Bisimulations over Parity Formulas

MSc Thesis (*Afstudeerscriptie*)

written by

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Abstract

This thesis is an investigation into how to define the notion of bisimulation over parity formulas. We provide and argue for a list of criteria against which we could judge how good such a definition is. In general, a notion of bisimulation should be sound, closed under union and composition, easily decidable and as close to being complete as possible. It should also guarantee the existence of a largest bisimulation, namely the bisimilarity relation. Particular to the situation with parity formulas, a good bisimulation should also have a 'relative flavor' in its handling of the priority condition. We propose four definitions of bisimulations over parity formulas and evaluate each of them according to those criteria. We especially argue for one of the four definitions to be the best by far, since it satisfies all qualitative criteria and lies in a relatively good position on the 'spectrum of completeness'. We also provide an adequate bisimilarity game for this notion of bisimulation which makes it easier to work with the notion.

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1 Introduction

The modal μ-calculus is the extension of basic modal logic with the addition of the least and greatest fixpoint operators. Its root in modal logic makes it a natural candidate when it comes to specifying properties of transition systems, and its usage of fixpoint operators endows it the taste of recursion. These two powerful traits combined with that fact that it is decidable allow it to become an exciting and fruitful area of research in logic and theoretical computer science.

One way to think of any formula in a logic is to think of it as a representation of a statement about one or a class of mathematical models whose truth and falsity can be constructed or recovered in an inductive way. The modal μ-calculus is no exception. Taking the standard modal μ-calculus as introduced by Kozen [1] as an example: its formulas are generated by recusive applications of connectives starting with atoms as the base, and its semantics is understood by seeing the atoms as given subsets of the state space and the connectives as set operations. The former inspires graph representations of formulas based on syntax such as subformula dag. The latter is called the algebraic semantics of modal μ-calculus since it treats connectives as (parameterized) algebraic operations on the power set algebra of the state space.

Formulas in modal μ-calculus can yet be given another interpretation, where the formula and the pointed transition system in which the formula is to be evaluated on are put together to generate a game arena for a two-player game, and we define the truth-value of the formula in terms of whether a given player has a winning strategy in the game. Due to the involvement of the game-theoretic concept in this definition, we call this interpretation the game semantics of modal μ-calculus.

An elegant result that bridges the two interpretations is the *Adequacy Theorem*, which states that these two ways of giving semantics to modal μ-calculus are actually equivalent. Given this equivalence, one may wonder the following question. What if we take the graph representation as the primary syntax and the game semantics as the primary semantics for modal μ-calculus? This is indeed the basic idea for parity formulas, which is the main subject of our investigation in this thesis.

Why Parity Formulas?

Parity formulas are graph-based modal formulas derived from the graph representations of modal μ-calculus. Besides the conceptual motivation we have just mentioned, there are several benefits we can speculate to get from considering parity formulas.

First, parity formulas themselves can be seen as coalgebra automata that run on Kripke models. Aside from literatures that discuss in general coalgebra automata, most work in the field focus on ω-automata and sometimes tree automata. It would be interesting to look at a class of automata that take in specifically Kripke models, and the study of parity formulas is a step towards that direction. Second, the use of parity formulas can help simplify proofs of properties of modal μ-formulas. Proofs that rely on the syntactic structure of modal μ-formulas are known to be tedious to read and write. Parity formulas give us a global view of the connections among different elements in the closure set and the set of subformulas. This may give us a better way to come up with or refer to the key notions that aid inductive proofs on the structure of formula, and make such proofs easier to write and more accessible to readers.

Third, parity formulas may provide an alternative perspective that avoid the flaws of existing methods for the study of modal μ-calculus. For one, parity formulas may help circumvent the known problem of size-explosion from α-conversion. α-conversion is the process of renaming bound variables. Two formulas are α-equivalent if one can be obtained from the other by renaming bound variables. Formulas that are α-equivalent are commonly accepted as 'identical' since the intensional meaning of a formula does not depend on the choice of name of the bound variables. However, as [2] pointed out, the size of the graph representations can grow exponentially after α-conversion. This poses problems when one tries to extend the

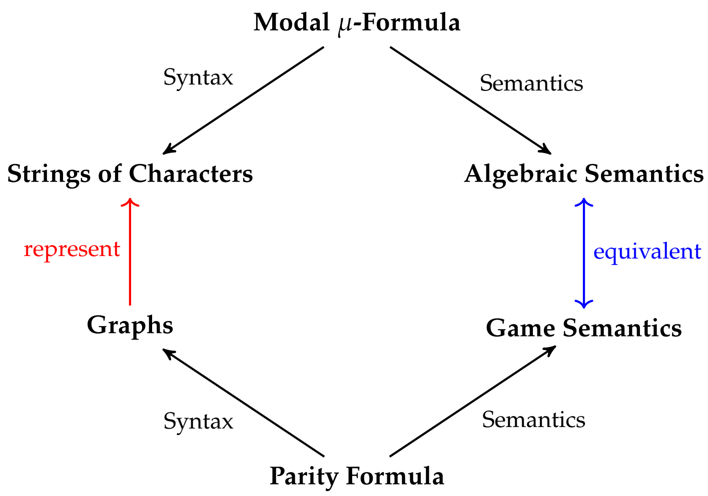


Figure 1.1: Inspiration for Parity Formulas

complexity results obtained from a sub-class of nicely behaved modal μ-formulas to the whole class. The study of parity formulas may give us some insight in resolving this issue. In parity formulas, the recursive power from the fix-point operation is not realized by variable binding but by looping in the graph. This might provide us with a way to give graph representation to modal μ-formulas that is invariant to α-conversion.

Finally, the study of parity formulas may give inspiration to studies on objects that share structural similarity. Besides serving as a different perspective to formulas in modal μ-calculus, parity formulas are also closely related to parity automata [3] and parity games, which we will introduce later in the chapter on preliminaries. Given the structural similarity, discovery made on the properties of parity formulas could be translated to these structures and provide insights on them.

Why Bisimulations?

Bisimulation is a recurrent notion in modal logic. The concept was first proposed by van Benthem [4] for talking about models, modal languages and invariance results. It has been used as a standard tool for investigations of modal expressivity. Bisimulations reflect, in a particularly simple and direct way, the locality of the modal satisfaction definition.

Following are some main goals for which one explores the notion of bisimulations.

- We want to establish identity between objects of potentially infinite size using only a finite collection of properties. For example, we can determine whether two pointed Kripke models satisfy the exact same set of formulas by observing whether there is a bisimulation between them. Alternatively, with the notion we will define in this thesis, one can use the existence of a bisimulation over two formulas as a sufficient condition for two formulas having the exact same class of models in which they are true.
- We can use bisimulations to capture behavioral equivalence over transition systems. Bisimulations establish similarity between two transition systems in global behaviors from their similarity in local behaviors. Sometimes, these global behaviors concerns infinite processes. In such cases, bisimulations can be seen as a tool with which we establish properties concerning infinity with a finite collection of facts.
- We investigate notions of bisimulation to come up with ways to take the quotient of a structure, and as a result, compress the structure while maintaining behavioral equivalence. If we have a notion of bisimulation that satisfies some additional nice properties, e.g. there exists a largest bisimulation which is also an equivalence relation, we can take the quotient of a structure using the largest bisimulation

from that structure to itself to get a behaviorally equivalent structure that is minimal. Such quotienting operations often prove useful in reducing computational costs and defining canonical structures.

This thesis is an investigation into how to define the notion of bisimulaitions over parity formulas. We will provide and argue for a list of criteria for a good notion of bisimulation that we consider to be faithful to the aforementioned reasons for studying bisimulations and parity formulas. We will propose four definitions of bisimulations over parity formulas and evaluate each of them according to those criteria.

Related Work

[5] proposes a way to resolve the issue of size-explosion from α-conversion with formulas in modal μ-calculus by defining a process called 'skeletal renaming' that sends any modal μ-formula to a unique representative of the α-equivalence class with the minimal size in terms of its graph representation. One can look into the functional relation induced by skeletal renaming, from a parity formula translation of a $\mu\mathsf{ML}$ formula ξ, to a parity formula translation of the skeletal representative of ξ. It is very well possible that the relation fits into some definitions of bisimulation we proposed in this these.

In Section 4 of [6], the authors implemented a sound definition of bisimulation over parity games, utilizing the idea of power bisimulation [7]. In a power bisimulation, for two nodes to be bisimilar, it is not enough for their successors to satisfy the canonical 'zig-zag' condition, but rather, there must be for both nodes in question a particular set of nodes that together satisfy the 'zig-zag' condition. This can be seen as an alternative method to deal with the global flavor of the priority condition that presents itself in both parity games and parity formulas.

In Section 6 of [8], a 'consequence game' between two coalgebra automata is proposed. The game is used to show that the coalgebras accepted by the left automata are included in that of the right automata. The game progresses similarly as the bisimilarity game that we will propose for one of the definition in Chapter 5. We can show the equivalence of the set of accepted coalgebras, in our setting, Kripke models, by playing two consequence games with symmetric positions of the two coalgebras. Given the close connection between bisimulations and bisimilarity, we might be able to derive a notion of bisimulations from this perspective.

Quite some notable work has been done on simulation, bisimulation and quotients over ω automata. [9] casts deterministic Muller automata that run on infinite words as coalgebras. They show that the coalgebraic bisimilarity induced coincides with language equivalence. [10] uses a unified parity-game framework to give efficient algorithms for calculating different kinds of simulation relations over Büchi automata. They also established that one of the simulation relation, called delayed simulation, preserves the automaton language upon taking quotients. The authors of [11] noted that automata language is not preserve if we generalize the delayed simulation onto parity automata, and so-called biased notions of delayed simulation are proposed as a remedy. [12] proposes several new approaches to reduce the state space of deterministic parity automata based on extracting information from structures within the automaton. It also establishes a framework to generalize the notion of quotient automata and uniformly describe their algorithms

Outline of the Thesis

- In Chapter 2, we prepare the readers with necessary preliminary knowledge on modal μ-calculus, games and parity formulas.
- In Chapter 3, we propose and argue for a list of criteria against which we could judge how good such a definition is. We proposed four definitions of bisimulations over parity formulas and evaluated them according to those criteria.
- Chapter 4 demonstrates some positive resuts when we consider Definition 3 on two special cases. One case concerns bisimulations that can be seen as functions, which we can also see as the morphisms over parity formulas. We show that when restricted to functional relations, Definition 3 has nice

category-theoretical properties to be considered the 'arrows' in the category of parity formulas. We also show that the expansion map by Kozen is indeed a morphism by our definition.
The other case concerns parity formulas with only one cluster. We show that in this restricted situation, Definition 3 is closed under union and composition, and thus, the largest bisimulation over any two such parity formulas always exists. In light of that, we provide a way to take the quotient of a parity formula with only one cluster.

- In Chapter 5, we prove that Definition 4 behaves nicely in terms of the proposed list of criteria. We show that Definition 4 is sound, closed under union and composition, and hence, guarantees the existence of the largest bisimulation, namely the bisimilarity relation. We provide an easy decision procedure for bisimilarity and an adequate bisimilarity game. We also demonstrate an alternative formulation of the bisimilarity relation using fixpoints.

Preliminaries 2

This chapter serves to introduce to the readers the necessary prerequisites to understand the rest of the thesis. The chapter is divided into three sections.

In the first section, we introduce the modal μ-calculus, an extension of propositional modal logic with the least and greatest fixpoint operators that has a close connection with the subject of study of this thesis. We will first introduce the syntax of modal μ-calculus, together with the syntactical notations we use in this thesis. Then, we show two graph representations of formulas in the modal μ-calculus. Finally, we give the algebraic semantics for modal μ- calculus.

In the second section, we define the concept of board games. We first give definitions to the essential elements of a board game, including the arena, players and (winning) strategies. Then, we introduce a particular type of board games call parity games. Finally, we give an equivalent semantics for modal μ-calculus from a game theoretic point of view employing the concepts previously introduced in this section.

In the third section, we define parity formulas. We give its syntax and semantics, as well as ways to translate a formula in modal μ-calculus to an equivalent parity formula.

Modal μ-calculus

Modal μ-calculus is an extension of propositional modal logic with least fixpoint operator μ and greatest fixpoint operator ν.

Syntax

Definition 2.1 *Given a set* P *of proposition letters, we define the sets* $\mathsf{Lit}(\mathsf{P})$ *and* $\mathsf{At}(\mathsf{P})$ *of literals and atomic formulas over* P *by setting* $\mathsf{Lit}(\mathsf{P}) := \{p, \overline{p} \mid p \in \mathsf{P}\}$ *and* $\mathsf{At}(\mathsf{P}) := \mathsf{Lit}(\mathsf{P}) \cup \{\top, \bot\}$*, respectively.*

Definition 2.2 *Given a set* D *of atomic actions, we define the collection* $\mu\mathsf{ML}$ *of modal fixpoint formulas as follows:*

$$\varphi := \bot \mid \top \mid x \mid \overline{x} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mu x.\varphi \mid \nu x.\varphi$$

where x is a propositional variable. There is a restriction on the formation of the formulas $\mu x.\varphi$ and $\nu x.\varphi$, namely, that the formula φ is positive in x. That is, all occurrences of x in φ may not be in the form of the negative literal $\overline{x}$.

For a finite set of propositional variables P, we let $\mu\mathsf{ML}(\mathsf{P})$ denote the set of $\mu\mathsf{ML}$-formulas φ of which all free variables belong to P.

Definition 2.3 *We define the set $Sfor_0(\xi)$ of direct subformulas of a formula $\xi \in \mu\mathsf{ML}$ via the following case distinction:*

$$\begin{array}{lll}
Sfor_0(\xi) & := \varnothing & \text{if } \xi \in \mathsf{At}(\mathsf{P}) \\
Sfor_0(\xi_0 \odot \xi_1) & := \{\xi_0, \xi_1\} & \text{where } \odot \in \{\wedge, \vee\} \\
Sfor_0(\heartsuit\xi) & := \{\xi\} & \text{where } \heartsuit \in \{\Diamond, \Box\} \\
Sfor_0(\eta x.\xi) & := \{\xi\} & \text{where } \eta \in \{\mu, \nu\}
\end{array}$$

and we write $\varphi \triangleleft_0 \xi$ if $\varphi \in Sfor_0(\xi)$.

Definition 2.4 *For any formula $\xi \in \mu$ML, Sfor(ξ) is the least set of formulas which contains ξ and is closed under taking direct subformulas. Element of the set Sfor(ξ) are called subformulas of ξ, and we write $\varphi \trianglelefteq \xi (\varphi \triangleleft \xi)$ if φ is a subformula (proper subformula, respectively) of ξ.*

Syntactically, the fixpoint operators are very similar to the quantifiers of first-order logic in the way they bind variables.

Definition 2.5 *Fix a formula φ. the set $FV(\varphi)$ and $BV(\varphi)$ of free and bound variables of φ are defined by the following induction on φ.*

$$
\begin{array}{llll}
FV(\top) & := \varnothing & BV(\top) & := \varnothing \\
FV(\bot) & := \varnothing & BV(\bot) & := \varnothing \\
FV(x) & := \{x\} & BV(x) & := \varnothing \\
FV(\overline{x}) & := \{x\} & BV(\overline{x}) & := \varnothing \\
FV(\varphi \wedge \psi) & := FV(\varphi) \cup FV(\psi) & BV(\varphi \wedge \psi) & := BV(\varphi) \cup BV(\psi) \\
FV(\varphi \vee \psi) & := FV(\varphi) \cup FV(\psi) & BV(\varphi \vee \psi) & := BV(\varphi) \cup BV(\psi) \\
FV(\Diamond\varphi) & := FV(\varphi) & BV(\Diamond\varphi) & := BV(\varphi) \\
FV(\Box\varphi) & := FV(\varphi) & BV(\Box\varphi) & := BV(\varphi) \\
FV(\eta x.\varphi) & := FV(\varphi)\backslash\{x\} & BV(\eta x.\varphi) & := BV(\varphi) \cup \{x\}
\end{array}
$$

Formulas like $x \vee \mu x.((p \vee x) \wedge \Box \nu x.\Diamond x)$ may be well-formed, but in practice they are very hard to read and to work with. In the sequel, we will often work with formulas in which every bound variable uniquely determines a subformula where it is bound, and almost exclusively with formulas in which no variable has both free and bound occurrences in φ.

Definition 2.6 *A formula $\varphi \in \mu$ML is tidy if $FV(\varphi) \cap BV(\varphi) = \varnothing$, and clean if in addition with every bound variable x of φ we may associate a unique subformula of the form $\eta x.\delta$. In the latter case, we let $\varphi_x = \eta_x x.\delta_x$ denote this unique subformula.*

Now we define the recurrent operation of substitution for formulas of modal μ-calculus.

Definition 2.7 *Let ψ,ξ and x be respectively two modal μ-calculus formulas and a propositional variable. We say that ψ is free for x in ξ if ξ is positive in x and for every variable $y \in FV(\psi)$, every occurrence of x in a subformula $\eta y.\chi$ of ξ is in the scope of a fixpoint operator λx in ξ, i.e., bound in ξ by some occurrence of λx.*

Definition 2.8 *Let $\{\psi_z \mid z \in Z\}$ be a set of modal μ-calculus formulas, indexed by a set of variables Z, let $\varphi \in \mu$ML be positive in each $z \in Z$, and assume that, for each $z \in Z$, ψ_z is free for z in φ. We inductively define the simultaneous substitution $[z/\psi_z \mid z \in Z]$ as the following operation on μML.*

$$
\begin{array}{ll}
\varphi[z/\psi_z \mid z \in Z] & := \begin{cases} \psi_p & \text{if } \varphi = p \in Z \\ \varphi & \text{if } \varphi \text{ is atomic but } \varphi \notin Z \end{cases} \\
\heartsuit\varphi[z/\psi_z \mid z \in Z] & := \heartsuit\varphi[z/\psi_z \mid z \in Z] \\
(\varphi_0 \odot \varphi_1)[z/\psi_z \mid z \in Z] & := \varphi_0[z/\psi_z \mid z \in Z] \odot \varphi_1[z/\psi_z \mid z \in Z] \\
(\eta x.\varphi)[z/\psi_z \mid z \in Z] & := \eta x.\varphi[z/\psi_z \mid z \in Z]
\end{array}
$$

In case Z is a singleton, say $Z = \{z\}$, we will simply write $\varphi[z/\psi_z]$

Definition 2.9 *Given a formula $\eta x.\chi \in \mu$ML, we call the formula $\chi[x/\eta x.\chi]$ its unfolding.*

Definition 2.10 *Let φ and ψ be μ-calculus formulas. We say that φ is a free subformula of ψ, notation: $\varphi \trianglelefteq_f \psi$, if $\psi = \psi_0[x/\varphi]$ for some formula ψ_0 such that $x \in FV(\psi_0)$ and φ is free for x in ψ_0.*

Proposition 2.1 *Let ξ be a clean formula and $x \in BV(\xi)$. For all $\varphi \trianglelefteq \xi$, if $x \in FV(\varphi)$, then $\varphi \trianglelefteq \eta_x.\delta_x$; if $x \in BV(\varphi)$, then $\eta_x.\delta_x \trianglelefteq \varphi$.*

Proof. Consider the subformula dag $(Sfor(\xi), \triangleright_0)$. Note that since ξ is clean, there is a unique subformula in $Sfor(\xi)$ of the form $\eta_x x.\delta'_x$ for some formula $\delta_x{}'$ and that subformula is $\eta_x x.\delta_x$. Since $x \in BV(\xi)$ and ξ is tidy, it follows that $x \notin FV(\xi)$. This means that for all $\xi \triangleright_0 \varphi_0 \triangleright_0 \ldots \triangleright_0 \varphi_n \triangleright_0 x$, there exists $0 \leq i \leq n$ such that $\varphi_i = \eta_x x.\delta_x$. Take any $\varphi \trianglelefteq \xi$ and suppose x occurs in φ. This means that there exists $\xi \triangleright_0 \varphi_0 \triangleright_0 \ldots \triangleright_0 \varphi_n \triangleright_0 x$, there exists $0 \leq j \leq n$ such that $\varphi_j = \eta_x x.\delta_x \varphi$. By the previous statement, there exists $0 \leq i \leq n$ such that $\varphi_i = \eta_x x.\delta_x$. If $i \leq j$, then $x \in BV(\varphi)$ and $\eta_x x.\delta_x$; otherwise, $j < i$, $x \in FV(\varphi)$ and $\varphi \trianglelefteq \eta_x x.\delta_x$. □

Definition 2.11 *Given a clean formula ξ, we define a dependency order $\leq_\xi$ on the set $BV(\xi)$, saying that y ranks higher than x if $x \leq_\xi y$. The relation $\leq_\xi$ is defined as the least partial order containing all pairs (x, y) such that $y \trianglelefteq \delta_x \trianglelefteq \delta_y$*

There are two canonical ways to provide a graph representation for a formula ξ in modal μ-calculus, one based on the set of subformulas of ξ, the other based on the closure set of ξ.

Graph Representations

Subformula Dag

Definition 2.12 *The subformula dag of a formula ξ is the directed acyclic graph $(Sfor(\xi), \triangleright_0)$, where $\triangleright_0$ is the converse of the directed subformula relation $\triangleleft_0$.*

Closure Graph

Definition 2.13 *Let $\to_C$ be the binary relation between tidy μ-calculus formulas given by the following exhaustive list:*

1. $(\varphi_0 \odot \varphi_1) \to_C \varphi_i$, *for any $\varphi_0, \varphi_1 \in \mu\mathrm{ML}$, $\odot \in \{\wedge, \vee\}$ and $i \in \{0, 1\}$;*
2. $\heartsuit\varphi \to_C \varphi$, *for any $\varphi \in \mu\mathrm{ML}$ and $\heartsuit \in \{\Diamond, \Box\}$;*
3. $\eta x.\varphi \to_C \varphi[x/\eta x.\varphi]$, *for any $\eta x.\varphi \in \mu\mathrm{ML}$, with $\eta \in \{\mu, \nu\}$.*

We call a $\to_C$-path $\psi_0 \to_C \psi_1 \to_C \ldots \to_C \psi_n$ a (finite) trace; similarly, an infinite trace is a sequence $(\psi_i)_{i\in\omega}$ such that $\psi_i \to_C \psi_{i+1}$ for all $i \in \omega$.

Definition 2.14 *We define the relation $\twoheadrightarrow_C$ as the relfexive and transitive closure of $\to_C$, and define the closure of a formula ψ as the set*

$$Clos(\psi) := \{\varphi \mid \psi \twoheadrightarrow_C \varphi\}.$$

Formulas in the set $Clos(\psi)$ are said to be derived from ψ. The closure graph of ψ is the directed graph $(Clos(\xi), \to_C)$.

Algebraic Semantics for Modal μ-calculus

Canonically, formulas of modal μ-calculus are evaluated on pointed Kripke models.

Definition 2.15 *A (labelled) transition system, LTS, or Kripke model of type* (P, D) *is a triple* $\mathbb{S} = \langle S, V, R\rangle$ *such that* S *is a set of objects called states or points,* $V : \mathsf{P} \to \mathcal{P}(S)$ *is a valuation, and* $R \subseteq S \times S$ *is a binary accessiblility relation.*

Elements of the set $R[s] := \{t \in S \mid (s,t) \in R\}$ *are called successors of* s*.*

A pointed labelled transition system (pointed Kripke model) is a pair $(\mathbb{S}, s)$ *consisting of a transition system* $\mathbb{S}$ *and a designated state* s *in* $\mathbb{S}$*.*

The following theorem is true in general for all complete lattices. However, due to the restricted application of this theorem in this thesis, we restrict the formulation to complete lattices induced by the powersets of sets.

Theorem 2.2 (Knaster-Tarski, Powerset) *Let* S *be a set and let* $\mathcal{P}S$ *denote the powerset of* S*. Let* $f : \mathcal{P}S \to \mathcal{P}S$ *be monotone. Then both the least and the greatest fixpoint exists for* f*, and these are given as*

$$LFP.f = \bigcap PRE(f) \qquad\qquad GFP.f = \bigcup POS(f).$$

where LFP stands for the least fixpoint, GFP stands for the greatest fixpoint, $PRE(f) := \{A \in \mathcal{P}S \mid f(A) \subseteq A\}$ *and* $POS(f) := \{A \in \mathcal{P}S \mid A \subseteq f(A)\}$*.*

Proof. We only prove the result for the least fixpoint, the proof for the greatest fixpoint is completely analogous.

Define $q := \bigwedge \mathrm{PRE}(f)$, then we have that $q \subseteq x$ for all prefixpoints x of f. From this it follows by monotonicity that $f(q) \subseteq f(x)$ for all $x \in \mathrm{PRE}(f)$, and hence by definition of prefixpoints, $f(q) \subseteq f(x)$ for all $x \in \mathrm{PRE}(f)$, and hence by definition of prefixpoints, $f(q) \subseteq x$ for all $x \in \mathrm{PRE}(f)$. In other words, $f(q)$ is a lower bound of the set $\mathrm{PRE}(f)$. Hence, by definition of q as the greatest such lower bound, we find $f(q) \subseteq q$, that is, q itself is a prefixpoint of f.

It now suffices to prove that $q \subseteq f(q)$, and for this we may show that $f(q)$ is a prefixpoint of f as well, since q is by definition a lower bound of the set of prefixpoints. But in fact, we may show that $f(y)$ is a prefixpoint of f for every prefixpoint y of f, by monotonicity of f it immediately follows from $f(y) \subseteq y$ that $f(f(y)) \subseteq f(y)$. □

In order to define the algebraic semantics of the modal μ-calculus, we need to consider formulas as operations on the power set of the state space of a transition system, and we have to prove that such operations indeed have least and greatest fixpoints. In order to make this precise, we need some preliminary definitions.

Definition 2.16 *Given an LTS* $\mathbb{S} = \langle S, V, R\rangle$ *and subset* $X \subseteq S$*, define the valuation* $V[x \mapsto X]$ *by putting*

$$V[x \mapsto X](y) := \begin{cases} V(y) & \text{if } y \neq x \\ X & \text{if } y = x \end{cases}$$

Then, the LTS $\mathbb{S}[x \mapsto X]$ *is given as the structure* $\langle S, V[x \mapsto X], R\rangle$*.*

Now inductively assume that $[\![\varphi]\!]^{\mathbb{S}}$ has been defined for all LTSs. Given a labelled transition system $\mathbb{S}$ and a propositional variable $x \in \mathsf{P}$, each formula φ induces a map $\varphi^{\mathbb{S}}_x : \mathcal{P}(S) \to \mathcal{P}(S)$ defined by

$$\varphi^{\mathbb{S}}_x(X) := [\![\varphi]\!]^{\mathbb{S}[x \mapsto X]}.$$

In particular, it is not the case that every formula has a least fixpoint. If we can guarantee that the induced function $\varphi^{\mathbb{S}}_x$ of φ is monotone, however, then the Knaster-Tarski theorem provides both least and greatest fixpoints of $\varphi^{\mathbb{S}}_x$. Precisely for this reason, in the definition of fixpoint formulas, we imposed the condition in the clauses for $\eta x.\varphi$, that x may only occur positively in φ. As we will see, this condition on x guarantees monotonicity of the function $\varphi_x^{\mathbb{S}}$.

Definition 2.17 *Given a* μML*-formula* φ *and a labelled transition system* $\mathbb{S} = \langle S, V, R\rangle$, *we define the extension* $[\![\varphi]\!]^{\mathbb{S}}$, *together with the map* $\varphi_x^{\mathbb{S}} : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$ *by the following simultaneous formula induction:*

$$
\begin{aligned}
[\![\bot]\!]^{\mathbb{S}} &= \varnothing \\
[\![\top]\!]^{\mathbb{S}} &= S \\
[\![p]\!]^{\mathbb{S}} &= V(p) \\
[\![\overline{p}]\!]^{\mathbb{S}} &= S \backslash V(p) \\
[\![\varphi \vee \psi]\!]^{\mathbb{S}} &= [\![\varphi]\!]^{\mathbb{S}} \cup [\![\psi]\!]^{\mathbb{S}} \\
[\![\varphi \wedge \psi]\!]^{\mathbb{S}} &= [\![\varphi]\!]^{\mathbb{S}} \cap [\![\psi]\!]^{\mathbb{S}} \\
[\![\Diamond\varphi]\!]^{\mathbb{S}} &= \langle R\rangle [\![\varphi]\!]^{\mathbb{S}} \\
[\![\Box\varphi]\!]^{\mathbb{S}} &= [R][\![\varphi]\!]^{\mathbb{S}} \\
[\![\mu x.\varphi]\!]^{\mathbb{S}} &= \bigcap PRE(\varphi_x^{\mathbb{S}}) \\
[\![\nu x.\varphi]\!]^{\mathbb{S}} &= \bigcup POS(\varphi_x^{\mathbb{S}})
\end{aligned}
$$

Here $\langle R\rangle(U) := \{s \in S \mid Rsu \text{ for some } u \in U\}$ *and* $[R](U) := \{s \in S \mid Rsu \text{ for all } u \in U\}$. *The map* $\varphi_x^{\mathbb{S}}$ *for* $x \in$ Prop, *is given by* $\varphi_x^{\mathbb{S}}(X) = [\![\varphi]\!]^{\mathbb{S}[x \mapsto X]}$.

Theorem 2.3 *Let* φ *be an* μML*-formula, in which* x *occurs only positively, and let* $\mathbb{S}$ *be a labelled transition system. Then* $[\![\mu x.\varphi]\!]^{\mathbb{S}} = LFP.\varphi_x^{\mathbb{S}}$ *and* $[\![\nu x.\varphi]\!]^{\mathbb{S}} = GFP.\varphi_x^{\mathbb{S}}$.

Proof. This is an immediate consequence of the Knaster-Tarski theorem, provided we can prove that $\varphi_x^{\mathbb{S}}$ is monotone in x if all occurences of x in φ are positive. The latter is easily proven by induction on the structure of φ. □

Board Games

The games that appear in this thesis can be classified as board or graph games. They are played by two agents, here to be called 0 and 1. A board game is played on a board or arena, which is nothing but a directed graph in which each node is marked with either 0 or 1. A match or play of the game consists of the two players moving a pebble or token across the board, following the edges of the graph. To regulate this, the collection of graph nodes, usually referred to as positions of the game, is partitioned into two sets, one for each player. Thus, with each position we may associate a unique player whose turn it is to move when the token lies on position p. A match or play of the game corresponds to a (finite or infinite) path through the graph. Furthermore, the winning conditions of a match are determined by the nature of this path.

Definition 2.18 *A board or arena is a structure* $\mathbb{B} = \langle B_0, B_1, E\rangle$, *such that* B_0 *and* B_1 *are disjoint sets, and* $E \subseteq B^2$, *where* $B := B_0 \cup B_1$. *We will make use of the notation* $E[p]$ *for the set of admissible or legitimate moves from a board position* $p \in B$, *that is,* $E[p] := \{q \in B \mid (p, q) \in E\}$. *Positions not in* $E[p]$ *will sometimes be referred to as illegitimate moves with respect to* p. *A position* $p \in B$ *is a dead end if* $E[p] = \varnothing$. *If* $p \in B$, *we let* Π_p *denote the (unique) player such that* $p \in B_{\Pi_p}$, *and say that* p *belongs to* Π_p, *or that it is* Π_p*'s turn to move at* p.

A match of the game may in fact be identified with the sequence of positions visited during play, and thus corresponds to a path through the graph.

Definition 2.19 *A path through a board $\mathbb{B} = \langle B_0, B_1, E \rangle$ is a (finite or infinite) sequence $\Sigma \in B^\infty$ such that $E\Sigma_i\Sigma_{i+1}$ whenever applicable, where $\Sigma \in B^\infty$ denotes the collection of all finite and infinite sequences of elements of B. A full or complete match or play through $\mathbb{B}$ is either an infinite $\mathbb{B}$-path, or a finite $\mathbb{B}$-path ending with a dead end. A partial match is a finite path through $\mathbb{B}$ that is not a full match; in other words, the last position of a partial match is not a dead end. We let PM_Π denote the set of partial matches such that Π is the player whose turn it is to move at the last position of the match.*

Each full or complete match is won by one of the players, and lost by their opponent; that is, there are no draws. A finite match ends if one of the players gets stuck, that is, is forced to move the token from a position without successors. Such a finite, completed, match is lost by the player who got stuck. If neither player ever gets stuck, an infinite match arises. The flavor of a board game is very much determined by the winning conditions of these infinite matches.

Definition 2.20 *Given a board $\mathbb{B}$, a winning condition is a map $W : B^\infty \to \{0, 1\}$. An infinite match Σ is won by $W(\Sigma)$. A board game is a structure $\mathcal{G} = \langle B_0, B_1, E, W \rangle$ such that $\langle B_0, B_1, E \rangle$ is a board, and W is a winning condition on B.*

Before players can actually start playing a game, they need a starting position.

Definition 2.21 *An initialized board game is a pair consisting of a board game $\mathcal{G}$ and a position q on the board of the game; such a pair is usually denoted $\mathcal{G}@q$.*

Central in the theory of games is the notion of a strategy. Roughly, a strategy for a player is a method that the player uses to decide how to continue partial matches when it is their turn to move. More precisely, a strategy is a function mapping partial plays for the player to new positions. It is a matter of definition whether one requires a strategy to always assign moves that are legitimate or not; here we will not make this requirement.

Definition 2.22 *Given a board game $\mathcal{G} = \langle B_0, B_1, E, W \rangle$ and a player Π, a Π-strategy, or a strategy for Π, is a map $f : PM_\Pi \to B$. In cases we are dealing with an initialized game $\mathcal{G}@q$, then we may take a strategy to be a map $f : PM_\Pi(q) \to B$. A match Σ is consistent with or guided by a Π-strategy f if for any Σ' which is a proper initial segment of Σ with ($last(\Sigma') \in B_\Pi$), the next position on Σ (after Σ') is indeed the element $f(\Sigma')$.*

Definition 2.23 *A Π-strategy is surviving in $\mathcal{G}@q$ if the moves that it prescribes to f-guided partial matches in $PM_\Pi@p$ are always admissible to Π, and winning for Π in $\mathcal{G}@p$ if in addition all f-guided full matches starting at p are won by Π. A position $q \in B$ is winning for Π if Π has a winning strategy for the game $\mathcal{G}@q$; the collection of all winning positions for Π in $\mathcal{G}$ is called the winning region for Π in $\mathcal{G}$, and denoted as $Win_\Pi(\mathcal{G})$.*

Intuitively, f being a surviving strategy in $\mathcal{G}@q$ means that Π never gets stuck in an f-guided match of $\mathcal{G}@q$, and so guarantees that Π can stay in the game forever.

It is easy to see that a position in a game $\mathcal{G}$ cannot be winning for both players. On the other hand, whether a position p is always a winning position for one of the players, is a rather subtle one. Observe that in such games the two winning regions partition the game board.

Definition 2.24 *The game $\mathcal{G}$ on board $\mathbb{B}$ is determined if $Win_0(\mathcal{G}) \cup Win_1(\mathcal{G}) = B$; that is, each position is winning for one of the players.*

In principle, when deciding how to move a match of a board game, players may use information about the entire history of the match played thus far. However, it will turn out to be advantageous to work with strategies that are simple to compute. Particularly nice are so called positional strategies, which only depend on the current position (i.e., the final position of the partial play).

Definition 2.25 *A strategy f is positional or history-free if $f(\Sigma) = f(\Sigma')$ for any Σ, Σ' with $last(\Sigma) = last(\Sigma')$.*

Parity Games

Now, we introduce a specific type of infinite board games called parity games. A parity game is played on a colored directed arena, where each node has been colored by a priority – one of finitely many natural numbers.

Definition 2.26 *A coloring of B is a function $\Gamma : B \longrightarrow C$ assigning to each position $p \in B$ a color $\Gamma(p)$ taken from some finite set C of colors. By putting $\Gamma(p_0p_1...) := \Gamma(p_0)\Gamma(p_1)...$ we can naturally extend such a coloring $\Gamma : B \longrightarrow C$ to a map $\Gamma : B^\omega \longrightarrow C^\omega$.*

Now if $\Gamma : B \longrightarrow C$ is a coloring, for any infinite sequence $\Sigma \in B^\omega$, the map $\Gamma \circ \Sigma \in C^\omega$ forms the associated sequence of colors. But then since C is finite, there must be some elements of C that occur infinitely often in this stream.

Definition 2.27 *Let $\mathbb{B}$ be a board and $\Gamma : B \longrightarrow C$ a coloring of B. Given an infinite sequence $\Sigma \in B^\omega$, we let $inf_\Gamma(\Sigma)$ denote the non-empty set of colors that occur infinitely often in the sequence $\Gamma \circ \Sigma$.*

Definition 2.28 *Let B be some set; a priority map on B is a coloring $\Omega : B \longrightarrow \omega$, that is, a map of finite range. A parity game is a board game $\mathcal{G} = (B_0, B_1, E, W_\Omega)$ in which the winning condition is given by*

$$W_\Omega : B^\omega \longrightarrow \{0, 1\}, \quad W_\Omega(\Sigma) := max(inf_\Omega(\Sigma)) \ mod\ 2.$$

Such a parity game is usually denoted as $\mathcal{G} = \langle B_0, B_1, E, \Omega \rangle$.

The key property that makes parity games so interesting is that they enjoy positional determinacy.

Theorem 2.4 (Positional Determinacy of Parity Games) *For any parity game $\mathcal{G}$ there are positional strategies f_0 and f_1 for 0 and 1, respectively, such that for every position q there is a player Π such that f_Π is a winning strategy for Π in $\mathcal{G}@q$.*

Here we omit the proof of this theorem, interested readers can refer to [13] or [14].

To end this subsection, the following is an auxiliary proposition that comes in handy in this thesis for proving that two matches in parity games have the same winner.

Proposition 2.5 *Let a, b be two function from ω to ω of finite range. If there exists $m \in \omega$ such that for all $i, j \geq m$ it holds that*

1) $a(i) \equiv_2 b(i)$
2) *if $a(i) \not\equiv_2 a(j)$, then $a(i) < a(j)$ if and only if $b(i) < b(j)$,*

then $max(inf(a)) \equiv_2 max(inf(b))$.

Proof. Let $n \geq m$ be such that $a(n) = \max(inf(a))$. Note that such an n always exists because $\max(inf(a)$ occurs infinitely many times in a. Since $a(n)$ is the maximal value that occurs in a for infinitely many times, it follows that there are only finitely many $i \in \omega$ such that $a(i) > a(n)$. Specifically, there are only finitely many $m \leq i < \omega$ such that $a(i) > a(n)$. This means that the set

$$\{i \in \omega \mid a(i) \not\equiv_2 a(n), a(i) > a(n), i \geq m\}$$

Table 2.1: Evaluation game for modal μ-calculus

Position	Player	Admissible moves
$(\varphi_1 \vee \varphi_2, s)$	$\exists$	$\{(\varphi_1, s), (\varphi_2, s)\}$
$(\varphi_1 \wedge \varphi_2, s)$	$\forall$	$\{(\varphi_1, s), (\varphi_2, s)\}$
$(\Diamond\varphi, s)$	$\exists$	$\{(\varphi, t) \mid t \in \sigma(s)\}$
$(\Box\varphi, s)$	$\forall$	$\{(\varphi, t) \mid t \in \sigma(s)\}$
$(\bot, s)$	$\exists$	$\varnothing$
$(\top, s)$	$\forall$	$\varnothing$
$(p, s), p \in V(s)$	$\forall$	$\varnothing$
$(p, s), p \notin V(s)$	$\exists$	$\varnothing$
$(\overline{p}, s), p \in V(s)$	$\exists$	$\varnothing$
$(\overline{p}, s), p \notin V(s)$	$\forall$	$\varnothing$
$(\eta_x x.\delta_x, s)$	-	$\{(\delta_x, s)\}$
(x, s),with $x \in BV(\xi)$	-	$\{(\delta_x, s)\}$

Table 2.2: Winning conditions of $\mathscr{E}(\xi, \mathbb{S})$

	$\exists$ wins Σ	$\forall$ wins Σ
Σ is finite	$\forall$ got stuck	$\exists$ got stuck
Σ is infinite	$\max(Unf^{\infty}(\Sigma))$ is a ν variable	$\max(Unf^{\infty}(\Sigma))$ is a μ variable

is of finite size.

