Paraconsistent and Paracomplete Zermelo-Fraenkel Set Theory

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Abstract

We present a novel treatment of set theory in a four-valued *paracomplete* and *paraconsistent* logic, i.e., a logic in which propositions can be neither true nor false, and can be both true and false. By prioritising a system with an ontology of non-classical sets that is easy to understand and apply in practice, our approach overcomes many of the obstacles encountered in previous attempts at such a formalization.

We propose an axiomatic system BZFC, obtained by analysing the ZFC-axioms and translating them to a four-valued setting in a careful manner. We introduce the *anti-classicality axiom* postulating the existence of non-classical sets, and prove a surprising results stating that the existence of a single non-classical set is sufficient to produce any other type of non-classical set.

We also look at bi-interpretability results between BZFC and classical ZFC, and provide an application concerning Tarski semantics, showing that the classical definition of the satisfaction relation yields a logic precisely reflecting the non-classicality in the meta-theory.

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1 Introduction

The classical Zermelo-Fraenkel axiom system ZFC is generally accepted as the foundation of all mathematics. ZFC is formalized in classical logic in which any statement is either true or false, and cannot be both at the same time. Throughout the course of the 20th century, there has been a continuous interest in foundations of mathematics formalized in various non-classical logics. The most notable examples are constructive set theories, such as IZF, CZF and CST, proposed by Harvey Friedman [11], John Myhill [18] and Peter Aczel [1] among others. These theories are based in intuitionistic logic and aim to formalize the constructive side of mathematics.

A more significant departure from classical logic would be a *paraconsistent* set theory, i.e., a set theory in which statements can be both true and false at the same

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time. For such a theory to be non-trivial, the underlying logic must, at the very least, fail to satisfy the *ex falso quodlibet* principle. Attempts at such a formalization have independently been proposed by various authors to a various degree of success, see, e.g., [16, 13] and the more recent [8]. Sometimes, a motivating factor was the desire to adopt some form of *full comprehension* as an axiom and avoid Russell's paradox.

In this paper, we are explicitly *not* concerned with full comprehension. In our view, Russell's paradox is a natural consequence of the distinction between formal language and meta-language, and not something that needs to be avoided. Instead, we prioritise an intuitive treatment of non-classical sets so as to make our theory accessible to the classical mathematician used to working in ZFC.

We propose a natural formalization of set theory in the logic BS4. This is a four-valued, paraconsistent and paracomplete predicate logic based on developments by Dunn [10], Belnap [6, 7] and [4]. The propositional fragment of BS4 appeared under the name CLoNs in [5], and our version is essentially due to Omori, Sano and Waragai [20, 23] (from where we also take its name). In the semantics of BS4, *truth* and *falsity* are formally separated, so a statement φ can be true and not false (1), false and not true (0), both true and false ($\mathfrak{b}$), or neither true nor false ($\mathfrak{n}$). We formulate an axiomatic system called BZFC, based on a careful generalisation of ZFC, together with the *anti-classicality axiom* postulating the existence of non-classical sets.

In our opinion, previous attempts at a similar approach have not been fully successful. We conjecture that this is, in part, due to an insufficiently careful treatment of the language of set theory. For example, in our logic BS4, there are two types of negations: the (native) *paraconsistent negation* and the (defined) *classical negation*. Likewise, there are two types of implications: the *native* implication, and the (defined) *strong* implication. When formulating the axioms of our set theory, a careful approach is needed to determine what the proper generalization of each axiom should be. For instance, we think the failure to properly address this issue is one of the reason that the approach in [8] has not been fully successful.

Our theory will come together with a clear and intuitive *ontology* where a non-classical set u can be described by a *positive* extension (the collection of all x such that $x \in u$ is true) and a *negative* extension (the collection of all x such that $x \in u$ is false),[1] and this can be expressed within the system. The universe of non-classical sets naturally extends the classical *von Neumann* universe of sets, and every model of BZFC contains within it a natural model of ZFC given by the "hereditarily classical" sets. On the other hand, starting in ZFC one can produce a natural model of BZFC, leading to an intuitive bi-interpretability between the two theories.[2]

This paper is almost entirely self-contained—in particular, no prior knowledge of paraconsistent logic is required. In Section 2 we provide the syntax and semantics of the logic BS4 which is used to set up the theory. In Sections 3 and 4 we first motivate

[1] Since the negative extension is, in general, a proper class, we will actually talk about the complement of the negative extension and call it the ?-extension, see Section 3.

[2] The situation is analogous to Peter Aczel's theory of non-well-founded sets [2], which can be viewed as another enrichment of the von Neumann universe, itself consistent relative to ZFC and containing within it a natural copy of a model of ZFC.

and then postulate the axiomatic system PZFC, and in Section 5 we add the *Anti-Classicality Axiom* to obtain the full theory BZFC. Sections 6, 7 and 8 are devoted to the construction of models and bi-interpretability with ZFC, and finally, Section 9 is devoted to a question of philosophical interest: how does formal model theory (using Tarski semantics) depend on the logical properties of the ambient meta-theory in which it is defined?

The work in this paper was carried out in the course of the Master's thesis of the second author [19]. On occasion, we will refer to the thesis which goes in more depth on some points, and contains more details which have been left out of this paper for clarity.

2 The Logic BS4

The logic BS4 is due to [23] with the exception that we take the contradictory constant $\perp$ as primary instead of the classicality operator $\circ$. We start by introducing BS4 from a semantic point of view. In this section, the meta-theory is classical mathematics, e.g., ZFC.[3]

2.1 Syntax and Semantics

The main idea behind BS4 is the separation of truth from falsity, i.e., if φ is a sentence and $\mathcal{M}$ a model, then φ can be or not be *true* in $\mathcal{M}$ and, independently, can be or not be *false* in $\mathcal{M}$. This is achieved by considering two satisfaction relations, $\models^T$ and $\models^F$, separating the inductive definition for the logical connectives and quantifiers, and considering two interpretation of all predicate symbols (a "true" and a "false" interpretation). For convenience we will consider vocabularies without function symbols.

Definition 2.1. (The Syntax of BS4)

The *syntax* of BS4 is the usual syntax of first order logic, except that we use "$\sim$" to denote negation. We also use the constant connective $\perp$.

Definition 2.2. (T/F-models)

Suppose τ is a vocabulary with constant and relation symbols. A *T/F-model* $\mathcal{M}$ consist of a *domain* M together with the following:

- An element $c^{\mathcal{M}} \in M$ for every constant symbol c.
- For every n-ary relation symbol R, a "positive" interpretation $(R^{\mathcal{M}})^+ \subseteq M^n$ and a "negative" interpretation $(R^{\mathcal{M}})^- \subseteq M^n$.
- A binary relation $=^+$ which coincides with the true equality relation; and a binary relation $=^-$ satisfying $a =^- b$ iff $b =^- a$.

[3]For more about formalizing the semantics of BS4 within BZFC, see Section 9.

Definition 2.3. (T/F-semantics for BS4)

Suppose τ is a vocabulary without function symbols and $\mathcal{M}$ a T/F-model. We define two relations, $\models^T$ and $\models^F$, by induction on the complexity of φ:

1. $\mathcal{M} \models^T (t = s)[a, b] \quad \Leftrightarrow \quad a =^+ b.$
   $\mathcal{M} \models^F (t = s)[a, b] \quad \Leftrightarrow \quad a =^- b.$

2. $\mathcal{M} \models^T R(t_1, \dots, t_n)[a_1 \dots a_n] \quad \Leftrightarrow \quad R^+(a_1, \dots, a_n)$ holds.
   $\mathcal{M} \models^F R(t_1, \dots, t_n)[a_1 \dots a_n] \quad \Leftrightarrow \quad R^-(a_1, \dots, a_n)$ holds.

3. $\mathcal{M} \models^T {\sim}\varphi \quad \Leftrightarrow \quad \mathcal{M} \models^F \varphi.$
   $\mathcal{M} \models^F {\sim}\varphi \quad \Leftrightarrow \quad \mathcal{M} \models^T \varphi.$

4. $\mathcal{M} \models^T \varphi \wedge \psi \quad \Leftrightarrow \quad \mathcal{M} \models^T \varphi$ and $\mathcal{M} \models^T \psi$.
   $\mathcal{M} \models^F \varphi \wedge \psi \quad \Leftrightarrow \quad \mathcal{M} \models^F \varphi$ or $\mathcal{M} \models^F \psi$.

5. $\mathcal{M} \models^T \varphi \vee \psi \quad \Leftrightarrow \quad \mathcal{M} \models^T \varphi$ or $\mathcal{M} \models^T \psi$.
   $\mathcal{M} \models^F \varphi \vee \psi \quad \Leftrightarrow \quad \mathcal{M} \models^F \varphi$ and $\mathcal{M} \models^F \psi$.

6. $\mathcal{M} \models^T \varphi \to \psi \quad \Leftrightarrow \quad$ if $\mathcal{M} \models^T \varphi$ then $\mathcal{M} \models^T \psi$.
   $\mathcal{M} \models^F \varphi \to \psi \quad \Leftrightarrow \quad \mathcal{M} \models^T \varphi$ and $\mathcal{M} \models^F \psi$.

7. $\mathcal{M} \models^T \varphi \leftrightarrow \psi \quad \Leftrightarrow \quad$ ($\mathcal{M} \models^T \varphi$ if and only if $\mathcal{M} \models^T \psi$).
   $\mathcal{M} \models^F \varphi \leftrightarrow \psi \quad \Leftrightarrow \quad$ ($\mathcal{M} \models^T \varphi$ and $\mathcal{M} \models^F \psi$) or ($\mathcal{M} \models^F \varphi$ and $\mathcal{M} \models^T \psi$).

8. $\mathcal{M} \models^T \exists x \varphi(x) \quad \Leftrightarrow \quad \mathcal{M} \models^T \varphi[a]$ for some $a \in M$.
   $\mathcal{M} \models^F \exists x \varphi(x) \quad \Leftrightarrow \quad \mathcal{M} \models^F \varphi[a]$ for all $a \in M$.

9. $\mathcal{M} \models^T \forall x \varphi(x) \quad \Leftrightarrow \quad \mathcal{M} \models^T \varphi[a]$ for all $a \in M$.
   $\mathcal{M} \models^F \forall x \varphi(x) \quad \Leftrightarrow \quad \mathcal{M} \models^F \varphi[a]$ for some $a \in M$.

10. $\mathcal{M} \models^T \bot \quad \Leftrightarrow \quad$ never.
    $\mathcal{M} \models^F \bot \quad \Leftrightarrow \quad$ always.

If $\mathcal{M} \models^T \varphi$ then we say that φ is *true* in $\mathcal{M}$, and if $\mathcal{M} \models^F \varphi$ then we say that φ is *false* in $\mathcal{M}$.

Definition 2.4. (Semantic consequence) If Σ is a set of formulas and φ another formula, then *semantic consequence* is defined by

$$\Sigma \vdash_{\mathrm{BS4}} \varphi$$

iff for every T/F-model $\mathcal{M}$, if $\mathcal{M} \models^T \Sigma$ then $\mathcal{M} \models^T \varphi$.

Remark 2.5. While most of the inductive steps in Definition 2.3 are unsurprising, there are two points that need to be addressed:

1. The symbol "$\perp$" should be understood as more than merely a contradiction. While many classical contradictions such as $\varphi \wedge \sim\varphi$ will be satisfiable, the symbol $\perp$ stands for a much stronger contradiction, i.e., one which is not satisfiable even in "paraconsistent" models. Some readers might find the addition of $\perp$ to the logic distasteful as it seems to run counter to the idea of paraconsistent logic. However, when dealing with a vocabulary S with only finitely many relation symbols, one can consider the following sentence:

$$\forall x \forall y (x = y \wedge x \neq y) \wedge \bigwedge_{R \in S} \forall x_1 \ldots \forall x_n (R(x_1, ..., x_n) \wedge \sim R(x_1, ..., x_n)).$$

   This sentence *is* satisfiable, but only in the trivial model consisting of exactly one object a, which is both equal and not equal to itself ($a =^+ a$ and $a =^- a$) and for which all relation-interpretations are *true* and *false*. Adding "$\perp$" to the language is equivalent to disregarding this trivial model. Since we focus on set theory, we will have no interest in such a model and thus have no reservations about $\perp$.

2. The truth-definition for the material implication $\varphi \to \psi$ is designed to reflect semantic consequence and guarantee that BS4 satisfies the deduction theorem. This definition may be debatable, because (together with $\perp$) it will allows us to define classical negation, and to refer not only to *truth* and *falsity*, but also to the *absence* of truth and/or falsity, from within the system. But this should not be seen as a drawback of the system. In fact, the original formulation of BS4 from [23] explicitly contained a "classicality" operator.

