Sahlqvist preservation for topological fixed-point logic

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Abstract

We introduce a new order-topological semantics for the positive modal mu-calculus over modal compact Hausdorff spaces, which are generalizations of descriptive frames. We define Sahlqvist sequents in this language and prove Esakia's lemma and Sahlqvist preservation theorem in this semantics. We show that every Sahlqvist sequent has a frame correspondent in first-order logic with fixed-point operators.
Keywords: Modal mu-calculus, order-topological semantics, Sahlqvist correspondence, canonicity.

1 Introduction

By *topological fixed-point logic* we mean a family of fixed-point logics that admit topological interpretations, and where the fixed-point operators are evaluated with respect to these topological interpretations. In this paper, which brings together the methods and results of [4] and [5], we concentrate on a variant of topological fixed-point logic whose models are modal compact Hausdorff space (MKH-spaces for short). These spaces were introduced in [4] as a generalization of modal spaces (descriptive frames), which are central order-topological structures appearing in modal logic. In [4] duality and various properties of MKH-spaces were studied for positive modal languages without any fixed-point operators. [5] studied topological fixed-point logic based on descriptive μ-frames. This is a restricted class of modal spaces (descriptive frames) that admits a topological interpretation of fixed-point operators. In this paper, we investigate topological semantics of fixed-point operators (we consider only the least fixed-point operator) similar to the ones discussed in [5], but in the framework of MKH-spaces of [4]. This way the methods of [5] are extended to a wider class of models and the language of [4] is expanded by incorporating (topological) fixed-point operators.

The duality between modal algebras and modal spaces [17] plays an important role in modal logic (see eg. [6, 19]). Modal algebras are obtained by extending Boolean algebras with a normal and additive unary operator. Modal spaces are Stone spaces (compact, Hausdorff and zero-dimensional spaces) equipped with a binary relation satisfying additional conditions. It is known that modal spaces are isomorphic to descriptive frames [22], [6, Chapter 5]. This duality is an extension of the celebrated Stone duality between Boolean algebras and Stone spaces [24]. Every system of modal logic is complete with respect to modal algebras and via this duality with respect to modal spaces e.g., [22], [6, Chapter 5].

Modal spaces also admit a coalgebraic representation. The Vietoris space of closed sets of a Stone space [30], is a standard construction in topology. The construction naturally extends to an endofunctor on a Stone space. It turns out that the category of modal spaces and continuous p-morphisms, is isomorphic to the category of coalgebras for the Vietoris functor on the category of Stone spaces and continuous maps [1, 20]. The Vietoris functor, however, can be defined in a more general setting of compact Hausdorff spaces.

An MKH-space is defined as a concrete realization of the Vietoris functor on a compact Hausdorff space. In particular, an MKH-space is a tuple (W, R) where W is a compact Hausdorff space and R is a continuous relation on W, meaning the corresponding map from W to its Vietoris space is continuous. An example of an MKH-space is the interval $[0, 1]$ with the binary relation $\leq$. It is well known that $[0, 1]$ is compact and Hausdorff, but not zero-dimensional. In [4] modal compact regular frames and modal DeVries algebras were introduced as algebraic structures dual to MKH-spaces, and a Sahlqvist preservation and correspondence result for the positive modal language was proved.

In this paper, we advance the study of MKH-spaces by extending the positive modal language of [4] with fixed-point operators. We introduce and compare the different semantics of positive modal language extended with a least fixed-point operator over MKH-spaces. In modal spaces formulas are evaluated as clopen (both closed and open) sets. Note that clopen subsets, in general, do not form a complete lattice. Thus, there may exist fixed-point formulas that cannot be interpreted on a modal space as an intersection of *clopen pre-fixed points*. To overcome this, *descriptive mu-frames* (modal mu-spaces) were introduced in [3] as those descriptive frames that admit a topological interpretation of the least fixed-point operator. The main motivation to study this semantics is that every axiomatic system of modal mu-calculus is complete with respect to descriptive mu-frames [3]. Moreover, powerful Sahlqvist correspondence and completeness results hold for mu-calculus over descriptive mu-frames [5]. Unlike descriptive frames, every least fixed-point formula can be interpreted in an MKH-space as the interior of the intersection of *open pre-fixed points*. This makes MKH-spaces a natural candidate to study topological semantics of fixed-point operators.

Sahlqvist correspondence and completeness theorem [21, 26, 27] is a cornerstone result in classical modal logic. The correspondence result states that every formula in the Sahlqvist class, which is a syntactically defined class of formulas, corresponds to an *elementary* (first-order definable) condition on frames. The first-order condition can be effectively obtained from the Sahlqvist formula. The completeness result states that every modal logic obtained by adding Sahlqvist formulas to the basic modal logic **K** is sound and complete with respect to a first-order definable class of Kripke frames. A simplified proof of Sahlqvist theorem was given by Sambin and Vaccaro [23] using order-topological methods. A crucial lemma in their proof of completeness is Esakia's lemma [12]. Using the lemma, the valuation of a positive formulaon a closed assignment can be expressed as an intersection of valuations of the formula on clopen assignments. Goranko and Vakarelov [14] generalize the results in [23] to the class of inductive formulas, which properly extend Sahlqvist formulas. In [9] Conradie and Palmigiano use duality theory to extend the results in [14] to distributive modal logic. In particular, they develop an Ackermann lemma [2] based algorithm for correspondence and canonicity of inductive formulas. For an overview of this approach, we refer to [8]. Recently, Sahlqvist theory has also been extended to the modal mu-calculus. A Sahlqvist correspondence theorem for the mu-calculus was shown in [29] by extending the classical Sahlqvist-van Benthem algorithm using the PIA formulas introduced in an earlier work by van Benthem [28]. A related work [7] extends the algorithmic-algebraic approach in [9] to intuitionistic modal mu-calculus. Finally, (as already mentioned above) Sahlqvist completeness and correspondence result for clopen semantics for modal mu-calculus on descriptive mu-frames was proved in [5].

The key contributions of this paper is a Sahlqvist preservation theorem for topological fixed-point logic over MKH-spaces. We define a Sahlqvist sequent in our language. By preservation, we mean the following: a Sahlqvist sequent in the language of the positive modal logic with a least fixed-point operator is valid under arbitrary open assignments if, and only if, it is valid under arbitrary set-theoretic assignments. Since we are no longer in the setting of zero-dimensional spaces, the Sahlqvist preservation result in [5] fails for the clopen semantics for the fixed-point operator. We overcome this by introducing an alternative topological semantics where the pre-fixed point of a map f is defined as an open set U such that $f(\overline{U}) \subseteq U$, where $\overline{U}$ is the topological closure of a set U. We call such sets *topological pre-fixed points.*

The fixed-point is then computed as an intersection of all topological pre-fixed points. For this new semantics and *shallow modal formulas* we prove an analogue of Esakia's lemma, from which our preservation result follows immediately. We show that the new semantics has a nice algebraic counterpart when restricted to shallow modal formulas. We also show that the Sahlqvist sequent in our language has a frame correspondent in LFP, which is first-order language extended with fixed-point operators with topological interpretations. We also provide a few examples of Sahlqvist sequents, their corresponding LFP-formulas and their semantics in MKH-spaces.

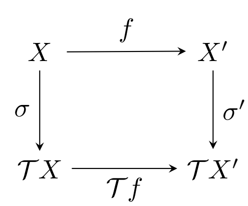
Finally, we note on an unfortunate overlap of terminology in modal logic and point-free topology: the meaning of the term "frame" in modal logic differs from its meaning in point-free topology. By now both terms are well established in the modal logic and point-free topology literature. We follow these standard terminology hoping that it will not generate any confusion. In particular, in Section 4 of the paper we use the term "frame" in the context of point-free topology and in Section 6 we refer to "frame conditions" which have a standard meaning in the modal logic literature.

The paper is organized as follows: in Section 2, we introduce preliminary definitions on Vietoris construction and MKH-spaces. In Section 3 we introduce and compare different semantics of the least fixed-point operator over MKH-spaces. In Section 4 we look into the algebraic semantics for our language. In Section 5 we show the Esakia's lemma and Sahlqvist preservation theorem. In Section 6, we prove a correspondence theorem for Sahlqvist sequents followed by examples in Section 7. We conclude and present directions for future research in Section 8.

2 Preliminaries

In this section we recall a few preliminary definitions from [4]. Let W be a non-empty set and $R \subseteq W \times W$ be a binary relation on W. For $w \in W$, define $R[w] = \{v \in W : wRv\}$ and $R^{-1}[w] = \{v \in W : vRw\}$. Also, for $S \subseteq W$, $R[S] = \{w \in W : R^{-1}[w] \cap S \neq \varnothing\}$ and $R^{-1}[S] = \{w \in W : R[w] \cap S \neq \varnothing\}$.

Definition 2.1 (**$\mathcal{T}$-Coalgebra).** Let C be a category and let $\mathcal{T} : \mathsf{C} \to \mathsf{C}$ be an endofunctor. A $\mathcal{T}$-coalgebra is a pair (X, σ), where $\sigma : X \to \mathcal{T}X$ is a morphism in C. A morphism between two coalgebras (X, σ) and (X', σ') is a morphism f in C such that the following diagram commutes:



Definition 2.2 (**Modal Space).** A modal space is a pair (W, R) where W is a Stone space and R is a binary relation on W satisfying (i) $R[x]$ is closed for each $x \in W$ and (ii) $R^{-1}[U]$ is clopen for each clopen $U \subseteq W$. For modal spaces, (W, R) and (W', R'), a function $f : W \to W'$ is a p-morphism if (i)wRw' implies $f(w)Rf(w')$ and (ii) $f(w)Rv$ implies there is $u \in W$ with wRu and $f(u) = v$. Let MS be the category of modal spaces and continuous p-morphisms.

Definition 2.3 (**Vietoris Space).** For a topological space W and $U \subseteq W$ an open set, consider the sets

$$\Box U = \{F \subseteq W : F \text{ is closed and } F \subseteq U\}$$
$$\Diamond U = \{F \subseteq W : F \text{ is closed and } F \cap U \neq \varnothing\}.$$

Then the Vietoris space $\mathcal{V}(W)$ of W is defined to have the closed sets of W as its points, and the collection of all sets $\Box U, \Diamond U$, where $U \subseteq W$ is open, as a subbasis for its topology.

It is a standard result in topology that if W is a Stone space, then so is $\mathcal{V}(W)$ (see, eg., [11], p. 380). Let Stone be the category of Stone spaces and continuous maps. The Vietoris construction $\mathcal{V}$ extends to a functor $\mathcal{V} : \mathsf{Stone} \to \mathsf{Stone}$, which sends a Stone space W to $\mathcal{V}(W)$ and a continuous map $f : W \to Y$ to $\mathcal{V}(f)$ where $\mathcal{V}(f)(F) = f[F]$ for all closed sets $F \subseteq W$. In considering $\mathcal{V}$-coalgebras, note that if R is a relation on W, then $\rho_R : W \to \mathcal{P}(W)$ given by $\rho_R(w) = R[w]$ is a well-defined continuous map from W to $\mathcal{V}(W)$ iff (W, R) is a modal space. This leads to the following theorem.

Theorem 2.4. *([1, 20, 12])* MS *is isomorphic to the category of* $\mathcal{V}$*-coalgebras on* Stone.

It is known that the Vietoris functor can be defined in the more general setting of compact Hausdorff spaces (see, e.g., [11], p. 244). The category of compact Hausdorff spaces and continuous maps is denoted by KHaus. The Vietoris construction yields a functor $\mathcal{V} : \mathsf{KHaus} \to \mathsf{KHaus}$ where a continuous map $f : W \to Y$ is taken to $\mathcal{V}(f)$ with $\mathcal{V}(f)(F) = f[F]$ for all closed sets $F \subseteq W$. It is natural to consider coalgebras for this functor. We first define the notion of a continuous relation on a compact Hausdorff space.