Now, we show that, for all $i \geq m$, we have $a(i) \not\equiv_2 a(n)$ and $a(i) > a(n)$ if and only if $b(i) \not\equiv_2 b(n)$ and $b(i) > b(n)$. Fix an arbitrary $i \geq m$

- $\Rightarrow$ Suppose $a(i) \not\equiv_2 a(n)$ and $a(i) > a(n)$. By condition 1) of the assumption in the proposition, $b(i) \equiv_2 a(i) \not\equiv_2 a(n) \equiv_2 b(n)$. By condition 2), we have that $b(i) > b(n)$.
- $\Leftarrow$ Suppose $b(i) \not\equiv_2 b(n)$ and $b(i) > b(n)$. By condition 1) of the assumption in the proposition, $a(i) \equiv_2 b(i) \not\equiv_2 b(n) \equiv_2 a(n)$. Given that we have shown $a(i) \not\equiv_2 a(n)$, by condition 2), we have $a(i) > a(n)$.

In other words,

$$\{i \geq m \mid a(i) \not\equiv_2 a(n), a(i) > a(n)\} = \{i \geq m \mid b(i) \not\equiv_2 b(n), b(i) > b(n)\}.$$

Note that

$$\begin{aligned}
|\{i \in \omega \mid b(i) \not\equiv_2 b(n), b(i) > b(n)\}| &= |\{i \geq m \mid b(i) \not\equiv_2 b(n), b(i) > b(n)\} \cup \{i < m \mid b(i) \not\equiv_2 b(n), b(i) > b(n)\}| \\
&= |\{i \geq m \mid b(i) \not\equiv_2 b(n), b(i) > b(n)\}| + |\{i < m \mid b(i) \not\equiv_2 b(n), b(i) > b(n)\}| \\
&\leq |\{i \in \omega \mid b(i) \not\equiv_2 b(n), b(i) > b(n), i \geq m\}| + m \\
&= |\{i \in \omega \mid a(i) \not\equiv_2 a(n), a(i) > a(n), i \geq m\}| + m
\end{aligned}$$

Therefore, $\{i \in \omega \mid b(i) \not\equiv_2 b(n), b(i) > b(n)\}$ is of finite size. This means that for any $k \in \omega$ such that $k \in inf(b)$ and $k > b(n)$, it most be the case that $k \equiv_2 b(n)$. Thus,

$$\max(inf(b)) \equiv_2 b(n) \equiv_2 a(n) = \max(inf(a)).$$

□

Game Semantics for modal μ-calculus

We now provide an equivalent semantics for modal μ-calculus in game-theoretic terms. We define the evaluation game $\mathscr{E}(\xi, \mathbb{S})$ associated with a (fixed) formula ξ and a (fixed) LTS $\mathbb{S}$ (player 0 and 1 are refered to as player $\forall$ and $\exists$). This game is an example of a board game.

Definition 2.29 *Given a clean modal μ-calculus formula ξ and a transition system $\mathbb{S}$, we define the evaluation game or model checking game $\mathcal{E}(\xi, \mathbb{S})$ as a board game with player $\exists$ and $\forall$ moving a token around positions of the form $(\varphi, s) \in Sfor(\xi) \times S$. The rules, determining the admissible moves from a given position, together with the player who is supposed to make this move, are given in Table 2.1. $\mathcal{E}(\xi, \mathbb{S})@(\xi, s)$ denotes the instantiation of this game where the stargin position is fixed as (ξ, s).*

Definition 2.30 *Let ξ be a clean μML-formula, and $\mathbb{S}$ a labelled transition system. A match of the game $\mathcal{E}(\xi, \mathbb{S})$ is a (finite or infinite) sequence of positions*

$$\Sigma = (\varphi_i, s_i)_{i \in \kappa}$$

(where κ is either a natural number or ω) which is in accordance with the rules of the evaluation game — that is, Σ is a path through the game graph given by the admissibility relation of Table 2.1. A full match is either an infinite match, or a finite match in which the player responsible for the last position got stuck. In practice, we will always refer to full matches simply as matches. A match that is not full is called partial.

Definition 2.31 *Given an infinite match Σ, we let $Unf^{\infty} \subseteq BV(\xi)$ denote the set of variables that are unfolded infinitely during Σ.*

Proposition 2.6 *Let ξ be a clean μML-formula, and $\mathbb{S}$ a labelled transition system. Then for any infinite match Σ of the game $\mathcal{E}(\xi, \mathbb{S})$, the set $Unf^{\infty}(\Sigma)$ has a highest ranking member in terms of the dependency order.*

Proof. Since Σ is an infinite match, the set $U := Unf^{\infty}(\Sigma)$ is not empty. Let y be an element of U which is maximal (with respect to the ranking order $\leq_{\xi}$) — such an element exists since U is finite.

We claim that

$$\text{from some moment on, } \Sigma \text{ only features subformulas of } \delta_y. \tag{2.1}$$

To prove this, note that since y is $\leq_{\xi}$-maximal in U, there must be a position in Σ such that y is unfolded to δ_y, while no variable $z >_{\xi} y$ is unfolded at any later position in Σ. But then a straightforward induction shows that all formulas featuring at later positions must be subformulas of δ_y: the key observation here is that if $z \trianglelefteq \delta_y$ unfolds to δ_z, and by assumption $z \not>_{xi} y$, then it must be the case that $\delta_z \trianglelefteq \delta_u$.

As a corollary of 2.1, we claim that

$$y \text{ is in fact the maximum of } U \text{ (with respect to } \leq_{\xi} \text{)}. \tag{2.2}$$

To see this, suppose for contradiction that there is a variable $x \in U$ which is not below y. It follows from 2.1 that $\delta_x \trianglelefteq \delta_y$, and without loss of generality we may assume x to be such that δ_x is a maximal subformula of δ_y such that $x \not\leq_{\xi} y$ (in the sense that $z \leq_{\xi} y$ for all $z \in U$ with $\delta_x \triangleleft \delta_z$). In particular, then we have $y \notin FV(\delta_x)$. But since y is unfolded infinitely often, there must be a variable $z \in FV(\delta_x)$ which allows Σ to 'leave' δ_x infinitely often; this means that $z \in U$, $\delta_x \trianglelefteq \delta_z$, but $\delta_z \ntrianglelefteq \delta_x$. From this it is immediate that $x \leq_{\xi} z$, while from $z \in U$ and 2.1 we obtain $\delta_z \trianglelefteq \delta_y$. It now follows from our maximality assumption on x that $z \leq_{\xi} y$. But then by transitivity of $\leq_{\xi}$ we find that $x \leq_{\xi} y$ indeed. In other words, we have arrived at the desired contradiction. This shows that 2.2 holds indeed, and from this the proposition is immediate. □

Definition 2.32 *Let ξ be a clean μML-formula. The winning conditions of the game $\mathcal{E}(\xi, \mathbb{S})$ are given in Table 2.2.*

In fact, the $\mathcal{E}(\xi, \mathbb{S})$ mentioned in the definition above can always be seen as a parity game. This is due to the fact that we can always map the dependency order of bound variables onto the natural numbers in an order- and parity-preserving way. We omit the detail of this mapping here, but interested reader can refer to [13].

We can now formulate the game-theoretic semantics of the modal μ-calculus as follows.

Definition 2.33 *Let ξ be a clean formula of the modal μ-calculus, and let $\mathbb{S}$ be a transition system of the appropriate type. Then we say that ξ is (game-theoretically) satisfied at s, notation $\mathbb{S}, s \models_g \xi$ if $(\xi, s) \in Win_\exists(\mathcal{E}(\xi, \mathbb{S}))$.*

Theorem 2.7 (Equivalence of the Algebraic Semantics and the Game Semantics) *Let ξ be a clean μML-formula. Then for any Kripke model $\mathbb{S}$ and any state s in $\mathbb{S}$:*

$$s \in [\![\xi]\!]^{\mathbb{S}} \iff (\xi, s) \in Win_\exists(\mathcal{E}(\xi, \mathbb{S})).$$

We omit the proof of this theorem. Interested readers can refer to [13].

Parity Formula

Syntax

Definition 2.34 *Let P be a finite set of proposition letters. A parity formula over P is a quintuple $\mathbb{G} = (V, E, L, \Omega, v_I)$ where*

1. *(V, E) is a finite, directed graph, with $|E[v]| \leq 2$ for every vertex v;*
2. *$L : V \longrightarrow \mathsf{At}(\mathsf{P}) \cup \{\wedge, \vee, \diamond, \square, \epsilon\}$ is a labelling function;*
3. *$\Omega : V \overset{\circ}{\longrightarrow} \omega$ is a partial map, the priority map of $\mathbb{G}$; and*
4. *v_I is a vertex in V, referred to as the initial node of $\mathbb{G}$;*

such that

1. *$|E[v]| = 0$ if $L(v) \in \mathsf{At}(\mathsf{P})$, and $|E[v]| = 1$ if $L(v) \in \{\diamond, \square, \epsilon\}$; and*
2. *every cycle of (V, E) contains at least one node in $\mathsf{Dom}(\Omega)$.*

A node $v \in V$ is called silent if $L(v) = \epsilon$, constant if $L(v) \in \{\top, \bot\}$, literal if $L(v) \in \mathsf{Lit}(\mathsf{P})$, atomic if it is either constant or literal, boolean if $L(v) \in \{\wedge, \vee\}$ and modal if $L(v) \in \{\diamond, \square\}$. Elements of $\mathsf{Dom}(\Omega)$ will be called states. We say that a proposition letter q occurs in $\mathbb{G}$ if $L(v) \in \{q, \overline{q}\}$ for some $v \in V$.

Example 2.1 Figure 2.1 shows the parity formula

$$\begin{aligned}\mathbb{G} = (V &= \{w, x, y, z\},\\ E &= \{(w, x), (x, y), (x, z), (y, w),\\ L &= \{(w, \epsilon), (x, \vee), (y, \square), (z, p)\},\\ \Omega &= \{(w, 0)\}\}\\ v_I &= w).\end{aligned}$$

This parity formula corresponds to the modal μ-formula $\mu x.\square x \vee p$.

Semantics

An example that demonstrates the following semantics can be found in Appendix A.

Definition 2.35 *Let $\mathbb{S} = (S, R, V)$ be a Kripke model for a set P of proposition letters, and let $\mathbb{G} = (V, E, L, \Omega, v_I)$ be a parity P-formula. The evaluation game $\mathcal{E}(\mathbb{G}, \mathbb{S})$ is the parity game (G, E, Ω') of which the board consists of the*

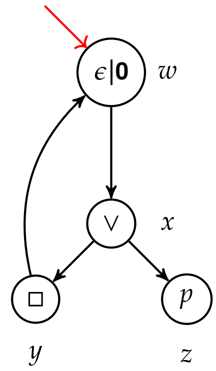


Figure 2.1: A parity formula that corresponds to the modal μ-formula $\mu x.\Box x \vee p$

Table 2.3: Evaluation game $\mathcal{E}(\mathbb{G}, \mathbb{S})$

Position (v, s)	Player	Admissible moves
$L(v) = p$ & $s \in V(p)$	$\forall$	$\varnothing$
$L(v) = p$ & $s \notin V(p)$	$\exists$	$\varnothing$
$L(v) = \overline{p}$ & $s \in V(p)$	$\exists$	$\varnothing$
$L(v) = \overline{p}$ & $s \notin V(p)$	$\forall$	$\varnothing$
$L(v) = \bot$	$\exists$	$\varnothing$
$L(v) = \top$	$\forall$	$\varnothing$
$L(v) = \epsilon$	-	$E[v] \times \{s\}$
$L(v) = \vee$	$\exists$	$E[v] \times \{s\}$
$L(v) = \wedge$	$\forall$	$E[v] \times \{s\}$
$L(v) = \Diamond$	$\exists$	$E[v] \times R[s]$
$L(v) = \Box$	$\forall$	$E[v] \times R[s]$

set $V \times S$, the priority map $\Omega' : V \times S \longrightarrow \omega$ is given by

$$\Omega'(v, s) := \begin{cases} \Omega(v) & \text{if } v \in \mathsf{Dom}(\Omega) \\ 0 & \text{otherwise} \end{cases}$$

and the game graph is given in Table 2.3. Note that we do not need to assign a player to positions that admit a single move only.

Definition 2.36 *A parity formula $\mathbb{G} = (V, E, L, \Omega, v_I)$ holds at or is satisfied by a pointed Kripke model $(\mathbb{S}, s)$, notation: $\mathbb{S}, s \models \mathbb{G}$, if the pair (v_I, s) is a winning position for $\exists$ in $\mathcal{E}(\mathbb{G}, \mathbb{S})$. We say two parity formulas $\mathbb{G}$ and $\mathbb{G}'$ are equivalent if they hold at the exact same class of pointed Kripke models, notation $\mathbb{G} \equiv \mathbb{G}'$.*

Definition 2.37 *Let $\mathbb{G} = (V, E, L, \Omega, v_I)$ be a parity formula, and let v be a vertex in V. Let $\mathbb{G}\langle v \rangle := (V, E, L, \Omega, v)$ denote the variant of $\mathbb{G}$ that takes v as its initial node.*

From μML Formula to Parity Formula

In this subsection, we discuss two ways to construct an equivalent parity formula given a clean modal μ-formula.

Parity Formulas on Subformula Dag

The following construction shows that for a clean formula, we can indeed obtain an equivalent parity formula which is based on its subformula dag. The basic idea underlying the construction is to view the evaluation games for clean formulas in μML as instances of parity games. Given an arbitrary formula $\xi \in \mu$ML, we then need to see which modifications are needed to turn the subformula dag $(Sfor(\xi), \triangleright_0)$ into a parity formula $\mathbb{H}_\xi$ such that, for any model $\mathbb{S}$, the evaluation games $\mathcal{E}(\xi, \mathbb{S})$ and $\mathcal{E}(\mathbb{H}_\xi, \mathbb{S})$ are more or less identical. Clearly, the fact that the positions of the evaluation game $\mathcal{E}(\xi, \mathbb{S})$ are given as the pairs in the set $Sfor(\xi) \times S$, means that we can take the set $V_\xi := Sfor(\xi)$ as the carrier of $\mathbb{H}_\xi$ indeed.

Looking at the admissible moves in the two games, it turns out that we cannot just take the converse direct subformula relation $\triangleright_0$ as the edge relation of $\mathbb{H}_\xi$: we need to add all back edges from the set

$$B_\xi := \{(x, \delta_x) \mid x \in BV(\xi)\},$$

where, as usual, we let δ_x denote the unique formula such that, for some $\eta \in \{\mu, \nu\}$ the formula $\eta x.\delta_x$ is a subformula of ξ. In fact, if we write D_ξ for the relation $\triangleright_0$, restricted to $Sfor(\xi)$, then we can take

$$E_\xi := D_\xi \cup B_\xi,$$

as the edge relation of $\mathbb{H}_\xi$. Furthermore, the labelling L_ξ is naturally defined via the following case distinction:

$$L_\xi(\varphi) := \begin{cases} \varphi & \text{if } \varphi \in \{\top, \bot\} \cup \{p, \overline{p} \mid p \in FV(\xi)\} \\ \odot & \text{if } \varphi \text{ is of the form } \varphi_0 \odot \varphi_1 \text{ with } \odot \in \{\vee, \wedge\} \\ \heartsuit & \text{if } \varphi \text{ is of the form } \heartsuit\psi \text{ with } \heartsuit \in \{\Box, \Diamond\} \\ \epsilon & \text{if } \varphi \text{ is of the form } \eta_x x.\delta_x \text{ with } \eta \in \{\mu, \nu\} \end{cases}$$

With this definition, it is easy to see that the boards of the two evaluation games $\mathcal{E}(\xi, \mathbb{S})$ and $\mathcal{E}(\mathbb{H}_{xi}, \mathbb{S})$ are isomorphic, for any labeled transition system $\mathbb{S}$. As the initial node v_ξ of $\mathbb{H}\xi$ we simply take

$$v_\xi := \xi.$$

In order to finish the definition of the parity formula $\mathbb{H}_\xi$ it is then left to come up with a suitable priority map Ω_ξ on V_ξ. Since the winning condition s of the evaluation game for the formula ξ are defined in terms of the priority ordering $\leq_\xi$ on the collection $BV(\xi)$ of bound variables of ξ, it seems natural to take these bound variables of ξ as the states of $\mathbb{H}_\xi$, that is, the nodes for which a priority is defined. It will be more convenient, however, to take the unfolding of these bound variables instead; that is, we take

$$\mathsf{Dom}(\Omega_\xi) := \{\delta_x \mid x \in BV(\xi)\}.$$

Definition 2.38 *Let ξ be a clean modal μ-formula. Define the its parity formula translation based on the subformula dag $\mathbb{H}_\xi := (V_\xi, E_\xi, L_\xi, \Omega_\xi, \xi)$.*

Note that for the sake of equivalence between ξ and $\mathbb{H}_\xi$, it is sufficient for the the priority map Ω to satisfy the following two conditions:

1. $\Omega(\delta_x) \leq \Omega(\delta_y)$ if and only if $x \leq_\xi y$, and
2. $\Omega(\delta_x)$ is even if and only if x is a ν-variable.

In light of this, we propose

$$\Omega_\xi(\delta_x) := \begin{cases} h_\xi(x) - h_\xi^{\uparrow}(x) & \text{if } h_\xi(x) - h_\xi^{\uparrow}(x) \text{ has the same parity as } \eta_x \\ h_\xi(x) - h_\xi^{\uparrow}(x) + 1 & \text{otherwise,} \end{cases}$$

where for any bound variable $x \in BV(\xi)$, $h^{\uparrow}_{\xi}(x)$ denote the maximal length of an alternating $<_{\xi}$-chain of fixpoint variables starting at x, and $h_{\xi}(x)$ denote the maximal length of an alternating $<_{\xi}$-chain in the cluster of x. Additionally, $ind(\mathbb{H}_{\xi}) = ad(\xi)$ in this construction. $\mathbb{H}_{\xi}$ can be shown to be equivalent to ξ. Interested readers can refer to [15].

Parity Formulas on Closure Graph

For an arbitrary tidy formula, we can find an equivalent parity formula that is based on the formula's closure graph.

The priority map that we will define on the closure graph of a tidy formula is in fact global in the sense that it can be defined uniformly for all (tidy) formulas, independently of any ambient formula. Furthermore, we will base this map on a partial order of fixpoint formulas, the closure priority relation $\sqsubseteq_C$ that we will introduce now. Recall that $\trianglelefteq_f$ denotes the free subformula relation introduced in Definition 2.10.

Definition 2.39 *We let $\equiv_C$ denote the equivalence relation generated by the relation $\to_C$, in the sense that: $\varphi \equiv_C \psi$ if $\varphi \twoheadrightarrow_C \psi$ and $\psi \twoheadrightarrow_C \varphi$. We will refer to the equivalence classes of $\equiv_C$ as (closure) clusters, and denote the cluster of a formula φ as $C(\varphi)$.*

Definition 2.40 *Define the closure priority relation $\sqsubseteq_C$ on fixpoint formulas by putting $\varphi \sqsubseteq_C \psi$ precisely if $\psi \twoheadrightarrow^{\psi}_C \varphi$, where $\twoheadrightarrow^{\psi}_C$ is the relation given by $\rho \twoheadrightarrow^{\psi}_C \sigma$ if there is a trace $\rho = \chi_0 \to_C \chi_1 \to_C \dots \to_C \chi_n = \sigma$ such that $\psi \trianglelefteq_f \chi_i$ for all $0 \leq i \leq n$. We write $\varphi \sqsubset_C \psi$ if $\varphi \sqsubseteq_C \psi$ and $\psi \not\sqsubseteq_C \varphi$.*

Definition 2.41 *An alternating $\sqsubset_C$-chain of length n is a sequence $(\eta_i x_i.\chi_i)_{0 \leq i \leq n}$ of tidy fixpoint formulas such that $\eta_i x_i.\chi_i \sqsubset_C \eta_{i+1} x_{i+1}.\chi_{i+1}$ and $\eta_{i+1} = \overline{\eta_i}$ for all $0 \leq i < n$. We say that such a chain starts at $\eta_1 x_1.\chi_1$ and leads up to $\eta_n x_n.\chi_n$.*

Given a tidy fixpoint formula ξ, we let $h^{\uparrow}(\xi)$ and $h^{\downarrow}(\xi)$ denote the maximal length of any alternating $\sqsubset_C$-chain starting at, respectively leading up to, ξ. Given a closure cluster C, we let $cd(C)$ denote the closure depth of C, i.e., the maximal length of any alternating $\sqsubset_C$-chain in C.

The global priority function $\Omega_g : \mu\mathsf{ML} \to \omega$ is defined cluster-wise, as follows. Take an arbitrary tidy fixpoint formula $\psi = \eta y.\varphi$, define

$$\Omega_g(\psi) := \begin{cases} cd(C(\psi)) - h^{\uparrow}(\psi) & \text{if } cd(C(\psi)) - h^{\uparrow}(\psi) \text{ has parity } \eta \\ cd(C(\psi)) - h^{\uparrow}(\psi) + 1 & \text{if } cd(C(\psi)) - h^{\uparrow}(\psi) \text{ has parity } \overline{\eta}. \end{cases}$$

Here we recall that we associate μ and ν with odd and even parity, respectively. If ψ is not of the form $\eta y.\varphi$, we leave $\Omega_g(\psi)$ undefined.

Definition 2.42 *Fix some tidy formula ξ. Let $\mathbb{C}_{\xi}$ be the closure graph $(Clos(\xi), \to_C)$ of ξ, expanded with the natural labelling L_C given by*

$$L_C(\varphi) = \begin{cases} \varphi & \text{if } \varphi \in \mathsf{At}(\mathsf{P}) \\ \heartsuit & \text{if } \varphi = \heartsuit\psi \\ \odot & \text{if } \varphi = \psi_0 \odot \psi_1 \\ \epsilon & \text{if } \varphi = \eta x.\psi \end{cases}$$

Let

$$\mathbb{G}_{\xi} := (\mathbb{C}_{\xi}, \Omega_g \restriction_{Clos(\xi)}, \xi).$$

$\mathbb{G}_{\xi}$ can be shown to be equivalent to ξ. Interested readers can refer to [15].

From Parity Formula to $\mu\mathsf{ML}$ Formula

In the previous section we saw constructions that, for a given $\mu\mathsf{ML}$ formula, produce equivalent parity formulas based on, respectively, the subformula dag and the closure graph of the original formula. It is also possible to give a construction that turns an arbitrary parity formula $\mathbb{G}$ in to an equivalent $\mu\mathsf{ML}$ formula $\xi_{\mathbb{G}} \in \mu\mathsf{ML}$. Basically, this construction takes a parity formula as a system of equations, and it solves these equations by a Gaussian elimination of variables. As a result, the transformation from parity formulas to $\mu\mathsf{ML}$ formulas can be seen as some sort of unraveling construction. We will not go into details about this construction here, and interested readers can refer to [13].

3 Criteria and Evaluation

In this chapter, we will introduce a list of criteria for judging whether a definition of bisimulation is a good one, as well as the four attempts at defining a good notion of bisimulation over parity formulas that appear in this thesis.

We first consider a list of properties we want a definition of bisimulation to have in general before delving into any details of the specific definitions. In Chapter 1, we mentioned that there are three main reasons for which we want to investigate the notion of bisimulation, namely to capture behavioral equivalence over transition systems, to establish identity of theory or extension, and to take the quotient of a structure. In our setting, these goals inspire the following criteria for a good definition of bisimulation $\mathcal{S}$.

1. $\mathcal{S}$ is a sufficient condition for modal equivalence.
2. There is a largest bisimulation between any two structures and it is an equivalence relation.
3. Given a relation B, it should be easily verifiable whether it is a bisimulation.
4. $\mathcal{S}$ should include as many pairs of modally equivalent nodes as possible.

Criterion 1 and 2 are intuitively some necessary conditions for achieving aforementioned goals. With criterion 3, we want to capture the nature of bisimulations as a tool to establish identity between infinite objects by a finite collection of facts, while leaving enough space for interpreting counts as 'easily verifiable'. If we think of Criterion 1 as expressing the requirement of the notion of bisimulation being 'sound', then Criterion 4 requires the notion to be as close to 'complete' as possible. However, completeness for any notion of bisimulation built on the matching of labels is impossible, as is illustrated by the following two parity formulas that are obviously modally equivalent while having two disjoint sets of labels.

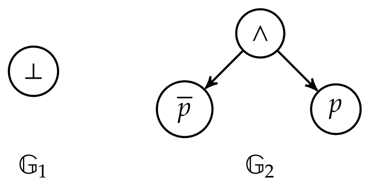


Figure 3.1: Two modally equivalent parity formula with disjoint labels

In the case where it is not immediately obvious whether Criterion 2 holds or not, we also propose the following two criteria which when combined are stronger than Criterion 2, a consequence of the fact that there are only finitely many bisimulations over any two parity formulas due to their finite nature.

5. $\mathcal{S}$ is closed under union.
6. $\mathcal{S}$ is closed under composition.

Given the fact that the semantics of parity formulas is built upon parity games, one requirement particular to the definition of bisimulation over *parity formulas* is that we want the treatment of priorities to have a 'relative flavor', since in parity games what matters is only the order between the priorities of different positions, and not their absolute value. Consider Example 3.1.

Example 3.1 Figure 3.2 shows two parity formulas that should be counted as the same, as there is a one-to-one correspondence between the nodes respecting the labeling, parity and relative order of the priorities.

Accordingly, we propose the following criterion.

7. In $\mathcal{S}$, the treatment of priorities has a 'relative flavor'.

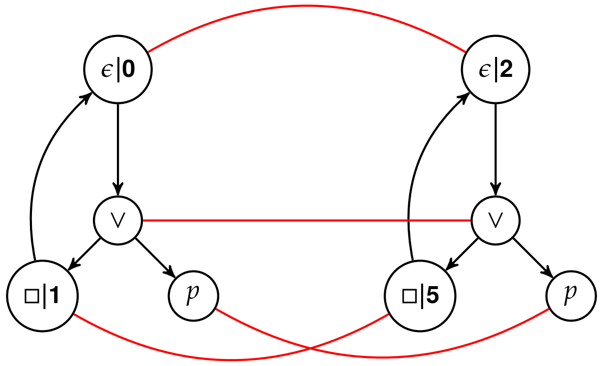


Figure 3.2: Two intuitively isomorphic parity formulas

In the rest of this chapter, we will introduce the aforementioned definitions of bisimulation and evaluate them against these criteria. Table 3.1 is an overview of the four definitions we discuss in this chapter. Table 3.2 summarizes the result of the evaluation.

Table 3.1: An overview of the definitions.

Definition 1	**Definition 2**	**Definition 3**	**Definition 4**
- modal equivalence	- Match on Labels - Match on Priority - Zig-zag	- Match on Labels - Match on Parities - Zig-zag - Order Preserving in Clusters	- Match on Labels - Match on Parities - Zig-zag from $A_{i+1} \backslash A_0$ to A_i - Zig-zag within A_0 - Order Preserving in A_0

Table 3.2: An overview of the evaluation of the definitions.

	Def. 1	Def. 2	Def. 3	Def. 4
Crit. 1	✓	✓	✓	✓
Crit. 2	✓	✓	✗	✓
Crit. 3	✗	✓	✓	✓
Crit. 4	See Figure 3.3			
Crit. 5	✓	✓	✗	✓
Crit. 6	✓	✓	✗	✓
Crit. 7	-	✗	✓	✓

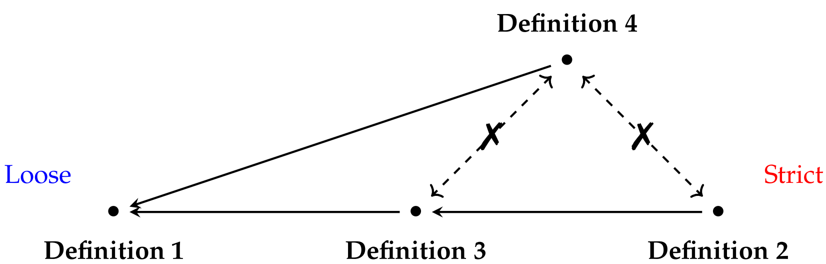


Figure 3.3: 'Completeness' of the definitions

Definition 1

The easiest way to ensure criterion 4 is satisfied is perhaps to take modal equivalence as the definition for bisimulation.

Definition 3.1 (Definition 1) *Let* $\mathbb{G} = (V, E, L, \Omega)$ *and* $\mathbb{G}'(V', E', L', \Omega')$ *be two parity formulas.* $B \subseteq V \times V'$ *is a bisimulation over* $\mathbb{G}$ *and* $\mathbb{G}'$ *if and only* $B = \{(v, v') \mid v \in V, v' \in V', \mathbb{G}\langle v\rangle \equiv \mathbb{G}'\langle v'\rangle\}$.

For this definition, criterion 2 and 5 are satisfied because there is exactly one bisimulation over any two parity formulas and criterion 6 is satisfied due to transitivity of identity.

However, this approach to defining bisimulations is almost tautological and does not provide any information about the structure of the parity formulas in question. It goes against the idea of bisimulation, which is to show infinite properties with a finite collection of facts, and to establish similarity in global behaviors from similarity in local behaviors. The result of this flaw is that it is unclear how to verify whether a given relation is indeed a bisimulation, i.e., criterion 3.

Definition 2

Another inspiration for defining bisimulations over parity formulas is how bisimulations are defined on Kripke models for the basic modal logic.

Definition 3.2 *Let* $\mathbb{S}_1 = \langle S_1, V_1, R_1\rangle$ *and* $\mathbb{S}_2 = \langle S_2, V_2, R_2\rangle$ *be two Kripke models.* Z *is a bisimulation over* $\mathbb{S}_1$ *and* $\mathbb{S}_2$ *if for any* $s_1 \in S_1$, $s_2 \in S_2$, $s_1 Z s_2$, *we have*

1) s_1 *and* s_2 *satisfy the same proposition letters;*
3) *for any* $s_1' \in S_1$, *if* $R_1 s_1 s_1'$ *then there exists* $s_2' \in S_2$ *such that* $s_1' Z s_2'$ *and* $R_2 s_2 s_2'$; *and*
4) *for any* $s_2' \in S_2$, *if* $R_2 s_2 s_2'$ *then there exists* $s_1' \in S_1$ *such that* $s_1' Z s_2'$ *and* $R_1 s_1 s_1'$.

If we see the valuation in Kripke models and the label and priority functions both as some sort of 'coloring', then we can translate the definition above into a definition of bisimulation over parity:

Definition 3.3 (Definition 2) *Let* $\mathbb{G} = (V, E, L, \Omega)$ *and* $\mathbb{G}' = (V', E', L', \Omega')$ *be two parity formulas.* $B \subseteq V \times V'$ *is a bisimulation between* $\mathbb{G}$ *and* $\mathbb{G}'$ *if for all* $(u, u') \in B$ *we have*

1) $L(u) = L'(u')$;
2) *if* $u \in \mathsf{Dom}(\Omega)$, *then* $u' \in \mathsf{Dom}(\Omega')$ *and* $\Omega(u) = \Omega'(u')$;
3) *if* $u' \in \mathsf{Dom}(\Omega')$, *then* $u \in \mathsf{Dom}(\Omega)$ *and* $\Omega(u) = \Omega'(u')$;
4) *for any* $v \in V$, *if* Euv *then there exists* $v' \in V'$ *such that* $(v, v') \in B$ *and* $E'u'v'$; *and*
5) *for any* $v' \in V'$, *if* $E'u'v'$ *then there exists* $v \in V$ *such that* $(v, v') \in B$ *and* Euv.

This definition satisfies criteria $1, 2, 3, 5$ and 6. Following is the proof for criteria $1, 5$ and 6. Criterion 2 then is a result of condition 5 and 6. Criterion 3 is true because we can simply enumerate the element in a given relation R and check for each of them if they satisfy the conditions in Definition 2.

Proposition 3.1 *Let* B *be a bisimulation between parity formulas* $\mathbb{G}$ *and* $\mathbb{G}'$ *in the sense of Definition* 2. *Then for any* $(v, v') \in B$ *it holds that*

$$\mathbb{G}\langle v\rangle \equiv \mathbb{G}'\langle v'\rangle.$$

Proof. This is a corollary of Proposition 3.5 and 3.4. □

Proposition 3.2 *Let B_1 and B_2 be two bisimulations between parity formulas $\mathbb{G}$ and $\mathbb{G}'$ in the sense of Definition 2. Then for any $B := B_1 \cup B_2$ is also a bisimulation between parity formulas $\mathbb{G}$ and $\mathbb{G}'$.*

Proof. Let $(u, u') \in B = B_1 \cup B_2$. Without loss of generality, assume $(u, u') \in B_1$.

1. By assumption, $L(u) = L'(u')$.
2. By assumption, if $u \in \mathsf{Dom}(\Omega)$, then $u' \in \mathsf{Dom}(\Omega')$ and $\Omega(u) = \Omega'(u')$.
3. Symmetric to the previous point
4. Let $v \in V$ be such that Euv. Then, by assumption, there exists $v' \in V'$ such that $(v, v') \in B_1 \subseteq B$ and $E'u'v'$.
5. Symmetric to the previous point. □

Proposition 3.3 *Let $\mathbb{G}$, $\mathbb{G}'$ and $\mathbb{G}''$ be three parity formulas. Let B_1 be a bisimulation between $\mathbb{G}$ and $\mathbb{G}'$ and let B_2 be a bisimulation between $\mathbb{G}'$ and $\mathbb{G}''$, both in the sense of Definition 2. Then, $B := B_1; B_2$ is a bisimulation between $\mathbb{G}$ and $\mathbb{G}''$.*

Proof. Let $(u, u'') \in B = B_1; B_2$. Then, there exists $u' \in V'$ such that $(u, u') \in B_1$. and $(u', u'') \in B_2$.

1. By assumption, $L(u) = L'(u')$ and $L'(u') = L''(u'')$. Thus, $L(u) = L''(u'')$.
2. Suppose $u \in \mathsf{Dom}(\Omega)$. Then, by assumption, $u' \in \mathsf{Dom}(\Omega')$ and $\Omega(u) = \Omega'(u')$. Again, by assumption, $u'' \in \mathsf{Dom}(\Omega'')$ and $\Omega'(u') = \Omega''(u'')$. This means that $\Omega(u) = \Omega''(u'')$.
3. Symmetric to the previous point.
4. Let $v \in V$ be such that Euv. Then, by assumption, there exists $v' \in V'$ such that $(v, v') \in B_1$ and $E'u'v'$. Again, by assumption, there exists $v'' \in V''$ such that $(v', v'') \in B_2$ and $E''u''v''$. This means that there exists $v'' \in V''$ such that $(v, v'') \in B_1; B_2 = B$ and $E''u''v''$.
5. Symmetric to the previous point. □

The main problem with Definition 2 is that it loses the 'relative flavor' of the priority map, that is, what matters is only the order between the priorities of different nodes, and not their absolute value. This loss in the 'relative flavor' is reflected in the fact that Definition 2 does not include enough pairs of nodes that are modally equivalent, i.e., Criterion 4. To see this, note that the relation in Example 3.1 is not a bisimulation in the sense of Definition 2 since the priorities do not match in terms of their exact value.

Definition 1 and 2 together show the two extremes in the trade-off in defining the notion of bisimulation between the requirement to include more pairs that are modally equivalent and the requirement to make the decision problem for bisimilarity as easy as possible in terms of computation. A better definition should find a subtle balance between these two extremes.

Definition 3

Since priorities only matter in infinite plays and any infinite play is eventually restricted to one cluster, one way to strike a balance between these two extremes, as is hinted in [13], is to relax the requirement on the priorities of the pairs in a bisimulation to only match on their parity and add another requirement that a bisimulation preserve the relative ordering of the priorities within a cluster.

Definition 3.4 (Definition 3) *Let $\mathbb{G} = (V, E, L, \Omega)$ and $\mathbb{G}' = (V', E', L', \Omega')$ be two parity formulas. $B \subseteq V \times V'$ is a bisimulation over $\mathbb{G}$ and $\mathbb{G}'$ if for all $(u, u') \in B$ we have*

1) $L(u) = L'(u')$;
2)' if $u \in \mathsf{Dom}(\Omega)$, then $u' \in \mathsf{Dom}(\Omega')$ and $\Omega(u) \equiv_2 \Omega'(u')$;

3)′ *if* $u \in \mathsf{Dom}(\Omega)$*, then* $u' \in \mathsf{Dom}(\Omega')$ *and* $\Omega(u) \equiv_2 \Omega'(u')$*;*

4) *for any* $v \in V$*, if* Euv *then there exists* $v' \in V'$ *such that* $(v, v') \in B$ *and* $E'u'v'$*;*

5) *for any* $v' \in V'$*, if* $E'u'v'$ *then there exists* $v \in V$ *such that* $(v, v') \in B$ *and* Euv*; and*

6) *for any* $u, v \in V, u', v' \in V'$ *such that* $(u, u'), (v, v') \in B$*,* $u \equiv_E v$*,* $u' \equiv_{E'} v'$ *and* $\Omega(u) \not\equiv_2 \Omega(v)$*, it holds that* $\Omega(u) < \Omega(v)$ *if and only if* $\Omega'(u') < \Omega'(v')$*.*

The following proposition on the relation between Definition 2 and 3 is easy to see, so we present it without proof.

Proposition 3.4 *Let* $\mathbb{G} = (V, E, L, \Omega)$ *and* $\mathbb{G}' = (V', E', L', \Omega')$ *be two parity formulas. Let* $B \subseteq V \times V'$ *be a bisimulation over* $\mathbb{G}$ *and* $\mathbb{G}'$ *in the sense of Definition 2. Then* B *is also a bisimulation in the sense of Definition 3.*

The advantages of Definition 3 are that it satisfies Criterion 1, as is shown by Proposition **??**, that it has the 'relative flavor', as is shown by condition 5), and that it improves Definition 2 in terms of Criterion 4, since the red relation in Example 3.1 is now indeed a bisimulation. The disadvantage of Definition 3 is that it is no longer closed under union and bisimulation, nor does it always allow the largest bisimulation, as is shown by Proposition 3.6.