Definition 2.6. (Truth value) For every φ and T/F-model $\mathcal{M}$ we define:

$$[\![\varphi]\!]^{\mathcal{M}} \quad := \quad \begin{cases} 1 & \text{if } \mathcal{M} \models^T \varphi \text{ and } \mathcal{M} \not\models^F \varphi \\ \mathfrak{b} & \text{if } \mathcal{M} \models^T \varphi \text{ and } \mathcal{M} \models^F \varphi \\ \mathfrak{n} & \text{if } \mathcal{M} \not\models^T \varphi \text{ and } \mathcal{M} \not\models^F \varphi \\ 0 & \text{if } \mathcal{M} \not\models^T \varphi \text{ and } \mathcal{M} \models^F \varphi \end{cases}$$

We can now view BS4 as a four-valued logic with truth tables for propositional connectives given in Table 1.

2.2 Defined connectives

We will need several defined connectives to make the presentation more smooth and intuitive.

First we look at material implication: notice that $\mathcal{M} \models^T \varphi \to \psi$ tells us that if φ is true in $\mathcal{M}$ then ψ is true in $\mathcal{M}$, but does not tell us that if ψ is false in $\mathcal{M}$ then φ is false in $\mathcal{M}$, as can easily be verified. Similarly, a bi-implication $\varphi \leftrightarrow \psi$ tells us that, in a model $\mathcal{M}$, φ is true iff ψ is true, but not that φ is false iff ψ is false. In particular, $\varphi \leftrightarrow \psi$ does not allow us (as we are used from classical logic) to substitute an arbitrary occurrences of φ with ψ within a larger formula.

For this reason, we define the following abbreviations, which we call *strong implication* and *strong bi-implication*, respectively.[4]

[4] The strong implication appears, e.g., in [22, Chapter XII].

	$\sim$
1	0
$\mathfrak{b}$	$\mathfrak{b}$
$\mathfrak{n}$	$\mathfrak{n}$
0	1

$\wedge$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$	0	0
$\mathfrak{n}$	$\mathfrak{n}$	0	$\mathfrak{n}$	0
0	0	0	0	0

$\vee$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	1	1	1
$\mathfrak{b}$	1	$\mathfrak{b}$	1	$\mathfrak{b}$
$\mathfrak{n}$	1	1	$\mathfrak{n}$	$\mathfrak{n}$
0	1	$\mathfrak{b}$	$\mathfrak{n}$	0

$\to$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{b}$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{n}$	1	1	1	1
0	1	1	1	1

$\leftrightarrow$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$	1	1
0	0	0	1	1

Table 1: Truth tables for the propositional connectives of BS4

- $\varphi \Rightarrow \psi$ abbreviates $(\varphi \to \psi) \wedge (\sim\psi \to \sim\varphi)$
- $\varphi \Leftrightarrow \psi$ abbreviates $(\varphi \leftrightarrow \psi) \wedge (\sim\varphi \leftrightarrow \sim\psi)$.

In particular, if $\varphi \Leftrightarrow \psi$ is true then any occurrence of φ may be substituted with ψ and vice versa. The distinction between regular and strong implication and bi-implication will play a crucial role in the generalization of the axioms of set theory.

Next, following up on Remark 2.5 we define *classical negation* as follows:

- $\neg\varphi$ abbreviates $\varphi \to \bot$

One can easily check that $\mathcal{M} \models^T \neg\varphi$ iff $\mathcal{M} \not\models^T \varphi$ while $\mathcal{M} \models^F \neg\varphi$ iff $\mathcal{M} \models^T \varphi$. So in a model $\mathcal{M}$, $\neg\varphi$ can be either true and not false (when $\mathcal{M} \models^T \varphi$) or false and not true (when $\mathcal{M} \not\models^T \varphi$). Using classical negation as a defined notion we can talk about presence and absence of truth and falsity (and, more generally, specify the truth value of a formula) from within the system. We will use the following important abbreviations:

- $!\varphi$ abbreviates $\sim\neg\varphi$
- $?\varphi$ abbreviates $\neg\sim\varphi$

We think of $!\varphi$ as *presence of truth* and $?\varphi$ as *absence of falsity.* The truth tables for $\sim, \neg, !$ and $?$ in Table 2 make this clear. Notice that $\neg\varphi$, $!\varphi$ and $?\varphi$ will always have truth value 1 or 0. Moreover, the truth value of $\neg\varphi$ and $!\varphi$ depends *only* on whether φ was *true* in the model, and completely disregards whether φ was *false.* Similarly, $?\varphi$ depends only on whether φ was *false* and disregards whether it was *true.*

A BS4-formula is *complete* if it can never obtain the truth-value $\mathfrak{n}$, and *consistent* if it can never obtain the truth-value $\mathfrak{b}$. It is called *classical* if it is both complete and consistent, i.e., if in every model it has truth-value 1 or 0. In particular, $\neg\varphi$, $!\varphi$ and $?\varphi$ are classical formulas for any φ. Notice also that classicality, completeness and consistency can each be expressed within the system, by $!\varphi \leftrightarrow ?\varphi$, $?\varphi \to !\varphi$ and $!\varphi \to ?\varphi$ respectively. We will use the following abbreviation in accordance to [4]:

- $\circ\varphi$ abbreviates $!\varphi \leftrightarrow ?\varphi$

φ	$\sim\varphi$	$\neg\varphi$	$!\varphi$	$?\varphi$
1	0	0	1	1
$\mathfrak{b}$	$\mathfrak{b}$	0	1	0
$\mathfrak{n}$	$\mathfrak{n}$	1	0	1
0	1	1	0	0

Table 2: Truth table for $\sim, \neg, !$ and $?$.

2.3 Proof system

A sound and complete proof calculus for the propositional fragment of BS4 is presented in [23]. We present a slightly modified but equivalent version, consisting of the following axioms and rules of inference:

- Axioms of classical predicate logic:

  1. $\varphi \to (\psi \to \varphi)$
  2. $(\varphi \to (\psi \to \chi)) \to$
     $((\varphi \to \psi) \to (\varphi \to \chi))$
  3. $\varphi \vee (\varphi \to \psi)$
  4. $(\varphi \wedge \psi) \to \varphi$
  5. $(\varphi \wedge \psi) \to \psi$
  6. $\varphi \to (\psi \to (\varphi \wedge \psi))$
  7. $\varphi \to (\varphi \vee \psi)$
  8. $\psi \to (\varphi \vee \psi)$
  9. $(\varphi \to \chi) \to ((\psi \to \chi) \to$
     $((\varphi \vee \psi) \to \chi))$
  10. $\bot \to \varphi$
  11. $\forall x \varphi(x) \to \varphi(t)$
  12. $\varphi(t) \to \exists x \varphi(x)$
  13. $x = x$
  14. $x = y \to (\varphi(x) \to \varphi(y))$

- Axioms for negation:

  15. $\sim\sim\varphi \leftrightarrow \varphi$
  16. $\sim(\varphi \wedge \psi) \leftrightarrow (\sim\varphi \vee \sim\psi)$
  17. $\sim(\varphi \vee \psi) \leftrightarrow (\sim\varphi \wedge \sim\psi)$
  18. $\sim(\varphi \to \psi) \leftrightarrow (\varphi \wedge \sim\psi)$
  19. $\sim\bot$
  20. $\sim\forall x \varphi \leftrightarrow \exists x \sim\varphi$
  21. $\sim\exists x \varphi \leftrightarrow \forall x \sim\varphi$
  22. $\sim(x = y) \to \sim(y = x)$.[5]

- The rules of inference:
  - From φ and $\varphi \to \psi$, infer ψ (modus ponens).
  - Infer $\varphi \to \forall x \psi$ from $\varphi \to \psi$, provided x does not occur free in φ.
  - Infer $\exists x \varphi \to \psi$ from $\varphi \to \psi$, provided x does not occur free in ψ.

[5] Axiom 22 does not occur in the original formulation in [23], but we need to add it to take care of the semantic requirement that $=^-$ is a symmetric relation.

Lemma 2.7. *The calculus described above is sound and complete with respect to T/F-semantics.*

Proof. This follows from [23, Corollary 5.15], after an adaptation to refer to $\perp$ as the primary symbol as opposed to the classicality operator $\circ$. We leave out the details. $\square$

In practice, we will reason informally within the system BS4 using arguments which can, in principle, be formalized in the calculus. We will often be referring to the defined connectives as well. Below, we specifically mention a few provable statements concerning defined connectives which will frequently be needed in later arguments.

Lemma 2.8. *The following statements are provable in* BS4*:*

- $\varphi \leftrightarrow\, !\varphi$
- $\sim\!\varphi \leftrightarrow\, \sim ?\varphi$

  (This describes the fact that ! talks only about *truth* and ? only about *faslity.*)
- $x = y \to (\varphi(x) \Leftrightarrow \varphi(y))$

  (This says that a true equality allows us to interchange terms).
- $\circ\varphi \quad \to \quad ((\varphi \Leftrightarrow\, !\varphi) \wedge (\varphi \Leftrightarrow\, ?\varphi))$
- $\circ\varphi \quad \to \quad (\sim\!\varphi \Leftrightarrow \neg\varphi)$
- $\circ\varphi \wedge \circ\psi \quad \to \quad ((\varphi \to \psi) \Leftrightarrow (\varphi \Rightarrow \psi))$

  (The last three points say that for *classical* formulas there is no distinctions between strong and weak implication, between native and classical negation, and that ! and ? may be dropped.)

3 Non-classical sets

Before delving into the axioms, it is helpful to think about the concept of a set in a paraconsistent and paracomplete setting from a naive point of view. In classical ZFC, a *set* x is identified with the class of its elements $\{y : y \in x\}$ and divides the entire universe of sets in two parts: those y that are *in* x, and those y that are *not in* x. In the context of paraconsistent and paracomplete logic, we can have a situation where a set y is both in x and not in x, or a situation where y is neither in x, nor is it the case that y is not in x.

Therefore, it seems natural to identify a set x with the pair consisting of a *positive* extension $\{y : y \in x\}$ and a *negative* extension $\{y : y \notin x\}$[6] without the added requirement that one is the complement of the other. In fact, it makes sense to call a

[6]We write $y \notin x$ instead of $\sim(y \in x)$.

set *consistent* if its positive and negative extensions do not intersect, *complete* if their union is the whole universe, and *classical* if it is both consistent and complete.

There is, however, an asymmetry between the positive and negative extensions: $\{y : y \in x\}$ is a set while $\{y : y \notin x\}$ is a proper class. Therefore, it turns out to be more appropriate to talk about the *complement* of the negative extension, i.e., the collection of all y for which the statement "$y \notin x$" is *not true*, or, equivalently, for which the statement "$y \in x$" is *not false*). We will refer to this collection as the ?-*extension* of x.

Above, we have naively referred to statements being *true* or *false*, which, a priori, seem to be meta-theoretic notions. However, recall that the operators ! and ? allow us to discuss truth and falsity from *within* the system as well. In particular, $!(y \in x)$ is *true* if and only if $y \in x$ is *true*, and $?(y \in x)$ is *true* if and only if $y \in x$ is *not false*. This motivates the following

Definition 3.1. Let x be a set:

- The !-*extension* of x is
$$x^! := \{y \ : \ !(y \in x)\}$$
- The ?-*extension* of x is
$$x^? := \{y \ : \ ?(y \in x)\}$$

For the time being, it is not clear that the above collections describe sets and not proper classes, but we shall set up the axiomatic framework in such a way that if x is a set, then both $x^!$ and $x^?$ are sets. The four boolean combinations of $x^!$ and $x^?$ determine the classes consisting of all y for which the statement "$y \in x$" has one of the four possible truth-values, as visualized in Figure 1.

We remark that the property of sets being complete, consistent and classical can be expressed within the system. In fact the following holds:[7]

- x is *complete* if $x^? \subseteq x^!$
- x is *consistent* if $x^! \subseteq x^?$
- x is *classical* if $x^! = x^?$

The intuition is that classical sets behave as we are used to in ZFC: in particular, it does not matter whether we consider the native negation $\sim$ or classical negation $\neg$, nor whether we use $\to$ or $\Rightarrow$. It is not hard to see that $(x^!)^! = (x^!)^? = x^!$ and $(x^?)^! = (x^?)^? = x^?$, so the !-extension and ?-extension themselves are classical sets.

