Frames and Labels

A Modal Analysis of Categorial Inference

Natasha Kurtonina

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Academisch Proefschrift

ter verkrijging van de graad van doctor aan de
Universiteit Utrecht,
op gezag van de Rector Magnificus
Prof. Dr. J.A. van Ginkel
ingevolge het besluit van het College van Decanen
in het openbaar te verdedigen
op dinsdag 20 juni 1995
des middags te 12.45 uur

door

Natalia Jakovlevna Kurtonina

geboren te Moskou.

Promotores:
Prof.dr. J.F.A.K. van Benthem
Prof.dr. M.J. Moortgat

CIP gegevens

Published by
LEd, Hooghiemstraplein 97, 3514 AX UTRECHT, The Netherlands.
OTS, Trans 10, 3512 JK UTRECHT, The Netherlands.

ISBN 90–7479528–5

Моим родителям

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Acknowledgements

It was a great privilege and a special pleasure to work with Prof. Johan van Benthem and Prof. Michael Moortgat. Their own contributions to the field and comprehensive view of exciting perspectives made the project possible. While Johan revealed to me step by step the beauty of modal semantical structures, Michael has managed to convince me completely that a lot of questions coming from linguistic applications have real charm and elegance. I always felt their generous and gentle support and I owe them a special debt of gratitude for the direct and vital help in bringing this book into existence. Thank you, Johan and Michael!

There are many other people to whom I want to express my deepest appreciation. Yde Venema always kindly agreed to read what I called proofs and papers, and shared his beautiful ideas with me. Dirk Roorda provided very useful comments and warm encouragemnt for the first paper I have written for this book. Leon Horsten helped me to get started when I was still in Belgium and kept giving a lot of friendly support when I moved to Holland. I am deeply indebted tó Dick de Jongh for his care about my work, for his help and nice suggestions. I am very grateful to Dick Oehrle for fruitful discussions and invaluable comments on the draft of this work. Back in Moscow, I would like to thank Prof. Smirnova Elena Dmitrievna who attracted me to philosophical logic and whose support still means so much to me.

This research has benefited a lot from seminars and conferences on subjects related to categorial logic. It was a very special pleasure to participate in "Logic and Linguistics" seminars in Utrecht , "snelkookpan" sessions in Amsterdam and the inspiring meetings and dinners with the 'Categorial Grammar Community' in Rome, London, Paris and Lisbon. I am therefore most grateful to all friends and colleagues who provided support and criticism over the past two years in one way or another. Although I list them together, each person's contribution is individually appreciated: Erik Aarts, Natasha Alechina, Michael Astroh, Paul Cortois, Herman Hendriks, Mark Hepple, Dirk Heylen, Rianne Janssen, Marianne Kalsbeek, Ruth Kempson, Alain Lecomte, Anne-Marie Mineur, Maarten Marx, Szabolcs Mikulás, Glyn Morrill, Michiel van Lambalgen, Benta-Helena van Lambalgen, Andreja Prijatelj, Maarten de Rijke, Volodja Shavrukov, Vera Stebletsova, Teresa Solias, Koen Versmissen, Kees Vermeulen, Domenico Zambella.

Finally, I would like to thank the Research Institute for Language and Speech (OTS) in Utrecht for enabling me to carry out this project and the Institute for Logic, Language and Computation (ILLC) in Amsterdam for its hospitality. The work on Chapters 5 and 6 was partially supported by the Esprit Basic Research Project DYANA-2.

Introduction

Categorial inference has been studied in the literature for many different reasons and from many different backgrounds, ranging from linguistics and philosophy to logic and computer science. In the last decade, various presentations of Categorial Grammar have emerged within the paradigm of substructural logics. Logical calculus that carry the epithet resources-conscious can serve not just as a deductive system, but also as a paradigm for sertain styles of grammatical inference. This dissertation is about a *modal semantic* framework for the study of categorial inference, allowing one to analyze structural rules as constraints on information structures. Three lines of research come together in the following chapters: Categorial Grammar, modal logic and labelled deduction.

Grammar logics: the Lambek paradigm

Among the logically oriented grammar formalisms, categorial systems have a long history, going back to the classical calculi of Ajdukiewicz 1935 and Bar-Hillel 1953. The work of Lambek in the late Fifties represents an important turning point in the field. Lambek redesigned the earlier calculi (which were basically rule-based systems with some small inventory of reduction schemata) as *grammar logics*: systems of inference specifically designed for reasoning about grammatical organization. The Lambek calculi were rediscovered in the early Eighties. Since then they have been the subject of active research in logic and linguistics. The advent of Linear Logic gave an extra impulse to this research: it soon became clear that the Lambek systems are in fact the prototypical examples of what we now call substructural or resource-conscious logics: logics with structure sensitive consequence relation. (Indeed, the Associative Lambek Calculus may be viewed just as the non-commutative multiplicative fragement of intuitionistic linear logic.) The triad of monographs Moortgat 1988, Van Benthem 1991 and Morrill 1994 offers a good perspective on the development of the Lambek paradigm on the interface of linguistics and logic. Here is a brief overview of the main themes of this paradigm.

The language of Categorial Grammar is obtained by closing some set of basic categories under a set of category-forming operators. As basic categories one could have familiar parts of speech such as nouns (N), noun phrases (NP), sentences (S), prepositional phrases (PP), etc. As category-forming operators one could have $/, \bullet, \backslash$, where the semantics of the product connective $\bullet$ tells us how to assemble linguistic expressions of category A and B into a structured configuration of category $A \bullet B$, whereas the slashes $/$ and $\backslash$ give us the means of disassembling such a structured configuration into its A and B components. The language of Categorial Grammar is now obtained if we assign basic categories to the *atomic* expressions, consider categories as formulas and category-forming operators as connectives. Finding the category assignment for *composite* configurations then becomes a problem of logical inference.

As an example of this kind of 'grammatical inference', consider the expression 'Andy talks to the sheriff'. Assume the categorial lexicon classifies the name 'Andy' as of category NP, the preposition 'to' as PP/NP, the determiner 'the' as NP/N, the noun 'sheriff' as N, and the verb in this construction as (NP\S) /PP. Using the basic rules of functional application

$$A/B \bullet B \vdash A \qquad\qquad B \bullet B\backslash A \vdash A$$

we conclude that the sequence

$$\mathrm{NP} \bullet ((\mathrm{NP}\backslash \mathrm{S})/\mathrm{PP}) \bullet (\mathrm{PP}/\mathrm{NP}) \bullet (\mathrm{NP}/\mathrm{N}) \bullet \mathrm{N}$$

corresponding to the construction 'Andy talks to the sheriff' as a whole implies category S. This is the basic mechanism proposed already by Ajdukiewicz. But in the Lambek setting one can do more using *hypothetical reasoning* in its full generality. If from 'Andy talks to NP' we can derive the category S, then withdrawing a hypothetical NP assumption, we infer that 'Andy talks to' is of category S/NP. Hypothetical resoning greatly enhances the expressivity of our logic. We could now lexically assign a relative pronoun like 'whom' the higher order category (N\N)/(S/NP), and infer that the complex expression 'the sheriff whom Andy talks to' is of category NP. The lexical assignment to 'whom' in this case triggers the bit of hypothetical reasoning illustrated above, when we try to find out whether the relative clause body 'Andy talks to' is indeed of category S/NP. Hypothetical reasoning thus establishes the link between the relative pronoun and the role it plays in the relative clause body in *deductive terms*, without introducing abstract syntax as we find it in the subscripted he_i placeholders of Montague Grammar, or the empty categories that populate Chomskyan syntax.

This simple example has enough structure to stress a number of important points, which highlight the need for the general logical approach to Categorial

Grammar. We saw above that the product connective stands for structural composition. What structural aspects should one take into account in the composition process? Does the *order* of the components affect syntactic well-formedness? Does the *hierarchical grouping* of the components carry grammatical meaning? In fact one can systematically generate a hierarchy of categorial category logics by considering the basic logic for $/$, $\bullet$, $\backslash$ together with different packages of structural rules characterizing the properties of linguistic composition such as expressed by the $\bullet$ connective. Linear order and hierarchical grouping are two very obvious structural aspects that can affect grammaticality. But one can consider many further structurally relevant linguistic dimensions.

An important theme in current categorial research is the shift of emphasis from individual category logics to communicating families of such systems. The reason for this shift is that the individual logics are not expressive enough for realistic grammar development: the grammar writer needs access to the combined inferential capacities of a family of logics. Motivation for the move to 'mixed' categorial architectures can be based on considerations of cross-linguistic variation, and on structural variation within one language system. As to the former, order sensitivity may be relevant for one language (say, English, where word order constraints are rather rigid) but much less relevant for another (say, Latin, where changes in word order do not directly affect grammaticality). As to the latter, even within a language with very free word order, such as Latin, it may be essential to rigidly control the placement of, for example, a preposition as coming before its nominal complement, although the phrasal ordering itself is very liberal. Linguistic applications of mixed styles of inference can be found in the recent publications of Moortgat, Morrill and Oehrle. In line with these developments we study different modes of categorial inference first from the perspective of logical semantics for categorial languages in Part I. Then in Part II we develop a uniform labelling discipline for the family of categorial logics. Finally Part III offers a systematic architecture for categorial grammars using different logical systems, allowing different structural rules, which can communicate via modal translations.

Categorial grammars as modal logics

For the modal connection, we read the category constructors as *binary* modal operators: the left and right slashes $/$, $\backslash$ are seen as directed implications $\rightarrow$, $\leftarrow$. The laws of function application then become the modal laws

$$A \bullet (A \rightarrow B) \vdash B \qquad\qquad (B \leftarrow A) \bullet A \vdash B$$

The basic semantic structures to be used in this thesis to interpret binary modalities are ternary Kripke frames, originally introduced in the field of relevant logic. The ternary accessibility relation interpreting the binary connectives, in the categorial setting, is to be thought of as structural composition of sentence parts. In order to unfold the categorial landscape our strategy will be to start with an arbitrary ternary relation and to look for the theory of categorial definability of its properties. The question that we are interested in may be formulated in two ways. First, given a categorial principle, we want to know which requirements have to be satisfied by our frames to make the principle valid. And conversely, given certain properties of ternary frames, we are also interested just when these are definable by means of some suitable set of categorial laws. The two directions meet in the notion of 'frame correspondence'. Our modal paradigm, then, offers a semantic characterization of structural rules. The different aspects of structure sensitivity of the grammatical consequence relation are captured by frame constraints restricting the accessibility relation, i.e., structural composition.

In applying the techniques of correspondence theory for standard modal logic we have to take into account the peculiarities of the categorial languages, which lack certain standard facilities, such as unrestricted Booleans. Questions like the following then arise:

- Can we characterize expressive power of categorial languages through appropriate model and frame constructions supporting preservation theorems?
- What are effective methods extracting perspicuous (first-order) frame correspondents from (second-order) categorial principles, and how does one disprove such definability?
- How does one prove frame completeness results for categorial logics, given the expressive peculiarities of categorial languages?

These issues will be studied in the first three chapters, over standard ternary models. In Chapter 4, they will also arise for the special class of two-dimensional binary relational models, where completeness can be obtained via the method of labelling.

What we have said so far concerns binary modalities interpreted via a ternary accessibility relation. The residuation pattern that holds together the binary families $\leftarrow, \rightarrow, \bullet$ can be generalized to families of n-ary connectives interpreted with respect to n+1-ary accessibility relations. On the other hand, the standard unary modalities will continue to play an important role, too (they may even be understood as multiplicatives, akin to structural composition). Unary operators will be the key devices to obtain a theory of systematic communication between substructural systems. This shows in the existence of faithful modal embeddings recovering the inferential capacity of one categorial logic in another. In Chapter 1 we show that non-associative Lambek Calculus can be faithfully embedded into bi-modal tense logic. In Chapter 6, we provide general theory of structural embeddings for categorial logics.

Categorial inference from a philosophical perspective

In a broader philosophical perspective, one can adopt an information-oriented view of ternary frames. The *worlds* or *states* then represent pieces of information: either highly structured ones, such as grammatical forms or linguistic signs, or more abstract ones. We can focus on the static structure of the information pieces, but also on information flow. The dynamic view on ternary frames suggests an interpretation with worlds standing for transitions between processes. Altogether, then, our semantics can be seen as modelling information structure via static representations, but it also models information processing via dynamic actions.

If the ternary relation Ra, bc is interpreted as abstract composition of actions — a is an outcome of composing b and c — then we can treat logical inference dynamically. It will say that $A_1, \ldots, A_n$ implies A if, for any sequence of information states $a_1, \ldots, a_n$ where each a_i supports A_i , the outcome a of sequential composition of $a_1, \ldots, a_n$ supports A. Thus, each premise invites us to change a corresponding information state, and the conclusion of logical inference acts like a program of transformation of the starting point of the first premise to the endpoint of the last one. This framework is introduced in Chapter 1. In Chapter 2 categorial axioms are systematically studied as semantic constraints on the accessibility relation in frames where they hold. Interpreting frames as structures of informational tokens then yields a correlation between categorial axioms corresponding to structural rules and desired properties of informational composition. For example, the intuitive semantic account of Contraction connects this structural rule with re-use of informational resources. Correspondence Theory makes this precise: it says in which exact frame-semantic sense the metaphor holds. Moreover, it shows how different proof-theoretic formulations of Contraction may correspond to different frame-semantic shades of meaning.

In the standard treatement of logical inference, information structures live in the semantic realm, and do not appear explicitly in our formalisms. An important novelty in current logical investigations, is the use of labeled deduction with explicit labels or signs encoding various kinds of relevant information. In Part II this explicit informational style of categorial inference will be the central theme. The key idea here is to replace the formula as the basic declarative unit by a pair $x : A$, consisting of a label x and a formula A. Sequents then assume the form

$$x_1 : A_1, \ldots, x_n : A_n \Rightarrow y : B$$

The label is to be thought of as an extra piece of information added to the formula. Rules of inference manipulate not just the formula, but the formula plus its label. We then obtain a whole scala of labelling regimes depending on the degree of autonomy between formula and label. Chapter 5 offers a general method of completeness proofs for labelled categorial systems.

A look ahead: chapter overview

Part I introduces ternary frame semantics and offers basic model theory and correspondence theory for categorial languages.

Chapter 1 is concerned with the logical, philosophical and linguistic motivations for ternary frame semantics and choices of languages over ternary models. It develops basic model theory using bisimulations as a fundamental tool. Finally, it offers ternary frame constructions which can be useful for disproving categorial definability of first order principles.

Chapter 2 is about Correspondence Theory for categorial principles.We prove a Sahlqvist-van Benthem theorem for categorial languages and develop some apsects of a general theory of definability. As an application, we obtain a semantic characterisation of structural rules from the perspective of correspondence theory. To refute first-order definability of categorial principles we propose two methods. The first is based on translating categorial formulas into some non-first-order definable standard modal formula. The second method is more direct: non-first-orderness of categorial formulas shows up in failures of the Lowenheim-Skolem theorem.

In Part II, we pass from the study of pure semantic expressive power to the combinatorics of categorial deduction.

Chapter 3 proposes an analysis of completeness theorems for categorial axiomatic systems in a perspective of filter representation. Moreover, we prove frame incompleteness theorem and distinguish a complete class of categorial Sahlqvist formulas.

Next, we consider labeled formats of deduction, where sequents may carry information about linguistic signs. In Chapter 4, we use the method of labelling to obtain a rather simple completeness proof of the Lambek Calculus with respect to binary relational semantics using suitable pair labels. Finally, in Chapter 5, we provide a more general labelling discipline for ternary frame semantics, relating this method to the earlier correspondence perspective of translation into fragments of first-order predicate logic.

Finally Part III deals with the control of resource management in categorial systems. We develop a theory of systematic communication between these systems. The communication is two-way: we show how one can fully recover the structural discrimination of a weaker logic from within a stronger one, and how one can reintroduce structural flexibility of stronger categorial logics within weaker ones. We show how unary modal operators can be used to obtain structural relaxation, or to impose structural constraints. From a logical point of view, our contribution consists in some general translation methods, plus a number of embedding theorems connecting the main calculi in the categorial landscape.

Part I
Categorial Structures and Frame Semantics

This Part presents a modal semantics for categorial structures, based on ternary possible worlds frames from relevant logic, and develops its model theory and correspondence theory along mathematical lines familiar from general modal logic. This achieves a systematic view of semantic variety in the landscape of all categorial logics.

Chapter 1

Ternary Frame Semantics and Its Model Theory

1.1 Introduction and Motivation

1.1.1 Model Structures

In standard modal logic, Kripke semantics employs sets of worlds with a binary accessibility relation (with interpretations such as 'modal alternatives', 'later points in time', 'extended information states'). The basic semantic structures to be used in this thesis are ternary Kripke frames, which may be defined as follows.

1.1.1. DEFINITION. *Ternary Frames and Models.*
A ternary frame consists of a non-empty domain W and a ternary accessibility relation R. Notation for frames: $F = \langle W, R^3 \rangle$. A ternary model $\mathcal{M}$ consists of a ternary frame plus a valuation V sending proposition letters to sets of worlds.

Ternary relational frames can have different interpretations. In relevant logic (cf. [Dunn 86], [Rout. & Meyer 73]), the basic intuition is that of 'information pieces', with ternary accessibility between them guiding the interpretation of implications:

$$\mathcal{M}, a \models A \rightarrow B \quad \Longleftrightarrow \quad \forall b \forall c (R^3 c, ab \;\&\; b \models A \Rightarrow c \models B)$$

This move avoids validity of the usual 'paradoxes of implication' rejected in the literature. One of the more unexpected mathematical realizations of a ternary accessibility relation is the operation of addition in vector spaces ([Urquhart 72]).

For the purposes of categorial logic and semantics, we also adopt the information-oriented view of ternary frames. The 'worlds' or 'states' represent pieces of information: either highly structured ones, such as grammatical forms or linguistic signs, or more abstract ones, as in Kripke-style semantics for intuitionistic logic (cf. [Troel. & van Dalen 88]). Another possible view, following the semantics of linear logic, is that of states as linguistic 'resources', in the style of [Moortgat 94]. But we can also focus on a different aspect of information, namely the manner of its flow. Thus, we can also think of ternary frames more dynamically, with worlds standing for transitions of possible processes. Altogether, then, our semantics can be seen as a model for information structure via static representations, but it also models information processing via dynamic actions. ([van Benthem 91a] has a more extensive presentation of this dual view.) Accordingly, the ternary relation R^3a, bc acquires different informal interpretations, too. With linguistic expressions, it might express that a is the concatenation of b and c (in that order), or perhaps the result of some more sophisticated merge of information structures (cf. [Vermeulen 94]). With resources or information pieces, one can think of 'pooling'. And finally, with transitions, the natural interpretation is that a is the result of performing b and c, successively. In general, henceforth, we shall think of ternary R^3a, bc as a relation of abstract 'composition': a is an outcome of composing b and c, or conversely, that a can be decomposed in b and c.

There exist natural variations on this ternary frame semantics. These will be used occasionally in what follows. For instance, one can fix a distinguished point of the domain, yielding ternary frames of the form $F = \langle W, 0, R\rangle$, where 0 is an element of W . In relevant logic, models over such frames are often required to have only restricted valuation satisfying:

$$\text{if } a \in V(p) \text{ and } Rb, a0, \text{ then } b \in V(p)$$
$$\text{if } a \in V(p) \text{ and } Rb, 0a, \text{ then } b \in V(p)$$

(Evaluation of formulas in relevant logic is then often restricted to cases where this 'actual world' 0 plays a distinguished role.) More generally, this example motivates the use of so-called general frames carrying certain plausible restrictions on the range of their 'admissible valuations'. (For instance, in intuitionistic logic, one requires that all sets of worlds assigned to propositions be upward closed under passing to richer information states.) But for most of our purposes, the most abstract bare frame semantics will do.

1.1.2 Information-Based Consequence

The above semantics allows us much greater freedom than usual in defining validity for different styles of grammatical inference. For instance, in classical logic, $A, B \vdash C$ is semantically valid if, whenever A and B are true at point a in a model $\mathcal{M}$, that same point a verifies C . This classical notion can be adopted as it stands in ternary frame semantics. But in the latter domain, the more natural notion of valid consequence seems the following one. The conclusion should be enforced by combining evidence supporting the various premises:

1.1.2. DEFINITION. *Dynamic Validity.*
$A, B \vdash_{\text{dyn}} C$ is valid in a ternary model, if A is true at point a and B is true at a possibly different point b with Rc, ab, then C is true at c .

This is quite different from ordinary consequence. For instance, the ordering of the premises A, B may matter to the outcome — and so may the multiplicity of their occurrence. This non-standard, resource-sensitive structural behavior — of course — is precisely what has been studied in the literature on categorial deduction, starting from [Lambek 58]. More generally, semantic validity of an inference $A_1, A_2, \ldots, A_n \vdash A$ is related to 'semantical trees' d with root x and leaves $x_1, \ldots, x_n$. If the leaves x_i $(1 \leq i \leq n)$ support the premises A_i , then x must support A . Thus, with information encoding done by propositions, logical inference does the job of information processing. An even more dynamic interpretation arises here when propositions themselves get a procedural interpretation as programmable actions transforming informational states. This time, think of ternary frames consisting of pairs $a = (a_0, a_1)$ with starting point a_0 and endpoint a_1. The earlier composition of states then becomes the following partial function:

$$Ra, bc \quad \text{iff} \quad a_0 = b_0, a_1 = c_1, b_1 = c_0$$

Our new notion of logical inference from $A_1, A_2, \ldots, A_n$ to A then says that, if the successive premises denote transitions

$$a_0a_1, a_1a_2, \ldots, a_{n-1}a_n$$

then the resulting total transition from input to output state a_0a_n supports B. Again, the non-standard structural behavior of categorial inference will be clear from this procedural view of the contributions of premises as forming a complex instruction for a sequence of actions.

As we shall see later, this notion of inference matches categorial deduction in the basic system of non-associative Lambek Calculus. But also its extensions now acquire concrete semantic content.

1.1.3 Special Frame Classes for Structural Rules

On the basis of this general model class, we can now model more specific categorial or dynamic logics by special constraints on the ternary relation. In particular, ternary frame semantics provides a semantical characterization of structural rules of deduction. This makes it a suitable tool for analyzing various inhabitants of the landscape of substructural logics (Lambek Calculus, relevance logics, intuitionistic logic) which differ from each other mostly in the presence or absence of certain structural rules. Here are central examples from the categorial literature, concerning the difference between 'directed' and 'non-directed' Lambek Calculus.

The structural rule of *Permutation* declares the inference $B, A \vdash C$ valid whenever $A, B \vdash C$ is. Semantically, this amounts to ignoring the order of composing two information states. Formally, the corresponding ternary frame constraint looks like this

$$\forall a \forall b \forall c (Ra, bc \Rightarrow Ra, cb)$$

Analogously, the structural rule of *Contraction*, which states that a sequent $A, A \vdash C$ is valid only if $A \vdash C$ is, expresses reflexivity of the ternary relation: $\forall a Ra, aa$.
Weakening declares the inference $B, A \vdash C$ valid whenever $A \vdash C$ is. Semantically this amounts to the identification of two points in a ternary relation: $Ra, bc \Rightarrow a = b$

Later on we shall develop a systematic Correspondence Theory which can deal with diverse categorial laws, providing a broad picture of various shade of structure sensitivity. To make the connections with categorial grammars more precise,though, we must introduce a formal modal language accessing our ternary models.

1.1.4 Modal Categorial Languages

The choice of an appropriate language over ternary models is dictated by intended applications. One obvious candidate here is a propositional modal logic with a binary modality $\bullet$ accessing the composition relation;

$$\mathcal{M}, a \models A \bullet B \iff \exists b \exists c (Ra, bc \& \mathcal{M}, b \models A \& \mathcal{M}, c \models B)$$

The usual language $ML(\bullet, \rightarrow, \leftarrow)$ of categorial grammar differs from this in two respects. On the one hand, it lacks the Booleans. But on the other hand, it has two functional slashes accessing 'converses' of the composition relation. These may be interpreted as follows:

$$\mathcal{M}, a \models A \rightarrow B \iff \forall b, c (Rc, ba \,\&\, \mathcal{M}, b \models A \Rightarrow \mathcal{M}, c \models B)$$

$$\mathcal{M}, a \models B \leftarrow A \iff \forall b, c (Rc, ab \,\&\, \mathcal{M}, b \models A \Rightarrow \mathcal{M}, c \models B)$$

Moreover, we are going to interpret sequents in such models. We say that sequent A $\vdash$ B is *true at point a* iff, if M, a $\models$ A then M, a $\models$ B. In addition we shall follow the usual conventions of modal logic. That is, 'truth in a model' means truth at all worlds in that model, and 'truth in a frame' means truth at all worlds under all valuations over that frame.

This categorial language again suggests an enrichment (cf. [Venema 91] of the original modal one, with two further binary existential modalities:

$$\begin{array}{lcl}\mathcal{M}, a \models A \bullet_1 B & \Longleftrightarrow & \exists b \exists c (Ra, bc \ \& \ \mathcal{M}, b \models A \ \& \ \mathcal{M}, c \models B) \\ \mathcal{M}, a \models A \bullet_2 B & \Longleftrightarrow & \exists b \exists c (Rc, ba \ \& \ \mathcal{M}, b \models A \ \& \ \mathcal{M}, c \models B) \\ \mathcal{M}, a \models A \bullet_3 B & \Longleftrightarrow & \exists b \exists c (Rc, ab \ \& \ \mathcal{M}, b \models A \ \& \ \mathcal{M}, c \models B)\end{array}$$

We can define the latter two with slashes and Boolean negation, and vice versa:

$$\begin{array}{lll}A \bullet_2 B & \text{is equivalent to} & \neg(A \rightarrow \neg B) \\ A \bullet_3 B & \text{is equivalent to} & \neg(\neg B \leftarrow A) \\ A \rightarrow B & \text{is equivalent to} & \neg(A \bullet_2 \neg B) \\ B \leftarrow A & \text{is equivalent to} & \neg(A \bullet_3 \neg B)\end{array}$$

Thus, the latter language ML($\bullet, \rightarrow, \leftarrow, \neg, \&$) may also be viewed as the natural Boolean completion of the basic categorial language, which can be treated via standard modal techniques. The point, however, is that categorial grammar is linked intrinsically with *fragments* of this language – and the peculiarities of the latter will give our treatment its special non-routine flavour.

Further operators

Over ternary models, futher operators may be plausible, beyond the above set. For instance, *Relevant Logic* has a non-boolean negation $\sim$, involving a new operation of 'reversal' for states. The relevant semantic clause is as follows:

$$\mathcal{M}, a \models \sim A \quad \text{iff} \quad \textit{not} \quad \mathcal{M}, r(a) \models A$$

To get the usual validities of relevant negation out of this, one has to impose futher constraints. E.g., the usual Double Negation laws require idempotence, in the form $\forall a \; rr(a) = a$.

The same move also occurs in the above-mentioned *Arrow Logic* (cf. [van Benthem 94], [Venema 93a]). Its language is designed to talk about transitions in their own rights, or more picturesquely, about graphical arrows. Besides ternary composition of arrows, 'arrow frames' contains set I of 'identity arrows' (lazy transitions that do not change states) and a reversal function r. The three modal operators of the language are $\bullet$, $\breve{}$, and id, using composition, converse and identity for their semantic interpretation:

$$\begin{array}{lll}\mathcal{M}, a \models A \bullet B & \text{iff} & \exists b \exists c (Ra, bc \ \& \ \mathcal{M}, b \models A \ \& \ \mathcal{M}, c \models B) \\ \mathcal{M}, a \models A^{\smile} & \text{iff} & \mathcal{M}, r(a) \models A \\ \mathcal{M}, a \models id & \text{iff} & I(a)\end{array}$$

The arrow language also has Boolean connectives, interpreted in the usual way. This framework also allows us to define the categorial slashes:

$$\begin{array}{lll}A \rightarrow B & \text{is equivalent to} & \neg(A^{\smile} \bullet \neg B) \\ B \leftarrow A & \text{is equivalent to} & \neg(\neg B \bullet A^{\smile})\end{array}$$

These definitions will validate the usual laws of the Lambek Calculus $A \bullet (A \rightarrow B) \vdash B$ and $(B \leftarrow A) \bullet A \vdash B$ provided that we impose the following restrictions on arrow frames:

$$\text{if } Ra, bc, \quad \text{then } Rc, r(b)a$$
$$\text{if } Ra, bc, \quad \text{then } Rb, ar(c)$$

These express the natural shifts of perspective in composition triangles. For instance, graphically, if an arrow a is composed out of matching arrows b and c, then we can also view b as staring with a and then returning via reverse of arrow c.

1.1.5 The Lambek Calculus as a minimal modal logic

Let us now apply our minimal modal logic in ML($\bullet, \rightarrow, \leftarrow$) to analyze the Lambek Calculus (cf. [Lambek 58], [Buszkowski 82], [van Benthem 88b], [Moortgat 88], [Moortgat 94]). The latter system has a double character. Logically (though with great historical injustice), it can be viewed as a multiplicative fragment of non-commutative intuitionistic linear logic. But linguistically, it is an essential engine for parsing with categorial grammar. Thus, a logical calculus can serve not just as a deductive system, but also as a paradigm for grammatical inference, with grammatical analysis becoming proof search for categorial logics.

The basic principle underlying categorial grammar is the function-argument structure inside linguistic expressions. In the basic sentence 'John reads', for example, 'John' can be thought as an argument, supplied to the functor 'reads'. We can make the following categorial type assignment to account for this. 'John' is of type e ('entity'), the sentence as a whole is of type t ('truth value'), and the verb 'runs' is of type $e\backslash t$ (a left-looking function from e to t). By the basic categorial rule of function application

$$a,\ a\backslash b\ \vdash\ b$$

combining e and $e\backslash t$ yields t. Now, think of the left slash $\backslash$ as a directed implication $\rightarrow$, and concatenate types for sentence parts using a product modality $\bullet$. That is, read categorial type constructors as modal operators. The modal analogue of this function application then becomes the modal formula

$$A \bullet (A \rightarrow B)\ \vdash\ B$$

which is clearly valid in all ternary models presented above (by checking the above semantic conditions). In Categorial Grammar there is also a functional slash / for right combination, and a corresponding functional rule

$$b/a, a \vdash b$$

In a similar way, the latter becomes the valid modal principle

$$(B \leftarrow A) \bullet A \vdash B$$

Similar observations hold for more complex modes of categorial combination from the literature. For instance, the directed Geach composition rule corresponds to the valid (in associative models which are going to be defined later) modal principle

$$(A \rightarrow B) \bullet (B \rightarrow C) \vdash (A \rightarrow C)$$

whereas its disharmonic variants would be invalid. (The latter fact may be shown by means of a simple ternary counter-example.) Likewise, the Montague lifting rule corresponds to the valid modal principle

$$B \vdash (B \leftarrow A) \rightarrow B$$

whereas its disharmonic variant becomes invalid. (one can even explain the invariants of [Roorda 91] in this modal way [Moortgat 94])

Using this modal perspective on the categorial type inference, we can now transcribe the basic categorial calculi as follows ([Došen 92a]). First, the *Non-Associative Lambek Calculus* (**NL**)

NL is axiomatized as follows

axioms

$A \vdash A$

$A \bullet (A \rightarrow B) \vdash B$	$(B \leftarrow A) \bullet A \vdash B$
$A \vdash B \rightarrow (B \bullet A)$	$A \vdash (A \bullet B) \leftarrow B$

rules

if $A \vdash B$ and $C \vdash D$ then	$A \bullet C \vdash B \bullet D$
	$B \rightarrow C \vdash A \rightarrow B$
	$C \leftarrow B \vdash D \leftarrow A$
if $A \vdash B$ and $B \vdash C$ then	$A \vdash C$

We define semantic validity of sequents A $\vdash B$ as their truth in all ternary models. 'Non-associative' means that we do not assume any equivalence between different bracketings of products.

1.1.3. THEOREM. **NL** is sound and complete with respect to the class of ternary models

Proof
Soundness follows by a simple inspection of the principles listed in the definition of **NL**. For the completeness part, define a simple Henkin model

$$\mathcal{M} = \langle W, R^3, \models \rangle,$$

where

- W is the set of all types (categorial formulas)
- Ra, bc iff $a \vdash b \bullet c$
- $a \in V(p)$ iff $\mathrm{a} \vdash \mathrm{p}$

(This construction can be kept so simple because the categorial language is such a small fragment of the full modal language). The usual Truth Lemma states that for any α,

$$\mathcal{M}, a \models \alpha \quad \text{iff} \quad a \vdash \alpha$$

This is proved by a straightforward induction on the construction of α, where the above principle show their role. The completeness result is a direct consequence.
□

NL can be also presented as a Gentzen calculus of sequents with one axiom and six inference rules. Here A, B, C stand for formulas, Y, X, X_1, X_2 for non-empty finite bracketed sequences of formulas.

$$A \Rightarrow A$$

$$\frac{Y \Rightarrow A \qquad X[B] \Rightarrow C}{X[(Y, A \to B)] \Rightarrow C} \qquad\qquad \frac{(A, X) \Rightarrow B}{X \Rightarrow A \to B}$$

$$\frac{Y \Rightarrow A \qquad X[B] \Rightarrow C}{X[(B \leftarrow A, Y)] \Rightarrow C} \qquad\qquad \frac{(X, A) \Rightarrow B}{X \Rightarrow B \leftarrow A}$$

$$\frac{X[(A, B)] \Rightarrow C}{X[A \bullet B] \Rightarrow C} \qquad\qquad \frac{X_1 \Rightarrow A \qquad X_2 \Rightarrow B}{(X_1, X_2) \Rightarrow A \bullet B}$$

Cut Rule

$$\frac{X \Rightarrow A \qquad Y[A] \Rightarrow C}{Y[X] \Rightarrow C}$$

Here sequences are bracketed, and this information can not be "flattened" to a mere sequence of commas separating the premises. Note, that no structural rules appear in this sequence formulation. Moreover no sequent with an empty antecedent is derivable. It is known from ([Kandulski 88]) that the Cut Rule is conservative in **NL**.