Definition 2.5 (**Continuous Relation**)**.** We say, a relation R on a compact Hausdorff space W is *point closed*, if the relational image $R[w]$ is a closed set for each $w \in W$. Further, R is *continuous* if it is point closed and the map $\rho_R : W \to \mathcal{V}(W)$, taking a point w to $R[w]$ is a continuous map from the space W to its Vietoris space $\mathcal{V}(W)$. In other words, R is continuous if (X, ρ_R) is a Vietoris coalgebra.

Proposition 2.6. *([4]) A relation R on a compact Hausdorff space W is continuous iff R satisfies the following conditions:*

1. *$R[w]$ is closed for each $w \in W$.*
2. *$R^{-1}[F]$ is closed for each closed $F \subseteq W$.*
3. *$R^{-1}[U]$ is open for each open $U \subseteq W$.*

Definition 2.7 (**Modal Compact Hausdorff space**)**.** A *modal compact Hausdorff space* or an *MKH-space* is a tuple (W, R) such that W is a compact Hausdorff space and R is a continuous relation on W. Let MKHaus be the category of MKH-spaces and continuous p-morphisms.

Theorem 2.8. *([4])* MKHaus *is isomorphic to the category of* $\mathcal{V}$*-coalgebras on* KHaus.

3 Topological fixed-point semantics

In this section, we discuss various semantics for the modal mu-calculus on modal compact Hausdorff spaces. We first recall the Knaster-Tarski theorem for complete lattices.

Theorem 3.1 (**Knaster-Tarski Theorem**)**.** *Let $(L, \leq)$ be a complete lattice and $f : L \to L$ be a monotone map, that is, for each $a, b \in L$, with $a \leq b$ we have $f(a) \leq f(b)$. The Knaster-Tarski theorem states that f has a least fixed-point $LFP(f)$, which can be computed as*

$$LFP(f) = \bigwedge \{a \in L : f(a) \leq a\}$$

The least fixed-point of f or $LFP(f)$ can be computed in another way. For an ordinal α, let $f^0(0) = 0$, $f^\alpha(0) = f(f^\beta(0))$ if $\alpha = \beta + 1$, and $f^\alpha(0) = \bigvee_{\beta \leq \alpha} f^\beta(0)$, if α is a limit ordinal. Then $LFP(f) = f^\alpha(0)$, for some ordinal α such that $f^{\alpha+1}(0) = f^\alpha(0)$.

We restrict our language to positive modal logic. Given a set Prop of countably infinite propositional variables, the *modal mu-formulas* in our language are inductively defined by the following rule

$$\varphi := \bot \mid \top \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mu x\varphi$$

where $p, x \in \mathsf{Prop}$. Note that we have only the least fixed-point operator in our language. An occurrence of x in φ is said to be bound if it is in the scope of a μx, and free, otherwise. We interpret formulas in our language over MKH-spaces. Given an MKH-space (W, R), let $\mathfrak{F} \subseteq \mathcal{P}(W)$ be such that $(\mathfrak{F}, \subseteq)$ is a sublattice[1] of $(\mathcal{P}(W), \subseteq)$. That is, $\varnothing, W \in \mathfrak{F}$ and if $U, V \in \mathfrak{F}$, then $U \cap V \in \mathfrak{F}$ and $U \cup V \in \mathfrak{F}$. We denote the (infinite) meets and joins in $\mathfrak{F}$ by $\bigwedge^{\mathfrak{F}}$ and $\bigvee^{\mathfrak{F}}$, respectively. If $(\mathfrak{F}, \subseteq)$ is complete, then infinite mets and joins always exist. As we will see below, $\bigwedge^{\mathfrak{F}}$ and $\bigvee^{\mathfrak{F}}$ may differ from set-theoretic intersection and union. An *assignment* h is a map from the set of propositional variables Prop to $\mathfrak{F}$. For each modal mu-formula φ, we denote by $[\![\varphi]\!]_h^{\mathfrak{F}}$, the set of points satisfying φ under assignment h. Given $S \subseteq W$, let $\langle R\rangle(S) = R^{-1}[S]$ and $[R](S) = W \backslash (R^{-1}[W \backslash S])$. We define the semantics of a modal mu-formula φ, by induction on the complexity of formulas as follows:

$$\begin{aligned}
[\![\bot]\!]_h^{\mathfrak{F}} &= \varnothing, \\
[\![\top]\!]_h^{\mathfrak{F}} &= W, \\
[\![p]\!]_h^{\mathfrak{F}} &= h(p), \\
[\![\varphi \wedge \psi]\!]_h^{\mathfrak{F}} &= [\![\varphi]\!]_h^{\mathfrak{F}} \cap [\![\psi]\!]_h^{\mathfrak{F}}, \\
[\![\varphi \vee \psi]\!]_h^{\mathfrak{F}} &= [\![\varphi]\!]_h^{\mathfrak{F}} \cup [\![\psi]\!]_h^{\mathfrak{F}}, \\
[\![\Diamond\varphi]\!]_h^{\mathfrak{F}} &= \langle R\rangle([\![\varphi]\!]_h^{\mathfrak{F}}), \\
[\![\Box\varphi]\!]_h^{\mathfrak{F}} &= [R]([\![\varphi]\!]_h^{\mathfrak{F}}),
\end{aligned}$$

where $p \in \mathsf{Prop}$.

Let $\varphi(x, p_1, \ldots, p_n)$ be a modal mu-formula. The semantics of φ is defined for all assignments h using the definition above. For a fixed assignment h, φ and h give rise to a map $f_{\varphi,h} : \mathfrak{F} \to \mathfrak{F}$ defined by $f_{\varphi,h}(U) = [\![\varphi]\!]_{h_x^U}^{\mathfrak{F}}$, where $U \in \mathfrak{F}$, $h_x^U(x) = U$ and $h_x^U(y) = h(y)$ for each propositional variable $y \neq x$. Since we have restricted our language to positive modal formulas, $f_{\varphi,h}$ is a monotone map with respect to the inclusion order. Assume that $(\mathfrak{F}, \subseteq)$ is a complete lattice. Therefore, by the Knaster-Tarski theorem, $f_{\varphi,h}$ has a least fixed-point. We define $[\![\mu x\varphi]\!]_h^{\mathfrak{F}}$ to be the least fixed-point of $f_{\varphi,h}$, which, is computed as follows

$$[\![\mu x\varphi]\!]_h^{\mathfrak{F}} = \bigwedge^{\mathfrak{F}}\{U \in \mathfrak{F} : [\![\varphi]\!]_{h_x^U}^{\mathfrak{F}} \subseteq U\}.$$

A set $U \in \mathfrak{F}$ such that $[\![\varphi]\!]_{h_x^U}^{\mathfrak{F}} \subseteq U$ is called a *pre-fixed point*.

Note that the powerset $(\mathcal{P}(W), \subseteq)$ is a complete lattice where meets and joins are set-theoretic intersections and unions. Therefore, if $\mathfrak{F} = \mathcal{P}(W)$, then

$$[\![\mu x\varphi]\!]_h^{\mathcal{P}(W)} = \bigcap\{U \in \mathcal{P}(W) : [\![\varphi]\!]_{h_x^U}^{\mathcal{P}(W)} \subseteq U\}.$$

In the complete lattice $(\mathsf{Cl}(W), \subseteq)$ of closed sets of a topological space, infinite meets are intersections and infinite joins are the closure of the union. Thus, if $\mathfrak{F} = \mathsf{Cl}(W)$, then

[1] Note that this requirement is not essential (see Remark 3.3), but it always holds in the examples that we consider in this paper. So we find it convenient to make this restriction.

$$[\![\mu x\varphi]\!]_h^{\mathsf{Cl}(W)} = \bigcap\{U \in \mathsf{Cl}(W) : [\![\varphi]\!]_{h_x^U}^{\mathsf{Cl}(W)} \subseteq U\}.$$

Finally, in the complete lattice $(\mathsf{Op}(W), \subseteq)$ of open sets of a topological space infinite meets are the interior of the intersection and joins are unions. Thus, if $\mathfrak{F} = \mathsf{Op}(W)$, then

$$[\![\mu x\varphi]\!]_h^{\mathsf{Op}(W)} = \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : [\![\varphi]\!]_{h_x^U}^{\mathsf{Op}(W)} \subseteq U\}\right),$$

where Int is the interior operator.

If $\mathfrak{F} = \mathcal{P}(W)$, then $[\![.]\!]_h^{\mathfrak{F}}$ is called *classical* or *set-theoretic semantics*. If $\mathfrak{F} = \mathsf{Cl}(W)$, then $[\![.]\!]_h^{\mathfrak{F}}$ is called *closed semantics*, and if $\mathfrak{F} = \mathsf{Op}(W)$, then $[\![.]\!]_h^{\mathfrak{F}}$ is called *open semantics*. The assignment h is called a *set-theoretic assignment* if $h(p) \in \mathcal{P}(W)$, *closed* if $h(p) \in \mathsf{Cl}(W)$, and *open* if $h(p) \in \mathsf{Op}(W)$, for each $p \in \mathsf{Prop}$.

The following example illustrates how to compute modal mu-formulas in MKH-spaces.

Example 3.2. Consider the interval $[0, 1] \subseteq \mathbb{R}$ with the subspace topology. It is an example of a compact Hausdorff space which is not totally disconnected. The only clopen sets are $[0, 1]$ and $\varnothing$. Consider the relation $\leq$ on this space which gives, $\leq [a] = [a, 1]$, which shows that $\leq$ is point closed. Also, for an open set $U \subseteq [0, 1]$ with supremum b we have $\langle\leq\rangle U = [0, b)$, which is open in the subspace topology. Checking that $\langle\leq\rangle$ of a closed set is closed is similar. Therefore, the relation $\leq$ satisfies the conditions of the Proposition 2.6, which shows $([0, 1], \leq)$ is an MKH-space. Moreover, it is not a modal space.

Consider a modal mu-formula, $\mu x(p \vee \Diamond x)$ with the open assignment of p given by $h(p) = (\frac{1}{3}, \frac{2}{3})$. The valuation for the formula is given by

$$[\![\mu x(\ p \vee \Diamond x)]\!]_h^{\mathsf{Op}(W)} = \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : h(p) \cup \langle\leq\rangle U \subseteq U\}\right).$$

As noted above, for an open set $U \subseteq [0, 1]$ with supremum b we have $\langle\leq\rangle U = [0, b)$. The only open sets U which satisfy $h(p) \cup \langle\leq\rangle U \subseteq U$, are the ones which are of the the form $[0, b)$ and contain $h(p)$. The interior of the intersection of all such sets will be the set $[0, \frac{2}{3})$, which is the least fixed-point of the formula.

Remark 3.3. The requirement that $(\mathfrak{F}, \subseteq)$ is a complete lattice is not necessary for interpreting fixed-point operators. It is sufficient to demand that the meet of the sets of type $\{U \in \mathfrak{F} : [\![\varphi]\!]_{h_x^U}^{\mathfrak{F}} \subseteq U\}$, for each φ and h, exist in $\mathfrak{F}$. The lattice $(\mathfrak{F}, \subseteq)$ may not be complete, but such meets may still exist in $\mathfrak{F}$. For example, for a modal space (W, R) the lattice $(\mathsf{Clop}(W), \subseteq)$ of its clopen sets may not be complete. *Descriptive mu-frames* are those modal spaces where meets of such sets are clopen, see [3], [5]. Descriptive mu-frames play an important role in the study of modal mu-calculus. They provide completeness for any axiomatic system of modal mu-calculus. Moreover, a version of Sahlqvist theorem holds for descriptive mu-frames [5]. We view MKH-spaces as generalizations of descriptive mu-frames. Similarly the results in this paper generalize the results of [5] to the case of MKH-spaces.

Remark 3.4. Also note that regular open (closed) sets of a topological space form a complete Boolean algebra [11]. These sets provide important topological structures for interpreting modal mu-formulas. Note that these Boolean algebras are not sublattices of the powerset Boolean algebra, see e.g., [11]. As already noted in the footnote in the previous page, the demand that $(\mathfrak{F}, \subseteq)$ is a sublattice of the powerset, is made only for convenience and could be easily dropped in order to accommodate interesting examples such as regular open (closed) sets. Since we do not consider regular open and closed sets in this paper, we are going to keep this restriction.