Proposition 3.5 *Let* B *be a bisimulation between parity formulas* $\mathbb{G}$ *and* $\mathbb{G}'$ *in the sense of Definition 3. Then for any* $(u, u') \in B$ *it holds that*

$$\mathbb{G}\langle u \rangle \equiv \mathbb{G}'\langle u' \rangle.$$

Proof. We need to show that, given a transition system $\mathbb{S}$ with initial state s_I, $\exists$ has a winning strategy in $\mathcal{E}(\mathbb{G}\langle u \rangle, \mathbb{S})$ if and only if $\exists$ has a winning strategy in $\mathcal{E}(\mathbb{G}'\langle u' \rangle, \mathbb{S})$. By symmetry and positional determinacy of parity games, it suffices to show that given a positional winning strategy for $\exists$ on $\mathcal{E}(\mathbb{G}\langle u \rangle, \mathbb{S})$, $g : V \times S \to V \times S$, $\exists$ has a winning strategy g' on $\mathcal{E}(\mathbb{G}'\langle u' \rangle, \mathbb{S})$. This strategy is constructed by 'shadow playing'. To be specific, during the game, $\exists$ keeps track of two matches: one is the match she plays with $\forall$ in $\mathcal{E}(\mathbb{G}'\langle u' \rangle, \mathbb{S})$ and the other is what we call a shadow match in $\mathcal{E}(\mathbb{G}\langle u \rangle, \mathbb{S})$ which she uses together with g to decide what to do in $\mathcal{E}(\mathbb{G}'\langle u' \rangle, \mathbb{S})$ when it is her turn. At each turn, $\exists$ updates the positions simultaneously for both games. We write (u_i, s_i) and (u'_i, s'_i) respectively to denote the updated positions after the i-th turn in $\mathcal{E}(\mathbb{G}\langle u \rangle, \mathbb{S})$ and $\mathcal{E}(\mathbb{G}'\langle u' \rangle, \mathbb{S})$. $\exists$ update the positions in a way such that for all applicable i,

1. $s_i = s'_i$ and $(u_i, u'_i) \in B$, and
2. $(u_0, s_0)(u_1, s_1)...(u_i, s_i)$ is a full match if and only if $(u'_0, s'_0)(u'_1, s'_1)...(u'_i, s'_i)$ is a full match.

$\exists$ starts from $(u_0, s_0) := (u, s_I)$ in $\mathcal{E}(\mathbb{G}\langle u \rangle, \mathbb{S})$ and $(u'_0, s'_0) := (u', s_I)$ in $\mathcal{E}(\mathbb{G}'\langle u' \rangle, \mathbb{S})$. Note that $s_0 = s'_0$ and $(u_0, u'_0) \in B$ by our assumption that $(u, u') \in B$. Suppose i turns have passed in $\mathcal{E}(\mathbb{G}'\langle u' \rangle, \mathbb{S})$. Assume $s_i = s'_i$ and $(u_i, u'_i) \in B$. Then, $L(u_i) = L'(u'_i)$ by the definition of B. We have the following cases.

1. $L(u_i) = L'(u'_i) \in \{\Diamond, \vee\}$ and (u_i, s_i) has at least one successor in $\mathcal{E}(\mathbb{G}\langle u \rangle, \mathbb{S})$. In this case, $\exists$ can make a move in $\mathcal{E}(\mathbb{G}'\langle u' \rangle, \mathbb{S})$. Let $(u_{i+1}, s_{i+1}) := g(v, s)$. Since (u_i, s_i) has at least one successor in $\mathcal{E}(\mathbb{G}\langle u \rangle, \mathbb{S})$ and g is a well-defined strategy, it follows that (u_{i+1}, s_{i+1}) is a successor of (u_i, s_i), and thus is a legitimate move from (u_i, s_i). Let $s'_{i+1} := s_{i+1}$ and u'_{i+1} be an element from $\{z \in V' \mid (u_{i+1}, z) \in B, E'u'_i z\}$ that we pick out in a certain way. Note that $\{z \in V' \mid (u_{i+1}, z) \in B, E'u'_i z\}$ is non-empty because of condition 5) in Definition 2. Note that we have $s_{i+1} = s'_{i+1}$ and $(u_{i+1}, u'_{i+1}) \in B$ and neither of $(u_0, s_0)(u_1, s_1)...(u_i, s_i)$ and $(u'_0, s'_0)(u'_1, s'_1)...(u'_i, s'_i)$ are full matches in this case.

2. $L(u_i) = L'(u'_i) \in \{\Box, \wedge\}$ and (u'_i, s'_i) has at least one successor in $\mathcal{E}(\mathbb{G}\langle u \rangle, \mathbb{S})$. In this case, $\forall$ can make a move in $\mathcal{E}(\mathbb{G}'\langle u' \rangle, \mathbb{S})$. Denote the position $\forall$ chooses as (u'_{i+1}, s'_{i+1}). Let $s_{i+1} := s'_{i+1}$ and u_{i+1} be an element from $\{z \in V \mid (z, u'_{i+1}) \in B, Eu_i z\}$ that we choose in a certain way. Note that $\{z \in V \mid (z, u'_{i+1}) \in B, Eu_i z\}$ is non-empty because of condition 4) in Definition 2. Note that we have $s_{i+1} = s'_{i+1}$ and $(u_{i+1}, u'_{i+1}) \in B$ and neither of $(u_0, s_0)(u_1, s_1)...(u_i, s_i)$ and $(u'_0, s'_0)(u'_1, s'_1)...(u'_i, s'_i)$ are full matches in this case.

3. $L(u_i) = L'(u_i') = \epsilon$. Let (u_{i+1}, s_{i+1}) and (u'_{i+1}, s'_{i+1}) be the unique successors of (u_i, s_i) and (u_i', s_i') respectively. This means that Eu_iu_{i+1}, $E'u_i'u'_{i+1}$, $s_i = s_{i+1}$ and $s_i' = s'_{i+1}$. Since $s_i = s_i'$, it follows that $s_{i+1} = s'_{i+1}$. Since $(u_i, u_i') \in B$ and Eu_iu_{i+1} by assumption, it follows that there exists $v' \in V'$ such that $E'u_i'v'$ and $(u_{i+1}, v') \in B$. Since u'_{i+1} is the only successor to u_i', it follows that $v' = u'_{i+1}$. Thus, $(u_{i+1}, u'_{i+1}) \in B$. That is, we have $s_{i+1} = s'_{i+1}$ and $(u_{i+1}, u'_{i+1}) \in B$ and neither of $(u_0, s_0)(u_1, s_1)...(u_i, s_i)$ and $(u_0', s_0')(u_1', s_1')...(u_i', s_i')$ are full matches in this case.
4. If none of the cases above are true, then it must be one of the following cases.
   - $L(v) = L'(v') \in \mathsf{At(P)}$.
   - $L(u_i) = L'(u_i') \in \{\Diamond, \vee\}$ and (u_i, s_i) has no successor in $\mathscr{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$.
   - $L(u_i) = L'(u_i') \in \{\Box, \wedge\}$ and (u_i', s_i') has no successor in $\mathscr{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$.

   We argue that in all of these cases, neither (u_i, s_i) has successors in $\mathscr{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$ nor (u_i', s_i') has successors in $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$, i.e., both $(u_0, s_0)(u_1, s_1)...(u_i, s_i)$ and $(u_0', s_0')(u_1', s_1')...(u_i', s_i')$ are full matches. This is obvious in the first case. The argument for the third case is similar to that of the second case, so we only show the argument for the latter.
   Now we show that if $L(u_i) = L'(u_i') \in \{\Diamond, \vee\}$ and (u_i, s_i) has no successor in $\mathscr{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$, then (u_i', s_i') has no successor in $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. We make the following distinction.
   - $L(u_i) = L'(u_i') = \Diamond$. Suppose (v', s') is a successor to (u_i', s_i') in $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. This means that $(s_i', s') = (s_i, s') \in R_{\mathbb{S}}$ and $Eu_i'v'$. Since $(u_i, u_i') \in B$, it follows that there exists $v \in V$ such that Eu_iv and $(v, v') \in B$. This means that (v, s') is a successor to (u_i, s_i) in $\mathscr{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$, which contradicts our assumption. Thus, (u_i', s_i') has no successor in $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$.
   - $L(u_i) = L'(u_i') = \vee$. Suppose (v', s') is a successor to (u_i', s_i') in $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. This means that $s_i' = s_i = s'$ and $Eu_i'v'$. Since $(u_i, u_i') \in B$, it follows that there exists $v \in V$ such that Eu_iv and $(v, v') \in B$. This means that (v, s') is a successor to (u_i, s_i) in $\mathscr{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$, which contradicts our assumption. Thus, (u_i', s_i') has no successor in $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$.

   Since (u_i, s_i) has no successor in $\mathscr{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$ and (u_i', s_i') has no successor in $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$, no move is possible in either game and the two games end simultaneously.

Now we show that g' is indeed a winning strategy. Suppose $\exists$ follows this strategy. As we have shown above, the two matches always end simultaneously. There are two possibilities.

1. The matches end after finitely many, say n, steps. Let $l = (u, s_I)(u_1, s_1)...(u_n, s_n)$ be the match in $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$ and $l' = (u', s_I')(u_1', s_1')...(u_n', s_n')$ be the match in $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. By a), $s_n = s_n'$ and $(u_n, u_n') \in A$. Thus (u_n, s_n) and (u_n', s_n') belong to the same player. Since both matches are full, it follows that the two matches are won by the same player.
2. The matches are infinite. Let $l = (u, s_I)(u_1, s_1)...$ be the match in $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$ and $l' = (u', s_I')(u_1', s_1')...$ be the match in $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. Since any infinite match is eventually restricted to one cluster on the first coordinate, it follows that there exists $n, n' \in \omega$ such that for all $i \geq n$ we have $u_i \in C(u_n)$ and for all $i \geq n'$ we have $u_i' \in C(u'_{n'})$. Without loss of generality, assume $n \geq n'$. Then for all $i \geq n$ we have $u_i \in C(u_n)$ and $u_i' \in C(u_n')$. Note that we have shown that $(u_i, u_i') \in B$ for all i applicable. By condition 5) of Definition 3, we have
   i) $\Omega(u_i, s_i) \equiv_2 \Omega'(u_i', s_i')$, and
   ii) if $\Omega(u_i, s_i) \not\equiv_2 \Omega(u_j, s_j)$, then $\Omega(u_i, s_i) < \Omega(u_j, s_j)$ if and only if $\Omega'(u_i', s_i') < \Omega'(u_j', s_j')$.

   By Proposition 2.5, l and l' have the same winner. Since l is a match where $\exists$ follows strategy g, it follows that $\exists$ is the winner in l. Thus, $\exists$ is also the winner in l'.

Thus, g' is a winning strategy for $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. □

Proposition 3.6 *Definition 3 is not closed under union or composition. Futhurmore, it does not always allow the largest bisimulation between two parity formulas. These two claims still hold true even if one relaxes condition 6) to*

6)′ For any infinite play $f : \omega \to V$ and $g : \omega \to V'$ on $\mathbb{G}$ and $\mathbb{G}'$ such that $(f(i), g(i)) \in B$ for any $i \in \omega$, we

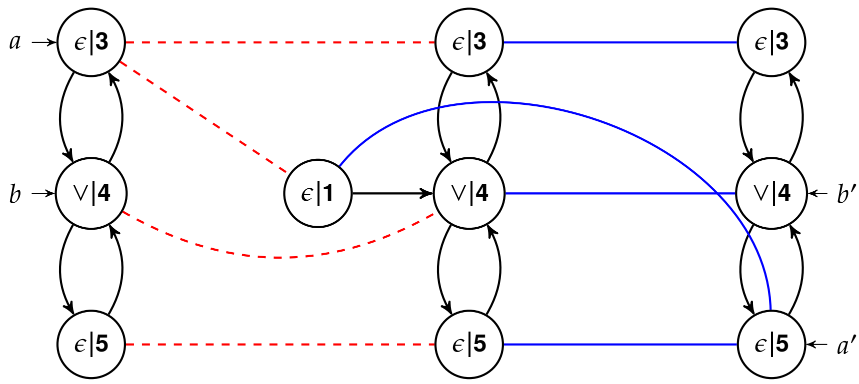


Figure 3.4: Counterexample to closure under composition

have that $max(inf(\Omega(f))) \equiv_2 max(inf(\Omega(g)))$.

Proof. Figure 3.4 presents a counterexample to closure under composition. In Figure 3.4, Bisimulation B is shown as the red (dashed) lines while bisimulation B' is shown as the blue (solid) lines. Note that $B_1; B_2$ is not a bisimulation since the composed line between a and a' and the composed line between b and b' together contradicts condition 6).

Figure 3.5 presents a counterexample to closure under union. In Figure 3.5, bisimulation B_1 is the union of the red (dashed) lines and gray (solid) lines and bisimulation B_2 is the union of the blue (dotted) lines and gray lines. Note that $B_1 \cup B_2$ is not a bisimulation because the red line between a and a' and the blue line between b and b' together contradicts condition 6).

One can replace the condition 6) with the weaker condition 6)′. However, this does not make bisimulations closed under unions or compositions and the two counter-examples above still serve their purpose. To see that, note the fact that since 5)′ is weaker than 5), it follows that B, B' and B_1, B_2 are still bisimulations under the weaker definition. In both examples, the play $f := (ab)^\omega$ and $g := (a'b')^\omega$ together satisfy the condition that $(f(i), g(i)) \in B; B'$ and $(f(i), g(i)) \in B_1 \cup B_2$ for all $i \in \omega$ respectively, but they have different winners.

The counterexample shown by Figure 3.5 also rules out in general the existence of the largest bisimulation between two parity formulas, since any relation that contains both the red line between a and a' and the blue line between b and b' violates condition 6) for being a bisimulation. This argument is still valid if we replace condition 6) with condition 6)′. □

Despite the disadvantage that Definition 3 is not satisfying Criterion 2, 5 and 6, it already shows some promising results when restricted to special cases. We dedicate Chapter 4 to discuss these positive results.

Definition 4

Definition 3 takes into account the 'relative flavor' of the priorities in parity formulas, and in doing so, includes more pairs of nodes that are modally equivalent in the definition. However, an obvious shortcoming of Definition 3 is that it is not closed under union and composition and as a result, does not allow the largest bisimulation over two parity formulas in general. In the next definition, we will maintain the relative treatment of priorities while trying to make the definition closed under union and composition.

To achieve this goal, we first take another look at the counterexamples presented in the proof of Proposition 3.6. One observation is that both examples run into problems because, following the bisimulation, it is possible to stay in the same cluster in one graph while going down to another cluster in the other. Since the definition does not have any restriction on the priorities of nodes that are in different clusters, such bisimulations, when united or composed, can have undesirable loops. However, we do not want to completely get rid of such links from our bisimulation, since they still express a sense of equivalence in terms of the game

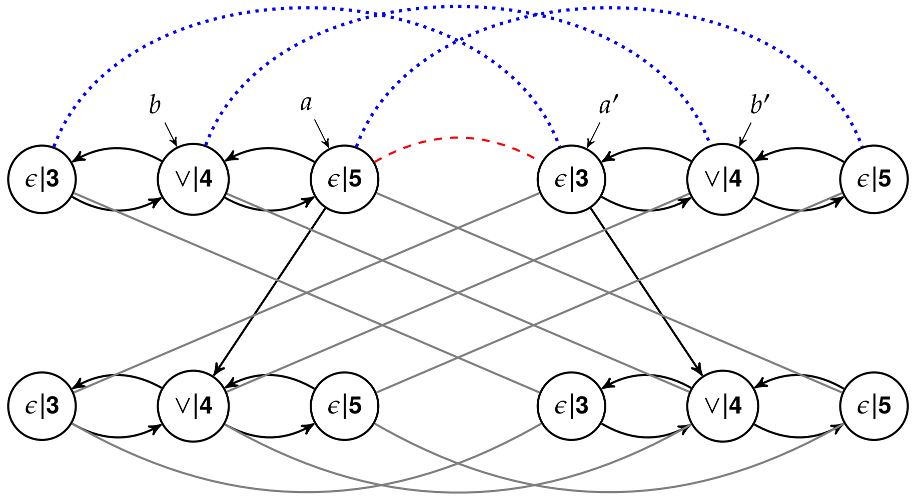


Figure 3.5: Counterexample to closure under union

semantics. It is safe to assume that such a dilemma results from mixing two different ideas of equivalence: equivalence as initial positions, and equivalence as positions that can be visited infinitely many times. To reconcile these two notions, we define a notion of bisimulation in a two-step manner. First, we define the non-trivial base case of induction, A_0, which is a collection of partial bisimulations restricted to pairs of clusters. In these partial bisimulations, relative order of priorities is respected. Then, on top of that, we define inductively the entire bisimulation ($\bigcup_{i\in\omega} A_i$) over the two parity formulas, where we no longer care about priorities. The precise definition is shown below.

Definition 3.5 (Definition 4) *Given two parity formulas $\mathbb{G} = (V, E, L, \Omega, v_I)$ and $\mathbb{G}' = (V', E', L', \Omega', v_I')$, a family of binary relations $(A_i)_{i\in\omega}$, with $A_i \subseteq V \times V'$ for all $i \in \omega$, is a bisimulation between $\mathbb{G}$ and $\mathbb{G}'$ if it satisfies the conditions 1 - 6 below. Here we use A to denote $\bigcup_{i\in\omega} A_i$.*

1. *$A_i \subseteq A_j$ for all $i, j \in \omega$ such that $i < j$;*
2. *$L(v) = L'(v')$ for all $(v, v') \in A$;*
3. *for all $(v, v') \in A_0$, neither $C(v)$ nor $C(v')$ are degenerate, $v \in \mathsf{Dom}(\Omega) \Leftrightarrow v' \in \mathsf{Dom}(\Omega')$, and if $v \in \mathsf{Dom}(\Omega)$, then $\Omega(v) \equiv_2 \Omega'(v')$;*
4. *for any $(u, u'), (v, v') \in A_0$ such that $v \in C(u)$, $v' \in C(u')$ and $\Omega(u) \not\equiv_2 \Omega(v)$, we have that $\Omega(u) < \Omega(v)$ if and only if $\Omega'(u') < \Omega'(v')$;*
5. *for all $i \in \omega$, $(v, v') \in A_{i+1} \backslash A_0$;*
   a) *for all $u \in V$ such that Evu, there exists $u' \in V'$ such that $E'v'u'$ and $(u, u') \in A_i$;*
   b) *for all $u' \in V'$ such that $E'v'u'$, there exists $u \in V$ such that Evu and $(u, u') \in A_i$; and*
6. *for all $(v, v') \in A_0$,*
   a) *for all $u \in C(v)$ such that Evu, there exists $u' \in C(v')$ such that $E'v'u'$ and $(u, u') \in A_0$;*
   b) *for all $u' \in C(v')$ such that $E'v'u'$, there exists $u \in C(v)$ such that Evu and $(u, u') \in A_0$;*
   c) *for all $u \in V$ such that Evu, there exists $u' \in V'$ such that $E'v'u'$ and $(u, u') \in A$;*
   d) *for all $u' \in V'$ such that $E'v'u'$, there exists $u \in V$ such that Evu and $(u, u') \in A$.*

Definition 4 has the 'relative flavor', i.e., Criterion 7, as is shown by condition 6). Its status in terms of Criterion 4 is shown in Figure 3.3. The following propositions explains its position in Figure 3.3.

Proposition 3.7 *There exist parity formulas $\mathbb{G} = (V, E, L, \Omega)$ and $\mathbb{G}' = (V', E', L', \Omega')$, a bisimulation $(A_i)_{i\in\omega}$*

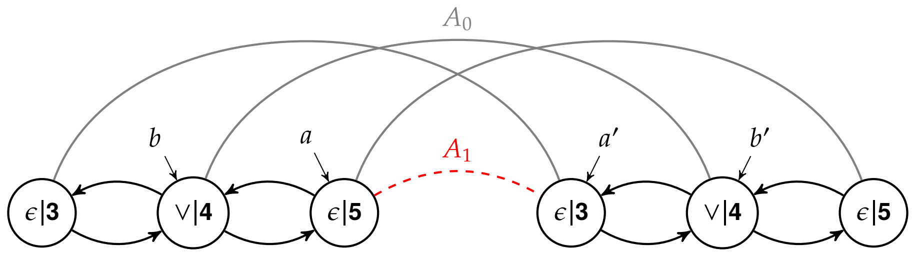


Figure 3.6: Example for Def. 4 ⇏ Def. 3

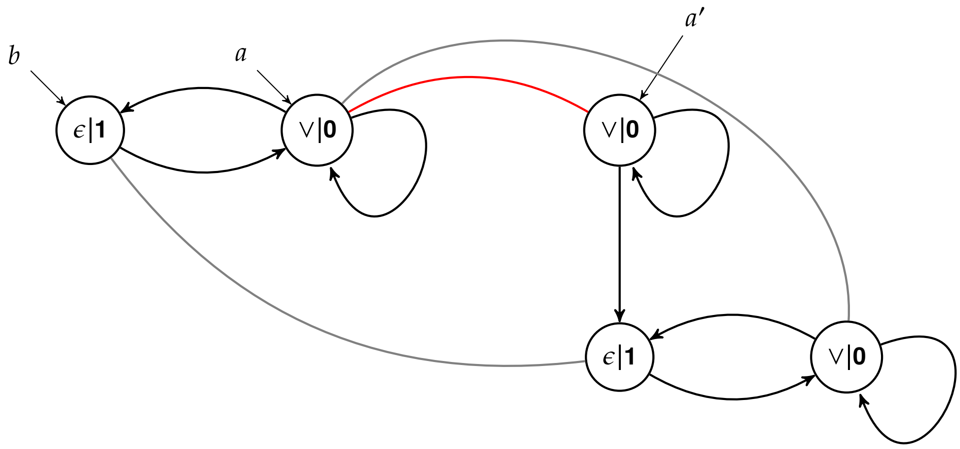


Figure 3.7: Example for Def. 2 ⇏ Def. 4

between them in the sense of Definition 4, and $(v, v') \in A$, such that there is no bisimulation B between $\mathbb{G}$ and $\mathbb{G}'$ in the sense of Definition 3 with $(v, v') \in B$.

Proof. Consider the two parity formulas in Figure 3.6. Let $(A_i)_{i\in\omega}$ be such that A_0 is the collection of the gray (solid) lines and A_{i+1} is A_0 together with the red (dashed) line. It is easy to check that $(A_i)_{i\in\omega}$ is indeed a bisimulation in the sense of Definition 4. However, note that if we include (a, a') in any bisimulation B in the sense of Definition 3, then to comply with condition 4)/5), we would need to include (b, b') in B as well. But having (a, a') and (b, b') in B at the same time contradicts condition 6). This means that no bisimulation in the sense of Definition 3. contains (a, a'). □

Proposition 3.8 *There exist parity formulas $\mathbb{G} = (V, E, L, \Omega)$ and $\mathbb{G}' = (V', E', L', \Omega')$ and a bisimulation B between them in the sense of Definition 2, and $(v, v') \in B$, such that there is no bisimulation $(A_i)_{i\in\omega}$ between $\mathbb{G}$ and $\mathbb{G}'$ in the sense of Definition 4 with $(v, v') \in A$.*

Proof. Consider the two parity formulas in Figure 3.7. Let B be the relation represented by both the gray lines and the red line. It is easy to check that B is indeed a bisimulation in the sense of Definition 2. Suppose there exists a bisimulation $(A_i)_{i\in\omega}$ in the sense of Definition 4 such that $(a, a') \in A$. Then either $(a, a') \in A_0$ or $(a, a') \in A_{i+1}$ for some $i \in \omega$.

Suppose $(a, a') \in A_0$. By condition 5a), when we go from a to b, we need to be able to go from a' to a successor b' of a' in the cluster of a' such that $(b, b') \in A_0$. Since $C(a') = \{a'\}$, it has to be the case that $b' = a'$. This means that the pair (b, a') has to be in A_0. But this contradicts the requirement that any pair in A has to match in both labels and parity. So the red line cannot be in A_0.

Suppose $(a, a') \in A_{i+1}$ for some $i \in \omega$. By condition 4b), when we go along the self-loop of a', we need to be able to go from a to a successor c of a such that $(c, a') \in A_i$. Since any pair in A has to match in both labels and parity, it follows that $c = a$. This means that the red line is in A_i. Since by assumption, the red line is not in A_0, it follows that the red line is in $A_i \backslash A_0$. If $i = 0$, then we have a contradiction. If $i > 0$, then repeat this argument for i more times and we will run into contradiction. So the red line cannot be in $A_{i+1} \backslash A_0$ for any $i \in \omega$.

To sum up, there is no bisimulation $(A_i)_{i\in\omega}$ between $\mathbb{G}$ and $\mathbb{G}'$ in the sense of Definition 4 with $(a, a') \in A$. □

Furthermore, Definition 4 also satisfies Criterion 1, 2, 3, 5 and 6. We dedicate Chapter 5 to proving and discussing these properties.

4 Definition 3: Special cases

In this chapter, we consider some positive results when considering Definition 3 on two special cases. One case concerns bisimulations that can be seen as functions, which we can also see as the morphisms over parity formulas. We show that when restricted to functional relations, Definition 3 has nice category-theoretical properties to be considered the 'arrows' in the category of parity formulas. We also show that the well-known expansion map by Kozen [1] is indeed a morphism by our definition. The other case concerns parity formulas with only one cluster. We show that in this restricted situation, Definition 3 is closed under union and composition, and thus, the largest bisimulation over any two such parity formulas always exists. This provides a way to take the quotient of a parity formula with only one cluster.

Case 1: Morphism

Let $\mathbb{G} = (V, E, L, \Omega, v_I)$ and $\mathbb{G}' = (V', E', L', \Omega', v_I')$ be two parity formulas and $B \subseteq V \times V'$ be a binary relation. We call B *functional* if, for each $v \in V$, there exists a unique v' such that $(v, v') \in B$. When restricted to functional relations, Definition 3 translates to the following.

Definition 4.1 *Let $\mathbb{G} = (V, E, L, \Omega, v_I)$ and $\mathbb{G}' = (V', E', L', \Omega', v_I')$ be two parity formulas. A morphism from $\mathbb{G}$ to $\mathbb{G}'$ is a map $f : V \longrightarrow V'$ satisfying the following conditions, for all $u, v \in V$:*

1) $L(u) = L'(f(u))$;
2) *if* Euv *then* $E'f(u)f(v)$;
3) *if* $E'f(u)v'$ *then* Euv *for some* $v \in V$ *such that* $f(v) = v'$;
4) $u \in \mathsf{Dom}(\Omega)$ *if and only if* $f(u) \in \mathsf{Dom}(\Omega')$;
5) *if* $u \in \mathsf{Dom}(\Omega)$, *then*
6) *if* $u \equiv_E v$ *and* $\Omega(u) \not\equiv_2 \Omega(v)$, *then* $\Omega(u) < \Omega(v)$ *if and only if* $\Omega'(f(u)) < \Omega'(f(v))$.

Now we show that Definition 4.1 has nice category-theoretical properties to be considered the 'arrows' in the category of parity formulas.

Proposition 4.1 *The following holds:*

1. *The identity function i is a morphism of parity games.*
2. *For any $u, v \in V$ such that $u \neq v$, if $u \equiv_E v$, then $f(u) \equiv_{E'} f(v)$.*
3. *Function composition works as composition of morphism of parity formulas.*
4. *Identity functions are identity morphisms and composition of morphisms is associative.*

Let us prove each statement separately.

1. The identity function i is a morphism of parity formulas.

   Proof. For any $u, v \in V$

   1) $L'(i(u)) = L(i(u)) = L(u)$
   2) $Euv \Rightarrow E(i(u))(i(v)) \Rightarrow E'(i(u))(i(v))$
   3) $E'(i(u))v \Rightarrow Euv$ while $i(v) = v$
   4) $u \in \dot{}$ if and only if $i(u) \in \dot{}$.
   5) If $u \in \dot{}$, then $\Omega(u) \equiv_2 \Omega(i(u)) \equiv_2 \Omega'(i(u))$
   6) If $u \equiv_E v$ and $\Omega(u) \not\equiv_2 \Omega(v)$, then it is obvious that $\Omega(u) < \Omega(v)$ if and only if $\Omega(i(u)) < \Omega(i(v))$. ☐

2. For any $u, v \in V$ such that $u \neq v$, if $u \equiv_E v$, then $f(u) \equiv_{E'} f(v)$.

Proof. Assume $u \equiv_E v$. Then there exists $u = w_0, w_1, \ldots, w_n = v, w_{n+1}, \ldots, w_{n+m} = u \in V$ such that $w_i E w_{i+1}$ for all $0 \leq i < n + m$. By condition 2) of Definition 3 this means that $f(w_i)E'f(w_{i+1})$ for all $0 \leq i < n + m$, which in turn means that $f(u) \equiv_{E'} f(v)$. □

3. Function composition works as composition of morphism of parity formulas.

   Proof. Let $\mathbb{G} = (V, E, L, \Omega, v_I), \mathbb{G}' = (V', E', L', \Omega', v'_I), \mathbb{G}'' = (V'', E'', L'', \Omega'', v''_I)$ be three parity formulas. Let $f : G \to G'$ and $g : G' \to G''$ be two morphisms of parity games. For any $u, v \in V$:

   1) $L''(gf(u)) = L'(f(u)) = L(u)$
   2) $Euv \Rightarrow E'f(u)f(v) \Rightarrow E''gf(u)gf(v)$
   3) Suppose $E''gf(u)v''$. Then $E'f(u)v'$ for some v' such that $g(v') = v''$. This in turn means that Euv for some v with $f(v) = v'$. Note that $gf(v) = v''$.
   4) Since both f and g are morphisms, we have $u \in \mathsf{Dom}(\Omega)$ if and only if $f(u) \in \mathsf{Dom}(\Omega')$ if and only if $gf(u) \in \mathsf{Dom}(\Omega'')$.
   5) Suppose $u \in \mathsf{Dom}(\Omega)$. Then by condition 4), $f(u) \in \mathsf{Dom}(\Omega')$ and $gf(u) \in \mathsf{Dom}(\Omega'')$. By condition 5, $\Omega(u) \equiv_2 \Omega'(f(u)) \equiv_2 \Omega''(gf(u))$
   6) Suppose $u \equiv_E v$ and $\Omega(u) \not\equiv_2 \Omega(v)$. Then from Proposition 3 part 2 it follows that $f(u) \equiv_{E'} f(v)$ and from Definition 3 condition 4) it follows that $\Omega'(f(u)) \not\equiv_2 \Omega'(f(v))$. Then $\Omega(u) < \Omega(v)$ if and only if $\Omega'(f(u)) < \Omega'(f(v))$ if and only if $\Omega''(gf(u)) < \Omega''(gf(v))$. □

4. Identity functions are identity morphisms and composition of morphisms is associative.

   Proof. If we take the identity funtion as the identity isomorphism, it is clear that composing it with any morphism of parity games from right or left returns that morphism. Also, the composition of morphisms of parity formulas are associative since functional composition is associative. □

Proposition 4.2 *Let $f : \mathbb{G} \to \mathbb{G}'$ be a morphism of parity formulas. Then for any node v in $\mathbb{G}$ it holds that*

$$\mathbb{G}\langle v \rangle \equiv \mathbb{G}'\langle f(v) \rangle.$$

Proof. This proposition is a corollary of Proposition 3.5 □

Expansion as Morphism

In this section, we show that the expansion map defined by Kozen [1] is indeed a morphism according to Definition 3.

Definition 4.2 *Let $\xi \in \mu\mathsf{ML}(\mathsf{P})$ and let $BV(\xi) = \{x_0, \ldots, x_n\}$, where we assume that $i < j$ if $x_i <_\xi x_j$. Define the expansion $\mathsf{exp}_\xi(\varphi)$ of some $\varphi \trianglelefteq \xi$ as:*

$$\mathsf{exp}_\xi(\varphi) := \varphi[x_0/\eta_{x_0}x_0.\delta_{x_0}]\ldots[x_n/\eta_{x_n}x_n.\delta_{x_n}].$$

For the proof that exp_ξ is indeed a function from $Sfor(\xi)$ to $Clos(\xi)$ please refer to [13]. In the rest of this section we write $x_1, \ldots, x_n$ with this specific order in which exp_ξ performs substitutions in mind.

Throughout this subsection, we assume $\xi \in \mu\mathsf{ML}(\mathsf{P})$ to be clean. First, we construct an equivalent parity formula $\mathbb{H}'_\xi$ from the subformula dag in a slightly different way than that of Definition 2.38. Let $\mathbb{H}'_\xi := (V_\xi, E_\xi, L_\xi, \Omega_\xi, v_I)$ where

- $V_\xi := Sfor(\xi) - BV(\xi)$.
- Use D_ξ to denote the relation $\triangleright_0$ restricted to $Sfor(\xi)$.

$$E_\xi := \{(a, b) \mid (a, b) \in D_\xi, b \notin BV(\xi)\} \cup \{(a, \eta_x x.\delta_x) \mid (a, x) \in D_\xi\}.$$

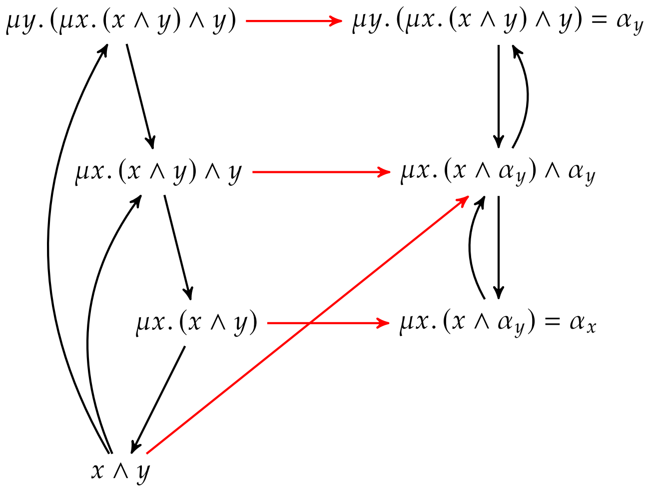


Figure 4.1: Example of an Expansion Map that is not injective

- $$L_\xi(\varphi) := \begin{cases} \varphi & \text{if } \varphi \in \{\top, \bot\} \cup \{p, \overline{p} \mid p \in FV(\xi)\} \\ \odot & \text{if } \varphi \text{ is of the form } \varphi_0 \odot \varphi_1 \text{ with } \odot \in \{\vee, \wedge\} \\ \heartsuit & \text{if } \varphi \text{ is of the form } \heartsuit\psi \text{ with } \heartsuit \in \{\Box, \Diamond\} \\ \epsilon & \text{if } \varphi \text{ is of the form } \eta_x x.\delta_x \text{ with } \eta \in \{\mu, \nu\} \end{cases}$$
- Given a bound variable $x \in BV(\xi)$, let $h_\xi^\uparrow(x)$ be the maximal length of an alternating $<_\xi$-chain of fixpoint variables starting at x. Furthermore, let $h_\xi(x)$ be the maximal length of an alternating $<_\xi$-chain with the corresponding fixpoint subformula of each variable from the chain being in the cluster of $\eta_x x.\delta_x$. Then we define

  $$\Omega_\xi(\eta_x x.\delta_x) := \begin{cases} h_\xi(x) - h_\xi^\uparrow(x) & \text{if } h_\xi(x) - h_\xi^\uparrow(x) \text{ has the same parity as } \eta_x \\ h_\xi(x) - h_\xi^\uparrow(x) + 1 & \text{otherwise} \end{cases}.$$

  For $\varphi \in V_\xi$ that is not of the shape of $\eta_x x.\delta_x$, $\Omega_\xi(\varphi)$ is undefined.
- $v_I := \xi$.

Let $\mathbb{G}_\xi$ be the equivalent parity formula constructed from the closure graph in Definition 2.42. Before we delve into the proof that is the goal of this subsectioin, we first provide an example of an expansion map which at the same time shows that even for clean formulas, the expansion map is not always an injection.

Example 4.1 Note that $\mathsf{exp}_\xi \upharpoonright_{V_\xi}$ is not necessarily injective. Consider $\xi = \mu y.(\mu x.(x \wedge y) \wedge y)$. Figure 4.1 shows $\mathbb{H}_\xi$, $\mathbb{G}_\xi$ and exp_ξ.

In the rest of this subsection, we show a collection of propositions that gradually build up to the final proposition in which we prove that $\mathsf{exp}_\xi \upharpoonright_{V_\xi}$ is a morphism from $\mathbb{H}_\xi$ to $\mathbb{G}_\xi$ for any clean modal μ-formula ξ.

Proposition 4.3 *For all* $0 \leq k < j \leq n$, *it holds that* $x_k \notin FV(\delta_{x_j})$.

Proof. Suppose $x_k \in FV(\delta_{x_j})$ for some $0 \leq k < j \leq n$. Then $x_k \in FV(\eta_{x_j} x_j.\delta_{x_j})$. Since ξ is clean, by Proposition 2.1 the fact that x_k occurs freely in $\eta_{x_j} x_j.\delta_{x_j}$ means that $x_k \trianglelefteq \eta_{x_j} x_j.\delta_{x_j} \trianglelefteq \eta_{x_k} x_k.\delta_{x_k}$. This means that $x_j <_\xi x_k$, and thus, $j < k$. However, by our assumption, $k < j$. Hence, contradiction. $\square$

Proposition 4.4 *Let* $x_0, \dots, y_i, y_{i+1}$ *be all distinct variables and* $\varphi, \psi_0, \dots, \psi_{i+1}$ *be formulas in* μML. *Then*

$$\varphi[y_0, \dots, y_i/\psi_0, \dots, \psi_i][y_{i+1}/\psi_{i+1}] = \varphi[y_0, \dots, y_i, y_{i+1}/\chi_0, \dots, \chi_i, \psi_{i+1}]$$

where for all j *with* $0 \leq j \leq i$, $\chi_j = \psi_i[y_{i+1}/\psi_{i+1}]$.