Let us now look more closely at the notion of *equality* of two sets and the extensionality principle.[8] In ZFC, two sets are equal precisely if the classes of their elements are the same; equivalently, two sets are different precisely if one set contains an element which the other does not, or vice versa.

[7] For now, the symbols $=$ and $\subseteq$ should be understood informally; this will be made precise later.

[8] Although we have already used notation such as "$x = y$" and even "$x \subseteq y$" without properly defining it, in fact we only used them for *classical* sets, in which case it means the same thing as in classical ZFC.

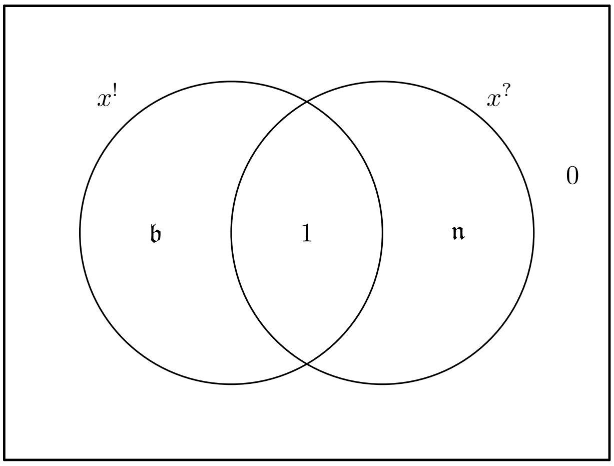

Figure 1: The four truth values of "$y \in x$" depending on the boolean combination of $x^!$ and $x^?$.

In our setting, $x = y$ and $x \neq y$ could both be true statements[9] or it could happen that neither $x = y$ nor $x \neq y$ are true statements. But we still require that "$x = y$" express the idea that x and y are the same set-theoretic objects, meaning that both its !-extension and ?-extension must be the same. Likewise, we still want "$x \neq y$" to express that there is something in x which is not in y, or vice versa. So the guiding principle behind the extensionality axiom must be the following:

- $x = y \quad \leftrightarrow \quad (x^! = y^! \wedge x^? = y^?)$
- $x \neq y \quad \leftrightarrow \quad \exists z((z \in x \wedge z \notin y) \vee (z \in y \wedge z \notin x))$

Figure 2 illustrates the situation in which two sets x and y are equal (because their !-extensions and ?-extensions are the same) but also unequal (because there is a set z such that $z \in x$ but $z \notin y$). In fact, if x is any inconsistent set, then $x = x$ and $x \neq x$.

It is customary in set theory to use notation such as $\{y : \varphi(y)\}$ to refer to the set (if it exists) of all objects y satisfying property φ. In our setting, where a set is determined by its !-extension and ?-extension, it makes sense to agree on the following:

Convention 3.2.

$$x = \{y : \varphi(y)\} \quad \text{abbreviates} \quad \forall y \, (y \in x \, \Leftrightarrow \, \varphi(y)).$$

The use of the strong implication means that $x^!$ is the set of all y for which $\varphi(y)$ is *true*, and $x^?$ is the set of all y for which $\varphi(y)$ is *not false*.

[9] Again, we write $x \neq y$ instead of $\sim(x = y)$.

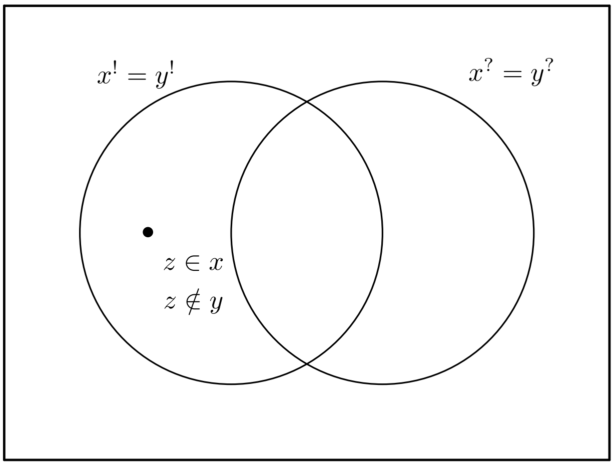

Figure 2: $x = y$ and $x \neq y$.

Notably, our theory does not, and does not attempt to, satisfy *full comprehension* or avoid Russell's paradox. In fact, the same proof as usual shows that $R := \{x : \neg(x \in x)\}$ is a proper class and not a set. Likewise, the universe of all sets does not form a set.

A last point of subtlety should be discussed: how do we understand notation such as $\{u\}$, for a set u? One might initially assume that this is $\{y : y = u\}$ (referring to Convention 3.2). However, a closer look reveals the following: if $\{y : y = u\}$ is a set, then we would like its ?-extension to also be a set. However, a bit of work following the definitions shows that this ?-extension would be the collection of all sets a such that $a^! \subseteq u^?$ and $u^! \subseteq a^?$. Since there is no upper bound on $a^?$, it would seem (following classical intuition) that there are class-many sets a satisfying this condition, meaning that the ?-extension of this set is, in fact, a proper class.[10]

Instead, we opt for the following:

Convention 3.3. If u is a set, then $\{u\}$ is the set $\{y : !(y = u)\}$; and more generally if $u_0, \ldots, u_n$ are sets, then $\{u_0, \ldots, u_n\}$ is the set $\{y : !(y = u_0) \vee \cdots \vee !(y = u_n)\}$. This is always a *classical* set whose !-extension and ?-extension are exactly the finite set consisting of the elements in question.

A similar phenomenon occurs with other set-theoretic operations, most notably the power set operation.

[10] We will provide a proper proof of this result in Lemma 5.5

4 The theory PZFC

We will now present the axiomatic system in this. The first step is to formulate an appropriate generalization of the ZFC axioms, which is the task of the current section. It will give rise to the theory PZFC, which is an intermediate step to the full theory BZFC presented in the next section.

Here it is important to have a careful look, and put in some additional effort, to make sure that the axioms are translated in accordance with the intuition described in the previous section. For this to succeed, it is usually not enough to merely re-interpret the ZFC axioms in our logic.

4.1 Extensionality

The first and most essential axiom needed to sustain an ontology of sets is the following version of the extensionality axiom.

$$\textbf{Extensionality:} \quad \boldsymbol{\forall x \forall y (x = y \ \Leftrightarrow\ \forall z (z \in x \Leftrightarrow z \in y))}.$$

Why do we choose strong rather than weak bi-implications? For the first one the choice is clear: extensionality seeks to define the *meaning* of the expression "$x = y$" in terms of the elements of x and y, and this needs to talk both about *truth* and *falsity*. Notice that the use of this strong implications allows us to *interchange* the expression "$x = y$" with the expression on the right-hand side within any given formula.

The second bi-implication is more interesting: recall that we want $x = y$ to express that both the !-extension and ?-extension of x and y coincide. The statement with a weak implication $\forall z(z \in x \leftrightarrow z \in y))$ would only say that the !-extensions coincide.

At this point it is instructive to define subsets:

$$x \subseteq y \quad \text{abbreviates} \quad \forall z\ (z \in x \Rightarrow z \in y)$$

Again, the use of the strong implication makes sure that $x \subseteq y$ says that both the !-extensions and ?-extensions of x are included in that of y. In particular, the Axiom of Extensionality can now be reformulated as follows: $\forall x \forall y (x = y \ \Leftrightarrow\ x \subseteq y \wedge y \subseteq x)$.

4.2 Comprehension

The second most important axiom is comprehension, which is as usual and again uses the strong bi-implication:[11]

$$\textbf{Comprehension:} \ \boldsymbol{\forall u \exists x \forall y\ (y \in x \Leftrightarrow y \in u \wedge \varphi(y))}$$

This axiom allows us to use traditional set-theoretic notation

$$x = \{y \in u : \varphi(y)\}.$$

As in Convention 3.2, this is an abbreviation for $y \in x \Leftrightarrow y \in u \wedge \varphi(y)$, and the Comprehension axiom guarantees that if u is a set then x is also a set.

[11] We suppress mention of parameters to simplify notation.

4.3 Classical Supersets

The development of our theory is greatly simplified by considering an axiom postulating that every set is contained within a classical superset. In Section 3 we said that a set was *classical* if its !-extension and ?-extension are the same. We do not know yet whether !-extensions and ?-extensions are sets, but we can express that C is a classical set with the sentence $\forall y \circ(y \in C)$, where $\circ$ is the classicality operator.

$$\textbf{Classical Superset:} \quad \forall \boldsymbol{x}\, \exists \boldsymbol{C}\, (\boldsymbol{x} \subseteq \boldsymbol{C} \ \wedge\ \forall \boldsymbol{y} \circ(\boldsymbol{y} \in \boldsymbol{C}))$$

Strictly speaking, this axiom is superfluous: it can be proved to follow from the remaining axioms in a roundabout way. However, adopting it at this stage allows us to frame the remaining axioms, and the theory in general, in a more intuitive way. In particular, it allows us to confirm several properties of sets we postulated in the previous section.

Lemma 4.1 (Cl. Superset + Comprehension)**.** *If x is a set, then its !-extension and ?-extension are sets.*

Proof. Let C be a classical set with $x \subseteq C$. Then

$$\begin{aligned} x^! &= \{y \in C\ : !(y \in x)\} \\ x^? &= \{y \in C\ : ?(y \in x)\} \end{aligned}$$

are both sets by the Comprehension axiom. □

Since $x^!$ and $x^?$ are classical, *a posteriori* we see that a particularly convenient classical superset of x is obtained by considering $x^! \cup x^?$. This is the *smallest* classical set containing x and we will sometimes call this the *realm* of x:

$$\mathrm{rlm}(x) := x^! \cup x^?.$$

The next lemma elaborates on the meaning of equality and the subset relation between sets; note that the bi-implications are native and not strong, so the equivalence relates to truth values of the statements in question.

Lemma 4.2 (Cl. Superset + Extensionality + Comprehension)**.**

1. $x \subseteq y \ \leftrightarrow\ x^! \subseteq y^! \wedge x^? \subseteq y^?$
2. $x \not\subseteq y \ \leftrightarrow\ \exists z(z \in x \wedge z \notin y) \ \leftrightarrow\ x^! \not\subseteq y^?$
3. $?(x \subseteq y) \ \leftrightarrow\ x^! \subseteq y^?$
4. $x = y \ \leftrightarrow\ x^! = y^! \wedge x^? = y^?$
5. $x \neq y \ \leftrightarrow\ \exists z((z \in x \wedge z \notin y) \vee (z \in y \wedge z \notin x)) \ \leftrightarrow\ x^! \not\subseteq y^? \vee y^! \not\subseteq x^?$
6. $?(x = y) \ \leftrightarrow\ x^! \subseteq y^? \wedge y^! \subseteq x^?$

Proof. We will provide detailed proofs of 1 and 2 in order to illustrate how reasoning within BS4 works. The remaining proofs are left to the reader.

1. We have the following sequence of BS4-provable bi-implications:

$$
\begin{array}{ll}
& x \subseteq y \\
\Leftrightarrow & \\
& \forall z\,((z \in x \to z \in y)\ \wedge\ (\sim(z \in y) \to \sim(z \in x)) \\
\overset{(*)}{\leftrightarrow} & \\
& \forall z\,((!(z \in x) \to !(z \in y))\ \wedge\ (?(z \in x) \to ?(z \in y)) \\
\overset{(**)}{\Leftrightarrow} & \\
& \forall z\,((z \in x^! \Rightarrow z \in y^!)\ \wedge\ (z \in x^? \Rightarrow z \in y^?) \\
\Leftrightarrow & \\
& x^! \subseteq y^! \wedge x^? \subseteq y^?.
\end{array}
$$

   Here $(*)$ is due to the truth-functional definition of the implications and the ! and ? operators, while $(**)$ is due to the definition of $x^!, y^!, x^?, y^?$ and the fact that these sets are classical (so $\to$ and $\Rightarrow$ are interchangeable). The first and last strong bi-implication is the definition of $\subseteq$.

