Relative Weak Factorization Systems

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis introduces and develops the notion of a relative weak factorization system. Motivated by research directions in type theory, we combine ideas from algebraic weak factorization systems with the concept of relative monads and comonads, to define a generalized, more flexible analogue of weak factorization systems, which is able to incorporate additional shapes of diagrams. We prove results regarding the properties of these systems and their relationships to existing notions of weak factorization systems.

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1 Introduction

The motivation for the work presented here arises from type theory. Type theory and category theory are closely linked; in particular, we can think of category theory as providing semantics for type theory (and type theory as providing syntax for category theory). As new forms of type theory are developed, we are interested in developing and understanding categorical models for them, and perhaps also interested in using possible categorical models to inspire or guide their development.

One intriguing possibility in type theory is the development of a *directed* type theory, and in particular directed homotopy type theory (see for example Licata and Harper [8] and North [7]). As homotopy type theory has provided a rich and unifying perspective on topological spaces, ∞-groupoids, and more, we imagine that a directed homotopy type theory would provide a rich and unifying perspective on directed topological spaces, (∞, n)-categories, and more.

There are various obstacles to creating directed analogues of homotopy type theory and its categorical models. One such obstacle is that it is not clear how dependent types should be modeled in directed categories. We would like to employ *fibrations* of some kind, but the existing definitions do not generalize straightforwardly to directed settings. In particular, consider identity types and their interpretations, which are very central concepts in homotopy type theory. The usual Martin-Löf identity type can be modeled in category theory using the notion of a *weak factorization system.* However, when considering a directed context, where symmetry does not always hold, the Martin-Löf rules are too strong, and weak factorization systems do not model the behavior we have in mind.

What we would like instead is an object with many of the properties of a weak factorization system, but built around factorizations of different shapes, and incorporating *two-sided fibrations.* This thesis explores one possible route toward this goal, via the novel concept of a *relative weak factorization system.* Chapters 2 and 3 introduce the background concepts of weak factorization systems, algebraic weak factorization systems, and relative monads. Chapter 4 begins combining these ideas, in the context of the usual form of factorization (that is, factorization of morphisms into composable pairs of morphisms). Chapters 5 and 6 further explore possible notions of relative weak factorization systems, and the relationships between relative weak factorization systems and their predecessors. Chapter 7 gives an abstract definition of a relative weak factorization system and proves some general results. Finally, chapter 8 explores the application of the general definition to the case of two-sided factorizations, as desired.

2 Algebraic weak factorization systems

In this preliminary chapter, we will describe the motivation for and development of algebraic weak factorization systems. This will provide a foundation and an analogue for our development of relative weak factorization systems in later chapters.

A *weak factorization system* on a category is a pair of classes of arrows, which have a factorization property and lifting properties with respect to each other. The lifting properties are formalized as follows:

Definition 1. For arrows ℓ and r in a category $\mathcal{C}$, we say that ℓ has the left lifting property against r, and equivalently r has the right lifting property against ℓ, if for every commutative square of the form

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, "\ell"'] & \bullet \arrow[d, "r"] \\ \bullet \arrow[r, "v"'] & \bullet \end{tikzcd}$$

there is an arrow s

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, "\ell"'] & \bullet \arrow[d, "r"] \\ \bullet \arrow[r, "v"'] \arrow[ur, dotted, "s"] & \bullet \end{tikzcd}$$

such that $s \circ \ell = u$ and $r \circ s = v$. The arrow s is called the solution to the lifting problem posed by the commutative square.

For a class $\mathcal{A}$ of arrows, we denote by $\mathcal{A}^{\pitchfork}$ the class of arrows that have the right lifting property against every arrow in $\mathcal{A}$, and by ${}^{\pitchfork}\mathcal{A}$ the class of arrows that have the left lifting property against every arrow in $\mathcal{A}$.

Definition 2. A *weak factorization system* on a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms such that

1. every morphism of $\mathcal{C}$ factors as a map in $\mathcal{L}$ followed by a map in $\mathcal{R}$
2. $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\pitchfork}$.

Weak factorization systems arose in the context of homotopy theory. When working with topological spaces, or other structures with a notion of homotopy, we often want to use homotopy rather than isomorphism as the most important form of equality. In category theory, this preference can be implemented via *localization* – that is, forcing all homotopies to be isomorphisms by adding formal inverses. However, the resulting localized categories can be hard to work with, and even basic results about them tend to require additional properties and/or structure. Quillen [6] introduced the additional structure of a *model category* in order to categorically state and prove the homotopy-theoretic equivalence of topological spaces and simplicial sets.

Quillen model categories can be thought of as the categorical contexts for doing homotopy theory. A Quillen model structure on a category consists of three classes of morphisms: *weak equivalences* (analogous to homotopies), *fibrations*, and *cofibrations*. These three classes form two interacting weak factorization systems.

In a weak factorization system, both the factorizations and the lifts are guaranteed to exist, but they are not necessarily unique, and are not in general chosen naturally. To address this, we will impose additional structure, first in the form of functorial factorizations, and then, following Grandis and Tholen [1], in the form of monad-comonad pairs, leading to the definition of an algebraic weak factorization system. For more on the development of algebraic weak factorization systems, see Rosický and Tholen [4] and Bourke and Garner [5], among others.

We begin by introducing some notation. For a category $\mathcal{C}$, we denote by $\mathcal{C}^{\to}$ the arrow category of $\mathcal{C}$, whose objects are morphisms of $\mathcal{C}$, and whose morphisms are commutative squares of $\mathcal{C}$. Similarly, $\mathcal{C}^{\to\to}$ is the category whose objects are composable pairs of morphisms in $\mathcal{C}$, and whose morphisms are triples of morphisms in $\mathcal{C}$ forming commutative diagrams of the following shape:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

Using these categories, we can consider the composition functor, which we will call $\mathsf{comp} : \mathcal{C}^{\to\to} \to \mathcal{C}^{\to}$. This functor takes an object of $\mathcal{C}^{\to\to}$ to the composition of its two composable morphisms, and takes a morphism of $\mathcal{C}^{\to\to}$ to the commutative square formed by the first and third of its component morphisms.

Now we can define a more explicit version of the factorization that appears in a weak factorization system.

Definition 3. A *functorial factorization* is a functor $T : \mathcal{C}^{\to} \to \mathcal{C}^{\to\to}$ that is a section of the composition functor comp.

We will often refer to a functorial factorization T in terms of its component functors (L, E, R). The functors $L, R : \mathcal{C}^{\to} \to \mathcal{C}^{\to}$ give the left and right halves of the factorization, and $E : \mathcal{C}^{\to} \to \mathcal{C}$ gives the object that a morphism factors through. That is, $T = (L, E, R)$ takes a commutative square to a commutative rectangle as follows:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] & & A \arrow[r, "u"] \arrow[d, "Lf"'] & X \arrow[d, "Lg"] \\
B \arrow[r, "v"'] & Y & \overset{T}{\longmapsto} & Ef \arrow[r, "E(u;v)"] \arrow[d, "Rf"'] & Eg \arrow[d, "Rg"] \\
& & & B \arrow[r, "v"] & Y.
\end{tikzcd}$$

We also want to define a more explicit version of the lifting that appears in a weak factorization system. It turns out that we can do this by fixing, for every map f in $\mathcal{L}$, a lift of f against Rf, and for every map g in $\mathcal{R}$, a lift of Lg against g.

$$
\begin{tikzcd}
A \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[d, "Rf"] \\
B \arrow[r, "1_B"'] \arrow[ur, dotted, "s"] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[d, "Lg"'] & X \arrow[d, "g"] \\
Eg \arrow[r, "Rg"'] \arrow[ur, dotted, "t"] & Y
\end{tikzcd}
$$

If we assume that each map in $\mathcal{L}$ or $\mathcal{R}$ has this additional structure of a distinguished lift against its own factorization, then we can construct by a general formula a solution to any lifting problem of a map in $\mathcal{L}$ against in $\mathcal{R}$. Suppose that as above, f is a map in $\mathcal{L}$ with a distinguished lift s, and g is a map in $\mathcal{R}$ with a distinguished lift t. Then any lifting problem $(u;v)$ of f against g

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\
B \arrow[r, "v"'] & Y
\end{tikzcd}
$$

has the solution $t \circ E(u;v) \circ s$

$$
\begin{tikzcd}
 & A \arrow[r, "u"] \arrow[d, "Lf"'] & X \arrow[r, "1_X"] \arrow[d, "Lg"'] & X \arrow[d, "g"] \\
A \arrow[ur, "1_A"] \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[r, dotted, "E(u;v)"] \arrow[d, "Rf"] & Eg \arrow[r, "Rg"'] \arrow[ur, dotted, "t"] \arrow[d, "Rg"'] & Y \\
B \arrow[r, "1_B"'] \arrow[ur, dotted, "s"] & B \arrow[r, "v"'] & Y \arrow[ur, "1_Y"'] &
\end{tikzcd}
$$

which we call the *canonical lift*.

We would like to describe these $\mathcal{L}$ and $\mathcal{R}$ maps and their distinguished lifts more categorically, and to do this we will consider how they relate to the components L and R of a functorial factorization. Both L and R are endofunctors on $\mathcal{C}^{\to}$, and they come with a *copointing* and a *pointing* respectively; that is, a natural transformation $\varepsilon : L \Rightarrow 1_{\mathcal{C}^{\to}}$ and a natural transformation $\eta : 1_{\mathcal{C}^{\to}} \Rightarrow R$.

$$
\begin{array}{cc}
\eta_f & \varepsilon_f \\
\begin{tikzcd}
A \arrow[r, "Lf"'] \arrow[d, "f"'] & Ef \arrow[d, "Rf"] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}
&
\begin{tikzcd}
A \arrow[r, "1_A"'] \arrow[d, "Lf"'] & A \arrow[d, "f"] \\
Ef \arrow[r, "Rf"'] & B
\end{tikzcd}
\end{array}
$$

Now consider an *algebra* for the pointed endofunctor (R, η). An algebra for (R, η) consists of an element f of $\mathcal{C}^{\to}$, plus a morphism $(m; n) : Rf \to f$, such that $(m; n) \circ \eta_f = 1_f$. In a diagram,

$$\begin{tikzcd}
A \arrow[r, "Lf"'] \arrow[d, "f"'] \arrow[rr, bend left, "1_A"] & Ef \arrow[r, dotted, "m"'] \arrow[d, "Rf"] & A \arrow[d, "f"] \\
B \arrow[r, "1_B"] \arrow[rr, bend right, "1_B"'] & B \arrow[r, dotted, "n"] & B.
\end{tikzcd}$$

Clearly this diagram forces $n = 1_B$, so finding an algebra structure for the map f reduces to finding a map m such that $m \circ Lf = 1_A$ and $f \circ m = Rf$. These are exactly the conditions for m being a solution to the lifting problem $(Lf; 1_B)$ of f against Rf, so the notion of an algebra for the pointed endofunctor can serve as a more categorical replacement for our earlier notion of distinguished lifts.

Definition 4. Given a functorial factorization $T = (L, E, R)$, we define the category $\mathcal{R}$ to be the category of algebras for the pointed endofunctor (R, η). That is, an object of $\mathcal{R}$ is an object $f : A \to B$ of $\mathcal{C}^{\to}$, plus an arrow $m : Ef \to A$ of $\mathcal{C}$ such that $m \circ Lf = 1_A$ and $f \circ m = Rf$; a morphism of $\mathcal{R}$ from (f, m) to (f', m') is a morphism $(u; v) : f \to f'$ of $\mathcal{C}^{\to}$ such that $m' \circ E(u; v) = u \circ m$.

Dually, we define $\mathcal{L}$ to be the category of coalgebras for the copointed endofunctor (L, ε).

For both $\mathcal{R}$ and $\mathcal{L}$, we may use the notation U for the obvious forgetful functor $\mathcal{R} \to \mathcal{C}^{\to}$ or $\mathcal{L} \to \mathcal{C}^{\to}$.

Now we have discussed explicit factorizations, and categories $\mathcal{L}$ and $\mathcal{R}$ whose objects have explicit lifts against one another. One property that is still missing an explicit analogue is the requirement that the left and right parts of a factorization should in fact fall into the left and right classes of maps (respectively). That is, for any arrow f of $\mathcal{C}$, we would like Lf to be an object of $\mathcal{L}$, and Rf to be an object of $\mathcal{R}$. We will now define an explicit structure that will ensure that this property always holds.

Definition 5. For an endofunctor R with a pointing η, a *multiplication* is a natural transformation $\mu : RR \Rightarrow R$ that satisfies

- $\mu \circ R\mu = \mu \circ \mu R$ (*associativity*) and
- $\mu \circ R\eta = \mu \circ \eta R = 1_R$ (*unit conditions*).

Dually, for an endofunctor L with a copointing ε, a *comultiplication* is a natural transformation $\delta : L \Rightarrow LL$ that satisfies

- $L\delta \circ \delta = \delta L \circ \delta$ (*coassociativity*) and
- $L\varepsilon \circ \delta = \varepsilon L \circ \delta = 1_L$ (*counit conditions*).

Definition 6. A triple (R, η, μ) such that μ is a multiplication for (R, η) is called a *monad*, and a triple (L, ε, δ) such that δ is a comultiplication for (L, ε) is called a *comonad.*

With all this in mind, we are ready to define a fully explicit analogue of a weak factorization system.

Definition 7. An *algebraic weak factorization system* on a category $\mathcal{C}$ consists of:

- a functorial factorization $T = (L, E, R) : \mathcal{C}^{\to} \to \mathcal{C}^{\to\to}$
- a multiplication μ for (R, η)
- a comultiplication δ for (L, ε).

Remark. Definitions of algebraic weak factorization systems sometimes also require the natural transformation $LR \Rightarrow RL$ composed of the squares

$$\begin{tikzcd} \bullet \arrow[r, "\delta_f"] \arrow[d, "LRf"'] & \bullet \arrow[d, "RLf"] \\ \bullet \arrow[r, "\mu_f"] & \bullet \end{tikzcd}$$

to be a distributive law of the comonad over the monad. Since we will not use this requirement in what follows, we omit it for simplicity.