For the semantic interpretation of sequents we define an extended valuation V over ternary models as follows:

- Let $a \in W$ and α in ML($\bullet, \leftarrow, \rightarrow$). Then $a \in V(\alpha)$ iff $a \models \alpha$
- If $a \in W$ and X, Y are non-empty sequences of formulas, then $a \in V([X,Y])$ iff there exist $b \in W$ and $c \in W$ such that $b \in V(X)$, $c \in V(Y)$ and Ra, bc
- Finally, $X \Rightarrow \overset{Y}{\cancel{A}}$ is true in a ternary model if $V(X) \subseteq V(Y)$

The non-associative Lambek calculus is a *minimal modal logic* in the landscape of categorial logics (the analogue of **K** in the hierarchy of standard modal logics). Its theorems do not define any frame constraints. More positively, the complete lack of structural rules makes the inference in **NL** resource sensitive. By contrast, already the usual Associative Lambek Calculus **L** makes semantic demands. When formulated in the above axiomatic format, it adds the following two principles:

$$A \bullet (B \bullet C) \vdash (A \bullet B) \bullet C$$
$$(A \bullet B) \bullet C \vdash A \bullet (B \bullet C)$$

This have a non-trivial frame content, to which we turn now.

1.1.6 Correspondence Arguments

The question that we are interested in may be approached in two ways. First, given a categorial principle, we are interested in the demand that its validity makes on ternary frames. This may be called semantic analysis of categorial logics. But conversely, given certain properties of ternary frames, we are also interested just when these are definable by means of some set of categorial laws. This may be called categorial definability of semantic properties. (We shall encounter both positive and negative examples of this.) The two directions meet in the notion of 'frame correspondence'.

1.1.4. DEFINITION. *Frame Correspondence*
A set of categorial formulas Φ *characterizes* or *defines* a class of ternary frames Σ iff $\Sigma = \{F | F \models \alpha \text{ for all } \alpha \in \Phi\}$
If Σ is also defined by some formula ψ (say, from first order logic, or second order logic), then we say that Σ *corresponds* to ψ. A class Σ is *categorially definable* if some set of categorial formulas defines it.

The statement that Φ characterizes Σ says that all formulas in Φ are true on each frame in Σ , while on each frame outside of Σ , at least one formula of Φ can be falsified. Now we turn to some examples.

Example Analyzing Categorial Laws : Contraction
In the preceding, we already gave a property of ternary frames corresponding

to the Permutation rule generating the non-directed Lambek Calculus LP Likewise, we can analyze the Contraction rule via corresponding axioms. The latter illustrates an interesting phenomenon. Variants in axiomatic formulations in different languages may have similar semantic import. Proofs for these, and the above, examples are very much like those found in correspondence theory for ordinary modal logic. We merely give a sketch of the style of argument at this stage.

In $ML(\bullet), \forall x Rx, xx$ is defined by $p \vdash p \bullet p$

First, it is easy to check that, if $\forall x Rx, xx$ holds in a frame F , then $p \vdash p \bullet p$ is true in F. Second, for the opposite direction, suppose that $F \models p \vdash p \bullet p$. By definition, this means that the latter formula is true everywhere in our frame, under every valuation. Now, consider any $w \in W$. Define so called 'minimal valuation' CV , which singles out the world w as follows:

$$u \in CV(p) \text{ iff } u = w.$$

Clearly, p is true at w under this valuation. But then, by the frame truth of $p \vdash p \bullet p$, we must also have the following truth:

$$\exists x \exists y\ (Rw, xy\ \&\ F, CV, x \models p\ \&\ F, CV, y \models p).$$

Given the definition of CV , the latter is equivalent to Rw,ww . □

The same strategy can be used for the following examples.

- In $ML(\bullet, \&)$,
  $\forall x Rx, xx$ is defined by $p\&q \vdash p \bullet q$

- In $ML(\rightarrow, \neg)$,
  $\forall x Rx, xx$ is defined by $p \rightarrow (\neg p) \vdash \neg p$
- In $ML(\neg, \rightarrow, \&)$
  $\forall x Rx, xx$ is defined by $p\&\neg q \vdash \neg(p \rightarrow q)$
- In $ML(\vee, \neg, \rightarrow)$,
  $\forall x Rx, xx$ is defined by $p \rightarrow q \vdash (\neg p) \vee q$

Finally, here are some examples of 'cautious contraction' together with their frame equivalents

$$(A \bullet A) \rightarrow B \vdash A \rightarrow B \qquad \forall abc(Rc, ba \Rightarrow \exists x(Rc, xa\&Rx, bb))$$
$$A \rightarrow (A \rightarrow B) \vdash A \rightarrow B \qquad \forall abc(Rc, ba \Rightarrow \exists y(Rc, by\&Ry, ba))$$

The latter two principles will only be equivalent to the earlier-mentioned reflexivity $\forall x Rx, xx$ in the presence of Associativity Laws.
The first Associativity principle $A \bullet (B \bullet C) \vdash (A \bullet B) \bullet C$ defines

$$\forall abcdx((Ra, bx\ \&\ Rx, cd) \Rightarrow \exists t(Ra, td\&Rt, bc))$$

The dual principle goes likewise.

We shall study these correspondence phenomena systematically later on, providing general methods and results.

1.2 Options in Formal Modelling

The preceding Section has given the main perspective from which we approach the analysis of categorial grammar and inference. In this Section, which can be skipped without loss of continuity, we prove some technical results concerning logical alternatives. First, we ask a question if structure sensitivity of a consequence relation can be captured not by frame constraints but rather by variations of truth conditions. Second, we show that decomposition of ternary relation is possible due to the faithful embedding of **NL** into tense logic.

1.2.1 Varying Truth Definitions

Ternary relation semantics provides frame conditions for various categorial axioms related to structural rules. But there is another degree of freedom in semantic modelling (cf. [van Benthem 85]). A natural question to ask is whether one can obtain the same effects over arbitrary ternary frames by varying the modal truth definition for the categorial type constructors. We shall investigate the latter option for the case of Permutation: $A \bullet B \vdash B \bullet A$.
A natural account of commutative composition says that $A \bullet B$ is true in world x iff x can be decomposed into some y and z such that, either x supports A and y supports B or x supports B and y supports A. But there is also another, more 'parallel' view on this order indifference, replacing 'either ... or' by 'both ... and'. These two views of composition describe two procedural interpretations of a commutative categorial product:

(i) combine two resources in an *arbitrary* order
(ii) combine them in *both* orders at once.

A choice between the two becomes essential if one wants to have both commutative and non-commutative products around in one multimodal categorial inference system. For instance, let $^\bullet$ be non-commutative Lambek product and $\circ$ the commutative one. On reading (i), the principle $A^\bullet B \vdash A \circ B$ has to be valid, whereas on reading (ii), it is invalid and its converse $A \circ B \vdash A^\bullet B$ must hold. Can we force a choice here by formal semantic considerations? The answer turns out negative.

The first interpretation of commutativity yields the following truth definition over the earlier ternary models:

$$\begin{array}{lll} M, a \models_1 A \bullet B & \text{iff} & \text{there are } b, c \text{ with } Ra, bc \text{ and} \\ & & M, b \models_1 A \quad \& \quad M, c \models_1 B \ \textit{or} \\ & & M, b \models_1 B \quad \& \quad M, c \models_1 A \\ M, a \models_1 A \rightarrow B & \text{iff} & \text{for all } b, c: \\ & & \text{if } Rc, ba \quad \& \quad M, b \models_1 A \text{then} M, c \models_1 B \\ & & \textit{and} \\ & & \text{if } Rc, ab \quad \& \quad M, b \models_1 A \text{ then } \ M, c \models_1 B \end{array}$$

We are going to compare the modal (categorial) theory of the class of all ternary models with the new truth definition with that of the old truth definition on the class of ternary frames satisfying the earlier frame constraint

$$\forall a \forall b \forall c (Ra, bc \ \Rightarrow \ Ra, cb) \qquad (*)$$

1.2.1. PROPOSITION. For any $ML(\bullet, \rightarrow)$ formula ϕ, the following two assertions are equivalent:
(i) α is universally valid on ternary models with non-standard evaluation $\models_1$
(ii) α is universally valid on ternary models satisfying $(*)$ with standard evaluation $\models$

Proof
(ii) $\Rightarrow$ (i). Suppose there is a model $M = \langle W, R, V\rangle$ which $\models_1$-falsifies ϕ at some world a. Consider $M^* = \langle W, R^*, V\rangle$ with R defined as follows:

$$\text{if } \langle abc\rangle \in R, \text{ then put } \langle abc\rangle \text{ and } \langle acb\rangle \text{ in } R^*$$

By a straightforward induction on modal formulas, one shows that

$$\forall \psi \forall a\, (M, a \models_1 \psi \quad \Longleftrightarrow \quad M^*, a \models \psi)$$

(i)$\Rightarrow$(ii) Suppose that ϕ is standardly falsified at some ternary model satisfying (*). By an easy induction, evaluation via $\models$ and $\models_1$ amount to the same thing. Hence, ϕ can not be universally $\models_1$-valid either. □

The second interpretation of commutativity inspires the following truth definition over the earlier ternary models:

$$
\begin{array}{lll}
M, a \models_2 A \bullet B & \text{iff} & \text{there are } b, c \text{ with } Ra, bc \ \ \textit{and} \ Ra, cb \\
 & & M, b \models_2 A \quad \& \quad M, c \models_2 B \\
M, a \models_2 A \to B & \text{iff} & \text{for all } b, c: \\
 & & \text{if } Rc, ba \quad \& \quad M, b \models_2 A \\
 & & \text{then } M, c \models_2 B
\end{array}
$$

Next, we compare $\models_2$ with standard evaluation over ternary frames.

1.2.2. PROPOSITION. For any $ML(\bullet, \to)$ formula ϕ, the following two assertions are equivalent:

(i) α is universally valid on ternary models with non-standard evaluation $\models_2$
(ii) α is universally valid on ternary models satisfying (*) with standard evaluation $\models$

Proof
(ii) $\Rightarrow$ (i). Suppose there is a model $M = \langle W, R, V\rangle$ which $\models_2$-falsifies ϕ at some world a. Consider $M^* = \langle W, R^*, V\rangle$ with R defined as follows:
if $\langle abc\rangle \in R$, and $\langle acb\rangle \in R$ then put $\langle abc\rangle$ and $\langle acb\rangle$ in R^*
Again, a straightforward induction shows that two models verify the same formulas at corresponding worlds. Thus, ϕ is not valid in the standard sense over commutative frames either.
(i)$\Rightarrow$(ii) Suppose that ϕ is standardly falsified at some ternary model satisfying (*). As before, evaluation via $\models$ and $\models_2$ amount to the same thing. Hence, ϕ can not be universally $\models_2$-valid either. □

Thus, the two views on commutative composition can not be distinguished in our modal framework. Of course, in a multimodal categorial system with *standard* truth definition, the relation between $A^\bullet B$ and $A \circ B$ can be fixed via correlations between their corresponding primitive ternary relations (being a commutative and a non-commutative one). But this is more ad-hoc, and not the outcome of our genuine semantic analysis.

In what follows, we shall stick with the standard truth definition, and let the frame constraints vary along the categorial landscape.

1.2.2 Trading Ternary Relations for Binary Ones

It is at least of purely modal interest to see if the work of ternary accessibility relation can be done by binary ones. So far, we have analyzed the Lambek Calculus **NL** and its kind as a modal logic with binary modalities. Can we also analyze it as a modal logic with only unary modalities? For the case of a modal logic having just one unary modality, probably no faithful embedding exists. But we can embed into a multimodal temporal logic as follows.

1.2.3. DEFINITION. *Minimal Bi-Tense Logic*
Let $K^t_{1,2}$ be a minimal tense logic having two 'forward looking operators' $\Box_1$, $\Box_2$ (with existential duals $\Diamond_1$, $\Diamond_2$) and two corresponding 'backward looking' operators $\Box_1^\downarrow$, $\Box_2^\downarrow$ (with existential duals $\Diamond_1^\downarrow$, $\Diamond_2^\downarrow$). The modal language of $K^t_{1,2}$ has its formulas build up from propositional letters according to the rule:

$$\phi ::= p|\neg\phi|\phi\&\psi|\Diamond_1\phi|\Diamond_2\phi|\Diamond_1^\downarrow\phi|\Diamond_2^\downarrow\phi|\Box_1\phi|\Box_2\phi|\Box_1^\downarrow\phi|\Box_2^\downarrow\phi$$

By standard methods, $K^t_{1,2}$ can be axiomatized using

Axioms

- all tautologies of classical propositional logic
- all modal distribution axioms
  $\Box_i(A \supset B) \supset (\Box_i A \supset \Box_i B)$ and $\Box_i^\downarrow(A \supset B) \supset (\Box_i^\downarrow A \supset \Box_i^\downarrow B)$
- all tense-logical conversion axioms
  $\Diamond_i \Box_i^\downarrow A \supset A$ and $A \supset \Diamond_i \Box_i^\downarrow A$

Rules

- modus ponens
- necessitation $A/\Box_i A$ and $A/\Box_i^\downarrow A$, where i=1,2.

A $K^t_{1,2}$ *model* is an ordinary bimodal model

$$M = \langle W,\ R_1^2,\ R_2^2,\ V\rangle$$

with truth definition (i=1,2)

$$\begin{array}{lcl} M,a \models \Box_i A & \Longleftrightarrow & \forall b(R_i ab \Rightarrow M,b \models A) \\ M,a \models \Box_i^\downarrow A & \Longleftrightarrow & \forall b(R_i ab \Rightarrow M,b \models A) \end{array}$$

A faithful embedding of the non-associative Lambek Calculus into $K^t_{1,2}$ runs as follows:

$$\begin{array}{lcl} p^\# & = & p \\ (A \bullet B)^\# & = & \Diamond_1(\Diamond_1 A^\# \& \Diamond_2 B^\#) \\ (A \to B)^\# & = & \Box_2^\downarrow(\Diamond_1 A^\# \supset \Box_1^\downarrow B^\#) \\ (A \leftarrow B)^\# & = & \Box_1^\downarrow(\Diamond_1 A^\# \supset \Box_1^\downarrow B^\#) \end{array}$$

1.2.4. THEOREM. *Embedding Theorem* The following assertions are equivalent:

(i) $A \vdash B$ is derivable in **NL**

(ii) $A^\# \vdash B^\#$ is derivable in $K^t_{1,2}$

Proof

The direction (i) $\Rightarrow$ (ii) can be proved by an easy induction on the length of **NL**-derivation for $A \vdash B$. For example, consider the axiom

$$A \vdash B \to (B \bullet A)$$

Its translation

$$A^{\#} \vdash \Box_2^{\downarrow}(\Diamond_1 B^{\#} \supset \Box_1^{\downarrow}\Diamond_1(\Diamond_1 B^{\#} \& \Diamond_2 A^{\#}))$$

is derivable in the minimal bi-tense logic $K^t_{1,2}$. For the converse direction (ii) $\Rightarrow$ (i), we use a semantical representation argument. Let $A \vdash B$ be underivable in **NL**. By the above completeness of **NL** with respect to ternary semantics, there exists a ternary model M where $A \vdash B$ fails. So, there exists a world $k \in W$ which verifies A, but falsifies B. We construct a $K^t_{1,2}$ model $M = \langle W^*, R_1, R_2, V^* \rangle$ where $A^{\#} \vdash B^{\#}$ fails, as follows:

- put k in W^*
- if $\langle a, bc \rangle \in R$, then take a *fresh* object x, and put $\langle ax \rangle$ and $\langle xb \rangle$ in R_1, $\langle xc \rangle$ in R_2, a, b, c, x in W^*
- set for all $a \in W \cap W^*$, $a \in V^*(p)$ iff $a \in V(p)$

$\Box$

1.2.5. CLAIM. For all categorial formulas A, and all $a \in W \cap W^*$,

$$M^*, a \models A^{\#} \iff M, a \models A$$

Proof

Induction on the length of A. The basic case is a direct consequence of the definition of M^*. We demonstrate only one typical clause of the inductive step, to illustrate this kind of elementary semantic argument over ternary models.

(1) Suppose $M^*, a \models \Box_2^{\downarrow}(\Diamond_1 A^{\#} \supset \Box_1^{\downarrow} B^{\#})$
We need to show that $M, a \models A \to B$.

(2) Suppose (a) $R^3 c, ba$ and (b) $M, b \models A$
We need to show that $M, c \models B$

(3) By the inductive hypothesis : $M^*, b \models A^{\#}$.
By the above construction of M^*, (2(a)) yields ;

(4) (a) $R_1 cx$ (b) $R_1 xb$ (c) $R_2 xa$

(5) By the truth definition:
$M^*, x \models \Diamond_1 A^{\#}$
From (1) and (4(c))
$M^*, x \models \Diamond_1 A\# \supset \Box_1^{\downarrow} B^{\#}$
Thus $M^*, x \models \Box_1^{\downarrow} B^{\#}$

(6) From (4(a)) we get $M^*, c \models B^{\#}$
and by inductive hypothesis $M, c \models B$.

Here is the converse argument. Again, we start by successively unpacking what needs to be shown.

(1) Suppose $M, a \models A \to B$.
We need to shaw that $M^*, a \models \Box_2^{\downarrow}(\Diamond_1 A^\# \supset \Box_1^{\downarrow} B^\#)$
(2) Suppose (a) R_2xa (b) $M^*, x \models \Diamond_1 A^\#$ (c) R_1bx
We need to show that $M^*, b \models B^\#$
(3) By (2(b)) there exists c such that
(a) R_1xc and (b) $M^*, c \models A^\#$
By inductive assumption :
(c) $M, c \models A$
(4) Note, that by the construction of M^*,
R_1bx, R_1xc, R_2xa
can 'come from' the unique triangle, namely Rb,ca.
Then, from (1) and (3(c)), $M \models B$, and therefore
$M^*, b \models B^\#$. □

The preceding Claim implies that any ternary **NL**-counter-model M for a sequent

$A \vdash B$ can be transformed into $K^t_{1,2}$ model M^* where $A^\# \vdash B^\#$ *fails*. Hence $A^\# \vdash B^\#$ is not derivable in $K^t_{1,2}$. This proves the faithful embedding.

Remark *Alternative Routes*
The first claim in this proof can also be shown via a semantical representation. One start from a bi-tense model, and defines the ternary relation in a suitable manner corresponding to the above translation. Also, the shape of the translation itself can be varied in several ways without affecting the outcome.

The preceding method of proof generalizes directly to the earlier full categorial language with Booleans. Recall that the latter may also be viewed as an ordinary modal language with a 'versatile' triple of existential modalities accessing the same ternary relation ([Venema 91]). Indeed, we have proved something more general:

1.2.6. THEOREM. (1) The minimal modal logic of one binary existential modality can be faithfully embedded in the minimal bimodal logic. (And likewise for poly-modal versions.)
(2) The minimal modal logic of versatile triples of existential modalities can be faithfully embedded into the corresponding minimal poly-tense logic with matching future and past operators, via the above translation.

We believe that a similar reduction is possible from quaternary to ternary relations, and so on. As stated before, this leaves the open question if one can also perform such an embedding into a non- temporal modal logic?

1.3 Basic Model Theory

In this Section, we develop some basic modal model theory for ternary frame semantics. First, we concentrate on the basic semantic invariance called 'bisimulation' and define some useful properties of ternary models. Next, we drop valuations, and study pure frames. In particular, we are looking for weak forms of model and frame constructions which can fit with weak categorial languages.

1.3.1 Models and Bisimulation

In modal logic and process theory, bisimulation is the basic notion of semantic equivalence ([vBen & Berg 93], [vBvESteb 94], [An.vB.Nem. 95]). This is a 'process equivalence' stating when two models represent the 'same structure'. We start with the basic mathematical concept as it applies to the full categorial language with Booleans.

1.3.1. DEFINITION. *A Bisimulation*
A bisimulation is a non-empty binary relation Z between two ternary models $M_1 = \langle W_1, R_1, V_1\rangle$ and $M_2 = \langle W_2, R_2, V_2\rangle$ such that the following holds:

1. Atomic harmony
   If Zwv , then w, v verify the same proposition letters
2. Ternary Zig
   (a) If $R_1 a, bc$ and Zax , then $\exists y \exists z(R_2 x, yz \;\&\; Zby \;\&\; Zcz)$
   (b) If $R_1 b, ac$ and Zax , then $\exists y \exists z(R_2 y, xz \;\&\; Zby \;\&\; Zcz)$
   (c) If $R_1 c, ba$ and Zax , then $\exists y \exists z(R_2 z, yx \;\&\; Zby \;\&\; Zcz)$
3. Ternary Zag
   (a) If $R_2 a, bc$ and Zax , then $\exists y \exists z(R_1 x, yz \;\&\; Zby \;\&\; Zcz)$
   (b) If $R_2 b, ac$ and Zax , then $\exists y \exists z(R_1 y, xz \;\&\; Zby \;\&\; Zcz)$
   (c) If $R_2 c, ba$ and Zax , then $\exists y \exists z(R_1 z, yx \;\&\; Zby \;\&\; Zcz)$

This notion says that structural decompositions made in one representation have matching decompositions in the other. Fixing such a notion of semantic equivalence amounts to deciding on a level of abstraction for processes or linguistic structures. The finer-grained and concrete one's view of what the relevant structure is, the more 'sensitive' will be the preservation clauses in the simulation.

bisimulation turns out to be a natural concept because it fits very well with modal languages. An easy induction establishes:

1.3.2. PROPOSITION. *From Bisimulation to Invariance*
Let M_1 and M_2 be two models, with a bisimulation Z between them such that

Zax . Then, for every formula ϕ in our full Boolean categorial language $ML(\bullet, \rightarrow, \leftarrow, \neg, \&)$,

$$M_1, a \models \phi \iff M_2, x \models \phi$$

1.3.3. PROPOSITION. *From Invariance to Bisimulation*
Let M_1 and M_2 be two finite models, with worlds a, x, resp., verifying the same formulas of $ML(\bullet, \rightarrow, \leftarrow, \neg, \&)$ Then there exists a bisimulation Z between these models such that Zax .

Proof
By a straightforward analogy with standard modal logic. (Cf. [van Benthem and Meyer Viol 94] or [Blackburn de Rijke and Venema 94]). As these references show, on infinite models, the situation is somewhat more complex. □

Now the obvious question is whether the same results hold for our categorial fragments, in particular, for $ML(\bullet, \rightarrow, \leftarrow)$ formulas. Proposition 1 evidently goes through, but Proposition 2 does not.

1.3.4. PROPOSITION. Modal equivalence on finite models with respect to $ML(\bullet, \rightarrow, \leftarrow)$ formulas does not guarantee bisimulation.

Proof
Consider the following two models:

$M_1 = \langle W_1, R_1, V_1\rangle$ such that $W_1 = \{a\}$, $R_1 a, aa$ and all atoms are true at a ;
$M_2 = \langle W_2, R_2, V_2\rangle$ such that $W_2 = \{x, y\}$, $R_2 x, xx$, $R_2 x, yx$, $R_2 x, xy$, all atoms are true at x, and the valuation in y is arbitrary

- **Claim 1**
  All categorial formulas α are true at x in M_2 .
  Proof
  Induction on α. We consider three inductive cases in M_2 . Suppose $x \not\models \alpha \rightarrow \beta$. Two cases occur: $y \models \alpha$, $x \not\models \beta$ or $x \models \alpha$, $x \not\models \beta$. In both cases, $x \not\models \beta$ gives a contradiction. The same argument works for $\beta \leftarrow \alpha$. Finally, $x \models \alpha \bullet \beta$, since Rx, xx and, by the inductive hypothesis, $x \models \alpha$, $x \models \beta$.

  A similar argument shows that
- **Claim 2**
  All categorial formulas α are true at a in M_1 .
  Nevertheless, we have this final observation:

- **Claim 3**
  M_1 and M_2 are not bisimilar.
  Proof
  Any bisimulation Z should link y to a . So, $R_1 a, aa$ and aZy should yield a ternary triangle in M_2 with y as a root. But no such triangle exists. □

Thus we have a question what kind of weaker simulation is needed to obtain full analogues to the above results for the categorial fragment. (Cf. the discussion of weaker modal languages for coarser process simulations in [Hennesy and Milner 85], or [van Benthem and Bergstra 93].)

Here we introduce a new notion of *directed categorial bisimulation* which is weaker than the original bisimulation. It helps to draw arrows between models when visualizing the clauses to come.

1.3.5. DEFINITION. *Directed Categorial Bisimulation*
Let Z be a directed binary relation between ternary models M, N, i.e., a set of ordered pairs (x, y) with x in M and y in N, or vice versa. Such a relation Z is called a directed categorial bisimulation if it satisfies the following conditions:

- if xZy and an atom p holds at x, then it also holds at y
- if xZy and Rx, uv, then there exist s, t in the model of y with Ry, st and uZs , vZt
- if xZy and Rs, ty, then there exist u, v in the model of x with Ru, vx and tZv and uZs
- if xZy and Rs, yt, then there exist u, v in the model of x with Ru, xv and tZv and uZs

Example *Directed Bisimulation Without Bisimulation*
Consider the two models employed in our earlier argument:

$M_1 = \langle W_1, R_1, V_1\rangle$ such that $W_1 = \{a\}$, $R_1 a, aa$ and all atoms are true at a ;
$M_2 = \langle W_2, R_2, V_2\rangle$ such that $W_2 = \{x, y\}$, $R_2 x, xx$, $R_2 x, yx$, $R_2 x, xy$, all atoms are true at x, and the valuation in y is arbitrary

We observed that no bisimulation exists here connecting a to x . But the earlier facts can all be explained by seeing that there is a directed categorial bisimulation between these two models:

$$\{(a, x), (x, a), (y, a)\}$$

□

1.3.6. PROPOSITION. *Directed Preservation*
For all categorial formulas ϕ , if xZy for some directed categorial bisimulation Z between two models M, N, then

$$M, x \models \phi \quad \text{implies} \quad N, y \models \phi$$

Proof
By induction on categorial formulas. The above three zigzag clauses were made for just these three cases. □

Actually, the above inductive invariance argument would also go through for conjunctions and disjunctions of categorial formulas. Thus we have come closer to the categorial fragment, but we are not yet quite there. But at least, we have the following converse.

1.3.7. PROPOSITION. *From Invariance to Directed Bisimulation*
Let M and N be two *finite* ternary models with x in M and y in N , ~~or vice versa~~, such that, for all $ML(\bullet, \to, \leftarrow, t, f, \&, \vee)$ formulas α , if $M, x \models \alpha$ then $N, y \models \alpha$. Then there exists a directed categorial bisimulation Z between M and N such that xZy .

Proof
We set aZb iff for any $ML(\bullet, \to, \leftarrow, t, f, \&, \vee)$ formula α, if $a \in V(\alpha)$, then $b \in V(\alpha)$ (in either direction between the two models M and N). Now, we must check all the above clauses.
(a) Suppose that uZu_1 and Ru, wv . We have to show that there exist w_1 and v_1 such that Ru_1, w_1, v_1 with wZw_1, vZv_1 . First note that $u \models t \bullet t$, whence $u_1 \models t \bullet t$. Thus, in the model of u_1 there is at least one triangle with u_1 as a root. Now suppose that, for

$$\langle u_1, a_1 b_1 \rangle, \ldots, \langle u_1, a_k b_k \rangle$$

it is not the case that wZa_i $(1 \leq i \leq k)$ and for

$$\langle u_1, c_1 d_1 \rangle, \ldots, \langle u_1, c_n d_n \rangle$$

it is not the case that vZd_i $(1 \leq i \leq n)$. Then there exist $ML(\bullet, \to, \leftarrow, t, f, \&, \vee)$ formulas $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_n$, such that

- $w \models \alpha_i$ but $a_i \not\models \alpha_i$, for all $1 \leq i \leq k$
- $v \models \beta_i$ but $d_i \not\models \beta_i$, for all $1 \leq i \leq n$

Therefore, $w \models \alpha_1 \& \ldots \& \alpha_k$ and $v \models \beta_1 \& \ldots \& \beta_n$ Hence $u \models (\alpha_1 \& \ldots \& \alpha_k) \bullet (\beta_1 \& \ldots \& \beta_n)$. But by the definition of these formulas,

$$u_1 \not\models (\alpha_1 \& \ldots \& \alpha_k) \bullet (\beta_1 \& \ldots \& \beta_n)$$

which contradicts the fact that uZu_1 .

(b) Suppose that vZv_1 and $Ru_1, w_1 v_1$. Note, that $v_1 \not\models t \to f$. Therefore $v \not\models t \to f$. Thus, there is a least one triangle in the model of v with v for a right daughter. Suppose now that for

$$\langle a_1, b_1 v \rangle, \ldots, \langle a_k, b_k v \rangle \in R,$$

it is not the case that $w_1 Z b_i$ $(1 \leq i \leq k)$, while, for

$$\langle d_1, c_1 v \rangle, \ldots, \langle d_n, c_n v \rangle \in R,$$

it is not the case that $d_i Z u_1 (1 \leq i \leq n)$. Here we assume that

$$\langle a_1, b_1 v\rangle, \ldots, \langle a_k, b_k v\rangle, \langle d_1, c_1 v\rangle, \ldots, \langle d_n, c_n v\rangle$$

are all triangles in the corresponding model with v for a right daughter. Then there exist $ML(\bullet, \rightarrow, \leftarrow, t, f, \&, \vee)$ formulas $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_n$, such that

- $w_1 \models \alpha_i$ but $b_i \not\models \alpha_i$, for all $1 \leq i \leq k$
- $d_i \models \beta_i$ but $u_1 \not\models \beta_i$, for all $1 \leq i \leq n$

Therefore $w_1 \models \alpha_1 \& \ldots \& \alpha_k$ and $u_1 \not\models (\beta_1 \vee \ldots \vee \beta_n)$.
Since $Ru_1, w_1 v_1$, we have

$$v_1 \not\models (\alpha_1 \& \ldots \& \alpha_k) \rightarrow (\beta_1 \vee \ldots \vee \beta_n).$$

Then, because of the link vZv_1 ,

$$v \not\models (\alpha_1 \& \ldots \& \alpha_k) \rightarrow (\beta_1 \vee \ldots \vee \beta_n).$$

But that means that there exists some ternary decomposition $Ra_i, b_i v$ or $Rd_j, c_j v$ witnessing this. Thus, either some b_i does support $\alpha_1 \& \ldots \& \alpha_k$, or some d_j fails to support $\beta_1 \vee \ldots \vee \beta_n$. Both cases yield a contradiction.

(c) The remaining case is quite similar. □

1.3.2 First-Order Translation

There is another broad perspective behind the preceding notions. Like standard modal languages, categorial languages can be viewed as as fragments of full first-order predicate languages via the following transcription of their semantic truth definition:

1.3.8. DEFINITION. *Standard Translation*

$$ST(p) = P(x)$$

$$ST(A \bullet B) = \exists y, z(Rx, yz \;\&\; [y/x]ST(A) \;\&\; [z/x]ST(B))$$

$$ST(A \rightarrow B) = \forall y, z(\neg(Rz, yx \;\&\; [y/x]ST(A)) \vee [z/x]ST(B))$$

$$ST(A \leftarrow B) = \forall y, z(\neg(Rz, xy \;\&\; [y/x]ST(A)) \vee [z/x]ST(B))$$

$$ST(\neg A) = \neg ST(A)$$

$$ST(A\&B) = ST(A) \;\&\; ST(B)$$

The obvious question now is: which fragments of first-order language corresponds to categorial languages? The answer is the following preservation result, which may be proved by standard modal techniques ([van Benthem 85], [de Rijke 93])

1.3.9. THEOREM. *Preservation for Bisimulation*
For any first-order formula ϕ with at most one variable x free, the following two assertions are equivalent:

1. ϕ is equivalent to the standard translation of some categorial formula in $ML(\bullet, \rightarrow, \leftarrow, \neg, \&)$
2. ϕ is invariant for bisimulations.

