt}(\mathsf{S4})$ to sets of intuitionistic formulas:

$$\rho(M) = \{\varphi : GMT(\varphi) \in M\}$$

The definition is suggested by the notion of modal companion, and one would want to say that $\rho(M)$ is a logic in $\Lambda(\mathsf{IPC})$, but this is again not immediate. A way to prove these facts goes by looking at the algebraic semantics of these logics, and corresponding to these syntactic assignments some semantic ones.

Let $(-)_\Box : \mathbf{S4} \to \mathbf{HA}$ be the functor which maps an S4-algebra $\mathbf{B}$ to $B_\Box$, and which, for every homomorphism of S4-algebras $f : B \to B'$ maps this to $f{\upharpoonright}_\Box : B_\Box \to B'_\Box$, the restriction of f. This is well-defined: if $a \in B_\Box$, then $\Box a = a$, so $\Box f(a) = f(\Box a) = f(a)$. As it turns out, this functor has some additional nice properties:

- It is a *right adjoint* functor, and hence, preserves all limits (including injective homomorphisms and products, which are specific kinds of limits);
- The right adjoint functor preserves surjective homomorphisms.

Its corresponding left adjoint functor $B(-) : \mathbf{HA} \to \mathbf{S4}$, assigns to each Heyting algebra H its Boolean envelope $B(H)$ (see Preliminaries), together with a $\Box$-modality defined as follows:

$$\Box a = \bigvee \{c \in H : c \leqslant a\}$$

This is well-defined [23, Construction 2.4.7]. On maps, given $f : \mathbf{H} \to \mathbf{H}'$ a Heyting algebra homomorphism, we let $B(f) : B(\mathbf{H}) \to B(\mathbf{H}')$ be the unique lift of this map, which can be shown to preserve the modality. Then we have that:

- The functor $B(-)$ is a left adjoint, and hence preserves all colimits (including surjective homomorphisms, which are coequalizers in the case of varieties) and also preserves finite products and injective homomorphisms.

Finally, the adjunction of these two functors has two very desirable properties:

[1]We note however that the current axiomatisation is, as far as we are aware, due to Segerberg [55], who noted Grzegorczyk's original axiom was equivalent to the present simpler one.

- The unit map η is an isomorphism.

- The counit map ε is pointwise injective.

Using these functors one can define appropriate maps on varieties. Given a variety $\mathbf{V} \in \Lambda(\mathbf{HA})$, consider:

$$\tau(\mathbf{V}) = \{\mathbf{B} \in \mathbf{S4} : \mathbf{B}_\Box \in \mathbf{V}\}.$$

The ambiguity in denoting this as τ is intentional, and is motivated by the following proposition:

Proposition 4.1.2. Let $L \in \Lambda(\mathsf{IPC})$ and $\mathbf{K} \in \Lambda(\mathsf{HA})$. Then:

1. $\tau(\mathbf{K})$ is a variety, and τ is a complete homomorphism on the lattice of varieties.

2. $\tau(\mathsf{Var}(L)) = \mathsf{Var}(\tau(L))$. Hence $\tau(\mathsf{Log}(\mathbf{K})) = \mathsf{Log}(\tau(\mathbf{K}))$ and τ is a complete homomorphism on the lattice of logics.

3. $\tau(L)$ is the least modal companion of L;

Proof. To see (1), suppose that $\mathbf{A} \in \tau(\mathbf{K})$. By Tarski's HSP theorem, assume that $\mathbf{A}$ is a homomorphic image of $\mathbf{B}$, which is a subalgebra of $\prod_{i\in I} \mathbf{B}_i$, and $(\mathbf{B}_i)_\Box \in K$. Since $(-)_\Box$ commutes with all limits and surjective homomorphisms, then $\mathbf{A}_\Box$ is a homomorphic image of $\mathbf{B}_\Box$ which is a subalgebra of $\prod_{i\in I}(\mathbf{B}_i)_\Box$. Hence $\mathbf{A}_\Box \in K$, and thus, $\mathbf{A} \in \tau(K)$.

To see that τ is a homomorphism on the lattice of varieties, notice that $\tau(\bigcap_{i\in I} \mathbf{K}_i) = \bigcap_{i\in I} \tau(\mathbf{K}_i)$ by definition: $\mathbf{A} \in \tau(\bigcap_{i\in I} \mathbf{K}_i)$ if and only if $\mathbf{A}_\Box \in \bigcap_{i\in I} \mathbf{K}_i$, if and only if $\mathbf{A}_\Box \in \mathbf{K}_i$ for each i, if and only if $\mathbf{A} \in \tau(\mathbf{K}_i)$ for each i, if and only if $\mathbf{A} \in \bigcap_{i\in I} \tau(\mathbf{K}_i)$. Moreover, if $\mathbf{A} \in \tau(\bigvee_{i\in I} \mathbf{K}_i)$, then $\mathbf{A}_\Box \in \bigvee_{i\in I} \mathbf{K}_i$; hence $\mathbf{A}_\Box$ is a homomorphic image of $\mathbf{B}$, which is a subalgebra of $\prod_{i\in I} \mathbf{C}_i$, where the factors are from the $\mathbf{K}_i$. Then because the unit is an isomorphism, $\mathbf{B} = \mathbf{B}'_\Box$ and $\prod_{i\in I} \mathbf{C}_i = \prod_{i\in I}(\mathbf{C}'_i)_\Box$, and $\mathbf{C}'_i \in \tau(\mathbf{K}_i)$ surely. Hence $\mathbf{A} \in \bigvee_{i\in I} \tau(\mathbf{K}_i)$. For the converse, suppose that $\mathbf{A} \in \bigvee_{i\in I} \tau(\mathbf{K}_i)$; then $\mathbf{A}$ is a homomorphic image of $\mathbf{B}$, a subalgebra of $\prod_{i\in I} \mathbf{C}_i$, and $(\mathbf{C}_i)_\Box \in \mathbf{B}_i$; then $\mathbf{A}_\Box$ is a homomorphic image of $\mathbf{B}_\Box$, a subalgebra of $\prod_{i\in I}(\mathbf{C}_i)_\Box$. Hence $\mathbf{A} \in \bigvee_{i\in I} \mathbf{B}_i$, so $\mathbf{A} \in \tau(\bigvee_{i\in I} \mathbf{K}_i)$.

To see (2), we see that for each $\mathbf{S4}$-algebra $\mathbf{B}$, $\mathbf{B} \in \tau(\mathsf{Var}(L))$ if and only if $\mathbf{B}_\Box \in \mathsf{Alg}(L)$, if and only if $\mathbf{B}_\Box \models L$ if and only if $\mathbf{B} \models \{GMT(\varphi) : \varphi \in L\}$, if and only if $\mathbf{B} \in \mathsf{Var}(\tau(L))$. Hence by algebraic completeness:

$$\begin{aligned} \tau(\mathsf{Log}(\mathbf{K})) = \mathsf{Log}(\tau(\mathbf{K})) &\iff \mathsf{Var}(\tau(\mathsf{Log}(\mathbf{K})) = \mathsf{Var}(\mathsf{Log}(\tau(\mathbf{K}))) = \tau(\mathbf{K}) \\ &\iff \tau(\mathsf{Var}(\mathsf{Log}(\mathbf{K}))) = \tau(\mathbf{K}). \end{aligned}$$

With this it is straightforward to see that τ is also a complete homomorphism on logics.

To see (3), notice that by definition if $\varphi \in L$ then $GMT(\varphi) \notin \tau(L)$. Conversely, if $\varphi \notin L$, let $\mathbf{H} \in \mathsf{Var}(L)$ be such that $\mathbf{H} \not\models \varphi$; because η is an isomorphism, we know that there is some $\mathbf{B}$ such that $\rho(\mathbf{B}) = \mathbf{H}$; hence, $\mathbf{B} \in \tau(\mathsf{Alg}(L))$. By (2), then $\mathbf{B} \in \mathsf{Alg}(\tau(L))$, i.e, $\mathbf{B} \models \tau(L)$, and $\mathbf{B} \not\models GMT(\varphi)$. This proves that $\tau(L)$ is a modal companion. It is clear to see that it must be least. ∎

We now turn to the other direction. Paralleling the assignment ρ above, we can define:

$$\rho(\mathbf{K}) := \{\mathbf{B}_\Box : \mathbf{B} \in \mathbf{K}\}.$$

The following proposition also establishes that this indeed gives us a well-defined map, and also that the map ρ on logics does give us genuine modal companions. We call the attention to the reader to the different properties of the adjunctions we will use here, compared to τ:

Proposition 4.1.3. Let $\mathbf{K} \in \Lambda(\mathsf{S4})$. Then:

1. $\rho(\mathbf{K})$ is a variety, and ρ is a surjective complete homomorphism on the lattice of varieties.
2. The map $\rho : \mathbf{NExt}(\mathsf{S4}) \to \Lambda(\mathsf{IPC})$ defined as $\rho(L) = \{\varphi : GMT(\varphi) \in L\}$ is a complete lattice homomorphism.
3. For all $N \in \mathbf{NExt}(\mathsf{S4})$, and $L \in \Lambda(\mathsf{IPC})$, N is a modal companion of L if and only if $\rho(N) = L$. Hence, for all $\mathbf{K} \in \mathsf{Var}(\mathsf{S4})$ and $\mathbf{P} \in \mathsf{Var}(\mathsf{HA})$ we have that if $\mathsf{Log}(K)$ is a modal companion of $\mathsf{Log}(\mathbf{P})$ then $\rho(\mathbf{K}) = \mathbf{P}$.

Proof. (1) Suppose that $\mathbf{A} \in \rho(\mathbf{K})$. By Tarski's HSP theorem, $\mathbf{A}$ is a homomorphic image of $\mathbf{B}$, which is a subalgebra of $\prod_{i\in I}(\mathbf{C}_i)_\Box$, where $\mathbf{C}_i \in \mathbf{K}$. Note that since $(-)_\Box$ preserves products, the latter is $(\prod_{i\in I}\mathbf{C}_i)_\Box$. Hence, since $B(-)$ preserves homomorphic images, $B(\mathbf{A})$ is a homomorphic image of $B(\mathbf{B})$ which is a subalgebra of $B((\prod_{i\in I}\mathbf{C}_i)_\Box)$; since the counit is an injective homomorphism, $B((\prod_{i\in I}\mathbf{C}_i)_\Box)$ is a subalgebra of $\prod_{i\in I}\mathbf{C}_i$, and so $B(\mathbf{B})$ is as well. Thus, $B(\mathbf{A}) \in \mathbf{K}$, since the latter is a variety. Since $(B(\mathbf{A}))_\Box \cong \mathbf{A}$, we have that $\mathbf{A} \in \rho(\mathbf{K})$. We leave the verification that this is a complete homomorphism to the reader. To see that it is surjective, simply note that if $\mathbf{K}$ is any variety of Heyting algebras, then $\tau(\mathbf{K}) = \{\mathbf{B} : \rho(\mathbf{B}) \in \mathbf{K}\}$ is such that $\rho\tau(\mathbf{K}) = \mathbf{K}$.

To see (2), first note that if $\mathbf{H} \in \rho(\mathsf{Var}(M))$ then $\mathbf{H} = \rho(\mathbf{B})$ where $\mathbf{B} \vDash M$; hence, if $GMT(\varphi) \in M$, then $\mathbf{H} \vDash \varphi$, so $\mathbf{H} \vDash \rho(M)$. Thus $H \in \mathsf{Var}(\rho(M))$. Conversely, assume that $\varphi \in \rho(M)$, and let $\mathbf{H}$ be such that $\rho(\mathbf{B}) = \mathbf{H}$ and $\mathbf{B} \vDash M$; since $\varphi \in \rho(M)$, then $GMT(\varphi) \in M$, so $\mathbf{B} \vDash GMT(\varphi)$, and hence, $\mathbf{H} \vDash \varphi$ so:

$$\rho(M) \subseteq \mathsf{Log}(\rho(\mathsf{Var}(M))$$

Hence by algebraic completeness, and the fact that $\rho(\mathsf{Var}(M))$ is a variety:

$$\mathsf{Var}(\rho(M)) \subseteq \mathsf{Var}(\mathsf{Log}(\rho(\mathsf{Var}(M)))) = \rho(\mathsf{Var}(M))$$

This shows the other inclusion. Hence we have that $\rho(\mathsf{Var}(M)) = \mathsf{Var}(\rho(M))$.

Thus, given $M \in \mathbf{NExt}(\mathsf{S4})$, $\mathsf{Log}(\mathsf{Var}(\rho(M))) = \mathsf{Log}(\rho(\mathsf{Var}(M))$. We will show that the latter is simply $\rho(M)$. Indeed, if $\varphi \in \rho(M)$, then $GMT(\varphi) \in M$; if $\mathbf{A} \in \rho(\mathsf{Var}(M))$, then for some $\mathbf{B} \in \mathsf{Var}(M)$, $\mathbf{B}_\Box = \mathbf{A}$. Since $GMT(\varphi) \in M$, by the results of 30, $\mathbf{A} \vDash \varphi$, so $\mathbf{A} \in \mathsf{Log}(\rho(\mathsf{Var}(M))$. Conversely, if $\varphi \notin \rho(M)$, then $GMT(\varphi) \notin M$, so let $\mathbf{A} \in \mathsf{Var}(M)$ be such that $\mathbf{A} \nvDash GMT(\varphi)$; we have that $\mathbf{A}_\Box \nvDash \varphi$ by the same result, and since $\mathbf{A}_\Box \in \rho(\mathsf{Var}(M))$, this shows that $\varphi \notin \mathsf{Log}(\rho(\mathsf{Var}(M))$. By arguments analogous to those above, we can show that ρ is a complete homomorphism.

(3) Note that given any such N and L, we have that $\rho(M) = L$ if and only if $\varphi \in L$ if and only if $GMT(\varphi) \in M$ if and only if M is a modal companion of L.

To see the last statement, assume that $\mathsf{Log}(K)$ is a modal companion of $\mathsf{Log}(P)$. Then $\rho(\mathsf{Log}(K)) = \mathsf{Log}(P)$. Hence $\mathsf{Log}(\rho(K)) = \mathsf{Log}(P)$, so by algebraic completeness, $\rho(K) = P$. ■

The final map that appears in this context is usually called σ. Whereas, by the contents of the previous propositions, τ is the *least* modal companion, σ outlines the greatest. Its definition on algebras is thus as follows:

$$\sigma(\mathbf{K}) := \mathbb{HSP}\{B(\mathbf{H}) : \mathbf{H} \in \mathbf{K}\}$$

However, a careful analysis of the Blok-Esakia theory, reveals that the greatest modal companion is the one structure which heavily depends on properties of **S4** and **HA**. One fact which can be established generically is the following:

Proposition 4.1.4. For any $L \in \Lambda(\mathsf{IPC})$, $\mathsf{Log}(\sigma(\mathsf{Var}(L))$ is the greatest modal companion of L.

Proof. First we note that this is a modal companion: assume that $\varphi \in L$. Since $\sigma(\mathsf{Var}(L))$ is generated by $B(\mathbf{H})$ for $\mathbf{H} \in \mathsf{Var}(L)$, if $\mathbf{H}$ is such an algebra, then $\mathbf{H} \models \varphi$, so $B(\mathbf{H}) \models GMT(\varphi)$ (given that $(B(\mathbf{H}))_\Box \cong \mathbf{H}$. Hence $GMT(\varphi) \in \mathsf{Log}(\sigma(\mathsf{Var}(L)))$. Conversely, if $\varphi \notin L$, let $\mathbf{H} \in \mathsf{Var}(L)$ be such that $\mathbf{H} \not\models \varphi$; hence $B(\mathbf{H}) \not\models GMT(\varphi)$. But since $B(\mathbf{H}) \in \sigma(\mathsf{Var}(L))$ we have that $GMT(\varphi) \notin \mathsf{Log}(\sigma(\mathsf{Var}(L)))$.

To see that it is the greatest modal companion, suppose that M is an arbitrary modal companion of L. To show that $M \subseteq \mathsf{Log}(\sigma(\mathsf{Var}(L)))$ it suffices to show that $\sigma(\mathsf{Var}(L)) \subseteq \mathsf{Var}(M)$. In turn to show this, it suffices to show that $\{B(\mathbf{H}) : \mathbf{H} \in \mathsf{Var}(L)\} \subseteq \mathsf{Var}(M)$. So let $\mathbf{H} \in \mathsf{Var}(L)$ be arbitrary. Since M is a modal companion of L, then $\rho(\mathsf{Var}(M)) = \mathsf{Var}(L)$. Hence $\mathbf{H} \cong \rho(\mathbf{B})$ for some $\mathbf{B} \in \mathsf{Var}(M)$. Moreover, we know that:

$$B(\mathbf{B}_\Box) \hookrightarrow \mathbf{B}$$

Maps injectively, via the counit map. Hence $B(\mathbf{B}_\Box) \in \mathsf{Var}(M)$. But then $\sigma(\mathbf{H}) \in \mathsf{Var}(M)$, which shows the result. ■

Unlike the remaining maps, the fact that $\sigma : \Lambda(\mathsf{IPC}) \to \mathbf{NExt}(\mathsf{Grz})$ as defined above is a homomorphism on the lattice of varieties, or that its definition on varieties matches up with its definition on logics, requires using "Blok's lemma". This is the result which says that for each $\mathbf{K} \in \Lambda(\mathbf{Grz})$, that is, each subvariety of the variety of **Grz**-algebras, $\mathbf{K}$ is generated by its elements of the form $B(\mathbf{H})$ for $\mathbf{H} \in \mathbf{HA}$.

This result is quite particular to Heyting algebras and IPC, and to the properties of the **Grz** axiom. On the other hand, it is plain to see from the facts we have about τ and ρ, that the categorical properties we noted – the unit of the adjunction being an isomorphism, the counit being injective, etc – are already enough to allow some properties – like FMP, tabularity, or decidability - to transfer between logical systems.

4.2 Sober Translations

The recollections of last section should give us a good idea of how to generalise basic Blok-Esakia theory for a great number of contextual translations. For that purpose, assume that $\mathbf{X}$ and $\mathbf{Y}$ are quasivarieties, where $\mathbf{X}$ is an algebraic semantics of a logic $\vdash_\mathbf{X}$, with set of equations $\mu_\mathbf{X}(x)$ witnessing this fact, and $\mathbf{Y}$ is the equivalent algebraic semantics of $\vdash_\mathbf{Y}$, with sets $\mu_\mathbf{Y}(x)$ and $\Delta_\mathbf{Y}(x,y)$. Let $\Lambda(\vdash_\mathbf{X})$ and $\Lambda(\vdash_\mathbf{Y})$ be the lattices of finitary extensions of the logics. We assume throughout that $\overline{\zeta} = \langle \zeta, \Theta, f \rangle$ is a selective translation from $\models_\mathsf{X}$ to $\models_\mathsf{Y}$. Correspondingly, let $\Lambda(\mathbf{X})$ and $\Lambda(\mathbf{Y})$ be the lattices of subquasivarieties of $\mathbf{X}$ and $\mathbf{Y}$.

We also need a special assumption which is met in all cases we consider. This is that, essentially, the translation commutes with the algebraization. More concretely, given any formula $\varphi \in \mathbf{Tm}_{\mathcal{L}_\mathbf{X}}(\mathcal{VAR})$, we assume that:

$$\zeta^*(\mu_\mathbf{X}(\varphi)) = \mu_\mathbf{Y}(\zeta_*(\varphi))$$

Throughout, we use θ to refer to the right adjoint functor associated to this translation, and $\mathcal{F}$ to denote the corresponding left adjoint functor.

First we will define two maps between $\vdash_{\mathbf{X}}$ and $\vdash_{\mathbf{Y}}$.

Definition 4.2.1. Let ρ be the following map: for $\vdash_* \in \Lambda(\vdash_{\mathbf{Y}})$:

$$\rho(\vdash_*) := \{(\Gamma, \varphi) \in \mathcal{P}(\mathbf{Tm}_{\mathcal{L}_{\mathbf{X}}}(\mathcal{VAR})) \times \mathbf{Tm}_{\mathcal{L}_{\mathbf{X}}}(\mathcal{VAR}) : \zeta_*[\Gamma] \vdash_* \zeta_*(\varphi)]\}$$

Also, define τ as the following map, for $\vdash^* \in \Lambda(\vdash_{\mathbf{X}})$:

$$\tau(\vdash^*) := \vdash_{\mathbf{Y}} \oplus \{(\zeta_*[\Gamma], \zeta_*(\varphi)) \in \mathcal{P}(\mathbf{Tm}_{\mathcal{L}_{\mathbf{Y}}}(\mathcal{VAR})) \times \mathbf{Tm}_{\mathcal{L}_{\mathbf{Y}}}(\mathcal{VAR}) : \Gamma \vdash^* \varphi\}$$

It is straightforward to see that τ as defined is a logic. These maps are related to the following definition:

Definition 4.2.2. Let $\vdash_* \in \Lambda(\vdash_{\mathbf{Y}})$ and $\vdash^* \in \Lambda(\vdash_{\mathbf{X}})$. We say that $\vdash_*$ is a ζ-*companion* of $\vdash^*$ if:

$$\Gamma \vdash^* \varphi \iff \zeta_*[\Gamma] \vdash_* \zeta_*(\varphi)$$

Given any $\vdash^*$, we denote by $\zeta(\vdash^*)$ the collection of ζ-companions of this logic.

Correspondingly we define the following on algebras: for $\mathbf{K}$ a subquasivariety of $\mathbf{X}$:

$$\tau(\mathbf{K}) := \{\mathbf{A} : \theta(\mathbf{A}) \in K\}$$

The following is analogous to Proposition 4.1.2. Its proof follows mostly the same way, using the hypothesis of strong selectivity to exploit the fact that the unit is an isomorphism, and by Proposition 3.3.8, the right adjoint θ preserves surjective homomorphsims.

Proposition 4.2.3. Let $\vdash^* \in \Lambda(\vdash_{\mathbf{X}})$ and $\mathbf{K} \in \Lambda(\mathbf{X})$. Assume that ζ is a strongly selective translation.

1. $\tau(\mathbf{K})$ is a quasivariety, and τ is a complete homomorphism on the lattices of quasivarieties.
2. $\tau(\mathsf{QVar}(\vdash^*)) = \mathsf{QVar}(\tau(\vdash^*))$. Hence $\tau(\mathsf{Log}(\mathbf{K})) = \mathsf{Log}(\tau(\mathbf{K}))$ and τ is a complete homomorphism on the lattice of logics.
3. $\tau(\vdash_*)$ is the least ζ-companion of $\vdash_*$.

Proof. (1) follows from the same arguments as before. For (2) assume that $\mathbf{A} \in \tau(\mathsf{QVar}(\vdash^*))$, hence $\theta(\mathbf{A}) \in \mathsf{QVar}(\vdash^*)$. If $(\Gamma, \varphi) \in \vdash^*$, then $\theta(\mathbf{A}) \vDash (\Gamma, \varphi)$. Assume that $\mathbf{A}, v \vDash \mu_{\mathbf{Y}}[\zeta_*[\Gamma]]$. hence, by assumption, $\mathbf{A}, v \vDash \zeta^*[\mu_{\mathbf{X}}[\Gamma]]$. So since the translation is selective, we can transfer this valuation to obtain some valuation v', such that $\theta(\mathbf{A}), v' \vDash \mu_{\mathbf{X}}[\Gamma]$. By assumption,then $\theta(\mathbf{A}), v' \vDash \mu_{\mathbf{X}}(\varphi)$, which by the reverse arguments shows that $\mathbf{A}, v \vDash \mu_{classY}[\zeta_*(\varphi)]$. This shows that $\mathbf{A} \vDash (\zeta_*[\Gamma], \zeta_*(\varphi))$. Similarly, if $\mathbf{A} \in \mathsf{QVar}(\tau(\vdash^*))$, then we show that $\theta(\mathbf{A}) \in \mathsf{QVar}(\vdash^*)$ using the converse arguments. Similar arguments also show that:

$$\tau(\mathsf{Log}(\mathbf{K})) = \mathsf{Log}(\tau(\mathbf{K}))$$

For (3), left to right is obvious. Now for the converse, assume that $\Gamma \nvdash^* \varphi$. Let $\mathbf{A} \in \mathsf{QVar}(\vdash^*)$ witness this. Since the translation is strongly selective, we know that there exists some algebra $\mathbf{B}$ such that $\theta(\mathbf{B}) \cong \mathbf{A}$. Hence, $\mathbf{B} \in \tau(\mathsf{QVar}(\vdash^*))$. By the same arguments as above, then $\mathbf{B} \nvDash (\zeta_*[\Gamma], \zeta_*(\varphi))$. But by (2) we have that then $\mathbf{B} \in \mathsf{QVar}(\tau(\vdash^*))$, hence, by completeness, $(\zeta_*[\Gamma], \zeta_*(\varphi)) \notin \tau(\vdash^*)$. ∎

Hence, all strongly selective translations admit a notion of a least ζ-companion. For such translations, for now, we cannot say much more. Hence we turn to the following definition:

Definition 4.2.4. Let $\langle \zeta, \Theta, f \rangle$ be a strongly selective translation between $\mathbf{X}$ and $\mathbf{Y}$. We say that $\langle \zeta, \Theta, f \rangle$ is a *sober translation* if it satisfies:

- (Injective Preservation) $\mathcal{F}$ preserves injective morphisms;
- (Sobriety) The counit ε is pointwise injective.

In light of the previous section we have:

Proposition 4.2.5. The Gödel-McKinsey-Tarski translation is sober.

Proof. We have noted in Example 3.3.6 that the translation is strongly selective. The fact that the counit of the adjunction is injective, and the left adjoint preserves injective morphisms was noted in pp.49, and is shown in [23, Construction 2.5.7]. ■

For sober translations we can show the following:

Proposition 4.2.6. Let $\langle \zeta, \Theta, f \rangle$ be a sober translation. For $\mathbf{K} \in \Lambda(\mathbf{Y})$, let:

$$\rho(\mathbf{K}) = \{\theta_\zeta(\mathbf{A}) : \mathbf{A} \in \mathbf{K}\}$$

Then $\rho(\mathbf{K})$ is a quasivariety, and the map $\rho : \Lambda(\mathbf{X}) \to \Lambda(\mathbf{Y})$ is a surjective complete homomorphism.

Proof. The proof runs exactly the same way as (1) in 4.1.3, except we use Maltsev's ISP_R theorem, instead of Tarski's HSP theorem. ■

We also have the definition of ρ on logics, analgous to the GMT case:

Proposition 4.2.7. For each $\vdash_* \in \Lambda(\vdash_\mathbf{Y})$, $\rho(\vdash_*)$ is a logic in $\Lambda(\vdash_\mathbf{X})$. Moreover, $\vdash_* \in \Lambda(\vdash_\mathbf{Y})$ is a ζ-companion of $\vdash^* \in \Lambda(\vdash_\mathbf{X})$ if and only if $\rho(\vdash_*) = \vdash^*$. Hence, for all $\mathbf{K} \in \Lambda(\mathbf{Y})$ and $\mathbf{P} \in \Lambda(\mathbf{X})$ we have that if $\mathsf{Log}(\mathbf{K})$ is a modal companion of $\mathsf{Log}(\mathbf{P})$ then $\rho(\mathbf{K}) = \mathbf{P}$.

Hence, for sober translations we have that all finitary extensions of $\vdash_\mathbf{X}$ have ζ-companions, and the syntactic maps witnessing this transformation have a concrete semantic meaning. Moreover, we also have the existence of greatest ζ-companions: for $\mathbf{K} \in \Lambda(\mathbf{X})$, define:

$$\sigma(\mathbf{K}) := \mathbb{ISP}_R\{\mathcal{F}(\mathbf{B}) : \mathbf{B} \in \mathbf{K}\}$$

Then we have the following:

Proposition 4.2.8. For any logic $\vdash_* \in \Lambda(\vdash_\mathbf{X})$, $\mathsf{Log}(\sigma(\mathsf{QVar}(\vdash_*)))$ is the greatest ζ-companion of $\vdash_*$.

As noted, properties like FMP and tabularity now could be studied in the relationship between these two systems: for instance, if $\mathbf{K} \in \Lambda(\mathbf{Y})$ is generated by its finite elements, clearly $\rho(\mathbf{K})$ will be as well; if $\mathbf{P} \in \Lambda(\mathbf{X})$ is generated by a single element, then $\tau(\mathbf{P})$ will be as well. In specific cases, a more detailed study could then be made paralleling the preservation results studied for the Blok-Esakia theory.

We conclude this section by showing that the KGG translation is not sober. First, we note that the following result is proven in [57, Lemma 4.2]:

Proposition 4.2.9. The unit of the adjunction $\mathcal{F} \vdash \mathsf{Reg}$, between Boolean algebras and Heyting algebras, is an isomorphism.

Hence the natural follow up question is whether such a translation is in fact sober. The following results would also easily follow from results at the end of this section; however the proof we provide illustrates some of the questions that we will encounter in Chapter 5, and so we opt to discuss it in greater detail.

For that purpose we recall some facts about the regularisation of a Heyting algebra. Throughout, given an Esakia space $(X, \leqslant, \tau)$, and U a clopen in X, let $\mathsf{Max}(U) = \{x \in U : \forall y \in X, \text{ if } x \leqslant y \text{ then } x = y\}$.

Lemma 4.2.10. Let H be a Heyting algebra, and X_H the dual Esakia space. Then:

1. For each $U \in \mathsf{ClopUp}(X_H)$, we have that $U = \varphi(a)$ is such that $a \in \mathsf{Reg}(H)$ if and only if: for all $x \in X_H$, if $\mathsf{Max}(\uparrow x) \subseteq U$, then $x \in U$.
2. For each $a, b \in \mathsf{Reg}(H)$ we have that $a = b$ if and only if $\varphi(a) \cap \mathsf{Max}(X_H) = \varphi(b) \cap \mathsf{Max}(X_H)$.
3. The assignment $k : \mathsf{Reg}(H) \to \mathsf{Clop}(\mathsf{Max}(X_H))$ given by $k(a) = \varphi(a) \cap \mathsf{Max}(X_H)$ is an isomorphism.

Proof. (1) First assume that $a \in \mathsf{Reg}(H)$. Let $U = \varphi(a)$. Assume that $\mathsf{Max}(\uparrow x) \subseteq U$, but $x \notin U$. Hence since $U = \neg\neg U$, there is some $x \leqslant y$, such that whenever $y \leqslant z$, then $z \notin U$. Since X_H is an Esakia space, let $y \leqslant z$ be maximal element; then we get a contradiction to $\mathsf{Max}(\uparrow x) \subseteq U$. Conversely, we show that $\neg\neg U = U$. Hence, assume that $x \in \neg\neg U$; hence whenever $x \leqslant y$, $y \leqslant z$ and $z \in U$. Hence in particular, this holds for all maximal elements seen by x, which implies by assumption that $x \in U$.

(2) The left to right direction is clear. If $a \neq b$, then by Esakia duality, $\varphi(a) \neq \varphi(b)$. Then without loss of generality, there is $x \in \varphi(a)$ such that $x \notin \varphi(b)$. Now if $\mathsf{Max}(\uparrow x) \subseteq \varphi(b)$, by regularity, $x \in \varphi(b)$, which does not hold. Hence, there is some y, such that $x \leqslant y$, and $y \notin \varphi(b)$, whereas obviously, $y \in \varphi(a)$.

