Modal Information Logics

MSc Thesis *(Afstudeerscriptie)*

written by

Søren Brinck Knudstorp
(born November 6th, 1995 in Køge, Denmark)

under the supervision of **prof. dr. Johan van Benthem** and **dr. Nick Bezhanishvili**, and submitted to the Examinations Board in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam*.

Date of the public defense:	**Members of the Thesis Committee:**
August 29th, 2022	dr. Maria Aloni (chair)
	prof. dr. Johan van Benthem
	dr. Nick Bezhanishvili
	dr. Alexandru Baltag
	dr. Yanjing Wang

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Contents

Abstract

The present thesis studies formal properties of a family of so-called modal information logics (MILs)—modal logics first proposed in van Benthem (1996) as a way of using possible-worlds semantics to model a theory of information. They do so by extending the language of propositional logic with a binary modality defined in terms of being the supremum of two states.

First proposed in 1996, MILs have been around for some time, yet not much is known: van Benthem (2017, 2019) pose two central open problems, namely (1) axiomatizing the two basic MILs of suprema on preorders and posets, respectively, and (2) proving (un)decidability.

The main results of the first part of this thesis are solving these two problems: (1) by providing an axiomatization [with a completeness proof entailing the two logics to be the same], and (2) by proving decidability. In the proof of the latter, an emphasis is put on the method applied as a heuristic for proving decidability 'via completeness' for semantically introduced logics; the logics lack the FMP w.r.t. their classes of definition, but not w.r.t. a generalized class.

These results are build upon to axiomatize and prove decidable the MILs attained by endowing the language with an 'informational implication'—in doing so a link is also made to the work of Buszkowski (2021) on the Lambek Calculus. Moreover, concluding the study of MILs on preorders and posets, it is shown that interpreting the modalities based on *minimal* upper bounds instead of *least* upper bounds does not alter the logics.

Broadening the study, the basic MIL of suprema on join-semilattices is axiomatized with an infinite scheme. This constitutes the by far most substantive part of the thesis. Accordingly—as to contribute to the toolbox of techniques for (modal) completeness proofs—an informal focus is also lend to accenting key ideas.

Finally, as an appendix, the (compactness and) decidability result(s) in Fine and Jago (2019) are significantly extended, chiefly via defining and proving a truthmaker analogue of the FMP.

Acknowledgements

Foremost, I would like to thank my supervisors: Johan and Nick. Johan, thank you for introducing me to the topics and problems of this thesis, for your genuine and thought-out comments and feedback, for your structured and structuring emails, and—not least—for your always encouraging words. Nick, beyond supervising me in writing my thesis, you have guided and mentored me throughout my master's: from courses, a project and TA'ing to PhDs, conferences and much, much more. For this, I am immensely grateful.

Additionally, I would like to thank my mom, my dad, my siblings, and my partner for all their love and support. You are the people that mean the very most to me.

Preface

In this thesis, preliminary definitions of and basic results about preorders, partial orders, join-semilattices, modal logics, homomorphisms, congruences and more are omitted and knowledge thereof is assumed.

Regarding notation, terminology and design choices, two things are worth mentioning: I (1) largely use the terminology of Blackburn et al. (2001), and (2) use margin notes for informal comments, typically providing intuition or recalling notation. I owe my thanks to Levin Hornischer for having shared his TeX-template with me. Any design choice of your liking is most likely due to him (and any that isn't is – of course – my responsibility).

Here's a margin note.

Introduction

This introduction is divided into two parts. First, we give a more general introduction, forwarding the logics of concern, placing them in the broader landscape of logics, and motivating their study. Second, we break down the thesis chapter-by-chapter, outlining the mathematical issues at hand and how they are solved, ending with a list of the main results achieved in this thesis.

Motivation and general introduction

The 'fusion connective' '$\circ$' characterized by its semantics

$$x \Vdash \varphi \circ \psi \quad \textbf{iff} \quad \text{there exist } y, z \text{ such that } y \Vdash \varphi;\ z \Vdash \psi;\ \text{and } x = \sup\{y, z\}$$

features in an array of logical systems: as 'intensional conjunction' or simply 'fusion' in relevance logics (Anderson et al. 1992); as regular conjunction in semantics for exact truthmaking (Fine 2017; van Fraassen 1969); as 'tensor disjunction' in the team semantics of Yang and Väänänen (2016, 2017); and as 'split disjunction' in the state-based semantics of Aloni (2022), to name some. Occurring in such a varied range of settings modeling all sorts of phenomena, it is maybe somewhat surprising that next to nothing is known about the logic(s) resulting from enriching classical propositional logic (CL) with the fusion connective; i.e., about the modal logic(s) with a binary modality for fusion.[1] In a nutshell, this thesis seeks to fill this gap.

It does so by studying formal properties of a family of so-called *modal information logics* (MILs), not least by providing axiomatizations and proving decidability results. The name owes to van Benthem (1996). Aiming to model a theory of information by using the possible-worlds semantics of modal logic, van Benthem (1996) introduces a modal logic of a single binary modality '$\langle \sup \rangle$' with the semantics of the fusion connective. The logic is motivated by construing the 'worlds' as information states; the relation as an ordering of the information

[1] We are aware that some readers might be stumbled by our almost casual identification of the (somewhat vague) term "logic(s) resulting from enriching classical propositional logic (CL) with the fusion connective" with (the more well-defined) "modal logic(s) with a binary modality for fusion". From one perspective, modal logic is a formal study of intensional notions modeled via 'possible worlds', but from another perspective modal logics are nothing but CL enriched with 'relational connectives'. When making the identification, we are tacitly alluding to this second perspective.

states; and the supremum modality '$\langle \sup \rangle$' as providing language for speaking of 'merge' (or 'fusion') of information states. Thus, besides from the more abstract 'CL-enriched-with-fusion' interpretation, the modal logics with a binary modality for fusion have a more concrete informational interpretation. This provides a second motivation for studying our logics of concern.

To explain our third and last main point of motivation, we must first get clear on a principal way in which MILs can differ, namely in their notion of 'fusion': on what class of structures do we want to interpret the '$\langle \sup \rangle$'-modality – what is our choice of frames? Rather general are preorders where the modality '$\langle \sup \rangle$' is defined in terms of quasi-least upper bounds; i.e., 'merges' are not unique but come in clusters. This defines the basic modal information logic denoted MIL_{Pre}. Both the informational and the CL-augmented-with-fusion interpretation further suggest examining the cases where (a) the relation is also anti-symmetric (resulting in posets) and, moreover, (b) any two worlds (or information states) have a unique merge (resulting in join-semilattices). We denote the corresponding logics as MIL_{Pos} and MIL_{Sem}, respectively.

Now for the last point of motivation, beginning with the logic MIL_{Sem}. Studying modal logics of algebraic structures has recently found a newborn interest (van Benthem and Bezhanishvili 2022; Wang and Wang 2022), so as a modal logic of join-semilattices with a modality for the join-operation, a study of MIL_{Sem} contributes to this line of research. As regards MIL_{Pre} and MIL_{Pos}, even if MIL_{Sem} also has been considered by van Benthem (1996, 2017, 2019, Forthcoming), in these papers it is the former two that takes centre stage of the three of them. In part, this is because of yet another interesting aspect of these two logics: using '$\langle \sup \rangle$' the past-looking modality 'P' becomes definable, so by being modal logics of preorders and posets, they mildly extend **S4**. Put in this light, MIL_{Pre} and MIL_{Pos} are quite natural extensions of **S4** obtained by adding vocabulary for describing further structure of preorders and posets. Thus, seen from a purely mathematical angle, these MILs can be motivated by an interest in seeking a modal perspective on either (a) algebraic structures or (b) rather ubiquitous mathematical structures (preorders and posets).[2]

We end this part of the introduction by setting out the remaining MILs we will be studying. To begin with, this manner of construing MIL_{Pre} and MIL_{Pos} as extensions of **S4** also invites investigation of kindred logics: what logics do we obtain if we likewise extend **S4** but through vocabulary for describing (quasi-)*minimal* upper bounds instead of (quasi-)*least* upper bounds? That is,

[2]This takes on a third perspective on modal logic, namely as a way of studying relational structures – including viewing algebraic structures as such. Needless to say, as with all other mathematical concepts, the abstraction involved permits multiple fruitful perspectives – counting many more than the mentioned three.

a '$\langle \text{sup} \rangle$'-formula will be satisfied also in case a world is the (quasi-)minimal upper bound even if not (quasi-)least. We denote these modal information logics MIL^{Min}_{Pre} and MIL^{Min}_{Pos} and will be particularly interested in how they relate to MIL_{Pre} and MIL_{Pos}, respectively. For good measure, let us mention that in terms of the informational viewpoint, they can be seen as formalizing settings in which there can be multiple incomparable 'merges'; and in terms of the CL-augmented-with-fusion viewpoint, they, in a similar vein, formalize how CL can be augmented with an 'incomparable fusion connective' while keeping the semantics for the propositional connectives classical.

The last kind of logics in the vast space of MILs we will engage with, is obtained by further enlarging the language with the modality '$\backslash$' with semantics

$$y \Vdash \varphi \backslash \psi \qquad \textbf{iff} \qquad \text{for all } x, z, \text{ if } z \Vdash \varphi \text{ and } x = \sup\{y, z\}, \text{ then } x \Vdash \psi.$$

In particular, we will be concerned with the logics attained by enriching MIL_{Pre} and MIL_{Pos} with '$\backslash$' and denote these as $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$, respectively. This extension was suggested in van Benthem (Forthcoming), and is motivated under the informational interpretation as an 'informational implication': an information state y 'satisfies' $\varphi \backslash \psi$ iff for all information states z and all merges $x = \sup\{y, z\}$ of information states y, z, if z satisfies φ (the antecedent), then the merge x satisfies ψ (the consequent).

Once again, to the best of our knowledge, this connective is not well-studied in a modal setting (if at all), even if connectives with this kind of semantics feature prominently in several logics: in fact, our informational interpretation is that of the relevance logic of Urquhart (1972, 1973) where '$\backslash$' is relevant implication; and the symbol '$\backslash$' is the (left) residual of the Lambek Calculus (Lambek 1958) – a logic we will make a junction with. It should also be noted that '$\backslash$' compliments '$\langle \text{sup} \rangle$' (or '$\circ$') very naturally: if, say, $x = \sup\{y, z\}$, then '$\backslash$' accesses this from the perspective of y (or z) while '$\langle \text{sup} \rangle$' accesses it from the perspective of x. It is thus not surprising that the 'intensional conjunction' of Urquhart (1972, 1973) and the 'product connective' of Lambek (1958) are analogues of '$\langle \text{sup} \rangle$'.

Wrapping up this general introduction, an unsurprising, nonetheless (perhaps the most) interesting, consequence of MILs essentially being (versions of) CL extended with a fusion connective,[3] is how they connect with other logics. Thus, the Lambek Calculus is not the only logic we will make a junction with: albeit in an appendix (B), we, among more, briefly study how the truthmaker logics of Fine (2017) relate to our MILs.

[3]And in the case of $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$, also with an informational implication.

Guide to chapters

Zooming in, in the order as they occur in this thesis, we explain the mathematical problems we will be addressing. Starting off, we examine MIL_{Pre} and MIL_{Pos}, motivated by two central open problems posed by van Benthem (2017, 2019, Forthcoming), namely (1) axiomatizing the logics and (2) proving (un)decidability. The first three chapters of this thesis are concerned with these two problems.

In Chapter 1, after having formally defined the logics, we, in particular, show that MIL_{Pre} lacks the finite model property (FMP) w.r.t. preorders. This proof extends to all above mentioned MILs on their respective classes of frames as well. Although this can be taken as a (clear) indication of undecidability, we end the chapter by explaining why this need not be, forwarding a method for proving decidability 'via completeness' when dealing with semantically introduced logics (like MILs).

'Semantically introduced' as contrasting logics introduced by a syntactic (or proof-theoretical) definition.

In Chapter 2, we provide an axiomatization of MIL_{Pre} and prove it to be sound and strongly complete. We do so by, given a consistent set, constructing a model for it. As the constructed models are, in fact, posets, we get as a corollary that $MIL_{Pre} = MIL_{Pos}$; thus, solving problem (1) for both logics in one go.

Following the method laid out in Chapter 1, in Chapter 3, we, first, use this axiomatization to find another class of structures $\mathcal{C}$ for which the logic also is complete. Second, we show that on this class of structures we do, in fact, have the FMP—allowing us to deduce decidability. As an appendix (A.2) to the chapter, we show that on $\mathcal{C}$ we have the tree model property (TMP). This concludes the first part of the thesis.

Next, in Chapter 4, we explore the conservative extensions $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$ obtained by adding the informational implication '$\backslash$'. Combining ideas from our study of $MIL_{Pre} = MIL_{Pos}$ with some new ones—among which some are ours and some, more interestingly, are due to work on the Lambek Calculus of Buszkowski (2021)—we (i) axiomatize the logics, (ii) show that $MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos}$, and (iii) prove them to be decidable. This crossing with the Lambek Calculus sheds one more illuminating light on modal information logics: $MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos}$ is the Lambek Calculus (augmented with CL) of suprema on preorders (or posets).

In Chapter 5, we investigate the logics MIL^{Min}_{Pre} and MIL^{Min}_{Pos}. Perhaps a bit surprising, we show that $MIL^{Min}_{Pre} = MIL^{Min}_{Pos} = MIL_{Pre}$ and also $MIL^{Min}_{\backslash\text{-}Pre} = MIL^{Min}_{\backslash\text{-}Pos} = MIL_{\backslash\text{-}Pre}$.

Chapter 6 concludes our study of modal information logics, primarily in axiomatizing MIL_{Sem}. While MIL_{Pre} and MIL_{Pos} coincide and are finitely axiomatizable, going one step further to join-semilattices triggers an explosion in complexity;

for instance, in axiomatizing this logic, we employ an infinite extension scheme. The completeness proof is through model construction, allowing for a corollary in terms of identifying MIL_{Sem} with the MIL of suprema on the subclass of join-semilattices having all finite bounded infima.

Lastly, Appendix B deserves mention too. First, through translation, compactness and recursive enumerability of a family of truthmaker logics (TMLs) are achieved. Second, a 'truthmaker FMP' is developed and proven, thus entailing decidability of many a truthmaker logic. Third and finally, how TMLs and MILs are related is explored through translations.

In summary, the main results achieved are:

- Axiomatizing MIL_{Pre} and deducing $MIL_{Pre} = MIL_{Pos}$.
- Proving MIL_{Pre} decidable.
- Axiomatizing $MIL^{Min}_{\backslash\text{-}Pre}$ and deducing $MIL^{Min}_{\backslash\text{-}Pre} = MIL^{Min}_{\backslash\text{-}Pos}$.
- Proving $MIL^{Min}_{\backslash\text{-}Pre}$ decidable.
- Showing $MIL^{Min}_{Pre} = MIL^{Min}_{Pos} = MIL_{Pre}$ and $MIL^{Min}_{\backslash\text{-}Pre} = MIL^{Min}_{\backslash\text{-}Pos} = MIL_{\backslash\text{-}Pre}$.
- Proving the FMP and decidability for a family of truthmaker logics (app).
- Axiomatizing MIL_{Sem}.

1. Modal Information Logics: Preliminaries

We start off this chapter by formally defining the basic modal information logics (section 1.1). Then we show lacks of properties related to that of decidability, most notably proving that all of the logics of concern lack the finite model property w.r.t. their respective classes of definition (section 1.2). Lastly, in section 1.3, we sketch a general method for proving decidability in cases like ours – a method which we will employ in Chapter 2 and 3.

1.1. Defining the logics

Definition 1.1 (Language)**.** The basic language $\mathcal{L}_M$ of modal information logic is defined using a countable set of proposition letters $\mathbf{P}$ and a binary modality $\langle \mathrm{sup} \rangle$. The formulas $\varphi \in \mathcal{L}_M$ are then given by the following BNF-grammar

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle \mathrm{sup} \rangle \varphi\psi,$$

where $p \in \mathbf{P}$ and $\bot$ is the falsum constant. ⊣

Modal information logics are defined by semantical means; i.e., as sets of $\mathcal{L}_M$-validities on classes of structures. The most general class of interest is that of preorders; formally, we define as follows:

Definition 1.2 (Frames and models)**.** A (Kripke) *preorder frame* for $\mathcal{L}_M$ is a pair $\mathbb{F} = (W, \leqslant)$ where

- W is a set; and
- $\leqslant$ is a preorder on W, i.e., reflexive and transitive.

A (Kripke) *preorder model* for $\mathcal{L}_M$ is a triple $\mathbb{M} = (W, \leqslant, V)$ where

- $(W, \leqslant)$ is a preorder frame; and
- V is a valuation on W, i.e., a function $V : \mathbf{P} \to \mathcal{P}(W)$. ⊣

For clarity, before defining other classes of structures we will be considering, we set out the basic modal information logic of preorders in full detail. Having defined the structures in which to interpret the $\mathcal{L}_M$-formulas, we are about to define the actual semantics. In order to do so, we provide the following definition generalizing the notion of supremum from partial orders to preorders:

Definition 1.3 (Supremum). Given a preorder frame $(W, \leqslant)$ and worlds $u, v, w \in W$, we say that w is *a quasi-supremum* (or simply *supremum*) of $\{u, v\}$ and write $w \in \sup\{u, v\}$ iff

- w is an upper bound of $\{u, v\}$, i.e., $u \leqslant w$ and $v \leqslant w$; and
- $w \leqslant x$ for all upper bounds x of $\{u, v\}$.

In general, $\sup\{u, v\}$ denotes the set of quasi-suprema of $\{u, v\}$, and if this happens to be a singleton $\{w\}$, we may write $w = \sup\{u, v\}$. ⊣

Note how $w \in \sup_{\leqslant}\{u, v\}$ on a preorder $\leqslant$ iff $[w] = \sup_{\leqslant_\sim}\{[u], [v]\}$ on its 'skeletal' partial order $\leqslant_\sim$.

Definition 1.4 (Semantics). Given a preorder model $\mathbb{M} = (W, \leqslant, V)$ and a world $w \in W$, *satisfaction* of a formula $\varphi \in \mathcal{L}_M$ at w in $\mathbb{M}$ (written '$\mathbb{M}, w \Vdash \varphi$' or '$w \Vdash \varphi$' for short) is defined using the following recursive clauses on φ:

$$\begin{array}{lll} \mathbb{M}, w \nVdash \bot, & & \\ \mathbb{M}, w \Vdash p & \textbf{iff} & w \in V(p), \\ \mathbb{M}, w \Vdash \neg\varphi & \textbf{iff} & \mathbb{M}, w \nVdash \varphi, \\ \mathbb{M}, w \Vdash \varphi \vee \psi & \textbf{iff} & \mathbb{M}, w \Vdash \varphi \;\; \textit{or} \;\; \mathbb{M}, w \Vdash \psi, \\ \mathbb{M}, w \Vdash \langle \sup \rangle \varphi\psi & \textbf{iff} & \text{there exist } u, v \in W \text{ such that } \mathbb{M}, u \Vdash \varphi;\ \mathbb{M}, v \Vdash \psi; \\ & & \qquad\qquad \text{and } w \in \sup\{u, v\}. \end{array}$$

Notions like *global truth*, *validity*, etc. are defined as usual in possible-worlds semantics (see, e.g., Blackburn et al. (2001, ch. 1)). ⊣

With these notions laid out, we can define the logic as follows:

Definition 1.5. The basic modal information logic of suprema on preorders is denoted by MIL_{Pre}, and defined as the set of $\mathcal{L}_M$-validities on the class of all preorder frames; that is,

$$MIL_{Pre} := \{\varphi \in \mathcal{L}_M : (W, \leqslant) \Vdash \varphi \;\; \textit{for all preorder frames } (W, \leqslant)\}. \qquad ⊣$$

Using the defined notions and semantics, we further define the basic modal information logics of suprema on posets and join-semilattices, respectively, namely

MIL_{Pos}, which is the logic of *poset frames*, i.e., frames $(W, \leqslant)$ where '$\leqslant$' is a partial order (viz. an antisymmetric preorder); and

MIL_{Sem}, which is the logic of frames $(W, \leqslant)$ where '$\leqslant$' is a join-semilattice, i.e., a partial order with all binary suprema.

I.e., for all $u, v \in W$, there is some $w \in W$ s.t. $w = \sup\{u, v\}$. Note how these additional logics arise from a uniqueness and existence requirement, respectively.

As an appetizer, we end this section remarking what is already known about how these logics relate.

Remark 1.6.

$$MIL_{Pre} \subseteq MIL_{Pos} \subsetneq MIL_{Sem}.$$

As the notation suggests, these inclusions follow from latter logics being semantically defined by restricting classes of frames for former logics.

The latter inclusion being strict is witnessed by, e.g., the associativity formula

$$\text{(As.)}\ \langle \sup \rangle (\langle \sup \rangle pq) r \leftrightarrow \langle \sup \rangle p (\langle \sup \rangle qr),$$

which is valid on join-semilattices but not on posets, as the reader can easily check. ⊣

The former inclusion is, in fact, an equality. This will be a corollary of our completeness proof in Chapter 2.

1.2. Road to decidability: negative results

Having formally set out these logics and semantics, we continue with some preliminary remarks. Objective being to get a feel for how the semantics works by stating a few minor – yet interesting – results, and, most notably, showing that the logics lack the FMP w.r.t. their respective frames of definition; viz., for instance, MIL_{Pre} does not have the FMP w.r.t. preorder frames. Foremost, we mention how to express the past-looking modality.

Remark 1.7. Besides the connectives '$\wedge$', '$\rightarrow$', '$\leftrightarrow$', '[sup]', and '$\top$' being definable in the standard way, the past-looking unary modality 'P' is definable as

$$P\varphi := \langle \sup \rangle \varphi \top.$$

Thus, as promised in the introduction, MIL_{Pre} and MIL_{Pos} are (quite natural) extensions of **S4**.

This can be seen by recalling the definition

$$\mathbb{M}, w \Vdash P\varphi \quad \textbf{:iff} \quad \text{there exists } v \leqslant w \text{ such that } \mathbb{M}, v \Vdash \varphi,$$

and observing that also

$$\mathbb{M}, w \Vdash \langle \sup \rangle \varphi \top \quad \textbf{iff} \quad \text{there exists } v \leqslant w \text{ such that } \mathbb{M}, v \Vdash \varphi. \quad \dashv$$

Since for any v, w: $w \in \sup\{w, v\}$ iff $v \leqslant w$.

Using this observation, the first contribution of our thesis is to show a lack of the FMP.

Proposition 1.8. *MIL_{Pre} does not have the FMP w.r.t. preorder frames.*

Proof. Consider the formula

$$\psi_N := HP\langle \sup \rangle pp \wedge HP\neg\langle \sup \rangle pp,$$

where $H := \neg P \neg$ is the dual of P. We claim that ψ_N only is satisfiable in infinite models.

The subscript 'N' is short for 'negative', as ψ_N witnesses a negative property.

First, we show that ψ_N is, indeed, satisfiable on an infinite model. Accordingly, let $\mathbb{M} := (\mathbb{Z}_-, \leqslant, V)$ where

- $\mathbb{Z}_-$ is the set of negative integers;
- $\leqslant$ is the less-than relation on the negative integers; and
- $V(p)$ is the set of even negative integers.

Then $\mathbb{M}$, clearly, is a preorder model, and for all $z \in \mathbb{Z}_-$:

$$\mathbb{M}, z \Vdash \langle \text{sup} \rangle pp \qquad \textbf{iff} \qquad z \text{ is even.}$$

Because for any z_1, z_2: $\sup\{z_1, z_2\} = \max\{z_1, z_2\}$.

Thus, for all $z \in \mathbb{Z}_-$:

$$\mathbb{M}, z \Vdash P\langle \text{sup} \rangle pp \wedge P\neg\langle \text{sup} \rangle pp.$$

But then ψ_N must be globally true in $\mathbb{M}$; in particular, ψ_N is satisfied in $\mathbb{M}$, proving the first part of the claim

Second, to see that ψ_N isn't satisfiable in any finite model, observe that for any preorder, if two points are situated in the same cluster, then they are suprema of the exact same (sets of) points. It follows that for any preorder model, points in the same cluster satisfy the exact same '$\langle \text{sup} \rangle$'-formulas (those are: formulas with '$\langle \text{sup} \rangle$' as main connective).

For clarity, recall that given a preorder $\leqslant$, w, v are said to be in the same cluster :iff $w \leqslant v \leqslant w$.

With this in mind, it is easy to see that the satisfaction of ψ_N necessitates the existence of an infinite, strictly descending chain: if some $w \Vdash \psi_N$ and some $v_i \leqslant w$ satisfies, say, $\langle \text{sup} \rangle pp$, then, in particular, there must be some $v_{i+1} \leqslant v_i$ s.t. $v_{i+1} \Vdash \neg\langle \text{sup} \rangle pp$, whence v_{i+1} must be in a cluster strictly below v_i. Thus, ψ_N cannot be satisfied in any finite model. □

Remark 1.9. The above proof applies to the classes of posets and join-semilattices as well since the frame $(\mathbb{Z}_-, \leqslant)$ was, in fact, a join-semilattice, hence neither do MIL_{Pos} nor MIL_{Sem} enjoy the FMP w.r.t. their respective classes of definition. ⊣

Beyond not having the FMP, there are even more indicators of undecidability. For the purpose of this thesis, these are not central, so we mention them without (elaborate) proof.

Remark 1.10. MIL_{Pre} does not have the tree model property (TMP) w.r.t. preorder frames. That is, there is a formula χ_N which is satisfiable, but not in a preorder frame $(W, \leqslant)$ where $(W, \geqslant)$ is a reflexive and transitive tree.[4] ⊣

Proof. The following formula is satisfiable but not in a (converse) tree

$$\begin{aligned}\chi_N := \; & p \wedge q \wedge \langle \text{sup} \rangle (p \wedge \neg q)(\neg p \wedge q) \\ & \wedge H\big([p \wedge \neg q] \to P(\neg p \wedge \neg q)\big) \wedge H\big([\neg p \wedge q] \to P(\neg p \wedge \neg q)\big) \\ & \wedge H\big(\langle \text{sup} \rangle (\neg p \wedge \neg q)^2 \to [\neg p \wedge \neg q]\big). \end{aligned}$$ □

Remark 1.11. Not having the TMP extends to MIL_{Pos} and MIL_{Sem} as well. This can be witnessed by the same formula. And for the case of MIL_{Sem}, the satisfaction of χ_N even implies the existence of a *tile*; i.e., four pairwise distinct worlds n, s, e, w s.t. $s \leqslant e, s \leqslant w, e \leqslant n, w \leqslant n$. ⊣

Observation 1.12. Our modal information logics are neither guarded nor packed (as, e.g., the guarded and packed fragments do enjoy the FMP). ⊣

1.3. Road to decidability: general idea

At first glance, the results of the previous section might make decidability appear unlikely. As it turns out, there is an alternative way of proving decidability circumventing these problems. In this section, we lay out our method for doing so. This will serve two purposes: by describing the method, we hope to (1), generally, elucidate how and when our method can work as a heuristic for proving decidability, and (2), specifically, help the reader get a better grasp of the underlying ideas and structure of the ensuing two chapters of this thesis.

To help explain this method of proving decidability 'via completeness', we go by example, designing a thought-experiment:

(1) Sem. def. logic: Imagine wanting to use the basic modal language of one unary modality to describe the structure of *posets*. That is, being interested in poset models $\mathbb{M} = (W, \leqslant, V)$ with semantics

$$\mathbb{M}, w \Vdash \Diamond \varphi \qquad \textbf{iff} \qquad \exists v \in W (w \leqslant v \wedge v \Vdash \varphi).$$

One might then wonder whether the problem of determining whether a formula φ is satisfiable/valid on the class of all poset frames is decidable.

[4]Consult, e.g., Blackburn et al. (2001, ch. 1, def. 1.7) for the definition of a tree and, in particular, a reflexive and transitive one.

Additionally, note how we define the TMP in terms of the converse relation '$\geqslant$'; this is motivated by the way in which '$\langle \text{sup} \rangle$' is backward-looking. Otherwise, for the case of '$\leqslant$', a formula like '$p \wedge \langle \text{sup} \rangle (q \wedge \neg p)(\neg q \wedge \neg p)$' already shows the lack of 'a TMP'. For now, this suffices: we return to this as an addendum to Chapter 3 in Appendix A.2.

(1.5) No FMP: For this, the FMP is highly useful. However, the formula

$$\Box(\Diamond p \wedge \Diamond \neg p)$$

is only satisfiable on infinite partial orders.

(2) Completeness: At first glance, this makes decidability seem farfetched. But, in fact, there is an alternative road to decidability, beginning by axiomatizing the logic. Having axiomatized the logic, one realizes that it is also complete with respect to *preorders* (because, in fact, what one gets is **S4**).

Instead of first axiomatizing, a direct 'p-morphic'-argument would work as well. In a way, this is the recipe Chapter 4 will follow.

(3) FMP on other class: And on *preorders*, one can prove the FMP (via, e.g., the Lemmon filtration), and then decidability follows easily.

Summarizing the method conveyed by this example, when dealing with logics introduced by a semantic definition (cf. *(1)*), not having the FMP (cf. *(1.5)*) might not be very telling. The reason being that the resulting logic can very well be complete w.r.t. to another, bigger class of structures (cf. *(2)*) for which it does have the FMP (cf. *(3)*).

In our case, we will follow this recipe for the cases of MIL_{Pre} and MIL_{Pos}. Having already gone through steps *(1)* and *(1.5)*, we proceed with step *(2)* in the coming chapter where we axiomatize the logics and show that $MIL_{Pre} = MIL_{Pos}$. Using this axiomatization, in Chapter 3, we find another, bigger class of structures (where the ternary relation of $\langle sup \rangle$ won't necessarily be the supremum relation of a preorder, but something more general) which is sound and complete w.r.t. the logic MIL_{Pre} and, importantly, do enjoy the FMP.

2. Axiomatizing MIL_{Pre}

While van Benthem (Forthcoming) obtains an axiomatization of a variant of MIL_{Pre} extended with nominals and the global modality, the very same paper also inquires finding an axiomatization without hybrid extensions. In this chapter, we answer this inquiry, providing a purely modal axiomatization. In section 2.1, we give a proof-theoretic description of MIL_{Pre}, prove it to be sound, and lay some groundwork for the completeness proof of section 2.2, which also allows us to conclude that $\mathit{MIL}_{Pre} = \mathit{MIL}_{Pos}$.

2.1. Soundness and preparatory lemmas

We begin by syntactically defining a normal modal logic (NML), suggestively called $\mathbf{MIL_{Pre}}$. Through a soundness and completeness proof, we then show $\mathbf{MIL_{Pre}}$ exactly is an axiomatization of our semantically defined logic MIL_{Pre}.

As a convention, we ***boldface*** *when having 'syntactic' presentations of logics in mind and italicize when having 'semantic' presentations of logics in mind.*

Definition 2.1 (Axiomatization)**.** We define $\mathbf{MIL_{Pre}}$ to be the least normal modal logic in the language of $\mathcal{L}_M$ containing the following axioms:

(Re.) is short for 'Reflexivity'; (4) is the transitivity axiom; (Co.) is short for 'Commutativity'; and (Dk.) is short for 'Don't know what to call this axiom'.

(Re.) $p \wedge q \to \langle \mathrm{sup} \rangle pq$

(4) $PPp \to Pp$ $(= \langle \mathrm{sup} \rangle (\langle \mathrm{sup} \rangle p\top)\top \to \langle \mathrm{sup} \rangle p\top$, cf. Remark 1.7)

(Co.) $\langle \mathrm{sup} \rangle pq \to \langle \mathrm{sup} \rangle qp$

(Dk.) $(p \wedge \langle \mathrm{sup} \rangle qr) \to \langle \mathrm{sup} \rangle pq$ ⊣

Having proof-theoretically defined the logic $\mathbf{MIL_{Pre}}$, we can promptly show it to be sound w.r.t. MIL_{Pre}.

Theorem 2.2 (Soundness)**.** $\mathbf{MIL_{Pre}} \subseteq \mathit{MIL}_{Pre}$.

Proof. Standard, tedious task checking that MIL_{Pre} is a normal modal logic and that (Re.), (4), (Co.), and (Dk.) all are valid on preorder frames. □

As oftentimes is the case, while proving soundness is straightforward, proving completeness is much more intricate. Our proof will be a construction using maximal consistent sets (MCSs) for which some preparatory observations and lemmas are needed.

First hurdle is that the $\langle \mathrm{sup} \rangle$-modality is in a general sense a 'logical modality': although accompanied by a *ternary* relation (namely the supremum relation) its interpretation is fixed given a *binary* relation (namely a preorder). For starters, this means that the standard construction of the canonical frame for $\mathbf{MIL_{Pre}}$ won't come equipped with a binary relation for interpreting the binary modality $\langle\text{sup}\rangle$—as is the case for the preorder frames of $\mathit{MIL_{Pre}}$—but with a ternary one. Fortunately, defining an underlying preorder from this ternary relation spells no trouble. This is summarized in the definition below.

Definition 2.3. We denote the set of all maximal consistent $\mathbf{MIL_{Pre}}$-sets by $W_{\mathbf{Pre}}$, and the ternary relation of the canonical $\mathbf{MIL_{Pre}}$-frame by $C_{\mathbf{Pre}}$.[5] That is $C_{\mathbf{Pre}}\Gamma\Delta\Theta$ holds just in case

$$\forall \delta \in \Delta, \theta \in \Theta \left(\langle \text{sup} \rangle \delta\theta \in \Gamma \right).$$

From $C_{\mathbf{Pre}}$, we define the following binary relation on the canonical frame:

$$\leqslant_{\mathbf{Pre}} := \{(\Delta, \Gamma) \in W_{\mathbf{Pre}} \times W_{\mathbf{Pre}} : \exists \Theta \left(C_{\mathbf{Pre}}\Gamma\Delta\Theta \right)\}. \qquad \dashv$$

We want to show that $\leqslant_{\mathbf{Pre}}$ actually is a preorder. To do so, we begin by making two observations.

Observation 2.4. Since $\mathbf{MIL_{Pre}}$ is an NML, we have all the usual lemmas regarding its canonical model. ⊣

Observation 2.5. The formula

(T) $p \to Pp$

is derivable in $\mathbf{MIL_{Pre}}$.

In fact, {(T), (4), (Co.), (Dk.)} is an alternative axiomatization of $\mathbf{MIL_{Pre}}$. ⊣

Proof. Some straightforward syntactical manipulations prove the claim; the key steps being

(Re.) ⇒ (T): uniformly substitute q for $\top$ in (Re.); and

(T) ⇒ (Re.): use (T) to get $p \wedge q \to p \wedge Pq$ and then use (Dk.). □

Using these observations, in the ensuing lemma, we prove that not only is $\leqslant_{\mathbf{Pre}}$ a preorder, but more 'supremum-like' properties hold of the canonical relation $C_{\mathbf{Pre}}$.

Lemma 2.6. *The following hold:*

(a) $C_{\mathbf{Pre}}\Gamma\Delta\Theta$ ***iff*** $C_{\mathbf{Pre}}\Gamma\Theta\Delta$

[5] Consult Blackburn et al. (2001, ch. 4) for basic definitions, results, and techniques regarding canonical models for modal logics; we have sought to align our notation and terminology with this.

(b) $\Delta \leqslant_{\mathbf{Pre}} \Gamma \quad \overset{(\mathrm{i})}{\textit{iff}} \quad C_{\mathbf{Pre}}\Gamma\Gamma\Delta \quad \overset{(\mathrm{ii})}{\textit{iff}} \quad \forall\delta \in \Delta : P\delta \in \Gamma.$

(c) $\leqslant_{\mathbf{Pre}}$ *is a preorder.*

(d) $C_{\mathbf{Pre}}\Gamma\Delta\Theta$ ***only if*** $\Delta \leqslant_{\mathbf{Pre}} \Gamma, \Theta \leqslant_{\mathbf{Pre}} \Gamma$.

Proof. Since (Re.), (4), (Co.), (Dk.) all are Sahlqvist, one can prove all but *(b)*(ii) via the Sahlqvist-van Benthem algorithm (cf. next chapter's Lemma 3.1). As often is the case, though, a direct argument is faster; we provide such here.

(a) Let $\{\theta, \delta\} \subseteq \mathcal{L}_M$ be arbitrary. Then – by (Co.), US of $\mathbf{MIL_{Pre}}$, and closure under MP of MCSs – we have

$$\langle \sup \rangle \theta\delta \in \Gamma \Leftrightarrow \langle \sup \rangle \delta\theta \in \Gamma,$$

which suffices to prove the claim.

'US' stands for 'uniform substitution', and 'MP' for 'modus ponens'.

(b) Right-to-left of (i) is immediate (using (a)). For left-to-right, suppose $C_{\mathbf{Pre}}\Gamma\Delta\Theta$ for some $\Theta \in W_{\mathbf{Pre}}$ and $\gamma \in \Gamma, \delta \in \Delta$. Since $\top \in \Theta$, we have that $\langle \sup \rangle \delta\top \in \Gamma$, hence $(\gamma \wedge \langle \sup \rangle \delta\top) \in \Gamma$ and so we get by (Dk.) (and US and MP of MCSs) that $\langle \sup \rangle \gamma\delta \in \Gamma$—as suffices. Regarding (ii), left-to-right follows by (a), while right-to-left is proven using (Dk.).

(c) *Reflexivity*. Let $\Gamma \in W_{\mathbf{Pre}}$ and $\gamma \in \Gamma$ be arbitrary. By *(b)*, it suffices to show that $P\gamma \in \Gamma$, but this follows by $\mathbf{MIL_{Pre}} \vdash p \to Pp$.

Transitivity. Suppose $\Gamma_1 \leqslant_{\mathbf{Pre}} \Gamma_2 \leqslant_{\mathbf{Pre}} \Gamma_3$ and $\gamma_1 \in \Gamma_1$. Then by applying *(b)* twice, we get that $PP\gamma_1 \in \Gamma_3$, hence since $\mathbf{MIL_{Pre}} \vdash PPp \to Pp$, we're done.

(d) Consequence of *(a)*. □

2.2. Completeness: constructing our model

Given the previous section's results – indicating that the canonical frame is well behaved – one might start wondering whether the canonical relation $C_{\mathbf{Pre}}$ is, in fact, the supremum relation on $\leqslant_{\mathbf{Pre}}$. If so, we would have completeness in our pocket. Unfortunately, this is far from being the case: not only is the canonical relation $C_{\mathbf{Pre}}$ not the supremum relation on $\leqslant_{\mathbf{Pre}}$, it is utterly wild.

As to not interlude the completeness proof, observations regarding the wildness of the canonical frame have been put off to Appendix A.1.

This forces us to make a rather complicated construction where we do not work with the canonical model per se. Instead, we construct our frame by recursively repairing so-called 'defects' and 'labeling' points of a *sub*set of our frame with MCSs for which we prove a truth lemma. This somewhat generalized approach is useful since it (a) allows for reuse of the same MCS – i.e., *different* points of the frame might get labeled with the *same* MCS – and (b) utilizes that, in the extreme, we only need a truth lemma for one MCS, namely the one extending a given consistent set; thus, we may and will include (non-labeled) points in our construction only to ensure that other (labeled) points satisfy formulas dictated by their MCS-label. That is, we do not care what formulas these points satisfy themselves—their role is entirely auxiliary.[6]

This explanation might seem awfully abstract at first. We recommend revisiting this paragraph while/after reading the rest of this chapter.

To be more concrete, when recursively constructing this frame, we make sure that at each stage, its corresponding 'approximating frame' is determined by a triple $(\mathfrak{l}, \leqslant, D)$ satisfying the definition (of $\mathbb{P}$) below. Specifically, in the recursive step from, say, n to $n+1$, we will make sure that if $(\mathfrak{l}_n, \leqslant_n, D_n) \in \mathbb{P}$ then also $(\mathfrak{l}_{n+1}, \leqslant_{n+1}, D_{n+1}) \in \mathbb{P}$.[7] This is needed for the colimit construction – i.e., the structure obtained after all finite stages in the recursive construction – to be of the right form.

Definition 2.7. Let W be countable set, and $\mathbb{P}$ the set of all triples $(\mathfrak{l}, \leqslant, D)$ such that

1. $\mathfrak{l}$ is a partial function from W to the set of all MCSs, $W_{\mathbf{Pre}}$.
2. $|\mathrm{dom}(\mathfrak{l})| < \aleph_0$.
3. $D \subseteq W, |D| < \aleph_0$.
4. $D \cap \mathrm{dom}(\mathfrak{l}) = \varnothing$.
5. $d \in D \wedge d \leqslant a \Rightarrow a = d$.
6. $\leqslant$ is a partial order on $\mathrm{dom}(\mathfrak{l}) \cup D$, and the diagonal on $W \setminus (\mathrm{dom}(\mathfrak{l}) \cup D)$.
7. If $y \leqslant x$ then $\mathfrak{l}(y) \leqslant_{\mathbf{Pre}} \mathfrak{l}(x)$ (whenever $x, y \in \mathrm{dom}(\mathfrak{l})$). $\dashv$

'$\mathrm{dom}(\mathfrak{l})$' *refers to the domain of* $\mathfrak{l}$.

$\mathfrak{l}$ *(short for 'label') labels worlds with MCSs, while* D*-worlds (short for 'dummy worlds') sole purpose is to ensure '*$\neg\langle\sup\rangle$*'-formulas are satisfied at* $\mathrm{dom}(\mathfrak{l})$*-worlds.*

As mentioned, the recursion is carried out by repeatedly repairing 'defects'. Since our goal will be to prove a truth lemma for labeled points, any defect is, in essence, either

(1) that a point x's MCS-label Γ dictates that x satisfy some formula $\langle\sup\rangle\varphi\psi$; or

I.e., $\langle\sup\rangle\varphi\psi \in \Gamma = \mathfrak{l}(x)$.

(2) that a point x's MCS-label Γ dictates that x satisfy some formula $\neg\langle\sup\rangle\varphi\psi$.

I.e., $\neg\langle\sup\rangle\varphi\psi \in \Gamma = \mathfrak{l}(x)$.

[6] It is worth noting that it is not that we *cannot* make a construction in which all points are labeled (as, essentially, is done in our later Lemma 4.20 and Proposition 4.23), but doing so would obscure the central idea making the construction work.

[7] Our framework is loosely that of Burgess (1982) with terminology borrowed from Blackburn et al. (2001, sec. 4.6). More generally, this is a 'step-by-step' construction for which an(other) excellent introduction is the exposition of the 'construction method C' in Jongh and Veltman (1999).

Although this captures the gist of what defects are, as it turns out, for the proof to work, the precise definitions must be more detailed than this. We proceed giving these.

Definition 2.8 ($\langle \text{sup} \rangle$-defect)**.** Let $(\mathfrak{l}, \leqslant, D) \in \mathbb{P}$. Then a pair $(\langle \text{sup} \rangle \chi\chi', x)$ denotes a $\langle \text{sup} \rangle$-*defect* (of $(\mathfrak{l}, \leqslant, D)$) :iff

$$\text{(i) } x \in \text{dom}(\mathfrak{l}), \qquad \text{(ii) } \langle \text{sup} \rangle \chi\chi' \in \mathfrak{l}(x),$$

and (iii) there are no $y, z \in \text{dom}(\mathfrak{l})$ s.t.

That is, a $\langle \text{sup} \rangle$-defect is the failure of a rather strict requirement on a $\text{dom}(\mathfrak{l})$-world x when $\langle \text{sup} \rangle \chi\chi' \in \mathfrak{l}(x)$. The 'upset requirements' on y, z, state that – besides from themselves – if they see a point that x does not, then that point is 'incompatible' with x.

$$\begin{aligned} &\chi \in \mathfrak{l}(y), && C_{\mathbf{Pre}}\mathfrak{l}(x)\mathfrak{l}(y)\mathfrak{l}(z), && \uparrow y = \uparrow x \cup \{y\} \cup (\uparrow y \cap \{w \mid \uparrow w \cap \uparrow x = \varnothing\}), \\ &\chi' \in \mathfrak{l}(z), && x = \sup\{y, z\}, && \uparrow z = \uparrow x \cup \{z\} \cup (\uparrow z \cap \{w \mid \uparrow w \cap \uparrow x = \varnothing\}), \end{aligned}$$

where $\uparrow w := \{v \mid w \leqslant v\}$. ⊣

Definition 2.9 ($\neg\langle \text{sup} \rangle$-defect)**.** Let $(\mathfrak{l}, \leqslant, D) \in \mathbb{P}$. Then a quadruple $(\neg\langle \text{sup} \rangle \psi\psi', x, y, z)$ is denoted a $\neg\langle \text{sup} \rangle$-*defect* (of $(\mathfrak{l}, \leqslant, D)$) :iff

$$\begin{aligned} &x \in \text{dom}(\mathfrak{l}), && x = \sup\{y, z\}, && \neg\langle \text{sup} \rangle \psi\psi' \in \mathfrak{l}(x), \\ &\psi \in \mathfrak{l}(y), && \psi' \in \mathfrak{l}(z). \end{aligned}$$ ⊣

Note that – by 5. – if $x \in \text{dom}(\mathfrak{l})$ and $x = \sup\{y, z\}$, then $y, z \in \text{dom}(\mathfrak{l})$.

With these defects defined, next up is repairing them. Before providing the actual repair lemmas demonstrating how to coherently repair each of the defects (making sure that if $(\mathfrak{l}_n, \leqslant_n, D_n) \in \mathbb{P}$, then also $(\mathfrak{l}_{n+1}, \leqslant_{n+1}, D_{n+1}) \in \mathbb{P}$), we give an example to convey intuition for the repairs and the general construction.

