COFINAL STABLE LOGICS

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Abstract. We generalize the $(\wedge, \vee)$-canonical formulas of [3] to $(\wedge, \vee)$-canonical rules, and prove that each intuitionistic multi-conclusion consequence relation is axiomatizable by $(\wedge, \vee)$-canonical rules. This provides an intuitionistic analogue of [6], and is an alternative of [19]. It also yields a convenient characterization of stable superintuitionistic logics introduced in [3].

The $(\wedge, \vee)$-canonical formulas are analogues of the $(\wedge, \to)$-canonical formulas of [5], which are the algebraic counterpart of Zakharyaschev's canonical formulas for superintuitionistic logics (si-logics for short). Consequently, stable si-logics are analogues of subframe si-logics. We introduce cofinal stable intuitionistic multi-conclusion consequence relations and cofinal stable si-logics, thus answering the question of what the analogues of cofinal subframe logics should be. This is done by utilizing the $(\wedge, \vee, \neg)$-reduct of Heyting algebras. We prove that every cofinal stable si-logic has the finite model property, and that there are continuum many cofinal stable si-logics that are not stable. We conclude with several examples showing the similarities and differences between the classes of stable, cofinal stable, subframe, and cofinal subframe si-logics.

1. Introduction

Superintuitionistic logics (si-logics for short) are propositional logics extending the intuitionistic propositional calculus **IPC**. Consistent si-logics are also known as intermediate logics as they are exactly the logics situated between **IPC** and the classical propositional calculus **CPC**. By the Gödel translation, si-logics are closely related to modal logics extending **S4**. A detailed account of si-logics and how they relate to modal logics can be found in [8].

Finding uniform axiomatizations for si-logics and modal logics has been a significant problem in the area. First general results in this direction were obtained by Jankov [17, 18], de Jongh [9], and Fine [13, 14]. They introduced formulas that provide a uniform axiomatization for large classes of si-logics and transitive modal logics. Zakharyaschev [23, 24] generalized these results by introducing canonical formulas, which axiomatize all si-logics and all transitive modal logics. Jeřábek [19] further generalized Zakharyaschev's approach by defining canonical multi-conclusion rules that axiomatize all intuitionistic multi-conclusion consequence relations and all transitive modal multi-conclusion consequence relations.

The algebraic counterparts of Zakharyaschev's canonical formulas for si-logics are the $(\wedge, \to)$-canonical formulas of [5]. The $(\wedge, \to)$-canonical formula of a finite subdirectly irreducible (s.i.) Heyting algebra A fully describes the disjunction-free reduct of A, but describes the behavior of the missing connective $\vee$ only partially. In fact, the algebraic content of Zakharyaschev's closed domain condition (CDC) is encoded by $D \subseteq A^2$, where the behavior of $\vee$ is fully described. The $(\wedge, \to)$-canonical formulas, though syntactically quite different, serve the same purpose as Zakharyaschev's canonical formulas in providing a uniform axiomatization of all si-logics.

One of the main technical tools in developing the theory of $(\wedge, \to)$-canonical formulas is Diego's theorem [10] that the disjunction-free reduct of Heyting algebras is locally finite. Another natural locally finite reduct of Heyting algebras is the bounded lattice reduct. This suggests a different approach to canonical formulas, which was developed in [3], where $(\wedge, \vee)$-canonical formulas were

2000 *Mathematics Subject Classification.* 03B55; 06D20.

Key words and phrases. Intuitionistic logic, intuitionistic multi-conclusion consequence relation, axiomatization, Heyting algebra, variety, universal class.

introduced. The $(\wedge, \vee)$-canonical formula of a finite s.i. Heyting algebra A fully describes the bounded lattice reduct of A, and only partially the behavior of the missing connective $\to$. In this case the CDC is encoded by $D \subseteq A^2$, where the behavior of $\to$ is described fully. As in the $(\wedge, \to)$-case, the $(\wedge, \vee)$-canonical formulas provide a uniform axiomatization of all si-logics.

If $D = A^2$, then both $(\wedge, \to)$- and $(\wedge, \vee)$-canonical formulas become equivalent to Jankov formulas, and axiomatize splitting si-logics and their joins. If $D = \varnothing$, then $(\wedge, \to)$-canonical formulas axiomatize subframe and cofinal subframe si-logics, depending on whether or not the behavior of negation is encoded in the formula [5]. If $D = \varnothing$, then $(\wedge, \vee)$-canonical formulas axiomatize a new class of logics, called stable si-logics [3]. Stable si-logics form a well-behaved class of logics having the finite model property (FMP). Many natural si-logics are stable and there is a continuum of stable si-logics.

Stable si-logics can be seen as the $(\wedge, \vee)$-counterpart of subframe si-logics. But what is the $(\wedge, \vee)$-counterpart of cofinal subframe si-logics? As we pointed out above, for $D = \varnothing$, cofinal subframe si-logics are axiomatized by $(\wedge, \to, \neg)$-canonical formulas, while subframe si-logics by $(\wedge, \to)$-canonical formulas. It is only natural to introduce cofinal stable si-logics as the logics axiomatized by $(\wedge, \vee, \neg)$-canonical formulas when $D = \varnothing$. For this we need to work with the pseudocomplemented lattice reduct of Heyting algebras, instead of the bounded lattice reduct like in the case of stable logics. Fortunately, the $(\wedge, \vee, \neg)$-reduct of Heyting algebras is locally finite, and hence the algebraic approach is applicable. However, the $(\wedge, \vee, \neg)$-reduct of Heyting algebras is much worse behaved than the $(\wedge, \vee)$-reduct, which results in subtle differences between stable si-logics and cofinal stable si-logics. For example, while stable si-logics are axiomatizable by stable formulas, cofinal stable si-logics are not axiomatizable by cofinal stable formulas, instead cofinal stable rules are required.

Because of this, we first generalize $(\wedge, \vee)$-canonical formulas to $(\wedge, \vee)$-canonical rules, which axiomatize all intuitionistic multi-conclusion consequence relations. The $(\wedge, \vee)$-canonical rules are the intuitionistic counterpart of the stable canonical rules of [6], and are an alternative of Jeřábek's canonical multi-conclusion rules [19]. We also indicate how to generalize $(\wedge, \to)$-canonical formulas to $(\wedge, \to)$-canonical rules, which provide another uniform axiomatization of all intuitionistic multi-conclusion consequence relations.

Next we generalize stable si-logics to intuitionistic stable multi-conclusion consequence relations, prove that intuitionistic stable multi-conclusion consequence relations have the FMP, and utilize $(\wedge, \vee)$-canonical rules to give a convenient characterization of stable si-logics.

Finally, we generalize $(\wedge, \vee)$-canonical rules to $(\wedge, \vee, \neg)$-canonical rules, which give rise to cofinal stable rules when $D = \varnothing$. We prove that cofinal stable rules axiomatize cofinal stable intuitionistic multi-conclusion consequence relations and cofinal stable si-logics. However, the corresponding formulas axiomatize a larger class of si-logics, properly containing the class of cofinal stable si-logics. A characterization of this class and whether each logic in the class has the FMP remain open problems.

The class of cofinal stable si-logics provides a new class of si-logics. It contains the class of all stable si-logics, and we show that there are continuum many cofinal stable si-logics that are not stable. We conclude the paper with several examples that indicate the similarities and differences between the classes of stable, cofinal stable, subframe, and cofinal subframe si-logics.

2. Intuitionistic multi-conclusion consequence relations

In this section we recall multi-conclusion rules and multi-conclusion consequence relations. For more details see [19, 16]. A *multi-conclusion rule* is an expression of the form Γ/Δ, where Γ and Δ are finite sets of formulas.

Definition 2.1. *An* intuitionistic multi-conclusion consequence relation *is a set $\mathcal{S}$ of multi-conclusion rules such that*

(1) $\varphi/\varphi \in \mathcal{S}$.
(2) $\varphi, \varphi \to \psi/\psi \in \mathcal{S}$.
(3) $/\varphi \in \mathcal{S}$ *for each theorem* φ *of* **IPC**.
(4) *If* $\Gamma/\Delta \in \mathcal{S}$, *then* $\Gamma, \Gamma'/\Delta, \Delta' \in \mathcal{S}$.
(5) *If* $\Gamma/\Delta, \varphi \in \mathcal{S}$ *and* $\Gamma, \varphi/\Delta \in \mathcal{S}$, *then* $\Gamma/\Delta \in \mathcal{S}$.
(6) *If* $\Gamma/\Delta \in \mathcal{S}$ *and* s *is a substitution, then* $s(\Gamma)/s(\Delta) \in \mathcal{S}$.

We denote the least intuitionistic multi-conclusion consequence relation by $\mathcal{S}_{\mathbf{IPC}}$, and the complete lattice of multi-conclusion consequence relations extending $\mathcal{S}_{\mathbf{IPC}}$ by $\mathrm{Ext}(\mathcal{S}_{\mathbf{IPC}})$. For a set $\mathcal{R}$ of multi-conclusion rules, let $\mathcal{S}_{\mathbf{IPC}} + \mathcal{R}$ be the least intuitionistic multi-conclusion consequence relation containing $\mathcal{R}$. If $\mathcal{S} = \mathcal{S}_{\mathbf{IPC}} + \mathcal{R}$, then we say that $\mathcal{S}$ *is axiomatized by* $\mathcal{R}$.

Let $\mathrm{Ext}(\mathbf{IPC})$ be the complete lattice of si-logics. For an intuitionistic multi-conclusion consequence relation $\mathcal{S}$, let $\Lambda(\mathcal{S}) = \{\varphi : /\varphi \in \mathcal{S}\}$, and for a si-logic L, let $\Sigma(L) = \mathcal{S}_{\mathbf{IPC}} + \{/\varphi : \varphi \in L\}$. Then $\Lambda : \mathrm{Ext}(\mathcal{S}_{\mathbf{IPC}}) \to \mathrm{Ext}(\mathbf{IPC})$ and $\Sigma : \mathrm{Ext}(\mathbf{IPC}) \to \mathrm{Ext}(\mathcal{S}_{\mathbf{IPC}})$ are order preserving maps such that $\Lambda(\Sigma(L)) = L$ for each $L \in \mathrm{Ext}(\mathbf{IPC})$ and $\mathcal{S} \supseteq \Sigma(\Lambda(\mathcal{S}))$ for each $\mathcal{S} \in \mathrm{Ext}(\mathcal{S}_{\mathbf{IPC}})$. We say that a si-logic L is *axiomatized* by a set $\mathcal{R}$ of multi-conclusion rules if $L = \Lambda(\mathcal{S}_{\mathbf{IPC}} + \mathcal{R})$.