The key property of MKH-spaces is that modal operators $\Box$ and $\Diamond$ can be interpreted on open sets. The next theorem shows that modal mu-formulas can also be interpreted on open sets of an MKH-space.

Theorem 3.5. *The open semantics of modal mu-formulas is well defined, that is, if h is an open assignment, then $[\![\varphi]\!]_h^{\mathsf{Op(W)}}$ is an open set, for any modal mu-formula φ.*

Proof. The proof is by induction on the complexity of φ. In the base case, when $\varphi = \top, \bot$ or $p \in \mathsf{Prop}$, $[\![\varphi]\!]_h^{\mathsf{Op}(W)}$ is an open set, since $\varnothing$, W are open sets and $h(p)$ is an open assignment. For the induction step if $\varphi = \varphi_1 \vee \varphi_2$ or $\varphi_1 \wedge \varphi_2$, $[\![\varphi]\!]_h^{\mathsf{Op}(W)}$ is also open since finite intersection and union of open sets is open. If $\varphi = \Diamond\psi$, $[\![\Diamond\psi]\!]_h^{\mathsf{Op}(W)} = \langle R\rangle([\![\psi]\!]_h^{\mathsf{Op}(W)})$, which is open by Proposition 2.6, as $[\![\psi]\!]_h^{\mathsf{Op}(W)}$ is open by induction hypothesis. The case when $\varphi = \Box\psi$ is similar and uses the fact that $[R]U$ is open for an open U. Finally, if $\varphi = \mu x\psi$, since we define the semantics of $\mu x\psi$ to be equal to the interior of an intersection of open sets, $[\![\varphi]\!]_h^{\mathsf{Op}(W)}$ will be an open set. □

In order to simplify the notation, instead of $[\![\varphi(p_1, \dots, p_n)]\!]_h^{\mathfrak{F}}$ with $h(p_i) = U_i, 1 \leq i \leq n$, we will sometimes simply write $\varphi(U_1, \dots, U_n)^{\mathfrak{F}}$ or just $\varphi(U_1, \dots, U_n)$ if it is clear from the context. We now show that the semantics for $\mu x\varphi$ defined above, gives the least fixed-point of φ.

Lemma 3.6. *Let (W, R) be an MKH-space, $\mathfrak{F} \subseteq \mathcal{P}(W)$ a complete lattice and h such an assignment that $[\![\varphi]\!]_h^{\mathfrak{F}} \in \mathfrak{F}$. Then the valuation function defined for modal mu-formulas is monotone, that is for $U, V \in \mathfrak{F}$ such that $U \subseteq V$, we have $[\![\varphi]\!]_{h_x^U}^{\mathfrak{F}} \subseteq [\![\varphi]\!]_{h_x^V}^{\mathfrak{F}}$.*

Proof. The above lemma can be proved by induction on the complexity of the formula φ. The basic modal cases are well known. For the case when $\varphi = \mu y\psi$, we want to show that for $U \subseteq V$ we have $[\![\mu y\psi]\!]_{h_x^U}^{\mathfrak{F}} \subseteq [\![\mu y\psi]\!]_{h_x^V}^{\mathfrak{F}}$. By induction hypothesis, we have $[\![\psi]\!]_{h_x^U}^{\mathfrak{F}} \subseteq [\![\psi]\!]_{h_x^V}^{\mathfrak{F}}$. This means that for each $C \in \mathfrak{F}$, if $[\![\psi]\!]_{h_x^V}^{\mathfrak{F}} \subseteq C$, then $[\![\psi]\!]_{h_x^U}^{\mathfrak{F}} \subseteq C$. Therefore, $\bigwedge\{C : [\![\psi]\!]_{f_y^C}^{\mathfrak{F}} \subseteq C\} \subseteq \bigwedge\{C : [\![\psi]\!]_{g_y^C}^{\mathfrak{F}} \subseteq C\}$, where $f = h_x^U$ and $g = h_x^V$. Hence, $[\![\mu x\psi]\!]_{h_x^U}^{\mathfrak{F}} \subseteq [\![\mu x\psi]\!]_{h_x^V}^{\mathfrak{F}}$, and $[\![\varphi]\!]_{h_x^U}^{\mathfrak{F}} \subseteq [\![\varphi]\!]_{h_x^V}^{\mathfrak{F}}$. □

Theorem 3.7. *For a modal mu-formula φ, the map given by $(U \mapsto [\![\varphi]\!]_{h_x^U}^{\mathsf{Op}(W)})$, where h is an open assignment, has the least fixed-point $[\![\mu x\varphi]\!]_h^{\mathsf{Op}(W)}$.*

Proof. We know that $\mathsf{Op}(W)$ is a complete lattice, and $[\![\varphi]\!]_{h_x^U}^{\mathsf{Op}(W)}$ is monotone as shown in the previous lemma. Therefore, from the Knaster-Tarksi theorem, it follows that $[\![\mu x\varphi]\!]_h^{\mathsf{Op}(W)}$ is its least fixed-point.

□

3.1 Open fixed-point semantics

In this section we focus on the open semantics for the least fixed-point operator. We first prove the following theorem which shows that if we restrict ourselves to open assignments, the interpretation of any modal mu-formula under the set-theoretic semantics is the same as in the open semantics.

Theorem 3.8. *Let (W, R) be an MKH-space and h be an open assignment. Then, for each for each modal mu-formula φ, $[\![\varphi]\!]_h^{\mathcal{P}(W)} = [\![\varphi]\!]_h^{\mathsf{Op}(W)}$.*

Proof. We prove the lemma by induction on the complexity of formulas. The cases $\varphi = \top$ or $\bot$, $\varphi = \varphi_1 \wedge \varphi_2$ or $\varphi_1 \vee \varphi_2$, $\varphi = \Diamond\psi$ or $\Box\psi$ are obvious. Now assume $\varphi = \mu x\psi$ and suppose the result holds for ψ. We let $f_{\psi,h}$ and $g_{\psi,h}$ be a map such that $f_{\psi,h}(U) = [\![\psi]\!]_{h_x^U}^{\mathsf{Op(W)}}$ and $g_{\psi,h}(U) = [\![\psi]\!]_{h_x^U}^{\mathcal{P}(W)}$.

We have seen earlier that the least fixed-point can also be computed as the limit of the following increasing sequence of sets,

$$\varnothing \subseteq f_{\psi,h}(\varnothing) \subseteq f^2_{\psi,h}(\varnothing) \subseteq \dots$$

$$\varnothing \subseteq g_{\psi,h}(\varnothing) \subseteq g^2_{\psi,h}(\varnothing) \subseteq \dots$$

By the induction hypothesis $f_{\psi,h}(U) = g_{\psi,h}(U)$, for each $U \in \mathsf{Op(W)}$. So $f^n_{\psi,h}(\varnothing) = g^n_{\psi,h}(\varnothing)$, for each $n \in \omega$. As h is an open assignment, each $f^n_{\psi,h}(\varnothing)$ is an open set. So their join is just the union. Thus, $f^\omega_{\psi,h}(\varnothing) = \bigcup_{n\in\omega} f^n_{\psi,h}(\varnothing) = \bigcup_{n\in\omega} g^n_{\psi,h}(\varnothing) = g^\omega_{\psi,h}(\varnothing)$. Continuing this process transfinitely we obtain that for each ordinal α we have $f^\alpha_{\psi,h}(\varnothing) = g^\alpha_{\psi,h}(\varnothing)$. This implies that $[\![\mu x\psi]\!]^{\mathsf{Op(W)}}_h = [\![\mu x\psi]\!]^{\mathcal{P}(W)}_h$.

□

Note that the above theorem holds only when h is open. In the following we will be dealing with assignments that in general are not open. For such assignments the above theorem may not hold as Example 3.9 below shows.

Example 3.9. Consider the interval $I = [0, 1] \subseteq \mathbb{R}$ with the subspace topology. Note that this is an MKH-space. We compute the fixed-point of the modal mu-formula $\varphi = \mu x(p \vee x)$ on this interval with an assignment $h(p) = [\frac{1}{2}, \frac{2}{3})$ which is not open. It is easy to see that the least fixed-point of φ, with the set-theoretic semantics is $[\frac{1}{2}, \frac{2}{3})$. In case of open semantics, the least fixed-point is the interval $(\frac{1}{2}, \frac{2}{3})$. So, this example shows that, if the assignment is not open, then the least fixed-point of a modal mu-formula may not be the same in set-theoretic and open semantics.

We now show that the semantics of the least fixed-point operator simplifies in the case of open semantics and open assignments. To this end, we define a new semantics $||\varphi||^{\mathfrak{F}}_h$, where $\mathfrak{F} \subseteq \mathcal{P}(W)$ is complete. It agrees with $[\![]\!]$ on all clauses except for the one for the fixed-point operator which we define as follows

$$||\mu x\varphi||^{\mathfrak{F}}_h = \bigcap\{U \in \mathfrak{F} : ||\varphi||^{\mathfrak{F}}_{h^U_x} \subseteq U\}.$$

Lemma 3.10. *Let (W, R) be an MKH-space, $\mathfrak{F} \subseteq \mathcal{P}(W)$ a complete lattice and h such an assignment that $||\varphi||^{\mathfrak{F}}_h \in \mathfrak{F}$. Then the valuation function defined for modal mu-formulas is monotone, that is for $U, V \in \mathfrak{F}$ such that $U \subseteq V$, we have $||\varphi||^{\mathfrak{F}}_{h^U_x} \subseteq ||\varphi||^{\mathfrak{F}}_{h^V_x}$.*

Proof. Similar to the proof of Lemma 3.6. □

Theorem 3.11. *Let (W, R) be an MKH-space, h an arbitrary assignment and $\mathfrak{F} \subseteq \mathcal{P}(W)$ a complete sublattice. Then*

1. *For each modal mu-formula φ if $[\![\varphi]\!]^{\mathfrak{F}}_h \in \mathfrak{F}$, then*

$$[\![\varphi]\!]^{\mathfrak{F}}_h = ||\varphi||^{\mathfrak{F}}_h \tag{1}$$

2. *For each modal mu-formula φ if $||\varphi||^{\mathfrak{F}}_h \in \mathfrak{F}$, then*

$$[\![\varphi]\!]^{\mathfrak{F}}_h = ||\varphi||^{\mathfrak{F}}_h \tag{2}$$

Proof. (1) We prove the theorem by induction on the complexity of φ. Suppose $\varphi = \mu x\psi$. Then, by definition $[\![\mu x\psi]\!]^{\mathfrak{F}}_h = \bigwedge^{\mathfrak{F}}\{U \in \mathfrak{F} : [\![\psi]\!]^{\mathfrak{F}}_{h^U_x} \subseteq U\}$. Since $[\![\mu x\psi]\!]^{\mathfrak{F}}_h \subseteq U$ for each pre-fixed point $U \in \mathfrak{F}$, we have that $[\![\mu x\psi]\!]^{\mathfrak{F}}_h \subseteq \bigcap\{U \in \mathfrak{F} : ||\psi||^{\mathfrak{F}}_{h^U_x} \subseteq U\}$. For the converse inclusion, note that that $||\mu x\psi||^{\mathfrak{F}}_h \subseteq U$ implies $||\psi||^{\mathfrak{F}}_{h^{||\mu x\psi||}_x} \subseteq ||\psi||^{\mathfrak{F}}_{h^U_x} \subseteq U$ using Lemma

3.10. But this implies that $||\psi||^{\mathfrak{F}}_{h_x^{||\mu x\psi||}} \subseteq \bigwedge^{\mathfrak{F}}\{U \in \mathfrak{F} : [\![\psi]\!]^{\mathfrak{F}}_{h_x^U} \subseteq U\} = [\![\mu x\psi]\!]^{\mathfrak{F}}_h$. So $||\psi||^{\mathfrak{F}}_h$ is a pre-fixed point. By our assumption, it also belongs to $\mathfrak{F}$. Hence, $\bigcap\{U \in \mathfrak{F} : ||\psi||^{\mathfrak{F}}_{h_x^U} \subseteq U\} \subseteq [\![\mu x\psi]\!]^{\mathfrak{F}}_h$. This finishes the proof of the theorem.