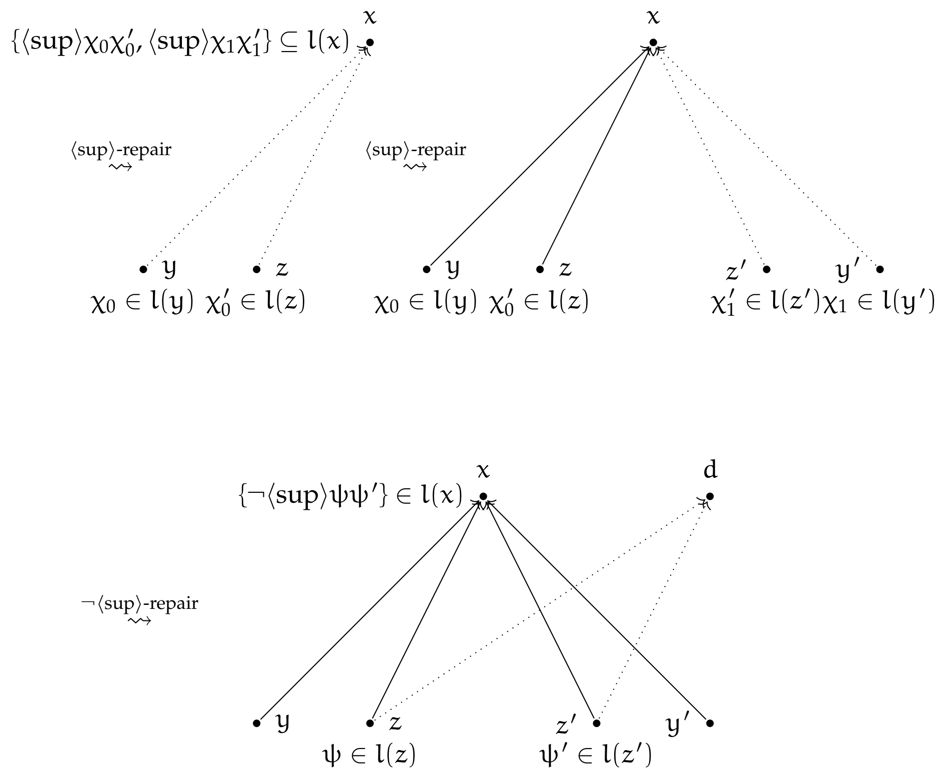
Example 2.10. Suppose $(\mathfrak{l}, \leqslant, D) \in \mathbb{P}$ and $(\langle \text{sup} \rangle \chi_0\chi_0', x)$ constitutes a $\langle \text{sup} \rangle$-defect; that is, (i) $x \in \text{dom}(\mathfrak{l})$, (ii) $\langle \text{sup} \rangle \chi_0\chi_0' \in \mathfrak{l}(x)$, and there are no y, z fulfilling (iii). Put crudely, the problem is that x's label $\mathfrak{l}(x)$ requires x to satisfy $\langle \text{sup} \rangle \chi_0\chi_0'$, but x is not the supremum of any y, z s.t. $\chi_0 \in \mathfrak{l}(y), \chi_0' \in \mathfrak{l}(z)$. To solve this, we simply add two fresh points y, z immediately below x. Then using the existence lemma of the canonical model for the case $\langle \text{sup} \rangle \chi_0\chi_0' \in \mathfrak{l}(x)$, we get two MCSs Γ_y, Γ_z s.t. $C_{\mathbf{Pre}}\mathfrak{l}(x)\Gamma_y\Gamma_z$. Setting $\mathfrak{l}(y) := \Gamma_y$ and $\mathfrak{l}(z) := \Gamma_z$, the defect has been repaired. The idea is illustrated in the top left corner of the figure below.

Regarding the existence lemma, recall Observation 2.4.

Further, if, say, $(\langle \text{sup} \rangle \chi_1\chi_1', x)$ also constitutes a $\langle \text{sup} \rangle$-defect, we simply repeat the process as illustrated in the top right corner of the figure below.

While these two repairs did solve the problems they intended to, they might have created new ones. If, say, $\neg\langle \text{sup} \rangle \psi\psi' \in \mathfrak{l}(x), \psi \in \mathfrak{l}(z)$ and $\psi' \in \mathfrak{l}(z')$, in solving these problems they have made $(\neg\langle \text{sup} \rangle \psi\psi', x, z, z')$ constitute a $\neg\langle \text{sup} \rangle$-defect. This is where we need the 'dummies': to repair this defect, we add a quasi-blind point d as an incomparable upper bound of $\{z, z'\}$ so that x no more is the supremum of $\{z, z'\}$ (cf. the bottom part of the figure). Since d is quasi-blind—and stays quasi-blind (viz. condition 5.)—whatever formula it satisfies is of absolutely no influence to the rest of the points: they cannot 'access' d. So, at bottom, adding dummies is a technique for altering the supremum relation without having to give second thought to what formulas the added points (the dummies) are to satisfy: they are entirely auxiliary (and, hence, do not get labeled, cf. condition 4.). And, most importantly, the alteration of a supremum relation caused by adding a dummy is sufficiently local to not mess up previously repaired defects; in this simplest of cases, we still have $x = \sup\{y, z\} = \sup\{y', z'\}$ after having added the dummy d.

Note that x does stay a minimal upper bound of $\{z, z'\}$; we will come back to this in Chapter 5.



⌟

We continue by making this basic intuition rigorous – starting with providing the repair lemmas.

Lemma 2.11 ($\langle\sup\rangle$-repair lemma)**.** *Suppose* $(\langle\sup\rangle\chi\chi', x)$ *is a* $\langle\sup\rangle$*-defect of some* $(l, \leqslant, D) \in \mathbb{P}$*. Then we can extend to* $(l', \leqslant', D') \in \mathbb{P}$ *by taking distinct* $y, z \in$

$W \setminus (\mathrm{dom}(l) \cup D)$ *s.t.*

$$l' := l \cup \{(y,\Gamma),(z,\Delta)\}, \qquad \leqslant' := \leqslant \cup \{(y,u),(z,u) | x \leqslant u\}, \qquad D' := D,$$
$$\chi \in \Gamma, \chi' \in \Delta, \qquad C_{Pre} l(x)\Gamma\Delta,$$

and y, z *witness that* $(\langle\sup\rangle\chi\chi', x)$ *does not constitute a* $\langle\sup\rangle$*-defect of* $(l', \leqslant', D')$.

Proof. Define as in the lemma by taking fresh $y \neq z$ and mapping them to Γ, Δ obtained via the existence lemma for $(\langle\sup\rangle\chi\chi', l(x))$. Then the last claim is easily checked to be satisfied, and $(l', \leqslant', D')$ also clearly satisfies 1.-6.; thus, it remains to show 7. Since $(l, \leqslant, D) \in \mathbb{P}$ – and having the definition of $\leqslant'$ in mind – it suffices to consider the subset

$$\{(y,u),(z,u) | x \leqslant u\} \subseteq \leqslant'$$

and the cases $y \leqslant' y, z \leqslant' z$. For these, we find:

($y \leqslant' y$) $l(y) \leqslant_{\mathbf{Pre}} l(y)$ follows by $\leqslant_{\mathbf{Pre}}$ being a preorder, hence reflexive, cf. Lemma 2.6 *(c)*.

($y \leqslant' x$) $l(y) \leqslant_{\mathbf{Pre}} l(x)$ follows by Lemma 2.6 *(d)*.

($y \leqslant' u$) For $u > x$, $l(y) \leqslant_{\mathbf{Pre}} l(u)$ follows by transitivity of $\leqslant_{\mathbf{Pre}}$.

($z \leqslant' z, x, u$) Same as for y. □

Observe how the axioms are being used via Lemma 2.6; each 'item' employs an axiom: first (Re.), then (Dk.), then (4), then (Co.). This elucidates their role, and why they are – even if rather weak – adequate: they need only 'encode' this lemma 2.6, which enables extending to $(l', \leqslant', D')$*, and then the 'dummies' do the rest.*

Lemma 2.12 ($\neg\langle\sup\rangle$-repair lemma)**.** *Suppose* $(\neg\langle\sup\rangle\psi\psi', x, y, z)$ *is a* $\neg\langle\sup\rangle$*-defect of some* $(l, \leqslant, D) \in \mathbb{P}$*. Then we can extend to* $(l', \leqslant', D') \in \mathbb{P}$ *by taking* $d \in W \setminus (\mathrm{dom}(l) \cup D)$*, letting*

$$l' := l, \qquad \leqslant' := \leqslant \cup \{(u,d),(v,d) | u \leqslant y, v \leqslant z\}, \qquad D' := D \cup \{d\},$$

and getting $x \neq \sup_{\leqslant'}\{y, z\}$.

This is where we add a dummy d*, whose sole purpose is to ensure that* $x \neq \sup_{\leqslant'}\{y,z\}$.

Proof. Extend to $(l', \leqslant', D')$ as described. It follows that $(l', \leqslant', D') \in \mathbb{P}$. To show

$$x \neq \sup{}_{\leqslant'}(y, z),$$

since $d \geqslant' y$ and $d \geqslant' z$, it suffices to show

$$d \not\geqslant' x.$$

To see this, observe that if $x = y$, since $z \leqslant x$, we would have by 7. that $l(z) \leqslant_{\mathbf{Pre}} l(x)$ hence (cf. Lemma 2.6 *(b)*)

$$C_{\mathbf{Pre}} l(x) l(y) l(z),$$

but then $(\neg\langle\sup\rangle\psi\psi', x, y, z)$ couldn't have been a $\neg\langle\sup\rangle$-defect. Same for $x = z$. Thus,

$$y < x \quad \text{and} \quad z < x,$$

whence $d \not\geqslant' x$ by definition of $\leqslant'$ and $\leqslant$ being a *partial* order by assumption (cf. condition 6.). □

With all of these preliminaries out of the way, we are finally in a position to construct the needed frame and prove completeness.

Theorem 2.13 (Completeness)**.** $\mathbf{MIL_{Pre}}$ *is strongly complete w.r.t.* MIL_{Pre}*. So, in particular,* $\mathbf{MIL_{Pre}} \supseteq MIL_{Pre}$*.*

Proof. Suppose Γ_0 is consistent. It suffices to show that Γ_0 is satisfiable. As previously mentioned, to show so, we will construct a model satisfying a truth lemma for labeled points by taking the colimit of a sequence of models getting ever closer to satisfying this truth lemma. We begin by extending Γ_0 to a maximal consistent set $\Gamma \supseteq \Gamma_0$, and letting $\leqslant_0$ be the diagonal on W, $D_0 := \varnothing$ and $l_0 := \{(x_0, \Gamma)\}$ for some $x_0 \in W$. Then 1.-7. are satisfied, where 7. follows by reflexivity of $\leqslant_{\mathbf{Pre}}$. We continue by constructing a sequence

$$(l_0, \leqslant_0, D_0), (l_1, \leqslant_1, D_0), \dots, (l_n, \leqslant_n, D_n), \dots$$

s.t. for all $i \in \omega$

$$l_i \subseteq l_{i+1}, \quad \leqslant_i \subseteq \leqslant_{i+1}, \quad D_i \subseteq D_{i+1},$$

using the repair lemmas repeatedly. We do so by enumerating the set of all pairs $(\langle\sup\rangle\chi\chi', x)$ and all quadruples $(\neg\langle\sup\rangle\psi\psi', x, y, z)$. Then at each stage $n+1$ we pick the least tuple constituting a defect to $(l_n, \leqslant_n, D_n)$, which we repair obtaining $(l_{n+1}, \leqslant_{n+1}, D_{n+1})$. Letting

Such an enumeration exists because (1) W is countable, and (2) there are countably many formulas.

$$(l_\omega, \leqslant_\omega, D_\omega) := \left(\bigcup_{n\in\omega} l_n, \bigcup_{n\in\omega} \leqslant_n, \bigcup_{n\in\omega} D_n \right),$$

we get that (1) $(l_\omega, \leqslant_\omega, D_\omega)$ satisfies 4.-7.; (2) l_ω is a (partial) function from W to the set of all MCSs; and (3) $(l_\omega, \leqslant_\omega, D_\omega)$ neither has any $\langle\sup\rangle$- nor $\neg\langle\sup\rangle$-defects. Only (3) isn't straightforward. To show this, we prove two claims, and in order to do so, we need the following observation.

Observation. Let $n \in \omega$ and $\{x, v\} \subseteq \mathrm{dom}(l_n)$ be arbitrary s.t.

$$\uparrow_n v = \uparrow_n x \cup \{v\} \cup (\uparrow_n v \cap \{w \mid \uparrow_n w \cap \uparrow_n x = \varnothing\}).$$

Then for all $m \geqslant n$:

$$\uparrow_m v = \uparrow_m x \cup \{v\} \cup (\uparrow_m v \cap \{w \mid \uparrow_m w \cap \uparrow_m x = \varnothing\}),$$

hence also

$$\uparrow_\omega v = \uparrow_\omega x \cup \{v\} \cup (\uparrow_\omega v \cap \{w \mid \uparrow_\omega w \cap \uparrow_\omega x = \varnothing\}).$$

This is easily seen by induction, using that each $(l_{m+1}, \leqslant_{m+1}, D_{m+1})$ is obtained from $(l_m, \leqslant_m, D_m)$ using either of the repair lemmas.

Claim (a). Suppose $(\langle\text{sup}\rangle\chi\chi', x)$ *does not constitute a defect for some* $(l_n, \leqslant_n, D_n)$ *at which (i)* $x \in \text{dom}(l_n)$ *and (ii)* $\langle\text{sup}\rangle\chi\chi' \in l_n(x)$*. Then this must be witnessed by some* y, z *(cf. Definition 2.8). We show that for all* $m \geqslant n$*:*

$(\langle\text{sup}\rangle\chi\chi', x)$ *does not constitute a defect for* $(l_m, \leqslant_m, D_m)$*, witnessed by* y, z*.*

A fortiori, neither does it for $(l_\omega, \leqslant_\omega, D_\omega)$*.*

By the observation and noting that $l_i \subseteq l_{i+1}$ for all $i \in \omega$, it suffices to show that for all $m \geqslant n$:

$$x = \sup_m\{y, z\}.$$

We prove this by induction on $m \geqslant n$. By assumption, this holds for $m = n$. Accordingly, suppose it holds for an arbitrary $m \geqslant n$. We show it holds for $m+1$. We have two cases, depending on the type of defect being repaired at stage $m+1$.

First, suppose the defect repaired was a $\langle\text{sup}\rangle$-defect for some world s. Since the corresponding introduced $\text{dom}(l_{m+1})$-worlds y_s, z_s have no proper $\leqslant_{m+1}$-predecessors, the claim follows. Reason being that, cf. the IH and the definition

$$\leqslant_{m+1} := \leqslant_m \cup \{(y_s, u), (z_s, u) | s \leqslant_m u\},$$

y_s and z_s are the only possible counterexamples to the claim.

Second, suppose $(l_{m+1}, \leqslant_{m+1}, D_{m+1})$ was obtained via $\neg\langle\text{sup}\rangle$-repairing some s, y_s, z_s by introducing the dummy d_s. Notice that, by IH and the definition

$$\leqslant_{m+1} := \leqslant_m \cup \{(u, d_s), (v, d_s) | u \leqslant_m y_s, v \leqslant_m z_s\},$$

the only possible counterexample to the claim is d_s. Accordingly, suppose $d_s \geqslant_{m+1} y, z$. Going by cases, we prove that this implies $d_s \geqslant_{m+1} x$:

- If $y_s \geqslant_m y, z$, then, by IH, $y_s \geqslant_m x$ so $d_s \geqslant_{m+1} x$.
- If $z_s \geqslant_m y, z$, then as above.

- If $y_s \geqslant_m y$ and $z_s \geqslant_m z$, then, by the observation, either (a) $y_s = y$ or (b) $y_s \geqslant_m x$ or (c) $\uparrow_m y_s \cap \uparrow_m x = \varnothing$. If (b), then $d_s \geqslant_{m+1} x$. And if (c), then note that as s is a $\leqslant_m$-upper bound of $\{y_s, z_s\}$, it must also be a $\leqslant_m$-upper bound of $\{y, z\}$, hence, by IH, $x \leqslant_m s$ – contradicting $\uparrow_m y_s \cap \uparrow_m x = \varnothing$. Thus, we may assume (a) $y_s = y$; and, analogously, $z_s = z$. But then $s = \sup_{\leqslant_m}\{y_s, z_s\} = \sup_{\leqslant_m}\{y, z\} = x$, hence $(s, y_s, z_s) = (x, y, z)$ couldn't have constituted a $\neg\langle\text{sup}\rangle$-defect because $C_{\mathbf{Pre}} l_m(x) l_m(y) l_m(z)$.

- If $z_s \geqslant_m y$ and $y_s \geqslant_m z$, then as above.

This exhausts all cases, showing $d_s \geqslant_{m+1} x$, which completes the induction. $\square_{(a)}$

Claim (b). Suppose $n \in \omega$ *and* $a, b \in (\mathrm{dom}(l_n) \cup D_n)$ *are s.t.* $a \not\geqslant_n b$. *Then for all* $m \geqslant n$, *we have that* $a \not\geqslant_m b$. *A fortiori,* $a \not\geqslant_\omega b$.

Follows by induction on m, noting that if $(l_{m+1}, \leqslant_{m+1}, D_{m+1})$ was obtained by $\langle\text{sup}\rangle$-repairing some x by introducing some y, z, we would have

$$\leqslant_{m+1} := \leqslant_m \cup \{(y, u), (z, u) \mid x \leqslant u\};$$

that is, there is no change in successors of b.

Likewise, if $(l_{m+1}, \leqslant_{m+1}, D_{m+1})$ was obtained by $\neg\langle\text{sup}\rangle$-repairing some x, y, z by introducing a dummy d, there is no change in predecessors of a. This exhaust the cases, hence proves the claim. $\square_{(b)}$

Using *(a)* and *(b)*, it is straightforward to see *(c)*: If some tuple *did* constitute a defect at some stage n, but no longer at some later stage $m > n$, then it didn't for all $k \geqslant m$.

With these claims at hand, we can show (3) that $(l_\omega, \leqslant_\omega, D_\omega)$ neither has $\langle\text{sup}\rangle$- nor $\neg\langle\text{sup}\rangle$-defects. For $\langle\text{sup}\rangle$, let

$$(\langle\text{sup}\rangle\chi\chi', x)_i$$

be an arbitrary pair in our enumeration s.t. $x \in \mathrm{dom}(l_\omega)$ and $\langle\text{sup}\rangle\chi\chi' \in l_\omega(x)$. Then $x \in \mathrm{dom}(l_n)$ for some $n \in \omega$, hence

$$x \in \mathrm{dom}(l_m), \langle\text{sup}\rangle\chi\chi' \in l_m(x)$$

for all $m \geqslant n$. If, on one hand, $(\langle\text{sup}\rangle\chi\chi', x)_i$ didn't constitute a defect to $(l_n, \leqslant_n, D_n)$ – using *claim (a)* (and the observation) – we get that it wouldn't for $(l_\omega, \leqslant_\omega, D_\omega)$ either. On the other, in case it did, it would no more no later than at stage $n + i + 1$ (cf. *(c)*), and henceforth – by *claim (c)* – remain repaired. Thus,

$(l_\omega, \leqslant_\omega, D_\omega)$ has no $\langle\sup\rangle$-defects.

For $\neg\langle\sup\rangle$, suppose towards contradiction that

$$(\neg\langle\sup\rangle\psi\psi', x, y, z)_i$$

denotes a $\neg\langle\sup\rangle$-defect. Then $x \geqslant_\omega y$ and $x \geqslant_\omega z$, so there is some $n \in \omega$ s.t.

$$x \geqslant_n y, z.$$

If

$$x \neq \sup_{\leqslant_m}\{y, z\}$$

for some $m \geqslant n$, there must be some $a \in (\mathrm{dom}(l_m) \cup D_m)$ s.t.

$$y, z \leqslant_m a \not\geqslant_m x,$$

but then – cf. *claim (b)* –

$$y, z \leqslant_\omega a \not\geqslant_\omega x\,,$$

which, in particular, shows $x \neq \sup_\omega\{y, z\}$—contradicting $(\neg\langle\sup\rangle\psi\psi', x, y, z)_i$ being a $\neg\langle\sup\rangle$-defect. Thus, we must have

$$x = \sup_{\leqslant_m}\{y, z\}$$

for all $m \geqslant n$, implying – and simultaneously contradicting – that the defect will be repaired no later than at stage $n + i + 1$ (cf. *(c)*). That is, there can be no $\neg\langle\sup\rangle$-defects either.

Finally, setting

$$V(p) := \{x \in \mathrm{dom}(l_\omega) : p \in l_\omega(x)\},$$

we show that for all $x \in \mathrm{dom}(l_\omega)$ and all $\varphi \in \mathcal{L}_M$:

$$(W, \leqslant, V), x \Vdash \varphi \qquad \textbf{iff} \qquad \varphi \in l_\omega(x).$$

This is our truth lemma.

The proof goes by induction on the complexity of formulas. Base case is by definition and Boolean cases are straightforward. For the $\langle\sup\rangle$-case, we get

$$\begin{aligned} x \Vdash \langle\sup\rangle\varphi_1\varphi_2 \quad &\overset{\textit{Def}}{\textbf{iff}} \quad \exists y, z\,[x = \sup_\omega\{y, z\}, y \Vdash \varphi_1, z \Vdash \varphi_2] \\ &\overset{\textit{(IH)}}{\textbf{iff}} \quad \exists y, z\,[x = \sup_\omega\{y, z\}, \varphi_1 \in l_\omega(y), \varphi_2 \in l_\omega(z)] \\ &\overset{\textit{(i)}}{\textbf{iff}} \quad \langle\sup\rangle\varphi_1\varphi_2 \in l_\omega(x), \end{aligned}$$

where we in the left-to-right direction of *(IH)* use – apart from the induction hypothesis itself – that $(l_\omega, \leqslant_\omega, D_\omega)$ satisfies 5.-6.; i.e., in particular, neither of the witnessing y, z are dummies nor in $W \setminus (\mathrm{dom}(l_\omega) \cup D_\omega)$, and so they must be in $\mathrm{dom}(l_\omega)$. Further, left-to-right of (i) holds by there being no $\neg\langle\sup\rangle$-defects, while right-to-left follows from there being no $\langle\sup\rangle$-defects.

This completes the induction, from which it follows that

$$(W, \leqslant_\omega, V), x_0 \Vdash \Gamma_0,$$

showing that Γ_0 is satisfiable in a preorder model and, thus, at long last, finalizing our proof of completeness. □

Corollary 2.14. $MIL_{Pre} = MIL_{Pos}$.

Proof. As noted in Remark 1.6, $MIL_{Pre} \subseteq MIL_{Pos}$, while the other inclusion follows from the frame constructed in the completeness proof being a *partial* order. □

3. Decidability of MIL_{Pre}

This chapter consists of two parts. In section 3.1, we show that $\mathbf{MIL_{Pre}}$ is complete w.r.t. another class of structures $\mathcal{C}$. Then, in section 3.2, we show that $\mathbf{MIL_{Pre}}$ has the FMP *w.r.t.* $\mathcal{C}$*-frames* and conclude that MIL_{Pre} (and MIL_{Pos}) are, after all, decidable—solving a problem posed in van Benthem (2017, 2019, Forthcoming). Lastly, as an addendum, in Appendix A.2, we show that $\mathbf{MIL_{Pre}}$ also enjoys the TMP w.r.t. $\mathcal{C}$-frames.

3.1. Reinterpreting $\langle \mathrm{sup} \rangle$ on generalized structures $\mathcal{C}$

Following the method of section 1.3, and with an axiomatization of MIL_{Pre} at hand, we continue our road to decidability by proving completeness relative to a different class of structures. These structures will be named $\mathcal{C}$*-frames*, alluding to our denoting this class of structures as $\mathcal{C}$.

Before we get that far, though, the first key observation to make is that there is nothing in the *syntactic* definition of $\mathbf{MIL_{Pre}}$ implying that the binary modality-symbol $\langle \mathrm{sup} \rangle$ need be interpreted in terms of the supremum relation on a pre-order. I.e., there is nothing a priori hindering us from reinterpreting $\mathbf{MIL_{Pre}}$ through reinterpreting the symbol $\langle \mathrm{sup} \rangle$.

Further, $\mathbf{MIL_{Pre}}$ being an NML means that there might be a canonical reinterpretation, namely the one reached through frame correspondence of $\mathbf{MIL_{Pre}}$ on the class of all pairs (W, C) where W is a set and C is an *arbitrary* ternary relation on W. And, indeed, that is how we proceed.

Lemma 3.1. *Let* (W, C) *be a frame for the modal language with a single binary modality. Then we have the following frame correspondences:*

(i) $(W, C) \Vdash (\mathrm{Re.})$ ***iff*** $(W, C) \vDash \forall w \, (Cwww)$

(ii) $(W, C) \Vdash (4)$ ***iff*** $(W, C) \vDash \forall w, v, x \, (Cwvx \wedge Cvuy \rightarrow \exists z \, [Cwuz])$

(iii) $(W, C) \Vdash (\mathrm{Co.})$ ***iff*** $(W, C) \vDash \forall w, v, u \, (Cwvu \rightarrow Cwuv)$

(iv) $(W, C) \Vdash (\mathrm{Dk.})$ ***iff*** $(W, C) \vDash \forall w, v, u \, (Cwvu \rightarrow Cwwv)$

Proof. Standard frame correspondence proofs work, using arguments similar to the ones in the proof of Lemma 2.6*(a)*, *(b)(i)*, *(c)* and *(d)*. Alternatively, the

Sahlqvist-van Benthem algorithm also applies because the formulas are Sahlqvist. □

Definition 3.2. We denote the first-order correspondents of (Re.), (4), (Co.) and (Dk.) as (Re.f), (4f′), (Co.f) and (Dk.f), respectively. ⊣

While (Re.f), (Co.f), and (Dk.f) all match neatly with (Re.), (Co.), and (Dk.), respectively, (4f′) is a bit ugly FO-correspondent of (4). However, as the following proposition shows, in the presence of the other axioms, the correspondence crystallizes.

Proposition 3.3. *Let (W, C) be a frame for the modal language with a single binary modality. Then TFAE:*

- $(W, C) \Vdash \mathbf{MIL_{Pre}}$
- $(W, C) \vDash (\mathrm{Re.f}) \wedge (\mathrm{Co.f}) \wedge (\mathrm{Dk.f}) \wedge \forall w, v, u\,(Cwwv \wedge Cvvu \to Cwwu)$

In other words, (4f′) and (4f) are equivalent modulo (Re.f), (Co.f) and (Dk.f), where

In fact, even modulo (Dk.f) and (Co.f).

$$(4\mathrm{f}) := \forall w, v, u\,(Cwwv \wedge Cvvu \to Cwwu)\,.$$

Proof. Straightforward consequence of Lemma 3.1. □

It now follows that we have obtained a different class of structures, namely $\mathcal{C}$, which is complete w.r.t. $\mathbf{MIL_{Pre}}$ – as summarized in the ensuing corollary.

Corollary 3.4. $\mathbf{MIL_{Pre}}$ *is sound and (strongly) complete w.r.t.*

$$\mathcal{C} := \{(W, C) \vDash (\mathrm{Re.f}) \wedge (\mathrm{Co.f}) \wedge (\mathrm{Dk.f}) \wedge (4\mathrm{f})\}\,.$$

In particular,

$$\mathbf{MIL_{Pre}} = \mathrm{Log}(\mathcal{C}),$$

where $\mathrm{Log}(\mathcal{C}) := \{\varphi \in \mathcal{L}_M \mid (W, C) \Vdash \varphi, (W, C) \in \mathcal{C}\}$ *denotes the NML of* $\mathcal{C}$.

Proof. The preceding proposition implies soundness, and then our earlier completeness theorem (2.13) gives us (strong) completeness. □

This corollary proven, we have arrived at the final step described in section 1.3: showing the FMP of $\mathbf{MIL_{Pre}}$ when reinterpreted on $\mathcal{C}$. Before proving this in the next section, we find it instructive to revisit the formula ψ_N from Proposition 1.8 and show that, although not satisfiable on a finite *preorder frame*, it is satisfiable on a finite $\mathcal{C}$-*frame*. We do this right after observing the following:

Observation 3.5. It is not hard to prove that for any $(W, C) \in \mathcal{C}$, $x \in W$, valuation V on (W, C) and formula φ, we have that

$$(W, C, V), x \Vdash P\varphi \quad \textbf{iff} \quad \exists y \in W\,(Cxy \wedge y \Vdash \varphi)\,,$$

and hence also

$$(W, C, V), x \Vdash H\varphi \quad \textbf{iff} \quad \forall y \in W\,(Cxy \rightarrow y \Vdash \varphi)\,. \qquad \dashv$$

Remark 3.6. Although

$$\psi_N := HP\langle\sup\rangle pp \wedge HP\neg\langle\sup\rangle pp$$

only is satisfiable on infinite preorder models under the standard interpretation of $\langle\sup\rangle$ (cf. Proposition 1.8), it is satisfiable on a finite $\mathcal{C}$-frame. $\dashv$

Proof. Set

$$W := \{w, v\}\,, \quad C := \{(w, w, w), (v, v, v), (w, w, v), (w, v, w), (v, v, w), (v, w, v)\},$$
$$V(p) := \{w\}.$$

(W, C) can be intuited as a two-point cluster in which '$x \notin \sup(y, y)$' when $x \neq y$.

We claim that $(W, C) \in \mathcal{C}$ and $(W, C, V), w \Vdash \psi_N$.

The former can be seen by a quick (yet tedious) check that (W, C) models the given first-order conditions [otherwise, our later Remark A.3.1 also implies this]. The latter can be seen by first noting that

$$\text{(a) } w \Vdash \langle\sup\rangle pp \qquad \textit{while} \qquad \text{(b) } v \nVdash \langle\sup\rangle pp,$$

since, respectively, (a) $Cwww$ and $w \Vdash p$, and (b) $\neg Cvww$ and $v \nVdash p$.

Moreover, using that $(W, C) \in \mathcal{C}$, we get

$$\begin{aligned} w \Vdash HP\langle\sup\rangle pp \quad & \textbf{iff} \quad \forall x \in W\,(Cwwx \rightarrow x \Vdash P\langle\sup\rangle pp) \\ & \textbf{iff} \quad \forall x \in W\,(Cwwx \rightarrow \exists y\,[Cxy \wedge y \Vdash \langle\sup\rangle pp])\,, \end{aligned}$$

hence also

$$w \Vdash HP\neg\langle\sup\rangle pp \quad \textbf{iff} \quad \forall x \in W\,(Cwwx \rightarrow \exists y\,[Cxy \wedge y \nVdash \langle\sup\rangle pp])\,.$$

With this spelt out, we find that $w \Vdash \psi_N$ as we have $Cwww$, $Cwwv$, $Cvvv$, $Cvvw$; i.e., the existential consequents are always fulfilled. $\square$

3.2. The finite model property

As promised, we go on proving that $\mathbf{MIL_{Pre}}$ enjoys the FMP w.r.t. $\mathcal{C}$ and then use this to deduce decidability of MIL_{Pre}. The proof of the FMP is done by employing a filtration-style argument. To this end, we define a notion extending the standard notion of a set of formulas being subformula closed.

Definition 3.7. We say that a set Σ of $\mathcal{L}_M$-formulas is $\mathcal{C}$-*closed* :iff

(Sub) it is subformula closed;

(Com) $\langle \sup \rangle \varphi\psi \in \Sigma$ implies $\langle \sup \rangle \psi\varphi \in \Sigma$; and

(S-P) $\langle \sup \rangle \varphi\psi \in \Sigma$ implies $P\varphi \in \Sigma$.

Moreover, for any set of formulas Σ_0, we say that Σ is the $\mathcal{C}$-*closure* of Σ_0 :iff it is the least $\mathcal{C}$-closed set of formulas extending Σ_0. ⊣

Note that the $\mathcal{C}$-closure of a set of formulas always exists.

An immediate consequence of the definition is the following lemma:

Lemma 3.8. *Suppose Σ_0 is a finite set of $\mathcal{L}_M$-formulas. Then its $\mathcal{C}$-closure $\Sigma \supseteq \Sigma_0$ is finite as well.*

Less immediate is how to use this notion for a filtration-style argument of the FMP. This is the content of the following theorem, whose proof contains the actual definition of a filtration through a $\mathcal{C}$-closed set of formulas.

Theorem 3.9. $\mathbf{MIL_{Pre}}$ *admits filtration w.r.t. the class $\mathcal{C}$. Thus,*

$$\mathbf{MIL_{Pre}} = \mathrm{Log}(\mathcal{C}_F),$$

where $\mathrm{Log}(\mathcal{C}_F)$ *denotes the NML of the class of finite $\mathcal{C}$-frames.*

Proof. Cf. Lemma 3.8 and the obvious inclusion $\mathrm{Log}(\mathcal{C}) \subseteq \mathrm{Log}(\mathcal{C}_F)$, it suffices to show that for any $\mathcal{C}$-model (W, C, V) and $\mathcal{C}$-closed set of formulas Σ, the following hold:

1. $(W_\Sigma, C^{\mathcal{C}}_\Sigma) \in \mathcal{C}$, where our filtered universe is

$$W_\Sigma := \{|x|_\Sigma : x \in W\}$$

$|x|_\Sigma$ denotes the equivalence class on the set of worlds W defined as satisfying the same Σ-formulas as x.

with relation

$$C^{\mathcal{C}}_\Sigma |x||y||z| \quad \textbf{:iff} \quad \forall \langle \sup \rangle \varphi\psi \in \Sigma \big([(y \Vdash \varphi, z \Vdash \psi) \Rightarrow x \Vdash \langle \sup \rangle \varphi\psi] \text{ and } [(y \Vdash P\varphi, z \Vdash P\psi) \Rightarrow x \Vdash P\varphi \wedge P\psi] \big).$$

2. For all $\varphi \in \Sigma, x \in W$:

$$(W_\Sigma, C^{\mathcal{C}}_\Sigma, V_\Sigma), |x| \Vdash \varphi \qquad \textbf{iff} \qquad (W, C, V), x \Vdash \varphi,$$

where $V_\Sigma(p) := \{|x| \in W_\Sigma : x \in V(p)\}$ for all $p \in \Sigma$.

We begin by proving 1.; i.e., showing $(W_\Sigma, C^{\mathcal{C}}_\Sigma) \in \mathcal{C}$. This we do as follows:

- $(W_\Sigma, C^{\mathcal{C}}_\Sigma) \vDash (\mathsf{Re.f})$ can be seen using $(W, C) \Vdash (\mathsf{Re.})$.[8]
- $(W_\Sigma, C^{\mathcal{C}}_\Sigma) \vDash (\mathsf{Co.f})$ can be seen using $(W, C) \Vdash (\mathsf{Co.})$ and the (Com)-closure.
- Showing $(W_\Sigma, C^{\mathcal{C}}_\Sigma) \vDash (\mathsf{Dk.f})$ is a bit more tricky. Accordingly, suppose $C^{\mathcal{C}}_\Sigma|x||y||z|$ and let $\langle\text{sup}\rangle\varphi\psi \in \Sigma$ be arbitrary. It then suffices to show

  $$(x \Vdash \varphi, y \Vdash \psi) \Rightarrow x \Vdash \langle\text{sup}\rangle\varphi\psi \quad \textit{and} \quad (x \Vdash P\varphi, y \Vdash P\psi) \Rightarrow x \Vdash P\varphi \wedge P\psi.$$

  For the former, since $\langle\text{sup}\rangle\psi\top = P\psi \in \Sigma$ by (Com)- and (S-P)-closure, we have that if $y \Vdash \psi$, then $x \Vdash \langle\text{sup}\rangle\psi\top$ because $C^{\mathcal{C}}_\Sigma|x||y||z|$. So if also $x \Vdash \varphi$, then using $(W, C) \Vdash (\mathsf{Dk.})$, we get $x \Vdash \langle\text{sup}\rangle\varphi\psi$.

  Further, for the latter, if $y \Vdash P\psi$, using $z \Vdash P\top$ because $(W, C) \Vdash (\mathsf{T})$ [and, again, $\langle\text{sup}\rangle\psi\top = P\psi \in \Sigma$ and $C^{\mathcal{C}}_\Sigma|x||y||z|$], we get $x \Vdash P\psi$.

- Lastly, to prove $(W_\Sigma, C^{\mathcal{C}}_\Sigma) \vDash (4\mathsf{f})$, suppose $C^{\mathcal{C}}_\Sigma|x||x||y|$, $C^{\mathcal{C}}_\Sigma|y||y||z|$ and $\langle\text{sup}\rangle\varphi\psi \in \Sigma$. We show that

  $$(x \Vdash \varphi, z \Vdash \psi) \Rightarrow x \Vdash \langle\text{sup}\rangle\varphi\psi \quad \textit{and} \quad (x \Vdash P\varphi, z \Vdash P\psi) \Rightarrow x \Vdash P\varphi \wedge P\psi.$$

  For the former, if $z \Vdash \psi$, then $C^{\mathcal{C}}_\Sigma|y||y||z|$ and $\langle\text{sup}\rangle\top\psi \in \Sigma$ imply $y \Vdash \langle\text{sup}\rangle\top\psi$, hence $y \Vdash P\psi$ by $(W, C) \Vdash (\mathsf{Co.})$. But then this along with $x \Vdash P\top$

[8] Alternatively, below we show that this filtration indeed satisfies the homomorphic filtration condition: $Cxyz \Rightarrow C^{\mathcal{C}}_\Sigma|x||y||z|$. From this and surjectivity of $x \mapsto |x|_\Sigma$, $(\mathsf{Re.f})$ follows.

On this node, it is worth (foot)noting that the culprit in hindering this inheritance argument for the three other FO-conditions are the implications in their respective definitions; e.g., for (Dk.f) we have $Cwvu \to Cwwv \equiv \neg Cwvu \vee Cwwv$, so when this implication holds by virtue of the first disjunct, namely '$\neg Cwvu$', we cannot likewise conclude $\neg C^{\mathcal{C}}_\Sigma|w||v||u|$.

This also explains that the filtration relation and the set of formulas we are filtering through have been defined to accommodate these three axioms. As for the transitivity axiom, we have drawn inspiration from the Lemmon filtration.

Lastly, we briefly indicate why FO-conditions with an *existential* in the consequent of an implication are much worse [e.g., (one-way) associativity: $\forall w, v, x, y, z([Cwvz \wedge Cvxy] \to \exists u[Cwxu \wedge Cuyz])$]. When we didn't have an existential in the consequent, the general idea was – to use the previous example – to ensure that in cases where $C^{\mathcal{C}}_\Sigma|w||v||u|$ while also $\neg Cwvu$, we always had $C^{\mathcal{C}}_\Sigma|w||w||v|$. Crucially, we had some concrete worlds to try to get a handle on: the worlds of the consequent appeared in the antecedent. As soon as we have an existential instead, this 'handle' goes down the drain.

While a bit of an aside, we find it an interesting general aspect worth pointing out, and it also connects with our later Remark A.3.2 and section 6.5 in general.

and $C^{\mathcal{C}}_{\Sigma}|x||x||y|$ imply that $x \Vdash P\psi$. So if also $x \Vdash \varphi$, then $(W, C) \Vdash (\mathrm{Dk.})$ implies $x \Vdash \langle \sup \rangle \varphi\psi$.

Further, if $z \Vdash P\psi$, using $z \Vdash P\top$, then $y \Vdash P\psi$, and in turn $x \Vdash P\psi$.

This completes our proof of 1. For proving 2., it suffices to show that $(W_{\Sigma}, C^{\mathcal{C}}_{\Sigma}, V_{\Sigma})$ is a filtration of (W, C, V) through Σ. That is, we need to check two conditions, namely

(F1) $Cxyz \Rightarrow C^{\mathcal{C}}_{\Sigma}|x||y||z|$; and

(F2) $C^{\mathcal{C}}_{\Sigma}|x||y||z| \Rightarrow \forall \langle \sup \rangle \varphi\psi \in \Sigma\, [(y \Vdash \varphi, z \Vdash \psi) \Rightarrow x \Vdash \langle \sup \rangle \varphi\psi]$.

(F2) follows by definition of our filtration relation. For (F1), suppose $Cxyz$ and $\langle \sup \rangle \varphi\psi \in \Sigma$. Then the only non-trivial part is to show that

$$(y \Vdash P\varphi, z \Vdash P\psi) \Rightarrow x \Vdash P\varphi \wedge P\psi.$$

Since $(W, C) \models (\mathrm{Dk.f}) \wedge (\mathrm{Co.f})$, we also have $Cxxy$ and $Cxxz$. Thus, if $y \Vdash P\varphi$ and $z \Vdash P\psi$, we get that

$$x \Vdash PP\varphi \wedge PP\psi,$$

hence from $(W, C) \Vdash (4)$, we get

$$x \Vdash P\varphi \wedge P\psi$$

as desired. □

Finally, we end this chapter by deducing decidability.[9]

Corollary 3.10. *MIL_{Pre} is decidable (and so is MIL_{Pos}).*

Proof. Cf. Theorem 2.13 and Corollary 3.4, we know that

$$MIL_{Pre} = \mathbf{MIL_{Pre}} = \mathrm{Log}(\mathcal{C}_F).$$

So since $\mathbf{MIL_{Pre}}$ is a finitely axiomatized NML admitting filtration w.r.t. $\mathcal{C}$, we get decidability. □

Similarly, using that our filtration argument establishes the strong finite model property, one can prove decidability.

[9] For the interested reader, in Appendix A.2 we show that the general heuristic regarding decidability and the FMP outlined in section 1.3 also applies to the TMP.

4. MIL with Informational Implication

With $MIL_{Pre} = MIL_{Pos}$ axiomatized and proven decidable, this chapter investigates their enrichments, $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$, with the 'informational implication' '$\backslash$'. The main goals are to provide an axiomatization and a decidability proof.

In section 4.1, we formally set out the logics of concern and briefly comment on the increased expressibility. In section 4.2, we, first, put forward an axiomatization and point out on an interesting junction with the Lambek Calculus. Before, second, pausing our investigation of $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$ per se, to show that the proposed axiomatization is sound and complete w.r.t. the class $\mathcal{C}$. Using this result, in section 4.3, we obtain soundness and completeness w.r.t. our poset frames through combining two representation results: the first achieved via an adaptation of 'bulldozing', and the second via supplementing the framework of section 2.2 with an additional defect. We deduce that $MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos}$. Lastly, in section 4.4, we modify the filtration technique of section 3.2 to attain decidability of $MIL_{\backslash\text{-}Pre}$.

4.1. Augmenting with '$\backslash$'

As noted, we seek to study the enrichment of the basic modal information logic(s), MIL_{Pre} and MIL_{Pos}, given by adding an 'informational implication' as a binary modality. In this section we cover some preliminaries, specifically, some definitions followed by a few comments on expressivity. We start with supplying the following pertinent definitions:

Definition 4.1 (Language)**.** The language $\mathcal{L}_{\backslash\text{-}M}$ is given by extending the basic language of modal information logic $\mathcal{L}_M$ with a binary modality symbol '$\backslash$'.

As a convention we use infix notation for '$\backslash$' instead of prefix/Polish notation; that is, we write '$\varphi\backslash\psi$', rather than '$\backslash\varphi\psi$' (as we, e.g., would do with '$\langle \sup \rangle$' and '$[\sup]$'). ⊣

Definition 4.2 (Semantics)**.** Given a preorder model $\mathbb{M} = (W, \leqslant, V)$, a world $v \in W$ and a formula $\varphi\backslash\psi \in \mathcal{L}_{\backslash\text{-}M}$ with main connective '$\backslash$', we let

$$\mathbb{M}, v \Vdash \varphi\backslash\psi \quad \textbf{iff} \quad \text{for all } u, w \in W, \text{ if } \mathbb{M}, u \Vdash \varphi \text{ and } w \in \sup\{u, v\}, \text{ then } \mathbb{M}, w \Vdash \psi.$$ ⊣

Notice how '$\backslash$' is the '$\Box$-ed' and not the '$\Diamond$-ed' half of a modality pair.

Definition 4.3 (Logic)**.** We denote the modal information logic on preorders in the enriched language of $\mathcal{L}_{\backslash\text{-M}}$ as $MIL_{\backslash\text{-}Pre}$, which – to be explicit – is defined as

$$MIL_{\backslash\text{-}Pre} := \{\varphi \in \mathcal{L}_{\backslash\text{-M}} : (W, \leqslant) \Vdash \varphi \text{ \textit{for all preorder frames} } (W, \leqslant)\}.$$

$MIL_{\backslash\text{-}Pos}$ is defined analogously. ⊣

Remark 4.4. As a minor interlude, as mentioned in the introduction, the choice of symbol '\' concurs with standard notation in the Lambek Calculus. With the semantics given, the reason becomes evident: the interpretation is the same (given a supremum relation). It is also worth pointing out that the commutativity of suprema implies that the other Lambek residual – typically denoted by '/' – collapses into '\' in the sense that $\varphi/\psi \equiv \psi\backslash\varphi$. Lastly, the modality '$\langle\text{sup}\rangle$' is interpreted (again, given a supremum relation) exactly as the binary product '·' is in the Lambek Calculus. In the next section, we expound this connection even further. ⊣

Now, recall that the primary results we are after are (1) axiomatizing $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$ and (2) showing them to be decidable. Once more, we will be following the heuristic laid out in section 1.3; however, this time our completeness theorem will not be proven via model constructions but via representation results. For this to work, we, needlees to say, must (a) have another class of structures for which we can prove the representation results, and (b) also already have the logic of this other class axiomatized. Regarding (a), a natural candidate arises: the $\mathcal{C}$-frames of the previous chapter. Before being able to (b) axiomatize the logic of this class (as we will in the next section), we must clarify how '\' is to be interpreted on $\mathcal{C}$-models. This is the content of the following definition:

Definition 4.5. Given a frame $(W, C) \in \mathcal{C}$, a valuation V on (W, C), a world $v \in W$ and a formula $\varphi\backslash\psi \in \mathcal{L}_{\backslash\text{-M}}$ with main connective '\', we let

$$(W, C, V), v \Vdash \varphi\backslash\psi \quad \textbf{iff} \quad \text{for all } u, w \in W, \text{ if } (W, C, V), u \Vdash \varphi \text{ and } Cwvu, \text{ then } (W, C, V), w \Vdash \psi.$$

⊣

To be precise, we explicate how this generalizes our definition on preorder frames.