(3) Now let $k : \mathsf{Reg}(H) \to \mathsf{Clop}(\mathsf{Max}(X_H))$ be the assignment $k(a) = \varphi(a) \cap \mathsf{Max}(X_H)$. Then k is injective, since if $a \neq b$, then $k(a) \neq k(b)$ by what we just showed. Moreover, this is easily seen to be surjective, and it is also a Boolean algebra homomorphism: $k(a \wedge b) = k(a) \cap k(b)$, and we can see that $k(\neg a) = \varphi(\neg a) \cap \mathsf{Max}(X_H) = \mathsf{Max}(X_H) - (\downarrow (\varphi(a))$. Then we claim the latter is equal to $\mathsf{Max}(X_H) - \varphi(a)$. Indeed, if $x \in k(\neg a)$, then $x \notin\downarrow (\varphi(a))$. Since x is maximal, then $x \notin \varphi(a)$, hence, $x \in \mathsf{Max}(X_H) - \varphi(a)$. Conversely, if $x \in \mathsf{Max}(X_H) - \varphi(a)$, then if $x \leqslant y$, then $x = y$, so $y \notin \varphi(a)$. This shows the claim. Moreover, k preserves the bounds, which shows that k is an isomorphism of Boolean algebras. ■

Now let $\mathsf{Reg} : \mathsf{HA} \to \mathsf{BA} : \mathcal{F}$ be the adjunction corresponding to the double negation translation. As far as we are aware, there are no descriptions of the left adjoint functor for this right adjoint[2] which are very explicit (but see [57] for an extended discussion). However, we can show that the translation is not sober using the above duality-theoretic properties.

Consider $\mathbf{2}$ the 2-element Boolean algebra. This is a Heyting algebra, and it is clear that $\mathsf{Reg}(\mathbf{2}) = \mathbf{2}$. Moreover, by Lemma 4.2.10, we obtain that any Heyting algebra which Esakia space has exactly two maximal elements (and only such Esakia spaces) will map via Reg to $\mathbf{2}$.

[2]By contrast, it is known that Reg is *left adjoint* to the full and faithful inclusion of Boolean algebras into Heyting algebras.

Now assume towards a contradiction that the translation is sober. Hence $\mathcal{F}(\mathsf{Reg}(\mathbf{2}))$ must be isomorphic to $\mathbf{2}$, since it is an adjunction: we know that $\mathsf{Reg}(\mathcal{F}(\mathsf{Reg}(\mathbf{2}))) = \mathbf{2}$, by the fact that this is an idempotent adjunction (since it is strongly selective), and hence it must have two elements in the dual Esakia space. But now consider the Heyting algebra in Figure 4.1

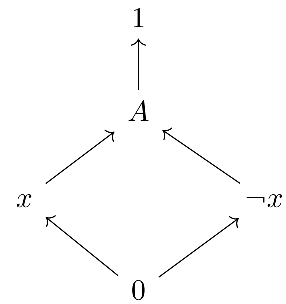


Figure 4.1: Hasse Diagram of $H(\mathbf{2})$

Dually, this corresponds to the frame known as a 2-fork. It is clear to see that $\mathsf{Reg}(H) = \mathbf{2}$ as well. By assumption on the adjunction we have a counit map $\varepsilon : \mathcal{F}(\mathsf{Reg}(H)) \to H$, which is dually, a surjective p-morphism from the 2-fork to the space with 2 elements (since we assume that $\mathcal{F}(\mathsf{Reg}(H)) = \mathcal{F}(\mathbf{2})$ is isomorphic to the 2-element Boolean algebra). But this is impossible, as can be obtained by inspecting the diagram in Figure 4.2, and noting that there cannot be a surjective p-morphism from a connected frame to a disconnected one.

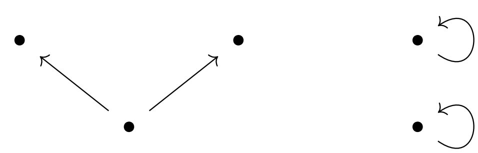


Figure 4.2: Impossibility of P-morphism from 2-fork to 2-Boolean algebra

Hence we have shown:

Corollary 4.2.11. *The KGG translation is not sober.*

The former proof illustrates our choice of the term *sobriety*: dually, the space X_H as seen above is *not* "sober" in so far as it contains points which are redundant from the point of view of the adjunction. This will become a theme in later chapters.

4.3 Polyatomic Logics as Generalised Companions

Despite the notion of ζ-companion corresponding very naturally to that found in Blok-Esakia theory, it seems to have some flaws. For starters, we do not have an understanding of how many selective translations are strongly selective or sober. On the other hand, even for sober translations, the description we gave of the greatest modal companion is in a sense purely existential – we do not have any semblance of an axiomatisation of it on the basis of the original logic. Hence, one might ask whether some natural construction could exist which could match up with ζ-companions but also be defined for a much larger scope of translations.

To assist us in this task, we recall from [6] two crucial notions: that of the *least* and *greatest* DNA-variant. It was proved that every intermediate logic has both of these variants. More concretely, the following was shown:

Definition 4.3.1. Let $L \in \Lambda(\mathsf{IPC})$. We say that L is:

- *DNA-minimal* if whenever S is an intermediate logic and $L^{\neg} = S^{\neg}$ then $L \subseteq S$.
- *DNA-maximal* if whenever S is an intermediate logic and $L^{\neg} = S^{\neg}$ then $S \subseteq L$.

Definition 4.3.2. Let L' be a DNA-logic. We define the *schematic fragment* of L', denoted $Schem(L')$ as follows:

$$Schem(L') = \{\varphi : \forall \psi, \varphi[\psi/p] \in L'\}$$

In other words, the schematic fragment is the smallest fragment of the DNA-logic which is closed under substitution. It can be shown that this is an intermediate logic, and moreover:

$$\mathsf{Var}(Schem(L')) = \mathsf{Var}(\{\langle H_{\neg}\rangle : \mathbf{H} \in \mathsf{DNA} - \mathsf{Var}(L')\})$$

In other words, the variety generated by the schematic fragment is precisely the variety generated by the regularly generated subalgebras of those $\mathbf{H}$ which belong to the DNA-variety of L'. More importantly, we have the following:

Proposition 4.3.3. For each intermediate logic L, L is DNA-maximal if and only if $L = Schem(L^{\neg})$.

Hence, schematic fragments provide a concrete syntactic description of the greatest DNA-variant. This is not of course a very concrete decription - in fact, it is shown in [6] that the schematic fragment of $\mathcal{IPC}$ is the well-known *Medvedev Logic*, which as far as we are aware has never been given a recursive axiomatisation. Nevertheless, knowing the properties associated to it can provide us with insight on the nature of the translation and the logics at play.

To see this, we first exemplify this for the GMT translation, once again. If indeed $\Box$-varieties and $\Box$-logics should serve as something like generalised modal companions, then we should hope that for this prototypical example they should coincide. Indeed, note that:

Proposition 4.3.4. For each variety $\mathbf{K} \in \Lambda(\mathbf{S4})$:

$$\tau(\rho(\mathbf{K})) = \mathbf{K}^{\uparrow}$$

Consequently, given two logics $M, N \in \mathbf{NExt}(\mathsf{S4})$ and a logic $L \in \Lambda(\mathcal{IPC})$, we have that M, N are both modal companions of L if and only if M and N have the same $\Box$-variant.

Proof. Indeed, if $\mathbf{B} \in \tau(\rho(\mathbf{K}))$, then $\mathbf{B}_{\Box} = \mathbf{C}_{\Box}$ where $\mathbf{C} \in \mathbf{K}$; hence, up to isomorphism, $\mathbf{C}_{\Box}$ includes in $\mathbf{B}_{\Box}$, so $B(\mathbf{C}_{\Box})$ maps injectively into $B(\mathbf{B}_{\Box})$, which since the counit is injective, is a subalgebra of $\mathbf{B}$. Since $B(\mathbf{C}_{\Box}) \in \mathbf{K}$, then $\mathbf{B}$ is a core superalgebra of $B(\mathbf{C}_{\Box})$, and so is in $\mathbf{K}^{\uparrow}$. Conversely, if $\mathbf{B} \in \mathbf{K}^{\uparrow}$, then $\mathbf{C} \leqslant \mathbf{B}$ where $\mathbf{C} \in \mathbf{K}$, and they share the same core; hence, $\mathbf{B}_{\Box} \in \rho(\mathbf{K})$, hence, $\mathbf{B} \in \tau(\rho(\mathbf{K}))$.

Now in light of the fact we just showed, $M^{\Box} = N^{\Box}$ (their Box variant) if and only if $\tau(\rho(\mathsf{Var}(M))) = \tau(\rho(\mathsf{Var}(N)))$. Since τ is injective, this holds if and only if $\rho(\mathsf{Var}(M)) = \rho(\mathsf{Var}(N))$. This holds if and only if:

$$\mathsf{Var}(\rho(M)) = \mathsf{Var}(\rho(N))$$

Which by algebraic completeness holds if and only if $\rho(M) = \rho(N)$. The latter holds if and only if M and N are modal companions of L. ∎

Hence, as expected, greatest modal companions should exactly correspond to the largest logic with the same $\Box$-variant. That is:

Proposition 4.3.5. If L is an intermediate logic, $\sigma(L) = Schem(\tau(L)^{\Box})$.

Proof. First note that $Schem(\tau(L)^{\Box})$ and $\tau(L)$ havethe same $\Box$-variant by definition; hence, they are both modal companions of L. Moreover, since $Schem(\tau(L)^{\Box})$ is the greatest having $\tau(L)^{\Box}$ as a $\Box$-variant, it will also be the greatest modal companion. Hence since $\sigma(L)$ is moreover the greatest modal companion, we have that $Schem(\tau(L)^{\Box}) = \sigma(L)$. ∎

If we now consider a world where we did not know the axiom Grz, we can imagine the possibilities of arriving to this axiom through the study of the schematic fragment. Namely, we could – as we will do in the next section – derive the following result:

Proposition 4.3.6. The lattice $\Lambda(\mathsf{IPC})$ is isomorphic to $\Lambda^f(\mathsf{S4})$.

Let us exemplify this briefly: consider for instance the logic $\mathsf{LC} \in \Lambda(\mathsf{IPC})$, which is axiomatised by the axiom $p \to q \vee q \to p$. It can be shown by semantic methods that $\tau(\mathsf{LC}) = \mathsf{S4.3}$, the system S4 together with the axiom $\Box(\Box p \to q) \vee \Box(\Box q \to p)$. The lattice of extensions of this logic has a countable, though somewhat complicated structure; by contrast, in light of the previous result, one has that $\Lambda(LC) \cong \Lambda^{\Box}(\mathsf{S4.3}) \cong \Lambda(\mathsf{Grz.3})$, which is known to be isomorphic to an infinite descending chain (see [14, pp.427]). Hence, the study of $\Box$-variant extensions could presumably, in a setting where the Grz axiom was not known, be carried out in a more straightforward fashion.

A similar but more striking example is the following: if one looks at CPC, its least modal companion is the system S5, which has infinitely many extensions. By contrast, $\mathsf{S5}^{\Box}$ has no proper extensions.

In the next few sections, following the structure of [6], we outline how the notion of a "DNA-logic" can be carried out in general. We provide most of the proofs, since the change of setting brings some subtleties we would wish to take into consideration, but refer the reader to the article above for some facts which carry to our setting immediately.

4.4 Connecting Companions and Variants

In this section we conclude our discussion by establishing the connection between our discussion of ζ-companions and PAt-logics. Assume throughout a contextual translation $\overline{\zeta} = \langle \zeta, \Theta, f \rangle$.

Proposition 4.4.1. Let $\overline{\zeta}$ be a strongly selective translation, and $\mathbf{K} \in \Lambda(\mathbf{X})$. Then $\tau(\mathbf{K})$ is a PAt-quasivariety. Moreover, the assignment:

$$\tau : \Lambda(\mathbf{X}) \to \Lambda^f(\mathbf{Y})$$

is injective.

Proof. By Proposition 4.2.3, we know that $\tau(\mathbf{K})$ is a quasivariety. Moreover, it is easy to see that it is closed under core superalgebras: if $\mathbf{A} \in \tau(\mathbf{K})$, and $\mathbf{A} \preceq \mathbf{B}$, where $\mathbf{A}^f = \mathbf{B}^f$, then $\theta(\mathbf{B}) \in \mathbf{K}$, so $\mathbf{B} \in \tau(\mathbf{K})$ by definition.

To see that τ is injective, notice that if $\mathbf{K} \neq \mathbf{K}'$, then let $\mathbf{A} \in \mathbf{K}$ and $\mathbf{A} \notin \mathbf{K}'$ be arbitrary. Then $\mathcal{F}(\mathbf{A}) \in \tau(\mathbf{K})$ by definition of τ and the fact that the unit is an isomorphism. By the same fact, we have $\mathcal{F}(A) \notin \tau(\mathbf{K}')$. So $\tau(\mathbf{K}) \neq \tau(\mathbf{K}')$. ■

Hence, we have that, semantically, for strongly selective translations, one can study least ζ-companions as logics, or as PAt-logics. If moreover we assume that the translation is sober, we have the following result:

Proposition 4.4.2. (PAt-Blok Esakia Isomorphism) Assume that $\overline{\zeta}$ is a sober translation. Then the assignment $\tau : \Lambda(\mathbf{X}) \to \Lambda^f(\mathbf{Y}) : \rho^f$ is a lattice isomorphism.

Proof. Let $\mathbf{K}$ be an arbitrary PAt-quasivariety in $\Lambda^f(\mathbf{Y})$. We will show that $\tau\rho(\mathbf{K}) = \mathbf{K}$. Indeed, first we see that if $\mathbf{A} \in \mathbf{K}$, then $\theta(\mathbf{A}) \in \rho(\mathbf{K})$, so by definition $\mathbf{A} \in \tau\rho(\mathbf{K})$. Conversely, if $\mathbf{A} \in \tau\rho(\mathbf{K})$, then $\theta(\mathbf{A}) \in \rho(\mathbf{K})$. By assumption, then $\theta(\mathbf{A}) = \theta(\mathbf{B})$ for some $\mathbf{B} \in K$. Hence note that $\mathcal{F}(\theta(\mathbf{A})) = \mathcal{F}(\theta(\mathbf{B}))$. Since the translation is sober, by sobriety, $\mathcal{F}(\theta(\mathbf{B})) \leqslant B$, so $\mathcal{F}(\theta(\mathbf{B})) \in K$. But by sobriety again, $\mathcal{F}(\theta(\mathbf{A})) \leqslant \mathbf{A}$, and so we get that $\mathbf{A}$ is a core superalgebra of $\mathcal{F}(\theta(\mathbf{B}))$ - i.e, $\mathbf{A} \in \mathbf{K}$, since $\mathbf{K}$ is a PAt-quasivariety. This shows that τ is surjective.

The fact that τ is an injective homomorphism between these lattices follows from the commuting diagram of pp.45, and Proposition 4.4.1. ■

Corollary 4.4.3. *If $\langle \zeta, \Theta, f \rangle$ is a sober translation, then $\Lambda(\vdash_{\mathbf{X}}) \cong \Lambda^f(\vdash_{\mathbf{Y}})$.*

This isomorphism can thus serve as a natural correspondence for the study of the relationship between two systems. In a sense, as noted in the introduction, it can be seen as "modding out" Blok's Lemma: since our Polyatomic Logics are by construction complete with respect to regularly generated algebras, this ensures the isomorphism goes through in that case. Nevertheless, this does not trivialise the situation – as we will have opportunity to see, sobriety is far from a straightforward property.

As a consequence of the former result, we also get for free that the Double Negation Translation could never be sober: it is trivial to observe that there is a single quasivariety of Boolean algebras, and correspondingly, a single logic, though it is known that there are infinitely many DNA-logics [6, Theorem 5.11].

For the rest of this section, we show that additionally, in the setting of sober translations we get that the natural counterpart of the map σ is defined. As we will see, we have a good grasp of it semantically:

Definition 4.4.4. Let $\langle \zeta, \Theta, f \rangle$ be a sober translation between $\mathbf{X}$ and $\mathbf{Y}$, inducing an adjunction $\theta : \mathbf{X} \to \mathbf{Y} : \mathcal{F}$. We define for each quasivariety $\mathbf{K} \in \Lambda(\mathbf{X})$, the collection

$$\sigma(K) = \mathsf{QVar}(\{\mathcal{F}(\mathbf{A}) : \mathbf{A} \in \mathbf{K}\})$$

Definition 4.4.5. Let $\vdash_f$ be a PAt-logic. We define $Schem(\vdash_f)$, its *schematic fragment* as:

$$Schem(\vdash_f) := \{(\Gamma, \varphi) : \forall \psi \in \mathcal{L}_{\mathbf{Y}}, \Gamma[\psi/p] \vdash_f \varphi[\psi/p]\}.$$

In other words, the schematic fragment is the collection of all formulas for which the PAt-logic is closed under substitution. The following explains the main properties of this:

Lemma 4.4.6. For each PAt-logic $\vdash_f$, $Schem(\vdash_f)$ is a logic in $\Lambda(\vdash_{\mathbf{Y}})$. Moreover, it is the greatest logic which has $\vdash_f$ as a PAt-variant.

Proof. The verification that this is a logic is straightforward. It is also clear that the schematic fragment will have $\vdash_f$ as its PAt-variant. Now suppose that $\vdash_*$ has $\vdash_f$ as a PAt-variant. Assume that $\Gamma \vdash_* \varphi$ is arbitrary, and suppose that ψ. Then for each ψ, we have $\Gamma[\psi/p] \vdash_* \varphi[\psi/p]$, hence, $\Gamma[\psi/p][f(q)/q] \vdash_* \varphi[\psi/p][f(q)/q]$. This means that $\Gamma[\psi/p] \vdash_f \varphi[\psi/p]$. Hence $(\Gamma, \varphi) \in Schem(\vdash_f$). ■

Moreover, schematic fragments are also companions. To simplify notation, given $\vdash_*$, we denote by:

$$\mathbf{Sch}(\vdash_*) := Schem(\mathsf{Log}^f(\mathsf{QVar}(\tau(\vdash_*))))$$

Lemma 4.4.7. For each $\vdash_* \in \Lambda(\vdash_{\mathbf{X}})$ we have that $\mathbf{Sch}(\vdash_*)$ is a ζ-companion of $\vdash_*$.

Proof. Suppose that $\Gamma \vdash_* \varphi$. Then by definition, $(\zeta_*[\Gamma], \zeta_*[\varphi]) \in \tau(\vdash_*)$. Now suppose that ψ is any formula. We want to show that if $\mathbf{A} \in Var(\tau(\vdash_*))$, then $\mathbf{A} \vDash_f (\zeta_*[\Gamma][\psi/p], \zeta_*[\varphi][\psi/p])$. Hence by definition, we want to check that $\mathbf{A} \vDash (\zeta_*[\Gamma][\psi/p][f(q)/q], \zeta_*[\psi/p][f(q)/q])$. But we know that $\tau(\vdash_*)$ is a logic, and hence is closed under uniform substitution; so since $\mathbf{A} \vDash (\zeta_*[\Gamma], \zeta_*[\varphi])$, the result follows.

Conversely, suppose that $(\Gamma, \varphi) \notin \vdash_*$. Suppose that $\mathbf{A} \in \mathsf{QVar}(\tau(\vdash_*))$ is an algebra such that $\mathbf{A} \nvDash (\zeta_*[\Gamma], \zeta_*[\varphi])$ in the context of Θ, by Proposition 3.3.1. Hence, we can assume the valuation witnessing this to be regular, i.e, $\mathbf{A} \nvDash_f (\zeta_*[\Gamma], \zeta_*[\varphi])$, which by Lemma 3.4.7 means that $\langle A^f \rangle \nvDash_f (\zeta_*[\Gamma], \zeta_*[\varphi])$. But now by Corollary 3.4.8, we have that there is a substitution σ, such that $\mathbf{A} \nvDash_f (\zeta_*[\Gamma][\sigma(p)/p], \zeta_*[\varphi][\sigma(p)/p])$. This means that $(T[X], T[\varphi]) \notin \mathbf{Sch}(\vdash_*)$. ■

We will now show that schematic fragments correspond exactly to the quasivarieties of the form $\sigma(\mathbf{K})$ as defined above.

Proposition 4.4.8. Let $\vdash_* \in \Lambda(\vdash_{\mathbf{Y}})$. Then $\vdash_*^f = Schem((\vdash_*)^f)$ if and only if $\mathsf{QVar}(\vdash_*) = \mathsf{QVar}(\mathcal{C})$ for some class $\mathcal{C}$ of regularly generated algebras.

Proof. First we prove right to left. Suppose that $\vdash_*$ is the logic of a class of regularly generated algebras. Note that by maximality of the schematic fragment amongst logics with the same variant, $\vdash_* \subseteq Schem((\vdash_*)^f)$, so we focus on the other inclusion. Suppose that $\Gamma \nvdash_* \varphi$. Hence by assumption, we can find $\mathbf{A}$, a regularly generated algebra, such that $\mathbf{A} \nvDash (\Gamma, \varphi)$; hence $\langle A^f \rangle \nvDash (\Gamma, \varphi)$, so by Corollary 3.4.8, $\mathbf{A} \nvDash_f (\Gamma[\sigma(p)/p], \varphi[\sigma(p)/p])$, and so $\mathbf{A} \nvDash (\Gamma[\sigma(p)/p][f(q)/q], \varphi[\sigma(p)/p][f(q)/q])$. Hence, since $\mathbf{A}$ is an algebra in $\mathsf{QVar}(\vdash_*)$, and by algebraic completeness, we have that $\Gamma[\sigma(p)/p][f(q)/q] \nvdash_* \varphi[\sigma(p)/p][f(q)/q]$. Thus, $\Gamma[\sigma(p)/p] \nvdash_*^f \varphi[\sigma(p)/p]$. Thus by definition:

$$(\Gamma, \varphi) \notin Schem((\vdash_*)^f)$$

As desired.

For the converse, assume that $\vdash_* = Schem((\vdash_*)^f)$. First define:

$$\mathsf{QVar}_R(\vdash_*) = \mathsf{QVar}(\{\langle B^f \rangle : \mathbf{B} \in \mathsf{QVar}(\vdash_*)\})$$

Then by definition:

$$\mathsf{QVar}_R(\vdash_*) \subseteq \mathsf{QVar}(\vdash_*)$$

So we show the other inclusion. For that purpose, we show that $\mathsf{Log}(\mathsf{QVar}_R)$ has $\vdash_*^f$ as its PAt-variant. Indeed we have:

$$\begin{aligned}(\mathsf{Log}(\mathsf{QVar}_R(\vdash_*)))^f &= \mathsf{Log}^f(\mathsf{QVar}(\{\langle B^f \rangle : B \in \mathsf{QVar}(\vdash_*)\})^{\uparrow}) \\ &= \mathsf{Log}^f(\mathsf{QVar}^f(\vdash_*)) \\ &= (\vdash_*)^f\end{aligned}$$

Where the first inclusion follows by the commutativity of the operators, the second follows from the fact that every PAt-quasivariety is generated by its regular elements and the final is by definition. Hence we conclude that $\mathsf{Log}(\mathsf{QVar}_R(\vdash_*))$ has $(\vdash_*)^f$ as its PAt-variant, whence we know that $\mathsf{Log}(\mathsf{QVar}_R(\vdash_*)) \subseteq Schem((\vdash_*)^f)$. Hence $\mathsf{QVar}(Schem((\vdash_*)^f)) \subseteq \mathsf{QVar}_R(\vdash_*)$, which shows the result. ■

Lemma 4.4.9. Let $\mathbf{K} \in \Lambda(\mathbf{Y})$, and suppose that $\mathsf{Log}(\mathbf{K})$ is a ζ-companion of $\vdash_* \in \Lambda(\mathbf{X})$. Suppose that $\mathbf{S} = \mathsf{QVar}(\vdash_*)$. Then $\sigma(\mathbf{S}) \subseteq \mathbf{K}$.

Proof. It suffices to show that $\{\mathcal{F}(\mathbf{A}) : \mathbf{A} \in \mathbf{S}\} \subseteq \mathbf{K}$. By Proposition 4.4.2, we have that $\rho(\mathbf{K}) = \mathbf{S}$. Hence if $\mathbf{A} \in \mathbf{S}$, then $\mathbf{A} = \theta(\mathbf{B})$ for some $\mathbf{B} \in \mathbf{K}$. But then $\mathcal{F}(\mathbf{A}) = \mathcal{F}(\theta(\mathbf{B})) \preceq \mathbf{B}$, given the translation is sober. So indeed, $\mathcal{F}(\mathbf{A}) \in \mathbf{K}$, as intended. ■

Lemma 4.4.10. Let $\mathbf{A} = \mathcal{F}(\mathbf{B})$. Then $\mathbf{A}$ is regularly generated.

Proof. First note that $\theta(\mathbf{A}) \cong \mathbf{B}$, since the unit is an isomorphism. Hence, consider $\langle \mathbf{B} \rangle$, the subalgebra generated in $\mathbf{A}$ by $\mathbf{B}$. Clearly we have that $\theta(\langle \mathbf{B} \rangle) \cong \mathbf{B}$. But then we have that $\mathbf{A} \cong \mathcal{F}(\mathbf{B})$ is a subalgebra of $\langle \mathbf{B} \rangle$, since the counit is injective. ■

Lemma 4.4.11. Suppose that $\mathbf{A}$ is regularly generated. Then $\mathbf{A} = \mathcal{F}(\theta(\mathbf{A}))$.

Proof. Suppose that $\mathbf{A}$ is regularly generated. Then first note that $\theta(\mathbf{A}) \subseteq \mathcal{F}(\theta(\mathbf{A})$, since $\theta(\mathbf{A}) \cong \theta(\mathcal{F}(\theta(\mathbf{A})) \subseteq \mathcal{F}(\theta(\mathbf{A}))$. Hence, $\mathcal{F}(\theta(\mathbf{A}))$ is a subalgebra of $\mathbf{A}$ which contains the regular elements; since $\mathbf{A}$ is regularly generated, $\mathbf{A} \cong \mathcal{F}(\theta(\mathbf{A}))$. ■

Using all of these lemmas we conclude the following:

Corollary 4.4.12. *Let $\langle \zeta, \Theta, f \rangle$ be a sober translation between $\mathbf{X}$ and $\mathbf{Y}$, then for each $\vdash_* \in \Lambda(\vdash_{\mathbf{X}})$ there is a greatest ζ-companion. This is exactly $\mathbf{Sch}(\vdash_*)$.*

Proof. By Lemma 4.4.7, we have that $\mathbf{Sch}(\vdash_*)$ is a ζ-companion. Moreover, by lemma 4.4.8, we have that $\mathsf{QVar}(\mathbf{Sch}(\vdash_*))$ is generated by a class of regularly generated algebras; hence in particular it is generated by all of its regularly generated algebras. By the previous lemma, we have that $\sigma(\mathsf{QVar}(\vdash_*)) \subseteq \mathsf{QVar}(\mathbf{Sch}(\vdash_*))$. Now note that the former are exactly the regularly generated algebras: if $\mathbf{A} \in \sigma(\mathsf{QVar}(\vdash_*))$ then $\mathbf{A} = \mathcal{F}(B)$, so by Lemma 4.4.10, it is regularly generated; and if $\mathbf{A}$ is regularly generated, then by Lemma 4.4.11, $\mathbf{A} \cong \mathcal{F}(\theta(\mathbf{A}))$, and we have that $\theta(\mathbf{A}) \in \mathsf{QVar}(\vdash_*)$, so $\mathbf{A} \in \sigma(\vdash_*)$. Hence:

$$\mathsf{QVar}(\mathbf{Sch}(\vdash_*)) = \mathsf{QVar}(\sigma(\mathsf{QVar}(\vdash_*))),$$

but by Lemma 4.4.9, we have that $\sigma(\mathsf{QVar}(\vdash_*)) \subseteq \mathbf{K}$ whenever $\mathbf{K}$ is a quasivariety which logic is a PAt companion of $\vdash_*$. Hence for all such quasivarieties:

$$\mathsf{QVar}(\mathbf{Sch}(\vdash_*)) \subseteq \mathbf{K},$$

but then by completeness:

$$\mathsf{Log}(\mathbf{K}) \subseteq \mathsf{Log}(\mathsf{QVar}(\mathbf{Sch}(\vdash_*))) = \mathbf{Sch}(\vdash_*).$$

This shows that **Sch** is the greatest ζ-companion, as was to show. ■

We conclude with a tentative definition:

Definition 4.4.13. Let $\overline{\zeta}$ be a sober translation. We say this is a *BE-translation* if there is some logic $\vdash_* \in \Lambda(\vdash_{\mathbf{Y}})$ such that for each $\Vdash \in \Lambda(\vdash_{\mathbf{Y}})$, $\Vdash \in \Lambda(\vdash_*)$ if and only if there is some $\vdash^* \in \Lambda(\vdash_{\mathbf{X}})$ such that $\mathbf{Sch}(\vdash^*) = \Vdash$.

As discussed in 51, the GMT is a BE-translation, and it and its close progeny (the translaton for bi-intuitionistic logic, amongst others), are the only examples we are aware of this kind. We remark that in this discussion we have only provided tentative steps in using PAt-logics for exploring the connections between logical systems: for instance, in [6], it is shown that the lattice of inquisitive extensions of the system KP is isomorphic to the lattice of Boolean algebras ordered by the $\leq$ order, where:

$$A \leq B \iff A \text{ is a homomorphic image of a subalgebra of } B$$

Such connections are much more fine-grained than can be found in homomorphisms of varieties, and seem to more naturally be discussed in the setting of Polyatomic logics rather than companions.

In the rest of the thesis, for reasons of space, we will not be able to explore deeply PAt-logics for the cases we will study. Nevertheless, we expect that the results from this chapter can highlight the importance of this connection, and bring attention to the potential of such tools to study translation relations between logical systems.

As a visual summary of our discussions, we have the following picture, found in Figure 4.3, capturing the various kinds of translations and logical relationships studied.

These translations are some of those we have previously encountered, and they also witness the strictness of some of these inclusions: K_2, the translation between Kleene algebras and Distributive lattices mentioned in [53], is a contextual translation but not selective (in any obvious sense). As we will see in the next chapter, the Goldblatt translation is selective, and it is not strongly selective. The double negation translation is strongly selective, and we showed above that it is not sober. The last class – BE-translations – is introduced to highlight the Blok-Esakia isomorphism, and its close progeny, through the GMT translation.

4.5 Chapter Summary

We summarise our contributions in this chapter as follows:

- Inspired by the GMT translation, we introduce the notion of a *sober translation*, for which a rich Blok-Esakia theory can be introduced.
- We show that the KGG translation is strongly selective, but not sober.
- We generalise the notions of schematic fragment and least DNA-variant to the Polyatomic case, and show that they correspond, for sober translations, to the least and greatest generalised companions.