(2) is similar to (1). □

Corollary 3.12. *Let (W, R) be an MKH-space. If h is an open assignment, then*

$$[\![\mu x\varphi]\!]^{\mathsf{Op}(W)}_h = \bigcap\{U \in \mathsf{Op}(W) : [\![\varphi]\!]^{\mathsf{Op}(W)}_{h_x^U} \subseteq U\}.$$

Proof. The result follows directly form Theorems 3.5 and 3.11. □

By Theorem 3.8, open semantics for open assignments coincides with the classical semantics. However, in this paper, we are more interested in topological semantics of fixed-point operators. Moreover, we aim at proving an analogue of the Sahlqvist theorem of [5]. For this purpose, it is essential to pove an analogue of Esakia's lemma. As we will show in Section 5.1 Esakia's lemma fails for the open semantics considered above. We remedy this by introducing a new topological semantics of fixed-point operators. For this we will first need to recall form [4] the algebraic semantics and duality for MKH-spaces.

4 Algebraic semantics

A duality between compact Hausdorff spaces and compact regular frames was established by Isbell [15] (see also [16]). In [4] Isbell duality was extended to a duality between modal compact Hausdorff spaces and modal compact regular frames. We briefly recall this duality and later show that the duality extends to the language of positive modal mu-calculus.

Definition 4.1 (Compact frames). A *frame* L is a complete lattice that satisfies $a \wedge \bigvee S = \bigwedge\{a \wedge s \mid s \in S\}$, where $S \subseteq L$. It is *compact* if whenever $\bigvee S = 1$, there is a finite subset $T \subseteq S$ with $\bigvee T = 1$. A map $f : L \to M$ between frames is a *frame homomorphism* if it preserves finite meets and arbitrary joins.

Suppose L is a frame. For each $a \in L$ there is a largest element of L whose meet with a is zero, called the pseudocomplement of a and written $\neg a$. For $a, b \in L$ we say a is well inside b and write $a \prec b$ if $\neg a \vee b = 1$. We say L is *regular* if $a = \bigvee\{b \mid b \prec a\}$ for each $a \in L$.

Given a topological space X, the collection $\mathsf{Op}(X)$ of all open sets of X is a frame. For a continuous map $f : X \to Y$ between spaces, define $\Omega f = f^{-1} : \mathsf{Op}(Y) \to \mathsf{Op}(X)$. It can be checked that Ω is a contravariant functor from the category of topological spaces to the category of frames. Given a frame L, a filter $F \subseteq L$ is called *complete* if $\bigvee A \in F$ implies that there is $a \in A$ such that $a \in F$. Te set The set $\mathfrak{p}L$ of complete filters forms a topological space with the basis $\alpha(a) = \{x \in \mathfrak{p}L \mid a \in x\}$ where $a \in L$.

For a frame homomorphism $h : L \to M$, the map $\mathfrak{p}h : \mathfrak{p}M \to \mathfrak{p}L$ sending a $x \in \mathfrak{p}M$ to $h^{-1}(x)$ is well defined and continuous. $\mathfrak{p}$ is a contravariant functor between the category of frames and the category of topological spaces. The functors Ω and $\mathfrak{p}$ give dual equivalence when we restrict them to appropriate subcategories.

Theorem 4.2 (Isbell). *The functors Ω and $\mathfrak{p}$ provide a dual equivalence between the category* KHaus *of compact Hausdorff spaces and continuous maps and the category* KRFrm *of compact regular frames and frame homomorphisms.*

Definition 4.3 (Modal compact regular frames). A *modal compact regular frame* (abbreviated: *MKR-frame*) is a triple $L = (L, \Box, \Diamond)$ where L is a compact regular frame, and $\Box, \Diamond$ are unary operations on L satisfying the following conditions.

1. $\Box$ preserves finite meets, so $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$.
2. $\Diamond$ preserves finite joins, so $\Diamond 0 = 0$ and $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$.
3. $\Box(a \vee b) \leq \Box a \vee \Diamond b$ and $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$.
4. $\Box, \Diamond$ preserve directed joins, so $\Diamond \bigvee S = \bigvee\{\Diamond s \mid s \in S\}, \Box \bigvee S = \bigvee\{\Box s \mid s \in S\}$ for any up-directed S.

For MKR-frames $L = (L, \Box, \Diamond)$ and $M = (M, \Box, \Diamond)$, an *MKR-morphism* from L to M is a frame homomorphism $h : L \to M$ that satisfies $h(\Box a) = \Box h(a)$ and $h(\Diamond a) = \Diamond h(a)$ for each $a \in L$. Let MKRFrm be the category whose objects are MKR-frames and whose morphisms are MKR-morphisms.

Definition 4.4 ([4])**.** For $\mathfrak{M} = (W, R)$ an MKH-space, $\Omega\mathfrak{M} = (\mathsf{Op}(W), [R], \langle R\rangle)$. For a continuous p-morphism $f : W \to V$ between MKH-spaces (W, R) and (W', R') define $\Omega f : \mathsf{Op}(W') \to \mathsf{Op}(W)$ by $\Omega f = f^{-1}$.

Definition 4.5. For $\mathcal{L} = (L, \Box, \Diamond)$ an MKR-frame, $\mathfrak{p}\mathcal{L} = (W, R)$ where $W = \mathfrak{p}L$ and R is a relation on W defined by PRQ iff $a \in Q$ implies $\Diamond a \in P$ for all $a \in L$ (alternatively, by $\Box a \in P$ implies $a \in Q$). For a modal frame homomorphism $h : L \to M$, between MKR-frames $\mathcal{L} = (L, \Box, \Diamond)$ and $\mathcal{M} = (M, \Box, \Diamond)$ we define $\mathfrak{p}h : \mathfrak{p}M \to \mathfrak{p}L$ as $(\mathfrak{p}h) = h^{-1}$.

Theorem 4.6 ([4])**.** *The functors Ω and $\mathfrak{p}$ defined above, provide a dual equivalence between* MKHaus *and* MKRFrm

The positive modal mu-formulas in our language can be interpreted over a modal compact regular frame $\mathcal{L} = (L, \Box, \Diamond)$. An algebra assignment h is a map from propositional variables to L. The semantics of propositional connectives are given in a standard way. The formulas $\Box\varphi$ and $\Diamond\varphi$ are interpreted using $\Box$ and $\Diamond$ in $\mathcal{L}$. Let h^a_x denote the map which agrees with h on all variables except for x and which maps x to a. The semantics of $\mu x\varphi$ is given by

$$[\mu x\varphi]^L_h = \bigwedge\{a \in L : [\varphi]_{h^a_x} \leq a\}$$

Using the Knaster-Tarski theorem, it is easy to see that $[\mu x\varphi]_h$ is the least fixed-point of the map given by $(a \mapsto [\varphi]_{h^a_x})$.

The next theorem shows that computing a modal mu-formula φ in (W, R) or algebraically in its dual frame yields the same result.

Theorem 4.7. *Let (W, R) be an MKH-space and $(\mathsf{Op}(W), \Box, \Diamond)$ be the dual MKR-frame. For each modal mu-formula φ and open assignment h, we have $[\varphi]^{\mathsf{Op}(W)}_h = [\![\varphi]\!]^{\mathsf{Op}(W)}_h$*

Proof. The proof is by induction on the complexity of φ. For the propositional and modal cases we refer to the modal Isbell duality in [4, Prop. 3.10]. If $\varphi = \mu x\psi(x, p_1, \ldots, p_n)$, by induction hypothesis $[\psi]^L_h = [\![\psi]\!]^{\mathsf{Op}(W)}_h$. Let $\mathbf{U} = \{U \in \mathsf{Op}(W) : [\![\psi]\!]_{h^U_x} \subseteq U\}$. The result now follows for the fact that $\bigwedge \mathbf{U} = \mathrm{Int}(\bigcap \mathbf{U})$, which is true because in $\mathsf{Op}(W)$ the meet is the interior of the intersection. $\Box$

We now introduce an alternative semantics for $\mu x\varphi$ as follows.

Definition 4.8. Let $(L, \Box, \Diamond)$ be an MKR-frame and h an assignment. For each modal formula φ we let $[\varphi]^{L'}_h = [\varphi]_h$ and we let

$$[\mu x\varphi]^{L'}_h = \bigwedge\{a \in L : \exists b \in L \text{ s.t. } a \prec b \text{ and } [\varphi]^{L'}_{h^b_x} \leq a\}$$

We will now define its topological counter-part.

Definition 4.9. Let (W, R) be an MKH-space and h an open assignment. For each modal formula φ we let $[\![\varphi]\!]^{\mathsf{Op}(W)'}_h = [\![\varphi]\!]^{\mathsf{Op}(W)}_h$ and we let

$$[\![\mu x\varphi]\!]^{\mathsf{Op}(W)'}_h = \mathrm{Int}\bigcap\{U \in \mathsf{Op}(W) : \exists V \in \mathsf{Op}(W) \text{ s.t. } \overline{U} \subseteq V \text{ and } [\![\varphi]\!]^{\mathsf{Op}(W)'}_{h^V_x} \subseteq U\}$$

The next theorem shows that the two new interpretations of the fixed-point operator coincide for MKH-spaces.

Theorem 4.10. *Let (W, R) be an MKH-space and $(\mathsf{Op}(W), \Box, \Diamond)$ be the dual MKR-frame. For any modal mu-formula φ we have $[\varphi]_h^{\mathsf{Op}(W)'} = [\![\varphi]\!]_h^{\mathsf{Op}(W)'}$*

Proof. We prove the theorem by induction on the complexity of φ. We only consider the case $\varphi = \mu x\psi$. First note that in the frame $\mathsf{Op}(W)$ for $U, V \in \mathsf{Op}(W)$ we have $U \prec V$ iff $\overline{U} \subseteq V$. The rest of the proof follows from duality and the fact that meets in $\mathsf{Op}(W)$ are the interior of the intersection. □

We will use this new algebraic interpretation of the fixed-point operator in the next section. In particular, we will give yet another (topological) interpretation of the fixed-point operator. But we will show that in some important cases the topological and algebraic interpretations of the fixed-point operator coincide.

5 Sahlqvist preservation

In this section, we define Sahlqvist sequents in our language and prove a preservation result for these sequents using Esakia's lemma. We begin by introducing an alternative topological semantics for the fixed-point operator.

5.1 An alternative fixed-point semantics

In case of classical modal logic, Esakia's lemma shows that in modal spaces the valuation of a positive formula φ on a closed set is equal to the intersection of valuations of φ on clopen sets containing this closed set [12], [23]. This was extended in [5] to positive modal mu-formulas and descriptive mu-frames. An analogue of Esakia's lemma for MKH-spaces and positive modal formulas was proved in [4]. In case of MKH-spaces clopen sets are replaced by open sets. First, we show that an analogue of Esakia's lemma does not hold for the open semantics defined in Section 3. This motivates an introduction of a new topological semantics for fixed-point operators for which a fixed-point analogue of Esakia's lemma will be shown in Section 5.2.

Example 5.1. Consider an MKH-space $([0, 1], \leq)$ and a modal mu-formula $\mu x(p \vee x)$, such that $h(p) = [\frac{1}{2}, 1]$. The least fixed-point of the formula is computed as the intersection of those open sets U, for which $h(p) \cup U \subseteq U$, or $[\frac{1}{2}, 1] \cup U \subseteq U$. This is equal to the interior of $[\frac{1}{2}, 1]$, which is $(\frac{1}{2}, 1]$.