Definition 4.6. Let $\mathcal{S}_{\text{Pre}}$ (resp. $\mathcal{S}_{\text{Pos}}$) be the class of pairs $(W, S_{\leqslant})$ where W is a set and $S_{\leqslant} \subseteq W^3$ is a ternary relation for which there is some preorder (resp. partial order) $\leqslant$ on W s.t. for all $w, v, u \in W$:

I.e., $S_{\leqslant}$ is the supremum relation induced by a preorder (resp. poset).

$$S_{\leqslant}wvu \qquad \textbf{iff} \qquad w \in \text{sup}_{\leqslant}\{u, v\}.$$

Then the semantics of '$\backslash$' on a preorder model $(W, \leqslant, V)$ comes down to

$$(W, S_{\leqslant}, V), v \Vdash \varphi \backslash \psi \quad \textbf{iff} \quad \text{for all } u, w \in W, \text{ if } (W, S_{\leqslant}, V), u \Vdash \varphi \text{ and } S_{\leqslant} wvu, \text{ then } (W, S_{\leqslant}, V), w \Vdash \psi,$$

where $S_{\leqslant}$ is the supremum relation induced by $\leqslant$. ⊣

As the last definition of this section, we set forth the logic of $\mathcal{C}$-frames in this extended language:

Definition 4.7. We write $\text{Log}_{\backslash}(\mathcal{C})$ for the logic of $\mathcal{C}$-frames in the language $\mathcal{L}_{\backslash\text{-M}}$; i.e., $\text{Log}_{\backslash}(\mathcal{C})$ denotes the set of $\mathcal{L}_{\backslash\text{-M}}$-validities on $\mathcal{C}$-frames. ⊣

With these definitions out of the way, we finish up this section with the promised comments on expressivity. First off, we show that with the additional vocabulary provided, we are not only able to express the past-looking unary modality 'P', but also the future-looking 'F'.

Remark 4.8. The future-looking unary modality 'F' (i.e., the standard '$\Diamond$') is definable as

$$F\varphi := \neg(\top \backslash \neg\varphi).$$

Notice that this places us in a (rather simple and natural) extension of *temporal* **S4**.

This can be seen by recalling the definition

$$\mathbb{M}, v \Vdash F\varphi \qquad \textbf{:iff} \qquad \exists w(v \leqslant w, w \Vdash \varphi),$$

and observing that also

$$\begin{aligned} \mathbb{M}, v \Vdash \neg(\top \backslash \neg\varphi) \qquad & \textbf{iff} \qquad \exists u, w(w \in \sup\{u, v\}, u \Vdash \top, w \nVdash \neg\varphi) \\ & \textbf{iff} \qquad \exists w(v \leqslant w, w \Vdash \varphi). \end{aligned}$$ ⊣

Finally, for good measure, observe that '$\backslash$' is not expressible in our simpler language $\mathcal{L}_{\text{M}}$. To see this, take, e.g., a two-chain $\{0, 1\}$ where $0 \leqslant 1$ and a one-chain $\{0'\}$; and let $0 \Vdash \neg p$, $1 \Vdash p$, and $0' \Vdash \neg p$. Then $0 \Vdash Fp$ while $0' \nVdash Fp$, but for all $\varphi \in \mathcal{L}_{\text{M}}$: $0 \Vdash \varphi$ iff $0' \Vdash \varphi$.

4.2. Axiomatizing $\text{Log}_{\backslash}(\mathcal{C})$

Now for the promised axiomatization of $\text{Log}_{\backslash}(\mathcal{C})$, which – via the representation results of the next section – entails that it even is an axiomatization of $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$.

Definition 4.9 (Axiomatization)**.** We define $\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}$ to be the least set of $\mathcal{L}_{\backslash\text{-}\mathcal{M}}$-formulas that (i) is closed under the axioms and rules of $\mathbf{MIL}_{\mathbf{Pre}}$; (ii) contains the K-axioms for $\backslash$; (iii) contains the axioms

(I1) $\langle\text{sup}\rangle p(p\backslash q) \to q$, and

(I2) $p \to q\backslash(\langle\text{sup}\rangle pq)$;

'(I1)' and '(I2)' are short for 'inverses': they capture how '$\langle\text{sup}\rangle$' and '$\backslash$' relate.

and (iv) is closed under the rule

($N_\backslash$) if $\vdash_{\backslash\text{-Pre}} \varphi$, then $\vdash_{\backslash\text{-Pre}} \psi\backslash\varphi$. ⊣

'($N_\backslash$)' is short for 'necessitation'. Observe that the other necessitation rule is not validity preserving. We, e.g., have $\text{Log}_\backslash(\mathfrak{C}) \Vdash \top$ but we do not have $\text{Log}_\backslash(\mathfrak{C}) \Vdash \top\backslash\bot$.

Before showing that $\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}$ is sound and strongly complete w.r.t. $\mathfrak{C}$-frames, some remarks are due.

Remark 4.10 (Lambek Calculus of suprema on preorders)**.** In its basic version, the Lambek Calculus only contains the three binary connectives '$\cdot$', '$\backslash$' and '$/$', of which the first matches our '$\langle\text{sup}\rangle$' and the last two, modulo (Co.), both match our '$\backslash$'. It is defined proof-theoretically with the constitutive rules of the connectives (when given in our language) being

(L1) if $\vdash \langle\text{sup}\rangle\varphi\psi \to \chi$, then $\vdash \psi \to \varphi\backslash\chi$; and its converse

(L1) if $\vdash \psi \to \varphi\backslash\chi$, then $\vdash \langle\text{sup}\rangle\varphi\psi \to \chi$.

'(L1)' and '(L2)' are short for 'Lambek'.

Unsurprisingly, both of these rules are derivable in our Hilbert system for $\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}$. We refer the reader to Buszkowski (2021) for a proof; in this paper, Buszkowski considers the extensions of both the associative and non-associative Lambek Calculus—which he denotes **L** and **NL**, respectively—with the classical propositional calculus, resulting in the logical systems **L-CL** and **NL-CL**, respectively. It is his proof of derivability of (L1) and (L2) in his Hilbert system for **NL-CL** that readily applies to our $\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}$. Reason being that $\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}$ turns out to be nothing but an extension of **NL-CL** with the axioms (Re.), (4), (Co.), and (Dk.)—shedding another interesting light on modal information logics and, especially, $MIL_{\backslash\text{-}Pre}$ (and $MIL_{\backslash\text{-}Pos}$) when having in mind that we end up proving that $MIL_{\backslash\text{-}Pos} = MIL_{\backslash\text{-}Pre} = \mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}$. In other words, $MIL_{\backslash\text{-}Pre}$ *is* the Lambek Calculus (augmented with CL) of suprema on preorders (or on posets). ⊣

Remark 4.11. Besides from Buszkowski (2021) being a recent gem in the literature on the Lambek Calculus extended with CL (i.e., essentially, studying it as a classical modal logic with three binary modalities), it has received some newborn attention: in Buszkowski and Farulewski (2009) **NL-CL** is denoted **BFNL**, and in Kaminski and Francez (2014) **L-CL** and **NL-CL** are denoted **PL** and **PNL**, respectively. ⊣

We continue with the pledged completeness proof.

Theorem 4.12. $\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}$ *is sound and strongly complete w.r.t. the class* $\mathcal{C}$*. Thus, in particular,* $\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}} = \mathrm{Log}_{\backslash}(\mathcal{C})$.

Proof. Soundness $\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}} \subseteq \mathrm{Log}_{\backslash}(\mathcal{C})$ is routine.

Nevertheless, for soundness, to understand how '\' and '⟨sup⟩' capture different aspects of the same relation, it might be instructive for the reader to check that (I1) and (I2) are valid on $\mathcal{C}$-frames.

For strong completeness, we define the canonical frame as we did in Definition 2.3, but now defined w.r.t. the language $\mathcal{L}_{\backslash\text{-}M}$ instead; i.e., we let $W_{\backslash\text{-}\mathbf{Pre}}$ denote the set of $\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}$-MCSs, and set $C_{\backslash\text{-}\mathbf{Pre}}\Gamma\Delta\Theta$:iff

$$\forall\delta \in \Delta, \theta \in \Theta\,(\langle\mathrm{sup}\rangle\delta\theta \in \Gamma)\,.$$

Note that Lindenbaum's Lemma and standard properties of MCSs hold, since our logic contains all classical propositional tautologies and is closed under MP and US. As in Lemma 2.6, we then get that $(W_{\backslash\text{-}\mathbf{Pre}}, C_{\backslash\text{-}\mathbf{Pre}}) \in \mathcal{C}$.

Thus, it suffices to show the standard truth lemma. The base and Boolean cases are straightforward by standard properties of MCSs, and since '[sup]' is a normal modality and $C_{\backslash\text{-}\mathbf{Pre}}$ is defined in terms of it(s dual), the corresponding inductive step of the truth lemma goes through. Therefore, it only remains to cover the inductive step for '\'.[10] To this end, the following two claims will suffice:

- *Claim:* If $\varphi\backslash\psi \in \Delta$, $\varphi \in \Theta$ and $C_{\backslash,\mathbf{Pre}}\Gamma\Delta\Theta$, then $\psi \in \Gamma$.

  Proof. Assume $\varphi\backslash\psi \in \Delta$, $\varphi \in \Theta$ and $C_{\backslash,\mathbf{Pre}}\Gamma\Delta\Theta$. By definition of $C_{\backslash,\mathbf{Pre}}$, we would have that $\langle\mathrm{sup}\rangle(\varphi\backslash\psi)\varphi \in \Gamma$. $\psi \in \Gamma$ then follows by (I1), (Co.), US, and MP of MCSs.

- *Claim:* If $\neg(\varphi\backslash\psi) \in \Delta$, then there are some Θ, Γ s.t. $\varphi \in \Theta, \neg\psi \in \Gamma$ and $C_{\backslash,\mathbf{Pre}}\Gamma\Delta\Theta$.

  This is the existence lemma for '\'.

  Proof. Assume $\neg(\varphi\backslash\psi) \in \Delta$. Then

  $$\Gamma_0 := \{\langle\mathrm{sup}\rangle\delta\varphi \mid \delta \in \Delta\} \cup \{\neg\psi\}$$

  is consistent because if not, then

  $$\vdash_{\backslash\text{-Pre}} \bigwedge_{i\leqslant k} \langle\mathrm{sup}\rangle\delta_i\varphi \to \psi$$

[10] For another, more elaborate proof of a truth lemma which resembles ours, see the one given for the canonical model of **NL-CL** (their **PNL**) in Kaminski and Francez (2014).
We provide our own proof and keep it brief, assuming familiarity with the techniques involved. This will be done in the terminology of Blackburn et al. (2001, ch. 4), which also is an excellent resource for an explication of arguments and details sufficiently similar to the ones we will omit.

for some finite $\{\delta_0, \dots, \delta_k\} \subseteq \Delta$, hence (a)

$$\vdash_{\backslash\text{-Pre}} \langle \sup \rangle \widehat{\delta} \varphi \to \psi$$

where $\widehat{\delta} := \bigwedge_{i \leqslant k} \delta_i$. Moreover, since $\widehat{\delta} \in \Delta$, we get by (I2), US, and MP of MCSs that (b) $\varphi \backslash (\langle \sup \rangle \widehat{\delta} \varphi) \in \Delta$. Thus, since all MCSs extend $\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}$ and the monotonicity rule

$$\text{if } \vdash_{\backslash\text{-Pre}} \alpha_0 \to \alpha_1, \text{ then } \vdash_{\backslash\text{-Pre}} \beta \backslash \alpha_0 \to \beta \backslash \alpha_1$$

is easily derived, we get by (a), (b), US and MP of MCSs that $\varphi \backslash \psi \in \Delta$ – contradiction. Consequently, Γ_0 must be consistent.

Now, let $\chi_0, \chi_1, \dots$ be an enumeration of all $\mathcal{L}_{\backslash\text{-M}}$-formulas, and define

$$\Theta_0 := \{\varphi\},$$

and

$$\Theta_{n+1} := \begin{cases} \Theta_n \cup \{\chi_n\}, & \textit{if } \{\langle \sup \rangle \delta (\widehat{\Theta_n} \wedge \chi_n) \mid \delta \in \Delta\} \cup \{\neg \psi\} \textit{ is consistent} \\ \Theta_n \cup \{\neg \chi_n\}, & \textit{otherwise.} \end{cases}$$

We claim that the set

$$\{\langle \sup \rangle \delta \widehat{\Theta_n} \mid \delta \in \Delta\} \cup \{\neg \psi\}$$

is consistent for all $n \in \omega$. For the base case, notice that Γ_0 being consistent precisely means that

$$\{\langle \sup \rangle \delta \widehat{\Theta_0} \mid \delta \in \Delta\} \cup \{\neg \psi\}$$

is consistent.

So assume

$$\{\langle \sup \rangle \delta \widehat{\Theta_n} \mid \delta \in \Delta\} \cup \{\neg \psi\}$$

is consistent for some $n \in \omega$. If

$$\{\langle \sup \rangle \delta (\widehat{\Theta_n} \wedge \chi_n) \mid \delta \in \Delta\} \cup \{\neg \psi\}$$

is consistent, we are done, so suppose not. Enumerating the formulas of Δ as $\delta_0, \delta_1, \dots$ and setting $\delta'_i := \{\delta_j : j \leqslant i\}$, there must then be some $k \in \omega$ s.t. for all $m \geqslant k$:

$$\vdash_{\backslash\text{-Pre}} \langle \sup \rangle \widehat{\delta'_m} (\widehat{\Theta_n} \wedge \chi_n) \wedge \neg \psi \to \bot. \qquad (*)$$

Furthermore, since by the IH

$$\{\langle \sup \rangle \delta \widehat{\Theta_n} \mid \delta \in \Delta\} \cup \{\neg \psi\}$$

is consistent, using Lindenbaum, we can extend it to an MCS Λ_n. For this MCS Λ_n, we must then have for all $i \in \omega$:

$$\langle \sup \rangle \widehat{\delta'_i}(\widehat{\Theta_n} \wedge [\chi_n \vee \neg \chi_n]) \in \Lambda_n.$$

So for all $i \in \omega$:

$$\langle \sup \rangle \widehat{\delta'_i}(\widehat{\Theta_n} \wedge \chi_n) \in \Lambda_n \quad \textit{or} \quad \langle \sup \rangle \widehat{\delta'_i}(\widehat{\Theta_n} \wedge \neg \chi_n) \in \Lambda_n.$$

Thus, combining this with (∗) [and having in mind that $\neg\psi \in \Lambda_n$], we get that for all $m \geqslant k$:

$$\langle \sup \rangle \widehat{\delta'_m}(\widehat{\Theta_n} \wedge \neg \chi_n) \in \Lambda_n.$$

But this entails that

$$\Big(\{\langle \sup \rangle \delta(\widehat{\Theta_n} \wedge \neg \chi_n) \mid \delta \in \Delta\} \cup \{\neg \psi\}\Big) \subseteq \Lambda_n,$$

wherefore $\{\langle \sup \rangle \delta \widehat{\Theta_{n+1}} \mid \delta \in \Delta\} \cup \{\neg \psi\}$ is consistent, as required for the induction proof.

From this, one easily sees that (1)

$$\Gamma_\omega := \bigcup_{n \in \omega} \{\langle \sup \rangle \delta \widehat{\Theta_n} \mid \delta \in \Delta\} \cup \{\neg \psi\}$$

is consistent, and (2)

$$\Theta := \bigcup_{n \in \omega} \Theta_n$$

is an MCS. Extending $\Gamma_\omega \subseteq \Gamma$ to an MCS, we get that $\varphi \in \Theta_0 \subseteq \Theta, \neg\psi \in \Gamma$ and $C_{\backslash,\mathbf{Pre}}\Gamma\Delta\Theta$, which precisely shows the claim.

With these claims at our disposal, the inductive step regarding '\' in a proof of the truth lemma is immediate (the two claims cover one direction each). Since this was the last obstacle for proving the truth lemma, and we have already noted that $(W_{\backslash\text{-}\mathbf{Pre}}, C_{\backslash\text{-}\mathbf{Pre}}) \in \mathcal{C}$, we can deduce strong completeness—finishing not only our proof, but also this section. □

4.3. Bulldozing and completeness-via-representation

With $\text{Log}_{\backslash}(\mathcal{C})$ axiomatized, next up is showing $\text{Log}_{\backslash}(\mathcal{C}) = MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos}$ via representation; i.e., via onto 'p-morphisms'.

Another commonly used term for 'p-morphism' is 'bounded morphism'.

Importantly, to find the technique of onto p-morphisms in our arsenal of validity-preserving techniques, when dealing with preorder frames, we have to define the 'back'- and 'forth'-conditions in terms of the accompanying ternary (and not binary) relations.[11] For ease of reference, let us spell this out:

Definition 4.13. Given any two frames $\{(W, C), (W', C')\} \subseteq \mathcal{C}$, a function

$$f : W' \to W$$

is denoted a *p-morphism* if it satisfies the following conditions:

(forth) if $C'x'y'z'$, then $Cf(x')f(y')f(z')$; and

($\langle\text{sup}\rangle$-back) if $Cf(x')yz$, then there exist $\{y', z'\} \subseteq W'$ s.t. $f(y') = y, f(z') = z$ and $C'x'y'z'$.

If f additionally satisfies

($\backslash$-back) if $Cxf(y')z$, then there exist $\{x', z'\} \subseteq W'$ s.t. $f(x') = x, f(z') = z$ and $C'x'y'z'$,

we denote it a *$\backslash$-p-morphism*.

When dealing with preorder frames $(W, \leqslant)$, [$\backslash$-]p-morphisms are defined in terms of the induced $(W, S_{\leqslant}) \in \mathcal{S}_{\text{Pre}} \subseteq \mathcal{C}$. ⊣

Note the symmetry in the two back clauses: this is caused by '$\backslash$' and '$\langle\text{sup}\rangle$' referring to the same relation, but from different perspectives.

Now to be clear, onto p-morphisms preserve validity (and, generally, consequences) of $\mathcal{L}_M$-formulas, while onto $\backslash$-p-morphisms even preserve validity (and consequences) of $\mathcal{L}_{\backslash\text{-}M}$-formulas. This means we have a formal framework for developing representation results. In this section (and in the next chapter), this is a substantial part of what we will be doing.[12]

First up is our plighted proof that any $\mathcal{C}$-frame (W, C) is the $\backslash$-p-morphic image of a poset frame $(W', S'_{\leqslant}) \in \mathcal{S}_{\text{Pos}}$, entailing that with $\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}$ we have achieved an axiomatization of both $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$. This representation is obtained by composing two other representations; the first of which generalizes 'bulldozing' from the usual unary modality setting to our binary modality setting.

[11] As also noted in Observation A.2.3.

[12] Regarding $\backslash$-p-morphisms, it is important to have in mind that they are also required to meet ($\langle\text{sup}\rangle$-back). A notion for simply meeting (forth) and ($\backslash$-back) would appear appropriate, but we will not be needing such since we do not deal with modal logics having only the modality '$\backslash$'. In general, of course, the results of this chapter have these modal logics as special cases; e.g., our decidability proof in the next section.

To explain how this works, we briefly (re-)observe the following (cf. Observation A.2.1):

Observation 4.14. For any $(W, C) \in \mathcal{C}$, let $\leqslant_C$ and $\leqslant'_C$ be given as follows:

$$\leqslant_C := \{(y, x) : Cxxy\}, \qquad \leqslant'_C := \{(y, x) : \exists z(Cxyz \vee Cxzy)\}.$$

Then, by definition of the class $\mathcal{C}$, it is not too hard to see that (a) $\leqslant_C = \leqslant'_C$, and (b) $\leqslant_C$ is a preorder on W.

Moreover, if C happened to be the supremum relation of some preorder $\leqslant$, then $\leqslant_C = \leqslant$. $\dashv$

I.e., $Cxyz$ *iff* $x \in \sup_{\leqslant}\{y, z\}$.

With this observed, we are ready for the first representation result, mending $\mathcal{C}$-frames (W, C) so that $\leqslant_C$ becomes a *partial* order.

Proposition 4.15 (Bulldozing). *Let* $(W, C) \in \mathcal{C}$. *Then* (W, C) *is the* $\backslash$*-p-morphic image of some* $(W', C') \in \mathcal{C}$ *for which* $\leqslant_{C'}$ *is a partial order.*

Proof. Let $(W, C) \in \mathcal{C}$ be arbitrary. We construct (W', C') by adapting the well-known bulldozing technique from the binary-relation setting to our ternary-relation setting. More precisely, let $\mathcal{K}$ denote the set of maximal non-degenerate clusters of (W, C) w.r.t. the preorder $\leqslant_C$. We then define the underlying set as

$$W' := \left(W \setminus \bigcup_{K \in \mathcal{K}} K\right) \cup \bigcup_{K \in \mathcal{K}} (K \times \mathbb{Z}),$$

and let the function

$$f : W' \to W$$

be given by

$$f(x) = \begin{cases} x, & x \in \left(W \setminus \bigcup_{K \in \mathcal{K}} K\right) \\ k, & x = (k, z) \in K \times \mathbb{Z}, K \in \mathcal{K} \end{cases}.$$

To define the relation C', fix some linear order $\leqslant^K$ for each $K \in \mathcal{K}$, and for all

$x, a, b \in W'$, let $C'xab$:iff

$$Cf(x)f(a)f(b) \quad \textit{and} \quad \left(\begin{array}{ll} \textit{(i)} & x \in W \setminus \bigcup_{K \in \mathcal{K}} K, \textbf{ or} \\ \textit{(ii)} & x = (k, z) \in K \times \mathbb{Z}; (K \times \mathbb{Z}) \cap \{a, b\} = \varnothing, \textbf{ or} \\ \textit{(iii)} & x = (k_x, z_x) \in K \times \mathbb{Z} \ni (k_a, z_a) = a; b \notin K \times \mathbb{Z}; [z_x > z_a \textit{ or } (z_x = z_a \textit{ and } k_x \geqslant^K k_a)], \textbf{ or} \\ \textit{(iv)} & x = (k_x, z_x) \in K \times \mathbb{Z} \ni (k_b, z_b) = b; a \notin K \times \mathbb{Z}; [z_x > z_b \textit{ or } (z_x = z_b \textit{ and } k_x \geqslant^K k_b)], \textbf{ or} \\ \textit{(v)} & \{x, a, b\} \subseteq K \times \mathbb{Z}; x = (k_x, z_x), a = (k_a, z_a), b = (k_b, z_b); \\ & [z_x > z_a \textit{ or } (z_x = z_a \textit{ and } k_x \geqslant^K k_a)]; [z_x > z_b \textit{ or } (z_x = z_b \textit{ and } k_x \geqslant^K k_b)] \end{array} \right).$$

We claim that (1) $(W', C') \in \mathcal{C}$; (2) (W, C) is a \-p-morphic image of (W', C') witnessed by f; and (3) $\leqslant_{C'}$ is a partial order.

We begin by proving **(1)** $(W', C') \in \mathcal{C}$. We have that

(Re.f) is satisfied because (a) $(W, C) \vDash (\mathrm{Re.f})$ by assumption and (b) for all $K \in \mathcal{K}$: $\leqslant^K$ is, as a (weak) linear order, in particular, reflexive;

(4f) can be seen to be satisfied by a straightforward, but tedious check using $(W, C) \vDash (4f)$. Only non-trivial case is when $C'xxa$ by virtue of *(iii)*: there one must observe that if $C'aab$ then $f(b)$ cannot be in the same cluster as $f(x)$ by maximality of clusters $K \in \mathcal{K}$;

(Co.f) is satisfied because (a) $(W, C) \vDash (\mathrm{Co.f})$ and (b) the definition of C' is symmetrical in the two last arguments; and

(Dk.f) is satisfied because (a) $(W, C) \vDash (\mathrm{Dk.f})$ and (b) if $C'xab$ holds by virtue of *(i)*, then $C'xxa$ holds by virtue of *(i)*; if $C'xab$ holds by virtue of *(ii)* or *(iv)*, then $C'xxa$ holds by virtue of *(iii)*; if $C'xab$ holds by virtue of *(iii)*, then $C'xxa$ holds by virtue of *(v)*; and if $C'xab$ holds by virtue of *(v)*, then $C'xxa$ holds by virtue of *(v)*.

Having proven (1), we continue by proving **(2)**. f is clearly (a) surjective and (b) a homomorphism. Therefore, it remains to show that (c) the back conditions are satisfied. Beginning with (⟨sup⟩-back), suppose $Cf(x)a'b'$ for arbitrary $x \in W'$, $\{a', b'\} \subseteq W$. We then have to find $a, b \in W'$ s.t. $C'xab$, $f(a) = a'$, and $f(b) = b'$. We go by cases:

(i) If $x \in \left(W \setminus \bigcup_{K \in \mathcal{K}} K\right)$, pick any $a \in f^{-1}(a')$ and $b \in f^{-1}(b')$ using surjectivity of f.

(ii) If $x = (k, z) \in K \times \mathbb{Z}$ and $\{a', b'\} \cap K = \varnothing$, pick any $a \in f^{-1}(a')$ and $b \in f^{-1}(b')$.

(iii) If $x = (k_x, z_x) \in K \times \mathbb{Z}$ and $a' \in K \not\ni b'$, set $a := (a', z_x - 1)$ and pick any $b \in f^{-1}(b')$.

(iv) If $x = (k_x, z_x) \in K \times \mathbb{Z}$ and $b' \in K \not\ni a'$, set $b := (b', z_x - 1)$ and pick any $a \in f^{-1}(a')$.

(v) If $x = (k_x, z_x) \in K \times \mathbb{Z}$ and $a' \in K \ni b'$, set $a := (a', z_x - 1)$ and $b := (b', z_x - 1)$.

This exhausts all cases, hence f satisfies the ($\langle\text{sup}\rangle$-back) condition, thus is a p-morphism. Continuing with (\-back), suppose $Cxf(a')b$ for some $a' \in W'$ and $\{x, b\} \subseteq W$. Again, we go by cases:

(i) If $a' \in W \setminus \bigcup_{K \in \mathcal{K}} K$ or $[a' = (k_a, z_a) \in K \times \mathbb{Z}$ and $x \notin K]$, then pick any $x' \in f^{-1}(x)$. Then $Cf(x')f(x')b$, so by the $\langle\text{sup}\rangle$-back condition and the definition of C', we can find a $b' \in W'$ s.t. $C'x'x'b'$ and $f(b') = b$. It follows that $C'x'a'b'$.

(ii) And if $a' = (k_a, z_a) \in K \times \mathbb{Z}$ and $x \in K$, then setting $x' := (x, z_a + 1)$, we, again, get that $Cf(x')f(x')b$, hence we can find a $b' \in W'$ s.t. $C'x'x'b'$ and $f(b') = b$. Thus, we get that $C'x'a'b'$, as required.

This covers all cases—completing our proof of f being an onto \-p-morphism.

Lastly, we show that **(3)** $\leqslant_{C'}$ is a partial order. Reflexivity and transitivity are consequences of $(W', C') \in \mathcal{C}$. To show anti-symmetry, let $x, y \in W'$ be arbitrary s.t. $C'xxy$ and $C'yyx$. We have to show that $x = y$. Going by cases we find that:

- If $\{x, y\} \subseteq \left(W \setminus \bigcup_{K \in \mathcal{K}} K\right)$, then $Cxxy$ and $Cyyx$ by definition of f and C', so since $\left(W \setminus \bigcup_{K \in \mathcal{K}} K\right)$ contains no non-degenerate clusters by definition, we must have $x = y$.
- If $x \in \left(W \setminus \bigcup_{K \in \mathcal{K}} K\right)$ and $y = (k, z) \in K \times \mathbb{Z}$, then $Cxxk$ and $Ckkx$ so $x \in K$—contradicting $x \in \left(W \setminus \bigcup_{K \in \mathcal{K}} K\right)$.
- If $y \in \left(W \setminus \bigcup_{K \in \mathcal{K}} K\right)$ and $x = (k, z) \in K \times \mathbb{Z}$, then as above.
- If $x = (k_x, z_x) \in K \times \mathbb{Z}$ and $y = (k_y, z_y) \in K' \times \mathbb{Z}$ for $K \neq K'$, then $Ck_xk_xk_y$ and $Ck_yk_yk_x$ so $k_x \in K'$—contradicting maximality of the clusters (which implies that whenever $K \neq K'$, we even have $K \cap K' = \varnothing$).
- If $x = (k_x, z_x) \in K \times \mathbb{Z} \ni (k_y, z_y) = y$, then $x = y$ follows by anti-symmetry of our lexicographical ordering (since the ordering of the integers is linear and so is $\leqslant^K$).

Thus, we've shown $\leqslant_{C'}$ to be anti-symmetric, which completes our proof of (3) $\leqslant_{C'}$ being a partial order, thus finalizing our bulldozing proof. □

Using this representation, we continue further mending $\mathcal{C}$-frames (W, C) into real poset frames (i.e., frames whose ternary relation is the supremum relation of a partial order). We do so through another representation, which is obtained by adopting the framework of the completeness proof of Chapter 2 (2.13). In brief, in the proof to come, we will also be constructing a poset frame recursively by repairing defects. However, this time, the defects will be determined by an onto function, which we iteratively extend seeking to make it an onto \-p-morphism. And, although the ⟨sup⟩- and ¬⟨sup⟩-defects only need minor revision, we do need to include a third kind of defect corresponding to (\-back).

Many of the arguments will be almost identical to the ones of the completeness proof of Chapter 2, and so will be omitted or only hinted at. But – although the general set-up is very similar – there are some differences, and since we will also reuse this set-up in the next chapter, it is worth spelling out. We proceed doing so.

Definition 4.16. Given any $(W, C) \in \mathcal{C}$, we let E be some set disjoint from W of cardinality $\max\{|W|, \aleph_0\}$, and $\mathbb{P}_{(W,C)}$ be the set of all quadruples $(f, D, X, \leqslant)$ such that

1.′ f is an onto function from $(W \cup D \cup X)$ to W;

2.′ $|D \cup X| < |E|$;

3.′ $(D \cup X) \subseteq E$;

4.′ $D \cap X = \varnothing$;

6.′ $\leqslant$ is a partial order on $(W \cup D \cup X)$; and

7.′ if $y \leqslant x$ then $f(y) \leqslant_C f(x)$. ⊣

The only significant change is the present deletion of what was condition 5. of Definition 2.7.

Next, we define the revised versions of the ⟨sup⟩- and ¬⟨sup⟩-defects and their complementary revised repair lemmas, before subsequently stating and proving the last defect/repair pair.

Definition 4.17 ((⟨sup⟩-back)-defect)**.** Let $(W, C) \in \mathcal{C}$ and $(f, D, X, \leqslant) \in \mathbb{P}_{(W,C)}$. Then a triple $(x', y, z) \in (W \cup E) \times W \times W$ denotes a *(⟨sup⟩-back)-defect* (of $(f, D, X, \leqslant)$) :iff

$$\text{(i) } x' \in (W \cup D \cup X), \qquad \text{(ii) } Cf(x')yz,$$

and (iii) there are no $y', z' \in (W \cup D \cup X)$ s.t. $x' = \sup\{y', z'\}$ and

$$f(y') = y, \qquad \uparrow y' = \uparrow x' \cup \{y'\} \cup (\uparrow y' \cap \{w' \mid \uparrow w' \cap \uparrow x' = \varnothing\}),$$
$$f(z') = z, \qquad \uparrow z' = \uparrow x' \cup \{y'\} \cup (\uparrow z' \cap \{w' \mid \uparrow w' \cap \uparrow x' = \varnothing\}). \qquad \dashv$$

Notice the similarity between ($\langle\sup\rangle$-back)-defects and $\langle\sup\rangle$-defects (2.8).

Definition 4.18 ((forth)-defect)**.** Let $(W, C) \in \mathcal{C}$ and $(f, D, X, \leqslant) \in \mathbb{P}_{(W,C)}$. Then a triple $(x', y', z') \in (W \cup E) \times (W \cup E) \times (W \cup E)$ is denoted a *(forth)-defect* (of $(f, D, X, \leqslant)$) :iff

$$\{x', y', z'\} \subseteq (W \cup D \cup X), \qquad x' = \sup\{y', z'\}, \qquad \neg Cf(x')f(y')f(z'). \qquad \dashv$$

And between (forth)-defects and $\neg\langle\sup\rangle$-defects (2.9).

Lemma 4.19 (($\langle\sup\rangle$-back)-repair lemma)**.** *Suppose (x', y, z) is a ($\langle\sup\rangle$-back)-defect of some $(f, D, X, \leqslant) \in \mathbb{P}_{(W,C)}$. Then we can extend to $(f', D, X', \leqslant') \in \mathbb{P}_{(W,C)}$ by taking distinct $y', z' \in E \setminus (D \cup X)$ and setting*

$$f' := f \cup \{(y', y), (z', z)\}, \qquad X' := X \cup \{y', z'\},$$
$$\leqslant' := \leqslant \cup \{(y', u), (z', u) \mid x' \leqslant u\} \cup \{(y', y'), (z', z')\}.$$

Then, witnessed by y' and z', (x', y, z) does not constitute a ($\langle\sup\rangle$-back)-defect of $(f', D, X', \leqslant')$.

Proof. Defining as described, the proof of $(f', D, X', \leqslant') \in \mathbb{P}_{(W,C)}$ resembles the one of Lemma 2.11: 1.'-6.' are obvious, and 7.' is shown using $Cf'(x')f'(y')f'(z')$ and $(W, C) \in \mathcal{C}$.

Moreover, the latter claim is immediate. □

Lemma 4.20 ((forth)-repair lemma)**.** *Suppose (x', y', z') is a (forth)-defect of some $(f, D, X, \leqslant) \in \mathbb{P}_{(W,C)}$. Then we can extend to $(f', D', X, \leqslant') \in \mathbb{P}_{(W,C)}$ by (a) taking $d' \in E \setminus (D \cup X)$, (b) letting*

$$f' := f \cup \{(d', f(x'))\}, \qquad D' := D \cup \{d'\},$$
$$\leqslant' := \leqslant \cup \{(u, d'), (v, d') \mid u \leqslant y', v \leqslant z'\} \cup \{(d', d')\},$$

and (c) getting $x' \neq \sup_{\leqslant'}\{y', z'\}$.

Proof. Extending to $(f', D', X, \leqslant')$ as described, it follows similarly to the proof of Lemma 2.12 that $(f', D', X, \leqslant')$ satisfies 1.'-7.' and $x' \neq \sup_{\leqslant'}\{y', z'\}$. Only two things are worth mentioning: (1) for proving 7.', we use that if $u <' d'$ then $u \leqslant x'$, hence $f'(u) = f(u) \leqslant_C f(x') = f'(d')$, and (2) for proving $x' \neq \sup_{\leqslant'}\{y', z'\}$, we need that $\leqslant$ is a *partial* order (this is where we use bulldozing). □

Now 'd'' is no longer short for 'dummy', but for 'duplicate' (of x'): $f'(d') = f(x')$. We stress: this is key.
(However, this is only a good intuition for the D-worlds introduced in this repair lemma—not for those in the next.)

Our third and last defect, naturally, bears much resemblance to the ($\langle\sup\rangle$-back)-defect. It is defined as follows:

Definition 4.21 ((\-back)-defect)**.** Let $(W, C) \in \mathcal{C}$ and $(f, D, X, \leqslant) \in \mathbb{P}_{(W,C)}$. Then a triple $(x, y', z) \in W \times (W \cup E) \times W$ denotes a *(\-back)-defect* (of $(f, D, X, \leqslant)$) :iff

$$\text{(i) } y' \in (W \cup D \cup X), \qquad \text{(ii) } Cxf(y')z,$$

and (iii) there are no $x', z' \in (W \cup D \cup X)$ s.t. $x' = \sup\{y', z'\}$ and

$$\begin{aligned} f(x') &= x, & \uparrow y' &= \uparrow x' \cup \{y'\} \cup (\uparrow y' \cap \{w' \mid \uparrow w' \cap \uparrow x' = \varnothing\}), \\ f(z') &= z, & \uparrow z' &= \uparrow x' \cup \{z'\} \cup (\uparrow z' \cap \{w' \mid \uparrow w' \cap \uparrow x' = \varnothing\}). \end{aligned} \qquad \dashv$$

This new defect is repaired in this fashion:

Lemma 4.22 ((\-back)-repair lemma)**.** *Suppose (x, y', z) is a (\-back)-defect of some $(f, D, X, \leqslant) \in \mathbb{P}_{(W,C)}$. Then we can extend to $(f', D', X', \leqslant') \in \mathbb{P}_{(W,C)}$ by taking distinct $x', z' \in E \setminus (D \cup X)$ and setting*

$$\begin{aligned} &f' := f \cup \{(x', x), (z', z)\}, \qquad D' := D \cup \{x'\}, \qquad X' := X \cup \{z'\}, \\ &\leqslant' := \leqslant \cup \{(u, x') \mid u \leqslant y'\} \cup \{(x', x'), (z', z'), (z', x')\}. \end{aligned}$$

Then, witnessed by x' and z', (x, y', z) does not constitute a \-back defect of $(f', D', X', \leqslant'$ $)$.

Proof. A matter of going over the definition. □

Employing these repairs, we are ready to prove the desired representation result.

Proposition 4.23. *Every $(W, C) \in \mathcal{C}$ for which $\leqslant_C$ is a partial order, is a \-p-morphic image of a poset frame.*

Proof. Let $(W, C) \in \mathcal{C}$ be arbitrary s.t. $\leqslant_C$ is a partial order. For the sake of simplicity, assume W is countable: as oftentimes is the case, the adjustments of the ensuing proof needed for the case where $|W| > \aleph_0$ are conceptually insignificant but notationally taxing. Besides, by a 'standard translation' and the Löwenheim-Skolem Theorem, $\mathcal{C}$ has the countable model property w.r.t. $\mathcal{L}_{\backslash\text{-}M}$-formulas, so, for instance, starting with a countable frame, we can bulldoze it into a countable $\mathcal{C}$-frame whose underlying preorder is a partial order.

The adjustments in case $|W| > \aleph_0$ are doing transfinite recursion and induction instead.

As in the completeness proof of Chapter 2, using the repair lemmas repeatedly, we will be constructing a sequence

$$(f_0, D_0, X_0, \leqslant_0), (f_1, D_1, X_1, \leqslant_1), \ldots$$

such that for all $i \in \omega$

$$(f_i, D_i, X_i, \leqslant_i) \in \mathbb{P}_{(W,C)}, \quad f_i \subseteq f_{i+1}, \quad D_i \subseteq D_{i+1}, \quad X_i \subseteq X_{i+1}, \quad \leqslant_i \subseteq \leqslant_{i+1}.$$

We begin the sequence by setting

$$f_0 := \mathrm{Id} : W \to W, \quad D_0 := X_0 := \varnothing, \quad \leqslant_0 := \leqslant_C.$$

Then $(f_0, D_0, X_0, \leqslant_0) \in \mathbb{P}_{(W,C)}$.

At each stage $n + 1$, we then pick the least tuple constituting a defect to $(f_n, D_n, X_n, \leqslant_n)$—according to a fixed enumeration of the set of all triples $(x', y, z) \in (W \cup E) \times W \times W$ and all triples $(x', y', z') \in (W \cup E)^3$ and all triples[13] $(x, y', z) \in W \times (W \cup E) \times W$—and repair it to obtain $(f_{n+1}, D_{n+1}, X_{n+1}, l_{n+1})$. Letting

$$(f_\omega, D_\omega, X_\omega, \leqslant_\omega) := \left(\bigcup_{n \in \omega} f_n, \bigcup_{n \in \omega} D_n, \bigcup_{n \in \omega} X_n, \bigcup_{n \in \omega} \leqslant_n \right),$$

we get that (1) $(f_\omega, D_\omega, X_\omega, \leqslant_\omega)$ satisfies 1.′ and 3.′-7.′, and (2) $(f_\omega, D_\omega, X_\omega, \leqslant_\omega)$ has no defects whatsoever. Again, only (2) is not straightforward, and, again, for proving (2) two claims and an observation are helpful.

Observation′. Let $n \in \omega$ and $\{x, v\} \subseteq (W \cup D_n \cup X_n)$ be arbitrary s.t.

$$\uparrow_n v' = \uparrow_n x' \cup \{v'\} \cup (\uparrow_n v' \cap \{w' \mid \uparrow_n w' \cap \uparrow_n x' = \varnothing\}).$$

Then for all $m \geqslant n$:

$$\uparrow_m v' = \uparrow_m x' \cup \{v'\} \cup (\uparrow_m v' \cap \{w' \mid \uparrow_m w' \cap \uparrow_m x' = \varnothing\}),$$

hence also

$$\uparrow_\omega v' = \uparrow_\omega x' \cup \{v'\} \cup (\uparrow_\omega v' \cap \{w' \mid \uparrow_\omega w' \cap \uparrow_\omega x' = \varnothing\}).$$

This follows by an easy induction, using that each $(f_{m+1}, D_{m+1}, X_{m+1}, \leqslant_{m+1})$ is obtained from $(f_m, D_m, X_m, \leqslant_m)$ using one of the repair lemmas.

[13] For simplicity of argument, we assume all $(x_0, y_0, x_0) \in (W \cup E) \times W \times W$ to be distinct from all $(x_1, y_1, z_1) \in (W \cup E)^3$ – and so forth.

Claim (a′). Let $n \in \omega$ and $\{x', y', z'\} \subseteq (W \cup D_n \cup X_n)$ be arbitrary s.t.

$$x' = \sup_n\{y', z'\}, \qquad Cf_n(x')f_n(y')f_n(z'),$$
$$\uparrow_n y' = \uparrow_n x' \cup \{y'\} \cup (\uparrow_n y' \cap \{w' \mid \uparrow_n w' \cap \uparrow_n x' = \varnothing\}),$$
$$\uparrow_n z' = \uparrow_n x' \cup \{z'\} \cup (\uparrow_n z' \cap \{w' \mid \uparrow_n w' \cap \uparrow_n x' = \varnothing\}).$$

Then for all $m \geqslant n$:

$$x' = \sup_m\{y', z'\};$$

a fortiori, $x' = \sup_\omega\{y', z'\}$.

We prove the claim by induction. By assumption, it holds for $m = n$, so assume it holds for some $m \geqslant n$. We show it holds for $m + 1$. This time we have three cases, depending on the type of defect being repaired at stage $m + 1$. The cases of a ($\langle$sup$\rangle$-back)-repair and (forth)-repair are the exact same as in Theorem 2.13.

Consequently, suppose stage $m + 1$ was obtained by ($\backslash$-back)-repairing some (s, y'_s, z_s) through introducing the worlds s', z'_s. Then s' is the only possible counterexample to $x' = \sup_{m+1}\{y', z'\}$, so assume $y' \leqslant_{m+1} s' \geqslant_{m+1} z'$. Then we must have $y' \leqslant_m y'_s \geqslant_m z'$, so by the IH $x' \leqslant_m y'_s$, hence $x' \leqslant_{m+1} s'$. $\square_{\textit{Claim (a')}}$

Claim (b′). Let $n \in \omega$ and suppose that $a, b \in (W \cup D_n \cup X_n)$ are s.t. $a \ngeqslant_n b$. Then for all $m \geqslant n$, we have that $a \ngeqslant_m b$. A fortiori, $a \ngeqslant_\omega b$.

Once again by induction on $m \geqslant n$ with no change concerning the cases of ($\langle$sup$\rangle$-back)-repairs and (forth)-repairs. Therefore, assume $(l_{m+1}, \leqslant_{m+1}, D_{m+1})$ was obtained by ($\backslash$-back)-repairing some (x, y', z) by introducing x', z'. Then there is no change in predecessors of a, which suffices for the claim. $\square_{\textit{Claim (b')}}$

Finally, from these claims we likewise get *(c)*: If some tuple *did* constitute a defect at some stage n, but no longer at some later stage $m > n$, then it didn't for all $k \geqslant m$.

Noteworthy is the overlap between our definitions of ($\langle$sup$\rangle$-back)-defects and ($\backslash$-back)-defects, which assures that *claim (a′)* applies to both types of defects. And using *(c)* along with *claim (a′)* and *(b′)* in an analogous manner to what we did in the completeness proof, we get that $(f_\omega, D_\omega, X_\omega, \leqslant_\omega)$ neither has (forth)-, ($\langle$sup$\rangle$-back)- nor ($\backslash$-back)-defects.

Lastly, the fact that there are no defects, entails that f_ω is a $\backslash$-p-morphism from $(W \cup D_\omega \cup X_\omega, \leqslant_\omega)$ to (W, C), so since f_ω also is onto, we've shown the desired. $\square$

At long last, combining the two representations, we can deduce that we have achieved the axiomatization we were seeking.

Theorem 4.24. *Every* $(W, C) \in \mathcal{C}$ *is a* *-p-morphic image of a poset frame.*

Thus, $\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}$ *is sound and strongly complete w.r.t. preorder frames, and, in particular,*

$$MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos} = \mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}.$$

Additionally, as a special case, we get another proof of $\mathbf{MIL}_{\mathbf{Pre}}$ *being sound and strongly complete w.r.t. preorder frames, and, particularly*

$$MIL_{Pre} = MIL_{Pos} = \mathbf{MIL}_{\mathbf{Pre}}.$$

Proof. The first assertion follows from propositions 4.15 and 4.23 because onto \\-p-morphisms are closed under composition.