2. We have the following sequence of bi-implications:

$$
\begin{array}{ll}
& \sim(x \subseteq y) \\
\Leftrightarrow & \\
& \sim\forall z(z \in x \Rightarrow z \in y) \\
\Leftrightarrow & \\
& \exists z\sim(z \in x \Rightarrow z \in y) \\
\overset{(*)}{\leftrightarrow} & \\
& \exists z(z \in x \wedge z \notin y)
\end{array}
$$

   where $(*)$ is because we are only looking at the *truth* condition of $\Rightarrow$. Further:

$$
\begin{array}{ll}
& \ldots \\
\overset{(**)}{\leftrightarrow} & \\
& \exists z(z \in x^! \wedge z \notin y^?) \\
\Leftrightarrow & \\
& x^! \not\subseteq y^?.
\end{array}
$$

   where $(**)$ is again due to the fact that ! refers to truth and ? to falsity. □

4.4 Replacement

The next axiom we consider is Replacement.[12] In ZFC the Replacement Axiom tells us that the image of a set under a class function is itself a set. It is not immediately clear how to generalize a *class function*. As a guiding principle we rely on the intuition that in practice, mathematicians apply Replacement when there is a pre-determined recipe by which each element of a given set is *replaced* by another element in a non-ambiguous way. We will call such a recipe an *operation*:

[12] As before, we suppress mention of parameters to simplify notation.

Definition 4.3. An *operation* is a formula $\varphi(x,y)$ such that

1. φ is classical, and
2. $\forall x \exists y(\varphi(x,y) \ \wedge \ \forall z\ (\varphi(x,z) \rightarrow !(y=z))$.

The reason we require φ to be classical is because the operation that maps x to y is supposed to occur in a well-defined way, and we are not interested in distinguishing between $\varphi(x,y)$ being *false* and $\varphi(x,y)$ being *not true*. Likewise, the reason we add the "!" is to make sure that the statement expressing that every input has at most one output is a classical sentence, since we would not know how to interpret a situation in which the statement "every input has at most one output" is both true and false.[13]

$$\textbf{Replacement:} \quad (\circ\varphi \wedge \forall \boldsymbol{x} \exists \boldsymbol{y}(\varphi(\boldsymbol{x},\boldsymbol{y}) \wedge \forall \boldsymbol{z}(\varphi(\boldsymbol{x},\boldsymbol{z}) \rightarrow !(\boldsymbol{y}=\boldsymbol{z}))) \rightarrow$$
$$\forall \boldsymbol{x} \exists \boldsymbol{y} \forall \boldsymbol{z}(\boldsymbol{z} \in \boldsymbol{y} \Leftrightarrow \exists \boldsymbol{w}(\boldsymbol{w} \in \boldsymbol{x} \wedge \varphi(\boldsymbol{w},\boldsymbol{z})))$$

After adopting this axiom, we can treat an operation φ as a class function F, and use notation such as

$$F[X] := \{y : \exists x\ (x \in X \wedge \varphi(x,y)\}$$

where X is a set.

4.5 Pairing

The Pairing Axiom is normally needed to get set-theory "started", e.g., by allowing the definition of ordered pairs, relations, functions, and so on. Recall the discussion in Convention 3.3 in Section 3 and that we want notation such as $\{u,v\}$ to stand for the *classical* set whose !-extension and ?-extension contain exactly the two objects u and v. This motivates the following

$$\textbf{Pairing:} \quad \forall \boldsymbol{u} \forall \boldsymbol{v} \exists \boldsymbol{x} \forall \boldsymbol{y}(\boldsymbol{y} \in \boldsymbol{x} \ \Leftrightarrow \ (!(\boldsymbol{y}=\boldsymbol{u}) \vee !(\boldsymbol{y}=\boldsymbol{v})))$$

As discussed in Section 3, this falls in line with the intuition of an unordered pair $\{u,v\}$ as a classical set. Referring to Lemma 4.2. (5) and (6), we now see that, if we had used the definition $\{x : x = u \vee x = v\}$ for the unordered pair, then the ?-extension would have been the collection of all x such that $x^! \subseteq u^?$ and $u^! \subseteq x^?$, or $x^! \subseteq v^?$ and $v^! \subseteq x^?$. This is problematic since there is no upper bound on the size of $x^?$. Indeed, in Lemma 5.5 we will prove that if there exists at least one incomplete set, then $\{x : x = u \vee x = v\}$ is a proper class.

4.6 Power Set

Suppose u is any set and we look at $\mathscr{P}(u) := \{x : x \subseteq u\}$. Referring to Lemma 4.2, we again notice that $?(x \subseteq u) \quad \leftrightarrow \quad x^! \subseteq u^?$, so the ?-extension of $\mathscr{P}(u)$ consists of sets x such that $x^! \subseteq u^?$, but with no special requirement on $x^?$. Again, it should be intuitively clear that there is a proper class of possible x satisfying this requirement. As with Pairing, we instead opt for the following definition:

[13] One can check that this will occur, for example, with the formula $\varphi(x,y) \ \equiv !(y=a)$ where a is any non-classical set.

$$\mathscr{P}^!(u) := \{x : !(x \subseteq u)\}$$

Now $\mathscr{P}^!(u)$ is a classical set containing exactly those sets x for which $x^! \subseteq u^!$ and $x^? \subseteq u^?$. This motivates the axiom:

$$\textbf{Power Set:} \;\; \forall \boldsymbol{u} \exists \boldsymbol{v} \forall \boldsymbol{x}(\boldsymbol{x} \in \boldsymbol{v} \;\Leftrightarrow\; \boldsymbol{!}(\boldsymbol{x} \subseteq \boldsymbol{u}))$$

4.7 The remaining axioms

We will now list the remaining four axioms since they are, mostly, non-problematic.

$$\textbf{Union:} \;\; \forall \boldsymbol{u} \exists \boldsymbol{x} \forall \boldsymbol{y}(\boldsymbol{y} \in \boldsymbol{x} \;\Leftrightarrow\; \exists \boldsymbol{z}\, (\boldsymbol{y} \in \boldsymbol{z} \wedge \boldsymbol{z} \in \boldsymbol{u})$$

After adopting this axiom we can use the abbreviation $\bigcup u := \{x : \exists z\,(y \in z \wedge z \in u)\}$, and this coincides with Convention 3.2.

$$\textbf{Infinity:} \;\; \exists \boldsymbol{x}\; (\varnothing \in \boldsymbol{x} \;\wedge \forall \boldsymbol{y}(\boldsymbol{y} \in \boldsymbol{x} \to \boldsymbol{y} \cup \{\boldsymbol{y}\} \in \boldsymbol{x})$$

This axiom, among other things, allows us to define numbers, the ordinal ω, and other transfinite ordinals.

$$\textbf{Foundation:} \;\; \forall \boldsymbol{x}(\forall \boldsymbol{y}(\boldsymbol{y} \in \boldsymbol{x}^! \cup \boldsymbol{x}^? \to \varphi(\boldsymbol{y})) \to \varphi(\boldsymbol{x})) \quad \to \quad \forall \boldsymbol{u} \varphi(\boldsymbol{u})$$

This axiom could more properly be called "set induction schema". It allows us to view the universe as being constructed by transfinite recursion, where each new level consists of those x for which the *realm* $x^! \cup x^?$ is a subset of the previous level. See [19, Section 4.12] for details.

$$\textbf{Choice:} \quad \forall \boldsymbol{u}(\forall \boldsymbol{x}(\boldsymbol{x} \in \boldsymbol{u} \to \exists \boldsymbol{y}(\boldsymbol{y} \in \boldsymbol{u})) \;\to$$
$$\exists \boldsymbol{f}(\mathbf{dom}(\boldsymbol{f}) = \boldsymbol{u} \;\wedge\; \forall \boldsymbol{x}(\boldsymbol{x} \in \boldsymbol{u} \to \boldsymbol{f}(\boldsymbol{x}) \in \boldsymbol{x})))$$

Notice that this is the standard formulation of the axiom of choice saying that if every element of u is non-empty, then there is a choice function f for u. Technically, we have not defined functions as sets in our theory; in fact, we have not even properly defined relations, Cartesian products or ordered pairs. But this can be done in a standard way after settling a few basic issues concerning ordered pairs. The interested reader may find the details in [19, Sections 4.8, 4.9 and Appendix A].

4.8 Summary

We summarize the theory we have built up.

Definition 4.4. The theory PZFC consist of the following BS4-axioms:

1. **Extensionality:** $\forall x \forall y(x = y \;\Leftrightarrow\; \forall z(z \in x \Leftrightarrow z \in y))$.
2. **Comprehension:** $\forall u \exists x \forall y\; (y \in x \Leftrightarrow y \in u \wedge \varphi(y))$.
3. **Classical Superset:** $\forall x\, \exists C\, (x \subseteq C \;\wedge\; \forall y\, \circ(y \in C))$.

4. **Replacement:** $(\circ\varphi \wedge \forall x \exists y(\varphi(x,y) \wedge \forall z(\varphi(x,z) \rightarrow !(y=z))) \rightarrow$
$\forall x \exists y \forall z(z \in y \Leftrightarrow \exists w(w \in x \wedge \varphi(w,z)))$.
5. **Pairing:** $\forall u \forall v \exists x \forall y(y \in x \;\Leftrightarrow\; (!(y=u) \vee !(y=v)))$.
6. **Power Set:** $\forall u \exists v \forall x(x \in v \;\Leftrightarrow\; !(x \subseteq u))$.
7. **Union:** $\forall u \exists x \forall y(y \in x \;\Leftrightarrow\; \exists z\,(y \in z \wedge z \in u)$.
8. **Infinity:** $\exists x\,(\varnothing \in x \;\wedge \forall y(y \in x \rightarrow y \cup \{y\} \in x)$.
9. **Foundation:** $\forall x(\forall y(y \in x^! \cup x^? \rightarrow \varphi(y)) \rightarrow \varphi(x)) \;\;\rightarrow\;\; \forall u \varphi(u)$.
10. **Choice:** $\forall u(\forall x(x \in u \rightarrow \exists y(y \in u)) \;\rightarrow\; \exists f(\mathrm{dom}(f) = u \;\wedge\; \forall x(x \in u \rightarrow f(x) \in x)))$.

Further, we make a note that the theory allows us to define, with relative ease, the following concepts needed for the standard development of mathematics: *ordered pairs*, *relations*, *functions*, *natural numbers*, *ordinals*, the *cumulative hierarchy* of sets, and the concept of *rank*. Definitions by recursion and proofs by induction are also non-problematic. We leave out the detailed construction of these concepts and refer the reader to [19, Chapter 4 and Appendix A].

5 The anti-classicality axiom and BZFC

None of the axioms of PZFC guarantee the existence of an inconsistent or incomplete set. In fact, the following should be clear:

Theorem 5.1. PZFC + $\forall x(x^! = x^?)$ *is equivalent to* ZFC.

Proof. If every set is classical, then there is no distinction between $\sim$ and $\neg$, nor between $\rightarrow$ and $\Rightarrow$. Likewise, ! and ? can be discarded. The axiom Classical Superset is trivially true. So what remains of PZFC is precisely the collection of ZFC axioms. $\square$

Since we are interested in theories with non-classical sets, we would like to adopt an axiom postulating their existence. Now we seem to be faced with a choice: exactly *which* non-classical sets do we want to postulate the existence of? Following a conservative approach, we might want to add an axiom that says only that *some* inconsistent set exists, and *some* incomplete set exists:

$$\exists x(x^! \not\subseteq x^?) \;\wedge\; \exists x(x^? \not\subseteq x^!)$$

On the other hand, a maximality approach might lead us to postulate the existence of *as many non-classical sets* as possible. The boldest statement of this kind would be to postulate that for *any* classical sets u, v, there exists a set x whose !-extension is exactly u and whose ?-extension is exactly v

$$\forall u \forall v(u \text{ and } v \text{ classical } \;\rightarrow\; \exists x\,(x^! = u \wedge x^? = v)).$$

An unexpected, yet surprisingly simple result now shows that in the presence of the other PZFC-axioms, any choice we make is equivalent, since the weakest of them (the conservative one) already implies the strongest (the maximizing one).

Theorem 5.2 (PZFC)**.** *Suppose there is an inconsistent and an incomplete set. Then for any classical sets u, v, there is x such that $x^! = u$ and $x^? = v$.*

Proof. From the assumption we have $a \in b \wedge a \notin b$ for some sets a, b and also $\neg(c \in d \vee c \notin d)$ for some sets c, d. Let us introduce the following abbreviation:

$$\begin{aligned} \phi_{\mathfrak{b}} &\equiv a \in b \\ \phi_{\mathfrak{n}} &\equiv c \in d \end{aligned}$$

Note that $\phi_{\mathfrak{b}}$ and $\sim\!\phi_{\mathfrak{b}}$ are both true, while $\phi_{\mathfrak{n}}$ and $\sim\!\phi_{\mathfrak{n}}$ are both not true.