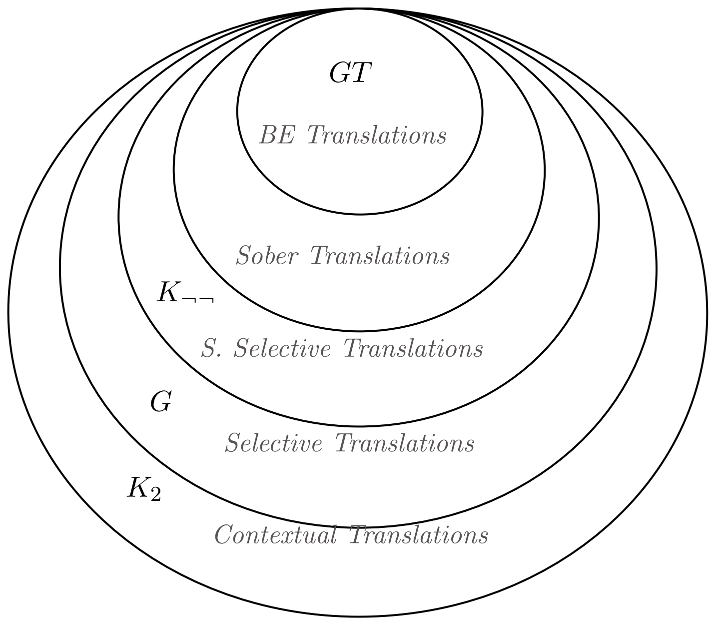


Figure 4.3: Types of Translations and Examples

- We introduce the concept of a Polyatomic Blok-Esakia isomorphism as a correspondence between logical systems, and show that it always holds for sober translations.

Chapter 5

Translations of Orthologic

In this chapter we investigate orthologic and the logics of ortholattices, through the lens of translations, adjunctions and PAt-logics developed in the previous chapter.

As noted in the introduction, this topic has received some attention, though not the same as either the KGG or GMT translations. Previous analysis which marked this topic were developed by Goldblatt [35, 36, 34] as well as Miyazaki [52], for general orthologic, and by several authors [18, 44, 48] for the special case of quantum logic.

In this and the next chapter, we take a close look at these developments, and relate them to our general picture. To do so, we begin by recalling the notions of orthologics, their associated algebraic semantics, and the duality associated to ortholattices. We recall in particular the investigations done by Miyazaki with respect to the *Goldblatt translation*, which relates orthologics and KTB normal modal logics. Using the proof of filtration introduced by Goldblatt, we are able to prove that the translation is selective.

We then turn to negative results. We first show – from basic facts since known about the lattices of orthologics and KTB logics – that a genuine Blok-Esakia theorem *cannot* hold between these two lattices. We then move further and show that the translation is not even sober, further evidencing the untractability of the translation. We conclude the chapter with an outline of the main difficulties found, through a discussion of the idea of sober algebras and spaces, and prepare our approach in the next chapter to overcome them.

5.1 Orthologics and Ortholattices

We begin with the notion of an *orthologic*, as introduced by Goldblatt [35]. Recall that $\mathcal{L}_O$, the language of ortholattices, is the same as the language of Boolean algebras, namely, the language with signature:

$$p \mid \neg p \mid p \wedge p \mid 0 \mid 1$$

Throughout, let $\mathcal{VAR}$ be a fixed set of countably many proposition letters.

Definition 5.1.1. Let $\vdash$ be a binary consequence relation in $\mathcal{L}_O$, i.e.:

$$\vdash \subseteq \mathcal{P}(\mathbf{Tm}_{\mathcal{L}_O}(\mathcal{VAR})) \times \mathbf{Tm}_{\mathcal{L}_O}(\mathcal{VAR})$$

Then we say that $\vdash$ is an *orthologic* if it is closed under uniform substitution, and satisfies the following axioms, for all $\varphi, \psi, \chi \in \mathcal{L}_O$:

1. For finite $\Gamma \subseteq \mathbf{Tm}_{\mathcal{L}_O}(\mathcal{VAR})$, $\Gamma \vdash \varphi$ if and only if $\bigwedge \Gamma \vdash \varphi$
2. $\varphi \wedge \psi \vdash \varphi$; $\varphi \wedge \psi \vdash \psi$
3. $\varphi \vdash \neg\neg\varphi$; $\neg\neg\varphi \vdash \varphi$
4. $\varphi \wedge \neg\varphi \vdash \psi$
5. If $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$
6. If $\varphi \vdash \psi$ and $\psi \vdash \chi$ then $\varphi \vdash \chi$
7. If $\varphi \vdash \psi$ then $\neg\psi \vdash \neg\varphi$

The former is exactly the concept as presented by Goldblatt. As we will note later, the peculiarities of this system will mean that the specific assumptions of our Chapter 3 might not be met, despite in all cases it being clear to see that the desired results could also be proved We will strive throughout to minimise such pathologies.

The following is an easy observation about orthologics:

Proposition 5.1.2. Let $(\vdash_i)_{i \in I}$ be a family of orthologics. Then $\bigcap_{i \in I} \vdash_i$ is an orthologic.

Henceforth, we denote by O the minimal orthologic, and by O arbitrary orthologics. We denote by $\Lambda(\mathsf{O})$ the lattice of orthologics.

Definition 5.1.3. Let $\mathbf{O}$ be an ortholattice. We write $\mathbf{O} \vDash (\varphi, \psi)$ to denote: for each valuation $v : T_{\mathcal{L}_O} \to \mathbf{O}$, $v(\varphi) \leqslant v(\psi)$.

Definition 5.1.4. Let O be an orthologic. Consider all pairs $(\varphi, \psi) \in O$, and let:

$$\mathsf{Var}(O) := \{\mathbf{O} : \mathbf{O} \vDash (\varphi, \psi)\}$$

In other words, we form $\mathsf{Var}(O)$ by taking the equational class containing $\varphi \leqslant \psi$ for $(\varphi, \psi) \in O$.

Similarly, given a class $\mathbf{K}$ of ortholattices, we consider:

$$\mathsf{Log}(\mathbf{K}) := \{(\varphi, \psi) \in \mathbf{Tm}_{\mathcal{L}_O}(\mathcal{VAR})^2 : \mathbf{K} \vDash (\varphi, \psi)\}$$

It is not hard to see that this is indeed an orthologic, since $\mathbf{K}$ is a variety of ortholattices. Indeed, we have the next proposition which follows by usual arguments.

Definition 5.1.5. Let O be an orthologic. We consider $\mathbf{Tm}_{\mathcal{L}_O}(\mathcal{VAR})$, the absolutely free algebra, and define an equivalence relation $\equiv_O$ there by:

$$\varphi \equiv_O \psi \iff (\varphi, \psi) \in O \text{ and } (\psi, \varphi) \in O$$

Then consider $\mathbf{F}(O) := \mathbf{Tm}_{\mathcal{L}_O}(\mathcal{VAR}) / \equiv_O$. We call this the *Lindenbaum-Tarski ortholattice* associated to O.

Denote by $\Lambda(\mathbf{Ort})$ the lattice of varieties of ortholattices. Then we have, as usual:

Proposition 5.1.6. The operators $Var : \Lambda(\mathsf{O}) \to \Lambda(\mathbf{Ort}) : Log$ establish a dual isomorphism between the lattice of orthologics and the lattice of varieties of ortholattices.

The former provides, then, an algebraic semantics (in the loose sense) to ortholattices and their varieties. However, the key results of Goldblatt [35] made use of a dual representation. We turn to this in the next section.

5.2 Orthospaces and Duality

In this section we briefly recall the duality of ortholattices using the space of filters. This was initiated by Goldblatt [35], who provided the original representation theorem; Bimbó [8] established the functoriality of this, though she left a gap, that was noted by Dmitrieva [20], in line with previous work on choice-free duality (see [7] for the original developments, and see also [49]). As such we refer to this simply as orthospace duality throughout. We refer to all stated articles for proofs of the statements in this section.

Definition 5.2.1. Let $(X, \perp)$ be a non-empty set, with $\perp$ an irreflexive and symmetric relation. Then we say that this is an orthoframe.

Whenever $x \perp y$, we say that x is orthogonal to y. If $x \perp y$ for every $y \in Y$, we denote this by $x \perp Y$.

Definition 5.2.2. Let $(X, \perp)$ be an orthoframe, and $A \subseteq X$. We define $A^{\perp}$ as:

$$A^{\perp} = \{x \in X : \forall y \in A, x \perp y\}$$

We say that $A^{\perp}$ is the *orthogonal complement* of A. We denote by A^* the set $(A^{\perp})^{\perp}$, and call this the *orthogonal closure*. We say that a set is *regular* if $A = A^*$. We define an operator:

$$\begin{aligned} R : \mathcal{P}(X) &\to \mathcal{P}(X) \\ \mathcal{A} &\mapsto \{A \in \mathcal{A} : A = A^*\} \end{aligned}$$

Thus, we say that $R(\mathcal{A})$ is the set of regular sets inside $\mathcal{A}$.

Let A be a subset of X. We say that A is $\perp$-closed iff for all $x \in X$:

$$\forall y \in X, y \perp A \to y \perp x \implies x \in A$$

In other words, if $A^{\perp} \subseteq \{x\}^{\perp}$ then $x \in A$.

It is a general fact that a relation between two sets induces a Galois connection between the power sets of those sets. In our case, this Galois connection turns out to be between the set and itself, and because of symmetry, both polarities coincide. This translates to the following useful properties (see for example [1, pp.38]):

Proposition 5.2.3. Given $(X, \perp)$ an orthoframe, and A, B subsets of X:

1. $A \subseteq B$ implies that $B^{\perp} \subseteq A^{\perp}$
2. $A \subseteq A^*$
3. $(A^*)^{\perp} = A^{\perp}$
4. $(\bigcup_{i \in I} A_i)^{\perp} = \bigcap_{i \in I} A_i^{\perp}$
5. $A^{\perp}$ is $\perp$-closed. So A^* is $\perp$-closed as well.

The concept of an orthoframe had been known in the theory of orthomodular lattices for a long time (see [44] for a detailed account). It was also known that this provided a representation for *complete* ortholattices. The topologisation of this concept allowed Goldblatt[36] to extend the representation to *all* ortholattices:

Definition 5.2.4. Let $(X, \leqslant, \perp, \tau)$ be a space such that $\leqslant$ is a partial order, $\perp$ is an irreflexive symmetric order, and τ is a compact topological space. Denote by $\mathsf{Clop}(X)$ the set of clopen sets, $\mathsf{ClopUp}(X)$ the set of clopen upsets, and in line with the above definition, $\mathsf{RClop}(X)$ and $\mathsf{RClopUp}(X)$ the set of regular clopen sets and regular clopen upsets, respectively. We say that X is a *maximal sober orthospace*[1] if for all $x, y \in X$

1. $x \not\leqslant y$ implies that there exists $O \in \mathsf{RClop}(X)$ such that $x \in O$ and $y \notin O$
2. $x \perp y$ implies there exists $O \in \mathsf{RClop}(X)$ such that $x \in O$ and $y \in O^{\perp}$
3. $x \perp y$ and $x \leqslant z$ implies that $z \perp y$
4. $O \in \mathsf{RClop}(X)$ implies that $O^{\perp}$ is in $\mathsf{RClop}(X)$
5. Whenever F is a filter in $\mathsf{RClop}(X)$, then there exists some $x \in X$ such that $F = F_x = \{O \in \mathsf{RClop}(X) : x \in O\}$.

If the structure satisfies 1-4, we call X simply an *orthospace.*

We note the following regarding the previous definition:

Definition 5.2.5. Let $(X, \perp)$ be an orthoframe. Define $x \preceq_{\perp} y$ as follows:

$$x \preceq_{\perp} y \iff \forall z, x \perp z \implies y \perp z$$

Lemma 5.2.6. Let $(X, \perp)$ be an orthoframe. Then $x \preceq_{\perp} y$ is a quasi-order. Moreover, if $(X, \leqslant, \perp, \tau)$ is an orthospace, then $\leqslant = \preceq_{\perp}$.

Proof. The relation is clearly reflexive, and transitive. To see that it is a partial order, suppose that $x \neq y$. Without loss of generality, then, $x \not\leqslant y$, so by Axiom 1, we have that there is some regular U such that $x \in U$ and $y \notin U$. by regularity, since $y \notin U$, there is some z such that $y \not\preceq z$, and $z \in U^{\perp}$. Since $x \in U$, then $x \perp z$, which implies that $x \not\preceq_{\perp} y$. Hence, by contraposition, we obtain that $\preceq_{\perp}$ is a partial order, and also that $\preceq_{\perp} \subseteq \leqslant$. Conversely, if $x \leqslant y$, by Axiom 3, we have that $x \preceq_{\perp} y$. ■

In this sense, the partial order is wholly induced by the irreflexive relation. This means that drawing the graphs corresponding to either the orthogonality or non-orthogonality relation canonically yields the partial order, so long as either relation corresponds to an orthospace.

We have the following fact:

Lemma 5.2.7. Given an orthospace $X = (X, \leqslant, \perp, \tau)$ the structure $(\mathsf{RClop}(X), \cap, ^{\perp}, \varnothing, X)$ is an ortholattice.

To move in the opposite direction, we make use of the space of filters:

Definition 5.2.8. Let O be an ortholattice. Denote by $\mathsf{Fil}(O)$ the set of all proper filters on O. We define an orthogonality relation on $\mathsf{Fil}(O)$ as follows:

$$F \perp G \iff \exists a \in O, a \in F \text{ and } a^{\perp} \in G$$

[1] These are called in [4] orthosober orthospaces. Our change of name reflects our concerns in the thesis with condition 5.

Then we have the following:

Lemma 5.2.9. Let O be an ortholattice, and $X_O = \mathsf{Fil}(O)$. For $a \in O$, denote by $\varphi(a)$ the set $\{F \in \mathsf{Fil}(O) : a \in F\}$. Then $(X_O, \subseteq, \perp, \tau)$ forms an orthosober orthospace where τ is given by the topology with subbasis:

$$\{\varphi(a) : a \in O\} \cup \{X_O - \varphi(a) : a \in O\}$$

From now on, denote by R the complement of the orthogonality relation:

$$xRy \iff x \not\perp y$$

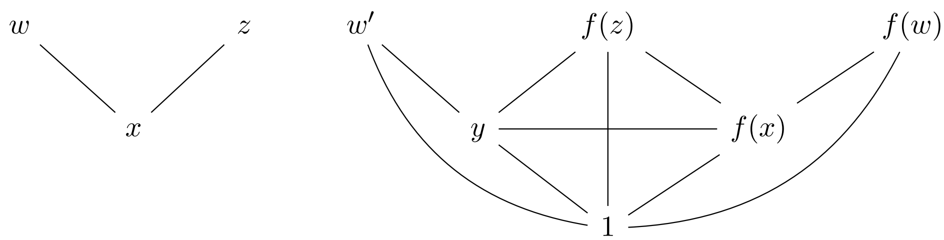
The morphisms of orthospaces are the following:

Definition 5.2.10. Let $f : X \to Y$ be a continuous map between orthospaces. We say that this is an *orthospace morphism* when:

1. If xRy then $f(x)Rf(y)$;
2. If $f(x)Ry$ then there exists some $z \in X$ such that xRz and $y \leqslant f(z)$.

We further say the orthospace morphism is *strong* if the last inequality is an equality.

Readers familiar with modal logic but not familiar with the literature on duality for relevant logics might find the inequality in the back condition odd. However, we note that this is not spurious. For instance, consider the following structures, where the lines refer to the non-orthogonality relation:



It is not hard to see that these are two orthospaces, and that the map defined by f is an orthospace morphism. However, this is *not* a p-morphism with respect to the non-orthogonality relation: indeed, $f(x)Ry$, and indeed, z is such that xRz and $y \leqslant f(z)$; but no node maps to y.

We have the following:

Lemma 5.2.11. Let $f : O \to O'$ be an ortholattice homomorphism. Then $f^{-1} : X_{O'} \to X_O$ is an orthospace morphism. Conversely, if $g : Y \to Y'$ is an orthospace morphism, then $g^{-1} : \mathsf{RClop}(Y') \to \mathsf{RClop}(Y)$ is an ortholattice homomorphism.

The following is the actual statement of the duality:

Theorem 5.2.12. ***(Orthospace duality)*** *The category of ortholattices with ortholattice homomorphisms is dually equivalent to the category of maximal sober orthospaces with orthospace morphisms.*

We now collect some facts we will later make use of regarding orthospaces in general. The first is the condition which Bimbó originally proved as part of her analysis, regarding orthospaces in general:

Lemma 5.2.13. Let $(X, \leqslant, \perp, \tau)$ be an arbitrary orthospace, and O be the dual ortholattice. Then X embeds into X_O through a topological embedding which both preserves and reflects the relation $\perp$.

Proof. See [8, Theorem 3.6]. ■

We also have the following properties which will come useful later:

Lemma 5.2.14. Let $(X, \perp, \leqslant, \tau)$ be an orthospace. Then:

1. If U is a clopen regular, then $x \in U$ if and only if whenever xRy then there is some z such that yRz and $z \in U$.

2. Every clopen regular set is an upset; hence $(X, \leqslant, \tau)$ is a Priestley space.

3. The class of clopen regulars is closed under intersection.

4. Every clopen upset of $(X, \perp, \leqslant, \tau)$ is a union of clopen regular sets.

Proof. (1) This is simply a matter of unfolding the definition that $U = U^*$.

(2) Let U be a clopen regular set. Suppose that $x \in U$ and $x \leqslant y$. Now assume that yRz. Then by the third condition on orthospaces, xRz, so by the first part, there is some w such that zRw and $w \in U$. Hence $y \in U$.

(3) Follows from the above duality. To see (4) assume that V is a clopen upset. Suppose that $x \notin V$ is arbitrary. Then whenever $y \in V$, $y \nleqslant x$; hence by the clopen regular separation property, there is some $U_{x,y}$ such that y belongs there and x does not. Hence:

$$V \subseteq \bigcup_{y \in V} U_{x,y}$$

By compactness we can obtain some $W_x = U_{x,y_0} \cup ... \cup U_{x,y_m}$ such that $V \subseteq W_x$. Then:

$$V = \bigcap_{x \notin V} W_x$$

So by compactness again, we obtain that:

$$V = W_{x_0} \cap ... \cap W_{x_n}$$

Now applying the distributivity of $\cap$ and $\cup$ in a judicious manner, one can write:

$$V = \bigcup_{i=1}^{n} \bigcap_{j=1}^{m} U_{x_i,y_j}$$

Since by (3) clopen regulars are closed under intersection this gives us the result. ■

5.3 Orthologic and KTB

As promised, we can use the structures we have just studied to provide a relational semantics to orthologics:

Definition 5.3.1. (Kripke semantics of Orthologic) Let $(X, \perp)$ be an orthoframe. A valuation v on this frame is a map $v : T_{\mathcal{L}_O}(\mathcal{VAR}) \to \mathcal{P}(X)$ such that $v(p)$ is a regular set for each $p \in \mathcal{VAR}$. We call $\mathfrak{X} = (X, \perp, v)$ a model, and define the satisfaction, at a point $x \in X$, as follows:

- $\mathfrak{X}, x \Vdash p$ iff $x \in v(p)$
- $\mathfrak{X}, x \Vdash \varphi \wedge \psi$ iff $\mathfrak{X}, x \Vdash \varphi$ and $\mathfrak{X}, x \Vdash \psi$
- $\mathfrak{X}, x \Vdash \neg\varphi$ iff whenever xRy, then $\mathfrak{X}, y \nVdash \varphi$.

We say that an orthoframe $(X, \perp)$ *validates* a sequent (φ, ψ) if and only if for all models $\mathfrak{X}$ on $(X, \perp)$, and all points $x \in X$, if $\mathfrak{X}, x \Vdash \varphi$ then $\mathfrak{X}, x \Vdash \psi$. This leads to the following:

Definition 5.3.2. Let O be an orthologic, and $\mathbf{K}$ a class of orthoframes. We say $Log(\mathbf{K}) = \{(\lambda, \psi) : \mathbf{K} \vDash (\lambda, \psi)\}$ is the *orthologic associated to* $\mathbf{K}$.

As usual in modal logic, there is a Galois connection between logics and classes of frames.

Definition 5.3.3. Let O be an orthologic. We say that this is *Kripke complete* if there is a class $\mathbf{K}$ of orthoframes such that:

$$(\lambda, \psi) \in O \iff \mathbf{K} \vDash (\lambda, \psi)$$

All following results were shown by Goldblatt in [35]. First, we have the following adequacy theorem:

Lemma 5.3.4. Let $\mathbf{O}$ be an ortholattice, and $v : T_{\mathcal{L}_O}(\mathcal{VAR}) \to O$ a valuation. Let Y be an orthospace such that $\mathsf{RClop}(Y) \cong \mathbf{O}$ via a map φ. Let $v' : T_{\mathcal{L}_O}(\mathcal{VAR}) \to \mathcal{P}(Y)$ be the valuation given by $v'(p) = \varphi(v(p))$. Then $\mathbf{O}, v \vDash (\lambda, \psi)$ if and only if $Y, v' \vDash (\lambda, \psi)$.

Proof. Note that the above definition is sound, and by induction on the construction of formulas, and the duality, we get for each $\chi \in \mathcal{L}_O$:

$$v'(\chi) = \varphi(v(\chi))$$

Indeed, $v'(p) = \varphi(v(p))$, and for conjunctions, this is obvious. If this holds for χ, note that $v'(\chi^{\perp}) = v'(\chi)^{\perp} = \varphi(v(\chi))^{\perp} = \varphi(v(\chi)^{\perp})$. Now assume that $\mathbf{O}, v \nvDash (\lambda, \psi)$. Since $v(\lambda) \nleq v(\psi)$ by duality, $\varphi(v(\lambda)) \nsubseteq \varphi(v(\psi))$; hence $v'(\lambda) \nsubseteq v'(\psi)$ so there is a point x in the first set, but not the second. Hence $(X_O, v'), x \Vdash \lambda$ but $(X_O, v'), x \nVdash \psi$. The converse follows by similar arguments. ∎

Corollary 5.3.5. *(Canonical Model of Orthologic) Let O be any orthologic. Then there is a model, $\mathfrak{M}_O = (W^O, \perp^O, v^O)$, such that $(\lambda, \psi) \in O$ if and only if for each $x \in W^O$ $\mathfrak{M}_O, x \Vdash (\lambda, \psi)$.*

Proof. Let O be the arbitrary orthologic, and $\mathbf{F}_O$ be the Lindenbaum-Tarski algebra associated to this orthologic. Let $v : T_{\mathcal{L}_O}(\mathcal{VAR}) \to \mathbf{F}_O$ be the valuation mapping $v(p) = [p]_O$. Then it is clear to see that $(\varphi, \psi) \in O$ if and only if $\mathbf{F}_O, v \vDash (\varphi, \psi)$. Hence, by Lemma 5.3.4, we get the desired result by taking the orthospace dual of $\mathbf{F}_O$. ∎

Corollary 5.3.6. *The minimal orthologic is Kripke complete.*

Proof. Soundness follows from verifying that the axioms outlined above in the definition of an orthologic are valid in the minimal orthologic. Completeness follows from Corollary 5.3.5. ■

Quite relevant for our purposes is the notion of *filtration*. This was shown by Goldblatt, and consists essentially of the following:

Theorem 5.3.7. *Let $\Sigma \subseteq \mathbf{Tm}_{\mathcal{L}_O}$ be a finite collection of well-formed formulas which is closed under subformulas, and such that for each propositional variable p_i, $p_i \in \Sigma$ if and only if $p_i^{\perp} \in \Sigma$. Let $\mathfrak{X} = (X, \perp, v)$ be an orthomodel. Then there exists an orthomodel $\mathfrak{X}_f = (X/\sim, \perp^f, v^f)$, such that for each $x \in X$, and for each $\varphi \in \Sigma$, $\mathfrak{X}, x \Vdash \varphi$ if and only if $\mathfrak{X}_f, [x] \Vdash \varphi$.*

Note that more is true: given a sequent (φ, ψ) where $\varphi, \psi \in \Sigma$, we have that $\mathfrak{X}, x \Vdash (\varphi, \psi)$ if and only if $\mathfrak{X}_f, [x] \Vdash (\varphi, \psi)$.

We are now ready to take on the main object of study of this chapter. We recall from Chapter 3 that the *Goldblatt translation* between Orthologic and KTB logic [36] is defined recursively as follows:

1. For each propositional variable p, $G(p) = \Box\Diamond p$;
2. $G(\psi \wedge \varphi) = G(\psi) \wedge G(\varphi)$
3. $G(\varphi^{\perp}) = \Box\neg G(\varphi)$

From these clauses and the DeMorgan laws of ortholattices one deduces semantically that $G(\varphi \vee \psi) \approx \Box\Diamond(G(\varphi) \vee G(\psi))$. Given a set of formulas Γ, define $G[\Gamma] := \{G(\varphi) : \varphi \in \Gamma\}$.

The first and foundational result about the Goldblatt translation was the following:

Theorem 5.3.8. *(Goldblatt, 1975)[36] For any pair of formulas $\varphi, \psi \in \mathcal{L}_O$, we have that:*

$$(\varphi, \psi) \in \mathsf{O} \iff G(\varphi) \to G(\psi) \in \mathsf{KTB}$$

Proof. Suppose that $G(\varphi) \to G(\psi) \notin \mathsf{KTB}$. By Kripke completeness of **KTB**, there is then a Kripke model $\mathfrak{M} = (W, R, V)$, and a world $x \in W$ such that $\mathfrak{M}, x \Vdash G(\Gamma)$ but $\mathfrak{M}, x \nVdash G(\varphi)$. Now consider the orthoframe given by $\mathfrak{M}' = (W, \perp_R, V')$ where $x \perp_R y$ if and only if $\neg(xRy)$, and:

$$V'(p) = \Box\Diamond V(p)$$

It is not hard to see that regular subsets for $\perp_R$ are exactly those of the form $\Box\Diamond U$ for some $U \subseteq W$. Hence, $\mathfrak{M}'$ is an orthoframe model. Moreover, we can show by induction that:

$$V'(p) = V(G(p))$$

Which shows that then $\mathfrak{M}', x \Vdash \Gamma$ and $\mathfrak{M}', x \nVdash \psi$. Hence, by soundness of orthologic with respect to orthoframes, $(\varphi, \psi) \notin \mathsf{O}$. The converse follows similarly, using Kripke completeness of orthologic. ■

We thus have that the translation is indeed correct. Later, Miyazaki [52] extended such results to a theory analogous to that of modal companions:

Definition 5.3.9. Let $O \in \Lambda(\mathsf{O})$ and $L \in \mathbf{NExt}(\mathsf{KTB})$. We say that L is a *KTB-companion* of O if:

$$(\varphi, \psi) \in O \iff G(\varphi) \to G(\psi) \in L$$

Theorem 5.3.10. *[52, Theorem 21] The following hold:*

1. *For each $L \in \mathbf{NExt}(\mathsf{KTB})$, there is a logic $O \in \Lambda(\mathsf{O})$ such that L is the modal companion of O; this assignment preserves Kripke completeness, tabularity and FMP.*

2. *For each orthologic $O \in \Lambda(\mathsf{O})$ with the FMP, there is a logic $L \in \mathbf{NExt}(\mathsf{KTB})$ such that L is the modal companion of O; this assignment preserves tabularity and FMP.*

The techniques used by Miyazaki to prove the first part mirror our functor θ, whilst for the second he uses a finite representation theorem, and arguments analogous to those used to establish orthospace duality. In light of our discussion of orthospaces, the correspondence can be captured in the following:

Proposition 5.3.11. Let $(X, \perp)$ be a finite orthospace. Then (X, R) is a KTB-space, and additionally:

$$(X, \perp) \Vdash (\varphi, \psi) \iff (X, R) \Vdash G(\varphi) \to G(\psi)$$

Proof. The fact that (X, R) will be a KTB space is trivial: since $(X, \perp)$ is finite, the space is discrete, hence every set is clopen, and since R is reflexive and symmetric, (X, R) will be a finite KTB space, i.e., a KTB frame. Given an orthomodel $(X, \perp, v)$, consider the same assignment in (X, R, v); hence $v(\chi) = v(G(\chi))$, by definition. Similarly, if (X, R, v) is a KTB model, define v' by letting $v'(p) = \Box\Diamond v(p)$. Then again we have that $v'(\chi) = v(G(\chi))$. This yields the result. ■

We will now move to approaching the Goldblatt translation as a contextual translation. The following is the basic result that we can expect:

Proposition 5.3.12. Let $(\mathbf{B}, \Box)$ be a **KTB**-algebra. Then the set:

$$\mathbf{O_B} := \{a \in B : \Box\Diamond a = a\}$$

Is an ortholattice with the operations induced by the Goldblatt translation. Moreover, for each set $\Phi \subseteq \mathsf{Eq}_{\mathcal{L}_O}$ of equations, we have:

$$\mathbf{O}_B \vDash \Phi \iff \mathbf{B} \vDash G^*(\Phi)$$

Proof. The verification that the regular elements form an ortholattice is straightforward; see for instance that for conjunction:

$$\Box\Diamond(a \wedge b) \leqslant \Box(\Diamond a \wedge \Diamond b) \leqslant \Box\Diamond a \wedge \Box\Diamond b \leqslant a \wedge b \leqslant \Box\Diamond(a \wedge b)$$

Where the inequalities follow from usual modal logical reasoning. The second part follows by the same arguments as Proposition 3.3.1. ■

Now let $\Theta = \{x \approx \Box\Diamond x\}$. Note that mapping variables to $\Box\Diamond$, and considering all valuations, is the same as operating modulo the context Θ. Hence the former can be seen as a translation in the sense presented in Chapter 3. We thus have the following:

Proposition 5.3.13. The translation $\langle G, \Theta, \Box\Diamond \rangle$ is a selective translation

Proof. Let Φ be a collection of equations in the language of ortholattices. By Proposition 5.3.12, we can prove the translation is contextual: indeed, assume that Φ is a set of equations in the language of KTB, and $G^*(\Phi) \not\models G_*(\lambda) \approx G_*(\psi)$. Let $\mathbf{B}$ be a KTB algebra, and v a valuation, such that $\mathbf{B} \models G^*(\Phi)$, and $\mathbf{B}, v \not\models G_*(\lambda) \neq G_*(\psi)$. Then by that Proposition, $\mathbf{O_B} \not\models \lambda \approx \psi$, whilst it satisfies Φ, by Proposition 5.3.12.