Proposition 1. *If $(T = (L, E, R), \mu, \varepsilon)$ is an algebraic weak factorization system on a category $\mathcal{C}$, then $(ob(U\mathcal{L}), ob(U\mathcal{R}))$ is a weak factorization system on $\mathcal{C}$.*

Proof. First we want to show that every morphism f of $\mathcal{C}$ factors as a map from the left class followed by a map from the right class. Since T is a functorial factorization, it is a section of the composition functor, so $f = (\mathsf{comp} \circ T)(f) = \mathsf{comp}(Lf; Rf) = Rf \circ Lf$. It remains to show that Rf carries an object of $\mathcal{R}$ and Lf carries an object of $\mathcal{L}$; as promised, we can use the (co)multiplication to do this.

The multiplication μ gives a map $\mu_f := (m; n)$ from RRf to Rf. By one of the unit conditions, we have $\mu_f \circ \eta_{Rf} = 1_{Rf}$:

$$\begin{tikzcd} Ef \arrow[r, "Lf"'] \arrow[rr, bend left, "1_{Ef}"] \arrow[d, "Rf"'] & ERf \arrow[r, "m"'] \arrow[d, "RRf"] & Ef \arrow[d, "Rf"] \\ B \arrow[r, "1_B"] \arrow[rr, bend right, "1_B"'] & B \arrow[r, "n"] & B. \end{tikzcd}$$

This diagram shows that we have $m \circ Lf = 1_{Ef}$. Furthermore, it shows that we must have $n = 1_B$, so also $Rf \circ m = RRF$. These two conditions show that (Rf, m) is an object of $\mathcal{R}$.

The proof that Lf carries an object of $\mathcal{L}$ is dual.

Now we want to show that $\mathrm{ob}(U\mathcal{L}) = {}^{\pitchfork}\mathrm{ob}(U\mathcal{R})$ and $\mathrm{ob}(U\mathcal{R}) = \mathrm{ob}(U\mathcal{L})^{\pitchfork}$. The canonical lift shows that every lifting problem of an object of $\mathcal{L}$ against an object of $\mathcal{R}$ has a solution; that is, $\mathrm{ob}(U\mathcal{L}) \subseteq {}^{\pitchfork}\mathrm{ob}(U\mathcal{R})$ and $\mathrm{ob}(U\mathcal{R}) \subseteq \mathrm{ob}(U\mathcal{L})^{\pitchfork}$. Now suppose that $f \in {}^{\pitchfork}\mathrm{ob}(U\mathcal{R})$; that is, any lifting problem of f against a carrier for an object of $\mathcal{R}$ has a solution. Then in particular, since we showed above that Rf carries an object of $\mathcal{R}$, the following lifting problem has some solution s:

$$\begin{tikzcd} A \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[d, "Rf"] \\ B \arrow[r, "1_B"'] \arrow[ur, dotted, "s"] & B. \end{tikzcd}$$

Since $Rf \circ s = 1$ and $s \circ f = Lf$, by definition 4 (f, s) is an element of $\mathcal{L}$. This shows that ${}^{\pitchfork}\mathrm{ob}(U\mathcal{R}) \subseteq \mathrm{ob}(U\mathcal{L})$, and the proof that $\mathrm{ob}(U\mathcal{L})^{\pitchfork} \subseteq \mathrm{ob}(U\mathcal{R})$ is dual. We may conclude that $\mathrm{ob}(U\mathcal{L}) = {}^{\pitchfork}\mathrm{ob}(U\mathcal{R})$ and $\mathrm{ob}(U\mathcal{R}) = \mathrm{ob}(U\mathcal{L})^{\pitchfork}$, as desired. $\square$

3 Relative monads and comonads

In this chapter, we introduce the concept of a *relative monad*, following Altenkirch, Chapman, and Uustalu [2]. This concept will allow us to handle the shape mismatch that we encounter when trying to incorporate two-sided factorizations into some kind of factorization system.

First we introduce an alternative presentation of a monad, sometimes called a *Kleisli extension system*, originally developed by Kleisli [3] in the course of proving that every monad arises from an adjunction.

Definition 8. A *Kleisli extension system* on a category $\mathcal{C}$ consists of:

- an endofunctor $T : \mathcal{C} \to \mathcal{C}$
- a natural transformation $\eta : 1_{\mathcal{C}} \Rightarrow T$, called the *unit*
- an operation $(-)^*$ called the *Kleisli extension*, giving for every morphism $k : A \to TB$ a morphism $k^* : TA \to TB$

satisfying

1. for every object A, $(\eta_A)^* = 1_{TA}$
2. for every morphism $f : A \to B$, $(\eta_B \circ f)^* = Tf$
3. whenever $k : A \to TB$, $k^* \circ \eta_A = k$
4. whenever $k : A \to TB$ and $\ell : B \to TC$, $\ell^* \circ k^* = (\ell^* \circ k)^*$.

Dually, a *Kleisli coextension system* on $\mathcal{C}$ consists of:

- an endofunctor $T : \mathcal{C} \to \mathcal{C}$
- a natural transformation $\varepsilon : T \Rightarrow 1_{\mathcal{C}}$, called the *counit*
- an operation $(-)_*$ called the *Kleisli coextension*, giving for every morphism $k : TA \to B$ a morphism $k_* : TA \to TB$

satisfying

1. for every object A, $(\varepsilon_A)_* = 1_{TA}$
2. for every morphism $f : A \to B$, $(f \circ \varepsilon_A)_* = Tf$
3. whenever $k : TA \to B$, $\varepsilon_B \circ k_* = k$
4. whenever $k : TA \to B$ and $\ell : TB \to C$, $\ell_* \circ k_* = (\ell \circ k_*)_*$.

Notably, unlike the presentation of a monad given in definition 6, the definition of a Kleisli extension system does not involve iteration of the endofunctor T. This feature of the Kleisli presentation is sometimes useful, particularly when studying monads in the context of computer science. The two presentations are equivalent, as we verify now.

Proposition 2. *There is an isomorphism between the monads of a category $\mathcal{C}$ and the Kleisli extension systems on $\mathcal{C}$.*

Dually, there is an isomorphism between the comonads of $\mathcal{C}$ and the Kleisli coextension systems on $\mathcal{C}$.

Proof. First, suppose that $(T, \eta, (-)^*)$ is a Kleisli extension system. Then we can construct a multiplication for (T, η) by $\mu_A := (1_{TA})^*$. Checking that these maps form a natural transformation, we see that indeed for any $f : A \to B$ we have

$$\begin{aligned} Tf \circ (1_{TA})^* &= (\eta_B \circ f)^* \circ (1_{TA})^* \\ &= ((\eta_B \circ f)^* \circ 1_{TA})^* \\ &= ((\eta_B \circ f)^*)^* \end{aligned} \qquad \begin{aligned} (1_{TB})^* \circ TTf &= (1_{TB})^* \circ (\eta_{TB} \circ (\eta_B \circ f)^*)^* \\ &= ((1_{TB})^* \circ \eta_{TB} \circ (\eta_B \circ f)^*)^* \\ &= (1_{TB} \circ (\eta_B \circ f)^*)^* \\ &= ((\eta_B \circ f)^*)^*. \end{aligned}$$

Then we want to check the associativity and unit conditions (see definition 5). We find that for any object A,

$$\begin{aligned} \mu_A \circ T(\mu_A) &= (1_{TA})^* \circ T((1_{TA})^*) \\ &= (1_{TA})^* \circ (\eta_{TA} \circ (1_{TA})^*)^* \\ &= ((1_{TA})^* \circ \eta_{TA} \circ (1_{TA})^*)^* \\ &= (1_{TA} \circ (1_{TA})^*)^* \\ &= ((1_{TA})^*)^* \end{aligned} \qquad \begin{aligned} \mu_A \circ \mu_{TA} &= (1_{TA})^* \circ (1_{TTA})^* \\ &= ((1_{TA})^* \circ 1_{TTA})^* \\ &= ((1_{TA})^*)^*, \end{aligned}$$

establishing associativity, and

$$\mu_A \circ \eta_{TA} = (1_{TA})^* \circ \eta_{TA} \qquad \mu_A \circ T(\eta_A) = (1_{TA})^* \circ (\eta_{TA} \circ \eta_A)^*$$

$$= 1_{TA} \qquad\qquad\qquad = ((1_{TA})^* \circ \eta_{TA} \circ \eta_A)^*$$

$$= (1_{TA} \circ \eta_A)^*$$

$$= (\eta_A)^*$$

$$= 1_{TA},$$

establishing the unit conditions.

Now suppose instead that (T, η, μ) is a monad. We can construct a Kleisli extension $(-)^*$ by letting $k^* := \mu_B \circ Tk$, for every $k : A \to TB$. Then we simply check the four conditions of a Kleisli extension system.

$$1. \qquad \begin{aligned}(\eta_A)^* &= \mu_A \circ T(\eta_A)\\ &= 1_{TA}.\end{aligned}$$

$$2. \qquad \begin{aligned}(\eta_B \circ f)^* &= \mu_B \circ T(\eta_B \circ f)\\ &= \mu_B \circ T(\eta_B) \circ Tf\\ &= 1_{TB} \circ Tf\\ &= Tf.\end{aligned}$$

$$3. \qquad \begin{aligned}k^* \circ \eta_A &= \mu_B \circ Tk \circ \eta_A\\ &= \mu_B \circ \eta_{TB} \circ k\\ &= 1_{TB} \circ k\\ &= k.\end{aligned}$$

$$4. \qquad \begin{aligned}(\ell^* \circ k)^* &= \mu_C \circ T(\ell^* \circ k)\\ &= \mu_C \circ T(\mu_C \circ T\ell) \circ Tk\\ &= \mu_C \circ T\mu_C \circ TT\ell \circ Tk\\ &= \mu_C \circ \mu_{TC} \circ TT\ell \circ Tk\\ &= \mu_C \circ T\ell \circ \mu_B \circ Tk\\ &= \ell^* \circ k^*.\end{aligned}$$

Lastly, note that these constructions are inverses. Starting from a Kleisli extension system $(T, \eta, (-)^*)$, constructing a monad (T, η, μ) and thence a Kleisli extension system $(T, \eta, (-)^{*'})$, we find that for any $k : A \to TB$, we have

$$\begin{aligned}k^{*'} &= \mu_B \circ Tk\\ &= (1_{TB})^* \circ (\eta_{TB} \circ k)^*\\ &= ((1_{TB})^* \circ \eta_{TB} \circ k)^*\\ &= k^*.\end{aligned}$$

Meanwhile, starting from a monad (T, η, μ), constructing a Kleisli extension system $(T, \eta, (-)^*)$ and thence a monad (T, η, μ'), we find that for any object A, $\mu'_A = (1_{TA})^* = \mu_A \circ T(1_{TA}) = \mu_A$, completing the proof. □

Now that we have a presentation of a monad which does not require iteration of an endofunctor, is it possible that the underlying functor need not be an endofunctor at all? In their paper "Monads need not be endofunctors" [2], Altenkirch, Chapman, and Uustalu give the motivating example of finite-dimensional vector spaces. They note that the finite-dimensional vector spaces on a semiring R can be given by:

- For every object m of the category **Fin** of finite cardinals, an object $\mathsf{Vec}(m)$ of **Set**, namely the set of functions from Jm to R, where J is the inclusion $\mathbf{Fin} \hookrightarrow \mathbf{Set}$.
- For every m in **Fin**, a morphism of sets $\eta_m : Jm \to \mathsf{Vec}(m)$, namely the function taking $i \in Jm$ to the ith basis vector.
- For every morphism of sets $k : Jm \to \mathsf{Vec}(n)$, a morphism of sets $k^* : \mathsf{Vec}(m) \to \mathsf{Vec}(n)$, namely the function corresponding to multiplication by the matrix k.

The authors remark that $(\mathsf{Vec}, \eta, (-)^*)$ resembles a Kleisli extension system, except that in place of an endofunctor, Vec forms a functor $\mathbf{Fin} \to \mathbf{Set}$, and the parallel functor $J : \mathbf{Fin} \to \mathbf{Set}$ is used in place of the identity, to repair the resulting mismatches in the types of η and $(-)^*$. The authors refer to this situation as Vec carrying a monad *relative to* the inclusion J, and from this starting point they develop a theory of *relative monads*.

From here on, we will use the following general definitions:

Definition 9. A *relative monad* on a functor $I : \mathcal{I} \to \mathcal{C}$ consists of

- a functor $T : \mathcal{I} \to \mathcal{C}$
- a natural transformation $\eta : I \Rightarrow T$, the *unit*
- a Kleisli extension $(-)^*$, giving for each morphism $k : IA \to TB$ in $\mathcal{C}$, a morphism $k^* : TA \to TB$

satisfying

1. for every object A in $\mathcal{I}$, $(\eta_A)^* = 1_{TA}$
2. for every morphism $f : A \to B$ in $\mathcal{I}$, $(\eta_B \circ If)^* = Tf$
3. whenever $k : IA \to TB$, $k^* \circ \eta_A = k$
4. whenever $k : IA \to TB$ and $\ell : IB \to TC$, $(\ell^* \circ k)^* = \ell^* \circ k^*$.

Dually, a *relative comonad* on a functor $J : \mathcal{J} \to \mathcal{C}$ consists of

- a functor $T : \mathcal{J} \to \mathcal{C}$
- a natural transformation $\varepsilon : T \Rightarrow J$, the *counit*

- a Kleisli coextension $(-)_*$, giving for each morphism $k : TB \to JA$ in $\mathcal{C}$, a morphism $k_* : TB \to TA$

satisfying

1. for every object A in $\mathcal{J}$, $(\varepsilon_A)_* = 1_{TA}$
2. for every morphism $f : A \to B$ in $\mathcal{J}$, $(Jf \circ \varepsilon_A)_* = Tf$
3. whenever $k : TB \to JA$, $\varepsilon_A \circ k_* = k$
4. whenever $k : TB \to JA$ and $\ell : TC \to JB$, $(k \circ \ell_*)_* = k_* \circ \ell_*$.