A Heyting algebra A *validates* a multi-conclusion rule Γ/Δ provided for every valuation v on A, if $v(\gamma) = 1$ for all $\gamma \in \Gamma$, then $v(\delta) = 1$ for some $\delta \in \Delta$. If A validates Γ/Δ, we write $A \models \Gamma/\Delta$. If $\Gamma = \{\phi_1, \dots, \phi_n\}$, $\Delta = \{\psi_1, \dots, \psi_m\}$, and $\phi_i(\underline{x})$ and $\psi_j(\underline{x})$ are the terms in the first-order language of Heyting algebras corresponding to the ϕ_i and ψ_j, then $A \models \Gamma/\Delta$ iff A is a model of the universal sentence $\forall \underline{x}\ (\bigwedge_{i=1}^n \phi_i(\underline{x}) = 1 \to \bigvee_{j=1}^m \psi_j(\underline{x}) = 1)$. Consequently, intuitionistic multi-conclusion consequence relations correspond to universal classes of Heyting algebras. It is well known (see, e.g., [7, Thm. V.2.20]) that a class of Heyting algebras is a universal class iff it is closed under isomorphisms, subalgebras, and ultraproducts.

On the other hand, si-logics correspond to equationally definable classes of Heyting algebras; that is, models of the sentences $\forall \underline{x}\ \phi(\underline{x}) = 1$ in the first-order language of Heyting algebras. It is well known (see, e.g., [7, Thm. II.11.9]) that a class of Heyting algebras is an equational class iff it is a variety (that is, it is closed under homomorphic images, subalgebras, and products).

For an intuitionistic multi-conclusion consequence relation $\mathcal{S}$, let $\mathcal{U}(\mathcal{S})$ be the universal class of Heyting algebras corresponding to $\mathcal{S}$, and for a universal class of Heyting algebras $\mathcal{U}$, let $\mathcal{S}(\mathcal{U})$ be the intuitionistic multi-conclusion consequence relation corresponding to $\mathcal{U}$. Then $\mathcal{S}(\mathcal{U}(\mathcal{S})) = \mathcal{S}$ and $\mathcal{U}(\mathcal{S}(\mathcal{U})) = \mathcal{U}$. This yields an isomorphism between $\mathrm{Ext}(\mathcal{S}_{\mathbf{IPC}})$ and the complete lattice $\mathsf{U}(\mathsf{HA})$ of universal classes of Heyting algebras (ordered by reverse inclusion).

Similarly, for a si-logic L, let $\mathcal{V}(L)$ be the variety of Heyting algebras corresponding to L, and for a variety $\mathcal{V}$ of Heyting algebras, let $L(\mathcal{V})$ be the si-logic corresponding to $\mathcal{V}$. Then $L(\mathcal{V}(L)) = L$ and $\mathcal{V}(L(\mathcal{V})) = \mathcal{V}$, yielding an isomorphism between $\mathrm{Ext}(\mathbf{IPC})$ and the complete lattice $\mathsf{V}(\mathsf{HA})$ of varieties of Heyting algebras (ordered by reverse inclusion).

Under this correspondence, for an intuitionistic multi-conclusion consequence relation $\mathcal{S}$, the variety $\mathcal{V}(\Lambda(\mathcal{S}))$ corresponding to the si-logic $\Lambda(\mathcal{S})$ is the variety generated by the universal class $\mathcal{U}(\mathcal{S})$.

3. $(\wedge, \vee)$-canonical rules

We generalize the $(\wedge, \vee)$-canonical formulas of [3] to $(\wedge, \vee)$-canonical rules, and show that each intuitionistic multi-conclusion consequence relation is axiomatizable by $(\wedge, \vee)$-canonical rules. This provides an intuitionistic analogue of similar results in [6]. Our proofs are modifications of those in [6, Sec. 5] and generalize those in [3, Sec. 3]. Therefore, we will only provide their sketches.

Definition 3.1. *Let A be a finite Heyting algebra and let $D \subseteq A^2$. For every $a \in A$ let p_a be a propositional letter, and define the* $(\wedge, \vee)$-canonical rule *$\rho(A, D)$ associated with A and D as $\rho(A, D) = \Gamma/\Delta$, where*

$$\begin{aligned}\Gamma = &\{p_0 \leftrightarrow 0\} \cup \{p_1 \leftrightarrow 1\} \cup \\ &\{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid a, b \in A\} \cup \\ &\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b \mid a, b \in A\} \cup \\ &\{p_{a\to b} \leftrightarrow p_a \to p_b \mid (a, b) \in D\}\end{aligned}$$

and

$$\Delta = \{p_a \leftrightarrow p_b \mid a, b \in A \text{ with } a \neq b\}.$$

Proposition 3.2. *Let A be a finite Heyting algebra, $D \subseteq A^2$, and B be an arbitrary Heyting algebra. Then $B \not\models \rho(A, D)$ iff there is a bounded lattice embedding $h : A \to B$ such that $h(a \to b) = h(a) \to h(b)$ for each $(a, b) \in D$.*

Proof. (Sketch) Suppose $B \not\models \rho(A, D)$ and let v be a valuation on B witnessing that $B \not\models \rho(A, D)$. Then $v(\varphi) = 1$ for all $\varphi \in \Gamma$ and $v(\delta) \neq 1$ for all $\delta \in \Delta$. It then follows that the map $h : A \to B$ defined by $h(a) = v(p_a)$ for all $a \in A$ is a bounded lattice embedding such that $h(a \to b) = h(a) \to h(b)$ for all $(a, b) \in D$. Conversely, if $h : A \to B$ is a bounded lattice embedding such that $h(a \to b) = h(a) \to h(b)$ for each $(a, b) \in D$, then the valuation v on B defined by $v(p_a) = h(a)$ for all $a \in A$ witnesses that $B \not\models \rho(A, D)$. □

Proposition 3.3. *If $\mathcal{S}_{\mathbf{IPC}} \not\vdash \Gamma/\Delta$, then there are finite Heyting algebras $A_1 \dots A_m$ and $D_1 \dots D_m$ with $D_i \subseteq A_i^2$ for $1 \leq i \leq m$, such that for each Heyting algebra B, we have $B \not\models \Gamma/\Delta$ iff there is $1 \leq i \leq m$ and a bounded lattice embedding $h : A_i \to B$ with $h(a \to b) = h(a) \to h(b)$ for all $(a, b) \in D_i$.*

Proof. (Sketch) Let Ξ be the set of all subformulas of formulas in $\Gamma \cup \Delta$. Then Ξ is finite. Let n be the cardinality of Ξ. Since the bounded lattice reduct of Heyting algebras is locally finite, up to isomorphism, there are only finitely many pairs (A, D) satisfying the following two conditions:

(i) A is a finite Heyting algebra that is at most n-generated as a bounded distributive lattice and $A \not\models \Gamma/\Delta$.
(ii) $D := \{(v(\varphi), v(\psi)) \mid \varphi \to \psi \in \Xi\}$, where v is a valuation on A witnessing $A \not\models \Gamma/\Delta$.

If $(A_1, D_1), \dots, (A_m, D_m)$ is their enumeration, then we show that this collection is as required. Suppose B is a Heyting algebra with $B \not\models \Gamma/\Delta$. If v is a valuation witnessing that $B \not\models \Gamma/\Delta$, then let A be the bounded sublattice of B generated by the set $v[\Xi]$, and let $D := \{(v(\varphi), v(\psi)) \mid \varphi \to \psi \in \Xi\}$. Then (A, D) is up to isomorphism one of (A_i, D_i) and the inclusion from A to B gives rise to a bounded lattice embedding $h : A_i \to B$ such that $h(a \to b) = h(a) \to h(b)$ for all $(a, b) \in D_i$. Conversely, let $h : A_i \to B$ be a bounded lattice embedding with $h(a \to b) = h(a) \to h(b)$ for all $(a, b) \in D_i$. Let v be the valuation on A_i that gave rise to $D_i = \{(v(\varphi), v(\psi)) \mid \varphi \to \psi \in \Sigma\}$. Then $h \circ v$ is a valuation on B witnessing that $B \not\models \Gamma/\Delta$. □

Proposition 3.4. *Every intuitionistic multi-conclusion consequence relation $\mathcal{S}$ is axiomatizable by $(\wedge, \vee)$-canonical rules. Moreover, if $\mathcal{S}$ is finitely axiomatizable, then it is axiomatizable by finitely many $(\wedge, \vee)$-canonical rules.*

Proof. (Sketch) Suppose $\mathcal{S} = \mathcal{S}_{\mathbf{IPC}} + \{\Gamma_i/\Delta_i \mid i \in I\}$ with $\Gamma_i/\Delta_i \notin \mathcal{S}_{\mathbf{IPC}}$. For every Γ_i/Δ_i pick $(A_{i_1}, D_{i_1}), \dots, (A_{i_{m_i}}, D_{i_{m_i}})$ according to Proposition 3.3. Then $\mathcal{S} = \mathcal{S}_{\mathbf{IPC}} + \{\rho(A_{i_j}, D_{i_j}) \mid i \in I, 1 \leq j \leq m_i\}$. □

Remark 3.5. The $(\wedge, \to)$-canonical formulas of [5] can also be generalized to $(\wedge, \to)$-canonical rules. This yields a slightly different axiomatization of all intuitionistic multi-conclusion consequence relations. Namely, let A be a finite Heyting algebra and let $D \subseteq A^2$. For every $a \in A$ let p_a be a propositional letter, and define the $(\wedge, \to, 0)$-*canonical rule* $\zeta(A, D)$ associated with A and

D as $\zeta(A,D) = \Gamma/\Delta$, where

$$\begin{aligned}\Gamma = &\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b \mid a,b \in A\} \cup \\ &\{p_{a\to b} \leftrightarrow p_a \to p_b \mid a,b \in A\} \cup \\ &\{p_{\neg a} \leftrightarrow \neg p_a \mid a \in A\} \cup \\ &\{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid (a,b) \in D\}\end{aligned}$$

and

$$\Delta = \{p_a \leftrightarrow p_b \mid a,b \in A \text{ with } a \neq b\}.$$

Then the results of [5] can be generalized to prove that each intuitionistic multi-conclusion consequence relation $\mathcal{S}$ is axiomatizable by $(\wedge,\to,0)$-canonical rules. Moreover, if $\mathcal{S}$ is finitely axiomatizable, then it is axiomatizable by finitely many $(\wedge,\to,0)$-canonical rules. In fact, the same way $(\wedge,\to)$-canonical formulas are an algebraic counterpart of Zakharyaschev's canonical formulas for si-logics, $(\wedge,\to)$-canonical rules are an algebraic counterpart of Jeřábek's canonical rules for intuitionistic multi-conclusion consequence relations.

4. Stable multi-conclusion consequence relations

Let A be a finite Heyting algebra and let $D \subseteq A^2$. Then there are two obvious extreme cases, when $D = A^2$ and when $D = \varnothing$. If $D = A^2$, then we call the $(\wedge,\vee)$-canonical rule $\rho(A,D)$ a *Jankov rule*. As follows from [19, Sec. 6] (see also [6, Sec. 7]), Jankov rules axiomatize splittings and join splittings in the lattice of all intuitionistic multi-conclusion consequence relations.