Proof. We prove this by induction on the size of φ:

- Base case.
  a) $\varphi = p$ for some $p \notin \{y_0, \dots, y_i\}$.

$$p[y_0, \dots, y_i/\psi_0, \dots, \psi_i][y_{i+1}/\psi_{i+1}] = p[y_{i+1}/\psi_{i+1}] = p[y_0, \dots, y_i, y_{i+1}/\chi_0, \dots, \chi_i, \psi_{i+1}]$$

  b) $\varphi = y_j$ for some $0 \leq j \leq i$.

$$y_j[y_0, \dots, y_i/\psi_0, \dots, \psi_i][y_{i+1}/\psi_{i+1}] = \psi_j[y_{i+1}/\psi_{i+1}] = \chi_j = y_j[y_0, \dots, y_i, y_{i+1}/\chi_0, \dots, \chi_i, \psi_{i+1}]$$

- Induction step.
  a) $\varphi = \varphi_0 \odot \varphi_1$ for some $\odot \in \{\wedge, \vee\}$.

$$\begin{aligned}
&\varphi[y_0, \dots, y_i/\psi_0, \dots, \psi_i][y_{i+1}/\psi_{i+1}] \\
&= \varphi_0[y_0, \dots, y_i/\psi_0, \dots, \psi_i][y_{i+1}/\psi_{i+1}] \odot \varphi_1[y_0, \dots, y_i/\psi_0, \dots, \psi_i][y_{i+1}/\psi_{i+1}] \\
&\overset{IH}{=} \varphi_0[y_0, \dots, y_i, y_{i+1}/\chi_0, \dots, \chi_i, \psi_{i+1}] \odot \varphi_1[y_0, \dots, y_i, y_{i+1}/\chi_0, \dots, \chi_i, \psi_{i+1}] \\
&= (\varphi_0 \odot \varphi_1)[y_0, \dots, y_i, y_{i+1}/\chi_0, \dots, \chi_i, \psi_{i+1}] \\
&= \varphi[y_0, \dots, y_i, y_{i+1}/\chi_0, \dots, \chi_i, \psi_{i+1}]
\end{aligned}$$

  b) $\varphi = \heartsuit\psi$ for some $\heartsuit \in \{\Box, \Diamond\}$.

$$\begin{aligned}
&\varphi[y_0, \dots, y_i/\psi_0, \dots, \psi_i][y_{i+1}/\psi_{i+1}] \\
&= \heartsuit\psi[y_0, \dots, y_i/\psi_0, \dots, \psi_i][y_{i+1}/\psi_{i+1}] \\
&= \heartsuit(\psi[y_0, \dots, y_i/\psi_0, \dots, \psi_i][y_{i+1}/\psi_{i+1}]) \\
&\overset{IH}{=} \heartsuit(\psi[y_0, \dots, y_i, y_{i+1}/\chi_0, \dots, \chi_i, \psi_{i+1}]) \\
&= \heartsuit\psi[y_0, \dots, y_i, y_{i+1}/\chi_0, \dots, \chi_i, \psi_{i+1}] \\
&= \varphi[y_0, \dots, y_i, y_{i+1}/\chi_0, \dots, \chi_i, \psi_{i+1}]
\end{aligned}$$

  c) $\varphi = \eta x.\psi$ for some $\eta \in \{\mu, \nu\}$.

$$\begin{aligned}
&\varphi[y_0, \dots, y_i/\psi_0, \dots, \psi_i][y_{i+1}/\psi_{i+1}] \\
&= \eta x.\psi[y_0, \dots, y_i/\psi_0, \dots, \psi_i][y_{i+1}/\psi_{i+1}] \\
&= \eta x.(\psi[x/q][y_0, \dots, y_i/\psi_0, \dots, \psi_i][y_{i+1}/\psi_{i+1}][q/x]) \\
&\overset{IH}{=} \eta_x x.(\psi[x/q][y_0, \dots, y_i, y_{i+1}/\chi_0, \dots, \chi_i, \psi_{i+1}][q/x]) \\
&= \eta x.\psi[y_0, \dots, y_i, y_{i+1}/\chi_0, \dots, \chi_i, \psi_{i+1}]
\end{aligned}$$

  where q is a new propositional letter. □

Proposition 4.5 *For any* $\varphi \in Sfor(\xi)$, $\mathsf{exp}_\xi(\varphi) = \varphi[x_1, x_2, \dots x_n/\mathsf{exp}_\xi(x_1), \mathsf{exp}_\xi(x_2), \dots, \mathsf{exp}_\xi(x_n)]$. *That is,* $\mathsf{exp}_\xi(\varphi)$ *can be obtained from* φ *by a uniform substitution that replaces the variable* x_i *with* $\mathsf{exp}_\xi(x_i)$ *for all* $0 \leq i \leq n$.

Proof. Denote $\varphi[x_0/\eta_{x_0}x_0.\delta_{x_0}]...[x_i/\eta_{x_i}x_i.\delta_{x_i}]$ by $\mathsf{exp}^i_\xi(\varphi)$ for all $0 \leq i \leq n$. We show by induction on i that for all $0 \leq i \leq n$ and $\varphi \in \mathit{Sfor}(\xi)$, it holds that

$$\mathsf{exp}^i_\xi(\varphi) = \varphi[x_0, ..., x_i/\mathsf{exp}^i_\xi(x_0), ..., \mathsf{exp}^i_\xi(x_i)].$$

- Base case, $i = 0$. It is easy to see that for all $\varphi \in \mathit{Sfor}(\xi)$, it holds that

$$\varphi[x_0/\eta_{x_0}x_0.\delta_{x_0}] = \varphi[x_0/(x_0[x_0/\eta_{x_0}x_0.\delta_{x_0}])].$$

- Induction step. Assume that the proposition holds for some $0 \leq i < n$. We show that the proposition also holds for $i + 1$. Note that

$$\begin{aligned}
\mathsf{exp}^{i+1}_\xi(\varphi) &= \mathsf{exp}^i_\xi(\varphi)[x_{i+1}/\eta_{x_{i+1}}x_{i+1}.\delta_{x_{i+1}}] \\
&= \varphi[x_0, ..., x_i/\mathsf{exp}^i_\xi(x_0), ..., \mathsf{exp}^i_\xi(x_i)][x_{i+1}/\eta_{x_{i+1}}x_{i+1}.\delta_{x_{i+1}}] \\
&\overset{Prop.4.4}{=} \varphi[x_0, ..., x_i, x_{i+1}/\mathsf{exp}^i_\xi(x_0)[x_{i+1}/\eta_{x_{i+1}}x_{i+1}.\delta_{x_{i+1}}], ..., \mathsf{exp}^i_\xi(x_i)[x_{i+1}/\eta_{x_{i+1}}x_{i+1}.\delta_{x_{i+1}}], \eta_{x_{i+1}}x_{i+1}.\delta_{x_{i+1}}] \\
&\overset{def}{=} \varphi[x_0, ..., x_i, x_{i+1}/\mathsf{exp}^{i+1}_\xi(x_0), ..., \mathsf{exp}^{i+1}_\xi(x_i), \eta_{x_{i+1}}x_{i+1}.\delta_{x_{i+1}}] \\
&\overset{*}{=} \varphi[x_0, ..., x_i, x_{i+1}/\mathsf{exp}^{i+1}_\xi(x_0), ..., \mathsf{exp}^{i+1}_\xi(x_i), \mathsf{exp}^{i+1}_\xi(x_{i+1})].
\end{aligned}$$

  Note that $*$ holds because

$$\begin{aligned}
\eta_{x_{i+1}}x_{i+1}.\delta_{x_{i+1}} &= x_{i+1}[x_{i+1}/\eta_{x_{i+1}}x_{i+1}.\delta_{x_{i+1}}] \\
&= x_{i+1}[x_0/\eta_{x_0}x_0.\delta_{x_0}]...[x_i/\eta_{x_i}x_i.\delta_{x_i}][x_{i+1}/\eta_{x_{i+1}}x_{i+1}.\delta_{x_{i+1}}] \\
&= \mathsf{exp}^{i+1}_\xi(x_{i+1})
\end{aligned}$$

The case $i = n$ is precisely the proposition we set out to prove. □

Corollary 4.6 $x_i \notin FV(\mathsf{exp}_\xi(\varphi))$ *for any* $x_i \in BV(\xi)$ *and* $\varphi \in \mathit{Sfor}(\xi)$.

Proof. Directly follows from Proposition 4.5. □

Proposition 4.7 *For all* $x_i \in BV(\xi)$*, it holds that* $\mathsf{exp}_\xi(x_i) = \mathsf{exp}_\xi(\eta_{x_i}x_i.\delta_{x_i})$.

Proof.

$$\begin{aligned}
\mathsf{exp}_\xi(x_i) &= x_i[x_0/\eta_{x_0}x_0.\delta_{x_0}]...[x_n/\eta_{x_n}x_n.\delta_{x_n}] \\
&= \eta_{x_i}x_i.\delta_{x_i}[x_{i+1}/\eta_{x_{i+1}}x_{i+1}.\delta_{x_{i+1}}]...[x_n/\eta_{x_n}x_n.\delta_{x_n}] \\
&\overset{Prop.4.3}{=} \eta_{x_i}x_i.\delta_{x_i}[x_0/\eta_{x_0}x_0.\delta_{x_0}]...[x_n/\eta_{x_n}x_n.\delta_{x_n}] \\
&= \mathsf{exp}_\xi(\eta_{x_i}x_i.\delta_{x_i})
\end{aligned}$$

□

Proposition 4.8 *For any* $\varphi, \psi \in V_\xi$*, if* $\varphi\ E_\xi\ \psi$*, then* $\mathsf{exp}_\xi(\varphi) \rightarrow_C \mathsf{exp}_\xi(\psi)$.

Proof. Suppose $\varphi\ E_\xi\ \psi$. There are the following cases:

1. $\varphi = \varphi_0 \odot \varphi_1$ and $\psi = \varphi_i$ for some $i \in \{0, 1\}$ and $\odot \in \{\vee, \wedge\}$. Then $\mathsf{exp}_\xi(\varphi) = \mathsf{exp}_\xi(\varphi_0) \odot \mathsf{exp}_\xi(\varphi_1)$ and $[\mathsf{exp}_\xi(\varphi_0) \odot \mathsf{exp}_\xi(\varphi_1)] \rightarrow_C \mathsf{exp}_\xi(\varphi_i)$.

2. $\varphi = \varphi_0 \odot \varphi_1$ and $\varphi_i = x$, $\psi = \eta_x x.\delta_x$ for some $x \in BV(\xi)$, $i \in \{0, 1\}$. Without loss of generality, assume $i = 1$. Then $\mathsf{exp}_\xi(\varphi) = \mathsf{exp}_\xi(\varphi_0) \odot \mathsf{exp}_\xi(x) \overset{Prop.4.7}{=} \mathsf{exp}_\xi(\varphi_0) \odot \mathsf{exp}_\xi(\eta_x x.\delta_x)$ and $[\mathsf{exp}_\xi(\varphi_0) \odot \mathsf{exp}_\xi(\eta_x x.\delta_x)] \to_C \mathsf{exp}_\xi(\eta_x x.\delta_x)$.
3. $\varphi = \heartsuit\varphi'$ and $\psi = \varphi'$ for some $\heartsuit \in \{\Box, \Diamond\}$. Then $\mathsf{exp}_\xi(\heartsuit\varphi') = \heartsuit\mathsf{exp}_\xi(\varphi')$ and $\heartsuit\mathsf{exp}_\xi(\varphi') \to_C \mathsf{exp}_\xi(\varphi')$.
4. $\varphi = \heartsuit x$ and $\psi = \eta_x x.\delta_x$ for some $\heartsuit \in \{\Box, \Diamond\}$ and $x \in BV(\xi)$. Then $\mathsf{exp}_\xi(\heartsuit x) = \heartsuit\mathsf{exp}_\xi(x) \overset{Prop.4.7}{=} \heartsuit\mathsf{exp}_\xi(\eta_x x.\delta_x)$ and $\heartsuit\mathsf{exp}_\xi(\eta_x x.\delta_x) \to_C \mathsf{exp}_\xi(\eta_x x.\delta_x)$.
5. $\varphi = \eta_{x_i} x_i.\delta_{x_i}$ and $\psi = \delta_{x_i}$ for some $x_i \in BV(\xi)$. Then

$$
\begin{aligned}
&\mathsf{exp}_\xi(\eta_{x_i} x_i.\delta_{x_i}) \\
&\overset{Prop.4.5}{=} \eta_{x_i} x_i.\delta_{x_i}[x_0, \ldots, x_n / \mathsf{exp}_\xi(x_0), \ldots, \mathsf{exp}_\xi(x_n)] \\
&= \eta_{x_i} x_i.(\delta_{x_i}[x_0, \ldots, x_{i-1}, x_{i+1} x_n / \mathsf{exp}_\xi(x_0), \ldots, \mathsf{exp}_\xi(x_{i-1}), \mathsf{exp}_\xi(x_{i+1}), \ldots, \mathsf{exp}_\xi(x_n)]) \\
&\to_C \delta_{x_i}[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n / \mathsf{exp}_\xi(x_0), \ldots, \mathsf{exp}_\xi(x_{i-1}), \mathsf{exp}_\xi(x_{i+1}), \ldots, \mathsf{exp}_\xi(x_n)][x_i / \mathsf{exp}_\xi(\eta_{x_i} x_i.\delta_{x_i})] \\
&\overset{Prop.4.4}{=} \delta_{x_i}[x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n / \mathsf{exp}_\xi(x_0)[x_i / \mathsf{exp}_\xi(\eta_{x_i} x_i.\delta_{x_i})], \ldots, \mathsf{exp}_\xi(x_{i-1})[x_i / \mathsf{exp}_\xi(\eta_{x_i} x_i.\delta_{x_i})], \\
&\quad \mathsf{exp}_\xi(\eta_{x_i} x_i.\delta_{x_i}), \mathsf{exp}_\xi(x_{i+1})[x_i / \mathsf{exp}_\xi(\eta_{x_i} x_i.\delta_{x_i})], \ldots, \mathsf{exp}_\xi(x_n)[x_i / \mathsf{exp}_\xi(\eta_{x_i} x_i.\delta_{x_i})]] \\
&\overset{Cor.4.6}{=} \delta_{x_i}[x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n / \mathsf{exp}_\xi(x_0), \ldots, \mathsf{exp}_\xi(x_{i-1}), \mathsf{exp}_\xi(\eta_{x_i} x_i.\delta_{x_i}), \mathsf{exp}_\xi(x_{i+1}), \ldots, \mathsf{exp}_\xi(x_n)] \\
&\overset{Prop.4.7}{=} \delta_{x_i}[x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n / \mathsf{exp}_\xi(x_0), \ldots, \mathsf{exp}_\xi(x_{i-1}), \mathsf{exp}_\xi(x_i), \mathsf{exp}_\xi(x_{i+1}), \ldots, \mathsf{exp}_\xi(x_n)] \\
&\overset{Prop.4.5}{=} \mathsf{exp}_\xi(\delta_{x_i}).
\end{aligned}
$$

Here we distinguish two different cases:

a) $\psi = \delta_{x_i}$. Then it is obvious that $\mathsf{exp}_\xi \delta_{x_i} = \mathsf{exp}_\xi(\psi)$.
b) $\psi \neq \delta_{x_i}$. Then by construction, δ_{x_i} must be of the form x_j for some $x_j \in BV(\xi)$. This means that $\psi = \eta_{x_j} x_j.\delta_{x_j}$ and thus,

$$\mathsf{exp}_\xi(\psi) = \mathsf{exp}_\xi(\eta_{x_j} x_j.\delta_{x_j}) \overset{Prop.4.7}{=} \mathsf{exp}_\xi(x_j) = \mathsf{exp}_\xi(\delta_{x_i}).$$

□

Proposition 4.9 *For any $\varphi \in V_\xi$ and $\psi \in Clos(\xi)$ such that $\mathsf{exp}_\xi(\varphi) \to_C \psi$, there is $\psi' \in V_\xi$ such that $\mathsf{exp}_\xi(\psi') = \psi$ and $\varphi\, E_\xi\, \psi'$.*

Proof. Suppose $\mathsf{exp}(\varphi) \to_C \psi$. There are the following cases:

1. $L_C(\mathsf{exp}_\xi(\varphi)) = \odot$ for some $\odot \in \{\wedge, \vee\}$. Since exp_ξ preserves the labels, it follows that φ is of the form $\varphi_0 \odot \varphi_1$ for some $\varphi_0, \varphi_1 \in Sfor(\xi)$. Then $\mathsf{exp}_\xi(\varphi) = \mathsf{exp}_\xi(\varphi_0) \odot \mathsf{exp}_\xi(\varphi_1)$. This means that $\psi = \mathsf{exp}_\xi(\varphi_i)$ for some $i \in \{0, 1\}$. If $\varphi\, E_\xi\, \varphi_i$, then we can let $\psi' = \varphi_i$; otherwise, by construction, φ_i must be of the form x for some $x \in BV(\xi)$, in which case $\psi = \mathsf{exp}_\xi(x) \overset{Prop.4.7}{=} \mathsf{exp}_\xi(\eta_x.\delta_x)$ and $\varphi\, E_\xi\, (\eta_x.\delta_x)$, so we can let $\psi' = \eta_x.\delta_x$.

2. $L_C(\mathsf{exp}_\xi(\varphi)) = \heartsuit$ for some $\heartsuit \in \{\Box, \Diamond\}$. Since exp_ξ preserves the labels, it follows that φ is of the form $\heartsuit\varphi'$ for some $\varphi' \in Sfor(\xi)$. Then $\mathsf{exp}_\xi(\varphi) = \heartsuit\mathsf{exp}_\xi(\varphi')$. This means that $\psi = \mathsf{exp}_\xi(\varphi')$. If $\varphi\, E_\xi\, \varphi'$, then we can let $\psi' = \varphi'$; otherwise, by construction, φ' must be of the form x for some $x \in BV(\xi)$, in which case $\psi = \mathsf{exp}_\xi(x) \overset{Prop.4.7}{=} \mathsf{exp}_\xi(\eta_x.\delta_x)$ and $\varphi\, E_\xi\, (\eta_x.\delta_x)$, so we can let $\psi' = \eta_x.\delta_x$.

3. $L_C(\mathsf{exp}_\xi(\varphi)) = \epsilon$. Then $\mathsf{exp}_\xi(\varphi)$ is of the form $\eta_{x_i}.\delta'_{x_i}$ for some $x_i \in BV(\xi)$ and some formula δ'_x. Since ξ is clean, it follows that $\varphi = \eta_{x_i}.\delta_{x_i}$. By Proposition 4.8, $\mathsf{exp}_\xi(\varphi) = \mathsf{exp}_\xi(\eta_x x.\delta_x) \to_C \mathsf{exp}_\xi(\delta_x)$. Since $L_C(\mathsf{exp}_\xi(\varphi)) = \epsilon$, $\mathsf{exp}_\xi(\varphi)$ only has one successor. So, $\psi = \mathsf{exp}_\xi(\delta_x)$. If $\varphi\, E_\xi\, \delta_{x_i}$, then we can let $\psi' = \delta_{x_i}$; otherwise, by construction, it has to be that $\delta_{x_i} = x_j$ for some $x_j \in BV(\xi)$, in which

case $\psi = \mathsf{exp}_\xi(x_j) \overset{Prop.4.7}{=} \mathsf{exp}_\xi(\eta_{x_j} x_j.\delta_{x_j})$. By construction, $(\varphi = \eta_{x_i} x_i.x_j)\, E_\xi\, (\eta_{x_j} x_j.x_j)$, so we can let $\psi' = \eta_{x_i} x_i.\delta_{x_i}$. □

Corollary 4.10 $\mathsf{exp}_\xi \upharpoonright_{V_\xi}$ *is surjective.*

Proof. Note that $\xi = \mathsf{exp}_\xi(\xi)$. Proposition 4.9 shows that $\mathsf{exp}_\xi[V_\xi]$ is closed under $\rightarrow_C$. Since $Clos(\xi)$ is characterized as the smallest set that contains ξ and is closed under $\rightarrow_C$, it follows that $Clos(\xi) \subseteq \mathsf{exp}_\xi[V_\xi]$. In other words, $\mathsf{exp}_\xi \upharpoonright_{V_\xi}$ is surjective. □

Proposition 4.11 *For any $\varphi \in V_\xi$, if $\mathsf{exp}_\xi(\varphi) = \mathsf{exp}_\xi(x)$ for some $x \in BV(\xi)$, then $\varphi = \eta_x x.\delta_x$.*

Proof. By Proposition 4.7, $\mathsf{exp}_\xi(\varphi) = \mathsf{exp}_\xi(\eta_x x.\delta_x)$. Then $\mathsf{exp}_\xi(\varphi) = \eta_x x.\delta'_x$ for some formula δ'_x. It follows that φ must be of the form $\eta_x x.\delta''_x$ for some formula δ''_x. Since ξ is clean, it must be that $\varphi = \eta_x x.\delta_x$. □

Proposition 4.12 *For $x, y \in BV(\xi)$ such that $x \neq y$, it holds that $\eta_x x.\delta_x \equiv_{E_\xi} \eta_y y.\delta_y$ if and only if $\mathsf{exp}_\xi(x) \equiv_C \mathsf{exp}_\xi(y)$.*

Proof. We show both directions:

$\Rightarrow$ Suppose $\eta_x x.\delta_x \equiv_{E_\xi} \eta_y y.\delta_y$. Then there exists path

$$(\eta_x x.\delta_x = \varphi_0)\, E_\xi \ldots E_\xi\, (\varphi_n = \eta_y y.\delta_y)\, E_\xi \ldots E_\xi\, (\varphi_{m+n} = \eta_x x.\delta_x).$$

By Proposition 4.8,

$$\mathsf{exp}_\xi(\eta_x x.\delta_x) \rightarrow_C \ldots \rightarrow_C \mathsf{exp}_\xi(\varphi_i) \rightarrow_C \ldots \rightarrow_C \mathsf{exp}_\xi(\eta_y y.\delta_y) \rightarrow_C \ldots \rightarrow_C \mathsf{exp}_\xi(\varphi_j) \rightarrow_C \ldots \rightarrow_C \mathsf{exp}_\xi(\eta_x x.\delta_x).$$

By Proposition 4.7,

$$\mathsf{exp}_\xi(x) \rightarrow_C \ldots \rightarrow_C \mathsf{exp}_\xi(\varphi_i) \rightarrow_C \ldots \rightarrow_C \mathsf{exp}_\xi(y) \rightarrow_C \ldots \rightarrow_C \mathsf{exp}_\xi(\varphi_j) \rightarrow_C \ldots \rightarrow_C \mathsf{exp}_\xi(x).$$

Thus, $\mathsf{exp}_\xi(x) \equiv_C \mathsf{exp}_\xi(y)$.

$\Leftarrow$ Suppose $\mathsf{exp}_\xi(x) \equiv_C \mathsf{exp}_\xi(y)$. Then there exists a trace

$$\mathsf{exp}_\xi(x) \rightarrow_C \ldots \rightarrow_C \mathsf{exp}_\xi(\varphi_i) \rightarrow_C \ldots \rightarrow_C \mathsf{exp}_\xi(y) \rightarrow_C \ldots \rightarrow_C \mathsf{exp}_\xi(\varphi_j) \rightarrow_C \ldots \rightarrow_C \mathsf{exp}_\xi(x).$$

By Proposition 4.9, there exists a path $\varphi_0\, E_\xi \ldots E_\xi\, \varphi_n\, E_\xi \ldots E_\xi\, \varphi_{m+n}$ such that $\mathsf{exp}_\xi(\varphi_0) = \mathsf{exp}_\xi(\varphi_{n+m}) = \mathsf{exp}_\xi x$ and $\mathsf{exp}_\xi(\varphi_n) = \mathsf{exp}_\xi(y)$. By Proposition 4.11, it follows that $\varphi_o = \varphi_{n+m} = \eta_x x.\delta_x$ and $\varphi_n = \eta_y y.\delta_y$. So, $\eta_x x.\delta_x \equiv_{E_\xi} \eta_y y.\delta_y$. □

Proposition 4.13 *For any $x, y \in BV(\xi)$, it follows that $x <_\xi y$ if and only if $\mathsf{exp}_\xi(x) \sqsubset_C \mathsf{exp}_\xi(y)$.*

Proof. We show both directions.

$\Rightarrow$: For this direction we show that given a path $\eta_y y.\delta_y \triangleright_0 \varphi_0 \triangleright_0 \ldots \triangleright_0 \varphi_m = \eta_x x.\delta_x$ witnessing that $x <_\xi y$, it holds that $\mathsf{exp}_\xi(y) = \mathsf{exp}_\xi(\eta_y y.\delta_y) \rightarrow_C \mathsf{exp}_\xi(\varphi_0) \rightarrow_C \ldots \rightarrow_C \mathsf{exp}_\xi(\eta_x x.\delta_x) = \mathsf{exp}_\xi(x)$ and $\mathsf{exp}_\xi(y) \trianglelefteq_f \mathsf{exp}_\xi(\varphi_i)$ for all $0 \leq i \leq m$.

We first show that for any $x, y \in BV(\xi)$ such that $x \neq y$ and $y \trianglelefteq \eta_x x.\delta_x \trianglelefteq \eta_y y.\delta_y$, it holds that $\mathsf{exp}_\xi(x) \sqsubset_C \mathsf{exp}_\xi(y)$. Here we use the fact that for any $\varphi \in Sfor(\xi)$ such that $y \in FV(\varphi)$, it holds that

$\mathsf{exp}_\xi(y) \trianglelefteqslant_f \mathsf{exp}_\xi(\varphi)$, that is, $\mathsf{exp}_\xi(y) \trianglelefteqslant \mathsf{exp}_\xi(\varphi)$ and $FV(\mathsf{exp}_\xi(y)) \cap BV(\mathsf{exp}_\xi(\varphi)) = \emptyset$. The former is true because by Proposition 4.5, $\mathsf{exp}_\xi(y)$ occurs as a subformula of $\mathsf{exp}_\xi(\varphi)$ where y is replaced with $\mathsf{exp}_\xi(y)$; the latter is true because $FV(\mathsf{exp}_\xi(y)) \subseteq FV(\xi)$ and $BV(\mathsf{exp}_\xi(\varphi)) \subseteq BV(\xi)$ and that ξ is tidy.

Since $y \trianglelefteqslant \eta_x x.\delta_x \trianglelefteqslant \eta_y y.\delta_y$, we have a path $\eta_y y.\delta_y \triangleright_0 \varphi_0 \triangleright_0 \ldots \triangleright_0 \varphi_m = \eta_x x.\delta_x$ such that $y \in FV(\varphi_i)$ for all $0 \le i \le m$. To see that y is free in all φ_i, $0 \le i \le m$, note that if y occurs in φ_m, it must occur in φ_i for all $0 \le i \le m$. Suppose y occurs bound in φ_i. Since ξ is clean, it must be that $\eta_y y.\delta_y \trianglelefteqslant \varphi_i$. This implies that $|\varphi_i| \ge |\eta_y y.\delta_y|$. However, since $\eta_y y.\delta_y \triangleright_0 \varphi_0 \triangleright_0 \ldots \triangleright_0 \varphi_i$, we have $|\varphi_i| < |\eta_y y.\delta_y|$. Contradiction. Thus, y is free in all φ_i, $0 \le i \le m$. By Proposition 4.8, $\mathsf{exp}_\xi(y) \to_C \mathsf{exp}_\xi(\varphi_0) \to_C \mathsf{exp}_\xi(\varphi_1) \to_C \ldots \to_C \mathsf{exp}_\xi(\varphi_m) = \mathsf{exp}_\xi(x)$. Furthermore, by what we have shown in the previous paragraph, $\mathsf{exp}_\xi(y) \trianglelefteqslant_f \mathsf{exp}_\xi(\varphi_i)$ for all $0 \le i \le m$. Thus, $\mathsf{exp}_\xi(x) \sqsubset_C \mathsf{exp}_\xi(y)$.

We have shown that for any $x, y \in BV(\xi)$ such that $x \ne y$ and $y \trianglelefteqslant \eta_x x.\delta_x \trianglelefteqslant \eta_y y.\delta_y$, it holds that $\mathsf{exp}_\xi(x) \sqsubset_C \mathsf{exp}_\xi(y)$. Suppose $x <_\xi y$. Then there is a chain of bound variables $x = y_0, y_1, \ldots, y_n = y$ such that $y_{i+1} \trianglelefteqslant \eta_{y_i} y_i.\delta_{y_i} \trianglelefteqslant \eta_{y_{i+1}} y_{i+1}.\delta_{y_{i+1}}$ for all $0 \le i < n$. This means that $\mathsf{exp}_\xi(x) = \mathsf{exp}_\xi(y_0) \sqsubset_C \mathsf{exp}_\xi(y_1) \sqsubset_C \ldots \sqsubset_C \mathsf{exp}_\xi(y_n) = \mathsf{exp}_\xi(y)$. Since $\sqsubset_C$ is transitive, it follows that $\mathsf{exp}_\xi(x) \sqsubset_C \mathsf{exp}_\xi(y)$.

$\Leftarrow$: Suppose $\mathsf{exp}_\xi(x) \sqsubset_C \mathsf{exp}_\xi(y)$. Let $\mathsf{exp}_\xi(y) = \psi_0 \to_C \ldots \to_C \psi_n = \mathsf{exp}_\xi(x)$ be one of the shortest traces in the closure graph witnessing $\mathsf{exp}_\xi(y) \twoheadrightarrow_C^{\mathsf{exp}_\xi(y)} \mathsf{exp}_\xi(x)$. By Proposition 4.9, there exists a path $\varphi_0\, E_\xi\, \varphi_1\, E_\xi \ldots E_\xi\, \varphi_n$ such that $\mathsf{exp}_\xi(\varphi_i) = \psi_i$ for all $0 \le i \le n$. By Proposition 4.11, $\varphi_0 = \eta_y y.\delta_y$ and $\varphi_n = \eta_x x.\delta_x$.We argue that $\eta_y y.\delta_y = \varphi_0 \triangleright_0 \varphi_1 \triangleright_0 \ldots \triangleright_0 \varphi_n = \eta_x x.\delta_x$. To see this, assume the contrary. Let $0 \le i < n$ be the smallest number such that $\varphi_i \not\triangleright_0 \varphi_{i+1}$. Then φ_{i+1} is of the form $\mathsf{exp}_\xi(\eta_z z.\delta_z) = \mathsf{exp}_\xi(z)$ for some $z \in BV(\xi) \cap FV(\varphi_i)$. We further distinguish the following three cases:

a) $z = y$. Then $\mathsf{exp}_\xi(y) = \psi_i \to_C \psi_{i+1} \to_C \ldots \to_C \psi_n = \mathsf{exp}_\xi(x)$ is a strictly shorter trace in the closure graph witnessing $\mathsf{exp}_\xi(y) \twoheadrightarrow_C^{\mathsf{exp}_\xi(y)} \mathsf{exp}_\xi(x)$, which contradicts our assumption.
b) $z \ne y$ and $z \in FV(\delta_y)$. This means that $z \trianglelefteqslant \eta_y y.\delta_y \trianglelefteqslant \eta_z z.\delta_z$ and thus, $y <_\xi z$. By the $\Rightarrow$ direction, it follows that $\mathsf{exp}_\xi(y) \sqsubset_C \mathsf{exp}_\xi(z)$. However, since $z \ne y$ and $\mathsf{exp}_\xi(y) = \psi_0 \to_C \ldots \to_C \psi_j = \mathsf{exp}_\xi(z)$ witnesses $\mathsf{exp}_\xi(y) \twoheadrightarrow_C^{\mathsf{exp}_\xi(y)} \mathsf{exp}_\xi(z)$, we have $\mathsf{exp}_\xi(z) \sqsubset_C \mathsf{exp}_\xi(y)$. Contradiction.
c) $z \ne y$ and $z \in BV(\delta_y)$. Since $z \in FV(\varphi_i)$ and $\varphi_j \triangleright_0 \varphi_{j+1}$ for all $0 \le j \le i$, it follows that there exists $0 < j \le i$ such that $\varphi_j = \eta_z z.\delta_z$ and $\varphi_{j+1} = \delta_z$. This means that $\mathsf{exp}_\xi(y) = \psi_0 \to_C \ldots \to_C \psi_j = \psi_{i+1} \to \psi_{i+2} \to_C \ldots \to_C \psi_n = \mathsf{exp}_\xi(x)$ is a strictly shorter trace in the closure graph witnessing $\mathsf{exp}_\xi(y) \twoheadrightarrow_C^{\mathsf{exp}_\xi(y)} \mathsf{exp}_\xi(x)$, which contradicts our assumption.

Note that since $\varphi_i \trianglelefteqslant \eta_y y.\delta_y$ and that z occurs in φ_i, it follows that z must occur in $\eta_y y.\delta_y$. So the three cases above exhaust all possibilities. Thus, $\eta_y y.\delta_y = \varphi_0 \triangleright_0 \varphi_1 \triangleright_0 \ldots \triangleright_0 \varphi_n = \eta_x x.\delta_x$.

Now we argue that for all $0 < i \le n$, there exists $0 \le j < i$ and $z \in FV(\varphi_i)$ such that $\varphi_j = \eta_z z.\delta_z$. Fix an i. Note that $|\varphi_i| < |\eta_y y.\delta_y| \le |\mathsf{exp}_\xi(y)|$ so $\mathsf{exp}_\xi(y) \not\trianglelefteqslant \varphi_i$. However, $\mathsf{exp}_\xi(y) \trianglelefteqslant \mathsf{exp}_\xi(\varphi_i)$, by assumption. As a consequence of Proposition 4.5, we know that the subformula dag of $\mathsf{exp}_\xi(\varphi_i)$ can be obtained from the subformula dag of φ_i by simultaneously replacing the node for x_i with the whole subformula dag of $\mathsf{exp}_\xi(x_i)$ for all $x_i \in FV(\varphi_i)$. Under this perspective, two possibilities arise in terms of how $\mathsf{exp}_\xi(y)$ appears as a subformula of $\mathsf{exp}_\xi(\varphi_i)$, or equivalently, how $\mathsf{exp}_\xi(y)$ appear as a subtree in the subformula dag of $\mathsf{exp}_\xi(\varphi_i)$:

a) The root note for the subtree $\mathsf{exp}_\xi(y)$ already exists in φ_i (including the x_i's which are later substituted). This means that there exists $\psi \trianglelefteqslant \varphi_i$ such that $\psi[y_1, \ldots, y_m / \mathsf{exp}_\xi(y_1), \ldots, \mathsf{exp}_\xi(y_m)] = \mathsf{exp}_\xi(y) = \mathsf{exp}_\xi(\eta_y y.\delta_y)$ where $\{y_1, \ldots, y_m\} = BV(\xi) \cap FV(\varphi_i)$. Then ψ is of the form $\eta_y y.\delta'_y$ for some formula δ'_y. Since $\psi \trianglelefteqslant \xi$ and ξ is clean, it must be the case that $\psi = \eta_y y.\delta_y$. Since φ_i is a strict subformula of $\eta_y y.\delta_y$ for all $0 < i \le n$, this is a contradiction.
b) The root note for the subtree $\mathsf{exp}_\xi(y)$ does not exist in φ_i. This means that there exist $z \in FV(\varphi_i)$ such that $\mathsf{exp}_\xi(y) \trianglelefteqslant \mathsf{exp}_\xi(z)$. Since z occurs in φ_i and $\varphi_i \trianglelefteqslant \eta_y y.\delta_y$, it follows that z also occurs in $\eta_y y.\delta_y$. Thus, $z \in FV(\eta_y y.\delta_y)$ or $z \in BV(\eta_y y.\delta_y)$.
   i. $z \in FV(\eta_y y.\delta_y)$. Then $y <_\xi z$, and by $\Rightarrow$, $\mathsf{exp}_\xi(y) \sqsubset \mathsf{exp}_\xi(z)$, which implies that $\mathsf{exp}_\xi(z) \trianglelefteqslant \mathsf{exp}_\xi(y)$. But we have assumed that $\mathsf{exp}_\xi(y) \trianglelefteqslant \mathsf{exp}_\xi(z)$. Putting these two statements together we have $\mathsf{exp}_\xi(y) = \mathsf{exp}_\xi(z)$, which implies that $\eta_y y.\delta_y = \eta_z z.\delta_z$. This contradicts our assumption that $z \in FV(\eta_y y.\delta_y)$.

ii. $z \in BV(\eta_y y.\delta_y)$. Since $z \in FV(\varphi_i)$ and $\varphi_j \triangleright_0 \varphi_{j+1}$ for all $0 \leq j \leq i$, it follows that there exists $0 \leq j < i$ such that $\varphi_j = \eta_z z.\delta_z$ and $\varphi_{j+1} = \delta_z$.

In light of this, we know that there exists a finite sequence $(x, n) = (y_0, k_0)(y_1, k_1)...(y_m, k_m) = (y, 0)$ of elements from $BV(\xi) \times \{0, 1, ...n\}$ such that

- $0 \leq k_{i+1} < k_i$,
- $\varphi_{k_i} = \eta_{y_i} y_i \delta_{y_i}$ for all $0 \leq i \leq m$, and
- $y_{i+1} \trianglelefteq_f \eta_{y_i} y_i.\delta_{y_i} \trianglelefteq \eta_{y_{i+1}} y_{i+1}.\delta_{y_{i+1}}$ for all $0 \leq i < m$.

Note that $x = y_0, y_1, ..., y_m = y$ is a chain of bound variables witnessing $x <_\xi y$.

In conclusion, we have shown that for any $x, y \in BV(\xi)$, it follows that $x <_\xi y$ if and only if $\mathsf{exp}_\xi(x) \sqsubset_C \mathsf{exp}_\xi(y)$. □

Proposition 4.14 *Given a clean* $\xi \in \mu\mathsf{ML}(\mathsf{P})$, exp_ξ *is a morphism from* $\mathbb{H}'_\xi$ *to* $\mathbb{G}_\xi$ *in the sense of Definition 3.*

Proof. We prove that exp_ξ satisfies every condition in Definition 3.

1) It follows directly from the definition of exp_ξ that $L(\varphi) = L(\mathsf{exp}_\xi(\varphi))$ for all $\varphi \in V_\xi$.
2) Precisely Proposition 4.8.
3) Precisely Proposition 4.9.
4) Let $v \in V_\xi$. Suppose $v \in \mathsf{Dom}(\Omega)$, then by definition $v = \eta_x x.\delta_x$ for some $x \in BV(\xi)$. Since exp_ξ only involves substitutions on v, it is easy to see that exp_ξ is still a fixpoint formula and by definition in the domain of Ω_g. Suppose $\mathsf{exp}_\xi(v) \in \mathsf{Dom}(\Omega_g)$. Then by definition, $\mathsf{exp}_\xi(v)$ is a fixpoint formula. Note that exp_ξ, consisting only of substitutions, does not change the outermost operator in a formula. This means that $v = x$ or $v = \eta_x x.\delta_x$ for some $x \in BV(\xi)$. Since the former is not in V_ξ, it follows that v is a fixpoint formula, and therefore, $v \in \mathsf{Dom}(\Omega_\xi)$.

5),6) Suppose $v \in \mathsf{Dom}(\Omega)$, then by definition $v = \eta_x x.\delta_x$ for some $x \in BV(\xi)$. Note that for all $x \in BV(\xi)$, it holds that

$$\Omega_\xi(\eta_x x.\delta_x) \equiv_2 \eta_x \equiv_2 \Omega_g(\mathsf{exp}_\xi(\eta_x x.\delta_x)).$$

Combining Proposition 4.11, Proposition 4.12 and Proposition 4.13, we observe the following.

- If $z_0 <_\xi z_1 <_\xi ... <_\xi z_m$ is an alternating $<_\xi$ chain such that $\eta_{z_i} z_i.\delta_{z_i}$ is in the cluster of $\eta_x x.\delta_x$ for all $0 \leq i \leq m$, then $\mathsf{exp}_\xi(z_0) \sqsubset_C \mathsf{exp}_\xi(z_1) \sqsubset_C ... \sqsubset_C \mathsf{exp}_\xi(z_m)$ is an alternating $\sqsubset_C$ chain in the cluster of $\mathsf{exp}_\xi(x) = \mathsf{exp}_\xi(\eta_x x.\delta_x)$;

- If $\eta_{z_0} z_0.\delta'_{z_0} \sqsubset_C \eta_{z_1} z_1.\delta'_{z_1} \sqsubset_C ... \sqsubset_C \eta_{z_m} z_m.\delta'_{z_m}$ is an alternating $\sqsubset_C$ chain in the cluster of $\mathsf{exp}_\xi(\eta_x x.\delta_x) = \mathsf{exp}_\xi(x)$ for some formula $\delta'_{z_0}, \delta'_{z_1}, ..., \delta'_{z_m}$, then $\eta_{z_i}.z_i\delta'_{z_i} = \mathsf{exp}_\xi(z_i)$ for all $0 \leq i \leq m$ and thus, $z_0 <_\xi z_1 <_\xi ... <_\xi z_m$ is an alternating $<_\xi$ chain such that $\eta_{z_i} z_i.\delta_{z_i}$ is in the cluster of $\eta_x x.\delta_x$ for all $0 \leq i \leq m$.

- If $\eta_x x.\delta_x = \eta_{z_0} z_0.\delta_{z_0} <_\xi \eta_{z_1} z_1.\delta_{z_1} <_\xi ... <_\xi \eta_{z_m} z_m.\delta_{z_m}$ is an alternating $<_\xi$-chain then $\mathsf{exp}_\xi(\eta_x x.\delta_x) = \mathsf{exp}_\xi(\eta_{z_0} z_0.\delta_{z_0}) = \mathsf{exp}_\xi(z_0) \sqsubset_C \mathsf{exp}_\xi(z_1) \sqsubset_C ... \sqsubset_C \mathsf{exp}_\xi(z_m)$ is an alternating $\sqsubset_C$ chain.