Soundness and strong completeness is the upshot of onto \\-p-morphisms preserving the consequence relation of a frame and the fact that $\mathcal{S}_{\mathrm{Pos}} \subseteq \mathcal{S}_{\mathrm{Pre}} \subseteq \mathcal{C}$; so also, in particular

$$MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos} = \mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}.$$

Lastly, since \\-p-morphisms are p-morphisms and $\mathcal{L}_{\mathrm{M}} \subseteq \mathcal{L}_{\backslash\text{-}\mathrm{M}}$, this also restricts to the special case of the basic modal information language. □

4.4. Decidability

The problem of axiomatizing our conservative extension(s), $MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos}$, of the basic modal information logic(s), $MIL_{Pre} = MIL_{Pos}$, solved, the biggest remaining problem is, arguably, that of decidability. As already mentioned, we continue being guided by the procedure outlined in section 1.3, thus showing decidability qua a proof of the FMP w.r.t. another class of frames, which, of course, is $\mathcal{C}$ anew. Albeit the $\mathcal{L}_{\mathrm{M}}$-filtration through a $\mathcal{C}$-closed set of formulas (cf. section 3.2) is *not* an $\mathcal{L}_{\backslash\text{-}\mathrm{M}}$-filtration—that is, through a $\mathcal{C}$-closed set of formulas it does not preserve satisfaction of $\mathcal{L}_{\backslash\text{-}\mathrm{M}}$-formulas, but only of $\mathcal{L}_{\mathrm{M}}$-formulas—we are not at a loss: only some minor modifications are needed.

Borrowing the idea of a *suitable* set of formulas from Buszkowski (2021), we define a notion extending our notion of a $\mathcal{C}$-closed set of formulas.

Definition 4.25. We say that a set Σ of $\mathcal{L}_{\backslash\text{-}\mathrm{M}}$-formulas is $\mathcal{C}$*-suitably closed* :iff

($\mathcal{C}$) it is $\mathcal{C}$-closed; and

(Suit) $\varphi\backslash\psi \in \Sigma$ implies $\langle\sup\rangle\varphi(\varphi\backslash\psi) \in \Sigma$.

Moreover, for any set of $\mathcal{L}_{\backslash\text{-}\mathrm{M}}$-formulas Σ_0, we say that Σ is the $\mathcal{C}$*-suitable closure* of Σ_0 :iff it is the least $\mathcal{C}$-suitably closed set of formulas extending Σ_0. ⊣

Note that the $\mathcal{C}$*-suitable closure of a set of formulas always exists.*

Afresh, an immediate consequence is:

Lemma 4.26. *For any finite set of $\mathcal{L}_{\backslash\text{-M}}$-formulas Σ_0, its $\mathcal{C}$-suitable closure $\Sigma \supseteq \Sigma_0$, too, is finite.*

As the last ingredient for achieving decidability, we show that when filtrating through $\mathcal{C}$-*suitably* closed sets of formulas, the $\mathcal{L}_{\text{M}}$-filtration of Theorem 3.9 lifts to an $\mathcal{L}_{\backslash\text{-M}}$-filtration:

Theorem 4.27. $\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}$ *admits filtration w.r.t. the class $\mathcal{C}$. Consequently,*

$$\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}} = \mathrm{Log}_{\backslash}(\mathcal{C}_{\mathrm{F}}),$$

where $\mathrm{Log}_{\backslash}(\mathcal{C}_{\mathrm{F}})$ denotes the logic of the class of finite $\mathcal{C}$-frames in the language of $\mathcal{L}_{\backslash\text{-M}}$.

Proof. Let (W, C, V) be an arbitrary $\mathcal{C}$-model; Σ an arbitrary $\mathcal{C}$-suitably closed set of formulas; and $(W_\Sigma, C^{\mathcal{C}}_\Sigma, V_\Sigma)$ be the filtration of (W, C, V) through Σ defined in Theorem 3.9. Then, as shown in the proof of said theorem, $(W_\Sigma, C^{\mathcal{C}}_\Sigma) \in \mathcal{C}$ and the filtration conditions (F1) and (F2) hold for the modality '$\langle\mathrm{sup}\rangle$'. Thus, because of Lemma 4.26 and the inclusion $\mathrm{Log}_{\backslash}(\mathcal{C}) \subseteq \mathrm{Log}_{\backslash}(\mathcal{C}_{\mathrm{F}})$, we need only show that the synonymous filtration conditions for the modality '$\backslash$' likewise are met.

The former, homomorphism condition is evidently the same, while the latter becomes

(F2') $C^{\mathcal{C}}_\Sigma|x||y||z| \Rightarrow \forall\varphi\backslash\psi \in \Sigma\,[(y \Vdash \varphi\backslash\psi, z \Vdash \varphi) \Rightarrow x \Vdash \psi]$.

Recall that '$\backslash$' is a '$\Box$-ed' modality; therefore, this presentation of the second filtration clause.

Consequently, all that remains to be proven is (F2').[14] So assume $C^{\mathcal{C}}_\Sigma|x||y||z|$, and let $\varphi\backslash\psi \in \Sigma$ be arbitary s.t. $y \Vdash \varphi\backslash\psi$ and $z \Vdash \varphi$. By (Suit), $\langle\mathrm{sup}\rangle\varphi(\varphi\backslash\psi) \in \Sigma$ so by (Com) we have that $\langle\mathrm{sup}\rangle(\varphi\backslash\psi)\varphi \in \Sigma$. But then (F2) entails that $x \Vdash \langle\mathrm{sup}\rangle(\varphi\backslash\psi)\varphi$, whence $x \Vdash \langle\mathrm{sup}\rangle\varphi(\varphi\backslash\psi)$ by $(W, C) \vDash (\mathrm{Co.f})$, so finally since $(W, C) \Vdash (\mathrm{I1})$, we have $x \Vdash \psi$ as required. □

Using this, we can conclude that the basic modal information logic of preorders (or posets) endowed with the informational implication is decidable.

Corollary 4.28. *$MIL_{\backslash\text{-}Pre}$ is decidable (and so is $MIL_{\backslash\text{-}Pos}$).*

Proof. We have shown that

$$MIL_{\backslash\text{-}Pre} = \mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}} = \mathrm{Log}_{\backslash}(\mathcal{C}_{\mathrm{F}}),$$

so since $\mathbf{MIL}_{\backslash\text{-}\mathbf{Pre}}$ is finitely axiomatizeable and complete w.r.t. a recursively enumerable (r.e.) class of finite frames [simply check for satisfaction of the

[14]The proof of Lemma 2 in Buszkowski (2021) pertains to showing the satisfaction of (F2') in our present setting, so the ensuing argument is only given for the sake of completeness of the current proof—we claim no originality whatsoever.

first-order formulas (Re.f), (4f), (Co.f), and (Dk.f)], we obtain decidability of $MIL_{\backslash\text{-}Pre}$. □

Closing off this chapter, we state the following corollary:

Corollary 4.29. *Let $\mathcal{L}_{\Diamond\text{-}M}$ be the extension of the basic language $\mathcal{L}_M$ with the unary modality '$\Diamond$', and let the semantics for '$\Diamond$' be the usual one, namely those of the forward-looking modality 'F' given in Remark 4.8. Then letting $MIL_{\Diamond\text{-}Pre}$ and $MIL_{\Diamond\text{-}Pos}$ be the MILs of this language on preorders and posets, respectively, we get that both are decidable.*

Proof. A decision procedure is given as follows: For any $\mathcal{L}_{\Diamond\text{-}M}$-formula φ, translate it into a formula $t(\varphi) \in \mathcal{L}_{\backslash\text{-}M}$ in accordance with Remark 4.8, and then use the decision procedure of the preceding corollary. □

5. MIL of Minimal Upper Bounds

So far we have been concerned with modal vocabulary for capturing the structure of (quasi-)*least* upper bounds on preorders and posets. In this chapter, we change the interpretation to that of (quasi-)*minimal* upper bounds, with the aim of (a) giving the resulting logics (MIL^{Min}_{Pre}, MIL^{Min}_{Pos}, $MIL^{Min}_{\backslash\text{-}Pre}$, and $MIL^{Min}_{\backslash\text{-}Pos}$) the attention they deserve on their own merits, and not least (b) studying how this different minimal-perspective relates to the more standard least-perspective.

In section 5.1, we define the logics of concern in the basic language of MIL, show soundness w.r.t. $\mathbf{MIL_{Pre}}$, and end with a cautionary remark regarding completeness. Section 5.2 continues in the spirit of the preceding chapter: via representation it is proven that $MIL^{Min}_{Pre} = MIL^{Min}_{Pos} = MIL_{Pre}$, which then is generalized to show $MIL^{Min}_{\backslash\text{-}Pre} = MIL^{Min}_{\backslash\text{-}Pos} = MIL_{\backslash\text{-}Pre}$.

5.1. Introducing the logics

Before defining the actual 'minimal MILs' of the basic language $\mathcal{L}_M$, for the sake of completeness, we first get clear on what a quasi-minimal upper bound is:

Definition 5.1. For any preorder $(W, \leqslant)$ and any $w, v, u \in W$, we say that w is a *quasi-minimal upper bound* – or simply a *minimal upper bound* – of $\{u, v\}$ and write $w \in \min\{u, v\}$:iff

- w is an upper bound of $\{u, v\}$, i.e. $u \leqslant w$ and $v \leqslant w$; and
- $x \nleqslant w$ or $w \leqslant x$, for all upper bounds x of $\{u, v\}$.

Note that if $\leqslant$ is a partial order, w is a quasi-minimal upper bound iff it is a minimal upper bound in the usual sense. ⊣

We proceed by defining these MILs of (quasi-)minimal upper bounds on preorders/posets.

Definition 5.2. We define $\mathcal{M}_{\text{Pre}}$ to be the class of pairs $(W, M_{\leqslant})$ where W is a set and $M_{\leqslant}$ is a ternary relation on W for which there is some preorder $\leqslant$ on W s.t for all $w, v, u \in W$:

I.e., $M_{\leqslant}$ is the minimal-upper-bound relation induced by a preorder (resp. poset).

$$M_{\leqslant}wvu \qquad \textbf{iff} \qquad w \in \min_{\leqslant}\{u, v\}.$$

$\mathcal{M}_{\text{Pos}}$ is defined analogously. ⊣

Definition 5.3. The modal information logics of minimal upper bounds on preorders and posets, respectively, are defined as follows:

$$MIL^{Min}_{Pre} := \{\varphi \in \mathcal{L}_M : (W, M_\leqslant) \Vdash \varphi \text{ for all } (W, M_\leqslant) \in \mathcal{M}_{Pre}\},$$

and

$$MIL^{Min}_{Pos} := \{\varphi \in \mathcal{L}_M : (W, M_\leqslant) \Vdash \varphi \text{ for all } (W, M_\leqslant) \in \mathcal{M}_{Pos}\}. \qquad \dashv$$

The logics having been defined, we can show soundness of $\mathbf{MIL_{Pre}}$.

Theorem 5.4 (Soundness). $\mathbf{MIL_{Pre}} \subseteq MIL^{Min}_{Pre}$.

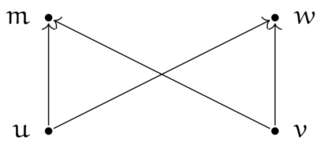
Proof. Routine check that MIL^{Min}_{Pre} is a normal modal logic validating (Re.), (4), (Co.), and (Dk.). □

As hinted at earlier, the converse inclusion – i.e., completeness – will be a consequence of a representation result, showing that any poset frame is a p-morphic image of an $\mathcal{M}_{Pos}$-frame. Before plunging into this, a warning might be in place.

Remark 5.5 (Caution). While it is the case that for any preorder $(W, \leqslant)$, we have that $S_\leqslant \subseteq M_\leqslant$ where

$$S_\leqslant := \{(w, v, u) \in W^3 : w \in \sup\{u, v\}\}, \quad M_\leqslant := \{(w, v, u) \in W^3 : w \in \min\{u, v\}\},$$

it is a (natural) misunderstanding to think that this implies $MIL^{Min}_{Pre} \subseteq MIL_{Pre}$. To conclude so—rather than, given any preorder, having the inclusion $S_\leqslant \subseteq M_\leqslant$—one would need the inclusion $\mathcal{S}_{Pre} \subseteq \mathcal{M}_{Pre}$. And this inclusion is easily seen to fail. To exemplify, consider the below depicted Hasse diagram of a preorder $(W, \leqslant)$:



While, of course, on one hand $S_\leqslant \subseteq M_\leqslant$. On the other—since, e.g., $\sup\{u, v\} \not\ni w \in \min\{u, v\}$—the induced ternary relations are distinct, so because they are induced by the same preorder, we get that $\mathcal{S}_{Pre} \ni (W, S_\leqslant) \notin \mathcal{M}_{Pre}$, which shows $\mathcal{S}_{Pre} \not\subseteq \mathcal{M}_{Pre}$ as desired. $\dashv$

5.2. Collapsing the minimal-u.b. relation

With this last clarification of the previous section, it perhaps, contrarily, now appears a bit surprising that we soon will show that $MIL_{Pos}^{Min} = MIL_{Pos}$—especially having in mind the way we repaired $\neg\langle\text{sup}\rangle$-defects (and (forth)-defects): by introducing incomparable upper bounds, so-called dummies (or duplicates), which do, indeed, ensure that if $x = \text{sup}_n\{y, z\}$, then $x \neq \text{sup}_{n+1}\{y, z\}$, but, contrariwise, do not change that x is a *minimal* upper bound of $\{y, z\}$. I.e., when dealing with the induced supremum relation of a preorder/poset, the dummies did repair the defects, but had it been the induced minimal-u.b. relation, dummy repairs would not have worked.

Recall Example 2.10.

Before embarking on the principal lemma allowing us to deduce $MIL_{Pos}^{Min} = MIL_{Pos}$, we provide some intuition for it—also gesturing at why the fact that $MIL_{Pos}^{Min} = MIL_{Pos}$ actually is not so surprising when further analyzed.

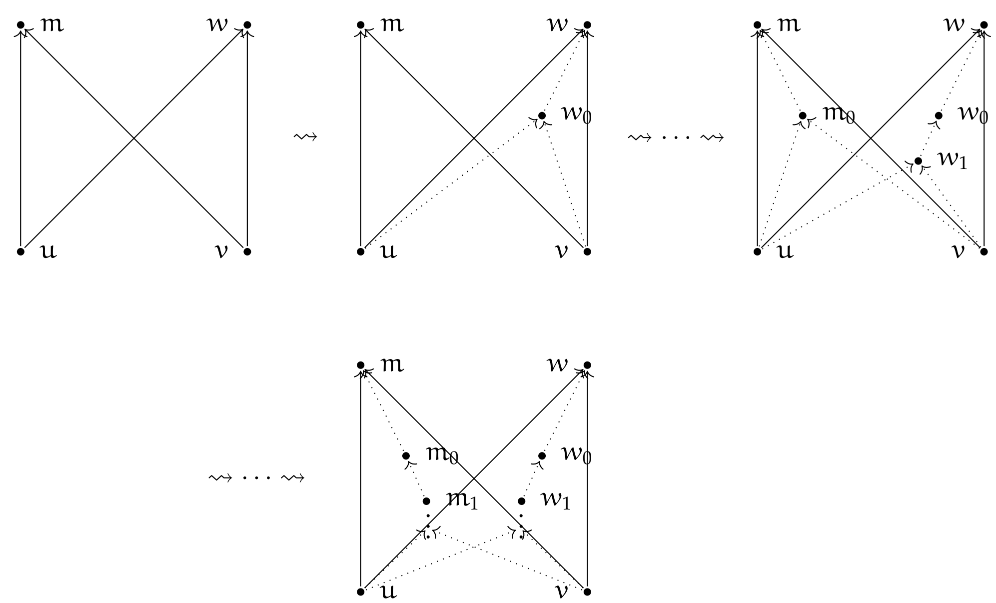
Intuition for representation construction. Given any poset frame $(W, \leqslant)$ we know that $S_{\leqslant} \subseteq M_{\leqslant}$. The basic idea then is to transform the frame so that $M_{\leqslant}$ collapses into $S_{\leqslant}$; i.e., we get an equality $S_{\leqslant} = M_{\leqslant}$, hence whether the binary modality refers to the supremum or the minimial-u.b. relation does not matter: the same formulas are satisfied.

We will make this transformation by exploiting the fact that there are two ways for an upper bounded set $\{u, v\}$ to not have a supremum:

(i) incomparable u.b.s; vs.

(ii) infinitely descending chain(s) of u.b.s.

Essentially, the idea is to transform all cases of (i) into (also being) cases of (ii). In this way, we transform all cases where $\text{sup}\{u, v\} \neq w \in \text{min}\{u, v\}$ into cases where $\text{sup}\{u, v\} = \text{min}\{u, v\} = \varnothing$, thus collapsing the relation $M_{\leqslant}$ into $S_{\leqslant}$. Put naively, we will be shooting in points as illustrated below:



This transformation will take place in a stepwise manner, adding one point of the infinitely descending chain at a time (first w_0, then w_1, etc.). We will need our transformation to be satisfaction-preserving, whence the worlds of a constructed infinitely descending chain $w_0, w_1, \ldots$ below, say, w are constructed as to satisfy the same formulas as w. Exactly because of this, the idea conveyed by the illustration above does not work in general; it conveys the very basic idea but is, as-such, too naive and runs into the following problems:

- w (or $\mathfrak{m}$) might be the supremum of some other worlds, hence only 'duplicating' w is not enough. Our solution will be to duplicate its downset, and placing each member of the duplicated downset right below the corresponding 'original'.
- When shooting in, say, w_0, not only do $\{u, v\}$ get a new upper bound, but so do, say, $\{y, z\}$ where $y \leqslant u$ and $z \leqslant v$. Problem then is that, although $\{u, v\}$ does not have a supremum, $\{y, z\}$ could have some supremum x. So for x to stay supremum of $\{y, z\}$, it does not suffice to make (everything below) u and v see w_0: we must also make x see w_0. Taking this line of argument to its ultimate conclusion, w_0 must be seen by any world in the least downset containing $\{u, v\}$ and closed under binary suprema.

We will be denoting this set A.

Hopefully, the depiction has illuminated the (naive) spirit of the argument, and the two bullet points some intuition for the actual, more complicated construction taking place. It is all made rigorous in the the following lemma:

We recommend revisiting the bulletpoints during/after studying the lemma below.

Lemma 5.6. *Let* $(W, \leqslant)$ *be a poset frame and* $\{w, u, v\} \subseteq W$ *s.t.* $w \in \min\{u, v\}$ *but* $w \neq \sup\{u, v\}$*. Then* $(W, \leqslant)$ *is the p-morphic image of a poset frame* $(W', \leqslant')$ *s.t.*

1. $W \subseteq W', |W'| \leqslant \max\{\aleph_0, |W|\}$;

2. $\leqslant' \cap (W \times W) = \leqslant$;

3. *if* $x = \sup\{y, z\}$*, then* $x = \sup'\{y, z\}$;

4. $w \notin \min'\{u, v\}$.

Proof. Let $W' := W \sqcup {\downarrow} w = \{(x, 0), (y, 1) \mid x \in W, y \in {\downarrow} w\}$, and

We write ${\downarrow} w := \{v \mid v \leqslant w\}$.

$$f : W' \to W$$

be the function given by

$$f(x, i) = x$$

for all $x \in W, i \in \{0, 1\}$.

To define the relation $\leqslant'$, first, let $A \subseteq W$ be the least downset containing $\{v, u\}$ and closed under binary suprema. I.e., $A = \bigcup_{n \in \omega} A_n$, where $A_0 = {\downarrow} v \cup {\downarrow} u$, and

$$A_{n+1} = {\downarrow}\big(A_n \cup \big\{ \sup\{b_n, c_n\} \mid \{b_n, c_n\} \subseteq A_n \big\}\big).$$

Then for all $(x, i), (y, j) \in W'$, we let $(y, j) \leqslant' (x, i)$ **iff**

(i) $i = 0$ and $y \leqslant x$, **or**

(ii) $j = i = 1$ and $y \leqslant x$, **or**

(iii) $j = 0, i = 1, y \in A$ and $x = w$.

We claim that (1) $(W', \leqslant')$ is a poset frame; (2) 1.-4. are satisfied; and (3) f is an onto p-morphism.

To prove so, we begin by proving the following claims in the order they appear:

(a) if $y \in A$, then $y < w$; and

(b) f is order-preserving.

To show *(a)*, note that since $w \in \min\{u, v\}$ but $w \neq \sup\{u, v\}$, there must be some $m \geqslant u, v$ s.t. m and w are incomparable. Using this, we show that for all $n \in \omega$, $A_n \subsetneq {\downarrow} w$ and $A_n \subsetneq {\downarrow} m$, which suffices to prove the claim.

Since w and m are incomparable upper bounds, we must have $d < e$ for $d \in \{u, v\}$ and $e \in \{w, m\}$, which shows the base case. Accordingly, assume for some $n \in \omega$ that $A_n \subsetneq {\downarrow} w$ and $A_n \subsetneq {\downarrow} m$, and let $\{b_n, c_n\} \subseteq A_n$ be arbitrary s.t. $\sup\{b_n, c_n\}$ exists. It then suffices to show that $\sup\{b_n, c_n\} < w$ and

$\sup\{b_n, c_n\} < m$. Since both w and m are upper bounds of $\{b_n, c_n\}$ by the IH, we have that $\sup\{b_n, c_n\} \leqslant w$ and $\sup\{b_n, c_n\} \leqslant m$. And because w and m are incomparable, the inequalities must be strict—finalizing our proof of *(a)*.

Now, to show *(b)*, assume $(y, j) \leqslant' (x, i)$. If $i = 0$ or $j = i = 1$, we have that $f(y, j) = y \leqslant x = f(x, i)$, showing the required. And if $j = 0, i = 1, y \in A$ and $x = w$, then, by the previous claim, $f(y, j) = y < w = x = f(x, i)$, which covers the last case.

With these claims proven, we're ready to tackle (1), (2), and (3), beginning with **(1)**: showing that $\leqslant'$ is a partial order.

Reflexivity follows by reflexivity of $\leqslant$ and (i), (ii).

Transitivity: Suppose $(z, k) \leqslant' (y, j) \leqslant' (x, i)$. By *(b)*, $z \leqslant y \leqslant x$, so, since $\leqslant$ is transitive, $z \leqslant x$. Thus, if $i = 0$ or $k = i = 1$, then $(z, k) \leqslant' (x, i)$ as required. And if $k = 0$ and $i = 1$, we must show $z \in A$ and $x = w$. If $j = 0$, then since $(y, j) \leqslant' (x, i)$, we have that $y \in A$ and $x = w$, so because $z \leqslant y$ and A is a downset, we also have $z \in A$. Lastly, if $j = 1$, then since $(z, k) \leqslant' (y, j)$, we have that $z \in A$ and $y = w$, so since $y \leqslant x$ and $x \in {\downarrow}w$, it must also be that $x = w$.

Anti-symmetry: Suppose $(y, j) \leqslant' (x, i)$ and $(x, i) \leqslant' (y, j)$. If $j = i$, we're done by anti-symmetry of $\leqslant$. Moreover, we cannot have $j \neq i$, since if, say, $j = 0, i = 1$, then $y \in A$ and $x = w$, so by *(a)*, $y < x$, but by *(b)* we also have $x \leqslant y$ $\lightning$.

Having proven (1), we continue by proving **(2)**, namely that 1.-4. are satisfied.

1. and 2. are clearly satisfied: It is only a matter of notational convenience that we have defined W' to be the disjoint union of W and ${\downarrow}w$; we could just as well have defined W' to be an 'actual extension' of W. To be clear, what is then meant in 3. and 4. is that: if $x = \sup\{y, x\}$, then $(x, 0) = \sup'\{(y, 0), (z, 0)\}$; and $(w, 0) \notin \min\{(u, 0), (v, 0)\}$. For the latter, simply note that $(w, 1) \leqslant' (w, 0)$, and since $\{u, v\} \subseteq A$, we also have $(u, 0), (v, 0) \leqslant' (w, 1)$; therefore, $(w, 0) \notin \min\{(u, 0), (v, 0)\}$. Regarding the former, if $x = \sup\{y, z\}$, we have that $(x, 0)$ is an upper bound of $\{(y, 0), (z, 0)\}$. Accordingly, suppose that (s, i) is an upper bound of $\{(y, 0), (z, 0)\}$. We then have to show that $(x, 0) \leqslant' (s, i)$. By *(b)* and $x = \sup\{y, z\}$, we know that $x \leqslant s$. So if $i = 0$, we are done. And if $i = 1$, we must show that $x \in A$ and $s = w$. Since $(s, 1)$ is an upper bound of $\{(y, 0), (z, 0)\}$, we have that $\{y, z\} \subseteq A$ and $s = w$. But then since A is closed under binary suprema and $x = \sup\{y, z\}$ by assumption, we also have $x \in A$, which completes the proof of (2).

Notice how condition 3. is met by reason of our definition of A, cf. the latter of the two bullet points preceding this lemma.

It remains to show **(3)**: f is an onto p-morphism. It is certainly onto, so we need only show that the back and forth clauses hold. We begin with the latter.

Forth. Suppose $(x, i) = \sup'\{(y, j), (z, k)\}$. By *(b)*, we know that x is an upper bound of $\{y, z\}$. So assume that s is also an upper bound. Then $(s, 0)$ is an upper bound of $\{(y, j), (z, k)\}$, hence $(x, i) \leqslant' (s, 0)$, so by another application of *(b)*, we get that $x \leqslant s$, which exactly shows that $f(x, i) = x = \sup\{y, z\} = \sup\{f(y, j), f(z, k)\}$.

Back. Suppose $f(x, i) = x = \sup\{y, z\}$ for some $y, z \in W$. Going by cases, we get:

- If $i = 0$, we have, by 3., that $(x, i) = \sup'\{(y, 0), (z, 0)\}$, which shows the required since, clearly, $f(y, 0) = y, f(z, 0) = z$.
- If $i = 1$, then $x \in {\downarrow}w$, hence $\{y, z\} \subseteq {\downarrow}w$. Thus, $\{(y, 1), (z, 1)\} \subseteq W'$. Besides, clearly, $f(y, 1) = y, f(z, 1) = 1$, so we need only show $(x, 1) = \sup'\{(y, 1), (z, 1)\}$. Since $x = \sup\{y, z\}$, we know that $(x, 1)$ is an upper bound of $\{(y, 1), (z, 1)\}$. So suppose (s, j) also is an upper bound of $\{(y, 1), (z, 1)\}$. Then s is an upper bound of $\{y, z\}$, therefore $x \leqslant s$, and, thus $(x, 1) \leqslant' (s, j)$ as required.

This completes our proof of **(3)**, hence of the lemma. □

With this key lemma at our disposal, we can deduce the succeeding two results.

Proposition 5.7. *Every poset frame $(W, \leqslant)$ is the p-morphic image of a poset frame $(W', \leqslant')$ satisfying*

$$\forall w', v', u' \in W' \, (w' \in \min\{u', v'\} \Rightarrow w' = \sup\{u', v'\}) \, .$$

I.e., $S_{\leqslant} = M_{\leqslant}$.

Proof. Follows by using a formal framework similar to the one presented in the completeness-via-representation proof of the previous chapter. To be a bit more concrete, the construction is as follows: w.l.o.g. we assume that $(W, \leqslant)$ is countable; then we enumerate all triples (w, v, u) possibly constituting a "min-defect": $w \in \min\{u, v\}$ but $w \neq \sup\{u, v\}$; and then we repeatedly use the previous lemma as our repair lemma to obtain a sequence $(W, \leqslant, \mathrm{Id}) = (W_0, \leqslant_0, f_0), (W_1, \leqslant_1, f_1), \ldots$ from which we define $(W_\omega, \leqslant_\omega, f_\omega)$.

Importantly, the f_ns are obtained by composing the onto p-morphisms obtained via the previous lemma, and it then remains to show that (1) their union f_ω is an onto p-morphism from $(W_\omega, \leqslant_\omega)$ to $(W, \leqslant)$; (2) $(W_\omega, \leqslant_\omega)$ is, in fact, a poset frame; and (3) $(W_\omega, \leqslant_\omega)$ satisfies the condition of the proposition description (i.e., $S_{\leqslant_\omega} = M_{\leqslant_\omega}$).

The argument for (1) goes: f_ω is clearly an onto function; the forth condition follows using 2. of the previous lemma and the forth condition of the f_ns; and the back condition follows using the back condition of the f_ns and 3. of the previous lemma. (2) is, as always, easily shown. Lastly, (3) follows by the enumeration of triples potentially constituting a min-defect and 2. and 4. of the foregoing lemma. □

Finally, we can deduce the following:

Theorem 5.8. $MIL_{Pre}^{Min} = MIL_{Pos}^{Min} = MIL_{Pre}$.

Proof. We have that

$$MIL_{Pre}^{Min} \subseteq MIL_{Pos}^{Min}$$

because the latter logic is defined by restricting the class of frames of the former logic. Moreover,

$$MIL_{Pre} \subseteq MIL_{Pre}^{Min}$$

since $MIL_{Pre} = \mathbf{MIL_{Pre}} \subseteq MIL_{Pre}^{Min}$, cf. Theorem 5.4. Thus, since we also know that $MIL_{Pre} = MIL_{Pos}$, it suffices to show that

$$MIL_{Pos}^{Min} \subseteq MIL_{Pos}.$$

We claim that this is an immediate implication of the proposition just proven. It showed that all poset frames $(W, \leqslant)$ are a p-morphic image of a poset frame $(W', \leqslant')$ satisfying

$$\forall w', v', u' \in W' \, (w' \in \min\{u', v'\} \Rightarrow w' = \sup\{u', v'\}),$$

which means $S_{\leqslant'} = M_{\leqslant'}$. That is, all poset frames $(W, \leqslant)$ are a p-morphic image of a frame $(W', M_{\leqslant'}) \in \mathcal{M}_{\mathrm{Pos}}$, exactly as required. □

Corollary 5.9. *MIL_{Pre}^{Min} is decidable (and so is MIL_{Pos}^{Min}).*

With this proven and the previous chapter in mind, a most natural follow-up is what happens when we include the informational implication '\' and consider the resulting logics $MIL_{\backslash\text{-}Pre}^{Min}$ and $MIL_{\backslash\text{-}Pos}^{Min}$. As is the content of the proceeding theorem, the short answer is: nothing really.

Theorem 5.10. $MIL_{\backslash\text{-}Pre}^{Min} = MIL_{\backslash\text{-}Pos}^{Min} = MIL_{\backslash\text{-}Pre}$.

Proof. Examining the proofs of Theorem 5.8, Proposition 5.7 and Lemma 5.6, it becomes clear that all we need to show is that the p-morphism f of Lemma 5.6 is, in fact, a \-p-morphism. And since we have shown it to be a p-morphism, it is enough to show the \-back clause.

\-*back.* Accordingly, suppose $x = \sup\{f(y, j), z\}$ for some $x, z \in W$. Going by cases, we get:

- If $j = 0$, we have, by 3. (cf. 5.6), that $(x, 0) = \sup'\{(y, 0), (z, 0)\}$ – showing the desired.

- If $j = 1$ and $x \in \downarrow w$, then also $z \in \downarrow w$. Thus, $\{(x, 1), (z, 1)\} \subseteq W'$, and $(x, 1)$ is an upper bound of $\{(y, 1), (z, 1)\}$. So suppose (s, i) also is an upper bound of $\{(y, 1), (z, 1)\}$. It then suffices to show that $(x, 1) \leqslant' (s, i)$. Since f is order-preserving, we have that $y \leqslant s \geqslant z$, so by assumption $x \leqslant s$, hence $(x, 1) \leqslant' (s, i)$.

- If $j = 1$ and $x \notin \downarrow w$, then we claim that $(x, 0) = \sup'\{(y, 1), (z, 0)\}$. Since $x = \sup\{f(y, j), z\}$ by assumption, $(x, 0)$ is an upper bound of $\{(y, 1), (z, 0)\}$. Assume (s, i) also is an upper bound of $\{(y, 1), (z, 0)\}$. Then $y \leqslant s \geqslant z$, so $x \leqslant s$. Thus, $s \notin \downarrow w$, hence we must have $i = 0$, whence $(x, 0) \leqslant' (s, i)$ as required.

Thus, we have shown the $\backslash$-back condition of the function f defined in Lemma 5.6, hence we may conclude that Lemma 5.6, Proposition 5.7 and Theorem 5.8 all extend to our present richer setting. □

Corollary 5.11. *$MIL^{Min}_{\backslash\text{-}Pre}$ is decidable (and so is $MIL^{Min}_{\backslash\text{-}Pos}$).*

One might conclude that, on preorders and posets, the landscape of MILs is both uniform and decidable:

$$MIL_{Pre} = MIL_{Pos} = MIL^{Min}_{Pre} = MIL^{Min}_{Pos}, \quad MIL_{\backslash\text{-}Pre} = MIL_{\backslash\text{-}Pos} = MIL^{Min}_{\backslash\text{-}Pre} = MIL^{Min}_{\backslash\text{-}Pos}$$

However, even if the suprema and minima interpretations neither come apart in the basic MIL-setting nor in the $\backslash$-augmented setting, our central proof method does suggest a setting where they might do.

To see this, summing up, our proof(s) fundamentally relied on there being two distinct ways for an upper bounded set $\{u, v\}$ not to have a supremum: (i) incomparable u.b.s vs. (ii) infinitely descending chain(s) of u.b.s. And, importantly, us being able to 'eliminate' (i)—or, rather, make sure that (ii) is the case whenever (i) is—so that any $\{u, v\}$ has a supremum iff it has a minimal upper bound. In light of this, it is not only (a) not surprising that beyond $MIL^{Min}_{Pre} = MIL_{Pre}$ also $MIL^{Min}_{\backslash\text{-}Pre} = MIL^{Min}_{\backslash\text{-}Pos} = MIL_{\backslash\text{-}Pre}$, but also (b) further suggestive of a place where the differing interpretations do result in differing logics; we end this chapter with a comment on this:

Remark 5.12. Suggested by the preceding paragraph, although the two interpretations (min. u.b. vs. least u.b.) result in the same logics when defined on *arbitrary* preorders/posets, the logics come apart when restricting to *finite* structures. This is witnessed by the formula

$$(Pp \wedge Pq) \to P\langle \sup \rangle pq,$$

which is easily seen valid on the class of finite $\mathcal{M}_{\text{Pre}}$-frames, but not on the class of finite $\mathcal{S}_{\text{Pre}}$-frames. Reason being that making sure that (ii) is the case whenever (i) is the case requires adding an *infinite* chain. $\dashv$

6. Axiomatizing *MIL_{Sem}*

This chapter is the most comprehensive chapter of this thesis—but for counting the sheer number of results proven because those are few. Having fairly thoroughly studied modal information logics on preorders and posets, we expand this line of inquiry to join-semilattices. The central goal being axiomatizing MIL_{Sem}, which, intriguingly, turns out to be an intricate manner. For this reason, the chapter begins with a section (6.1) solely concerned with providing intuition and highlighting key ideas of the subsequent two sections. Section 6.2 then proceeds to supply the actual infinite axiomatization and prove soundness, before the completeness proof of section 6.3. Using the completeness proof, in section 6.4, it is shown that MIL_{Sem} also is the logic of join-semilattices with all lower-bounded binary meets, but not all binary meets simpliciter (i.e., lattices). This concludes our actual study of modal information logics; section 6.5 ends the chapter with a direction for future work.[15]

6.1. Axiomatization: conceptual solution

We recommend further consulting this section both during and after reading the subsequent sections 6.2 and 6.3.

Arguably, the axiomatization of this chapter contains the most complex ideas of the thesis. While all accompanying proofs are verifiable without prior insight into these ideas, doing so could feel like navigating by GPS: one follows the instructions given and does end up in the right place, but is not really sure how one got there. This is not to say that the proofs themselves do not convey these ideas – they must do – but untangling some of the ideas from the get-go amplifies the salient points and, like studying the map before starting the car, provides intuition for where we are heading.

For that reason, we continue by informally highlighting key ideas, showing how they solve particular problems in the setting of attempting an axiomatization. The method being working out what axioms are needed to construct a model satisfying some MCS $\Gamma_0 \supseteq X_0$ extending some consistent set X_0.[16]

[15] As the only chapter of the thesis, this chapter presumes familiarity with basic notions and results from universal algebra. If ever needed, the reader may consult Burris and Sankappanavar (1981).

[16] We assume familiarity with such axiomatization practice, but, even so, we are aware that what follows still (probably) requires significant effort to properly understand. In this vein, let us finally stress that what follows is not clear-cut mathematical arguments, but heuristic guidance. It is not intended to be 'literally true' but 'metaphorically helpful'—hopefully not least for applying similar ideas and drawing inspiration in other axiomatization settings.

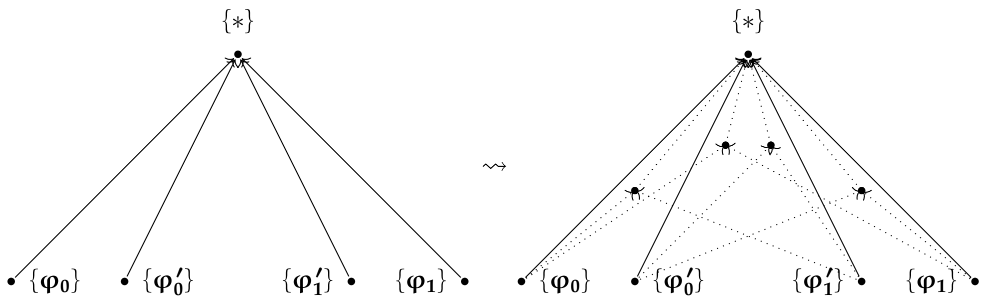
Starting off, we define the single-state join-semilattice $(\{\{*\}\}, \{(\{*\}, \{*\})\})$ and label $\mathfrak{l}(\{*\}) = \Gamma_0$. Constructing our model stepwise, the objective is then to prove a truth lemma. If, say, $\{\langle \sup \rangle \varphi_0 \varphi_0', \langle \sup \rangle \varphi_1 \varphi_1'\} \subseteq \Gamma_0$, we would want to add corresponding points—for convenience called $\{\boldsymbol{\varphi_0}\}, \{\boldsymbol{\varphi_0'}\}, \{\boldsymbol{\varphi_1}\}, \{\boldsymbol{\varphi_1'}\}$—as in the left part of the figure below, and label them according to the existence lemma so that $\varphi_0 \in \mathfrak{l}(\{\boldsymbol{\varphi_0}\})$, etc.

For later convenience, we let the single state be a singleton $\{*\}$ itself.

The first complication then becomes that although, e.g., $\{*\} = \sup\{\{\boldsymbol{\varphi_0'}\}, \{\boldsymbol{\varphi_1'}\}\}$, we need not have $C_{\mathbf{Sem}} \Gamma_0 \mathfrak{l}(\{\boldsymbol{\varphi_0'}\}) \mathfrak{l}(\{\boldsymbol{\varphi_1'}\})$. Therefore, we would want a formula $\pi_1 \in MIL_{Sem}$ somehow enabling us to add a point, $\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\}$, and label it s.t. not only $C_{\mathbf{Sem}} \mathfrak{l}(\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\}) \mathfrak{l}(\{\boldsymbol{\varphi_0'}\}) \mathfrak{l}(\{\boldsymbol{\varphi_1'}\})$ but also $C_{\mathbf{Sem}} \Gamma_0 \mathfrak{l}(\boldsymbol{\varphi_0}) \mathfrak{l}(\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\})$ and $C_{\mathbf{Sem}} \Gamma_0 \mathfrak{l}(\boldsymbol{\varphi_1}) \mathfrak{l}(\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\})$. Taking this argument a step further, we would want π_1 to enable freely generating a join-semilattice modulo the requirements $\{*\} = \sup\{\boldsymbol{\varphi_0}, \boldsymbol{\varphi_0'}\}$ and $\{*\} = \sup\{\boldsymbol{\varphi_1}, \boldsymbol{\varphi_1'}\}$ (i.e., the RHS semilattice of the figure below) so that whenever $x = \sup\{y, z\}$, it is also the case that $C_{\mathbf{Sem}} \mathfrak{l}(x) \mathfrak{l}(y) \mathfrak{l}(z)$.

'$C_{\mathbf{Sem}}$' referring to the ternary relation of the canonical frame of the sought axiomatization.

Observe that adding dummies would not work: we are constructing a join-semilattice, so $\{\boldsymbol{\varphi_0'}\}$ and $\{\boldsymbol{\varphi_1'}\}$ must have a join, and this must be below the upper bound $\{*\}$.



Now, it is obviously false that whenever some $w \Vdash \langle \sup \rangle \varphi_0 \varphi_0' \wedge \langle \sup \rangle \varphi_1 \varphi_1'$ in some join-semilattice model, the sub-join-semilattice generated by w and witnesses for $\{\langle \sup \rangle \varphi_0 \varphi_0', \langle \sup \rangle \varphi_1 \varphi_1'\}$ is *isomorphic* to the RHS join-semilattice. But it is true that this sub-join-semilattice will be the (join-semilattice) *homorphic image* of the RHS join-semilattice. Moreover, this can adequately be encoded into the formula π_1 and will suffice for dealing with this first complication. This helps explain the following parts of the coming axiomatization:

- The axioms, like π_1, will be implications that can be intuited as follows: given the satisfaction of some formulas (the antecedent), a certain sub-join-semilattice is the homomorphic image of a certain other join-semilattice which is freely generated modulo some specified requirements (the consequent).
- To define formulas like π_1, we must, first, define this "certain other join-semilattice" which is "freely generated modulo specified requirements". In

particular, we will formalize this by taking freely generated join-semilattices $\mathcal{P}(S) \setminus \{\varnothing\}$ and quotienting out under the least congruence relations $\sim$ meeting the given requirements.

Continuing the stepwise construction, suppose, say, $\langle \sup \rangle \psi\psi' \in l(\{\boldsymbol{\varphi_0}\})$. Again, simply adding corresponding worlds $\{\boldsymbol{\psi}\}, \{\boldsymbol{\psi'}\}$ labeled using the existence lemma for $l(\{\boldsymbol{\varphi_0}\})$ does not do the job. Because then, for instance, $\{*\} = \sup\{\{\boldsymbol{\psi'}\}, \{\boldsymbol{\varphi'_0}\}\}$ while we need not have $C_{\mathbf{Sem}} l(\{*\}) l(\{\boldsymbol{\psi'}\}) l(\{\boldsymbol{\varphi'_0}\})$. Once more, the solution must be to have some formula $\pi_2 \in MIL_{Sem}$ enabling us to construct an extended join-semilattice freely generated modulo the obvious requirements so that, crucially, $x = \sup\{y, z\}$ implies $C_{\mathbf{Sem}} l(x) l(y) l(z)$. This brings about a second (minor) complication: since $\langle \sup \rangle \psi\psi' \in l(\{\boldsymbol{\varphi_0}\})$, it is instinctive to want to find a formula $\pi_2 \in MIL_{Sem}$ ascertaining this when 'evaluated at' $l(\{\boldsymbol{\varphi_0}\})$; however, our join-semilattice interpretation of $\mathcal{L}_M$-formulas can, clearly, only express properties of worlds below any given world of evaluation. Thus, there can be no formula π_2 expressing the desired when evaluated at $l(\{\boldsymbol{\varphi_0}\})$. Fortunately, a solution can be found: $l(\{*\}) = \Gamma_0$ is all-seeing (backwardly), so we should (and will) be able to express the desired with a formula π_2 evaluated at $l(\{*\}) = \Gamma_0$. Before going any further, let us summarize the key take-aways.

- To achieve the truth lemma, we will need to unboundedly extend the join-semilattice under construction. This explains one way in which our axiomatization will be infinite: having, e.g., defined the RHS join-semilattice $\big(\mathcal{P}(S_1) \setminus \{\varnothing\}, \cup\big)/_{\sim_1}$ using the formula π_1, if, e.g., $\langle \sup \rangle \psi\psi' \in l(\{\boldsymbol{\varphi_0}\})$, we will need to construct an *extended* join-semilattice $\big(\mathcal{P}(S_2) \setminus \{\varnothing\}, \cup\big)/_{\sim_2}$ using a formula π_2. And then an extended one using a formula π_3, etc. That is, we must be able to ascertain that an ever-increasing sub-join-semilattice is the homormorphic image of a correspondingly ever-increasing join-semilattice freely generated modulo ever-more specified requirements.
- In a sense, the item above explains a way in which we must include axioms for each 'depth' $n \in \omega$. On top of that, we must also include axioms for each 'width' $n \in \omega$: the join-semilattice freely generated modulo requirements of $\{*\} = \sup\{\boldsymbol{\varphi_0}, \boldsymbol{\varphi'_0}\}$ and $\{*\} = \sup\{\boldsymbol{\varphi_1}, \boldsymbol{\varphi'_1}\}$ is obviously smaller than the one generated modulo requirements of $\{*\} = \sup\{\{\boldsymbol{\varphi_0}\}, \{\boldsymbol{\varphi'_0}\}\}, \{*\} = \sup\{\{\boldsymbol{\varphi_1}\}, \{\boldsymbol{\varphi'_1}\}\}$ and $\{*\} = \sup\{\{\boldsymbol{\varphi_2}\}, \{\boldsymbol{\varphi'_2}\}\}$, etc.
- When constructing the model to ensure that $x = \sup\{y, z\}$ always implies $C_{\mathbf{Sem}} l(x) l(y) l(z)$, we have to label all points with MCSs obtained by evaluating the formulas $\pi_1, \pi_2, \ldots$ at the top MCS $l(\{*\}) = \Gamma_0$.

Visually, I like to imagine the formulas $\pi_1, \pi_2, \ldots$ as ascertaining we can set an 'image resolution' of level $1, 2, \ldots$, respectively—both regarding the sub-join-semilattices and the complementary join-semilattices of which they are homomorphic images.