We can also define algebras and coalgebras for these relative monads and comonads:

Definition 10. An *algebra* for $(T, \eta, (-)^*)$ consists of an object X in $\mathcal{C}$ and an operation χ which gives, for each morphism $k : IA \to X$ in $\mathcal{C}$, a morphism $\chi k : TA \to X$, such that (1) for every A in $\mathcal{I}$ and $k : IA \to X$, $\chi k \circ \eta_A = k$, and also (2) whenever $k : IA \to X$ and $\ell : IB \to TA$, $\chi(\chi k \circ \ell) = \chi k \circ \ell^*$.

Dually, a *coalgebra* for $(T, \varepsilon, (-)_*)$ consists of an object X in $\mathcal{C}$ and an operation ψ which gives, for each morphism $k : X \to JA$ in $\mathcal{C}$, a morphism $\psi k : X \to TA$, such that (1) for every A in $\mathcal{J}$ and $k : X \to JA$, $\varepsilon_A \circ \psi k = k$, and also (2) whenever $k : X \to JA$ and $\ell : TA \to JB$, $\psi(\ell \circ \psi k) = \ell_* \circ \psi k$.

For the purposes of relative weak factorizations systems, we will often be interested in a stripped-down version of these relative definitions, corresponding to the concepts of pointed and copointed endofunctors.

Definition 11. A *relative pointing* for a functor $T : \mathcal{I} \to \mathcal{C}$, relative to a functor $I : \mathcal{I} \to \mathcal{C}$, is a unit $\eta : I \Rightarrow T$.

Dually, a *relative copointing* for a functor $T : \mathcal{J} \to \mathcal{C}$, relative to a functor $J : \mathcal{J} \to \mathcal{C}$, is a counit $\varepsilon : T \Rightarrow J$.

Definition 12. An *algebra for the relative pointing* (T, η) consists of

- an object X in $\mathcal{C}$, together with
- a section χ of $\eta_A^* : \hom_{\mathcal{C}}(TA, X) \to \hom_C(IA, X)$ which is natural in A.

Dually, a *coalgebra for the relative copointing* (T, ε) consists of

- an object X in $\mathcal{C}$, together with
- a section ψ of $(\varepsilon_A)_* : \hom_{\mathcal{D}}(X, TA) \to \hom_{\mathcal{D}}(X, JA)$ which is natural in A.

Note that requiring χ to be a section of $\eta_A^* : \hom_{\mathcal{C}}(TA, X) \to \hom_C(IA, X)$ is equivalent to requiring χ to be an operation which gives, for each morphism $k : IA \to X$ in $\mathcal{C}$, a morphism $\chi k : TA \to X$, such that for every A in $\mathcal{I}$ and $k : IA \to X$, $\chi k \circ \eta_A = k$. Then we have additionally required that this section be natural in A. Naturality of the operations χ and ψ is implied by the definitions of algebras for relative monads and coalgebras for relative comonads (definition 10), but must be explicitly assumed when defining algebras for relative pointings and coalgebras for relative copointings.

4 Relative weak factorization systems for composable pairs

In this chapter we will begin to explore the idea of a *relative weak factorization system*. A relative weak factorization system will be a structure that generalizes the idea of a weak factorization system, whose development is guided by trying to incorporate relative (co)monads into weak factorization systems, analogously to the way that the theory of algebraic weak factorization systems incorporates ordinary (co)monads into weak factorization systems.

We begin by considering factorizations of arrows in a category $\mathcal{C}$ into composable pairs. Suppose we have a functorial factorization $T = (L, E, R) : \mathcal{C}^{\to} \to \mathcal{C}^{\to\to}$. We will consider T *relative* to two trivial factorization functors, which we will call I and J. I maps an arrow f of $\mathcal{C}$ to f after an identity map:

$$A \xrightarrow{f} B \qquad \overset{I}{\mapsto} \qquad A \xrightarrow{1_A} A \xrightarrow{f} B$$

which we denote by $I : f \mapsto (1; f)$. Dually, J maps f to f followed by an identity map, denoted $(f; 1)$. I and J act on morphisms of $\mathcal{C}^{\to}$, i.e. commutative squares of $\mathcal{C}$, in the obvious way:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] & Y \end{tikzcd} \qquad \overset{I}{\mapsto} \qquad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "1_A"'] & X \arrow[d, "1_X"] \\ A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"] & Y \end{tikzcd}$$

and J dually. These trivial factorizations I and J prove to be left and right adjoints of the composition functor $\mathsf{comp} : \mathcal{C}^{\to\to} \to \mathcal{C}^{\to}$.

$$\begin{tikzcd} \mathcal{C}^{\to} \arrow[rr, bend left, "I"] \arrow[rr, bend right, "J"'] & & \mathcal{C}^{\to\to} \arrow[ll, "\mathsf{comp}"'] \end{tikzcd}$$

Proposition 3. *I is left adjoint to* comp*, with $1_{1_{\mathcal{C}^{\to}}}$ as the unit of the adjunction.*

Dually, J is right inverse to comp*, with $1_{1_{\mathcal{C}^{\to}}}$ as the counit of the adjunction.*

Proof. First, note that I is a factorization, in the sense of being a section of comp, and therefore $1_{1_{\mathcal{C}^{\to}}}$ is indeed a natural transformation from $1_{\mathcal{C}^{\to}}$ to $\mathsf{comp} \circ I = 1_{\mathcal{C}^{\to}}$, and can serve as the unit of the adjunction.

For the counit of the adjunction, we need a natural transformation $\varepsilon' : I \circ \mathsf{comp} \Rightarrow 1_{\mathcal{C}^{\to}}$. Let ε' be composed of the following maps from $I(\mathsf{comp}(f;g)$ to $(f;g)$:

$$\varepsilon'_{(f;g)}$$

$$\begin{tikzcd} A \arrow[r, "1_A"'] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] \arrow[d, "g\circ f"'] & B \arrow[d, "g"] \\ C \arrow[r, "1_C"'] & C \end{tikzcd}$$

To check that these maps form a natural transformation, we verify that for any map $(\alpha, \beta, \gamma) : (f, g) \to (m, n)$ of composable pairs,

$$\begin{tikzcd} A \arrow[r, "1_A"'] \arrow[d, "1_A"'] & A \arrow[r, "\alpha"'] \arrow[d, "f"] & X \arrow[d, "m"] \\ A \arrow[r, "f"'] \arrow[d, "g\circ f"'] & B \arrow[r, "\beta"'] \arrow[d, "g"] & Y \arrow[d, "n"] \\ C \arrow[r, "1_C"'] & C \arrow[r, "\gamma"'] & Z \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "\alpha"'] \arrow[d, "1_A"'] & X \arrow[r, "1_X"'] \arrow[d, "f"] & X \arrow[d, "m"] \\ A \arrow[r, "\alpha"'] \arrow[d, "g\circ f"'] & X \arrow[r, "m"'] \arrow[d, "g"] & Y \arrow[d, "n"] \\ C \arrow[r, "\gamma"'] & Z \arrow[r, "1_Z"'] & Z \end{tikzcd}$$

which holds because $\beta \circ f = m \circ \alpha$ is a requirement of $(\alpha, \beta, \gamma) : (f, g) \to (m, n)$ being a map of composable pairs.

It remains only to show the triangle identities. Since the unit is an identity natural transformation, the desired triangle identities simplify to $\varepsilon'_{If} = 1_{If}$ and $\mathsf{comp}(\varepsilon'_{(f;g)}) = 1_{\mathsf{comp}(f;g)}$. And indeed, both ε'_{If} and 1_{If} are the map

$$\begin{tikzcd} A \arrow[r, "1_A"'] \arrow[d, "1_A"'] & A \arrow[d, "1_A"] \\ A \arrow[r, "1_A"'] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, "1_B"'] & B \end{tikzcd}$$

while both $\mathsf{comp}(\varepsilon'_{(f;g)})$ and $1_{\mathsf{comp}(f;g)}$ are simply the map

$$\begin{tikzcd} A \arrow[r, "1_A"'] \arrow[d, "g\circ f"'] & A \arrow[d, "g\circ f"] \\ C \arrow[r, "1_C"'] & C \end{tikzcd}$$

completing the proof. $\square$

The ordinary definition of an algebraic weak factorization system uses the fact that a functorial factorization automatically comes equipped with a pointing $1 \Rightarrow R$ and a copointing $L \Rightarrow 1$. Here, we use the fact that a functorial factorization also automatically comes equipped with a relative pointing $\eta : I \Rightarrow T$, for T relative to I, and a relative copointing $\varepsilon : T \Rightarrow J$, for T relative to J. Namely, η and ε are given by

$$\eta_f \quad \begin{tikzcd} A \arrow[r, "1"'] \arrow[d, "1"'] & A \arrow[d, "Lf"] \\ A \arrow[r, "Lf"'] \arrow[d, "f"'] & Ef \arrow[d, "Rf"] \\ B \arrow[r, "1"'] & B \end{tikzcd} \qquad \varepsilon_f \quad \begin{tikzcd} A \arrow[r, "1"'] \arrow[d, "Lf"'] & A \arrow[d, "f"] \\ Ef \arrow[r, "Rf"'] \arrow[d, "Rf"'] & B \arrow[d, "1"] \\ B \arrow[r, "1"'] & B \end{tikzcd}$$

From this, we can already define left and right classes of maps, as follows:

Definition 13. We define a *fibration* in this setting to be an algebra for (T, η).
Dually, we define a *cofibration* to be a coalgebra for (T, ε).

That is, a fibration is a composable pair $(m; n)$ together with an operation χ, and χ gives, for any morphism of composable pairs $k : If \to (m; n)$, a morphism natural in f of composable pairs $\chi k : Tf \to (m; n)$, satisfying $\chi k \circ \eta_f = k$.

Remark. Here we are using the terms *fibration* and *cofibration* in the general sense of "right map" and "left map"; they are not the fibrations and cofibrations of a particular Quillen model structure.

Let's investigate the consequences of this definition. For $(m; n)$ to be a fibration, we are demanding that whenever we have a map of the form

$$\begin{tikzcd} A \arrow[r, "k_0"'] \arrow[d, "1"'] & X \arrow[d, "m"] \\ A \arrow[r, "k_1"'] \arrow[d, "f"'] & Y \arrow[d, "n"] \\ B \arrow[r, "k_2"'] & Z \end{tikzcd}$$

we can construct a map of the form

$$\begin{tikzcd} A \arrow[r, "\chi k_0"'] \arrow[d, "Lf"'] & X \arrow[d, "m"] \\ Ef \arrow[r, "\chi k_1"'] \arrow[d, "Rf"'] & Y \arrow[d, "n"] \\ B \arrow[r, "\chi k_2"'] & Z \end{tikzcd}$$

Also, this construction must satisfy $\chi k \circ \eta_f = k$:

$$
\begin{tikzcd}
 & \eta_f & & \chi k & \\
A \arrow[r, "1"'] \arrow[d, "1"'] & A \arrow[r, "\chi k_0"'] \arrow[d, "Lf"'] & X \arrow[d, "m"] \\
A \arrow[r, "Lf"'] \arrow[d, "f"'] & Ef \arrow[r, "\chi k_1"'] \arrow[d, "Rf"'] & Y \arrow[d, "n"] \\
B \arrow[r, "1"'] & B \arrow[r, "\chi k_2"'] & Z
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & k & \\
A \arrow[r, "k_0"'] \arrow[d, "1"'] & X \arrow[d, "m"] \\
A \arrow[r, "k_1"'] \arrow[d, "f"'] & Y \arrow[d, "n"] \\
B \arrow[r, "k_2"'] & Z
\end{tikzcd}
$$

so in particular, $\chi k_0 = k_0$ and $\chi k_2 = k_2$, with only χk_1 left to be constructed. Furthermore, the construction can depend only on k_0 and k_2, because already in the diagram of k, k_1 is determined by $k_1 = m \circ k_0$. Therefore, constructing χk from k amounts to finding, naturally in f, a map χk_1 that fills the following diagram:

$$
\begin{tikzcd}
A \arrow[r, "k_0"'] \arrow[d, "Lf"'] & X \arrow[d, "m"] \\
Ef \arrow[r, dotted] \arrow[d, "Rf"'] & Y \arrow[d, "n"] \\
B \arrow[r, "k_2"'] & Z
\end{tikzcd}
$$

Note that any map χk_1 filling this diagram satisfies $\chi k_1 \circ Lf = m \circ k_0 = k_1$, so $\chi k := (k_0, \chi k_1, k_2)$ always satisfies $\chi k \circ \eta_f = k$.

Diagrams of the above shape will play a role in our theory analogous to the role played by square lifting problems in the ordinary theory of algebraic weak factorization systems. For this reason, we will call a diagram like this a lifting problem, and call a solution to it a lift (even though we draw the desired arrow horizontally). Thus, we can think of χ as a system of solutions to lifting problems of a particular form.

Proposition 4. *A composable pair $(m; n)$ has a fibration structure if and only if the lifting problem $(1; 1)$ of $T(n \circ m)$ on the left against $(m; n)$ on the right has a solution.*

Dually, a composable pair $(u; v)$ has a cofibration structure if and only if the lifting problem $(1; 1)$ of $(u; v)$ on the left against $T(n \circ m)$ on the right has a solution.