Let $\mathcal{A} = \{U \in \mathsf{Op}(W) : [\frac{1}{2}, 1] \subseteq U\}$. Then $[\frac{1}{2}, 1] = \bigcap \mathcal{A}$. Let $\varphi = \mu x(p \vee x)$. If Esakia's lemma were true, we would have

$$[\![\mu x(p \vee x)]\!]^{\mathsf{Op}(W)}_{h_p^{[\frac{1}{2},1]}} = \bigcap \{[\![\mu x(p \vee x)]\!]^{\mathsf{Op}(W)}_{h_p^U} : U \in \mathcal{A}\}.$$

It is easy to check that with $h'(p) = A \in \mathcal{A}$, the least fixed-point of the formula $\mu x(p \vee x)$ is equal to A itself. The intersection of all the least fixedpoints, or A's in this case, is the closed set $[\frac{1}{2}, 1]$. So, we have

$$\left(\frac{1}{2}, 1\right] = [\![\mu x(p \vee x)]\!]^{\mathsf{Op}(W)}_{h_p^{[\frac{1}{2},1]}} \supsetneq \bigcap \{[\![\mu x(p \vee x)]\!]^{\mathsf{Op}(W)}_{h_p^U} : U \in \mathcal{A}\} = \left[\frac{1}{2}, 1\right]$$

Therefore, Esakia's lemma fails for modal mu-formulas for open semantics.

We remedy this by introducing an alternative semantics for fixed-point operator. For an important class of modal mu-formulas this semantics will coincide with the semantics introduced in the previous section. We first introduce an alternative notion of a pre-fixed-point of a modal formula φ.

Definition 5.2. Let (W, R) be an MKH-space and h be an open assignment. The Boolean and modal operators for the *topological semantics* $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ are interpreted in the same way as in the case of open semantics. Finally, for a formula φ with free variable x, we set

$$[\![\mu x\varphi]\!]_h^{\overline{\mathsf{Op}(W)}} = \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : [\![\varphi]\!]_{h_x^{\overline{U}}}^{\overline{\mathsf{Op}(W)}} \subseteq U\}\right),$$

where $\overline{U}$ is the closure of U.

The difference between topological and the open semantics is that the pre-fixed points in the topological semantics are taken with respect to the closure of a set. Sets U such that $[\![\varphi]\!]_{h_x^{\overline{U}}}^{\overline{\mathsf{Op}(W)}} \subseteq U$ will be called *topological pre-fixed points.*

Example 5.3. Consider the interval $I = [0, 1]$ with the usual metric topology. We compute the valuation of fixed-point operator according to the topological semantics defined above. Consider a modal mu-formula $\mu x(p \vee x)$ and an open assignment $h(p) = (\frac{1}{3}, \frac{2}{3})$. As we saw in Example 3.2, $[\![\mu x(p \vee x)]\!]_h^{\mathsf{Op}(I)} = (\frac{1}{3}, \frac{2}{3})$. For the new semantics we have

$$[\![\mu x(p \vee x)]\!]_h^{\overline{\mathsf{Op}(I)}} = \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(I) : \left(\frac{1}{3}, \frac{2}{3}\right) \cup \overline{U} \subseteq U\}\right).$$

It can be checked that the only open $U \subseteq [0, 1]$ which satisfies $(\frac{1}{3}, \frac{2}{3}) \cup \overline{U} \subseteq U$, is $U = [0, 1]$. Now this is a pre-fixed point but not the least fixed-point, in the sense that it is not the least open pre-fixed point. We have seen earlier in the Example 3.2 that the set $(\frac{1}{3}, \frac{2}{3})$ is the least open pre-fixed point for the formula $\mu x(p \vee x)$.

The following lemma shows that the topological semantics $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ is well-defined.

Lemma 5.4. *Let (W, R) be an MKH-space and h an open assignment. Then for each modal mu formula φ $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ is an open set.*

Proof. We want to show that if we restrict ourselves to open assignments, then the open semantics $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ is an open set. It is easy to see this for the cases when φ is a modal formula, since the valuation function is the same as in the case of usual semantics. In the case when $\varphi = \mu x\psi$, $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ is still open since we define it to be the interior of intersection of sets U such that $[\![\psi]\!]_{h_x^{\overline{U}}}^{\overline{\mathsf{Op}(W)}} \subseteq U$. ☐

The following lemma connects the topological semantics with the algebraic semantics discussed in the previous section.

Lemma 5.5. *For an MKH-space (W, R), if $F_1, \dots, F_n$ are closed sets and $\varphi(x_1, \dots, x_n)$ is a modal formula*[2]*, then $[\![\varphi]\!]_{h_{x_1,\dots,x_n}^{F_1,\dots,F_n}}^{\overline{\mathsf{Op}(W)}}$ is a closed set.*

Proof. The above lemma can be proved by induction on complexity of φ. For the base case when $\varphi = p, \bot$ or $\top$, the lemma follows trivially. If $\varphi = \varphi_1 \vee \varphi_2$ or $\varphi_1 \wedge \varphi_2$, the lemma holds since finite union and intersection of closed sets is closed. If $\varphi = \Box\psi$ or $\varphi = \Diamond\psi$, the lemma is true because of the conditions on R in Proposition 2.6. ☐

Theorem 5.6. *For an MKH-space (W, R), $U \subseteq W$ is a topological pre-fixed of a modal formula $\varphi(x)$ as defined above iff there exists an open V such that $\overline{U} \subseteq V$ and $[\![\varphi]\!]_{h_x^V}^{\overline{\mathsf{Op}(W)}} \subseteq U$.*

[2] φ does not have any fixed-point operator

Proof. Note that φ is a modal formula and does not contain any fixed-point operators. The direction from right to left is easy. If there is an open V such that $\overline{U} \subseteq V$, by monotonicity, $[\![\varphi]\!]_{h_x^{\overline{U}}}^{\overline{\mathsf{Op}(W)}} \subseteq [\![\varphi]\!]_{h_x^{V}}^{\overline{\mathsf{Op}(W)}} \subseteq U$. For the converse direction, since $\varphi(x)$ is a modal formula, Esakia's lemma for positive modal formulas and MKH-spaces ([4, Lemma 7.8]) holds for it. So, we have

$$[\![\varphi]\!]_{h_x^{\overline{U}}}^{\overline{\mathsf{Op}(W)}} = \bigcap\{[\![\varphi]\!]_{h_x^{\overline{V'}}}^{\overline{\mathsf{Op}(W)}} : \overline{U} \subseteq V' \ \&\ V \in \mathsf{Op}(W)\}.$$

Then, $[\![\varphi]\!]_{h_x^{\overline{U}}}^{\overline{\mathsf{Op}(W)}} \subseteq U$ implies that $\bigcap\{[\![\varphi]\!]_{h_x^{\overline{V'}}}^{\overline{\mathsf{Op}(W)}} : \overline{U} \subseteq V' \ \&\ V \in \mathsf{Op}(W)\} \subseteq U$. As φ is a modal formula, $[\![\varphi]\!]_{h_x^{\overline{V'}}}^{\overline{\mathsf{Op}(W)}}$ is a closed set using Lemma 5.5. Therefore, by compactness of W, there is an open V with $\overline{U} \subseteq V$ such that $[\![\varphi]\!]_{h_x^{\overline{V}}}^{\overline{\mathsf{Op}(W)}} \subseteq U$. But then $[\![\varphi]\!]_{h_x^{V}}^{\overline{\mathsf{Op}(W)}} \subseteq [\![\varphi]\!]_{h_x^{\overline{V}}}^{\overline{\mathsf{Op}(W)}} \subseteq U$. So, we found V with $\overline{U} \subseteq V$ such that $[\![\varphi]\!]_{h_x^{V}}^{\overline{\mathsf{Op}(W)}} \subseteq U$. □

We restrict the syntax of modal mu-formulas so that we only have a modal formula in the scope of a fixed-point connective.

Definition 5.7 (**Shallow modal mu-formula**)**.** A *shallow modal mu-formula* is a modal mu-formula such that only a modal formula (without fixed-point operators) can occur in the scope of the least fixed-point operator.

Example 5.8. A simple example of a shallow modal mu-formula is $\mu x(\Diamond p \vee x)$. We cannot have the formula $\mu x \mu y(\Diamond p \vee x) \wedge (\Box p \vee y)$ in our language since the nesting of fixed-point operators is not allowed by the syntax, but we can have $\mu x(\Diamond p \vee x) \wedge\ \mu y(\Box p \vee y)$. To see more concrete cases, one can check that the computational tree logic (CTL), linear temporal logic (LTL) and propositional dynamic logic (PDL) have shallow fixed-point connectives. For example, the iteration diamond $\langle \alpha^* \rangle$ of the PDL can be expressed as the least fixed-point of the modal formula $p \vee \langle \alpha \rangle x$, that is, $\mu x\,(p \vee \langle \alpha \rangle x)$. We note, however, that both PDL and CTL do allow for nesting of operators, even if each operator is "shallow".

The following theorem connects the topological semantics with the algebraic semantics discussed in the previous section.

Theorem 5.9. *Let (W, R) be an MKH-space. Then for each shallow modal mu-formula φ and an open assignment h we have $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}} = [\![\varphi]\!]^{\mathsf{Op}(W)'}$.*

Proof. We prove the lemma by induction on the complexity of the formula φ. The only case that needs to be checked is $\varphi = \mu x \psi$, where ψ is a modal formula. But then by Theorem 5.6, and the definitions of $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ and $[\![\varphi]\!]^{\mathsf{Op}(W)'}$, we immediately obtain that $[\![\mu x \psi]\!]_h^{\overline{\mathsf{Op}(W)}} = [\![\mu x \psi]\!]^{\mathsf{Op}(W)'}$.

□

Theorem 5.10. *Let (W, R) be an MKH-space and h be an open assignment. Then for each modal μ-formula $\varphi(x)$, $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ is monotone. That is, for $U \subseteq V$, s.t. $U, V \in$ $\mathsf{Op}(W)$*

$$U \subseteq V \ \textit{ implies } \ [\![\varphi]\!]_{h_x^{U}}^{\overline{\mathsf{Op}(W)}} \subseteq [\![\varphi]\!]_{h_x^{V}}^{\overline{\mathsf{Op}(W)}}$$

Proof. We prove the lemma by induction on the complexity of φ and show the induction step only for the case when $\varphi = \mu y \psi(y, x)$. By induction hypothesis, the lemma holds for

ψ, that is, for all $U, V \subseteq W$ and $C \in \mathsf{Op}(W)$, we have

$$\begin{aligned}
U \subseteq V \Rightarrow\ & [\![\psi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{C},U}_{y,x}} \subseteq [\![\psi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{C},V}_{y,x}} \\
\Rightarrow\ & \text{If } [\![\psi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{C},V}_{y,x}} \subseteq C, \text{ then } [\![\psi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{C},U}_{y,x}} \subseteq C \\
\Rightarrow\ & \{C : [\![\psi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{C},V}_{y,x}} \subseteq C\} \subseteq \{C : [\![\psi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{C},U}_{y,x}} \subseteq C\} \\
\Rightarrow\ & \bigcap\{C : [\![\psi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{C},U}_{y,x}} \subseteq C\} \subseteq \bigcap\{C : [\![\psi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{C},V}_{y,x}} \subseteq C\} \\
\Rightarrow\ & \mathrm{Int}\left(\bigcap\{C : [\![\psi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{C},U}_{y,x}} \subseteq C\}\right) \subseteq \mathrm{Int}\left(\bigcap\{C : [\![\psi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{C},V}_{y,x}} \subseteq C\}\right) \\
\Rightarrow\ & [\![\mu y\psi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{U}_{x}} \subseteq [\![\mu y\psi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{V}_{x}}.
\end{aligned}$$

□

We have already seen in the Example 5.3 that the alternative semantics of the formula $\mu x\varphi$ does not give the least fixed-point of φ. In the following lemma, we show that if h is an open assignment, then $[\![\mu x\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h}$, gives a pre-fixed point of φ. This is similar to [5], where the semantics of the least fixed-point operator is the standard semantics, which is not necessarily the least fixed-point.