Now conversely, assume that $\Phi \not\models \lambda \approx \gamma$. Hence, without loss of generality, $\Phi \not\models (\lambda, \gamma)$. Let $\mathbf{O}$ be an ortholattice such that $\mathbf{O} \models \Phi$ and $\mathbf{O} \not\models (\lambda, \gamma)$, and let v be a valuation. Hence, by Lemma 5.3.4, we have that $Y, v \not\models (\lambda, \gamma)$, for Y the dual orthospace of $\mathbf{O}$, at a point x, and $Y, v \models (\mu, \iota), (\iota, \mu)$ for all $\mu \approx \iota \in \Phi$. Now consider:

$$\Sigma = \{G_*(\varphi) \leqslant G_*(\psi) : (\varphi, \psi) \in \Phi\} \cup \{G_*(\lambda) \leqslant G_*(\gamma)\}.$$

Let Σ_0 be a finite subset of Σ, say $\Sigma_0 = \{G_*(\varphi_i) \leqslant G_*(\psi_i) : i \leqslant n\} \cup \{G_*(\lambda) \leqslant G_*(\gamma)\}$. Let $F_0 = \{\varphi_i, \psi_i : G_*(\varphi_i) \leqslant G_*(\psi_i) \in \Sigma_0\} \cup \{\lambda, \gamma\}$, and F the subformula closure of F_0. Let (Y_F, v_F) be the filtration of Y, v along F. Hence, by hypothesis on Y, we have that:

$$\forall G_*(\varphi_i) \to G_*(\psi_i), Y_F, v_F \Vdash (\varphi_i, \psi_i) \text{ and } Y_F, v_F \nVdash (\lambda, \gamma)$$

By Proposition 5.3.11, we have that $(Y_F, v_F) \Vdash G_*(\varphi_i) \to G_*(\psi_i)$, and $(Y_F, v_F) \nVdash G_*(\lambda) \to G_*(\gamma)$. By duality, this means that there is a valuation v on a KTB algebra $\mathbf{B}$, such that $\mathbf{B}, v \models G_*(\varphi_i) \to G_*(\psi_i)$. By definition of satisfaction, this means that $\mathbf{B}, v \models G_*(\varphi_i) \leqslant G_*(\psi_i)$., and also $\mathbf{B}, v \not\models G_*(\lambda) \leqslant G_*(\gamma)$. Hence, we can find a KTB algebra and a valuation satisfying Σ_0.

By compactness of first order logic, we have that since each finite subset of Σ is satisfiable in a KTB algebra, the whole of Σ is. Hence, there is a KTB algebra $\mathbf{B}$, such that $\mathbf{B} \models G^*(\Phi)$, and $\mathbf{B} \not\models G_*(\lambda) \approx G_*(\gamma)$. Hence, this implies that $G^*(\Phi) \not\models G_*(\lambda) \approx G_*(\gamma)$. This shows that the translation is faithful.

Finally, notice that the term $\Box\Diamond$ is a selector, essentially because $\Box\Diamond\Box\Diamond p = \Box\Diamond p$. ∎

Having this established, we can define versions of the map τ: for $O \in \Lambda(\mathsf{O})$, $\tau(O) = \mathbf{KTB} \oplus \{G(\varphi) \to G(\psi) : (\varphi, \psi) \in O\}$; as well as ρ: $\rho(L) = \{(\varphi, \psi) : G(\varphi) \to G(\psi) \in L\}$. On algebras, we have $\tau(\mathbf{K}) = \{\mathbf{B} : \mathbf{O_B} \in \mathbf{K}\}$ and $\rho(\mathbf{M}) = \{\mathbf{O}_B : \mathbf{B} \in \mathbf{M}\}$. Using these, one can make many questions related to our discussion in Chapter 4. Given what we have discussed about Polyatomic Blok-Esakia isomorphisms, we can naturally ask whether the Goldblatt translation is strongly selective, or even sober, and whether we can describe the schematic fragment of the PAt-variants as laid out above.

Our results from now on will be mostly negative, and we will seek to identify and emphasise the sources of problems. To work towards this, we begin by giving a dual description of the transformations we have encountered.

Definition 5.3.14. Let $\theta^* : \mathbf{KTBS} \to \mathbf{OrtS}$ be the assignment going from the category of **KTB**-spaces with continuous p-morphisms, to the category of orthospaces with orthospace morphisms, defined as follows: given a KTB space (X, R, τ) take the relation, $\preceq$:

$$x \preceq y \iff \forall z \in X (yRz \implies xRz)$$

This forms a preorder. Let $\equiv_{\preceq}$ be the induced equivalence relation. Let $\equiv_{\preceq}$ be defined as:

$$[x]R_{\equiv}[y] \iff \exists x' \in [x], y' \in [y], x'Ry'$$

And similarly for $\preceq_{\equiv}$. Then we denote by $X^* = (X/\equiv_{\preceq}, R_{\equiv}, \preceq_{\equiv}, \tau_{\equiv})$, where $\tau_{\equiv}$ is the quotient topology.

Lemma 5.3.15. Let (X, R, τ) be a KTB-space. Then X^* as defined above is an orthospace, when we define $[x]\perp[y]$ if and only if $\neg([x]R_{\equiv}[y])$. Moreover, the map $q : X \to X^*$, when X^* is understood as a KTB space, is a surjective p-morphism.

Proof. (Sketch) It is a straightforward, using the fact that the relation R is reflexive and symmetric, to see that the definitions given are sound. Now let $q : X \to X^*$ be the quotient map; to obtain the result, after observing that this preserves the relations, it suffices to show that the clopen regulars on X are saturated, i.e, if U is a clopen set and $\Box\Diamond U = U$, then $q^{-1}q(U) = U$. Indeed, if $x \in q^{-1}q(U)$, then $q(x) = q(z)$ where $z \in U$; hence for some $x \equiv x'$ and $z \equiv z'$, $z' \leq x'$; but then $z' \in U$, so $x' \in U$, i.e, $x \in U$ as well. Using this one can then show the axioms in a straightforward way. ■

The reader will have noted that the former transformation essentially only removes clusters associated to the induced quasi-order, and collapses them. This means that, for instance, a cluster of 2-elements in the modal logic sense will be collapsed to a single element, as exemplified in Figure 5.1:

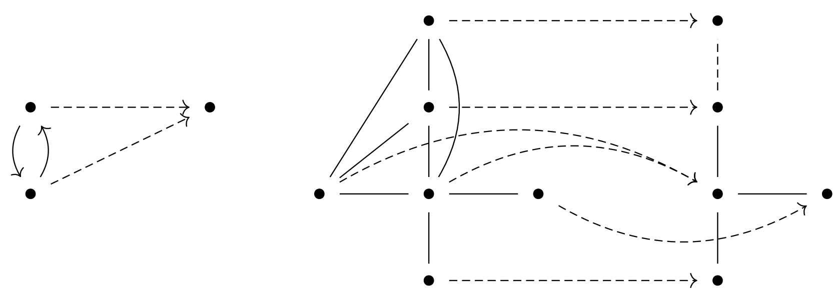


Figure 5.1: Collapsing of clusters

We can also define the action of this transformation on morphisms:

Lemma 5.3.16. Let $f : X \to X'$ be a p-morphism between KTB-spaces. Then $\dot{f} : \theta^*(X) \to \theta^*(X')$ defined by $[x] \mapsto [f(x)]$ is a strong orthospace morphism.

Proof. This is well-defined since f preserves the order $\leq$: if $x \equiv y$, suppose that $f(y)Rz$; then yRw where $f(w) = z$; then xRw, so $f(x)Rz$, i.e, $f(x) \leq f(y)$, and similarly, $f(y) \leq f(x)$, i.e, $f(x) \equiv f(y)$. It is easily seen to be continuous and order preserving since f is. It also satisfies the back condition: if $f([x])R[y]$, there is some $x' \equiv f(x)$ such that $x'Ry'$ and $y' \leq y$, i.e, $f(x)Ry$; then because f is a p-morphism, there is some z such that xRz, and $f(z) = y$. Hence $[x]R[z]$ and $[f(z)] = [y]$, i.e, $f([z]) = [y]$. ■

More importantly, we get that this corresponds to the θ operation:

Proposition 5.3.17. Let $\mathbf{B}$ be an arbitrary $\mathbf{KTB}$ algebra and $X_{\mathbf{B}}$ its dual space. Then $\mathsf{RClop}(X^*_{\mathbf{B}}) \cong \mathbf{O_B}$. Hence, $X^*_{\mathbf{B}}$ is an orthospace embedding into the orthospace dual of $\mathbf{O_B}$. Moreover, if $f : \mathbf{B} \to \mathbf{B}'$ is a homomorphism, then $(f\restriction_{\mathbf{O_B}})^{-1} = \dot{f}^{-1}$

Proof. For each $a \in \mathbf{O_B}$, let $f(a) = \{[x] \in X_B^* : x \in \varphi(a)\}$. Note that because a is regular, this is a clopen regular subset of $X_{\mathbf{B}}^*$, and is well-defined: if $x \in \varphi(a)$ and $x \equiv y$, then $a \in x$; now if $a \notin y$, because a is regular, $\Diamond\Box\neg a \in y$; then yRz, and $\Box\neg a \in z$, but also, xRz, so zRx, a contradiction.

Additionally, this assignment is injective: if $a \neq b$ are both regular, let $a \in x$ and $b \notin x$ where $x \in X_{\mathbf{B}}$. Then we have that $[x] \in f(a)$ and $[x] \notin f(b)$, which shows that $f(a) \neq f(b)$. Moroever, if U is a clopen regular subset of $X_{\mathbf{B}}^*$, by definition of the quotient topology, $q^{-1}(U)$ is clopen, and since q is a p-morphism, also regular:

- First, note that $\Box\Diamond q^{-1}(U) = q^{-1}(\Box\Diamond U)$: if $q(x) \in \Box\Diamond U$, and xRy, then $q(x)Rq(y)$, so by definition, $q(y)Rz$ and $z \in U$; hence, there is some z' such that yRz', and $q(z') = z$. Hence, $x \in \Box\Diamond q^{-1}(U)$. Conversely, if $x \in \Box\Diamond q^{-1}(U)$, and $q(x)Ry$, by assumption, for some y', xRy', so $y'Rz$ and $z \in q^{-1}(U)$; so $q(y')Rq(z')$, and $q(z') \in U$. So $x \in q^{-1}(\Box\Diamond U)$.
- Then notice that $\Box\Diamond q^{-1}(U) = q^{-1}(\Box\Diamond U) = q^{-1}(U)$, since U is assumed to be regular.

Hence, $q^{-1}(U) = \varphi(a)$, where a is a regular element. Thus, note that $f(a) = \{[x] : x \in \varphi(a)\} = q[\varphi(a)] = q[q^{-1}(U)] = U$, as intended. Finally, to see that this is a homomorphism, note that $f(a \wedge b) = f(a) \wedge f(b)$, clearly, and $f(a^{\perp}) = \{[x] : x \in \varphi(\Box\neg a)\} = \Box\neg\{[x] : x \in \varphi(a)\}$: if $x \in \varphi(\Box\neg a)$, then $\Box\neg a \in x$; so if $x \equiv x'$, since this is a regular element, also $\Box\neg a \in x'$. Similarly, if $y \equiv y'$, then $a \in y$ if and only if $a \in y'$ because this is regular. Hence, if $[x]R[y]$, then for some $x' \equiv x$ and $y \equiv y'$ we have that $x'Ry'$, so $\Box\neg a \in x'$, and hence, $a \notin y$. So $[y] \notin f(a)$. Conversely, if $[x] \in \Box\neg f(a)$, assume that $\Box\neg a \notin x$; hence $\Diamond a \in x$, so xRy, and $a \in y$; hence $[x]R[y]$ and $a \in y$, so $[y] \in f(a)$, a contradiction. This shows the first statement. The second statement follows by similar arguments on the maps. ■

Using these transformations, we can highlight some aspects which distinguishes this translation from the GMT case. There, recall that the class of spaces such that the corresponding transformation ρ acts as an identity, is modally definable (by the **Grz** axiom), even whilst it is not elementary. However, we have:

Proposition 5.3.18. The class of KTB spaces such that θ^* acts as an identity is not closed under p-morphic images.

Proof. Consider the KTB frame, $\mathbf{3} - \mathbf{f}$, consisting of three points in a chain, and consider the map as drawn in Figure 5.2.

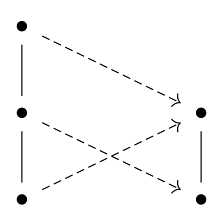


Figure 5.2: Collapse of frame

It is straightforward to verify that the map as defined is a p-morphism. But we see that the first frame does not have any clusters in the $\leq$-sense, whilst the second does. ■

Hence, our hopes that an isomorphism motivated by a natural axiom such as **Grz** might be diminished. So let us diminish them further, by removing the possibility of a more complicated isomorphism. To see this, consider the algebras corresponding to the **KTB**-algebra which is a single cluster, and the **KTB**-algebra which is a 2-cluster, as drawn above. As mentioned before, these will be identified by θ^*. Now we note the following two facts, respectively from ortholattice and KTB theory:

Fact 1. (cf. [11, Corollary 3.6]) The bottom of lattice of varieties of ortholattices consists of the following varieties: the trivial variety, covered by the Boolean algebras, covered by the variety generated by MO_2, and the variety generated by Benzene.

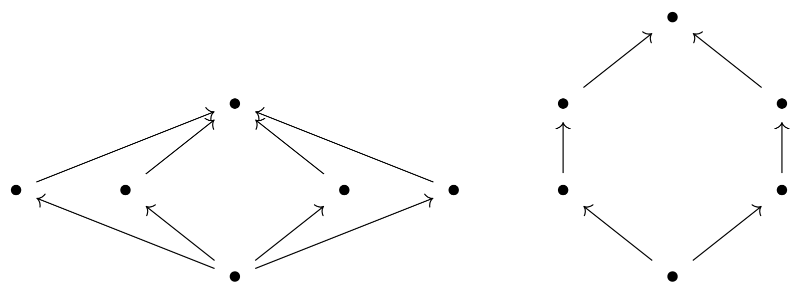


Figure 5.3: MO_2 and Benzene

Fact 2. (cf. Miyazaki,[46, Theorem 2.2]) The top of the lattice of KTB logics consists of the following logics: the trivial logic, covered by the logic generated by a single reflexive point, covered by the logic generated by a 2-cluster.

By this latter fact, we have that the identification made by θ above collapses two algebras which come from distinct varieties. Hence, the map cannot correspond, in the top part, to an isomorphism. But what we noted from ortholattices implies more is true. Since any isomorphism would have to preserve the cover relation, the existence of such a map is impossible for purely combinatorial reasons.

Corollary 5.3.19. *There is no logic* $L \in \mathbf{NExt}(\mathbf{KTB})$ *such that there is an isomorphism between* $\Lambda(\mathsf{O})$ *and* $\mathbf{NExt}(L)$.

This leaves the possibility that we could have a Polyatomic Blok-Esakia isomorphism. We will now rule this possibility out. Recall that by a Polyatomic Blok-Esakia isomorphism, we mean that the lattices $\Lambda(\mathsf{O})$ and $\Lambda^{\Box\Diamond}(\mathsf{KTB})$ are isomorphic through the specific maps τ and ρ, and it is assumed additionally that ρ is a homomorphism on the lattice $\Lambda(\mathbf{KTB})$ of varieties. We will prove that such a situation cannot happen.

To see this, consider the orthologic generated by *Benzene*, the ortholattice ocurring on the right hand side in Figure 5.3[2]. Now consider the following two KTB frames: Note that θ^* acts as an identity on both of these frames (i.e, they contain no $\leq$-clusters), and both of them are dual to the Benzene ortholattice. The right hand side frame is the maximally sober orthospace frame

[2]This is a very well-known algebra in the theory of orthomodular lattices [11], since it is the forbidden subalgebra which characterises orthomodular lattices within ortholattices

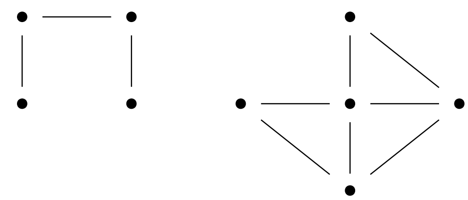


Figure 5.4: Two Benzene frames

which is dual to Benzene, whilst the left hand side is another orthospace which has Benzene as the ortholattice of clopen regulars. Indeed, we have the following:

Lemma 5.3.20. The previous two frames are the only two orthospaces which have Benzene as the ortholattice of clopen regulars.

Proof. By inspection we see that the right hand side orthospace, call it X_{Be} is indeed the maximal sober one. By Lemma 5.2.13, we know that if Y is an orthospace with Benzene as the ortholattice of clopen regulars, then Y embeds through a topological embedding that both preserves and reflects the relation $\perp$ (correspondingly the relation R) into X_{Be}. This means that the only possible orthospaces are those which remove a point and all relations from that point to the points it relates to. Now, manually, we can check that removing any point other than the central one from X_{Be} yields a frame which has a cluster, and hence cannot be an orthospace. So Y_{Be} is the only possible candidate. ∎

Now let $L(X_{Be})$ and $L(Y_{Be})$ be the KTB-logics of each of these frames. Let O_{Be} be the orthologic generated by Benzene. Then we have the following:

Lemma 5.3.21. Both $L(X_{Be})$ and $L(Y_{Be})$ are KTB-companions of O_{Be}.

Proof. Suppose that (λ, ψ) is an arbitrary sequent in O_{Be}. Let v be a valuation on X_{Be}, respectively Y_{Be}; then consider the valuation v' given by $v'(p) = \Box\Diamond v(p)$. Then we have that:

$$(X_{Be}, v'), x \Vdash (\lambda, \psi) \iff (X_{Be}, v), x \Vdash G(\lambda) \to G(\psi)$$

This follows easily by induction, and the definition of the translation. Moreover, by Lemma 5.3.4, we have that since $(\lambda, \psi) \in O_{Be}$, then $(X_{Be}, v'), x \Vdash (\lambda, \psi)$ for any x; and similar for Y_{Be} (we note that there we did not assume that the dual space was maximally sober). Hence $G(\lambda) \to G(\psi) \in L(X_{Be})$ and $L(Y_{Be})$.

Conversely, suppose that $(\lambda, \psi) \notin O_{Be}$. By assumption, then Be, the Benzene algebra, is such that $Be \nvDash (\lambda, \psi)$. So $X_{Be} \nvDash (\lambda, \psi)$, and $Y_{Be} \nvDash (\lambda, \psi)$. Transferring this to KTB using the usual arguments then yields that $G(\lambda) \to G(\psi) \notin L(X_{Be}) \cup L(Y_{Be})$. ∎

Moreover, one can see that these logics are distinct by using the following observation (see Miyazaki [51, Lemma 2.6] for a stronger result implying this):

Lemma 5.3.22. Every finite KTB algebra whose dual is a connected graph is a simple algebra (and hence, subdirectly irreducible).

This follows from the fact that by standard Jonsson-Tarski duality, generated subframes correspond to homomorphic images. Moreover, since the duals of KTB algebras have a symmetric relation, this means any generated subframe will necessarily encompass the whole frame.

Now, given this, we have that the two frames above are subdirectly irreducible, so by Jonssons' Lemma (see Preliminaries), noting that in the case of finite algebras ultraproducts disappear, if they were equal, we would have these would be p-morphic images of generated subframes of each other. Since all generated subframes are the frames themselves, they would have to be p-morphic images of each other – something which one can manually verify does not happen, in either case. Indeed, this implies that there is no p-morphism from X_{Be} to Y_{Be} or vice-versa, whether surjective or not.

Hence, by the above facts, if X is some KTB space and $\theta^*(X)$ is the dual of Benzene, then X is obtained by some cluster-expansion of either X_{Be} or Y_{Be}. Moreover, note that if X is arbitrary in such conditions, then either X_{Be} or Y_{Be} is a p-morphic of X, by Lemma 5.3.15.

Now assume that there is a greatest KTB-companion of the orthologic of Benzene. Let $L \in$ **NExt**(KTB) be such a logic, and $\mathsf{Var}(L)$ be the dual variety. Hence we have that:

$$\mathsf{Var}(L) \subseteq \mathsf{Var}(X_{Be}) \text{ and } \mathsf{Var}(L) \subseteq \mathsf{Var}(Y_{Be}).$$

Since it is a companion, and we assume that ρ is a homomorphism, we have that this is enough to get that $\rho(\mathsf{Var}(L)) = \mathsf{Var}(Be)$ (through arguments similar to those sketched in Chapter 4). Hence, there is some $\mathbf{B} \in \mathsf{Var}(L)$ such that $\theta(\mathbf{B}) \cong Be$, and this corresponds dually to some KTB space X.

Now if the first case holds, then X has X_{Be} as a p-morphic image, and we have a contradiction, since that would mean that X_{Be} would be in $Var(Y_{Be})$. A similar argument shows that Y_{Be} cannot be the p-morphic image of X. Hence, we have just proved:

Corollary 5.3.23. *The Polyatomic Blok-Esakia fails for the Goldblatt translation.*

From the previous proof one can extract, using a very similar idea, a stronger result:

Proposition 5.3.24. The Goldblatt translation is not strongly selective.

Proof. We make use again of Be and the two spaces above, X_{Be} and Y_{Be}. Let their algebras be denoted by A and B, respectively.

Now assume that the translation was strongly selective, and let $\mathcal{F}(Be)$ be the image under the left adjoint. Denote by Z the unique dual KTB space of this algebra. Since this is an adjunction, we have counit maps which are KTB homomorphisms:

$$\varepsilon_A : \mathcal{F}(Be) \to A \text{ and } \varepsilon_B : \mathcal{F}(Be) \to B$$

Dualising, this means that there are p-morphisms $f_A : X_{Be} \to Z$, and $f_B : Y_{Be} \to Z$. Now, since we assume the translation to be strongly selective, we have that $\mathbf{O}_{\mathcal{F}(Be)} \cong Be$. Hence, by Proposition 5.3.17, we have that $\theta^*(Z)$ is an orthospace which dualises to Be. By the above arguments, then Z must be a cluster expansion of either X_{Be} or Y_{Be}.

First assume that Z is a cluster expansion of X_{Be}. Hence, we have a p-morphism $k : Z \to X_{Be}$, which collapses all the clusters. Since $f_B : Y_{Be} \to Z$ is a p-morphism, note that then we have a p-morphism $k \circ f_B : Y_{Be} \to X_{Be}$. But as noted above, there is no p-morphism between these two structures. Similar arguments show that if Z is a cluster expansion of Y_{Be}, then we get a contradiction. Hence, by reductio, we conclude that the Goldblatt translation is not strongly selective. ∎

At this point, the hope for a reasonable Blok-Esakia theory of the kinds we discussed in Chapter 4 seems very slim. Hence it might be good to take stock on the problems we have here encountered so far. These can be summarily divided into two classes:

- **Regularity problems**: In a (PAt-) Blok-Esakia theory one would want regularly generated algebras to form a nice enough class. This implies that the class should be modally definable, or at least, definable by some quasi-equation, universal formula or other such logical apparatus, over the target signature (e.g, modal algebras).

- **Sobriety problems**: In a (PA) Blok-Esakia theory, one would want the functors to lose as little information as possible.

We note that as far as regularity goes, very little seems to be possible to do in our current setting. The lack of closure under subalgebras of the class of regularly generated KTB algebras means that a genuine Blok-Esakia isomorphism would likely require some definition involving formulas of a higher complexity than universal ones.

As far as sobriety goes, the situation is rather peculiar. The proofs we gave showing that sobriety fails, made use of the fact that there were multiple KTB spaces which dualised to the same orthospace – just as in Corollary 4.2.11 regarding the KGG translation. However, unlike that case, we could establish some bounds, namely, the maximal sober orthospace as a universal frame embedding all other such frames.

As such, it seems there could be some hope to eliminate these extraneous frames through some logical expedient, and with that, obtain some form of isomorphism between logical systems. A natural choice, in light of this translation, could be to consider a suitably defined fragment of KTB consisting of all maximally sober KTB spaces; that is, consider only the KTB models which dualise to maximally sober orthospaces. This approach would be justified in having these models be somewhat analogous to the *UV-spaces* of choice-free duality [7, 49]. This can presumably be done if one can find an appropriate logical description of such a fragment in algebraic terms. We have not been able to do so.

Instead, in the next chapter we will take a different route, and look for *minimally sober* spaces. As we will see these present different challenges, and require us to rework our machinery. Nevertheless, we think that this approach is justified by recent approaches to duality of bounded lattices [30, 29], and might be independently interesting.

5.4 Chapter Summary

We summarise our principal contributions in this chapter as follows:

- We show that the Goldblatt translation is a selective translation.

- We develop the maps naturally corresponding to the adjunction induced by the Goldblatt translation in a dual setting, and use them to study KTB companions.

- We show the impossibility of an isomorphism of the same kind as in the classic Blok-Esakia theorem, in Corollary 5.3.19, that the Polyatomic Blok-Esakia isomorphism fails, in Corollary 5.3.23, and that the Goldblatt translation is not strongly selective in Proposition 5.3.24.

Chapter 6

Sober Representations and Orthoimplicative Logic

In this chapter we continue our investigations of chapter 5 into the logic of ortholattices, and the relationships this establishes with other logical systems.

As noted in the previous chapter, some of the difficulties faced in studying the translation of orthologics into other systems can be attributed to the duality these ortholattices hold with orthospaces - whether through the fact that maximal sobriety is required to get an actual duality, or the fact that ortholattices seem to lack syntactic resources to capture natural operations one can consider on the dual space. In this chapter, we outline an alternative dual representation. This makes use of a notion of *quasi-prime filter*, which provides some advantages, in our setting, over the orthospace one (e.g, it restricts to Stone duality in the Boolean algebra case). Using this notion, we construct a simple representation of the "distributive envelope" of an ortholattice [29]. We then study *compatible ortholattices*, which admit an implicative structure, relating these to the Weak Heyting algebras of Celani and Jansana [13]. We prove some basic properties of the resulting systems, exploiting the advantages of our representation.

We then turn our focus to the relationship between these new "orthoimplicative systems" and **KTB**. Motivated by our representation, we identify a fragment of the system KTB admitting a sober translation, which we call "sober KTB", and provide a translation, analogous to the Goldblatt translation, to these systems. This allows us to obtain a PAt-Blok-Esakia isomorphism between the lattice of orthoimplicative logics and the lattice of sober KTB.

6.1 Admissible joins and Quasi-Prime Filters

Let us take stock of the representation studied in the last section. We begin with an ortholattice, and get from it a dual space consisting of all filters. Now, since Boolean algebras are ortholattices as well, this means that the dual of a Boolean algebra will include many more filters than just the ultrafilters - indeed, what one gets is the so called *upper Vietoris space* (see [7], see also [49] for the case of ortholattices), which represents a Boolean algebra by *all* of its filters. Whilst having the advantage of being choice-free, it is somewhat unsatisfactory if one wants to connect this construction to the established Priestley and Stone-type dualities for distributive lattices.

Similar concerns have long been present in the duality of orthomodular lattices. Indeed, already in 1986 Iturrioz [42] proved a representation theorem for these lattices which made use of filters sharing the center of an orthomodular lattice, which is a Boolean algebra, and her representation does restrict to the Boolean algebra case. However, her approach does not seem easily adapted to the context of general ortholattices, since the notions of center and commuting elements are only truly valuable in the presence of orthomodularity.

Our approach stems from ideas developed by Gehrke and van Gool [29]. These authors studied the free distributive lattices generated by a (possibly non-distributive) bounded lattice. Naturally, in the case of non-distributive lattices, one cannot then expect for this extension to be a homomorphism, since it cannot preserve both joins and meets. However, the key observation of the authors, stemming from the theory of canonical extensions, was that in such a construction one needs to preserve as much of the original structure as possible. That is, if a join was already distributive, for example, if $a \vee b$ is such that for all c:

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c),$$

then we would want this join to be preserved by the extension map.

We call a finitary – not necessarily just binary – join with this property an *admissible join*. A trivial, but important, case, is the order relation:

Lemma 6.1.1. If $a \leqslant b$ in a lattice, then the join $a \vee b$ is admissible.

Proof. Simply note that if c is arbitrary, then:

$$a \wedge c \leqslant b \wedge c$$

Hence $(a \wedge c) \vee (b \wedge c) = (b \wedge c) = (a \vee b) \wedge c$. ■

This motivates the following definition, introduced in [29]:

Definition 6.1.2. Let L be a lattice, and F a filter. We say that F is *quasi-prime* if whenever $M \subseteq L$ is a finite subset where $M = \{m_i : i \leqslant n\}$, and $\bigvee M \in F$ is an admissible join, then for some i, $a_i \in F$.

Example 6.1.3. Consider the lattice in Figure 6.1:

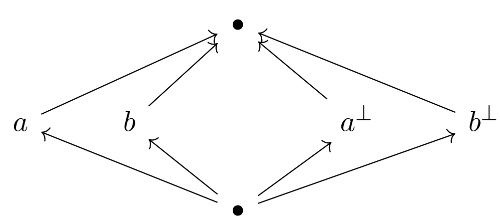


Figure 6.1: The lattice MO_2

It is easy to see that the space of all filters consists of all principal proper filters on this lattice. However, note that the filter $\{1\}$ is not quasi-prime: $\bigvee\{a, b, a^\perp, b^\perp\}$ is an admissible join, and is equal to 1, but the above filter contains no element from the set.[1]

[1]The former example also illustrates that, in generla, binary admissible joins do not suffice.

In a distributive lattice, all joins will be admissible, so quasi-prime filters will simply coincide with prime filters. Hence, as we will see below, our duality will indeed simply become Stone-Priestley duality in the case of distributive ortholattices, i.e., Boolean algebras.

It is of course important that quasi-prime filters exist in a way that makes them useful. For that we will need one more definition:

Definition 6.1.4. Let $\mathbf{O}$ be an ortholattice and $I \subseteq O$ a downwards closed set. We say that I is an *admissible ideal* if whenever $M \subseteq I$ is a subset with an admissible join, then $\bigvee M \in I$.

Proposition 6.1.5. Let $\mathbf{O}$ be an ortholattice, and $S \subseteq O$ a downwards closed subset; then the smallest admissible ideal containing S exists, and is:

$$\mathsf{AdId}(S) := \{a \in O : \exists M \subseteq S, \text{ such that } \bigvee M \text{ is admissible and } a \leqslant \bigvee M\}$$

Proof. It is not hard to see by the definitions that the intersection of admissible ideals is an admissible ideal. Hence let $\mathsf{AdId}(S)$ be the least admissible ideal containing S. Let $K = \{a : a \leqslant \bigvee M, M \subseteq S,\ M \text{ has an admissible join }\}$; first note that this is an admissible ideal. It is clearly downwards closed. If $M \subseteq K$ is a subset with an admissible join, let $M = \{m_i : i \leqslant n\}$. For each i, by hypothesis, $m_i \leqslant \bigvee N_i$, where the latter is an admissible join. Hence:

$$m_i = m_i \wedge \bigvee N_i = \bigvee_j m_i \wedge n_j$$

Where the last equation uses admissibility. Moreover, note that $\bigvee_j m_i \wedge n_j$ is an admissible join:

$$(\bigvee_j m_i \wedge n_j) \wedge c = (\bigvee N_j) \wedge m_i \wedge c = \bigvee_j m_i \wedge n_j \wedge c$$

Since S is downwards closed, and $n_j \in S$, then $m_i \wedge n_j \in S$ as well. Thus:

$$\bigvee M = \bigvee_{i=1}^{n} \bigvee_j m_i \wedge n_j$$

Which is thus an admissible join of elements from S.

It is easy to show that K will be contained in every admissible ideal. Hence, $K \subseteq \mathsf{AdId}(S)$, which is least. This concludes the proof. ∎

We are now ready to prove the appropriate version of the prime filter theorem for our purposes:

Theorem 6.1.6. *(Quasi Prime filter theorem) Let* $\mathbf{O}$ *be an ortholattice,* $F, I \subseteq O$*. Assume that* F *is a filter and* I *is an admissible ideal, such that* $F \cap I = \varnothing$*. Then there exists some* $H \subseteq O$*, a quasi-prime filter, such that* $F \subseteq H$ *and* $H \cap I = \varnothing$*.*

Proof. The proof is analogous to the proof of the prime filter theorem. Consider:

$$P := \{G \subseteq O : G \text{ is a filter }, G \cap I = \varnothing\}$$

Note that this is non-empty since F is contained in it. Moreover, it is not hard to see that the set is inductive. So by Zorn's Lemma, let H be a maximal element there.