Proof. Suppose $(m; n)$ has a fibration structure χ. Then for any morphism of composable pairs $k : If \to (m; n)$, there is a morphism of composable pairs $\chi k : Tf \to (m; n)$, satisfying $\chi k \circ \eta_f = k$. As discussed above, χ will always satisfy $\chi k_0 = k_0$ and $\chi k_2 = k_2$, and so χk_1 will be a solution to the lifting problem $(k_0; k_2)$ of Tf against $(m; n)$. Therefore, since $(1, m, 1) : I(n \circ m) \to (m; n)$

is a map of composable pairs, χm is a solution to the lifting problem $(1;1)$ of $T(n \circ m)$ against $(m;n)$.

$$
\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "1"'] & X \arrow[d, "m"] \\
X \arrow[r, "m"] \arrow[d, "m\circ n"'] & Y \arrow[d, "n"] \\
Z \arrow[r, "1"] & Z
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "L(n\circ m)"'] & X \arrow[d, "m"] \\
E(n\circ m) \arrow[r, dotted, "\chi m"] \arrow[d, "R(n\circ m)"'] & Y \arrow[d, "n"] \\
Z \arrow[r, "1"] & Z
\end{tikzcd}
$$

Conversely, suppose the lifting problem $(1;1)$ of $T(n \circ m)$ against $(m;n)$ has a solution $\bar{\chi}$.

$$
\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "L(n\circ m)"'] & X \arrow[d, "m"] \\
E(n\circ m) \arrow[r, "\bar{\chi}"] \arrow[d, "R(n\circ m)"'] & Y \arrow[d, "n"] \\
Z \arrow[r, "1"] & Z
\end{tikzcd}
$$

To show that $(m;n)$ has a fibration structure, we want to find a natural solution to every lifting problem of the form

$$
\begin{tikzcd}
A \arrow[r, "k_0"] \arrow[d, "Lf"'] & X \arrow[d, "m"] \\
Ef \arrow[r, dotted] \arrow[d, "Rf"'] & Y \arrow[d, "n"] \\
B \arrow[r, "k_2"] & Z.
\end{tikzcd}
$$

Applying $T \circ \mathsf{comp}$ to this second diagram and combining it with the first, we have

$$
\begin{tikzcd}
A \arrow[r, "k_0"] \arrow[d, "Lf"'] & X \arrow[r, "1"] \arrow[d, "L(n\circ m)"] & X \arrow[d, "m"] \\
Ef \arrow[r, "E(k_0;k_2)"] \arrow[d, "Rf"'] & E(n\circ m) \arrow[r, "\bar{\chi}"] \arrow[d, "R(n\circ m)"] & Y \arrow[d, "n"] \\
B \arrow[r, "k_2"'] & Z \arrow[r, "1"'] & Z
\end{tikzcd}
$$

which shows that $\chi k_1 := \bar{\chi} \circ E(k_0;k_2)$ is a solution to the proposed lifting problem.

It remains to show that this solution is natural in f. Suppose that $(h_0; h_1) : g \to f$ is a morphism of $\mathcal{C}^{\to}$. Then indeed we have

$$\begin{tikzcd}
C \arrow[r, "h_0"] \arrow[d, "Lg"'] & A \arrow[r, "k_0"] \arrow[d, "Lf"'] & X \arrow[d, "m"] \\
E(g) \arrow[r, "E(h_0;h_1)"] \arrow[d, "Rg"'] & E(f) \arrow[r, "\bar{\chi}\circ E(k_0;k_2)"] \arrow[d, "Rf"'] & Y \arrow[d, "n"] \\
B \arrow[r, "h_1"'] & B \arrow[r, "k_2"'] & Z
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
C \arrow[r, "k_0\circ h_0"] \arrow[d, "Lg"'] & X \arrow[d, "m"] \\
E(g) \arrow[r, "\bar{\chi}\circ E(k_0\circ h_0;k_2\circ h_1)"] \arrow[d, "Rg"'] & Y \arrow[d, "n"] \\
D \arrow[r, "k_2\circ h_1"'] & Z
\end{tikzcd}$$

by the functoriality of E. ☐

Proposition 5. *For any arrow f, $Tf = (Lf; Rf)$ has both a fibration structure and a cofibration structure.*

Proof. This follows directly from proposition 4, because the lifting problem $(1; 1)$ of $T(Rf \circ Lf) = Tf = (Lf; Rf)$ on the left against $(Lf; Rf)$ on the right can trivially be filled by an identity map, as can the lifting problem $(1; 1)$ of $(Lf; Rf)$ on the left against $T(Rf \circ Lf) = Tf = (Lf; Rf)$ on the right. ☐

Proposition 6. *Every lifting problem of a fibration against a cofibration has a solution.*

Proof. Suppose that $(m; n)$ has a fibration structure, $(u; v)$ has a cofibration structure, and we have a lifting problem:

$$\begin{tikzcd}
Q \arrow[r, "a_0"] \arrow[d, "u"'] & X \arrow[d, "m"] \\
R \arrow[d, "v"'] & Y \arrow[d, "n"] \\
S \arrow[r, "a_2"] & Z
\end{tikzcd}$$

By proposition 4, there is a lift $\bar{\psi}$ of $(u; v)$ against $T(v \circ u)$ and a lift $\bar{\chi}$ of $T(n \circ m)$ against $(m; n)$:

$$\begin{tikzcd}
Q \arrow[r, "1"] \arrow[d, "u"'] & Q \arrow[d, "L(v\circ u)"] \\
R \arrow[r, "\bar{\psi}"] \arrow[d, "v"'] & E(v\circ u) \arrow[d, "R(v\circ u)"] \\
S \arrow[r, "1"] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "L(n\circ m)"'] & X \arrow[d, "m"] \\
E(n\circ m) \arrow[r, "\bar{\chi}"] \arrow[d, "R(n\circ m)"'] & Y \arrow[d, "n"] \\
Z \arrow[r, "1"] & Z
\end{tikzcd}$$

These maps, combined with $T(a_0; a_2)$, form a solution to the proposed lifting problem.

$$\begin{tikzcd}
Q \arrow[r, "1"] \arrow[d, "u"'] & Q \arrow[r, "a_0"] \arrow[d, "L(v \circ u)"] & X \arrow[r, "1"] \arrow[d, "L(n \circ m)"] & X \arrow[d, "m"] \\
R \arrow[r, "\bar{\psi}"] \arrow[d, "v"'] & E(v \circ u) \arrow[r, "E(a_0; a_2)"] \arrow[d, "R(v \circ u)"] & E(n \circ m) \arrow[r, "\bar{\chi}"] \arrow[d, "R(n \circ m)"] & Y \arrow[d, "n"] \\
S \arrow[r, "1"] & S \arrow[r, "a_2"] & Z \arrow[r, "1"] & Z
\end{tikzcd}$$

□

5 The Kleisli extension and coextension

In the previous chapter, we defined fibrations as algebras for relative pointings, and defined cofibrations as coalgebras for relative copointings. It is natural to wonder whether we should instead consider full relative monads and relative comonads, and their algebras and coalgebras. The answer is that we can, but the Kleisli extensions and coextensions turn out to be trivial, so typically there is no reason to include the extra complication.

Consider the following results:

Proposition 7. *The only Kleisli extension* $(\text{-})^*$ *making* $(T, \eta, (\text{-})^*)$ *a monad relative to* $I : \mathcal{C}^{\to} \to \mathcal{C}^{\to\to}$ *is* $T \circ \mathsf{comp}$.

Dually, the only Kleisli coextension $(\text{-})_*$ *making* $(T, \varepsilon, (\text{-})_*)$ *a comonad relative to* $J : \mathcal{C}^{\to} \to \mathcal{C}^{\to\to}$ *is* $T \circ \mathsf{comp}$.

Proof. A Kleisli extension $(\text{-})^*$ gives, for every map of the form

$$\begin{tikzcd}
A \arrow[r, "k_0"'] \arrow[d, "1"'] & C \arrow[d, "Lg"] \\
A \arrow[r, "k_1"'] \arrow[d, "f"'] & Eg \arrow[d, "Rg"] \\
B \arrow[r, "k_2"'] & D
\end{tikzcd}$$

a map of the form

$$\begin{tikzcd}
A \arrow[r, "k_0^*"'] \arrow[d, "Lf"'] & C \arrow[d, "Lg"] \\
Ef \arrow[r, "k_1^*"'] \arrow[d, "Rf"'] & Eg \arrow[d, "Rg"] \\
B \arrow[r, "k_2^*"'] & D
\end{tikzcd}$$

in such a way that the following conditions are satisfied:

1. $(\eta_f)^* = 1_{Tf}$
2. $(\eta_g \circ Ih)^* = Th$
3. $k^* \circ \eta_f = k$
4. $(\ell^* \circ k)^* = \ell^* \circ k^*$.

Condition (3) requires

$$
\begin{tikzcd}
 & \eta_f & & k^* & \\
A \arrow[rr, "1"'] \arrow[d, "1"'] & & A \arrow[rr, "k_0^*"'] \arrow[d, "Lf"'] & & C \arrow[d, "Lg"] \\
A \arrow[rr, "Lf"'] \arrow[d, "f"'] & & Ef \arrow[rr, "k_1^*"'] \arrow[d, "Rf"'] & & Eg \arrow[d, "Rg"] \\
B \arrow[rr, "1"'] & & B \arrow[rr, "k_2^*"'] & & D
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & k & \\
A \arrow[rr, "k_0"'] \arrow[d, "1"'] & & C \arrow[d, "Lg"] \\
A \arrow[rr, "k_1"'] \arrow[d, "f"'] & & Eg \arrow[d, "Rg"] \\
B \arrow[rr, "k_2"'] & & D
\end{tikzcd}
$$

so in particular it forces $k_0^* = k_0$ and $k_2^* = k_2$, with only k_1^* left to be constructed. Furthermore, the construction depends only on k_0 and k_2, because already in the diagram of k, k_1 is determined by $k_1 = Lg \circ k_0$. Therefore, constructing k^* from k is equivalent to finding a map that fills the following diagram:

$$
\begin{tikzcd}
A \arrow[r, "k_0"'] \arrow[d, "Lf"'] & C \arrow[d, "Lg"] \\
Ef \arrow[r, dotted] \arrow[d, "Rf"'] & Eg \arrow[d, "Rg"] \\
B \arrow[r, "k_2"'] & D
\end{tikzcd}
$$

This shows that the Kleisli extension (-)* is equivalent to a coherent system of solutions to lifting problems of a particular form: whenever $(k_0; k_2) : f \to g$ is a map of arrows, $(k_0; k_2)^* : Ef \to Eg$ is a solution to the lifting problem of Tf against Tg.

Now consider condition (2). Suppose that $(h_0; h_1) : f \to g$ is a morphism of

$\mathcal{C}^{\rightarrow}$. Then condition (2) tells us that

$$\begin{tikzcd}
& Ih & & \eta_g & \\
A \arrow[rr, "h_0"'] \arrow[d, "1"'] & & C \arrow[rr, "1"'] \arrow[d, "1"'] & & C \arrow[d, "Lg"] \\
A \arrow[rr, "h_0"'] \arrow[d, "f"'] & & C \arrow[rr, "Lg"'] \arrow[d, "g"'] & & Eg \arrow[d, "Rg"] \\
B \arrow[rr, "h_1"'] & & D \arrow[rr, "1"'] & & D
\end{tikzcd}
\quad \overset{(\text{-})^*}{\longmapsto} \quad
\begin{tikzcd}
& Th & \\
A \arrow[rr, "h_0"'] \arrow[d, "Lf"'] & & C \arrow[d, "Lg"] \\
Ef \arrow[rr, "E(h_0;h_1)"'] \arrow[d, "Rf"'] & & Eg \arrow[d, "Rg"] \\
B \arrow[rr, "h_1"'] & & D.
\end{tikzcd}$$

That is, the solution $(h_0;h_1)^*$ to the lifting problem $(h_0;h_1)$ must in fact be $(h_0;h_1)^* = E(h_0;h_1)$, for any lifting problem $(h_0;h_1)$.

This requirement completely determines the Kleisli extension. For any map of arrows $(k_0;k_2) : f \to g$, the lifting problem $(k_0;k_2)$ of Tf against Tg must be given the solution $E(k_0;k_2)$; that is, the Kleisli extension must be given by $T \circ \mathsf{comp}$.

$$\begin{tikzcd}
A \arrow[rr, "k_0"] \arrow[d, "Lf"'] & & C \arrow[d, "Lg"] \\
E(f) \arrow[d, "Rf"'] & & E(g) \arrow[d, "Rg"] \\
B \arrow[rr, "k_2"] & & D
\end{tikzcd}
\quad \overset{T \circ \mathsf{comp}}{\longmapsto} \quad
\begin{tikzcd}
A \arrow[rr, "k_0"] \arrow[d, "Lf"'] & & C \arrow[d, "Lg"] \\
E(f) \arrow[rr, "E(k_0;k_2)"] \arrow[d, "Rf"'] & & E(g) \arrow[d, "Rg"] \\
B \arrow[rr, "k_2"] & & D
\end{tikzcd}$$

The remaining conditions that the lift must satisfy are simply $(1,1)^* = 1$, from condition (1) above, and $(\ell_0,\ell_2)^* \circ (k_0,k_2)^* = (\ell_0 \circ k_0, \ell_2 \circ k_2)^*$, from condition (4) above. $T \circ \mathsf{comp}$ satisfies both of these conditions, as the functoriality of E gives $E(1;1) = 1$ and $E(\ell_0;\ell_2) \circ E(k_0;k_2) = E(\ell_0 \circ k_0; \ell_2 \circ k_2)$. $\square$

Proposition 8. *Suppose* $(\text{-})^*$ *is any Kleisli extension making* $(T,\eta,(\text{-})^*)$ *a monad relative to* $I : \mathcal{C}^{\rightarrow} \to \mathcal{C}^{\rightarrow\rightarrow}$. *Then a composable pair* $(m;n)$ *carries an algebra for* $(T,\eta,(\text{-})^*)$ *if and only if it carries an algebra for* (T,η).