Notice that all formulas in this fragments have their quantifiers occurring restricted by ternary atoms. A general study of such restricted-quantifier fragments of first-order logic is made in [An.vB.Nem. 95], where it is shown how their model theory and proof theory is much simpler than that of full first-order logic. In particular, complete proof calculi need not invoke the structural rule of Contraction. These analogies with categorial inference will be discussed further in Chapter 5 below.

For the moment, we just note that under these translations, categorial derivations can also be viewed as first-order deductions with variables ranging over information states.

Example *Left Function Application*
The Lambek-derivable sequent $A \bullet (A \rightarrow B) \vdash B$ corresponds to the valid first-order inference

$$\exists yz(Rx, yz \;\&\; A(y) \;\&\; \forall uv((Ru, vz \;\&\; A(v)) \rightarrow B(u))) \models B(x).$$

The latter can be proved by completely standard means.

Example *Geach's Principle*
The well-known L-derivable sequent of 'function composition'

$$(A \rightarrow B) \bullet (B \rightarrow C) \vdash (A \rightarrow C)$$

is not derivable in NL . This has to do with its expressing Associativity of ternary composition, as we shall see in the correspondence analysis of Chapter 2. These facts show up as follows under first-order translation. The Geach sequent translates into the first-order formula

$\exists yz((Rx, yz \;\&\; \forall uv((Ru, vy \;\&\; A(v)) \rightarrow B(u))$

$\& \; \forall st((Rs, tz \;\&\; B(t)) \rightarrow C(s)) \models$

$\forall mn((Rm, nx \;\&\; A(n)) \rightarrow C(m))$

The latter formula is not first-order valid as it stands. It requires a certain form of associativity for R to become so.

Incidentally, this style of analysis provides a powerful means of finding counter-examples to putative categorial principles: translate, and search for standard first-order counter-examples in ternary models via any known technique (e.g., semantic tableaus).

1.3.3 Frame Constructions

Next, we concentrate on ternary frames, and their categorially definable structural properties which do not refer to particular valuations. Again, this is a known area from modal logic. For the full modal language $ML(\bullet, \rightarrow, \leftarrow, \neg, \&)$ with Booleans, we have a number of familiar frame constructions preserving frame truth. These include so-called generated subframes, disjoint unions, p–morphic images and ultrafilter extensions. (Cf. [van Benthem 85], or [Blackburn de Rijke and Venema 94].) Such constructions can be used to show that certain frame properties are not modally definable: namely, if they fail to be preserved under them. In some cases, these properties even characterize the modal language over frames, witness the following definability result from [Goldblatt and Thomason 74]: the modally definable first-order classes of frames are precisely those which are preserved under taking generated subframes, disjoint unions, p–morphic images and whose complement is closed under ultrafilter extensions.

For classes of finite frames, the first three conditions even suffice. With our weaker categorial language without Booleans, this theory essentially generalizes. We first consider the most evident cases, namely constructions preserving frame truth which are not sensitive to the presence or absence of booleans. Next, we add a notion of weak filter extension for categorial language without booleans and prove the corresponding antipreservation theorem.

1,3.10. Definition. *Generated Subframes*
Let $F = \langle W, R\rangle$ be a ternary frame. $F' = \langle W', R'\rangle$ is a *generated subframe* of F if it is a subframe in the ordinary model-theoretic sense (i.e., W' is a subset of W , and R' is the restriction of R to that subset), which satisfies three additional closure properties:

$$\begin{array}{l}
\text{if } x \in W' \text{ and } Rx, yz, \text{ then } y, z \in W' \\
\text{if } y \in W' \text{ and } Rx, yz, \text{ then } x, z \in W' \\
\text{if } z \in W' \text{ and } Rx, yz, \text{ then } y, x \in W'
\end{array}$$

1.3.11. Proposition. If F' is a generated subframe of F and $w \in W'$, then, for each formula ϕ in $ML(\bullet, \rightarrow, \leftarrow)$, if $F \models \phi$, then $F' \models \phi$

Proof

This result, and the following ones, are easy uniform consequences of the earlier invariance for bisimulation. Suppose that ϕ fails in F' . I.e., there exists a valuation V' and a $w \in W'$ with $\langle F', V'\rangle, w \not\models \phi$. But then, the identity relation between points in W' and W is a bisimulation between the two models $\langle F', V"\rangle$ and $\langle F, V'\rangle$ connecting w to itself. Therefore, ϕ also fails at w in F , and hence $F \models \phi$. □

Corollary

$\neg\forall x Rx, xx$ ('non-reflexivity') is not definable in $ML(\bullet, \rightarrow, \leftarrow)$.

Proof

Consider two frames F_1 and F_2 defined by

- $F_1 = \langle W_1, R_1\rangle$, where $W_1 = \{x\}$ and $R_1 x, xx$
- $F_2 = \langle W_2, R_2\rangle$ with $W_2 = \{a, b, c, x\}$ and $R_2 a, bc$; $R_2 x, xx$

F_1 is a reflexive generated subframe of the non-reflexive F_2 . This refutes modal definability, by the preceding result. □

Likewise, the ***disjoint union*** of a family of ternary frames can be defined taking the disjoint union of their domains and their ternary accessibility relations. Note that each of these frames lies embedded as a generated subframe of the disjoint union. As a consequence of the previous result, we then have:

1.3.12. PROPOSITION. A $ML(\bullet, \rightarrow, \leftarrow)$ formula ϕ is true in a disjoint union of ternary frames iff ϕ is true in all of them.

Corollary

$\forall b \forall c \exists a Ra, bc$ ('existence of compositions') is not definable in $ML(\bullet, \rightarrow, \leftarrow)$.

Proof

Take a disjoint union of the single reflexive point frame $\langle\{x\}; \{\langle x, x, x\rangle\}\rangle$ with itself. □

1.3.13. DEFINITION. *P–Morphism*

Let $F = \langle W, R\rangle$, $F' = \langle W', R'\rangle$ be ternary Kripke frames. A function $f : W \rightarrow W'$ is a p–morphism from F to F' if

1. if Ra, bc , then $R'f(a), f(b)f(c)$
2.
   - if $R'f(a), xy$, then there are $b, c \in W$ such that $f(b) = x$; $f(c) = y$ and Ra, bc
   - if $R'x, f(b)y$, then there are $a, c \in W$ such that $f(a) = x$; $f(c) = y$ and Ra, bc
   - if $R'x, yf(c)$, then there are $a, b \in W$ such that $f(a) = x$; $f(b) = y$ and Ra, bc

1.3.14. PROPOSITION. If F' is a p-morphic image of F , then for any $ML(\bullet, \rightarrow, \leftarrow)$ formula ϕ , if $F \models \phi$ then $F' \models \phi$.

Proof
Let F' refute ϕ in some world w under some valuation. Use inverse images of the p–morphism f to copy this valuation onto F . The result is a bisimulation between the two resulting models. Then, again by bisimulation invariance, ϕ will be refuted in any inverse f–image of w. □

Corollary
$\forall x \neg Rx, xx$ ('irreflexivity') is not $ML(\bullet, \rightarrow, \leftarrow)$ definable.

Proof
Compare F_1 and F_2 defined as follows:

- $F_1 = \langle W_1, R_1 \rangle$, with $W_1 = \{a, b, c\}$
  and $R_1 a,\ ac; R_1 b,\ ba; R_1 c, cb$
- $F_2 = \langle W_2, R_2 \rangle$, with $W_2 = \{x\}$ and $R_2 x, xx$.

Identification with x is a p-morphism from the irreflexive frame F_1 to the reflexive frame F_2. □

Finally, we consider the least obvious extension of standard modal preservation properties.

1.3.15. DEFINITION. *Weak Filter*
Let I be a non-empty set. A weak filter D over I is a set of non-empty subsets of I such that
(i) $I \in D$
(ii) if $X \in D$ and $X \subseteq Y$, then $Y \in D$

Next, we define three useful operations on subsets X, Y of any ternary frame F, obtained by obvious lifts (borrowed from the completeness arguments):

$$\begin{array}{rcl} X \circ Y & = & \{a | \exists bc(Ra, bc \,\&\, b \in X \,\&\, c \in Y)\} \\ X \rhd Y & = & \{a | \forall bc((Rc, ba \,\&\, b \in X\ ) \Rightarrow\ c \in Y)\} \\ X \lhd Y & = & \{a | \forall bc((Rc, ab \,\&\, b \in X\ ) \Rightarrow\ c \in Y)\} \end{array}$$

It is easy to see that:

$$X \circ Y \subseteq Z \iff Y \subseteq X \rhd Z \iff X \subseteq Z \lhd Y$$

The motivation for these operations has to do with our truth definition. Let $V(\phi)$ be the set of worlds in a model where a formula ϕ holds. Then, we have the following equalities:

$$\begin{array}{lcl} V(A \bullet B) & = & V(A) \circ V(B) \\ V(A \rightarrow B) & = & V(A) \rhd V(B) \\ V(A \leftarrow B) & = & V(A) \lhd V(B) \end{array}$$

1.3.16. DEFINITION. *Weak Filter Extensions*
Let $F = \langle W, R \rangle$ be a ternary frame. The weak filter extension wfe(F) of F is the frame $F' = \langle W', R' \rangle$ where W' is the set of all weak filters over W and R' is defined as follows:

$R'u_1, u_2u_3 \Longleftrightarrow \forall X, Y(X \in u_2 \ \& \ Y \in u_3 \ \Rightarrow X \circ Y \in u_1)$

1.3.17. LEMMA. Let V be a valuation on $F = \langle W, R \rangle$, and let V' be the valuation on wfe(F) defined by

$u \in V'(p)$ iff $V(p) \in u$.

Then, for all $u \in W'$ and all categorial formulas ϕ,

$u \in V'(\phi)$ iff $V(\phi) \in u$.

Proof
Induction on ϕ. We start with the case where $\phi = A \bullet B$:

1. Suppose $u \in V'(A \bullet B)$
2. There are u_1, u_2 such that

$$u_1 \in V'(A) \qquad u_2 \in V'(B) \qquad R'u, u_1u_2$$

3. By the inductive hypothesis, $V(A) \in u_1$ and $V(B) \in u_2$
   Hence, by the definition of R' : $V(A) \circ V(B) \in u$.
   By a preceding observation, then, $V(A \bullet B) \in u$.

Next, conversely,

1. Suppose $V(A \bullet B) \in u$.
2. We need to find two suitable weak filters u_1, u_2. Set:

$$u_1 = \{X | V(A) \subseteq X\} \qquad u_2 = \{X | V(B) \subseteq X\}$$

3. We show that $R'u, u_1u_2$. Let $X \in u_1$ and $Y \in u_2$.
   Hence, $V(A) \subseteq X$, $V(B) \subseteq Y$ and therefore $V(A) \circ V(B) \subseteq X \circ Y$.
   It follows that $V(A \bullet B) \subseteq X \circ Y$, and from (1) plus the definition of a weak filter, we have $X \circ Y \in u$.
4. Altogether, the constructed u_1, u_2 satisfy what is needed:

$$u_1 \in V'(A) \qquad u_2 \in V'(B) \qquad R'u, u_1u_2$$

We consider one other case, that of $\phi = A \to B$, giving the main steps without extensive annotation:

1. Suppose $V(A \to B) \in u$
2. Suppose $R'u_2, u_1 u$ and $u_1 \in V'(A)$
3. By the inductive hypothesis, $V(A) \in u_1$, hence
4. $V(A) \circ V(A \to B) \in u_2$
5. Since $V(A) \circ V(A \to B) \subseteq V(B)$, $V(B) \in u_2$
6. Therefore, $u_2 \in V'(B)$ and $u \in V'(A \to B)$

Conversely,

1. Suppose $V(A \to B) \notin u$
2. We have to construct weak filters u_1, u_2 such that
   $u_1 \in V'(A), \quad R'u_2, u_1 u \quad \text{and} \quad u_2 \notin V'(B)$
3. Define
   $u_1 = \{X \subseteq W | V(A) \subseteq X\}$
   $u_2 = \{Z \subseteq W | \exists X, Y \subseteq W (X \circ Y \subseteq Z \;\&\; X \in u_1 \;\&\; Y \in u)\}$
4. $V(A) \in u_1$,therefore $u_1 \in V'(A)$
5. Note that, if $X \in u_1$ and $Y \in u$, then $X \circ Y \in u_2$. Therefore $R'u_2, u_1 u$
6. We show that $V(B) \notin u_2$ by contraposition: Suppose $V(B) \in u_2$.
   Then there are $X, Y \subseteq W$ such that
   $X \circ Y \subseteq V(B) \qquad X \in u_1 \qquad Y \in u$
7. Next, since $V(A) \subseteq X$, clearly $V(A) \circ Y \subseteq X \circ Y$
   and $V(A) \circ Y \subseteq V(B)$, whence
   $Y \subseteq V(A) \rhd V(B)$ and $Y \subseteq V(A \to B)$
   hence $V(A \to B) \in u$

   In all, then:
   $V(B) \notin u_2 \Rightarrow u_2 \notin V'(B) \Rightarrow u \notin V'(A \to B)$. □

Now, we can prove the desired 'antipreservation result' for weak filter extensions:

1.3.18. THEOREM. For all frames F and all categorial formulas ϕ, if ϕ is true in wfe(F) , then ϕ is true in F .

Proof
Suppose there exists a valuation V on F refuting ϕ in some world. I.e., $W - V(\phi)$ is non-empty: say, it generates the weak filter u . By the preceding Lemma, we have that u is not in $V'(\phi)$: i.e., ϕ is not true in wfe(F) either. □

Corollary The first-order formula $\forall a \exists b Rb, ba$ is not categorially definable.
Proof
Consider the ternary frame $F = \langle W, R\rangle$, where $W = N$ (the natural numbers) and Ra, bc iff $a < (b + c)$. Clearly, $\forall a \exists b Rb, ba$ is false in F : consider the number 0 . By contrast, we have:

Claim $\forall a \exists b Rb, ba$, ba is true in wfe($F$).
Proof Let a be an arbitrary weak filter. Define b as $\{N\}$, which is a weak filter. We still have to check that $R'b, ba$. Suppose that $X \in b$ and $Y \in a$. (Note that $X = N$ and that Y must be non-empty.) So, we need that $N \circ Y = N$. But this is trivial, using the definitions of $\circ$ and R.

□

The preceding analysis has given a broader version of modal ultrafilter extensions, matching our weaker categorial language. Similar generalizations might be possible for the three more traditional frame constructions of generated subframes, disjoint unions and pmorphic images. More generally, we have this question: Can one generalize the Goldblatt-Thomason Definability Theorem to this categorial setting?

Remark *Categorial Sequents*
Our categorial analysis will eventually access ternary frames a little differently. For, we shall be interpreting categorial sequents $A \vdash B$, whose truth condition referred to truth at all worlds. This corresponds to a (modest) use of the so-called 'universal modality' (de Rijke 1993), which induces a slightly modified perspective on ternary frames (cf. van Benthem 1989 on the resulting form of the Goldblatt-Thomason theorem).

Chapter 2

Correspondence Theory For Categorial Principles

2.1 Introduction

The aim of this Chapter is to demonstrate some methods of modal Correspondence Theory for use in categorial analysis. In particular, we prove a Sahlqvist – van Benthem theorem for categorial languages. Moreover, we develop some general theory of definability.

2.1.1 Correspondence Phenomena

We now resume the semantic analysis of the frame import of structural principles in categorial logics. In the preceding Chapter, we considered frame truth of categorial formulas by themselves. But now, we want to analyze the sequents $A \vdash B$ that make up categorial calculi. These were true in a ternary frame F, if for all valuations V , all worlds verifying the antecedent in the model $\langle F, V\rangle$ also verify the consequent. There is no obvious reduction between the two notions, say, via some deduction theorem.

Example *Sequents versus Formulas*
The sequent $A \vdash A \bullet A$ corresponds to the frame property of reflexivity, as we have seen before: $\forall x Rx, xx$. Its closest formula relative

$$A \to A \bullet A$$

corresponds to $\forall xyz(Rx, yz \to Rx, yy)$. Reductions will only occur if we introduce additional vocabulary into the categorial formalism, such as an 'identity type constant' I satisfying

$$A \bullet I = I \bullet A = A.$$

The latter will allow removal of all antecedents up to I, using Residuation Principle

$$A \bullet B \vdash C \iff B \vdash A \rightarrow C \iff A \vdash C \leftarrow B$$

Let us consider some main steps in the Categorial Hierarchy.

Associativity and Function Composition
Consider Geach Composition, in one of its many equivalent forms:

$$(A \rightarrow B) \vdash (C \rightarrow A) \rightarrow (C \rightarrow B).$$

One can derive this as follows:

First, note that

$$(C \bullet (C \rightarrow A)) \bullet (A \rightarrow B) \vdash B$$

is **NL**-derivable. Then by Associativity and Cut

$$C \bullet ((C \rightarrow A) \bullet (A \rightarrow B)) \vdash B$$

By residuation

$$(A \rightarrow B) \vdash (C \rightarrow A) \rightarrow (C \rightarrow B)$$

But there is also a converse route. Substitute the following:

$$B := (A \bullet B) \bullet C \quad A := A \bullet B \quad C := A$$

Then the Geach Rule yields:
$((A \bullet B) \rightarrow ((A \bullet B) \bullet C)) \vdash (A \rightarrow (A \bullet B)) \rightarrow (A \rightarrow ((A \bullet B) \bullet C))$.
Now note that the following are **NL**-derivable:

$$C \vdash (A \bullet B) \rightarrow ((A \bullet B) \bullet C)$$
$$B \vdash A \rightarrow (A \bullet B)$$

Using Cut and Residuation, one obtains:

$$B \bullet C \vdash (A \rightarrow ((A \bullet B) \bullet C)).$$

Again, by Residuation this gives

$$A \bullet (B \bullet C) \vdash (A \bullet B) \bullet C.$$

Another way of arriving at this insight, however, is to analyze the semantic meaning of these laws.Both express that the composition relation is associative, in the sense that

$$\forall abcde((Rc, ba \,\&\, Ra, de) \Rightarrow \exists x(Rc, xe \,\&\, Rx, bd)).$$

Permutation and Contraction
Similar points can be made concerning the law of Permutation, the structural principle which leads to the non-directed Lambek Calculus LP. Its natural product formulation is $A \bullet B \vdash B \bullet A$. But there are also slash formulations, such as the principle which says that functions do not care about the provenance of

their arguments: $A \to B \vdash B \leftarrow A$. Again, we shall analyze these two, and show that their semantic content is the same. On the other hand, we will show that there exists a variety of principles expressing Contraction, in forms like

$$A \vdash A \bullet A \quad \text{or} \quad (A \to (A \to B) \vdash (A \to B).$$

Peirce's Law
Finally, the principle which collapses resource-sensitive categorial logics into classical conditional logic is Peirce's Law

$$((A \to B) \to A) \to A \vdash A.$$

We shall study this principle in some detail below.

2.1.2 Correspondence Theory

There is an extensive modal theory of frame correspondence (cf. [van Benthem 84, 85]. Here are some basics. Any modal formula $\phi(p_1, \ldots, p_k)$ has a *second-order* frame correspondent via the earlier standard translation, which may be written as follows:

$$\forall P_1 \ldots \forall P_k \forall x ST(\phi).$$

Here, the second-order prefix reflects the quantification over all (relevant) valuations in a frame, and the first-order one that over all its worlds. But in many cases, a simple first-order equivalent is available. For instance, the well-known S4 axioms correspond to reflexivity and transitivity of accessibility. On the other hand, the 'Löb Axiom' in provability logic expresses transitivity and well-foundedness of (converse) accessibility, and hence it is essentially second-order. We call a modal formula *first-order* if it defines a first-order frame property. Model-theoretically, modal formulas are first-order if and only if they are preserved under ultrapowers. More syntactically, there are also methods for extracting first-order equivalents from (suitable) modal formulas. A powerful one is the Sahlqvist-van Benthem algorithm, which produces first-order substitutions for the above second-order quantifiers $\forall P_1 \ldots \forall P_k$ turning the matrix formula $ST(\phi)$ into a first-order equivalent for the original modal ϕ (modulo some logical transformations), provided that the latter has a suitable form described in the Sahlqvist- van Benthem Theorem.

There is also a converse direction of interest. Given any first-order frame property, one can ask whether it is modally definable. Here, the earlier frame preservation properties provide constraints (cf. Chapter 1), and indeed the Goldblat-Thomason Theorem provided a complete though not syntactically explicit description. For further issues and results, we refer to the above three references. In particular, current interest revolves around correspondence for stronger modal formalisms, with respect to stronger standard languages (including higher-order ones), often using connections with Universal Algebra (cf. also [Marx 94]).

2.2 A Sahlqvist-van Benthem Theorem for Categorial Formulas

Our aim now is to find a version of Sahlqvist-van Benthem algorithm, which when input a categorial formula in $ML(\bullet, \rightarrow, \leftarrow)$ of a certain form, reduces it to an equivalent *first-order* property of *ternary* relations via suitable instantiations. We start with examples with formulas in $ML(\bullet)$ to illustrate the method in action. Then we propose two ways of treating implicative categorial sequents, one by reduction and one by direct analysis. These motivate two parts to the general Sahlqvist-van Benthem Theorem for categorial language.

2.2.1 Examples with Product

We start with some examples involving only the categorial product $\bullet$. An algorithm for this class should deal with at least the following axioms related to the structural rules:

$(p \bullet q) \bullet r \vdash p \bullet (q \bullet r)$	*Associativity*
$p \bullet q \vdash q \bullet p$	*Permutation*
$p \vdash p \bullet p$	*Contraction*
$p \bullet q \vdash p$	*Weakening*

A small point of notation. In this Chapter, in line with the modal literature, we shall often consider single sequents written using proposition letters (e.g., $p \bullet q \vdash q \bullet p$) , rather than their schematic versions indicated with capital letters (compare $A \bullet B \vdash B \bullet A$). The reason is that frame truth for both amounts to the same thing, due to its second-order quantification over all valuations

2.2.1. PROPOSITION. If both α and β are in $ML(\bullet)$, then the sequent $\alpha \vdash \beta$ corresponds to some first order formula δ which is effectively obtainable from $\alpha \vdash \beta$.

Proof
This is a straightforward adaptation of the of minimal substitution algorithm in van Benthem 85]. At this stage, we merely illustrate how it works by some heuristic examples. □

Example *Permutation*
What does $p \bullet q \vdash q \bullet p$ correspond to? A useful heuristic is to think of some 'minimal valuation' making its antecedent true, and then see what the latter

says about the consequent. Now, a minimal verification of $p \bullet q$ is some situation Rx, yz in a model with $y \models p$ and $z \models q$. Nothing more is needed than truth of p and q at just these worlds. If this is to imply $q \bullet p$, then there must exist u, v satisfying: (*i*) Rx, uv, (*ii*) $u \models q$, (*iii*) $v \models p$. This can only be the case iff $u = z$ and $v = y$. I.e., frame truth of our sequent implies the already familiar $\forall xyz(Rx, yz \to Rx, zy)$. Moreover, it is easy to check the converse: the latter frame property implies that the sequent $p \bullet q \vdash q \bullet p$ holds under any valuation. □

Evidently, this style of reasoning can even serve to extract a first-order equivalent from the given sequent, if none were known beforehand. Here is a more extensive explanation of the main steps of this heuristic.

Example *Associativity*
Consider the sequent $(p \bullet q) \bullet r \vdash p \bullet (q \bullet r)(^*)$. The following is a heuristic 'derivation' giving its first-order frame correspondent.

1. Suppose (*) is true in some arbitrary F. That is, under all valuations, we have $\forall a(a \models (p \bullet q) \bullet r \Rightarrow a \models p \bullet (q \bullet r))$. Together with truth conditions for $\bullet$, this transcribes into
2. $$\forall a(\exists bcde(Ra, bc \,\&\, Rb, de \,\&\, d \models p \,\&\, e \models q \,\&\, c \models r) \Rightarrow$$
$$\Rightarrow \exists xyzv(Ra, xy \,\&\, Ry, zv \,\&\, x \models p \,\&\, z \models q \,\&\, v \models r)).$$
This is logically equivalent to the (more) universal form
$$\forall abcde((Ra, bc \,\&\, Rb, de \,\&\, d \models p \,\&\, e \models q \,\&\, c \models r) \Rightarrow$$
$$\Rightarrow \exists xyzv(Ra, xy \,\&\, Ry, zv \,\&\, x \models p \,\&\, z \models q \,\&\, v \models r)).$$
3. Pick up the canonical valuation CV such that
$$u \in CV(p) \text{ iff } u = d$$
$$u \in CV(q) \text{ iff } u = e$$
$$u \in CV(r) \text{ iff } u = c$$
For convenience, we shall write '$s \models^* p$' for '$s \in CV(p)$' henceforth. Then by instantiation from [2],
$$\forall abcde((Ra, bc \,\&\, Rb, de \,\&\, d \models^* p \,\&\, e \models^* q \,\&\, c \models^* r) \Rightarrow$$
$$\Rightarrow \exists xyzv(Ra, xy \,\&\, Ry, zv \,\&\, x \models^* p \,\&\, z \models^* q \,\&\, v \models^* r)).$$
Simplifying, we get the equivalent form
$$\forall abcde((Ra, bc \,\&\, Rb, de) \Rightarrow$$
$$\exists xyzv(Ra, xy \,\&\, Ry, zv \,\&\, x = d \,\&\, z = e \,\&\, v = c)).$$

This is equivalent to the earlier associativity of composition:

$$\forall abcde((Ra, bc \;\&\; Rb, de) \Rightarrow \exists y(Ra, dy \;\&\; Ry, ec)).$$

To show that there is a way 'back', consider any valuation V on our frame F, and suppose that

4. (a) $\forall abcde((Ra, bc \;\&\; Rb, de \;\&\; d \models^* p \;\&\; e \models^* q \;\&\; c \models^* r) \Rightarrow$

$$\Rightarrow \exists xyzv(Ra, xy \;\&\; Ry, zv \;\&\; x \models^* p \;\&\; z \models^* q \;\&\; v \models^* r)).$$

   is true in F

   (b) $a \models (p \bullet q) \bullet r$ for the $\models$ associated with V Thus, there are b, c, d and e such that

$$Ra, bc \;\&\; Rb, de \;\&\; d \models p \;\&\; e \models q \;\&\; c \models r$$

   We have to show that $a \models p \bullet (q \bullet r)$.

5. Define a new valuation CV as above, whose corresponding evaluation is again denoted by $\models^*$. Then we have $a \models^* (p \bullet q) \bullet r$. This again reflects the special nature of our antecedents, which can be verified 'minimally'.
6. By [4.a], we must also have $a \models^* p \bullet (q \bullet r)$.
7. Next, we exploit our special antecedent. All proposition letters p, q, r occur positively in $(p \bullet q) \bullet r$. Therefore, the latter is semantically *monotone* . I.e., it will stay true in passing from a valuation where it holds in our frame to a valuation assigning (possibly) larger sets of worlds to proposition letters. But this is precisely the relationship between CV and the original valuation V. Therefore, we also have $a \models p \bullet (q \bullet r)$, and we are done. □

The first-order equivalent 'derived' in this way may also be obtained from the second order translation of (*) :

$$\forall P \forall Q \forall R \forall a(\exists bcde(Ra, bc \;\&\; Rb, de \;\&\; P(d) \;\&\; Q(e) \;\&\; R(c)) \Rightarrow$$
$$\exists xyzv(Ra, xy \;\&\; Ry, zv \;\&\; P(x) \;\&\; Q(z) \;\&\; R(v))).$$

Here P, Q, R denote the sets of possible worlds where p, q, r hold. The above critical valuation CV then provides three syntactical substitutions to be performed in the (rearranged) matrix, namely: $[u = d/Pu], [u = e/Qu], [u = c/Ru]$. Further transformations are as in the above argument.

The above analysis actually yields something a little bit stronger. It will work for all categorial sequents $\alpha \vdash \beta$ such that α is in $ML(\bullet)$ and β is an arbitrary categorial formula in $ML(\bullet, \rightarrow, \leftarrow)$ without negative occurrences of any proposition letter. To define the latter, we follow the usual convention (cf. Moortgat 1988):

Polarity of Occurrence

A proposition letter p occurs positively in p itself, not at all in other q . Positive occurrences in both components of a positive (resp. negative) product, and in consequents of positive (resp. negative) implications, remain positive. (resp. negative) Occurrences switch polarity positive-negative in antecedents. Thus, in positive (resp. negative) function type $A \to B$ or $B \leftarrow A$, B occurs positively (resp. negatively) and A negatively (resp. positively). For example, in the Montague Raising of A, being the compound of positive $B \leftarrow (A \to B)$, the first occurrence of B as well as that of A are positive, while the second occurrence of B is negative.

2.2.2 Examples With Slashes and Products

Next, we consider some principles with function type constructors. Consider again some intuitive counterparts to structural rules:

$p \to q \vdash (r \to p) \to (r \to q)$	*Associativity*
$p \to q \vdash q \leftarrow p$	*Permutation*
$p \to (p \to q) \vdash p \to q$	*Contraction*
$q \to r \vdash q \to (p \to r)$	*Weakening*

At first sight, it seems harder to treat principles like this, as no obvious 'minimal valuation' is suggested by their antecedent. Nevertheless, there is a lot one can do by a change of viewpoint. On the one hand, one can try to reduce sequents with slashes to its product equivalents. For example, as said, $p \to q \vdash q \leftarrow p$ has the same semantical meaning as $p \bullet q \vdash q \bullet p$. On the other hand, one can use a direct analysis to analyze sequents with slashes , starting with a *contraposition*.. Later on after we will raise a question if these two methods complement each other, or if one of them is subsumed by the other.

Example *Direct Derivation of First-Order Equivalents*
Consider weak contraction sequent $p \to (p \to q) \vdash p \to q$.

1. Suppose that $F \models p \to (p \to q) \vdash p \to q$. That is, for *any* valuation V and any point a , if $a \notin V((p \to q))$, then $a \notin V(p \to (p \to q))$. Let P and Q denote the sets of possible worlds where p and q hold, respectively. This is reflected in the corresponding second-order formula

$$\forall P \forall Q \forall a[\exists b \exists c(Rc, ba \,\&\, P(b) \,\&\, \neg Q(c)) \Rightarrow \\ \exists x \exists y(Ry, xa \,\&\, P(x) \,\&\, \exists u \exists v(Rv, uy \,\&\, P(u) \,\&\, \neg Q(v))]$$

2. Consider the antecedent of this formula. Move its existential quantifiers to the front as universal quantifiers (the justification is provided by predicate logic):

$$\forall P \forall Q \forall a \forall b, c[(Rc, ba \,\&\, P(b) \,\&\, \neg Q(c)) \Rightarrow \\ \exists x \exists y(Ry, xa \,\&\, P(x) \,\&\, \exists uv(Rv, uy \,\&\, P(u) \,\&\, \neg Q(v))]$$

3. Fix a fresh variable w . Define a special 'minimal valuation' V^*, which will provide the required first-order condition:

$$\begin{array}{rcl} V^*(p) & = & \{b\} \\ V^*(\neg q) & = & \{c\} \end{array}$$

This corresponds to the syntactic substitution

$$\begin{array}{ll} P^*(w) & w = b \\ Q^*(w) & w \neq c \end{array}$$

4. The required frame condition is obtained by instantiation:

$$\begin{array}{l} \forall a \forall b \forall c[(Rc, ba \mathbin{\&} P^*(b) \mathbin{\&} \neg Q^*(c)) \Rightarrow \\ \exists x \exists y (Ry, xa \mathbin{\&} P^*(x) \mathbin{\&} \exists u \exists v (Rv, uy \mathbin{\&} P^*(u) \mathbin{\&} \neg Q^*(v))] \end{array}$$

This is already a first order formula, which can be presented as

$$\begin{array}{l} \forall a \forall b \forall c(Rc, ba \Rightarrow \\ \exists x \exists y \exists u \exists v (Ry, xa \mathbin{\&} x = b \mathbin{\&} Rv, uy \mathbin{\&} u = b \mathbin{\&} v = c)) \end{array}$$

which is again equivalent to

$$\forall a \forall b \forall c(Rc, ba \Rightarrow \exists y(Ry, ba \mathbin{\&} Rc, by))$$

As universal second-order formulas imply all their instantiations, this shows that the original contraction sequent implies this frame property. But also conversely, it is easy to see that the sequent $p \rightarrow (p \rightarrow q) \vdash p \rightarrow q$ is true at each frame satisfying the latter. Even so, to illustrate our general proof method, we provide some explicit steps leading 'backwards'. Consider any valuation. Assume that

$$Rc, ba \mathbin{\&} P(b) \mathbin{\&} \neg Q(c)$$

and show that

$$\exists x \exists y (Ry, xa \mathbin{\&} P(x) \mathbin{\&} \exists u \exists v (Rv, uy \mathbin{\&} P(u) \mathbin{\&} \neg Q(v))$$

First, we can define the special valuation V^* as above. This has the following effects:

$$\begin{array}{c} Rc, ba \mathbin{\&} P^*(b) \mathbin{\&} \neg Q^*(c) \\ \text{for any } x \text{ , if } P^*(x), \text{ then } P(x) \\ \text{for any } x \text{ , if } \neg Q^*(x), \text{ then } \neg Q(c). \end{array}$$

Moreover, P, $\neg Q$ occur monotone positively in our consequent. Therefore, we see that

$$\exists x \exists y(Ry, xa \,\&\, P^*(x) \,\&\, \exists u \exists v(Rv, uy \,\&\, P^*(u) \,\&\, \neg Q^*(v))]$$

and together with semantic monotonicity, this implies

$$\exists x \exists y(Ry, xa \,\&\, P(x) \,\&\, \exists u \exists v(Rv, uy \,\&\, P(u) \,\&\, \neg Q(v))$$

A similar analysis may be performed on other principles with function type constructors related to structural rules. However they also can be analyzed via a detour to the broader modal language with all Booleans, or alternatively, with three matching categorial products. For convenience, we repeat their definitions:

$$\begin{array}{lcl} \mathcal{M}, a \models A \bullet_1 B & \Longleftrightarrow & \exists b \exists c(Ra, bc \;\&\; \mathcal{M}, b \models A \;\&\; \mathcal{M}, c \models B) \\ \mathcal{M}, a \models A \bullet_2 B & \Longleftrightarrow & \exists b \exists c(Rc, ba \;\&\; \mathcal{M}, b \models A \;\&\; \mathcal{M}, c \models B) \\ \mathcal{M}, a \models A \bullet_3 B & \Longleftrightarrow & \exists b \exists c(Rc, ab \;\&\; \mathcal{M}, b \models A \;\&\; \mathcal{M}, c \models B) \end{array}$$

Example *From Slashes to Mixed Products*
Consider $p \bullet q \vdash q \bullet p$. A negated functional consequent $\neg(q \leftarrow p)$ is equivalent to a product $p \bullet_3 \neg q$, and a negated antecedent $\neg(p \rightarrow q)$ to $p \bullet_2 \neg q$. Thus, by contraposition, the above principle is also equivalent to

$$p \bullet_3 \neg q \vdash p \bullet_2 \neg q$$

As we are dealing with a universal quantification over valuations here, working with atoms or their negations makes no difference. Thus, this sequent is frame equivalent with

$$p \bullet_3 q \vdash p \bullet_2 q$$

The latter principle, too, expresses commutation, as can be seen using the methods of the earlier subsection on products. □

The latter explanation again raises a new issue. How can this implication between mixed products be equivalent with a pure sequent axiom for the remaining product $p \bullet_1 q \vdash q \bullet_1 p$? We shall not analyze this matter in detail here, but merely state that a precise algorithm for this shift can be worked out from the above semantic analysis.