We check that H is quasi-prime. Indeed, suppose that $\bigvee M \in H$ is an admissible join, where $M = \{m_i : i \leqslant n\}$, but no m_i is in H. So consider in O, for each i:

$$H_{m_i} := \uparrow\{m_i \wedge c : c \in H\}$$

Then since H was maximal, each of these filters must intersect with I. So for each i, there is some $c_i \in H$, such that that $m_i \wedge c_i \in I$. Let $d = \bigwedge_{i=1}^{n} c_i$, and note that $d \in H$, hence $\bigvee M \wedge d \in H$. Now note that since $\bigvee M$ is admissible then so is $\bigvee_{i=1}^{n} m_i \wedge d$. Indeed, indeed if e is arbitrary:

$$\begin{aligned}(\bigvee_{i=1}^{n} m_i \wedge d) \wedge c &= (\bigvee M) \wedge d \wedge c \\ &= (\bigvee_{i=1}^{n} m_i \wedge d \wedge c)\end{aligned}$$

So since I is an admissible ideal, and $m_i \wedge d \in I$, $(\bigvee_{i=1}^{n} m_i \wedge d) \in I$ as well. By admissibility, then, $\bigvee M \wedge d \in I$. But this is also in H, which is disjoint from I; contradiction. ■

We now make use of our quasi prime filters to provide the desired representation theorem. We note that the arguments proving the following are very similar to those given in [8], and hence, we give here the proof of the two cases where the quasi-prime filter representation deviates from orthospace duality. A detailed proof of the full duality can be found in the Appendix.

Definition 6.1.7. Let $(X, \leqslant, R, \tau)$ be an orthospace. We say that this is a *slim orthospace* if it satisfies the following condition:

- (Admissibility) For each finite subset $U_0, ..., U_n, C \in \mathsf{RClop}(X)$, if:

$$\text{if } (\bigcup_{i=1}^{n} U_i)^* \cap C \subseteq (\bigcup_{i=1}^{n} U_i \cap C)^* \text{ then } (\bigcup_{i=1}^{n} U_i)^* = \bigcup_{i=1}^{n} U_i$$

In words, a slim orthospace makes precise the following intuition: if a join of a lattice is distributive, then it should be represented using the "real join" - where this means, through the idea that lattices are carved out of a power set lattice, the union. Using these ideas, we have the following:

Theorem 6.1.8. *Let* $\mathbf{O}$ *be an ortholattice, and* $X_{\mathbf{O}}$ *its orthospace dual. Let* $Y_{\mathbf{O}} := \{x \in X_{\mathbf{O}} : x$ *is a quasi-prime filter*$\}$*. Then* $Y_{\mathbf{O}}$ *with the subspace topology, the restriction to* $\perp$*, and the restriction of* $\leqslant$*, is a slim orthospace such that* $\mathsf{RClop}(Y_{\mathbf{O}}) \cong O$*.*

Let $\varphi' : \mathbf{O} \to \mathsf{RClop}(Y_{\mathbf{O}})$ be the map taking U to $\{x \in Y_{\mathbf{O}} : U \in x\}$. Notice that since $Y_{\mathbf{O}}$ is a subset of X, this will form a subbasis of the subspace topology. The point of the proof of orthospace duality which requires some more elaboration lies in the proof of compactness.

Lemma 6.1.9. The space $Y_{\mathbf{O}}$ as defined above is compact.

Proof. We use Alexander's Subbase Lemma. So suppose that:

$$Y_O := \bigcup_{a \in C} \varphi(a) \cup \bigcup_{b \in D} Y_O - \varphi(b)$$

Suppose towards a contradiction that there is no finite subcover. Then, for each collection $a_0, ..., a_n \in C$ and $b_0, ..., b_m \in D$, we have:

$$Y_O \neq \bigcup_{i=1}^{n} \varphi(a_i) \cup \bigcup_{j=1}^{m} X - \varphi(b_j)$$

Hence, there exists some quasi-prime filter $F_{\overline{a},\overline{b}}$ in the first set which is not in the second. Spelling this out, all of the $b_i \in F_{\overline{a},\overline{b}}$ and none of a_i are there. Thus, consider:

$$\mathsf{Fil}(\{b : b \in D\}) \text{ and } \mathsf{AdId}(\{a : a \in C\})$$

We claim that these sets are disjoint. Indeed, suppose not. Then this means that:

$$b_0 \wedge ... \wedge b_n \leqslant \bigvee M$$

Where $M = \{m_i : i \leqslant k\} \subseteq \downarrow\{a : a \in C\}$, by Lemma 6.1.5, and the join is admissible. Thus, for each i, $m_i \leqslant a_i$ for some $a_i \in C$. Let $\overline{a} = (a_0, ..., a_k)$ be the elements in these conditions. Let $F_{\overline{a},\overline{b}}$ be the filter corresponding to this as well as the sequence $\overline{b} = (b_0, ..., b_n)$. By hypothesis, $b_0 \wedge ... \wedge b_n \in F_{\overline{a},\overline{b}}$. But then $\bigvee M \in F_{\overline{a},\overline{b}}$, so there is some m_i in that filter, by admissibility and the fact that this filter is quasi-prime. But then by upwards closure, there is some $a_i \in F_{\overline{a},\overline{b}}$, a contradiction.

Because these sets are disjoint, by the Quasi-Prime Filter theorem we have that there is some $x \in Y_O$ which extends $\{U : U \in C\}$), and is disjoint from the admissible ideal noted above. Hence, there is some x such that:

$$x \in \bigcup_{b \in D} \varphi(b) \cup \bigcup_{a \in C} Y_O - \varphi(a) = Y_O - \big(\bigcup_{a \in C} \varphi(a) \cup \bigcup_{b \in D} Y_O - \varphi(b)\big) = \varnothing$$

Which is a contradiction. Hence, by contradiction, we have that Y_O is compact. ■

We now move to checking that the function φ as defined above is an embedding preserving the orthocomplement. This follows essentially by the same arguments of Goldblatt in [36, Theorem 4]:

Lemma 6.1.10. The function $\varphi : \mathbf{O} \to \mathsf{RClop}(Y_{\mathbf{O}})$ is an embedding, and moreover, whenever $\langle a_i : i \leqslant n \rangle \subseteq O$ are such that $\bigvee a_i$ is admissible, then:

$$\varphi(\bigvee a_i) = \bigcup_{i=1}^{n} \varphi(a_i)$$

Proof. The fact that φ is injective follows from the quasi-prime filter theorem: if $a \nleqslant b$, then consider $\uparrow a$ and $\downarrow b$. The former is clearly a filter, and the latter is an ideal, hence clearly an admissible ideal, and the two are disjoint. So by the Quasi-prime filter, we can separate them, i.e, $\varphi(a) \nleqslant \varphi(b)$.

We now move to showing the orthocomplement is preserved, that is $\varphi(a)^{\perp} = \varphi(a^{\perp})$ (note we take the restriction of the operator $\perp$, so this ranges only over quasi-prime filters). One inclusion is clear: if $a^{\perp} \in x$, then if $a \in y$, clearly $x \perp y$. For the other inclusion, suppose that $a^{\perp} \notin x$. Then we show that:

$$\mathsf{Fil}(\{a\}) \text{ and } \mathsf{Id}(\{c : c^{\perp} \in x\} \text{ are disjoint}$$

Indeed, suppose not. Then $a \leqslant c_0 \vee ... \vee c_n$. But then by applying $\perp$ to both sides we get:

$$c_0^\perp \wedge ... \wedge c_n^\perp \leqslant a^\perp$$

Since all the former are in x, $a^\perp$ would also be in x, a contradiction. Hence by the Quasi-Prime filter theorem, there is some y containing a, such that whenever $a^\perp \in x$, $a \notin y$ - that is, xRy. This shows that $x \notin \varphi(a)^\perp$.

To show the last statement, note that since $a_i \leqslant \bigvee_i a_i$, whenever $a \in x$, then $\bigvee_i a_i$, so $\bigcup_{i=1}^n \subseteq \varphi(\bigvee_i a_i)$. On the other hand, if $\bigvee_i a_i \in x$, and the join is admissible, ten for some i, $x \in \varphi(a_i)$. This shows the result. ■

Moreover, we can prove surjectivity:

Lemma 6.1.11. The map φ is surjective, i.e, any clopen regular of $Y_\mathbf{O}$ is of the form $\varphi(a)$ for some $a \in \mathbf{O}$.

Proof. The proof is the usual one: if K is a clopen regular in $Y_\mathbf{O}$, and $K = Y_\mathbf{O}$ then $K = \varphi(1)$. Otherwise, let $x \notin K$. By regularity, let $y \in K^\perp$ such that xRy. Then for each $z \in K$ we have that there is some V_z such that $v_z \in z$ and $v_z^\perp \in y$. Then:

$$K \subseteq \bigcup_{z \in K} \varphi(v_z)$$

By compactness, $K \subseteq \varphi(v_{z_0}) \cup ... \cup \varphi(v_{z_n}) \subseteq \varphi(v_{z_0} \vee ... \vee v_{z_n})$. By contrast, $(v_{z_0} \vee ... \vee v_{z_n})^\perp \in y$. Since xRy, $(v_{z_0} \vee ... \vee v_{z_n}) \notin x$. Hence $x \in \neg\varphi(v_{z_0} \vee ... \vee v_{z_n}) \subseteq Y - K$. Let $Z_x = v_{z_0} \vee ... \vee v_{z_n}$. Then $Y - K = \bigcup_{x \notin K} \varphi(z_x)$. By compactness we can extract a finite subcover that will get us the desired representation of K. ■

Then we can finally show:

Lemma 6.1.12. The space $Y_\mathbf{O}$ is a slim orthospace such that $\mathsf{RClop}(Y_\mathbf{O}) \cong \mathbf{O}$.

Proof. The proof that this space is compact has already been given. The two separation conditions follow by definition: if $x \perp y$ then $a \in x$ and $a^\perp \in y$, so $x \in \varphi(a)$ and $\varphi(a^\perp) = \varphi(a)^\perp \ni y$. The upwards closure condition is automatically satisfied. Finally, the admissibility condition is automatic by the fact that, as just showed, the lattice $\mathbf{O}$ and $\mathsf{RClop}(\mathbf{O})$ are isomorphic via φ. ■

The morphisms of slim orthospaces will just be orthospace morphisms. However, to get a duality, following [29], the algebraic morphisms need to be tweaked:

Definition 6.1.13. Let L and L' be lattices. Suppose that $f : L \to L'$ is a lattice homomorphism. We say that f is join-admissible if whenever $\bigvee M$ is an admissible join, then $\bigvee_{m \in M} f(m)$ is an admissible join.

These morphisms are discussed in [29, Example 3.11], where the authors show that the condition of preservation of admissibility is necessary. With it, we can obtain a duality:

Theorem 6.1.14. *The categories* $\mathbf{Ort}^{ad}$ *of ortholattices with admissible homomorphisms and* $\mathbf{SOrtS}$ *of slim orthospaces are dually equivalent.*

Proof. See Appendix. ■

Moreover, the former results also have, as a corollary, that slim orthospaces are minimal amongst orthospaces representing a given ortholattice:

Proposition 6.1.15. Let $\mathbf{O}$ be an ortholattice, and X an orthospace representing $\mathbf{O}$. Then $Y_{\mathbf{O}}$ embeds into X through a continuous map that preserves and reflects the relation $\perp$.

Proof. See Appendix. ∎

All of these results make use of the crucial notion of a *distributive envelope*. This is what motivated the above representation, and what we turn to now. Since in this chapter we will have to often consider the "distributive" join, we denote this as $\cup$, and reserve $\vee$ for the join coming from the ortholattice, i.e, $\vee = (\cup)^{\perp\perp}$.

We begin by reviewing some ideas and facts proved by Gehrke and van Gool which we will make use of:

Definition 6.1.16. Let $\mathbf{L}$ be a bounded lattice, and $\mathbf{L}$ a bounded distributive lattice. Let $e : \mathbf{L} \to \mathbf{D}$ be an injective map preserving meets, admissible joins and the bounds. We say that $(\mathbf{D}, e)$ is a (join)-*distributive envelope* if for each $\mathbf{D'}$, a bounded distributive lattice, such that $f : \mathbf{L} \to \mathbf{D'}$ is an injective map preserving meets, admissible joins and the bounds, there is some $g : \mathbf{D} \to \mathbf{D'}$ where $g \circ e = f$.

In other words, a distributive envelope is the free distributive extension of $\mathbf{L}$. For an arbitrary lattice $\mathbf{L}$, denote by $D^\wedge(L)$ the unique distributive envelope of $\mathbf{L}$. Gehrke and van Gool [29] proved the existence (and uniqueness) of such a distributive envelope, and provided a topological duality for these. However, for arbitrary lattices, the representation requires making use of a topological polarity, connecting two Priestley spaces. The key observation simplifying this in our case is that for ortholattices, since the join can be defined in terms of the meet, we can afford this simpler representation.

The study by Gherke and van Gool reveals the distributive envelope is a functorial construction, enjoying nice properties:

Theorem 6.1.17. *The map* $^\wedge : \mathbf{Lat} \to \mathbf{DLat}$*, between the category of bounded lattices with join-admissible homomorphisms and distributive lattices, assigning to each distributive lattice its distributive envelope, is a functor, left adjoint to the inclusion from* $\mathbf{DLat}$ *to* $\mathbf{Lat}$*. Moreover, if* $f : \mathbf{L} \to \mathbf{D}$ *is a map from* $\mathbf{L}$ *to a bounded distributive lattice* $\mathbf{D}$ *which preserves all meets and admissible joins, the unique lift,* $\hat{f} : D^\wedge(\mathbf{L}) \to \mathbf{D}$ *is given by:*

$$\hat{f} : D^\wedge(\mathbf{L}) \to \mathbf{D}$$
$$\bigcup_{i=1}^{n} a_i \mapsto \bigcup_{i=1}^{n} f(a_i)$$

Moreover, they provided a very useful characterisation of the distributive envelope. For that, recall that we say that a map $e : \mathbf{L} \to \mathbf{L'}$ between bounded lattices is *join-dense* if every element in $\mathbf{L'}$ can be written as a join of elements $e(a)$ for $a \in \mathbf{L}$:

Proposition 6.1.18. Let $\mathbf{L}$ be a lattice and $(\mathbf{D}, e)$ a distributive lattice, and a map $e : \mathbf{L} \to \mathbf{D}$. Then $(\mathbf{D}, e)$ is the distributive envelope if and only if:

1. e is injective;

2. e preserves all meets and admissible joins;

3. e is join dense;

Proof. For a proof, see [29, Corollary 3.15]. ■

With this we can obtain the following fact, capitalising on our representation theorem:

Proposition 6.1.19. Let Y be a slim orthospace. Then $\mathsf{ClopUp}(Y)$ is isomorphic to $D^\wedge(\mathsf{RClop}(Y))$.

Proof. Consider the inclusion of $\mathsf{RClop}(Y)$ into $\mathsf{ClopUp}(Y)$. By the same arguments as in the previous proposition, this is seen to be injective, preserve all meets, and by the slim orthospace condition, to preserve admissible joins. It is moreover join dense by Lemma 5.2.14. Hence by the universal property of the distributive envelope, the inclusion factors through the distributive envelope. More explicitly, we get a universal isomorphism:

$$h : D^\wedge(\mathsf{RClop}(Y)) \to \mathsf{ClopUp}(Y)$$
$$\bigcup_{i=1}^{n} \varphi(U_i) \to \bigcup_{i=1}^{n} U_i$$

This was to show. ■

As such we get the following useful tool:

Lemma 6.1.20. Suppose that Y is a slim orthospace. If $U, V_0, ..., V_n$ are clopen regulars, and $U \subseteq V_0 \cup ... \cup V_n$. Then if x is a quasi-prime filter over $\mathsf{RClop}(Y)$, and $U \in x$, then there is some i such that $V_i \in x$.

Proof. Let x be an arbitrary quasi-prime filter over $\mathsf{RClop}(Y)$. Now, since $U \subseteq V_0 \cup ... \cup V_n$, this implies by the isomorphism in Proposition 6.1.19 that $\varphi(U) \subseteq \varphi(V_0) \cup ... \cup \varphi(V_n)$. If $U \in x$, then $x \in \varphi(U)$, so $x \in \varphi(V_i)$ for some i, hence, $V_i \in x$. ■

We now look a bit further into the properties of the functors at play here, and investigate their universal algebraic and categorical properties.

Definition 6.1.21. Let $\mathbf{D}$ be a bounded distributive lattice. We say that an operation $(-)^\perp : \mathbf{D} \to \mathbf{D}$ is an *orthonegation* if it satisfies for every $a, b \in D$:

- $a \leqslant (a^\perp)^\perp$ and $a^{\perp\perp\perp} = a^\perp$
- $(a^\perp \vee b^\perp)^\perp = (a^\perp)^\perp \wedge (b^\perp)^\perp$
- $(a^\perp \wedge b^\perp)^\perp = ((a^\perp)^\perp \vee (b^\perp)^\perp)^{\perp\perp}$

In this case we say that $(\mathbf{D}, ^\perp)$ is an orthonegated lattice.

The following is immediate in virtue of our representation:

Proposition 6.1.22. If $\mathbf{O}$ is an ortholattice, then $(D^\wedge(\mathbf{O}), ^\perp)$, where $^\perp$ is calculated in the dual Priestley space, is an orthonegated lattice.

Definition 6.1.23. Let $F : \mathbf{O}^\wedge \to \mathbf{PDLat}^\wedge$ be the map assigning to each ortholattice its distributive envelope with the associated orthonegation, and to each admissible morphism $f : \mathbf{O} \to \mathbf{O}'$ the unique lift:

$$\hat{f} : D^\wedge(O) \to D^\wedge(O')$$
$$\bigcup_{i=1}^{n} a_i \mapsto \bigcup_{i=1}^{n} f(a_i)$$

We call this the **Distributivisation** of $\mathbf{O}$.

To check that the former makes sense, we will need a little fact about dualising maps:

Lemma 6.1.24. Suppose that $f : O \to O'$ is an admissible ortholattice homomorphism. Then $f^{-1} : Y_{O'} \to Y_O$ is an orthospace morphism.

Proof. See Appendix. ∎

This allows us to show the following:

Lemma 6.1.25. The map F as defined above is a well-defined functor, which moreover preserves injective morphisms.

Proof. The following argument stems essentially from [29, Corollary 2.13], where we also check preservation of the orthonegation. The definition of F on objects is given. Now suppose that $f : \mathbf{O} \to \mathbf{O}'$ is an ortholattice homomorphism. We check that the lift is well-defined. Indeed, suppose that $\bigcup_{i=1}^{n} a_i = \bigcup_{j=1}^{k} b_j$. Then by Lemma 6.1.20 this refers to $\bigcup_{i=1}^{n} \varphi(a_i) = \bigcup_{j=1}^{k} \varphi(b_j)$. So now suppose that $x \in Y_O$ is a quasi-prime filter, and $x \in \varphi(f(b_j))$. Hence $f(b_j) \in x$, so by the previous lemma, $b_j \in f^{-1}[x]$, so $f^{-1}[x] \in \varphi(b_j)$. But we have that:

$$\varphi(b_j) \subseteq \bigcup_{i=1}^{n} \varphi(a_i)$$

So $f^{-1}[x] \in \varphi(a_i)$ for some i. So $x \in \varphi(f(a_i))$. A similar argument shows the other inclusion, and shows the map is well-defined.

Now if f is injective: if $\bigcup_{i=1}^{n} f(a_i) = \bigcup_{j=1}^{k} f(b_j)$, then suppose that $a_i \in x$; since f is injective, $x = f^{-1}[x']$, so $f(a_i) \in x'$; the rest of the argument then follows inverting the arguments above.

It is clear from the definition that this will be a distributive lattice homomorphism. To see that it also preserves pseudo-negation one can see that:

$$\begin{aligned}
\overline{f}(\bigcup_{i=1}^{n} a_i)^\perp &= \overline{f}(\bigcap_{i=1}^{n} a_i^\perp) \\
&= f(\bigwedge_{i=1}^{n} a_i^\perp) \\
&= \bigwedge_{i=1}^{n} f(a_i)^\perp \\
&= (\bigcup_{i=1}^{n} f(a_i))^\perp
\end{aligned}$$

∎

Thus we can capture the following universal property of the distributive envelope, analogously to the work of Gehrke and van Gool, and which we will need later:

Corollary 6.1.26. *If* $\mathbf{O}$ *is an ortholattice, and* $f : \mathbf{O} \to \mathbf{D}$ *is a map to an orthonegated lattice, preserving meets, admissible joins and the orthonegation, then there exists a unique map* $g : D^\wedge(\mathbf{O}) \to \mathbf{D}$ *such that* $f = g \circ i$*, where* $i : \mathbf{O} \to D^\wedge(\mathbf{O})$ *is the natural inclusion.*

Proof. Simply note that the universal map given above will, according to the previous lemma, lift to a map preserving the pseudo-negation. ■

For future use, we also note an extra fact about categorical constructions and admissible homomorphisms:

Proposition 6.1.27. Let $\mathbf{O}$ and $\mathbf{O}'$ be ortholattices, and $\mathbf{D}$ and $\mathbf{D}'$ distributive lattices with a pseudo-negation:

1. Let $f : \mathbf{O} \to \mathbf{O}'$ be a surjective homomorphism of ortholattices. Then f is admissible.

2. Let $f : \mathbf{O} \to \prod_{i \in I} \mathbf{O}_i$ be a subdirect embedding. Then f is admissible.

Proof. First we show (1): suppose that $\bigvee M$ are some elements in $\mathbf{O}$ which form an admissible join, and let $c \in O$ be arbitrary. Then by surjectivity, $c = f(d)$ for each $d \in D$. Hence:

$$\begin{aligned}
(\bigvee_{m \in M} f(m)) \wedge c &= (\bigvee_{m \in M} f(m)) \wedge f(d) \\
&= f((\bigvee M) \wedge d) \\
&= f(\bigvee_{m \in M} m \wedge d) \\
&= \bigvee_{m \in M} f(m) \wedge f(d)
\end{aligned}$$

Where the equalities follow from the admissibility of M and.

To see (4) we use a similar argument. Suppose that M is an admissible subset. By subdirectness, for each $i \in I$, $\pi_i \circ f$ is surjective. Hence, using the same argument as above coordinatewise will get us the result. ■

Having these results, one is left with some natural questions. One line of inquiry runs as follows. Given an orthonegated lattice $(\mathbf{D}, ^\perp)$, one can look at:

$$\mathbf{D}_\perp = \{a \in D : a = a^{\perp\perp}\}$$

we can show that this is an ortholattice. This is essentially by the same arguments as in Proposition 5.3.12, where the conditions of an orthonegation ensure that $\mathbf{D}_\perp$ satisfies the equations of an ortholattice. It it not hard then to conceive of this yielding a translation between orthologic and some logic of orthonegated lattices. So we can ask: could such a hypothetical translation in this setting have the above distributivisation functor as its left adjoint?

Consider the distributive lattice with a negation given in Figure 6.2:

Which we call the "2-Fork" algebra, L. Here we have that $\neg x = y$, and $\neg y = x$, whilst $\neg A = \neg 1 = 0$ and $\neg 1 = 0$. Then note that $x, y, 0, 1$ are all $\neg\neg$ fixed points, whilst A is not. Hence,

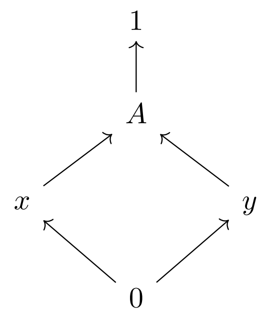


Figure 6.2: 2-Fork

if we take the ortholattice of fixed points, this turns out to in fact be a Boolean algebra. In such an algebra, all joins are admissible, trivially. However, the inclusion map cannot preserve all these joins: since $x \vee y$ in $\theta(L)$ will be 1, whilst in L it must be A. Hence there is no hope in general to return from the $\neg$-fixed points to the original algebra through a map which preserves all admissible joins.

A different question: we know that as a distributive lattice, $D^\wedge(\mathbf{O})$ can be Booleanised. Since we know that the elements of $D^\wedge(\mathbf{O})$ are finite unions of elements from $\mathbf{O}$, we also know the shape of the elements from $B(D^\wedge(\mathbf{O}))$ - they are finite unions of finite differences of elements from $\mathbf{O}$. This is justified by, up to isomorphism, such elements being the clopens from the slim orthospace. However, there is no general guarantee that given an *arbitrary clopen* U, $U^\perp$ is clopen as well. In other words, we do not know whether the orthonegation of $D^\wedge(\mathbf{O})$ can be extended to $B(D^\wedge(\mathbf{O}))$. This leads us to the following definition:

Definition 6.1.28. Let $\mathbf{O}$ be an ortholattice. We say that $\mathbf{O}$ is *compatible* if whenever $U \subseteq Y_{\mathbf{O}}$ is a clopen subset, then $U^\perp$ is clopen as well.

In essence, this implicit question is, as we will see, intimately related to the question of how we can extend the expressive power of a logic of ortholattices by some operations. In the next sections we take a closer look at this question.

6.2 Orthoimplicative Systems

In this section we tackle the question of providing an algebraic structure which captures the notion of a compatible ortholattice in an algebraic way. As we will see, the underlying description is quite complicated, and requires heavy use of admissibility of specific joins. We do this essentially by providing a structure, which we deem an *orthoimplicative system*, which is deeply related to Weak Heyting algebras. Indeed, readers familiar with [13] will notice our proofs are essentially more complicated versions of the arguments found in that paper.

Additionally, the logical complexity of the class is somewhat unusual, in that these are "*Inductive Rule*" classes, i.e., axiomatised by Π_2 first order formulas. The latter have recently received attention in [2], in the effort to axiomatise a calculus for compact Hausdorff spaces; our work here relies heavily on these ideas.

To motivate our developments, we recall that the quest to find an adequate implication connective has long marked research into quantum logic (see [44, Chapter 14] for an extensive discussion).

Most interest has focused on term-definable implications, though recent work has started to focus on signature expansions [27]. The key difficulties seem to be that the desiderata for implications are in many senses conflicting: one would want the implication to satisfy the residuation laws, as in the intuitionistic setting, but this is known to bring us back to Boolean algebras. Already in 1981, Dalla Chiara in [18] noticed that the Kripke semantic approach to ortholattices allowed for another connective, modeled as:

$$\Box(U \to V)$$

Where U and V are propositions, and $\Box$ is the induced operator from the non-orthogonality relation. This was noted to have many if not all of the properties which were desired in such an implication; however, this was, as far as we are aware, not pursued further.

As it turns out, our goal to capture compatible ortholattices turns out to be the same as providing an axiomatisation of this implication relation, and its close relatives. Given the intricate nature of the technical details, we focused on keeping in the main text only the crucial details, and as before, direct the reader to the Appendix, where full proofs can be found.

Throughout, as before, we let R denote the non-orthogonality relation on any orthospace under consideration.

Definition 6.2.1. Let $\mathbf{O} = (O, \wedge, \vee, (\multimap_n)_{n\in\omega}, ^\perp, 0, 1)$ be an algebra with a family of implications. We say that $\mathbf{O}$ is an *orthoimplicative system* if it satisfies:

1. $(O, \wedge, \vee, ^\perp, 0, 1)$ is an ortholattice.
2. For each n, $a \multimap_n (0, ..., 0) = a^\perp$
3. $a \multimap_n (b_0, ..., b_m) \leqslant a \multimap_{n+1} (b_0, ..., b_m, c)$
4. $a \multimap_k (a, b_0, ..., b_{k-1}) = 1$
5. $a \multimap_n (b_0, ..., b_n) \wedge a \multimap_m (c_0, ..., c_m) = a \multimap_{n\times m} (b_i \wedge c_j)_{(i,j)\in n\times m}$
6. $(a \vee b) \multimap_m (b_0, ..., b_m) \leqslant a \multimap_m (b_0, ..., b_m) \wedge b \multimap_m (b_0, ..., b_m)$
7. $a \multimap_k (b, c_0, ..., c_{k-1}) \wedge b \multimap_m (d_0, ..., d_m) \leqslant a \multimap_{k+m} (d_0, ..., d_m, c_0, ..., c_{k-1})$
8. $a \wedge (a \multimap_1 b) \leqslant b$
9. $a \multimap_{n+1} ((a \multimap (b_0, ..., b_n))^\perp, b_0, ..., b_n) = 1$
10. For each $i \leqslant n$, $a \multimap_n (b_0, ..., b_i, b_{i+1}, ..., b_n) = a \multimap_n (b_0, ..., b_{i+1}, b_i, ..., b_n)$.
11. For each n, $a \multimap_{n+1} (c, c, b_0, ..., b_{n-1}) = a \multimap_n (c, b_0, ..., b_{n-1})$

As well as the *admissibility axioms*:

- (A1) Whenever $\bigvee M$ is an admissible join, then $(\bigvee M) \multimap_m (b_0, ..., b_m) = \bigwedge_{m\in M} n \multimap_m (b_0, ..., b_m)$
- (A2) Whenever $C = \{c_0, ..., c_n\}$ and $D = \{d_0, ..., d_k\}$ are such that $c_i = \bigvee_{j=1}^{k} c_i \wedge d_j$ and this is an admissible join, then for all $e \in O$:

$$e \multimap_n (c_0, ..., c_n) \leqslant e \multimap_k (d_0, ..., d_k)$$

- (A3) For all $a, b_0, ..., b_n$ if $a \multimap_k (b_0, ..., b_n) = 1$ then $a = \bigvee_{i=1}^{k} a \wedge b_i$ is an admissible join.

Example 6.2.2. Let $\mathbf{B} = (B, \wedge, \vee, \neg, 0, 1)$ be a Boolean algebra. Then for each n, and each $a, b_0, ..., b_n \in B$ let:

$$a \multimap_n (b_0, ..., b_n) := \neg a \vee b_0 \vee ... \vee b_n$$

Then $\mathbf{B}$ is an orthoimplicative system. Note that in this setting, the admissibility axioms are trivial, since every join is admissible.

Example 6.2.3. More generally, let $\mathbf{O}$ be a compatible ortholattice. Then by slim orthospace duality, $Y_{\mathbf{O}}$ is such that whenever U is clopen, $U^{\perp}$ is clopen. Then define, for $a, b_0, ..., b_n \in \mathbf{O}$

$$\varphi(a \multimap_n (b_0, ..., b_n)) := \Box_R(\neg\varphi(a) \cup \varphi(b_0) \cup ... \cup \varphi(b_n))$$

where $\neg\varphi(a) = Y_O - \varphi(a)$ and $\Box_R Z = \{y : \forall w(yRw \text{ implies } w \in Z\}$. Then $(\mathbf{O}, (\multimap_n)_{n \in \omega})$ is an orthoimplicative system.

The latter example is the example which guided our definitions, and as we will see, it encompasses all possible orthoimplicative systems.