If $D = \varnothing$, then we call $\rho(A,D)$ a *stable rule* and denote it by $\rho(A)$. In this section we show that stable rules axiomatize stable multi-conclusion consequence relations, thus generalizing the results of [3, Sec. 6], and providing intuitionistic analogues of the results of [6, Sec. 7].

Definition 4.1.

(1) *We call a class $\mathcal{K}$ of Heyting algebras* stable *provided for all Heyting algebras A, B, if A is isomorphic to a bounded sublattice of B and $B \in \mathcal{K}$, then $A \in \mathcal{K}$.*

(2) *We call an intuitionistic multi-conclusion consequence relation $\mathcal{S}$* stable *provided its corresponding universal class $\mathcal{U}(\mathcal{S})$ is stable.*

The next proposition is immediate from the previous section.

Proposition 4.2. *Let A be a finite Heyting algebra. For every Heyting algebra B, we have $B \not\models \rho(A)$ iff there is a bounded lattice embedding $h : A \to B$.*

Suppose B is a Heyting algebra, v is a valuation on B, and Γ is a set of formulas. We write $v(\Gamma) = 1$ if $v(\gamma) = 1$ for all $\gamma \in \Gamma$, and $v(\Gamma) \neq 1$ if $v(\gamma) \neq 1$ for all $\gamma \in \Gamma$. The next lemma generalizes [3, Lem. 3.6].

Lemma 4.3. *If a Heyting algebra B refutes a multi-conclusion rule Γ/Δ, then there is a finite Heyting algebra A such that A is a bounded sublattice of B and A refutes Γ/Δ.*

Proof. Since $B \not\models \Gamma/\Delta$, there is a valuation v_B on B such that $v_B(\Gamma) = 1$ and $v_B(\Delta) \neq 1$. Let Ξ be the set of subformulas of formulas in $\Gamma \cup \Delta$, and let A be the bounded sublattice of B generated by $v_B[\Xi]$. Since Ξ is finite and bounded distributive lattices are locally finite, we see that A is finite. Therefore, A is a finite Heyting algebra, where $a \to_A b = \bigvee\{x \in A \mid a \wedge x \leq b\}$ for all $a, b \in A$. It is then easy to see that for all $a, b \in A$, we have $a \to_A b \leq a \to_B b$ and $a \to_A b = a \to_B b$ provided $a \to_B b \in A$. Define the valuation v_A on A by $v_A(p) = v_B(p)$ for $p \in \Xi$ and $v_A(p)$ is an arbitrary element of A otherwise. It is then straightforward that $v_A(\varphi) = v_B(\varphi)$ for all $\varphi \in \Xi$. Thus, $v_A(\Gamma) = 1_A$ and $v_A(\Delta) \neq 1_A$, and hence $A \not\models \Gamma/\Delta$. □

As an immediate corollary, we obtain that stable multi-conclusion consequence relations have the FMP.

Proposition 4.4. *Every stable multi-conclusion consequence relation has the FMP.*

Proof. Suppose $\mathcal{S}$ is a stable multi-conclusion consequence relation and $\mathcal{S} \not\vdash \Gamma/\Delta$. Then there is $B \in \mathcal{U}(\mathcal{S})$ with $B \not\models \Gamma/\Delta$. By Lemma 4.3, there is a bounded lattice embedding $A \hookrightarrow B$ with A finite and $A \not\models \Gamma/\Delta$. Since $\mathcal{U}(\mathcal{S})$ is stable, $A \in \mathcal{U}(\mathcal{S})$. Thus, $\mathcal{S}$ has the FMP. □

We conclude this section by proving that stable multi-conclusion consequence relations are exactly the multi-conclusion consequence relations that are axiomatizable by stable rules. This generalizes [3, Thm. 6.11] and is an intuitionistic analogue of [6, Thm. 7.3].

Proposition 4.5. *An intuitionistic multi-conclusion consequence relation $\mathcal{S}$ is stable iff it is axiomatizable by stable rules.*

Proof. First suppose that $\mathcal{S}$ is stable. We show that $\mathcal{S} = \mathcal{S}_{\mathbf{IPC}} + \{\rho(A) \mid A \text{ finite}, A \notin \mathcal{U}(\mathcal{S})\}$. Let $B \models \mathcal{S}$. If there is finite $A \notin \mathcal{U}(\mathcal{S})$ such that $B \not\models \rho(A)$, then by Proposition 4.2, there is a bounded lattice embedding $A \hookrightarrow B$. Since $\mathcal{U}(\mathcal{S})$ is stable, $A \in \mathcal{U}(\mathcal{S})$, a contradiction. Therefore, $B \models \{\rho(A) \mid A \text{ finite}, A \notin \mathcal{U}(\mathcal{S})\}$. Conversely, if $B \not\models \mathcal{S}$, then there is a rule Γ/Δ in $\mathcal{S}$ such that $B \not\models \Gamma/\Delta$. By Lemma 4.3, there is a finite Heyting algebra A such that A is a bounded sublattice of B and $A \not\models \Gamma/\Delta$. Therefore, $A \not\models \mathcal{S}$. By Proposition 4.2, $B \not\models \rho(A)$. Thus, $B \not\models \mathcal{S}_{\mathbf{IPC}} + \{\rho(A) \mid A \text{ finite}, A \notin \mathcal{U}(\mathcal{S})\}$.

Next suppose that $\mathcal{S} = \mathcal{S}_{\mathbf{IPC}} + \{\rho(A_i) \mid i \in I\}$. Let A, B be Heyting algebras such that A is isomorphic to a bounded sublattice of B and $B \in \mathcal{U}(\mathcal{S})$. If $A \notin \mathcal{U}(\mathcal{S})$, then there is $i \in I$ such that $A \not\models \rho(A_i)$. By Proposition 4.2, there is a bounded lattice embedding $A_i \hookrightarrow A$. Therefore, there is a bounded lattice embedding $A_i \hookrightarrow B$, so applying Proposition 4.2 again yields $B \not\models \rho(A_i)$. Thus, $B \notin \mathcal{U}(\mathcal{S})$, a contradiction. This shows that $A \in \mathcal{U}(\mathcal{S})$, so $\mathcal{U}(\mathcal{S})$ is a stable class, and hence $\mathcal{S}$ is stable. □

5. Stable logics

In the previous section we saw that stable multi-conclusion consequence relations provide a generalization of stable si-logics introduced in [3]. As we will see now, stable multi-conclusion consequence relations also aid in obtaining a number of convenient characterizations of stable si-logics.

Recall from [3] that for a finite s.i. Heyting algebra A and $D \subseteq A^2$, the $(\wedge, \vee)$-*canonical formula associated with A and D* is $\gamma(A, D) = \bigwedge \Gamma \to \bigvee \Delta$. If $D = \varnothing$, then $\gamma(A, D)$ is denoted by $\gamma(A)$, and is called the *stable formula associated with A*.

A si-logic L is *stable* provided for arbitrary s.i. Heyting algebras A, B, if A is isomorphic to a bounded sublattice of B and $B \models L$, then $A \models L$. By [3, Sec. 6], stable si-logic have the FMP, and a si-logic L is stable iff it is axiomatizable by stable formulas.

In [3, Thm. 6.3] it is proved that if A, B are s.i. Heyting algebras with A finite, then $B \not\models \gamma(A)$ iff A is isomorphic to a bounded sublattice of B. We show that in the theorem the condition that B is s.i. can be weakened to B being well-connected.

Proposition 5.1. *Let A, B be Heyting algebras with A finite and s.i. and B well-connected. Then $B \not\models \gamma(A)$ iff A is isomorphic to a bounded sublattice of B.*

Proof. First suppose that A is isomorphic to a bounded sublattice of B. The same argument as in the proof of [3, Thm. 6.3] shows that there is a valuation v on B refuting $\gamma(A)$ so that $v(\gamma) = 1_B$ for each $\gamma \in \Gamma$ and $v(\delta) \neq 1_B$ for each $\delta \in \Delta$. Therefore, since B is well-connected and Γ, Δ are finite, $v(\bigwedge \Gamma) = \bigwedge\{v(\gamma) : \gamma \in \Gamma\} = 1_B$ and $v(\bigvee \Delta) = \bigvee\{v(\delta) : \delta \in \Delta\} \neq 1_B$. Thus, $B \not\models \gamma(A)$.

For the converse, we prove a stronger statement, that for an arbitrary B, from $B \not\models \gamma(A)$ it follows that A is isomorphic to a bounded sublattice of B.

Claim 5.2. *Let A, B be Heyting algebras with A finite and s.i. If $B \not\models \gamma(A)$, then A is isomorphic to a bounded sublattice of B.*

Proof of claim. If $B \not\models \gamma(A)$, then by [3, Lem. 3.6], there is a finite Heyting algebra B' such that B' is a bounded sublattice of B and $B' \not\models \gamma(A)$. By [3, Thm. 3.4], there is a homomorphic image C of B' and a bounded lattice embedding $f : A \to C$. Since B' is finite, applying [3, Lem. 6.2] yields a bounded lattice embedding $f' : A \to B'$. Thus, A is isomorphic to a bounded sublattice of B. □

This completes the proof of the proposition. □

We are ready to prove the main result of the section. For a variety $\mathcal{V}$ of Heyting algebras, let $\mathcal{V}_{wc}$ be the class of well-connected members of $\mathcal{V}$.

Theorem 5.3. *For a si-logic L and the corresponding variety $\mathcal{V}$ of Heyting algebras, the following are equivalent:*

(1) *L is stable.*
(2) *L is axiomatizable by stable formulas.*
(3) *L is axiomatizable by stable rules.*
(4) *$\mathcal{V}_{wc}$ is stable.*
(5) *$\mathcal{V}$ is generated by a stable universal class.*
(6) *$\mathcal{V}$ is generated by a stable class.*

Proof. (1) $\Leftrightarrow$ (2): This is proved in [3, Thm. 6.11].

(2) $\Rightarrow$ (4): By (2), there is a family $\{A_i : i \in I\}$ of finite s.i. Heyting algebras such that $L = \mathbf{IPC} + \{\gamma(A_i) : i \in I\}$. Let A, B be Heyting algebras with $B \in \mathcal{V}_{wc}$ and A isomorphic to a bounded sublattice of B. Since bounded sublattices preserve well-connectedness, we see that A is also well-connected. Therefore, it remains to prove that $A \in \mathcal{V}$. If not, then there is $i \in I$ such that $A \not\models \gamma(A_i)$. By Proposition 5.1, A_i is isomorphic to a bounded sublattice of A. By transitivity, A_i is isomorphic to a bounded sublattice of B. Since B is well-connected, applying Proposition 5.1 again yields $B \not\models \gamma(A_i)$, a contradiction. Thus, $A \in \mathcal{V}_{wc}$.