Instead, we conclude this subsection with some illustrations motivating our eventual categorially useful Sahlqvist-van Benthem Theorem. Informally speaking, there are two ways of dealing with categorial slash sequents. The first uses contraposition as above, the second involves a reduction to ML($\bullet$) formulas. Here are some illustrations of these reductions.

Example *Weak Contraction Principle: from slashes to products*

Claim For any frame F,

$$F \models s \to (s \to t) \vdash s \to t \text{ iff } F \models p \bullet q \vdash p \bullet (p \bullet q)$$

Proof

For left to right direction, suppose

$F \models s \to (s \to t) \vdash s \to t.$

By substituting $p \bullet (p \bullet q)$ for t and p for s,

$F \models p \to (p \to (p \bullet (p \bullet q))) \vdash p \to (p \bullet (p \bullet q))$

Next, note that $q \vdash p \to (p \to (p \bullet (p \bullet q)))$ is valid n any frame. Therefore

$F \models q \vdash p \to (p \bullet (p \bullet q)).$

Finally, by residuation on frames

$F \models p \bullet q \vdash p \bullet (p \bullet q).$

For right to left direction , suppose

$F \models p \bullet q \vdash p \bullet (p \bullet q)$

By substituting $s \to (s \to t)$ for q and s for p

$F \models s \bullet (s \to (s \to t)) \vdash s \bullet (s \bullet (s \to (s \to t))),$

whence

$F \models s \bullet (s \to (s \to t)) \vdash t.$

Then, by residuation on frames

$F \models s \to (s \to t) \vdash s \to t.$

□

By similar reasoning one obtains:

$$\begin{array}{lcl} F \models p \to q \vdash q \leftarrow p & \Longleftrightarrow & F \models q \bullet p \vdash p \bullet q \\ F \models p \to (q \to r) \vdash q \to (p \to r) & \Longleftrightarrow & F \models q \bullet (p \bullet r) \vdash p \bullet (q \bullet r) \\ F \models q \to r \vdash q \to (p \to r) & \Longleftrightarrow & F \models q \bullet (p \bullet r) \vdash q \bullet r \end{array}$$

The next result provides a generalization:

2.2.2. PROPOSITION. *Switching between Products and Slashes*
Let q, $p_i\ y_j\ (1 \leq i, j \leq m), s_i\ x_j\ (1 \leq i, j \leq n)$, z be propositional variables such that

- $s_k = s_j$ iff $x_k = x_j$ for $1 \leq k, j \leq n$
- $p_k = p_j$ iff $y_k = y_j$ for $1 \leq k, j \leq m$.

Then

$$F \models p_1 \to (p_2 \to (\ldots (p_m \to q)) \ldots) \vdash s_1 \to (s_2 \to (\ldots (s_n \to q)) \ldots)$$

iff

$$F \models x_n \bullet (x_{n-1} \bullet (\ldots (x_1 \bullet z)) \ldots) \vdash y_1 \bullet (y_2 \bullet (\ldots (y_m \bullet z)) \ldots)$$

Proof

From up to down, make the following substitution:
$y_1 \bullet (y_2 \bullet (\ldots (y_m \bullet z)) \ldots)/q \;\; x_i/s_i \; (1 \leq i \leq n) \;\; y_i/p_i \; (1 \leq i \leq m)$
From down to up, make the following substitution:
$p_1 \rightarrow (p_2 \rightarrow (\ldots (p_m \rightarrow q)) \ldots)/z \;\; s_i/x_i \;\; p_i/y_i$.
It follows that the two sequents have the same first-order frame correspondent. Similar reasoning applies to formulas with $\leftarrow$. □

We have found two ways of treating implicative categorial sequents, one by reduction and one by direct analysis. These motivate two parts to the general Sahlqvist-van Benthem Theorem stated below. Before getting there, we raise two questions:

i Is every implicative Sahlqvist formula reducible to some Sahlqvist equivalent with a pure product antecedent?

ii Does every implicative first-order definable allow a direct substitution analysis via the above 'contraposition'?

For the first question, consider the categorial sequent

$$(q \rightarrow p) \rightarrow (p \rightarrow q) \vdash p \rightarrow q$$

We doubt that it has a product equivalent of the described kind. Nevertheless, it is first-order: reasoning by contraposition and substitutions ends up with an equivalent

$$\forall abc(Rc, ba \Rightarrow \exists xy(Ry, xa \& Rc, by \& \forall zu(Rz, ux \rightarrow (u = c \vee z = b)))).$$

For the second question, consider the following implicative version of Associativity:

$$p \rightarrow q \vdash (r \rightarrow p) \rightarrow (r \rightarrow q).$$

We have not been able to find a first-order equivalent for it through our contraposition/substitution analysis. Nevertheless, in an earlier subsection, we showed that it was **NL**–inter-derivable with the usual product form of Associativity:

$$x \bullet (y \bullet z) \vdash (x \bullet y) \bullet z.$$

The later, of course, had an obvious first-order equivalent and therefore, so has the former.

2.2.3 A Sahlqvist-van Benthem theorem for categorial formulas

Let us now formulate a broadly useful categorial version of the Sahlqvist-van Benthem Theorem. What should be its syntactic conditions? In standard modal logic, Sahlqvist formulas are of the form $\Box^m(A_1 \supset A_2)$, where A_2 is syntactically positive, and where A_1 is obtained from proposition letters and their negations, applying $\&, \vee, \Box, \Diamond$ in such a way, that no positive occurrence of a variable is in a subformula of the form $B_1 \vee B_2$ or $\Diamond B$ within the scope of a universal modality $\Box$. The relevance of all these restrictions becomes clear when analyzing the correctness proof for the Sahlqvist-van Benthem substitution algorithm. Moreover, this formulation is the most general one, in the sense that non-first-order examples may be provided for each violation of these syntactic restrictions (cf. [van Benthem 85]. Our version will come in two parts. One of them was already treated in the preceding Section.

2.2.3. THEOREM. *Product Version*
If $\alpha \vdash \beta$ is a categorial sequent with an antecedent α constructed entirely out of atoms and product, and a consequent β in which atoms occur only positively , then there exists a first-order formula δ which corresponds effectively to $\alpha \vdash \beta$.

To formulate the second version, we need an auxiliary notion.

2.2.4. DEFINITION. *Nice Formulas*
Nice categorial formulas are constructed using the following rules:

- proposition letters are nice
- if χ is nice and ξ is in $ML(\bullet)$, then $\xi \rightarrow \chi$ and $\chi \leftarrow \xi$ are nice.

Here is a further technical notion:

- ξ is consistent if no proposition letter occurs both positively and negatively in ξ
- ξ is monotone with respect to χ if, for every proposition letter p, either p occurs positively in ξ and χ does not contain only negative occurrences of p, or p occurs negatively in ξ and χ does not contain positive occurrences of p

We might also have used these notions to slightly generalize the preceding result. Moreover, the proof to follow essentially works for the above case as well.

2.2.5. THEOREM. *Slash Version*
If $\alpha \vdash \beta$ is a categorial sequent with a nice consequent β and an antecedent α which is consistent and monotone with respect to β, then there is a first-order formula δ which corresponds effectively to $\alpha \vdash \beta$.

Proof
Let $\alpha \vdash \beta$ be a sequent of the described form We describe the procedure to be

followed, plus its semantic correctness, relying on earlier examples for its intuitive motivation.

1. Translate $\alpha \vdash \beta$ into $\neg ST(\beta) \supset \neg ST(\alpha)$.
2. Now, we clean up this sequent. First, rewrite $ST(\alpha)$, $ST(\beta)$, so that no two quantifiers occur with the same bound variable. Next, remove predicate letters which do not occur on both sides:
   - if P occurs positively (negatively) in $\neg ST(\beta)$ but it does not occur in $\neg ST(\alpha)$, then replace it by $\top(\bot)$
   - if P occurs positively (negatively) in $\neg ST(\alpha)$ but it does not occur in $\neg ST(\beta)$, then replace it by $\bot(\top)$.

   The result is still a frame-equivalent sequent, as may be seen from standard (modal) logic.
3. Since β is nice, there are only existential quantifiers, with distinct bound variables in $\neg ST(\beta)$. Move all these existential quantifiers to the front of the whole formula, by the following prenex principle of first-order logic (x does not occur free in B):

   $$\exists x A(x) \supset B \quad \text{iff} \quad \forall x (A(x) \supset B)$$

   This is possible because only occurrences of & have to be 'crossed' inside $\neg ST(\beta)$. The result is a formula β' , which leads to a new equivalent of the form:

   $$\forall y_1 \ldots \forall y_k (\beta' \supset \neg ST(\alpha))$$

4. Fix a variable u not occurring in $\neg ST(\beta) \supset \neg ST(\alpha)$. Define a 'canonical valuation' CV for any occurrence $|p|$ of p in β' :

   $CV(|p|, \beta')$: $u = a$, if $P(a)$ occurs in β'.

   $CV(|p|, \beta')$: $u \neq a$, if $\neg P(a)$ occurs in β'.

   Let $CV(p, \beta')$ be the disjunction of all formulas $CV(|p|, \beta')$.
5. Now, the required L_0-equivalent δ is obtained by substituting the formula $[z/u]CV(p, \beta')$ for each atom Pz (representing an occurrence of the proposition letter p) in the above form

   $$\forall y_1 \ldots \forall y_k (\beta' \supset \neg ST(\alpha))$$

   As second-order universal quantifiers imply all their (first-order) substitution instances, we have the following implication for frame truth:

   for any frame F, if $F, w \models a \vdash b$, then $F, w \models \delta$.

   Here, the special world w displayed is assigned to the main free variable of δ , corresponding to the 'current world of evaluation'.

The remainder of the proof consists in showing the converse:

for any frame F, if $F, w \models \delta$, then $F \models \alpha \vdash \beta$.

1. Suppose that $F, w \models \delta$, .
2. Let V be any valuation on F with $\langle F, V\rangle, w \not\models \beta$.
3. By the truth definition, we have that $\langle F, V\rangle, w \models \exists y_1 \ldots \exists y_k \beta'$, and hence, for some $w_1, \ldots, w_k \in W$, $\langle F, V\rangle \models \beta'[w_1, ..., w_k]$.
4. Define V^* as follows:

$$a \in V^*(p) \quad \text{iff} \quad F \models CV(p, \beta')[w_1, \ldots, w_k, a],$$

where a is assigned to u .
5. Given the form of our formulas, and the definition of CV, we have that $\langle F, V*\rangle, w \not\models \beta$. Pushing the substitutions into β' , and then appealing to the truth of δ (assumption [1]), we obtain that the substituted form of $\neg ST(\alpha)$ must hold. Pushing substitutions into the valuation again, we get that $\langle F, V^*\rangle, w \not\models \alpha$.
6. What remains to be shown, however, is that $\langle F, V\rangle, w \not\models \alpha$ for the original valuation V of assumption [2]. Here, we need our special assumptions of monotonicity and consistency. We will be done if we can show that, for our relevant proposition letters, either p occurred positively, and we have if $x \models^* p$, then $x \models p$ – or p occurred negatively, and we have: if $x \not\models^* p$, then $x \not\models p$. But this requires a simple verification.
7. (a) Suppose that $x \models^* p$. Then p occurs negatively in α (recall that we are reasoning by contraposition). Since α is monotone with respect to β , β has only negative occurrences of p . Therefore, $\neg ST(\beta)$ contains only positive occurrences of P. Thus the canonical valuation for p is defined in such a way that $u \models^* p$ iff $u = y_1$ or $\ldots$or $u = y_n$ for each y_j $(1 \leq j \leq n)$, such that $y_j \models p$. Then $x = y_i$ for some $1 \leq i \leq n$, and therefore $x \models p$.
   (b) Suppose $x \not\models^* p$. Then p occurs positively in α . Since α is monotone with respect to β , β has some positive occurrence of p . Therefore $\neg ST(\beta)$ contains some negative occurrences of P . Thus the canonical valuation for p is defined in such a way that $u \models^* p$ iff $u = y_1$ or $\ldots$or $u = y_n$ or $u \neq z_1$ or $\ldots$or $u \neq z_k$ for each $z_j(1 \leq j \leq k)$, such that $z_j \not\models p$. Since $x^* \not\models^* p$, we have $x \neq y_1$ and $\ldots$and xy_n and $x = z_1$ and...and $x = z_k$. Hence $x \not\models p$.

□

Another way of establishing the preceding Theorem would be by direct translation into the full Boolean categorial language, and then inspecting the resulting syntactic forms involving the three matching categorial products $\bullet_1, \bullet_2, \bullet_3$. Our notions of monotonicity and consistency will still be essential (think for example of the translation of the Sahlqvist formula $(p \to q) \to (q \to p) \vdash q \to p)$. We conclude with some questions

(1) Is the above a most general Sahlqvist Theorem for the categorial fragment $ML(\bullet, \to, \leftarrow)$?
(2) Can one provide an effective syntactic description of all first-order definable forms in $ML(\bullet, \to, \leftarrow)$ (with or without a substitution method)?

2.2.4 Categorial Applications. Informational Paradigm and Correspondence Theory

We want to stress once more the purpose of our present analysis. The general picture for categorial logics is like Correspondence Theory for standard modal logic. Modal axioms express semantic constraints on the accessibility relation in frames where they hold. Interpreting frames as structures of informational tokens, as we did in Chapter 1, then yields a correlation between categorial axioms corresponding to structural rules and desired properties of informational composition. For example, the intuitive account of Contraction connects this structural rule with 're-use' of informational resources. Our correspondences make this precise: they say in which exact frame-semantic sense the metaphor holds. And they do more. Different proof-theoretic formulations of Contraction may correspond to different frame-semantic shades of meaning. To make the intended informational content of the composition relation more visual, we introduce an arbitrary place relation $R^n a, x$ as follows, describing 'composition trees':

Let $a \in W$ and x be a bracketed string of elements of W of length $n-1$. We set

$$
\begin{array}{llcl}
(n=2) & R^2 a, b & \Longleftrightarrow & a = b \\
(n\rangle 2) & R^n a, xy & \Longleftrightarrow & \exists b \exists c (R_3 a, bc \& R_k b, x \& R_m c, y) \\
 & & & \text{where } k + m = n - 1.
\end{array}
$$

For example, $Ra, (bc)(de)$ is an abbreviation for the first-order description

$$\exists x \exists y (Ra, xy \;\&\; Rx, bc \;\&\; Ry, de).$$

Now, we state a table of correspondences, obtained by the previous our general method. First, we consider categorial variants of Contraction, together with their corresponding frame properties:

$p \vdash p \bullet p$	$\forall x Rx, xx$
$p \to (p \to q) \vdash p \to q$	$\forall a \forall b \forall c(Ra, bc \Rightarrow Ra, b(bc))$
$(p \bullet p) \to q) \vdash p \to q$	$\forall a \forall b \forall c(Ra, bc \Rightarrow Ra, (bb)c)$
$(q \leftarrow p) \leftarrow p \vdash q \leftarrow p$	$\forall a \forall b \forall c(Ra, bc \Rightarrow Ra, (bc)c)$
$(q \leftarrow (p \bullet p) \vdash q \leftarrow p$	$\forall a \forall b \forall c(Ra, bc \Rightarrow Ra, (b(cc))$
$p \to (q \leftarrow p) \vdash p \to q$	$\forall a \forall b \forall c(Ra, bc \Rightarrow Ra, (bc)b)$
$(p \to q) \leftarrow p \vdash q \leftarrow p$	$\forall a \forall b \forall c(Ra, bc \Rightarrow Ra, c(bc))$

Various restrictions on the *order* of informational tokens can be expressed by categorial axioms related to the Permutation rule. Here is the list of some axioms and their frame correspondents expressing the variety of the relaxations of order sensitivity.

$p \to q \vdash q \leftarrow p$	$\forall a \forall b \forall c(Ra, bc \Rightarrow Ra, cb)$
$p \to (q \to r) \vdash q \to (p \to r)$	$\forall a \forall b \forall c(Ra, b(cd) \Rightarrow Ra, c(bd))$
$(p \bullet q) \to r \vdash (q \bullet p) \to r$	$\forall a \forall b \forall c(Ra, (bc)d \Rightarrow Ra, (cb)d)$
$(r \leftarrow q) \leftarrow p \vdash (r \leftarrow p) \leftarrow q$	$\forall a \forall b \forall c(Ra, (bc)d \Rightarrow Ra, (bd)c))$
$r \leftarrow (q \bullet p) \vdash r \leftarrow (p \bullet q)$	$\forall a \forall b \forall c(Ra, b(cd) \Rightarrow Ra, b(dc))$

Finally, the semantical interpretation of the Weakening rule is related to 'diminishing sizes ' of composition trees.

$p \vdash p \leftarrow q$	$\forall a \forall b \forall c(Ra, bc \Rightarrow a = c)$
$p \vdash q \to p$	$\forall a \forall b \forall c(Ra, bc \Rightarrow a = b)$
$p \bullet (q \bullet r) \vdash p \bullet r$	$\forall a \forall b \forall c(Ra, b(cd) \Rightarrow Ra, bc)$
$(p \bullet q) \bullet r \vdash p \bullet r$	$\forall a \forall b \forall c(Ra, (bc)d) \Rightarrow Ra, bc)$

A similar fine-structure emerges with variants of Associativity.

2.2.5 Peirce's Law

In this section we provide a semantical analysis of Peirce's Law, being the sequent axiom

$$(p \to q) \to p \vdash p$$

which turn out to play a special role in Correspondence Theory for categorial logics. This principle has a reputation as a destroyer of structure sensitivity. In intuitionistic correspondence theory (Rodenburg 1986), it imposes a domain restriction on binary Kripke frames to single points: $\forall xy(Rxy \Rightarrow y = x)$. What about its categorial version in ternary frame semantics?

As it turns out, we can compute a first-order frame condition corresponding to the sequent $(p \to q) \to p \vdash p$ by the method of substitutions in our second Sahlqvist Theorem.

Claim
Peirce's Law corresponds to the first-order frame condition

$$\forall a \exists b (Ra, ba \mathbin{\&} \forall x \forall y (Ry, xb \Rightarrow x = a)).$$

This is a strong semantic restriction, though not yet one to single points only.
Proof
Assume for any valuation V

$$\text{if } a \notin V(p), \text{ then } a \notin V((p \to q) \to p).$$

Therefore, if $a \notin V(p)$, then

$$\exists b \exists c (Rc, ba \mathbin{\&} b \in V(p \to q) \mathbin{\&} c \notin V(p)),$$

whence

$$\forall a \exists b \exists c (Rc, ba \mathbin{\&} \forall x \forall y (Ry, xb \Rightarrow (x \notin V(p) \vee y \in V(q)) \mathbin{\&} c \notin V(p)).$$

Define the canonical valuation V^* as follows:

$$V^*(p) = \{c | c \neq a\}$$
$V^*(q)$ is the empty set.

By instantiation, we get

$$\forall a \exists b \exists c [Rc, ba \mathbin{\&} \forall x \forall y (Ry, xb \Rightarrow (x = a \vee \bot) \mathbin{\&} c = a].$$

This reduces to the above equivalent

$$\forall a \exists b [Ra, ba \mathbin{\&} \forall x \forall y (Ry, xb \Rightarrow x = a)] \qquad (\ddagger)$$

which is the first-order formula we were searching for.

To show that $(p \to q) \to p \vdash p$ is true in any frame where this property holds, assume that $M, a \models (p \to q) \to p$, but not $M, a \models p$. By ($\ddagger$) , there exists some world $b \in W$ such that Ra, ba. Therefore, if $M, b \models (p \to q)$, then $M, a \models p$. But a does not support p , whence $p \to q$ is not true in b . In other words, there exist $x \in W$ and $y \in W$ such that Ry, xb and $M, x \models p$, but y does not support q . By ($\ddagger$) again, Ry, xb implies $x = a$. But this is impossible since p is true in x , but not in a. □

To conclude our first-order analysis of Peirce's Law, here is an interesting contrast. Compare the following two sequents:

(i) $(p \to p) \to p \vdash p$
(ii) $(p \to q) \to p \vdash p$

The latter is a Sahlqvist formula, but the former (weaker) is not. Later on we will show that (i) is not first order definable at all. The reason is the 'wrong mixture' of positive and negative occurrences. (Looking at the matter backwards, also in modal logic, the phenomenon is known that one can sometimes strengthen a non-first-order principle to a first-order one by un-identifying some occurrences of the same proposition letter.)

Does Peirce's Law really define a restriction of the frame domain? To obtain this effect, one also needs the other structural rules. Consider the earlier first-order conditions corresponding to Associativity and Permutation:

Permutation
$\forall a, b, c(Ra, bc \Rightarrow Ra, cb)$ (†)

Associativity
$\forall a, b(\exists c(Ra, bc \& Rc, de) \Leftrightarrow \exists d(Ra, te \& Rt, bd))$ (††)

Peirce's Law
$\forall a \exists b(Ra, ba \ \& \ \forall x \forall y(Ry, xb \Rightarrow x = a))$ († † †)

2.2.6. PROPOSITION. In any frame where (†) – († † †) hold, one also has

$$\forall a \forall v \forall w(Ra, vw \Rightarrow v = w = a).$$

Proof
Suppose that Ra, vw. By († † †), there is a point $b \in W$ such that Ra, ba with the described property. Moreover, by (††) , this implies $\exists t(Ra, tw \ \& \ Rt, bv)$. Next, by Permutation (†), we pass from Rt, bv to Rt, vb and then by († † †), conclude to $v = a$. Therefore, we have $\forall a \forall v \forall w(Ra, vw \Rightarrow v = a)$. Now, applying (††) once more, we get $\forall a \forall v \forall w(Ra, vw \Rightarrow w = a)$. □

2.3 Disproving First-Order Definability

Through the standard translation of Chapter 1, categorial formulas define constraints on the composition relation in ternary frames. In this Section, we are interested in non-first-order definable constraints. Two ways of refuting first-order definability are presented. The first is based on translating categorial formulas into some non-first-order definable standard modal formula. The second method is more direct: non-first-orderness of categorial formulas shows up in failures of the Löwenheim-Skolem theorem.

2.3.1 Reduction to Modal Cases

To refute first-order definability of a categorial formula, one can try to 'decompose' a ternary relation. Here is the general heuristic:

- Translate the relevant categorial formula α into some standard modal formula α^* which is known to be non-first-order definable
- Find a suitable construction of a ternary frame $F^3 = \langle W, R^3 \rangle$ such that R^3 is defined in terms of R_2 , where $F_2 = \langle X, R_2 \rangle$ is an arbitrary binary frame, and prove that $F_3 \models \alpha$ iff $F_2 \models \alpha^*$.

Now, if α were first-order in ternary frame semantics, then α^*. would be first-order in binary frame semantics. Quod non.

Example *A Peirce Variant*
Consider the sequent $(p \to p) \to p \vdash p$

- Define a translation $^\sharp$ into modal logic as follows:

  $p^\sharp = p$
  $(A \to B)^\sharp = \Box(A^\sharp \supset \Box B^\sharp)$

  This takes care of categorial formulas. To treat categorial sequents, replace the outer sequent arrow by a material implication. Thus, the sequent $(p \to p) \to p \vdash p$ translates into the modal formula $\Box(\Box(p \supset \Box p) \supset \Box p) \supset p$: a variant of the Grzegorczyk Axiom which is known to be non-first-order.
- Let $F^2 = \langle X, R^2 \rangle$ be any binary frame. Define a ternary frame $F^3 = \langle W, R^3 \rangle$ by setting

$$W = X$$
$$R^3 a, bc \quad \text{iff} \quad R^2 cb \text{ and } R^2 ba$$

  Claim If α is in $ML(\to)$, then $F^3 \models \alpha$ iff $F^2 \models \alpha^\sharp$
  Proof
  Given a ternary (binary) model where $\alpha(\alpha^\sharp)$ fails, just copy the valuation to define the corresponding binary (ternary) model. Then proceed by a straightforward induction on the construction of α . The point is that $A \to B$ will mean exactly the same as its translation, under this definition of R^3 □
- As a consequence, the sequent $(p \to p) \to p \vdash p$ is non-first-order definable on the special ternary frames defined above. This is a direct consequence of the non-first-orderness over binary frames of $\Box(\Box(p \supset \Box p) \supset \Box p) \supset p$. But then, a fortiori, $(p \to p) \to p \vdash p$ cannot be first-order over the class of all ternary frames. □

Example *One Atom Permutation*
Consider the formula $p \to p \vdash p \leftarrow p$, arising out the earlier first-order sequent $p \to q \vdash q \leftarrow p$ by substituting p for q .

- Define a translation $^\sharp$ into tense logic as follows:

$p^\sharp = p$
$(A \to B)^\sharp = \Box(A^\sharp \supset \Box B^\sharp)$ $(A \leftarrow B)^\sharp = \Diamond^\downarrow A^\sharp \supset \Box B^\sharp$
Then $(p \to p)^\sharp \vdash (p \leftarrow p)^\sharp$ obtains the form

$$\Box(p \supset \Box p) \supset (\Diamond^\downarrow p \supset \Box p).$$

- Now, van Benthem (1983) has an elegant proof showing that the modal formula

$$\Box(p \supset \Box p) \supset (\Diamond p \supset \Box p)$$

is not preserved under certain L_0-elementary extensions of the frame $F = \langle W, R \rangle$, where

  - $W = IN$ (i.e., the natural numbers)
  - $R = \{\langle 0, n\rangle, \langle n, 0\rangle, \langle n, n+1\rangle \langle n+1, n\rangle | n = 1, 2, 3, ...\}$

even though it holds in this frame itself. Now, note that this frame is symmetric. Hence, $x \models \Diamond A$ iff $x \models \Diamond^\downarrow A$, for any $x \in W$. It follows that $\Box(p \supset \Box p) \vdash \Diamond^\downarrow p \supset \Box p$ is not preserved under L_0-elementary extensions of F . Hence it cannot be first-order definable either.
- Define again $R^3 a, bc$ as $R^2 cb$ & $R^2 ba$. An easy extension of the earlier argument shows that, for all binary frames F^2 , $F^3 \models \alpha$ iff $F^2 \models \alpha^\sharp$, for all α in $ML(\to, \leftarrow)$. As an immediate consequence, we get non-first-order definability of $p \to p \vdash p \leftarrow p$.

$\Box$

Remark
This temporal reduction is related to that used in Chapter 1 to reduce ternary to binary modalities.

Example *A Categorial McKinsey Axiom*
One can also start from non-first-order modal formulas, and find categorial equivalents. An example is the well-known McKinsey Axiom $\Box\Diamond p \vdash \Diamond\Box p$. Let t be a new basic type, standing for Boolean 'true'. I.e., it holds everywhere in ternary models. Here is an obvious categorial relative of the modal version:

$$t \to \neg(t \to \neg p) \vdash \neg(t \to \neg(t \to p)).$$

To analyze this, define a translation $^\sharp$ as follows:

$p^\sharp = p$
$(A \to B)^\sharp = \Box(A^\sharp \supset B^\sharp)$
$(t)^\sharp = \top$
$(\neg A)^\sharp = \neg A^\sharp$

Clearly, this takes $t \to \neg(t \to \neg p) \vdash \neg(t \to \neg(t \to p))$ to the McKinsey Axiom. Moreover, the two formulas will evaluate to the same thing under the following construction of ternary frames from binary ones: define $R_3 a, bc$ as $R_2 cb$ & $R_2 ba$. It follows that our categorial McKinsey Axiom cannot be first-order definable. □

2.3.2 Frame Arguments

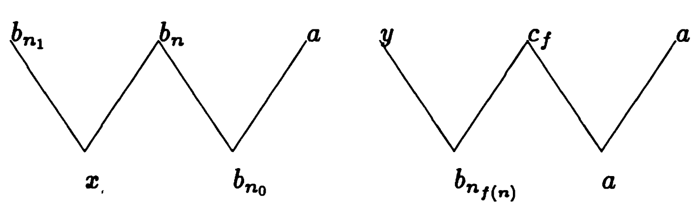
Another general method for refuting first-order definability of a categorial principle a proceeds as follows. One shows that, over the class of ternary frames, a lacks some model-theoretic feature which all first-order formulas possess. A well-known technique from the modal case is the use of the upward Lowenheim-Skolem theorem on elementary subframes. We will show how this technique is available for categorial principles, too, by means of three examples. We start again with the earlier Peircean variant.

2.3.1. PROPOSITION. $(p \to p) \to p \vdash p$ is not first-order definable.

Proof

We show that this sequent is not preserved under passing to elementary subframes. Our argument here treats so-called 'local validity' of sequents, in one single world but it can be modified so as to deal with 'global validity' in all worlds. Consider the following *uncountable* ternary frame $F = \langle W, R \rangle$:

Fig.1



$$
\begin{array}{lcl}
W & = & \{a, x, y\} \cup \{b_n, c_f, b_{n_0} | n \in N\} \cup \{c_f | f : N \to \{1, 0\}\} \\
R & = & \{\langle b_{n_0}, b_n, a\rangle | n \in N\} \cup \{\langle x, c_f, b_n\rangle | n \in N\} \cup \\
 & & \{\langle b_{n_{f(n)}}, y, c_f\rangle | n \in N, f : N \to \{1, 0\}\} \cup \\
 & & \{\langle a, c_f, a\rangle | f : N \to \{1, 0\}\}
\end{array}
$$

Claim 1 $F, a \models (p \to p) \to p \vdash p$

Proof

1. Consider any valuation V on F . Suppose that we have $\langle F, V\rangle, a \models (p \to p) \to p$. (For convenience, we will often drop explicit reference to frame and valuation.)
2. Since for all n , $\langle b_{n_0}, b_n, a\rangle \in R$, it follows that $b_n \not\models p \to p$ or $b_{n_0} \models p$. Also, because for any n , $\langle x, b_{n_1}, b_n\rangle$ is the only triangle having b_n as a 'right daughter', we have $b_n \not\models p \to p$ iff $b_{n_1} \models p$ and $x \not\models p$. Therefore, we must have $b_{n_1} \models p$ or $b_{n_0} \models p$, for all n . Now, pick up any choice function $f^* : N \to \{1, 0\}$ such that $b_{n_{f^*(n)}} \models p$ for all n.
3. Since for all n, $\langle b_{n_{f^*(n)}}, yc_{f^*}\rangle$ is the only triangle where c_{f^*} is the right daughter and $\forall n\ (b_{n_{f^*(n)}} \models p)$, clearly $c_{f^*} \models p \to p$. Then $a \models p$, by [1], [3] and the fact that $\langle a, c_{f^*}, a\rangle \in R$.
   We are done. □

Claim 2 There exists a countable elementary subframe F' of F such that F', $a \not\models (p \to p) \to p \vdash p$.