Continuing, although solving one problem, this last solution of evaluating at $l(\{*\})$ inevitably constructs another (major) problem: having first labeled, e.g.,

$\{\boldsymbol{\varphi_0}\}$ via evaluating the formula π_1 at Γ_0, we now relabel $\{\boldsymbol{\varphi_0}\}$ via evaluating another formula π_2 at Γ_0. How then are we to ascertain that π_2 and π_1 agree on the labeling; i.e., that, e.g., $l_2(\{\boldsymbol{\varphi_0}\}) = l_1(\{\boldsymbol{\varphi_0}\})$? If we by using formulas somehow could 'name' the MCSs of the labeling induced by π_1, we could construct π_2 using these 'names' as to ensure that the labeling of $\{\boldsymbol{\varphi_0}\}$ induced by π_2 agrees with the labeling of π_1; thus, solving the problem. Evidently, (without nominals) there can be no way of doing so when dealing with MCSs. There is an alternative, though: while an MCS Θ is equivalently defined as an *infinite* conjunction $\widehat{\Theta}$, a finite set of formulas Θ_F is equivalently defined as a *finite* conjunction $\widehat{\Theta_F}$; i.e., in some sense, going finite facilitates 'naming'. This suggests the following idea:

We write $\widehat{\Theta_F} := \bigwedge_{\theta \in \Theta_f} \theta$.

- Instead of starting out with some (possibly infinite) consistent set X_0, we go for weak completeness and start with a consistent formula φ which we extend to the least subformula-closed set Φ containing $\{\varphi\}$. We then label our worlds according to which Φ-formulas they satisfy instead of with MCSs. In this way, using finite conjunctions, we can contain the labeling in the formula π_1, and then also in the extended formula π_2, etc. We then get that (1) $x = \sup\{y, z\}$ implies $C_{\mathbf{Sem}}\Gamma_x\Gamma_y\Gamma_z$ for some $\Gamma_i \supseteq l(i)$, and, importantly, (2) $l_1(x) = l_2(x)$.

Strong completeness will be an easy consequence of weak completeness, in any case (as we also will show).

Notice that (1) is simply the 'finite version' of the previous: $x = \sup\{y, z\}$ *implies* $C_{\mathbf{Sem}}l(x)l(y)l(z)$.

Yet again, solving one problem we have caused another: how can π_1 also contain the information determining what Φ-formulas the worlds are to satisfy *and* still be valid: that, say, some $w \Vdash \langle \sup \rangle \varphi_0 \varphi_0'$ does not determine what Φ-formulas the witnessing φ_0- and φ_0'-worlds satisfy. Key here is that Φ is finite, so there are only finitely many 'names' over Φ, and we do know that the witnessing φ_0- and φ_0'-worlds must have some 'Φ-name'. Therefore, the consequent of π_1 will not state that one particular sub-join-semilattice is the homomorphic image of one particular other join-semilattice, but instead disjunctively quantify over all such options induced by all possible Φ-names. This brings us to our final point of elaboration:

- If the consequents of the formulas $\pi_1, \pi_2, \ldots$ consist of disjunctions defining *distinct* join-semilattices, which disjunct shall we choose when stepwise extending our join-semilattice as to satisfy the truth lemma? To answer this, it is helpful recalling how π_2 is to 'extend' π_1. Essentially, we want π_2 to encode how a bigger sub-join-semilattice must also be the homomorphic image of another bigger join-semilattice. So since each disjunct of the consequent of π_1 encodes how a sub-join-semilattice is the homomorphic image of another join-semilattice, π_2 must encode the extended claim for each disjunct. To do so, π_2 must, in particular, split each disjunct of π_1 into further disjunctions to quantify over all possible Φ-names for the 'new worlds' of the extended join-semilattices. And so forth as for $\pi_3, \ldots$.

What we are left with is a tree where each node at each layer i, in particular, defines a join-semilattice (the one that a given sub-join-semilattice must be the homomorphic image of) and also a corresponding disjunct of a formula π_i, and the edges mark 'extension' of both join-semilattices and formulas. The main observations are then (1) the tree is finitely branching, and (2) we can assure that at each layer at least one disjunct must be 'satisfied', allowing for an infinite subtree where König's Lemma applies to supply an infinite chain of join-semilattice models of which its colimit is our satisfying join-semilattice model.

So instead of the more common 'deterministic', linear construction of a model, as in, e.g., Theorem 2.13, we construct a tree of models and then let König do the work of picking out a linearly constructed model.

This concludes our 'study guide' for the axiomatization to come. While it is in no way exhaustive and should be treated as nothing but an informal heuristical guide, we hope that by having called attention to particular features, we have (1) made what comes more edible, and (2) called to the fore ideas potentially applicable in other axiomatization settings.

6.2. Soundness

We proceed with the actual definitions and proofs. To axiomatize MIL_{Sem}, we use an infinite extension scheme for which we need the following definition:

Definition 6.1. Given any set of formulas $\Phi \subseteq \mathcal{L}_M$, we say that $A \subseteq \mathcal{L}_M$ is a *maximal theory* over Φ just in case there is some $\Phi' \subseteq \Phi$ s.t.

$$A = \Phi' \cup \{\neg\varphi \in \mathcal{L}_M \mid \varphi \in (\Phi \setminus \Phi')\}.$$

We write $\mathsf{MT}(\Phi)$ for the set of all maximal theories over Φ. ⊣

Think of $\mathsf{MT}(\Phi)$ as the set of all possible (and impossible) 'Φ-names'.

Besides from this definition, in order to define the formulas, we will need auxiliary constructions, namely trees. We continue by defining these.

Definition 6.2 (Extension scheme: trees)**.** For any finite, subformula-closed set of formulas $\Phi \subseteq \mathcal{L}_M$ and maximal theory $A \in \mathsf{MT}(\Phi)$, we define a tree $\mathbf{T}^{\Phi,A}$ with layers $\mathbf{T}^{\Phi,A}_k$ for all $k \in \omega$. We do so recursively, going layer by layer specifying the immediate successors of any $t_k \in \mathbf{T}^{\Phi,A}_k$ as a set $\mathbf{T}^{\Phi,A}_{k+1}(t_k)$ and then setting

$$\mathbf{T}^{\Phi,A}_{k+1} := \bigcup_{t_k \in \mathbf{T}^{\Phi,A}_k} \mathbf{T}^{\Phi,A}_{k+1}(t_k).$$

For layer $k = 0$, it will be notationally convenient to first define

$$S_{-1} := \sim_{-1} := l_{-1} := \varnothing, \qquad \mathbf{T}^{\Phi,A}_{-1} := \{(S_{-1}, \sim_{-1}, l_{-1})\},$$

'l' is, again, short for 'labeling', while each S_k and $\sim_k$ will define a join-semilattice.

then abbreviate $t_{-1} := (S_{-1}, \sim_{-1}, l_{-1})$ and use it to define

$$S_0 := \{*\}, \qquad l_0 : \mathcal{P}(S_0) \setminus \{\varnothing\} \to MT(\Phi), \{*\} \mapsto A,$$
$$\sim_0 = \{(\{*\}, \{*\})\}, \qquad \mathbf{T}_0^{\Phi,A}(t_{-1}) := \{(S_0, \sim_0, l_0)\}, \qquad \mathbf{T}_0^{\Phi,A} := \{(S_0, \sim_0, l_0)\},$$

for some symbol $*$. I.e., at layer $k = 0$, our tree consists of the root $(S_0, \sim_0, l_0)$.

For layer $k = n + 1$, assume that for all $t_n = (S_n, \sim_n, l_n) \in \mathbf{T}_n^{\Phi,A}$ there is a unique $t_{n-1} = (S_{n-1}, \sim_{n-1}, l_{n-1}) \in \mathbf{T}_{n-1}^{\Phi,A}$ such that $t_n \in \mathbf{T}_n^{\Phi,A}(t_{n-1})$. Further, assume that these satisfy the following conditions:

- $S_n \supseteq S_{n-1}$, $\sim_n \supseteq \sim_{n-1}$, $l_n \supseteq l_{n-1}$;
- $\sim_n$ is a congruence relation on join-semilattice $\big(\mathcal{P}(S_n) \setminus \{\varnothing\}, \cup\big)$, and $\sim_{n-1}$ is a congruence relation on join-semilattice $\big(\mathcal{P}(S_{n-1}) \setminus \{\varnothing\}, \cup\big)$;
- $l_n : \mathcal{P}(S_n) \setminus \{\varnothing\} \to MT(\Phi)$, $l_{n-1} : \mathcal{P}(S_{n-1}) \setminus \{\varnothing\} \to MT(\Phi)$; and
- if $Y \sim_n Y'$, then $l_n(Y) = l_n(Y')$.

For each

$$t_n = (S_n, \sim_n, l_n) \in \mathbf{T}_n^{\Phi,A},$$

we continue defining its set of successors $\mathbf{T}_{n+1}^{\Phi,A}(t_n)$. Accordingly, suppose $t_n \in \mathbf{T}_n^{\Phi,A}(t_{n-1})$ for some

$$t_{n-1} = (S_{n-1}, \sim_{n-1}, l_{n-1}) \in \mathbf{T}_{n-1}^{\Phi,A},$$

and consider their corresponding set

$$N(t_n) := \big\{Y \in \big(\mathcal{P}(S_n) \setminus \{\varnothing\}\big) \setminus \big(\mathcal{P}(S_{n-1}) \setminus \{\varnothing\}\big) \mid \neg\exists Y' \big(Y \sim_n Y', Y' \in \big(\mathcal{P}(S_{n-1}) \setminus \{\varnothing\}\big)\big)\big\}.$$

Clearly, if $Y \in N(t_n)$, then $[Y]_n \subseteq N(t_n)$. Therefore, we can (a) consider each of these equivalence classes $[Y]_n \subseteq N(t_n)$, (b) let

'N' is short for 'new worlds'.

$$\langle \sup \rangle \varphi_0^Y \varphi_0^{Y\prime}, \ldots, \langle \sup \rangle \varphi_{m_Y}^Y \varphi_{m_Y}^{Y\prime}$$

denote all formulas $\chi \in l_n(Y)$ with main connective '$\langle \sup \rangle$' and (c) define

$$S_{n+1}^{[Y]_n\prime} := \big\{\boldsymbol{\varphi}_0^{\mathbf{Y}}, \boldsymbol{\varphi}_0^{\mathbf{Y}\prime}, \ldots, \boldsymbol{\varphi}_{m_Y}^{\mathbf{Y}}, \boldsymbol{\varphi}_{m_Y}^{\mathbf{Y}\prime}\big\}$$

assuring that all such **boldface** symbols for formulas are pairwise distinct.

Using this, we further define

$$S_{n+1}' := \bigcup_{[Y]_n \subseteq N(t_n)} S_{n+1}^{[Y]_n\prime},$$

and

$$S_{n+1} := S'_{n+1} \cup S_n.$$

Then $\big(\mathcal{P}(S_{n+1}) \setminus \{\varnothing\}, \cup\big)$ is a join-semilattice on which we let $\sim_{n+1}$ be the least congruence relation satisfying

1. if $Y \sim_n Y'$, then $Y \sim_{n+1} Y'$; and
2. $\forall [Y]_n \subseteq N(t_n)\ \forall \left\{\boldsymbol{\varphi}_i^Y, \boldsymbol{\varphi}_i^{Y}{}'\right\} \subseteq S_{n+1}^{[Y]_n}{}' : \left\{\boldsymbol{\varphi}_i^Y, \boldsymbol{\varphi}_i^{Y}{}'\right\} \sim_{n+1} Y.$

Lastly, while the elements of $\mathbf{T}_{n+1}^{\Phi,A}(t_n)$ (i.e. the successors of t_n) agree on the first two components (namely S_{n+1} and $\sim_{n+1}$), they disagree on the last: the labeling. This we define by considering[17]

$$L_{n+1}(t_n) := \Big\{ l_{n+1} : \mathcal{P}(S_{n+1}) \setminus \{\varnothing\} \to \mathsf{MT}(\Phi) \mid l_{n+1}\restriction(\mathcal{P}(S_n) \setminus \{\varnothing\}) = l_n, [Y \sim_{n+1} Y' \Rightarrow l_{n+1}(Y) = l_{n+1}(Y')],$$
$$\forall [Y]_n \subseteq N(t_n)\ \forall \left\{\boldsymbol{\varphi}_i^Y, \boldsymbol{\varphi}_i^{Y}{}'\right\} \subseteq S_{n+1}^{[Y]_n}{}' : \varphi_i^Y \in l_{n+1}\left(\left\{\boldsymbol{\varphi}_i^Y\right\}\right), \varphi_i^{Y}{}' \in l_{n+1}\left(\left\{\boldsymbol{\varphi}_i^{Y}{}'\right\}\right) \Big\},$$

and then letting

$$\mathbf{T}_{n+1}^{\Phi,A}(t_n) := \{(S_{n+1}, \sim_{n+1}, l_{n+1}) \mid l_{n+1} \in L_{n+1}(t_n)\}.$$

In case $S'_{n+1} = \varnothing$, we would get $S_{n+1} = S_n, \sim_{n+1} = \sim_n$. This pseudo-problem can be circumvented in multiple ways; w.l.o.g. we act as if any such t_{n+1} is formally distinct from any other node previously constructed (formally, the disjoint union, e.g., achieves this, but this clutters notation).

Defining so for all $t_n \in \mathbf{T}_n^{\Phi,A}$, we set

$$\mathbf{T}_{n+1}^{\Phi,A} := \bigcup_{t_n \in \mathbf{T}_n^{\Phi,A}} \mathbf{T}_{n+1}^{\Phi,A}(t_n),$$

which completes the recursive step, whence the definition as well. $\dashv$

To see that the recursive step actually was well-defined, we assert that our assumption used when defining the successor nodes was granted.

Lemma 6.3. *For any finite, subformula-closed set of formulas $\Phi \subseteq \mathcal{L}_M$, maximal theory $A \in \mathsf{MT}(\Phi)$, and $t_n \in \mathbf{T}_n^{\Phi,A}$, there is a (unique) $t_{n-1} \in \mathbf{T}_{n-1}^{\Phi,A}$ such that*

$$t_n \in \mathbf{T}_n^{\Phi,A}(t_{n-1}).$$

Moreover, whenever $t_n \in \mathbf{T}_n^{\Phi,A}(t_{n-1})$, then

$$t_n = (S_n, \sim_n, l_n) \quad \textit{and} \quad t_{n-1} = (S_{n-1}, \sim_{n-1}, l_{n-1})$$

for some $S_n, \sim_n, l_n, S_{n-1}, \sim_{n-1}, l_{n-1}$ such that

- $S_n \supseteq S_{n-1},\ \sim_n \supseteq \sim_{n-1},\ l_n \supseteq l_{n-1};$

[17] In this chapter—for clarity, but compromising aesthetics—we have allowed for overflows.

- *$\sim_n$ is a congruence relation on join-semilattice $(\mathcal{P}(S_n)\setminus\{\varnothing\},\cup)$, and $\sim_{n-1}$ is a congruence relation on join-semilattice $(\mathcal{P}(S_{n-1})\setminus\{\varnothing\},\cup)$;*
- *$l_n : \mathcal{P}(S_n)\setminus\{\varnothing\} \to \mathsf{MT}(\Phi)$, and $l_{n-1} : \mathcal{P}(S_{n-1})\setminus\{\varnothing\} \to \mathsf{MT}(\Phi)$; and*
- *if $Y \sim_n Y'$, then $l_n(Y) = l_n(Y')$.*

Proof. Follows by an easy induction, scrutinizing the preceding definition. □

Even if immediate by definition, explicitly including the following lemma underlines and elucidates important aspects of the previous definition:

Lemma 6.4. *For any finite, subformula-closed set of formulas $\Phi \subseteq \mathcal{L}_M$, maximal theory $A \in \mathsf{MT}(\Phi)$, and $t_n = (S_n, \sim_n, l_n) \in \mathbf{T}^{\Phi,A}_n$ we have that*

- *S_n is finite; and*
- *if $[Y]_n \subseteq N(t_n)$ and $\langle \sup\rangle \varphi^Y_i \varphi^{Y\prime}_i \in l_n(Y)$, then for all successors of t_n – i.e., all $t_{n+1} = (S_{n+1}, \sim_{n+1}, l_{n+1}) \in \mathbf{T}^{\Phi,A}_n(t_n)$ – it is the case that*

$$\varphi^Y_i \in l_{n+1}\left(\{\boldsymbol{\varphi}^{\mathbf{Y}}_{\mathbf{i}}\}\right), \qquad \varphi^{Y\prime}_i \in l_{n+1}\left(\{\boldsymbol{\varphi}^{\mathbf{Y}\prime}_{\mathbf{i}}\}\right),$$

and

$$\left[\{\boldsymbol{\varphi}^{\mathbf{Y}}_{\mathbf{i}}\}\right]_{n+1} \cup_{\sim_{n+1}} \left[\{\boldsymbol{\varphi}^{\mathbf{Y}\prime}_{\mathbf{i}}\}\right]_{n+1} = [Y]_{n+1},$$

where '$\cup_{\sim_{n+1}}$' refers to the join of the join-semilattice $(\mathcal{P}(S_{n+1})\setminus\{\varnothing\},\cup)/_{\sim_{n+1}}$.

Proof. Doing an induction, the proof becomes immediate by definition. □

Although it will not be needed until our much later completeness proof, it is instructive to have explicated this second assertion of the lemma to understand our tree definition. Now, put in the light of a model construction for completeness, informally speaking, it spells out that whenever a world $[Y]_n$ is 'new' (i.e. $[Y]_n \subseteq N(t_n)$), if the labeling function l_n requires $[Y]_n$ to satisfy some $\langle \sup\rangle \varphi^Y_i \varphi^{Y\prime}_i \in l_n(Y)$, then this requirement is met in all successor steps t_{n+1}; i.e., in all successor models.

To gain even more familiarity with these trees, we additionally prove the following lemma, although it, too, only will be needed in the completeness proof:

Lemma 6.5. *For any $\mathbf{T}^{\Phi,A}$, sequence $t_0, \ldots, t_n$ satisfying $t_{i+1} \in \mathbf{T}^{\Phi,A}_{i+1}(t_i)$, and $Y \in \mathcal{P}(S_n)\setminus\{\varnothing\}$, there are $X \in \mathcal{P}(S_n)\setminus\{\varnothing\}$ and $m \leqslant n$ such that*

$$Y \sim_n X \quad \text{and} \quad X \in N(t_m).$$

In essence, this lemma asserts that all worlds will have been 'new worlds' at some stage m, hence at stage $m+1$ their '$\langle \sup\rangle$-defects' will have been fixed.

Proof. We prove the claim by induction on n. For $n = 0$, it holds because $\sim_0$ is reflexive and

$$\mathcal{P}(S_0) \setminus \{\varnothing\} = \{\{*\}\} = N(t_0).$$

For $n > 0$, if $Y \in N(t_n)$, we are done by reflexivity of $\sim_n$. And if not, by definition of $N(t_n)$, there is some $Y' \in \mathcal{P}(S_{n-1}) \setminus \{\varnothing\}$ such that $Y \sim_n Y'$. By IH, there must be some $X \in \mathcal{P}(S_{n-1}) \setminus \{\varnothing\}$ and $m \leqslant (n-1)$ such that $Y' \sim_{n-1} X$ and $X \in N(t_m)$. But then $Y' \sim_n X$, so – by transitivity of $\sim_n$ – $Y \sim_n X$, which proves the claim. □

Picking up where we left, we define the formulas of our extension scheme using the auxiliary trees.

Definition 6.6 (Extension scheme: formulas)**.** For any finite, subformula-closed set of formulas $\Phi \subseteq \mathcal{L}_M$, maximal theory $A \in MT(\Phi)$ and $k \in \omega$, we define a formula $\pi_k^{\Phi,A}$.

To do so, we will need to first define formulas $\alpha^{t_k}_{[S_k]_k}$, for each $t_k \in \mathbf{T}_k^{\Phi,A}$, which, in turn, are build from other formulas $\alpha^{t_k}_{[Y]_k}$.

Accordingly, for each $t_k = (S_k, \sim_k, l_k)$ and $[Y]_k \in \big(\mathcal{P}(S_k) \setminus \{\varnothing\}, \cup\big)/_{\sim_k}$, define the set $C_k[Y]_k$ consisting of the pairs of congruence classes $([Y_a]_k, [Y_b]_k)$ *strictly* below $[Y]_k$ and of which it is the join; that is,

$$C_k[Y]_k := \{([Y_a]_k, [Y_b]_k) \mid [Y]_k = [Y_a]_k \cup_{\sim_k} [Y_b]_k, [Y_a]_k \neq [Y]_k \neq [Y_b]_k\},$$

and use it to define the formula

$$\alpha^{t_k}_{[Y]_k} := \widehat{l_k(Y)} \wedge \bigwedge_{([Y_a]_k,[Y_b]_k) \in C_k[Y]_k} \langle \sup \rangle \alpha^{t_k}_{[Y_a]_k} \alpha^{t_k}_{[Y_b]_k} \wedge \bigwedge_{[Y]_k >_k [X]_k} P\alpha^{t_k}_{[X]_k}.$$

The part '$\widehat{l_k(Y)}$' is our 'naming' facilitated by 'finiteness'; the latter two parts deal with the encoding of being the homomorphic image of the join-semilattice $\big(\mathcal{P}(S_k) \setminus \{\varnothing\}, \cup\big)/_{\sim_k}$.

To see that the formulas $\alpha^{t_k}_{[Y]_k}$ are well-defined, notice that

$$([Y_a]_k, [Y_b]_k) \in C_k[Y]_k$$

implies that

$$[Y]_k >_k [Y_a]_k, [Y_b]_k.$$

Consequently, since the partial order $>_k$ induced by the join-semilattice $\big(\mathcal{P}(S_k) \setminus \{\varnothing\}, \cup\big)/_{\sim_k}$ is well-founded (because it is on a *finite* set), the formulas are well-defined.

Using these formulas, we set

$$\pi_k^{\Phi,A} := \hat{A} \to \bigvee_{(S_k,\sim_k,l_k) \in \mathbf{T_k}^{\Phi,A}} \alpha^{(S_k,\sim_k,l_k)}_{[S_k]_k},$$

I.e., each node $t_k = (S_k, \sim_k, l_k) \in \mathbf{T}_k^{\Phi,A}$ corresponds to a disjunct in the consequent of $\pi_k^{\Phi,A}$.

which completes our definition. $\dashv$

Definition 6.7. For any finite, subformula-closed set of formulas $\Phi \subseteq \mathcal{L}_M$ and maximal theory $A \in MT(\Phi)$, we write $\Pi^{\Phi,A} := \{\pi_k^{\Phi,A} \mid k \in \omega\}$. $\dashv$

Having defined the formulas of our extension scheme, we are in a place to proof-theoretically define a logic, $\mathbf{MIL_{Sem}}$, which we will show to be an axiomatization of MIL_{Sem}.

Definition 6.8 (Axiomatization)**.** We define $\mathbf{MIL_{Sem}}$ to be the least normal modal logic in the language of $\mathcal{L}_M$ which (a) extends $\mathbf{MIL_{Pre}}$ and (b) contains $\Pi^{\Phi,A}$ for all finite subformula-closed sets of formulas $\Phi \subseteq \mathcal{L}_M$ and maximal theories $A \in MT(\Phi)$. $\dashv$

Prior to proving soundness, let us show how the associativity axiom

$$\text{(As.)}\ \langle \sup \rangle(\langle \sup \rangle pq)r \leftrightarrow \langle \sup \rangle p(\langle \sup \rangle qr),$$

of Remark 1.6 is derived in $\mathbf{MIL_{Sem}}$. Because it is not derivable in $\mathbf{MIL_{Pre}}$, it must be derivable by virtue of the tree formulas. Thus, its derivation serves as an obvious way of further elucidating this extension scheme – this time by giving a concrete example.

Example 6.9. To show that $\mathbf{MIL_{Sem}} \vdash$ (As.), it is enough to show one direction

$$\mathbf{MIL_{Sem}} \vdash \langle \sup \rangle(\langle \sup \rangle pq)r \rightarrow \langle \sup \rangle p(\langle \sup \rangle qr),$$

since the converse direction then follows by (Co.), propositional tautologies, US and MP. To this end, let $\Phi := \{\langle \sup \rangle(\langle \sup \rangle pq)r, (\langle \sup \rangle pq), p, q, r\}$, and observe that Φ is subformula closed and finite. Further, note that

$$\mathbf{MIL_{Sem}} \vdash \langle \sup \rangle(\langle \sup \rangle pq)r \rightarrow \bigvee\{\hat{A} \mid \langle \sup \rangle(\langle \sup \rangle pq)r \in A \in MT(\Phi)\}.$$

Thus, it suffices to show that for any maximal theory $A \in MT(\Phi)$ which contains $\langle \sup \rangle(\langle \sup \rangle pq)r$, we have that

$$\mathbf{MIL_{Sem}} \vdash \hat{A} \rightarrow \langle \sup \rangle p(\langle \sup \rangle qr).$$

To show this, given any $A \in MT(\Phi)$ s.t. $\langle \sup \rangle(\langle \sup \rangle pq)r \in A$, we will be constructing (some of) the formula $\pi_2^{\Phi,A}$.[18] Consequently, we have to construct layer 2 of the corresponding tree; that is, $\mathbf{T}_2^{\Phi,A}$. Following the recursive construction,

[18] For the sake of keeping the example within reasonable length, we will be omitting quite some details and only include the steps illuminating the bare workings.

the base case is given as follows:

$$S_{-1} :=\sim_{-1}:= l_{-1} := \varnothing, \qquad \mathbf{T}^{\Phi,A}_{-1} := \{(S_{-1}, \sim_{-1}, l_{-1})\},$$

and

$$\begin{aligned} &S_0 := \{*\}, && l_0 : \mathcal{P}(S_0) \setminus \{\varnothing\} \to \mathsf{MT}(\Phi), \{*\} \mapsto A, \\ &\sim_0 = \{(\{*\}, \{*\})\}, \quad && \mathbf{T}^{\Phi,A}_0(t_{-1}) := \{(S_0, \sim_0, l_0)\}, \qquad \mathbf{T}^{\Phi,A}_0 := \{(S_0, \sim_0, l_0)\}, \end{aligned}$$

where $t_{-1} := (S_{-1}, \sim_{-1}, l_{-1})$. For layer 1, since $\mathbf{T}^{\Phi,A}_0$ is a singleton, we only need to construct the successors of $t_0 = (S_0, \sim_0, l_0)$. Since $N(t_0) = \{\{*\}\}$ and $\langle\text{sup}\rangle(\langle\text{sup}\rangle pq)r \in A = l_0(\{*\})$, we know that

$$\{\langle\mathbf{sup}\rangle\mathbf{pq}, \mathbf{r}\} \subseteq S_1^{[\{*\}]_0}{}' \subseteq S_1,$$

hence

$$\{\langle\mathbf{sup}\rangle\mathbf{pq}, \mathbf{r}\} \sim_1 \{*\},$$

and for all $l_1 \in L_1(t_0)$:

$$\langle\text{sup}\rangle pq \in l_1\left(\{\langle\mathbf{sup}\rangle\mathbf{pq}\}\right), \quad r \in l_1\left(\{\mathbf{r}\}\right).$$

Knowing this regarding $\mathbf{T}^{\Phi,A}_1$ is adequate for our purposes. Because now for any $t_1 = (S_1, \sim_1, l_1) \in \mathbf{T}^{\Phi,A}_1$, we find that

$$\{\langle\mathbf{sup}\rangle\mathbf{pq}\} \in N(t_1),$$

hence by similar reasoning to before: for all $t_1 = (S_1, \sim_1, l_1) \in \mathbf{T}^{\Phi,A}_1$,

$$\{\mathbf{p}, \mathbf{q}\} \sim_2 \{\langle\mathbf{sup}\rangle\mathbf{pq}\}, \qquad \forall l_2 \in L_2(t_1) : p \in l_2\left(\{\mathbf{p}\}\right), q \in l_2\left(\{\mathbf{q}\}\right).$$

Again this is adequate concerning $\mathbf{T}^{\Phi,A}_2$. It is then not too hard to check that for all $t_2 \in \mathbf{T}^{\Phi,A}_2$,

$$\begin{aligned} &p \in_\wedge \alpha^{t_2}_{[\{\mathbf{p}\}]_2}, \quad q \in_\wedge \alpha^{t_2}_{[\{\mathbf{q}\}]_2}, \quad r \in_\wedge \alpha^{t_2}_{[\{\mathbf{r}\}]_2}, \\ &([\{\mathbf{q}\}]_2, [\{\mathbf{r}\}]_2) \in C_2[\{\mathbf{q}, \mathbf{r}\}]_2, \quad ([\{\mathbf{p}\}]_2, [\{\mathbf{q}, \mathbf{r}\}]_2) \in C_2[S_2]_2, \end{aligned}$$

whence

$$\langle\text{sup}\rangle\alpha^{t_2}_{[\{\mathbf{q}\}]_2}\alpha^{t_2}_{[\{\mathbf{r}\}]_2} \in_\wedge \alpha^{t_2}_{[\{\mathbf{q},\mathbf{r}\}]_2} \quad \textit{and} \quad \langle\text{sup}\rangle\alpha^{t_2}_{[\{\mathbf{p}\}]_2}\alpha^{t_2}_{[\{\mathbf{q},\mathbf{r}\}]_2} \in_\wedge \alpha^{t_2}_{[S_2]_2},$$

where the occurrence of '$\in_\wedge$' in, e.g., $p \in_\wedge \alpha^{t_2}_{[\{\mathbf{p}\}]_2}$ denotes that 'p' occurs as a conjunct of a conjunction '$\wedge$' occurring as a main connective in '$\alpha^{t_2}_{[\{\mathbf{p}\}]_2}$'. Therefore, by normality, we get that

So, e.g., $\varphi \in_\wedge [q_0 \wedge \varphi \wedge (q_2 \vee q_3)]$.

$$\mathbf{MIL_{Sem}} \vdash \alpha^{t_2}_{[S_2]_2} \to \langle \text{sup} \rangle p(\langle \text{sup} \rangle qr).$$

Lastly, since (1) this holds for all $t_2 \in \mathbf{T}_2^{\Phi,A}$, and (2)

$$\mathbf{MIL_{Sem}} \ni \pi_2^{\Phi,A} := \hat{A} \to \bigvee_{(S_2,\sim_2,l_2) \in \mathbf{T_2}^{\Phi,A}} \alpha^{(S_2,\sim_2,l_2)}_{[S_2]_2},$$

we get that

$$\mathbf{MIL_{Sem}} \vdash \hat{A} \to \langle \text{sup} \rangle p(\langle \text{sup} \rangle qr)$$

as sufficed. ⌟

Now for showing soundness. For this, we will need a preliminary result, which we have split into two lemmas: the latter being what we need, and the former being helpful in proving the latter.

Lemma 6.10. *Suppose $(W, \vee)$ has a greatest element w and is a sub-join-semilattice of $(V, \vee)$. Further, suppose there is an onto (join-semilattice) homomorphism*

$$h : \big(\mathcal{P}(S) \setminus \{\varnothing\}, \cup\big)/_\sim \quad \to \quad \big(W, \vee\big)$$

for some set S, and that

$$h([X_0]_\sim) = u_0 \vee v_0$$

for some $[X_0]_\sim \in \big(\mathcal{P}(S) \setminus \{\varnothing\}, \cup\big)/_\sim$ and $\{u_0, v_0\} \subseteq V$. Then the following hold.

- *The structure*

  $$\big(W', \vee\big) := \big(\big\{x_1 \vee \cdots \vee x_k : k \geqslant 1, \{x_1, ..., x_k\} \subseteq W \cup \{u_0, v_0\}\big\}, \vee\big),$$

  is a sub-join-semilattice of $(V, \vee)$ with greatest element w.

- *Define*

  $$S' := \{\mathbf{u_0}, \mathbf{v_0}\}$$

  assuring $\mathbf{u_0} \neq \mathbf{v_0}$ (note that we need not have $u_0 \neq v_0$) and $S' \cap S = \varnothing$, and let $\sim'$ be the least congruence relation on the join-semilattice $\big(\mathcal{P}(S \cup S') \setminus \{\varnothing\}, \cup\big)$ satisfying

  0. *if $X \sim Y$ then $X \sim' Y$; and*
  1. *$X_0 \sim' \{\mathbf{u_0}, \mathbf{v_0}\}$.*

Then sending

$$[\{\mathbf{u_0}\}]_{\sim'} \mapsto u_0, [\{\mathbf{v_0}\}]_{\sim'} \mapsto v_0$$

and, for all $\varnothing \neq X \subseteq S$*, sending*

$$[X]_{\sim'} \mapsto h([X]_\sim)$$

defines a partial function which canonically extends to an onto homorphism

$$h' : \big(\mathcal{P}(S \cup S') \setminus \{\varnothing\}, \cup\big)/_{\sim'} \quad \to \quad \big(W', \vee\big).$$

Proof. For the former claim, all we need is that (1) $w \geqslant h([X_0]_\sim) = u_0 \vee v_0$ and (2) the set

$$\big\{x_1 \vee \cdots \vee x_k : k \geqslant 1, \{x_1, ..., x_k\} \subseteq W \cup \{u_0, v_0\}\big\}$$

is closed under taking joins, which both clearly are the case.

For the latter, observe that sending

$$\{\mathbf{u_0}\} \mapsto u_0, \{\mathbf{v_0}\} \mapsto v_0,$$

and for all $\varnothing \neq X \subseteq S$

$$X \mapsto h([X]_\sim)$$

defines a partial function since $\mathbf{u_0} \neq \mathbf{v_0}$ and $S \cap S' = \varnothing$. In fact, it generates the function

$$f' : \big(\mathcal{P}(S \cup S') \setminus \{\varnothing\}, \cup\big) \to \big(W', \vee\big)$$

which is well-defined by being defined on all atoms; a homomorphism because h is; and onto because (a) it is onto on W (because h is), (b) it is onto on $\{u_0, v_0\}$, and (c) it is a homomorphism and W' consists of finite joins of elements from $W \cup \{u_0, v_0\}$.

Thus, f' induces another congruence relation $\sim_{f'}$ on $\big(\mathcal{P}(S \cup S') \setminus \{\varnothing\}, \cup\big)$ given as

$$X \sim_{f'} Y \qquad \textbf{iff} \qquad f'(X) = f'(Y).$$

Importantly, $\sim_{f'}$ satisfies the defining conditions *0.-1.* of $\sim'$. *0.* because if $X \sim Y$, then

$$f'(X) = h([X]_\sim) = h([Y]_\sim) = f'(Y),$$

and *1.* because

$$f'(X_0) = h([X_0]_\sim) = u_0 \vee v_0 = f'(\{\mathbf{u_0}\}) \vee f'(\{\mathbf{u_0}\}) = f'(\{\mathbf{u_0}\} \cup \{\mathbf{u_0}\}) = f'(\{\mathbf{u_0}, \mathbf{v_0}\}).$$

Thus,

$$\sim' \subseteq \sim_{f'},$$

which suffices to prove the claim. □

Generalizing this result, we prove the needed lemma.

Lemma 6.11. *Suppose* $(W, \vee)$ *has a greatest element* w *and is a sub-join-semilattice of* $(V, \vee)$*. Further, suppose there is an onto homomorphism*

$$h : \big(\mathcal{P}(S) \setminus \{\varnothing\}, \cup\big)/_{\sim} \quad \rightarrow \quad (W, \vee)$$

for some set S*, and that*

1. $h([X_0]_\sim) = u_0^0 \vee v_0^0 = \cdots = u_{n_0}^0 \vee v_{n_0}^0$

$\vdots \qquad \vdots \qquad \vdots$

m. $h([X_m]_\sim) = u_0^m \vee v_0^m = \cdots = u_{n_m}^m \vee v_{n_m}^m,$

for some

$$\big\{[X_0]_\sim, \ldots, [X_m]_\sim\big\} \subseteq \big(\mathcal{P}(S) \setminus \{\varnothing\}, \cup\big)/_\sim$$

and

$$R' := \{u_0^0, v_0^0, \ldots, u_{n_0}^0, v_{n_0}^0, \ldots, u_0^m, v_0^m, \ldots, u_{n_m}^m, v_{n_m}^m\} \subseteq V.$$

Then the following hold.

- *The structure*

$$\big(\big\{x_1 \vee \cdots \vee x_k : k \geqslant 1, \{x_1, ..., x_k\} \subseteq W \cup R'\big\}, \vee\big),$$

  is a sub-join-semilattice of $(V, \vee)$ *with greatest element* w.

- *Define*

$$S' := \{\mathbf{u_0^0}, \mathbf{v_0^0}, \ldots, \mathbf{u_{n_0}^0}, \mathbf{v_{n_0}^0}, \ldots, \mathbf{u_0^m}, \mathbf{v_0^m}, \ldots, \mathbf{u_{n_m}^m}, \mathbf{v_{n_m}^m}\}$$

  assuring all elements are pairwise distinct (this need not be the case for $u_0^0, v_0^0, \ldots, u_{n_m}^m, v_{n_m}^m$*, so essentially we're considering a multiset) and* $S' \cap S = \varnothing$*, and let* $\sim'$ *be the least congruence relation on the join-semilattice* $\big(\mathcal{P}(S \cup S') \setminus \{\varnothing\}, \cup\big)$ *satisfying*

  0. *if* $X \sim Y$ *then* $X \sim' Y$;
  1. $X_0 \sim' \{\mathbf{u_0^0}, \mathbf{v_0^0}\} \sim' \cdots \sim' \{\mathbf{u_{n_0}^0}, \mathbf{v_{n_0}^0}\}$;

  $\vdots \qquad \vdots \qquad \vdots$

  m. $X_m \sim' \{\mathbf{u_0^m}, \mathbf{v_0^m}\} \sim' \cdots \sim' \{\mathbf{u_{n_m}^m}, \mathbf{v_{n_m}^m}\}$.

Then sending

$$[\{\mathbf{u_0^0}\}]_{\sim'} \mapsto u_0^0, [\{\mathbf{v_0^0}\}]_{\sim'} \mapsto v_0^0, \quad \ldots, \quad [\{\mathbf{u_{n_0}^0}\}]_{\sim'} \mapsto u_{n_0}^0, [\{\mathbf{v_{n_0}^0}\}]_{\sim'} \mapsto v_{n_0}^0, \quad \ldots,$$
$$[\{\mathbf{u_0^m}\}]_{\sim'} \mapsto u_0^m, [\{\mathbf{v_0^m}\}]_{\sim'} \mapsto v_0^m, \quad \ldots, \quad [\{\mathbf{u_{n_m}^m}\}]_{\sim'} \mapsto u_{n_m}^m, [\{\mathbf{v_{n_m}^m}\}]_{\sim'} \mapsto v_{n_m}^m,$$

and, for all $\varnothing \neq X \subseteq S$, sending

$$[X]_{\sim'} \mapsto h([X]_{\sim})$$

defines a partial function which canonically extends to an onto homorphism

$$h' : \big(\mathcal{P}(S \cup S') \setminus \{\varnothing\}, \cup\big)/_{\sim'} \quad \to \quad \big(W', \vee\big).$$

Proof. As before, the former of the claims is basically by definition.

For the latter, we use the previous lemma repeatedly. To be precise, first, we list

$$\{\mathbf{u_0^0}, \mathbf{v_0^0}\}, \ldots, \{\mathbf{u_{n_0}^0}, \mathbf{v_{n_0}^0}\}, \ldots, \{\mathbf{u_0^m}, \mathbf{v_0^m}\}, \ldots, \{\mathbf{u_{n_m}^m}, \mathbf{v_{n_m}^m}\}$$

so that

$$\begin{aligned} S \cup S' &= S \cup \big(\{\mathbf{u_0^0}, \mathbf{v_0^0}\} \cup \cdots \cup \{\mathbf{u_{n_0}^0}, \mathbf{v_{n_0}^0}\} \cup \cdots \cup \{\mathbf{u_0^m}, \mathbf{v_0^m}\} \cup \cdots \cup \{\mathbf{u_{n_m}^m}, \mathbf{v_{n_m}^m}\}\big) \\ &= \big(\cdots\big(\cdots\big(\cdots\big(S \cup \{\mathbf{u_0^0}, \mathbf{v_0^0}\}\big) \cup \cdots \cup \{\mathbf{u_{n_0}^0}, \mathbf{v_{n_0}^0}\}\big) \cup \cdots \cup \{\mathbf{u_0^m}, \mathbf{v_0^m}\}\big) \cup \cdots \cup \{\mathbf{u_{n_m}^m}, \mathbf{v_{n_m}^m}\}\big). \end{aligned}$$

Second, iteratively defining congruence relations

$$\sim_0^0 \subseteq \cdots \subseteq \sim_{n_0}^0 \subseteq \cdots \subseteq \sim_0^m \subseteq \cdots \subseteq \sim_{n_m}^m$$

and corresponding target join-semilattices $\big(W_0^0, \vee\big), \ldots, \big(W_{n_m}^m, \vee\big)$ as given in the previous lemma – and using that the corresponding homomorphisms 'extend' each other – that exact lemma gives us an onto homomorphism $h_{n_m}^m$. It is then an easy matter to check that $\sim_{n_m}^m = \sim'$, $\big(W_{n_m}^m, \vee\big) = \big(W', \vee\big)$, and that $h_{n_m}^m$ satisfies the given conditions. □

Using this lemma, we continue by proving soundness.

Theorem 6.12 (Soundness). **$\mathbf{MIL_{Sem}}$** $\subseteq$ *MIL_{Sem}*.

Proof. We show that for any finite, subformula-closed set of formulas $\Phi \subseteq \mathcal{L}_M$, maximal theory $A \in MT(\Phi)$, and $k \in \omega$, it is the case that

$$\pi_k^{\Phi,A} \in MIL_{Sem}.$$

Therefore, suppose $(W, \vee, V) = \mathbb{M}$ is a join-semilattice model, and w is a world such that

$$\mathbb{M}, w \Vdash \hat{A}.$$

Further, let $k \in \omega$ be arbitrary. We prove the claim by constructing a sequence of tuples

$$t_{-1} = (S_{-1}, \sim_{-1}, l_{-1}), t_0 = (S_0, \sim_0, l_0), t_1 = (S_0, \sim_0, l_0), \ldots, t_k = (S_k, \sim_k, l_k)$$

such that for all $0 \leqslant i \leqslant k$

$$t_i \in \mathbf{T}_i^{\Phi, A}(t_{i-1}),$$

and then showing that

$$\mathbb{M}, w \Vdash \alpha_{[S_k]_k}^{t_k}.$$

The sequence of tuples will be constructed using a concurrent construction of onto homomorphisms

$$h_i : \big(\mathcal{P}(S_i) \setminus \{\varnothing\}, \cup\big)/_{\sim_i} \to \big(W_i, \vee\big),$$

where $\big(W_i, \vee\big)_{0 \leqslant i \leqslant k}$ constitutes a chain of sub-join-semilattices of $\big(W, \vee\big)$ all with greatest element w, and such that

$$l_i(Y) = ||h_i([Y]_i)|| \cap (\Phi \cup \neg\Phi)$$

for all $Y \in \mathcal{P}(S_i) \setminus \{\varnothing\}$.

Recall that given any $\mathbb{M}, x$, we denote and abbreviate the truthset of x (w.r.t. $\mathbb{M}$) as $||x|| := \{\varphi \mid \mathbb{M}, x \Vdash \varphi\}$.

Base case. t_{-1} and t_0 are already given, and the unique function

$$h_0 : \big(\mathcal{P}(S_0) \setminus \{\varnothing\}, \cup\big)/_{\sim_0} \to \big(\{w\}, \vee\big)$$

is obviously an onto homomorphism such that

$$l_0(Y) = ||h_0([Y]_0)|| \cap (\Phi \cup \neg\Phi)$$

for all $Y \in \mathcal{P}(S_0) \setminus \{\varnothing\}$.

Recall that, by their very definitions (6.2), $S_0 := \{*\}$ and $l_0(\{*\}) := A$.

Recursive step. Suppose for some $i < k$ that we have constructed

$$t_{-1} = (S_{-1}, \sim_{-1}, l_{-1}), t_0 = (S_0, \sim_0, l_0), t_1 = (S_0, \sim_0, l_0), \ldots, t_i = (S_i, \sim_i, l_i)$$

and a corresponding chain of onto homomorphisms and sub-join-semilattices. By definition, we know that

$$S_{i+1} := S'_{i+1} \cup S_i,$$

where

$$S'_{i+1} := \bigcup_{[Y]_i \subseteq N(t_i)} S^{[Y]_i}_{i+1}{}',$$

and

$$S^{[Y]_i}_{i+1}{}' := \{\boldsymbol{\varphi}^Y_0, \boldsymbol{\varphi}^Y_0{}', \ldots, \boldsymbol{\varphi}^Y_{n_Y}, \boldsymbol{\varphi}^Y_{n_Y}{}'\}$$

arise from the formulas $\chi \in l_i(Y)$ with main connective '$\langle \text{sup} \rangle$' for all $[Y]_i \subseteq N(t_i)$. Enumerating representatives of these equivalence classes as $Y_1, \ldots, Y_m$, we get a list

$$\left\{\boldsymbol{\varphi}^{Y_1}_0, \boldsymbol{\varphi}^{Y_1}_0{}'\right\}, \ldots, \left\{\boldsymbol{\varphi}^{Y_1}_{n_{Y_1}}, \boldsymbol{\varphi}^{Y_1}_{n_{Y_1}}{}'\right\}, \ldots, \left\{\boldsymbol{\varphi}^{Y_m}_0, \boldsymbol{\varphi}^{Y_m}_0{}'\right\}, \ldots, \left\{\boldsymbol{\varphi}^{Y_m}_{n_{Y_m}}, \boldsymbol{\varphi}^{Y_m}_{n_{Y_m}}{}'\right\}$$

such that

$$S_{i+1} = S_i \cup \left(\left\{\boldsymbol{\varphi}^{Y_1}_0, \boldsymbol{\varphi}^{Y_1}_0{}'\right\} \cup \cdots \cup \left\{\boldsymbol{\varphi}^{Y_1}_{n_{Y_1}}, \boldsymbol{\varphi}^{Y_1}_{n_{Y_1}}{}'\right\} \cup \cdots \cup \left\{\boldsymbol{\varphi}^{Y_m}_0, \boldsymbol{\varphi}^{Y_m}_0{}'\right\} \cup \cdots \cup \left\{\boldsymbol{\varphi}^{Y_m}_{n_{Y_m}}, \boldsymbol{\varphi}^{Y_m}_{n_{Y_m}}{}'\right\}\right).$$

Since, by IH,

$$l_i(Y) = \|h_i([Y]_i)\| \cap (\Phi \cup \neg\Phi)$$

for all $Y \in \mathcal{P}(S_i) \setminus \{\varnothing\}$, we, in particular, have that for all $1 \leqslant j \leqslant m$:

$$h_i([Y_j]_i) \Vdash \bigwedge_{a \leqslant n_{Y_j}} \langle \text{sup} \rangle \varphi^{Y_j}_a \varphi^{Y_j}_a{}',$$

hence it must be witnessed by some

$$u^j_a \Vdash \varphi^{Y_j}_a, \qquad v^j_a \Vdash \varphi^{Y_j}_a{}'$$

such that $h_i([Y_j]_i) = u^j_a \vee v^j_a$.