(4) $\Rightarrow$ (5): The class $\mathcal{V}_{wc}$ is universal since it is definable over $\mathcal{V}$ by the universal formula $\forall xy(x \vee y = 1 \to (x = 1 \vee y = 1))$. By (4), $\mathcal{V}_{wc}$ is stable. Since $\mathcal{V}_{wc}$ contains all s.i. members of $\mathcal{V}$, we see that $\mathcal{V}_{wc}$ generates $\mathcal{V}$.

(5) $\Rightarrow$ (6): This is obvious.

(6) $\Rightarrow$ (2): Suppose that $\mathcal{V}$ is generated by a stable class $\mathcal{K}$. First we show that L has the FMP.

Claim 5.4. *If $\varphi \notin L$, then there is a finite $A \in \mathcal{K}$ with $A \not\models \varphi$. In particular, L has the FMP.*

Proof of claim. If $\varphi \notin L$, then there is $B \in \mathcal{K}$ with $B \not\models \varphi$. By [3, Lem. 3.6], there is a finite Heyting algebra A such that A is a bounded sublattice of B and $A \not\models \varphi$. Since $\mathcal{K}$ is stable, $A \in \mathcal{K}$. Thus, L has the FMP. □

Let $L' = \mathbf{IPC} + \{\gamma(A) \mid A \text{ is finite, s.i., and } A \notin \mathcal{V}\}$. We claim that $L = L'$. By Claim 5.4, L has the FMP, and by [3, Sec. 6], L' has the FMP. Therefore, it is sufficient to show that for each finite s.i. Heyting algebra B, we have $B \models L$ iff $B \models L'$. Suppose B is a finite s.i. Heyting algebra. If $B \not\models L'$, then there is a finite s.i. Heyting algebra $A \notin \mathcal{V}$ such that $B \not\models \gamma(A)$.

Claim 5.5. $\gamma(A) \in L$.

Proof of claim. If $\gamma(A) \notin L$, then since $\mathcal{V}$ is generated by $\mathcal{K}$, there is $C \in \mathcal{K}$ with $C \not\models \gamma(A)$. By Claim 5.2, A is isomorphic to a bounded sublattice of C. Since $\mathcal{K}$ is stable, $A \in \mathcal{K}$, which contradicts to $A \notin \mathcal{V}$. Thus, $\gamma(A) \in L$. □

By Claim 5.5, $\gamma(A) \in L$. Therefore, $B \not\models L$, showing that $L' \subseteq L$. Conversely, suppose that $B \not\models L$. By definition, $\gamma(B) \in L'$. Since $B \not\models \gamma(B)$, it follows that $B \not\models L'$. This shows that $L \subseteq L'$. Thus, $L = L'$, which yields that L is axiomatizable by stable formulas.

(3) $\Rightarrow$ (5): Suppose L is axiomatizable by stable rules. Then there is a family $\{A_i : i \in I\}$ of finite Heyting algebras such that $L = \Lambda(\mathcal{S})$, where $\mathcal{S} = \mathcal{S}_{\mathbf{IPC}} + \{\rho(A_i) \mid i \in I\}$. Therefore, $\mathcal{V}$ is generated by $\mathcal{U}(\mathcal{S})$, which by Proposition 4.5 is a stable universal class.

(5) $\Rightarrow$ (3): Suppose $\mathcal{V}$ is generated by a stable universal class $\mathcal{U}$. Let $\mathcal{S}$ be the intuitionistic multi-conclusion consequence relation corresponding to $\mathcal{U}$. By Proposition 4.5, $\mathcal{S} = \mathcal{S}_{\mathbf{IPC}} + \{\rho(A) \mid A \text{ finite}, A \notin \mathcal{U}\}$. Therefore, since $\mathcal{V}$ is generated by $\mathcal{U}$, we see that $L = \Lambda(S)$. Thus, L is axiomatizable by stable rules. $\square$

Remark 5.6. Let L be a si-logic and let $\mathcal{V}$ be the corresponding variety of Heyting algebras. By Theorem 5.3, L is stable iff $\mathcal{V}$ is generated by a stable class. This, however, does not mean that $\mathcal{V}$ itself is a stable class. In fact, the variety of all Heyting algebras is the only nontrivial stable variety of Heyting algebras. This can be seen as follows. Suppose $\mathcal{V}$ is a nontrivial stable variety of Heyting algebras. Since $\mathcal{V}$ is nontrivial, it contains the variety of Boolean algebras. Let A be an arbitrary Heyting algebra, and let $\mathcal{B}(A)$ be the free Boolean extension of A (see, e.g., [1, Ch. V.4]). Then $\mathcal{B}(A)$ is a Boolean algebra and A is isomorphic to a bounded sublattice of $\mathcal{B}(A)$. Therefore, $\mathcal{B}(A) \in \mathcal{V}$, and since $\mathcal{V}$ is stable, $A \in \mathcal{V}$. Thus, $\mathcal{V}$ is the variety of all Heyting algebras.

Remark 5.7. Let $\{A_i : i \in I\}$ be a family of finite s.i Heyting algebras. Then $\mathbf{IPC} + \{\gamma(A_i) \mid i \in A\} = \Lambda\left(\mathcal{S}_{\mathbf{IPC}} + \{\rho(A_i) \mid i \in I\}\right)$; that is, the si-logic axiomatized by the stable formulas $\{\gamma(A_i) : i \in I\}$ is the same as the si-logic axiomatized by the corresponding stable rules $\{\rho(A_i) : i \in I\}$. This can be seen as follows. By [3, Thm. 6.3] and Proposition 4.2, for every s.i. Heyting algebra B, we have $B \models \gamma(A_i)$ iff $B \models \rho(A_i)$. Therefore, the universal class $\mathcal{U}$ corresponding to the intuitionistic multi-conclusion consequence relation $\mathcal{S}_{\mathbf{IPC}} + \{\rho(A_i) \mid i \in I\}$ and the variety $\mathcal{V}$ corresponding to the si-logic $\mathbf{IPC} + \{\gamma(A_i) \mid i \in A\}$ contain the same s.i. members. Thus, $\mathcal{V}$ is generated by $\mathcal{U}$, yielding the result. As we will see in Remark 7.14, this does not generalize to a similar statement about $(\wedge, \vee)$-canonical formulas and $(\wedge, \vee)$-canonical rules.

6. Cofinal stable multi-conclusion consequence relations

While the theories of $(\wedge, \vee)$- and $(\wedge, \to)$-canonical rules and formulas have many similarities, there is one important difference. Namely, in the $(\wedge, \to)$-case, since $\to$ is preserved, preserving 0 implies preserving $\neg$. This is not true in the $(\wedge, \vee)$-case. Therefore, in the $(\wedge, \to)$-case, when 0 is preserved, putting $D = \varnothing$ yields cofinal subframe rules and formulas. To obtain subframe rules and formulas, along with $D = \varnothing$, we also have to take out $\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\}$ from Γ. Since in the $(\wedge, \vee)$-case $\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\}$ is not part of Γ, by taking $D = \varnothing$ we only obtain stable rules and formulas, which are analogues of subframe rules and formulas. In order to obtain analogues of cofinal subframe rules and formulas, we have to add $\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\}$ to Γ. Thus, instead of working with bounded distributive sublattices of Heyting algebras, we will have to work with pseudocomplemented sublattices. This is what we do next.

We recall that a *pseudocomplemented distributive lattice* is a bounded distributive lattice L with an additional unary operation $\neg : L \to L$ satisfying $a \wedge b = 0$ iff $a \leq \neg b$ (see, e.g., [1, Ch. VIII]). Let PDLat be the class of pseudocomplemented distributive lattices. Clearly PDLat is contained in the class DLat of bounded distributive lattices and contains the class HA of Heyting algebras.

It is well known that PDLat is a variety. We will require that PDLat is a locally finite variety. Since we were unable to find a proof of this result in the literature, we give a proof of it below.

Theorem 6.1. PDLat *is locally finite.*

Proof. We utilize the criterion of local finiteness from [2, Thm. 3.7] stating that a variety $\mathcal{V}$ of finite signature is locally finite iff for each $n \in \mathbb{N}$ there is $m(n) \in \mathbb{N}$ such that the cardinality of all n-generated s.i. members of $\mathcal{V}$ is bounded above by $m(n)$. It is well known (see, e.g., [1, Thm. VIII.5.1]) that s.i. members of PDLat are of the form $B \oplus 1$, where B is a Boolean algebra and $- \oplus 1$ is the operation of adjoining a new top. We claim that the cardinality of each n-generated

s.i. $A \in \mathsf{PDLat}$ is bounded above by $m(n) = 2^{2^n} + 1$. Indeed, suppose $A \cong B \oplus 1$ is generated by $g_1, \dots, g_n$. Without loss of generality we may assume that $g_1, \dots, g_n \in B$. Therefore, $g_1, \dots, g_n$ generate B as a Boolean algebra. Thus, the cardinality of B is bounded above by 2^{2^n}. This yields that the cardinality of A is bounded above by $2^{2^n} + 1$. Consequently, the criterion applies. □

We are ready to define cofinal stable classes of Heyting algebras and cofinal stable rules.

Definition 6.2.

(1) *We call a class $\mathcal{K}$ of Heyting algebras* cofinal stable *provided for Heyting algebras A, B, if A is isomorphic to a pseudocomplemented sublattice of B and $B \in \mathcal{K}$, then $A \in \mathcal{K}$.*

(2) *Let A be a finite Heyting algebra. The* cofinal stable rule associated with A *is the rule $\sigma(A) = \Gamma/\Delta$, where*

$$\begin{aligned}\Gamma = \{&p_0 \leftrightarrow 0\} \cup \\ &\{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid a, b \in A\} \cup \\ &\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b \mid a, b \in A\} \cup \\ &\{p_{\neg a} \leftrightarrow \neg p_a \mid a \in A\}\end{aligned}$$

and

$$\Delta = \{p_a \leftrightarrow p_b \mid a, b \in A \text{ with } a \neq b\}.$$

Remark 6.3. There is no need to add $p_1 \leftrightarrow 1$ to Γ because $p_{\neg 0} \leftrightarrow \neg 0$ is contained in $\{p_{\neg a} \leftrightarrow \neg p_a \mid a \in A\}$.

Using that PDLat is locally finite, the statements and proofs of Section 4 generalize to the setting of cofinal stable rules. Namely, Proposition 4.2 generalizes to the following proposition.

Proposition 6.4. *Suppose A, B are Heyting algebras with A finite. Then $B \not\models \sigma(A)$ iff A is isomorphic to a pseudocomplemented sublattice of B.*

Also, Proposition 4.5 generalizes to the following proposition.

Proposition 6.5. *An intuitionistic multi-conclusion consequence relation $\mathcal{S}$ is axiomatizable by cofinal stable rules iff the corresponding universal class $\mathcal{U}(\mathcal{S})$ of Heyting algebras is cofinal stable.*

We leave the proofs out since they are virtually the same as the relevant proofs in Section 4, but use local finiteness of PDLat instead of local finiteness of DLat.