Some remarks are in order. The former is, as the reader might suspect, obtained from the axioms of a Weak Heyting algebras by essentially splitting the right coordinate of a Weak Heyting implication into infinitely many connectives. Moreover, the former axiomatisation is *not* equational, due to the admissibility Axioms. (A3) is essentially a collection of quasi-equations, whilst (A1) and (A2), through a more careful writing, can be seen to be collections of Π_2 formulas, indeed, so-called special Horn Formulas: for example, (A2) says, for every n:

$$\forall c_0, ..., c_n, d_0, ..., d_k, e((c_i = \bigvee c_i \wedge d_j \ \&\ \forall f(f \wedge (\bigvee c_i \wedge d_j) \leqslant \bigvee(f \wedge c_i \wedge d_j))) \\ \implies e \multimap_n (c_0, ..., c_n) \leqslant e \multimap_n (d_0, ..., d_k))$$

We will later take stock of this situation. For now, we first define the following relation between filters:

$$xTy \iff \forall n \in \omega, \forall a, b_0, ..., b_n, a \multimap_n (b_0, ..., b_n) \in x \text{ and } a \in y \implies \exists i \leqslant n, b_i \in y$$

One can then show the following:

Proposition 6.2.4. Let $\mathbf{O}$ be an orthoimplicative system. Let $Y_{\mathbf{O}}$ be the dual slim orthospace. Then:

$$\varphi(a \multimap_k (b_0, ..., b_k)) = \Box_T(\neg\varphi(a) \cup \varphi(b_0) \cup ... \cup \varphi(b_k))$$

Proof. See Appendix. ■

We also have that the relation T as defined above is simply the relation R:

Proposition 6.2.5. If $\mathbf{O}$ is an orthoimplicative system, and $Y_{\mathbf{O}}$ is the dual slim orthospace, then T is reflexive and $T = R$. Hence $\varphi(a \multimap_k (b_0, ..., b_k)) = (\varphi(a) - \varphi(b_0) - ... - \varphi(b_k))^{\perp}$

Proof. (1) First suppose that $a \multimap_n (b_0, ..., b_n) \in y$ and $a \in y$. By Axiom 9, we have that:

$$a \multimap_{n+1} ((a \multimap (b_0, ..., b_n))^{\perp}, b_0, ..., b_n) = 1$$

By (A3), then:

$$\bigvee_{i=1}^{n} (a \wedge b_i) \ \vee \ (a \wedge (a \multimap (b_0, ..., b_n))^{\perp})$$

Is an admissible join. Hence either one of $b_i \in y$, or $(a \multimap (b_0, ..., b_n))^{\perp} \in y$; but since $(a \multimap (b_0, ..., b_n)) \in y$, the latter would be a contradiction. So there is some i such that $b_i \in y$, as intended. Thus, we conclude that yTy.

(2) First suppose that xTy. Assume that $a \in x$. Then if $a^{\perp} \in y$, for any of the equivalent definitions of $a^{\perp}$, then since:

$$a^{\perp} = a \multimap 0$$

This would imply that $0 \in y$, a contradiction. So $a^{\perp} \notin y$. Hence xRy.

Conversely, assume that xRy, and $a \multimap_n (b_0, ..., b_n) \in x$, and $a \in y$. Now, because xRy, we have that $(a \multimap_n (b_0, ..., b_n))^{\perp} \notin y$. By Axiom 10, we have that:

$$a \multimap_{n+1} ((a \multimap_n (b_0, ..., b_n))^{\perp}, b_0, ..., b_n) \in y$$

So since by (1), yTy, we must have that for some i, $b_i \in y$. This was to show. ■

With these facts, we can show that the notion of orthoimplicative system axiomatises the compatible ortholattices. First, note that in light of the above, if an ortholattice admits an orthoimplicative system structure, it does so uniquely: if **O** was an ortholattice with two orthoimplicative structures, by duality they would yield two relations T_0 and T_1. In light of the above proposition, $T_0 = R = T_1$. Hence, by duality, the derived operators $\Box_{T_0}$ and $\Box_{T_1}$ must be the same, meaning that the actions of the arrows must agree. Moreover, orthoimplicative systems are intimately related to special classes of Weak Heyting algebras:

Definition 6.2.6. Let $\mathbf{H} = (H, \wedge, \vee, \multimap, 0, 1)$ be a Weak Heyting algebra. We say that **H** is a *WH-symmetric algebra*, if **H** satisfies the reflexivity axiom, as well as:

$$a \leqslant (a \multimap b)^{\perp} \cup b$$

This allows us to prove the following characterisation theorem, which ensures that these are the appropriate notions to work with:

Theorem 6.2.7. *(Characterisation of Compatible Ortholattices) Let* **O** *be an ortholattice. Then the following are equivalent:*

1. **O** *admits an orthoimplicative system structure.*
2. $D^{\wedge}(\mathbf{O})$ *admits a unique WH-symmetric algebra structure, with the dual relation being given by the non-orthogonality relation.*
3. **O** *is compatible.*

Proof. See the Appendix. ■

For orthoimplicative systems, we can moreover explicitly describe the implications, using the distributive envelope structure: if **O** is such a structure, then for each $a, b_0, ..., b_k \in O$:

$$a \multimap (b_0, ..., b_k) := \bigvee \{c \in O : a \leqslant c^{\perp} \cup b_0 \cup ... \cup b_k\}$$

Indeed, the symmetry axiom ensures that $a \multimap (b_0, ..., b_k) \leqslant \bigvee\{c \in O : a \leqslant c^{\perp} \cup b_0 \cup ... \cup b_k\}$. Conversely, if $a \leqslant c^{\perp} \cup b_0 \cup ... \cup b_k$, this means:

$$\varphi(a) \subseteq \varphi(c)^{\perp} \cup \varphi(b_0) \cup ... \cup \varphi(b_k)$$

Hence:

$$\varphi(a) - \varphi(b_0) - ... - \varphi(b_k) \subseteq \varphi(c)^{\perp}$$

And so:

$$\varphi(c) \subseteq (\varphi(a) - \varphi(b_0) - ... - \varphi(b_k))^{\perp} = \varphi(a \multimap_k (b_0, ..., b_k))$$

Which establishes the equation.

We saw in Proposition 6.1.26, that the distributive envelope enjoys a universal property. It is not hard to see that this extends to orthoimplicative systems and their corresponding WH-envelopes.

Corollary 6.2.8. *Let* **O** *be an orthoimplicative system and* **D** *a WH-symmetric algebra and* f *:* **O** $\to$ **D** *an admissible map such that:*

$$f(a \multimap_k (b_0, ..., b_k)) = f(a) \multimap \bigcup_{i=1}^{k} f(b_i)$$

Then there is a unique WH-homomorphism $\overline{f} : D^{\wedge}(\mathbf{O}) \to \mathbf{D}$ *extending* f *such that* $\overline{f} \circ i = f$ *where* $i : \mathbf{O} \to D^{\wedge}(\mathbf{O})$ *is the inclusion.*

Proof. We already know that the lift $\overline{f}$ is a homomorphism of distributive lattices. Now given $\bigcup_{i=1}^{k} a_i$ and $\bigcup_{i=1}^{k} b_i$ note that:

$$\begin{aligned}
\overline{f}(\bigcup_{i=1}^{k} a_i \multimap \bigcup_{i=1}^{k} b_i) &= \bigcap_{i=1}^{k} \overline{f}(a_i \multimap \bigcup_{i=1}^{k} b_i) \\
&= \bigcap_{i=1}^{k} \overline{f}(a_i \multimap_k (b_0, ..., b_k)) \\
&= \bigcap_{i=1}^{k} f(a_i \multimap_k (b_0, ..., b_k) \\
&= \bigcap_{i=1}^{k} f(a_i) \multimap \bigcup_{i=1}^{k} f(b_i) \\
&= \bigcup_{i=1}^{k} f(a_i) \multimap \bigcup_{i=1}^{k} f(b_i)
\end{aligned}$$

This was to show. ■

We now look briefly at general questions of duality for this setting.

Definition 6.2.9. Let Y be a slim orthospace. We call (Y,T), where $T \subseteq Y \times Y$ is a point-closed binary relation (i.e., for each $x \in Y$, $T[x]$ is closed), an *expanded orthospace*. We say that an expanded orthospace is *full* if $T = R$ and additionally, whenever U is clopen, then $U^{\perp}$ is closed.

The former definition is motivated by the following fact:

Lemma 6.2.10. Let $X = (X, \leqslant, R, \tau)$ be an orthospace and $U \subseteq X$ a closed set. Then $U^{\perp}$ is open.

Proof. Suppose that $x \in U^{\perp}$. Then for each $y \in U$, by the separation axioms of orthospaces, there is some $W_{x,y}$ such that $x \in W_{x,y}$ and $y \in W_{x,y}^{\perp}$, where $W_{x,y}$ is a clopen regular. Hence:

$$U \subseteq \bigcup_{y \in U} W_{x,y}^{\perp}$$

Since U is closed, by compactness, we can extract a finite subcover, i.e:

$$U \subseteq W_{x,y_0}^{\perp} \cup ... \cup W_{x,y_n}^{\perp}$$

Hence, $W_{x,y_0} \cap ... \cap W_{x,y_n} \subseteq U^{\perp}$, which is an open set, and $x \in W_{x,y_0} \cap ... \cap W_{x,y_n}$. This establishes that $U^{\perp}$ is open. ∎

As far as the converse proposition – that whenever U is open, then $U^{\perp}$ is closed – we see that the above proof carries out under the additional hypothesis that U is regular. The general case appears to require further analysis.

We recall that an orthospace morphism $f : X \to X'$ is called *strong* if it satisfies for all $x, y \in X$:

- R-forth: if xRy then $f(x)Rf(y)$
- R-back: $f(x)Ry$ implies that $\exists z, xRz$ and $f(z) = y$

For orthoimplicative systems, strong orthospace morphisms are indeed the right duals to homomorphisms:

Proposition 6.2.11. Let $f : \mathbf{O} \to \mathbf{O}'$ be a homomorphism between orthoimplicative systems. Then $f^{-1} : Y_{\mathbf{O}'} \to Y_{\mathbf{O}}$ is a strong orthospace morphism. Conversely, if $g : Y \to Y'$ is a strong orthospace morphism between full orthospaces, then $g^{-1} : \mathsf{RClop}(Y') \to \mathsf{RClop}(Y)$ is a homomorphism between orthoimplicative systems.

Proof. See Appendix. ∎

We conclude with some logical problems associated to this class of structures. The axiomatisation we have given of orthoimplicative systems uses Π_2-formulas which are also special Horn sentences. This is outside most classical formalisms dealing with algebraic logic and universal algebra, but has not gone undiscussed. Notably, work related to the symmetric calculus of subordination algebras [2] and some specific investigations of these structures [3] have marked some recent analysis of such settings. This setting is also a particularly well-behaved (in being Π_2) fragment of so-called *geometric theories*, heavily studied in categorical logic [47] and more recently in proof theory [54].

We begin by recalling some classic concepts from model theory:

Definition 6.2.12. Let φ be a first order formula. We say that φ is a *Horn formula* if φ is in prenex normal form and:

$$\varphi = Q_0 x_0 ... Q_n x_n \big(\mathop{\&}_{i=1}^{k} \psi_i \big)$$

Where each Q_i is a quantifier, and ψ_i is a formula of the form:

$$\chi_0 \vee ... \vee \chi_n$$

in which each χ_i is atomic or negated atomic, and at most one χ_i is atomic.

Definition 6.2.13. Let φ be a first order formula. We say that φ is a *Special Horn formula* if it is of the form:

$$\mathop{\&}_{i=1}^{n} \forall \overline{x} (\varphi_i \to \psi_i)$$

where each φ_i is positive, and each ψ_i is atomic.

Proposition 6.2.14. Let $\mathbf{K}$ be an elementary class (of algebras). Then:

- K is axiomatised by Horn formulas if and only if it is closed under reduced products.
- K is axiomatised by special Horn sentences if and only if it is closed under subdirect products.

Proof. See for instance [15, pp.418-419]. ■

Since we are mostly interested in equational classes of orthoimplicative systems, we would want to work with something approximating varieties. However, naturally, the fact that we carry three inductive rules with us implies closure under homomorphic images or subalgebras cannot be expected in general. The notion we are looking for is that of a *relative variety*:

Definition 6.2.15. Let $\mathbf{K}$ be an elementary class of algebras, and $\mathbf{S} \subseteq \mathbf{K}$ another class. We say that $\mathbf{S}$ is a *relative variety* with respect to $\mathbf{K}$, if $\mathbf{S}$ is closed under subdirect products and homomorphic images which belong to $\mathbf{K}$.

Given a theory T in first order logic, We say that $\mathbf{S}$ is a *relative equational class* with respect to T, if there exists a collection of equations Φ in the language, such that for each $\mathbf{A}$ a model of T, $\mathbf{A} \in \mathbf{S}$ if and only if $\mathbf{A} \models \Phi$.

Note we work with subdirect products, rather the more usual subalgebras and products combination, in light of Hall's theorem (see preliminaries), and in light of the pathologies associated to subalgebra embeddings, such as the fact that they may not be admissible. We make the connection explicit in the case we are interested in. We recall a small universal algebraic fact:

Lemma 6.2.16. Let $\mathbf{K}$ be any class. Then for some cardinal κ, if $|X| \geqslant \kappa$, then $\mathbf{F}_{\mathbf{K}}(X) \in \mathbb{IP}_S(\mathbf{K})$.

Proof. See for instance [12, Theorem 11.11]. ■

Theorem 6.2.17. *The collection $\Lambda_A(\mathbf{OIS})$ of relative varieties of orthoimplicative systems forms a complete lattice. This is dually isomorphic to $\Lambda(\mathsf{OIL})$, the lattice of relative equational classes, relative to the axiomatisation given above.*

Proof. First we show that a class is a relative variety if and only if it is a relative equational class. The if part is clear: if $\mathbf{K}$ is a relative equational class, then since subdirect products and homomorphisms preserve equations, we have that it is a relative variety. For the converse, we use a standard argument, together with the lemma above. If $\mathbf{K}$ is a relative variety, then consider:

$$\mathsf{Eq}(K)$$

The set of equations satisfied by all members of $\mathbf{K}$. Then clearly if $\mathbf{A} \in \mathbf{K}$, $\mathbf{A} \models \mathsf{Eq}(\mathbf{K})$. Now if $\mathbf{B}$ is an orthoimplicative system, assume that $\mathbf{B} \models \mathsf{Eq}(\mathbf{K})$. Let κ be a sufficiently large cardinal to satisfy the former lemma, and X a set such that $|X| = \max(|B|, \kappa)$. Hence, let

$$f : X \to B$$

be an arbitrary surjective map. By the universal mapping property, f has a lift to $\overline{f} : \mathbf{Tm}_{\mathcal{L}_{OIS}}(X) \to \mathbf{B}$. Since $\mathbf{B} \models \mathsf{Eq}(\mathbf{K})$, then this factors through the algebra $\mathbf{F}_{\mathbf{K}}(X)$, i.e, there is a surjective homomorphism $g : \mathbf{F}_{\mathbf{K}}(X) \to \mathbf{B}$. By the above lemma, we have that $\mathbf{F}_{\mathbf{K}}(X) \in \mathbb{IP}_S(\mathbf{K})$, and by assumption the class is closed under subdirect products. Hence we have that $\mathbf{B}$ is a homomorphic image of that algebra, ensuring that $\mathbf{B} \in \mathbf{K}$ by closure under homomorphic images. ■

We have thus laid all of the groundwork for our return, in the next section, to the relationship between Ortholattices and KTB. We close this section with some remarks about compatibility of ortholattices and the naturality of the structures so discussed.

What we have proved in this section is that we can axiomatise compatibility in a stronger signature, but have not discussed how such structures can be found in ortholattices. It is obvious by the notion of compatibility that finite ortholattices will be compatible, and it can be shown that compatible ortholattices are closed under products (See Appendix), though we have not managed to prove or refute closure under homomorphic images or subalgebras.

However, it might be that many natural structures will admit an orthoimplicative system structure. As an encouraging example, the most paradigmatic orthomodular lattice – the lattice of closed subspaces of a Hilbert space – can be seen to admit an implication satisfying the equational axioms above, using the explicit definition we outlined. We leave the details of whether this means such a lattice forms a genuine orthoimplicative system, and a proof that such a space indeed forms a compatible ortholattice, for further work.

6.3 Orthoimplicative Logic and Sober KTB

In this brief section, we outline how the structures we met in the previous section can be described logically. We show that such systems are conservative over many classes of orthologics – such as those with the FMP – which ensures that our results still extend those of Miyazaki. We then explain how the developments of the previous chapter can be paralleled in the case of KTB, and construct a calculus for KTB, where the models are exactly the soberly generated **KTB**-algebras. We discuss logical admissibility of Π_2-rules, and provide natural examples of KTB-extensions which are conservative (such as the whole logic **KTB**)) and others which are not (such as tabular logics generated by non-sober KTB frames).

Definition 6.3.1. Let $\mathcal{L}_{OIS}$ be the language consisting of $(\wedge, (\multimap_n)_{n\in\omega}, 0, 1)$. We say that O, a set of pairs (φ, ψ) of formulas in $\mathbf{Tm}_{\mathcal{L}_{OIS}}(\mathcal{VAR})$ is an orthoimplicative logic if:

1. O is an orthologic;

2. For each axiom from 1-11 of an orthoimplicative system, of the form $\lambda \approx \gamma$, O contains the pairs (λ, γ) and (γ, λ);

3. O is closed under uniform substitution.

4. O is closed under the admissibility rules:

   (a) For each finite subset Φ, if for each formula ψ, $(\bigvee \Phi) \wedge \psi \vdash \bigvee_{\varphi \in \Phi} \varphi \wedge \psi$ (we say that Φ is admissible) then $\bigvee \Phi \multimap_m (b_0, ..., b_m) \dashv\vdash \bigwedge_{\varphi \in \Phi} \varphi \multimap_m (b_0, ...,_m)$

   (b) For each pair of finite subsets Φ and Ψ, if $\varphi \in \Phi$ is such that $\varphi \dashv\vdash \bigvee_{\psi \in \Psi} \varphi \wedge \psi$ and Φ is admissible, then for each χ $\chi \multimap_n (\varphi_0, ..., \varphi_n) \vdash \chi \multimap_k (\psi_0, ..., \psi_k)$.

   (c) For all $\varphi, \psi_0, ..., \psi_n$ if $\top \dashv\vdash \varphi \multimap_n (\psi_0, ..., \psi_n)$ then $\varphi \dashv\vdash \bigvee_{\psi \in \Psi} \psi \wedge \varphi$ and $\{\psi \wedge \varphi : \psi \in \Psi\}$ is admissible.

Where we write $\varphi \vdash_O \psi$ to mean, as usual, $(\varphi, \psi) \in O$.

It is trivial to see that arbitrary intersections of OIL (Orthoimplicative Logic) will again be OIL. More interestingly, the Lindenbaum-Tarski techniques can still be applied.

Lemma 6.3.2. Let $\mathbf{Tm}_{\mathcal{L}_{OIS}}(\mathcal{VAR})$ be the term algebra on the signature of orthoimplicative systems, and O an orthoimplicative logic. Then $\mathbf{F}(\mathcal{VAR}) := \mathbf{Tm}_{\mathcal{L}_{OIS}}(\mathcal{VAR})/\equiv_O$, the algebra quotiented by the equivalence relation generated by interderivability in the logic, is an orthoimplicative system.

Proof. First notice that with this definition, $\equiv_O$ is still a congruence of the algebra. We know this will be an ortholattice, and by usual arguments, it will be clear that it will satisfy axioms 1-10. The fact that it satisfies the admissibility axioms follows from the admissibility rules. We prove (A1), and the arguments for the others are wholly similar.

Suppose that $\bigvee[\Phi]$ is an admissible join. Hence, for each ψ a formula, we have that:

$$\bigvee[\Phi] \wedge [\psi] \leqslant \bigvee_{\varphi \in \Phi}[\varphi] \wedge [\psi]$$

Hence, since the relation is a congruence:

$$[(\bigvee \Phi) \wedge \psi] \leqslant [\bigvee_{\varphi \in \Phi} \varphi \wedge \psi]$$

Hence, this means, by the lattice structure:

$$\bigvee \Phi \wedge \psi \vdash \bigvee_{\varphi \in \Phi} \varphi \wedge \psi$$

Since this holds for all ψ, by the A1-rule, then we get that:

$$[\bigvee \Phi \multimap_m (b_0, ..., b_m)] = [\bigwedge_{\varphi \in \Phi} \varphi \multimap_m (b_0, ..., b_m)]$$

■

To provide an algebraization result, we make use of our work from last section:

Lemma 6.3.3. Let O be an orthoimplicative logic. Then $\mathsf{Var}(O)$, the class of algebras validating this, is a relative variety of orthoimplicative systems.

Proof. Immediate, since such a logic will extend the basic orthoimplicative logic by some sequents of the form (φ, ψ), and these are simply equations of the form $\varphi \wedge \psi = \varphi$. ■

Corollary 6.3.4. *Every orthoimplicative logic is sound and complete with respect to the relative variety of orthoimplicative systems validating the logic.*

We now reserve the symbol OIL for the minimal orthoimplicative logic. We also note the following fact connecting orthologics and orthoimplicative logics:

Definition 6.3.5. Let O be an orthoimplicative logic. We define $\overline{O}$, the orthologic reduct, to be:

$$\overline{O} := \{(\varphi, \chi) : \varphi, \chi \in \mathbf{Tm}_{\mathcal{L}_O}(\mathcal{VAR}), (\varphi, \chi) \in O\}.$$

It is clear by definition that every orthoimplicative logic is conservative over its orthologic reduct. Conversely, given an axiomatic extension $\mathsf{O} \oplus T$, where T is a collection of ortholattice formulas, we can consider $\mathsf{OIL} \oplus T$. Then we get the following:

Corollary 6.3.6. *For each orthologic $O = \mathsf{O} \oplus T$, $\mathsf{Var}(O)$ is a variety generated by compatible ortholattices if and only if $\mathsf{OIL} \oplus T$ is conservative over O.*

Proof. The left to right direction of this is clear. Conversely, suppose that $\mathsf{OIL} \oplus T$ is conservative over $\mathsf{O} \oplus T$. Then for each $(\varphi, \psi) \notin \mathsf{O} \oplus T$, let A_φ be an orthoimplicative system satisfying $\mathsf{OIL} \oplus T$ but not (φ, ψ). Consider $S = \{A_\varphi : (\varphi, \psi) \notin \mathsf{OIL} \oplus T\}$ be the collection of ortholattice reducts of these structures. Then look at the orthologic of S. If $(\varphi, \psi) \in O$, then $(\varphi, \psi) \in \mathsf{OIL} \oplus T$, hence it is valid on all A_φ; conversely, if $(\varphi, \psi) \notin O$, by construction this fails in some A_φ. Thus, $\mathsf{Log}(S) = O$, so the variety of O is generated by S. ■

We now move to a discussion of *sober* KTB algebras and their logic. Recall that in Chapter 5 that we showed that the Goldblatt translation could not be full by showing that there were two logics which both mapped to the same logic - namely, two logics generated by different representations of Benzene. There, we claimed that in Figure 5.4, the right hand side was the maximally sober frame; it is now not hard to see that the left hand side is the minimally sober one, i.e, the slim orthospace dual. If we want to pick only the minimally sober ones, it turns out it is enough to require closure under a collection of Π_2-rules:

Definition 6.3.7. Let $(B, \Box)$ be a KTB algebra. We say that $\mathbf{B}$ is *sober* if it satisfies the following:

- (KTB-sob) If $M = \{m_0, ..., m_k\}$ is a subset of B such that for all $b \in B$:

$$\Box\Diamond(\bigvee_{m \in M} \Box\Diamond m) \wedge \Box\Diamond b \leqslant \Box\Diamond(\bigvee \Box\Diamond m \wedge \Box\Diamond c)$$

  Then $\Box\Diamond \bigvee_{m \in M} \Box\Diamond m = \bigvee_{m \in M} \Box\Diamond m$.

Lemma 6.3.8. Let $\mathbf{B}$ be a sober KTB algebra. Then:

1. $O_{\mathbf{B}}$ embeds into $\mathbf{B}$ through a map which preserves meets and admissible joins.

2. $X_{\mathbf{B}}^*$ is a slim orthospace.

Proof. (1) is clear: the fact that the inclusion preserves meets is obvious by definition of $O_{\mathbf{B}}$, whilst preservation of admissible joins follows from the rule KTB-sob.

To see (2), let $[x] \in X_B^*$. We will show that this is a quasi-prime filter with respect to $O_{\mathbf{B}}$. So let $\bigvee M$ be admissible in $O_{\mathbf{B}}$, and assume that $\bigvee M \in [x]$; by (1), then, $i(\bigvee M) \in B$, the mapping through the inclusion, sends this to $\bigcup_{m \in M} m$ understood as an element of $\mathbf{B}$. Since $\bigvee M \in [x]$, then by definition $\Box\Diamond(\bigcup M) \in x$; but by assumption, $\bigcup_{m \in M} m = \Box\Diamond(\bigcup M)$, hence, there is some $m \in M$ such that $m \in [x]$. This was to show. ∎

The former thus repairs one of the issues we had found with the previous situation in **KTB**: we could start with a **KTB** algebra, turn it into a **KTB** space, translate it into an orthospace, and then obtain the induced ortholattice; but if we took the dual of that ortholattice, the orthospace obtained need not be the same. Since we know that slim orthospaces are unique, and minimal, this cannot happen for sober **KTB** algebras.

Definition 6.3.9. Let $\mathbf{K}$ be a class of KTB algebras. We say that $\mathbf{K}$ is *sober* if all $\mathbf{B} \in \mathbf{K}$ are sober.

We note that again, given the underlying formula is a special Horn sentence, the class of sober KTB algebras will be closed under subdirect products. Moreover similar arguments to those sketched in the previous section yield:

Proposition 6.3.10. There is a dual isomorphism between the lattice of relative varieties, relative to sober KTB algebras, and relative equational classes relative to the theory of sober KTB algebras.

Moreover, adding the admissibility rule to the calculus of KTB yields a logic, which we denote KTB^s. Using similar arguments to those of this section, we have that the relative varieties are dual to the logics extending KTB^s. Hence we have a perfect parallelism between orthoimplicative logic and sober KTB logic, which will be the focus of our next, and last, section.

Before that, we turn to a natural question which is better discussed in the setting of KTB sober algebras, given the availability of tools and literature. The rule we just added states, explicitly, for a given logic L:

- If M is a subset which is admissible, in the sense that for each ψ we can prove that:

$$\vdash_L (\bigvee_{\chi \in M} \Box\Diamond\chi) \wedge \Box\Diamond\psi \to \Box\Diamond(\bigvee_{\chi \in M} \Box\Diamond\chi \wedge \Box\Diamond\psi)$$

  then:

$$\vdash_L \Box\Diamond(\bigvee_{\chi \in M} \Box\Diamond\chi) \leftrightarrow \bigvee_{\chi \in M} \Box\Diamond\chi.$$

Now we note the following which uses similar ideas to Corollary 6.3.6:

Lemma 6.3.11. Let $L \in \mathbf{NExt}(\mathsf{KTB})$ be arbitrary such that $L = \mathsf{KTB} \oplus T$. The following are equivalent:

- $\mathsf{Var}(L)$ is generated by sober KTB algebras;

- $L^s = \mathsf{KTB}^s \oplus T$ is conservative over L
- The KTB-sob rule is admissible in L.

Proof. The equivalence of (1) and (2) is essentially as in the above corollary. Now we show that (1) implies (3): assume that we have $\vdash_L (\bigvee_{\chi \in M} \Box\Diamond\chi) \wedge \Box\Diamond\psi \to \Box\Diamond(\bigvee_{\chi \in M} \Box\Diamond\chi \wedge \Box\Diamond\psi)$. Let $A \in Var(L)$ be a sober generator. Then for each valuation v:

$$A \models (\bigvee_{\chi \in M} \Box\Diamond v(\chi)) \wedge \Box\Diamond v(\psi) \to \Box\Diamond(\bigvee_{\chi \in M} \Box\Diamond v(\chi) \wedge \Box\Diamond v(\psi)) \approx 1$$

Hence by definition in A:

$$(\bigvee_{\chi \in M} \Box\Diamond v(\chi)) \wedge \Box\Diamond v(\psi) \leqslant \Box\Diamond(\bigvee_{\chi \in M} \Box\Diamond v(\chi) \wedge \Box\Diamond v(\psi))$$

Since this holds for all valuations, given any valuation v, we can let p' be a proposition letter not ocurring in any $\chi \in M$, and let $v'(p')$ be any element in A, with the same result. Since A is sober, then:

$$\bigvee_{\chi \in M} \Box\Diamond v(\chi) = \Box\Diamond(\bigvee_{\chi \in M} \Box\Diamond v(\chi))$$

Hence, since the sober elements generate $Var(L)$:

$$\vdash_L \bigvee_{\chi \in M} \Box\Diamond\chi \leftrightarrow \Box\Diamond(\bigvee_{\chi \in M} \Box\Diamond\chi)$$

This was to show.

Conversely, assume that the KTB-sob rule is admissible in L. Then $\mathsf{KTB}^s \oplus T = \mathsf{KTB} \oplus T$, since we know that no application of the rule can yield new theorems. Hence (3) implies (2). ■

These rules can be put into a very specific shape, as discussed in [2, 3]:

Definition 6.3.12. A Π_2-rule is a rule of the form:

$$(\rho)\,\frac{F(\overline{\varphi}/\overline{x}, \overline{p}) \to \chi}{G(\overline{\varphi}/\overline{x}) \to \chi}$$

Where $F(\overline{x}, \overline{p})$, $G(\overline{x})$ and χ are formulas, possibly with open variables, in the language of modal algebras.

To see that our KTB rules are of this shape, consider the following formulas

$$F(x_0, ..., x_n, y) \coloneqq \Box\Diamond(\bigvee_{i=1}^{n} \Box\Diamond x_i) \wedge \Box\Diamond y \to \Box\Diamond(\bigvee_{i=1}^{n} \Box\Diamond x_i \wedge \Box\Diamond y) \text{ and}$$

$$G(x_0, ..., x_n) \coloneqq \Box\Diamond(\bigvee_{i=1}^{n} \Box\Diamond x_i) \leftrightarrow \bigvee_{i=1}^{n} \Box\Diamond x_i$$

In [3], a series of techniques were used to recognise admissibility of Π_2-rules. We recall one of the ones used. For it, recall that a modal logic L is said to have the interpolation property if, whenever $\vdash_L \varphi \to \psi$, then you can find a formula $\chi \in Lang(\varphi) \cap Lang(\psi)$, (where $Lang(\chi)$ denotes the set of all terms involving variables from χ) the shared language, such that $\vdash_L \varphi \to \chi$ and $\vdash_L \chi \to \psi$.