Let u and v be classical sets and define

$$x := \{z \in u \cup v \;\;:\;\; z \in (u \cap v) \;\vee\; (z \in (u \setminus v) \wedge \phi_{\mathfrak{b}}) \;\vee\; (z \in (v \setminus u) \wedge \phi_{\mathfrak{n}})\}$$

Then we have:

$$\begin{aligned} & z \in x \\ \leftrightarrow\; & \\ & z \in (u \cap v) \;\vee\; (z \in (u \setminus v) \wedge \phi_{\mathfrak{b}}) \;\vee\; (z \in (v \setminus u) \wedge \phi_{\mathfrak{n}}) \\ \leftrightarrow\; & \\ & z \in (u \cap v) \;\vee\; z \in (u \setminus v) \\ \leftrightarrow\; & \\ & z \in u. \end{aligned}$$

And:

$$\begin{aligned} & z \notin x \\ \leftrightarrow\; & \\ & z \notin (u \cap v) \;\wedge\; (z \notin (u \setminus v) \vee \sim\!\phi_{\mathfrak{b}}) \;\wedge\; (z \notin (v \setminus u) \vee \sim\!\phi_{\mathfrak{n}}) \\ \leftrightarrow\; & \\ & z \notin (u \cap v) \;\wedge\; z \notin (v \setminus u) \\ \leftrightarrow\; & \\ & z \notin v. \end{aligned}$$

It follows that $z \in x^! \leftrightarrow z \in u$ and $z \in x^? \leftrightarrow z \in v$. This completes the proof. $\square$

Definition 5.3. We let the *anti-classicality axiom* be the statement:

$$\textbf{ACLA:} \;\; \exists \boldsymbol{x}(\boldsymbol{x}^! \not\subseteq \boldsymbol{x}^?) \;\wedge\; \exists \boldsymbol{x}(\boldsymbol{x}^? \not\subseteq \boldsymbol{x}^!)$$

and consider the system

$$\mathrm{BZFC} \;\equiv\; \mathrm{PZFC} + \mathrm{ACLA}$$

This is the main axiomatic system under consideration in this paper.

Remark 5.4. Some readers might be interested in a finer distinction and consider a theory that has only incomplete but not inconsistent sets. The corresponding theory would then be PZFC $+\, \exists x(x^? \not\subseteq x^!) + \forall x(x^! \subseteq x^?)$. A proof similar to the above would then show that this implies that for any classical sets u, v with $u \subseteq v$, there is x such that $x^! = u$ and $x^? = v$. Likewise, if one is interested in a theory that has only inconsistent but not incomplete sets one can look at PZFC$+\exists x(x^! \not\subseteq x^?)+\forall x(x^? \subseteq x^!)$. This implies that for any classical sets u, v with $v \subseteq u$, there is x such that $x^! = u$ and $x^? = v$. We will briefly return to this finer distinction in Section 9, but will not pursue it further in detail.

As a nice application of ACLA, we can internally define truth values in a succinct way. Let $\Omega := \mathscr{P}^!(\{\varnothing\})$. This is the classical set containing sets x such that $x^! \subseteq \{\varnothing\}$ and $x^? \subseteq \{\varnothing\}$. There are four possible combinations for such x, and ACLA guarantees us that all four of them exist and are members of Ω. We can give them names as follows:

- $x = 1$ if $x^! = x^? = \{\varnothing\}$
- $x = 0$ if $x^! = x^? = \varnothing$
- $x = \mathfrak{n}$ if $x^! = \varnothing$ and $x^? = \{\varnothing\}$
- $x = \mathfrak{b}$ if $x^! = \{\varnothing\}$ and $x^? = \varnothing$

Then $\Omega = \{1, \mathfrak{b}, \mathfrak{n}, 0\}$ is called the *set of truth values*, and for any formula φ, we can define the *truth value* of φ by:

$$[\![\varphi]\!] := \{\varnothing : \varphi\} \in \Omega.$$

We now have that $\varnothing \in [\![\varphi]\!]$ is true precisely if φ is true, and false precisely if φ is false. In other words, for any formula φ, from the point of view of the meta-theory:

$$[\![\ \varnothing \in [\![\varphi]\!]\ ]\!] \ = \ [\![\varphi]\!].$$

This justifies the term "truth value" of φ and the fact that Ω is the set of truth values.

Finally, recall that when we formulated the Pairing and Power Set axioms, we mentioned that a careless translation of the definition would result in a proper class. We can now make this assertion precide.

Lemma 5.5 (BZFC)**.** *For sets $u, u_1, \ldots u_n$, the collections $\{y : y \subseteq u\}$, $\{y : y = u\}$ and $\{y : y = u_1 \vee \cdots \vee y = u_n\}$ are proper classes.*

Proof. We will only prove the first case since the others are similar. We know that $\{x : \neg(x \in x)\}$ is a proper class, and from this it easily follows that the collection Cl of all classical sets forms a proper class.

Suppose there is a set $X = \{y : y \subseteq u\}$. By Lemma 4.1 we know that $X^?$ is also a set. By Lemma 4.2 (3), we know that $y \in X^? \Leftrightarrow y^! \subseteq u^?$. Consider the operation

$$F : \begin{array}{ccc} X^? & \to & \mathrm{Cl} \\ y & \mapsto & y^? \end{array}$$

Formally, this operation is given by $\varphi(y, w) \ \equiv\ !(w = y^?)$ which is a classical formula and satisfies the conditions for the Replacement axiom. Moreover, by ACLA the operation is surjective, i.e., for every classical w there is $y \in X^?$ such that $\varphi(y, w)$ holds (take y with $y^! = u^?$ and $y^? = w$). But then $F[X^?] = \mathrm{Cl}$, and since $X^?$ is a set, Cl should be a set, which is a contradiction. □

The above argument is somewhat informal, so some readers may have wondered what it means to says that something is a proper class, or whether a proof by contradiction is valid in the paraconsistent setting. Formally, what we have proven is that if a set X such as above exists, then, via the Russell set, we derive $\perp$.

6 A model of BZFC

In this section we construct a natural T/F-model for BZFC within classical ZFC. Normally speaking this would yield a *relative consistency* proof, i.e., a proof that if ZFC is consistent then BZFC is consistent. Of course BZFC is, by design, *inconsistent*, so instead we will talk of *non-triviality:*

Definition 6.1. A BS4-theory Γ is called *non-trivial* if $\Gamma \not\vdash_{\mathrm{BS4}} \bot$.

Since BZFC implies a strong maximality principle with regards to the existence of all possible sets, this suggests a natural model in which sets are constructed recursively, adding all possible pairs of the form (extension, ?-extension) made from previously constructed sets, at each step.

Definition 6.2 (ZFC)**.** By induction on ordinals define:

- $W_0 = \varnothing$
- $W_{\alpha+1} := \mathscr{P}(W_\alpha) \times \mathscr{P}(W_\alpha)$
- $W_\lambda = \bigcup_{\alpha<\lambda} W_\alpha$ for limit λ
- $\mathbb{W} := \bigcup\limits_{\alpha \in \mathrm{Ord}} W_\alpha$.

Define the positive and negative interpretations of $\in$ and $=$ by:

- $(a, b) \in^+ (c, d)$ iff $(a, b) \in c$
- $(a, b) \in^- (c, d)$ iff $(a, b) \notin d$
- $(a, b) =^+ (c, d)$ iff $(a, b) = (c, d)$
- $(a, b) =^- (c, d)$ iff $\exists z \in a \setminus d$ or $\exists z \in c \setminus b$.

The following properties of $\mathbb{W}$ parallel the V-hierarchy and follow immediately from the definition.

Lemma 6.3 (ZFC)**.**

1. *If $\alpha < \beta$ then $W_\alpha \subseteq W_\beta$.*
2. *If $(a, b) \in W_\alpha$, $a' \subseteq a$, and $b' \subseteq b$ then $(a', b') \in W_\alpha$.*
3. *$x \in \mathbb{W}$ iff $x = (a, b)$ for some $a, b \subseteq \mathbb{W}$.*

Proof. Induction on the definition. □

Now $(\mathbb{W}, \in^+, \in^-, =^+, =^-)$ may be considered a T/F-model in the language of set theory according to Definition 2.3, except that $\mathbb{W}$ is a proper class. Therefore, "$\mathbb{W} \models \varphi$" must be understood via *relativization*: for every φ we define $\varphi^{\mathbb{W},T}$ and $\varphi^{\mathbb{W},F}$ by syntactic induction, generalizing from Definition 2.3 in the obvious way. For example

$$(x = y)^{\mathbb{W},T} \quad \equiv \quad x = y$$

$$(x = y)^{\mathbb{W},F} \qquad \equiv \qquad x =^{-} y$$

$$\cdots$$

$$(\exists x\varphi)^{\mathbb{W},T} \qquad \equiv \qquad \exists x \left(x \in \mathbb{W} \wedge \varphi^{\mathbb{W},T}\right)$$

$$(\exists x\varphi)^{\mathbb{W},F} \qquad \equiv \qquad \forall x \left(x \in \mathbb{W} \to \varphi^{\mathbb{W},F}\right)$$

and so on. We leave the details to the reader. By $\mathbb{W} \models \varphi$ we will always mean $\varphi^{\mathbb{W},T}$, and for a theory Γ, $\mathbb{W} \models \Gamma$ means that for every φ in Γ, there is a proof of $\varphi^{\mathbb{W},T}$.

Theorem 6.4 (ZFC)**.** *$\mathbb{W} \models$ BZFC.*

Proof. We will prove Extensionality and Comprehension in some detail and leave the rest to the reader.

Written out in full, the relativization (Extensionality)$^{\mathbb{W},T}$ reads as follows

$$\forall (a, b) \in \mathbb{W}\ \forall (c, d) \in \mathbb{W} :$$

$$(a, b) =^{+} (c, d) \quad \leftrightarrow \quad \forall z \in \mathbb{W}((z \in^{+} (a, b) \leftrightarrow z \in^{+} (c, d)) \wedge (z \in^{-} (a, b) \leftrightarrow z \in^{-} (c, d))$$

$$\wedge$$

$$(a, b) =^{-} (c, d) \quad \leftrightarrow \quad \exists z \in \mathbb{W}((z \in^{+} (a, b) \wedge z \in^{-} (c, d)) \vee (z \in^{+} (c, d) \wedge z \in^{-} (a, b))$$

Assume (a, b) and (c, d) are arbitrary, and we show that the two equivalences hold. For the first we have

$$\begin{aligned}
& \forall z \in \mathbb{W}((z \in^{+} (a, b) \leftrightarrow z \in^{+} (c, d)) \wedge (z \in^{-} (a, b) \leftrightarrow z \in^{-} (c, d)) \\
\overset{(*)}{\leftrightarrow}\ & \forall z \in \mathbb{W}((z \in a \leftrightarrow z \in c) \wedge (z \notin b \leftrightarrow z \notin d)). \\
\overset{(**)}{\leftrightarrow}\ & \forall z((z \in a \leftrightarrow z \in c) \wedge (z \notin b \leftrightarrow z \notin d)) \\
\leftrightarrow\ & a = c \text{ and } b = d \\
\leftrightarrow\ & (a, b) =^{+} (c, d)
\end{aligned}$$

where $(*)$ refers to the definition of $\in^{+}$ and $\in^{-}$, and $(**)$ is because $\mathbb{W}$ is "transitive" in the sense that $a, b, c, d \in \mathbb{W}$ only contain sets which are also in $\mathbb{W}$,

For the second equivalence we have

$$\begin{aligned}
& \exists z \in \mathbb{W}((z \in^{+} (a, b) \wedge z \in^{-} (c, d)) \vee (z \in^{+} (c, d) \wedge z \in^{-} (a, b)) \\
\leftrightarrow\ & \exists z((z \in a \wedge z \notin d) \vee (z \in c \wedge z \notin b)) \\
\leftrightarrow\ & (a, b) =^{-} (c, d)
\end{aligned}$$

Next we look at (Comprehension)$^{\mathbb{W},T}$ which is the following statement:[14]

$$\forall (a, b) \in \mathbb{W}\ \exists (c, d) \in \mathbb{W}\ \forall z \in \mathbb{W}$$

[14] Again we suppress parameters, noting that if there are parameters these are understood to be in $\mathbb{W}$.

$$((z \in^+ (c,d) \leftrightarrow (z \in^+ (a,b) \wedge \varphi(z)^{\mathbb{W},T}) \wedge (z \in^- (c,d) \leftrightarrow (z \in^- (a,b) \vee \varphi(z)^{\mathbb{W},F}))$$

Suppose $(a,b) \in \mathbb{W}$ and φ is given. Define

$$c := \{z \in a : \varphi(z)^{\mathbb{W},T}\}$$

$$d := \{z \in b : \neg\varphi(z)^{\mathbb{W},F}\}^{15}$$

Then $(c,d) \in \mathbb{W}$ by construction and for all z we have $z \in c \leftrightarrow (z \in a \wedge \varphi(z)^{\mathbb{W},T})$ and $z \notin d \leftrightarrow (z \notin b \vee \varphi(z)^{\mathbb{W},F})$. This is exactly the statement above as needs to be proved. $\square$

Corollary 6.5. *If* ZFC *is consistent then* BZFC *is non-trivial.*

Definition 6.6 (ZFC)**.** For every $x \in V$, inductively define

$$\check{x} := (\{\check{y} : y \in x\},\ \{\check{y} : y \in x\}).$$

Also let $\check{V} := \{\check{x} : x \in V\}$.