Proof
By the downward Lowenheim-Skolem theorem, F has a *countable* elementary substructure F' whose domain contains at least the worlds $a, x, y, b_n, b_{n_1}, b_{n_0} (n \in N)$ (and of course, some worlds $c_f (f : N \to \{1, 0\})$. Because F' is countable, some of the uncountably many worlds c_f from F will be missing in its domain. Fix one of them, say c_g. Define a valuation V' by setting

$$V'(p) = \{y\} \cup \{b_{n_{g(n)}} | n \in N\}$$

Then $a \not\models p$. Let us show that, nevertheless, $a \models (p \to p) \to p$: which refutes the validity of the sequent $(p \to p) \to p \vdash p$ in F', a.

1. Suppose $a \not\models (p \to p) \to p$.
2. The definition of R' brings two cases to consider:
   (a) There exists $k \in N$ such that $b_k \models p \to p$, but $b_{k_0} \not\models p$.
   (b) There exists $f^* : N \to 1, 0$ such that $c_{f^*} \models p \to p$, and $a \not\models p$.

   Consider the first case. Because $b_{k_0} \not\models p$, the definition of V' says that $g(k) = 1$, and hence $b_{k_1} \models p$. Now, since $\langle x, b_{k_1}, b_k\rangle \in R'$ and $b_k \models p \to p$, we get $x \models p$: but that would contradict the definition of V'.
   Consider the second case. Since for all n, $\langle b_{n_{f^*(n)}}, y, c_{f^*}\rangle \in R'$ and $y \models p$, we have $b_{n_{f^*(n)}} \models p$ for all n. But then f^* and g would coincide, which is impossible: as c_g was missing from F' .

□

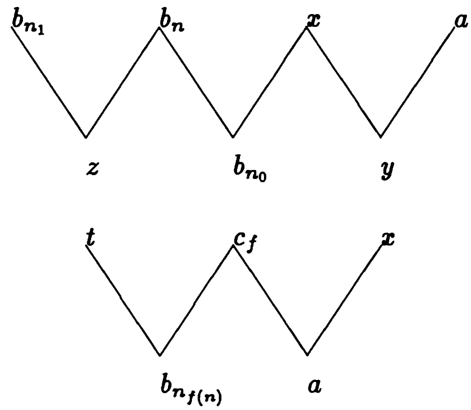
We conclude that $(p \to p) \to p \vdash p$ is not (locally) first-order definable, not being preserved under elementary subframes.

The following arguments will be sketches in a formal shorthand

2.3.2. PROPOSITION. $\neg p \vdash ((p \to p) \to p) \to p$ is not first-order definable.

Proof
Consider the following uncountable frame:

Fig 2



$$\begin{array}{lcl} W & = & \{a, x, y, z, t\} \cup \{b_n, b_{n_1}, b_{n_0} | n \in N\} \cup \{c_f | f : N \to \{1,0\}\} \\ R & = & \{\langle y, x, a\rangle\} \cup \{\langle b_{n_0}, b_n, x\rangle | n \in N\} \cup \\ & & \{\langle z, b_{n_1}, b_n\rangle | n \in N\} \cup \\ & & \{\langle b_{n_{f(n)}}, t, c_f\rangle | n \in N, f : N \to \{1,0\}\} \cup \\ & & \{\langle a, c_f, x\rangle | f : N \to \{1,0\}\} \end{array}$$

Validity in F .
Suppose $a \models \neg p$ and $a \not\models ((p \to p) \to p) \to p$. Since a occurs as a 'right daughter' only in one triangle in R , clearly $x \models ((p \to p) \to p)$. That implies $\forall n(b_n \not\models p \to p$ or $b_{n_0} \models p)$. Because for any n, $\langle z, b_{n_1}, b_n\rangle$ is the only triangle having b_n as a right daughter, we have $\forall n(b_{n_1} \models p$ or $b_{n_0} \models p)$. Now, pick any function $f^* : N \to \{1,0\}$ such that $b_{n_{f^*(n)}} \models p$ for all n . Immediately, $c_{f^*} \models p \to p$ and hence $a \models p$ - a contradiction.
Invalidity in F' .
In the countable elementary subframe F', define

$$V'(p) = \{b_{n_{g(n)}}\} \cup \{t\},$$

where c_g is a missing world. To show that $\neg p \vdash ((p \to p) \to p) \to p$ is not true at the point a, proceed as follows:

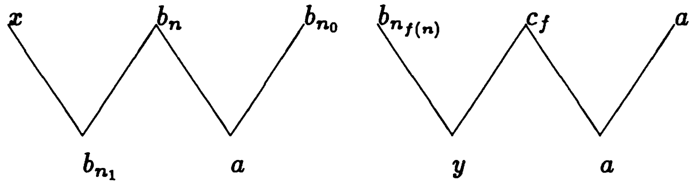
1. Suppose $a \models ((p \to p) \to p) \to p$. Then, by the definition of V', $x \not\models ((p \to p) \to p)$.
2. Consider two cases

(a) $\exists k \in N(b_k \models p \to p \,\&\, b_{k_0} \not\models p)$
(b) $\exists f^*(c_{f^*} \models p \to p \,\&\, a \not\models p)$

As above, in both cases we end up with a contradiction. □

2.3.3. PROPOSITION. $p \vdash (p \to p) \bullet p$ is not first-order definable.

Proof
Consider the following uncountable ternary frame:
Fig 3



$$
\begin{array}{lcl}
W & = & \{a, x, y\} \cup \{b_n, b_{n_1}, b_{n_0} | n \in N\} \cup \{c_f | f : N \to \{1, 0\}\} \\
R & = & \{\langle a, b_n, b_{n_0}\rangle | n \in N\} \cup \{\langle b_{n_1}, x, b_n\rangle | n \in N\} \cup \\
 & & \{\langle y, b_{n_{f(n)}}, c_f\rangle | n \in N, f : N \to \{1, 0\}\} \cup \\
 & & \{\langle a, c_f, a\rangle | f : N \to \{1, 0\}\}
\end{array}
$$

Validity in F .
Suppose that $a \models p$, but $a \not\models (p \to p) \bullet p$. Then either $b_n \not\models p \to p$, or $b_{n_0} \not\models p$.Since $\langle b_{n_1}, x, b_n\rangle$ is the only triangle where b_n is the right daughter, $\forall n$ ($b_n \not\models p \to p$ iff $b_{n_1} \not\models p$ and $x \models p$). Therefore, $\forall n$ ($b_{n_1} \not\models p$ or $b_{n_0} \not\models p$). Now, pick any function $f^* : N \to \{1, 0\}$ such that $b_{n_{f^*(n)}} \not\models p$ for all n . Then immediately $c_{f^*} \models p \to p$. Now recall that $a \models p$, so $a \models (p \to p) \bullet p$ –a contradiction.
Invalidity in F' .
For the elementary subframe F', choose one missing world, say c_g. Set

$$V'(p) = W - \{y\} - \{b_{n_{g(n)}} | n \in N\}.$$

Therefore $a \models p$. Suppose $a \models (p \to p) \bullet p$. That brings two cases to consider:

[i] $\exists k \in N(b_k \models p \to p \,\&\, b_{k_0} \models p)$, and
[ii] $\exists f^*(c_{f^*} \models p \to p \,\&\, a \models p)$.

If [i] holds, then $g(k) = 1$, whence $b_{k_0} \not\models p$. Therefore $x \not\models p$, but that contradicts the definition of V'.
If [ii] holds, then, since $y \not\models p$, we have $\forall n(b_{n_{f^*(n)}} \not\models p)$. That means $g = f^*$, which is impossible. □

2.3.3 Further Issues in Correspondence

We conclude with some issues for further research. Evidently, there are further mathematical depths of Correspondence Theory that have been left unexplored here as their relevance to our intended categorial applications is (too) remote. Or, starting with the latter, we have not probed all peculiarities of the categorial fragment. In particular, there might be much more than shown here to working with weak modal languages without Booleans. Finally, we have not considered correspondences for *derived rules of inference*, stating when one or more sequents imply another. Some of these are admissible in, say, the Lambek Calculus, other are not and one can also determine what non-valid ones would demand (cf. van Benthem 1985 for a first attempt in modal logic).

On the other hand, Correspondence theory can be extended to various relevant richer languages. For instance, one can give a correspondence-style analysis of the richer vocabularies found in Relevant Logic (living around the Lambek Calculus enriched by Contraction).

Example *Relevant Negation*
Recall the major innovation in this field, over and above product and implication, viz. the 'reversal negation' defined as follows:

$$M, a \models \sim \phi \iff M, r(a) \not\models \phi,$$

where r is an abstract 'reversal map' on our ternary frames. Just as with our earlier categorial laws, key principles of relevant negation may then express structural constraints on these notions. Here are some illustrations, which may be obtained via the earlier minimal substitution analysis for locating first-order equivalents. On the functional view, e.g., the Double Negation law will express that r is idempotent:

$$A \vdash \sim\sim A \text{ defines that } \forall a(rr(a) = a).$$

A valid rule of inference, without further ado, will be Contraposition from $A \vdash B$ to $\sim B \vdash \sim A$. If the former sequent holds everywhere in a model, so does the latter.

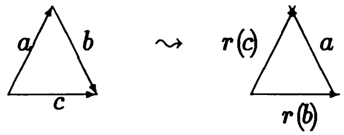
Next, recall the truth condition for relevant implication:

$$M, a \models A \to B \iff \forall b \forall c(Rc, ab \& M, b \models A \to M, c \models B)$$

Clearly, Contraposition of the form

$$A \to B \vdash \sim B \to \sim A$$

defines $Rc, ab \Rightarrow Rr(b), r(c)a$. This constraint expresses the natural shift of perspective in composition triangles, i.e. graphically if an arrow c is composed out of matching arrows a and b, then the reverse of b can be viewed as starting with the reverse of c and then returning via arrow a:



Finally, a typical relevant invalidity is Ex Falso Sequitur Quodlibet , which expresses that reversal is an identity map:

$$A, \sim A \vdash B \text{ defines that } \forall a r(a) = a$$

and we revert to classical Boolean negation.

Yet a richer similarity type is found in Arrow Logic (cf. Venema 1994, Marx 1995). Thus, van Benthem 1991 analyzes the Lambek Calculus as a dynamic logic of arrows. What these amount to in our current perspective, may be illustrated as follows.

Example *Categorial Logic as Arrow Logic*
As said, in Arrow Logic, one reads the categorial slashes as defined operators, namely

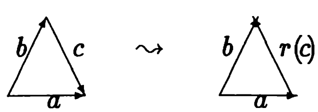
$$A \rightarrow B = \neg(A^{\smile} \bullet \neg B)$$

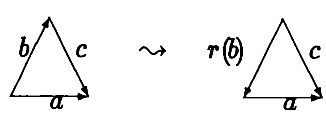
$$B \leftarrow A = \neg(\neg B \bullet A^{\smile}).$$

Recall that $\neg$ is Boolean complement, $\bullet$ is (arrow-style) relational composition, and $^{\smile}$ stands for (arrow-style) relational converse, referring to a reversal function r on arrows. Amongst others, this allows a natural decomposition of the above relevant negation $\sim A$ into $\neg(A)^{\smile}$ (Boolean complement of reversal). Under this reading, the categorial laws of Function Application express what may also be called natural 'perspective rotations' in composition triangles:

$$A \bullet (A \rightarrow B) \vdash B \text{ corresponds to } \forall xyz : Rx, yz \rightarrow Rz, r(y)x$$

$$(B \leftarrow A) \bullet A \vdash B \text{ corresponds to } \forall xyz : Rx, yz \rightarrow Ry, xr(z).$$





The precise dynamic content of Categorial Grammar in this arrow style remains to be investigated.

Finally, we mention possible extensions of categorial languages with modalities or even temporal operators, as will be used extensively in the richer architectures of our final Chapter. The present theory easily extends to such settings, with frames now carrying not just a composition relation, but also various accessibilities for further modal operators. For instance, one can add a universal modality $\Box$ which refers to a binary accessibility relation R^2 in the usual way. (Chapter 3 has the resulting logic.) The latter semantic structure will be superimposed on our ternary frames, and one gets at least two kinds of correspondence. There are 'pure modal principles', such as the universally valid inference rule ' from $A \vdash B$ to $\Box A \vdash \Box B$.' But there can also be 'mixed principles' relating the modality to the categorial base language, such as (perhaps)

$$\Box(A \to B) \vdash \Box A \to \Box B.$$

By our earlier substitution technique, suitably but straightforwardly generalized, the latter axiom will express the following interaction between the R^3 and the R^2 frame structure:

$$\forall abc(R^3c, ba \;\&\; R^2cz \to \exists xy(R^3z, yx \;\&\; R^2ax \;\&\; R^2by)).$$

Part II
Categorial Inference and Labelled Deduction

In this Part, we pass from the study of pure semantic expressive power to the combinatorics of categorial deduction. In Chapter 3, we consider the traditional axiomatic format of the field, analysing completeness theorems in a perspective of filter representations. Next, we consider labelled formats of deduction, where sequents may carry information about linguistic signs. In Chapter 4, we prove that the Lambek Calculus is complete for binary relational semantics using suitable pair labels. Finally, in Chapter 5, we provide a more general labelling for ternary frame semantics, relating this method to the earlier correspondence perspective of translation into fragments of first-order predicate logic.

Chapter 3

Axiomatics and Completeness Proofs

3.1 General Background

In this Chapter, we will analyze completeness proofs for categorial logics. This is a vast area, where much has been done already, and we shall only add some results 'rounding off' the literature at various points. The field lies in between two poles. At the top, there is the well-understood completeness theory for modal propositional logics, which correspond to categorial logics in a vocabulary with Booleans (cf. [Goldblatt 89], [Venema 91], [Roorda 91]). This tradition works mostly with Henkin models having maximally consistent sets of formulas for their worlds. At the bottom end lies the completeness theory for categorial logics, which makes do with very simple representations, where formulas themselves serve as worlds. Key publications in this area, to which our treatment is deeply indebted, include [Buszkowski 86], [Došen 92a]. Finally, there are logics in between, such as intuitionistic logic, or lower down, relevant logic [Rout. & Meyer 73]. Their completeness theory involves the use of prime filters for constructing worlds, or related constructs dealing also with relevant negation. For these approaches, standard references are available, which are presupposed here: [Troel. & van Dalen 88] provide a compendium of intuitionistic logic, and the masterful survey [Dunn 86] provides all necessary insight into relevant logic.

3.2 Base Representation: Sahlqvist Completeness and Incompleteness

Formulas and the canonical model construction

3.2.1. THEOREM. **NL** is sound and complete with respect to the class of ternary models

Proof
Soundness follows by a simple inspection of the principles listed in the definition of **NL**. For the completeness part, define a simple Henkin model

$$\mathcal{M} = \langle W, R^3, \models \rangle$$

where

- W is the set of all types (categorial formulas)
- Ra, bc iff $a \vdash b \bullet c$
- $a \in V(p)$ iff a $\vdash$ p

3.2.2. LEMMA. *Truth Lemma*

$$\mathcal{M}, a \models \alpha \quad \text{iff} \quad a \vdash \alpha$$

Proof
Induction on α . The statement is clear for atoms, by definition. For *products*, argue as follows. (i) If $M, a \models B \bullet C$, then there exist b, c with Ra, bc such that $M, b \models B$, $M, c \models C$. By the inductive hypothesis, $b \vdash B$ and $c \vdash C$ are derivable, and hence so is $b \bullet c \vdash B \bullet C$. As $Ra,\ bc$, it follows that $a \vdash B \bullet C$ is derivable. (ii) If $a \vdash B \bullet C$ is derivable, then we have Ra, BC . Moreover, by the inductive hypothesis: $M, C \models C$, $M, B \models B$. It follows that $M, a \models B \bullet C$. For *slashes*, we consider one case. (i) Suppose that $M, a \models B \to C$. As before, we have $M, B \models B$. Moreover, evidently, $R(B \bullet a), Ba$. Therefore, $M, B \bullet a \models C$, and by the inductive hypothesis, $B \bullet a \vdash C$ is derivable. But then, we can also derive the sequent $A \vdash B \to C$. (ii) Conversely, suppose that $a \vdash B \to C$ is derivable. Consider any b, c such that Rc, ba and $M, b \models B$. By the inductive hypothesis, $b \vdash B$ is derivable. Thus we have the following derivable in **NL** : $c \vdash b \bullet A$, $c \vdash B \bullet (B \to C)$, $c \vdash C$. By the inductive hypothesis, then, $M, c \models C$, as required. The argument for the other slash is symmetric. □

Now, completeness follows because non-provable sequents $A \vdash B$ will fail at the world A . □

This representation admits of obvious variations for further calculi upward in the Categorial Hierarchy. For, example to prove completeness of the associative

Lambek Calculus **L**, suppose Ra, bc and Rc, de. It terms of **L**-derivability, this means $a \vdash b \bullet c$, $c \vdash d \bullet e$, hence by Cut and Associativity Postulate $a \vdash (b \bullet d) \bullet e$. It provides the required principle

$$\exists t(Ra, te \ \&\ Rt, bd).$$

The other direction goes likewise.

By very similar arguments, we can establish the completeness of the Lambek Calculus with permutation **LP**. Adding the axiom $A \bullet B \vdash B \bullet A$ leads to completeness for ternary frames which satisfy permutation of arguments $\forall abc : (Ra, bc \rightarrow Ra, cb)$. Likewise, we can treat the case with Contraction $A \vdash A \bullet A$ added (**LC** or **LPC**) , where the frame satisfies reflexivity: $\forall a Ra, aa$. The first hitch occurs with Weakening $A \bullet B \vdash A$. No obvious frame argument establishes the required property $\forall abc(Ra, bc \rightarrow b = a)$. We shall give an explanation for these observations later on.

Sahlqvist Analysis

In modal logic, the semantic correspondence version of Sahlqvist theorems as in Chapter 2 comes with a completeness version, allowing us to predict frame completeness for a wide class of logics axiomatized in a suitable manner (cf. [Sahlqvist 75], [Venema 91]). The latter result may be stated in several ways. For our purposes, we use the following version. If a categorial extension of **NL** uses additional sequent axioms which are in Sahlqvist form, then it is complete with respect to the class of ternary frames satisfying the first-order constraint computed by the Sahlqvist-van Benthem algorithm. In standard modal logic the proof of this result proceeds by analysis of canonical Henkin models for such logics, showing how the frame constraints emerges on the underlying Henkin frame. An elegant presentation is [Sambin & Vacc. 89]. [Kracht 93b] obtains correspondence and completeness results together in a unifying approach towards definability in standard modal logic. But in our categorial case, there is a difficulty.

Sahlqvist Incompleteness

Consider the following natural, almost 'classical' categorial logic **LPW**. It consists of the non-associative Lambek Calculus with the following additional axioms:

$A \bullet B \vdash B \bullet A$	*Permutation*
$A \bullet B \vdash A$	*Weakening*

By the earlier correspondence analysis, ternary **LPW**. frames have a relation R satisfying the following first-order conditions:

$\forall a, b, c(Ra, bc \Rightarrow Ra, cb)$	*permutation*
$\forall a, b, c(Ra, bc \Rightarrow a = b)$	*weakening*

Taken together, these imply triviality of composition:

$$\forall a, b, c(Ra, bc \Rightarrow a = b = c)$$

A Peirce Variant

Consider the formula $((p \to q) \to q) \to (p \to q) \vdash p \to q$, arising out of Peirce's Law $(p \to q) \to p) \vdash p$ by substituting $p \to q$ for p.

Claim.

$((p \to q) \to q)) \to (p \to q) \vdash p \to q$ is valid in all **LPW** models.

Proof

Suppose that $a \models ((p \to q) \to q) \to (p \to q)$. We must check that $a \models p \to q$. Suppose otherwise, i.e., $a \not\models p \to q$. Then there are b and c such that Rc, ba, $b \models p$ and $c \not\models q$. By triviality of composition, we conclude that Ra, aa, $a \models p$ and $a \not\models q$. Next, observe that $a \models (p \to q) \to q$ (if not, triviality of composition gives a contradiction). Since $a \models ((p \to q) \to q) \to (p \to q)$, we conclude that $a \models p \to q$, which contradicts our assumption. □

This observation motivates the following result.

3.2.3. THEOREM. **LPW** is frame incomplete.

Proof

It suffices to show that our Peirce Variant is not derivable in **LPW**. For this purpose, we introduce a simple proof-theoretic invariant, namely, the following 'positive counter':

$p^\sharp$ is some arbitrary positive natural number;

$$\begin{array}{lcll} (A \bullet B)^\sharp & = & A^\sharp + B^\sharp & \\ (A \to B)^\sharp & = & B^\sharp - A^\sharp, & \text{if } B^\sharp > A^\sharp \\ & & 1, & \text{otherwise} \\ (B \leftarrow A)^\sharp & & \text{likewise} & \end{array}$$

Note that all formulas get positive counters in this way. (For categorial uses of this numerical technique, cf. [van Benthem 91b], [Roorda 91], [Pentus 92b].)

Claim

If $\alpha \vdash \beta$ is derivable in **LPW**, then $\alpha^\sharp \geq \beta^\sharp$.

Proof

By induction on the length of the derivation of $\alpha \vdash \beta$. One first has to check all principles of **NL**. For instance, consider the rule

$$A \bullet B \vdash C \quad \Longleftrightarrow \quad B \vdash A \to C.$$

Suppose that $A^\sharp + B^\sharp \geq C^\sharp$. Then either $A^\sharp < C^\sharp$, and $(A \to C)^\sharp = C^\sharp - A^\sharp \leq B^\sharp$, or $A^\sharp \geq C^\sharp$, and $(A \to C)^\sharp = 1$, which is always smaller than or equal to

$B^\sharp$. The argument in the opposite direction is similar. Clearly, Permutation and Weakening axioms preserve our numerical invariant. □

Now, it suffices to show that our Peirce's Variant

$$((p \to q) \to q)) \to (p \to q) \vdash p \to q$$

fails this test. Set

$$p^\sharp = 1 \quad q^\sharp = 3$$

Then $(((p \to q) \to q)) \to (p \to q))^\sharp$ works out to 1 , which is smaller than $(p \to q)^\sharp$. □

Thus, in Kracht's terms, completeness theory and correspondence theory for categorial inference are not always married. One possible remedy here is increasing the inferential power of categorial logics, using additional rules. (Cf. the general Sahlqvist completeness theorem in [Venema 91] for rich modal languages, using so-called 'irreflexivity rules'.) We leave this modal route for categorial logics unexplored here. Instead, we will be looking at least for a *subclass* of Sahlqvist axioms which do guarantee frame completeness. In view of the earlier examples, these should still include many familiar categorial axioms.

Sahlqvist Completeness

The heuristics of the earlier completeness proofs over the simple formula representation with (some) additional sequent axioms suggests an analogy with the substitution arguments found in Chapter 2. The class of axioms for which this works well may be described as follows.

3.2.4. DEFINITION. *Weak Sahlqvist Axioms*

A weak Sahlqvist axiom is a sequent of the form $\alpha \vdash \beta$, where α is a pure product formula, associated in any order, without repetitions of proposition letters, and β is also a pure product formula containing at least one $\bullet$, all of whose atoms occur in α.

Note, that due to the equivalence between product and slash formulas pointed out in Chapter 2, the set of sequents equivalent to weak Sahlqvist axioms is not restricted to ML($\bullet$). Here are some examples:

$$p \to q \vdash p \leftarrow q$$
$$(p \to (q \to r) \vdash q \to (p \to r)$$
$$(p \bullet q) \bullet r \vdash p \bullet (q \bullet r)$$
$$(p \bullet q) \bullet r \vdash p \bullet r$$
$$(r \leftarrow p) \leftarrow p \vdash r \leftarrow p$$
$$p \leftarrow r \vdash p \leftarrow (r \bullet q)$$

Sahlqvist Completeness Theorem gives the completeness result not just for any particular sequent, but the whole *class* of sequents equivalent to weak Sahlqvist axioms. Moreover, it proves, that adding a weak Sahlqvist axiom to a canonical logic does not disturb canonicity. By 'canonical' logics, we mean those which hold in the underlying frame of their canonical model as constructed in the above completeness proof.

3.2.5. THEOREM. *Sahlqvist Completeness*
If $\alpha \vdash \beta$ is a weak Sahlqvist axiom, then

(i) **NL** $+ \ \alpha \vdash \beta$ is frame complete for the first-order frame condition corresponding to $\alpha \vdash \beta$

(ii) X $+ \ \alpha \vdash \beta$ is a canonical logic, whenever X is a canonical categorial logic.

In order to prove the Theorem one has to show, that a first order frame correspondent of an arbitrary weak Sahlqvist formula holds in the canonical model. First, we imitate the main steps of the proof reconsidering Associativity Axiom in the canonical model construction and then present the general proof as such.

Consider the weak Sahlqvist axiom $(p \bullet q) \bullet r \vdash p \bullet (q \bullet r)$.

1. Suppose that $(p \bullet q) \bullet r \vdash p \bullet (q \bullet r)$ is true in the canonical model M . Then for *all* formulas A, B, C

$$M \models (A \bullet B) \bullet C \vdash A \bullet (B \bullet C)$$

2. That is $\forall A \forall B \forall C \forall a(a \models (A \bullet B) \bullet C \Rightarrow a \models A \bullet (B \bullet C))$.
Together with the truth condition for $\bullet$, this leads to

$$\forall A \forall B \forall C \forall a(\exists bcde(Ra, bc \ \&\ Rb, de \ \&\ d \models A \ \&\ e \models B \ \&\ c \models C) \Rightarrow \\ \exists xyzv(Ra, xy \ \&\ Ry, zv \ \&\ x \models A \ \&\ z \models B \ \&\ v \models C)).$$

This is equivalent to

$$\forall A \forall B \forall C \forall abcde((Ra, bc \ \&\ Rb, de \ \&\ d \models A \ \&\ e \models B \ \&\ c \models C) \Rightarrow \\ \exists xyzv(Ra, xy \ \&\ Ry, zv \ \&\ x \models A \ \&\ z \models B \ \&\ v \models C)).$$

3. Now, set

$$A = d \qquad B = e \qquad C = c$$

Then, by instantiation from the last formula, one gets

$$\forall abcde((Ra, bc \ \&\ Rb, de \ \&\ d \models d \ \&\ e \models e \ \&\ c \models c) \Rightarrow \\ \exists xyzv(Ra, xy \ \&\ Ry, zv \ \&\ x \models d \ \&\ z \models e \ \&\ v \models c)).$$

Using the trivial consequence of the Truth Lemma 3.2.2 that

$$\forall k(k \in W \Rightarrow k \models k),$$

we can simplify this to obtain

$$\forall abcde((Ra, bc \ \&\ Rb, de) \Rightarrow \\ \exists xyzv(Ra, xy \ \&\ Ry, zv \ \&\ x \vdash d \ \&\ z \vdash e \ \&\ v \vdash c)).$$

By the definition of the canonical model

$$Rk, nm \iff k \vdash n \bullet m,$$

Altogether this implies

$$\forall abcde((Ra, bc \;\&\; Rb, de) \Rightarrow \exists y(Ra, dy \;\&\; Ry, ec)).$$

Crucially, the latter first-order formula is exactly the same frame condition as was achieved in Chapter 2 by the application of the minimal substitution algorithm.

For the general proof of the Theorem, we apply the same trick with small combinatorial complications, following the successive steps of the Sahlqvist-van Benthem algorithm, applied to weak Sahlqvist axiom $\alpha \vdash \beta$.

(i) Pure product formulas α have a standard translation $ST(\alpha)$ consisting only of atoms, & and $\exists$. Thus, one can move all existential quantifiers up front.

(ii) Next, compute the canonical valuation CV. The essential point is that each unary atom Pu occurs only once (because of the non-iteration of proposition letters in α). Therefore, each such atom will have just one identity atom $u = a$ for its substituendum. (Note, that the definition of the antecedent of weak Sahlqvist formulas forbids occurrence of disjunctions in the canonical valuations)

(iii) Now, the usual equivalent will be obtained by substituting these identity atoms into the corresponding positions of the consequent $ST(\beta)$.

Let us now analyze what happens in parallel in the canonical model.

1. Suppose $\alpha \vdash \beta$ is true in the canonical model M. Let $p_1, \ldots, p_n$ be distinct propositional variables which occur in α and $\alpha' \vdash \beta'$ be obtained from $\alpha \vdash \beta$ by substituting $\alpha_1, \ldots, \alpha_n$ for $p_1, \ldots, p_n$. Then, for any $x_0 \in W$

$$[a] \qquad \forall\alpha_1 \ldots \forall\alpha_n(x_0 \models \alpha' \Rightarrow x_0 \models \beta')$$

2. By truth conditions, decode $x_0 \models \alpha'$ and$x_0 \models \beta'$ providing that no two quantifiers have the same bound variables. This yields

$$[b] \qquad \forall\alpha_1 \ldots \forall\alpha_n(\Theta \Rightarrow \Lambda)$$

with

- Θ is equivalent to $\exists x_1 \ldots \exists x_{n+1}(\bigwedge Rx_i, x_j x_k \;\&\; \bigwedge(x_m \models \alpha_m))$, where $1 \leq i, j, k \leq n+1$ and $1 \leq m \leq n$ and $x_1, \ldots, x_{n+1}$ are distinct;
- Λ is equivalent to $\exists y_1 \ldots \exists y_t(\bigwedge Ry_i, y_j y_k \;\&\; \bigwedge(y_s \models \alpha_m))$ where $1 \leq t, i, j, k \leq n+1$ and $1 \leq s, m \leq n$. Again, $y_1, \ldots, y_t$ are distinct, moreover if $y_s \models \alpha_m$ occurs in Λ, then y_s is a *leaf* of exactly one triangle which belongs to $\bigwedge Ry_i, y_j y_k$.

3. Since α is in ML($\bullet$), there are only existential quantifiers with distinct bound variables in Θ By standard technique of first order logic, one gets

$$[c] \qquad \forall\alpha_1 \ldots \forall\alpha_n \forall x_1, \ldots \forall x_k(\Theta' \Rightarrow \Lambda)$$

where $\Theta' = \bigwedge Rx_i, x_j x_k \;\&\; \bigwedge(x_m \models \alpha_m)$.

4. Pick up $\alpha_1, \ldots, \alpha_n$ such that $x_i = \alpha_i$ $(1 \leq i \leq n)$. Eliminate $\forall\alpha_1 \ldots \forall\alpha_n$ by the corresponding instantiation. This leads to

$$[c] \qquad \forall x_1, \ldots \forall x_k((\bigwedge Rx_i, x_j x_k \;\&\; \bigwedge(x_m \models x_m)) \Rightarrow \\ \exists y_1, \ldots \exists y_t(\bigwedge Ry_i, y_j y_k \;\&\; \bigwedge(y_s \models x_m)))$$

Next, by Truth Lemma

$$[d] \qquad \forall x_1, \ldots \forall x_k((\bigwedge Rx_i, x_j x_k \;\&\; \bigwedge(x_m \vdash x_m)) \Rightarrow \\ \exists y_1, \ldots \exists y_t(\bigwedge Ry_i, y_j y_k \;\&\; \bigwedge(y_s \vdash x_m))).$$

Recall, that if $y_s \models x_m$ is in Λ, then y_s occurs uniquely as a leaf in one triangle of $\bigwedge Ry_i, y_j y_k$. Thus, $Ry_p, y_s y_r$ (or $Ry_p, y_r y_s$) by the definition of ternary relation in the canonical model, can be transcribed in $y_p \vdash y_s \bullet y_r$ (or $y_p \vdash y_s \bullet y_r$); taken together with $y_s \vdash x_m$, this yields $Ry_p, x_m y_r$ (or $Ry_p, y_r x_m$). Finally, [d] leads to the required first order frame condition [e] which turns our to be valid in the canonical model:

$$[e] \qquad \forall x_1, \ldots \forall x_k((\bigwedge Rx_i, x_j x_k) \Rightarrow [\exists y_1, \ldots \exists y_t \bigwedge Ry_i, y_j y_k]^*$$

where $[\exists y_1, \ldots \exists y_t \bigwedge Ry_i, y_j y_k]^*$ is obtained from $\exists y_1, \ldots \exists y_t \bigwedge Ry_i, y_j y_k$ by substituting x_m for y_s, for each $y_s \models x_m$ occurring in Λ.

3.3 A Hierarchy of Filters

Once again, the point about categorial completeness proofs is this. As we are dealing with weak fragments of a full modal language, we are in a spectrum of model constructions, where worlds may be constructed in ways ranging from single formulas to infinite maximally consistent sets of formulas. Our aim is to show this systematically, by means of the following notions.

3.3.1. DEFINITION. *Types of Filters*

The following notions are all relative to some fixed categorial logic driving sequent deduction.