Remark 6.6. Not only stable rules have an obvious generalization to cofinal stable rules, but also $(\wedge, \vee)$-canonical rules have an obvious generalization to $(\wedge, \vee, \neg)$-canonical rules. Suppose A is a finite Heyting algebra and $D \subseteq A^2$. For every $a \in A$ let p_a be a propositional letter, and define the $(\wedge, \vee, \neg)$-*canonical rule* $\zeta(A, D)$ associated with A and D as $\sigma(A, D) = \Gamma/\Delta$, where

$$\begin{aligned}\Gamma = \{&p_0 \leftrightarrow 0\} \cup \\ &\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b \mid a, b \in A\} \cup \\ &\{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid a, b \in A\} \cup \\ &\{p_{\neg a} \leftrightarrow \neg p_a \mid a \in A\} \cup \\ &\{p_{a\to b} \leftrightarrow p_a \to p_b \mid (a, b) \in D\}\end{aligned}$$

and

$$\Delta = \{p_a \leftrightarrow p_b \mid a, b \in A \text{ with } a \neq b\}.$$

All proofs from Section 3 have an obvious translation to the $(\wedge, \vee, \neg)$-setting by utilizing the local finiteness of the pseudocomplemented bounded lattice reduct instead of the bounded lattice reduct. Consequently, $(\wedge, \vee, \neg)$-canonical rules provide yet another uniform axiomatization of all intuitionistic multi-conclusion consequence relations.

7. Cofinal stable logics

The same way we generalized stable rules to cofinal stable rules, we can generalize stable formulas to cofinal stable formulas. However, unlike the stable case, it is no longer true that cofinal stable logics are axiomatizable by cofinal stable formulas. This mismatch is essentially due to the fact that the analogue of [3, Lem. 6.2] fails for the $(\wedge, \vee, \neg)$-reducts of Heyting algebras. Nevertheless, we will show that cofinal stable logics are axiomatizable by cofinal stable rules.

Let A be a finite s.i. Heyting algebra, and let $\sigma(A) = \Gamma/\Delta$ be the cofinal stable rule associated with A. We define the *cofinal stable formula associated with* A as $\delta(A) = \bigwedge \Gamma \to \bigvee \Delta$. Then [3, Thm. 3.4] has an obvious generalization to the following proposition.

Proposition 7.1. *Let A, B be Heyting algebras with A finite and s.i. Then $B \not\models \delta(A)$ iff there is a s.i. homomorphic image C of B such that A is isomorphic to a pseudocomplemented sublattice of C.*

If B is well-connected and we work with $\gamma(A)$ instead of $\delta(A)$, then as is shown in Proposition 5.1, the homomorphic image C can be omitted from the consideration. We will see that this is no longer true if we work with $\delta(A)$. However, one implication of the proposition remains true, with the same proof.

Lemma 7.2. *Let A, B be Heyting algebras with A finite and s.i. and B well-connected. If A is isomorphic to a pseudocomplemented sublattice of B, then $B \not\models \delta(A)$.*

To see that the converse of Lemma 7.2 does not hold, we recall the representation theorem for pseudocomplemented distributive lattices. Since pseudocomplemented distributive lattices are situated between bounded distributive lattices and Heyting algebras, their representation is closely related to the Priestley representation of bounded distributive lattices and the Esakia representation of Heyting algebras.

We call a subset U of a (partially) ordered set X an *upset* if $x \in U$ and $x \leq y$ imply $y \in U$; we define *downsets* dually. For $U \subseteq X$, let $\uparrow U$ be the upset and $\downarrow U$ be the downset generated by U. If $U = \{x\}$, then we write $\uparrow x$ and $\downarrow x$, respectively. A *Priestley space* is a compact ordered space X such that $x \not\leq y$ implies that there is a clopen (closed and open) upset of X containing x and missing y. A map between Priestley spaces is a *Priestley morphism* if it is continuous and order preserving. By Priestley duality [20, 21], the category DLat of bounded distributive lattices and bounded lattice homomorphisms is dually equivalent to the category Pries of Priestley spaces and Priestley morphisms.

An *Esakia space* is a Priestley space in which $\downarrow U$ is open for each open set U. A map $f : X \to Y$ between Esakia spaces is an *Esakia morphism* if it is a continuous p-morphism, where we recall that f is a p-morphism provided $f(\uparrow x) = \uparrow f(x)$. By Esakia duality [11], the category HA of Heyting algebras and Heyting homomorphisms is dually equivalent to the category Esa of Esakia spaces and Esakia morphisms.

Since pseudocomplemented distributive lattices are situated between Heyting algebras and bounded distributive lattices, dual spaces of pseudocomplemented distributive lattices are situated between Esakia spaces and Priestley spaces. They were characterized by Priestley in [22]. For a subset U of an ordered set X, let $\max(U)$ be the set of maximal points of U ($m \in \max(U)$ provided $(\forall y \in U)(x \leq y \Rightarrow x = y)$). We call a Priestley space X a *PC-space* if $\downarrow U$ is open for each open upset U. Clearly each Esakia space is a PC-space, but the converse is not true in general. We call a map $f : X \to Y$ between ordered spaces a *q-morphism* if $\max{\uparrow} f(x) = f(\max{\uparrow} x)$. By [22], the category PDLat of pseudocomplemented distributive lattices and pseudocomplemented lattice homomorphisms is dually equivalent to the category PCS of PC-spaces and continuous q-morphisms.

The next lemma gives several characterizations of q-morphisms.

Lemma 7.3. *Let $f : X \to Y$ be a Priestley morphism between PC-spaces. The following are equivalent.*

(1) *f is a q-morphism.*
(2) *For all $x \in X$, we have* $\max{\uparrow} f(x) \subseteq f(\max{\uparrow} x)$.
(3) *For all $x \in X$ and $y \in Y$, from $f(x) \leq y$ it follows that there is $x' \in X$ with $x \leq x'$ and $y \leq f(x')$.*
(4) *For every upset $A \subseteq Y$, we have $f^{-1}(\downarrow A) = \downarrow f^{-1}(A)$.*
(5) *For every $y \in Y$, we have $f^{-1}(\downarrow\uparrow y) = \downarrow f^{-1}(\uparrow y)$.*

Proof. (1) $\Rightarrow$ (2): This is obvious.

(2) $\Rightarrow$ (3): Suppose $x \in X$, $y \in Y$, and $f(x) \leq y$. Then $y \in \uparrow f(x)$. Since Y is a Priestley space, there is $y' \in \max{\uparrow} f(x)$ with $y \leq y'$. By (2), $y' \in f(\max{\uparrow} x)$. Therefore, there is $x' \in X$ such that $x \leq x'$ and $f(x') = y' \geq y$. This shows (3).

(3) $\Rightarrow$ (1): Let $x \in X$. We first show that $\max{\uparrow} f(x) \subseteq f(\max{\uparrow} x)$. Suppose $y \in \max{\uparrow} f(x)$. By (3), there is $x' \in X$ with $x \leq x'$ and $y \leq f(x')$. Since y is a maximal point, $y = f(x')$. As X is a Priestley space, there is $x'' \in \max{\uparrow} x$ with $x' \leq x''$. Since f is order preserving and y is maximal, $f(x'') = y$. Thus, $y \in f(\max{\uparrow} x)$. Next we show that $f(\max{\uparrow} x) \subseteq \max{\uparrow} f(x)$. First note that every map satisfying (3) maps maximal points to maximal points. For let $x \in X$ be maximal and suppose x is mapped to a non-maximal $y \in Y$. Then there is y' in Y with $y < y'$. However, since x is maximal, there is no x' with $x \leq x'$ that is mapped to y'. So Condition (3) is violated for x. Now suppose $y \in f(\max{\uparrow} x)$. Then there is $x' \in \max{\uparrow} x$ such that $f(x') = y$. Since f is order preserving, $f(x) \leq f(x')$. By the above, x' is mapped to a maximal point, so $f(x') = y \in \max{\uparrow} f(x)$. Thus, f is a q-morphism.

(3) $\Rightarrow$ (4): Let $A \subseteq Y$ be an upset. Since f is order preserving, $\downarrow f^{-1}(A) \subseteq f^{-1}(\downarrow A)$. We show $f^{-1}(\downarrow A) \subseteq \downarrow f^{-1}(A)$. Suppose $x \in f^{-1}(\downarrow A)$. Then there is $y \in A$ with $f(x) \leq y$. By (3), there is $x' \geq x$ with $y \leq f(x')$. Since A is an upset, $f(x') \in A$, yielding $x \in \downarrow f^{-1}(A)$.

(4) $\Rightarrow$ (5): This is obvious since $\uparrow y$ is an upset.

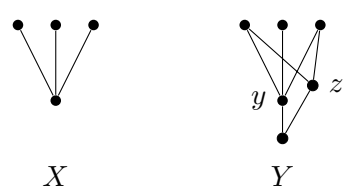
(5) $\Rightarrow$ (3): Let $x \in X$, $y \in Y$, and $f(x) \leq y$. Then $f(x) \in \downarrow\uparrow y$ so $x \in f^{-1}(\downarrow\uparrow y)$. By (5), $x \in \downarrow f^{-1}(\uparrow y)$. This implies that there is $x' \geq x$ with $y \leq f(x')$, which proves (3). □

Remark 7.4. Conditions (1) and (2) are not *locally* equivalent; that is, it is not true that for a given $x \in X$ we have $\max{\uparrow} f(x) = f(\max{\uparrow} x)$ iff $\max{\uparrow} f(x) \subseteq f(\max{\uparrow} x)$.

We are ready to see that the converse of Lemma 7.2 does not hold. For this we must find Heyting algebras A, B such that A is finite and s.i., B is well-connected, $B \not\models \delta(A)$, and yet A is not isomorphic to a pseudocomplemented sublattice of B. By Proposition 7.1, $B \not\models \delta(A)$ iff there is a s.i. homomorphic image C of B such that A is isomorphic to a pseudocomplemented sublattice of C. Thus, we want A to be isomorphic to a pseudocomplemented sublattice of a s.i. homomorphic image C of B, and yet not to be isomorphic to a pseudocomplemented sublattice of B itself. We do this by passing to the dual picture.

Dually finite Heyting algebras correspond to finite posets (partially ordered sets). If A is a finite Heyting algebra and X is its dual finite poset, then homomorphic images of A correspond to upsets of X, subalgebras of A to p-morphic images of X, pseudocomplemented sublattices of A to q-morphic images of X, and bounded sublattices of A to order preserving images of X (see [11, 22, 21]). In addition, A is s.i. iff A is well-connected, which happens iff X is *rooted* (that is, $X = \uparrow x$ for some $x \in X$) [12, 4]. Therefore, it is sufficient to find two finite rooted posets X, Y such that X is a q-morphic image of an upset U of Y, but X is not a q-morphic image of Y.

Example 7.5. Let X and Y be the finite rooted posets drawn below.