Consequently, cf. the preceding lemma and the IH, these witnessing worlds together with W_i generate a sub-join-semilattice $(W_{i+1}, \vee)$ with greatest element w that is a homomorphic image through

$$h_{i+1} : \big(\mathcal{P}(S_{i+1}) \setminus \{\varnothing\}, \cup\big)/_{\sim_{i+1}} \to \big(W_i, \vee\big),$$

where

$$\left[\left\{\boldsymbol{\varphi}^{Y_j}_a\right\}\right]_{\sim_{i+1}} \mapsto u^j_a, \quad \left[\left\{\boldsymbol{\varphi}^{Y_j}_a{}'\right\}\right]_{\sim_{i+1}} \mapsto v^j_a$$

for all $1 \leqslant j \leqslant m$ and $a \leqslant n_{Y_j}$. Because of this latter fact, and since also (a) h_{i+1} 'extends' h_i, and (b) $(\Phi \cup \neg\Phi)$ is subformula closed (because Φ is), letting

$$l_{i+1} : \big(\mathcal{P}(S_{i+1}) \setminus \{\varnothing\}, \cup\big) \to MT(\Phi)$$

be given by

$$l_{i+1}(Y) = \|h_{i+1}([Y]_{i+1})\| \cap (\Phi \cup \neg\Phi),$$

defines an element $t_{i+1} \in \mathbf{T}^{\Phi,\mathcal{A}}_{i+1}(t_i)$, which finishes the recursive step and, thus, our construction of the sequence.

It therefore remains to show that

$$w \Vdash \alpha^{(S_k,\sim_k,l_k)}_{[S_k]_k}.$$

We do so by induction on the join-semilattice $\big(\mathcal{P}(S_k) \setminus \{\varnothing\}, \cup\big)/_{\sim_k}$ showing that for all $[Y]_k$:

$$h_k([Y]_k) \Vdash \alpha^{(S_k,\sim_k,l_k)}_{[Y]_k}.$$

For atoms $[Y]_k$ of $\big(\mathcal{P}(S_k) \setminus \{\varnothing\}, \cup\big)/_{\sim_k}$, using that

$$l_k(Y) = \|h_k([Y]_k)\| \cap (\Phi \cup \neg\Phi),$$

this amounts to

$$h_k([Y]_k) \Vdash \|h_k([Y]_k)\| \cap (\Phi \cup \neg\Phi),$$

which obviously is the case. So let $[Y]_k$ be arbitrary so that the claim holds for all points $[Y_a]_k <_k [Y]_k$ and $[Y_b]_k <_k [Y]_k$. Again, this is well-defined since $<_k$ is well-founded (because our join-semilattice is finite). Clearly,

$$h_k([Y]_k) \Vdash \widehat{l_k(Y)},$$

while

$$h_k([Y]_k) \Vdash \bigwedge_{([Y_a]_k,[Y_b]_k) \in C_k[Y]_k} \langle \sup \rangle \alpha^{t_k}_{[Y_a]_k} \alpha^{t_k}_{[Y_b]_k} \wedge \bigwedge_{[Y]_k >_k [X]_k} P\alpha^{t_k}_{[X]_k}$$

follows by (1) induction hypothesis, (2) h_k being a join-semilattice homomorphism – hence preserving joins and (weak) order – and (3) $\big(W_k, \bigvee\big)$ being a sub-join-semilattice of $(W, \bigvee)$.

This finalizes the induction; a fortiori, since h_k is onto and $[S_k]_k$ and w are the greatest elements of $\big(\mathcal{P}(S_k) \setminus \{\varnothing\}, \cup\big)$ and $\big(W_k, \bigvee\big)$, respectively, we get

$$w = h_k([S_k]_k) \Vdash \alpha^{(S_k,\sim_k,l_k)}_{[S_k]_k},$$

as desired. □

6.3. Completeness

Having shown soundness, next task is to show completeness. As with soundness, we begin with some preliminary results; the first of which we have split into two lemmas: once again, the latter being what we need, and the former being useful in proving the latter.

Lemma 6.13. *Suppose* $S, S', \{a, b\}$ *are sets such that* $\{a, b\} \cap S = \varnothing, S' = \{a, b\} \cup S$, *and* $\sim$ *is a congruence relation on*

$$\big(\mathcal{P}(S) \setminus \{\varnothing\}, \cup\big).$$

Then mapping

$$[Y]_{\sim} \mapsto [Y]_{\sim'}$$

for all $\varnothing \neq Y \subseteq S$, *defines an embedding*

$$e : \big(\mathcal{P}(S) \setminus \{\varnothing\}, \cup\big)/_{\sim} \to \big(\mathcal{P}(S') \setminus \{\varnothing\}, \cup\big)/_{\sim'},$$

where $\sim'$ *is the least congruence relation satisfying*

1. *if* $X \sim Y$ *then* $X \sim' Y$; *and*
2. $\{a, b\} \sim' Z$,

for some $\varnothing \neq Z \subseteq S$.

Proof. First, note that since $S \subseteq S'$ and $\sim \subseteq \sim'$ (cf. 1.), the map

$$e : [Y]_{\sim} \mapsto [Y]_{\sim'}$$

is well-defined.

Second, to see that e is a homomorphism, observe that for any $[X]_{\sim}, [Y]_{\sim}$, we have that

$$\begin{aligned} e\left([X]_{\sim} \cup_{\sim} [Y]_{\sim}\right) &= e\left([X \cup Y]_{\sim}\right) = [X \cup Y]_{\sim'} = [X]_{\sim'} \cup_{\sim'} [Y]_{\sim'} \\ &= e\left([X]_{\sim}\right) \cup_{\sim'} e\left([Y]_{\sim}\right). \end{aligned}$$

Third, we have to show that e is injective; i.e., the converse of condition 1., namely that

$$\textit{for all } X, Y \in \mathcal{P}(S) \setminus \{\varnothing\}, \textit{ if } X \sim' Y \textit{ then } X \sim Y.$$

To show so, it suffices to find another relation $\sim'' \supseteq \sim'$ for which this holds. Consequently, consider the relation on $\mathcal{P}(S') \setminus \{\varnothing\}$ given by $X \sim'' Y$ **:iff**

Although not needed for the time being, we will even show that $\sim'' = \sim'$, *since this will be of use in the (much) later Lemma 6.23* *either* *(i)* $(X \cap \{a,b\}) = (Y \cap \{a,b\})$, $\big((X \setminus \{a,b\}) \sim (Y \setminus \{a,b\})$ *or* $(X \setminus \{a,b\}) = (Y \setminus \{a,b\}) = \varnothing\big)$

or *(ii)* $\big(((X \setminus \{a,b\}) \cup Z) \sim (X \setminus \{a,b\})$ *or* $X \supseteq \{a,b\}\big)$, $\big(((Y \setminus \{a,b\}) \cup Z) \sim (Y \setminus \{a,b\})$ *or* $Y \supseteq \{a,b\}\big)$, $((X \setminus \{a,b\}) \cup Z) \sim ((Y \setminus \{a,b\}) \cup Z)$.

for all $\{X, Y\} \subseteq \mathcal{P}(S') \setminus \{\varnothing\}$.

Observe that the 'or' in (i) is exclusive.

We show that $\sim''$ extends $\sim'$. First, observe that *(i)* implies condition *1.* in the definition of $\sim'$, and *(ii)* implies condition *2.* in the definition of $\sim'$. Thus, for showing $\sim'' \supseteq \sim'$, it is enough to show that $\sim''$ is a congruence relation. *Reflexivity* is an easy consequence of *(i)* and $\sim$ being reflexive, and *symmetry* is an easy consequence of $\sim$ being symmetric. For *transitivity*, suppose

$$X \sim'' Y \sim'' W.$$

If $X \overset{(i)}{\sim''} Y \overset{(i)}{\sim''} W$ (that is, $X \sim'' Y \sim'' W$ by virtue of fulfilling condition *(i)*), then $X \overset{(i)}{\sim''} W$ follows by transitivity of $\sim$, so suppose instead that $X \overset{(ii)}{\sim''} Y \overset{(ii)}{\sim''} W$. Then

$$((X \setminus \{a,b\}) \cup Z) \sim ((Y \setminus \{a,b\}) \cup Z) \sim ((W \setminus \{a,b\}) \cup Z),$$

hence $X \overset{(ii)}{\sim''} W$ follows by transitivity of $\sim$. Now suppose $X \overset{(i)}{\sim''} Y \overset{(ii)}{\sim''} W$. Then $Y \overset{(ii)}{\sim''} W$ implies that either (1) $Y \supseteq \{a,b\}$ or (2) $((Y \setminus \{a,b\}) \cup Z) \sim (Y \setminus \{a,b\})$. In case of (1), also $X \supseteq \{a,b\}$ because $X \overset{(i)}{\sim''} Y$. If even $Y = \{a,b\}$, we have $X = Y$, hence

$$((X \setminus \{a,b\}) \cup Z) = ((Y \setminus \{a,b\}) \cup Z) \sim ((W \setminus \{a,b\}) \cup Z),$$

so $X \overset{(ii)}{\sim''} W$. And if $(X \setminus \{a,b\}) \sim (Y \setminus \{a,b\})$, then since $Z \sim Z$, we get

$$((X \setminus \{a,b\}) \cup Z) \sim ((Y \setminus \{a,b\}) \cup Z) \sim ((W \setminus \{a,b\}) \cup Z)$$

by $\sim$ being a congruence relation, hence, in particular, compatible with '$\cup$'. Thus, $X \overset{(ii)}{\sim''} W$. In case of (2), we find that

$$((X \setminus \{a,b\}) \cup Z) \sim ((Y \setminus \{a,b\}) \cup Z) \sim (Y \setminus \{a,b\}) \sim (X \setminus \{a,b\}).$$

Thus, $((X \setminus \{a,b\}) \cup Z) \sim (X \setminus \{a,b\})$. Moreover, transitivity and $((Y \setminus \{a,b\}) \cup Z) \sim ((W \setminus \{a,b\}) \cup Z)$ entail that

$$((X \setminus \{a,b\}) \cup Z) \sim ((W \setminus \{a,b\}) \cup Z),$$

showing $X \overset{(ii)}{\sim''} W$. Finally, for the remaining case, $X \overset{(ii)}{\sim''} Y \overset{(i)}{\sim''} W$, we use symmetry to get that $W \overset{(i)}{\sim''} Y \overset{(ii)}{\sim''} X$, for which the previous proof shows $W \overset{(ii)}{\sim''} X$, so by symmetry $X \overset{(ii)}{\sim''} W$. Thus, we've shown $\sim''$ to be transitive.

Lastly, we have to show that $\sim''$ is *compatible* with taking unions. Accordingly, suppose

$$X_1 \sim'' Y_1, \qquad X_2 \sim'' Y_2.$$

Then we are to show $(X_1 \cup X_2) \sim'' (Y_1 \cup Y_2)$. If $X_1 \overset{(i)}{\sim''} Y_1$ and $X_2 \overset{(i)}{\sim''} Y_2$, then using that

$$(X_1 \setminus \{a, b\}) \sim (Y_1 \setminus \{a, b\}), (X_2 \setminus \{a, b\}) \sim (Y_2 \setminus \{a, b\})$$

entail

$$\begin{aligned}((X_1 \cup X_2) \setminus \{a, b\}) &= (X_1 \setminus \{a, b\}) \cup (X_2 \setminus \{a, b\}) \\ &\sim (Y_1 \setminus \{a, b\}) \cup (Y_2 \setminus \{a, b\}) = ((Y_1 \cup Y_2) \setminus \{a, b\})\end{aligned}$$

by '$\cup$'-compatibility of $\sim$, we find that $(X_1 \cup X_2) \overset{(i)}{\sim''} (Y_1 \cup Y_2)$ follows easily. And if $X_1 \overset{(ii)}{\sim''} Y_1$ and $X_2 \overset{(ii)}{\sim''} Y_2$, then to see that $(X_1 \cup X_2) \overset{(ii)}{\sim''} (Y_1 \cup Y_2)$, simply observe that (1)

$$\begin{aligned}(((X_1 \cup X_2) \setminus \{a, b\}) \cup Z) &= \big(((X_1 \setminus \{a, b\}) \cup Z) \cup ((X_2 \setminus \{a, b\}) \cup Z)\big) \\ &\sim \big(((Y_1 \setminus \{a, b\}) \cup Z) \cup ((Y_2 \setminus \{a, b\}) \cup Z)\big) \\ &= (((Y_1 \cup Y_2) \setminus \{a, b\}) \cup Z),\end{aligned}$$

and (2a) $(X_i \supseteq \{a, b\} \Rightarrow (X_1 \cup X_2) \supseteq \{a, b\})$, while (2b)

$$((X_1 \setminus \{a, b\}) \cup Z) \sim (X_1 \setminus \{a, b\}), ((X_2 \setminus \{a, b\}) \cup Z) \sim (X_2 \setminus \{a, b\})$$

entail

$$\begin{aligned}(((X_1 \cup X_2) \setminus \{a, b\}) \cup Z) &= ((X_1 \setminus \{a, b\}) \cup Z) \cup ((X_2 \setminus \{a, b\}) \cup Z) \\ &\sim (X_1 \setminus \{a, b\}) \cup (X_2 \setminus \{a, b\}) \sim ((X_1 \cup X_2) \setminus \{a, b\}).\end{aligned}$$

And, analogously, for Y_1, Y_2, the arguments of (2a) and (2b) go through, hence we get $(X_1 \cup X_2) \overset{(ii)}{\sim''} (Y_1 \cup Y_2)$. Therefore, suppose $X_1 \overset{(i)}{\sim''} Y_1$ and $X_2 \overset{(ii)}{\sim''} Y_2$. Then

$$\begin{aligned}\big(((X_1 \cup X_2) \setminus \{a, b\}) \cup Z\big) &= \big((X_1 \setminus \{a, b\}) \cup ((X_2 \setminus \{a, b\}) \cup Z)\big) \\ &\sim \big((Y_1 \setminus \{a, b\}) \cup ((Y_2 \setminus \{a, b\}) \cup Z)\big) = \big(((Y_1 \cup Y_2) \setminus \{a, b\}) \cup Z\big)\end{aligned}$$

because (1)

$$((X_2 \setminus \{a, b\}) \cup Z) \sim ((Y_2 \setminus \{a, b\}) \cup Z),$$

and (2)

$$\textit{either}\ \ (X_1 \setminus \{a, b\}) \sim (Y_1 \setminus \{a, b\}) \qquad \textit{or}\ \ (X_1 \setminus \{a, b\}) = (Y_1 \setminus \{a, b\}) = \varnothing.$$

Thus, if $(X_1 \cup X_2) \supseteq \{a, b\}$ and $(Y_1 \cup Y_2) \supseteq \{a, b\}$, we're done. So suppose $(X_1 \cup X_2) \not\supseteq \{a, b\}$. Then

$$((X_2 \setminus \{a, b\}) \cup Z) \sim (X_2 \setminus \{a, b\}),$$

hence

$$\begin{aligned}(((X_1 \cup X_2) \setminus \{a, b\}) \cup Z) &= (X_1 \setminus \{a, b\}) \ \cup\ ((X_2 \setminus \{a, b\}) \cup Z) \\ &\sim\ (X_1 \setminus \{a, b\}) \ \cup\ (X_2 \setminus \{a, b\}) = ((X_1 \cup X_2) \setminus \{a, b\}).\end{aligned}$$

The case $\{Y_1, Y_2\} \not\supseteq \{a, b\}$ is analogous; thus, $(X_1 \cup X_2) \overset{\textit{(ii)}}{\sim''} (Y_1 \cup Y_2)$. Finally, the remaining case, $X_1 \overset{\textit{(ii)}}{\sim''} Y_1$ and $X_2 \overset{\textit{(i)}}{\sim''} Y_2$, is covered by the exact same reasoning using commutativity of '$\cup$'.

Summarizing, this shows that $\sim'' \supseteq \sim'$, so to show injectivity of e, it suffices to show that

$$\textit{for all } X, Y \in \mathcal{P}(S) \setminus \{\varnothing\},\ \textit{if } X \sim'' Y \textit{ then } X \sim Y.$$

Accordingly, let $X, Y \in \mathcal{P}(S) \setminus \{\varnothing\}$ be arbitrary such that $X \sim'' Y$. Notice that then $(X \cap \{a, b\}) = \varnothing = (Y \cap \{a, b\})$. So if $X \overset{\textit{(i)}}{\sim''} Y$, we have both $(X \cap \{a, b\}) = (Y \cap \{a, b\})$ and – since $X \neq \varnothing \neq Y$ –

$$X = (X \setminus \{a, b\}) \sim (Y \setminus \{a, b\}) = Y$$

as required.

And if $X \overset{\textit{(ii)}}{\sim''} Y$, we have

$$X = (X \setminus \{a, b\}) \sim ((X \setminus \{a, b\}) \cup Z) \sim ((Y \setminus \{a, b\}) \cup Z) \sim ((Y \setminus \{a, b\}) = Y,$$

completing our proof of injectivity, hence completing our proof of the lemma.

As stated, even though it is of no concrete use for the time being, we will show that the converse inclusion holds as well; that is, $\sim'' \subseteq \sim'$. We show this because not only does an unraveled characterization of $\sim'$ as $\sim''$ provide some intuition, but, more importantly, it will also be of explicit reference in the proof of Lemma 6.23.

Consequently, first, suppose $X \overset{(i)}{\sim''} Y$. Then $(X \cap \{a, b\}) = (Y \cap \{a, b\})$. So if $(X \setminus \{a, b\}) = (Y \setminus \{a, b\}) = \varnothing$, we must have $X = Y$, hence $X \sim' Y$ by reflexivity. And if $(X \setminus \{a, b\}) \sim (Y \setminus \{a, b\})$, we get by compatibility with unions and $\sim' \supseteq \sim$ that

$$\begin{aligned} X = (X \setminus \{a, b\}) \cup (X \cap \{a, b\}) = (X \setminus \{a, b\}) \cup (Y \cap \{a, b\}) \\ \sim' (Y \setminus \{a, b\}) \cup (Y \cap \{a, b\}) = Y. \end{aligned}$$

Second, suppose $X \overset{(ii)}{\sim''} Y$. If (1) $X \supseteq \{a, b\}$, then

$$\begin{aligned} X = X \cup \{a, b\} \sim' X \cup Z = (X \setminus \{a, b\}) \cup Z \cup \{a, b\} \\ \sim' (X \setminus \{a, b\}) \cup Z \cup Z = (X \setminus \{a, b\}) \cup Z \end{aligned}$$

by repeated use of $\{a, b\} \sim' Z$ and compatibility with unions. And if (2) $((X \setminus \{a, b\}) \cup Z) \sim (X \setminus \{a, b\})$, then, using that

$$Z = Z \cup Z \sim' Z \cup \{a, b\} \supseteq Z \cup (X \cap \{a, b\}),$$

so

$$Z \sim' Z \cup (Z \cup (X \cap \{a, b\})) = Z \cup (X \cap \{a, b\}),$$

we get

$$\begin{aligned} X = (X \setminus \{a, b\}) \cup (X \cap \{a, b\}) \sim' ((X \setminus \{a, b\}) \cup Z) \cup (X \cap \{a, b\}) \\ = (X \setminus \{a, b\}) \cup (Z \cup (X \cap \{a, b\})) \sim' (X \setminus \{a, b\}) \cup Z. \end{aligned}$$

I.e., in either case we get that

$$X \sim' (X \setminus \{a, b\}) \cup Z.$$

Analogously, we get

$$Y \sim' (Y \setminus \{a, b\}) \cup Z,$$

whence

$$X \sim' (X \setminus \{a, b\}) \cup Z \sim (Y \setminus \{a, b\}) \cup Z \sim' Y,$$

so $X \sim' Y$ as required, finishing our proof of $\sim'' \subseteq \sim'$ and, therefore, allowing us to conclude $\sim'' = \sim'$. □

Employing this lemma, we prove the following:

Lemma 6.14. *Let $\Phi \subseteq \mathcal{L}_M$ be a finite, subformula-closed set of formulas and A a maximal theory over Φ. Then for any $n \in \omega, t_n = (S_n, \sim_n, l_n) \in \mathbf{T}_n^{\Phi,A}$ and $t_{n+1} = (S_{n+1}, \sim_{n+1}, l_{n+1}) \in \mathbf{T}_{n+1}^{\Phi,A}(t_n)$, mapping*

$$[Y]_{\sim_n} \mapsto [Y]_{\sim_{n+1}}$$

defines an embedding

$$e : \big(\mathcal{P}(S_n) \setminus \{\varnothing\}, \cup\big)/_{\sim_n} \to \big(\mathcal{P}(S_{n+1}) \setminus \{\varnothing\}, \cup\big)/_{\sim_{n+1}}.$$

I.e., this lemma formalizes and proves how the generated join-semilattices extend each other (cf. section 6.1).

Proof. As in the soundness proof, we know that

$$S_{n+1} := S'_{n+1} \cup S_n,$$

where

$$S'_{n+1} := \bigcup_{[Y]_n \subseteq N(t_n)} S_{n+1}^{[Y]_n\,\prime},$$

and

$$S_{n+1}^{[Y]_n\,\prime} := \{\boldsymbol{\varphi}_0^Y, \boldsymbol{\varphi}_0^{Y\prime}, \dots, \boldsymbol{\varphi}_{n_Y}^Y, \boldsymbol{\varphi}_{n_Y}^{Y}{}'\}$$

arise from the formulas $\chi \in l_n(Y)$ with main connective '$\langle \sup \rangle$' for all $[Y]_n \subseteq N(t_n)$. Therefore, we can enumerate representatives of these equivalence classes as $Y_1, \dots, Y_m$ and using each of their corresponding enumerations of pairs of formulae $\{\varphi_a^{Y_j}, \varphi_a^{Y_j}{}'\}$, we get a list

$$\Big\{\boldsymbol{\varphi}_0^{Y_1}, \boldsymbol{\varphi}_0^{Y_1\prime}\Big\}, \dots, \Big\{\boldsymbol{\varphi}_{n_{Y_1}}^{Y_1}, \boldsymbol{\varphi}_{n_{Y_1}}^{Y_1}{}'\Big\}, \dots, \Big\{\boldsymbol{\varphi}_0^{Y_m}, \boldsymbol{\varphi}_0^{Y_m\prime}\Big\}, \dots, \Big\{\boldsymbol{\varphi}_{n_{Y_m}}^{Y_m}, \boldsymbol{\varphi}_{n_{Y_m}}^{Y_m}{}'\Big\}$$

such that

$$\begin{aligned} S_{n+1} &= S_n \cup \Big(\Big\{\boldsymbol{\varphi}_0^{Y_1}, \boldsymbol{\varphi}_0^{Y_1\prime}\Big\} \cup \dots \cup \Big\{\boldsymbol{\varphi}_{n_{Y_1}}^{Y_1}, \boldsymbol{\varphi}_{n_{Y_1}}^{Y_1}{}'\Big\} \cup \dots \cup \Big\{\boldsymbol{\varphi}_0^{Y_m}, \boldsymbol{\varphi}_0^{Y_m\prime}\Big\} \cup \dots \cup \Big\{\boldsymbol{\varphi}_{n_{Y_m}}^{Y_m}, \boldsymbol{\varphi}_{n_{Y_m}}^{Y_m}{}'\Big\}\Big) \\ &= \Bigg(\cdots\Big(\cdots\Big(\cdots\Big(S_n \cup \Big\{\boldsymbol{\varphi}_0^{Y_1}, \boldsymbol{\varphi}_0^{Y_1\prime}\Big\}\Big) \cup \dots \cup \Big\{\boldsymbol{\varphi}_{n_{Y_1}}^{Y_1}, \boldsymbol{\varphi}_{n_{Y_1}}^{Y_1}{}'\Big\}\Big) \\ &\quad \cup \dots \cup \Big\{\boldsymbol{\varphi}_0^{Y_m}, \boldsymbol{\varphi}_0^{Y_m\prime}\Big\}\Big) \cup \dots \cup \Big\{\boldsymbol{\varphi}_{n_{Y_m}}^{Y_m}, \boldsymbol{\varphi}_{n_{Y_m}}^{Y_m}{}'\Big\}\Bigg) \end{aligned}$$

Iteratively defining corresponding congruence relations

$$\sim_0^{Y_1} \subseteq \dots \subseteq \sim_{n_{Y_1}}^{Y_1} \subseteq \dots \subseteq \sim_0^{Y_m} \subseteq \dots \subseteq \sim_{n_{Y_m}}^{Y_m}$$

as given in the previous lemma, and using that embeddings compose, that exact lemma gives us that

$$[Y]_{\sim_n} \mapsto [Y]_{\sim^{Y_m}_{n_{Y_m}}}$$

is an embedding from

$$\big(\mathcal{P}(S_n) \setminus \{\varnothing\}, \cup\big)/_{\sim_n}$$

to

$$\big(\mathcal{P}(S_{n+1}) \setminus \{\varnothing\}, \cup\big)/_{\sim^{Y_m}_{n_{Y_m}}}.$$

Moreover, an easy induction shows that

$$\sim^{Y_m}_{n_{Y_m}} = \sim_{n+1},$$

which completes the proof. □

In addition to the preceding lemma, we will also need the following:

Lemma 6.15. *Let* $\mathbf{T}^{\Phi,A}$, $t_n = (S_n, \sim_n, l_n) \in \mathbf{T}^{\Phi,A}_n$ *and* $t_{n+1} = (S_{n+1}, \sim_{n+1}, l_{n+1}) \in \mathbf{T}^{\Phi,A}_{n+1}(t_n)$ *be arbitrary. Then for any* $\varnothing \neq Y \subseteq S_n$, *it is the case that*

$$\alpha^{t_{n+1}}_{[Y]_{n+1}} \to \alpha^{t_n}_{[Y]_n} \in \mathbf{MIL_{Sem}}.$$

Note how this shows in what way the axioms 'extend' each other.

Proof. We prove the claim by induction on the join-semilattice $\big(\mathcal{P}(S_n), \bigvee\big)/_{\sim_n}$. If $[Y]_n$ is an atom, then

$$\alpha^{t_n}_{[Y]_n} = \widehat{l_n(Y)},$$

so since $l_{n+1} \supseteq l_n$, it follows by uniform substitution on a propositional tautology that

$$\alpha^{t_{n+1}}_{[Y]_{n+1}} \to \alpha^{t_n}_{[Y]_n} \in \mathbf{MIL_{Sem}}.$$

Now suppose it holds for all $[Y_a]_n <_n [Y]_n$ and $[Y_b]_n <_n [Y]_n$. By the preceding lemma,

$$[X]_n \mapsto [X]_{n+1}$$

defines an embedding, hence

$$([Y_a]_n, [Y_b]_n) \in C_n[Y]_n \quad \Rightarrow \quad ([Y_a]_{n+1}, [Y_b]_{n+1}) \in C_{n+1}[Y]_{n+1}$$

and

$$[Y]_n >_n [X]_n \quad \Rightarrow \quad [Y]_{n+1} >_{n+1} [X]_{n+1},$$

from which it easily follows by (1) IH, (2) $l_{n+1} \supseteq l_n$, and (3) normality of $\mathbf{MIL_{Sem}}$ that

$$\alpha^{t_{n+1}}_{[Y]_{n+1}} \to \alpha^{t_n}_{[Y]_n} \in \mathbf{MIL_{Sem}},$$

exactly as required. ∎

With these results at hand, we embark on the actual completeness proof.

Theorem 6.16 ((Weak) Completeness)**.** $\mathbf{MIL_{Sem}} \supseteq MIL_{Sem}$.

Proof. Suppose $\varphi \in \mathcal{L}_M$ is $\mathbf{MIL_{Sem}}$-consistent, and let Φ be the least subformula-closed set of formulas containing φ. Extend $\{\varphi\}$ to a $\mathbf{MIL_{Sem}}$-MCS Γ_0, and let

$$X_0 := \Gamma_0 \cap (\Phi \cup \neg\Phi).$$

Then Φ is finite, and X_0 is a maximal theory over Φ containing φ. We show that X_0 is satisfiable on a join-semilattice model. We construct this model by constructing a tree in which every node is a join-semilattice embedding into all of its successor nodes. The tree will be infinite, but finitely branching, hence König's Lemma asserts the existence of an infinite branch, and taking the colimit of the models on the infinite branch then gives us our join-semilattice model.

Since Φ is finite, and X_0 is a maximal theory over Φ, we get a tree

$$\mathbf{T}^{\Phi,X_0} := \bigcup_{n\in\omega} \mathbf{T}_n^{\Phi,X_0}.$$

While this is a finitely branching, infinite tree, it is not quite what we seek: it is too 'big' in the sense of containing 'incoherent' nodes or, more precisely, 'incoherent' generated subtrees. To explicate, in order to show that X_0 is satisfiable, we will need a truth lemma, which – loosely speaking – asserts that for any node t_n at layer n the truth lemma is (partially) satisfied up to 'depth' n. To ensure this, we have to get rid of some nodes and we, therefore, cut the tree by restricting the layers as follows:

$$\mathbf{T}_n^{\Phi,X_0\prime} := \left\{(S_n, \sim_n, l_n) \in \mathbf{T}_n^{\Phi,X_0} \mid \alpha_{[S_n]_n}^{(S_n,\sim_n,l_n)} \in \Gamma_0\right\}$$

This is the way in which we label all points with MCSs by evaluating the formulas at the top MCS, Γ_0, cf. section 6.1.

for all $n \in \omega$. To see that this, indeed, defines a subtree $\mathbf{T}^{\Phi,X_0\prime} \subseteq \mathbf{T}^{\Phi,X_0}$, it suffices to show that if $\mathbf{T}_{n+1}^{\Phi,X_0\prime} \ni t_{n+1} \in \mathbf{T}_{n+1}^{\Phi,X_0}(t_n)$ then $t_n \in \mathbf{T}_n^{\Phi,X_0\prime}$. However, this follows by (1) closure under modus ponens of MCSs; (2) the preceding Lemma 6.15 which, in particular, implies

$$\alpha_{[S_n]_{n+1}}^{t_{n+1}} \to \alpha_{[S_n]_n}^{t_n} \in \mathbf{MIL_{Sem}} \subseteq \Gamma_0;$$

and (3) the easily provable fact that $[S_n]_{n+1} = [S_{n+1}]_{n+1}$.

Moreover, $\mathbf{T}^{\Phi,X_0\prime}$ is clearly finitely branching (because $\mathbf{T}^{\Phi,X_0}$ is), but also infinite since

$$\Pi^{\Phi,X_0} \subseteq \mathbf{MIL_{Sem}} \subseteq \Gamma_0, \qquad X_0 \subseteq \Gamma_0,$$

and MCSs enjoy the disjunction (and converse conjunction) property and are closed under modus ponens. I.e., this entails that for each $n \in \omega$, there is a $t_n \in \mathbf{T}^{\Phi,X_0}_n{}'$, so $|\mathbf{T}^{\Phi,X_0}_n{}'| \geqslant \aleph_0$.

Thus, by König's Lemma, there must be an infinite branch

$$t_0 = (S_0, \sim_0, l_0), \dots, t_n = (S_n, \sim_n, l_n), \dots,$$

for which we have that $t_{n+1} \in \mathbf{T}^{\Phi,X_0}_{n+1}(t_n)$ and $\alpha^{(S_n,\sim_n,l_n)}_{[S_n]_n} \in \Gamma_0$ for all $n \in \omega$. Our satisfying join-semilattice will be the colimit of the join-semilattices corresponding to each t_n and the embeddings between them.

To explicate how the colimit is constructed, first note that, cf. Lemma 6.14, for each $m \leqslant n$, we have an embedding

$$e_{m,n} : \big(\mathcal{P}(S_m) \setminus \{\varnothing\}\big)/_{\sim_m} \to \big(\mathcal{P}(S_n) \setminus \{\varnothing\}\big)/_{\sim_n}$$

given by

$$e_{m,n}([Y]_m) = [Y]_n$$

I.e., we tacitly use that embeddings compose, and for $m = n$, $e_{m,n}$ becomes the identity.

for all $[Y]_m \in \big(\mathcal{P}(S_m) \setminus \{\varnothing\}\big)/_{\sim_m}$. Thus, we can take the disjoint union

$$W' := \bigsqcup_{n \in \omega} \big(\mathcal{P}(S_n) \setminus \{\varnothing\}\big)/_{\sim_n} = \left\{([Y]_n, n) \mid n \in \omega, [Y]_n \in \big(\mathcal{P}(S_n) \setminus \{\varnothing\}, \cup\big)/_{\sim_n}\right\},$$

quotient out under the equivalence relation

$$([X]_m, m) \sim ([Y]_n, n) \qquad \textbf{iff} \qquad e_{m,n}([X]_m) = [Y]_n \text{ or } e_{n,m}([Y]_n) = [X]_m,$$

write $\big[[Y]_n, n\big]$ for an equivalence class $\big[([Y]_n, n)\big]_\sim$, and let

$$W := W'/\sim$$

be the underlying set of a join-semilattice $(W, \vee)$ for which

$$\big[[X]_k, k\big] = \big[[Y]_m, m\big] \vee \big[[Z]_n, n\big] \qquad \textbf{:iff} \qquad [X]_p = [Y]_p \cup_{\sim_p} [Z]_p, \text{ where } p = \max\{k, m, n\}.$$

It is then (tedious but) straightforward to check that (1) this object $(W, \vee)$ is, indeed, a join-semilattice; (2) it – together with the maps

$$e^W_n : \big(\mathcal{P}(S_n) \setminus \{\varnothing\}\big)/_{\sim_n} \to (W, \vee)$$

given by sending $[Y]_n \mapsto \big[[Y]_n, n\big]$ – is the colimit of the chain; and (3) e^W_n even are embeddings.

While this defines our join-semilattice frame, we still need to prove three preparatory claims before defining the valuation and subsequently proving a truth lemma.

Claim (a). Suppose $\langle \sup \rangle \psi\psi' \in l_n(Y)$ *for some* t_n*. Then there exist* $[Y_a]_{n+1}, [Y_b]_{n+1}$ *such that*

$$[Y]_{n+1} = [Y_a]_{n+1} \cup_{\sim_{n+1}} [Y_b]_{n+1}$$

and $\psi \in l_{n+1}(Y_a), \psi' \in l_{n+1}(Y_b)$*.*

Proof of *claim (a)*. We first prove the claim for the subcase where $Y \in N(t_n)$. In those cases, we have that

$$\{\psi, \psi'\} \subseteq S_{n+1}^{[Y]_n}{}',$$

so, by definition of $\sim_{n+1}$,

$$\{\psi\} \cup \{\psi'\} = \{\psi, \psi'\} \sim_{n+1} Y.$$

Thus,

$$[Y]_{n+1} = [\{\psi\}]_{n+1} \cup_{\sim_{n+1}} [\{\psi'\}]_{n+1},$$

and since $l_{n+1} \in L_{n+1}(t_n)$, it must satisfy

$$\psi \in l_{n+1}(\{\psi\}), \quad \psi' \in l_{n+1}(\{\psi'\})$$

as required.

Using this, we prove the general claim. By Lemma 6.5, we know that there are $X \in \mathcal{P}(S_n) \setminus \{\varnothing\}$ and $m \leqslant n$ such that $Y \sim_n X$ and $X \in N(t_m)$. So there are $[Y_a]_{m+1}, [Y_b]_{m+1}$ such that

$$[X]_{m+1} = [Y_a]_{m+1} \cup_{\sim_{m+1}} [Y_b]_{m+1}$$

and $\psi \in l_{m+1}(Y_a), \psi' \in l_{m+1}(Y_b)$. But then

$$\begin{aligned} [Y]_{n+1} &= e_{m+1,n+1}([X]_{m+1}) = e_{m+1,n+1}([Y_a]_{m+1} \cup_{\sim_{m+1}} [Y_b]_{m+1}) \\ &= e_{m+1,n+1}([Y_a]_{m+1}) \cup_{\sim_{n+1}} e_{m+1,n+1}([Y_b]_{m+1}) = [Y_a]_{n+1} \cup_{\sim_{n+1}} [Y_b]_{n+1}, \end{aligned}$$

and since $l_{n+1} \supseteq l_{m+1}$, this proves the claim. $\square_{\textit{Claim (a)}}$

Claim (b). For all $n \in \omega$ *and* $\varnothing \neq Y \subseteq S_n$*, there is an MCS* Γ_Y *such that*

$$\Gamma_Y \ni \alpha^{t_n}_{[Y]_n}.$$

Hence, in particular, $\Gamma_Y \supseteq \mathfrak{l}_n(Y)$.

Proof of *claim (b)*. To prove this, we go by cases. If $[Y]_n = [S_n]_n$ then $\Gamma_0 \ni \alpha^{t_n}_{[S_n]_n} = \alpha^{t_n}_{[Y]_n}$.

And if $[Y]_n \neq [S_n]_n$ then $[Y]_n <_n [S_n]_n$, hence

$$P\alpha^{t_n}_{[Y]_n} \in_\wedge \alpha^{t_n}_{[S_n]_n} \in \Gamma_0,$$

where the occurrence of '$\in_\wedge$' in '$P\alpha^{t_n}_{[Y]_n} \in_\wedge \alpha^{t_n}_{[S_n]_n}$' denotes that '$P\alpha^{t_n}_{[Y]_n}$' occurs as a conjunct of a conjunction '$\bigwedge$' occurring as a main connective in '$\alpha^{t_n}_{[S_n]_n}$'.

Recall the notation of Example 6.9.

Thus, $P\alpha^{t_n}_{[Y]_n} \in \Gamma_0$, so, by the existence lemma for the canonical $\mathbf{MIL_{Sem}}$-frame, there is some $\Gamma_Y \ni \alpha^{t_n}_{[Y]_n}$.

Lastly, the last part of the claim follows from $\mathfrak{l}_n(Y) \subseteq_\wedge \alpha^{t_n}_{[Y]_n} \in \Gamma_Y$. $\square_{\textit{Claim (b)}}$

Claim (c). Suppose

$$\big[[X]_k, k\big] = \big[[Y]_m, m\big] \vee \big[[Z]_n, n\big].$$

That is,

$$[X]_p = [Y]_p \cup_{\sim_p} [Z]_p, \textit{ where } p = \max\{k, m, n\}.$$

Then there are MCSs $\Gamma_X \supseteq \mathfrak{l}_p(X), \Gamma_Y \supseteq \mathfrak{l}_p(Y), \Gamma_Z \supseteq \mathfrak{l}_p(Z)$ *such that*

$$C_{\mathbf{Sem}}\Gamma_X\Gamma_Y\Gamma_Z,$$

where $C_{\mathbf{Sem}}$ *denotes the ternary relation of the canonical* $\mathbf{MIL_{Sem}}$*-frame.*

Proof of *claim (c)*. We prove the claim going by cases.

- If $[X]_p = [Y]_p = [Z]_p$, then, by claim (b), there is $\Gamma_X \supseteq \mathfrak{l}_p(X)$, so since (Re.) is Sahlqvist hence canonical, the claim follows.
- If $[X]_p = [Y]_p \neq [Z]_p$, then $P\alpha^{t_p}_{[Z]_p} \in_\wedge \alpha^{t_p}_{[X]_p} \in \Gamma_X$, so it follows by the existence lemma and canonicity of (Dk.).
- If $[X]_p = [Z]_p \neq [Y]_p$, then same as before and using canonicity of (Co.).
- If $[Y]_p \neq [X]_p \neq [Z]_p$, then $\langle \sup \rangle \alpha^{t_p}_{[Y]_p} \alpha^{t_p}_{[Z]_p} \in_\wedge \alpha^{t_p}_{[X]_p} \in \Gamma_X$, so, by the existence lemma, there are Γ_Y, Γ_Z such that $C_{\mathbf{Sem}}\Gamma_X\Gamma_Y\Gamma_Z$ and $\mathfrak{l}_p(Y) \subseteq_\wedge \alpha^{t_p}_{[Y]_p} \in \Gamma_Y, \mathfrak{l}_p(Z) \subseteq_\wedge \alpha^{t_p}_{[Z]_p} \in \Gamma_Z$. $\square_{\textit{Claim (c)}}$

With these claims proven and our satisfying join-semilattice frame defined, we continue by defining the satisfying valuation:

$$V(p) := \left\{ \big[[X]_k, k\big] \in W : p \in \mathfrak{l}_k(X) \right\}.$$

Note that this is well-defined because (a) $l_n \supseteq l_m$ for all $n \geqslant m$ and (b) $X \sim_n X'$ implies $l_n(X) = l_n(X')$. We then wish to show that for all $\psi \in (\Phi \cup \neg\Phi)$ and all $\big[[X]_k, k\big] \in W$:

$$\big(W, \vee, V\big), \big[[X]_k, k\big] \Vdash \psi \qquad \textbf{iff} \qquad \psi \in l_k(X).$$

This is our truth lemma.

The proof goes by induction on the complexity of formulas. Base case is by definition and Boolean cases are straightforward using properties of MCSs (enabled by claim (b)) and having in mind that $(\Phi \cup \neg\Phi)$ is subformula closed because Φ is.

For the $\langle\sup\rangle$-case, we have

$$\begin{aligned}
\big[[X]_k, k\big] \Vdash \langle\sup\rangle\psi_1\psi_2 \quad & \textbf{iff} \quad \exists\big[[Y]_m, m\big], \big[[Z]_n, n\big] \textit{ s.t. } [X]_p = [Y]_p \cup_{\sim_p} [Z]_p, \textit{ where } p = \max\{k, m, n\}, \\
& \qquad \big[[Y]_m, m\big] \Vdash \psi_1, \textit{ and } \big[[Z]_n, n\big] \Vdash \psi_2 \\
& \overset{(IH)}{\textbf{iff}} \quad \exists\big[[Y]_m, m\big], \big[[Z]_n, n\big] \textit{ s.t. } [X]_p = [Y]_p \cup_{\sim_p} [Z]_p, \textit{ where } p = \max\{k, m, n\}, \\
& \qquad \psi_1 \in l_m(Y) \textit{ and } \psi_2 \in l_n(Z) \\
& \textbf{iff} \quad \exists\big[[Y]_m, m\big], \big[[Z]_n, n\big] \textit{ s.t. } [X]_p = [Y]_p \cup_{\sim_p} [Z]_p, \textit{ where } p = \max\{k, m, n\}, \\
& \qquad \psi_1 \in l_p(Y) \textit{ and } \psi_2 \in l_p(Z) \\
& \overset{(i)}{\textbf{iff}} \quad \exists k' \in \omega : \langle\sup\rangle\psi_1\psi_2 \in l_{k'}(X) \\
& \textbf{iff} \quad \langle\sup\rangle\psi_1\psi_2 \in l_k(X),
\end{aligned}$$

where we in the left-to-right direction of *(i)* use that – cf. claim (c) – there are MCSs $\Gamma_X \supseteq l_p(X), \Gamma_Y \supseteq l_p(Y), \Gamma_Z \supseteq l_p(Z)$ such that $C_{\mathbf{Sem}}\Gamma_X\Gamma_Y\Gamma_Z$, hence $\psi_1 \in l_p(Y), \psi_2 \in l_p(Z)$ imply $\langle\sup\rangle\psi_1\psi_2 \in l_p(X)$. Further, right-to-left of *(i)* holds by claim (a).

This completes the induction, from which it follows that

$$(W, \vee, V), \big[[S_0]_0, 0\big] \Vdash X_0$$

showing that $\varphi \in X_0$ is satisfiable in a join-semilattice model and, thus, at long last, finalizing our proof of completeness. □

As an immediate follow-up, we deduce strong completeness.

Corollary 6.17. $\mathbf{MIL_{Sem}}$ *is strongly complete w.r.t.* MIL_{Sem}.

Proof. Since $\mathbf{MIL_{Sem}}$ is sound and weakly complete w.r.t. MIL_{Sem}, it suffices to show compactness of MIL_{Sem}: for any set $\Gamma \subseteq \mathcal{L}_M$, if all finite subsets of Γ are satisfiable then Γ is satifisable. However, this is an easy consequence of (1) compactness of first-order logic; (2) a standard translation for $\mathcal{L}_M$-formulas; and (3) the fact that being a join-semilattice is a first-order definable property. $\square$

Remark 6.18. An alternative, more overt way of proving strong completeness is the following:

- Suppose $\Gamma \subseteq \mathcal{L}_M$ is $\mathbf{MIL}_{\mathbf{Sem}}$-consistent.
- For each finite subset $\Gamma_F \subseteq \Gamma$, get a join-semilattice model using the colimit construction of the completeness proof.
- Obtain a join-semilattice model of all of Γ using the standard ultrafilter-compactness construction. $\dashv$

In addition to the question of strong completeness we just addressed, our (weak) completeness proof naturally sparks some further questions and inquiries.

Firstly, while we did explicitly use (Re.), (Co.), (Dk.) and, of course, formulas from the infinite extension scheme in the completeness proof, the reader might have noticed that we did not explicitly use (4). Although this seems peculiar, it is actually not, as is explained by the following remark:

Remark 6.19. Let Λ be an arbitrary NML for the modal language with a single binary modality containing the axiom (As.). Then $\Lambda \vdash (4)$.