- A set of formulas Δ is a *weak filter* iff
  if $\alpha \in \Delta$ and $\alpha \vdash \beta$, then $\beta \in \Delta$
- A *filter* is a weak filter satisfying
  $\alpha \;\&\; \beta \in \Delta$ iff $\alpha \in \Delta$ and $\beta \in \Delta$

- A *prime filter* is a filter satisfying
  $\alpha \vee \beta \in \Delta$ iff $\alpha \in \Delta$ or $\beta \in \Delta$
- An *ultrafilter* is a prime filter satisfying
  $\neg\alpha \in \Delta$ iff $\alpha \notin \Delta$

Our plan is as follows. First, we reprove the completeness theorem for the non-associative base calculus NL using weak filters. (This may be compared with the earlier semantic construction of "weak filter extensions" in Chapter 1.) Next,we show how other types of filters determine canonical model construction when categorial language is enriched with different connectives. All these results will concern ternary frame semantics. Improvements to binary frame semantics are possible for certain categorial logics, and some of these will be considered in Chapter 4.

3.3.1 Weak Filter Representation

We reprove completeness for the basic non-associative Lambek calculus by a construction which lifts the earlier formula-based canonical model to one whose worlds are sets of formulas. Essentially, what we shall do is apply the weak filter extensions of Chapter 1 to the former. The advantage of this is that we set up a better platform for future generalizations. We repeat the result

3.3.2. THEOREM. *Completeness for* **NL**
NL is sound and complete with respect to the class of all ternary models.

Proof
As usual, soundness is trivial. Completeness follows by a Truth Lemma for the following new construction. The **NL**-Henkin Model is a quadruple

$$M = \langle W, R^3, V\rangle$$

where

- W is the set of all weak filters of formulas
- $R^3 a, bc$ iff $b \circ c \subseteq a$,
  where $\circ$ is a binary operation on sets defined as follows:
  $\alpha \in x \circ y$ iff there are β, γ such that
  $\beta \in x$, $\gamma \in y$ and $\beta \bullet \gamma \vdash \alpha$
- $a \in V(p)$ iff $p \in a$.

We need two observations concerning weak filters. Let $|A|$ denote the set of all α such that $A \vdash \alpha$.

Fact
For any $ML(\bullet, \rightarrow, \leftarrow)$ formula A, $|A| \in W$.

Fact
If $a \in W$ and $b \in W$, then $a \circ b \in W$

3.3.3. LEMMA. For any $ML(\bullet, \rightarrow, \leftarrow)$ formula α and any $a \in W$, $M, a \models \alpha$ iff $\alpha \in a$.

Proof
By induction on α. The base case directly follows from the definition of V. We demonstrate two cases.

$\alpha = A \bullet B$

From right to left,

1. Suppose $A \bullet B \in a$.
2. The crucial observation is that, then, $|A| \circ |B| \subseteq a$. To see this, suppose that $\alpha \in |A| \circ |B|$. So, there exist α_1, α_2 such that $\alpha_1 \in |A|, \alpha_2 \in |B|$ and $\alpha_1 \bullet \alpha_2 \vdash \alpha$. Hence, by Cut, $A \bullet B \vdash \alpha$, and by [1] we have that $\alpha \in a$.
3. By definition, $|A| \circ |B| \subseteq a$ means that $Ra, |A||B|$. Now, by the inductive hypothesis, $|A| \in A$ and $|B| \in B$. Hence we have that $\exists b \exists c(Ra, bc \;\; \& \;\; M, b \in A \;\&\; M, c \in B)$, i.e., $M, a \in A \bullet B$.

From left to right,

1. Suppose that $M, a \in A \bullet B$.
2. By the inductive hypothesis and the definition of R, there exist x and y such that $[i] x \circ y \subseteq a$, [ii] $A \in x$, $[iii] B \in y$. By the definition of $\circ$, this immediately leads to $A \bullet B \in a$.

$\alpha = A \rightarrow B$

From left to right,

1. Suppose that $M, a \in A \rightarrow B$.
2. By the truth definition, $\forall b \forall c(Rc, ba \;\&\; M, b \in A \Rightarrow M, c \in B)$. Since $R|A|oa, |A|a$ and $M, |A| \models A$ (by the inductive hypothesis), this implies $M, |A|oa \models B$. By the inductive hypothesis once more, $B \in |A| \circ a$. This means that some $C \bullet D \vdash B$ is derivable, where $A \vdash C$ is derivable and $D \in a$. Hence also $A \bullet D \vdash B$ is derivable. It follows that D derives $A \rightarrow B$, whence $A \rightarrow B$ belongs to the weak filter a, as required.

From right to left,

1. Suppose $A \rightarrow B \in a$.
2. Suppose Rc, ba and $M, b \models A$. By the inductive hypothesis, one gets $A \in b$, and hence by the definition of $R, A \bullet (A \rightarrow B) \in c$. The latter derives B , whence $B \in c$. By the inductive hypothesis, $M, c \in A$, and hence $M, a \in A \rightarrow B$.

□

The Completeness Theorem now follows in the usual way.

3.3.2 Extension with a Minimal Modality

M. Moortgat in [Moortgat 94] shows that the formula-based canonical model provides a completeness proof for an extension of the Lambek calculus by axioms for a minimal temporal logic. But adding just one standard unary modality $\Box$ (without its temporal dual $\Diamond^{\downarrow}$) make a difference for the canonical domain. We need the preceding weak filter construction to get things right. here are the main steps that need to be added.
The minimal modal extension $\mathbf{NL}^m$ of $\mathbf{NL}$ is obtained by adding just one principle:

Modalization Rule
if $A \vdash B$ then $\Box A \vdash \Box B$

Various further extensions arise in obvious ways. As for semantic structures (cf. Chapter 2), a NL^m frame is a triple $F = \langle W, R^3, R^2 \rangle$, where $\langle W, R^3 \rangle$ is a ternary frame and R^2 is a binary accessibility relation. A NL^m model is a quadruple $M = \langle W, R^3, R^2, V \rangle$ such that $\langle W, R^3, R^2 \rangle$ is a NL^m frame with a propositional valuation V. The inductive truth definition is as usual, with one extra clause for $\Box$:

$$M, a \models \Box A \iff \forall b(R^2ab \Rightarrow M, b \models A)$$

Definitions of 'weak filter', etcetera, remain the same.

3.3.4. THEOREM. $\mathbf{NL}^m$ is sound and complete with respect to the class of all $\mathbf{NL}^m$ models.

Proof
We merely indicate the additional steps, as compared with the previous completeness argument. In the Henkin Model, we need one additional clause for modal accessibility:

$$R^2ab \iff \forall b(\Box A \in a \Rightarrow A \in b)$$

In the proof of the Truth Lemma, we need one additional case for modalized formulas:

$\alpha = \Box A$
From left to right,

1. Suppose that $M, a \models \Box A$.
2. Suppose that $\Box A \notin a$.

3. Define a new class k of $ML(\rightarrow, \bullet, \leftarrow, \Box)$ formulas as follows:

$$k = \{\alpha | \Box\alpha \in a\}$$

4. By definition, Rak, and therefore from [1] combined with the truth definition and the inductive hypothesis, $A \in k$. On the other hand, $A \notin k$ by the definition of k. This contradiction shows that $\Box A \in a$.

From right to left,

1. Suppose that $\Box A \in a$.
2. Suppose that Rab. Then by the definition of R, $A \in b$. Next, by the inductive hypothesis, $M, b \models A$, and therefore $M, a \models A$.

Completeness for other standard modal extensions of the Lambek Calculus, such as the above T and K4 versions, require only one additional step, namely proving that the corresponding frame condition is valid in the canonical model. This can be shown by the usual arguments of standard modal logic.

3.3.3 Filters and prime filters in Henkin model

Extended Vocabularies
So far,we managed with weak filters. Next, we want to introduce further type-forming connectives. We start with a straightforward extension, which does not really affect the earlier proofs.

Conjunction and filters
Consider the extension $\mathbf{NL}_{\&}$ of the Lambek calculus $\mathbf{NL}$ with the following axioms and inference rule:

$$A\&B \vdash A \qquad\qquad A\&B \vdash B$$
$$\text{if } A \vdash B \text{ and } A \vdash C, \qquad \text{then } A \vdash B\&C$$

In ternary models, & gets the obvious interpretation as Boolean conjunction. This will make these new principles obviously sound.

3.3.5. Theorem. $\mathbf{NL}_{\&}$ is complete for ternary frame semantics.

Proof
The only adaptation needed in previous arguments is a change in the Henkin worlds. These now become *filters*, that is, sets of $ML(\bullet, \rightarrow, \leftarrow, \&)$ formulas X such that

X is a weak filter
for all $ML(\bullet, \rightarrow, \leftarrow, \&)$ formulas A and B ,
$A \in X$ and $B \in X$ iff $A\&B \in X$.

The rest of the completeness proof for $\mathbf{NL}_{\&}$ is straightforward. □

Disjunctions and Prime Filters

The next natural question to ask is whether there is a similar complete categorial inference system with the other Booleans. But here, some complications arise. By technical algebraic methods, [Andreka & Mikulas 93] prove a negative result, at least for binary frame semantics (cf. Chapter 4). Informally, the latter semantics with categorial product plus all Boolean connectives, has no simple complete proof system. Of course, this does not disqualify attempts via ternary frame semantics. We shall consider the full Boolean language in a later Section. For the moment, we shall rather follow the tradition of Relevant Logic. We can introduce disjunction of types into our logics, with standard axioms, and obtain completeness via a so-called 'prime filter' construction. We shall not perform this extension separately here.

Adding "Inverse" Negation

Instead, we note at once that the prime filter construction can be generalized to accommodate at least one well-known kind of negation. More precisely, one can construct a sound and complete categorial logic with a set of connectives $\{\bullet, \rightarrow, \leftarrow, \&, \vee, \sim\}$, where $\sim$ is the special "inverse negation" from Relevant Logic already encountered in Chapter 2. Its semantic explication was that a negation $\sim A$ is supported by a world x in a ternary model iff the frame inverse x^* of x does not support A.

The main deductive calculus $LC_{\&,\vee,\sim}$ that we shall study here can be obtained from $LC_{\&}$ by adding the following axioms:

$$
\begin{array}{ll}
A \vdash A \vee B & B \vdash A \vee B \\
\sim\sim A \vdash A & A \vdash \sim\sim A \\
\sim A \,\&\, \sim B \vdash \sim (A \vee B) & \sim (A \vee B) \vdash \sim A \,\&\, \sim B \\
\sim A \vee \sim B \vdash \sim (A \,\&\, B) & \sim (A \,\&\, B) \vdash \sim A \vee \sim B
\end{array}
$$

plus two new inference rules:

$$A \vdash B \text{ and } C \vdash B \text{ imply } A \vee C \vdash B$$

$$A \vdash B \text{ implies } \sim B \vdash \sim A$$

Next, we turn to semantic interpretation.

3.3.6. DEFINITION. *$LC_{\&,\vee,\sim}$ Models*

A $LC_{\&,\vee,\sim}$ is a quadruple $M = \langle W, R, ^*, V\rangle$, where W is a non-empty set closed under a unary operation *, R is a ternary relation, V is a valuation function, and the following postulate of Idempotence holds: for all worlds x, $x^{**} = x$.

It is easy to see that the calculus $LC_{\&,\vee,\sim}$ is sound for its models. Before turning to its completeness, we note some useful properties of the prime filters needed in the Henkin construction. Recall the earlier definition: these are filters that split their disjunctions.

3.3.7. LEMMA. *Special Lindenbaum Lemma*
Let a, b be filters and c a prime filter containing $a \circ b$.
Then there exist prime filters a' and b' such that
$a \subseteq a'$, $b \subseteq b'$ and $a' \circ b' \subseteq c$.

Proof
Let a, b be filters and let c be a prime filter, satisfying $a \circ b \subseteq c$. We will show that, for any filter a and prime filter c with $a \circ b \subseteq c$, there exists an extension of a to a prime filter a' such that $a' \circ b \subseteq c$. The prime extension for b then goes similarly. The crucial step in our argument appeals to Zorn's Lemma, applied to the family of filters a' extending a which satisfy $a' \circ b \subseteq c$ ordered under inclusion. There must be maximal such filters, and the following Claim then does the job.

Claim Any maximal filter k satisfying $k \circ b \subseteq c$ is prime.

Proof

1. Suppose that k is a maximal filter satisfying $k \circ b \subseteq c$.
2. Suppose that $\alpha \vee \beta \in k$, but $\alpha \notin k$ and $\beta \notin k$.
3. We need an auxiliary definition. Let x be a set of formulas and α a formula such that $\alpha \notin x$. Define a new set $[x, \alpha]$ via:

$$\delta \in [x, \alpha] \text{ iff } \delta \in x \text{ or } \alpha \vdash d$$

   Here, k is strictly smaller than $[k, \alpha]$,because $\alpha \notin k$.Since k is a maximal filter satisfying $k \circ b \subseteq c$, we may conclude

   (a) $[k, \alpha] \circ b \not\subseteq c$, and likewise,
   (b) $[k, \beta] \circ b \not\subseteq c$.

4. It follows that there is some formula $\psi \in [k, \alpha] \circ b$ with $\psi \notin c$,and likewise, some formula $\chi \in [k, \beta] \circ b$ with $\chi \notin c$.
5. By the definition of $[k, \alpha] \circ b$, there exist ψ_1, ψ_2 such that

$$\psi_1 \in [k, \alpha] \quad \psi_2 \in b \quad \psi_1 \bullet \psi_2 \vdash \psi$$

   In particular, either $\psi_1 \in k$ or $\alpha \vdash \psi_1$. But the first is impossible. Otherwise, combining the preceding facts, we get $\psi \in k \circ b$, and hence $\psi \in c$: contradicting the assumption in [4]. Therefore, $\alpha \vdash \psi_1$. Analogously, we conclude that there exist χ_1, χ_2 such that

$$\chi_1 \in [k, \beta] \quad \chi_2 \in b \quad \chi_1 \bullet \chi_2 \vdash \chi$$

where $\beta \vdash \chi_1$.

6. Now we derive some successive consequences of [6]:
$\psi_1 \vdash \psi \leftarrow \psi_2$
$\alpha \vdash \psi \leftarrow \psi_2$
$\alpha \vdash (\psi \vee \chi) \leftarrow (\psi_2 \,\&\, \chi_2)$
as well as
$\chi_1 \vdash \chi \leftarrow \chi_2$
$\beta \vdash \chi \leftarrow \chi_2$
$\beta \vdash (\psi \vee \chi) \leftarrow (\psi_2 \,\&\, \chi_2)$

7. Together, in our calculus, the preceding sequents derive $\alpha \vee \beta \vdash (\psi \vee \chi) \leftarrow (\psi_2 \,\&\, \chi_2)$, and therefore $(\alpha \vee \beta) \bullet (\psi_2 \,\&\, \chi_2) \vdash \psi \vee \chi$.Now recall that $\alpha \vee \beta \in k$, while $\psi_2 \,\&\, \chi_2 \in b$(filter property). Then, since $k \circ b \subseteq c$,we get $\psi \vee \chi \in c$. As c is a prime filter, either $\psi \in c$ or $\chi \in c$, which is a contradiction with [4]

□

Now, we are ready to prove our main result.

3.3.8. THEOREM. $LC_{\&,\vee,\sim}$ is complete for its ternary models.

Proof
The canonical Henkin model is a quadruple $M = \langle W, R, *, V\rangle$ such that

W is the set of all *prime filters*
$a^* = \{\alpha | \sim \alpha \notin a\}$
$R^3 a, bc$ iff $b \circ c \subseteq a$
$a \in V(p)$ iff $p \in a$.

Truth Lemma
For all $a \in W$, and all $ML(\bullet, \rightarrow, \leftarrow, \&, \vee, \sim)$ formulas α, $a \models \alpha$ iff $\alpha \in a$.

Proof
Induction on the length of α. The base case is a direct consequence of the definition of the valuation V. The case of conjunctions $A\&B$ is taken care of by the filter requirement as before. Likewise, the case of disjunctions $A \vee B$ follows by the splitting requirement on prime filters. Next, we consider the crucial inductive step for products $A \bullet B$. One direction is as before. Suppose that $a \models A \bullet B$. Then, by the truth definition, the definition of R, and the inductive hypothesis, there exist prime filters x and y such that [i] $x \circ y \subseteq a$, [ii] $A \in x$, [iii] $B \in y$. This immediately gives $A \bullet B \in a$. Conversely, suppose that $A \bullet B \in a$. We want to show that $a \models A \bullet B$. First note, that, as before, $|A| \circ |B| \subseteq a$. This time, these do not suffice, as there is no guarantee that $|A|$ and $|B|$ are prime filters. Here, we appeal to the earlier Lindenbaum Lemma, which gives prime filters x and y with $|A| \subseteq x$, $|B| \subseteq y$ and $x \circ y \subseteq a$. Therefore, $\exists x \exists y(Ra, xy \,\&\, A \in x \,\&\, A \in y)$,

and by the inductive hypothesis, we get $a \models A \bullet B$, as desired. We omit the verification steps for categorial implications. □

To make the completeness result follow from the Truth Lemma, one has to prove that our prime-filter based Henkin model is in fact an $LC_{\&,\vee,\sim}$ model. First, we observe that the set of all prime filters is closed under its unary operation *.

Claim If $a \in W$, then $a^* \in W$

Proof
Suppose that $a \in W$. Recall that $a^* = \{\alpha | \sim \alpha \notin a\}$. We must check the requirements for being a prime filter. (i) To show that a^* is deductively closed, suppose $\alpha \in a^*$ and $\alpha \vdash \beta$. If $\beta \notin a^*$, then $\sim \beta \in a$. Since a is deductively closed and $\sim \beta \vdash \sim \alpha$, we may conclude that $\sim \alpha \in a$: which is a contradiction. (ii) Next, suppose that α, β both belong to a^*. I.e., $\sim \alpha \notin a$ and $\sim \beta \notin a$. Then we also have $\sim (\alpha \;\&\; \beta) \notin a$ and hence the conjunction $\alpha \;\&\; \beta \in a^*$. (Otherwise, because $\sim (\alpha \;\&\; \beta)$ derives $\sim \alpha \vee \sim \beta$, the latter disjunction would be in a , whence the prime filter a has $\sim \alpha \in a$ or $\sim \beta \in a$: quod non.) (iii) Finally, suppose that $\alpha \vee \beta \in a^*$, but $\alpha \notin a^*$ and $\beta \notin a^*$. Then we have $\sim \alpha \in a$ and $\sim \beta \in a$, and therefore (by one of the above De Morgan axioms) $\sim \alpha \;\&\; \sim \beta \in a$. That means $\sim (\alpha \vee \beta) \in a$ (by deductive closure), but then also $\alpha \vee \beta \notin a^*$. □

Claim The Henkin model is idempotent.

Proof
The following chain of equivalences is justified by the definition of * in the Henkin model, plus the Double Negation laws of our calculus:

$$A \in a^{**} \text{ iff } \sim A \notin a^* \text{ iff } \sim\sim A \in a \text{ iff } A \in a.$$

□

Here is our final argument. Suppose that $A \vdash B$ is not derivable in $LC_{\&,\vee,\sim}$. Then clearly $B \notin |A|$. Now, the family x of filters extending $|A|$ and not containing B is partially ordered by inclusion $\subseteq$. Therefore, by Zorn's Lemma, there exists a maximal filter k satisfying these conditions.

Claim k is a prime filter.

Proof
Suppose that $\alpha \vee \beta \in k$, but $\alpha \notin k$ and $\beta \notin k$. Construct $[k, \alpha]$ and $[k, \beta]$ as in the proof of an earlier Claim. Since k is maximal, $[k, \alpha]$ and $[k, \beta]$ do not satisfy the requirement of omitting B, whence $B \in [k, \alpha]$ and $B \in [k, \beta]$. Because $B \notin k$, this implies that $\alpha \vdash B$ and $\beta \vdash B$. But then also (in our deductive calculus)

$\alpha \vee \beta \vdash B$. But as we assumed that $\alpha \vee \beta \in k$, this gives $B \in k$: a contradiction. □

For this prime filter k which does not contain B , the Truth Lemma says that $k \models A$, but $k \not\models b$. Therefore, the non-derivable sequent $A \vdash B$ is not valid in all $LC_{\&,\vee,\sim}$ models. □

With this method of proof, we can also analyze further categorial logics, having additional axioms for negation and disjunction. In particular, a more general Sahlqvist completeness theorem is lurking in the background here, whose statement we must omit. As an example, Idempotence itself was a Sahlqvist axiom, and the above argument for its validity on the Henkin frame can easily be generalized to cover other axioms.

3.3.4 The Full Modal Case

So far,we managed with weak filters, filters and prime filters. Next, in this Section, we look at the full modal calculus with Booleans. In a natural sense, this is the 'limit' of our considerations. Lambek Calculus was studied from this perspective in [Roorda 91], and much of what we have done so far consists in a more delicate analysis of the 'categorial fine-structure' of this broader modal system.

Recall the basic definitions (cf. also chapter 1). We work with a language with three inter-locked products and all Booleans, which is 'versatile' in the sense of [Venema 91]. This language is interpreted over ternary models as follows:

$$\begin{array}{lcl}
\mathcal{M}, a \models A \bullet_1 B & \Longleftrightarrow & \exists b \exists c (Ra, bc \;\&\; \mathcal{M}, b \models A \;\&\; \mathcal{M}, c \models B) \\
\mathcal{M}, a \models A \bullet_2 B & \Longleftrightarrow & \exists b \exists c (Rc, ba \;\&\; \mathcal{M}, b \models A \;\&\; \mathcal{M}, c \models B) \\
\mathcal{M}, a \models A \bullet_3 B & \Longleftrightarrow & \exists b \exists c (Rc, ab \;\&\; \mathcal{M}, b \models A \;\&\; \mathcal{M}, c \models B)
\end{array}$$

In particular, this system allows us to define the two categorial slashes via negations and products:

$$\begin{array}{lll}
A \to B & \text{is equivalent to} & \neg(A \bullet_2 \neg B) \\
B \leftarrow A & \text{is equivalent to} & \neg(A \bullet_3 \neg B)
\end{array}$$

The minimal modal logic for this language, with respect to ternary frame semantics is as follows:

(i) all axioms and rules of classical propositional logic

(ii) residuation laws

$$A \bullet_1 B \vdash C \;\text{ iff } B \vdash A \to C \;\text{ iff } B \vdash C \leftarrow A$$

(iii) modal distribution of product over disjunction

$$A \bullet_i (B \vee C) \vdash (A \bullet_i B) \vee (A \bullet_i C)$$

(iv) "necessitation"

$$\text{if } \vdash A \text{ then } \vdash \neg(\neg A \bullet iB)$$

where $1 \leq i \leq 3$.
Below we drop indeces and agree to use $\bullet$ for $\bullet_i$.

One virtue of this system is that it is an ordinary modal logic, of a well-understood kind - which can be used to understand many categorial patterns in standard terms, for example

$$(A \to C) \ \& \ (B \to) \vdash (A \vee B) \to C$$

More generally, with practice, one will come to recognize modal principles in categorial ones, and also vice versa.

As a culminating point for the 'Filter Hierarchy', we present an outline of the standard proof for the basic completeness result. Here, at last, worlds will be constructed as ultrafilters, in order to deal with Boolean negation.

3.3.9. THEOREM. The minimal modal logic of three versatile products is complete for ternary frame semantics.

Proof

The *Henkin Model* is a quadruple $M = \langle W, R^3, V\rangle$ where

- W is the set of ultrafilters (maximal consistent sets of formulas)
- R^3a, bc iff $b \circ c \subseteq a$,
- $a \in V(p)$ iff $p \in a$.

The crucial step in our argument appeals to the following

Claim.
If $A \bullet B \in w$ then there exist maximal consistent sets x and y such that

$$A \in x, \quad B \in y \quad \text{and} \quad Rw, xy$$

Let $\gamma_1, ..., \gamma_n, ...$ be an enumeration of all ML($\bullet$, &, $\vee$, $\neg$) formulas.
Set $x_0 = \{\alpha\}$ $y_0 = \{B\}$
Let $\bigwedge x_n (\bigwedge y_n)$denote the conjunction of all formulas which occur in $x_n(y_n)$. Let $\gamma_n = \alpha$
Note, that if $\bigwedge x_n \bullet \bigwedge y_n \in w$, then
$(\bigwedge x_n \wedge (\alpha \vee \neg\alpha)) \bullet (\bigwedge y_n \wedge (\alpha \vee \neg\alpha)) \in w$
By modal distributivity and properties of ultrafilters

$$(\bigwedge x_n \wedge \alpha^*) \bullet (\bigwedge y_n \wedge \alpha^{**}) \in w$$

for some α^*, α^{**} equal α or $\neg\alpha$
Define $x_{n+1} = x_n \cup \{\alpha^*\}$, $y_{n+1} = y_n \cup \{\alpha^{**}\}$

Next, suppose that x_{n+1} is inconsistent.
Then $\bigwedge x_n \vdash \neg\alpha^*$ and therefore
$\vdash \neg(\bigwedge x_n \wedge \alpha^*)$. By"necessitation" rule we have

$$\vdash \neg((\bigwedge x_n \wedge \alpha^*) \bullet (\bigwedge y_n \wedge \alpha^{**}))$$

and that contradicts consistency of w.

In the countable limit, this construction yields the desired pair of successors for w . □

Arrow Logic as a categorial logic

As it was mentioned already, two procedures are involved in the semantic interpretation of the relevant negation A, namely, Boolean negation and 'converse'. Arrow Logic contains explicit connectives for both of these, namely classical negation $\neg$, and a notion of converse $\breve{}$ interpreted as follows:

$A\breve{}$ is supported by x iff the inverse of x supports A.

So, as mentioned already, $\sim A$ can be read as $\neg A\breve{}$, whence we have the following translations :

$$A \rightarrow B = \sim (\sim B \bullet A) = \neg(\neg B\breve{} \bullet A)\breve{} = \neg(A\breve{} \bullet \neg B)$$
$$B \leftarrow A = \sim (A \bullet \sim B) = \neg(A \bullet \neg B\breve{})\breve{} = \neg(\neg B \bullet A\breve{})$$

Thus, Arrow Logic can be viewed as a categorial logic. The presence of full Boolean expressivity in Arrow logic determines *ultrafilter* constructions in the canonical model. We refer the reader to [van Benthem 194], [Venema 94] and [Marx 95] for more detailed completeness proofs for Arrow Logic. What we shall add here is a more finely structured analysis of the 'amount of Arrow Logic' needed to embed the Lambek Calculus.

Chapter 4

Relational Completeness of the Lambek Calculus

Introduction

This chapter introduces a labelled format of categorial deduction which inspires a special method of proving completeness of the Lambek Calculus with respect to two-dimensional dynamic semantics. The chapter is an almost literal reproduction of the paper 'The Lambek Calculus, Relational Semantics and the Method of Labelling' (see [Kurtonina 94]).

Content

1. The Lambek Calculus, Relational semantics and the method of labelling: the general idea behind the completeness proof.
2. Weak completeness of the Lambek Calculus with respect to Relational semantics.
3. Strong completeness of the Lambek Calculus with respect to Relational semantics.
4. Concluding remarks and open questions.

Relational Semantics as it is introduced in [van Benthem 88a], [van Benthem 91b] and [Busz. & Orl. 86] provides a *dynamic* interpretation for the Lambek Calculus: valuation of formulae is carried out on pairs of points, which can be treated as informational states. Propositions can be understood as programs of transformation of informational states. In [Andréka & Mikulás 94] the corresponding Completeness Theorem is obtained by algebraic methods.

The purpose of this paper is to propose a new, rather simple completeness proof by using the method of labelling. The general theory of labelled deductive systems is proposed by D.Gabbay. In the labelled version of the Lambek Calculus a pair of labels is attached to every formula and the way of labelling reflects corresponding semantical truth conditions. The straightforward parallelism between syntax and semantics enables one to obtain the completeness result.

4.1 The Lambek Calculus, Relational Semantics and the method of labelling: the general idea behind the Completeness proof.

The language of the Lambek Calculus **LC** is a propositional language with countable set of propositional letters and the binary connectives $\bullet, \rightarrow, \leftarrow$. We use $A, B, C, ..., A_1, B_1$ etc. as schematic letters for formulae. Expressions of the form $A_1, ..., A_n \Rightarrow B$ will be called sequents.

The Lambek Calculus
LC can be presented by the following axiom and inference rules where A, B, C stand for formulae and X, X_1, X_2, Y, Z stand for finite sequences of formulae including the empty sequence.

$$A \Rightarrow A$$

$$\frac{X \Rightarrow A \qquad Y, B, Z \Rightarrow \alpha}{Y, X, A \rightarrow B, Z \Rightarrow \alpha} \qquad \frac{A, X \Rightarrow B}{X \Rightarrow A \rightarrow B}$$

$$\frac{X \Rightarrow A \qquad Y, B, Z \Rightarrow \alpha}{Y, B \leftarrow A, X, Z \Rightarrow \alpha} \qquad \frac{X, A \Rightarrow B}{X \Rightarrow B \leftarrow A}$$

$$\frac{X, A, B, Y \Rightarrow \alpha}{X, A \bullet B, Y \Rightarrow \alpha} \qquad \frac{X_1 \Rightarrow A \qquad X_2 \Rightarrow B}{X_1, X_2 \Rightarrow A \bullet B}$$

Adding Cut Rule

$$\frac{X \Rightarrow A \qquad Y\ A, Z \Rightarrow C}{X, Y, Z \Rightarrow C}$$

is conservative.

Relational Semantics

(i) A Relational *Frame* is a pair $\langle K, D\rangle$ where K is a non-empty set and $D = K \times K$

(ii) A Relational *Model* adds a valuation function ψ which can be extended as follows: let a, b, c be elements of K

$$\begin{array}{lll} ab \models p & \text{iff} & ab \in \psi(p) \\ ab \models A \to B & \text{iff} & \forall c(ca \models A \Rightarrow cb \models B) \\ ab \models B \leftarrow A & \text{iff} & \forall c(bc \models A \Rightarrow ac \models B) \\ ab \models A \bullet B & \text{iff} & \exists c(ac \models A \;\&\; cb \models B) \end{array}$$

(iii) A sequent $A_1, \ldots, A_n \Rightarrow A$ (where $n > 0$) is ***true*** in a Relational Model M iff, for any $a, b \in K$, if $ab \models A_1 \bullet \ldots \bullet An$, then $ab \models A$
(iv) A sequent $\Rightarrow A$ is ***true*** in a Relational Model M iff, for any $a \in K$, $aa \in A$
(v) A sequent ϕ is ***valid*** in Relational Semantics iff it is true in all Relational models.

If **LC** is considered just as a sequent calculus then the Completeness Theorem says that every semantically valid sequent ϕ is derivable. We will call this theorem the *Weak* Completeness Theorem. But **LC** can be also treated as a 'sequent-axiomatic' system with a notion of an inference of a sequent ϕ from some sequents premises $\phi_1, \ldots, \phi_n$. In this case the Completeness Theorem says that, if $\phi_1, \ldots, \phi_n$ semantically entails ϕ, then ϕ is derivable from the premises $\phi_1, \ldots, \phi_n$. We will call this theorem the *Strong* Completeness Theorem.

To obtain the completeness result we are going to use the *Labelled* version of the Lambek Calculus. By a labelled formula we mean an expression of the form 'a pair: formula': '$ab : A$'. By a *labelled* sequent we mean an expression of the form

$$ac_1 : A_1,\ c_1c_2 : A_2,\ \ldots\ c_{n-1}b : A_n \ \Rightarrow\ ab : A$$

*The Labelled version of the Lambek Calculus (***LLC***)*
Let K' be a set of labels, $D' = K' \times K'$; $a, b, c \in K'$, α a labelled formula and X, X_1, X_2, Y, Z finite sequences of labelled formulas.
LLC is defined by the following axiom-scheme and inference rules

$$ab : A \ \Rightarrow\ ab : A$$

$$\frac{X \ \Rightarrow\ ca : A \quad Y,\ cb : B,\ Z, \Rightarrow\ \alpha}{Y, X,\ ab : A \to B,\ Z, \Rightarrow\ \alpha} \qquad \frac{ca : A,\ X \ \Rightarrow\ cb : B}{X \ \Rightarrow\ ab : A \to B}\ (\star)$$

$$\frac{X \ \Rightarrow\ bc : A \quad Y,\ ac : B,\ Z \ \Rightarrow\ \alpha}{Y,\ ab : B \leftarrow A,\ X,\ Z \ \Rightarrow\ \alpha} \qquad \frac{X, A \ \Rightarrow\ B}{X \ \Rightarrow\ ab : B \leftarrow A}\ (\star)$$

$$(\star)\ \frac{X,\ ac : A,\ cb : B,\ Y \ \Rightarrow\ \alpha}{X,\ ab : A \bullet B,\ Y \ \Rightarrow\ \alpha} \qquad \frac{X_1 \ \Rightarrow\ ac : A \quad X_2 \ \Rightarrow\ cb : B}{X_1, X_2 \ \Rightarrow\ ab : A \bullet B}$$

In the rules marked with $(\star)$ c is fresh.