Clearly X is isomorphic to the upset $\uparrow y$ of Y. Therefore, X is a q-morphic image of an upset of Y. On the other hand, we show that X is not a q-morphic image of Y. First observe that an onto q-morphism $f : Y \to X$ would map the maximal points of Y onto the maximal points of X. Now, $z \in Y$ sees two maximal points of Y, so since f is order preserving, $f(z)$ would need to be the root of X. But as X has three maximal points, the maximal points above z are not mapped onto the maximal points above $f(z)$. Thus, f cannot be a q-morphism.

We next define cofinal stable si-logics by directly generalizing the definition of stable si-logics of [3].

Definition 7.6. *We call a si-logic L* cofinal stable *provided for all s.i. Heyting algebras A, B, if A is isomorphic to a pseudocomplemented sublattice of B and $B \models L$, then $A \models L$.*

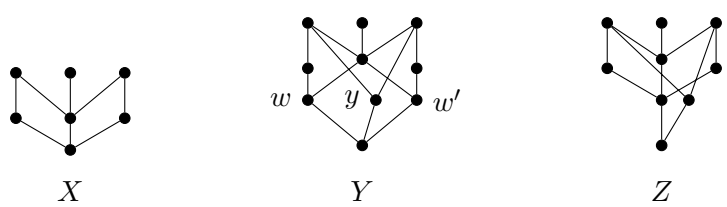
Proposition 7.7. *If L is cofinal stable, then L is axiomatizable by cofinal stable formulas.*

Proof. We prove that $L = \mathbf{IPC} + \{\delta(A) \mid A \text{ finite and s.i. }, A \not\models L\}$. Suppose that B is a s.i Heyting algebra with $B \models L$. Let A be a finite s.i. Heyting algebra with $A \not\models L$. If $B \not\models \delta(A)$, then by Proposition 7.1, there is a s.i. homomorphic image C of B such that A is isomorphic to a pseudocomplemented sublattice of C. Since $B \models L$ and C is a homomorphic image of B, we see that $C \models L$. As L is cofinal stable, from $C \models L$ it follows that $A \models L$. The obtained contradiction proves that $B \models \delta(A)$. Conversely, suppose that B is a s.i. Heyting algebra with $B \not\models L$. Then there is $\varphi \in L$ with $B \not\models \varphi$. As in the proof of [3, Lem. 3.6], but using that the variety of pseudocomplemented distributive lattices is locally finite, we can construct a finite s.i. Heyting algebra A such that A is a pseudocomplemented sublattice of B and $A \not\models \varphi$. Therefore, $A \not\models L$. Using Proposition 7.1 again yields $B \not\models \delta(A)$. □

Remark 7.8. The proof above also shows that all cofinal stable logics have the FMP.

Unlike the case of stable logics, the converse of Proposition 7.7 does not hold in general, as the following example shows.

Example 7.9. Let X, Y, Z be the finite rooted posets drawn below.



Let A, B, C be the dual Heyting algebras of X, Y, Z, respectively, and let $L := \mathbf{IPC} + \delta(A)$. Then L is axiomatized by a cofinal stable formula. On the other hand, we show that L is not cofinal stable. First observe that X is not a q-morphic image of Y. Indeed, if there were an onto q-morphism $f : Y \to X$, then f would map the maximal points of Y onto the maximal points of X. But $y \in Y$ sees exactly two maximal points, while X has no point with this property. This violates the q-morphism condition at y, a contradiction. Since each rooted upset of Y has smaller cardinality than X, it follows that there is no rooted upset U of Y such that X is a q-morphic image of U. From this we conclude by Proposition 7.1 that $B \models \delta(A)$, so $B \models L$. On the other hand, Z is a q-morphic image of Y, witnessed by the map that identifies w and w' in Y. Therefore, C is isomorphic to a pseudocomplemented sublattice of B. It is also obvious that X is isomorphic to an upset of Z, so by Proposition 7.1, $C \not\models \delta(A)$. Thus, $C \not\models L$. Since $B \models L$, C is isomorphic to a pseudocomplemented sublattice of B, and $C \not\models L$, we conclude that L is not a cofinal stable logic.

Remark 7.10. Although as pointed out in Remark 7.8 cofinal stable logics have the FMP, we do not know whether all logics axiomatized by cofinal stable formulas have the FMP. Neither do we know a convenient characterization of this class of logics.

As follows from Example 7.9, Theorem 5.3 does not fully generalize to the setting of cofinal stable logics as the analogue of Condition (2) of the theorem is not equivalent to Condition (1). Nevertheless, we will prove that all the conditions of the theorem but Condition (2) are still equivalent in the cofinal stable setting. In fact, some of the proofs will have a straightforward generalization, while others will require additional work.

Theorem 7.11. *Let L be a si-logic and let $\mathcal{V}$ be the corresponding variety of Heyting algebras. The following are equivalent.*

(1′) *L is a cofinal stable logic.*
(2′) *$\mathcal{V}_{wc}$ is cofinal stable.*
(3′) *$\mathcal{V}$ is generated by a cofinal stable universal class $\mathcal{U}$.*
(4′) *$\mathcal{V}$ is generated by a cofinal stable class $\mathcal{K}$.*
(5′) *L is axiomatizable by cofinal stable rules.*

Proof. The proof of (2′) ⇒ (3′) is the same as the proof of (4) ⇒ (5) in Theorem 5.3, and the proof of (3′) ⇒ (4′) is obvious.

(4′) ⇒ (2′): Let $\mathcal{V}$ be generated by a cofinal stable class $\mathcal{K}$. Suppose B, C are Heyting algebras such that $C \in \mathcal{V}_{wc}$ and B is isomorphic to a pseudocomplemented sublattice of C. Then B is also well-connected, so it remains to prove that $B \in \mathcal{V}$. If $B \notin \mathcal{V}$, then there is $\varphi \in L$ with $B \not\models \varphi$. We can find a finite pseudocomplemented sublattice B' of B with $B' \not\models \varphi$. Clearly $B' \not\models L$. Since B is well-connected, so is B', and as B' is finite, it is s.i. Next we require an analogue of Claim 5.5. But the proof in the stable case makes use of [3, Lem. 6.2], whose analogue, as we saw, does not hold in the cofinal stable case. Nevertheless, we can derive the desired result using the following claim instead.

Claim 7.12. *Suppose A, B are finite Heyting algebras such that A is isomorphic to a pseudocomplemented sublattice of a homomorphic image of B. Then there is a finite Heyting algebra C such that C is isomorphic to a pseudocomplemented sublattice of B and A is a homomorphic image of C.*

Proof of claim. Suppose A is isomorphic to a pseudocomplemented sublattice of a homomorphic image of B. Since B is finite, homomorphic images of B are up to isomorphism quotients of B by principal filters. Clearly $B/{\uparrow}b \cong [0, b]$, and the isomorphism $h : B/{\uparrow}b \to [0, b]$ is given by $h(x) = b \wedge x$. Therefore, we may assume that A is isomorphic to a pseudocomplemented sublattice of $[0, b]$, and we identify A with its image in $[0, b]$. Let $C = h^{-1}(A)$. It is obvious that C is a pseudocomplemented sublattice of B. We claim that the restriction of h to C is a Heyting homomorphism onto A. For this it is sufficient to check that h preserves Heyting implication. Let $x, y \in C$. Since C is finite and h commutes with joins,

$$h(x \to_C y) = \bigvee_A \{h(c) \mid c \in C \text{ and } x \wedge c \leq y\} = \bigvee_A \{c \wedge b \mid c \in C \text{ and } x \wedge c \leq y\}$$

and

$$h(x) \to_A h(y) = \bigvee_A \{z \in A \mid (x \wedge b) \wedge z \leq y\}.$$

It is easy to see that $\{c \wedge b \mid c \in C \text{ and } x \wedge c \leq y\} \subseteq \{z \in A \mid (x \wedge b) \wedge z \leq y\}$. For the reverse inclusion, let $z \in A$ with $(x \wedge b) \wedge z \leq y$. Since $z \in A \subseteq [0, b]$, we see that $z = z \wedge b = h(z)$, so $z \in C$ and $x \wedge z = x \wedge (b \wedge z) \leq y$. This shows that $\{z \in A \mid (x \wedge b) \wedge z \leq y\} \subseteq \{c \wedge b \mid c \in C \text{ and } x \wedge c \leq y\}$. Therefore, $h(x \to_C y) = h(x) \to_A h(y)$. Thus, the restriction of h to C is a Heyting homomorphism and it is clearly onto. □

Claim 7.13. *If $\mathcal{V}$ is generated by a cofinal stable class $\mathcal{K}$, then for every finite s.i. Heyting algebra A, from $A \not\models L$ it follows that $\delta(A) \in L$.*

Proof of claim. Suppose A is a finite s.i. Heyting algebra and $A \not\models L$. Since $\mathcal{V}$ is generated by a cofinal stable class $\mathcal{K}$, a similar argument to Claim 5.4 yields that each non-theorem of L is refuted on a finite member of $\mathcal{K}$. In particular, L has the FMP. Therefore, if $\delta(A) \notin L$, then there is a finite $B \in \mathcal{K}$ with $B \not\models \delta(A)$. By Proposition 7.1, there is a s.i. homomorphic image B' of B such that A is isomorphic to a pseudocomplemented sublattice of B'. By Claim 7.12, there is a Heyting algebra A' such that A' is isomorphic to a pseudocomplemented sublattice of B and A is a homomorphic image of A'. Since $B \in \mathcal{K}$ and $\mathcal{K}$ is cofinal stable, $A' \in \mathcal{K}$, so $A' \models L$. As A is a homomorphic image of A', we see that $A \models L$. The obtained contradiction proves that $\delta(A) \in L$. □

We are ready to complete the proof of $(4') \Rightarrow (2')$. Since B' is a finite s.i. Heyting algebra such that $B' \not\models L$, by Claim 7.13, $\delta(B') \in L$. As C is well-connected and B' is isomorphic to a pseudocomplemented sublattice of C, Lemma 7.2 yields $C \not\models \delta(B')$. Since $\delta(B') \in L$, this contradicts $C \models L$. The obtained contradiction proves that $B \in \mathcal{V}$.

$(3') \Rightarrow (5')$: Suppose $\mathcal{V}$ is generated by a cofinal stable universal class $\mathcal{U}$. Let $\mathcal{S}$ be the intuitionistic multi-conclusion consequence relation corresponding to $\mathcal{U}$. By Proposition 6.5, $\mathcal{S} = \mathcal{S}_{\mathbf{IPC}} + \{\sigma(A) \mid A \text{ finite}, A \notin \mathcal{U}\}$. Therefore, since $\mathcal{V}$ is generated by $\mathcal{U}$, we see that $L = \Lambda(S)$. Thus, L is axiomatizable by cofinal stable rules.