Lemma 6.7 (ZFC)**.** *The mapping*

$$i : \begin{array}{ccc} V & \to & \check{V} \subseteq \mathbb{W} \\ x & \mapsto & \check{x} \end{array}$$

is an isomorphism between $(V, \in, \notin, =, \neq)$ *and* $(\check{V}, \in^+, \in^-, =^+, =^-)$.

Proof. Easy consequence of the definitions. $\square$

In view of the above, $\mathbb{W}$ can be viewed as a natural *extension* of V, enriching the classical von Neumann universe with paraconsistent and paracomplete sets.

7 Hereditarily classical sets.

Starting in BZFC, we can also construct a natural model of ZFC: this is the class of "hereditarily classical" sets.

Definition 7.1 (BZFC)**.** By induction on ordinals define:[16]

- $\mathrm{HCL}_0 = \varnothing$
- $\mathrm{HCL}_{\alpha+1} := \{X \subseteq \mathrm{HCL}_\alpha : X \text{ is classical}\}$
- $\mathrm{HCL}_\lambda = \bigcup_{\alpha<\lambda} \mathrm{HCL}_\alpha$ for limit λ
- $\mathbb{HCL} := \displaystyle\bigcup_{\alpha \in \mathrm{Ord}} \mathrm{HCL}_\alpha$.

[15] An equivalent definition is $c = \{z \in a : \mathbb{W} \models\ !\varphi(z)\}$ and $d = \{z \in b : \mathbb{W} \models\ ?\varphi(z)\}$.

[16] By the remark in Section 4.8, ordinals and transfinite recursion can be carried out in BZFC in a manner analogous to ZFC.

$\mathbb{HCL}$ is a transitive proper class, and for all x, we have that $x \in \mathbb{HCL}$ if and only if x is classical and $x \subseteq \mathbb{HCL}$. If φ is a formula in the language of set theory, then $\varphi^{\mathbb{HCL}}$ refers to the standard relativization where quantification is replaced by quantification over $\mathbb{HCL}$. As usual, $\mathbb{HCL} \models \Gamma$ means that for every φ in Γ, there is a proof of $\varphi^{\mathbb{HCL}}$.

Theorem 7.2 (BZFC)**.** $\mathbb{HCL} \models$ ZFC.

Proof. It is clear that $\mathbb{HCL} \models$ every set is classical; so, by the discussion in Section 5 it suffices to show that $\mathbb{HCL} \models$ PZFC. Most of the axioms are straightforward since none of them postulate the existence of non-classical sets without assuming the existence of non-classical sets.

We will only show (Comprehension)$^{\mathbb{HCL}}$. Suppose u is a herditarily classical set and φ any formula. It is enough to show that $x := \{y \in u : \varphi^{\mathbb{HCL}}(y, a_1, \dots, a_n)\}$ is a hereditarily classical set if the parameters $a_1, \dots, a_n$ are hereditarily classical.[17] Since $x \subseteq u$ and $u \subseteq \mathbb{HCL}$ we know that $x \subseteq \mathbb{HCL}$, so it remains to show that x itself is classical. But from the fact that $y, a_1, \dots a_n$ are classical sets, it follows by an easy induction that φ is a classical formula, i.e., $!\varphi \leftrightarrow ?\varphi$. It follows that $x^! = x^?$, so x is classical. $\square$

Corollary 7.3. *If* BZFC *is non-trivial, then* ZFC *is consistent.*

In analogy to Definition 6.6, we can now consider an embedding from the universe of BZFC (which we also refer to as V, hopefully not leading to confusion) to a natural copy of it within $\mathbb{HCL}$.

Definition 7.4 (BZFC)**.** For every set x, inductively define:[18]

$$\hat{x} := (\{\hat{y} : y \in x^!\}, \{\hat{y} : y \in x^?\}).$$

Also let $\hat{V} := \{\hat{x} : x \in V\}$. For $\hat{x}, \hat{y} \in \hat{V}$ define

- $\hat{x} E^+ \hat{y} \leftrightarrow x \in y$
- $\hat{x} E^- \hat{y} \leftrightarrow x \notin y$
- $\hat{x} \approx^+ \hat{y} \leftrightarrow x = y$
- $\hat{x} \approx^- \hat{y} \leftrightarrow x \neq y$

Now $\hat{x}$ are hereditarily classical sets, $\hat{V}$ is a proper class of hereditarily classical sets, and $E^+, E^-, \approx^+, \approx^-$ are hereditarily classical, class-sized binary relations on $\hat{V}$.

Lemma 7.5 (BZFC)**.** *The mapping*

$$j : \begin{array}{ccc} V & \to & \hat{V} \subseteq \mathbb{HCL} \\ x & \mapsto & \hat{x} \end{array}$$

is an isomorphism between $(V, \in, \notin, =, \neq)$ *and* $(\hat{V}, E^+, E^-, \approx^+, \approx^-)$.

[17]Here we are explicit about parameters $a_1, \dots, a_n$ since otherwise φ might fail to be a classical formula.

[18]We have not formally defined the *ordered pair* in BZFC. Interested readers may consult [19, Appendix A]. However, in this case, $\{\hat{y} : y \in x^!\}$ and $\{\hat{y} : y \in x^?\}$ are classical sets, so we can also consider the ordered pair via the standard ZFC-definition.

Proof. Follows immediately from the definitions. $\square$

8 Bi-interpretability

We can go further and look at the precise interaction between ZFC and BZFC. Not only are the two theories bi-interpretable, but the corresponding model constructions allow us to naturally translate between ZFC and BZFC-theorems.

Recall that in Lemma 6.7 we defined an isomorphism between V and $\check{V}$ via $i : x \mapsto \check{x}$. The next result shows that $\check{V}$ is the same thing as what $\mathbb{W}$ believes $\mathbb{HCL}$ to be. The notation $\mathbb{HCL}^{\mathbb{W}}$ refers to the class $\mathbb{HCL}$ relativized to $\mathbb{W}$, defined in ZFC.

Lemma 8.1 (ZFC)**.** $\mathbb{HCL}^{\mathbb{W}} = \check{V}$.

Proof. To show $\supseteq$ take $\check{x} \in \check{V}$. By construction $\check{x} = (\{\check{y} : y \in x\}, \{\check{y} : y \in x\})$. Inductively, we may assume that each $\check{y}$ appearing above is in $\mathbb{HCL}^{\mathbb{W}}$, therefore $\mathbb{W} \models \check{x} \subseteq \mathbb{HCL}$. Moreover, $\check{x}$ has the same !-extension as ?-extension, therefore $\mathbb{W} \models (\check{x}$ is classical). Together, this implies $\mathbb{W} \models \check{x} \in \mathbb{HCL}$.

To show $\subseteq$, suppose $x \in \mathbb{HCL}^{\mathbb{W}}$, i.e., $x \in \mathbb{W}$ and $\mathbb{W} \models (x$ is hereditarily classical). Let $a, b \in \mathbb{W}$ be such that $x = (a, b)$. Since $\mathbb{W} \models x$ is classical, in particular $\mathbb{W} \models x^! = x^?$, therefore $a = b$. Moreover, $\mathbb{W} \models x \subseteq \mathbb{HCL}$ which implies that $a \subseteq \mathbb{HCL}^{\mathbb{W}}$. Inductively, we may assume that every element in a is of the form $\check{z}$ for some $z \in V$. Let $\tilde{a} := \{z : \check{z} \in a\}$. Then $x = (\{\check{z} : z \in \tilde{a}\}, \{\check{z} : z \in \tilde{a}\}) = (a, a)$ by construction. But then, by definition, $x = \check{(\tilde{a})}$. $\square$

Corollary 8.2. ZFC $\vdash \varphi$ *iff* BZFC $\vdash (\mathbb{HCL} \models \varphi)$.

Proof. Suppose ZFC $\vdash \varphi$. Since BZFC $\vdash (\mathbb{HCL} \models$ ZFC), clearly BZFC $\vdash (\mathbb{HCL} \models \varphi)$. Conversely, suppose BZFC $\vdash (\mathbb{HCL} \models \varphi)$. By Theorem 6.4, ZFC $\vdash (\mathbb{W} \models$ BZFC), so ZFC $\vdash (\mathbb{W} \models (\mathbb{HCL} \models \varphi))$, so ZFC $\vdash (\mathbb{HCL}^{\mathbb{W}} \models \varphi)$. But $\mathbb{HCL}^{\mathbb{W}}$ is $\check{V}$, which is isomorphic to V by Lemma 6.7. Therefore ZFC $\vdash \varphi$. $\square$

Conversely, if we start in BZFC, look at the model $\mathbb{HCL}$, and apply Definition 6.2 to obtain $\mathbb{W}$ relativized to $\mathbb{HCL}$, we obtain the natural copy of the BZFC-universe V under the embedding $\hat{x}$, as defined in Definition 7.4.

Lemma 8.3 (BZFC)**.** $\mathbb{W}^{\mathbb{HCL}} = \hat{V}$.

Proof. Suppose $\hat{x} \in \hat{V}$. Then $\hat{x} = (\{\hat{y} : y \in x^!\}, \{\hat{y} : y \in x^?\})$, and since every $\hat{y}$ appearing here is in $\hat{V}$, inductively we can assume that it is also in $\mathbb{W}^{\mathbb{HCL}}$. By definition $\hat{x}$ is hereditarily classical, and since the ordered pair is classical, we have $\mathbb{HCL} \models (\hat{x} = (a, b)$ and $a, b \subseteq \mathbb{W})$. By Lemma 6.3 (3) applied in $\mathbb{HCL}$, this means $\mathbb{HCL} \models \hat{x} \in \mathbb{W}$.

Conversely, suppose $w \in \mathbb{W}^{\mathbb{HCL}}$. Again by Lemma 6.3 (3) we know that $\mathbb{HCL} \models (w = (u, v)$ and $u, v \subseteq \mathbb{W})$. Then $u, v \subseteq \mathbb{W}^{\mathbb{HCL}}$ so, inductively, we have $u, v \subseteq \hat{V}$. Therefore, let $\tilde{u} = \{y : \hat{y} \in u\}$ and $\tilde{v} = \{y : \hat{y} \in v\}$. Then $\tilde{u}$ and $\tilde{v}$ are classical sets (because u, v are), so by ACLA there exists a set x such that $x^! = \tilde{u}$ and $x^? = \tilde{v}$. Then we have

$$\hat{x} \;=\; (\{\hat{y} : y \in x^!\}, \{\hat{y} : y \in x^?\}) \;=\; (\{\hat{y} : y \in \tilde{u}\}, \{\hat{y} : y \in \tilde{v}\}) \;=\; (u, v) \;=\; w$$

This shows that $w \in \hat{V}$. $\square$

Corollary 8.4. BZFC $\vdash \varphi$ *iff* ZFC $\vdash (\mathbb{W} \models \varphi)$.

Proof. Suppose BZFC $\vdash \varphi$. Since ZFC $\vdash (\mathbb{W} \models$ BZFC), clearly ZFC $\vdash (\mathbb{W} \models \varphi)$. Conversely, suppose ZFC $\vdash (\mathbb{W} \models \varphi)$. By Theorem 7.2, BZFC $\vdash (\mathbb{HCL} \models$ ZFC), so BZFC $\vdash (\mathbb{HCL} \models (\mathbb{W} \models \varphi))$, so BZFC $\vdash \left(\mathbb{W}^{\mathbb{HCL}} \models \varphi\right)$. But $\mathbb{W}^{\mathbb{HCL}}$ is isomorphic to V, so BZFC $\vdash \varphi$. $\square$

The philosophical significance of these results is that classical ZFC and our theory BZFC are not at odds with each other, but can be viewed as complementing each other in a natural way. If our philosophical position favours the existence of paracomplete and paraconsistent sets, we can view BZFC as describing a universe that is an enlargement of the class $\mathbb{HCL}$ of hereditarily classical sets. ZFC can then be viewed as the theory of $\mathbb{HCL}$, and all of classical mathematics as taking place within $\mathbb{HCL}$. If we are only interested in classical mathematics, we can stay within $\mathbb{HCL}$, but whenever we encounters a phenomenon that is better described by paracomplete or paraconsistent sets, we can switch to BZFC and take full advantage of the anti-classicality axiom.