Proof. Suppose that $(m;n)$ carries an algebra for $(T,\eta,(\text{-})^*)$; that is, we have an extension χ which gives, for any morphism $k : (1;f) \to (m;n)$, a morphism $\chi k : Tf \to (m;n)$, satisfying (1) $\chi k \circ \eta_f = k$ and (2) whenever $k : (1;f) \to (m;n)$ and $\ell : (1;g) \to Tf$, $\chi(\chi k \circ \ell) = \chi k \circ \ell^*$. Then this extension χ also makes $(m;n)$ an algebra for (T,η), since we defined an algebra for a relative pointing as a strictly weaker concept than an algebra for a relative monad, requiring condition (1) but not condition (2).

Conversely, suppose that $(m;n)$ carries an algebra for (T,η). By proposition

4, there is a map $\bar{\chi}$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "L(n\circ m)"'] & X \arrow[d, "m"] \\
E(n\circ m) \arrow[r, "\bar{\chi}"] \arrow[d, "R(n\circ m)"'] & Y \arrow[d, "n"] \\
Z \arrow[r, "1"] & Z
\end{tikzcd}$$

We now define an extension χ by $\chi k_1 := \bar{\chi} \circ (k_0, k_2)^*$. That is, given $k : (1; f) \to (m; n)$, we construct $\chi k : Tf \to (m; n)$ as

$$\begin{tikzcd}
A \arrow[r, "k_0"] \arrow[d, "Lf"'] & X \arrow[r, "1"] \arrow[d, "L(n\circ m)"] & X \arrow[d, "m"] \\
Ef \arrow[r, "(k_0;k_2)^*"] \arrow[d, "Rf"'] & E(n\circ m) \arrow[r, "\bar{\chi}"] \arrow[d, "R(n\circ m)"] & Y \arrow[d, "n"] \\
B \arrow[r, "k_2"'] & Z \arrow[r, "1"'] & Z
\end{tikzcd}$$

It remains only to check that χ satisfies conditions (1) and (2).

$$\begin{aligned}
\chi k \circ \eta_f &= (k_0; \bar{\chi} \circ (k_0; k_2)^*; k_2) \circ (1; Lf; 1) \\
&= (k_0; \bar{\chi} \circ (k_0; k_2)^* \circ Lf; k_2) \\
&= (k_0; m \circ k_0; k_2) \\
&= (k_0; k_1; k_2) && \text{diagram of } k \\
&= k
\end{aligned}$$

$$\begin{aligned}
\chi(\chi k \circ \ell) &= \chi(k_0 \circ \ell_0; \bar{\chi} \circ (k_0; k_2)^* \circ \ell_1; k_2 \circ \ell_2) \\
&= (k_0 \circ \ell_0; \bar{\chi} \circ (k_0 \circ \ell_0; k_2 \circ \ell_2)^*; k_2 \circ \ell_2) \\
&= (k_0 \circ \ell_0; \bar{\chi} \circ (k_0; k_2)^* \circ (\ell_0; \ell_2)^*; k_2 \circ \ell_2) \\
&= (k_0; \bar{\chi} \circ (k_0; k_2)^*; k_2) \circ (\ell_0; (\ell_0; \ell_2)^*; \ell_2) \\
&= \chi k \circ \ell^*
\end{aligned}$$

□

This result shows that considering a Kleisli extension (or coextension) does not change the class of algebras (or coalgebras). Therefore, Kleisli (co)extensions are not a promising avenue for the purposes of creating an analogue to weak factorization systems, which were developed around classes of left and right maps.

6 Relative weak factorization systems as a generalization of weak factorization systems

We would like our new notion of relative weak factorization systems to be a generalization of the existing notions of weak factorization systems. In a sense, this result comes for free from our definitions in chapter 4 and discussion in chapter 5: the only structure we needed to assume in order to discuss a system of fibrations and cofibrations based on relative (co)pointings was a functorial factorization, which any algebraic notion of a weak factorization system will demand. However, we also came across a more elaborate result, which may shed more light on the sense in which relative weak factorizations systems are a generalization of algebraic weak factorization systems.

Recall that the only Kleisli extension $(\text{-})^*$ making $(T, \eta, (\text{-})^*)$ a monad relative to $I : \mathcal{C}^{\to} \to \mathcal{C}^{\to\to}$ is $T \circ \mathsf{comp}$, and likewise the only Kleisli coextension $(\text{-})_*$ making $(T, \varepsilon, (\text{-})_*)$ a comonad relative to $J : \mathcal{C}^{\to} \to \mathcal{C}^{\to\to}$ is $T \circ \mathsf{comp}$. Therefore, in an uninteresting sense, any category with a algebraic weak factorization system (T, μ, δ) also has a relative weak factorization system, namely $(T, T \circ \mathsf{comp}, T \circ \mathsf{comp})$. However, when working with a relaxed notion of relative weak factorization system that did not require $(h_0, h_1)^* = E(h_0, h_1) = (h_0, h_1)_*$, we noticed that the constructions $(k_0, k_2)^* := \mu_m \circ E(Lm \circ k_0, k_2)$ and $(k_0, k_2)_* := E(k_0, k_2 \circ Rf) \circ \delta_f$, invoking more of the machinery of an algebraic weak factorization system, sufficed to yield most of the properties of a relative weak factorization system. Indeed, in our examples of interest, such as those arising from Moore structure, we found that we always had $\mu_m \circ E(Lm \circ k_0, k_2) = E(k_0, k_2) = E(k_0, k_2 \circ Rf) \circ \delta_f$.

Suppose $(T = (L, E, R), \mu, \delta)$ is an algebraic weak factorization system on $\mathcal{C}$, and consider a map of arrows $(k_0, k_2) : f \to m$. We wish to find $(k_0, k_2)^*$ filling

$$\begin{tikzcd}
A \arrow[r, "k_0"'] \arrow[d, "Lf"'] & X \arrow[d, "Lm"] \\
Ef \arrow[r, dotted] \arrow[d, "Rf"'] & Em \arrow[d, "Rm"] \\
B \arrow[r, "k_2"'] & Y.
\end{tikzcd}$$

Take $(k_0, k_2)^* := \mu_m \circ E(Lm \circ k_0, k_2)$, which fills

$$
\begin{tikzcd}
A \arrow[r, "k_0"'] \arrow[d, "1"'] & X \arrow[d, "Lm"] & \\
A \arrow[r, "Lm\circ k_0"'] \arrow[d, "Lf"'] & Em \arrow[d, "LRm"] \arrow[dr, bend left, "1"] & \\
Ef \arrow[r, "E(Lm\circ k_0, k_2)"'] \arrow[d, "Rf"'] & ERm \arrow[r, "\mu_m"'] \arrow[d, "RRm"] & Em \arrow[d, "Rm"] \\
B \arrow[r, "k_2"'] & Y \arrow[r, "1"'] & Y
\end{tikzcd}
$$

where $\mu_m \circ LRm = 1$ follows from $\mu \circ \eta R = 1$, a unit law of the monad.

We check two conditions. First, we have

$$
\begin{aligned}
(1,1)^* &= \mu_m \circ E(Lm \circ 1, 1) \\
&= \mu_m \circ E(Lm, 1) \\
&= 1
\end{aligned}
$$

by the other unit law of the monad. And second, for maps of arrows $(k_0, k_2) : f \to m$ and $(\ell_0, \ell_2) : m \to n$, we have

$$
\begin{aligned}
(\ell_0, \ell_2)^* \circ (k_0, k_2)^* &= \mu_n \circ E(Ln \circ \ell_0, \ell_2) \circ \mu_m \circ E(Lm \circ k_0, k_2) \\
&= \mu_n \circ \mu_{Rn} \circ E(E(Ln \circ \ell_0, \ell_2), \ell_2) \circ E(Lm \circ k_0, k_2) && \text{naturality of } \mu \\
&= \mu_n \circ \mu_{Rn} \circ E(E(Ln \circ \ell_0, \ell_2) \circ Lm \circ k_0, \ell_2 \circ k_2) \\
&= \mu_n \circ \mu_{Rn} \circ E(LRn \circ Ln \circ \ell_0 \circ k_0, \ell_2 \circ k_2) \\
&= \mu_n \circ E(\mu_n, 1) \circ E(LRn \circ Ln \circ \ell_0 \circ k_0, \ell_2 \circ k_2) && \text{associativity of monad} \\
&= \mu_n \circ E(\mu_n \circ LRn \circ Ln \circ \ell_0 \circ k_0, \ell_2 \circ k_2) \\
&= \mu_n \circ E(Ln \circ \ell_0 \circ k_0, \ell_2 \circ k_2) && \text{unit law of monad} \\
&= (\ell_0 \circ k_0, \ell_2 \circ k_2)^*
\end{aligned}
$$

This completes the construction of the Kleisli extension. The construction of the Kleisli coextension, $(k_0, k_2)_* := E(k_0, k_2 \circ Rf) \circ \delta_f$, is dual:

$$
\begin{tikzcd}
A \arrow[r, "1"'] \arrow[d, "Lf"'] & A \arrow[r, "k_0"'] \arrow[d, "LLf"'] & X \arrow[d, "Lm"] \\
Ef \arrow[r, "\delta_f"'] \arrow[dr, bend right, "1"'] & ELf \arrow[r, "E(k_0, k_2\circ Rf)"'] \arrow[d, "RLf"'] & Em \arrow[d, "Rm"] \\
 & Ef \arrow[r, "k_2\circ Rf"'] \arrow[d, "Rf"'] & Y \arrow[d, "1"] \\
 & B \arrow[r, "k_2"'] & Y
\end{tikzcd}
$$

and the conditions follow similarly, from the properties of the comonad.

We are also interested in the relationship between the (co)fibrations defined in definition 13, which are composable pairs, and the (co)fibrations of an ordinary weak factorization system, which are maps. The relationship is the following:

Proposition 9. *$(1, g)$ carries an algebra for the relative pointing (T, η) if and only if g carries an algebra for the pointed endofunctor (R, η).*

Proof. From proposition 4 we know that $(1, g)$ is a fibration if and only if there is a map filling

$$\begin{tikzcd}
C \arrow[r, "1"'] \arrow[d, "Lg"'] & C \arrow[d, "1"] \\
Eg \arrow[r, dotted] \arrow[d, "Rg"'] & C \arrow[d, "g"] \\
D \arrow[r, "1"'] & D
\end{tikzcd}$$

while g carries an algebra for (R, η) if and only if there is a map filling

$$\begin{tikzcd}
C \arrow[r, "Lg"'] \arrow[rr, bend left, "1"] \arrow[d, "g"'] & Eg \arrow[r, dotted] \arrow[d, "Rg"] & C \arrow[d, "g"] \\
D \arrow[r, "1"'] & D \arrow[r, "1"'] & D
\end{tikzcd}$$

□

Dually, $(g, 1)$ carries a coalgebra for the relative copointing (T, ε) if and only if g carries a coalgebra for the copointed endofunctor (L, ε).

This result shows another sense in which relative weak factorization systems are a generalization of weak factorization systems. In a weak factorization system, the right maps are exactly those which carry an algebra for the pointed endofunctor. Each such map g appears among the fibrations of our relative weak factorization system as $(1, g)$, along with new, additional fibrations that are not of that form.

7 A general definition of relative weak factorization systems

We will now try to distill the key features that made the system in chapter 4 work. We will give a more abstract definition of a relative weak factorization system, which will then be applied in chapter 8 to the two-sided case, which was the original motivation for this research direction.

Suppose we have two categories $\mathcal{C}$ and $\mathcal{D}$, a functor $K : \mathcal{D} \to \mathcal{C}$, and left and right adjoints $I, J : \mathcal{C} \to \mathcal{D}$ for K.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "I" {name=A}] \arrow[r, bend right=50, "J"' {name=B}] & \mathcal{D} \arrow[l, "K"' {name=C}]
\arrow[from=A, to=C, phantom, "\perp"]
\arrow[from=C, to=B, phantom, "\perp"]
\end{tikzcd}$$

To disambiguate, let the unit and counit of the adjunction $I \dashv K$ be $\eta' : 1_{\mathcal{C}} \Rightarrow KI$ and $\varepsilon' : IK \Rightarrow 1_{\mathcal{D}}$, and let the unit and counit of the adjunction $K \dashv J$ be $\eta'' : 1_{\mathcal{D}} \Rightarrow JK$ and $\varepsilon'' : KJ \Rightarrow 1_{\mathcal{C}}$. Now, suppose further that η' and ε'' are identity natural transformations, so that I and J are sections of K. Note that these conditions do hold for the setting described in section 4, where K is the composition functor $\mathsf{comp} : \mathcal{C}^{\to\to} \to \mathcal{C}^{\to}$. We will call a situation like this a *factorization setting.*

Definition 14. A *factorization setting* consists of a functor K with and left and right adjoints I and J such that the unit of $I \dashv K$ and the counit of $K \dashv J$ are identity natural transformations.

Proposition 10. *In a factorization setting $I \dashv K \dashv J$, we have $K\varepsilon' = 1_K = K\eta''$, $\varepsilon' I = 1_I$, and $\eta'' J = 1_J$.*

Proof. Suppose $I \dashv K \dashv J$ is a factorization setting, where $K : \mathcal{D} \to \mathcal{C}$. Then because $\eta' = 1_{1_{\mathcal{C}}} = \varepsilon''$, the triangle identities simplify as follows:

$$\begin{array}{lcl}
\varepsilon' I \circ I\eta' = 1_I & \implies & \varepsilon' I = 1_I \\
K\varepsilon' \circ \eta' K = 1_K & \implies & K\varepsilon' = 1_K \\
\varepsilon'' K \circ K\eta'' = 1_K & \implies & K\eta'' = 1_K \\
J\varepsilon'' \circ \eta'' J = 1_J & \implies & \eta'' J = 1_J.
\end{array}$$

□

A factorization setting also allows us to consider lifting problems.