$(5') \Rightarrow (3')$: Suppose L is axiomatizable by cofinal stable rules. Then there is a family $\{A_i : i \in I\}$ of finite Heyting algebras such that $L = \Lambda(\mathcal{S})$, where $\mathcal{S} = \mathcal{S}_{\mathbf{IPC}} + \{\sigma(A_i) \mid i \in I\}$. Therefore, $\mathcal{V}$ is generated by $\mathcal{U}(\mathcal{S})$, which by Proposition 6.5 is a cofinal stable universal class.

The proof of $(2') \Rightarrow (1')$ is easy.

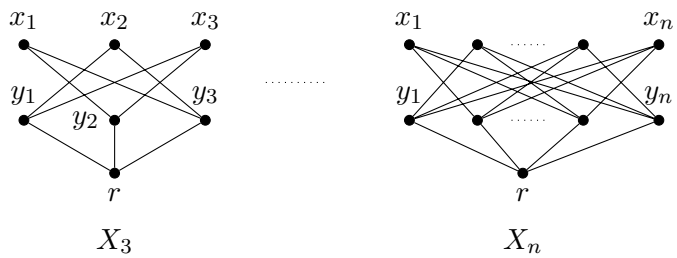
$(1') \Rightarrow (4')$: Let $\mathcal{V}_{fsi}$ be the class of finite s.i. members of $\mathcal{V}$. We show that $\mathcal{V}_{fsi}$ is a cofinal stable class generating $\mathcal{V}$. Since L is cofinal stable, as we pointed out in Remark 7.8, L has the FMP. Therefore, $\mathcal{V}$ is generated by $\mathcal{V}_{fsi}$. To see that $\mathcal{V}_{fsi}$ is cofinal stable, let A, B be Heyting algebras such that A is isomorphic to a pseudocomplemented sublattice of B and $B \in \mathcal{V}_{fsi}$. Since B is finite, B is well-connected, so A is well-connected, and as A is finite, we conclude that A is s.i. Therefore, by $(1')$, $A \in \mathcal{V}_{fsi}$, showing that $\mathcal{V}_{fsi}$ is cofinal stable. □

Remark 7.14. The analogue of Remark 5.7 does not hold in the cofinal stable case. To see this, let X, Y, Z be the finite rooted posets of Example 7.9 and let A, B, C be their dual Heyting algebras. It is easy to see that X is not a q-morphic image of Z. Therefore, $C \models \sigma(A)$, so $C \models \Lambda(\mathcal{S}_{\mathbf{IPC}} + \sigma(A))$. On the other hand, we saw in Example 7.9 that $C \not\models \delta(A)$. Thus, $\mathbf{IPC} + \delta(A) \neq \Lambda(\mathcal{S}_{\mathbf{IPC}} + \sigma(A))$. The same example shows that Remark 5.7 cannot be generalized to $(\wedge, \vee)$-canonical formulas and rules. In fact, the same reasoning as above shows that $\mathbf{IPC} + \gamma(A, D) \neq \Lambda(\mathcal{S}_{\mathbf{IPC}} + \rho(A, D))$, where $D := \{(a, 0) \mid a \in A\} \subseteq A^2$.

8. Cardinality

It is clear that each stable multi-conclusion consequence relation is a cofinal stable multi-conclusion consequence relation, and that each stable logic is a cofinal stable logic, but the converse is not true in general. It was shown in [3, Thm. 6.13] that there are continuum many stable logics. We show that there are continuum many cofinal stable logics that are not stable logics. It follows that there are continuum many cofinal stable multi-conclusion consequence relations that are not stable multi-conclusion consequence relations.

Consider the sequence of finite posets drawn below. It is well known that no member of this sequence is a p-morphic image of an upset of some other member. We will show that the same result holds if we replace p-morphisms with q-morphisms.



In each X_n we have $x_i \leq y_j$ iff $i \neq j$ and r is the root.

Lemma 8.1. *For $n, m \geq 3$, if $n \neq m$, then X_n is not a q-morphic image of any upset of X_m.*

Proof. Let $m, n \geq 3$ with $n \neq m$. Clearly if $m < n$, then X_n cannot be a q-morphic image of an upset of X_m. Suppose $n < m$. Let r_n be the root of X_n and r_m be the root of X_m. Also, let x_i, y_i be the elements of X_n and x'_i, y'_i be the elements of X_m. Suppose there is an onto q-morphism f from an upset U of X_m onto X_n. First we show that $\max(X_m) \subseteq U$. Since f is an onto q-morphism, the maximal elements of U have to be mapped onto the maximal elements of X_n, so $f(\max U) = \max(X_n)$. As $n \geq 3$, this implies that U contains y'_i, y'_j for $i \neq j$. But $\max(X_m) \subseteq \uparrow\{y'_i, y'_j\} \subseteq U$. Next we observe that no $y'_i \in U$ is mapped to some x_j. For suppose there is $y'_i \in U$ with $f(y'_i) = x_j$. Since y'_i sees all but one maximal elements of X_m, we see that f maps all but one maximal elements to x_j. As X_n has at least 3 maximal elements, this contradicts to the fact that $f(\max X_m) = \max(X_n)$. Therefore, each $y'_i \in U$ is mapped to some y_j or to r_n. Now suppose f maps two maximal elements x'_i and x'_j to the same maximal element x_k. If $y'_i \in U$, then $\max{\uparrow} f(y'_i) = f(\max{\uparrow} y'_i) = \max(X_n)$. This means that f maps y'_i to the root r_n. A similar argument gives that if $y'_j \in U$, then $f(y'_j) = r_n$. Since $m > n$, by the pigeonhole principle, there is at least one maximal element of X_n that has two f-preimages. Therefore, there are at most $n-1$ maximal elements of X_n that have a unique f-preimage. This in turn means that there are at most $n-1$ candidates $y_i \in U$ that f could map to some element of depth 2 of X_n. But then f cannot be onto. The obtained contradiction proves that there is no q-morphism from an upset of X_m onto X_n. □

On the other hand, we have the following lemma.

Lemma 8.2. *If $m \geq 2n$, then there is an onto order preserving map $f : X_m \to X_n$.*

Proof. Using the same notation as in the previous lemma, define $f : X_m \to X_n$ as follows. For $i \leq n$, send x'_i to x_i, x'_{n+i} to y_i, and send the rest of the points in X_m to the root r_n of X_n. It is straightforward to see that such f is an onto order preserving map. □

We are ready to prove the main result of this section. To simplify notation, for $n \geq 3$, we denote the cofinal stable formula of the dual Heyting algebra of X_n by $\delta(X_n)$.

Proposition 8.3. *There are continuum many cofinal stable si-logics that are not stable.*

Proof. For $n \geq 3$, suppose $\mathcal{K}_n$ is the class of q-morphic images of X_n. Let $I = \{n \in \mathbb{N} \mid n \geq 3\}$, and for $J \subseteq I$, let $L(J)$ be the logic of $\bigcup_{n \in J} \mathcal{K}_n$. Since $\bigcup_{n \in J} \mathcal{K}_n$ is closed under q-morphic images, the corresponding class of finite Heyting algebras is cofinal stable. Thus, $L(J)$ is a cofinal stable logic.

Claim 8.4. *For $S, T \subseteq I$, if $S \neq T$, then $L(S) \neq L(T)$.*

Proof of claim. We may assume without loss of generality that there is $n \in S \setminus T$. Then $X_n \models L(S)$, which implies that $\delta(X_n) \notin L(S)$ since $X_n \not\models \delta(X_n)$. We prove that $\delta(X_n) \in L(T)$. If $\delta(X_n) \notin L(T)$, then there is $m \in T$ and $Y \in \mathcal{K}_m$ with $Y \not\models \delta(X_n)$. By the dual statement of Proposition 7.1, there is a rooted upset Z of Y such that X_n is a q-morphic image of Z. Now $Y \in \mathcal{K}_m$ means that Y is a q-morphic image of X_m. Let $f : X_m \to Y$ be an onto q-morphism, and let $U = f^{-1}(Z)$. Then U is

an upset of X_m and the restriction of f to U is a q-morphism onto Z. Therefore, Z is a q-morphic image of an upset of X_m, and hence X_n is a q-morphic image of an upset of X_m. Since $n \notin T$, we have $m \neq n$. This contradicts Lemma 8.1. Thus, $\delta(X_n) \in L(T)$, and so $L(S) \neq L(T)$. □

Let $\Delta := \{J \subseteq I \mid J \text{ and } I \setminus J \text{ are infinite}\}$. By Claim 8.4, $\{L(J) \mid J \in \Delta\}$ provides continuum many cofinal stable logics. It remains to be shown that $L(J)$ is not stable for each $J \in \Delta$. Let $J \in \Delta$. Then there is $n \in I \setminus J$. By the proof of Claim 8.4, $\delta(X_n) \in L(J)$. Therefore, since $X_n \not\models \delta(X_n)$, we see that $X_n \not\models L(J)$. Now there is $m \in J$ with $m \geq 2n$. As $m \in J$, we have $X_m \models L(J)$. By Lemma 8.2, X_n is a stable image of X_m. Because $X_n \not\models L(J)$, we conclude that $L(J)$ is not stable. □

Remark 8.5. Let Y_n be obtained from X_n by removing the root. Then the same proof as above, but using the cofinal stable rules $\sigma(Y_n)$ instead of the cofinal stable formulas $\delta(X_n)$ yields a continuum of cofinal stable multi-conclusion consequence relations of depth 2. This shows that cofinal stable multi-conclusion consequence relations have much more complicated structure than cofinal stable logics as there are only countably many cofinal stable logics of depth 2.

9. Examples

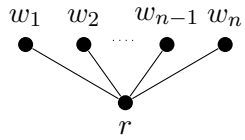
In this final section we give examples comparing cofinal stable, stable, subframe, and cofinal subframe si-logics. In the previous section we saw that there are continuum many cofinal stable logics that are not stable. In this section we show that each logic of depth 2 is a cofinal stable logic, while none of these but one is a stable logic. Examples of stable logics that are neither splitting nor subframe logics, as well as examples of splitting and subframe logics that are not stable logics were given in [3, Thm. 7.4]. We give an example of a stable logic that is not a cofinal subframe logic. We also give an example of a cofinal stable logic that is neither stable nor cofinal subframe logic.

For a finite poset X, let $\mathbf{depth}(X)$ denote the depth of X. We recall that $\mathbf{depth}(X) = n$ if X contains a chain of length n, but no chain of bigger length. Let $\mathbf{BD}_n$ be the si-logic of all finite ordered sets of depth $\leq n$ and let $\mathbf{LC}_n$ be the si-logic of finite chains of depth $\leq n$. Then $\mathbf{CPC} = \mathbf{BD}_1 = \mathbf{LC}_1$, and a si-logic L is of depth n iff $\mathbf{BD}_n \subseteq L \subseteq \mathbf{LC}_n$.

It was shown in [3, Thm. 7.4] that each $\mathbf{BD}_n$, for $n \geq 2$, is not stable.