On the other hand, if we are determined to preserve a classical meta-theory, then we can view BZFC as the theory of the T/F-model $\mathbb{W}$ and all of paraconsistent and paracomplete set theory, as layed out in this paper, as formal statements in $\mathbb{W}$. All of these statements are ultimately provable in ZFC.

9 Tarski semantics in BZFC

A discussion that frequently arises in the philosophy of logic is the extent to which the meta-theory affects the formal theory under consideration. For example, one may argue that *excluded middle* is a tautology in classical logic only because *excluded middle* is covertly assumed in the meta-theory:

$$\mathcal{M} \models \varphi \vee \neg\varphi \quad \Leftrightarrow \quad \mathcal{M} \models \varphi \text{ or } \mathcal{M} \not\models \varphi$$

Likewise, in classical logic $\varphi \wedge \neg\varphi$ can never be satisfied by a model, because $\mathcal{M} \models \varphi \wedge \neg\varphi$ would lead to $\mathcal{M} \models \varphi$ and $\mathcal{M} \not\models \varphi$, which is false in the meta-theory.

Non-classical logics are usually set up by considering other (non-Tarski) semantics, so that the formal system in question may exhibit various levels of non-classicality while the meta-theory remains classical. This is the case, for example, with Kripke semantics for intuitionistic logic, and with our own T/F-models for BS4 from Section 2.

But we could approach the question differently: what is the logic generated by considering standard *Tarski* semantics, but set up within a *non-classical meta-theory*? The purpose of this section is to show that, indeed, Tarski semantics in BZFC give rise to the logic BS4, i.e., BZFC proves that BS4 is sound and complete with respect to *classical* Tarski semantics. This result makes concrete use of the anti-classicality axiom; in fact, we shall see that the logic generated by Tarski semantics precisely reflects the level of non-classicality in the meta-theory.

In this section, we work in BZFC or PZFC. For clarity, we will refer to models defined below as *Tarski models* and the semantics as *Tarski semantics*, to keep them apart from T/F-models and T/F-semantics. We will also be explicit with using the notation $\models$ to refer to the Tarski satisfaction relation, and $\models^T$ and $\models^F$ to refer to the T/F-satisfaction relations.

Definition 9.1 (PZFC)**.** The language of first order logic is assumed to be coded by *classical* sets. A Tarski model $\mathcal{M}$ consists of a *classical set* M as a domain, and (not necessarily classical) interpretations for all constant and relation symbols. For simplicity, assume there are no function symbols. Inductively define:

1. $\mathcal{M} \models (t = s)[a, b] \quad \Leftrightarrow \quad a = b.$
2. $\mathcal{M} \models R(t_1, \ldots, t_n)[a_1, \ldots, a_n] \quad \Leftrightarrow \quad R^{\mathcal{M}}(a_1, \ldots a_n).$
3. $\mathcal{M} \models {\sim}\varphi \quad \Leftrightarrow \quad \mathcal{M} \not\models \varphi.$[19]
4. $\mathcal{M} \models \varphi \wedge \psi \quad \Leftrightarrow \quad \mathcal{M} \models \varphi$ and $\mathcal{M} \models \psi$.
5. $\mathcal{M} \models \varphi \vee \psi \quad \Leftrightarrow \quad \mathcal{M} \models \varphi$ or $\mathcal{M} \models \psi$.
6. $\mathcal{M} \models \varphi \to \psi \quad \Leftrightarrow \quad (\mathcal{M} \models \varphi \quad \to \quad \mathcal{M} \models \psi).$
7. $\mathcal{M} \models \varphi \leftrightarrow \psi \quad \Leftrightarrow \quad (\mathcal{M} \models \varphi \quad \leftrightarrow \quad \mathcal{M} \models \psi).$
8. $\mathcal{M} \models \exists x \varphi(x) \quad \Leftrightarrow \quad \mathcal{M} \models \varphi[a]$ for some $a \in M$.
9. $\mathcal{M} \models \forall x \varphi(x) \quad \Leftrightarrow \quad \mathcal{M} \models \varphi[a]$ for all $a \in M$.
10. $\mathcal{M} \models \bot \quad \Leftrightarrow \quad \bot.$

This definition should be understood in the framework of PZFC, with all equivalences as strong implications. Note that while M is a classical set, the relation-interpretations $R^{\mathcal{M}}$ are not necessarily classical. For example, it could happen that $R^{\mathcal{M}}(a)$ is both true and false, in which case we have $\mathcal{M} \models R(x)[a]$ and $\mathcal{M} \not\models R(x)[a]$, and subsequently $\mathcal{M} \models (R(x) \wedge {\sim}R(x))[a]$.

Remark 9.2. The reader may easily verify that the defined connectives are also translated the way we would expect, namely:

$$\begin{aligned} \mathcal{M} \models \neg\varphi \quad &\Leftrightarrow \quad \neg(\mathcal{M} \models \varphi) \\ \mathcal{M} \models !\varphi \quad &\Leftrightarrow \quad !(\mathcal{M} \models \varphi) \\ \mathcal{M} \models ?\varphi \quad &\Leftrightarrow \quad ?(\mathcal{M} \models \varphi). \end{aligned}$$

Had we, for example, chosen $\neg$ as the primitive connective in BS4 instead of $\bot$, then Definition 9.1 would have given rise to the same logic.

Linguistically, it is useful to distinguish *truth* in a Tarski model from *satisfaction* by a Tarski model. We say "$\mathcal{M}$ satisfies φ" to express "$\mathcal{M} \models \varphi$" and "$\varphi$ is true in M" to express $!(\mathcal{M} \models \varphi)$ (which is the same as "$\mathcal{M} \models !\varphi$"). Satisfaction captures the entire truth value of φ in $\mathcal{M}$, and we can use the notation concerning truth values from Section 5 to define the truth value of φ in a Tarski model $\mathcal{M}$:

$$[\![\varphi]\!]_{\mathcal{M}} := \{\varnothing : \mathcal{M} \models \varphi\}$$

[19] Here $\mathcal{M} \not\models \varphi$ is an abbreviation for ${\sim}(\mathcal{M} \models \varphi)$.

Definition 9.3 (PZFC)**.** For a set of formulas Σ and a formula φ, the Tarski semantic consequence relation is defined by $\Sigma \models \varphi$ if for all Tarski models $\mathcal{M}$,

$$\mathcal{M} \models \Sigma \ \rightarrow \ \mathcal{M} \models \varphi.$$

Theorem 9.4 (BZFC)**.** BS4 *is sound and complete with respect to Tarski semantics:*

$$\Sigma \models \varphi \quad \leftrightarrow \quad \Sigma \vdash_{\mathrm{BS4}} \varphi.$$

Remark 9.5. We should note here that the bi-implication is not a strong one, i.e., soundness and completeness only talks about *truth* in a model, not *satisfaction* by the model. Indeed, note that $\vdash_{\mathrm{BS4}}$ is a classical relation, while the Tarski-consequence relation $\models$ is not. For example, take any φ such that some model $\mathcal{N} \models \varphi \wedge {\sim}\varphi$, and consider the classical excluded middle $\varphi \vee \neg\varphi$. Then for every $\mathcal{M}$ we have $\mathcal{M} \models \varphi \vee \neg\varphi$, but one can verify that $\mathcal{N} \not\models \varphi \vee \neg\varphi$. This means that "$\varphi \vee \neg\varphi$" both *is* and *is not* a tautology according to Tarski semantics.

We will now prove Theorem 9.4. We could proceed by repeating the standard soundness and completeness proof within BZFC, but here we will take a short-cut by recalling the fact that BS4 is sound and complete with respect to T/F-semantics (Lemma 2.7), and showing how T/F-models can be "simulated" by Tarski models and vice versa, which seems to be an interesting phenomenon in its own right.

First we adapt Definition 2.3 to PZFC:

Definition 9.6 (PZFC)**.** A T/F-model is defined as in Definition 2.3, with the added condition that the domain M, all interpretations $(R^{\mathcal{M}})^+$ and $(R^{\mathcal{M}})^-$ are classical subsets of M^n, and $=^+$ and $=^-$ are classical subsets of $M \times M$. Moreover, it is required that for $a, b \in M$ we have $a =^+ b \ \leftrightarrow \ !(a = b)$.[20]

Lemma 9.7 (PZFC)**.** BS4 *is sound and complete with respect to T/F-semantics.*

Proof. Since all relevant sets and relations are classical, the original proof of Lemma 2.7 can be repeated inside PZFC, yielding the desired result. We leave out the details here. $\square$

The above Lemma essentially restates what we already knew about BS4 when we considered it inside a classical meta-theory. Now the trick is "simulate" T/F-models by Tarski models and vice versa.

Lemma 9.8 (PZFC)**.** *For every Tarski model $\mathcal{M}$, there exists a T/F-model $\mathcal{M}^{\pm}$ such that for every φ:*

$$\mathcal{M} \models \varphi \ \leftrightarrow \ \mathcal{M}^{\pm} \models^T \varphi$$
$$\mathcal{M} \not\models \varphi \ \leftrightarrow \ \mathcal{M}^{\pm} \models^F \varphi$$

[20] In other words, $=^+$ is a classical relation such that $a =^+ b$ is true precisely when $a = b$ is *true*, and false precisely when $a = b$ is *not true*. On the other hand $=^-$ is a classical and symmetric binary relation, but need not have anything to do with the meta-theoretic $a \neq b$.

Proof. Let $\mathcal{M}^{\pm}$ have the same domain as $\mathcal{M}$. For every relation symbol R define classical relations $(R^{\mathcal{M}^{\pm}})^{+}$ and $(R^{\mathcal{M}^{\pm}})^{-}$ (including equality)

$$(R^{\mathcal{M}^{\pm}})^{+} := \{(a_1, \dots a_n) \in M^n \ : !R(a_1, \dots, a_n)\}$$

$$(R^{\mathcal{M}^{\pm}})^{-} := \{(a_1, \dots a_n) \in M^n \ : \ \neg ?R(a_1, \dots, a_n)\}$$

This definition makes sure that we have

$$\begin{aligned} \mathcal{M} \models R[a_1 \dots a_n] \ &\leftrightarrow \ \mathcal{M}^{\pm} \models^{T} R[a_1 \dots a_n] \\ \mathcal{M} \not\models R[a_1 \dots a_n] \ &\leftrightarrow \ \mathcal{M}^{\pm} \models^{F} R[a_1 \dots a_n] \end{aligned}$$

An induction on the complexity of φ then shows that the two equivalences hold for all sentences. To exhibit an example:

$$\mathcal{M} \models {\sim}\varphi \ \Leftrightarrow \ \mathcal{M} \not\models \varphi \ \overset{\text{IH}}{\leftrightarrow} \ \mathcal{M}^{\pm} \models^{F} \varphi \ \leftrightarrow \ \mathcal{M}^{\pm} \models^{T} {\sim}\varphi$$

$$\mathcal{M} \not\models {\sim}\varphi \ \Leftrightarrow \ \mathcal{M} \models \varphi \ \overset{\text{IH}}{\leftrightarrow} \ \mathcal{M}^{\pm} \models^{T} \varphi \ \leftrightarrow \ \mathcal{M}^{\pm} \models^{F} {\sim}\varphi.$$

We leave the details to the reader. □

Lemma 9.9 (BZFC)**.** *For every T/F-model $\mathcal{N}$, there exists a Tarski model $\mathcal{N}^4$ such that for every φ:*

$$\begin{aligned} \mathcal{N} \models^{T} \varphi \ &\leftrightarrow \ \mathcal{N}^4 \models \varphi \\ \mathcal{N} \models^{F} \varphi \ &\leftrightarrow \ \mathcal{N}^4 \not\models \varphi \end{aligned}$$

Proof. Here we first need to take care of the equality relation, since the negative relation $=^{-}$ of $\mathcal{N}$ is not a priori related to the meta-theoretic $\neq$ relation for sets. We define a *non-classical equivalence relation* $\equiv$ on N, as follows, appealing to the Anti-Classicality Axiom:[21]

$$a \equiv b \ \leftrightarrow \ a = b$$

$$a \not\equiv b \ \leftrightarrow \ a =^{-} b$$

For every $a \in N$, let $[a] := \{b \in N : a \equiv b\}$. This is the $\equiv$-equivalence class of a, but note that we have $a \in [b] \ \leftrightarrow \ a = b$ and $a \notin [b] \ \leftrightarrow \ a =^{-} b$. In particular, we could have both $a \in [b]$ and $a \notin [b]$.