- If $\mathsf{exp}_\xi(\eta_x x.\delta_x) = \eta_{z_0} z_0.\delta'_{z_0} \sqsubset_C \eta_{z_1} z_1.\delta'_{z_1} \sqsubset_C ... \sqsubset_C \eta_{z_m} z_m.\delta'_{z_m}$ is an alternating $\sqsubset_C$ chain for some formula $\delta'_{z_0}, \delta'_{z_1}, ..., \delta'_{z_m}$, then $\eta_{z_i}.z_i\delta'_{z_i} = \mathsf{exp}_\xi(z_i)$ for all $0 \leq i \leq m$ and thus, $\eta_x x.\delta_x = \eta_{z_0} z_0.\delta_{z_0} <_\xi \eta_{z_1} z_1.\delta_{z_1} <_\xi ... <_\xi \eta_{z_m} z_m.\delta_{z_m}$ is an alternating $<_\xi$ chain.

These observations sum up to the fact that $cd(C(\mathsf{exp}_\xi(x))) = h_\xi(x)$ and $h^\uparrow(\mathsf{exp}_\xi(x)) = h^\uparrow_\xi(x)$ for all $x \in BV(\xi)$. Recall the definition for Ω_ξ and Ω_C, we conclude that $\Omega_\xi(\eta_x x.\delta_x) = \Omega_g(\mathsf{exp}_\xi(\eta_x x.\delta_x))$ for all $x \in BV(\xi)$. □

Case 2: Parity Formulas with One Cluster

In this section, we show that the notion of bisimulation given in Definition 3 is closed under unions and compositions for parity formulas that consist of only one cluster. These properties together ensure the existence of the largest bisimulation, i.e., the bisimilarity relation over any two parity formulas that have only one cluster. In the end of this chapter, we provide a way to take the quotient of such a parity formula with the help of the bisimilarity relation.

Proposition 4.15 *Let $\mathbb{G} = (V, E, L, \Omega, v_I)$, $\mathbb{G}' = (V', E', L', \Omega', v_I')$, $\mathbb{G}'' = (V'', E'', L'', \Omega'', v_I'')$ be three parity formulas such that for all $u, v \in V$, $u', v' \in V'$ and $u'', v'' \in V''$, we have that $u \equiv_E v$, $u' \equiv_{E'} v'$ and $u'' \equiv_{E''} v''$. Let $B \subseteq V \times V'$ and $B' \subseteq V' \times V''$ be two bisimulations of parity formulas. Then the composition of the two relations $B; B'$ is also a bisimulation.*

Proof. Let $(u, u'') \in B; B'$. Note that this means that there exists $u' \in V'$ such that $(u, u') \in B$ and $(u', u'') \in B'$.

1) $L(u) = L'(u') = L''(u'')$.
2) Let $v \in V$ such that Euv. This means that there exists $v' \in V'$ such that $E'u'v'$ and $(v, v') \in B$. It follows that there exists $v'' \in V''$ such that $E''u''v''$ and $(v', v'') \in B'$. Note that $(v, v'') \in B; B'$.
3) Let $v'' \in V''$ such that $E''u''v''$. Then there exists $v' \in V'$ such that $E'u'v'$ and $(v', v'') \in B'$. It follows that there exists $v \in V$ such that Euv and $(v, v') \in B$. Note that $(v, v'') \in B; B'$.
4) By condition 4, $u \in \mathsf{Dom}(\Omega)$ if and only if $u' \in \mathsf{Dom}(\Omega')$ if and only if $u'' \in \mathsf{Dom}(\Omega'')$.
5) Suppose $u \in \mathsf{Dom}(\Omega)$. Then by condition 4, $u' \in \mathsf{Dom}(\Omega')$ and $u'' \in \mathsf{Dom}(\Omega'')$. By condition 5, $\Omega(u) \equiv_2 \Omega'(u') \equiv_2 \Omega''(u'')$
6) Suppose $(v, v'') \in B; B'$. Then there exist $v' \in V'$ such that $(v, v') \in B$ and $(v', v'') \in B'$. Assume that $\Omega(u) \not\equiv_2 \Omega(v)$. By condition 2) of the definition of bisimulation, we have that $\Omega'(u') \not\equiv_2 \Omega'(v')$. Since each of the three parity formulas only consists of one cluster, it follows that $u \equiv_E v$, $u' \equiv_{E'} v'$ and $u'' \equiv_{E''} v''$. Then, by condition 5) of the definition of bisimulations, it follows that $\Omega(u) < \Omega(v)$ if and only if $\Omega'(u') < \Omega'(v')$ if and only if $\Omega(u'') < \Omega(v'')$. □

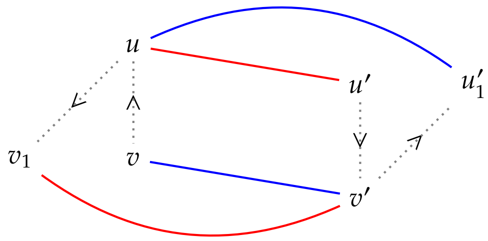
Proposition 4.16 *Let $\mathbb{G} = (V, E, L, \Omega, v_I)$, $\mathbb{G}' = (V', E', L', \Omega', v_I')$ be two parity formulas such that for all $u, v \in V$ and $u', v' \in V'$, we have that $u \equiv_E v$ and $u' \equiv_{E'} v'$. Let $B_1, B_2 \subseteq V \times V'$ be two bisimulations of parity formulas. Then the union $B_1 \cup B_2$ is also a bisimulation.*

Proof. Let $(u, u') \in B_1 \cup B_2$. Without loss of generality, assume $(u, u') \in B_1$.

1) By definition, $L(u) = L'(u')$.
2) For any $v \in V$ such that Euv, there exists $v' \in V'$ such that $(v, v') \in B_1 \subseteq B_1 \cup B_2$ and $E'u'v'$.
3) For any $v'' \in V'$ such that $Eu'v'$, there exists $v \in V$ such that $(v, v') \in B_1 \subseteq B_1 \cup B_2$ and Euv.
4) By condition 4, $u \in \mathsf{Dom}(\Omega)$ if and only if $u' \in \mathsf{Dom}(\Omega')$.
5) By condition 5, if $u \in \mathsf{Dom}(\Omega)$, then $\Omega(u) = \Omega'(u')$.
6) Suppose $v \in V$, $v' \in V'$, $(v, v') \in B_1 \cup B_2$, $u \equiv_E v$, $u' \equiv_{E'} v'$ and $\Omega(u) \not\equiv_2 \Omega(v)$. If $(v, v') \in B_1$, then condition 5) is met by definition. Thus, assume without loss of generality that $(v, v') \in B_2$, $\Omega(u) < \Omega(v)$ and $\Omega'(u') > \Omega'(v')$, and that

$$\Omega(u) = min\{\Omega(w) \mid (w, w') \in B_1, (x, x') \in B_2, \Omega(w) < \Omega(x), \Omega'(w') > \Omega'(x') \text{ for some } w, x \in V,\ w', x' \in V'\}.$$

Since $u' \twoheadrightarrow_{E'} v'$, it follows that there exists $v_1 \in V$ such that $u \twoheadrightarrow_E v_1$ and $(v_1, v') \in B_1$. Since B_1 is a bisimulation and $u \equiv_E v_1$, it follows that $\Omega(u) > \Omega(v_1)$.
Since $v \twoheadrightarrow_E u$, it follows that there exists $u_1' \in V'$ such that $v' \twoheadrightarrow_{E'} u_1'$ and $(u, u_1') \in B_2$. Since B_2 is a bisimulation and $u_1' \equiv_{E'} v'$, it follows that $\Omega'(u_1') < \Omega'(v')$.
Note that $(v_1, v') \in B_1$, $(u, u_1') \in B_2$, $\Omega(v_1) < \Omega(u)$, $\Omega'(v') > \Omega'(u_1')$, which contradicts our assumption that u has the smallest priority among nodes in v that has such a property.



Thus, for any $u, v \in V, u', v' \in V'$ such that $(u, u'), (v, v') \in B_1 \cup B_2$, $u \equiv_E v$, $u' \equiv_{E'} v'$ and $\Omega(u) \not\equiv_2 \Omega(v)$, it holds that $\Omega(u) < \Omega(v)$ if and only if $\Omega'(u') < \Omega'(v')$. □

Now that we have shown that bisimulations between parity formulas with one cluster are preserved under finite union, We can define the bisimilarity relation over $\mathbb{G}$, a parity formula that has only one cluster, as the biggest bisimulation between $\mathbb{G}$ and itself.

Definition 4.3 *Let $\mathbb{G} = (V, E, L, \Omega, v_I)$ be a parity formula with only one cluster. Define $\sim_{\mathbb{G}} \subseteq V \times V$ to be the union of all bisimulations from $\mathbb{G}$ to itself. Since there are only finitely many such bisimulations, their union is still a bisimulation. When it is clear in the context, we omit the subscript and only write $\sim$.*

It is easy to see that

1. The graph of the identity function on V, $\mathsf{graph}(id_V)$, is a bisimulation, and
2. if $R \subseteq V \times V$ is a bisimulation from $\mathbb{G}$ to itself, then $R^{-1} := \{(t, s) \mid t, s \in V, (s, t) \in R\}$ is also a bisimulation from $\mathbb{G}$ to itself.

These properties together with the fact that bisimulations are closed under composition mean that $\sim$ is an equivalence relation. This means that we can define a quotient parity formula out of $\mathbb{G}$.

Definition 4.4 *Let $\mathbb{G}$ be a parity formula with one cluster. Define the quotient parity formula*

$$\mathbb{G}/\sim := (V/\sim, E_\sim, L_\sim, \Omega_\sim, [v_I]_\sim)$$

where $E_\sim := \{([s]_\sim, [t]_\sim) \mid s, t \in V, Est\}$, $L_\sim([s]_\sim) = L(s)$ *and*

$$\Omega_\sim([s]_\sim) = \begin{cases} min(\{\Omega(t) \mid t \in V, s \sim t\}) & s \in \mathsf{Dom}(\Omega) \\ undefined & otherwise \end{cases}$$

.

Since $\sim$ is a bisimulation, we have that $L(s) = L(t)$ for any $s, t \in V, s \sim t$. So this is well-defined.

Proposition 4.17 *The graph of the quotient function $[\cdot]_\sim$ is a bisimulation.*

Proof. We show this by checking that it satisfies all the conditions of Definition 3.

1) $L(s) = L_\sim([s]_\sim)$ for all $s \in V$ by definition.
2) For any $s, t \in V$ such that Est, by definition $E_\sim([s]_\sim)([t]_\sim)$.
3) For any $s, t \in V$ such that $E_\sim([s]_\sim)([t]_\sim)$, there exists $s', t' \in V$ such that $Es't'$, $s \sim s'$ and $t \sim t'$. This means that there exists $t'' \in V$ such that Est'' and $t' \sim t''$. By transitivity of $\sim$, we have $t \sim t''$, or equivalently, $[t]_\sim = [t'']_\sim$.
4) Suppose $s \in \mathsf{Dom}(\Omega)$, then by definition $[s]_\sim \in \mathsf{Dom}(\Omega_\sim)$. Suppose $[s]_\sim \in \mathsf{Dom}(\Omega_\sim)$, then there exists $t \in \mathsf{Dom}(\Omega)$ such that $s \sim t$, $t \in \mathsf{Dom}(\Omega)$ and $\Omega_\sim([s]) = \Omega(t)$. Since $\sim$ is a bisimulation in the sense of Definition 3, it follows from condition 4 that $s \in \mathsf{Dom}(\Omega)$.

5) For all $s \in \mathsf{Dom}(\Omega)$, by definition there exists $t \in \mathsf{Dom}(\Omega)$ such that $s \sim t$ and $\Omega_\sim([s]) = \Omega(t)$. This means that $\Omega_\sim([s]) = \Omega(t) \equiv_2 \Omega(s)$.
6) Let $s, t \in \mathsf{Dom}(\Omega)$ be such that $\Omega(s) \not\equiv_2 \Omega(t)$. We show that $\Omega(s) < \Omega(v)$ if and only if $\Omega_\sim([s]_\sim) < \Omega_\sim([t]_\sim)$.
   - $\Leftarrow$: Note that there exists $s', t' \in V$ such that $s \sim s'$, $t \sim t'$ and $\Omega(s') = \Omega_\sim([s]_\sim) < \Omega_\sim([t]_\sim) = \Omega(t')$. The fact that $\sim$ is a bisimulation implies that $\omega(s) < \Omega(t)$.
   - $\Rightarrow$: Note that there exists $s', t' \in V$ such that $s \sim s'$, $t \sim t'$, $\Omega(s') = \Omega_\sim([s]_\sim)$ and $\Omega_\sim([t]_\sim) = \Omega(t')$. The fact that $\sim$ is a bisimulation implies that $\omega(s') < \Omega(t')$. Then it follows that $\Omega_\sim([s]_\sim) < \Omega_\sim([t]_\sim)$. □

5 Definition 4: Properties and Bisimilarity Game

This chapter is dedicated to proving the properties of Definition 4 shown in Figure 3.2. Recall Definition 4 provided in Chapter 3.

Definition 5.1 (Definition 4) *Given two parity formulas* $\mathbb{G} = (V, E, L, \Omega, v_I)$ *and* $\mathbb{G}' = (V', E', L', \Omega', v'_I)$*, a family of binary relations* $(A_i)_{i\in\omega}$*, with* $A_i \subseteq V \times V'$ *for all* $i \in \omega$*, is a bisimulation between* $\mathbb{G}$ *and* $\mathbb{G}'$ *if it satisfies the condition 1 - 6 below. Here we use* A *to denote* $\bigcup_{i\in\omega} A_i$*.*

1. $A_i \subseteq A_j$ *for all* $i, j \in \omega$ *such that* $i < j$*;*
2. $L(v) = L'(v')$ *for all* $(v, v') \in A$*;*
3. *for all* $(v, v') \in A_0$*, neither* $C(v)$ *nor* $C(v')$ *are degenerate,* $v \in \mathsf{Dom}(\Omega) \Leftrightarrow v' \in \mathsf{Dom}(\Omega')$*, and if* $v \in \mathsf{Dom}(\Omega)$*, then* $\Omega(v) \equiv_2 \Omega'(v')$*;*
4. *for any* $(u, u'), (v, v') \in A_0$ *such that* $v \in C(u)$*,* $v' \in C(u')$ *and* $\Omega(u) \not\equiv_2 \Omega(v)$*, we have that* $\Omega(u) < \Omega(v)$ *if and only if* $\Omega'(u') < \Omega'(v')$*;*
5. *for all* $i \in \omega$*,* $(v, v') \in A_{i+1} \backslash A_0$*;*
   a) *for all* $u \in V$ *such that* Evu*, there exists* $u' \in V'$ *such that* $E'v'u'$ *and* $(u, u') \in A_i$*;*
   b) *for all* $u' \in V'$ *such that* $E'v'u'$*, there exists* $u \in V$ *such that* Evu *and* $(u, u') \in A_i$*; and*
6. *for all* $(v, v') \in A_0$*,*
   a) *for all* $u \in C(v)$ *such that* Evu*, there exists* $u' \in C(v')$ *such that* $E'v'u'$ *and* $(u, u') \in A_0$*;*
   b) *for all* $u' \in C(v')$ *such that* $E'v'u'$*, there exists* $u \in C(v)$ *such that* Evu *and* $(u, u') \in A_0$*;*
   c) *for all* $u \in V$ *such that* Evu*, there exists* $u' \in V'$ *such that* $E'v'u'$ *and* $(u, u') \in A$*;*
   d) *for all* $u' \in V'$ *such that* $E'v'u'$*, there exists* $u \in V$ *such that* Evu *and* $(u, u') \in A$*.*

This chapter is divided into five sections. In section *Soundness*, we show that Definition 4 provides a sufficient condition for model equivalence between parity formulas. In section *Union*, we prove that Definition 4 is closed under union. Since parity formulas are by definition finite, it follows that closure under union ensures the existence of the largest bisimulation between any two parity formulas, which we call the bisimilarity relation. In section *Fixpoint Formulation & Decision Procedure*, we provide an alternative formulation of the bisimilarity relation which leads to a decision procedure for bisimilarity. In section *Bisimilarity Game*, we provide a two-player game, parameterized by two parity formulas, in the form of parity games in which $\exists$ has a winning strategy if and only if the two parity formulas are bisimilar in the sense of Definition 4. Finally, with the help of the bisimilarity game, we show in section *Composition* that Definition 4 is also closed under composition.

Soundness

In this section, we show that Definition 4 provides a sufficient condition for model equivalence between parity formulas.

Proposition 5.1 *Given two parity formulas* $\mathbb{G} = (V, E, L, \Omega, v_I)$ *and* $\mathbb{G}' = (V', E', L', \Omega', v'_I)$*, and a bisimulation* $(A_i)_{i\in\omega}$ *from* $\mathbb{G}$ *to* $\mathbb{G}'$*, it follows that for any* $(u, u') \in A$ *we have*

$$\mathbb{G}\langle u\rangle \equiv \mathbb{G}'\langle u'\rangle.$$

Proof. We need to show that, given a transition system $\mathbb{S}$ with initial state s_I, $\exists$ has a winning strategy in $\mathcal{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$ if and only if $\exists$ has a winning strategy in $\mathcal{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. By symmetry and positional determinacy of parity games, it suffices to show that given a positional winning $\exists$-strategy on $\mathcal{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$, $g : V \times S \to V \times S$,

$\exists$ has a winning strategy g' on $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. This strategy is constructed by 'shadow playing'. To be specific, during the game, $\exists$ keeps track of the position of the token in two matches: one is the match she plays with $\forall$ in $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$ and the other is what we call a shadow match in $\mathscr{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$ which she plays according to the aforementioned winning strategy. $\exists$ decides what to do in $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$ based on the position of the two tokens as well as g when it is her turn. At each turn, $\exists$ updates the positions simultaneously for both games. We write (u_i, s_i) and (u'_i, s'_i) respectively to denote the updated positions after the i-th turn in $\mathscr{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$ and $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. $\exists$ update the positions in a way such that for all applicable i,

a) $s_i = s'_i$ and $(u_i, u'_i) \in A$,
b) if $(u_i, u'_i) \in A_{j+1} - A_0$ for some $j \in \omega$ and neither of the games has ended, then $(u_{i+1}, u'_{i+1}) \in A_j$, and
c) if $(u_i, u'_i) \in A_0$, $u_{i+1} \in C(u_i)$ and $u'_{i+1} \in C(u'_i)$, then $(u_{i+1}, u'_{i+1}) \in A_0$.
d) (u_i, s_i) is a dead end if and only if (u'_i, s'_i) is a dead end.

$\exists$ starts from $(u_0, s_0) := (u, s_I)$ in $\mathscr{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$ and from $(u'_0, s'_0) := (u', s_I)$ in $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. Note that $s_0 = s'_0$ and $(u_0, u'_0) \in A$ by our assumption that $(u, u') \in A$. Suppose the positions after i turns are (u_i, s_i) and (u'_i, s'_i). We can assume $s_i = s'_i$ and $(u_i, u_i) \in A$. Then $L(u_i) = L'(u'_i)$ and we have the following cases.

1. $L(u_i) = L'(u'_i) \in \{\Diamond, \vee\}$ and (u_i, s_i) has at least one successor in $\mathscr{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$. In this case, by the definition of the evaluation game, it is $\exists$' turn make a move in $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. Let $(u_{i+1}, s_{i+1}) := g(u_i, s_i)$. Since (u_i, s_i) has at least one successor in $\mathscr{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$ and g is a well-defined strategy, it follows that (u_{i+1}, s_{i+1}) is a successor of (u_i, s_i), and thus is a legitimate move from (u_i, s_i). We make the following distinction.
   - If $(u_i, u'_i) \in A_0$ and $u_{i+1} \in C(u_i)$, then let $s'_{i+1} := s_{i+1}$ and $u'_{i+1} := c_{V'}(\{z \in V' \mid (u_{i+1}, z) \in A_0, E'u'_i z\})$. Note that we can use $c_{V'}$ here because of condition 6a) in Definition 3.5.
   - If $(u_i, u'_i) \in A_0$ and $u_{i+1} \notin C(u_i)$, then let $s'_{i+1} := s_{i+1}$ and $u'_{i+1} := c_{V'}(\{z \in V' \mid (u_{i+1}, z) \in A, E'u'_i z\})$. Note that we can use $c_{V'}$ here because of condition 6c) in Definition 3.5.
   - If neither of the two cases above is true, then let $(u_i, u_i) \notin A_0$. Since $(u_i, u_i) \in A$, it follows that $(u_i, u'_i) \in A_{j+1}$ for some $j \in \omega$. In this case, $s'_{i+1} := s_{i+1}$ and $u'_{i+1} := c_{V'}(\{z \in V' \mid (u_{i+1}, z) \in A_j, E'u'_i z\})$. Note that we can use $c_{V'}$ here because of condition 5a) in Definition 3.5.

   Note that the conditions a)-c) are respected in this case.
2. $L(u_i) = L'(u'_i) \in \{\Box, \wedge\}$ and (u'_i, s'_i) has at least one successor in $\mathscr{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$. In this case, $\forall$ can make a move in $\mathscr{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. Denote the position $\forall$ chooses as (u'_{i+1}, s'_{i+1}). We make the following distinction.
   - If $(u_i, u'_i) \in A_0$ and $u'_{i+1} \in C(u'_i)$, then $s_{i+1} := s'_{i+1}$ and $u_{i+1} := c_V(\{z \in V \mid (z, u'_{i+1}) \in A_0, Eu_i z\})$. Note that we can use c_V here because of condition 6b) in Definition 3.5.
   - If $(u_i, u'_i) \in A_0$ and $u'_{i+1} \notin C(u'_i)$, then $s_{i+1} := s'_{i+1}$ and $u_{i+1} := c_V(\{z \in V \mid (z, u'_{i+1}) \in A, Eu_i z\})$. Note that we can use c_V here because of condition 6d) in Definition 3.5.
   - If neither of the two cases above is true, then $(u_i, u'_i) \notin A_0$. Since $(u_i, u'_i) \in A$, it follows that $(u_i, u'_i) \in A_{j+1}$ for some $j \in \omega$. In this case, $s_{i+1} := s'_{i+1}$ and $u_{i+1} := c_V(\{z \in V \mid (z, u'_{i+1}) \in A_j, Eu_i z\})$. Note that we can use c_V here because of condition 6b) in Definition 3.5.

   Note that the conditions a)-c) are respected in this case.
3. $L(u_i) = L'(u'_i) = \epsilon$. Let (u_{i+1}, s_{i+1}) and (u'_{i+1}, s'_{i+1}) be the unique successors of (u_i, s_i) and (u'_i, s'_i) respectively. This means that $Eu_i u_{i+1}$, $E'u'_i u'_{i+1}$, $s_i = s_{i+1}$ and $s'_i = s'_{i+1}$.
   - Since $s_i = s'_i$, it follows that $s_{i+1} = s'_{i+1}$. Since $(u_i, u'_i) \in A$ and $Eu_i u_{i+1}$, it follows that there exists $v' \in V'$ such that $E'u'_i v'$ and $(u_{i+1}, v') \in A$. Since u'_{i+1} is the only successor of u'_i, it follows that $v' = u'_{i+1}$. Thus, $(u_{i+1}, u'_{i+1}) \in A$.
   - If $(u_i, u_i) \in A_{j+1} - A_0$ for some $j \in \omega$, then by condition 5a) of definition 3.5, there exists $u' \in V'$ such that $E'u'_i u'$ and $(u_{i+1}, u') \in A_j$. Since u'_{i+1} is the unique successor of u'_i, it follows that $u' = u'_{i+1}$ and $(u_{i+1}, u'_{i+1}) \in A_j$.
   - If $(u_i, u_i) \in A_0$ and $u_{i+1} \in C(u_i)$, then by condition 6a) of definition 3.5, there exists $u' \in V'$ such that $E'u'_i u'$ and $(u_{i+1}, u') \in A_0$. Since u'_{i+1} is the unique successor of u'_i, it follows that $u' = u'_{i+1}$ and $(u_{i+1}, u'_{i+1}) \in A_0$.

   Note that the conditions a)-c) are respected in this case.

4. If none of the cases above are true, then it must be one of the following cases.
   - $L(v) = L'(v') \in \mathsf{At}(\mathsf{P})$.
   - $L(u_i) = L'(u'_i) \in \{\Diamond, \vee\}$ and (u_i, s_i) has no successor in $\mathcal{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$.
   - $L(u_i) = L'(u'_i) \in \{\Box, \wedge\}$ and (u'_i, s'_i) has no successor in $\mathcal{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$.

   We argue that in all of these cases, neither (u_i, s_i) has successors in $\mathcal{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$ nor (u'_i, s'_i) has successors in $\mathcal{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. This is obvious in the first case. The argument for the third case is similar to that of the second case so we omit it. Now we show that if $L(u_i) = L'(u'_i) \in \{\Diamond, \vee\}$ and (u_i, s_i) has no successor in $\mathcal{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$, then (u'_i, s'_i) has no successor in $\mathcal{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. We make the following distinction.

   - $L(u_i) = L'(u'_i) = \Diamond$. Suppose (v', s') is a successor to (u'_i, s'_i) in $\mathcal{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. This means that $(s'_i, s') = (s_i, s') \in R_{\mathbb{S}}$ and $Eu'_i v'$. Since $(u_i, u'_i) \in A$, it follows that there exists $v \in V$ such that $Eu_i v$ and $(v, v') \in A$. This means that (v, s') is a successor to (u_i, s_i) in $\mathcal{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$, which contradicts our assumption. Thus, (u'_i, s'_i) has no successor in $\mathcal{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$.

   - $L(u_i) = L'(u'_i) = \vee$. Suppose (v', s') is a successor to (u'_i, s'_i) in $\mathcal{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. This means that $s'_i = s_i = s'$ and $Eu'_i v'$. Since $(u_i, u'_i) \in A$, it follows that there exists $v \in V$ such that $Eu_i v$ and $(v, v') \in A$. This means that (v, s') is a successor to (u_i, s_i) in $\mathcal{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$, which contradicts our assumption. Thus, (u'_i, s'_i) has no successor in $\mathcal{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$.

   Since (u_i, s_i) has no successor in $\mathcal{E}(\mathbb{G}\langle u\rangle, \mathbb{S})$ and (u'_i, s'_i) has no successor in $\mathcal{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$, no move is possible in either game and the two games end simultaneously.

Now we show that g' is indeed a winning strategy. Suppose $\exists$ follows this strategy. As we have shown above, the two matches always end simultaneously. There are two possibilities.

1. The matches end after finitely many, say, n, steps. Let $l = (u, s_I)(u_1, s_1)...(u_n, s_n)$ be the match in $\mathcal{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$ and $l' = (u', s'_I)(u'_1, s'_1)...(u'_n, s'_n)$ be the match in $\mathcal{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. By a), $s_n = s'_n$ and $(u_n, u'_n) \in A$. The latter implies that $L(u_n) = L'(u'_n)$ Thus (u_n, s_n) and (u'_n, s'_n) belong to the same player. Since both matches are full, it follows that the two matches are won by the same player.

2. The matches are infinite. Let $l = (u, s_I)(u_1, s_1)...$ be the match in $\mathcal{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$ and $l' = (u', s'_I)(u'_1, s'_1)...$ be the match in $\mathcal{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. Since any infinite match is eventually restricted to one cluster on the first coordinate, it follows that there exists $n, n' \in \omega$ such that for all $i \geq n$ we have $u_i \in C(u_n)$ and for all $i \geq n'$ we have $u'_i \in C(u'_{n'})$. Without loss of generality, assume $n \geq n'$. Then for all $i \geq n$ we have $u_i \in C(u_n)$ and $u'_i \in C(u'_n)$. Note that we have shown that $(u_i, u'_i) \in A$ for all $i \in \omega$. This means that there exists $m \in \omega$ such that $(u_n, u'_n) \in A_m$. By b), there exists $0 \leq j \leq m$ such that $(u_{n+j}, u'_{n+j}) \in A_0$. By c), $(u_i, u'_i) \in A_0$ for all $i \geq n + j$. By condition 4 of Definition 3.5, we have

   i) $\Omega(u_i, s_i) \equiv_2 \Omega'(u'_i, s'_i)$, and
   ii) if $\Omega(u_i, s_i) \not\equiv_2 \Omega(u_j, s_j)$, then $\Omega(u_i, s_i) < \Omega(u_j, s_j)$ if and only if $\Omega'(u'_i, s'_i) < \Omega'(u'_j, s'_j)$.

   By Proposition 2.5, l and l' have the same winner. Since l is a match where $\exists$ follows strategy g, it follows that $\exists$ is the winner in l. Thus, $\exists$ is also the winner in l'.

Thus, g' is a winning strategy for $\mathcal{E}(\mathbb{G}'\langle u'\rangle, \mathbb{S})$. □

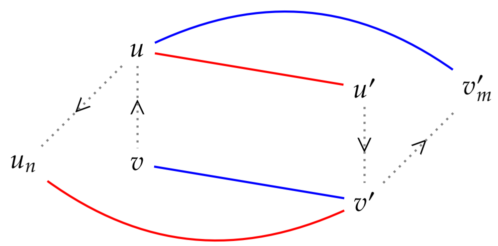
Union

In this section, we show that Definition 4 is closed under union.

Proposition 5.2 *Given two parity formulas $\mathbb{G} = (V, E, L, \Omega, v_I)$ and $\mathbb{G}' = (V', E', L', \Omega', v'_I)$, and two bisimulations $(A^1_i)_{i\in\omega}, (A^2_i)_{i\in\omega}$ from $\mathbb{G}$ to $\mathbb{G}'$, it follows that $(A'_i)_{i\in\omega}$ with $A'_i := A^1_i \cup A^2_i$ is also a bisimulation.*

Proof. We show that $(A'_i)_{i\in\omega}$ satisfies all the conditions in Definition 3.5.

1. For all $i, j \in \omega$ such that $i < j$, since $A^1_i \subseteq A^1_j$ and $A^2_i \subseteq A^2_j$, we have that
$$A'_i = A^1_i \cup A^2_i \subseteq A^1_j \cup A^2_j = A'_j.$$
2. Since $L(v) = L'(v')$ for all $(v, v') \in A^1$ and $(v, v') \in A^2$, it follows that $L(v) = L'(v')$ for all $(v, v') \in A^1 \cup A^2 = A'$.
3. Since for all $(v, v') \in A^1_0$ and $(v, v') \in A^2_0$ we have $\Omega(v) \equiv_2 \Omega(v')$ and neither $C(v)$ nor $C(v')$ are degenerate, it follows that for all $(v, v') \in A^1_0 \cup A^2_0 = A'_0$, $\Omega(v) \equiv_2 \Omega(v')$ and neither $C(v)$ nor $C(v')$ are degenerate.
4. Suppose $u, v \in V$, $u', v' \in V'$, $u \in C(v)$, $u' \in C(v')$, $(u, u'), (v, v') \in A^1_0 \cup A^2_0$ and $\Omega(u) \not\equiv_2 \Omega(v)$. Without loss of generality, suppose $(u, u') \in A^1_0$. If $(v, v') \in A^1_0$, then by assumption $\Omega(u) < \Omega(v)$ if and only if $\Omega'(u') < \Omega'(v')$. Otherwise, $(v, v') \in A^2_0$. Without loss of generality, suppose $\Omega(u) < \Omega(v)$. To argue towards contradiction, suppose $\Omega'(u') > \Omega'(v')$. Without loss of generality, also suppose $\Omega(u) = min(\{\Omega(w) \mid w \in D_u\})$ where
$$D_u := \{w \in C(u) \mid (w, w') \in A^1_0, (x, x') \in A^2_0, \Omega(w) < \Omega(x), \Omega'(w') > \Omega'(x') \text{ for some } x \in C(u),\ w', x' \in C(u')\}).$$
   - Since $u' \in C(v')$ and $(u, u') \in A^1_0$, it follows that there exist two paths $u' \rightarrow_{E'} u'_1 \rightarrow_{E'} \ldots \rightarrow_{E'} u'_n = v'$ and $u \rightarrow_E u_1 \rightarrow_E \ldots \rightarrow_E u_n$ for some $n \in \omega$ such that $u_i \in C(u)$, $u'_i \in C(u')$ and $(u_i, u'_i) \in A^1_0$ for all $1 \le i \le n$. In particular, $u_n \in C(u)$, $v' \in C(u')$ and $(u_n, v') \in A^1_0$. Since A^1_0 is a bisimulation, it follows that $\Omega(u) > \Omega(u_n)$.
   - Since $v \in C(u)$ and $(v, v') \in A^2_0$, it follows that there exist two paths $v' \rightarrow_{E'} v'_1 \rightarrow_{E'} \ldots \rightarrow_{E'} v'_m$ and $v \rightarrow_E v_1 \rightarrow_E \ldots \rightarrow_E v_m = u$ for some $m \in \omega$ such that $v_i \in C(v)$, $v'_i \in C(v')$ and $(v_i, v'_i) \in A^2_0$ for all $1 \le i \le m$. In particular, $u \in C(v)$, $v'_m \in C(v')$ and $(u, v'_m) \in A^2_0$. Since A^2_0 is a bisimulation, it follows that $\Omega(v'_m) < \Omega(v')$.
   - So far, we have gathered enough facts to show that $u_n \in D_u$. However, $\Omega(u_n) < \Omega(u)$, which contradicts our assumption that u has the smallest priority in D_u (red for A^1_0 and blue for A^2_0 in the demonstration below).



   Thus, $\Omega'(u') \not> \Omega'(v')$. Since $\Omega'(u') \equiv_2 \Omega(u) \not\equiv_2 \Omega(v) \equiv_2 \Omega'(v')$, it follows that $\Omega'(u') \neq \Omega'(v')$. This means that $\Omega'(u') < \Omega'(v')$.
5. Let $i \in \omega$ and $(v, v') \in (A^1_{i+1} \cup A^2_{i+1}) - (A^1_0 \cup A^2_0)$. Without loss of generality, suppose $(v, v') \in A^1_{i+1} - (A^1_0 \cup A^2_0)$, then $(v, v') \in A^1_{i+1} - A^1_0$.
   a) For all $u \in V$ such that Evu, by assumption there exists $u' \in V'$ such that $E'v'u'$ and $(u, u') \in A^1_i \subseteq A^1_i \cup A^2_i = A'_i$.
   b) Similar to (a).
6. Let $(v, v') \in A^1_0 \cup A^2_0$. Without loss of generality, suppose $(v, v') \in A^1_0$.
   a) For all $u \in C(v)$ such that Evu, by assumption there exists $u' \in C(v')$ such that $E'v'u'$ and $(u, u') \in A^1_0 \subset A^1_0 \cup A^2_0 = A'_0$.
   b) Similar to (a).

c) For all $u \in V$ such that Evu, by assumption there exists $u' \in V'$ such that $E'v'u'$ and $(u, u') \in A^1 \subseteq A^1 \cup A^2 = A'$.

d) Similar to (c). □

Let $\mathbb{G} = (V, E, L, \Omega, v_I)$ and $\mathbb{G}' = (V', E', L', \Omega', v'_I)$ be two parity formulas. Suppose $(A^1_i)_{i\in\omega}$ and $(A^2_i)_{i\in\omega}$ are two bisimulations from $\mathbb{G}$ to $\mathbb{G}'$. By the proof of Proposition 5.2, we know that $(A^1_i \cup A^2_i)_{i\in\omega}$ is also a bisimulation. Since there are only finitely many bisimulations from $\mathbb{G}$ to $\mathbb{G}'$, it follows that there exists a largest bisimulation over $\mathbb{G}$ to $\mathbb{G}'$.

Definition 5.2 (Bisimilarity) *Let $\mathbb{G} = (V, E, L, \Omega, v_I)$ and $\mathbb{G}' = (V', E', L', \Omega', v'_I)$ be two parity formulas. Denote that largest bisimulation over $\mathbb{G}$ and $\mathbb{G}'$ by $(\mathscr{A}_i)^{\mathbb{G},\mathbb{G}'}_{i\in\omega}$. When the context is clear, we just write $(\mathscr{A}_i)_{i\in\omega}$ and $\mathscr{A}$. Note that, for any bisimulation $(A_i)_{i\in\omega}$ over $\mathbb{G}$ and $\mathbb{G}'$, we have $A_i \subseteq \mathscr{A}_i^{\mathbb{G},\mathbb{G}'}$ for all $i \in \omega$.*

Definition 5.3 *Let $\mathbb{G} = (V, E, L, \Omega, v_I)$ and $\mathbb{G}' = (V', E', L', \Omega', v'_I)$ be two parity formulas. Define $r_{\mathbb{G},\mathbb{G}'} : V \times V' \to \omega \cup \{+\infty\}$ such that for all $(v, v') \in V \times V'$,*

$$r_{\mathbb{G},\mathbb{G}'}(v, v') := \begin{cases} min(\{i \in \omega \mid (v, v') \in \mathscr{A}_i\}) & \text{if } (v, v') \in \mathscr{A}^{\mathbb{G},\mathbb{G}'} \\ +\infty & \text{otherwise} \end{cases}$$

When $\mathbb{G}$ and $\mathbb{G}'$ are clear, we also write $r_{\mathbb{G},\mathbb{G}'}$ as r. Note that $r(v, v') = 0$ implies that $C(v)$ and $C(v')$ are non-degenerate but not the other way around.

Definition 5.4 *Let $\mathbb{G} = (V, E, L, \Omega, v_I)$ be a parity formula.*

- *Let $\leq_{\mathbb{G}}$ denote the pre-order over V such that for all $u, v \in V$, $u \leq_{\mathbb{G}} v$ if and only if vE^*u.*
- *Let $<_{\mathbb{G}}$ denote the strict partial order over V such that for all $u, v \in V$, $u <_{\mathbb{G}} v$ if and only if $u \leq_{\mathbb{G}} v$ and $u \notin C(v)$.*

Note that since G is acyclic, it follows that, for any $a, b \in N$, if aRb then $b <_G a$. Let $\mathbb{G}' = (V', E', L', \Omega', v'_I)$ be another parity formula.