To see this, recall that

$$(4) = PPp \to Pp = \langle \mathrm{sup} \rangle (\langle \mathrm{sup} \rangle p \top) \top \to \langle \mathrm{sup} \rangle p \top,$$

and use that

$$\langle \mathrm{sup} \rangle (\langle \mathrm{sup} \rangle p \top) \top \overset{(As.)}{\to} \langle \mathrm{sup} \rangle p (\langle \mathrm{sup} \rangle \top \top) \quad \text{and} \quad \langle \mathrm{sup} \rangle \top \top \to \top. \qquad \dashv$$

Secondly, after axiomatizing MIL_{Pre}, we could promptly deduce that $MIL_{Pre} = MIL_{Pos}$ because the constructed frame of the completeness proof was not only a preorder but, in fact, a poset. Likewise, it is reasonable to inquire whether our constructed model, beyond being a member of the class of join-semilattices, is a member of some interesting subclass of structures. This is the subject matter of the next section.

Thirdly, having axiomatized MIL_{Sem}, echoing our work in the setting of preorders, axiomatizing the augmented $MIL_{\backslash\text{-}Sem}$ is an interesting follow-up. We have not attempted so but leave it for future work.

Lastly, exploring the connected properties of decidability, finite axiomatizability and the FMP w.r.t. the class

$$\mathcal{C}_{\mathbf{Sem}} := \{(W, C) \mid (W, C) \Vdash \mathbf{MIL}_{\mathbf{Sem}}\},$$

constitutes an extremely intriguing direction for future work. In the last section of this chapter, we expand on this and draw similarities with longstanding open problems in relevance logic.

6.4. Join-semilattices with finite (bounded) infima

As outlined, in this section, similar to how we showed $MIL_{Pre} = MIL_{Pos} \subsetneq MIL_{Sem}$, we explore how far the logic MIL_{Sem} stays constant when considering subclasses of join-semilattices. In particular, we show that the logic stays constant just short of the class of lattices: we can have all lower-bounded binary meets, but not all binary meets simpliciter. First, we show the former, for which it suffices to prove that the model of the completeness proof is finite lower bounded complete; that is:

Definition 6.20. A poset $(W, \leqslant)$ is *finite lower bounded complete* :iff any finite subset $A \subseteq W$ which has a lower bound, has a greatest lower bound. ⊣

For context, a poset is (upper) bounded complete :iff any subset which has some upper bound has a least upper bound.

To prove finite lower bounded completeness of the colimit, we will need that (1) any object of the diagram is lower bounded complete, and (2) the embeddings, $e_{m,n}$, preserve infima. This is the content of the two ensuing lemmas, respectively. When proving these, we will frequently be needing the following observation:

Observation 6.21. For any $n \in \omega$ and any $[X]_n \in \big(\mathcal{P}(S_n) \setminus \{\varnothing\}, \cup\big)/_{\sim_n}$, the following hold:

- $\bigcup\{X' \subseteq S_n \mid X' \sim_n X\} \sim_n X$; and
- if $X' \sim_n X$, then $X' \subseteq \bigcup\{X' \subseteq S_n \mid X' \sim_n X\}$.

While the latter is obvious, the former holds because $\mathcal{P}(S_n) \setminus \{\varnothing\}$ is finite and $\sim_n$ is a a congruence relation on $\big(\mathcal{P}(S_n) \setminus \{\varnothing\}, \cup\big)$ – hence compatible with unions. ⊣

Lemma 6.22. *For any* $n \in \omega$ *and any two elements* $\{[X]_n, [Y]_n\} \subseteq \big(\mathcal{P}(S_n) \setminus \{\varnothing\}, \cup\big)/_{\sim_n}$, *we have:* $\inf\{[X]_n, [Y]_n\}$ *exists* ***iff***

$$\bigcup\{X' \subseteq S_n \mid X' \sim_n X\} \cap \bigcup\{Y' \subseteq S_n \mid Y' \sim_n Y\} \neq \varnothing.$$

And if so, then

$$\inf\{[X]_n, [Y]_n\} = \Big[\bigcup\{X' \subseteq S_n \mid X' \sim_n X\} \cap \bigcup\{Y' \subseteq S_n \mid Y' \sim_n Y\}\Big]_n.$$

A fortiori, all $\big(\mathcal{P}(S_n) \setminus \{\varnothing\}, \cup\big)/_{\sim_n}$ *are lower bounded complete.*

Proof. ($\Rightarrow$) First, suppose $\inf\{[X]_n, [Y]_n\} = [I]_n$ exists. Then, in particular,

$$\left[\bigcup\{X' \subseteq S_n \mid X' \sim_n X\}\right]_n = [X]_n = [X]_n \cup_{\sim_n} [I]_n = [X \cup I]_n,$$

cf. the foregoing observation. Hence,

$$\bigcup\{X' \subseteq S_n \mid X' \sim_n X\} \sim_n X \cup I,$$

so

$$\bigcup\{X' \subseteq S_n \mid X' \sim_n X\} \supseteq I \neq \varnothing.$$

Similarly,

$$\bigcup\{Y' \subseteq S_n \mid Y' \sim_n Y\} \supseteq I \neq \varnothing.$$

Thus,

$$\bigcup\{X' \subseteq S_n \mid X' \sim_n X\} \cap \bigcup\{Y' \subseteq S_n \mid Y' \sim_n Y\} \supseteq I \neq \varnothing,$$

which not only shows the left-to-right direction of the first claim, but also implies the second claim; that is,

$$\inf\{[X]_n, [Y]_n\} = [I]_n = \left[\bigcup\{X' \subseteq S_n \mid X' \sim_n X\} \cap \bigcup\{Y' \subseteq S_n \mid Y' \sim_n Y\}\right]_n$$

because join-semilattice homomorphisms preserve (weak) order.

($\Leftarrow$) Second, suppose

$$\bigcup\{X' \subseteq S_n \mid X' \sim_n X\} \cap \bigcup\{Y' \subseteq S_n \mid Y' \sim_n Y\} \neq \varnothing.$$

Then the congruence class of this intersection exists, and it is a lower bound of $\{[X]_n, [Y]_n\}$. To see that it is the greatest lower bound, given any lower bound $[I]_n$, simply replicate the reasoning above. This completes the proof of the biimplication.

Lastly, given any two elements $[X]_n, [Y]_n$ which have some lower bound $[I]_n$, the previous reasoning also shows that

$$\bigcup\{X' \subseteq S_n \mid X' \sim_n X\} \cap \bigcup\{Y' \subseteq S_n \mid Y' \sim_n Y\} \supseteq I \neq \varnothing,$$

hence

$$\inf\{[X]_n, [Y]_n\} = \left[\bigcup\{X' \subseteq S_n \mid X' \sim_n X\} \cap \bigcup\{Y' \subseteq S_n \mid Y' \sim_n Y\}\right]_n.$$

Consequently, $\left(\mathcal{P}(S_n) \setminus \{\varnothing\}, \cup\right)/_{\sim_n}$ is "binary lower bounded complete", so since $\mathcal{P}(S_n) \setminus \{\varnothing\}$ is finite, it follows by a standard induction argument that it is lower bounded complete since also $\inf(\varnothing) = [S_n]_n$. $\square$

Lemma 6.23. *For all $m \leqslant n$, the embedding $e_{m,n}$ preserves infima.*

Proof. Fix some $m < n$. Again, since $\mathcal{P}(S_m) \setminus \{\varnothing\}$, $\mathcal{P}(S_n) \setminus \{\varnothing\}$ are finite, it suffices to show that $e_{m,n}$ preserves binary infima. Therefore, cf. the preceding lemma, we need only show that for any $\{[X]_m, [Y]_m\} \subseteq \big(\mathcal{P}(S_m) \setminus \{\varnothing\}, \cup\big)/_{\sim_m}$, if

$$\bigcup\{X' \subseteq S_m \mid X' \sim_m X\} \cap \bigcup\{Y' \subseteq S_m \mid Y' \sim_m Y\} \neq \varnothing,$$

then

$$(\bigcup\{X' \subseteq S_m \mid X' \sim_m X\} \cap \bigcup\{Y' \subseteq S_m \mid Y' \sim_m Y\}) \sim_n (\bigcup\{X' \subseteq S_n \mid X' \sim_n X\} \cap \bigcup\{Y' \subseteq S_n \mid Y' \sim_n Y\})\,,$$

since the congruence class of the left-hand side is $\inf_m\{[X]_m, [Y]_m\}$, and the congruence class of the right-hand side is $\inf_n\{[X]_n, [Y]_n\}$ (because the right-hand side extends the left-hand side and, consequently, must be non-empty).

Assuming the antecedent, we will prove the consequent by showing that

$$\begin{aligned}(\bigcup\{X' \subseteq S_m \mid X' \sim_m X\} \cap \bigcup\{Y' \subseteq S_m \mid Y' \sim_m Y\}) \sim_{m+1} (\bigcup\{X' \subseteq S_{m+1} \mid X' \sim_{m+1} X\} \cap \bigcup\{Y' \subseteq S_{m+1} \mid Y' \sim_{m+1} Y\}) \sim_{m+2} \cdots \\ \sim_n \quad (\bigcup\{X' \subseteq S_n \mid X' \sim_n X\} \cap \bigcup\{Y' \subseteq S_n \mid Y' \sim_n Y\})\,.\end{aligned}$$

Therefore, let $k \in \{m, \dots, n-1\}$ be arbitrary. As in the proof of Lemma 6.14, we can construct a sequence of sets with corresponding congruence relations

$$\sim_k \subseteq \sim_0^{Y_1} \subseteq \cdots \subseteq \sim_{n_{Y_1}}^{Y_1} \subseteq \cdots \subseteq \sim_0^{Y_p} \subseteq \cdots \subseteq \sim_{n_{Y_p}}^{Y_p} = \sim_{k+1},$$

where the congruence relations $\sim_0^{Y_1}, \dots, \sim_{n_{Y_p}}^{Y_p}$ are defined as in the proof of Lemma 6.13; i.e., in terms of their predecessor in this chain of congruence relations. Then, by (1) letting $\sim_j \in \left\{\sim_k, \sim_0^{Y_1}, \dots, \sim_{n_{Y_p}-1}^{Y_p}\right\}$ be arbitrary; (2) denoting its successor in the above chain $\sim_{j+1}$; and (3) denoting their matching sets S_j and S_{j+1}, respectively; it is enough to show that (4)

$$(\bigcup\{X' \subseteq S_j \mid X' \sim_j X\} \cap \bigcup\{Y' \subseteq S_j \mid Y' \sim_j Y\}) \sim_{j+1} (\bigcup\{X' \subseteq S_{j+1} \mid X' \sim_{j+1} X\} \cap \bigcup\{Y' \subseteq S_{j+1} \mid Y' \sim_{j+1} Y\})\,.$$

For this purpose, we begin by proving that

$$(\bigcup\{X' \subseteq S_j \mid X' \sim_j X\} \cap \bigcup\{Y' \subseteq S_j \mid Y' \sim_j Y\}) = (\bigcup\{X' \subseteq S_{j+1} \mid X' \sim_{j+1} X\} \cap \bigcup\{Y' \subseteq S_{j+1} \mid Y' \sim_{j+1} Y\}) \setminus \{a, b\},$$

where 'a' and 'b' denote the **boldface** formula-symbols for which $S_{j+1} = S_j \cup \{a, b\}$ and $\sim_{j+1}$ is defined to be the least congruence relation extending $\sim_j$ and satisfying $\{a, b\} \sim_{j+1} Z$ for some $\varnothing \neq Z \subseteq S_j$. Using the alternative characterization of $\sim_{j+1}$ given in the proof of Lemma 6.13, we prove this going inclusion by inclusion.

I.e., 'the $\sim''$ characterization'.

"$\subseteq$" is a consequence of $\sim_j \subseteq \sim_{j+1}$ and $\{a, b\} \cap S_j = \varnothing$. For "$\supseteq$", suppose $X'' \sim_{j+1} X$. If $X'' \overset{(i)}{\sim}_{j+1} X$, then also $X'' \sim_j X$. And if $X'' \overset{(ii)}{\sim}_{j+1} X$, then since $X \cap \{a, b\} = \varnothing$, we have that

$$X = X \setminus \{a, b\} \sim_j (X \setminus \{a, b\}) \cup Z \sim_j (X'' \setminus \{a, b\}) \cup Z.$$

Therefore, in any case,

$$X'' \setminus \{a, b\} \subseteq \bigcup\{X' \subseteq S_j \mid X' \sim_j X\}.$$

Analogously, for any $Y'' \sim_{j+1} Y$, we find that

$$Y'' \setminus \{a, b\} \subseteq \bigcup\{Y' \subseteq S_j \mid Y' \sim_j Y\},$$

so we get "$\supseteq$" as well, whence we have proven the equality, showing that these two intersections agree on the complement of $\{a, b\}$.

Using this, we find that if they also agree on $\{a, b\}$, then

$$\left(\bigcup\{X' \subseteq S_j \mid X' \sim_j X\} \cap \bigcup\{Y' \subseteq S_j \mid Y' \sim_j Y\}\right) \overset{(i)}{\sim}_{j+1} \left(\bigcup\{X' \subseteq S_{j+1} \mid X' \sim_{j+1} X\} \cap \bigcup\{Y' \subseteq S_{j+1} \mid Y' \sim_{j+1} Y\}\right).$$

So suppose they don't. Then we must have

$$\left(\bigcup\{X' \subseteq S_{j+1} \mid X' \sim_{j+1} X\} \cap \bigcup\{Y' \subseteq S_{j+1} \mid Y' \sim_{j+1} Y\}\right) \cap \{a, b\} \neq \varnothing,$$

so, in particular, there is some $X' \sim_{j+1} X$ s.t. $X' \cap \{a, b\} \neq \varnothing$. Thus, because $X \cap \{a, b\} = \varnothing$, we must have $X' \overset{(ii)}{\sim}_{j+1} X$, so

$$X = X \setminus \{a, b\} \sim_j (X \setminus \{a, b\}) \cup Z = X \cup Z.$$

Consequently,

$$Z \subseteq \bigcup\{X' \subseteq S_j \mid X' \sim_j X\},$$

and, likewise,

$$Z \subseteq \bigcup\{Y' \subseteq S_j \mid Y' \sim_j Y\},$$

so

$$Z \subseteq \left(\bigcup\{X' \subseteq S_j \mid X' \sim_j X\} \cap \bigcup\{Y' \subseteq S_j \mid Y' \sim_j Y\}\right).$$

Therefore,

$$\left(\bigcup\{X' \subseteq S_j \mid X' \sim_j X\} \cap \bigcup\{Y' \subseteq S_j \mid Y' \sim_j Y\}\right) \setminus \{a, b\} = \Big(\left(\bigcup\{X' \subseteq S_j \mid X' \sim_j X\} \cap \bigcup\{Y' \subseteq S_j \mid Y' \sim_j Y\}\right) \setminus \{a, b\}\Big) \cup Z,$$

and

$$\bigcup\{X' \subseteq S_j \mid X' \sim_j X\} \cap \bigcup\{Y' \subseteq S_j \mid Y' \sim_j Y\} = (\bigcup\{X' \subseteq S_{j+1} \mid X' \sim_{j+1} X\} \cap \bigcup\{Y' \subseteq S_{j+1} \mid Y' \sim_{j+1} Y\}) \setminus \{a, b\}$$
$$= \big((\bigcup\{X' \subseteq S_{j+1} \mid X' \sim_{j+1} X\} \cap \bigcup\{Y' \subseteq S_{j+1} \mid Y' \sim_{j+1} Y\}) \setminus \{a, b\}\big) \cup Z,$$

which exactly imply that

$$(\bigcup\{X' \subseteq S_j \mid X' \sim_j X\} \cap \bigcup\{Y' \subseteq S_j \mid Y' \sim_j Y\}) \overset{(ii)}{\sim}_{j+1} (\bigcup\{X' \subseteq S_{j+1} \mid X' \sim_{j+1} X\} \cap \bigcup\{Y' \subseteq S_{j+1} \mid Y' \sim_{j+1} Y\}),$$

completing our proof of the embeddings $e_{m,n}$ preserving infima. □

With these lemmas at our disposal, we are ready to prove finite lower bounded completeness of the colimit.

Lemma 6.24. *The colimit is finite lower bounded complete; i.e., any finite set of worlds which has a lower bound has a greatest lower bound.*

Proof. Since $\inf(\varnothing) = \big[[S_0]_0, 0\big]$, it suffices to show that it is binary lower bounded complete; that is, whenever two worlds $\big[[X]_k, k\big], \big[[Y]_m, m\big]$ have some lower bound $\big[[Z]_n, n\big]$, they have a greatest lower bound. Accordingly, suppose

$$\big[[X]_k, k\big], \big[[Y]_m, m\big] \geqslant \big[[Z]_n, n\big].$$

Then, by definition,

$$[X]_{p_1} \geqslant_{p_1} [Z]_{p_1} \qquad \text{and} \qquad [Y]_{p_2} \geqslant_{p_2} [Z]_{p_2},$$

where $p_1 = \max\{k, n\}$ and $p_2 = \max\{m, n\}$. Hence, for $p = \max\{p_1, p_2\}$, we find that

$$[X]_p, [Y]_p \geqslant_p [Z]_p,$$

so

$$\inf{}_p\{[X]_p, [Y]_p\} = \big[\bigcup\{X' \subseteq S_p \mid X' \sim_p X\} \cap \bigcup\{Y' \subseteq S_p \mid Y' \sim_p Y\}\big]_p,$$

and, particularly,

$$\big[\inf{}_p\{[X]_p, [Y]_p\}, p\big] \leqslant \big[[X]_k, k\big], \big[[Y]_m, m\big].$$

Moreover, for any lower bound $\big[[B]_b, b\big] \leqslant \big[[X]_k, k\big], \big[[Y]_m, m\big]$, we find that

$$[B]_l \leqslant_l [X]_l, [Y]_l$$

for $\mathfrak{l} = \max\{\mathfrak{b}, \mathfrak{p}\}$, so by the foregoing lemma

$$[B]_{\mathfrak{l}} \leqslant_{\mathfrak{l}} \inf{}_{\mathfrak{l}}\{[X]_{\mathfrak{l}}, [Y]_{\mathfrak{l}}\} = \left[\bigcup\{X' \subseteq S_{\mathfrak{p}} \mid X' \sim_{\mathfrak{p}} X\} \cap \bigcup\{Y' \subseteq S_{\mathfrak{p}} \mid Y' \sim_{\mathfrak{p}} Y\}\right]_{\mathfrak{l}},$$

whence

$$\left[[B]_{\mathfrak{b}}, \mathfrak{b}\right] \leqslant \left[\inf{}_{\mathfrak{p}}\{[X]_{\mathfrak{p}}, [Y]_{\mathfrak{p}}\}, \mathfrak{p}\right],$$

which completes the proof by allowing us to conclude

$$\inf\left\{\left[[X]_{\mathfrak{k}}, \mathfrak{k}\right], \left[[Y]_{\mathfrak{m}}, \mathfrak{m}\right]\right\} = \left[\inf{}_{\mathfrak{p}}\{[X]_{\mathfrak{p}}, [Y]_{\mathfrak{p}}\}, \mathfrak{p}\right]$$

as desired. □

With this shown, we may now conclude the following:

Corollary 6.25. *Let $MIL_{FLBC\text{-}Sem}$ denote the modal information logic of finite lower bounded complete join-semilattices in the language $\mathcal{L}_M$. Then*

$$MIL_{Sem} = MIL_{FLBC\text{-}Sem}.$$

Although the term 'finite lower bounded complete join-semilattice' is clumsy, its denotation nearly rhymes with the denotation of the term 'lattice': we have binary joins and binary *bounded* meets—can we have binary meets simpliciter? The answer is no, as we now demonstrate.

Remark 6.26 ($MIL_{Sem} \subsetneq MIL_{Lat}$)**.** The formula

$$PHp \wedge PH\neg p$$

is satisfiable in a join-semilattice, but not in any lattice, whence – already in the language $\mathcal{L}_M$ – the modal information logic of lattices properly extends the modal information logic of join-semilattices. ⊣

Albeit it would not be entirely uninteresting to axiomatize the MIL of suprema on lattices, it seems (a lot) more fitting to have vocabulary for not only talking about joins (i.e., our '$\langle \sup \rangle$') but also for meets (i.e., an additional '$\langle \inf \rangle$'-modality). In fact, this exact study has been initiated in a very recent paper, namely in Wang and Wang (2022, sec. 5) where the authors axiomatize the MIL over lattices in the language $\mathcal{L}_M$ extended with an '$\langle \inf \rangle$'-modality and nominals. While we have not tried—given the symmetry between, on one hand: join-semilattices and the '$\langle \sup \rangle$'-modality and, on the other: lattices and both the '$\langle \sup \rangle$'- and '$\langle \inf \rangle$'-modality—it seems likely that our methods from this chapter adapt to axiomatize this MIL of lattices without the hybrid extension of nominals. We leave this for future work.

Having mentioned the '$\langle \text{inf} \rangle$'-modality, we end by including this remark:

Remark 6.27. All results obtained also obtain in a dual form if one substitutes the modality of suprema '$\langle \text{sup} \rangle$' with the modality of infima '$\langle \text{inf} \rangle$' and substitutes all mentions of suprema/joins with infima/meets (and alike). $\dashv$

6.5. On decidability and S: a direction for future work

Out of several potential future lines of research, we find one particularly enticing, deserving more than a mention in passing, namely the one centering around the problem of decidability of MIL_{Sem} and the ancillary problems of finite axiomatizability and the FMP w.r.t. $\mathcal{C}_{\mathbf{Sem}} = \{(W, C) \mid (W, C) \Vdash \mathbf{MIL_{Sem}}\}$. To us, there is something alluring about the (seemingly) paradoxical increase in complexity caused by chaffing away most of the posets to only consider the more well-behaved ones with all binary joins. For this reason, we use this section to set forth this line of research amplifying salient points—not least what appears to be a deep connection with relevance logic.

Practically foreseen by van Benthem (Forthcoming), the problem of decidability of MIL_{Sem} and its two ancillary problems resound enduring open problems in the relevance logic of positive join-semilattice semantics, usually denoted **S** and introduced by Urquhart (1972, 1973). Thus, getting clear on the problems of one logic, might very well necessitate/entail getting clear on the problems of the other. Accordingly, we continue by detailing this connection—primarily through exhibiting how these three problems of MIL_{Sem} all have counterparts in **S**. Secondarily, but first off, it is in place to set out **S** as to illuminate the similarities already occurring in the very definitions of the logics.

S is defined semantically with the frames, like MIL_{Sem}, being join-semilattices but, unlike MIL_{Sem}, with a least element 0. The formulas are formed standardly based on propositional variables and the three connectives '$\wedge$', '$\vee$' and '$\backslash$' with semantics as in this thesis; valuations are standard (i.e., with the powerset as codomain for all propositional variables). **S** is then defined as the set of formulas satisfied at 0 in all models. Besides from the similarity in frames [join-semilattices vs. join-semilattices with least element] and connectives ['$\langle \text{sup} \rangle$' vs. '$\backslash$'], it is worth noting that Urquhart (1972, 1973), like van Benthem (1996), motivates the semantics with an informational interpretation, namely (basically) the one we mentioned in the introduction.

In the relevance logic literature, the symbol '$\to$' is typically used instead of our '$\backslash$'.

This displays similarity in the basic framework; now for the counterparts of the three mentioned problems and some comments on how one might approach these. *First*, the problem of axiomatizing **S** was solved by Fine (1976) and worked out in detail by Charlwood (1981). Interestingly, the axiomatization, like our $\mathbf{MIL_{Sem}}$, is infinite,[19] and it is (to the best of our knowledge) an open problem whether **S** is finitely axiomatizable. Plausibly, this partly owes to the fact that it hardly could be—the same goes for $\mathbf{MIL_{Sem}}$ which we (strongly) conjecture to not be finitely axiomatizable.[20] Although showing $\mathbf{MIL_{Sem}}$ and/or **S** to not be finitely axiomatizable would be rather unsurprising per se, it could—beyond being novel—very well serve as an excellent stepping stone in a proof of (un)decidability: perhaps especially through a resulting better understanding of $\mathcal{C}_{\mathbf{Sem}}$-frames and/or some generalized class of **S**-frames.

Second, this brings us to the **S**-counterpart of whether $\mathbf{MIL_{Sem}}$ has the FMP w.r.t. $\mathcal{C}_{\mathbf{Sem}}$-frames. Most prominently, it is still an open problem whether **S** has the FMP w.r.t. its class of definition.[21] Less prominently, since **S**, too, is semantically defined and already has been axiomatized, the heuristic of section 1.3 could pertain; that is, using the axiomatizing to reinterpret the formulas on another class of structures for which one can prove the FMP.

Regarding $\mathbf{MIL_{Sem}}$ and the FMP w.r.t. $\mathcal{C}_{\mathbf{Sem}}$-frames, we have not attempted a proof/refutation yet, but scratching the surface two things become clear that are worth mentioning: (1) the formula ψ_N from Proposition 1.8 is satisfiable on a finite $\mathcal{C}_{\mathbf{Sem}}$-frame—i.e., ψ_N does not immediately eradicate hope of the FMP w.r.t. $\mathcal{C}_{\mathbf{Sem}}$—and (2) the Theorem 3.9 filtration fails for $\mathcal{C}_{\mathbf{Sem}}$-frames—i.e., contrarily, we do not immediately get the FMP either (see Appendix A.3 for quick proofs of these claims). In fact, (2) was already suggested by results covered in the overview paper by Kurucz et al. (1995), where the authors gather and prove (un)decidability results for various modal logics with a binary modality—including that the least NML of a single associative binary modality is undecidable.

Recall the mention of associativity and filtration in footnote 8 (3.2).

On this node, we, *third*, begin directly commenting on the last, and most

[19] Contrariwise, there seems to be no similarity qua proof methods.

[20] We have not worked on this problem in particular, but we have a clear strategy in mind: It suffices to show that MIL_{Sem} is not axiomatizable by any finite subset of our infinite axiomatization, and any such subset must have finite 'depth' and 'width' (cf. 6.1). Using this, it should then be possible to construct a counterframe validating these axioms but not an axiom of greater depth (and width). In the process of axiomatizing MIL_{Sem}, we constructed a $\mathcal{C}$-frame validating (As.) but not an axiom of greater depth; if the reader is interested in pursuing this, feel free to reach out and we will happily send our proof to draw inspiration from for proving the general case.

[21] In a recent paper, Yale Weiss (2021) shows that after augmenting with a connective '$\neg_i$' for intuitionistic negation, the resulting logic lacks the FMP.

central, of the three problems: decidability. As for $\mathbf{MIL_{Sem}}$, the fact that it extends the least associative NML is cause for concern (or hope) that it is undecidable: albeit the techniques of Kurucz et al. (1995) do not directly apply to MIL_{Sem}, their general idea of encoding the Post correspondence problem might still work; i.e., encoding the quasi-equational theory of semigroups (which is undecidable). This would be an algebraic manner of attempting an undecidability proof. For a relational manner, 'tiling' might work, cf. Remark 1.11 where we noted that the formula χ_N when satisfied in a join-semilattice implies the existence of a tile. Lastly, even with said strategies for proving undecidability in mind, our intuition on whether MIL_{Sem} is decidable is not strong at all: it might also be decidable.[22] To the contrary, the fact that many a relevant logic have been proven undecidable while **S** has escaped these techniques (Urquhart 1984), suggests that **S** maybe is decidable, as Urquhart (2016) conjectures. So if MIL_{Sem} and **S** truly are intimately connected and Urquhart (2016) is right, then MIL_{Sem} could be decidable as well.

This concludes our exposition of this research direction—hopefully serving not only as inspiration for others, but also as an initial guide for such study.

[22] It should be (foot)noted that in Appendix B, we show that the (tiny and) positive first-degree fragment of MIL_{Sem} is decidable.

Conclusion and Future Work

Our exploration of modal information logics has come to an end. We summarize this inquiry, clarify where it leaves us, and point to future lines of research.

First, we examined the basic modal information logics of suprema on preorders and posets, namely MIL_{Pre} and MIL_{Pos}. We showed that – even if they do not enjoy the FMP w.r.t. their frames of definition – they are decidable. This was shown 'via completeness' by (1) axiomatizing them; (2) deducing that they are one and the same logic; and (3) obtaining another class of frames $\mathcal{C}$ complete w.r.t. the logic(s), which, importantly, did enjoy the FMP.

Second, we tackled these same problems but for the enriched logics $MIL_{\setminus\text{-}Pre}$ and $MIL_{\setminus\text{-}Pos}$ and achieved analogous results. However now, already having a clear candidate for a generalized class of frames, namely $\mathcal{C}$, we could axiomatize this logic first, and subsequently solve the problem of axiomatization of $MIL_{\setminus\text{-}Pre}$ and $MIL_{\setminus\text{-}Pos}$ through representation.

Third, we considered modal information logics of minimal upper bounds on preorders and posets, and showed that no change occurs in the resulting logics; on preorders and posets, the landscape of MILs is decidable and uniform:

$$MIL_{Pre} = MIL_{Pos} = MIL^{Min}_{Pre} = MIL^{Min}_{Pos}, \quad MIL_{\setminus\text{-}Pre} = MIL_{\setminus\text{-}Pos} = MIL^{Min}_{\setminus\text{-}Pre} = MIL^{Min}_{\setminus\text{-}Pos}$$

Fourth and last, we expanded our inquiry to include the class of join-semilattices, and so from the savanna of preorders and posets a mountain appeared: we axiomatized MIL_{Sem} using an infinite extension scheme and left the problem of decidability wide open.

This brings us to directions for further research. In no particular order, we conclude this thesis by mentioning a few:

- Proving (un)decidability of MIL_{Sem} and solving the ancillary problems of finite axiomatizability and the FMP w.r.t. $\mathcal{C}_{\mathbf{Sem}}$, cf. section 6.5.

- Applying the axiomatization ideas of this thesis in other settings—not least those of Chapter 6, which, in particular, could be directly transferable to axiomatize the MIL of lattices with both a '$\langle \text{sup} \rangle$'- and an '$\langle \text{inf} \rangle$'-modality; thus, continuing the work of Wang and Wang (2022), which achieves an axiomatization of this logic with nominals.

- Further examining how MILs relate to other logics, including those mentioned in the introduction. This could shed new perspicuous lights on not only MILs but also the logics of comparison (cf. 4.10 and B.4).

Appendix

A. Various addenda

A.1. Wildness of canonical frames

As an informal addendum to Chapter 2, we briefly remark that the canonical relation $C_{\mathbf{Pre}}$ of the canonical frame for $\mathbf{MIL_{Pre}}$ is not the supremum relation of $\leqslant_{\mathbf{Pre}}$.

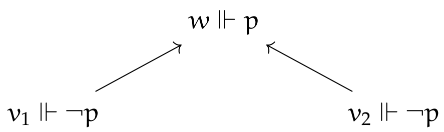
Remark A.1.1. The following hold:

1. There are MCSs Γ, Δ s.t. $C_{\mathbf{Pre}}\Gamma\Delta\Delta$ even if $\Gamma \not\leqslant_{\mathbf{Pre}} \Delta$. In other words, although Γ and Δ aren't in the same cluster ($\Gamma \not\leqslant_{\mathbf{Pre}} \Delta$), Γ 'claims' to be the 'supremum' of Δ.

2. In fact, there are continuum many such MCSs Γ_i all claiming to be the supremum of Δ. ⊣

Poor Δ :(

Proof. Consider the model depicted below where the worlds satisfy all and only the proposition letters shown.



Then $||v_1||, ||v_2||, ||w||$ are MCSs where $||x|| := \{\varphi \in \mathcal{L}_M \mid x \Vdash \varphi\}$. Moreover, $Pp \notin ||v_1|| = ||v_2||$, so (a) since $p \in ||w||$ we have that $\Gamma := ||w|| \not\leqslant_{\mathbf{Pre}} ||v_1|| =: \Delta$, and (b) since $w = \sup\{v_1, v_2\}$ we also have $C_{\mathbf{Pre}}\Gamma\Delta\Delta$, which proves the first claim.

For the second, simply change the valuation of w for proposition letters $q \neq p$ to get the same results for different MCSs Γ_i. Since there are countably many proposition letters (so continuum many subsets of proposition letters), we get continuum many MCSs claiming to be supremum of $\Delta = ||v_1|| = ||v_2||$. □

Observation A.1.2. The above depicted frame is, in fact, a join-semilattice, hence the claims extend to the canonical frame for MIL_{Sem}. ⊣

Evidently, this technique can be adapted to quickly show many more 'absurdities'; for instance, an infinite strictly ascending chain of MCSs all claiming to be the supremum of two MCSs below. Even if amusing, by now, this becomes too much of a sidetrack (even for an appendix), so we leave this as an activity for the reader.

A.2. The tree model property

Having shown that $\mathbf{MIL_{Pre}}$ – although not enjoying the FMP w.r.t. preorder frames – does enjoy the FMP w.r.t. the generalized $\mathcal{C}$-frames (cf. section 3.2), an analogous inquiry regarding the tree model property arises most naturally. However, given that trees are defined in terms of a *binary* relation, and $\mathcal{C}$-frames (W, C) come equipped with a *ternary* relation, one might wonder whether 'to have the tree model property w.r.t. $\mathcal{C}$-frames' even is a well-formed predicate. We begin by showing that it is: all $\mathcal{C}$-frames (W, C) canonically induce binary relations $\leqslant_C$.

Observation A.2.1. For any $(W, C) \in \mathcal{C}$, let $\leqslant_C$ and $\leqslant'_C$ be given as follows:

$$\leqslant_C := \{(y, x) : Cxxy\}, \qquad \leqslant'_C := \{(y, x) : \exists z(Cxyz \vee Cxzy)\}.$$

Then, by definition of the class $\mathcal{C}$, it is not too hard to see that (a) $\leqslant_C = \leqslant'_C$, and (b) $\leqslant_C$ is a preorder on W.

Moreover, if C happened to be the supremum relation of some preorder $\leqslant$, then $\leqslant_C = \leqslant$. ⊣

I.e., $Cxyz$ *iff* $x \in \sup_{\leqslant}\{y, z\}$.

As alluded to, this enables us to state what it should mean for $\mathbf{MIL_{Pre}}$ to have the tree model property w.r.t. $\mathcal{C}$-frames, namely: whenever a formula $\varphi \in \mathcal{L}_M$ is satisfiable, it is satisfiable in some frame (W, C) where $(W, \geqslant_{C'})$ is a (reflexive and transitive) tree.

As we did when concerned with the FMP, before showing the TMP in general, we revisit the formula χ_N from Remark 1.10.

Remark A.2.2. Although

$$\begin{aligned}\chi_N := \ & p \wedge q \wedge \langle \sup \rangle (p \wedge \neg q)(\neg p \wedge q) \\ & \wedge H\big([p \wedge \neg q] \to P(\neg p \wedge \neg q)\big) \wedge H\big([\neg p \wedge q] \to P(\neg p \wedge \neg q)\big) \\ & \wedge H\big(\langle \sup \rangle (\neg p \wedge \neg q)^2 \to [\neg p \wedge \neg q]\big)\end{aligned}$$

witnesses that $\mathbf{MIL_{Pre}}$ lacks the TMP w.r.t. preorder (and poset) frames, it is satisfiable on a $\mathcal{C}$-frame whose underlying preorder $\leqslant_C$ defines a (converse reflexive and transitive) tree. ⊣

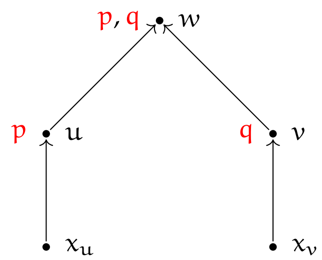
Proof. Let

$$
\begin{aligned}
W &:= \{w, v, u, x_v, x_u\}, \qquad V(p) := \{w, u\}, V(q) := \{w, v\}, \\
C &:= \{(w,w,w), (v,v,v), (u,u,u), (x_v,x_v,x_v), (x_u,x_u,x_u); (w,w,v), (w,v,w); \\
&\quad (w,w,u), (w,u,w); (w,w,x_v), (w,x_v,w); (w,w,x_u), (w,x_u,w); \\
&\quad (v,v,x_v), (v,x_v,v); (u,u,x_u), (u,x_u,u); (w,v,u), (w,u,v)\}.
\end{aligned}
$$

We claim that (1) $(W, C) \in \mathcal{C}$; (2) $(W, \geqslant_C)$ is a (reflexive and transtive) tree; and (3) $(W, C, V), w \Vdash \chi_N$.

Showing (1) is routine, and regarding (2), it is straightforward to check that the Hasse diagram of $\leqslant_C$ is the one depicted below, which certainly is a converse tree.

Notice how this isn't a preorder frame: e.g., $w = \sup_{\leqslant_C}\{u, x_v\}$ yet $\neg Cwux_v$.



Lastly, to see that (3) holds, (i) recall Observation 3.5, (ii) have the Hasse diagram in mind, and (iii) observe that

$$
\begin{aligned}
&V(p \wedge q) = \{w\}, && V(p \wedge \neg q) = \{u\}, && V(\neg p \wedge q) = \{v\}, \\
&V(\neg p \wedge \neg q) = \{x_u, x_v\}, && Cwuv.
\end{aligned}
$$

We define $V(\varphi) := \{y \mid y \Vdash \varphi\}$.

Then $(W, C, V), w \Vdash \chi_N$ follows by an easy check. □

When proving the general case – i.e. the TMP – we will be using the satisfaction-preserving technique of taking generated submodels. Because of this, ahead of the actual proof and for the sake of completeness, we include the following observation:

Observation A.2.3. The standard results regarding generated submodels (and -frames), p-morphisms, etc. apply to $\mathcal{C}$-models (and -frames).

Another commonly used term for 'p-morphism' is 'bounded morphism'.

OBS: When we are dealing with preorder frames $(W, \leqslant, V)$, the notion of p-morphism must be defined in terms of the supremum relation induced by $\leqslant$ – and *not* in terms of $\leqslant$ per se; otherwise, we do not get the sought invariance result. That said, coincidentally, defining the notion of generated submodel in terms of $\geqslant$ does give us satisfaction-preservation because, by pure happenstance, the resulting model is identical with the one defined in terms of taking the generated submodel w.r.t. the supremum relation. $\dashv$

With this observation in mind, we close off this addendum by showing that our discussion in section 1.3 applies just as well to the case of the TMP: when dealing with semantically introduced logics not having the TMP (w.r.t. the class of structures of definition) need not be very telling: although our logic does not enjoy the TMP w.r.t. preorder/poset frames, it does enjoy the TMP w.r.t. $\mathcal{C}$-frames.

Proposition A.2.4. $\mathbf{MIL}_{\mathbf{Pre}}$ *has the TMP w.r.t.* $\mathcal{C}$*-frames.*

Proof. Suppose φ is satisfiable. Then $\{\varphi\}$ is consistent, so we can apply our completeness proof (2.13) to obtain a poset model $(W, \leqslant_\omega, V)$ and world $x_0 \in W$ s.t.

$$(W, \leqslant_\omega, V), x_0 \Vdash \varphi.$$

Now, let $S_{\leqslant_\omega}$ be the supremum relation induced by $\leqslant_\omega$; that is,

$$S_{\leqslant_\omega} wvu \qquad \textbf{:iff} \qquad w = \sup_\omega\{v, u\},$$

and define

$$C' := S_{\leqslant_\omega} \cap (\downarrow_\omega x_0 \times \downarrow_\omega x_0 \times \downarrow_\omega x_0), \qquad V'(p) := V(p) \cap \downarrow_\omega x_0, \text{ for all } p \in \mathbf{P}.$$

Then, cf. the preceding observation,

$$(\downarrow_\omega x_0, C', V'), x_0 \Vdash \varphi.$$

Thus, it suffices to show that (1) $(\downarrow_\omega x_0, \geqslant_{C'})$ is a tree, and (2) $(\downarrow_\omega x_0, C') \in \mathcal{C}$.

We begin with (1). Observe that $\leqslant_{C'} = \leqslant_\omega \cap (\downarrow_\omega x_0 \times \downarrow_\omega x_0)$, so we are to show that

$$(\downarrow_\omega x_0, \geqslant_\omega \cap (\downarrow_\omega x_0 \times \downarrow_\omega x_0))$$

is a tree. We do so by showing, by induction, that for all $n \in \omega$:

$$(\downarrow_n x_0, \geqslant_n \cap (\downarrow_n x_0 \times \downarrow_n x_0)) \text{ is a tree with root } x_0.$$

For $n = 0$, this is clear. Moreover, the induction step is an easy matter as well: use the IH and that stage $n+1$ is obtained by either $\langle\text{sup}\rangle$- or $\neg\langle\text{sup}\rangle$-repairing.

Regarding (2), from the definition of C' as being the restriction of $S_{\leqslant_\omega}$ to the $\downarrow_\omega x_0$-worlds and the fact that $(W, C) \vDash (\text{Re.f}) \wedge (4\text{f}) \wedge (\text{Co.f}) \wedge (\text{Dk.f})$, we immediately get $(\downarrow_\omega x_0, C') \vDash (\text{Re.f}) \wedge (4\text{f}) \wedge (\text{Co.f}) \wedge (\text{Dk.f})$, which completes the proof and concludes this addendum. $\square$

A.3. On the FMP of $\mathbf{MIL_{Sem}}$ w.r.t. $\mathcal{C}_{\mathbf{Sem}}$.

This addendum contains proofs of the results mentioned in section 6.5, namely (1) that the formula ψ_N from Proposition 1.8 does not witness non-FMP for $\mathbf{MIL_{Sem}}$ w.r.t. $\mathcal{C}_{\mathbf{Sem}}$-frames, and (2), to the contrary, that the filtration of Theorem 3.9 fails for $\mathcal{C}_{\mathbf{Sem}}$-frames.

Remark A.3.1. Although

$$\psi_N := \mathsf{HP}\langle \mathsf{sup}\rangle \mathsf{pp} \wedge \mathsf{HP}\neg\langle \mathsf{sup}\rangle \mathsf{pp}$$

only is satisfiable on infinite join-semilattices (cf. Remark 1.9), it is satisfiable on a finite $\mathcal{C}_{\mathbf{Sem}}$-frame. ⊣

Proof. It suffices to show that the frame (W, C) from Remark 3.6 is a $\mathcal{C}_{\mathbf{Sem}}$-frame. To this end, we consider the chain $(\mathbb{Z}_-, \leqslant)$ from Proposition 1.8 consisting of the negative integers with the less-than relation. We show that (W, C) is the p-morphic image of $(\mathbb{Z}_-, \leqslant)$, which proves the claim because (a) onto p-morphisms preserve validities, and (b) $(\mathbb{Z}_-, \leqslant)$ is a join-semilattice, whence, in particular, a $\mathcal{C}_{\mathbf{Sem}}$-frame.

Consequently, let

$$f : \mathbb{Z}_- \to W$$

be the function given by setting

$$f(z) = \begin{cases} w & \textit{if } |z| \textit{ is odd} \\ v & \textit{otherwise} \end{cases}.$$

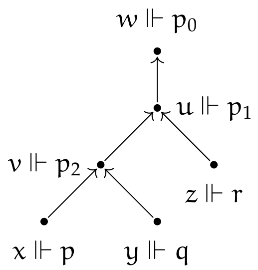
Then f is clearly onto. Further, it meets the forth condition because $z = \sup\{x, y\}$ implies $z \in \{x, y\}$. Lastly, if $Cf(z)ab$ for some $\{a, b\} \subseteq \{w, v\}$, then also $f(z) \in \{a, b\}$, so w.l.o.g. $f(z) = a$. Besides, we have that either $f(z) = b$ or $f(z-1) = b$, so in any case we find some $z' \in \mathbb{Z}_-$ s.t. $z = \sup\{z, z'\}$, $f(z') = b$ and $Cf(z)f(z)f(z')$. This shows the back condition, thus completing the proof. □

Remark A.3.2. Let (W, C, V) be a $\mathcal{C}_{\mathbf{Sem}}$-model, Σ a $\mathcal{C}$-closed set of formulas, and $(W_\Sigma, C^{\mathcal{C}}_\Sigma, V_\Sigma)$ the filtration of (W, C, V) through Σ as defined in Theorem 3.9. Then, even if we do have $(W_\Sigma, C^{\mathcal{C}}_\Sigma) \in \mathcal{C}$, we need not have $(W_\Sigma, C^{\mathcal{C}}_\Sigma) \in \mathcal{C}_{\mathbf{Sem}}$. ⊣

Proof. In order to show this, we provide a counterexample showing that the filtration does not preserve associativity even given the additional conditions demanded by $(W, C) \in \mathcal{C}_{\mathbf{Sem}}$. Accordingly, let

$$\Sigma := \{\top, p_0, p_1, p_2, p, q, r, \langle \sup \rangle qr, \langle \sup \rangle rq, Pq, \langle \sup \rangle \top q, Pr, \langle \sup \rangle \top r,$$
$$\langle \sup \rangle pp_1, \langle \sup \rangle p_1 p, Pp, \langle \sup \rangle \top p, Pp_1, \langle \sup \rangle \top p_1\},$$

and let $(W, \leqslant, V)$ be the join-semilattice model depicted below where the worlds satisfy all and only the proposition letters shown in the figure.



Then Σ is $\mathcal{C}$-closed, and $(W, \leqslant, V)$ is, in particular, a $\mathcal{C}_{\mathbf{Sem}}$-model. However, in the filtrated model, we have $C^{\mathcal{C}}_{\Sigma}|w||v||z|$ and $C^{\mathcal{C}}_{\Sigma}|v||x||y|$, but there is no $|a|$ s.t. $C^{\mathcal{C}}_{\Sigma}|a||y||z|$ and $C^{\mathcal{C}}_{\Sigma}|w||x||a|$; that is, the filtrated model do not validate (As.). To expound, only for $|a| = |u|$ do we get $C^{\mathcal{C}}_{\Sigma}|a||y||z|$, but $C^{\mathcal{C}}_{\Sigma}|w||x||u|$ fails because $x \Vdash p$ and $u \Vdash p_1$ yet $w \nVdash \langle \sup \rangle pp_1$. $\square$

B. Truthmaker logics

The present appendix does two things. *Foremost*, it studies formal properties of truthmaker logics (TMLs), specifically compactness and decidability. While compactness and decidability of a variant of truthmaker logic already were proven in Fine and Jago 2019, an alternative proof method general enough to obtain compactness and decidability of a family of truthmaker logics is put forward. The fundamental results enabling these proofs are (1) 'standard translations' into first-order logic, and (2) a truthmaker analogue of the finite model property. *Second*, it studies how TMLs relate to MILs through translations using the one of van Benthem (2019), with the aim of (1) casting an interesting light on either logic in terms of the other, and (2) allowing for transfer of results.