Cut Rule

$$\frac{X \Rightarrow ab : A \qquad Y,\ ab : A,\ Z, \Rightarrow C}{X,\ Y,\ Z, \Rightarrow C}$$

4.1.1. LEMMA. *Renaming Lemma*
If a sequent $ac_1 : A_1,\ c_1c_2 : A_2,\ \ldots\ c_{n-1}b : A_n \ \Rightarrow\ ab : A$ is derivable in LLC, then any substitution of a fresh label d for $c_i (i \leq n-1)$ or a or b does not disturb derivability.

Proof
By induction of the length of the derivation □

4.1.2. DEFINITION. *Canonical sequence of labels*
A sequence of labels $\langle a, c_1, \ldots, c_n, b\rangle$ is a canonical one iff all its members are distinct if $n > 0$ and $a = b$ if $n = 0$.

The following Lemma establishes the obvious connection between **LC** and **LLC** that we need for the completeness proof.

4.1.3. LEMMA. If for any canonical sequence of labels $\langle a, c_1, ..., c_n, b\rangle$ a sequent

$$ac_1 : A_1,\ c_1c_2 : A_2,\ \ldots, c_{n-1}b : A_n \ \Rightarrow\ ab : A$$

is derivable in **LLC**, then

$$A_1,\ A_2, \ldots,\ A_n \ \Rightarrow\ A$$

is derivable in **LC**.

Proof
If there is a derivation of

$$ac_1 : A_1,\ c_1c_2 : A_2,\ \ldots, c_{n-1}b : A_n \ \Rightarrow\ ab : A$$

in **LLC**, then dropping labels we immediately have a derivation of

$$A_1,\ A_2, \ldots,\ A_n \ \Rightarrow\ A$$

in **LC**. □

The general idea of the completeness proof can be expressed as follows:

(a) Suppose $A_1,\ A_2, \ldots,\ A_n \ \Rightarrow\ A$ (where $n \geq 0$) is not derivable in **LC** ;
(b) According to Lemma 4.1.3 there is a canonical sequence of labels $\langle a, c_1, ..., c_n, b\rangle$ such that the sequence

$$ac_1 : A_1,\ c_1c_2 : A_2,\ \ldots, c_{n-1}b : A_n \ \Rightarrow\ ab : A$$

is not derivable in **LLC**;

(c) We mark with T all left labelled formulas of the labelled sequent and with F -the right formula. The resulting $T - F$ set is

$$\Delta_0 = \{Tac_1 : A_1,\ Tc_1c_2 : A_2\ , \ldots,\ Tc_{n-1}b : A_n,\ Fab : A\}$$

(d) To prove completeness theorem one has to construct a model such that

$$ac_1 \models A_1,\ \ c_1c_2 \models A_2,\ \ \ldots\ \ , c_{n-1}b \models A_n, \quad \text{but} \ \ ab \not\models A$$

4.2 Weak completeness of the Lambek Calculus with respect to Relational Semantics

The method of construction of the canonical model requires the description of some properties of the $T - F$ set, i.e. a set of labelled formulas marked with T or F. We say that a labelled formula α is a T-member (F-member) of $T - F$ set Δ iff $T\alpha \in \Delta$ ($F\alpha \in \Delta$).

Properties of $T - F$ sets

- The $T - F$ set Δ is *consistent* iff, for any α, if $T\alpha \in D$, then $F\alpha$ does not belong to Δ.
- The $T - F$ set is *complete* iff, for any α, if $T\alpha \notin \Delta$, then $F\alpha \in \Delta$.
- The $T - F$ set is *saturated* (or Henkin complete) iff (i)-(iii) hold:
    (i) If $Fab : A \to B$ belongs to Δ , then there is c such that $Tca : A$ and $Fcb : B$ belong to Δ.
    (ii) If $Fab : A \leftarrow B$ belongs to Δ, then there is c such that $Tbc : A$ and $Fac : B$ belongs to Δ.
    (iii) If $Tab : A \bullet B$ belongs to Δ, then there is c such that $Tac : A$ and $Tcb : B$ belong to Δ .
- The $T - F$ set is *deductively closed* iff if $\gamma_1, \ldots, \gamma_k$ are T-members of Δ and the sequent $\gamma_1, \ldots, \gamma_k \Rightarrow \gamma$ is derivable, then $T\gamma$ belongs to Δ.

4.2.1. LEMMA. If Δ is a consistent, complete, saturated, closed $T - F$ set, then there is a model $M = \langle K, D, \phi\rangle$ such that for any $xy \in D$

$$Txy : \alpha \in \Delta \quad \text{iff} \quad xy \models \alpha$$

Proof
Define $M = \langle K, D, \phi\rangle$ as follows:

- K is a set of labels that occur in Δ
- $D = K \times K$
- ϕ is such that $xy \models p$ iff $Txy : p \in \Delta$

The first case: $\alpha = A \rightarrow B$

Suppose $ab \models A \rightarrow B$ but $Tab : A \rightarrow B \notin \Delta$. Since Δ is complete, $Fab : A \rightarrow B \in \Delta$ and since Δ is saturated, there is c such that $Tca : A \in \Delta$ but $Tcb : B \notin \Delta$. By the inductive hypothesis we easily get contradiction. Suppose $Tab : A \rightarrow B \in \Delta$ and $ca \models A$. By the inductive hypothesis and the closeness of Δ conclude $cb \models B$ and therefore $ab \models A \rightarrow B$.

The second case ($\alpha = A \leftarrow B$) is analogous.

The third case: $\alpha = A \bullet B$.

Suppose $ab \models A \bullet B$. Then there is c such that $ac \models A$ and $cb \models B$. By the inductive hypothesis and the closeness of Δ we get $Tab : A \bullet B \in \Delta$. Suppose $Tab : A \bullet B \in \Delta$. Since Δ is saturated, there is c such that $Tac : A \in \Delta$ and $Tcb : B \in \Delta$. By the inductive hypothesis we can easily get $ab \models A \bullet B$. □

Lemmas 4.1.3 and 4.2.1 enable one to claim, that if a sequent

$$ac_1 : A_1,\ c_1c_2 : A_2,\ \ldots, c_{n-1}b : A_n \Rightarrow ab : A$$

is not derivable in **LLC** and the corresponding $T - F$ set

$$\Delta_0 = \{Tac_1 : A_1,\ Tc_1c_2 : A_2\ , \ldots,\ Tc_{n-1}b : A_n,\ Fab : A\}$$

can be extended to a consistent, complete, saturated, closed $T - F$ set, then there is a model where $A_1,\ A_2, \ldots,\ A_n \Rightarrow A$ is falsified.

To describe how Δ_0 can be extended to a consistent, complete, saturated, closed $T - F$ set we need the following definition:

- The $T - F$ set Δ is deeply consistent (d.c.) iff, if $\gamma_1, \ldots, \gamma_k$ are T-members of Δ and the sequent $\gamma_1, \ldots, \gamma_k \Rightarrow \gamma$ is derivable in **LLC**, then $F\gamma$ does not belong to Δ.

Remark

If Δ is not closed, then consistency and deep consistency do not coincide: the second notion is stronger then the first one.

4.2.2. CLAIM. If a sequent $\gamma_1, \ldots, \gamma_k \Rightarrow \gamma$ is not derivable in **LLC** and the corresponding sequence of labels is a canonical one, then the $T - F$ set $\Delta_0 = \{T\gamma_1, \ldots, T\gamma_k, F\gamma\}$ is deeply consistent.

Proof

Since γ is the only F-formula that belongs to Δ_0, the fact that Δ_0 is not d.c. means that there is $m < k$ such that $\gamma_1, \ldots, \gamma_m \Rightarrow \gamma$ is derivable in **LLC**. Then the second label of γ is equal to the second label of γ_m, but if m is different from k that is not possible, because the sequence of labels that corresponds to $\gamma_1, \ldots, \gamma_k \Rightarrow \gamma$ is a canonical one. □

In order to show how any d.c. set can be extended to a saturated d.c. set we are going to describe the method of saturation with Henkin witnesses.

By adding Henkin witnesses we mean the following procedure.
Suppose Δ is a $T - F$ set and V_Δ is a set of all labels that occur in Δ.
(i). If $Fab : A \to B \in \Delta$, then add a new label c to V_Δ and new labelled formulas $Tca : A$ and $Fcb : B$ to Δ.
(ii). If $Fab : A \leftarrow B \in \Delta$, the add a new label c to V_Δ and new labelled formulas $Tbc : A$ and $Fac : B$ to Δ.
(iii). If $Tab : A \bullet B \in \Delta$, then add a new label c to V_Δ and new labelled formulas $Tac : A$ and $Tcb : B$ to Δ.

4.2.3. LEMMA. Adding Henkin witnesses does not disturb deep consistency.

Proof
The first case:$Fab : A \to B \in \Delta$
Suppose Δ is d.c. set and $Fab : A \to B \in \Delta$. Let us admit that adding a new label c to V_Δ and new formulas $Tca : A$ and $Fcb : B$ we get a set $\Delta + Tca : A + Fcb : B$ (here and below by $\Delta + \alpha$ we mean $\Delta \cup \{\alpha\}$), which is not deeply consistent. That means that there are $\gamma_1, \ldots, \gamma_k, \gamma$ such that

(i) $\gamma_1, \ldots, \gamma_k$ are T-members of $\Delta + ca : A$
(ii) the sequent $\gamma_1, \ldots, \gamma_k \Rightarrow \gamma$ is derivable in LLC
(iii) $F\gamma \in \Delta + Fcb : B$

If there is no $ca : A$ among $\gamma_1, \ldots, \gamma_k$ in (ii), then according to the properties of labels there is no c in the label part of γ since $c \notin V_\Delta$. That means $F\gamma \in \Delta$. (see (iii)) and Δ is not deeply consistent.
If there is $ca : A$ among $\gamma_1, \ldots, \gamma_k$ in (ii), then the only possibility is that γ_1 is $ca : A$, otherwise we could find a formula among $\gamma_1, \ldots, \gamma_k$ with c as the second label, but that is impossible since $Tca : A$ is the only member of $\Delta + Tca : A$ which contains c. But if γ_1 is $ca : A$, then according to the property of labels the first label of γ is c. That means that γ is in fact $cb : B$ because that is the only F member of $\Delta + Tca : A + Fcb : B$ which contains c.
To sum it all up, (ii) has a form

$$ca : A,\ \gamma_2,\ \ldots,\ \gamma_k \Rightarrow cb : B$$

Since c is fresh

$$\gamma_2, \ldots, \gamma_k \Rightarrow ab : A \to B$$

is also derivable
Altogether (i),(iii) in conjunction with our assumptions that Δ is a d.c. set and $Fab : A \to B \in \Delta$ infer a contradiction.

The second case: $Fab : A \leftarrow B \in \Delta$
By analogous argument.

The third case:$Tab : A \bullet B \in \Delta$
Suppose Δ is a d.c. set and $Tab : A \bullet B \in \Delta$, but adding Henkin witnesses does disturb deep consistency and the resulting set $\Delta + Tac : A + Tcb : B$ is not d.c. That means that there are $\gamma_1, \ldots, \gamma_k, \gamma$ such that

(i) $\gamma_1, \ldots, \gamma_k$ are T-members of $\Delta + Tac : A + Tcb : B$
(ii) the sequent $\gamma_1, \ldots, \gamma_k \Rightarrow \gamma$ is derivable in LLC
(iii) $F\gamma \in \Delta + Tac : A + Tcb : B$, and therefore $F\gamma \in \Delta$

We get four possibilities (a)-(d) to consider:

(a) Neither $ac : A$, nor $cb : B$ are among $\gamma_1, \ldots, \gamma_k$ in (ii). From (i)-(iii) immediately conclude that Δ is not d.c.
(b) $ac : A$ is among $\gamma_1, \ldots, \gamma_k$ but $cb : B$ is not. Actually this is not possible; if it were so, then the second label of γ would be c (see (ii)), but that contradicts (iii), because $c \notin V_\Delta$.
(c) likewise, it is not possible that $cb : B$ is among $\gamma_1, \ldots, \gamma_k$ but $ac : A$ is not .
(d) both ac:A and cb:B are among $\gamma_1, \ldots, \gamma_k$. Note that this is possible only if ac:A and cb:B are neighbors in $\gamma_1, \ldots, \gamma_k$. First, let us assume that the pair $\langle ab : A, bc : B\rangle$ occurs only once in $\gamma_1, \ldots, \gamma_k$ Then $\gamma_1, \ldots, \gamma_k \Rightarrow \gamma$ has a form

$$\gamma_1, \ldots, \gamma_i, \ ac : A, cb : B, \ \gamma_{i+2}, \ldots, \gamma_k \Rightarrow \gamma,$$

therefore the sequent

$$\gamma_1, \ldots, \gamma_i, \ ab : A \bullet B, \ \gamma_{i+2}, \ldots, \gamma_k \Rightarrow \gamma$$

is derivable, and from (i) and (iii) we find that Δ is not deeply consistent. If $\langle ac : A, cb : B\rangle$ occurs more then once in $\gamma_1, \ldots, \gamma_k$, then c has to be renamed once which is possible due to the Renaming Lemma.

□

It remains to demonstrate that any d.c. set can be extended to some complete d.c. set.

4.2.4. LEMMA. Let Δ be a deeply consistent set and α an arbitrary labelled formula such that $\Delta + T\alpha$ is not deeply consistent. Then $\Delta + F\alpha$ is deeply consistent.

Proof

1. Suppose $\Delta + T\alpha$ is not d.c.
2. Suppose $\Delta + F\alpha$ is not d.c.
3. From (1) there are $\beta_1, \ldots, \beta_k, \beta$ such that

   (3.1.) $\beta_1, \ldots, \beta_k$ are T-members of $\Delta + T\alpha$,
   (3.2.) the sequent $\beta_1, \ldots, \beta_{i-1}, \ \alpha, \ \beta_{i+1}, \ldots, \beta_k \Rightarrow \beta$ is derivable,
   (3.3.) $F\beta \in \Delta + T\alpha$ and therefore $F\beta \in \Delta$.

   Note, that α has to occur in $\beta_1, \ldots, \beta_k$, otherwise Δ would not be deeply consistent.
4. From (2) there are $\psi_1, \ldots, \psi_n, \psi$ such that

   (4.1.) $\psi_1, \ldots, \psi_n$ are T-members of $\Delta + F\alpha$, and therefore of Δ
   (4.2.) the sequent $\psi_1, \ldots, \psi_n \Rightarrow \psi$ is derivable,
   (4.3.) $F\psi \in \Delta + F\alpha$
5. Note, that ψ is actually α, otherwise by (4.3.) $F\psi \in \Delta$, and by (4.1.) and (4.2.) Δ is not deeply consistent.
   That means that (4.2.) has actually a form $\psi_1, \ldots, \psi_n \Rightarrow \alpha$.
6. Next, we can apply of the Cut rule to (5) and (3.2.) as many times as necessary until we are able to conclude that there is a sequence X of T-members of Δ such that $X \Rightarrow \beta$ is derivable. Clearly, by (3.3.) Δ is not deeply consistent then.

$\square$

Now we are ready to prove the main lemma.

4.2.5. LEMMA. Every deeply consistent set can be extended to some complete, saturated, consistent, closed $T - F$ set.

Proof

Suppose Δ_0 is a d.c. set.
Let $\langle \alpha_i / i \in N \rangle$be some usual enumeration of labelled formulas.
Define a sequence of $T - F$ sets $\langle X_i / i \in N \rangle$ by setting
$X_0 = \Delta_0$ and
X_{n+1} is defined as follows:

- If $X_n + T\alpha_n$ is d.c., then to obtain X_{n+1} add $T\alpha_n$ and its Henkin witnesses (if they exist) to X_n.
- If $X_n + T\alpha_n$ is not d.c., then to obtain X_{n+1} add $F\alpha_n$ to X_n and its Henkin witnesses (if they exist).

Lemmas 3,4 infer that if X_n is d.c., X_{n+1} is also d.c. Since X_0 is d.c., each $X_i (i \in N)$ is d.c..Let $X = \cup X_i$ where $i \in N$. Clearly, X is *complete* and *saturated.* Next, X is also *consistent*, otherwise there would be k such that X_k is not d.c. To show that X is closed, assume that $\alpha_1, \ldots, \alpha_k$ are T members of X and $\alpha_1, \ldots, \alpha_k \Rightarrow \alpha$ is a derivable sequent. If $T\alpha \notin X$ then by completeness of Δ find $F\alpha \in X$. That means that X is not d.c. and therefore is not consistent. Thus $T\alpha \in X$ and that shows that X is closed. □

4.2.6. THEOREM. *Weak Completeness Theorem*
If $A_1, \ldots, A_n \models A$ $(0 \leq n)$, then the sequent $A_1, \ldots, A_n \Rightarrow A$ is derivable in the Lambek Calculus.

Proof
Suppose $A_1, \ldots, A_n \Rightarrow A$ is not derivable in non-labelled version of the Lambek Calculus. If $0 < n$, then according to Lemma 4.1.3 there is a canonical sequence of labels

$$\langle a, c1_, ..., c_n, b \rangle$$

such that

$$ac_1 : A_1,\ c_1c_2 : A_2,\ \ldots, c_{n-1}b : A_n \Rightarrow ab : A$$

is not derivable in LLC That means that the corresponding $T - F$ set

$$\Delta_0 = \{Tac_1 : A_1,\ Tc_1c_2 : A_2\ , \ldots,\ Tc_{n-1}b : A_n,\ Fab : A\}$$

is deeply consistent . According to the main lemma , Δ_0 can be extended to some consistent, complete, saturated, closed $T - F$ set Δ. By lemma 4.2.1 we can claim the existence of a model where $xy \models \alpha$ iff $Txy : \alpha \in \Delta$. Thus $A_1, \ldots, A_n \not\models A$: a contradiction. If a sequence $\Rightarrow A$ is not derivable in the non-labelled version of the Lambek Calculus, then the $T - F$ set $\{Faa : A\}$ is deeply consistent and therefore $\Rightarrow A$ can not be semantically valid. □

4.3 Strong Completeness of the Lambek Calculus with Respect to Relational semantics.

As it was mentioned already, the Lambek Calculus can be considered as an "axiomatic-sequent" calculus with the notion of derivability of a sequent from a set of sequents.

- Let ϕ be a sequent and Γ be a set of sequents. Γ *infers* ϕ iff there is a sequence of sequents $\delta_1, \ldots, \delta_n$ such that each δ_i is either an axiom of LC, or $\delta_i \in \Gamma$, or δ_i can be obtained from $\delta_1, \ldots, \delta_{i-1}$ by inference rules of LC and δ_n is ϕ.
- We say, that Γ *semantically entails* ϕ iff in every model where all members of Γ are true, ϕ is also true.

- The Lambek calculus is *strongly complete* with respect to the Relational semantics iff Γ infers ϕ whenever Γ semantically entails ϕ

The Strong Completeness Theorem will be proved by contraposition, here are our main assumptions: Let $\Gamma = \{\phi_1, \ldots, \phi_n\}$ We suppose that

- Γ semantically entails ϕ ;
- Γ does not infer ϕ

Adding new axioms

Define two new systems: LC_Γ and LLC_Γ LC_Γ can be obtained from LC by adding $\phi_1, \ldots, \phi_n$ as axioms. To get LLC_Γ which is the labelled version the labelled version of LC_Γ add all labelled versions of $\phi_1, \ldots, \phi_n$ to LLC. By the labelled version of $A_1, \ldots, A_n \Rightarrow A$ we mean the any labelled sequent

$$xy : A_1 \bullet A_2 \bullet \ldots \bullet A_n \Rightarrow xy : A$$

Thus each non-labelled sequent has infinitely many labelled versions.

To obtain strong completeness from weak completeness we have to reformulate the notions of closed and d.c. set, namely replace derivability in LLC by derivability in LLC_Γ in their definitions .

- $T-F$ set Δ is *deeply consistent* (d.c.) iff for any $\alpha_1, \ldots, \alpha_n, \alpha$ if $\alpha_1, \ldots, \alpha_n$ are T-members of Δ and $\alpha_1, \ldots, \alpha_n \Rightarrow \alpha$ is derivable in LLC_Γ, then $F\alpha \notin \Delta$.
- $T-F$ set Δ is *closed* iff for any $\alpha_1, \ldots, \alpha_n, \alpha$ if $\alpha_1, \ldots, \alpha_n$ are T-members of Δ and the sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \alpha$ is derivable, then $T\alpha \in \Delta$.

It is not difficult to see that under these new definitions Lemmas 4.2.1, 4.2.3, 4.2.4 are still valid.

4.3.1. THEOREM. *Strong Completeness Theorem*
If a sequence of sequents Γ semantically entails a sequent ϕ, then Γ infers ϕ .

Proof

1. Suppose Γ semantically entails ϕ, where $\Gamma = \{\phi_1, \ldots, \phi_m\}$ and ϕ is $A_1, \ldots, A_n \Rightarrow A$ with $(0 \leq n, m)$
2. Suppose Γ does not infer ϕ.
3. That means that $A_1, \ldots, A_n \Rightarrow A$ is not derivable in the Lambek Calculus, otherwise it were deducible from Γ in LC_Γ.
4. According to Lemma 4.2.1 then there exists a construction of a consistent, complete, saturated, closed $T-F$ set Δ and a model M which falsifies ϕ with the following remarkable property: $Txy : \alpha \in \Delta$ iff $xy \models \alpha$.
5. Let us show that each $\phi_i \in \Gamma$ is true in M.

(a) . If ϕ_i is of the form $B_1, \ldots, B_n \Rightarrow B$ with $k > 0$ and $M \not\models \phi_i$, then there exist a, b such that $ab \models B_1 \bullet \ldots \bullet B_k$, but $ab \not\models B$. Hence $Tab : B_1 \bullet \ldots \bullet B_k \in \Delta$, but $Tab : B \notin \Delta$, which is not possible since $ab : B_1 \bullet \ldots \bullet B_k \Rightarrow ab : B$ is an axiom of LLC_G, and Δ is closed and consistent.

(b) Likewise if ϕ_i is of the form $\Rightarrow B$ and $M \not\models \phi_i$, then there exists a point c with $cc \not\models B$. Therefore $Tcc : B \notin \Delta$, but that is not possible since $\Rightarrow cc : B$ is an axiom of LLC_G.

□

4.4 Concluding remarks and open questions

The technique of labelling might provide a rather general method of completeness proof, since the Lambek Calculus can have different labelled versions which are related to different kinds of semantics. In the next chapter we are going to present a slightly different method of completeness proof using another version of the method of labelled deduction. The advantages and the boundaries of that method could become more clear if the following open questions can be answered:

1. To what degree the procedure of adding Henkin witnesses depends on a particular kind of semantics?
2. The completeness proof described above provides the construction of the infinite canonical model. It is still an open question if the Lambek calculus with Associativity has the finite model property. Can the labelling technique help?
3. What is the precise effect of adding structural rules to LLC?
4. H.Andréka and S.Mikulás showed that one can not have a completeness result for the Lambek Calculus enriched by disjunction with respect to Relational Semantics. What about adding other connectives and modal operators?

Chapter 5

Labelled Categorial Deduction

5.1 Labelling for categorial type systems

In the previous chapter we have studied a particular logic in the categorial hierarchy - the associative Lambek calculus - and we have developed a labelled deductive approach to obtain completeness for the "dynamic" relational semantics. In the present chapter we adapt and generalize our methods in order to offer a labelling perspective on categorial type logics as they are currently used in linguistic description. An important theme in current categorial research is the shift of emphasis from individual type logics to communicating families of such systems. The reason for this shift is that the individual logics are not expressive enough for realistic grammar development: the grammar writer needs access to the combined inferential capacities of a family of logics. See [Morrill 94], [Moortgat 94] for discussion and motivation. In line with these developments, our main objective will be to develop a uniform labelling discipline for the family of resource logics **NL, L, NLP** and **LP** and a number of generalizations that will be discussed in depth in Chapter 6.

Let us situate our approach with respect to related work, before starting with the technicalities. The technique of labelling has been used in the categorial literature before, for various reasons. [Buszkowski 86] used labelling as an auxiliary device to obtain his completeness results for the Lambek calculus. V. Sanchez in [Sanchez 90] relied on labelling for semantic reasons in his work on a categorially-driven theory of natural language reasoning. [Moortgat 91] added string labelling to type formulas for syntactic reasons, viz. to overcome the expressive limitations of the standard sequent language in capturing discontinuous forms of linguistic composition. In a proof theoretic study of categorial logics, [Roorda 91] introduced labelling to enforce the well-formedness conditions on proof nets. The programmatic introduction of Labelled Deductive Systems as a general framework for the study of structure sensitive consequence relations in [Gabbay 92] made it possible to re-evaluate these scattered earlier proposals. From 1991 on, labelling has been on the categorial agenda on a more systematic level. For recent studies, we refer to [Morrill 94], [Oehrle 94], [Hepple 94], [Venema 94], to mention just a few.

Consider a consequence relation

$$A_1, \ldots, A_n \Rightarrow B$$

representing the fact that a conclusion B can be derived from a database of assumptions $A_1, \ldots, A_n$. As we have seen in the previous chapter, the central idea of the labelled deductive approach is to replace the formula as the basic declarative unit by a pair $x : A$, consisting of a label x and a formula A. Sequents then assume the form

$$x_1 : A_1, \ldots, x_n : A_n \Rightarrow y : B$$

The label is to be thought of as an extra piece of information added to the formula. Rules of inference manipulate not just the formula, but the formula plus its label. We then obtain a whole scala of labelling regimes depending on how we make precise the intuitive notion of an extra piece of information added to a formula - depending on the degree of autonomy between the formula and the label.

At the conservative end of the spectrum one can find semantic lambda term labelling in the sense of the Curry-Howard correspondence. The labels, in this application, simply record the history of the proof - they do not make an independent contribution. At the other end of the spectrum are the labelling systems where in the declarative unit $x : A$ the label x and the formula A each make their own irreducible contribution. Such systems can best be seen as combinations of two logics: the formula logic and the logic governing the labelling algebra.

Our proposals are somewhere in between these two extremes. In the application to linguistic reasoning which is the subject of this chapter, a consequence relation represents a grammaticality judgement: the derivability of an expression of type B from a database of assumptions, i.e. expressions of type $A_1, \ldots, A_n$. Derivability, in this linguistic sense, has to take into account the structure of the assumptions - for example: their linear order, and the way they are grouped into hierarchical units or constituents. We rely on the labelling algebra to make the structure of our linguistic database explicit. In Chapter 1, we have seen how one obtains the systems **L, NLP, LP** from the pure residuation logic **NL** by removing order sensitivity, constituent structure sensitivity or both. In order to develop uniform algorithmic proof theory for this family of type logics we start from the multiset sequent presentation of their common denominator, the Lambek-Van Benthem system **LP**, and impose t he extra syntactic fine-tuning in terms of the labelling regime.

Labelling, in this sense, is not uncommon in current categorial work, see for example [Morrill 94] for descriptive and computational applications. But the labelling systems in use are related to groupoid interpretation, not to the more general ternary frame semantics which is the subject of this thesis. From the perspective of ternary frame semantics, the groupoid models are obtained by specializing the accessibility relation Rx, yz to $x = y + z$, where the binary operation $+$ imposes functionality. In developing labelling for the abstract ternary frame semantics, we implement the labelling discipline in such a way that we can manipulate the three components of the triples $(xyz) \in R$, rather than assuming that x is determined by $y + z$.

Let us finally insist here on the importance of the completeness result for the labelled calculus. Labels, just like formulas, are pieces of syntax. Even if the rules of a labelled system *look* very much *like* imitating truth conditions we will never have a guarantee that the labelling is really appropriate for the intended interpretation unless we present a completeness proof, which requires an explicit statement of the relation between the labels and the objects of the domain of interpretation. That there are real issues here is shown in the discussion of [Venema 94].

Outline of the chapter

The chapter is organized as follows. In Section 5.2 we study abstract labelling. We introduce uniform labelled sequent presentation for the logics **NL, NLP, L, LP** and prove completeness with respect to the ternary frame semantics for these logics. Labelled sequent calculus is presented in two formats which we show to be equivalent. The first format decorates formulas with atomic labels and enriches the sequent language with an explicit book-keeping component to record the structure of the database. The second format introduces a language of tree-terms over the atomic labels, and formulates the labelling discipline as a term assignment system. To close the section, we discuss a number of generalizations, showing how our results can be applied to multimodal architectures, and to logics extended with one-place multiplicative operators. Section 5.3 then moves to concrete labelling. We use the labelling algebra to add sortal refinements to the type formulas: the sort labels filter out theorems that would b e derivable in an unlabelled setting. In Section 5.4. we discuss some connections between labelling and Correspondence Theory, linking the material of the present chapter with the themes of Chapters 1 and 2.

5.2 Ternary frame semantics and labelling

5.2.1 Labelled sequent presentation

Let us introduce the labelled sequent presentation informally before starting with the definitions. The new declarative unit, as we said before, now is the labelled formula $x : A$, rather than the bare formula A. What kind of labels do we want to use, and how do we want to interpret them? In our first version of the labelled calculus the labels are all taken from some set of *atomic* markers. In a sense to be made precise below, the labels refer to elements of the domain of interpretation W. In order to keep track of the way the labels are configured into a structured database, we add an explicit book-keeping component to the sequent language. Labelled sequents then will be of the general form

$$[\delta];\ a_1 : A_1, a_2 : A_2, \ldots, a_n : A_n \Rightarrow a : A,$$

where each label $a_i (1 \leq i \leq n)$ is a witness of a piece of information attached to A_i and d fixes the configuration of the labels in a tree such that the succedent label a is its root and the antecedent labels a_i the leaves. Because the configuration is fixed in δ the sequent antecedent can be treated simply as a *multiset* of labelled formulae. Now for the definitions.

5.2.1. DEFINITION. *Labels and trees.*
Let **Lab** be a set of atomic labels. We define by simultaneous induction the set of trees **T** over **Lab** and functions

$$\text{root}: \mathbf{T} \to \mathbf{Lab}$$
$$\text{leaves}: \mathbf{T} \to P(\mathbf{Lab})$$
$$\text{nodes}: \mathbf{T} \to P(\mathbf{Lab})$$

If $x \in \mathbf{Lab}$, then x is a tree such that

$$\text{leaves}(x) = \{x\}$$
$$\text{root}(x) = x$$
$$\text{nodes}(x) = \{x\}$$

If $\langle a, bc\rangle \in \mathbf{Lab}^3$ and a, b, c are distinct,
then $\langle a, bc\rangle$ is a tree such that

$$\text{root}(\langle a, bc\rangle) = a$$
$$\text{leaves}(\langle a, bc\rangle) = \{b, c\}$$
$$\text{nodes}(\langle a, bc\rangle) = \{a, b, c\}$$

If δ_1 and δ_2 are trees such that

[i] $\text{root}(\delta_2) \in \text{leaves}(\delta_1)$ and
[ii] $\text{nodes}(\delta_1) \cap \text{nodes}(\delta_2)=\{\text{root}(\delta_2)\}$

then $\xi=(\delta_1\delta_2)$ is a tree and

root(ξ)=root(δ_1)
leaves(ξ)=(leaves(δ_1) $\cup$ leaves(δ_2))-{root(δ_2)}
nodes(ξ)=nodes(δ_1) $\cup$ nodes(δ_2)

Example
$\xi=((\langle a, bc\rangle\langle c, de\rangle)\langle d, kn\rangle)$ is a tree with
root(ξ)=a
leaves(ξ)=$\{b, k, n, e\}$
nodes(ξ)=$\{a, b, c, d, e, k, n\}$

Thus, each tree turns out to be a bracketed string of triangles. As usual we often drop the most external pair of brackets. The size (l) of a tree can be defined as follows:

$l(a) = 0$
$l(\langle a, bc\rangle) = 1$
$l(\xi\chi) = l(\xi) + l(\chi)$

Now we can give precise definitions of a labelled formula and a labelled sequent.

- $a : A$ is a labelled formula if A is a formula and $a \in$ **Lab**;
- $[\delta];\ a_1 : A_1, a_2 : A_2, \ldots, a_n : A_n \Rightarrow a : A$ is a labelled sequent if $a : A, a_i : A_i (1 \leq i \leq n)$ are labelled formulas, and $\delta \in \mathbf{T}$ with
  - root(δ)=a
  - leaves(δ)=$\{a_1 \ldots a_n\}$

Note that in a labelled sequent

$$[\delta];\ a_1 : A_1, a_2 : A_2, \ldots, a_n : A_n \Rightarrow a : A$$

$a_1, \ldots, a_n, a$ are distinct since according to the definition all nodes of δ are distinct. Moreover the presence of δ is not conservative, since the rules of the labelled Lambek calculus include not only manipulations with formulas and labels, but also with trees.