Proposition 9.1. *Suppose L is a si-logic containing $\mathbf{BD}_2$. Then L is cofinal stable, and it is stable iff $\mathbf{LC}_2 \subseteq L$.*

Proof. It is well known (see, e.g., [15]) that $\mathbf{BD}_2$ is the logic of all finite forks $\mathfrak{F}_n$ drawn below, that $\mathbf{LC}_2$ is the logic of the two element chain $\mathfrak{F}_1$, and that each L in the interval $[\mathbf{BD}_2, \mathbf{LC}_2]$ is the logic of the n-fork $\mathfrak{F}_n$ for some $n \geq 1$.



$\mathfrak{F}_n$

It follows from the proof of [3, Thm. 7.4] that no si-logic L that contains $\mathbf{BD}_2$ and is properly contained in $\mathbf{LC}_2$ is stable. The only other si-logics containing $\mathbf{BD}_2$ are $\mathbf{LC}_2, \mathbf{CPC}$, and the inconsistent logic. It is easy to see that each of these is stable. We prove that each L containing $\mathbf{BD}_2$ is cofinal stable. Since q-morphisms map maximal elements to maximal elements, it is immediate to see that every q-morphic image of the n-fork is either the one-point poset or the m-fork for $m \leq n$. Therefore, for each $L \supseteq \mathbf{BD}_2$, the class of finite rooted L-frames is closed under q-morphic

images. This means that the class of finite s.i. L-algebras is cofinal stable, and it generates the variety corresponding to L. Thus, by Theorem 7.11, $L \supseteq \mathbf{BD}_2$ is cofinal stable. $\square$

We next show that $\mathbf{BD}_2$ can be axiomatized by the cofinal stable formula of the dual Heyting algebra of the three-element chain $\mathfrak{C}_3 = \{w_1, w_2, w_3\}$, where $w_1 < w_2 < w_3$. For convenience, we write $\delta(\mathfrak{C}_3)$ for the cofinal stable formula of the Heyting algebra dual to $\mathfrak{C}_3$. (Clearly this Heyting algebra is the four-element chain.)

Lemma 9.2. *Suppose X is a finite rooted poset. Then* $\mathbf{depth}(X) \leq 2$ *iff* $\mathfrak{C}_3$ *is not a q-morphic image of X.*

Proof. We prove that $\mathbf{depth}(X) \geq 3$ iff $\mathfrak{C}_3$ is a q-morphic image of X. First suppose that $\mathbf{depth}(X) \geq 3$. Then X contains a chain $x_1 < x_2 < x_3$. Define $f : X \to \mathfrak{C}_3$ by setting

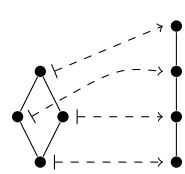
$$f(y) = \begin{cases} w_1 & \text{if } y \in {\downarrow}x_1 \\ w_2 & \text{if } y \in {\downarrow}x_2 \setminus {\downarrow}x_1, \\ w_3 & \text{otherwise.} \end{cases}$$

It is easy to see that f is an onto q-morphism (in fact, it is an onto p-morphism). Conversely, suppose $f : X \to \mathfrak{C}_3$ is an onto q-morphism. We show that X contains a chain of three elements. Since f is onto, the root r of X is mapped by f to w_1. Using again that f is onto, we find $x > r$ with $f(x) = w_2$. Since $f(x) = w_2 < w_3$ and $w_3 \in \max(\mathfrak{C}_3)$, there is $y > x$ such that $f(y) = w_3$. Thus, $r < x < y$ is a three element chain in X. $\square$

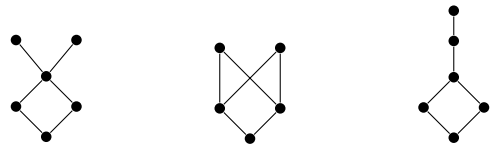
Proposition 9.3. $\mathbf{BD}_2 = \mathbf{IPC} + \delta(\mathfrak{C}_3)$.

Proof. Suppose $\mathfrak{F}_n$ is the n-fork. By Lemma 9.2, $\mathfrak{C}_3$ is not a q-morphic image of a rooted upset of $\mathfrak{F}_n$. By the dual statement of Proposition 7.1, $\mathfrak{F}_n \models \delta(\mathfrak{C}_3)$. Since $\mathbf{BD}_2$ is the logic of all n-forks, we conclude that $\mathbf{IPC} + \delta(\mathfrak{C}_3) \subseteq \mathbf{BD}_2$. Conversely, suppose X is an Esakia space such that $X \not\models \mathbf{BD}_2$. Since $\mathbf{BD}_2 = \mathbf{IPC} + \mathbf{bd_2}$, where $\mathbf{bd_2} = q \vee (q \to (p \vee \neg p))$, we see that $X \not\models \mathbf{bd_2}$. By dualizing the end of the proof of Proposition 7.7, we can construct a finite rooted poset Y such that Y is a continuous q-morphic image of X and $Y \not\models \mathbf{bd_2}$. It follows that $\mathbf{depth}(Y) \geq 3$. By Lemma 9.2, $\mathfrak{C}_3$ is a q-morphic image of Y. Therefore, $\mathfrak{C}_3$ is a continuous q-morphic image of X. Thus, by the dual statement of Proposition 7.1, $X \not\models \delta(\mathfrak{C}_3)$. As every si-logic L is complete with respect to the Esakia spaces validating L, we conclude that $\mathbf{BD}_2 \subseteq \mathbf{IPC} + \delta(\mathfrak{C}_3)$. $\square$

For $n \geq 3$, the logics $\mathbf{BD}_n$ are neither stable nor cofinal stable. The following picture shows an onto q-morphism from a rooted poset of depth 3 to a rooted poset of depth 4. We infer that the class of finite rooted posets of depth 3 is not closed under q-morphic images, which entails that $\mathbf{BD}_3$ is not cofinal stable. Clearly similar examples can be constructed to show that $\mathbf{BD}_n$ is not cofinal stable for all $n \geq 3$.



We next construct a tabular stable logic that is not a cofinal subframe logic. Let $\mathfrak{F}, \mathfrak{G}$, and $\mathfrak{H}$ be the posets drawn below. We let $St(\mathfrak{F}) := \{\mathfrak{K} \mid \mathfrak{K} \text{ is an order preserving image of } \mathfrak{F}\}$ and $\mathbf{L}_{St(\mathfrak{F})}$ be the logic of $St(\mathfrak{F})$. Since the class of Heyting algebras corresponding to the class $St(\mathfrak{F})$ is stable, $\mathbf{L}_{St(\mathfrak{F})}$ is a stable logic.



$\mathfrak{F}$ $\mathfrak{G}$ $\mathfrak{H}$

Lemma 9.4. *The logic $\mathbf{L}_{St(\mathfrak{F})}$ is not a cofinal subframe logic.*

Proof. Suppose $\mathbf{L}_{St(\mathfrak{F})}$ is a cofinal subframe logic. It is easy to see that $\mathfrak{G}$ is a cofinal subframe of $\mathfrak{F}$, so $\mathfrak{G} \models \mathbf{L}_{St(\mathfrak{F})}$. Since $\mathfrak{G}$ is rooted and $St(\mathfrak{F})$ is finite, the dual statement of Jónsson's Lemma yields that $\mathfrak{G}$ is isomorphic to an upset of a p-morphic image of a member of $St(\mathfrak{F})$. But $St(\mathfrak{F})$ is closed under order preserving images, hence it is closed under p-morphic images. Therefore, there is an onto order preserving map $f : \mathfrak{F} \to \mathfrak{K}$ such that $\mathfrak{G}$ is an upset of $\mathfrak{K}$. Since $\mathfrak{K}$ is an order preserving image of $\mathfrak{F}$, we see that $\mathfrak{K}$ has at most two maximal points. But $\mathfrak{G}$ is an upset of $\mathfrak{K}$, so $\mathfrak{K}$ has precisely two maximal points, and f maps the maximal points of $\mathfrak{F}$ to the maximal points of $\mathfrak{K}$. Note that $\mathfrak{F}$ has only one element of depth 2, which is a cover of the maximal elements of $\mathfrak{F}$. It follows that $\mathfrak{K}$ also has only one element of depth 2. However, $\mathfrak{G}$ has two elements of depth 2. Thus, $\mathfrak{G}$ cannot be an upset of $\mathfrak{K}$. The obtained contradiction proves that $\mathbf{L}_{St(\mathfrak{F})}$ is not a cofinal subframe logic. □

Our final example is a tabular cofinal stable logic that is neither cofinal subframe nor stable. Let $cSt(\mathfrak{F}) = \{\mathfrak{K} \mid \mathfrak{K}$ is a q-morphic image of $\mathfrak{F}\}$, and let $\mathbf{L}_{cSt(\mathfrak{F})}$ be the logic of $cSt(\mathfrak{F})$. Since the class of Heyting algebras corresponding to the class $cSt(\mathfrak{F})$ is cofinal stable, $\mathbf{L}_{cSt(\mathfrak{F})}$ is a cofinal stable logic.

Lemma 9.5. *The logic $\mathbf{L}_{cSt(\mathfrak{F})}$ is neither cofinal subframe nor stable.*

Proof. Since $cSt(\mathfrak{F}) \subseteq St(\mathfrak{F})$, the same proof as in Lemma 9.4 gives that $\mathbf{L}_{cSt(\mathfrak{F})}$ is not a cofinal subframe logic. In order to see that $\mathbf{L}_{cSt(\mathfrak{F})}$ is not stable, observe that $\mathfrak{H}$ is an order preserving image of $\mathfrak{F}$, so if $\mathbf{L}_{cSt(\mathfrak{F})}$ were stable, we would have $\mathfrak{H} \models \mathbf{L}_{cSt(\mathfrak{F})}$, and the same argument as in the proof of Lemma 9.4 would yield that $\mathfrak{H}$ is an upset of a member of $cSt(\mathfrak{F})$. Clearly $\mathfrak{H}$ is not an upset of $\mathfrak{F}$, and every proper q-morphic image of $\mathfrak{F}$ contains at most five elements. Since $\mathfrak{H}$ has six elements, $\mathfrak{H}$ is not an upset of a member of $cSt(\mathfrak{F})$. Thus, $\mathbf{L}_{cSt(\mathfrak{F})}$ is not stable. □

The following table summarizes the above comparison of stable, cofinal stable, subframe, and cofinal subframe logics.

	subframe	**cof. subframe**	**stable**	**cof. stable**
$\mathbf{LC}_n$	✓	✓	✓	✓
$\mathbf{BD}_2$	✓	✓	-	✓
$\mathbf{BD}_n, n \geq 3$	✓	✓	-	-
$\mathbf{L}_{St(\mathfrak{F})}$	-	-	✓	✓
$\mathbf{L}_{cSt(\mathfrak{F})}$	-	-	-	✓

Acknowledgments

The authors would like to thank David Gabelaia and Mamuka Jibladze for inspiring and helpful discussions that took place during TOLO IV (International Workshop on Topological Methods in Logic) in Tbilisi, June 23–27, 2014.

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