Theorem 6.3.13. *Let L be a modal logic system with the interpolation property. Then a Π_2-rule ρ as above is admissible if and only: whenever $\vdash_L F(\varphi/x, p) \wedge G(\varphi/x) \to \chi$ then $\vdash_L G(\varphi/x) \to \chi$.*

Proof. See [3, Theorem 3.2]. ∎

We apply this theorem to our case:

Corollary 6.3.14. *Let $L \in \mathbf{NExt}(\mathsf{KTB})$ be a system with the interpolation property. Then the KTB-sob rule is admissible in L.*

Proof. Assume that $\vdash_L F(\overline{\varphi}, p) \wedge G(\overline{\varphi}) \to \chi$, for some χ and $\overline{\varphi}$, where these formulas are the ones defined above. Let $\mathbf{A}$ be an algebra in $\mathsf{Var}(L)$, and v any valuation. Now, by assumption, then:

$$\mathbf{A} \vDash F(\overline{\varphi}, p) \wedge G(\overline{\varphi}) \to \chi$$

Now, notice that in $\mathbf{A}$, for any valuation, $G(\overline{\varphi}) \leqslant F(\overline{\varphi}, p)$: indeed, if the join is distributive, then the statement of F follows, regardless of the value taken by the extra parameter. Hence:

$$A, v \vDash G(\overline{\varphi}) \to \chi$$

This shows conservativity. Hence by Theorem 6.3.13, we have that L admits the rule. ∎

Corollary 6.3.15. *The variety of* KTB *algebras is generated by sober* KTB *algebras.*

With these preliminaries, we are ready to tackle the question of the relationship between ortho-logic and KTB logic.

6.4 The Goldblatt Translation Revisited

Definition 6.4.1. Let $\mathcal{L}_{OIS}$ be the language of orthoimplicative logic, and $\mathcal{L}_{KTB}$ the language of KTB logic. Define the second Goldblatt translation as follows:

1. $G_2(0) = 0$
2. $G_2(p) = \Box\Diamond p$ for any proposition p;
3. $G_2(\varphi \wedge \psi) = G_2(\varphi) \wedge G_2(\psi)$
4. $G_2(\varphi \multimap_n (\psi_0, ..., \psi_n)) = \Box(\neg G_2(\varphi) \vee G_2(\psi_0) \vee ... \vee G_2(\psi_n))$

From this translation we obtain the semantically equivalent clauses $G_2(\varphi^{\perp}) = \Box\neg G_2(\varphi)$ and $G_2(\varphi \vee \psi) = \Box\Diamond(G_2(\varphi) \vee G_2(\psi))$. Let $\Theta = \{x \approx \Box\Diamond x\}$. Then we have:

Proposition 6.4.2. The tuple $(G_2, \Theta, \Box\Diamond)$ satisfies the conditions of being a contextual translation between the relative equational consequence of orthoimplicative systems and sober KTB algebras, and has a selector term.

Proof. It is not hard to see that the Θ equations are compatible with the equational consequence, and the proposed term is a selector. Moreover, the translation is contextual: if $G_2[\Phi] \not\models G_2(\lambda) \approx G_2(\gamma)$, for some collection of equations $\Phi \cup \{\lambda \approx \gamma\}$ in the language of orthoimplicative systems, let B be a sober KTB algebra witnessing this. Then by arguments we have seen before, $O_{\mathbf{B}} = \{a : \Box\Diamond a\}$, equipped with the induced operations, is an ortholattice with an implication which refutes the equation in some valuation. Moreover, we have that $O_{\mathbf{B}}$ is orthoimplicative, because $\mathbf{B}$ is sober and in **KTB**: if $\bigvee M$ is an admissible join in $O_{\mathbf{B}}$, by definition, then $\Box\Diamond(\bigvee M)$ is admissible in B; hence $\bigvee M = \bigcup M$, whereby the axioms (A1) and (A2) follow as easy properties of Boolean algebras with modal operators. Axiom (A3) as well as Axioms 8 and 9 follow from the fact that the modality is symmetric and reflexive. ■

Our work in the previous chapters now allows us to explicitly describe the adjunction which is related to this translation.

Definition 6.4.3. Let **OIS** be the category of orthoimplicative systems with implicative maps, and **KTBsob** the category of sober KTB algebras with admissible homomorphisms. Let **KTBSsob** be the category of sober KTB spaces with p-morphisms, and **OrtS** be the category of slim orthospaces with strong orthospace morphisms. We define a functor:

$$\theta : \mathbf{KTBs} \to \mathbf{OIS}$$

Which takes $\mathbf{B}$ to $O_{\mathbf{B}}$, and acts as the restriction on maps. Similarly, we have its dual:

$$\theta^* : \mathbf{KTB} \to \mathbf{OrtS}$$

Which takes a sober KTB space X to X^*, and acts as the induced map on maps.

We also define a functor:

$$BD(-) : \mathbf{OIS} \to \mathbf{KTBsob}$$

Which on objects takes an orthoimplicative system O to the Boolean envelope of $D^\wedge(O)$, its distributive envelope, and on maps, takes the unique lift of all maps involved. Dually, we have:

$$BD^* : \mathbf{OrtS} \to \mathbf{KTBSsob}$$

Which acts by sending a slim orthospace to its Boolean space reduct, and acts as the identity on maps.

We have already studied the maps θ and θ^* and have that they are duals by Proposition 5.3.17 and Lemma 6.3.8. We will thus concentrate for now on showing that the other two maps are well-defined functors and that they are duals of each other.

Lemma 6.4.4. The maps BD and BD^* defined above are well-defined functors. Moreover, we have that:

- If $\mathbf{O}$ is an orthoimplicative system, then $X_{BD(O)} \cong Clop(BD^*(Y_{\mathbf{O}}))$
- If Y is a slim orthospace, then $Clop(BD^*(Y)) \cong BD(\mathsf{RClop}(Y))$

Proof. First we show that BD is a functor. Given $\mathbf{O}$, note that $BD^\wedge(\mathbf{O})$ is a well-defined Boolean algebra. By what we proved in Section 5.1, and facts mentioned in the preliminaries, $BD^\wedge(\mathbf{O})$ will be isomorphic to the set of clopens of $Y_{\mathbf{O}}$. Hence its elements are of the form:

$$a = \bigwedge_{i=1}^{k} c_i \vee -d_i^0 \vee ... \vee -d_i^j$$

where $c_i, d_i^k \in O$. Hence we define:

$$\Box a = \bigwedge_{i=1}^{k} c_i \multimap_j (d_i^0, ..., d_i^j)$$

We have that $\Box$ will be a symmetric and reflexive modality (since the relation R is reflexive and symmetric), hence, it is a KTB algebra. Dually, the space $Y_{\mathbf{O}}$ seen as a Boolean space with the non-orthogonality relation R, is a modal space, since R is point closed, and when U is clopen in Y_O, $\Box U$ is clopen (and regular).

It is clear to see that the proposed definitions on maps make BD and BD^* into functors. Now suppose that $\mathbf{O}$ is an arbitrary orthoimplicative system. Then by what we just mentioned, $BD(\mathbf{O})$ is isomorphic to the clopens of $Y_{\mathbf{O}}$. The second statement is similar. ■

We now consider two candidate maps:

Definition 6.4.5. Let $\eta_{\mathbf{O}} : \mathbf{O} \to \theta(BD(\mathbf{O}))$ be the map sending $a \in O$ to $a \in \theta(BD(O))$ for $\mathbf{O}$ an orthoimplicative system. Let $\varepsilon_{\mathbf{B}} : BD(\theta(B)) \to B$ be the unique map induced by the inclusion of $\theta(B)$ into B.

Lemma 6.4.6. η and ε are natural transformations. Moreover, η is a natural isomorphism, and ε is pointwise injective.

Proof. The proof of naturality is straightforward once we show the remaining facts (for η this is trivial, since it is an isomorphism, and for ε this follows from the universal property of Booleanisation and Distributivisation). To see that η is a well-defined isomorphism, note that if $a \in O$, then $a \in BD^\wedge(O)$, and moreover, $\Box\Diamond a = a^{\perp\perp} = a$. Moreover, if $a \in BD^\wedge(O)$ and $a = \Box\Diamond a$, then by duality, $\varphi(a)$ is a clopen regular in $BD^*(Y_O)$; but by construction, then $a \in O$. Hence we have that η is bijective, and it is clearly a homomorphism.

To see the statement for ε, consider $i : \theta(B) \to B$ the inclusion. Since B is sober, this inclusion preserves all meets and admissible joins. Hence, by the universal property of the distributivisation, there is a unique lift $\bar{i} : D^\wedge(\theta(B)) \to B$ which is injective, since i was injective. In turn, as proved in the preliminaries, this means that there is a unique lift $\hat{i} : BD^\wedge(\theta(B)) \to B$ which is again injective. This is what we define ε_B to be, and hence the result follows. ■

With this we can now show that θ and BD form an adjunction:

Theorem 6.4.7. *The maps BD : **OIS** $\to$ **KTBsob** : θ form an adjunction, where moreover BD preserves injective maps, ε is pointwise injective, θ preserves surjective maps and η is an isomorphism.*

$$\begin{array}{ccc} BD(\mathbf{O}) & \xrightarrow{BD(\eta_{\mathbf{O}})} & BD(\theta(BD(\mathbf{O}))) \\ & \searrow^{\mathbf{1}_{BD(\mathbf{O})}} & \downarrow \varepsilon_{BD(\mathbf{O})} \\ & & BD(\mathbf{O}) \end{array}$$

Figure 6.3: Left Adjoint Triangle Identity

Proof. To see this forms an adjunction it suffices to check the triangle identities on the maps η and ε. To see that:

Simply note that by uniqueness of the map $\varepsilon_{BD(\mathbf{O})}$, and the fact that $\eta_{\mathbf{O}}$ is an iso, we have that $1_{BD(\mathbf{O})} \circ (BD(\eta_{\mathbf{O}}))^{-1} = \varepsilon_{BD(\mathbf{O})}$, which immediately yields equality.

Similarly, to see as in Figure 6.4:

$$\begin{array}{ccc} \theta(\mathbf{B}) & \xrightarrow{\eta_{\theta(\mathbf{B})}} & \theta(BD(\theta(\mathbf{B}))) \\ & \searrow^{1_{\theta(\mathbf{B})}} & \downarrow \theta(\varepsilon_{\mathbf{B}}) \\ & & \theta(\mathbf{B}) \end{array}$$

Figure 6.4: Right Adjoint Triangle Identity

Note that if $a \in \theta(B)$, then $\eta_{\theta(B)}(a) = a$, and since the counit maps is an inclusion, $\theta(\varepsilon_B) \circ \eta_{\theta(B)}(a) = a$. ∎

Hence, we have that the extended Goldblatt translation is sober, in the sense developed in Chapter 4. Hence, carrying out minimal changes for the case of relative varieties, we get a theory of companions for this setting, and a Polyatomic Blok-Esakia theorem. We briefly spell out what this amounts to, where proofs will proceed exactly as in Chapter 4:

Definition 6.4.8. Given an orthoimplicative logic O and a sober KTB logic $L \in \mathbf{NExt}(\mathsf{KTB}^s)$ we say that O is a *Goldblatt companion* of L if:

$$O \vdash (\varphi, \psi) \iff L \vdash G_2(\varphi) \to G_2(\psi)$$

We define the maps ρ, τ and σ on relative varieties of algebras as before; we also define the maps ρ and τ on logics, in the same way as described on Chapter 5. We denote by $L^{\Box\Diamond}$ the $\Box\Diamond$-variant of the sober KTB logic L. Carrying out all arguments from Chapter 4 relativised to the sobriety condition, we then have finally:

Corollary 6.4.9. ***(PAt-Blok Esakia Theorem for* OIL *and* KTBs*)*** *The following hold:*

- *The map ρ is a surjective homomorphism on varieties of algebras.*
- *The map τ is an isomorphism between the lattice of orthoimplicative logics and the lattice of $\Box\Diamond$-variants of sober* KTB*-logics.*
- *The greatest Goldblatt companion of an orthoimplicative logic is the sober logic generated, as a relative variety by:*

$$\{\sigma(\mathbf{B}) : \mathbf{B} \in Var(L)\}$$

We thus have the sought out PAt-Blok Esakia theorem. Using the results from the previous section, we also note that this result encompasses all of the results found in the literature: since all varieties of ortholattices with the FMP are compatibly generated, all of them are conservatively translated to orthoimplicative logic, and hence the second Goldblatt translation extends the first. This recovers Theorem 5.3.10 by Miyazaki, and shows that our result is a genuine extension of the work done so far on the theory of KTB companions.

6.5 Chapter Summary

We summarise our principal contributions in this chapter as follows:

- We use the notion of quasi-prime filter to provide a new representation for ortholattices. This is done with the novel notion of a slim orthospace, which is shown to be minimal amongst orthospaces. This also yields a universal *distributivisation* functor.
- We study orthoimplicative systems as algebras with natural slim orthospace duals. We provide an axiomatisation of these using Π_2-formulas, and study the descriptions of these in model theoretic terms. We introduce Orthoimplicative Logic, which corresponds on the logic side to these structures.
- We introduce a sober version of KTB, and discuss the context of applicability of these systems. We discuss admissibility of the non-standard rules in this case, and show that the key rule is admissible in the case of KTB and other systems.
- We provide explicit descriptions of the adjunction witnessing the second Goldblatt translation in the case of orthoimplicative systems and sober KTB algebras.
- We show that the second Goldblatt translation is sober, and prove a Polyatomic Blok-Esakia isomorphism between Orthoimplicative Logic and Sober KTB.

Chapter 7

Conclusions and Further Work

In this thesis, we introduced the concepts of Polyatomic logics, and initiated a generalised study of "Blok-Esakia theory" for a large class of translations. Building on ideas from [53] and [6], we presented a framework for this study, outlining the concepts of selective, strongly selective, and sober translations, developing the concept of Polyatomic logics, and proving algebraic completeness theorems for such logics. Using these tools, we undertook a systematic study of the Goldblatt translation, finding that it fails to induce a classic Blok-Esakia isomorphism and fails to have the Polyatomic Blok-Esakia isomorphism. We then identified an adequate conservative expansion of minimal orthologic – orthoimplicative logic – as well as a conservative expansion of KTB – sober KTB – which are connected by a Polyatomic Blok-Esakia isomorphism.

The work presented provides only the basics of the underlying theory, and leaves many natural continuations for this research. We highlight some of these below.

We did not elaborate the study of the $\Box\Diamond$-logics associated to the Goldblatt translations, and did not investigate many natural properties of orthoimplicative systems and their logics. Properties such as FMP, Kripke completeness, canonicity, for both Orthoimplicative Logic and the dual $\Box\Diamond$-logics, would be the first natural continuation of the work presented here. We expect that standard techniques could be adapted to this case, with the role of the distributive envelope and Weak Heyting algebras should be emphatic in establishing these. In a similar vein, *preservation results* are an important theme of the GMT translation which we did not touch on here. Whilst certain properties follow from our results immediately (e.g., FMP and tabularity being preserved by the maps σ or ρ in the extended Goldblatt translation), others might require more careful analysis (e.g., Kripke completeness being preserved by the three maps). More broadly, for other polyatomic logics, including DNA-logic, the development of techniques like universal models and Jankov-De Jongh formulas can be expected to provide useful insights.

Related to all of these questions, the investigation of the regularly generated sober KTB algebras seem like a natural continuation of the research from chapter 6. The existence of a Polyatomic Blok-Esakia isomorphism leaves open the question of whether this translation is in fact BE-translation – that is, whether we can axiomatise the greatest companions. In this sense, we expect that the methods from [17] could be useful in establishing possible isomorphisms, though more work is necessary to understand the classes of sober and regularly generated KTB frames (i.e, the order theoretic reducts of slim orthospaces).

In addition to the contributions to the study of translations, in Chapter 5 we presented a new duality for ortholattices, which can be exploited further. For instance, we have the important case of *orthomodular lattices* – which we notably did not discuss throughout the thesis – and the associated systems of Quantum Logic. For instance, we have that the elements of the center, that is, those elements $a \in \mathbf{O}$ such that for every $b \in \mathbf{O}$:

$$a = (a \wedge b) \vee (a \wedge b^{\perp})$$

are such that $a \vee a^{\perp}$ is an admissible join. This implies that quasi-prime filters behave with respect to these elements in a way similar to ultrafilters. A broad research topic would be to look into the applicability of these ideas to the study of orthomodular lattices, modular ortholattices and Quantum Logics, which have various longstanding open problems [11]. In a similar vein, these methods should be applicable to modal ortholattices [41], and could allow for representations connecting these systems with classical modal logic.

Many questions which are naturally related to those in this thesis can be asked for adjacent translations, and we expect, can be addressed by methods such as the ones presented. One example is the following questions: intuitionistic logic embeds into classical logic (through an inclusion), and classical logic embeds into intuitionistic logic (via the KGG translation); orthologic embeds into classical logic (through an inclusion); so in what way should classical logic embed into orthologic? An intuitive idea, inspired by our Kripkean approach, is that this should somehow "collapse" the various states representing compatible options, making the phenomenon classical. We believe that a translation into specific *modal* orthologics, should be possible, so as to capture this intuition.

Finally, one could pursue a research program into Inductive Rule classes, which have the same kind of axiomatisation as Orthoimplicative Logic. This seems like a broad setting in which to develop algebraic logic, as its complexity is just above that of universal sentences, whilst preserving some of the universal algebraic desiderata (notably, Lindenbaum-Tarski constructions). We expect that a systematic investigation of this could unify and generalise many studies [58, 37, 54, 3] in the use of rules for axiomatising non-classical logical systems, and shed new light on the expressive power of such axiomatisations.

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Chapter 8

Appendix

The purpose of this appendix is to provide detailed proofs of some facts which are used in the thesis, but whose proofs are either standard or quite lengthy and unremarkable from the point of view of the main argument.

8.1 Slim Orthospace Duality

In Chapter 5, we proved the representation theorem of ortholattices inside slim orthospaces. We can show that this is in fact a full duality, by arguments analogous to those outlined in [8], which is what we do in this part of the appendix.

Our first focus will be on the method of working within distributive envelopes.

Lemma 8.1.1. The map ε defined above is a homeomorphism which preserves and reflects the $\perp$ relation.

Proof. We begin by showing that indeed ε is well-defined, namely, that its image is a quasi-prime filter:

- (Filter): if $U \in \varepsilon(x)$ and $U \subseteq V$, then $x \in U$ implies that $x \in V$, so $V \in \varepsilon(x)$; similar for closure under conjunction.
- (Quasi-primeness): if $(\bigcup_{i=1}^{n} U_i)^*$ is an admissible join, by the slim-orthospace property, $(\bigcup_{i=1}^{n} U_i)^* = \bigcup_{i=1}^{n} U_i$. Hence if it belongs to $\varepsilon(x)$, then $x \in \bigcup_{i=1}^{n} U_i$, so $x \in U_i$ for some i, i.e, $U_i \in \varepsilon(x)$.

Moreover, we see that it is continuous. For this it suffices to show that the pre-image of subbasic open sets is again open, and indeed we have, for $U \in RClop(Y_O)$:

$$\begin{aligned}\varepsilon^{-1}(\varphi(U)) &= \{x : \varepsilon(x) \in \varphi(U)\} \\ &= \{x : U \in \varepsilon(x)\} \\ &= \{x : x \in U\} \\ &= U\end{aligned}$$

And similar for $X - U$. Moreover, the map is injective: if $x \neq y$, then by Axiom 1 of an orthospace, there is some regular clopen set U such that $x \in U$ and $y \notin U$, hence $U \in \varepsilon(x)$ and $U \notin \varepsilon(y)$. It is

$\perp$ preserving and reflecting: by Axiom 2 $x \perp y$ if and only if there is some U such that $x \in U$ and $y \in U^\perp$, if and only if $U \in \varepsilon(x)$ and $U^\perp \in \varepsilon(y)$, if and only if $\varepsilon(x) \perp \varepsilon(y)$. Finally, we have that ε is surjective: suppose that $Y_{RClop(Y)} - \varepsilon(Y)$ is non-empty. Let z be an element there. Since this space is Boolean, by total disconnectedness, and the fact that $\varepsilon(Y)$ is closed, we can find a clopen set V such that $z \in V$ and $V \cap \varepsilon(Y) = \varnothing$. By usual representations, we may assume that:

$$V = \bigcup_{i=1}^{n} A_i - B_i^0 - ... - B_i^m$$

Where A, B are all clopen regulars. Hence we may assume without loss of generality that $z \in \varphi(U) - \varphi(V_0) - ... - \varphi(V_n)$. Hence we have that:

$$\varepsilon^{-1}(\varphi(U) - \varphi(V_0) - ... - \varphi(V_n)) = \varnothing$$

Which implies that $U \subseteq V_0 \cup ... \cup V_n$. Bu by Lemma 6.1.20, this means that:

$$\varphi(U) - \varphi(V_0) - ... - \varphi(V_n) = \varnothing$$

Which contradicts the fact that z is there. ■

Next we consider the case of the morphisms:

Lemma 8.1.2. Let $f : O \to O'$ be an admissible ortholattice homomorphism, and $g : Y \to Y'$ an orthospace morphism between slim orthospaces. Then:

- $f^{-1} : Y_{O'} \to Y_O$ is an orthospace morphism.
- $g^{-1} : RClop(Y') \to RClop(Y)$ is an admissible ortholattice homomorphism.
- $(f^{-1})^{-1} \cong f$ and $(g^{-1})^{-1} \cong g$

Proof. (1) Notice that because f is admissible, then f^{-1} is well-defined: it surely sends filters to filters, and moreover, if $x \in Y_{O'}$ is quasi-prime, then so is $f^{-1}[x]$: if $\bigvee a \in f^{-1}[x]$ is an admissible join, then $f(\bigvee a) \in x$, and by assumption, this is admissible as well.
It is straightforward to see that f^{-1} is continuous. Moroever, if xRy, then $f^{-1}[x]Rf^{-1}[y]$: if $a \in f^{-1}[x]$, then $f(a) \in x$, so $f(a)^\perp \notin y$, hence $a^\perp \notin f^{-1}[y]$ using preservation of $\perp$ by f. To see the back condition, suppose that $f^{-1}[x]Ry$. Then look at:

$$Fil(f[y]) \text{ and } Id(\{c^\perp : c \in x\})$$

Note these must be disjoint; otherwise, for some $a_0, ..., a_n \in y$, $f(a_0) \wedge ... \wedge f(a_n) \leqslant c_0^\perp \vee ... \vee c_m^\perp$. Hence:

$$f(a_0 \wedge ... \wedge a_n) \leqslant (c_0 \wedge ... \wedge c_n)^\perp$$

Hence $c_0 \wedge ... \wedge c_n \leqslant f((a_0 \wedge ... \wedge a_n)^\perp)$. Since the former is in x, so is the latter, so $(a_0 \wedge ... \wedge a_n)^\perp \in f^{-1}[x]$. But then $(a_0 \wedge ... \wedge a_n) \notin y$, which is a contradiction. So by the Quasi-prime filter theorem, there is some z such that $f[y] \subseteq z$, and whenever $c \in x$, $c^\perp \notin z$. So xRz, and moreover, $y \leqslant f^{-1}[z]$, as intended.
(2) To see that g^{-1} is an ortholattice homomorphism is clear. We show that it is admissible. If $\bigvee_{i=1}^n U_i$ is admissible, then by the slim orthospace condition, this is equal to $\bigcup_{i=1}^n U_i$. Then surely

$g^{-1}[\bigcup_{i=1}^{n} U_i] = \bigcup_{i=1}^{n} g^{-1}[U_i]$ will be such that all clopen regulars distribute over it.
(3) We show this fact for f, whilst for g this is analogous. Indeed, we show that if $a \in O$ is arbitrary, $\varphi(f(a)) = (f^{-1})^{-1}(\varphi(a))$. Suppose that $x \in \varphi(f(a))$. This is true if and only if $f(a) \in x$, if and only if $a \in f^{-1}(x)$, if and only if $f^{-1}(x) \in \varphi(a)$, if and only if $x \in (f^{-1})^{-1}(\varphi(a))$. ∎

With this we have shown the following.

Theorem 8.1.3. *The categories* $\mathbf{Ort}^{ad}$ *of ortholattices with admissible homomorphisms and* **SOrtS** *of slim orthospaces are dually equivalent.*

The former also allows us to prove, as claimed in Chapter 5, that slim orthospaces are minimal:

Proposition 8.1.4. Let O be an ortholattice, and X an orthospace representing O. Then Y_O embeds into X through a map that preserves and reflects the relation $\perp$.

Proof. Let $x \in Y_O$ be arbitrary. Since X represents O, we can, without loss of generality, denote its clopen regulars by $\varphi(b)$ for $b \in O$. So consider in X the intersection $\bigcap_{a\in x} \varphi(a) \cap \bigcap_{b\notin x} X - \varphi(b)$. First we claim that this is non-empty. Indeed, if it were, by compactness, some finite subfamily would be empty, i.e, $\varphi(a_0) \cap ... \cap \varphi(a_n) \cap X - \varphi(b_0) \cap ... \cap X - \varphi(b_m) = \varnothing$. Hence:

$$\varphi(a_0) \cap ... \cap \varphi(a_n) \subseteq \varphi(b_0) \cup ... \cup \varphi(b_m)$$

By the above isomorphism Lemma, we get then that in the distributive envelope:

$$a_0 \wedge ... \wedge a_n \leqslant b_0 \cup ... \cup b_m$$

But then, we have that x contains all the a_i, so it must contain one of the b_i, a contradiction.
Moreover, we claim that it can contain at most one element. Indeed, if x and y are both in that intersection, by assumption, there is some V a clopen regular such that $w \in V$ and $y \notin V$. I.e, $w \in \varphi(a)$ and $y \notin \varphi(a)$, so $a \in w$ and $a \notin y$, a contradiction. Hence mapping $\chi(x)$ to the unique element is well-defined.
Notice that the argument we have given now also shows that χ is injective. It is moreover continuous: $\chi^{-1}(\varphi(a)) = \{x : \chi(x) \in \varphi(a)\} = \{x : a \in \chi(x)\} = \varphi(a)$. And by similar arguments to those sketched above, we see that the map preserves and reflects the relation $\perp$. ∎

Hence, we can genuinely talk about the space of quasi-prime filters as the *minimally sober* orthospace.

8.2 Axiomatisation of Orthoimplicative Systems

The purpose of this section is to provide proofs for many of the claims on orthoimplicative systems, which were introduced in Chapter 5. To work up to this, we will need to gather some tools relating the distributive envelope to the original ortholattice. Throughout, given any $H \subseteq D^\wedge(\mathbf{O})$, define:

$$\overline{H} := \{a \in O : a \in H\}$$

We call the former the *restriction* of H to the ortholattice.

Proposition 8.2.1. Let $\mathbf{O}$ be an ortholattice, and $D^\wedge(\mathbf{O})$ its distributive envelope. Suppose that $F \subseteq O$ is a subset, and $F = \overline{H}$, where H is an upwards closed subset of $D^\wedge(\mathbf{O})$. The following are equivalent:

1. H is a prime filter.

2. H is a filter which is prime with respect to admissible joins from F.

3. F is a quasi-prime filter.

Proof. (1) implies (2) since admissible joins in $\mathbf{O}$ coincide with the joins from $D^\wedge(\mathbf{O})$. To see that (2) implies (3), suppose that H is a filter which is prime with respect to admissible joins from O. Then it is straightforward to see that $\overline{H}$ is upwards closed, since the order agrees, and closed under meets. Moreover, if $\bigvee M$ is an admissible join, then we know that $\bigvee M \in D^\wedge(O)$ is the same element, hence, if $\bigvee M \in H$ then $m \in H$ by the primeness assumption, so $m \in \overline{H}$.
Now to see (3) implies (1), note that by definition H is upwards closed. It is a filter, since if $a \leqslant \bigcup c_i$ and $b \leqslant \bigcup d_j$, where these are elements from $D^\wedge(\mathbf{O})$, then $a \wedge b \leqslant \bigcup c_i \wedge d_j$. Now we will show primeness. To see that, assume that for some $a_0, ..., a_n \in F$, $a_0 \wedge ... \wedge a_n \leqslant \bigcup_{i=1}^{k} c_i$. Since F is a filter, write the former as a. Then consider:

$$a = \bigvee_{i=1}^{k} a \wedge c_i$$

Then we claim that this is an admissible join. Indeed, if $e \in O$ is arbitrary, note that:

$$e \wedge \bigvee_{i=1}^{k} a \wedge c_i = e \wedge a$$

By duality:

$$\varphi(e) \cap \varphi(a) = \varphi(e) \cap \bigcup_{i=1}^{k} \varphi(a) \cap \varphi(c_i) = \bigcup_{i=1}^{k} \varphi(a) \cap \varphi(e) \cap \varphi(b_i)$$

Since this equality holds, and $\varphi(e) \cap \varphi(a)$ is clopen regular:

$$\bigcup_{i=1}^{k} \varphi(a) \cap \varphi(e) \cap \varphi(b_i) = \bigvee_{i=1}^{k} \varphi(a) \cap \varphi(e) \cap \varphi(b_i)$$

Which by duality again yields that $e \wedge a = \bigvee_{i=1}^{k} e \wedge a \wedge c_i$. This shows the desired equality. Now, since $a \in F$, and this is quasi-prime, then $a \wedge c_i \in F$ for some i; so since F is upwards closed, $c_i \in F$. Hence, $c_i \in H$, establishing that H is prime. ∎

We now begin by proving some elementary properties of orthoimplicative systems:

Lemma 8.2.2. Let $\mathbf{O}$ be an orthoimplicative system.

- If $a \leqslant b$ then $c \multimap_k (a, d_0, ..., d_{k-1}) \leqslant c \multimap_k (b, d_0, ..., d_{k-1})$.

- If $a \leqslant b$ then $b \multimap_k (c_0, ..., c_{k-1}) \leqslant a \multimap_k (c_0, ..., c_{k-1})$

Proof. Assume that $a \leqslant b$. Then $a \wedge b = a$, so $c \multimap a = c \multimap (a \wedge b) = c \multimap a \wedge c \multimap b$, by Axiom 5. Next note that under the same hypothesis, we have that:

$$\begin{aligned} b \multimap_k (c_0, ..., c_{k-1}) &= (a \vee b) \multimap_k (c_0, ..., c_{k-1}) \\ &\leqslant a \multimap_k (c_0, ..., c_{k-1}) \wedge b \multimap_k (c_0, ..., c_{k-1}) \end{aligned}$$

Which follows by Axiom 6. This shows the result. ∎

Recall that we defined the following relation between filters:

$$xTy \iff \forall n \in \omega, \forall a, b_0, ..., b_n, a \multimap_n (b_0, ..., b_n) \in x \text{ and } a \in y \implies \exists i \leqslant n, b_i \in y$$

And we also have the following relation between filters in $D^\wedge(O)$:

$$x\tilde{T}y \iff \forall n \in \omega, \forall a, b_0, ..., b_n \in O, a \multimap_n (b_0, ..., b_n) \in x \text{ and } a \in y \implies \bigcup_{i=1}^{n} b_i \in y$$

The key tool we will use to produce prime filters is an adaptation from [13].