Theorem 5.11. *The topological semantics for the fixed-point operator* $[\![\mu x\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h}$ *under an open assignment* h*, gives a pre-fixed point of the formula* φ*.*

Proof. In order to show that $[\![\mu x\varphi(x, p_1, \dots, p_n)\}]\!]^{\overline{\mathsf{Op}(W)}}_{h}$ is a pre-fixed point, we need to show that $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{S}_{x}} \subseteq S$, where $S = \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : [\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{U}}_{x}} \subseteq U\}\right)$. Let $\mathbf{U} = \{U \in \mathsf{Op}(W) : [\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{U}}_{x}} \subseteq U\}$. Since $S \subseteq U \subseteq \overline{U}$, for all $U \in \mathbf{U}$, we have $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{S}_{x}} \subseteq [\![\varphi]\!]^{\overline{\mathsf{Op}W)}}_{h^{\overline{U}}_{x}} \subseteq U$. So $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{S}_{x}} \subseteq \bigcap\{U \in \mathsf{Op}(W) : [\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{U}}_{x}} \subseteq U\}$. By Lemma 5.4, $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{S}_{x}}$ is open. So $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{S}_{x}} \subseteq \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : [\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{U}}_{x}}\right) = S$. Therefore, S is a pre-fixed point. □

5.2 Esakia's lemma

In this section, we work with only shallow modal mu-formulas. We prove an Esakia's lemma for MKH-spaces which will be used later to prove a Sahlqvist theorem for the shallow modal fixed-point formulas. Let W be any set. Recall that a set $\mathbb{F} \subseteq \mathcal{P}(W)$ is *downward directed* if for each $F, F' \in \mathbb{F}$, there exists $F'' \in \mathbb{F}$ such that $F'' \subseteq F \cap F'$.

Lemma 5.12. ***(Esakia's lemma)*** *Let* (W, R) *be an MKH-space. Let* $F, F_1, \dots, F_n \subseteq W$ *be closed sets and let* $\mathcal{A} \subseteq \mathsf{Op}(W)$ *be a downward directed family of open sets such that* $\bigcap \mathcal{A} = F$*. Then, for each positive shallow modal* μ*-formula* $\varphi(x, x_1, \dots, x_n)$*, we have*

$$[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{F,\vec{F}}_{x,\vec{x}}} = \bigcap\{[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{C,\vec{F}}_{x,\vec{x}}} : C \in \mathcal{A}\}$$

where $\vec{F} = (F_1, \dots F_n)$ *and* $\vec{x} = (x_1, \dots, x_n)$*.*

Proof. Throughout this proof, we adopt the following simplified notation: we use $\varphi(F, \vec{F})$ instead of $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{F,\vec{F}}_{x,\vec{x}}}$.

First, note that $\varphi(F,\vec{F}) = \bigcap\{\varphi(C,\vec{F}) : C \in \mathcal{A}\}$ follows from $\varphi(F,\vec{F}) = \bigcap\{\varphi(\overline{C},\vec{F}) : C \in \mathcal{A}\}$, where $\overline{C}$ is the closure of C, as a result of the following claim.

Claim. *$\varphi(F,\vec{F}) = \bigcap\{\varphi(\overline{C},\vec{F}) : C \in \mathcal{A}\}$, implies $\varphi(F,\vec{F}) = \bigcap\{\varphi(C,\vec{F}) : C \in \mathcal{A}\}$.*

Proof of Claim. From Lemma 5.10, we have that φ is monotone. So, if $F \subseteq C \subseteq \overline{C}$, then $\varphi(F,\vec{F}) \subseteq \varphi(C,\vec{F}) \subseteq \varphi(\overline{C},\vec{F})$, which implies $\varphi(F,\vec{F}) \subseteq \bigcap\{\varphi(C,\vec{F}) : C \in \mathcal{A}\} \subseteq \bigcap\{\varphi(\overline{C},\vec{F}) : C \in \mathcal{A}\}$. Therefore, if we show $\varphi(F,\vec{F}) = \bigcap\{\varphi(\overline{C},\vec{F}) : C \in \mathcal{A}\}$, we get $\varphi(F,\vec{F}) = \bigcap\{\varphi(C,\vec{F}) : C \in \mathcal{A}\}$ □

We show $\varphi(F,\vec{F}) = \bigcap\{\varphi(\overline{C},\vec{F}) : C \in \mathcal{A}\}$ by induction on the complexity of φ. For the cases not involving the fixed-point operator, we refer to the proof of [4, Lemma 7.8]. For the case when $\varphi = \mu x\psi(x,y,\vec{x})$, we need to show

$$\mu x\psi(x,F,\vec{F}) = \bigcap\{\mu x\psi(x,\overline{C},\vec{F}) : C \in \mathcal{A}\}$$

For each $C \in \mathcal{A}$, we have $F \subseteq C \subseteq \overline{C}$, which implies $\mu x\psi(x,F,\vec{F}) \subseteq \mu x\psi(x,\overline{C},\vec{F})$ using Lemma 5.10. Therefore, $\mu x\psi(x,F,\vec{F}) \subseteq \bigcap\{\mu x\psi(x,\overline{C},\vec{F}) : C \in \mathcal{A}\}$.

For the other direction, suppose $w \in \bigcap\{\mu x\psi(x,\overline{C},\vec{F}) : C \in \mathcal{A}\}$. This implies that $w \in \mu x\psi(x,\overline{C},\vec{F})$, for each $C \in \mathcal{A}$. As a result, $w \in \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : \psi(\overline{U},\overline{C},\vec{F}) \subseteq U\}\right)$, using the definition of the alternative semantics for the least fixed-point operator. Therefore, there exists a neighborhood U_w of w such that $U_w \subseteq \bigcap\{U \in \mathsf{Op}(W) : \psi(\overline{U},\overline{C},\vec{F}) \subseteq U\}$. So, for each $C \in \mathcal{A}$, and each $V \in \mathsf{Op}(W)$ with $\psi(\overline{V},\overline{C},\vec{F}) \subseteq V$ we have $U_w \subseteq V$.

Assume $U \in \mathsf{Op}(W)$ is such that $\psi(\overline{U},F,\vec{F}) \subseteq U$. By the induction hypothesis, $\psi(\overline{U},F,\vec{F}) = \bigcap\{\psi(\overline{U},\overline{C},\vec{F}) : C \in \mathcal{A}\}$. Hence, $\bigcap\{\psi(\overline{U},\overline{C},\vec{F}) : C \in \mathcal{A}\} \subseteq U$. By Lemma 5.5, each $\psi(\overline{U},\overline{C},\vec{F})$ is a closed set. Therefore, as U is open, by compactness, there exist finitely many $C_1,\dots,C_k \in \mathcal{A}$ such that $\bigcap_{i=1}^k \psi(\overline{U},\overline{C_i},\vec{F}) \subseteq U$. As $\mathcal{A}$ is downward directed, there exists a $C \in \mathcal{A}$ such that $C \subseteq \bigcap_{i=1}^k C_i$ which implies $\overline{C} \subseteq \overline{\bigcap_{i=1}^k C_i} \subseteq \bigcap_{i=1}^k \overline{C_i}$.

Finally, by Lemma 5.10, $\psi(\overline{U},\overline{C},\vec{F}) \subseteq U$ which implies $U_w \subseteq U$. Therefore, it follows that $w \in \mu x\psi(x,F,\vec{F})$.

□

Corollary 5.13. *Let (W,R) be an MKH-space, $\vec{F} = (F_1,\dots,F_n)$,$\vec{G} = G_1,\dots,G_k \subseteq W$ be closed sets and $\varphi(\vec{x},\vec{y})$ be a modal mu-formula, where $\vec{x} = (x_1,\dots,x_n)$ and $\vec{y} = (y_1,\dots,y_k)$. Then,*

1. $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{F_1,\dots,F_n,\vec{G}}_{x_1,\dots,x_n,\vec{y}}} = \bigcap\{[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{C_1,\dots,C_n,\vec{G}}_{x_1,\dots,x_n,\vec{y}}} : F_i \subseteq C_i \in \mathsf{Op}(W), 1 \leq i \leq n\}.$
2. $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{F_1,\dots,F_n,\vec{G}}_{x_1,\dots,x_n,\vec{y}}} = \bigcap\{[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{C_1,\dots,C_n,\vec{G}}_{x_1,\dots,x_n,\vec{y}}} : F_i \subseteq C_i \in A_i, 1 \leq i \leq n\}$*, where $A_i \subseteq \mathsf{Op}(W)$ is downward directed and $\bigcap A_i = F_i$, for each $1 \leq i \leq n$.*

Proof. The result follows from Lemma 5.12 by a trivial induction. □

Remark 5.14. From the proof of the Esakia's Lemma, one can see why do we need to restrict our syntax to shallow modal mu-formulas. In order to use the compactness property to get a finite intersection, from an infinite intersection, we need the set $S = [\![\psi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{U},\overline{C},\vec{F}}_{x,y,\vec{x}}}$ to be closed. If ψ contains fixed-points, S may not necessarily be a closed set.

To see this let ψ be the formula $\mu x(p \vee x)$. We consider the space $\mathbb{N}$ of natural numbers with the discrete topology. The Alexandroff one-point compactification $\alpha\mathbb{N}$ of

$\mathbb{N}$ is a compact Hausdorff (and also zero-dimensional) space. This space is obtained by adding ∞ to $\mathbb{N}$. A set U is open in $\alpha\mathbb{N}$ if $U \subseteq \mathbb{N}$ or $U = V \cup \{\infty\}$ for a cofinite subset $V \subseteq \mathbb{N}$. Let $h(p) = \{n \in \mathbb{N} : n \text{ is even}\} \cup \{\infty\}$ be a closed valuation. Then it is easy to check that the evaluation of the formula $\mu x(p \vee x)$ under the alternative semantics is equal to the set $\mathrm{Int}(\{n \in \mathbb{N} : n \text{ is even}\} \cup \{\infty\}) = \{n \in \mathbb{N} : n \text{ is even}\}$. Obviously this is open but not a closed set. This justifies why we work with shallow modal mu-formulas ensuring that ψ does not have any fixed-point operators and S is a closed set as a result of Lemma 5.5. The above example underlines once again the non-standard nature of this semantics. Note that in the standard semantics the evaluation of the formula $\mu x(p \vee x)$ is equal to the evaluation of the atom p.

5.3 Sahlqvist formulas

In this section, we define a Sahlqvist formula and Sahlqvist sequent in our language. We then prove a version of Sahlqvist preservation result using the Esakia's lemma proved in the previous section for shallow modal fixed-point logic. In fact, with an analogue of the Esakia's lemma at hand the proof follows the standard patter of a proof of Sahlqvist theorem via topological frame see e..g, [23], [14], [5], [13], [4]. Thus, we will only underline the main steps. The details can be found in any of the above reference.

Definition 5.15. Let (W, R) be an MKH-space and h an assignment. For each formulas φ and ψ we say that $\varphi \vdash$ is true in W under h if $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}} \subseteq [\![\psi]\!]_h^{\overline{\mathsf{Op}(W)}}$. We say that $\varphi \vdash \psi$ is *topologically valid* in (W, R) and write $W \models \varphi \vdash \psi$ if $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}} \subseteq [\![\psi]\!]_h^{\overline{\mathsf{Op}(W)}}$ for each open assignment h. We say that $\varphi \vdash \psi$ is *valid* in (W, R) and write $\mathcal{M}W \models \varphi \vdash \psi$ if $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}} \subseteq [\![\psi]\!]_h^{\overline{\mathsf{Op}(W)}}$ for each assignment h.

Definition 5.16. (**Sahlqvist antecedent**) Define $\Box^0 p = p$ and $\Box^{n+1} p = \Box^n p$. A *boxed atom* is a formula of the form $\Box^n \bot$, $\Box^n \top$, or $\Box^n p$ for some propositional variable p and $n \geq 0$. A *Sahlqvist antecedent* is obtained from boxed atoms by applying $\wedge$ and $\Diamond$.