- *Let $\leq_{\mathbb{G},\mathbb{G}'}$ denote the pre-order over $V \times V'$ such that for all $(u, u'), (v, v') \in V \times V'$, $(u, u') \leq_{\mathbb{G},\mathbb{G}'} (v, v')$ if and only if $u \leq_{\mathbb{G}} v$ and $u' \leq_{\mathbb{G}'} v'$.*
- *Let $=_{\mathbb{G},\mathbb{G}'}$ denote the equivalence relation over $V \times V'$ such that for all $(u, u'), (v, v') \in V \times V'$, $(u, u') =_{\mathbb{G},\mathbb{G}'} (v, v')$ if and only if $(u, u') \leq_{\mathbb{G},\mathbb{G}'} (v, v')$ and $(v, v') \leq_{\mathbb{G},\mathbb{G}'} (u, u')$.*
- *Let $<_{\mathbb{G},\mathbb{G}'}$ denote the strict partial order over $V \times V'$ such that for all $(u, u'), (v, v') \in V \times V'$, $(u, u') <_{\mathbb{G},\mathbb{G}'} (v, v')$ if and only if $(u, u') \leq_{\mathbb{G},\mathbb{G}'} (v, v')$ and $(u, u') =_{\mathbb{G},\mathbb{G}'} (v, v')$.*
- *Let $<_{\mathbb{G},\mathbb{G}',\omega}$ denote the partial order over $(V \times V') \times \omega$ such that, for any $u, x \in V$, $u', x' \in V'$ and $m, n \in \omega$, we have $((u, u'), m) <_{\mathbb{G},\mathbb{G}',\omega} ((x, x'), n)$ if and only if $(u, u') <_{\mathbb{G},\mathbb{G}'} (x, x')$ or $((u, u') =_{\mathbb{G},\mathbb{G}'} (x, x')$ and $m < n)$.*

Lemma 5.3 *For any parity formulas $\mathbb{G}$ and $\mathbb{G}'$, $<_{\mathbb{G}}$, $<_{\mathbb{G},\mathbb{G}'}$ and $<_{\mathbb{G},\mathbb{G}',\omega}$ are all well-founded.*

Proof. Since V and V' are finite, it follows that strict partial order $<_{\mathbb{G}}$ and $<_{\mathbb{G}'}$ are well-founded. Since $<_{\mathbb{G}}$ and $<_{\mathbb{G}'}$ are well-founded, it follows that their Cartesian product $<_{\mathbb{G},\mathbb{G}'}$ is also well-founded. Since both $<_{\mathbb{G},\mathbb{G}'}$ and $<$ on natural numbers are well-founded, it follows that their lexicographic product is also well-founded. □

Fixpoint Formulation & Decision Procedure

Recall the definition of bisimulation for Kripke models:

Definition 5.5 *Let $\mathbb{S}_1 = \langle S_1, V_1, R_1\rangle$ and $\mathbb{S}_2\langle S_2, V_2, R_2\rangle$ be two Kripke models. Z is a bisimulation over $\mathbb{S}_1$ and $\mathbb{S}_2$ if for any $s_1 \in S_1$, $s_2 \in S_2$, $s_1 Z s_2$, we have*

1) *$V_1(s_1) = V_2(s_2)$;*
3) *for any $s_1' \in S_1$, if $R_1 s_1 s_1'$ then there exists $s_2' \in S_2$ such that $s_1' Z s_2'$ and $R_2 s_2 s_2'$; and*
4) *for any $s_2' \in S_2$, if $R_2 s_2 s_2'$ then there exists $s_1' \in S_1$ such that $s_1' Z s_2'$ and $R_1 s_1 s_1'$.*

It is well-known that given any Kripke models $\mathbb{S}_1 = \langle S_1, V_1, R_1\rangle$ and $\mathbb{S}_2\langle S_2, V_2, R_2\rangle$, there exists a largest bisimulation, i.e. the bisimilarity relation, over $\mathbb{S}_1$ and $\mathbb{S}_2$. Denote this largest bisimulation by $\mathfrak{Z}^{\mathbb{S}_1,\mathbb{S}_2}$, or when the context is clear, $\mathfrak{Z}$. It has been proven that the bisimilarity relation has the following equivalent fixpoint account.

Definition 5.6 *Let $\mathbb{S}_1 = \langle S_1, V_1, R_1\rangle$ and $\mathbb{S}_2 = \langle S_2, V_2, R_2\rangle$ be two Kripke models. Define a function $g_{\mathbb{S}_1\mathbb{S}_2} : \mathcal{P}(S_1 \times S_2) \longrightarrow \mathcal{P}(S_1 \times S_2)$ (g for short) as follows. For any $A \subseteq S_1 \times S_2$ and $(s_1, s_2) \in S_1 \times S_2$, we have $(s_1, s_2) \in g(A)$ if and only if*

1) *$V_1(s_1) = V_2(s_2)$;*
3) *for any $s_1' \in S_1$, if $R_1 s_1 s_1'$ then there exists $s_2' \in S_2$ such that $s_1' A s_2'$ and $R_2 s_2 s_2'$; and*
4) *for any $s_2' \in S_2$, if $R_2 s_2 s_2'$ then there exists $s_1' \in S_1$ such that $s_1' A s_2'$ and $R_1 s_1 s_1'$.*

Proposition 5.4 *Let $\mathbb{S}_1 = \langle S_1, V_1, R_1\rangle$ and $\mathbb{S}_2 = \langle S_2, V_2, R_2\rangle$ be two Kripke models. Then,*

$$\mathfrak{Z}^{\mathbb{S}_1,\mathbb{S}_2} = GFP.\lambda A.g(A).$$

Similarly, we can give a fixpoint account for the bisimilarity relation induced by Definition 4. However, note that this fixpoint account features a least fixpoint operator instead of a greatest one.

Definition 5.7 *Given two parity formulas $\mathbb{G} = (V, E, L, \Omega, v_I)$, $\mathbb{G}' = (V', E', L', \Omega', v_I')$ and a family of base bisimulations $(B_i)_{1\le i\le n}$ between $\mathbb{G}$ and $\mathbb{G}'$, let $B := \bigcup_{i=1}^{n} B_i$. Define the function $f_{\mathbb{G},\mathbb{G}'} : \mathcal{P}^2(V \times V') \longrightarrow \mathcal{P}(V \times V')$ (f for short) as follows. For any $A \subseteq V \times V'$ and $(v, v') \in V \times V'$, we have $(v, v') \in f(B, A)$ if and only if*

1. a) *$L(v) = L(v')$, and*
   b) *for all $u \in V$ such that Evu there is $u' \in V'$ such that $E'v'u'$ and $(u, u') \in A$, and*
   c) *for all $u' \in V'$ such that $E'v'u'$ there is $u \in V$ such that Evu and $(u, u') \in A$,*

   or
2. a) *$(v, v') \in B$, and*
   b) *for all $(w, w') \in B\upharpoonright_{C(v)\times C(v')}$*
      i. *for any $u \in V\backslash C(v)$ such that Ewu, there is $u' \in V'$ such that $(u, u') \in A$ and $E'w'u'$, and*
      ii. *for any $u' \in V'\backslash C(v')$ such that $E'w'u'$, there is $u \in V$ such that $(u, u') \in A$ and Ewu.*

The following proposition is easy to see. We present it without proof.

Proposition 5.5 *Let B be the union of a family of base bisimulations. $\lambda A.f(B, A)$ is a monotone function.*

Proposition 5.6 *Let $\mathbb{G} = (V, E, L, \Omega, v_I)$ and $\mathbb{G}' = (V', E', L', \Omega', v_I')$ be two parity formulas. Then,*

$$\mathcal{A}^{\mathbb{G},\mathbb{G}'} = LFP.\lambda A.f(\mathcal{A}_0^{\mathbb{G},\mathbb{G}'}, A).$$

Proof. We prove both inclusions.

1. $\mathcal{A} \subseteq \text{LFP}.\lambda A.f(\mathcal{A}_0, A)$. It suffices to show that for all $(v, v') \in \mathcal{A}$ there exists a natural number n such that $(v, v') \in f^n(\mathcal{A}, \emptyset)$. To show this, we first define a function $\# : \mathcal{A} \to \omega$ such that for any $(v, v') \in \mathcal{A}$, we have

$$\#(v, v') := \begin{cases} \max(\{\#(u, u') \mid (u, u') \in \mathcal{A}, u \notin C(v) \text{ or } u' \notin C(v'), \text{there exists} \\ \qquad (w, w') \in \mathcal{A}_0 \restriction_{C(v)\times C(v')} \text{ s.t. } Ewu \text{ and } E'w'u'\}) + 1, & (v, v') \in \mathcal{A}_0 \\ \max(\{\#(u, u') \mid Evu, E'v'u', (u, u') \in \mathcal{A}_i\}) + 1, & (v, v') \in \mathcal{A}_{i+1} \backslash \mathcal{A}_0 \end{cases}$$

Here we assume $\max(\emptyset) = 0$. We show that # is indeed well-defined. Suppose the contrary. Since $<_{\mathbb{G},\mathbb{G}',\omega}$ is well-founded, let $(v, v') \in \mathcal{A} \backslash \mathsf{Dom}(\#)$ that is minimal with respect to $<_{\mathbb{G},\mathbb{G}',\omega}$.

a) Suppose $(v, v') \in \mathcal{A}_0$. Note that by assumption all the elements in

$$\{(u, u') \mid Ewu, E'w'u', w \in C(v), w' \in C(v'), [u \notin C(v) \text{ or } u' \notin C(v')]\}$$

are strictly smaller with respect to $<_{\mathbb{G},\mathbb{G}',\omega}$, we have that

$$\{(u, u') \mid Ewu, E'w'u', w \in C(v), w' \in C(v'), [u \notin C(v) \text{ or } u' \notin C(v')]\} \subseteq \mathsf{Dom}(\#).$$

Then, $\#(v, v')$ is also defined, which contradicts our assumption.

b) Suppose $(v, v') \in \mathcal{A}_{i+1} \backslash \mathcal{A}_0$ for some $i \in \omega$. Note that by assumption all the elements in $\{(u, u') \mid Evu, E'v'u', (u, u') \in \mathcal{A}_i\}$ are strictly smaller with respect to $<_{\mathbb{G},\mathbb{G}',\omega}$, we have that

$$\{(u, u') \mid Evu, E'v'u', (u, u') \in \mathcal{A}_i\} \subseteq \mathsf{Dom}(\#).$$

Then, $\#(v, v')$ is also defined, which contradicts our assumption.

Thus, # is well-defined. Now we claim that for all $(v, v') \in \mathcal{A}$ we have

$$(v, v') \in f^{\#(v,v')}(\mathcal{A}_0, \emptyset).$$

We show this by induction on $\#(v, v')$.

a) Base case. Note that $f^0(\mathcal{A}, \emptyset) = \emptyset = \{(v, v') \mid \#(v, v') = 0\}$.

b) Induction step. Suppose for all (u, u') in $\{(u, u') \mid \#(u, u') \leq n\}$, we have $(u, u') \in f^{\#(u,u')}(\mathcal{A}, \emptyset)$. Let $(v, v') \in \mathcal{A}$ be such that $\#(v, v') = n + 1$.

i. Suppose $(v, v') \in \mathcal{A}_0$. Define

$$D_{v,v'} := \{(u, u') \in \mathcal{A} \mid u \notin C(v) \text{ or } u' \notin C(v'), \text{there exists} \\ (w, w') \in \mathcal{A}_0 \restriction_{C(v)\times C(v')} \text{ such that } Ewu \text{ and } E'w'u'\}.$$

By the definition of #, $\#(u, u') \leq n$ for all $(u, u') \in D_{v,v'}$. By the induction hypothesis, $(u, u') \in f^{\#(u,u')}(\mathcal{A}_0, \emptyset)$ for all $(u, u') \in D_{v,v'}$. Recalling basic knowledge of fixpoint theory, we know that $f^m(\mathcal{A}_0, \emptyset) \subseteq f^k(\mathcal{A}_0, \emptyset)$ for any $m \leq k$. This means that $D_{v,v'} \subseteq f^n(\mathcal{A}_0, \emptyset)$.

Since $(\mathcal{A}_i)_{i\in\omega}$ is a bisimulation in the sense of Definition 4, it follows from condition 6 of Definition 4 that for all $(w, w') \in \mathcal{A}_0 \restriction_{C(v)\times C(v')}$, it holds that

(-) for any $u \in V \backslash C(v)$ such that Ewu, there is $u' \in V'$ such that $(u, u') \in \mathcal{A}$ and $E'w'u'$, and

(-) for any $u' \in V' \backslash C(v')$ such that $E'w'u'$, there is $u \in V$ such that $(u, u') \in \mathcal{A}$ and Ewu.

From the definition of $D_{v,v'}$, we can deduce that for all $(w, w') \in \mathcal{A}_0 \restriction_{C(v)\times C(v')}$, it holds that

(-) for any $u \in V \backslash C(v)$ such that Ewu, there is $u' \in V'$ such that $(u, u') \in D_{v,v'}$ and $E'w'u'$, and

(-) for any $u' \in V' \backslash C(v')$ such that $E'w'u'$, there is $u \in V$ such that $(u, u') \in D_{v,v'}$ and Ewu.

Note that we have shown that $D_{v,v'} \subseteq f^n(\mathcal{A}_0, \emptyset)$. Given the assumption that $(v, v') \in \mathcal{A}_0$ we have the following facts.

* $(v, v') \in \mathcal{A}_0$, and
* for all $(w, w') \in B \restriction_{C(v)\times C(v')}$

(–) for any $u \in V\backslash C(v)$ such that Ewu, there is $u' \in V'$ such that $(u,u') \in f^n(\mathcal{A}_0,\varnothing)$ and $E'w'u'$, and

(–) for any $u' \in V'\backslash C(v')$ such that $E'w'u'$, there is $u \in V$ such that $(u,u') \in f^n(\mathcal{A}_0,\varnothing)$ and Ewu.

By the definition of $\lambda A.f(\mathcal{A}_0,A)$, this means that

$$(v,v') \in f^{(n+1)=\#(v,v')}(\mathcal{A}_0,\varnothing).$$

ii. Suppose $(w,w') \in \mathcal{A}_{i+1\backslash}\mathcal{A}_0$ for some $i \in \omega$. Define

$$D_{v,v'} := \{(u,u') \mid Evu, E'v'u', (u,u') \in \mathcal{A}_i\}$$

By the definition of #, $\#(u,u') \leq n$ for all $(u,u') \in D_{v,v'}$. By the induction hypothesis, $(u,u') \in f^{\#(u,u')}(\mathcal{A}_0,\varnothing)$ for all $(u,u') \in D_{v,v'}$. Recalling basic knowledge of fixpoint theory, we know that $f^m(\mathcal{A}_0,\varnothing) \subseteq f^k(\mathcal{A}_0,\varnothing)$ for any $m \leq k$. This means that $D_{v,v'} \subseteq f^n(\mathcal{A}_0,\varnothing)$.

Since $(\mathcal{A}_i)_{i\in\omega}$ is a bisimulation in the sense of Definition 4, it follows from condition 5 of Definition 4 that for all $(w,w') \in \mathcal{A}_0 \restriction_{C(v)\times C(v')}$, it holds that

* for all $u \in V$ such that Evu, there exists $u' \in V'$ such that $E'v'u'$ and $(u,u') \in \mathcal{A}_i$;
* for all $u' \in V'$ such that $E'v'u'$, there exists $u \in V$ such that Evu and $(u,u') \in \mathcal{A}_i$;

From the definition of $D_{v,v'}$, we can deduce that for all $(w,w') \in \mathcal{A}_0 \restriction_{C(v)\times C(v')}$, it holds that

* for all $u \in V$ such that Evu, there exists $u' \in V'$ such that $E'v'u'$ and $(u,u') \in D_{v,v'}$;
* for all $u' \in V'$ such that $E'v'u'$, there exists $u \in V$ such that Evu and $(u,u') \in D_{v,v'}$;

Note that we have shown that $D_{v,v'} \subseteq f^n(\mathcal{A}_0,\varnothing)$. Given the assumption that $(v,v') \in \mathcal{A}$ we have the following facts.

* $L(v) = L(v')$, and
* for all $u \in V$ such that Evu there is $u' \in V'$ such that $E'v'u'$ and $(u,u') \in f^n(\mathcal{A}_0,\varnothing)$, and
* for all $u' \in V'$ such that $E'v'u'$ there is $u \in V$ such that Evu and $(u,u') \in f^n(\mathcal{A}_0,\varnothing)$,

By the definition of $\lambda A.f(\mathcal{A}_0,A)$, this means that

$$(v,v') \in f^{(n+1)=\#(v,v')}(\mathcal{A}_0,\varnothing).$$

Therefore, $\mathcal{A} \subseteq \text{LFP}.\lambda A.f(\mathcal{A}_0,A)$.

2. $\mathcal{A} \supseteq \text{LFP}.\lambda A.f(\mathcal{A}_0,A)$. It suffices to show that $\mathcal{A}$ is a postfixpoint, i.e., $\mathcal{A} \supseteq f(\mathcal{A}_0,\mathcal{A})$. Suppose the contrary. Then there exists $(v,v') \in f(\mathcal{A}_0,\mathcal{A})\backslash\mathcal{A}$. Since $\mathcal{A}_0 \subseteq \mathcal{A}$, it follows that condition 1 of Definition 4.11 is true for (v,v'), that is,

   a) $L(v) = L(v')$, and
   b) for all $u \in V$ such that Evu there is $u' \in V'$ such that $E'v'u'$ and $(u,u') \in \mathcal{A}$, and
   c) for all $u' \in V'$ such that $E'v'u'$ there is $u \in V$ such that Evu and $(u,u') \in \mathcal{A}$.

   Let
   $$n := \max\{r(u,u') \mid Evu, E'v'u', (u,u') \in \mathcal{A}\}.$$
   where r is the rank in Definition 5.3.
   Define $(A_i)_{i\in\omega}$ as follows:
   $$A_i := \begin{cases} \mathcal{A}_i & i \leq n \\ \mathcal{A}_i \cup \{(v,v')\} & i > n \end{cases}$$

   It is easy to see that $(A_i)_{i\in\omega}$ is a bisimulation that is strictly larger than $(\mathcal{A}_i)_{i\in\omega}$, which contradicts that fact that the latter is the largest bisimulation between $\mathbb{G}$ and $\mathbb{G}'$. Thus, $\mathcal{A} \supseteq f(\mathcal{A}_0,\mathcal{A})$, and therefore, $\mathcal{A} \supseteq \text{LFP}.\lambda A.f(\mathcal{A}_0,A)$.

□

This fixpoint account gives us the following decision procedure for bisimilarity. Let $\mathbb{G} = (V, E, L, \Omega, v_I)$ and $\mathbb{G}' = (V', E', L', \Omega', v_I')$ be two parity formulas. Let UC be the set of non-degenerate clusters in $\mathbb{G}$ and UC' the set of non-degenerate clusters in $\mathbb{G}'$.

1. For each $C \in UC$, $C' \in UC'$, calculate the biggest base bisimulation over $C \times C'$, $B_{C,C'}$. This can be done by a width-first search in the rooted search tree $\mathcal{T} = (T, R, r)$ where $T = \mathcal{P}(C \times C')$, $R[S] = \{S' \subseteq S \mid S' = S \backslash \{s\}, s \in S\}$ and $r = C \times C'$, the root of the tree.
2. Inductively calculate $f^n(B, \varnothing)$, where $B = \bigcup_{C \in UC, C' \in UC'} B_{C,C'}$, until we reach a fixpoint.

Bisimilarity Game

In this section, we provide a two-player game, parameterized by two parity formulas, in the form of parity games in which $\exists$ has a winning strategy if and only if the two parity formulas are bisimilar in the sense of Definition 4.

Bisimilarity Game: Preliminaries

Before we delve into the details of the bisimilarity game, we introduce some concepts and their properties, which will help us simplify the definition of the game and the proofs of its properties. First, we introduce the following concepts from relation lifting.

Definition 5.8 *Let S' and T' be two sets. Let $R \subseteq S' \times T'$.*

- *Define $\overrightarrow{\mathcal{P}}R$ to be the collection of $(S, T) \subseteq S' \times T'$ such that, $S \subseteq S'$, $T \subseteq T'$ and, for any $x \in S$, there is $y \in T$ such that $(x, y) \in R$.*
- *Define $\overleftarrow{\mathcal{P}}R$ to be the collection of $(S, T) \subseteq S' \times T'$ such that, $S \subseteq S'$, $T \subseteq T'$ and, for any $y \in T$, there is $x \in S$ such that $(x, y) \in R$.*
- *Let $\overline{\mathcal{P}}R$ denote $(\overrightarrow{\mathcal{P}}R) \cap (\overleftarrow{\mathcal{P}}R)$.*

Following are some properties of $\overleftarrow{\mathcal{P}}$, $\overrightarrow{\mathcal{P}}$ and $\overline{\mathcal{P}}$. These properties are easy to verify, so we present them here without proof.

Proposition 5.7 *Let S' and T' be two sets. Let $S, S_1, S_2 \subseteq S'$, $T, T_1, T_2 \subseteq T'$ and $R, R_1, R_2 \subseteq S' \times T'$.*

1. *If $(S, T_1) \in \overrightarrow{\mathcal{P}}R$ and $T_1 \subseteq T_2$, then $(S, T_2) \in \overrightarrow{\mathcal{P}}R$.*
2. *If $(S_1, T) \in \overleftarrow{\mathcal{P}}R$ and $S_1 \subseteq S_2$, then $(S_2, T) \in \overleftarrow{\mathcal{P}}R$.*
3. *If $(S_1, T) \in \overrightarrow{\mathcal{P}}R_1$ and $(S_2, T) \in \overrightarrow{\mathcal{P}}R_2$, then $(S_1 \cup S_2, T) \in \overrightarrow{\mathcal{P}}(R_1 \cup R_2)$.*
4. *If $(S, T_1) \in \overleftarrow{\mathcal{P}}R_1$ and $(S, T_2) \in \overleftarrow{\mathcal{P}}R_2$, then $(S, T_1 \cup T_2) \in \overleftarrow{\mathcal{P}}(R_1 \cup R_2)$.*
5. *If $(S, T) \in \overrightarrow{\mathcal{P}}R_1$ and $R_1 \subseteq R_2$, then $(S, T) \in \overrightarrow{\mathcal{P}}R_2$.*
6. *If $(S, T) \in \overleftarrow{\mathcal{P}}R_1$ and $R_1 \subseteq R_2$, then $(S, T) \in \overleftarrow{\mathcal{P}}R_2$.*
7. *$\overline{\mathcal{P}}$ is monotone, that is, given $R_1 \subseteq R_2 \subseteq S_1' \times S_2'$, if $(S_1, S_2) \in \overline{\mathcal{P}}R_1$, then $(S_1, S_2) \in \overline{\mathcal{P}}R_2$.*

Let P' be another set. Let $P \subseteq P'$ and $Q \subseteq T' \times P'$.

8. *If $(S, T) \in \overleftarrow{\mathcal{P}}R$ and $(T, P) \in \overleftarrow{\mathcal{P}}Q$, then $(S, P) \in \overleftarrow{\mathcal{P}}(R; Q)$.*
9. *If $(S, T) \in \overrightarrow{\mathcal{P}}R$ and $(T, P) \in \overrightarrow{\mathcal{P}}Q$, then $(S, P) \in \overrightarrow{\mathcal{P}}(R; Q)$.*
10. *If $(S, T) \in \overline{\mathcal{P}}R$ and $(T, P) \in \overline{\mathcal{P}}Q$, then $(S, P) \in \overline{\mathcal{P}}(R; Q)$.*

Now, we define the following three kinds of partial bisimulations.

Definition 5.9 (Local Bisimulation) *Let* $\mathbb{G} = (V, E, L, \Omega, v_I)$ *and* $\mathbb{G}' = (V', E', L', \Omega', v_I')$ *be two parity formulas. Let* $v \in V$ *and* $v' \in V'$*. Let* $\mathsf{LocB}(v, v')$ *denote the collection of* $R \subseteq V \times V'$ *such that* $(E[v], E'[v']) \in \overline{\mathcal{P}}R$*. We call elements of* $\mathsf{LocB}(v, v')$ local bisimulations *at* (v, v').

Definition 5.10 (Base Bisimulation) *Let* $\mathbb{G} = (V, E, L, \Omega, v_I)$ *and* $\mathbb{G}' = (V', E', L', \Omega', v_I')$ *be two parity formulas. Let* C *and* C' *be non-degenerate clusters in* $\mathbb{G}$ *and* $\mathbb{G}'$*, respectively. Let* $\mathsf{BaseB(C, C')}$ *be the collection of* $B \subseteq C \times C'$ *such that, for all* $(u, u') \in B$,

1. $L(u) = L'(u')$;
2. $u \in \mathsf{Dom}(\Omega)$ *if and only if* $u' \in \mathsf{Dom}(\Omega')$;
3. *if* $u \in \mathsf{Dom}(\Omega)$ *and* $u' \in \mathsf{Dom}(\Omega')$*, then* $\Omega(u) \equiv_2 \Omega'(u')$;
4. $(E[u] \cap C, E'[u'] \cap C') \in \overline{\mathcal{P}}B$;

and that, for all $(u, u'), (w, w') \in B$,

5. *if* $\Omega(u) \not\equiv_2 \Omega(w)$*, then* $\Omega(u) < \Omega(w)$ *if and only if* $\Omega'(u') < \Omega'(w')$.

We call elements of $\mathsf{BaseB}(C, C')$ base bisimulations *at* (C, C').

Definition 5.11 (Exit Bisimulation) *Let* $\mathbb{G} = (V, E, L, \Omega, v_I)$ *and* $\mathbb{G}' = (V', E', L', \Omega', v_I')$ *be two parity formulas. Let* $B \in \mathsf{BaseB}(C, C')$ *for some non-degenerate cluster* C *and* C' *in* $\mathbb{G}$ *and* $\mathbb{G}'$*, respectively. Let* $\mathsf{ExitB}(B)$ *denote the collection of* $R \subseteq V \times V'$ *such that for all* $(u, u') \in B$*, it holds that* $(E[u]\backslash C, E'[u']) \in \overrightarrow{\mathcal{P}}R$ *and* $(E[u], E'[u']\backslash C) \in \overleftarrow{\mathcal{P}}R$*. We call elements of* $\mathsf{ExitB}(B)$ *exit bisimulations at* B.

The following propositions demonstrate some useful properties of these partial bisimulations in terms of how they combine with, or convert to, each other.

Proposition 5.8 *Let* $\mathbb{G} = (V, E, L, \Omega, v_I)$*,* $\mathbb{G}' = (V', E', L', \Omega', v_I')$*, and* $\mathbb{G}'' = (V'', E'', L'', \Omega'', v_I'')$ *be three parity formulas. Let* $v \in V$*,* $v' \in V'$ *and* $v'' \in V''$*. Let* $R_1 \in \mathsf{LocB}(v, v')$ *and* $R_2 \in \mathsf{LocB}(v', v'')$*. Then* $R_1; R_2 \in \mathsf{LocB}(v, v'')$

Proof. Since $R_1 \in \mathsf{LocB}(v, v')$ and $R_2 \in \mathsf{LocB}(v', v'')$, it follows from the definition of local bisimulations that $(E[v], E'[v']) \in \overline{\mathcal{P}}R_1$ and $(E[v'], E'[v'']) \in \overline{\mathcal{P}}R_2$. By 10. of Proposition 5.8, this means that $(E[v], E''[v'']) \in \overline{\mathcal{P}}(R_1; R_2)$, which is precisely the definition for $R_1; R_2 \in \mathsf{LocB}(v, v'')$. ☐

Proposition 5.9 *Let* $\mathbb{G} = (V, E, L, \Omega, v_I)$*, and* $\mathbb{G}' = (V', E', L', \Omega', v_I')$ *be two parity formulas. Let* C *and* C' *be non-degenerate clusters in* $\mathbb{G}$ *and* $\mathbb{G}'$*, respectively. Let* $B \in \mathsf{BaseB}(C, C')$*. Then for all* $(u, u') \in B$*, we have* $B \in \mathsf{BaseB}(C(u), C(u'))$.

Proof. Fix $(u, u') \in B$. By the definition of base bisimulations, $B \subseteq C \times C$. This means that $u]inC$ and $u' \in C'$, or equivalently, $C = C(u)$ and $C(u')$. Therefore, $B \in \mathsf{BaseB}(C(u), C(u'))$. ☐

Proposition 5.10 *Let* $\mathbb{G} = (V, E, L, \Omega, v_I)$*, and* $\mathbb{G}' = (V', E', L', \Omega', v_I')$ *be two parity formulas. Let* C *and* C' *be non-degenerate clusters in* $\mathbb{G}$ *and* $\mathbb{G}'$*, respectively. Let* $B \in \mathsf{BaseB}(C, C')$ *and* $R \in \mathsf{ExitB}(B)$*. Then for all* $(u, u') \in B$*, we have* $(R \cup B) \in \mathsf{LocB}(u, u')$.

Proof. Since $(u, u') \in B$, it follows from Prop. 5.9 that

$$(E[u] \cap C(u), E'[u'] \cap C(u')) \in \overline{\mathcal{P}}B. \tag{5.1}$$

Given the definition of exit bisimulations, we have

$$(E[u]\backslash C(u), E'[u']) \in \overrightarrow{\mathcal{P}}R \tag{5.2}$$

and

$$(E[u], E'[u'] \backslash C(u')) \in \overleftarrow{\mathcal{P}}R. \tag{5.3}$$

Combining (5.1) with the definition of $\overline{\mathcal{P}}$, we have

$$(E[u] \cap C(u), E'[u'] \cap C(u')) \in \overrightarrow{\mathcal{P}}B \tag{5.4}$$

and

$$(E[u] \cap C(u), E'[u'] \cap C(u')) \in \overleftarrow{\mathcal{P}}B. \tag{5.5}$$

Combining (5.4) and 1. from Prop. 5.7, we have

$$(E[u] \cap C(u), E'[u']) \in \overrightarrow{\mathcal{P}}B. \tag{5.6}$$

Combining (5.5) and 2. from Prop. 5.7, we have

$$(E[u], E'[u'] \cap C(u')) \in \overleftarrow{\mathcal{P}}B. \tag{5.7}$$

Combining (5.2), (5.6) and 3. from Prop. 5.7, we have

$$(E[u], E'[u']) \in \overrightarrow{\mathcal{P}}(R \cup B). \tag{5.8}$$

Combining (5.3), (5.7) and 4. from Prop. 5.7, we have

$$(E[u], E'[u']) \in \overleftarrow{\mathcal{P}}(R \cup B). \tag{5.9}$$

Combining (5.8), (5.9) and the definition of $\overline{\mathcal{P}}$, we have

$$(E[u], E'[u']) \in \overline{\mathcal{P}}(R \cup B), \tag{5.10}$$

which is precisely the definition for $(R \cup B) \in \mathsf{LocB}(u, u')$.

□

Proposition 5.11 *Let $\mathbb{G} = (V, E, L, \Omega, v_I)$, $\mathbb{G}' = (V', E', L', \Omega', v_I')$, and $\mathbb{G}'' = (V'', E'', L'', \Omega'', v_I'')$ be three parity formulas. Let C, C' and C'' be non-degenerate clusters in $\mathbb{G}$, $\mathbb{G}'$ and $\mathbb{G}''$, respectively. Let $B_1 \in \mathsf{BaseB}(C, C')$ and $B_2 \in \mathsf{BaseB}(C', C'')$. Then $B_1; B_2 \in \mathsf{BaseB}(C, C'')$*

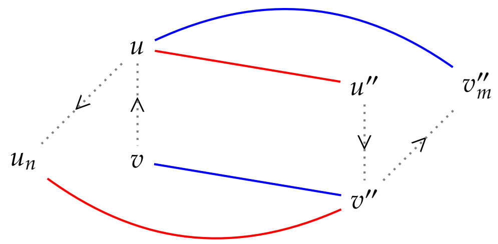
Proof. We prove that $B_1; B_2 \in \mathsf{BaseB}(C, C'')$ by show that $B_1; B_2$ satisfies all the conditions from the definition of base bisimulations. Fix $(u, u'), (v, v') \in B_1$, $(u', u''), (v', v'') \in B_2$. In this proof, when we mention the 'condition x' we mean the condition x of the definition of base bisimulations unless it is explicitly stated otherwise.

1. Note that $L(u) = L'(u')$ and $L'(u') = L''(u'')$. This means that $(L(u) = L''(u''))$.
2. Note that $u \in \mathsf{Dom}(\Omega)$ if and only if $u' \in \mathsf{Dom}(\Omega')$ if and only if $u'' \in \mathsf{Dom}(\Omega'')$.
3. Suppose $u \in \mathsf{Dom}(\Omega)$ and $u'' \in \mathsf{Dom}(\Omega'')$. Since B_1 is a base bisimulation, it follows from condition 2 that $u' \in \mathsf{Dom}(\Omega')$. By condition 3, $\Omega(u) \equiv_2 \Omega'(u') \equiv_2 \Omega''(u'')$.
4. By condition 4, we have $(E[u] \cap C, E'[u'] \cap C') \in \overline{\mathcal{P}}B_1$ and $(E'[u'] \cap C', E'[u''] \cap C'') \in \overline{\mathcal{P}}B_2$. By 10. from Proposition 5.7, $(E[u] \cap C, E''[u''] \cap C'') \in \overline{\mathcal{P}}(B_1; B_2)$.
5. We show that if $\Omega(u) < \Omega(v)$, then $\Omega''(u'') < \Omega''(v'')$. The argument for $\Omega''(u'') < \Omega''(v'')$ implying $\Omega(u) < \Omega(v)$ is complete symmetric. Let $\Omega(u) \not\equiv_2 \Omega(v)$ and $\Omega(u) < \Omega(v)$. To argue towards contradiction, suppose $\Omega''(u'') > \Omega''(v'')$. Without loss of generality, assume $\Omega(u) = min(\{\Omega(y) \mid y \in D\})$ where

$$D := \{y \in C \mid \exists x \in C,\ y'', x'' \in C'' \text{ s.t. } (y, y''), (x, x'') \in B_1; B_2, \Omega(y) < \Omega(x), \Omega''(y'') > \Omega''(x'')\}.$$

   - Since $u'' \in C''$, it follows from the definition of clusters that there exists a path $u'' \to_{E''} u_1'' \to_{E''} \dots \to_{E''} u_n'' = v''$ such that $u_i'' \in C''$ for all $1 \leq i \leq n$. Since $(u', u'') \in B_2$, it follows from condition 4 that there exist a path $u' \to_{E'} u_1' \to_{E'} \dots \to_{E'} u_n'$ such that $u_i' \in C'$ and $(u_i', u_i'') \in B_2$ for all $1 \leq i \leq n$. In particular, $u_n' \in C'$, $v'' \in C''$ and $(u_n', v'') \in B_2$. By condition 5, $\Omega(u') > \Omega(u_n')$. Since $(u, u') \in B_1$, it follows from condition 4 that there exists a path $u \to_E u_1 \to_E \dots \to_E u_n$ such that $u_i \in C$, and $(u_i, u_i') \in B_1$ for all $1 \leq i \leq n$. In particular, $u_n \in C$, $u_n' \in C'$ and $(u_n, u_n') \in B_1$. By condition 5, $\Omega(u) > \Omega(u_n)$. Since $(u_n, u_n') \in B_1$ and $(u_n', v'') \in B_2$, it follows that $(u_n, v'') \in B_1; B_2$.

- Since $v \in C$ it follows from the definition of clusters that there exists a path $v \to_E v_1 \to_{E'} \dots \to_E v_m = u$ such that $v_i \in C$ for all $1 \le i \le n$. Since and $(v, v') \in B_1$, it follows from condition 4 that there exists a path $v' \to_{E'} v'_1 \to_{E'} \dots \to_{E'} v'_m$ such that $v'_i \in C'$ and $(v_i, v'_i) \in B_1$ for all $1 \le i \le m$. In particular, $u \in C$, $v'_m \in C'$ and $(u, v'_m) \in B_1$. Since B_1 is a bisimulation, it follows that $\Omega(v'_m) < \Omega(v')$. Since $(v', v'') \in B_2$, it follows from condition 4 that there exists a path $v'' \to_{E''} v''_1 \to_{E''} \dots \to_{E''} v''_m$ such that $v''_i \in C''$, and $(v'_i, v''_i) \in B_2$ for all $1 \le i \le m$. In particular, $v''_m \in C''$, $v'_m \in C'$ and $(v'_m, v''_m) \in B_2$. By condition 5, $\Omega(v''_m) < \Omega(v'')$. Since $(u, v'_m) \in B_1$ and $(v'_m, v''_m) \in B_2$, it follows that $(u, v''_m) \in B_1; B_2$.
- In the two bullet points above we have shown that $\Omega(u) > \Omega(u_n)$, $(u_n, v'') \in B_1; B_2$, $\Omega(v''_m) < \Omega(v'')$ and $(u, v''_m) \in B_1; B_2$. This means that $u_n \in D$. However, $\Omega(u_n) < \Omega(u)$, which contradicts our assumption that u has the smallest priority in D (red for B_1 and blue for B_2 in the demonstration below).



Thus, $\Omega''(u'') \not> \Omega''(v'')$. Since $\Omega''(u') \equiv_2 \Omega(u) \not\equiv_2 \Omega(v) \equiv_2 \Omega''(v'')$, it follows that $\Omega''(u'') \neq \Omega''(v'')$. This means that $\Omega''(u'') < \Omega''(v'')$.

To sum up, we have shown that $B_1; B_2 \in \mathsf{BaseB}(C, C'')$. □

Proposition 5.12 *Let $\mathbb{G} = (V, E, L, \Omega, v_I)$, $\mathbb{G}' = (V', E', L', \Omega', v'_I)$, and $\mathbb{G}' = (V'', E'', L'', \Omega'', v''_I)$ be three parity formulas. Let C, C' and C'' be non-degenerate clusters in $\mathbb{G}$, $\mathbb{G}'$ and $\mathbb{G}''$, respectively. Let $B_1 \in \mathsf{BaseB}(C, C')$ and $B_2 \in \mathsf{BaseB}(C', C'')$. Let $R_1 \in \mathsf{ExitB}(B_1)$ and $R_2 \in \mathsf{ExitB}(B_2)$. Then*

$$\big(R_1; (B_2 \cup R_2)\big) \cup \big((B_1 \cup R_1); R_2\big) \in \mathsf{ExitB}(B_1; B_2).$$

Proof. Let $(u, u') \in B_1$ and $(u', u'') \in B_2$. From the definition of exit bisimulations, we have

$$(E[u] \backslash C, E'[u']) \in \vec{\mathcal{P}} R_1 \tag{5.11}$$

and

$$(E'[u'], E''[u''] \backslash C) \in \overleftarrow{\mathcal{P}} R_2 \tag{5.12}$$

By the proof of Proposition 5.10, we have

$$(E[u], E'[u']) \in \overline{\mathcal{P}}(B_1 \cup R_1),$$

which implies

$$(E[u], E'[u']) \in \overleftarrow{\mathcal{P}}(B_1 \cup R_1), \tag{5.13}$$

and

$$(E'[u'], E''[u'']) \in \overline{\mathcal{P}}(B_2 \cup R_2),$$

which implies

$$(E'[u'], E''[u'']) \in \vec{\mathcal{P}}(B_2 \cup R_2), \tag{5.14}$$

Combining (5.11), (5.14) and 9. from Prop. 5.7, we have

$$(E[u] \backslash C, E''[u'']) \in \vec{\mathcal{P}}(R_1; (B_2 \cup R_2)) \tag{5.15}$$

Combining (5.12), (5.13) and 8. from Prop. 5.7, we have

$$(E[u], E''[u''] \backslash C) \in \overleftarrow{\mathcal{P}}((B_1 \cup R_1); R_2) \tag{5.16}$$

Combining (5.15) and 5. from Prop. 5.7, we have

$$(E[u], E''[u''] \backslash C) \in \overleftarrow{\mathcal{P}}\big((R_1; (B_2 \cup R_2)) \cup ((B_1 \cup R_1); R_2)\big). \tag{5.17}$$

Combining (5.16) and 6. from Prop. 5.7, we have

$$(E[u]\backslash C, E''[u'']) \in \overrightarrow{\mathcal{P}}\big((R_1;(B_2 \cup R_2)) \cup ((B_1 \cup R_1);R_2)\big) \tag{5.18}$$

Note that (5.17) and (5.18) is precisely the definition for

$$\big(R_1;(B_2 \cup R_2)\big) \cup \big((B_1 \cup R_1);R_2\big) \in \mathsf{ExitB}(B_1;B_2).$$

□

Bisimilarity Game: Definition

We now come to the definition of the bisimilarity game.