Definition 15. In a factorization setting $I \dashv K \dashv J$, where $K : \mathcal{D} \to \mathcal{C}$, a *lifting problem* is an arrow $h : Kd_0 \to Kd_1$ in $\mathcal{C}$ between objects in the image of K, and a *solution* to that lifting problem is an arrow $\ell_h : d_0 \to d_1$ in $\mathcal{D}$ such that $K\ell_h = h$.

Note again that we saw an instance of this definition in chapter 4, where a lifting problem specifies two elements of $\mathcal{C}^{\to\to}$ and a map between their compositions, and a solution is a map between the two composable pairs which respects the given map between their compositions.

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[d] \arrow[r, dotted] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
$$

Given a factorization setting, for a relative weak factorization system we simply need something corresponding to a factorization. That is, any section T of the functor K, which corresponds to composition.

Definition 16. A *relative weak factorization system* consists of a factorization setting $I \dashv K \dashv J$ and a section of K.

Suppose T is a section of K for a factorization setting $I \dashv K \dashv J$. We would like T to have a relative pointing with respect to I and a relative copointing with respect to J, and indeed it always does: a pointing for T relative to I is simply a natural transformation $u : I \Rightarrow T$, and we can always construct such a natural transformation from the counit ε' of $I \dashv K$. We have $\varepsilon' : IK \Rightarrow 1_{\mathcal{D}}$, so $\varepsilon' T : IKT \Rightarrow T$, and since T is a section of K, in fact $\varepsilon' T : I \Rightarrow T$, so we simply let $u := \varepsilon' T$. Dually, from the unit η'' of $K \dashv J$ we have $\eta'' T : T \Rightarrow J$, so there is a copointing $v := \eta'' T$.

From here, we can define classes of fibrations and cofibrations: fibrations will be algebras for (T, u), and cofibrations will be coalgebras for (T, v).

Definition 17. For a relative weak factorization system $(T, I \dashv K \dashv J)$, where $K : \mathcal{D} \to \mathcal{C}$, a *fibration* consists of

- an object d in $\mathcal{D}$, together with
- a section χ of $u_c^* : \hom_{\mathcal{D}}(Tc, d) \to \hom_{\mathcal{D}}(Ic, d)$ which is natural in c.

Dually, a *cofibration* consists of

- an object e in $\mathcal{D}$, together with
- a section ψ of $(v_c)_* : \hom_{\mathcal{D}}(e, Tc) \to \hom_{\mathcal{D}}(e, Jc)$ which is natural in c.

Here u_c^* is precomposition by $u_c = \varepsilon'_{Tc}$, and $(v_c)_*$ is postcomposition by $v_c = \eta''_{Tc}$. It follows that for any morphism $k : Ic \to d$, χk will satisfy $\chi k \circ u_c = k$, showing that (d, χ) is an algebra for (T, u) as in definition 12, and likewise (e, ψ) is a coalgebra for (T, v).

We can collect these objects into categories of fibrations and cofibrations. We denote by Fib the category of fibrations; that is, an object of Fib is a fibration as in definition 17, and a morphism $(d, \chi) \to (d', \chi')$ of Fib is a morphism $f : d \to d'$ in $\mathcal{D}$ such that for any $k : Ic \to d$, we have $\chi'(f \circ k) = f \circ \chi(k)$.

Dually, we have a category Cof of cofibrations; an object of Cof is a cofibration as in definition 17, and a morphism $(e, \psi) \to (e', \psi')$ of Cof is a morphism $f : e \to e'$ in $\mathcal{D}$ such that for any $k : e' \to Jc$, we have $\psi(k \circ f) = \psi'(k) \circ f$.

Now we want to establish the connection between (co)fibrations and lifting problems. The existence of a fibration structure or cofibration structure for an object of $\mathcal{D}$ will prove to be equivalent to a lifting condition, which can be thought of either as a solution to one specific lifting problem, or as a coherent system of solutions to related lifting problems. We will establish the equivalence of these two forms of the lifting condition first.

Definition 18. Given a category $\mathcal{A}$ and a functor $A : \mathcal{A} \to \mathcal{D}$, a *right lifting structure* for an object d of $\mathcal{D}$ against A (or "against $\mathcal{A}$" when the functor is understood) is a section ℓ of $K : \hom_{\mathcal{D}}(Aa, d) \to \hom_{\mathcal{C}}(KAa, Kd)$ which is natural in a.

Dually, a *left lifting structure* for an object e of $\mathcal{D}$ against A (or "against $\mathcal{A}$" when the functor is understood) is a section ℓ of $K : \hom_{\mathcal{D}}(e, Aa) \to \hom_{\mathcal{C}}(Ke, KAa)$ which is natural in a.

That is, we have a right lifting structure for d against A whenever we have solution ℓh to each lifting problem of the form $h : KAa \to Kd$, and also these solutions collectively satisfy $\ell(h \circ KAf) = \ell(h) \circ Af$, for any $h : KAa \to Kd$ and $f : a' \to a$.

We will denote by $A^{\pitchfork}$ the category of objects of $\mathcal{D}$ with right lifting structures against A. That is, an object of $A^{\pitchfork}$ is some (d, ℓ), where d is an object of $\mathcal{D}$ and ℓ is a right lifting structure for d against A. A morphism $(d, \ell) \to (d', \ell')$ of $A^{\pitchfork}$ is a morphism $f : d \to d'$ of $\mathcal{D}$ such that the following diagram commutes for all a:

$$\begin{tikzcd}
\hom_{\mathcal{C}}(KAa, Kd) \arrow[r, "\ell"] \arrow[d, "Kf\circ(-)"'] & \hom_{\mathcal{D}}(Aa, d) \arrow[d, "f\circ(-)"] \\
\hom_{\mathcal{C}}(KAa, Kd') \arrow[r, "\ell'"] & \hom_{\mathcal{D}}(Aa, d')
\end{tikzcd}$$

Dually, the category of objects of $\mathcal{D}$ with right lifting structures against A is denoted ${}^{\pitchfork}A$.

We will now show that in the special case of lifting against the functor T, such a lifting structure is actually determined by a single lift, a solution to the identity lifting problem $1_{Kd} : KTKd \to Kd$ (or dually, a solution to $1_{Ke} : Ke \to KTKe$). For this purpose we define a category $\mathcal{D}_{\ell 1}$ of objects d of $\mathcal{D}$ with solutions to the lifting problem $1_{Kd} : KTKd \to Kd$. An element of $\mathcal{D}_{\ell 1}$ is some (d, i) such that $i : TKd \to d$ is a solution to $1_{Kd} : KTKd \to Kd$, and a morphism $(d, i) \to (d', i')$ of $\mathcal{D}_{\ell 1}$ is a morphism $f : d \to d'$ of $\mathcal{D}$ such that $f \circ i = i' \circ TKf$. The category ${}_{\ell 1}\mathcal{D}$ of objects e of $\mathcal{D}$ with solutions to the lifting problem $1_{Ke} : Ke \to KTKe$ is defined dually.

Proposition 11. *There is an isomorphism of categories over $\mathcal{D}$ between $T^{\pitchfork}$ and $\mathcal{D}_{\ell 1}$.*

Dually, there is an isomorphism of categories over $\mathcal{D}$ between ${}^{\pitchfork}T$ and ${}_{\ell 1}\mathcal{D}$.

Proof. There is a functor $F : T^{\pitchfork} \to \mathcal{D}_{\ell 1}$, taking (d, ℓ) to $(d, \ell(1_{Kd}))$. By definition, $\ell : \hom_{\mathcal{C}}(KTc, Kd) \to \hom_{\mathcal{D}}(Tc, d)$ is a section of K, so in particular, given the identity map $1_{Kd} : KTKd \to Kd$ we can produce $\ell(1_{Kd}) : TKd \to d$, which has the property that $K\ell(1_{Kd}) = 1_{Kd}$. Thus $\ell(1_{Kd})$ is a solution to the lifting problem $1_{Kd} : KTKd \to Kd$, so $(d, \ell(1_{Kd}))$ is an object of $\mathcal{D}_{\ell 1}$.

On morphisms, F simply takes $f : (d, \ell) \to (d', \ell')$ to $f : (d, \ell(1_{Kd})) \to (d', \ell'(1_{Kd'}))$. Since f is a morphism of $T^{\pitchfork}$, we have $f \circ \ell h = \ell'(Kf \circ h)$ for any $h : KTc \to Kd$, so in particular

$$\begin{aligned} f \circ \ell(1_{Kd}) &= \ell'(Kf \circ 1_{Kd}) \\ &= \ell'(1_{Kd} \circ Kf) \\ &= \ell'(1_{Kd} \circ KTKf) \\ &= \ell'(1_{Kd}) \circ TKf, \end{aligned}$$

showing that f is a morphism of $\mathcal{D}_{\ell 1}$.

There is also a functor $G : \mathcal{D}_{\ell 1} \to T^{\pitchfork}$, taking (d, i) to (d, ℓ), where ℓ is given by $\ell(h) := i \circ Th$. To check that this does define a right lifting structure for d against T whenever i is a solution to $1_{Kd} : KTKd \to Kd$, we need to check that ℓ is a section of K and that it is natural in c. Indeed we find

$$\begin{aligned} K\ell h &= K(i \circ Th) \\ &= Ki \circ KTh \\ &= 1_{Kd} \circ h \\ &= h \end{aligned}$$

and

$$\begin{aligned} \ell(h \circ KTg) &= \ell(h \circ g) \\ &= i \circ T(h \circ g) \\ &= i \circ Th \circ Tg \\ &= \ell(h) \circ Tg \end{aligned}$$

as desired.

On morphisms, G takes $f : (d, i) \to (d', i')$ to $f : (d, \ell) \to (d', \ell')$, where $\ell(h) := i \circ Th$ and $\ell'(h) := i' \circ Th$. Since f is a morphism of $\mathcal{D}_{\ell 1}$, we have $f \circ i = i' \circ TKf$, so for any $h : KTc \to Kd$ we find

$$\begin{aligned} \ell'(Kf \circ h) &= i' \circ T(Kf \circ h) \\ &= i' \circ TKf \circ Th \\ &= f \circ i \circ Th \\ &= f \circ \ell(h), \end{aligned}$$

showing that f is a morphism of $T^{\pitchfork}$.

Lastly, we note that FG and GF are both identity functors. FG maps (d, i) to $(d, \ell(1_{Kd}))$, where ℓ is defined by $\ell(h) := i \circ Th$, so in fact $\ell(1_{Kd}) =$

$i \circ T(1_{Kd}) = i$. Meanwhile GF maps (d, ℓ) to (d, ℓ'), where ℓ' is defined by $\ell'(h) := \ell(1_{Kd}) \circ Th$, so in fact for any h,

$$\begin{aligned}
\ell'(h) &= \ell(1_{Kd}) \circ Th \\
&= \ell(1_{Kd} \circ KTh) \\
&= \ell(h).
\end{aligned}$$

This shows an isomorphism between the categories $T^{\pitchfork}$ and $\mathcal{D}_{\ell 1}$. $\square$

Now we will prove a generalization of proposition **??**, establishing the equivalence of (co)fibration structure and lifting structure.

Proposition 12. *There is an isomorphism of categories over $\mathcal{D}$ between $T^{\pitchfork}$ and Fib.*

Dually, there is an isomorphism of categories over $\mathcal{D}$ between ${}^{\pitchfork}T$ and Cof.

Proof. By definition, a fibration structure for an object d of $\mathcal{D}$ is a section of $u_c^* : \hom_{\mathcal{D}}(Tc, d) \to \hom_{\mathcal{D}}(Ic, d)$ which is natural in c, and a right lifting structure for d against T is a section of $K : \hom_{\mathcal{D}}(Tc, d) \to \hom_{\mathcal{C}}(c, Kd)$ which is natural in c. So to show that Fib and $T^{\pitchfork}$ are equivalent we begin by identifying a natural isomorphism that completes this diagram

$$\begin{tikzcd}
 & \hom_{\mathcal{D}}(Tc, d) \arrow[dl, "u_c^*"'] \arrow[dr, "K"] & \\
\hom_{\mathcal{D}}(Ic, d) \arrow[rr, dotted, "\simeq"] & & \hom_{\mathcal{C}}(c, Kd)
\end{tikzcd}$$

in $[\mathcal{C}, \mathrm{Set}]$.

The adjunction $I \dashv K$ gives exactly an isomorphism $\hom_{\mathcal{D}}(Ic, d) \to \hom_{\mathcal{C}}(c, Kd)$ that is natural in c. Specifically, the isomorphism is given by

$$Ic \xrightarrow{k} d \qquad \mapsto \qquad c \xrightarrow{\eta'_c} KIc \xrightarrow{Kk} Kd$$

and by definition 14 we have $\eta'_c = 1_c$, so in fact the isomorphism is given by K. Its inverse is given by

$$c \xrightarrow{g} Kd \qquad \mapsto \qquad Ic \xrightarrow{Ig} IKd \xrightarrow{\varepsilon'_d} d.$$

We check that the diagram commutes:

$$\begin{tikzcd}
 & \hom_{\mathcal{D}}(Tc, d) \arrow[dl, "u_c^*"'] \arrow[dr, "K"] & \\
\hom_{\mathcal{D}}(Ic, d) \arrow[rr, "K"] & & \hom_{\mathcal{C}}(c, Kd)
\end{tikzcd}$$

That is, considering an element $f : Tc \to d$ of $\hom_{\mathcal{D}}(Tc, d)$, we want to check that $K(f \circ u_c) = Kf$. And indeed, by proposition 10 we have $K(f \circ u_c) = Kf \circ K\varepsilon'_{Tc} = Kf$.

Now we can see that there is a functor $F : T^{\pitchfork} \to \text{Fib}$ taking (d, ℓ) to $(d, \ell \circ K)$. Since ℓ is a section of $K : \hom_{\mathcal{D}}(Tc, d) \to \hom_{\mathcal{C}}(c, Kd)$ natural in c, and $K : \hom_{\mathcal{D}}(Ic, d) \to \hom_{\mathcal{C}}(c, Kd)$ is an isomorphism natural in c which completes the diagram, $\ell \circ K$ is a section of $u_c^* : \hom_{\mathcal{D}}(Tc, d) \to \hom_{\mathcal{D}}(Ic, d)$, so $(d, \ell \circ K)$ is an object of Fib.