Definition 5.17. (**Sahlqvist sequent**) A sequent $\varphi \vdash \psi$ is called a *Sahlqvist sequent* if φ is a Sahlqvist antecedent and ψ is a shallow modal mu-formula in our language.

Theorem 5.18 (**Sahlqvist preservation**). *Let (W, R) be an MKH-space and $\varphi \vdash \psi$ be a* Sahlqvist sequent. *Then the following are equivalent*

1. $W \models \varphi \vdash \psi$.
2. $\mathcal{M}W \models \varphi \vdash \psi$.

Proof. (Sketch) Obviously, (2) implies (1). Now suppose $\mathcal{M}W \not\models \varphi \vdash \psi$. Then there exists a set-theoretic assignment f and a point $w \in W$ such that $w \in [\![\varphi]\!]_f^{\overline{\mathsf{Op}(W)}}$ and $w \notin [\![\psi]\!]_f^{\overline{\mathsf{Op}(W)}}$. But since φ is Sahlqvist, there is a minimal closed assignment g such that $w \in [\![\varphi]\!]_g^{\overline{\mathsf{Op}(W)}}$ and $w \notin [\![\psi]\!]_g^{\overline{\mathsf{Op}(W)}}$. By the Esakia lemma there exists an open assignment h such that $w \notin [\![\psi]\!]_h^{\overline{\mathsf{Op}(W)}}$. Finally, by monotonicity, $x \in [\![\psi]\!]_h^{\overline{\mathsf{Op}(W)}}$. Thus, $W \not\models \varphi \vdash \psi$. $\square$

6 Sahlqvist correspondence

The aim of this section is to show that every Sahlqvist sequent is equivalent to a frame condition, which can be expressed in a first-order language with a least fixed-point operator (LFP). The language LFP [10] has a countably infinite set of variables, a binary relation symbol R, and a unary predicate P, for each propositional variable $p \in \mathsf{Prop}$. A formula χ in LFP is said to be an LFP-*frame condition* if it does not contain free variables or predicate symbols.

Let $\mathcal{M} = (W, R)$ be an MKH-space and h be an open assignment. We interpret formulas in LFP over (W, R), such that $P^{\mathcal{M}} = h(p) \in \mathsf{Op}(W)$ for every $p \in \mathsf{Prop}$. Let g be a first-order assignment of variables. The satisfaction of a LFP formula ξ, denoted by $(\mathcal{M}, h, g) \models \xi$, is defined in a standard way using induction on ξ. For a LFP formula $\xi(v, X)$, where v is a first-order variable and X is a unary predicate, let $h_x^{\overline{U}}$ denote the assignment of the variable x to the set $\overline{U}$ and g_u^w denote the first-order assignment mapping variable v to $w \in W$. Let $F(U) = \{w \in W : (\mathcal{M}, h_x^{\overline{U}}, g_v^w) \models \xi(v, X)\}$. The semantics of $(\mu(X, v)\xi(v, X)\varphi))(u)$, can be defined as follows

$$(\mathcal{M}, h, g) \models (\mu(X, v)\xi(v, X))(u) \text{ iff } g(u) \in \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : F(\overline{U}) \subseteq U\}\right)$$

Definition 6.1 (**Standard translation**)**.** Let u, v be first-order variables. The *standard translation* of a modal mu-formula into the language $\mathsf{FO} + \mathsf{LFP}$ is inductively defined as follows

- $ST_u(\bot) = \bot$,
- $ST_u(\top) = \top$,
- $ST_u(p) = P(u)$, where $p \in \mathsf{Prop}$,
- $ST_u(\varphi \wedge \psi) = ST_u(\varphi) \wedge ST_u(\psi)$,
- $ST_u(\varphi \vee \psi) = ST_u(\varphi) \vee ST_u(\psi)$,
- $ST_u(\Diamond\varphi) = \exists v(R(u, v) \wedge ST_v(\varphi)$,
- $ST_u(\Box\varphi) = \forall v(R(u, v) \to ST_v(\varphi))$,
- $ST_u(\mu x\varphi) = (\mu(X, v)ST_v(\varphi))(u)$
- $ST_u(\varphi \vdash \psi) = ST_u(\varphi) \to ST_u(\psi)$

Proposition 6.2. *Let $\mathcal{M} = (W, R)$ be an MKH-space, h be an open assignment and φ be a modal mu-formula. For each $w \in W$ and a first-order assignment g_u^w mapping variable v to w, we have,*

1. $w \in [\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ *iff* $(\mathcal{M}, h, g_u^w) \models ST_u(\varphi)$
2. $\forall h \left(w \in [\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}\right)$ *iff* $(\mathcal{M}, g_u^w) \models \forall P_1 \ldots \forall P_n\, ST_u(\varphi)$
3. $\forall h \forall w \left(w \in [\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}\right)$ *iff* $\mathcal{M} \models \forall P_1 \ldots \forall P_n \forall u\, ST_u(\varphi)$

Proof. The Proposition easily follows from an induction on complexity of φ. $\square$

Theorem 6.3. *Let (W, R) be an MKH-space and $\varphi \vdash \psi$ be a Sahlqvist sequent. Then there is a frame condition $\chi(\varphi, \psi)$ in LFP such that*

$$(W, R) \models \chi(\varphi, \psi) \textit{ iff } \varphi \vdash \psi \textit{ is valid in } (W, R)$$

Proof. We give an algorithm to effectively compute the first order frame correspondent $\chi(\varphi, \psi)$ of $\varphi \vdash \psi$.

Step 1 Since $\varphi \vdash \psi$ is valid in (W, R), $\forall w \in W$,

$$w \in [\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}} \Rightarrow w \in [\![\psi]\!]_h^{\overline{\mathsf{Op}(W)}}$$

Fix $w \in W$. Let $p_1, \ldots, p_n \in \mathsf{Prop}$ be the set of propositional variables occurring in φ. We compute the minimal assignment $h_0(p_i)$, $1 \leq i \leq n$ for each propositional variables as follows: let $\beta_1, \ldots, \beta_{m_i}$ be the boxed atoms in φ which contain p_i, with $\beta_j = \Box^{d_j} p_i, 1 \leq j \leq m_i$ and $d_j \geq 0$. Let $R^0[w] = \{w\}$ and $R^n[w] = \{w' \in W :$

$\exists w_1, \ldots, w_n$ s.t. $wRw_1R \ldots Rw_n$ and $w_n = w'\}$ for $n \geq 1$. The minimal valuation for p_i is equal to $h_0(p_i) = R^{d_1}[w] \cup \ldots \cup R^{d_{m_i}}[w]$.

Step 2 Let h_0 be the minimal assignment computed in Step 1. The syntactic shape of the Sahlqvist formula ensures that we have the following equivalence.

Claim. *If φ is a Sahlqvist antecedent, then*

$$\forall h \left(w \in [\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}} \Rightarrow w \in [\![\psi]\!]_h^{\overline{\mathsf{Op}(W)}} \right) \quad \textit{iff} \quad \forall h_0 \left(w \in [\![\varphi]\!]_{h_0}^{\overline{\mathsf{Op}(W)}} \Rightarrow w \in [\![\psi]\!]_{h_0}^{\overline{\mathsf{Op}(W)}} \right) \quad (3)$$

Proof of Claim. The direction from left to right is clear. We prove the converse by contraposition. Suppose there exists an arbitrary assignment h such that $w \in [\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ and $w \notin [\![\psi]\!]_h^{\overline{\mathsf{Op}(W)}}$. We show that there exists a minimal valuation h_0 such that $w \in [\![\varphi]\!]_{h_0}^{\overline{\mathsf{Op}(W)}}$ and $w \notin [\![\psi]\!]_{h_0}^{\overline{\mathsf{Op}(W)}}$, using an induction on the complexity of φ.

The base case with $\varphi = \bot$ is trivial. If $\varphi = \Box^n p$, it is easy to check that $w \in [\![\Box^n p]\!]_h^{\overline{\mathsf{Op}(W)}}$ if, and only if, $w \in [\![\Box^n p]\!]_{h_0}^{\overline{\mathsf{Op}(W)}}$, where $h_0(p) = R^n[w]$ is the minimal valuation computed in Step 1. Since ψ is a positive formula and $h_0(p) \subseteq h(p)$, it follows that $w \notin [\![\psi]\!]_{h_0}^{\overline{\mathsf{Op}(W)}}$. If $\varphi = \varphi_1 \wedge \varphi_2$, by induction hypothesis, there exist minimal valuations $g_0(p) \subseteq h(p)$ and $k_0(p) \subseteq h(p)$ for φ_1 and φ_2 respectively. Let $h_0(p) = g_0(p) \cup k_0(p)$, which implies $h_0(p) \subseteq h(p)$. Hence, $w \notin [\![\psi]\!]_{h_0}^{\overline{\mathsf{Op}(W)}}$ If $\varphi = \Diamond\varphi_1$, the minimal valuation h_0 such that $w \in [\![\varphi]\!]_{h_0}^{\overline{\mathsf{Op}(W)}}$ and $w \notin [\![\psi]\!]_{h_0}^{\overline{\mathsf{Op}(W)}}$, is the same as the minimal valuation for φ_1. □

Step 3 We showed in Step 2 that a Sahlqvist sequent is valid under an arbitrary assignment if and only if it is valid under a minimal assignment. As it is shown below, the minimal assignment h_0 computed in Step 1 is first-order definable. Hence, it ensures that the frame condition corresponding to a Sahlqvist sequent is in LFP.

Let $\chi'(\varphi, \psi) = \forall P_1 \ldots \forall P_n \forall u \; ST_u(\varphi \vdash \psi)$. Suppose $h_0(p_i) = R^{d_1}[w] \cup \ldots \cup R^{d_{m_i}}[w]$ for $p_i \in \mathsf{Prop}$. The LFP condition $\chi(\varphi, \psi)$ is obtained from $\chi'(\varphi, \psi)$ by replacing $\forall P_i$ with $\forall z_i$, where z_i is a fresh first order variable, and each atomic formula of the form $P_i(v)$ with an LFP formula $\theta_i = \exists v_0, \ldots, v_n[z_i = v \wedge \bigwedge_{j=0}^{n-1} v_j R v_{j+1} \wedge v_n = v]$, which says 'there exists an R-path from z_i to v in n steps'.

Claim. *The* LFP *sentence $\chi(\varphi, \psi)$ is the frame condition for $\varphi \vdash \psi$.*

Proof of Claim. The minimal valuation for all the propositional variables in φ computed above are first-order definable. Hence, it follows using Proposition 6.2.3 that $\chi(\varphi, \psi)$ is an LFP frame condition. □

The proof of the theorem follows from the claim. □

Example 6.4. Consider the sequent $\Diamond p \vdash \Box\Diamond^* p$, where $\Diamond^* p = \mu x(p \vee \Diamond x)$. The standard translation of the sequent is given as follows

$$\begin{aligned} ST_u(\Diamond p \vdash \Box\Diamond^* p) =& \exists v_1(R(u, v_1) \wedge P(v_1)) \rightarrow \forall v_2(R(u, v_2) \rightarrow \\ & \mu(X, v_3)(P(v_3) \vee \exists v_4(R(v_3, v_4) \wedge X(v_4)))(v_2)) \end{aligned}$$

The propositional variable p does not occur in scope of any box in the antecedent. Hence, the minimal valuation for p is $h_0(p) = \{w\}$. According to the algorithm in Theorem 6.3, the LFP frame condition $\chi(\Diamond p, \Box\Diamond^* p)$ is obtained by replacing all occurrences of $P(v_i)$ with $z_i = v_i$, where z_i is a new variable

$$\begin{aligned} \chi(\Diamond p, \Box\Diamond^* p) =& \forall z_1 \forall z_2 \forall u \exists v_1 \forall v_2 (R(u, v_1) \wedge (z_1 = v_1)) \rightarrow (R(u, v_2) \rightarrow \\ & \mu(X, v_3)((z_2 = v_3) \vee \exists v_4(R(v_3, v_4) \wedge X(v_4)))(v_2)) \end{aligned}$$