Definition 5.12 (Bisimilarity Game) *Let $\mathbb{G} = (V, E, L, \Omega, v_I)$ and $\mathbb{G}' = (V', E', L', \Omega', v'_I)$ be two parity formulas. The bisimilarity game $\mathcal{B}(\mathbb{G}, \mathbb{G}')$ is a parity game played by two players: $\exists$ (0) and $\forall$ (1). Positions and possible moves of each player are shown in Table 5.1. The game starts from position (v_I, v'_I). A player loses if it is their turn to move and the set of admissible moves is empty. Infinite matches are always won by $\forall$.*

Table 5.1: Positions and possible moves of each player in $\mathcal{B}(\mathbb{G}, \mathbb{G}')$

<table>
<tr><th colspan="3">Position</th><th>Player</th><th>Admissible moves</th></tr>
<tr><td rowspan="3">$(v, v') \in V \times V'$</td><td colspan="2">$L(v) \neq L'(v')$</td><td>$\exists$</td><td>$\varnothing$</td></tr>
<tr><td rowspan="2">$L(v) = L'(v')$</td><td>$C(v)$ or $C(v')$ degenerate</td><td>$\exists$</td><td>$\{(\varnothing, R) \mid R \in \mathsf{LocB}(v, v')\}$</td></tr>
<tr><td>$C(v)$ and $C(v')$ non-degenerate</td><td>$\exists$</td><td>$\{(\varnothing, R) \mid R \in \mathsf{LocB}(v, v')\}$ $\cup$ $\{(B, R) \mid (v, v') \in B \in \mathsf{BaseB}(C(v), C(v')),$ $R \in \mathsf{ExitB}(B)\}$</td></tr>
<tr><td colspan="3">$(B, R), B, R \subseteq V \times V'$</td><td>$\forall$</td><td>$R$</td></tr>
</table>

To form an intuitive understanding of this definition, recall the definition for local bisimulations and the bisimilarity game for Kripke models (c.f. Def. 3.2):

Definition 5.13 (Local Bisimulation) *Let $\mathbb{S} = \langle S, V, R\rangle$ and $\mathbb{S}'\langle S', V', R'\rangle$ be two Kripke models. Let $s \in V$ and $v' \in V'$. Let $\mathsf{LocB}^K(s, s')$ denote the collection of $Z \subseteq S \times S'$ such that $(R[s], R'[s']) \in \overline{\mathcal{P}}Z$. We call elements of $\mathsf{LocB}^K(s, s')$* local bisimulations *at (s, s').*

Definition 5.14 *Let $\mathbb{S} = \langle S, V, R\rangle$ and $\mathbb{S}'\langle S', V', R'\rangle$ be two Kripke models. The bisimilarity game $\mathcal{B}((\mathbb{S}, s_I), (\mathbb{S}', s'_I))$ is a parity game played by two players: $\exists$ and $\forall$. Positions and possible moves of each player are shown in Table 5.2. The game starts from position (s_I, s'_I). A player loses if it is their turn to move and the set of admissible moves is empty. Infinite matches are always won by $\exists$.*

Table 5.2: Positions and possible moves of each player in $\mathcal{B}(\mathbb{S}, \mathbb{S}')$

<table>
<tr><th colspan="2">Position</th><th>Player</th><th>Admissible moves</th></tr>
<tr><td rowspan="2">$(s, s') \in S \times S'$</td><td>$V(s) \neq V'(s')$</td><td>$\exists$</td><td>$\varnothing$</td></tr>
<tr><td>$V(s) = V'(s')$</td><td>$\exists$</td><td>$\{Z \subseteq S \times S' \mid Z \in \mathsf{LocB}^K(s, s')\}$</td></tr>
<tr><td colspan="2">$Z \subseteq S \times S'$</td><td>$\forall$</td><td>Z</td></tr>
</table>

One can notice some immediate similarities between the two definitions for bisimulation. To start with, both games process in an alternatioin between $\exists$'s turns and $\forall$'s turns, with similar admissible moves for each player: ∃ provides a partial bisimulation at each of her turn, and ∀ pick one element from that partial bisimulation. Also, local bismulations feature in both games as admissible moves for ∃. Furthermore, at a position that is a pair whose labeling/valuation does not complete match, ∃ immediately loses. These similarities reflect the shared idea behind the two definitions, which is to depict equivalence in global behaviors by a collection of equivalence in local behaviors.

However, the differences between these two definitions could help us understand Def. 5.12 even better. First, note that infinite games are always won by ∀ in Def. 5.12 while they are always won by ∃ in Def. 5.14. Second, sometimes it is allowed for ∃ to propose a base bisimulation together with a exit bismulation. These differences illustrate the dissimilar approaches of handling behaviors of infinite nature. Def. 5.14 is a classic example of defining bisimulation by coinduction, and as we have mentioned in the previous chapter, the resulting bisimilarity relation is a greatest fixpoint. This is in syn with the fact that infinite matches are won by ∃.

In contrast, Bisimilarity in the sense of Definition 4 is defined by induction with base bisimulations, which are defined by neither induction nor coinduction as the base case. As we have mentioned in the previous section, this results in the bisimilarity relation in the sense of Definition 4 being a least fixpoint. This means that from any two bisimilar nodes, ∃ can always force the token to a position that is in a base bisimulation or a dead end within finitely many steps. In the former case, ∃ can use an exit bisimulation exit at least one of the two clusters that the base bisimulation is based on. This can only happen finitely many times given the finite partial order among clusters in a parity formula that we will establish in the next section. In the latter case, ∀ loses immediately for being stuck. This is why infinite matches are always won by ∀ in Def. 5.12.

Bisimilarity Game: Strategy Graph

Before we delve into the details of the proof of adequacy for the bisimilarity game, we first define some auxilary concepts and proof their properties.

Definition 5.15 *Let* $\mathbb{G} = (V, E, L, \Omega, v_I)$ *and* $\mathbb{G}' = (V', E', L', \Omega', v_I')$ *be two parity formulas. Given a positional* ∃*-strategy,* $g : V \times V' \to (\mathcal{P}(V \times V'))^2$ *in* $\mathcal{B}(\mathbb{G}, \mathbb{G}')$*, define the strategy graph*

$$G = (N, B : N \to \mathcal{P}(V \times V'), R : N \to \mathcal{P}(V \times V'))$$

where $N = \{(u, u') \in V \times V' \mid (v_I, v_I')R^*(u, u')\}$ *and for each* $(u, u') \in N$*,* $B(u, u') = \pi_1(g(u, u'))$ *and* $R(u, u') = \pi_2(g(u, u'))$*.*

Note that for any $(u, u') \in N$, $R(u, u')$ is precisely the set of admissible moves for ∀ after ∃ make the suggested move $g(u, u') = (B(u, u'), R(u, u'))$. This means that N is precisely the collection of possibly positions for ∃ given that she plays according to strategy g. One implication of this observation is that if g is a winning strategy, then all the positions in N are winning for ∃. Another implication is that if g is a winning strategy, then G_g is acyclic.

Definition 5.16 *Let* $G = (N, B, R)$ *be an acyclic strategy graph.*

- *Let* $\leq_G$ *denote that pre-order such that, for any* $a, b \in N$*,* $a \leq_G b$ *if and only if* bR^*a*.*
- *Let* $=_G$ *denote that equivalence relation such that, for any* $a, b \in N$*,* $a =_G b$ *if and only if* $a = b$*.*
- *Let* $<_G$ *be the strict partial order such that, for any* $a, b \in N$*,* $a <_G b$ *if and only if* $a \leq_G b$ *and* $a \neq_G b$*.*

Let $G' = (N', B', R')$ *be another acyclic strategy graph.*

- *Let* $<_{G,G'}$ *denote the partial order over* $N \times N'$ *such that, for any* $(a, a'), (b, b') \in N \times N'$*, we have* $(a, a') <_{G,G'} (b, b')$ *if and only if* ($a \leq_G b$ *and* $a' <_{G'} b'$) *or* ($a <_G b$ *and* $a' \leq_{G'} b'$)*.*

Lemma 5.13 *Let $G = (N, B, R)$ be a strategy graph induced by a positional winning strategy. Then G is acyclic and $<_G$ is well-founded. Let $G' = (N', B', R')$ be a strategy graph induced by a positional winning strategy. Then $<_{G,G'}$ is well-founded.*

Proof. Since g is a winning $\exists$-strategy and every infinite match (which corresponds to a branch in G) is won by $\forall$, it follows that G has no infinite branches, and therefore is acyclic. Since N is finite and $<_G$ is a pre-order, it follows that $<_G$ is well-founded. Since both $<_G$ and $<_{G'}$ are well-founded, $<_{G,G'}$ is also well-founded. □

Bisimilarity Game: Adequacy

For the rest of this section, we prove the adequacy of the bisimularity game.

Proposition 5.14 *Let $\mathbb{G} = (V, E, L, \Omega, v_I)$ and $\mathbb{G}' = (V', E', L', \Omega', v_I')$ be two parity formulas and let $v \in V$ and $v' \in V'$. If $(v, v') \in \mathcal{A}^{\mathbb{G},\mathbb{G}'}$, then $(v, v') \in \mathsf{Win}_\exists(\mathcal{B}(\mathbb{G}\langle v \rangle, \mathbb{G}'\langle v' \rangle))$. In particular, if $(v, v') \in \mathcal{A}_0^{\mathbb{G},\mathbb{G}'}$, then$\exists$ has a positional winning strategy in $\mathcal{B}(\mathbb{G}\langle v \rangle, \mathbb{G}'\langle v' \rangle)$ that starts with a move (B, R) where $B \neq \varnothing$.*

Proof. Suppose $(v, v') \in \mathcal{A}$. The following describes a positional winning strategy for $\exists$,

$$g_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{P}^2(V \times V'),\ (x, x') \mapsto (B_{x,x'}, R_{x,x'}).$$

Suppose the token is at position $(x, x') \in V \times V'$ and $r_{\mathbb{G},\mathbb{G}'}(x, x') < +\infty$.

1. If $r_{\mathbb{G},\mathbb{G}'}(x, x') = i + 1$ for some $i \in \omega$, then $\exists$ chooses the move $(B_{x,x'} = \varnothing, R_{x,x'})$ where

$$R_{x,x'} := \{(u, u') \in \mathcal{A}_i \mid u \in E[x], u' \in E'[x']\}.$$

   By condition 5 of Definition 3.5, this is an admissible move. Note that here for any $(u, u') \in R_{x,x'}$, we have $r_{\mathbb{G},\mathbb{G}'}(u, u') < +\infty$, $(u, u') \leq_{\mathbb{G},\mathbb{G}'} (x, x')$ and $r_{\mathbb{G},\mathbb{G}'}(u, u') < r_{\mathbb{G},\mathbb{G}'}(x, x')$.

2. If $r_{\mathbb{G},\mathbb{G}'}(x, x') = 0$, then $\exists$ chooses the move $(B_{x,x'}, R_{x,x'})$ where $B_{x,x'} := \mathcal{A}_0 \upharpoonright_{C(x) \times C(x')}$ and

$$R_{x,x'} := \bigcup_{(u,u') \in B} \Big((E[u] \backslash C(u) \times E'[u']) \cup (E[u] \times E'[u'] \backslash C(u')\Big) \cap \mathcal{A}.$$

   By condition 3 and 6 of Definition 3.5, this is an admissible move. Note that here for any $(u, u') \in R_{x,x'}$, we have $r_{\mathbb{G},\mathbb{G}'}(u, u') < +\infty$ and $(u, u') <_{\mathbb{G},\mathbb{G}'} (x, x')$.

Since $(v, v') \in \mathcal{A}$, it follows that $r_{\mathbb{G},\mathbb{G}'}(v, v') < +\infty$. As we have seen, at any position $(x, x') \in \mathcal{A}$, the strategy above gives a move $(B_{x,x'}, R_{x,x'})$ such that $R_{x,x'} \subseteq \mathcal{A}$, so no matter which element $(u, u') \in R_{x,x'}$ $\forall$ chooses, we will have $r_{\mathbb{G},\mathbb{G}'}(u, u') < +\infty$. Thus, $g_{\mathcal{A}}$ is a well-defined strategy.

Now we show that this is a winning strategy. We have already shown that this strategy provides an admissible move at any reachable position where it is $\exists$'s turn to move. What is left to show is that the game always ends within finitely many steps. Note that at any position (x, x'), for all $(u, u') \in R_{x,x'}$, we have $((u, u'), r_{\mathbb{G},\mathbb{G}'}(u, u')) <_{\mathbb{G},\mathbb{G},\omega} ((x, x'), r_{\mathbb{G},\mathbb{G}'}(x, x'))$. Since $<_{\mathbb{G},\mathbb{G},\omega}$ is well-founded, it follows that each match is finite.

Finally, let us now consider the special case where $(v, v') \in \mathcal{A}_0$. In this case, $r_{\mathbb{G},\mathbb{G}'} = 0$. By the case b) above, we have $B := \mathcal{A}_0 \upharpoonright_{C(v) \times C(v')}$. Since $(v, v') \in \mathcal{A}_0 \upharpoonright_{C(v) \times C(v')}$, we know that $B \neq \varnothing$. This means that the positional winning strategy above provides (B, R) for some $B \neq \varnothing$ as the first move in $\mathcal{B}(\mathbb{G}\langle v \rangle, \mathbb{G}'\langle v' \rangle)$. Thus, if $(v, v') \in \mathcal{A}_0$, then $\exists$ has a positional winning strategy in $\mathcal{B}(\mathbb{G}\langle v \rangle, \mathbb{G}'\langle v' \rangle)$ that starts with a move (B, R) where $B \neq \varnothing$. □

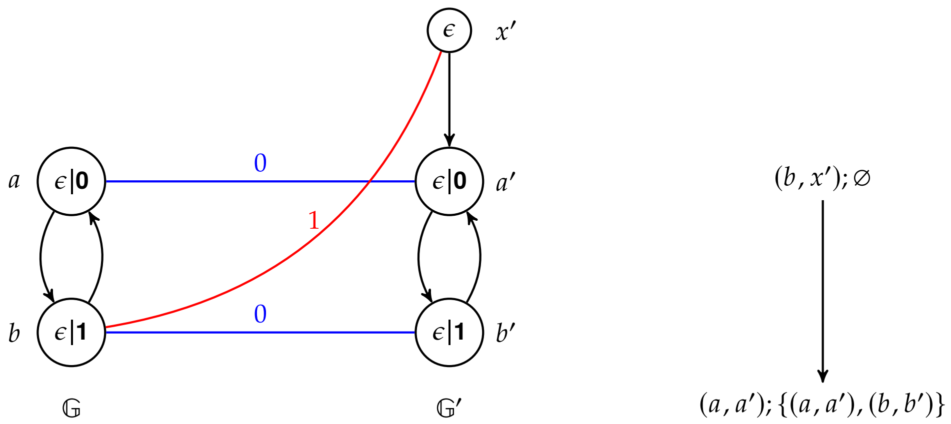


Figure 5.1: $\mathscr{A}^{\mathbb{G},\mathbb{G}'}$ and a strategy tree for $g_{\mathscr{A}^{\mathbb{G},\mathbb{G}'}}$

Example 5.1 Consider $\mathbb{G}$, $\mathbb{G}'$ and $\mathbb{G}''$ in Figure 5.1 and 5.2. The bisimilarity relation $\mathscr{A}^{\mathbb{G},\mathbb{G}'}$, $\mathscr{A}^{\mathbb{G}',\mathbb{G}''}$ are shown with the number denoting the rank of the link. The strategy tree for $g_{\mathscr{A}^{\mathbb{G},\mathbb{G}'}}$ and $g_{\mathscr{A}^{\mathbb{G}',\mathbb{G}''}}$ are shown in the same figures. The nodes in the strategy tree are shown in the format '$(u, u'); B(u, u')$', and R is represented by the solid directed arrows.

Proposition 5.15 *Let $\mathbb{G} = (V, E, L, \Omega, v_I)$ and $\mathbb{G}' = (V', E', L', \Omega', v'_I)$ be two parity formulas and let $v \in V$ and $v' \in V'$. Then $(v, v') \in \mathscr{A}^{\mathbb{G},\mathbb{G}'}$ if $(v, v') \in \mathsf{Win}_\exists(\mathscr{B}(\mathbb{G}\langle v\rangle, \mathbb{G}'\langle v'\rangle))$. In particular, $(v, v') \in \mathscr{A}_0^{\mathbb{G},\mathbb{G}'}$ if $\exists$ has a positional winning strategy in $\mathscr{B}(\mathbb{G}\langle v\rangle, \mathbb{G}'\langle v'\rangle)$ that starts with a move (B, R) where $B \neq \varnothing$.*

Proof. Suppose that $\exists$ has a positional winning strategy $g : V \times V' \longrightarrow (\mathscr{P}(V \times V'))^2$ in $\mathscr{B}(\mathbb{G}\langle v\rangle, \mathbb{G}'\langle v'\rangle)$. Let $G = (N, B, R)$ be the strategy graph induced by g. Define a bisimulation $(A_i)_{i\in\omega}$ from $\mathbb{G}$ to $\mathbb{G}'$ as follows.

$$A_0 := \bigcup_{(u,u')\in N} B(u, u'), \quad A_{i+1} := \{(u, u') \in N \mid R(u, u') \subseteq A_i\} \cup A_0$$

First, note that $A = N \cup \bigcup_{(u,u')\in N} B(u, u')$. We show this from both directions.

- Since G is acyclic and connected, and all the $(u, u') \in N$ that have no successors in G are in A_1 by definition, it is easy to show by induction on the structure of G that $(u, u') \in A$ for all $(u, u') \in N$. That is, $N \subseteq A$. By definition, $B(u, u') \subseteq A_0$ for all $(u, u') \in N$. Thus,
$$A \supseteq N \cup \bigcup_{(u,u')\in N} B(u, u').$$
- By definition, $A_0 \subseteq \bigcup_{(u,u')\in N} B(u, u')$ and $A_i \subseteq N \cup A_0$ for all $i > 0$. Thus
$$A \subseteq N \cup \bigcup_{(u,u')\in N} B(u, u').$$

We claim that $(A_i)_{i\in\omega}$ is a bisimulation in the sense of Definition 3.5.

1. It suffices to show that $A_i \subseteq A_{i+1}$ for all $i \in \omega$. We show this by induction on i. The base case where $i = 0$ is true by definition. For the induction step, suppose $(u, u') \in A_i$ for some $i > 0$. We make the following two case distinctions.
   - $(u, u') \in A_0$. Then by definition $(u, u') \in A_{i+1}$.
   - $(u, u') \notin A_0$. Since $(u, u') \notin A_0$ and $(u, u') \in A_i$, it must be the case that $i > 0$ and $R(u, u') \subseteq A_{i-1}$. By the induction hypothesis, $A_{i-1} \subseteq A_i$. This means that $R(u, u') \subseteq A_i$ and thus, $(u, u') \in A_{i+1}$.

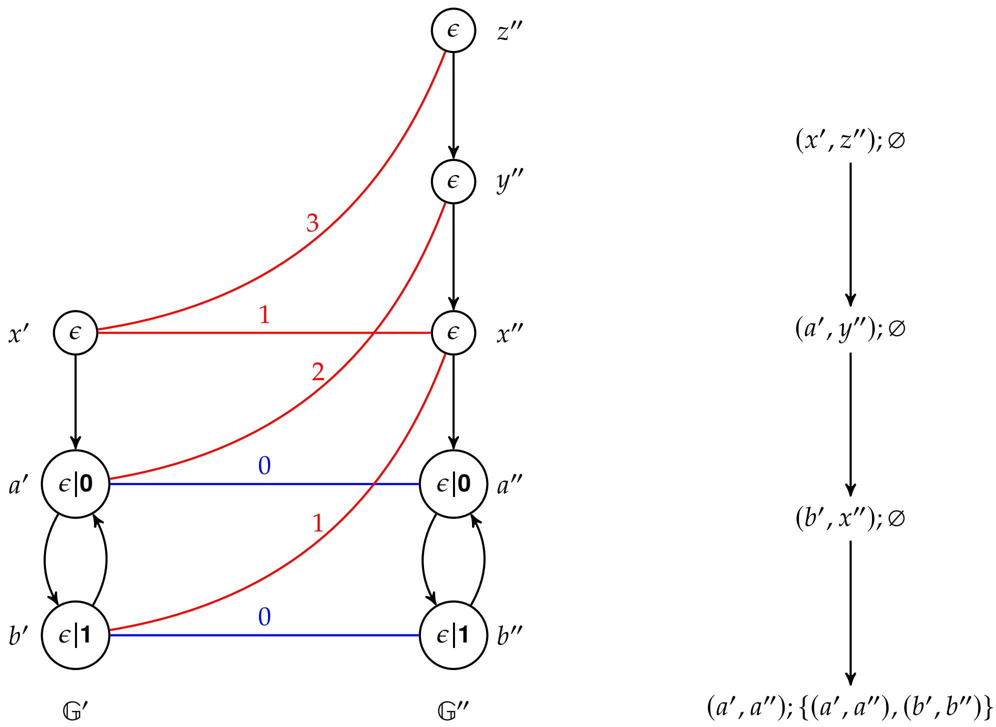


Figure 5.2: $\mathscr{A}^{\mathbb{G}',\mathbb{G}''}$ and a strategy tree for $g_{\mathscr{A}^{\mathbb{G}',\mathbb{G}''}}$

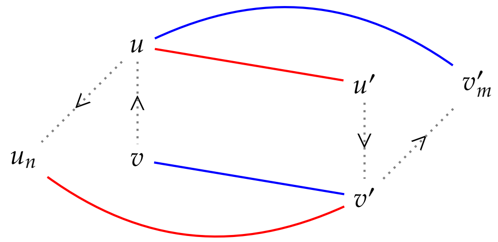
2. Note that if the token arrives at a position (u, u') where $L(u) \neq L'(u')$, then $\exists$ loses. As we have observed, if $G = (N, B, R)$ is the strategy graph induced by a winning strategy, then all positions in N are winning. This means that $L(u) = L'(u')$ for all $(u, u') \in N$. Also, from the definitioin of base bisimulations, it follows that that $L(u) = L'(u')$ for all $(u, u') \in B(w, w')$ for all $(w, w') \in N$. Since $A = N \cup \bigcup_{(w,w') \in N} B(w, w')$, it follows that $L(u) = L'(u')$ for all $(u, u') \in A$.

3. Let $(u, u') \in A_0$. According to the definition of $(A_i)_{i \in \omega}$, since A_0 is the union of base bisimulations, it must be the case that there exists $B \in \mathsf{BaseB}(C(u), C(u'))$ such that $(u, u') \in B$. By the definition of base bisimulations, we have that neither $C(u)$ nor $C(u')$ are degenerate, $u \in \mathsf{Dom}(\Omega) \Leftrightarrow u' \in \mathsf{Dom}(\Omega')$, and if $u \in \mathsf{Dom}(\Omega)$, then $\Omega(u) \equiv_2 \Omega'(u')$

4. Suppose $(u, u'), (x, x') \in A_0$, $u \in C(x)$, $u' \in C(x')$, and $\Omega(u) \not\equiv_2 \Omega(x)$. By the definition of A_0, there exist $n_1, n_2 \in N$ such that $(u, u') \in B(n_1)$ and $(x, x') \in B(n_2)$. Without loss of generality, suppose $\Omega(u) < \Omega(x)$. To argue towards contradiction, assume $\Omega'(u') > \Omega'(x')$. Without loss of generality, also suppose $\Omega(u) = min(\{\Omega(w) \mid w \in D_u\})$ where

$$D_u := \{w \in C(u) \mid (w, w') \in B(n_1), (x, x') \in B(n_2), \Omega(w) < \Omega(x), \Omega'(w') > \Omega'(x') \text{ for some } x \in C(u),\ w', x' \in C(u')\}).$$

   - Since $u' \in C(x')$ and $(u, u') \in B(n_1)$, it follows that there exist two paths $u' \to_{E'} u'_1 \to_{E'} \ldots \to_{E'} u'_n = x'$ and $u \to_E u_1 \to_E \ldots \to_E u_n$ for some $n \in \omega$ such that $u_i \in C(u)$, $u'_i \in C(u')$ and $(u_i, u'_i) \in B(n_1)$ for all $1 \leq i \leq n$. In particular, $u_n \in C(u)$, $x' \in C(u')$ and $(u_n, x') \in B(n_1)$. By the last condition for base bisimulations, $\Omega(u) > \Omega(u_n)$.
   - Since $x \in C(u)$ and $(x, x') \in B(n_2)$, it follows that there exist two paths $x' \to_{E'} v'_1 \to_{E'} \ldots \to_{E'} v'_m$ and $x \to_E v_1 \to_E \ldots \to_E v_m = u$ for some $m \in \omega$ such that $v_i \in C(x)$, $v'_i \in C(x')$ and $(v_i, v'_i) \in B(n_2)$ for all $1 \leq i \leq m$. In particular, $u \in C(x)$, $v'_m \in C(x')$ and $(u, v'_m) \in B(n_2)$. By the last condition for base bisimulations, $\Omega(x') > \Omega(v'_m)$.
   - So far, we have gathered enough facts to show that $u_n \in D_u$. However, $\Omega(u_n) < \Omega(u)$, which contradicts our assumption that u has the smallest priority in D_u (red for $B(n_1)$ and blue for $B(n_2)$ in the demonstration below).



Thus, $\Omega'(u') \not> \Omega'(x')$. Since $\Omega'(u') \equiv_2 \Omega(u) \not\equiv_2 \Omega(v) \equiv_2 \Omega'(x')$, it follows that $\Omega'(u') \neq \Omega'(x')$. This means that $\Omega'(u') < \Omega'(x')$.

5. Suppose $(u, u') \in A_{i+1} - A_0$. By definition, $R(u, u') \subseteq A_i$. Note that , for all $(w, w') \in N$, if $B(w, w') \neq \emptyset$, then by the definition of the admissible moves for $\exists$, we have $(w, w') \in B(w, w')$. This means that for all $(w, w') \in N$ such that $B(w, w') \neq \emptyset$, we have $(w, w') \in A_0$. Since $(u, u') \ni nA_0$, it follows that $B(u, u') = \emptyset$. By definition, this means that $R(u, u') \in \mathsf{LocB}(u, u')$, and thus, $(E[u], E'[u']) \in \overline{\mathcal{P}}(R(u, u'))$. Given that $\overline{\mathcal{P}}$ is monotone and that $R(u, u') \subseteq A_i$, we have $(E[u], E'[u']) \in \overline{\mathcal{P}}(A_i)$, which is precisely condition 5.

6. Suppose $(u, u') \in A_0$. According to the definition of A_0, there exists $(w, w') \in N$ such that $(u, u') \in B(w, w')$. Since $B(w, w')$ is non-empty, it follows that $B(w, w')$ is a base bisimulation. Furthermore, by Proposition 5.9, $B(w, w')$ is a base bisimulation at $(C(u), C(u'))$. By the condition 4 of the definition for base bisimulations, we have

$$(E[u] \cap C(u), E'[u'] \cap C(u')) \in \overline{\mathcal{P}}(B(w, w')).$$

Given the fact that $\overline{\mathcal{P}}$ is monotone and that $B(w, w') \subseteq A_0$, we have

$$(E[u] \cap C(u), E'[u'] \cap C(u')) \in \overline{\mathcal{P}}A_0.$$

This means that the conditions 6a) and 6b) are satisfied.
Now we show that the conditions 6c) and 6d) are also satisfied. Since $B(w, w') \neq \emptyset$, it follows from the admissive moves that $R(w, w') \in \mathsf{ExitB}(B(w, w'))$. Since $(u, u') \in B(w, w')$, it follows from the proof of Proposition 5.10 that

$$(E[u], E'[u']) \in \overline{\mathcal{P}}(R(w, w') \cup B(w, w')). \tag{5.19}$$

Note that

$$R(w, w') \cup B(w, w') \subseteq N \cup B(w, w') \subseteq N \cup \bigcup_{(x,x') \in N} B(x, x') = A \tag{5.20}$$

Combining 5.19 and 5.20 and 5. from Propositioin 5.7, we have

$$(E[u], E'[u']) \in \overline{\mathcal{P}}A.$$

This means that the conditions 6c) and 6d) are satisfied.

To sum up, we have shown that $(A_i)_{i \in \omega}$ is a bisimulation in the sense of Definition 4. Note that

$$(v, v') \in N \subseteq A \subseteq \mathcal{A}.$$

Thus, $(v, v') \in \mathcal{A}$ if $(v, v') \in \mathsf{Win}_\exists(\mathcal{B}(\mathbb{G}\langle v \rangle, \mathbb{G}'\langle v' \rangle))$.

Finally, let us now consider the special case where $B(v, v') \neq \emptyset$. In this case, by the definition of base bisimulations, we have $(v, v') \in B(v, v')$, and by the definition of $(A_i)_{u \in \omega}$, we have $B(v, v') \subseteq A_0$. Note that $A_0 \subseteq \mathcal{A}_0$. Thus, $(v, v') \in \mathcal{A}_0$ if $\exists$ has a positional winning strategy in $\mathcal{B}(\mathbb{G}\langle v \rangle, \mathbb{G}'\langle v' \rangle)$ that starts with a move (B, R) where $B \neq \emptyset$. ☐

Composition

Now that we have proven the adequacy of the bisimilarity game, we can show that Definition 4 is closed under composition. An example that demonstrates the construction described in proof of the following proposition can be found in Appendix B.

Proposition 5.16 *Given three parity formulas $\mathbb{G} = (V, E, L, \Omega, v_I)$, $\mathbb{G}' = (V', E', L', \Omega', v_I')$ and $\mathbb{G}'' = (V'', E'', L'', \Omega'', v_I'')$, we have $\mathcal{A}^{\mathbb{G},\mathbb{G}'}; \mathcal{A}^{\mathbb{G}',\mathbb{G}''} \subseteq \mathcal{A}^{\mathbb{G},\mathbb{G}''}$.*

Proof. We show an equivalent proposition: $\mathcal{A}^{\mathbb{G},\mathbb{G}'}; \mathcal{A}^{\mathbb{G}',\mathbb{G}''} \subseteq \mathcal{A}^{\mathbb{G},\mathbb{G}''}$. Let $(v, v') \in \mathcal{A}^{\mathbb{G},\mathbb{G}'}$ and $(v', v'') \in \mathcal{A}^{\mathbb{G}',\mathbb{G}''}$.

Strategy. Like some of the proofs we have seen in this thesis, we prove the statement by constructing a winning $\exists$-strategy in $\mathcal{B}(\mathbb{G}\langle v\rangle, \mathbb{G}''\langle v''\rangle)$ employing the technique of 'shadow playing'. Recall that 'shadow playing' is a technique used for constructing winning strategy for a player Π in a two-player game, which we call the 'main game', utilizing given winning strategies for Π in one or more 'shadow games'. While playing a match in the main game, player Π also simultaneously updates the position of the token in each of the shadow games. At each of her turns, she plays according to the current positions of the tokens in the shadow games as well as the given winning strategies for the shadow games. After either player has moved the token in the main game, she moves the token in the shadow games accordingly. Specific ways in which player Π performs these actions may differ given the different circumstances under which the shadow playing technique is applied.

In this specific case, the main game is $\mathcal{B}(\mathbb{G}\langle v\rangle, \mathbb{G}''\langle v''\rangle)$ and we utilize two shadow games, $\mathcal{B}(\mathbb{G}\langle v\rangle), \mathbb{G}'\langle v'\rangle)$ and $\mathcal{B}(\mathbb{G}'\langle v'\rangle, \mathbb{G}''\langle v''\rangle))$. Since $(v, v') \in \mathcal{A}^{\mathbb{G},\mathbb{G}'}$ and $(v', v'') \in \mathcal{A}^{\mathbb{G}',\mathbb{G}''}$, it follows from Proposition 5.14 that $\exists$ has a positional winning strategy $g : V \times V' \longrightarrow \mathcal{P}^2(V \times V')$ in $\mathcal{B}(\mathbb{G}\langle v\rangle, \mathbb{G}'\langle v'\rangle)$ and a positional winning strategy $g' : V' \times V'' \longrightarrow \mathcal{P}^2(V' \times V'')$ in $\mathcal{B}(\mathbb{G}'\langle v'\rangle, \mathbb{G}''\langle v''\rangle)$. Recall that in a bisimilarity game, $\exists$'s and $\forall$'s turns alternates and the first turn always belongs to $\exists$. In our construction, the way $\exists$ updates the positions of tokens in the shadow games depends on $\forall$'s moves in the main game, so $\exists$ does not update the positions of the tokens after her own moves, but only after $\forall$'s moves. When she updates the position of either of the two shadow tokens, she either does not move it, or she moves it from a $\exists$-position to another $\exists$-position, omitting the intermediate situation where the tokens are at $\forall$-positions. The key to this construction is that we make a shadow token wait in the base bisimulation after it arrives at one, if the other shadow token has not arrived at a base bisimulation. The specific structure of a match in $\mathcal{B}(\mathbb{G}'\langle v'\rangle, \mathbb{G}''\langle v''\rangle)$ where $\exists$ employs the technique of 'shadow playing' is shown in Figure 5.3. We call each node in Figure 5.3 a *stage*. The following is a summary of the stages in Figure 5.3.

INIT:	$\exists$ sets the initial positions for the two shadow tokens, (x, x') and (y', y''), as well as a helper node $u_0' \in V$.
PLAY$_\exists$:	$\exists$ makes a move from position (u, u'') in the main game according to $g(x, x')$ and $g'(y, y')$.
PLAY$_\forall$:	$\forall$ makes a move.
UPDATE:	$\exists$ picks out u'. With the help of u', $\exists$ updates (x, x') and (y', y'').

We use (u_i, u_i'') to denote the position of the token in the main game after $\forall$ has made i moves, and we use $(x_i.x_i')$ and (y_i', y_i'') to denote positions of the shadow tokens after i times UPDATE. We call the process of going from stage 2 to stage 3, then to stage 4, and then back to stage 2, one *step*. Now we flesh out the details of INIT, PLAY and UPDATE in the $i + 1$'th step. Let $G_g = (N_g, B_g, R_g)$ and $G_{g'} = (N_{g'}, B_{g'}, R_{g'})$ be the strategy graph induced by g and g', respectively.

INIT: At the start of the game, $\exists$ sets $x_0 := v$, $x_0' = y_0' := v'$, $y_0'' := v''$ and $u_0' = v'$. Note that in the begining of the the main game, $u_0 = v$ and $u_0'' = v''$.

PLAY$_\exists$: In the $i + 1$'th step, $\exists$ moves the token to (B^i, R^i). Here we assume

a) $(x_i, x_i') \in N_g$ and $(y_i', y_i'') \in N_{g'}$;

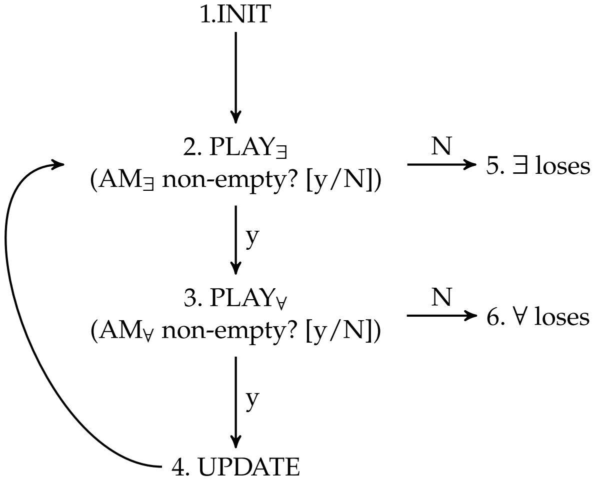


Figure 5.3: Structure of a match with 'shadow playing'; AM_Π=the set of admissible moves for player Π

With these assumptions, we can define

$$B^i_g := B_g(x_i, x'_i), \quad B^i_{g'} := B_{g'}(y'_i, y''_i), \quad R^i_g := R_g(x_i, x'_i), \quad R^i_{g'} := R_{g'}(y'_i, y''_i)$$

We also assume

b) if $(x_i, x'_i) \neq (u_i, u'_i)$, then $(u_i, u'_i) \in B^i_g$, and if $(y'_i, y''_i) \neq (u'_i, u''_i)$, then $(u'_i, u''_i) \in B^i_{g'}$;
c) if $B^i_g \neq \emptyset$, then $(u_i, u'_i) \in B^i_g$; if $B^i_{g'} \neq \emptyset$, then $(u'_i, u''_i) \in B^i_{g'}$.

It is easy to see that these assumptions are true when $i = 0$.

a) True by definition.
b) True since the antecidents are true.
c) If $B_g(v, v') = B^0_g \neq \emptyset$, then by the definition of admissible moves, we have $(u_0, u'_0) = (v, v') \in B_g(v, v') = B^0_g$.

Later, in the description of UPDATE, we will see that if these assumptions hold true for $i = n$, then they also hold true for $i = n + 1$.

Define $B^i := B^i_g; B^i_{g'}$. We make the following case distinction for defining R^i. We also show for each case that (B^i, R^i) is an admissible move for $\exists$ at (u_i, u''_i).

- $B^i_g = B^i_{g'} = \emptyset$. Define $R^i := R^i_g; R^i_{g'}$.
  Since $B^i_g = B^i_{g'} = \emptyset$, it follows from $b)$ that $(x_i, x'_i) = (u_i, u'_i)$ and $(y'_i, y''_i) = (u'_i, u''_i)$. By the definition of the admissible moves, $R^i_g \in \mathsf{LocB}(x_i, x'_i)$ and $R^i_{g'} \in \mathsf{LocB}(y'_i, y''_i)$. Combining these two observations, we have $R^i_g \in \mathsf{LocB}(u_i, u'_i)$ and $R^i_{g'} \in \mathsf{LocB}(u'_i, u''_i)$. By Propostion 5.8, $R^i \in \mathsf{LocB}(u_i, u''_i)$ and therefore, (B^i, R^i) an admissible move for $\exists$ at (u_i, u''_i).
- $B^i_g, B^i_{g'} \neq \emptyset$. Define $R^i := (R^i_g; (B^i_{g'} \cup R^i_{g'})) \cup ((B^i_g \cup R^i_g); R^i_{g'})$.
  Since $B^i_g \neq \emptyset$, it follows from $c)$ that $(u_i, u'_i) \in B^i_g$. By Proposition 5.9, $B^i_g \in \mathsf{BaseB}(C(u_i), C(u'_i))$. Similarly, $B^i_{g'} \in \mathsf{BaseB}(C(u'_i), C(u''_i))$. By Proposition 5.11, $B^i i \in \mathsf{BaseB}(C(u_i), C(u''_i))$. By Proposition 5.12, $R^i \in \mathsf{ExitB}(B_i)$. Therefore, (B^i, R^i) an admissible move for $\exists$ at (u_i, u''_i).
- $B^i_g = \emptyset, B^i_{g'} \neq \emptyset$. Define $R^i := R^i_g; (B^i_{g'} \cup R^i_{g'})$.
  i. Since $B^i_{g'} \neq \emptyset$, it follows form $c)$ that $(u'_i, u''_i) \in B^i_{g'}$. By Proposition 5.10, $B^i_{g'} \cup R^i_{g'} \in \mathsf{LocB}(u'_i, u''_i)$.
  ii. Since $B^i_g = \emptyset$, it follows from $b)$ that $(x_i, x'_i) = (u_i, u'_i)$, and from the definition of admissible moves that $R^i_g \in \mathsf{LocB}(x_i, x'_i)$. This means that $R^i_g \in \mathsf{LocB}(u_i, u'_i)$.