On morphisms, F takes $f : (d, \ell) \to (d', \ell')$ to $f : (d, \ell \circ K) \to (d', \ell' \circ K)$. Since f is a morphism of $T^{\pitchfork}$, we have $f \circ \ell h = \ell'(Kf \circ h)$ for any $h : KTc \to Kd$. In particular, for any $k : Ic \to d$, we have $Kk : KIc = c = KTc \to Kd$, so we can use this property of f to show

$$\begin{aligned}(\ell' \circ K)(f \circ k) &= \ell'(K(f \circ k)) \\ &= \ell'(Kf \circ Kk) \\ &= f \circ \ell(Kk) \\ &= f \circ (\ell \circ K)(k),\end{aligned}$$

showing that f is a morphism of Fib.

Similarly, there is a functor $G : \text{Fib} \to T^{\pitchfork}$ taking (d, χ) to $(d, \chi \circ (\varepsilon'_d)_* \circ I)$. Since χ is a section of $u_c^* : \hom_{\mathcal{D}}(Tc, d) \to \hom_{\mathcal{D}}(Ic, d)$ natural in c, and $(\varepsilon'_d)_* \circ I : \hom_{\mathcal{C}}(c, Kd) \to \hom_{\mathcal{D}}(Ic, d)$ is an isomorphism natural in c which completes the diagram, $\chi \circ (\varepsilon'_d)_* \circ I$ is a section of $K : \hom_{\mathcal{D}}(Tc, d) \to \hom_{\mathcal{C}}(c, Kd)$, so $(d, \chi \circ (\varepsilon'_d)_* \circ I)$ is an object of $T^{\pitchfork}$.

On morphisms, G takes $f : (d, \chi) \to (d', \chi')$ to $f : (d, \chi \circ (\varepsilon'_d)_* \circ I) \to (d', \chi' \circ (\varepsilon'_{d'})_* \circ I)$. Since f is a morphism of Fib, we have $\chi'(f \circ k) = f \circ \chi(k)$ for any $k : Ic \to d$. Using this property we can show that for any $h : KTc \to Kd$,

$$\begin{aligned}(\chi' \circ (\varepsilon'_{d'})_* \circ I)(Kf \circ h) &= \chi'(\varepsilon'_{d'} \circ I(Kf \circ h)) \\ &= \chi'(\varepsilon'_{d'} \circ IKf \circ Ih) \\ &= \chi'(f \circ \varepsilon'_d \circ Ih) \\ &= f \circ \chi(\varepsilon'_d \circ Ih) \\ &= f \circ (\chi \circ (\varepsilon'_d)_* \circ I)(h),\end{aligned}$$

which shows that f is a morphism of $T^{\pitchfork}$.

Lastly, we note that FG and GF are both identity functors. FG maps (d, χ) to $(d, \chi \circ (\varepsilon'_d)_* \circ I \circ K)$, and for any $k : Ic \to d$ we have

$$\begin{aligned}(\chi \circ (\varepsilon'_d)_* \circ I \circ K)(k) &= \chi(\varepsilon'_d \circ IKk) \\ &= \chi(k \circ \varepsilon'_{Ic}) \\ &= \chi(k)\end{aligned}$$

by proposition 10. Meanwhile GF maps (d, ℓ) to $(d, \ell \circ K \circ (\varepsilon'_d)_* \circ I)$, and for any $h : c \to Kd$ we have

$$\begin{aligned}(\chi \circ \ell \circ K \circ (\varepsilon'_d)_* \circ I)(h) &= \ell(K(\varepsilon'_d \circ Ih)) \\ &= \ell(K\varepsilon'_d \circ KIh) \\ &= \ell(h)\end{aligned}$$

by proposition 10.

This shows an isomorphism between $T^{\pitchfork}$ and Fib. $\square$

Proposition 13. *For any object $c \in \mathcal{C}$, TK is both a fibration structure and a cofibration structure for Tc.*

Proof. The lifting problem $1_{KTc} : KTKTc \to KTc$ has the solution $1_{Tc} : TKTc \to Tc$. By proposition 11, this solution extends to a right lifting structure ℓ for Tc against T, where ℓ is given by $\ell(h) := 1_{Tc} \circ Th = Th$, so in fact T is a right lifting structure for Tc against T. It then follows from proposition 12 that TK is a fibration structure for Tc. $\square$

Proposition 14. *There is a fully faithful functor from Fib to Cof$^{\pitchfork}$ over $\mathcal{D}$.*

Dually, there is a fully faithful functor from Cof to ${}^{\pitchfork}$Fib over $\mathcal{D}$.

Proof. We construct a functor $F : \mathrm{Fib} \to \mathrm{Cof}^{\pitchfork}$. Given an object (d, χ) of Fib, we construct a lifting structure ℓ for d against Cof as follows: for a lifting problem $h : Ke \to Kd$, where (e, ψ) is an object of Cof, the lift is given by $\ell(h) := \chi(\varepsilon'_d) \circ Th \circ \psi(\eta''_e)$.

$$e \xrightarrow{\psi(\eta''_e)} TKe \xrightarrow{Th} TKd \xrightarrow{\chi(\varepsilon'_d)} d$$

First, using proposition 12 we see that when (d, χ) is a fibration, $\chi \circ (\varepsilon'_d)_* \circ I$ is a lifting structure for d against T, and then using proposition 11 we see that $(\chi \circ (\varepsilon'_d)_* \circ I)(1_{Kd}) = \chi(\varepsilon'_d)$ is a solution to the lifting problem $1_{Kd} : KTKd \to Kd$. Dually, $\psi(\eta''_e)$ is a solution to the lifting problem $1_{Ke} : Ke \to KTKe$. Therefore, as a whole $\ell(h) = \chi(\varepsilon'_d) \circ Th \circ \psi(\eta''_e)$ is indeed a solution to the lifting problem $h : Ke \to Kd$.

Next we want to show that ℓ is natural in (e, ψ). Suppose that $f : (e, \psi) \to (e', \psi')$ is a morphism in Cof; that is, $f : e \to e'$ is a morphism of $\mathcal{D}$ and for any $k : e' \to Jc$, $\psi(k \circ f) = \psi'(k) \circ f$. Then for any $h : Ke' \to Kd$, we have

$$\begin{aligned}
\ell(h \circ Kf) &= \chi(\varepsilon'_d) \circ T(h \circ Kf) \circ \psi(\eta''_e) \\
&= \chi(\varepsilon'_d) \circ Th \circ TKf \circ \psi(\eta''_e) \\
&= \chi(\varepsilon'_d) \circ Th \circ \psi(JKf \circ \eta''_e) \\
&= \chi(\varepsilon'_d) \circ Th \circ \psi(\eta''_{e'} \circ f) \\
&= \chi(\varepsilon'_d) \circ Th \circ \psi'(\eta''_{e'}) \circ f \\
&= \ell(h) \circ f,
\end{aligned}$$

showing that ℓ is indeed a lifting structure for d against Cof.

On morphisms, F takes $f : (d, \chi) \to (d', \chi')$ to $f : (d, \ell) \to (d', \ell')$. Since f is a morphism of Fib, we have $\chi'(f \circ k) = f \circ \chi(k)$ for any $k : Ic \to d$. Using this property we can show that for any $(e, \psi) \in \text{Cof}$ and $h : Ke \to Kd$,

$$\begin{aligned}
\ell'(Kf \circ h) &= \chi'(\varepsilon'_{d'}) \circ T(Kf \circ h) \circ \psi(\eta''_e) \\
&= \chi'(\varepsilon'_{d'}) \circ TKf \circ Th \circ \psi(\eta''_e) \\
&= \chi'(\varepsilon'_{d'} \circ IKf) \circ Th \circ \psi(\eta''_e) \\
&= \chi'(f \circ \varepsilon'_d) \circ Th \circ \psi(\eta''_e) \\
&= f \circ \chi(\varepsilon'_d) \circ Th \circ \psi(\eta''_e) \\
&= f \circ \ell(h)
\end{aligned}$$

which shows that f is a morphism of $\text{Cof}^{\pitchfork}$.

F is clearly faithful. To see that F is also full, we want to show that any morphism $f : d \to d'$ of $\mathcal{D}$ that is also a morphism $f : (d, \ell) \to (d', \ell')$ of $\text{Cof}^{\pitchfork}$ was already a morphism $f : (d, \chi) \to (d', \chi')$ of Fib. So, consider some $f : d \to d'$ such that for any $(e, \psi) \in \text{Cof}$ and $h : Ke \to Kd$, we have $\chi'(\varepsilon'_{d'}) \circ T(Kf \circ h) \circ \psi(\eta''_e) = f \circ \chi(\varepsilon'_d) \circ Th \circ \psi(\eta''_e)$. We want to show that for any $k : Ic \to d$, $\chi'(f \circ k) = f \circ \chi(k)$. Note that for any $k : Ic \to d$, we have $\varepsilon'_d \circ IKk = k \circ \varepsilon'_{Ic} = k$ by proposition 10. Furthermore, by proposition 13, TK is a cofibration structure for Tc, so we can pick $(Tc, TK) \in \text{Cof}$ and $h := Kk$, which maps from $KTc = c = KIc$ to Kd. Now the fact that f is a morphism of $\text{Cof}^{\pitchfork}$ becomes $\chi'(\varepsilon'_{d'}) \circ T(Kf \circ Kk) \circ TK(\eta''_{Tc}) = f \circ \chi(\varepsilon'_d) \circ TKk \circ TK(\eta''_{Tc})$; which then, since we know from above that η''_{Tc} is a solution to an identity lifting problem, becomes simply $\chi'(\varepsilon'_{d'}) \circ T(Kf \circ Kk) = f \circ \chi(\varepsilon'_d) \circ TKk$. We therefore have

$$\begin{aligned}
f \circ \chi(k) &= f \circ \chi(\varepsilon'_d \circ IKk) \\
&= f \circ \chi(\varepsilon'_d) \circ TKk \\
&= \chi'(\varepsilon'_{d'}) \circ T(Kf \circ Kk)) \\
&= \chi'(\varepsilon'_{d'} \circ IK(f \circ k)) \\
&= \chi'(f \circ k),
\end{aligned}$$

as desired. ∎

Proposition 15. *There is a functor from $\text{Cof}^{\pitchfork}$ to Fib over $\mathcal{D}$.*

Dually, there is a functor from ${}^{\pitchfork}\text{Fib}$ to Cof over $\mathcal{D}$.

Proof. By propositions 11 and 12, it suffices to provide a functor $G : \text{Cof}^{\pitchfork} \to \mathcal{D}_{\ell 1}$ over $\mathcal{D}$.

Consider some object (d, ℓ) of $\text{Cof}^{\pitchfork}$. By proposition 13, TKd is a cofibration, so ℓ yields a solution to the lifting problem $1_{Kd} : KTKd \to Kd$. We let $G(d, \ell) := (d, \ell(1_{Kd}))$.

On morphisms, G takes $f : (d, \ell) \to (d', \ell')$ to $f : (d, \ell(1_{Kd})) \to (d', \ell'(1_{Kd'}))$. Since f is a morphism of $\text{Cof}^{\pitchfork}$, we have $f \circ \ell h = \ell'(Kf \circ h)$ for any $h : Ke \to Kd$ where e has a cofibration structure. Using this property we see

$$\begin{aligned}
\ell'(1_{Kd'}) \circ TKf &= \ell'(1_{Kd'} \circ KTKf) \\
&= \ell'(Kf \circ 1_{Kd}) \\
&= f \circ \ell(1_{Kd}),
\end{aligned}$$

showing that f is also a morphism of $\mathcal{D}_{\ell 1}$. $\square$

8 Relative weak factorization systems for two-sided factorization

Having defined these general structures, we will now apply them to a more specific situation, in order to investigate one way to build *two-sided weak factorization systems* around two-sided factorizations, and define notions of two-sided (co)fibrations.

Let *Span* and *Sprout* be the categories generated by the graphs

$$\begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \bullet \arrow[d] & \\ & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet \end{tikzcd}$$

respectively. Fixing a category $\mathcal{C}$, we will then refer to the objects of $\mathcal{C}^{Span}$ as the *spans* of $\mathcal{C}$, and the objects of $\mathcal{C}^{Sprout}$ as the *sprouts* of $\mathcal{C}$.

The sprout

$$\begin{tikzcd} & W \arrow[d, "\ell"] & \\ & X \arrow[dl, "m"'] \arrow[dr, "n"] & \\ Y & & Z, \end{tikzcd}$$

or $(\ell; m, n)$ for short, can be *composed* into the span

$$\begin{tikzcd} & W \arrow[dl, "m\circ\ell"'] \arrow[dr, "n\circ\ell"] & \\ Y & & Z, \end{tikzcd}$$

or $(m\circ\ell, n\circ\ell)$ for short. Therefore, *factoring* a span (f, g) into a sprout consists of finding a sprout $(\ell; m, n)$ such that $m \circ \ell = f$ and $n \circ \ell = g$.