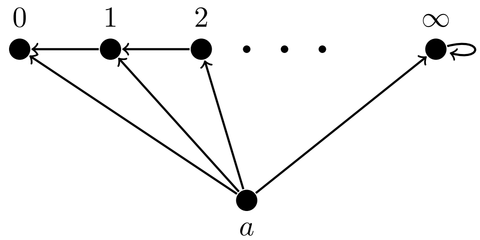


Figure 1: Alexandroff compactification of $\mathbb{N}$ with an isolated point

Example 6.5. Consider the sequent $\Diamond\Box\bot \vdash \Box\Diamond^*\Box\bot$, where $\Diamond^*\Box\bot = \mu x(\Box\bot \vee \Diamond x)$. Since there are no propositional variables in the sequent, its first order correspondence is obtained from its standard translation by quantifying over the free variable.

$$\chi(\Diamond\Box\bot,\ \Box\Diamond^*\Box\bot) = \forall u \exists v_1(R(u,v_1) \wedge \forall v_2(R(v_1,v_2) \to \bot)) \to \forall v_3(R(u,v_3) \to \mu(X,v_4)(\forall v_5(R(v_4,v_5) \to \bot) \vee \exists v_6(R(v_4,v_6) \wedge X(v_6)))(v_3))$$

which simplifies to

$$\chi(\Diamond\Box\bot,\ \Box\Diamond^*\Box\bot) = \forall u \exists v_1(R(u,v_1) \wedge \forall v_2(\neg R(v_1,v_2))) \to \forall v_3(R(u,v_3) \to \mu(X,v_4)(\forall v_5(\neg R(v_4,v_5)) \vee \exists v_6(R(v_4,v_6) \wedge X(v_6)))(v_3)) \tag{4}$$

We now give a semantic interpretation of the sequent. Consider the space of $\mathbb{N}$ of natural numbers with the discrete topology. The Alexandroff one-point compactification of this space obtained by adding ∞ is a compact and Hausdorff space. We further add an isolated point a to the space after compactification, as seen in Figure 1. Let $W = \mathbb{N} \cup \{\infty, a\}$ with the topology described above. The relation $R = \{(n, n-1) : n \in \mathbb{N} \text{ and } n \geq 1\} \cup \{\infty, \infty\} \cup \{a, 0\} \cup \{a, \infty\}$ on W makes (W, R) an MKH-space.

The antecedent $\Diamond\Box\bot$ of the sequent is valid at points a and 1. The classical semantics of the formula $\Diamond^*\Box\bot$ in the consequent is given as

$$[\![\Diamond^*\Box\bot]\!]^{\mathsf{Op}(W)} = \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : \left(\{0\} \cup R^{-1}(U)\right) \subseteq U\}\right)$$

For any open $U = \{0, 1, \ldots, k\} \cup \{a\}$, where $k \in \mathbb{N}$, $R^{-1}(U) = \{0, \ldots, k, k+1\} \cup \{a\}$. Hence, the open sets U which satisfy the condition $\left(\{0\} \cup R^{-1}(U)\right) \subseteq U$ are $\{0, a\} \cup \mathbb{N}$ and $\{0, a, \infty\} \cup \mathbb{N}$. As a result, $[\![\Diamond^*\Box\bot]\!]^{\mathsf{Op}(W)} = \{0, a\} \cup \mathbb{N}$.

In our closure semantics the semantics of $\Diamond^*\Box\bot$ is

$$[\![\Diamond^*\Box\bot]\!]^{\overline{\mathsf{Op}(W)}} = \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : \left(\{0\} \cup R^{-1}(\overline{U})\right) \subseteq U\}\right)$$

The closure of the open set $\{0, a\} \cup \mathbb{N}$ is $\{0, a, \infty\} \cup \mathbb{N}$. Therefore, it does not satisfy the condition $\left(\{0\} \cup R^{-1}(\overline{U})\right) \subseteq U$. The only open set which satisfies the condition is $U = \mathbb{N} \cup \{a, \infty\}$. Hence, $\Diamond^*\Box\bot$ is valid everywhere in (W, R). As a result, the sequent $\Diamond\Box\bot \vdash \Box\Diamond^*\Box\bot$ is valid. Therefore, it follows from Theorem 6.3 that the LFP frame condition $\chi(\Diamond\Box\bot \vdash \Box\Diamond^*\Box\bot)$ obtained above is valid on (W, R).

7 Conclusion and future work

In this paper, we studied different topological semantics of the least fixed-point operator on MKH-spaces. We showed that for an open assignment, set-theoretic and open semantics coincide. We gave an interpretation of the least fixed-point operator on compact regular frame and showed that the duality between compact Hausdorff spaces and compact regular locales extends to the language with the least fixed-point operator. For Sahlqvist preservation, we introduced a new topological semantics for the least fixed-point operator as the intersection of topological pre-fixed-points. In the new semantics, we proved that Esakia'a lemma holds for the class of shallow fixed-point formulas which do not have any nesting of fixed-point operators. As a consequence of Esakia's lemma, we obtained our main preservation result which states that a Sahlqvist sequent in out language is valid under open assignments on an MKH-space if, and only if it is valid under arbitrary assignments. We also showed that a Sahlqvist sequent is valid in an MKH-space, if and only if the condition expressible in LFP corresponding to the the sequent is valid on the space. Finally, using examples we illustrated that the alternative topological semantics for the least fixed-point operator is different from the usual semantics over MKH-spaces.

We summarize the different semantics introduced for the least fixed-point operator over topological spaces in Table 1. In Table 2, we list the results regarding the comparison of the different fixed-point semantics. Finally, we list our main results in Table 3.

One criticism of the semantics considered in the paper might be that it is specially tailored for proving Esakia's lemma and obtaining the Sahlqvist preservation result this way. Although this might be a valid criticism, we note that the fixed-point operators considered in the paper are new and topological in nature. These operators often differ from the classical fixed-point operators and thus enrich the realm and expressivity of the existing fixed-point operators. We also believe that this point of view opens up a wider perspective for other (topological) interpretations of fixed-point operators (e.g., via regular open or closed sets, convex sets, polygons, rectangles, etc.).

We conclude with a few open problems and future directions that can be explored. An interesting problem is whether our results hold for the greatest fixed-point operator and formulas with mixed fixed-point operators. Also regular open sets play an important role in semantics of spatial logics, and are also suitable for modal mu-calculus with negation. Therefore, the fixed-point semantics for regular open sets is an interesting and, for now, unexplored area that deserves attention.

The completeness of Kozen's axiomatization [18] over MKH-spaces is another open problem. In [3] Kozen's axiomatization was shown to complete with respect to descriptive mu-frames, or equivalently with respect to modal mu-algebras. In our case, the algebraic structures which provide the semantics are compact regular frames. These structures have infinitary operations, while our language has connectives of finite arity. This leads to a major question on what should be the logical counterpart of these structures. Does this have to be an infinitary logic or the infinitary operations of compact regular frames can be encoded in a finitary logic?

Another possible direction is to explore the expressivity results for our language with fixed-point operator over compact Hausdorff spaces (see eg., [25]). It would be interesting to find examples of standard topological properties which can be expressed with the alternative fixed-point semantics and e.g., to find an analogue of the Goldblatt-Thomasson theorem [6, Section 3.8].

Pre-condition	Semantics	Reference
$\mathfrak{F} \subseteq \mathcal{P}(W)$ complete	$[\![\mu x\varphi]\!]_h^{\mathfrak{F}} = \bigwedge^{\mathfrak{F}}\{U \in \mathfrak{F} : [\![\varphi]\!]_{h_x^U}^{\mathfrak{F}} \subseteq U\}$	Page 5
$\mathfrak{F} \subseteq \mathcal{P}(W)$ complete	$\vert\vert\mu x\varphi\vert\vert_h^{\mathfrak{F}} = \bigcap\{U \in \mathfrak{F} : \vert\vert\varphi\vert\vert_{h_x^U}^{\mathfrak{F}} \subseteq U\}$	Page 8
L frame	$[\mu x\varphi]_h^L = \bigwedge\{a \in L : [\varphi]_{h_x^a} \leq a\}$	Page 10
L frame	$[\mu x\varphi]_h^{L'} = \bigwedge\{a \in L : \exists b \in L \text{ s.t. } a \prec b \text{ and } [\varphi]_{h_x^b}^{L'} \leq a\}$	Page 10
h open	$[\![\mu x\varphi]\!]_h^{\mathsf{Op}(W)'} = \mathrm{Int}(\bigcap\{U \in \mathsf{Op}(W) : \exists V \in \mathsf{Op}(W) \text{ s.t. } \overline{U} \subseteq V \text{ and } [\![\varphi]\!]_{h_x^V}^{\mathsf{Op}(W)'} \subseteq U\})$	Def. 4.9
h open	$[\![\mu x\varphi]\!]_h^{\overline{\mathsf{Op}(W)}} = \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : [\![\varphi]\!]_{h_x^{\overline{U}}}^{\overline{\mathsf{Op}(W)}} \subseteq U\}\right)$	Def. 5.2

Table 1: Different fixed-point semantics

Pre-condition	Result	Reference
h open	$[\![\varphi]\!]_h^{\mathcal{P}(W)} = [\![\varphi]\!]_h^{\mathsf{Op}(W)}$	Theorem 3.8
$\mathfrak{F} \subseteq \mathcal{P}(W)$ complete sublattice, $[\![\varphi]\!]_h^{\mathfrak{F}} \in \mathfrak{F}$	$[\![\varphi]\!]_h^{\mathfrak{F}} = \vert\vert\varphi\vert\vert_h^{\mathfrak{F}}$	Theorem 3.11.1
$\mathfrak{F} \subseteq \mathcal{P}(W)$ complete sublattice, $\vert\vert\varphi\vert\vert_h^{\mathfrak{F}} \in \mathfrak{F}$	$[\![\varphi]\!]_h^{\mathfrak{F}} = \vert\vert\varphi\vert\vert_h^{\mathfrak{F}}$	Theorem 3.11.2
h open	$[\varphi]_h^{\mathsf{Op}(W)} = [\![\varphi]\!]_h^{\mathsf{Op}(W)}$	Theorem 4.7
h open	$[\varphi]_h^{\mathsf{Op}(W)'} = [\![\varphi]\!]_h^{\mathsf{Op}(W)'}$	Theorem 4.10
φ shallow, h open	$[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}} = [\![\varphi]\!]_h^{\mathsf{Op}(W)'}$	Theorem 5.9

Table 2: Comparing fixed-point semantics

Pre-condition	Result	Reference
(W,R) MKH-space, $F, F_1, \dots, F_n \subseteq W$ closed, $\vec{F} = (F_1, \dots, F_n)$ $\mathcal{A} \subseteq \mathsf{Op}(W)$ downward directed family of opens, $\bigcap \mathcal{A} = F$, $\varphi(x, x_1, \dots, x_n)$ positive shallow modal mu-formula	$[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{F,\vec{F}}_{x,\vec{x}}} = \bigcap\{[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{C,\vec{F}}_{x,\vec{x}}} : C \in \mathcal{A}\}$	Esakia's lemma (Lemma 5.12)
(W,R) MKH-space, $\varphi \vdash \psi$ Sahlqvist sequent $W \models \varphi \vdash \psi$ if $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_h \subseteq [\![\psi]\!]^{\overline{\mathsf{Op}(W)}}_h$ for each assignment h. $\mathcal{M}W \models \varphi \vdash \psi$ if $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_h \subseteq [\![\psi]\!]^{\overline{\mathsf{Op}(W)}}_h$ for each open assignment h.	$W \models \varphi \vdash \psi$ iff $\mathcal{M}W \models \varphi \vdash \psi$	Sahlqvist preservation theorem (Theorem 5.18)
(W,R) MKH-space, $\varphi \vdash \psi$ Sahlqvist sequent, $\chi(\varphi, \psi)$ LFP frame-condition	$(W,R) \models \chi(\varphi,\psi)$ iff $W \models \varphi \vdash \psi$	Sahlqvist correspondence theorem (Theorem 6.3)

Table 3: Main results

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