This work is structured as follows: B.1 defines several TMLs; B.2 achieves compactness and recursive enumerability through standard translations; B.3 develops and proves the FMP for TMLs and concludes decidability; and B.4 investigates how TMLs connect with MILs through translations.

B.1. Defining the logics

Before presenting the actual compactness and decidability proofs of this first part of the appendix, we set the stage by formally laying out various truthmaker logics.

Definition B.1.1 (Language)**.** The language $\mathcal{L}_T$ of truthmaker logics is defined using a countable set of proposition letters $\mathbf{P}$. The formulas $\varphi \in \mathcal{L}_T$ are then given by the following BNF-grammar

$$\varphi ::= \ p \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi,$$

where $p \in \mathbf{P}$. $\dashv$

Truthmaker logics are defined semantically and rather commonly in terms of join-semilattices.[23] Formally we define as follows:

Definition B.1.2 (Frames and models)**.** A *frame* for $\mathcal{L}_T$ is a pair $\mathbb{F} = (S, \leqslant)$ where

- S is a set; and
- $\leqslant$ is a join-semilattice on W.

A *model* for $\mathcal{L}_T$ is a quadruple $\mathbb{M} = (S, \leqslant, V^+, V^-)$ where

[23]Quite frequently, TMLs are also defined on other structures than join-semilattices; we will get back to this later.

- $(S, \leqslant)$ is a frame; and
- V^+ and V^- are valuations on S, that is: functions $V^+, V^- : \mathbf{P} \to \mathcal{P}(S)$. $\dashv$

In some truthmaker logics valuations do not go to the powerset, but have additional requirements. The following alterations are frequently used in the literature:

- Valuations are required to be closed under binary joins.
- Valuations must be such that no state both 'truthmakes' and 'falsitymakes' a propositional letter; i.e., $V^+(p) \cap V^-(p) = \varnothing$ for all $p \in \mathbf{P}$.
- Each proposition letter is required to be made true and/or false somewhere; i.e., $V^+(p) \neq \varnothing$ for all $p \in \mathbf{P}$ and/or $V^-(p) \neq \varnothing$ for all $p \in \mathbf{P}$.

Besides differing on the valuations, truthmaker logics can differ on the actual semantics. One version is the following:

Definition B.1.3 (Semantics)**.** Given a model $\mathbb{M} = (S, \leqslant, V^+, V^-)$ and a state $s \in S$, *truthmaking* and *falsitymaking* of a formula $\varphi \in \mathcal{L}_T$ at s in $\mathbb{M}$ (written $\mathbb{M}, s \Vdash^+ \varphi$ and $\mathbb{M}, s \Vdash^- \varphi$, respectively) are defined by the following recursive clauses:

$$
\begin{array}{lll}
\mathbb{M}, s \Vdash^+ p & \textbf{iff} & s \in V^+(p). \\
\mathbb{M}, s \Vdash^- p & \textbf{iff} & s \in V^-(p). \\
\mathbb{M}, s \Vdash^+ \neg\varphi & \textbf{iff} & \mathbb{M}, s \Vdash^- \varphi. \\
\mathbb{M}, s \Vdash^- \neg\varphi & \textbf{iff} & \mathbb{M}, s \Vdash^+ \varphi. \\
\mathbb{M}, s \Vdash^+ \varphi \wedge \psi & \textbf{iff} & \text{there exist } u, v \in S \text{ such that } \mathbb{M}, u \Vdash^+ \varphi,\ \mathbb{M}, v \Vdash^+ \psi, \\
 & & \qquad\qquad \text{and } s = \sup\{u, v\}. \\
\mathbb{M}, s \Vdash^- \varphi \wedge \psi & \textbf{iff} & \mathbb{M}, s \Vdash^- \varphi \ \textit{ or } \ \mathbb{M}, s \Vdash^- \psi. \\
\mathbb{M}, s \Vdash^+ \varphi \vee \psi & \textbf{iff} & \mathbb{M}, s \Vdash^+ \varphi \ \textit{ or } \ \mathbb{M}, s \Vdash^+ \psi. \\
\mathbb{M}, s \Vdash^- \varphi \vee \psi & \textbf{iff} & \text{there exist } u, v \in S \text{ such that } \mathbb{M}, u \Vdash^- \varphi,\ \mathbb{M}, v \Vdash^- \psi, \\
 & & \qquad\qquad \text{and } s = \sup\{u, v\}.
\end{array}
$$

Notions like *global truth(making)*, *validity*, *consequence*, etc. are defined as usual in possible-worlds semantics. In particular, consequence $\Gamma \Vdash^+ \varphi$ for $\Gamma \subseteq \mathcal{L}_T \ni \varphi$ is defined distributively. That is, $\Gamma \Vdash^+ \varphi$ holds iff whenever $\mathbb{M}, s \Vdash^+ \gamma$ for all $\gamma \in \Gamma$, $\mathbb{M}, s \Vdash^+ \varphi$.[24] $\dashv$

[24] Note that decidability of the distributive reading has as a special case decidability of the collective reading, which is where $\Gamma \Vdash^+ \varphi$ holds iff whenever $\mathbb{M}, s \Vdash^+ \bigwedge_{\gamma \in \Gamma} \gamma$, $\mathbb{M}, s \Vdash^+ \varphi$ (this is, of course, only defined for finite Γ [which suffices for decidability], but the definition can be canonically extended to the infinite case).

This semantics is 'non-inclusive'. Alterations, we are aware of, are:

- 'Inclusive' semantics where (1) $\mathbb{M}, s \Vdash^+ \varphi \wedge \psi$ also suffices for $\mathbb{M}, s \Vdash^+ \varphi \vee \psi$, and, analogously, (2) $\mathbb{M}, s \Vdash^- \varphi \vee \psi$ also suffices for $\mathbb{M}, s \Vdash^- \varphi \wedge \psi$.
- Semantics where the disjunction is defined in terms of infimum instead – mirroring how conjunction is defined in terms of supremum.[25]

B.2. Compactness and r.e.

With these various definitions laid out, we, first, present the compactness and decidability results proved by Fine and Jago (2019) before, second, presenting our results and methods of proof.

Theorem B.2.1 (Compactness (Fine and Jago 2019))**.** *The truthmaker logic of join-semilattices with valuations closed under binary joins and inclusive semantics is compact; that is, **if** $\Gamma \Vdash^+ \varphi$, **then** $\Gamma_F \Vdash^+ \varphi$ for some finite $\Gamma_F \subseteq \Gamma$.*

Theorem B.2.2 (Decidability (Fine and Jago 2019))**.** *The truthmaker logic of join-semilattices with valuations closed under binary joins and inclusive semantics is decidable; that is, for finite Γ_F it is decidable whether $\Gamma_F \Vdash^+ \varphi$.*[26]

In what follows, we (1) provide another way of obtaining these results, and (2) show how this method generalizes to prove compactness and decidability of several truthmaker logics. We begin with compactness.

Essentially, the idea is the following: Since being a join-semilattice is first-order definable and the truth- and falsitymaking clauses of the truthmaker logics are as well, we get standard translations into FOL, and then compactness of FOL implies compactness of the truthmaker logics.

Spelt out a bit more, the key things are: (a) the translation uses a double recursion trick [as the one van Benthem (2019) use to translate TML into MIL] to reduce two consequence relations (truth- and falsitymaking) to one consequence relation; (b) contrary to truthmaker logics, the object logic FOL can speak about 'not truthmaking' (hence, also 'not falsitymaking') through regular first-order negation; and (c) everything is first-order definable.

We now present the translation into FOL, which can be thought of as the standard translation for TMLs. Or, alternatively, as basically the composition of the translation given in van Benthem (2019) from TMLs into MILs with a standard translation of MILs into FOL.

[25] Note that, under the 'disjunction-as-infimum' definition, it might be natural to require the frame to not only be a join-semilattice but also a meet-semilattice, or, in other words: a lattice.

[26] It should be noted that the proofs of these two theorems given in Fine and Jago 2019 (1) do somewhat generalize to other truthmaker logics, and (2) are of independent interest, nevertheless.

Definition B.2.3. The target FO-language is with equality, contains a binary relation symbol '$\leqslant$', and two unary predicate symbols 'P^T', 'P^F' for each propositional letter $p \in \mathbf{P}$. The translation is then given by these double recursive clauses:

$$
\begin{array}{rcl}
ST_x^+(p) & = & P^T x \\
ST_x^-(p) & = & P^F x \\
ST_x^+(\neg\phi) & = & ST_x^-(\phi) \\
ST_x^-(\neg\phi) & = & ST_x^+(\phi) \\
ST_x^+(\phi \wedge \psi) & = & \exists y, z\big(x = \sup\{y, z\} \wedge ST_y^+(\phi) \wedge ST_z^+(\psi)\big) \\
ST_x^-(\phi \wedge \psi) & = & ST_x^-(\phi) \vee ST_x^-(\psi) \\
ST_x^+(\phi \vee \psi) & = & ST_x^+(\phi) \vee ST_x^+(\psi) \\
ST_x^-(\phi \vee \psi) & = & \exists y, z\big(x = \sup\{y, z\} \wedge ST_y^-(\phi) \wedge ST_z^-(\psi)\big)
\end{array}
$$

where $x = \sup\{y, z\}$ is short for $y \leqslant x \wedge z \leqslant x \wedge \forall u([y \leqslant u \wedge z \leqslant u] \rightarrow x \leqslant u)$. ⊣

Examining the translation, the succeeding proposition is almost self-explanatory (see Blackburn et al. (2001, ch. 2) for similar results in the setting of modal logics).

Proposition B.2.4 (Correspondence). *For all models $\mathbb{M}$ and all $\varphi \in \mathcal{L}_T$, we have:*

(Loc.)	*For all states* $s \in \mathbb{M}$:	(i)	$\mathbb{M}, s \Vdash^+ \varphi$	***iff***	$\mathbb{M} \vDash ST_x^+(\varphi)[s]$; *and*
		(ii)	$\mathbb{M}, s \Vdash^- \varphi$	***iff***	$\mathbb{M} \vDash ST_x^-(\varphi)[s]$.
(Glo.)		(i′)	$\mathbb{M} \Vdash^+ \varphi$	***iff***	$\mathbb{M} \vDash \forall x ST_x^+(\varphi)$.
		(ii′)	$\mathbb{M} \Vdash^- \varphi$	***iff***	$\mathbb{M} \vDash \forall x ST_x^-(\varphi)$.

To be clear, on the right-hand side of the 'iff's, strictly speaking, '$\mathbb{M}$' refers to the corresponding first-order definition of the truthmaker model $\mathbb{M}$.

Proof. An easy induction shows (i) and (ii), which then imply (i′) and (ii′), respectively. □

Definition B.2.5. Denote the FO-formula defining being a join-semilattice J; i.e., J is the conjunction of the formulas for refl., tr., anti-symm., and having all binary joins. ⊣

Theorem B.2.6 (Compactness). *All versions of truthmaker logics mentioned above are compact.*

Proof. First, we give the compactness proof for the truthmaker logic of join-semilattices with valuations going to the powerset and the non-inclusive semantics of Definition B.1.3. Second, we outline how the proof is modified to apply to other truthmaker logics.

Let $(\Gamma \cup \{\varphi\}) \subseteq \mathcal{L}_T$ be arbitrary, and set $ST^+_x(\Gamma) := \{ST^+_x(\gamma) \mid \gamma \in \Gamma\}$. Then

$$\Gamma \Vdash^+ \varphi \quad \overset{(i)}{\textbf{iff}} \quad ST^+_x(\Gamma) \cup \{J\} \vDash ST^+_x(\varphi)$$
$$\overset{(c)}{\textbf{iff}} \quad ST^+_x(\Gamma_F) \cup \{J\} \vDash ST^+_x(\varphi) \quad \overset{(i)}{\textbf{iff}} \quad \Gamma_F \Vdash^+ \varphi$$

where Γ_F is a finite subset of Γ obtained via compactness of FOL in the step *(c)*, and *(i)* follows from the first assertion of the above-stated proposition. This shows compactness.

Note that a symmetric argument argument shows compactness w.r.t. the falsitymaking consequence '$\Vdash^-$'.

The reasons this proof lifts to all of the truthmaker variants mentioned previously (and more) are:

(Sem) The various sorts of semantics for the connectives all admit a standard translation so that Proposition B.2.4 holds.

(Val) For any propositional variable $p \in \mathbf{P}$, all of the listed potential conditions on its valuation can be defined by a first-order formula, denote it V_p. Thus, for the proof to go through, it is simply a matter of changing the first-order premise '$\{J\}$' to '$\{J\} \cup \{V_p \mid p \in \mathbf{P}\}$' in the just-proven sequence of 'iff's. □

Now for decidability of the truthmaker logics. First off, we observe that we achieve recursive enumerability (r.e.) through our standard translations connecting truthmaker logics to first-order logic.

Proposition B.2.7 (Recursive enumerability)**.** *All versions of truthmaker logics mentioned above are recursively enumerable; that is, there is an effective procedure for enumerating the pairs* (Γ_F, φ) *s.t.* $\Gamma_F \Vdash^+ \varphi$ *for finite* Γ_F.

Proof. Once again, we begin by covering the case of our set out truthmaker logic, before explaining how the proof generalizes to the other mentioned TMLs.

For this, simply observe that for any such (Γ_F, φ), we have that

$$\Gamma_F \Vdash^+ \varphi \quad \textbf{iff} \quad ST^+_x(\Gamma_F) \cup \{J\} \vDash ST^+_x(\varphi)$$
$$\overset{(ii)}{\textbf{iff}} \quad \vDash \bigwedge(ST^+_x(\Gamma_F) \cup \{J\}) \to ST^+_x(\varphi),$$

where (ii) follows by there being finitely many premises and the first-order semantics for conjunction and implication; i.e., essentially, the deduction theorem of FOL in a semantic disguise.

Now, since FOL is r.e. (and this procedure of constructing the formula $\bigwedge(ST^+_x(\Gamma_F) \cup \{J\}) \to ST^+_x(\varphi)$ from a pair (Γ_F, φ), evidently, is effective), we have attained r.e.

Albeit the argument of (Sem) still go through to account for why this proof generalizes to other TMLs, the argument given in (Val) pertaining to potential requirements on valuations does not generalize straight away. The problem is that the set $\{V_p \mid p \in \mathbf{P}\}$ is infinite. Fortunately, we can restrict this set to the propositional variables occurring in $\Gamma_F \cup \{\varphi\}$, thus obtaining a finite set of formulas instead, which is adequate for the proof to apply to these truthmaker logics restricting the admissible valuations. □

B.3. FMP and co-r.e.

Now, to establish decidability, it remains to prove co-r.e.; viz., giving an effective procedure for enumerating the pairs (Γ_F, φ) s.t. $\Gamma_F \nVdash^+ \varphi$ for finite Γ_F. This will be our main concern in this section.

In many a logic, not least in modal logic, the most common way of establishing co-r.e. is by means of proving the FMP. Mirroring this, we develop and prove what, arguably, is the truthmaker analogue of the FMP.

Before doing so, notice that a direct analogue of the FMP, namely that whenever a formula is made true (resp. false) [or truth-refuted (resp. falsity-refuted)] it is made true (resp. false) [or truth-refuted (resp. falsity-refuted)] in a finite model, is trivial and unhelpful for the purpose at hand: the single-state model making true and false all propositional letters [or none at all], makes true and false all formulas [or none at all] in general. And, importantly, this does nothing for proving co-r.e. Instead, we must prove an 'FMP' that—just like the FMP of, e.g., modal logic—allows for a model-theoretical proof of co-r.e. via some sort of finite-model checking. To do so, we need two preparatory lemmas and a definition.

Lemma B.3.1. *Suppose* $\mathbb{M}_0 = (S_0, \leqslant_0, V_0^+, V_0^-)$ *and* $\mathbb{M}_1 = (S_1, \leqslant_1, V_1^+, V_1^-)$ *are models s.t. (i)* $(S_1, \leqslant_1)$ *is a sub-join-semilattice of* $(S_0, \leqslant_0)$*, and (ii) for all* $p \in \mathbf{P}$*:*

$$V_1^+(p) = V_0^+(p) \cap S', \qquad V_1^-(p) = V_0^-(p) \cap S'.$$

Then for all formulas $\varphi \in \mathcal{L}_T$ *and all states* $s_1 \in S_1$*, we have that*

$$\mathbb{M}_0, s_1 \nVdash^+ \varphi \quad \Rightarrow \quad \mathbb{M}_1, s_1 \nVdash^+ \varphi$$

and

$$\mathbb{M}_0, s_1 \nVdash^- \varphi \quad \Rightarrow \quad \mathbb{M}_1, s_1 \nVdash^- \varphi.$$

Proof. By induction on $\varphi \in \mathcal{L}_T$. Base cases are by definition and the inductive steps follow by use of the IH and $(S_1, \leqslant_1)$ being a sub-join-semilattice of $(S_0, \leqslant_0)$.

To illuminate, we cover the inductive case of not truthmaking $\varphi = \psi \wedge \chi$. So suppose

$$\mathbb{M}_0, s_1 \nVdash^+ \psi \wedge \chi,$$

and let $(u_1, v_1) \in S_1 \times S_1$ be arbitrary s.t. $s_1 = \sup_1\{u_1, v_1\}$. Since $(S_1, \leqslant_1)$ is a sub-join-semilattice of $(S_0, \leqslant_0)$, the inclusion mapping $\iota : S_1 \hookrightarrow S_0$ is a join-semilattice homomorphism, hence $s_1 = \sup_0\{u_1, v_1\}$. But then since $\mathbb{M}_0, s_1 \nVdash^+ \psi \wedge \chi$, we must have that

$$\mathbb{M}_0, u_1 \nVdash^+ \psi \quad \textit{or} \quad \mathbb{M}_0, v_1 \nVdash^+ \chi,$$

whence, by the IH,

$$\mathbb{M}_1, u_1 \nVdash^+ \psi \quad \textit{or} \quad \mathbb{M}_1, v_1 \nVdash^+ \chi,$$

which suffices for the claim since (u_1, v_1) was arbitrary. □

Observe that this proof goes through for all of the previously mentioned semantics.[27]

Definition B.3.2. For any model $\mathbb{M}$, state $s \in \mathbb{M}$ and formula $\gamma \in \mathcal{L}_T$ s.t. $\mathbb{M}, s \Vdash^+ \gamma$ (resp. $\mathbb{M}, s \Vdash^- \gamma$), we define a set $\mathsf{T}(\gamma, s)$ (resp. $\mathsf{F}(\gamma, s)$), which we denote a *T-selection w.r.t.* (γ, s) (resp. *F-selection*), by the following recursive clauses:

The intuition for $\mathsf{T}(\gamma, s)$ (resp. $\mathsf{F}(\gamma, s)$) is that it is a set of states by virtue of which $s \Vdash^+ \gamma$ (resp. $s \Vdash^- \gamma$).

$$\begin{aligned}
&\mathsf{T}(\gamma, s) = \{s\} && \textbf{iff} \quad \gamma = p.\\
&\mathsf{F}(\gamma, s) = \{s\} && \textbf{iff} \quad \gamma = p.\\
&\mathsf{T}(\gamma, s) = \{s\} \cup \mathsf{F}(\varphi, s) && \textbf{iff} \quad \gamma = \neg\varphi.\\
&\mathsf{F}(\gamma, s) = \{s\} \cup \mathsf{T}(\varphi, s) && \textbf{iff} \quad \gamma = \neg\varphi.\\
&\mathsf{T}(\gamma, s) = \{s\} \cup \mathsf{T}(\varphi, u) \cup \mathsf{T}(\psi, v) && \textbf{iff} \quad \gamma = \varphi \wedge \psi \text{ and } \mathbb{M}, u \Vdash^+ \varphi,\\
& && \qquad \mathbb{M}, v \Vdash^+ \psi, s = \sup\{u, v\}.\\
&\mathsf{F}(\gamma, s) = \begin{cases} \{s\} \cup \mathsf{F}(\varphi, s), & \mathbb{M}, s \Vdash^- \varphi \\ \{s\} \cup \mathsf{F}(\psi, s), & \text{otherwise} \end{cases} && \textbf{iff} \quad \gamma = \varphi \wedge \psi.\\
&\mathsf{T}(\gamma, s) = \begin{cases} \{s\} \cup \mathsf{T}(\varphi, s), & \mathbb{M}, s \Vdash^+ \varphi \\ \{s\} \cup \mathsf{T}(\psi, s), & \text{otherwise} \end{cases} && \textbf{iff} \quad \gamma = \varphi \vee \psi.\\
&\mathsf{F}(\gamma, s) = \{s\} \cup \mathsf{F}(\varphi, u) \cup \mathsf{F}(\psi, v) && \textbf{iff} \quad \gamma = \varphi \vee \psi \text{ and } \mathbb{M}, u \Vdash^- \varphi,\\
& && \qquad \mathbb{M}, v \Vdash^- \psi, s = \sup\{u, v\}.
\end{aligned}$$

Clearly, this need not define unique sets because, e.g., the truthmaking case of $\gamma = \varphi \wedge \psi$ might be satisfied by multiple choices of u, v; for the purpose of what

[27] If one also deals with infima and, e.g., requires the underlying frames to be lattices, one shall assume $(S', \leqslant')$ to be a sub*lattice* of $(S, \leqslant)$.

we are to prove, this is irrelevant: any choice will do, so no reason to complicate the definition.[28] $\dashv$

Lemma B.3.3. *For any model $\mathbb{M}$, state $s \in \mathbb{M}$ and formula $\gamma \in \mathcal{L}_T$ s.t. $\mathbb{M}, s \Vdash^+ \gamma$ (resp. $\mathbb{M}, s \Vdash^- \gamma$), the corresponding set $T(\gamma, s)$ (resp. $F(\gamma, s)$) contains $\{s\}$ and is finite.*

Proof. By induction on $\gamma \in \mathcal{L}_T$. $\square$

With these results at hand, we can prove our truthmaker analogue of the FMP.

Proposition B.3.4 (Truthmaker FMP)**.** *For any model $\mathbb{M}_0 = (S_0, \leqslant_0, V_0^+, V_0^-)$, state $s \in S_0$, and finite set of formulas $\Gamma_F \subseteq \mathcal{L}_T$ s.t.*

$$\mathbb{M}_0, s \Vdash^+ \Gamma_F,$$

there is a finite submodel $\mathbb{M}_1$ s.t. (a)

$$\mathbb{M}_1, s \Vdash^+ \Gamma_F,$$

and (b) for all $\varphi \in \mathcal{L}_T$:

$$\mathbb{M}_0, s \nVdash^+ \varphi \quad \Rightarrow \quad \mathbb{M}_1, s \nVdash^+ \varphi.$$

Notice that this proposition implies the analogous proposition stated in terms of falsitymaking, qua negating all formulas.

Proof. For each $\gamma \in \Gamma_F$, choose a set $T(\gamma, s)$ according to the previous definition, and let $(S_1, \leqslant_1)$ be the sub-join-semilattice generated by $\bigcup_{\gamma \in \Gamma_F} T(\gamma, s)$. Since Γ_F is finite, the set of generators $\bigcup_{\gamma \in \Gamma_F} T(\gamma, s)$ is finite by the preceding lemma, hence S_1 is finite.

Further, as in Lemma B.3.1, define V_1^+ and V_1^- as the restrictions of V_0^+ and V_0^-, respectively. Then $\mathbb{M}_1 := (S_1, \leqslant_1, V_1^+, V_1^-)$ is a model.[29] By Lemma B.3.1, we have that (b) for all $\varphi \in \mathcal{L}_T$,

$$\mathbb{M}_0, s \nVdash^+ \varphi \quad \Rightarrow \quad \mathbb{M}_1, s \nVdash^+ \varphi.$$

It remains to show (a)

$$\mathbb{M}_1, s \Vdash^+ \Gamma_F.$$

To show so, we prove that for all formulas $\varphi \in \mathcal{L}_T$ and all generator states $s' \in \bigcup_{\gamma \in \Gamma_F} T(\gamma, s)$: if there is some T-selection $T(\varphi, s') \subseteq \bigcup_{\gamma \in \Gamma_F} T(\gamma, s)$ (resp.

[28]For other semantics (e.g. inclusive), we make a likewise modification of this definition.

[29]In case we require, e.g., all $V^+(p) \neq \varnothing$, the proof goes through by simply adding a state $s_{p^+} \in V_0^+(p)$ to the set of generators for all propositional letters occurring in the formulas $\Gamma_F \cup \{\varphi\}$.

F-selection $F(\varphi, s') \subseteq \bigcup_{\gamma \in \Gamma_F} T(\gamma, s)$), then

$$\begin{array}{lcl} \mathbb{M}_0, s' \Vdash^+ \varphi & \Rightarrow & \mathbb{M}_1, s' \Vdash^+ \varphi \\ (\text{resp. } \mathbb{M}_0, s' \Vdash^- \varphi & \Rightarrow & \mathbb{M}_1, s' \Vdash^- \varphi). \end{array}$$

The proof is by structural induction on φ. The base cases follow by definition of V_1^+ and V_1^-, and all of the Boolean cases are straightforward as well. Again, for clarity, we consider the case of truthmaking $\varphi = \psi \wedge \chi$.

Accordingly, suppose $s' \in \bigcup_{\gamma \in \Gamma_F} T(\gamma, s)$ and there is some T-selection $T(\psi \wedge \chi, s') \subseteq \bigcup_{\gamma \in \Gamma_F} T(\gamma, s)$. By definition of a T-selection, there must be some $\{u', v'\} \subseteq S_0$ s.t. (i) $\mathbb{M}_0, u' \Vdash^+ \psi$; (ii) $\mathbb{M}_0, v' \Vdash^+ \chi$; (iii) $s' = \sup_0\{u', v'\}$; and (iv)

$$T(\psi \wedge \chi, s') = \{s'\} \cup T(\psi, u') \cup T(\chi, v').$$

So by (i), (ii) and the IH, we get that

$$\mathbb{M}_1, u' \Vdash^+ \psi \qquad \textit{and} \qquad \mathbb{M}_1, v' \Vdash^+ \chi.$$

Moreover, cf. the preceding lemma and (iv), we have that

$$u' \in T(\psi, u') \subseteq T(\psi \wedge \chi, s') \subseteq \bigcup_{\gamma \in \Gamma_F} T(\gamma, s) \supseteq T(\psi \wedge \chi, s') \supseteq T(\chi, v') \ni v',$$

whence $\{u', v', s'\} \subseteq S_1$, and, thus, $s' = \sup_1\{u', v'\}$ because of (iii) and the fact that $(S_1, \leqslant_1)$ is a sub-join-semilattice of $(S_0, \leqslant_0)$. Consequently, $\mathbb{M}_1, s' \Vdash \psi \wedge \chi$ – exactly as we wanted. This completes the induction, which entails (a) as a special case; consequently, completing the proof. □

Again, using the canonical modifications, this proof works for all truthmaker logics mentioned so far. With this proven, we got co-r.e., hence also decidability, in our pocket.

Theorem B.3.5 (Decidability)**.** *All versions of truthmaker logics mentioned above are decidable.*

Proof. Since we have proven r.e. in Proposition B.2.7, it suffices to provide an effective procedure for the case of $\Gamma_F \nVdash^+ \varphi$ for finite Γ_F. This is done as follows:

1. Enumerate all finite join-semilattices.
2. For each such finite join-semilattice, check the finitely many valuations ($\Gamma_F \cup \{\varphi\}$ is finite, so only finitely many proposition letters occurs in $\Gamma_F \cup \{\varphi\}$) and states for whether we witness $\Gamma_F \nVdash^+ \varphi$.

This suffices because, by the previous proposition, we have that *if* $\mathbb{M}_0, s \Vdash^+ \Gamma_F$ and $\mathbb{M}_0, s \nVdash^+ \varphi$, *then* there is a finite model $\mathbb{M}_1$ s.t. $\mathbb{M}_1, s \Vdash^+ \Gamma_F$ and $\mathbb{M}_1, s \nVdash^+ \varphi$. □

Coming to a close, we end with a few remarks on (a) other truthmaker logics that might be of philosophical interest and for which our proof methods apply, and (b) a general limitation of our proof methods.

Remark B.3.6 (TMLs with non-FO-conditions)**.** We begin with the latter: in both our compactness and decidability proof we have made use of the conditions being first-order definable. An immediate limitation, thus, becomes when conditions are not; our proof methods for compactness and r.e. do not, e.g., apply to a truthmaker logic where the frames are taken to be posets with all (non-empty) joins because it is not first-order definable. And this is, indeed, a version of TMLs quite frequently considered in the literature – so as regards to (compactness and) r.e. our proof method falls short. That said, our proof of the FMP does not rely on something being FO-definable, and for, e.g., the case of posets with all (non-empty) joins we do get the FMP, hence co-r.e. ⊣

Remark B.3.7 (TMLs of preorders and posets)**.** Inversely, there are also cases where our compactness and r.e. proofs apply, but our proof of the FMP does not. If one were to consider a truthmaker logic where the frames simply are taken to be preorders or posets (if one, e.g., wants to consider a philosophical setting where any pair of states need not have a 'fusion'), our proofs adapt straightforwardly to obtain both compactness and r.e.; however, our proof of the FMP does not. Problem being that these frames are not algebraic, so we cannot generate subframes algebraically. Fortunately, there is a way of circumventing this problem combining a few ideas from this thesis. We sketch it here.

As in the FMP proof, for each $\gamma \in \Gamma_F$, choose a set $T(\gamma, s)$, but instead let

$$(S_1, \leqslant_1) := \left(\bigcup_{\gamma \in \Gamma_F} T(\gamma, s), \leqslant_0 \cap \left(\bigcup_{\gamma \in \Gamma_F} T(\gamma, s) \right)^2 \right).$$

Now, whenever $(x, y, z) \in S_1^3$ is s.t. $\sup_1\{y, z\} = x \neq \sup_0\{y, z\}$, add a 'dummy' d (cf. Lemma 2.11) seen by all and only the states in the least downset containing $\{y, z\}$ and closed under suprema (cf. Lemma 5.6). This results in the desired finite model. ⊣

Remark B.3.8 (Language enrichment)**.** As a last remark of this section, another kind of truthmaker logic for which our proofs straightforward are seen to apply are those defined by enriching the vocabulary with a binary connective $\wedge_c$

with the semantics of classical conjunction. Philosophically this enrichment can be motivated by, for instance, wanting to examine the interplay between the statements

Also including constants for falsum '⊥' and verum '⊤', stays within the realm of our proof methods.

(1) "[s exactly makes true φ] and [s exactly makes true ψ]"

(2) "s exactly makes true [φ and ψ]"

I.e., the pertinent point of discussion is whether/when exact truthmaking distributes over conjunction.

Without the extra connective "$\wedge_c$", these would be 'formalized' as

(1′) $\mathbb{M}, s \Vdash^{+} \varphi$ and $\mathbb{M}, s \Vdash^{+} \psi$

(2′) $\mathbb{M}, s \Vdash^{+} \varphi \wedge \psi$.

But with the extra connective, we enable the object language to make this distinction by formalizing (1) as

(1″) $\mathbb{M}, s \Vdash^{+} \varphi \wedge_c \psi$. ⊣

B.4. Fragments of MILs

The last two remarks of the foregoing section were not only interesting in their own right, they also served as teasers for the forthcoming and last study of this appendix: exploring how TMLs relate to MILs.

Our goal is two-fold: (1) to elucidate how TMLs can be seen as $\{\vee, \langle \sup \rangle\}$-fragments of MILs or, vice versa, how MILs can be seen as augmenting TMLs with classical negation '$\neg_c$', and (2) to instantiate how this connection allows for transfer of results by showing that the positive first-degree fragment of MIL_{Sem} is decidable.

We begin by presenting the translation of van Benthem (2019) from TMLs to MILs.

Definition B.4.1. The target language is $\mathcal{L}_M$ but where we have two propositional variables for each propositional variable $p \in \mathcal{L}_T$, namely p^T and p^F.[30] The translation is then given by these double recursive clauses:

$$
\begin{array}{rclcrcl}
(p)^{+} & = & p^{T}, & & (p)^{-} & = & p^{F}, \\
(\neg\varphi)^{+} & = & \varphi^{-}, & & (\neg\varphi)^{-} & = & \varphi^{+}, \\
(\varphi \wedge \psi)^{+} & = & \langle \sup \rangle \varphi^{+}\psi^{+}, & & (\varphi \wedge \psi)^{-} & = & \varphi^{-} \vee \psi^{-}, \\
(\varphi \vee \psi)^{+} & = & \varphi^{+} \vee \psi^{+}, & & (\varphi \vee \psi)^{-} & = & \langle \sup \rangle \varphi^{-}\psi^{-}.
\end{array}
$$
⊣

Inspecting the translation,[31] as van Benthem (2019) notes, we see that

[30] From a purely mathematical perspective, we can w.l.o.g. think of these as corresponding to whether p^T [resp. p^F] was even [odd] in our original enumeration of the propositional variables of $\mathcal{L}_M$.

[31] For, e.g., inclusive semantics the translation modifies canonically.

Proposition B.4.2 (Correspondence). *For all models $\mathbb{M}$ and all $\varphi \in \mathcal{L}_T$, we have:*

(Loc.) *For all states* $s \in \mathbb{M}$:	(i)	$\mathbb{M}, s \Vdash^+ \varphi$	***iff***	$\mathbb{M}, s \Vdash (\varphi)^+$; *and*
	(ii)	$\mathbb{M}, s \Vdash^- \varphi$	***iff***	$\mathbb{M}, s \Vdash (\varphi)^-$.
(Glo.)	(i')	$\mathbb{M} \Vdash^+ \varphi$	***iff***	$\mathbb{M} \Vdash (\varphi)^+$.
	(ii')	$\mathbb{M} \Vdash^- \varphi$	***iff***	$\mathbb{M} \Vdash (\varphi)^-$.

To be perfectly clear, on the right-hand side of the 'iff's, '$\mathbb{M}$' refers to the corresponding MIL-definition of the truthmaker model $\mathbb{M}$, and '$\Vdash$' to MIL-satisfaction.

It, thus, becomes obvious that for 'complementary' TMLs and MILs,[32] we get the following proposition (as stated in van Benthem (2019)):

Proposition B.4.3. *For all* $(\Gamma \cup \{\varphi\}) \subseteq \mathcal{L}_T$*:*

$$\Gamma \Vdash^{\pm} \varphi \qquad \textbf{\textit{iff}} \qquad (\Gamma)^{\pm} \Vdash (\varphi)^{\pm},$$

where $(\Gamma)^{\pm} := \{(\gamma)^{\pm} \mid \gamma \in \Gamma\}$.

With these results re-capped, we explore this translation a bit more. As stated, most glaring is that it, in a way, licenses us to characterize TMLs as the $\{\vee, \langle sup \rangle\}$-fragment of MILs—or MILs as augmenting TMLs with classical negation. To explicate this a bit further, consider the following translation:

Definition B.4.4. Let $\mathcal{L}_M^{\{p^T, p^F, \vee, \langle sup \rangle\}} \subseteq \mathcal{L}_M$ be the fragment of the language of basic modal information logic restricted to the propositional letters, connective '$\vee$' and modality '$\langle sup \rangle$'. Then for all $\varphi \in \mathcal{L}_M^{\{p^T, p^F, \vee, \langle sup \rangle\}}$, we recursively define its translation $(\varphi)^{\bullet}$ into $\mathcal{L}_T$ as follows:

$$(p^T)^{\bullet} = p, \qquad (p^F)^{\bullet} = \neg p,$$
$$(\langle sup \rangle \varphi\psi)^{\bullet} = \varphi^{\bullet} \wedge \psi^{\bullet}, \qquad (\varphi \vee \psi)^{\bullet} = \varphi^{\bullet} \vee \psi^{\bullet}. \quad \dashv$$

Once again, we can immediately deduce a correspondence proposition.

Proposition B.4.5 (Correspondence). *For all models $\mathbb{M}$ and all $\varphi \in \mathcal{L}_M^{\{p^T, p^F, \vee, \langle sup \rangle\}}$, we have:*

(Loc.) *For all states* $s \in \mathbb{M}$:	(i)	$\mathbb{M}, s \Vdash \varphi$	***iff***	$\mathbb{M}, s \Vdash^+ (\varphi)^{\bullet}$.
(Glo.)	(i')	$\mathbb{M} \Vdash \varphi$	***iff***	$\mathbb{M} \Vdash^+ (\varphi)^{\bullet}$.

Symmetric results for falsitymaking are achieved by a symmetric translation.

[32] I.e., whenever the logics are defined on the same class of structures with the same admissible valuations, and we do not have, say, disjunction in the TML defined in terms of infima without also having an infimum-modality in the MIL. So, for example, the main TML of the previous chapter is complementary to MIL_{Sem} but not to MIL_{Pre}.

Thus, the translations $(\cdot)^+$ *and* $(\cdot)^\bullet$ *are, essentially, each other's 'inverses':*

$$\text{For all } \varphi \in \mathcal{L}_T \text{ and all } \mathbb{M}, s: \quad \mathbb{M}, s \Vdash^+ \varphi \quad \textbf{iff} \quad \mathbb{M}, s \Vdash^+ \left(\varphi^+\right)^\bullet.$$

$$\text{For all } \varphi \in \mathcal{L}_M^{\{p^T, p^F, \vee, \langle \text{sup} \rangle\}} \text{ and all } \mathbb{M}, s: \quad \mathbb{M}, s \Vdash \varphi \quad \textbf{iff} \quad \mathbb{M}, s \Vdash (\varphi^\bullet)^+.$$

Corollary B.4.6 (Characterization)**.** *TMLs are (at least, in a mathematical precise sense) the* $\{\vee, \langle \text{sup} \rangle\}$*-fragments of MILs, or alternatively, MILs arise from augmenting TMLs with classical negation.*

Besides providing a perspicuous view on both TMLs and MILs, these translations are also mathematically conducive; e.g., enabling us to transfer decidability results. We close off this appendix by quickly exemplifying this, showing that the 'positive first-degree' fragment of *MIL*$_{Sem}$ is decidable.

Definition B.4.7. We say that a formula $\varphi \in \mathcal{L}_M^{\{\top, \bot, \langle \text{sup} \rangle, \wedge, \vee\}}$ is *positive*; that is, if it is constructed from propositional letters, $\top$, and $\bot$ by applying $\vee$, $\langle \text{sup} \rangle$, $\wedge$.

Further, we call a formula $\varphi \to \psi \in \mathcal{L}_M$ a *positive first-degree formula* :iff φ and ψ are positive.

Lastly, we define the *positive first-degree fragment* (of $\mathcal{L}_M$) as the set of positive first-degree formulas. ⊣

Corollary B.4.8. *MIL*$_{Sem}$ *restricted to the positive first-degree fragment (of* $\mathcal{L}_M$*) is decidable.*

Proof. Given any such formula $\varphi \to \psi$, we have that

$$MIL_{Sem} \Vdash \varphi \to \psi \qquad \textbf{iff} \qquad (\varphi)^\bullet \Vdash^+ (\psi)^\bullet$$

where $(\cdot)^\bullet$ refers to the translation given previously, canonically extended for $\top$, $\bot$ and $\wedge$ [which goes to $\top$, $\bot$ and $\wedge_c$; i.e., we use the last remark of the previous section]. Since this translation is effective, and we have shown the translated problem to be decidable, we have proven the claim. □

Bibliography

Aloni, M. (2022). "Logic and conversation: The case of free choice." In: *Semantics and Pragmatics* (cit. on p. 2).

Anderson, A. R., N. D. Belnap, and J. M. Dunn (1992). *Entailment, Vol. II: The Logic of Relevance and Necessity*. Princeton University Press. DOI: doi:10.1515/9781400887071 (cit. on p. 2).

Blackburn, P., M. d. Rijke, and Y. Venema (2001). *Modal Logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press. DOI: 10.1017/CBO9781107050884 (cit. on pp. 1, 8, 11, 14, 16, 35, 111).

Burgess, J. (10/1982). "Axioms for tense logic. I. "Since" and "until"." In: *Notre Dame Journal of Formal Logic* 23. DOI: 10.1305/ndjfl/1093870149 (cit. on p. 16).

Burris, S. N. and H. P. Sankappanavar (1981). *A Course in Universal Algebra*. Graduate Texts in Mathematics. Springer (cit. on p. 60).

Buszkowski, W. (03/2021). "Lambek Calculus with Classical Logic." In: *Natural Language Processing in Artificial Intelligence—NLPinAI 2020*, pp. 1–36. DOI: 10.1007/978-3-030-63787-3_1 (cit. on pp. iii, 5, 34, 47 sq.).

Buszkowski, W. and M. Farulewski (2009). "Nonassociative Lambek Calculus with Additives and Context-Free Languages." In: *Languages: From Formal to Natural. Lecture Notes in Computer Science*. Ed. by K. M. K. S. Grumberg O. and S. Wintner. Vol. 5533. Springer, Berlin, Heidelberg. DOI: 10.1007/978-3-642-01748-3_4 (cit. on p. 34).

Charlwood, G. (1981). "An axiomatic version of positive semilattice relevance logic." In: *Journal of Symbolic Logic* 46, pp. 233 –239 (cit. on p. 98).

Fine, K. (1976). "Completeness for the semilattice semantics with disjunction and conjunction (abstract)." In: *Journal of Symbolic Logic* 41(2), p. 560 (cit. on p. 98).

Fine, K. (2017). "Truthmaker Semantics." In: *A Companion to the Philosophy of Language*. John Wiley Sons, Ltd. Chap. 22, pp. 556–577. DOI: https://doi.org/10.1002/9781118972090.ch22 (cit. on pp. 2, 4).

Fine, K. and M. Jago (2019). "Logic for Exact Entailment." In: *The Review of Symbolic Logic* 12.3, 536–556. DOI: 10.1017/S1755020318000151 (cit. on pp. iii, 108, 110).

Jongh, D. and F. Veltman (1999). "Intensional Logics." In: Available at: https://staff.fnwi.uva.nl/f.j.m.m.veltman/papers/FVeltman-intlog.pdf (cit. on p. 16).

Kaminski, M. and N. Francez (2014). "Relational Semantics of the Lambek Calculus Extended with Classical Propositional Logic." In: *Stud Logica* 102, 479—497. DOI: 10.1007/s11225-013-9474-7 (cit. on pp. 34 sq.).

Kurucz, Á. et al. (1995). "Decidable and undecidable logics with a binary modality." In: *Journal of Logic, Language and Information* 4, pp. 191–206 (cit. on pp. 98 sq.).

Lambek, J. (1958). "The Mathematics of Sentence Structure." In: *American Mathematical Monthly* 65, pp. 154–170 (cit. on p. 4).

Urquhart, A. (1972). "Semantics for relevant logics." In: *Journal of Symbolic Logic* 37, pp. 159 –169 (cit. on pp. 4, 97).

— (1973). "The Semantics of Entailment." PhD thesis. University of Pittsburgh (cit. on pp. 4, 97).

— (1984). "The undecidability of entailment and relevant implication." In: *Journal of Symbolic Logic* 49, pp. 1059 –1073 (cit. on p. 99).

— (2016). "Relevance Logic: Problems Open and Closed." In: *The Australasian Journal of Logic* 13 (cit. on p. 99).

Van Benthem, J. (1996). "Modal Logic as a Theory of Information." In: *Logic and Reality. Essays on the Legacy of Arthur Prior*. Ed. by J. Copeland. Clarendon Press, Oxford, pp. 135–168 (cit. on pp. iii, 2 sq., 97).

— (10/2017). "Constructive agents." In: *Indagationes Mathematicae* 29. DOI: 10.1016/j.indag.2017.10.004 (cit. on pp. iii, 3, 5, 25).

Van Benthem, J. (2019). "Implicit and Explicit Stances in Logic." In: *Journal of Philosophical Logic* 48.3, pp. 571–601. DOI: 10.1007/s10992-018-9485-y (cit. on pp. iii, 3, 5, 25, 108, 110, 118 sq.).

— (Forthcoming). "Relational Patterns, Partiality, and Set Lifting in Modal Semantics." In: *Kripke volume in the series 'Outstanding Contributions to Logic'*. Ed. by Y. Weiss. Springer.
Preprint available at: https://eprints.illc.uva.nl/id/eprint/1773/ (cit. on pp. 3 sqq., 13, 25, 97).

Van Benthem, J. and N. Bezhanishvili (2022). "Modal Structures in Groups and Vector Spaces." In: Preprint available at: https://eprints.illc.uva.nl/id/eprint/1871/1/Vectors-paper.pdf (cit. on p. 3).

Van Fraassen, B. C. (1969). "Facts and Tautological Entailments." In: *The Journal of Philosophy* 66, pp. 477–487 (cit. on p. 2).

Wang, X. and Y. Wang (2022). "Tense Logics over Lattices." In: *WoLLIC 2022* (cit. on pp. 3, 96, 100).

Weiss, Y. (2021). "A Conservative Negation Extension of Positive Semilattice Logic Without the Finite Model Property." In: *Studia Logica* 109, pp. 125–136 (cit. on p. 98).

Yang, F. and J. Väänänen (2016). "Propositional logics of dependence." In: *Annals of Pure and Applied Logic* 167.7, pp. 557–589. DOI: 10.1016/j.apal.2016.03.003 (cit. on p. 2).

— (07/2017). "Propositional Team Logics." In: *Annals of Pure and Applied Logic* 168, 1406–1441. DOI: 10.1016/j.apal.2017.01.007 (cit. on p. 2).