There are two trivial factorizations of spans into sprouts, which will exist whenever $\mathcal{C}$ is a cartesian category: we can precompose by an identity map,

$$
\begin{tikzcd}
A \arrow[dd, shift right, "f"'] \arrow[dd, shift left, "g"] & & A \arrow[d, "1_A"] \\
 & \overset{I}{\mapsto} & A \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\
B, C & & B, C
\end{tikzcd}
$$

or we can map into the product of the two codomains, and postcompose by the two projection maps.

$$
\begin{tikzcd}
A \arrow[dd, shift right, "f"'] \arrow[dd, shift left, "g"] & & A \arrow[d, "\langle f,g \rangle"] \\
 & \overset{J}{\mapsto} & B \times C \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\
B, C & & B, C
\end{tikzcd}
$$

Proposition 16. *Let* $\mathsf{comp} : \mathcal{C}^{Sprout} \to \mathcal{C}^{Span}$ *be the composition functor taking* $(\ell; m, n)$ *to* $(m \circ \ell, n \circ \ell)$*, let* $I : \mathcal{C}^{Span} \to \mathcal{C}^{Sprout}$ *be the factorization taking* $(f, g) \mapsto (1_A; f, g)$*, and let* $J : \mathcal{C}^{Span} \to \mathcal{C}^{Sprout}$ *be the factorization taking* $(f, g) \mapsto (\langle f, g\rangle; p_0, p_1)$*. Then* comp*,* I*, and* J *form a factorization setting.*

Proof. We wish to show that $I \dashv \mathsf{comp}$, with the unit of the adjunction being an identity natural transformation, and that $\mathsf{comp} \dashv J$, with the counit of the adjunction being an identity natural transformation.

For the adjunction $I \dashv \mathsf{comp}$, we let $\eta' := 1_{1_{\mathcal{C}^{Span}}} : 1_{\mathcal{C}^{Span}} \Rightarrow \mathsf{comp} \circ I$, and we let $\varepsilon' : I \circ \mathsf{comp} \Rightarrow 1_{\mathcal{C}^{Sprout}}$ be the natural transformation consisting of maps of sprouts of the following form:

$$
\varepsilon'_{(\ell;m,n)}
$$

$$
\begin{tikzcd}
W \arrow[r, "1"'] \arrow[d, "1"'] & W \arrow[d, "\ell"] \\
W \arrow[r, "\ell"'] \arrow[d, shift right, "m\circ\ell{,}n\circ\ell"'] \arrow[d, shift left] & X \arrow[d, shift right] \arrow[d, shift left, "m{,}n"] \\
Y, Z \arrow[r, shift left] \arrow[r, shift right, "1{,}1"'] & Y, Z.
\end{tikzcd}
$$

We then check the triangle identities, in the reduced forms found in proposition 10. It is clear from the diagram above that $\mathsf{comp}\varepsilon' = 1_{\mathsf{comp}}$, because the top and bottom horizontal arrows are all identities. To check that $\varepsilon' I = 1_I$, we note that for any span (f,g), $\varepsilon'_{I(f,g)}$ is simply

$$\begin{tikzcd}
A \arrow[r, "1"'] \arrow[d, "1"'] & A \arrow[d, "1"] \\
A \arrow[r, "1"'] \arrow[d, shift right, "{f,g}"'] \arrow[d, shift left] & A \arrow[d, shift left, "{f,g}"] \arrow[d, shift right] \\
B, C \arrow[r, shift left] \arrow[r, shift right, "{1,1}"'] & B, C
\end{tikzcd} \qquad = 1_{I(f,g)}.$$

For the adjunction $\mathsf{comp} \dashv J$, we let $\varepsilon'' := 1_{1_{\mathcal{C}^{Span}}} : \mathsf{comp} \circ J \Rightarrow 1_{\mathcal{C}^{Span}}$, and we let $\eta'' : 1_{\mathcal{C}^{Sprout}} \Rightarrow J \circ \mathsf{comp}$ be the natural transformation consisting of maps of sprouts of the following form:

$$\eta''_{(\ell;m,n)}$$

$$\begin{tikzcd}
W \arrow[r, "1"'] \arrow[d, "\ell"'] & W \arrow[d, "{\langle m\circ\ell, n\circ\ell\rangle}"] \\
X \arrow[r, "{\langle m,n\rangle}"'] \arrow[d, shift right, "{m,n}"'] \arrow[d, shift left] & Y \times Z \arrow[d, shift left, "{p_0,p_1}"] \arrow[d, shift right] \\
Y, Z \arrow[r, shift left] \arrow[r, shift right, "{1,1}"'] & Y, Z.
\end{tikzcd}$$

We then check the triangle identities. As above, it is clear from the diagram that $\mathsf{comp}\eta'' = 1_{\mathsf{comp}}$. To check that $\eta'' J = 1_J$, we note that for any span (f,g), $\eta''_{J(f,g)}$ turns out to be

$$\begin{tikzcd}
A \arrow[r, "1"'] \arrow[d, "{\langle f,g\rangle}"'] & A \arrow[d, "{\langle f,g\rangle}"] \\
B \times C \arrow[r, "{\langle p_0,p_1\rangle = 1}"'] \arrow[d, shift right, "{p_0,p_1}"'] \arrow[d, shift left] & B \times C \arrow[d, shift left, "{p_0,p_1}"] \arrow[d, shift right] \\
B, C \arrow[r, shift left] \arrow[r, shift right, "{1,1}"'] & B, C
\end{tikzcd} \qquad = 1_{J(f,g)}.$$

This shows that comp, I, and I form a factorization setting.

$$\begin{tikzcd}
\mathcal{C}^{Span} \arrow[r, bend left=40, "I"] \arrow[r, bend right=40, "J"'] \arrow[r, phantom, shift left=3ex, "\perp"] \arrow[r, phantom, shift right=3ex, "\perp"] & \mathcal{C}^{Sprout} \arrow[l, "\mathsf{comp}"']
\end{tikzcd}$$

□

Now that we have a factorization setting, we can consider lifting problems in this setting. By definition 15, a lifting problem in this setting will be a commutative diagram of the following form:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r, dotted] \arrow[d, shift left] \arrow[d, shift right] & \bullet \arrow[d, shift left] \arrow[d, shift right] \\
\bullet, \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet, \bullet
\end{tikzcd}$$

where the dashed arrow gives a solution to the lifting problem.

Definition 19. A *two-sided weak factorization system* for a cartesian category $\mathcal{C}$ consists of the factorization setting $I \dashv \mathsf{comp} \dashv J$ plus a section of comp.

By definition, two-sided weak factorization systems are examples of relative weak factorization systems. We refer to their fibrations and cofibrations, as defined in definition 17, as *two-sided fibrations* and *two-sided cofibrations*. Therefore, we inherit the properties of relative weak factorization systems, fibrations, and cofibrations discussed in section 7.

Suppose $(T, I \dashv \mathsf{comp} \dashv J)$ is a two-sided weak factorization system for $\mathcal{C}$. For notational convenience, we break $T : \mathcal{C}^{Span} \to \mathcal{C}^{Sprout}$ into three component functors: $\lambda : \mathcal{C}^{Span} \to \mathcal{C}^{\to}$, $E : \mathcal{C}^{Span} \to \mathcal{C}$, and $\rho : \mathcal{C}^{Span} \to \mathcal{C}^{Span}$.

$$\begin{tikzcd}
A \arrow[dd, shift right, "f"'] \arrow[dd, shift left, "g"] & & A \arrow[d, "\lambda(f,g)"] \\
 & \overset{T}{\longmapsto} & E(f,g) \arrow[d, shift left, "\rho(f,g)"] \arrow[d, shift right] \\
B, C & & B, C
\end{tikzcd}$$

Propositions 11 and 12 together tell us that the following categories are isomorphic:

1. The category of two-sided fibrations.

2. The category of sprouts $(\ell; m, n)$ with a solution to the following lifting

problem:

$$\begin{tikzcd}
W \arrow[r, "1"'] \arrow[d, "\lambda(m\circ\ell, n\circ\ell)"'] & W \arrow[d, "\ell"] \\
E(m\circ\ell, n\circ\ell) \arrow[r, dotted] \arrow[d, shift right, "\rho(m\circ\ell, n\circ\ell)"'] \arrow[d, shift left] & X \arrow[d, shift right] \arrow[d, shift left, "m,n"] \\
Y, Z \arrow[r, shift left] \arrow[r, shift right, "1,1"'] & Y, Z.
\end{tikzcd}$$

3. The category of sprouts with a coherent system of solutions to lifting problems against T-images.

Proposition 13 tells us that for any span (f, g), $T(f, g)$ carries both a two-sided fibration and a two-sided cofibration, with the common structure $T \circ \mathsf{comp}$.

Proposition 14 yields the canonical lift of any two-sided cofibration against any two-sided fibration. Suppose $((\ell; m, n), \chi)$ is a two-sided fibration, and $((q; r, s), \psi)$ is a two-sided cofibration. That is, for any morphism of sprouts $k : I(f, g) \to (\ell; m, n)$, χ yields a morphism of sprouts $\chi k = (k_0; \chi k_1; k_2, k_3) : T(f, g) \to (\ell; m, n)$, naturally in (f, g).

$$\begin{tikzcd}
A \arrow[r, "k_0"'] \arrow[d, "1"'] & W \arrow[d, "\ell"] \\
A \arrow[r, "k_1"'] \arrow[d, shift right, "f,g"'] \arrow[d, shift left] & X \arrow[d, shift right] \arrow[d, shift left, "m,n"] \\
B, C \arrow[r, shift left] \arrow[r, shift right, "k_2,k_3"'] & Y, Z
\end{tikzcd}
\quad \overset{\chi}{\mapsto} \quad
\begin{tikzcd}
A \arrow[r, "k_0"'] \arrow[d, "\lambda(f.g)"'] & W \arrow[d, "\ell"] \\
E(f, g) \arrow[r, "\chi k_1"'] \arrow[d, shift right, "\rho(f,g)"'] \arrow[d, shift left] & X \arrow[d, shift right] \arrow[d, shift left, "m,n"] \\
B, C \arrow[r, shift left] \arrow[r, shift right, "k_2,k_3"'] & Y, Z
\end{tikzcd}$$

Here we know that we must have $\chi k_0 = k_0$, $\chi k_2 = k_2$, and $\chi k_3 = k_3$ because χ is a section of precomposition by $u_{(f,g)} = \varepsilon'_{T(f,g)}$,which is simply the following map of sprouts:

$$\begin{tikzcd}
A \arrow[r, "1"'] \arrow[d, "1"'] & A \arrow[d, "\lambda(f,g)"] \\
A \arrow[r, "\lambda(f,g)"'] \arrow[d, shift right, "f,g"'] \arrow[d, shift left] & E(f, g) \arrow[d, shift right] \arrow[d, shift left, "\rho(f,g)"] \\
B, C \arrow[r, shift left] \arrow[r, shift right, "1,1"'] & B, C.
\end{tikzcd}$$

Likewise, for any morphism of sprouts $k : (q; r, s) \to J(f; g)$, ψ yields a morphism of sprouts $\psi k = (k_0; \psi k_1; k_2, k_3) : (q; r, s) \to T(f; g)$, naturally in

(f, g).

$$
\begin{tikzcd}
P \arrow[r, "k_0"'] \arrow[d, "q"'] & A \arrow[d, "\langle f,g\rangle"] & & P \arrow[r, "k_0"'] \arrow[d, "q"'] & A \arrow[d, "\lambda(f,g)"] \\
Q \arrow[r, "k_1"'] \arrow[d, shift right, "r,s"'] \arrow[d, shift left] & B \times C \arrow[d, shift right] \arrow[d, shift left, "p_0,p_1"] & \overset{\chi}{\longmapsto} & Q \arrow[r, "\psi k_1"'] \arrow[d, shift right, "r,s"'] \arrow[d, shift left] & E(f,g) \arrow[d, shift right] \arrow[d, shift left, "\rho(f,g)"] \\
R, S \arrow[r, shift left] \arrow[r, shift right, "k_2,k_3"'] & B, C & & R, S \arrow[r, shift left] \arrow[r, shift right, "k_2,k_3"'] & B, C
\end{tikzcd}
$$

Then proposition 14 tells us that any lifting problem $h : \mathsf{comp}(q; r, s) \to \mathsf{comp}(\ell; m, n)$ has the canonical solution $\chi(\varepsilon'_{(\ell;m,n)}) \circ Th \circ \psi(\eta''_{(q;r,s)})$.

$$
\begin{tikzcd}
P \arrow[r, "1"'] \arrow[d, "q"'] & P \arrow[r, "h_0"] \arrow[d, "\rho(r\circ q,s\circ q)"'] & W \arrow[r, "1"'] \arrow[d, "\rho(m\circ\ell,n\circ\ell)"] & W \arrow[d, "\ell"] \\
Q \arrow[r, "\psi\langle r,s\rangle"'] \arrow[d, shift right, "r,s"'] \arrow[d, shift left] & E(r\circ q, s\circ q) \arrow[r, "E(h_0;h_2,h_3)"] \arrow[d, shift right, "\rho(r\circ q,s\circ q)"'] \arrow[d, shift left] & E(m\circ\ell, n\circ\ell) \arrow[r, "\chi\ell"'] \arrow[d, shift right] \arrow[d, shift left, "\rho(m\circ\ell,n\circ\ell)"] & X \arrow[d, shift right] \arrow[d, shift left, "m,n"] \\
R, S \arrow[r, shift left] \arrow[r, shift right, "1,1"'] & R, S \arrow[r, shift left] \arrow[r, shift right, "h_2,h_3"'] & Y, Z \arrow[r, shift left] \arrow[r, shift right, "1,1"'] & Y, Z
\end{tikzcd}
$$

Lastly, proposition 15 provides the proof that any sprout which has a coherent system of solutions to lifting problems against two-sided cofibrations carries a two-sided fibration, and likewise that that any sprout which has a coherent system of solutions to lifting problems against two-sided fibrations carries a two-sided cofibration.

9 Conclusion

In this thesis we have traced the development of algebraic weak factorization systems, presented the concept of relative monads and comonads, and combined these ideas to define relative weak factorization systems, a generalized, more flexible analogue of weak factorization systems, which is able to incorporate additional shapes of diagrams. We have proved preliminary results about these systems, showing ways in which they do indeed function as analogues of weak factorization systems, as well as some ways in which they do not have all the properties we might have desired. There is much more to do in further examining the behavior of these systems, and their suitability as potential models of identity types and other dependent types in a directed type theory. Furthermore, other possible visions of categorical models should be created and explored, to support and inform the creation of useful, insightful directed type theories.

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