  By Propostion 5.8, $R^i = R^i_g; (B^i_{g'} \cup R^i_{g'}) \in \mathsf{LocB}(u_i, u''_i)$, and therefore, (B^i, R^i) is an admissible move for $\exists$ at (u_i, u''_i).

- $B^i_g \neq \emptyset, B^i_{g'} = \emptyset$. Define $R^i := (B^i_g \cup R^i_g); R^i_{g'}$.
  i. Since $B^i_{g'} \neq \emptyset$, it follows form *c*) that $(u_i, u'_i) \in B^i_g$. By Proposition 5.10, $B^i_g \cup R^i_g \in \mathsf{LocB}(u_i, u'_i)$.
  ii. Since $B^i_{g'} = \emptyset$, it follows from *b*) that $(y'_i, y''_i) = (u'_i, u''_i)$, and from the definition of admissible moves that $R^i_{g'} \in \mathsf{LocB}(y'_i, y''_i)$. This means that $R^i_{g'} \in \mathsf{LocB}(u'_i, u''_i)$.

  By Propostion 5.8, $R^i = (B^i_g \cup R^i_g); R^i_{g'} \in \mathsf{LocB}(u_i, u''_i)$, and therefore, (B^i, R^i) is an admissible move for $\exists$ at (u_i, u''_i).

PLAY$\forall$: $\forall$ picks $(u_{i+1}, u''_{i+1}) \in R^i$.

UPDATE. In the $i+1$'th step, $\exists$ picks out a $u'_{i+1} \in V'$ in a certain way. With the help of u'_{i+1}, $\exists$ updates the positions of the two shadow tokens to (x_{i+1}, x'_{i+1}) and (y'_{i+1}, y''_{i+1}). The following case distinction are made for the definition of u'_{i+1}. Note that since $(u_{i+1}, u''_{i+1}) \in R^i \neq \emptyset$, it follows that, in each case distinction, a node in V' that satisfies the description always exists.

- $B^i_g = B^i_{g'} = \emptyset$. Let u'_{i+1} be a node in V' such that $(u_{i+1}, u'_{i+1}) \in R^i_g$ and $(u'_{i+1}, u''_{i+1}) \in R^i_{g'}$.
- $B^i_g, B^i_{g'} \neq \emptyset$. Let u'_{i+1} be a node in V' such that $(u_{i+1}, u'_{i+1}) \in R^i_g$ and $(u'_{i+1}, u''_{i+1}) \in B^i_{g'} \cup R^i_{g'}$, or $(u_{i+1}, u'_{i+1}) \in B^i_g \cup R^i_g$ and $(u'_{i+1}, u''_{i+1}) \in R^i_{g'}$.
- $B^i_g = \emptyset, B^i_{g'} \neq \emptyset$. Let u'_{i+1} be a node in V' such that $(u_{i+1}, u'_{i+1}) \in R^i_g$ and $(u'_{i+1}, u''_{i+1}) \in B^i_{g'} \cup R^i_{g'}$.
- $B^i_g \neq \emptyset, B^i_{g'} = \emptyset$. Let u'_{i+1} be a node in V' such that $(u_{i+1}, u'_{i+1}) \in B^i_g \cup R^i_g$ and $(u'_{i+1}, u''_{i+1}) \in R^i_{g'}$.

Now we update (x, x') and (y', y'') with the help of u'_{i+1}.

$$(x_{i+1}, x'_{i+1}) := \begin{cases} (u_{i+1}, u'_{i+1}), & (u_{i+1}, u'_{i+1}) \in R^i_g \\ (x_i, x'_i), & (u_{i+1}, u'_{i+1}) \in B^i_g \backslash R^i_g, \end{cases} \quad (y'_{i+1}, y''_{i+1}) := \begin{cases} (u'_{i+1}, u''_{i+1}), & (u'_{i+1}, u''_{i+1}) \in R^i_{g'} \\ (y'_i, y''_i), & (u'_{i+1}, u''_{i+1}) \in B^i_{g'} \backslash R^i_{g'}. \end{cases}$$

Now we show that if *a*)-*c*) are true for $i = n$, then they are also true for $i = n + 1$. Suppose *a*)-*d*) are true for $i = n$. We only show the first half of each statement since the argument for the other half is complete analoguous.

a) We make the following two case distinctions.
   i. $(u_{i+1}, u'_{i+1}) \in R^i_g$. In this case, $(x_{i+1}, x'_{i+1}) = (u_{i+1}, u'_{i+1})$. Since $(u_{i+1}, u'_{i+1}) \in R^i_g \subseteq N_g$, it follows that $(x_{i+1}, x'_{i+1}) \in N_g$.
   ii. $(u_{i+1}, u'_{i+1}) \in B^i_g \backslash R^i_g$. By the induction hypothesis, $(x_i, x'_i) \in N_g$, and $(x_{i+1}, x'_{i+1}) = (x_i, x'_i) \in N_g$.
b) Suppose $(x_{i+1}, x'_{i+1}) \neq (u_{i+1}, u'_{i+1})$, then $(x_{i+1}, x'_{i+1}) = (x_i, x'_i)$ and $(u_{i+1}, u'_{i+1}) \in B^i_g \backslash R^i_g$. The latter implies that $B^i_g \neq \emptyset$. This means that $B^{i+1}_g = B_g(x_{i+1}, x'_{i+1}) = B_g(x_i, x'_i) = B^i_g \neq \emptyset$.
c) Suppose $B^{i+1}_g \neq \emptyset$. In other words, $B_g(x_{i+1}, x'_{i+1}) \neq \emptyset$. By the definition of admissible moves, this means that $(x_{i+1}, x'_{i+1}) \in B_g(x_{i+1}, x'_{i+1})$. We make the following two case distinctions.
   i. $(u_{i+1}, u'_{i+1}) \in R^i_g$. In this case, $(x_{i+1}, x'_{i+1}) = (u_{i+1}, u'_{i+1})$. Then

   $$(u_{i+1}, u'_{i+1}) = (u_{i+1}, u'_{i+1}) \in B_g(x_{i+1}, x'_{i+1}) = B^{i+1}_g$$

   ii. $(u_{i+1}, u'_{i+1}) \in B^i_g \backslash R^i_g$. In this case, $(x_{i+1}, x'_{i+1}) = (x_i, x'_i)$. Then

   $$(u_{i+1}, u'_{i+1}) \in B^i_g = B_g(x_i, x'_i) = B_g(x_{i+1}, x'_{i+1}) = B^{i+1}_g.$$

Correctness. We claim that the strategy we have just described is a winning strategy. We have already shown that this strategy provides an admissible move at any reachable position where it is $\exists$'s turn to move. What is left to show is that the game always ends within finitely many steps. We show this by pointing out that $\big((x_i, x'_i), (y'_i, y''_i)\big)_{i \text{ applicable}}$ forms a strictly decreasing sequence in terms of $<_{G_g, G_{g'}}$.

Since both g and g' are winning strategies for $\exists$, as we have observed before, both G_g and $G_{g'}$ are acyclic. By definition, $(z, z') <_{G_g} (w, w')$ for all $(z, z'), (w, w') \in N_g$ such that $(w, w') R_g (z, z')$. Recall that $R^i_g := R_g(x_i, x'_i)$. When comparing (x_{i+1}, x'_{i+1}) and (x_i, x'_i), we distinguish the following two cases.

1. If $(u_{i+1}, u'_{i+1}) \in R^i_g$, then $(u_{i+1}, u'_{i+1}) \in R_g$, and therefore, $(x_{i+1}, x'_{i+1}) <_{G_g} (x_i, x'_i)$.
2. If $(u_{i+1}, u'_{i+1}) \in B^i_g \backslash R^i_g$, then $(x_{i+1}, x'_{i+1}) = (x_i, x'_i)$, which implies $(x_{i+1}, x'_{i+1}) =_{G_g} (x_i, x'_i)$.

Similarly, we can deduce that

3. If $(u'_{i+1}, u''_{i+1}) \in R^i_{g'}$, then $(y'_{i+1}, y''_{i+1}) <_{G_{g'}} (y'_i, y''_i)$.
4. If $(u_{i+1}, u'_{i+1}) \in B^i_{g'} \backslash R^i_{g'}$, then $(y'_{i+1}, y''_{i+1}) =_{G_{g'}} (y'_i, y''_i)$.

Finally, observe that, due to the definition of R^i, for all applicable i, either $(u_{i+1}, u'_{i+1}) \in R^i_g$ or $(u'_{i+1}, u''_{i+1}) \in R^i_{g'}$. Summarizing these facts, we have arrive at the conclusion that, for all i,

$$\big((x_{i+1}, x'_{i+1}), (y'_{i+1}, y''_{i+1})\big) <_{G_g, G_{g'}} \big((x_i, x'_i), (y'_i, y''_i)\big).$$

Since $<_{G_g, G_{g'}}$ is well-founded, the match ends after finitely many steps. □

6 Conclusion and Future Work

This thesis is an investigation into how to define the notion of bisimulations over parity formulas. We provided and argued for a list of criteria against which we could judge how good such a definition is. In general, a notion of bisimulation should be sound, closed under union and composition, easily decidable and as close to being complete as possible. It should also guarantee the existence of a largest bisimulation, namely the bisimilarity relation. Particular to the situation with parity formulas, a good bisimulation should also have a 'relative flavor' in its handling of the priority condition. We proposed four definitions of bisimulations over parity formulas and evaluated them according to those criteria. We especially argued for Definition 4 to be the best by far, since it satisfies all qualitative criteria and lies in a relatively good position on the 'spectrum of completeness'. We also provided an adequate bisimilarity game for Definition 4 which makes it easier to work with the notion.

In terms of future work, it would be interesting to look into whether Definition 4 could give us a way to take quotients of, or minimize, parity formulas. It also remains to be investigated what notion of equivalence it would give us if we call two $\mu\mathsf{ML}$ formulas equivalent if their parity formula translation are bisimilar. Given the close relation among parity formulas, parity games and parity automata, it would not be surprising if Definition 4 could be translated to the setting of the latter two.

Recall that, in Section 4 of [6], the authors implemented a sound definition of bisimulation over parity games, utilizing the idea of power bisimulation [7]. In a power bisimulation, for two nodes to be bisimilar, it is not enough for their successors to satisfy the canonical 'zig-zag' condition, but rather, there must be for both nodes in question a particular set of nodes that together satisfy the 'zig-zag' condition. It is left to be investigated whether we can employ power bisimulation similarly to provide a notion of bisimulation over parity formulas.

The final note is on a more conceptual level. Note that the best notion we have come up with so far uses inductive definition and a non-trivial base case. It is not clear how the base case itself could be represented in an inductive or co-inductive way. This goes against the strong association between bisimulation and co-induction. What this shows is that parity formulas are not coalgebras and the priority condition of parity formulas is intrinsically a global condition. More works needs to be done to capture the essence of such structures, whose winning/accepting condition has a global flavor.

Appendix

A

Example for Semantics of Parity Formulas

Consider the parity formula $\mathbb{G}$ shown in Example 2.1. As we have mentioned, this parity formula is equivalent to the modal μ-formula $\mu x.\Box x \vee p$. The latter is usually interpreted to mean 'on any infinite path, p eventually holds. Consider pointed Kripke models

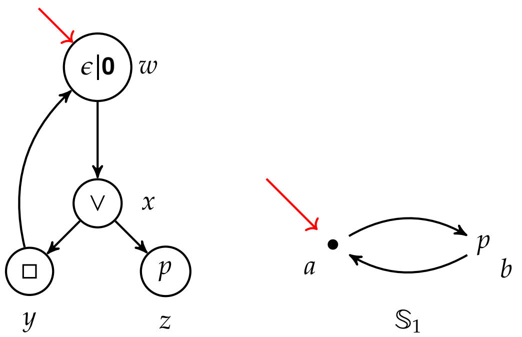
$$\mathbb{S}_1 = (S_1 = \{a, b\}, V_1 = \{(p, \{b\})\}, R_1 = \{(a, b), (b, a)\})$$

and

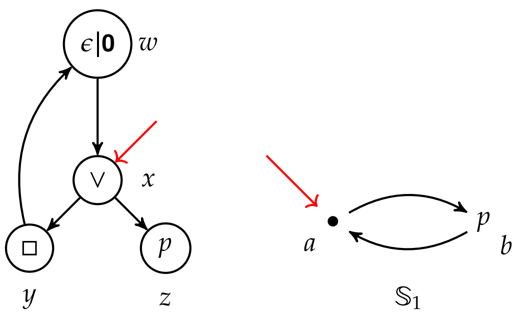
$$\mathbb{S}_2 = (S_2 = \{c, d\}, V_2 = \varnothing, R_2 = \{(c, d), (d, c)\}).$$

We show $\mathcal{E}(\mathbb{G}, (\mathbb{S}_1, a))$ and $\mathcal{E}(\mathbb{G}, (\mathbb{S}_2, c))$ as follows.

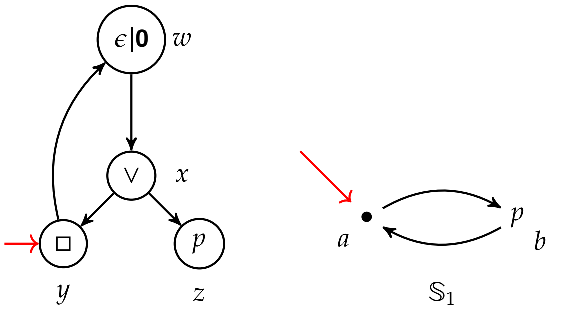
This is the start position (w, a) for $\mathcal{E}(\mathbb{G}, (\mathbb{S}_1, a))$.



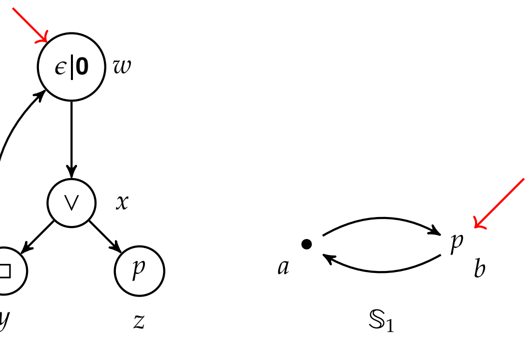
Since $L(w) = \epsilon$, by definition it is neither of the players' turn and the token moves on automatically to the only legitimate successor (x, a).



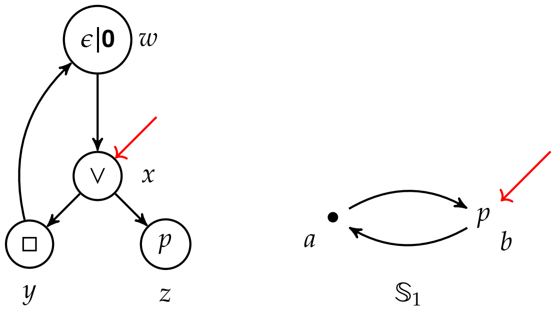
Since $L(x) = \vee$, it is $\exists$'s turn, and she can choose to move the token to (y, a) or (z, a). Note that (z, a) is a dead end. Since $L(z) = p$ and $a \notin V(p)$, it follows that it is $\exists$'s turn again at (z, a). This means that $\exists$ will lose immediately for being stuck. So any winning strategy f, if there is any, will suggests that $\exists$ goes to (y, a). So the token moves to (y, a).



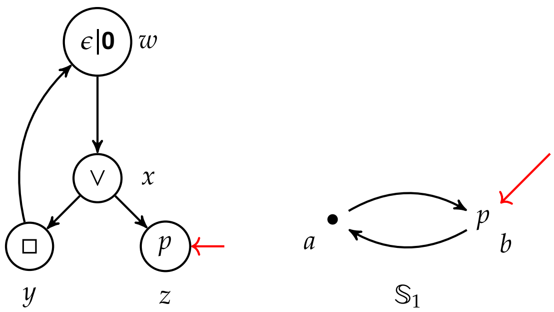
Since $L(y) = \Box$, it follows that it is ∀'s turn now. Note that (y, a) has only one successor (w, b), so ∀ moves the token to (w, b).



Since $L(w) = \epsilon$, by definition it is neither of the players' turn and the token moves on automatically to the only legitimate successor (x, b).



Since $L(x) = \vee$, it is ∃'s turn, and she can choose to move the token to (y, b) or (z, b). Note that (z, b) is a dead end. Since $L(z) = p$ and $b \in V(p)$, it follows that it is ∀'s turn again at (z, b). This means that at (z, b), ∃ will win immediately for making ∀ stuck.

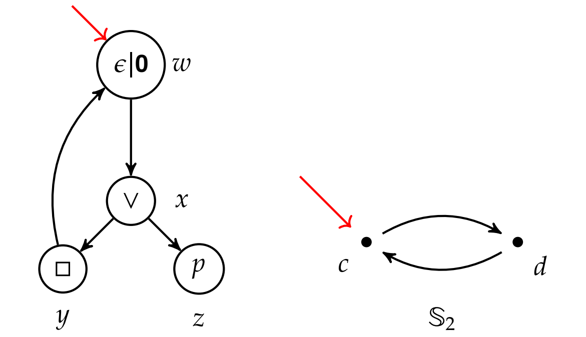


So ∃ moves the token to (z, b) and wins.

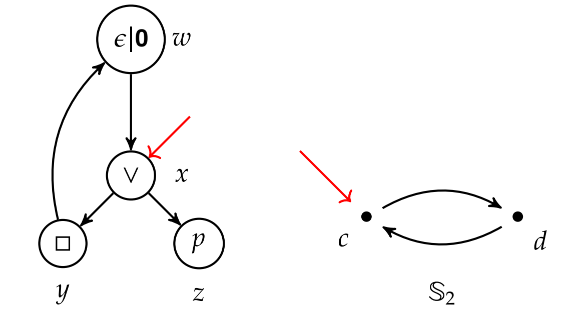
This analysis shows that $\mathbb{S}_1, a \models \mathbb{G}$. This fits our expectation because $\mathbb{S}_1, a \models \mu x.\Box x \vee p$.

Now let us look at $\mathscr{E}(\mathbb{G}, (\mathbb{S}_2, c))$.

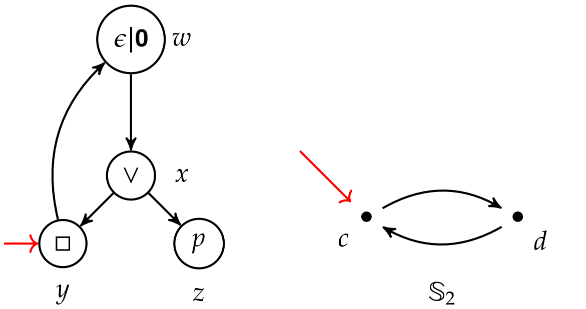
Following is the start position (w, c) for $\mathscr{E}(\mathbb{G}, (\mathbb{S}_2, c))$. Since $L(w) = \epsilon$, by definition it is neither of the players' turn and the token moves on automatically to the only legitimate successor (x, c).



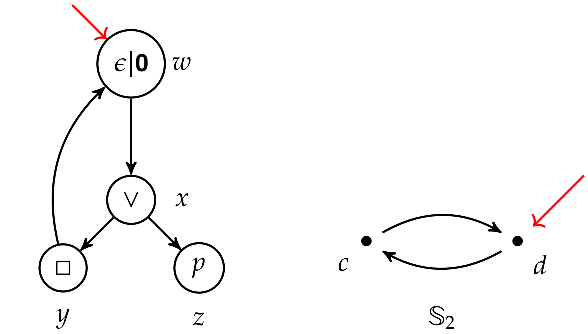
Since $L(x) = \vee$, it is $\exists$'s turn, and she can choose to move the token to (y, c) or (z, c). Note that (z, c) is a dead end. Since $L(z) = p$ and $c \notin V(p)$, it follows that it is $\exists$'s turn again at (z, c). This means that $\exists$ will lose immediately for being stuck. So any winning strategy f, if there is any, will suggests that $\exists$ goes to (y, c). So the token moves to (y, c).



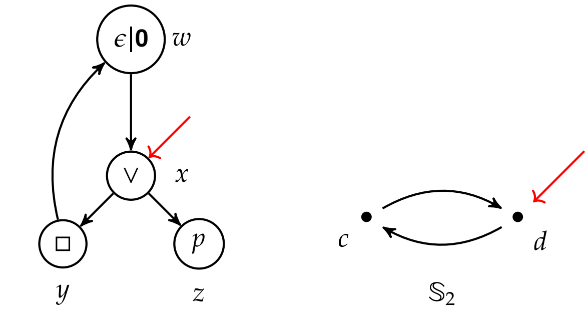
Since $L(y) = \Box$, it follows that it is $\forall$'s turn now. Note that (y, c) has only one successor (w, d), so $\forall$ moves the token to (w, d).



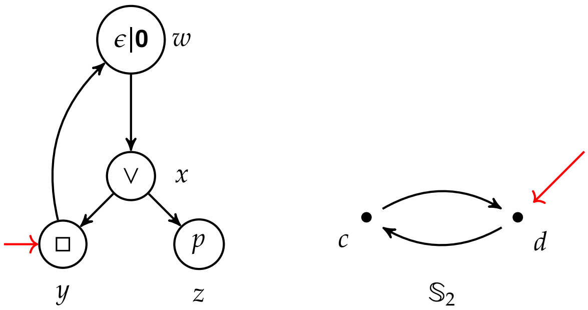
Since $L(w) = \epsilon$, by definition it is neither of the players' turn and the token moves on automatically to the only legitimate successor (x, d).



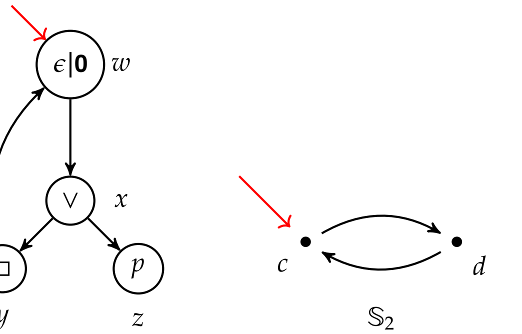
Since $L(x) = \vee$, it is $\exists$'s turn, and she can choose to move the token to (y, d) or (z, d). Note that (z, d) is a dead end. Since $L(z) = p$ and $d \notin V(p)$, it follows that it is $\exists$'s turn again at (z, d). This means that $\exists$ will lose immediately for being stuck. So any winning strategy f, if there is any, will suggests that $\exists$ goes to (y, d). So the token moves to (y, d).



Since $L(y) = \Box$, it follows that it is $\forall$'s turn now. Note that (y, d) has only one successor (w, c), so the $\forall$ moves the token to (w, c).



Note that now we are at the starting position again, and $\exists$ has no way to break out from this repetition without losing. Also note that the only priority visited in this repetition is 0.



This means that $\forall$ (player 0) can force an infinite match where the biggest priority visited infinitely many times is 0. This means that $\forall$,instead of $\exists$, has a winning strategy at (w, c), and thus $\mathbb{S}_2, c \not\models \mathbb{G}$. This fits our prediction because $\mathbb{S}_2, c \not\models \mu x.\Box x \vee p$.

B

Example for Composing Winning Strategies

As an example of how the construction in Proposition 5.16 works, we revisit $\mathbb{G}$, $\mathbb{G}'$ and $\mathbb{G}''$ from Example 5.1, and illustrate how to build a winning strategy for $\exists$ in $\mathcal{B}(\mathbb{G}, \mathbb{G}'')@(b, z'')$ using $g_{\mathcal{A}^{\mathbb{G},\mathbb{G}'}}$ (g for short) and $g_{\mathcal{A}^{\mathbb{G}',\mathbb{G}''}}$ (g' for short) as the positional winning strategy in $\mathcal{B}(\mathbb{G}, \mathbb{G}'')@(b, x')$ and $\mathcal{B}(\mathbb{G}', \mathbb{G}'')@(x', z'')$. Figure B.1, B.2, B.3 and B.4 show the positions of the token (x, x'), (y', y''), (u, u'') and the helper node u', as well as where the two shadow tokens are located with respect to the strategy graph of the two positional winning strategy.

INIT: $\exists$ sets $x_0 := b$, $x'_0 = y'_0 := x'$, $y''_0 := z''$ and $u'_0 = x'$. Note that in the begining of the the main game, $u_0 = b$ and $u''_0 = z''$. This is shown in Figure B.1.

Step 1:

PLAY$_\exists$: Since $B^0_g = B_g(x_0, x'_0) = \emptyset$ and $B^0_{g'} = B_{g'}(y'_0, y''_0) = \emptyset$, $\exists$ moves the token to $(\emptyset, R^0)$, where

$$R^0 := R^0_g; R^0_{g'} = \{(a, a')\}; \{(a, y'')\} = \{(a, y'')\}.$$

PLAY$_\forall$: Since R^0 is a singleton, $\forall$ has no choice but move the token to its only element, (a, y''). This means that $u_1 = a$ and $u''_1 = y''$.

UPDATE: Since $B^0_g = B^0_{g'} = \emptyset$, $\exists$ needs to find u'_1 such that $(a, u'_1) \in R^0_g = \{(a, a')\}$ and $(u'_1, a'') \in R^0_{g'} = \{(a', y'')\}$. It is easy to see that it has to be the case that $u'_1 = a'$. Since $(u_1, u'_1) = (a, a') \in R^0_g$ and $(u'_1, u''_1) = (a', y'') \in R^0_{g'}$, we let $(x_1, x'_1) := (u_1, u'_1) = (a, a')$ and $(y'_1, y''_1) := (u'_1, u''_1) = (a', y'')$. This is shown in Figure B.2.

Step 2:

PLAY$_\exists$: Since $B^1_g = B_g(x_1, x'_1) \neq \emptyset$ and $B^1_{g'} = B_{g'}(y'_1, y''_1) = \emptyset$, $\exists$ moves the token to $(\emptyset, R^1)$, where

$$R^1 := (B^1_g \cup R^1_g); R^1_{g'} = \{(a, a'), (b, b')\}; \{(b', x'')\} = \{(b, x'')\}.$$

PLAY$_\forall$: Since R^1 is a singleton, $\forall$ has no choice but move the token to its only element, (b, x''). This means that $u_2 = b$ and $u''_2 = x''$.

UPDATE: Since $B^1_g \neq \emptyset$ and $B^1_{g'} = \emptyset$, $\exists$ needs to find u'_2 such that $(u_2 = b, u'_1) \in B^1_g \cup R^1_g = \{(a, a'), (b, b')\}$ and $(u'_2, u''_2 = x'') \in R^1_{g'} = \{(b', x'')\}$. It is easy to see that it has to be the case that $u'_2 = b'$. Since $(u_2, u'_2) = (b, b'') \in B^1_g \backslash R^1_g$, it follows that $(x_2, x'_2) := (x_1, x'_1) = (a, a')$. Since $(u'_2, u''_2) = (b', x'') \in R^1_{g'}$, it follows that $(y'_2, y''_2) := (u'_2, u''_2) = (b', x'')$. This is shown in Figure B.3.

Step 3:

PLAY$_\exists$: Since $B^2_g = B_g(x_2, x'_2) \neq \emptyset$ and $B^2_{g'} = B_{g'}(y'_2, y''_2) = \emptyset$, $\exists$ moves the token to $(\emptyset, R^2)$, where

$$R^2 := (B^2_g \cup R^2_g); R^2_{g'} = \{(a, a'), (b, b')\}; \{(a', a'')\} = \{(a, a'')\}.$$

PLAY$_\forall$: Since R^2 is a singleton, $\forall$ has no choice but move the token to its only element, (a, a''). This means that $u_3 = a$ and $u''_3 = a''$.

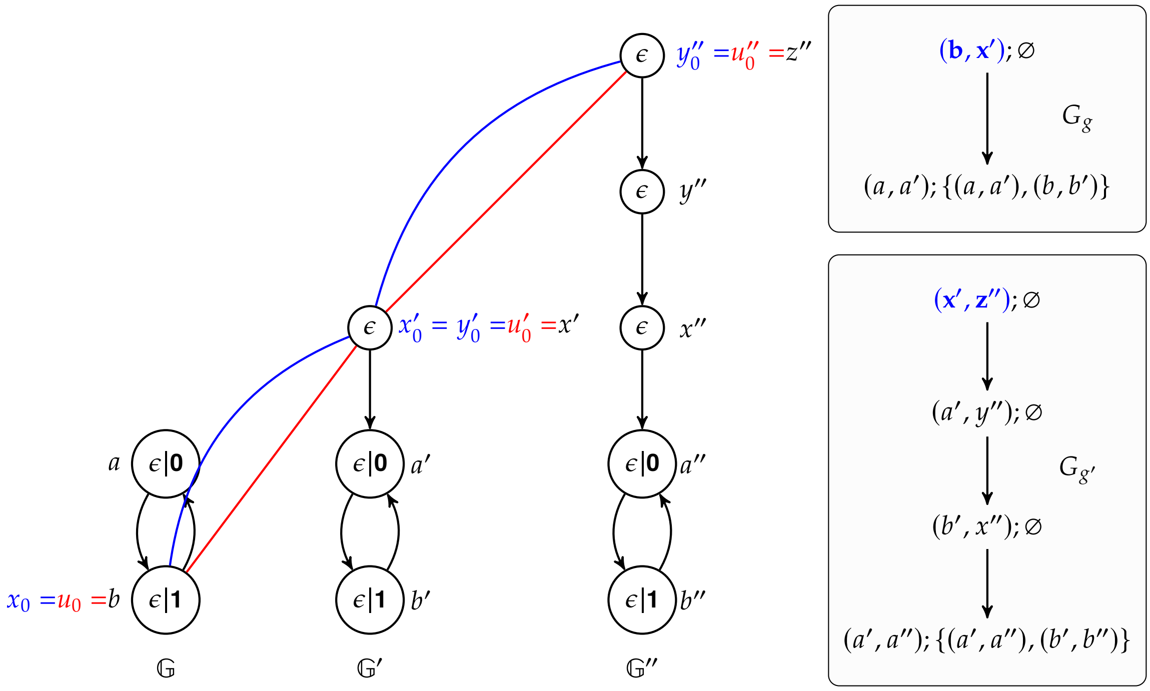


Figure B.1: Composition: INIT

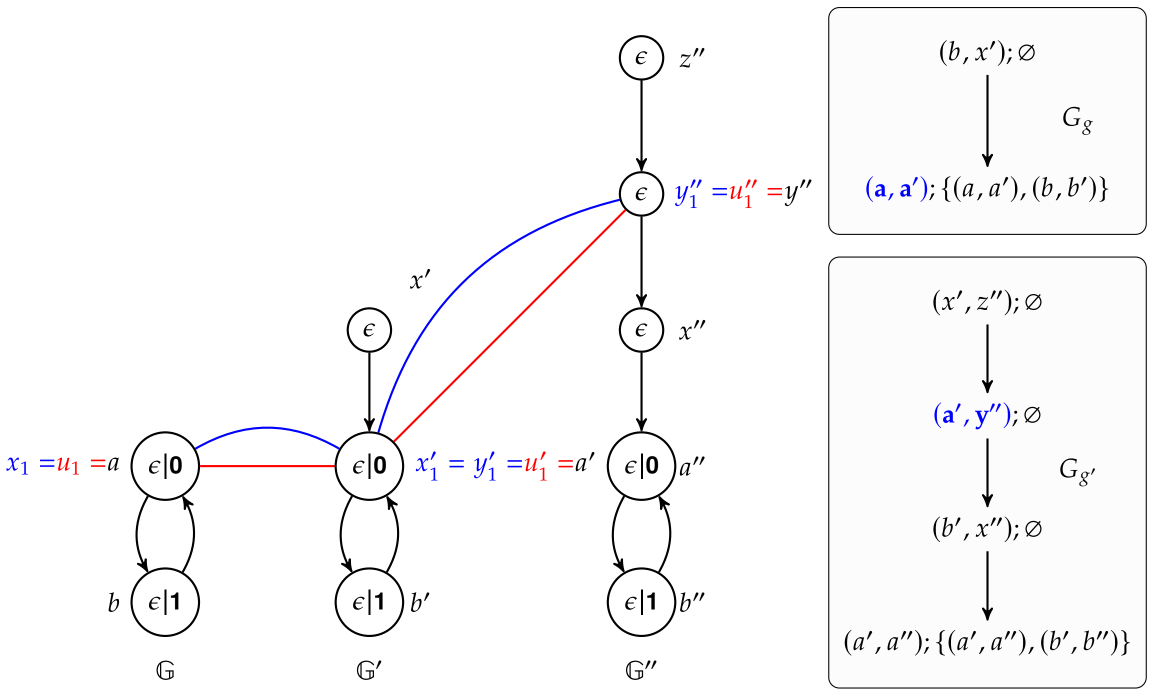


Figure B.2: Composition: step 1

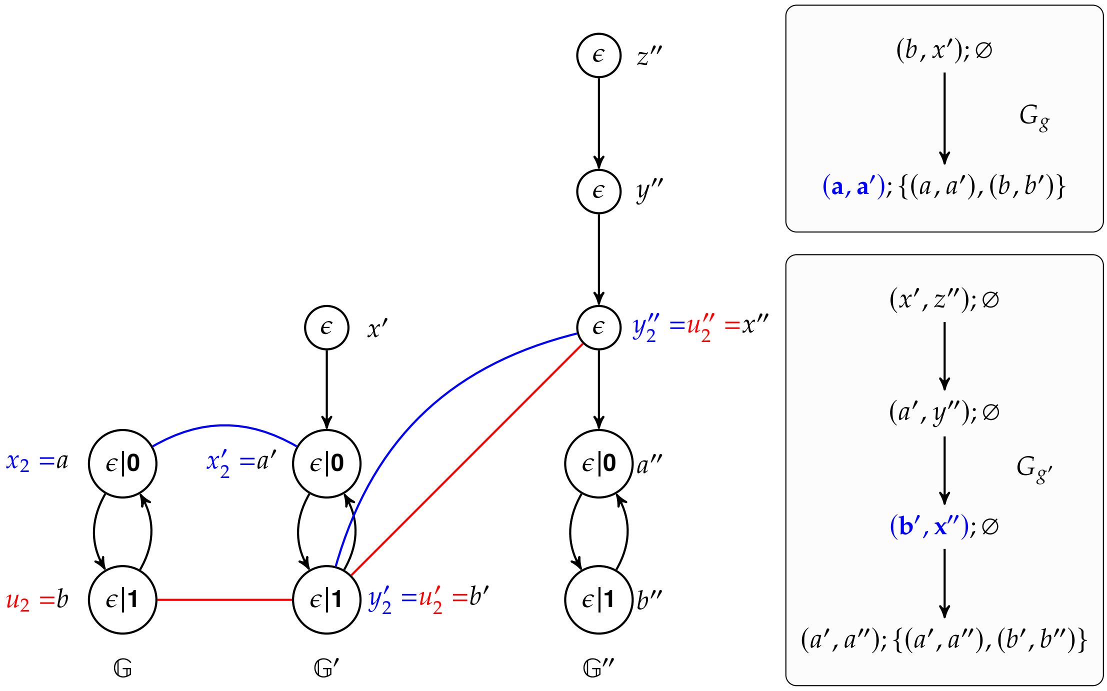


Figure B.3: Composition: step 2

UPDATE: Since $B^2_g \neq \emptyset$ and $B^2_{g'} = \emptyset$, $\exists$ needs to find u'_3 such that $(u_3 = a, u'_3) \in B^2_g \cup R^2_g = \{(a, a'), (b, b')\}$ and $(u'_3, u''_3 = a'') \in R^2_{g'} = \{(a', a'')\}$. It is easy to see that it has to be the case that $u'_2 = a'$. Since $(u_3, u'_3) = (a, a') \in B^1_g \backslash R^1_g$, it follows that $(x_3, x'_3) := (x_2, x'_2) = (a, a')$. Since $(u'_3, u''_3) = (a', a'') \in R^2_{g'}$, it follows that $(y'_3, y''_3) := (u'_3, u''_3) = (a', a'')$. This is shown in Figure B.4.

Step 4:

PLAY$_\exists$: Since $B^3_g = B_g(x_2, x'_2) \neq \emptyset$ and $B^3_{g'} = B_{g'}(y'_3, y''_3) \neq \emptyset$, $\exists$ moves the token to (B^3, R^3), where

$$B^3 := B^3_g; B^3_{g'} = \{(a, a;), (b, b')\}; \{(a', a''), (b', b'')\} = \{(a, a''), (b, b'')\}$$

$$R^3 := (R^3_g; (B^3_{g'} \cup R^3_{g'})) \cup ((B^3_g \cup R^3_g); R^3_{g'}) = (\emptyset; (B^3_{g'} \cup R^3_{g'})) \cup ((B^3_g \cup R^3_g); \emptyset) = \emptyset.$$

PLAY$_\forall$: Since R^3 is a singleton, the set of admissible moves for $\forall$ is empty and $\forall$ loses for getting stuck.

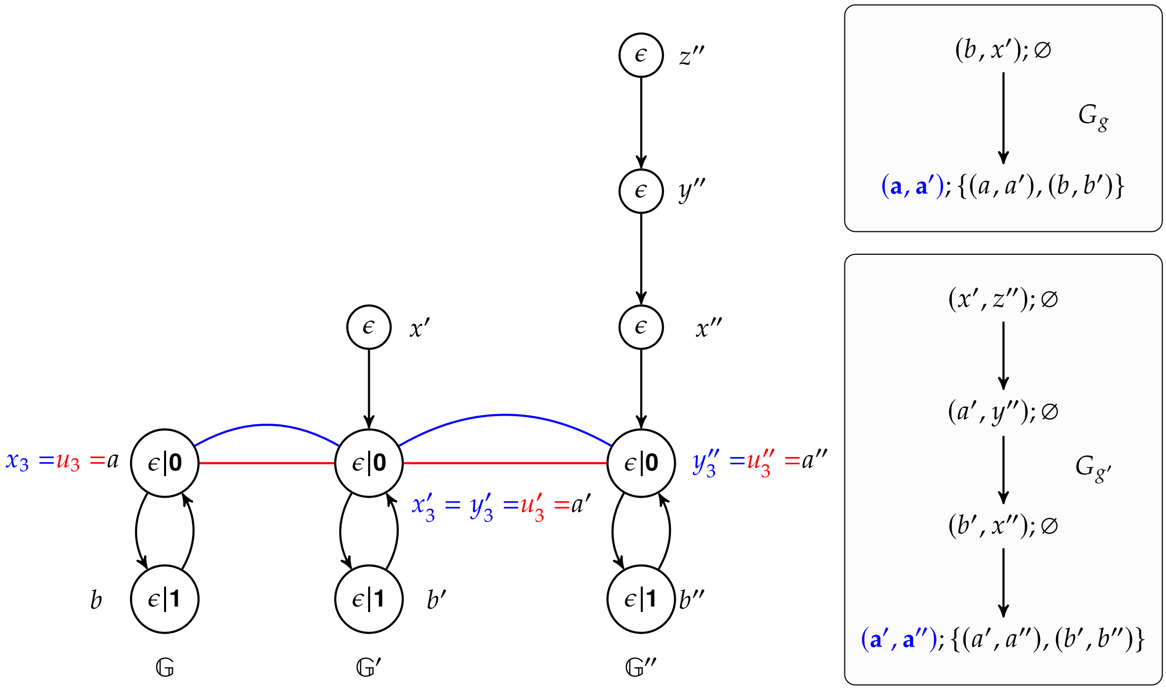


Figure B.4: Composition: step 3

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