Definition 8.2.3. If **O** is an orthoimplicative system, $F, X \subseteq O$, we define:

$$D_F(X) := \{c \in D^\wedge(O) : \exists c_0, ..., c_i \in O,\ c = \bigcup_{i=1}^{n} c_i \text{ and } \exists Y \subseteq X,\ Y \text{ is finite, } \bigwedge Y \multimap_n (c_0, ..., c_n) \in F\}$$

Lemma 8.2.4. For **O** an orthoimplicative system, $X \subseteq O$ a subset, and $F \subseteq O$ a quasi-prime filter, the following hold:

1. $D_F(X)$ is a filter.
2. $X \subseteq D_F(X)$
3. $(Fil^{D(O)}(F), D_F(X)) \in \tilde{T}$
4. $D_F(\overline{D_F(X)}) = D_F(X)$

Proof. (1) Suppose that $c \in D_F(X)$, and $c \leqslant d$. Hence for some $Y \subseteq X$, $\bigwedge Y \multimap_k (c_0, ..., c_n) \in F$. By arguments similar to Lemma 8.2.1, we have that since $c \leqslant d$, if $c = \bigcup c_i$ and $d = \bigcup_{j=1}^{k} d_j$, then $c_i \leqslant d$ and so $c_i = \bigvee_{j=1}^{k} c_i \wedge d_j$ is an admissible join. Hence, by Axiom A2, we have that:

$$\bigwedge Y \multimap_k (c_0, ..., c_k) \leqslant \bigwedge Y \multimap_m (d_0, ..., d_m)$$

Since F is a filter, and $\bigwedge Y \multimap_k (c_0, ..., c_k) \in F$, then $\bigwedge Y \multimap_m (d_0, ..., d_m) \in F$. So $d \in D_F(X)$ by definition.

Now suppose that $c, d \in D_F(X)$, where $c = a_0 \cup ... \cup a_{k-1}$ and $d = b_0 \cup ... \cup b_{m-1}$. Then $\bigwedge Z \multimap_k (a_0, ..., a_{k-1}) \in F$, and $\bigwedge Y \multimap_m (b_0, ..., b_{m-1})$. By the same Lemma, then, $f = \bigwedge Z \wedge \bigwedge Y$ is such that:

$$f \multimap_k (b_0, ..., b_{k-1}) \in F \text{ and } f \multimap_m (a_0, ..., a_{m-1}) \in F$$

Hence their meet is in F. Hence by Axiom 5, we have that $f \multimap_{k \times m} ((d_i \wedge c_j)$. Since $d \cap c = \bigcup_{(i,j) \in k \times m} d_i \times c_j$, this shows that $d \wedge c \in D_F(X)$ as intended.

(2) Note that by Axiom 4, $X \subseteq D_F(X)$ obviously holds: if $a \in X$, then $a \multimap_1 a = 1$, which is in F.

(3) Let $a, b_0, ..., b_n \in O$ and $a \multimap_n (b_0, ..., b_n) \in Fil^{D^\wedge(O)}(F)$, and assume that $a \in D_F(X)$. Since $Fil^{D^\wedge(O)}(F)$ agrees with F on the ortholattice reduct (given that F is a filter), we have that $a \multimap_n (b_0, ..., b_n) \in F$. Now by definition, since $a \in D_F(X)$ there is some $\bigwedge Y \multimap_1 a \in F$; so $\bigwedge Y \multimap_n (b_0, ..., b_n) \in F$ by Axiom 7. Hence $\bigcup_{i=1}^{n} b_i \in D_F(X)$ by definition.

(4) Finally suppose that $a \in D_F(\overline{D_F(X)})$. Then by assumption, there is $Y \subseteq \overline{D_F(X)}$ such that $\bigwedge Y \multimap_k (a_0, ..., a_k) \in F$. In turn, by definition, $Y = \{y_0, ..., y_m\}$, and each $y_i \in D_F(X)$. Hence, for some finite $Z_i \subseteq X$, we have that $\bigwedge Z_i \multimap_1 y_i$. Hence, using Lemma 8.2.2, and letting $W = \bigwedge_{i=1}^{m} Z_i$, we have that:

$$W \multimap_1 y_i \in F$$

So, by Axiom 5, we have that $W \multimap \bigwedge Y \in F$. But then by Axiom 7 we obtain that $W \multimap_k (a_0, ..., a_k) \in F$, which implies that $\bigcup_{i=1} a_i \in D_F(X)$, as desired. ■

Lemma 8.2.5. (Existence Lemma for Orthoimplicative Systems) Let $\mathbf{O}$ be an orthoimplicative system, and F a quasi-prime filter, and I an ideal in $D^\wedge(\mathbf{O})$ such that $D_F(X) \cap I = \varnothing$. Then there exists some G such that $D_F(X) \subseteq G$, G is a quasi-prime filter, $(F, G) \in T$, and $G \cap \overline{I} = \varnothing$.

Proof. Let as usual:

$$P = \{H \subseteq D^\wedge(O) : H \text{ is a filter } (Fil^{D^\wedge(O)}(F), H) \in \tilde{T} \text{ and } H \cap I = \varnothing\}$$

Now by assumption, and the last lemma, we have that P is non-empty, and it is easily seen that such a set is inductive. So let H' be a maximal element in P, and let $H = \overline{H'}$. We claim that H' is closed under admissible joins from O. For suppose not. Then there is $M \subseteq O$ such that $M = \{m_0, ..., m_k\}$, $\bigvee M$ is admissible, $\bigvee M \in H'$, and for no i do we have that $m_i \in H'$. Note that for each $m_i \in M$, $D_{m_i} = D_F(H \cup \{m_i\})$ is such that $(Fil^{D^\wedge(O)}(F), D_{m_i}) \in \tilde{T}$: if $a \multimap (b_0, ..., b_n) \in Fil^{D^\wedge(O)}(F)$, and $a \in D_{m_i}$, then by definition, for some $z \in H$ (by closure under meets), we have $z \wedge m_i \multimap_1 a \in F$, and hence $z \wedge m_i \multimap_1 a \in Fil^{D^\wedge(O)}(F)$. So by Axiom 7, $z \wedge m_i \multimap (b_0, ..., b_n) \in Fil^{D^\wedge(O)}(F)$. This in turn implies, since F is a filter, that $z \wedge m \multimap (b_0, ..., b_n) \in F$, which by definition means that $\bigcup_{j=1} b_j \in D_{m_i}$.

Hence, for each such m_i, we have that $D_{m_i} \cap I \neq \varnothing$. Hence for some $c_{m_i} \in H$ (by closure):

$$c_{m_i} \wedge m_i \multimap_{k_{m_i}} (e_0, ..., e_{k_{m_i}}) \in F$$

Where $\bigcup_{j=1}^{k_{m_i}} e_j \in I$. Since this is true for each m then, using repeatedly Axiom 3, we obtain that:

$$c_{m_i} \wedge m_i \multimap_p (l_0, ..., l_p) \in F$$

Where the latter is a list containing all sequences for each m_i. By Axioms 10 and 11, we know that it does not matter the order or the multiplicity in such a list. Moreover, let $c' = \bigwedge_{m_i \in M} c_{m_i}$. Then we have:

$$c' \wedge m_i \multimap_p (l_0, ..., l_p) \in F$$

For each m_i, and so:

$$\bigwedge_{m_i \in M} (c' \wedge m_i \multimap_p (l_0, ..., l_p)) \in F$$

Then we note that:

$$\bigvee_{m_i \in M} c' \wedge m_i$$

Is an admissible join, by arguments we have used before. So by Axiom A1:

$$\bigvee_{m \in M} (c' \wedge m) \multimap_p (l_0, ..., l_p) \in F$$

But then by admissibility:

$$\bigvee M \wedge c' \multimap_p (l_0, ..., l_p) \in F$$

But now we have that $\bigvee M \wedge c' \in H$, so $\bigcup_{i=1}^{p} l_i \in D_F(H)$. By Lemma 8.2.4, namely (3) and (4), we have that $D_F(H) = H'$ Hence $\bigcup_{i=1}^{p} l_i \in H'$. On the other hand, by assumption, for each i, $\bigcup_{j=0}^{k_{m_i}} e_j \in I$. So since I is an ideal, and $\bigcup_{i=1}^{p} l_i = \bigcup_{i=1}^{k} \bigcup_{j=0}^{k_{m_i}} e_j$, we have that $\bigcup_{i=1}^{p} l_i \in I$ - a contradiction to $I \cap H' = \varnothing$.

So finally consider H. Then by Lemma 8.2.1, and the fact that $H = \overline{H'}$, H is a quasi-prime filter, and $H \cap I = \varnothing$. We show that $(F, H) \in T$: whenever $a \multimap_n (b_0, ..., b_n) \in F$, and $a \in H$, then $a \multimap_n (b_0, ..., b_n) \in Fil^{D(O)}(F)$ and $a \in H'$, by assumption. Hence $\bigcup_{i=1} b_i \in H'$. Since by Lemma 8.2.1 we have that H' is a prime filter, there is some $b_i \in H'$, which shows that $b_i \in H$. This was to show. ■

To provide our characterisation, we briefly recall the concept of a WH-space ([13]):

Definition 8.2.6. Let $(X, \leqslant, S)$ be a set equipped with a partial order $\leqslant$ and a relation R. We say that $(X, \leqslant, S)$ is a *WH-frame* if $\leqslant \circ S \subseteq S$.

We say that a relational topological space $(X, \leqslant, S, \tau)$ is a *WH-space* if:

1. $(X, \leqslant, S)$ is a WH-frame;
2. $(X, \leqslant, \tau)$ is a Priestley space;
3. $(X, \leqslant, S)$ is a Modal space.

We say that a map $f : X \to X'$ between WH-spaces is a WH-morphism if it is (1) continuous, (2) $\leqslant$-order preserving, (3) an R p-morphism.

For a proof of the following see [13]:

Theorem 8.2.7. *The categories* **WH** *of Weak Heyting algebras with Weak Heyting homomorphisms and* **WHS** *of WH spaces with WH-morphisms are dually equivalent.*

The following is an easy correspondence result:

Lemma 8.2.8. For each WH-algebra $\mathbf{H} = (H, \wedge, \vee, \multimap, 0, 1)$, we have that:

1. (B) $H \models a \leqslant ((a \multimap b) \multimap 0) \vee b$ if and only if R is symmetric.
2. (T) $H \models a \wedge (a \multimap b) \leqslant b$ if and only if R is reflexive.

Proof. First suppose that $\mathbf{H} \vDash a \leqslant (a \multimap b) \multimap 0 \vee b$. Suppose that xSy. We will show that ySx. Indeed, suppose that $a \multimap b \in y$ and $a \in x$. By the axiom and primeness, either b or $(a \multimap b) \multimap 0 \in x$; but if the latter was the case, then since xSY we would have $0 \in y$, a contradiction. So $b \in x$, as desired.

Conversely, suppose that $\mathbf{H} \nvDash a \leqslant (a \multimap b) \multimap 0 \vee b$. Let x be a prime filter containing the first term but not the second. Then by our remark above, there must be some y such that xSy, $a \multimap b \in y$ and $0 \notin y$ (trivially); also $a \in x$, but $b \notin x$, hence $\neg(ySx)$.

The reflexivity axiom has a very similar argument. ∎

All of this allows us to prove the following:

Proposition 8.2.9. Let $\mathbf{O}$ be an orthoimplicative system. Let $Y_{\mathbf{O}}$ be the dual slim orthospace. Then:

$$\varphi(a \multimap_k (b_0, ..., b_k)) = \Box_T(\neg\varphi(a) \cup \varphi(b_0) \cup ... \cup \varphi(b_k))$$

Proof. One inclusion is obvious: if $a \multimap_n (b_0, ..., b_n) \in x$, and xTy and $a \in y$, then by definition there is some i such that $b_i \in y$. Now assume that $a \multimap_n (b_0, ..., b_n) \notin x$. Consider:

$$D_x(\{a\}) \text{ and } \mathsf{Id}^{D^\wedge(O)}(\{b_0, ..., b_n\}).$$

Indeed, we claim these must be disjoint subsets of $D^\wedge(O)$. If not, there is some $c \in D_x(\{a\})$ such that $c \leqslant b_0 \cup ... \cup b_n$. Since we have that $c = \bigcup_{j=1}^m c_j$, using the same argument as we have in prvious lemmas, we can show that $c_j = \bigvee_{i=1}^n c_j \wedge b_i$, and that this is an admissible join. Hence by Axiom A2:

$$a \multimap_m (c_0, ..., c_m) \leqslant a \multimap_n (b_0, ..., b_n)$$

Then since $a \multimap_m (c_0, ..., c_m) \in x$, this means we have that then:

$$a \multimap_k (b_0, ..., b_n) \in x$$

Which is a contradiction. Hence, by Lemma 8.2.5, we have that there exists some y such that xTy, $a \in y$, and $b_i \notin y$ for any i. This means that $x \notin \Box_T(\varphi(a) \to \varphi(b_0) \cup ... \cup \varphi(b_k))$. ∎

All of this work culminates, as presented in Chapter 5, in the following:

Theorem 8.2.10. *(Characterisation of Compatible Ortholattices) Let* $\mathbf{O}$ *be an ortholattice. Then the following are equivalent:*

1. $\mathbf{O}$ *admits an orthoimplicative system structure.*

2. $D^\wedge(\mathbf{O})$ *admits a unique WH-symmetric algebra structure, with the dual relation being given by the non-orthogonality relation.*

3. $\mathbf{O}$ *is compatible.*

Proof. To see that (1) implies (2), we define the structure on $D^\wedge(\mathbf{O})$ using duality. Indeed, for each $U = \bigcup_{i=1}^k \varphi(a_i)$ and $V = \bigcup_{i=1}^k \varphi(b_i)$, clopen upsets in Y_O, let:

$$U \implies V := \bigcap_{i=1}^{k} \Box_R(\neg\varphi(a_i) \cup V)$$

By the above Proposition 6.2.4, this is a clopen regular element, and represents precisely $\bigwedge a_i \multimap_k (b_0, ..., b_k)$. By duality, this is a well-defined notion. It is not hard to see that this will yield a WH-implication structure on $D^\wedge(O)$. Moreover, because R is reflexive and symmetric, $D^\wedge(\mathbf{O})$ will be a WH-symmetric algebra by Lemma 8.2.8. The argument that shows that the structure is unique runs parallel to the Lemma above establishing that $T = R$.

The fact that (2) implies (3) follows because, if $D^\wedge(\mathbf{O})$ admits such a structure, by WH-algebra duality with WH-frames, we have that $Y_{\mathbf{O}}$, the dual WH-space, will be a modal space with respect to the relation R.

Finally, to see that (3) implies (1), it suffices to show that if O is compatible, then it admits an orthoimplicative system structure. We do so using duality. Indeed, define, for $a, b_0, ..., b_k \in O$:

$$\varphi(a \multimap_k (b_0, ..., b_k)) := \Box(\varphi(a) \to \varphi(b_0) \cup ... \cup \varphi(b_k))$$

The reader can easily check that this satisfies Axioms 1-7, 10 and 11. Axiom 8 is valid since the relation R is reflexive, and Axiom 9 is valid since R is symmetric. So we need to check the admissibility axioms:

- (A1) if $\bigvee_{m \in M} \varphi(m)$ is admissible, then $\bigvee_{m \in M} \varphi(m) = \bigcup_{m \in M} \varphi(m)$. Thus:

$$\begin{aligned} \varphi(\bigvee M \multimap_k (b_0, ..., b_k)) &= \Box(\bigcup_{m \in M} \varphi(m) \to \varphi(b_0) \cup ... \cup \varphi(b_k)) \\ &= \Box(\bigcap_{m \in M} (\varphi(m) \to \varphi(b_0) \cup ... \cup \varphi(b_k))) \\ &= \bigcap_{m \in M} \Box(\varphi(m) \to \varphi(b_0) \cup ... \cup \varphi(b_k)) \\ &= \bigcap_{m \in M} \varphi(m \multimap_k (b_0, ..., b_k)) \\ &= \varphi(\bigwedge_{m \in M} m \multimap_k (b_0, ..., b_k)) \end{aligned}$$

- (A2) Assume that $C = \{c_0, ..., c_n\}$ and $D = \{d_0, ..., d_k\}$ are in the stated conditions. Then note that this means simply that:

$$C \subseteq D$$

  Hence the result follows by a straightforward calculation using the WH-implication and the fact that in WH-algebras, when $c \leqslant d$, then $e \multimap c \leqslant e \multimap d$.

- (A3) Assume that $a \multimap_n (b_0, ..., b_n) = 1$. Dually, this means that:

$$\Box(\varphi(a) \to \varphi(b_0) \cup ... \cup \varphi(b_n)) = Y_O$$

  Since $\Box$ is a reflexive modality, we have that:

$$\varphi(a) \to \varphi(b_0) \cup ...\varphi(b_n) = Y_O$$

  So by classical reasoning, $\varphi(a) \subseteq \varphi(b_0) \cup ... \cup \varphi(b_n)$. Then by an argument we have met before, $\bigvee a \wedge b_i$ is an admissible join, as it is equal to a.

This shows that $\mathbf{O}$ admits an orthoimplicative system structure, as desired. ■

We now show some of the facts mentioned in Chapter 5 related to morphisms.

Proposition 8.2.11. Let $\mathbf{O}, \mathbf{O'}$ be orthoimplicative systems, Y, Y' full orthospaces. Then:

1. If $f : \mathbf{O} \to \mathbf{O'}$ is a homomorphism between orthoimplicative systems, then $f^{-1} : Y_{\mathbf{O'}} \to Y_{\mathbf{O}}$ is a strong orthospace morphism.

2. If $g : Y \to Y'$ is a strong orthospace morphism between full orthospaces, then $g^{-1} : RClop(Y') \to RClop(Y)$ is a homomorphism between orthoimplicative systems.

Proof. (1) We already know, by the orthospace duality, that f^{-1} is an orthospace morphism. Now assume that xTy; if $a \multimap_k (b_0, ..., b_k) \in f^{-1}[x]$. Then $f(a) \multimap_k (f(b_0), ..., f(b_k)) \in x$. Hence by definition, $f(b_i) \in y$, hence, $b_i \in f^{-1}[y]$. Next suppose that $f^{-1}[x]Ry$. Consider:

$$\mathsf{Fil}(f[x]) \text{ and } \mathsf{AdId}(f[O - y] \cup \{c : c^{\perp} \in x\}).$$

We claim these two sets are disjoint. For otherwise, we would have, for some $M \subseteq\downarrow (f[O-y] \cup \{c : c^{\perp} \in x\})$ which is an admissible join:

$$f(a) \leqslant \bigvee M$$

Now by assumption, for each $m \in M$, $m \leqslant f(b_i)$ or $m \leqslant c$, and the join is admissible. Hence, by duality:

$$\varphi(f(a)) \subseteq \bigcup_{m \in M} m \subseteq \bigcup_{i=1}^{n} \varphi(b_i) \cup \bigcup_{j=1}^{k} \varphi(c_j)$$

By similar arguments to what we showed before, then $f(a) = \bigvee_{i=1}^{n} f(a) \wedge f(b_i) \vee \bigvee_{j=1}^{k} c_j \wedge f(a)$ is an admissible join. Since $\mathbf{O}$ is orthoimplicative, then:

$$\bigwedge_{j=1}^{k} c_j^{\perp} \leqslant f(a) \multimap_k (f(b_0), ..., f(b_n))$$

Now since $\bigwedge_{j=1}^{k} c_j^{\perp} \in x$, then the latter is as well. But since f preserves the implication connective $f(a \multimap_k (b_0, ..., b_k)) = f(a) \multimap_k (f(b_0), ..., f(b_k))$. Thus by hypothesis, the former is in x. Thus, $a \multimap_k (b_0, ..., b_k) \in f^{-1}[x]$. Since $a \in y$, by definition, for some i, $b_i \in y$. But this is a contradiction. By reductio, we obtain that the two sets above are disjoint, hence, by the prime filter theorem for distributive lattices, let z be such that $f[y] \subseteq z$ and $z \cap f[O - y] = \varnothing$, and whenever $c \in x$, then $c^{\perp} \notin z$. Thus finally look at $\overline{z}$. Then:

- $\overline{z}$ is a quasi-prime filter, by Lemma 8.2.1.

- $xR\overline{z}$

- $f^{-1}[\overline{z}] = y$: indeed, if $a \in y$, then $f(a) \in f[y]$, hence $f(a) \in z$; but since $f(a) \in O'$, then $f(a) \in \overline{z}$, hence $a \in f^{-1}[\overline{z}]$. Conversely, if $a \notin y$, then $f(a) \in f[O - y]$, so $f(a) \notin z$, hence clearly $f(a) \notin \overline{z}$.

This shows the result.

For (2), we show that:

$$g^{-1}[\varphi(a \multimap_n (b_0, ..., b_n)] = \Box_R(g^{-1}[\varphi(a)] \to g^{-1}[\varphi(b_0)] \cup ... \cup g^{-1}[\varphi(b_n)])$$

Indeed, if $g(x) \in \varphi(a \multimap_n (b_0, ..., b_n)$, then $a \multimap_n (b_0, ..., b_n) \in g(x)$; because this is R-monotone, if xRy, then $g(x)Rg(y)$, so if $a \in g(y)$, then for some i, $b_i \in g(y)$. Conversely, if $x \in \Box(g^{-1}[\varphi(a)] \rightarrow g^{-1}[\varphi(b_0)] \cup ... \cup g^{-1}[\varphi(b_n)])$, assume that $g(x)Ry$. Because this is a strong orthospace morphism, then for some z, xRz and $g(z) = y$. Hence, if $a \in y$, then $z \in g^{-1}[\varphi(a)]$, so $z \in g^{-1}[\varphi(b_i)]$ for some i, i.e, $b_i \in y$. We know that if $a \multimap_n (b_0, ..., b_n) \notin g(x)$, then by the Existence Lemma proved above, there would be some y such that $g(x)Ry$ and $a \in y$ whilst $b_i \notin y$. Hence this is not possible, i.e, $a \multimap_n (b_0, ..., b_n) \in g(x)$. This shows the result. ■

8.3 Compatibility of Products

In this section we provide a direct proof that the product of orthoimplicative systems is again an orthoimplicative system. The arguments hereby contained are also useful to prove that structures such as the lattice of closed subspaces of a Hilbert space is a compatible ortholattice, hence, we provide full proofs as much as possible.

Let $(Y_{\mathbf{O}_i})_{i \in I}$ be a family of spaces of quasi-prime filters of the orthoimplicative systems $\mathbf{O}_i$. Let $\mathbf{Z} := \bigsqcup_{i \in I} Y_{\mathbf{O}_i}$ be the disjoint union. Define a family $\mathsf{Reg}(\mathbf{Z})$ as follows: $S \subseteq Z$ is called regular if for each i, $S(i) = W$ that is the i'th coordinate of this set, where $W \subseteq Y_{\mathbf{O}_i}$ is a clopen regular. Define a relation $\perp'$ on this structure pointwise: given a subset $A \subseteq Z$, we let $x \perp' y$ if and only if x and y are in the same coordinate of $Y_{\mathbf{O}_i}$, and $x \perp_{Y_{\mathbf{O}_i}} y$. Using this relation, note that $Reg(\mathbf{Z})$ forms an ortholattice, under intersection, pointwise orthocomplement, and the implication $\Box(x \rightarrow y_0 \cup ... \cup y_n)$, which we denote by $\multimap_n$. Then note the following:

Lemma 8.3.1. For any $U, V_0, ..., V_n \in \mathsf{Reg}(Z)$ we have $(U \multimap_n (V_0, ..., V_n)) \in \mathsf{Reg}(Z)$.

Proof. This follows almost by definition, once we note that because the elements have disjoint parts, the subtractions are also taken pointwise, i.e:

$$(U - V_0 - ... - V_n) = \bigcup_{i \in I} U(i) - V_0(i) - ... - V_n(i)$$

Now if $x \in (U - V_0 - ... - V_n)^{\perp}$ then whenever $y \in (U - V_0 - ... - V_n)$, and x and y are in the same coordinate, that is, $y \in U(i) - V_0(i) - ... - V_n(i)$ for some i, we have $x \perp_{Y_{O_i}} y$. Thus $x \in \bigcup_{i \in I}(U(i) - V_0(i) - ... - V_n(i))^{\perp}$. The converse is immediate since each x can be in a single coordinate. ■

It is moreover easy to see that:

Proposition 8.3.2. The algebras $(\mathsf{Reg}(Z), \cap, (\multimap)_{n \in \omega}, \perp)$ and $\prod_{i \in I} \mathbf{O}_i$ are isomorphic.

Proof. Define the map $p : \prod_{i \in I} \mathbf{O}_i \rightarrow \mathsf{Reg}(\mathbf{Z})$ which assigns to $(a_i)_{i \in I}$ the sequence $(\varphi(a_i))_{i \in I}$; by our definitions above this is seen to be an isomorphism. ■

To proceed, we will show that this structure allows us to work with $\mathsf{Reg}(Z)$ in our calculations. We denote by $\mathsf{Fin}(\mathsf{Reg}(Z))$ the set of all $U = \bigcup_{i=1}^{k} U_i$ where $U_i \in \mathsf{Reg}(Z)$, and the union is taken over $\mathsf{Reg}(Z)$. Then we have:

Lemma 8.3.3. The structure $(\mathsf{Fin}(\mathsf{Reg}(Z)), \cup, \cap, \perp)$ is isomorphic to $D^{\wedge}(\mathsf{Reg}(\mathbf{Z}))$.

Proof. We apply Proposition 6.1.18. We have that the inclusion of $\mathsf{Reg}(Z)$ is injective and dense. We show it preserves admissible joins. Indeed, assume that $\bigvee_{j=1}^{k} U_j$ is admissible. Then we have that for each coordinate of the disjoint union $i \in I$, $\bigvee_{j=1} U_j(i)$ is admissible as well. Since the spaces are slim orthospaces, then $\bigvee_{j=1} U_j(i) = \bigcup_{j=1} U_j(i)$. So we can see that $\bigvee_{j=1}^{k} U_j = \bigcup_{j=1}^{k} U_j$. ■

Now let $\coprod_{i\in I} Y_{\mathbf{O}_i}$ be the space of quasi-prime filters on $(\mathsf{Reg}(Z), \cap, \implies, \bot)$. We will now prove the following crucial lemma.

Lemma 8.3.4. Let U' and $V_0', ..., V_n'$ be clopen regulars in $\coprod_{i\in I} Y_{\mathbf{O}_i}$, i.e, $U' = \varphi(U)$ and $V_i' = \varphi(V_i)$. Then:

$$(\varphi(U) - \varphi(V_0) - ... - \varphi(V_n))^{\bot} = \varphi((U \multimap_n (V_0, ..., V_n)))$$

Proof. Let $x \in \varphi(U \multimap_n (V_0, ..., V_n))$. Now suppose that $y \in \varphi(U) - \varphi(V_0) - ...\varphi(V_n)$. By the hypothesis, then $U \in y$ and $V_i \notin y$. Let $\hat{y}$ be the prime filter extension. Now note that in $\mathsf{Reg}(Z)$ we have that $U \subseteq (U - V_0 - ... - V_n)^{\bot\bot} \cup V_0 \cup ... \cup V_n$. Indeed, note that:

$$\begin{aligned}(U - V_0 - ... - V_n)^{\bot\bot} \cup V_0 \cup ... \cup V_n &\subseteq U^{\bot\bot} \cup V_0 \cup ... \cup V_n \\ &\subseteq U \cup V_0 \cup ... \cup V_n \\ &\subseteq (U - V_0 - ... - V_n) \cup V_0 \cup ... \cup V_n \\ &\subseteq (U - V_0 - ... - V_n)^{\bot\bot} \cup V_0 \cup ... \cup V_n\end{aligned}$$

Hence since clearly $U \subseteq U \cup V_0 \cup ... \cup V_n$ we have this fact. But now, since $U \in \hat{y}$, and $V_i \notin \hat{y}$, we must have $(U - V_0 - ... - V_n)^{\bot\bot} \in \hat{y}$; hence, since $(U \multimap_n (V_0, ..., V_n)) = (U - V_0 - ... - V_n)^{\bot} \in \mathsf{Reg}(Z)$ by our previous lemma, clearly its orthogonal complement will be as well. But now, we must have that $x \bot y$. This shows one inclusion.

Conversely, suppose that $x \notin \varphi((U - V_0 - ... - V_n)^{\bot})$. Then we will have that over $D^\wedge(\mathsf{Reg}(Z))$:

$$\mathsf{Fil}(\{U\}) \cap \mathsf{Id}(\{C^{\bot} : C \in x\} \cup \{V_0, ..., V_n\}) \neq \varnothing$$

For otherwise, we will have that:

$$U \subseteq (C_0)^{\bot} \cup ... \cup (C_n)^{\bot} \cup V_0 \cup ... \cup V_n$$

Hence since these are sets:

$$U - V_0 - ... - V_n \subseteq (C_0)^{\bot} \cup ... \cup (C_n)^{\bot}$$

Applying $\bot$ on each side:

$$(C_0)^{\bot\bot} \cap ... \cap (C_n)^{\bot\bot} \subseteq (U - V_0 - ... - V_n)^{\bot} = U \implies_n (V_0, ..., V_n)$$

Since the former are in x by regularity, we have the latter is as well, since by the previous lemma, it is clopen regular. But this is a contradiction. So by the Prime filter theorem, we can find some prime filter y extending the above filter, and disjoint from the filter. Let $\overline{y}$ be its restriction to a quasi-prime filter in O. Then whenever $C \in x$, $C^{\bot} \notin \overline{y}$, by our Proposition 8.2.1, so $xR\overline{y}$; also $U \in \overline{y}$ and $V_i \notin \overline{y}$ for any i. This shows that $x \notin (\varphi(U) - \varphi(V_0) - ... - \varphi(V_n))^{\bot}$. ■

Corollary 8.3.5. *The following hold:*

1. *The structure* $\prod_{i\in I} \mathbf{O}_i$ *in the signature* $(\wedge, \vee, (\multimap_n)_{n\in\omega}, 0, 1)$ *is an orthoimplicative system so long as all* $\mathbf{O}_i$ *are.*

2. *The structure* $\prod_{i\in I} \mathbf{O}_i$ *is the signature* $(\wedge, \vee, ^\perp, 0, 1)$ *is a compatible ortholattice so long as all* $\mathbf{O}_i$ *are.*

Proof. By the fact that the images under φ form a subbasis, and by compactness, we know that if A is a clopen subset of an orthospace, it is of the form:

$$A = \bigcup_{j=1}^{n} \varphi(U^j) - \varphi(V_0^j) - ... - \varphi(V_k^j)$$

Also we know by definition of the $\perp$ that:

$$A^\perp = \bigcap_{j=1}^{n} (\varphi(U^j) - \varphi(V_0^j) - ... - \varphi(V_k^j)^\perp$$

Now by Lemma 8.3.1 and Lemma 8.3.4, together with our hypothesis, we know that for each j:

$$(\varphi(U^j) - \varphi(V_0^j) - ... - \varphi(V_k^j))^\perp = \varphi((U^j - V_0^j - ... - V_k^j)^\perp)$$

Hence, that each such is a clopen regular. Hence $A^\perp$ is an intersection of clopens, and hence, a clopen. Thus, we have that $\prod_{i\in I} O_i$ is compatible as an ortholattice. Moreover, note that:

$$\varphi(U \multimap_n (V_0, ..., V_n)) = \square(\varphi(U) \multimap_n \varphi(V_0) \cup ... \cup \varphi(V_n))$$

By Lemma 8.3.4. Hence the implication coincides with the reduct of the WH-implication from the extension of $\prod_{i\in I} \mathbf{O}_i$, i.e, the product is orthoimplicative.
The second statement follows easily from the above lemmas and a similar argument as the one. ■