Claim 9.10. *For all $a, b \in N$ we have*

$$[a] = [b] \leftrightarrow a = b$$

$$[a] \neq [b] \leftrightarrow a =^{-} b.$$

[21]This is a specific case of a non-classical equivalence relation; for more details, see [19, Section 4.8].

Proof. If $a = b$ then clearly $[a] = [b]$. Conversely, if $[a] = [b]$ then for all z we have $z \in [a] \leftrightarrow z \in [b]$; but clearly $a \in [a]$, therefore $a \in [b]$, so $a = b$.

Now suppose $a =^- b$. By definition $a \not\equiv b$ so, again by definition, $a \notin [b]$. But then we have $a \in [a]$ but $a \notin [b]$, so (by Extensionality) we have $[a] \neq [b]$. Conversely, suppose $[a] \neq [b]$. Then there exists a $z \in [a]$ such that $z \notin [b]$ (or vice versa). Then $z = a$ and $z =^- b$, but then also $a =^- b$ (or vice versa). □(Claim)

We let the domain of $\mathcal{N}^4$ consist of the $\equiv$-equivalence classes of $\mathcal{N}$, i.e., $N^4 = \{[a] : a \in N\}$. Notice that we have

$$\mathcal{N} \models^T (t = s)[a, b] \;\leftrightarrow\; a = b \;\leftrightarrow\; [a] = [b] \;\leftrightarrow\; \mathcal{N}^4 \models (t = s)[[a], [b]]$$

$$\mathcal{N} \models^F (t = s)[a, b] \;\leftrightarrow\; a =^- b \;\leftrightarrow\; [a] \neq [b] \;\leftrightarrow\; \mathcal{N}^4 \not\models (t = s)[[a], [b]]$$

For every relation symbol R, define the interpretation $R^{\mathcal{N}^4}$ by:

$$R^{\mathcal{N}^4}([a_1] \dots [a_n]) \;\leftrightarrow\; (R^{\mathcal{N}})^+(a_1 \dots a_n)$$

$$\sim\!R^{\mathcal{N}^4}([a_1] \dots [a_n]) \;\leftrightarrow\; (R^{\mathcal{N}})^-(a_1 \dots a_n)$$

which is again possible by appealing to the Anti-Classicality Axiom. Also note that this is well-defined by the usual arguments.

Finally, we leave it to the reader to verify, by induction on the complexity of φ, that for all $a_1, \dots, a_n \in N$ we have

$$\mathcal{N} \models^T \varphi[a_1 \dots a_n] \;\;\leftrightarrow\;\; \mathcal{N}^4 \models \varphi[[a_1] \dots [a_n]]$$

$$\mathcal{N} \models^F \varphi[a_1 \dots a_n] \;\;\leftrightarrow\;\; \mathcal{N}^4 \not\models \varphi[[a_1] \dots [a_n]]$$

In particular this is true for all sentences φ, completing the proof. □

Proof of Theorem 9.4 (BZFC). By Lemma 9.7, BS4 is sound and complete with respect to T/F-semantics. But by Lemmas 9.8 and 9.9, T/F-models can be replaced by Tarski-models and vice versa. Thus T/F-semantics are equivalent to Tarski-semantics, which completes the proof. □

Notice that while Lemma 9.8 only required PZFC, in Lemma 9.9 we really used BZFC in an essential way. Clearly, if all sets are classical, then BS4 is *not* complete with respect to Tarski semantics (e.g., $(\varphi \land \sim\!\varphi) \to \bot$ is true in all models but not BS4-provable).

Indeed, as the reader will probably have guessed, it turns out that the logic that is sound and complete with respect to Tarski semantics is the one reflecting the precise level of non-classicality in the ambient meta-theory. To make this more precise, we briefly return to Remark 5.4: recall that if one is interested in a set theory which has only inconsistent but not incomplete sets, or only incomplete but not inconsistent sets, one can consider two fragments of the Anti-Classicality Axiom:

- $\mathbf{ACLA_{cons}}$ **:** $\forall \boldsymbol{x}(\boldsymbol{x}^! \subseteq \boldsymbol{x}^?) \land \exists \boldsymbol{x}(\boldsymbol{x}^? \not\subseteq \boldsymbol{x}^!)$

  (all sets are consistent, but there is an incomplete set).

- **ACLA$_{\text{comp}}$:** $\forall x(x^? \subseteq x^!) \wedge \exists x(x^! \not\subseteq x^?)$
  (all sets are complete, but there is an inconsistent set).

We can now look at three-valued logics which arise if we adapt Definitions 2.2–2.4 to deal with only incompleteness or only inconsistency.

Definition 9.11 (PZFC)**.**

- K3$^{\to}$ is the logic corresponding to T/F-models $\mathcal{M}$ such that, for all R (including equality) $(R^{\mathcal{M}})^+ \cap (R^{\mathcal{M}})^- = \varnothing$.
- LFI1 is the logic corresponding to T/F-models $\mathcal{M}$ such that, for all R (including equality) $(R^{\mathcal{M}})^+ \cup (R^{\mathcal{M}})^- = M$ (the whole domain).

The reason for adopting the names K3$^{\to}$ and LFI1 is that these are essentially the logics that appeared in [9] and [12], respectively.[22]

The following theorem, which tells us exactly which logic is sound and complete with respect to Tarski semantics, provably in PFZC, follows by methods similar to the above.We leave the details to the reader ($\vdash_{\text{FOL}}$ refers to classical logic).

Theorem 9.12 (PZFC)**.**

1. ACLA $\leftrightarrow$ $(\Sigma \models \varphi \ \leftrightarrow\ \Sigma \vdash_{\text{BS4}} \varphi)$.

2. ACLA$_{\text{cons}}$ $\leftrightarrow$ $(\Sigma \models \varphi \ \leftrightarrow\ \Sigma \vdash_{\text{K3}^{\to}} \varphi)$.

3. ACLA$_{\text{comp}}$ $\leftrightarrow$ $(\Sigma \models \varphi \ \leftrightarrow\ \Sigma \vdash_{\text{LFI1}} \varphi)$.

4. $\forall x(x^! = x^?)$ $\leftrightarrow$ $(\Sigma \models \varphi \ \leftrightarrow\ \Sigma \vdash_{\text{FOL}} \varphi)$.

5. Exactly one of the above four options holds.

If we restrict attention to formulas that do not contain implications or $\perp$, we obtain similar results for much more familiar systems, namely the ones usually referred to as K3, LP and FDE (the "$\to$"- and "$\perp$"-free fragments of K3$^{\to}$, LFI1, and BS4, respectively, see [14, 21, 3]).

Theorem 9.13 (PZFC)**.** *Suppose Σ and φ do not contain "$\to$" or "$\perp$". Then:*

1. ACLA $\leftrightarrow$ $(\Sigma \models \varphi \ \leftrightarrow\ \Sigma \vdash_{\text{FDE}} \varphi)$.

2. ACLA$_{\text{cons}}$ $\leftrightarrow$ $(\Sigma \models \varphi \ \leftrightarrow\ \Sigma \vdash_{\text{K3}} \varphi)$.

3. ACLA$_{\text{comp}}$ $\leftrightarrow$ $(\Sigma \models \varphi \ \leftrightarrow\ \Sigma \vdash_{\text{LP}} \varphi)$.

4. $\forall x(x^! = x^?)$ $\leftrightarrow$ $(\Sigma \models \varphi \ \leftrightarrow\ \Sigma \vdash_{\text{FOL}} \varphi)$.

5. Exactly one of the above four options holds.

[22] These papers only considered a propositional version, but the essence is the same.

10 Conclusion

We conclude by reflecting on and discussing our results, as well as outlining some future research directions.

While other authors have presented formalizations of paraconsistent and paracomplete set theory, we believe that our approach establishes a more complete and fully developed theory than any previous attempts. The system BZFC has clear intuitive underpinnings making it easy to reason about paraconsistent and paracomplete phenomena *informally*, as well as a well-defined *ontology* of non-classical sets allowing us to construct models and reason about them from a classical perspective if desired. Moreover, our approach (based in part on the careful analysis of the axioms in Section 4) seems to overcome many obstacles previous authors were faced with. For instance, in the approach of Carnielli and Coniglio [8], most of the axioms are translated using the native implications $\rightarrow$ and $\leftrightarrow$ instead of the strong implications. As a result, the authors struggle with providing a coherent model theory.

The non-classicality axiom ACLA and Theorem 5.2 have likewise, to our knowledge, not appeared anywhere previously. A proper understanding of this axiom and its consequences was vital for the study of Tarski semantics in BZFC, as we have done in Section 9.4.

Finally, we feel that the bi-interpretability results from Sections 7 and 8 are extremely helpful, as mathematicians are free to switch between viewing BZFC as the theory of a particular structure within ZFC, or viewing ZFC as the theory of a particular structure within BZFC.

A legitimate concern may be: what is the overall motivation for studying paraconsistent and paracomplete set theory, and BZFC in particular?

We can say with certainty that BZFC does *not* help to avoid Russell's paradox. As mentioned, we do not think that Russell's paradox is something that should be avoided. Nevertheless, there are other interesting "contradictions" in set theory and mathematics, such as *Kunen's inconsistency* [15] which puts an upper bound on the hierarchy of large cardinals. It is not inconceivable that BZFC may help to shed new light on this and related phenomena.

We also feel that the results on Tarski semantics in Section 9.4, and to a lesser extent the bi-interpretability results, may lead to new insights in the philosophy of mathematics and logic, such as formal theories of truth.

On a more down-to-earth level, applications of BZFC are conceivable in computer science, for example in the study of databases or structures with incomplete or inconsistent information. Say, we want an automated system to derive logical consequences based on data in a large database. Presumably, this system should not be able to derive any sentence whatsoever merely from the fact that $R(a) \wedge {\sim}R(a)$ holds, which could be due to a wrong entry in the database.

Finally, an answer to the above concern may be that there are applications of BZFC that we do not know yet. Peter Aczel's non-well-founded set theory [2] was initially instigated out of curiosity, but has subsequently found applications in reasoning about circular phenomena in computer science. The Axiom of Determinacy was initially introduced in [17] as a "...theory which seems very interesting although its consistency is problematic", but eventually led to some of the biggest breakthroughs in descriptive set theory, large cardinals and inner model theory.

Above all, with this paper we hope to instigate more research in the area of paraconsistent mathematics, dragging it out of the realm of speculative attempts into a more mainstream part of serious set theory. Some suggestions for future research are listed below.

1. **Algebraic approach.** The logic BS4 and set theory in BS4 can also be approached from an algebraic point of view, via so-called *twist algebras*. This can even be used to study an analogue of Boolean-valued models (closely related to forcing over classical models of ZFC) in the PZFC- or BZFC-context. A large part of this has already been done, see [19, Chapter 3 and Chapter 8], but there are more things that can be studied.

2. **Computability theory.** One natural application of BZFC comes from computability theory. Let us say that a Turing machine M *computes* a set $A \subseteq \mathbb{N}$ if for all n:

   - $n \in A \ \leftrightarrow\ M$ halts on input n and outputs some non-0 value.
   - $n \notin A \ \leftrightarrow\ M$ halts on input n and outputs 0.

   Moreover, let us say that M *recognizes* a set $A \subseteq \mathbb{N}$ if for all n:

   - $n \in A \ \leftrightarrow\ M$ halts on input n and outputs some non-0 value.

   In classical mathematics, every Turing machine recognizes a set, but not every Turing machine computes a set (since it may not halt on some input). Thus, one cannot use sets $A \subseteq \mathbb{N}$ to represent decisions of a machine.

   Let us revisit the situation in BZFC (or even $\mathrm{PZFC} + \mathrm{ACLA}_{\mathrm{cons}}$). Using the same definition as above, it follows from Theorem 5.2 that every Turing machine computes a set. If a given machine does not halt on some input n, it just means that the machine computes an incomplete set A (for which $\neg(n \in A \vee n \notin A)$). In particular, the decision process of a machine can be completely described by subsets of $\mathbb{N}$.

3. **Cardinal arithmetic.** Finally, we will need a new notion of *cardinality* together with new type of *cardinal numbers* to describe the size of our non-classical sets. It is clear that this notion should capture the structure of non-classical sets, i.e., it should capture the size of the !-extensions, ?-extensions, and the overlap, in one go. We expect that this will lead to a rich theory of cardinals and cardinal arithmetic.

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