Labelled sequent calculus
Let α be a labelled formula, X, Y, Z finite multisets of labelled formulas, and let δ_1, δ_2, be members of $\mathbf{T}$.
$\mathbf{NL}^{lab}$ has one axiom

$$[a];\ a : A\ \Rightarrow\ a : A$$

and the following inference rules provided that all involved sequents are well-defined:

$$\frac{[\delta_1];\ X \Rightarrow b : A \qquad [\delta_2];\ c : B, Y \Rightarrow \alpha}{[(\delta_2\langle c, ba\rangle)\delta_1];\ a : A \rightarrow B,\ X, Y \Rightarrow \alpha}$$

$$\frac{[\langle c, ba\rangle d]; b : A, X \Rightarrow c : B}{[\delta];\ X \Rightarrow a : A \rightarrow B}$$

$$\frac{[\delta_1];\ X \Rightarrow b : A \qquad [\delta_2];\ c : B, Y \Rightarrow \alpha}{[(\delta_2\langle c, ab\rangle)\delta_1];\ a : A \leftarrow B, X, Y \Rightarrow\ \alpha}$$

$$\frac{[\langle c, ab\rangle d];\ X, b : A \Rightarrow\ c : B}{[\delta];\ X \Rightarrow a : B \leftarrow A}$$

$$\frac{[d\langle a, bc\rangle];\ b : A, c : B, X \Rightarrow \alpha}{[\delta];\ a : A \bullet B, X \Rightarrow \alpha}$$

$$\frac{[\delta_1];\ X_1 \Rightarrow b : A \quad [\delta_2]; X_2\ \Rightarrow\ c : B}{[(\langle a, bc\rangle\delta_1)\delta_2];\ X_1, X_2 \Rightarrow a : A \bullet B}$$

$\mathbf{L}^{\mathbf{lab}}$ can be obtained from $\mathbf{NL}^{\mathbf{lab}}$ by adding the Associativity Rule

$$\frac{[\delta];\ X \Rightarrow \alpha}{[\delta'];\ X \underset{\bullet}{\Rightarrow} \alpha}$$

where δ' is obtained from δ by replacing some subtree $(\langle a, bc\rangle\langle c, de\rangle)$ by a tree $(\langle a, te\rangle\langle t, bd\rangle)$ or vice versa provided that t (resp. c) is fresh.

$\mathbf{NLP}^{\mathbf{lab}}$ can be obtained from $\mathbf{NL}^{\mathbf{lab}}$ by adding the Permutation Rule

$$\frac{[\delta];\ X \Rightarrow \alpha}{[\delta'];\ X \Rightarrow \alpha}$$

where δ' is obtained from δ by replacing some subtree $(\langle a, bc\rangle)$ by a tree $(\langle a, cb\rangle)$.

Finally, $\mathbf{LP}^{\mathbf{lab}}$ is obtained from $\mathbf{NL}^{\mathbf{lab}}$ by adding both Associativity and Permutation.

The following sequents give an example of theorems of $\mathbf{NL}^{\mathbf{lab}}$.
[i] $[\{\langle x, bc\rangle, \langle d, ax\rangle\}]; a : r \leftarrow q, b : p, c : p \rightarrow q \Rightarrow d : r$
[ii] $[\{\langle x, bc\rangle, \langle n, ax\rangle\}]; a : r \leftarrow q, b : p, c : p \rightarrow q \Rightarrow n : r$

Note that the derivation of [ii] can be obtained from the derivation of [i] by renaming d for n.

5.2.2. LEMMA. *Renaming Lemma*
This Lemma consists of two claims.

Claim 1.
If a sequent $[\delta];\ x_1 : X_1, x_2 : X_2, \ldots, x_n : X_n \Rightarrow x : X$ is derivable, then $[\delta']; x_1 : X_1, \ldots, \mathbf{y} : X_i, \ldots, x_n : X_n \Rightarrow x : X$ where δ' is obtained from δ by replacing x_i by $\mathbf{y}$ is also derivable.

Claim 2.
If a sequent $[\delta];\ x_1 : X_1, x_2 : X_2, \ldots, x_n : X_n \Rightarrow \mathbf{x} : X$ is derivable then $[\delta];\ x_1 : X_1, x_2 : X_2, \ldots, x_n : X_n \Rightarrow \mathbf{y} : X$ where δ' is obtained from δ by replacing x by $\mathbf{y}$ is also derivable.

Both Claims can be proved by straightforward induction on the length of the derivation. □

Interpreting labelled sequents

To obtain an interpretation for our labelled sequents we add to a ternary model $\langle W, R, V\rangle$ a function * which assigns exactly one element of W to each $a \in$ **Lab**.
A sequent $[\delta];\ a_1 : A_1, a_2 : A_2, \ldots, a_n : A_n \Rightarrow a : A$ is true in M if $a^* \models A$ whenever $a_i^* \models A_i (1 \leq i \leq n)$ and for each triangle $\langle x, yz\rangle$ which occurs in δ, Rx*,y*z* . A sequent ϕ is semantically valid if it is true in all models.
Note that although on the syntactic level the labels on the nodes of each triangle are distinct, we do not impose such distinctness as a semantic requirement. Since * is an arbitrary function it might very well be that in some models the same elements of the domain are assigned to distinct node labels of the syntactic tree. Thus on the semantic level the definition of a ternary frame realizes an *arbitrary* ternary accessibility relation as required.

As usual it is easy to prove soundness, i.e. each sequent derivable in $\mathbf{NL}^{\mathbf{lab}}$ is semantically valid. Completeness is the subject of the following section.

5.2.2 Completeness for $\mathbf{NL}^{\mathbf{lab}}$

The general idea behind the completeness proof can be expressed as follows:

1. Suppose $[\delta];\ a_1 : A_1, a_2 : A_2, \ldots, a_n : A_n \Rightarrow a : A$ is not derivable in $\mathbf{NL}^{\mathbf{lab}}$.
2. Mark all labelled formulas on the left hand side with T and $a : A$ with F. The resulting T-F set is

$$\Delta_0 = \{Ta_1 : A_1, \ldots, Ta_n : A_n, Fa : A\}.$$

Construct a model such that each a_i^* supports A_i, but a^* does not support A. In other words, extend Δ_0 to Δ and prove that $x^* \models X$ iff $Tx : X \in \Delta$.

To realize this idea of the completeness proof we need to identify some properties of $T - F$ sets, i.e sets of labelled formulas marked with T or F.

Properties of T-F sets
Let Δ be a $T - F$ set, V_Δ the set of all labels that occur in Δ, Δ_R the set of all triangles associated with Δ, and $\delta \in \mathbf{T}$ (i.e. δ is a tree).
We loosely say that $\delta \subseteq \Delta_R$ if all triangles that occur in δ are members of Δ_R.

- Δ is *deeply consistent* (d.c.) iff whenever $\delta \subseteq \Delta_R$, $\gamma_1, \ldots, \gamma_k$ are T-members of Δ and $[\delta]; \gamma_1, \ldots, \gamma_k \Rightarrow \gamma$ is derivable, then $F\gamma \notin \Delta$.
- Δ is *h-complete* (Henkin complete) iff
  (i) if $Fa : A \to B \in \Delta$, then there are $x, y \in V_\Delta$ such that $\langle y, xa \rangle \in \Delta_R$, $Tx : A \in \Delta$ and $Fy : B \in \Delta$;
  (ii) if $Fa : A \leftarrow B \in \Delta$, then there are $x, y \in V_\Delta$ such that $\langle y, ax \rangle \in \Delta_R$, $Tx : A \in \Delta$ and $Fy : B \in \Delta$;
  (iii) if $Ta : A \bullet B \in \Delta$, then there are $x, y \in V_\Delta$ such that $\langle a, xy \rangle \in \Delta_R$, $Tx : A \in \Delta$ and $Ty : B \in \Delta$.
- Δ is *r-complete* (relatively complete) iff
  (i) if $Fa : A \bullet B \in \Delta$ and $\langle a, xy \rangle \in \Delta_R$, then either $Fx : A \in \Delta$ or $Fy : B \in \Delta$;
  (ii) if $Ta : A \to B \in \Delta$ and $\langle y, xa \rangle \in \Delta_R$, then either $Fx : A \in \Delta$ or $Ty : B \in \Delta$;
  (iii) if $Ta : A \leftarrow B \in \Delta$ and $\langle y, ax \rangle \in \Delta_R$, then either $Fx : A \in \Delta$ or $Ty : B \in \Delta$.
- Δ is *nice* iff it is d.c., h-complete, r-complete
- Let Δ be a T-F set, $x \in V_\Delta$ and A be a non-labelled formula. We say that A and x are *linked* in Δ iff $Tx : A \in \Delta$ or $Fx : A \in \Delta$.

Henkin model

5.2.3. LEMMA. If Δ is a nice $T - F$ set, then there exists a model $M' = \langle W, R, ^*, V \rangle$ such that if $x \in V_\Delta$ and X are linked in Δ, then $Tx : X \in \Delta$ iff $x^* \models X$ in M'.

Proof
Define M' as follows:
$W = V_\Delta$ (the set of all labels that occur in Δ);

Rx, yz iff $\langle x, yz \rangle \in \Delta_R$.

for all $x \in V_\Delta$, $x^* = x$

V is such that $Tx : p \in \Delta$ iff $x^* \models p$ in M'.

The lemma is proved by induction on the length of X. In the atomic case the claim is a direct consequence of the definition. We have to take care of three cases when proving the inductive step.

The first case: $X = A \bullet B$.

1. Suppose $Ta : A \bullet B \in \Delta$.
2. Since Δ is h-complete there are $x, y \in V_\Delta$ such that $\langle a, xy \rangle \in \Delta_R$, $Tx : A \in \Delta$ and $Ty : B \in \Delta$.
3. Therefore $Ra, xy; x \models A$ and $y \models B$ (by inductive assumption) and $a \models A \bullet B$
4. Thus $a^* \models A \bullet B$

1. Suppose $a^* \models A \bullet B$ and therefore $a \models A \bullet B$.
2. Then there are $x, y \in W$ such that $Ra, xy; x \models A$ and $y \models B$.
3. Since a and $A \bullet B$ are linked in Δ, either $Ta : A \bullet B \in \Delta$ or $Fa : A \bullet B \in \Delta$. Let us suppose that $Fa : A \bullet B \in \Delta$.
4. Since Δ is r-complete, $Fa : A \bullet B \in \Delta$ and $\langle a, xy \rangle \in \Delta_R$ (because Ra, xy) imply that either $Fx : A \in \Delta$ or $Fy : B \in \Delta$; in both cases we get a contradiction with [2.].
   Thus $Ta : A \bullet B \in \Delta$.

The second case: $X = A \rightarrow B$

1. Suppose $Ta : A \rightarrow B \in \Delta$.
2. Suppose Rc, ba and $b \models A$
3. Then $\langle c, ba \rangle \in \Delta_R$ and $Tb : A \in \Delta$
4. Since Δ is r-complete, [1.] [2.] and [3.] imply $Tc : B \in \Delta$, and therefore by the inductive assumption $c \models B$
5. Hence, $a^* \models A \rightarrow B$

1. Suppose $a^* \models A \rightarrow B$
2. Suppose $Ta : A \rightarrow B \notin \Delta$. Then since a and $A \rightarrow B$ are linked in Δ, $Fa : A \rightarrow B \in \Delta$
3. By h-completeness of Δ, there are $b, c \in V_\Delta$ such that $Tb : A \in \Delta, Fc : B \in \Delta$ and $\langle c, ba \rangle \in \Delta_R$
4. By inductive assumption, get a contradiction.
   Therefore $Ta : A \rightarrow B \in \Delta$

The third case can be left to the reader □

Lemma 5.2.3 enables one to claim that if a sequent $u : A \Rightarrow u : B$ is not derivable and the corresponding $T - F$ set $\{Tu : A, Fu : B\}$ can be extended to a nice one, then there exists a model where $u : A \Rightarrow u : B$ is falsified. Now we have to define how to make the relevant extensions of d.c. sets. We start with the definition of saturation with h-witnesses and then r-witnesses.
Harmless witnesses

Formulas of the form $Fa : A \rightarrow B$, $Fa : A \leftarrow B$, $Ta : A \bullet B$ will be called *h-formulas* (Henkin formulas).

Let Δ be a d.c. set. By *adding h-witnesses* we refer to the following procedure:

(i) if $Fa : A \rightarrow B \in \Delta$, then add new labels b and c to V_Δ; $\langle c, ba \rangle$ to Δ_R and new labelled formulas $Tb : A$ and $Fc : B$ to Δ;
(ii) if $Fa : A \leftarrow B \in \Delta$, then add new labels b and c to V_Δ ; $\langle c, ab \rangle$ to Δ_R and new labelled formulas $Tb : A$ and $Fc : B$ to Δ;
(iii) if $Ta : A \bullet B \in \Delta$, then add new labels b and c to V_Δ; $\langle a, bc \rangle$ to Δ_R and new labelled formulas $Tb : A$ and $Tc : B$ to Δ.

We say that in (i)-(iii) the point a *generates* points b and c. The set of successors of a point x (Σ_x) is defined by (iv)-(v):

(iv) if x generates y, then $y \in \Sigma_x$;
(v) if $u \in \Sigma_x$ and u generates w, then $w \in \Sigma_x$.

To saturate some $T - F$ set Δ with r-witnesses perform (i)-(iii)
(i) if $Fa : A \bullet B \in \Delta$ and $\langle a, xy \rangle \in \Delta_R$, then add $Fx : A$ to Δ if it does not disturb d.c. of Δ, otherwise add $Fy : B$;
(ii) if $Ta : A \rightarrow B \in \Delta$ and $\langle y, xa \rangle \in \Delta_R$, then add $Fx : A$ to Δ if it does not disturb d.c. of Δ, otherwise add $Ty : B$;
(iii) if $Ta : A \leftarrow B \in \Delta$ and $\langle y, ax \rangle \in \Delta_R$, then add $Fx : A$ to Δ if it does not disturb d.c. of Δ, otherwise add $Ty : B$. Formulas of the form

$Fa : A \bullet B$, $Ta : A \rightarrow B$, $Ta : A \leftarrow B$ will be called *r-formulas.* We say that an r-formula $Fa : A \bullet B$ (resp. $Ta : A \rightarrow B, Ta : A \leftarrow B$) is *active in* Δ if there are $b, c \in V_\Delta$ such that $\langle a, bc \rangle \in \Delta_R$ (resp. $\langle c, ba \rangle \in \Delta_R$, $\langle c, ab \rangle \in \Delta_R$).

Let Δ be a $T - F$ set, V_Δ be a set of labels that occur in Δ and Δ_R be a set of triangles associated with Δ such that each member of Δ_R is a result of some h-decomposition. Then the following propositions hold.

5.2.4. Proposition. If $x \in V_\Delta$ and X is the set of all successors of x, then all members of X are distinct.

Proof
Direct consequence of the definition of adding h-witnesses. □

5.2.5. PROPOSITION. If $\delta \subseteq \Delta_R$, then δ is generated by a single point.

Proof
Straightforward induction on the size of δ.
Indeed, if $\delta = \langle a, bc \rangle$, then it is generated by a single point according to the definition of adding h-witnesses.

Claim
If $\delta = \xi_1 \xi_2$, and x is their common point, then either ξ_1 or ξ_2 is generated by x
Proof
If in both ξ_1 and ξ_2 x is a generated point, then x has to be generated twice: as a daughter and as a root, which is not possible, since by the definition of adding h-witnesses every point in Δ_R is uniquely generated. Thus, at least in one of this trees x is a generator.

Next, suppose $\delta = \xi_1 \xi_2$, x is their common point, and ξ_i is generated by x, while ξ_j $(i, j = 1, 2)$ is generated by y. Then either $x = y$ or x is a successor of y or vice versa. In all this cases $\delta = \xi_1 \xi_2$, is generated by a single point. □

5.2.6. PROPOSITION. Let $\delta_1, \delta_2 \subseteq \Delta_R$ with root $(\delta_2) \in$ leaves (δ_1).
Then $\delta_1 \delta_2 \subseteq \Delta_R$

Proof. We have to show that $\delta_1 \delta_2$ is a well defined tree, or in other words, that δ_1 and δ_2 have no other points in common besides x. Reasoning by analogy with the proof of the Proposition 5.2.5, conclude, that x generates δ_1 or δ_2 or both of them. If x generates δ_1 and y generates δ_2, then no matter if x is a successor of y or vice versa, the successor would always generate fresh points , therefore δ_1 and δ_2 can not share any points besides x. If x generates both δ_1 and δ_2, then in one case the set of successors would be generated by x as a root (and therefore, by a formula of the form $Ta : A \bullet B$) while the second set would be generated by x as a daughter (and therefore by formula of the form $Fa : A \rightarrow B$ or $Fa : A \leftarrow B$), thus two latter sets can not share any members. □

5.2.7. LEMMA. Let Δ be a d.c. set, and each member of Δ_R is a result of some h-decomposition. If β is an active r-formula in Δ, then there always exists an r-witness of β which can be added to Δ without disturbing its deep consistency.

Proof

The first case: $Fa : A \bullet B \in \Delta$

Let β be $Fa : A \bullet B$ and $\langle a, bc \rangle \in \Delta_R$. Suppose that neither $\Delta + Fb : A$, nor $\Delta + Fc : B$ is d.c. Then clearly

(1) there exist $\delta_1 \subseteq \Delta_R$ and T-members of Δ $\gamma_1, \ldots, \gamma_n$ such that

$$(*) \quad [\delta_1];\ \gamma_1, \ldots, \gamma_n \Rightarrow \gamma$$

is derivable in $\mathbf{NL}^{\mathbf{lab}}$ and γ is an F-member of $\Delta + Fb : A$. Since Δ is d.c., γ is nothing else but $b : A$;, and $(*)$ has actually the form

$$(*) \quad [\delta_1];\ \gamma_1, \ldots, \gamma_n \Rightarrow b : a$$

Thus, b is the root of δ_1

(2) there exist $\delta_2 \subseteq \Delta_R$ and T-members of Δ $\alpha_1, \ldots, \alpha_n$ such that

$$(**) \quad [\delta_2];\ \alpha_1, \ldots, \alpha_n \Rightarrow \alpha$$

is derivable in $\mathbf{NL}^{\mathbf{lab}}$ and α is an F-member of $\Delta + Fc : B$. Once again, since Δ is d.c., α is nothing else but $c : B$; whence

$$(**) \quad [\delta_2];\ \alpha_1, \ldots, \alpha_n \Rightarrow c : B$$

is derivable and therefore c is the root of δ_2

According to Proposition 5.2.6 $(\langle a, bc\rangle\delta_1)\delta_2$ is a well defined tree, thus by (1) and (2)

$$[(\langle a, bc\rangle\delta_1)\delta_2];\ \gamma_1, \ldots, \gamma_k, \alpha_1, \ldots, \alpha_n \Rightarrow A \bullet B$$

is derivable. Since $(\langle a, bc\rangle\delta_1)\delta_2 \subseteq \Delta_R$ and $\gamma_1, \ldots, \gamma_k, \alpha_1, \ldots, \alpha_n$ are T-members of Δ but $a : A \bullet B$ is an F-member of Δ, Δ can not be d.c..

The second case: $Ta : A \to B \in \Delta$
Let $Ta : A \to B \in \Delta$ and $\langle c, ba\rangle \in \Delta_R$. Suppose neither $\Delta + Fb : A$ nor $\Delta + Tc : B$ is d.c. Then clearly

(1) there exist $\delta_1 \subseteq \Delta_R$ and T-members of Δ $\gamma_1, \ldots, \gamma_k$ such that

$$[\delta_1];\ \gamma_1, \ldots, \gamma_k \Rightarrow \gamma$$

is derivable in $\mathbf{NL}^{\mathbf{lab}}$ and γ is an F-member of $\Delta + Fb : A$. Since Δ is d.c., γ is obviously $b : A$;

(2) there exist $\delta_2 \subseteq \Delta_R$ and T-members of $\Delta + Tc : B$ $\alpha_1, \ldots, \alpha_n$ such that

$$[\delta_2];\ \alpha_1, \ldots, \alpha_n \Rightarrow \alpha$$

is derivable in $\mathbf{NL}^{\mathbf{lab}}$ and α is an F-member of Δ. Since Δ is d.c., there exists an index i such that α_i coincides with $c : B$;

Clearly c is a leaf of δ_2 and b is a root of δ_1. According to Proposition 5.2.6 $(\delta_2\langle c, ba\rangle)\delta_1$ is a well defined tree. Therefore the sequent

$$[(\delta_2\langle c, ba\rangle)\delta_1];\ \alpha_1, \ldots, \alpha_{i-1}, \gamma_1, \ldots, \gamma_k, a : A \to B, \alpha_{i+1}, \ldots, \alpha_n \Rightarrow \alpha$$

turns out to be derivable and Δ can not be d.c..

The last case ($\alpha := Tb : A \leftarrow B$) is left to the reader. □

5.2.8. LEMMA. Adding h-witnesses does not disturb d.c..

Proof.
Suppose $Fa : A \to B$ belongs to a d.c. set Δ. Suppose that adding $\langle c, ba\rangle$ (where b and c are fresh) to Δ_R and adding $Tb : A$ and $Fc : B$ to Δ does disturb d.c.. Therefore, there exists a derivable sequent

$$(\sharp)\ \ [\delta];\ \gamma_1, \ldots, \gamma_n \Rightarrow \gamma$$

such that $\delta \subseteq \Delta_R \cup \{\langle c, ba\rangle\}$; $\gamma_1, \ldots, \gamma_n$ are T-members of $\Delta + Tb : A$ and γ is an F-member of $\Delta + Fc : B$. Note, that if $\langle c, ba\rangle$ does not occur in δ, then Δ is not d.c.. Thus $\delta = \langle c, ba\rangle\delta_1$, for some $\delta_1 \subseteq \Delta_R$, moreover b and c can occur only once in a tree, generated by δ. Therefore γ coincides with $c : B$ and γ_1 coincides with $b : A$, hence the sequent ($\sharp$) has actually the form

$$[\langle c, ba\rangle\delta_1];\ b : A, \gamma_2, \ldots, \gamma_n \Rightarrow c : B$$

Thus

$$[\delta_1];\ \gamma_2, \ldots, \gamma_n \Rightarrow a : A \to B$$

is also derivable, and since $\delta_1 \subseteq \Delta_R$, $\gamma_1, \ldots, \gamma_n$ are T-members of Δ, and $a : A \to B$ is an F-member of Δ, Δ can not be d.c.. In the case of adding h-witnesses of $Fa : B \leftarrow A$ or $Ta : A \bullet B$ our argument would not be much different. □

From deeply consistent set to a nice set. Completeness proof
Recall our initial assumption: $a : A \Rightarrow a : B$ is not derivable in NL^{lab}. Define $\Delta_0, \ldots \Delta_n, \ldots (n \in N)$ as follows:

$\Delta_0 = \{Ta : A, Fa : B\}$.
Δ_{n+1}: add all possible h-witnesses to Δ_n,
corresponding triangles to Δ_{R_n}
Δ_{n+2}: add all possible r-witnesses to Δ_{n+1}.

By Lemma 5.2.7 and Lemma 5.2.8, $\Delta = \cup\Delta_i (i \in N)$ is nice. Now our Completeness Theorem becomes just a direct consequence of the previous results.

Completeness theorems for $\mathbf{NLP^{lab}}$, $\mathbf{L^{lab}}$ and $\mathbf{LP^{lab}}$ require the additional proof that taking the associative or permutational closure of some d.c. set does not disturb its deep consistency. But this is guaranteed due to the presence of the corresponding structural rule in the sequent presentation of the labelled version of the Lambek Calculus. Note, that the Cut rule was not used in our completeness proof, which means that we have obtained semantical proof of Cut Elimination theorem. For the constructive procedures of Cut Elimination we refer to [Kurtonina 95]

5.2.3 Labelling with Kripke tree terms

The labelling preceding subsection decorated the formulas always with atomic labels: the structure of the database was accounted for by adding to the sequent language an explicit representation of the tree configuration of the atomic labels. The alternative labelling regime to be introduced below has a term language to build structured labels out of the atomic labels. The term language now directly captures the structure of the database, so that we can remove the book-keeping component from the sequent language.

5.2.9. DEFINITION. *Elementary tree terms, tree terms, proper tree terms.*

(i) If $x \in \mathbf{Lab}$, then x is an elementary tree term
(ii) If χ is an elementary tree term, then χ is a tree term;
(iii) If ξ, χ are tree terms and $x \in \mathbf{Lab}$,, then $r(x, \xi, \chi)$ is a tree term.

A tree term t is called *proper* if all its elementary subterms are distinct.
We define the *size* (l) of the tree term as follows:
(i) if x is an elementary tree term, then $l(x) = 0$
(ii) $l(r(x, \xi, \chi)) = l(\xi) + l(\chi) + 1$

Example $r(a, r(x, r(y, bc), d)e)$ is a proper tree term which corresponds to the following tree: $(\langle a, xe\rangle\langle x, y, d\rangle)\langle y, b, c\rangle$.

To prove that each proper tree term corresponds to some tree and vice versa, one proceeds by induction on the length of a tree term for one direction and on the length of a tree for the other one.

5.2.10. DEFINITION. *Labelled sequents*
Let a, b, c be elementary tree terms, and t, u, v proper tree terms. A labelled sequent is an expression of the form

$$a_1 : A_1, a_2 : A_2, \ldots, a_n : A_n \Rightarrow t : A$$

where each $a_i (1 \leq i \leq n)$ is an elementary tree term and t is a proper tree term.

Labelled sequent calculus.
Let X, Y be finite multisets of formulas labelled with elementary tree terms. And let a, b, c be elementary tree terms, t, u, v proper tree terms as before. We write

$t = t'[u/v]$ if t is obtained from the proper tree term t' by replacing the subterm v by u.

The labelled sequent presentation for the basic system $\mathbf{NL^{lab}}$ has one axiom-scheme and the following inference rules:

$$a : A \;\Rightarrow\; a : A$$

$$\frac{X \Rightarrow t_1 : A \qquad c : B, Y \Rightarrow t_2 : \alpha}{a : A \rightarrow B,\ X, Y \Rightarrow t : \alpha} \quad \text{where } t = t_2[r(c, t_1 a)/c]$$

$$\frac{b : A, X \Rightarrow r(c, bt) : B}{X \Rightarrow t : A \rightarrow B}$$

$$\frac{X \Rightarrow t_1 : A \qquad c : B, Y \Rightarrow t_2 : \alpha}{a : A \leftarrow B, X, Y \Rightarrow\ t : \alpha} \quad \text{where } t = t_2[r(c, a t_1)/c]$$

$$\frac{b : A, X \Rightarrow r(c, bt) : B}{X \Rightarrow a : B \leftarrow A}$$

$$\frac{b : A, c : B, X \Rightarrow t_1 : \alpha}{a : A \bullet B, X \Rightarrow t : \alpha} \quad \text{where } t = t_1[a/r(a, bc)]$$

$$\frac{X_1 \Rightarrow t_1 : A \qquad X_2 \;\Rightarrow\; t_2 : B}{X_1, X_2 \Rightarrow r(a, t_1 t_2) : A \bullet B}$$

As before, one obtains labelled presentation for the systems **L, NLP** and **LP** by adding Associativity, Permutation or their combination.

Associativity Rule

$$\frac{X \Rightarrow t : A}{X \Rightarrow t' : A}$$

where t' can be obtained from t by replacing a subterm $r(a, t_1, r(b, t_2 t_3))$ by a tree term $r(a, r(c, t_1 t_2) t_3)$, provided that c is fresh.

Permutation Rule

$$\frac{X \Rightarrow t : A}{X \Rightarrow t' : A}$$

where t' can be obtained from t by replacing a subterm $r(a, t_1 t_2)$ by a tree term $r(a, t_2 t_1)$.

Clearly via translation of proper tree terms into trees and vice versa one can easily prove the equivalence of the two formulations of the Lambek Calculus. A direct consequence of this fact is *soundness* and *completeness* of the Lambek Calculus with tree terms as labels with respect to ternary relation semantics.

5.2.4 Generalizations

In the preceding section we have looked at the systems **NL, L, NLP, LP** and presented uniform labelled sequent calculus for these logics, with completeness results for the relevant classes of ternary frames. The methods used are quite general: they can be straightforwardly applied to a number of related systems. Two generalizations seem especially relevant in view of current linguistic applications: the move from unimodal to multimodal architectures, and the introduction of unary multiplicative operators in addition to the familiar binary ones. We discuss these in turn.

Multimodal architectures
In the preceding paragraphs, we have studied the systems **NL, L, NLP, LP** in isolation: each of these systems characterizes a distinct resource management regime in terms of a package of structural rules - rules for the manipulation of labels in our labelled presentation. It has been argued in the linguistic literature (see for example [Moort. & Morrill 91], [Moort. & Oehrle 94], [Morrill 94] and [Moortgat 94] that for purposes of actual grammar development, one wants to have access to the combined inferential capacity of these various systems.
On the model-theoretic level, such a mixed style of inference requires a move from unimodal frames $\langle W, R\rangle$ to multimodal frames $\langle W, \{R_i\}_{i\in I}\rangle$. We now distinguish a family of accessibility relations: each of these R_i can have its own individual resource management properties, or if R_i and R_j have the same resource management regime they can still be kept distinct in virtue of their indexes i and j.
On the syntactic level, we also index the connectives with $i \in I$, so that we can interpret each $\bullet_i$ (and its residual implications $\rightarrow_i$ and $\leftarrow_i$) in terms of its own accessibility relation R_i. Structural postulates, and the corresponding frame conditions, are relativized to the mode indexes. Apart from the standard structural options differentiating **NL, L, NLP** and **LP**, the multimodal architecture supports mixed forms, where Associativity or Commutativity apply when two modes are in construction with each other. Such mixed structural principles greatly enhance the linguistic expressivity of the framework. See [Moort. & Oehrle 94], [Morrill 94] for concrete illustrations. We will treat the labelled version of such principles in a moment.

Our framework for labelled deduction directly accommo-dates the multimodal categorial architecture. We sketch the necessary changes for the tree term labelling. In the definition of tree terms, we now have a family of term constructors r_i instead of the one r of the unimodal setting.

5.2.11. DEFINITION. *Multimodal systems: elementary tree terms, tree terms, proper tree terms.*
Let **Lab**, be a set of atomic markers, as before, and **I** a set of resource manage-

ment mode indices. (i) If $x \in \mathbf{Lab}$, then x is an elementary tree term (ii) If χ is an elementary tree term, then χ is a tree term; (iii) If ξ, χ are tree terms , $x \in \mathbf{Lab}$, and $i \in \mathbf{I}$, then $r_(x, \xi, \chi)$ is a tree term.

Similarly, in the definition of the multimodal labelled sequent calculus, we harmonize the mode information on the connectives and on the associated tree term labels. Below the logical rules for the connectives.

$$\frac{X \Rightarrow t_1 : A \qquad c : B, Y \Rightarrow t_2 : \alpha}{a : A \rightarrow_i B,\ X, Y \Rightarrow t : \alpha}\ \text{where } t = t_2[r_i(c, t_1 a)/c] \qquad \frac{b : A, X \Rightarrow r_i(c, bt) : B}{X \Rightarrow t : A \rightarrow_i B}$$

$$\frac{X \Rightarrow t_1 : A \qquad c : B, Y \Rightarrow t_2 : \alpha}{a : A \leftarrow_i B, X, Y \Rightarrow\ t : \alpha}\ \text{where } t = t_2[r_i(c, at_1)/c] \qquad \frac{b : A, X \Rightarrow r_i(c, bt) : B}{X \Rightarrow a : B \leftarrow_i A}$$

$$\frac{b : A, c : B, X \Rightarrow t_1 : \alpha}{a : A \bullet_i B, X \Rightarrow t : \alpha}\ \text{where } t = t_1[a/r_i(a, bc)] \qquad \frac{X_1 \Rightarrow t_1 : A \qquad X_2 \Rightarrow t_2 : B}{X_1, X_2 \Rightarrow r_i(a, t_1 t_2) : A \bullet_i B}$$

The structural rules, in the multimodal setting, are mode restricted: they refer to specific resource management mode labels. Our earlier versions of Associativity and Permutation would now assume the following form (for obvious mode labels).

Associativity Rule

$$\frac{X \Rightarrow t : A}{X \Rightarrow t' : A}$$

where t' can be obtained from t by replacing a subterm $r_{ass}(a, t_1, r_{ass}(b, t_2 t_3))$ by a tree term $r_{ass}(a, r_{ass}(c, t_1 t_2) t_3)$, provided that c is fresh.

Permutation Rule

$$\frac{X \Rightarrow t : A}{X \Rightarrow t' : A}$$

where t' can be obtained from t by replacing a subterm $r_{perm}(a, t_1 t_2)$ by a tree term $r_{perm}(a, t_2 t_1)$.

Apart from these familiar unimodal structural options, our language is now expressive enough to also formulate mixed version for situations where different modes are in construction with each other. As an illustration, we present versions of Mixed Associativity and Mixed Commutativity for communication between modes i and j. The structural postulates are as follows.

Mixed Commutativity	$A \bullet_i (B \bullet_j C) \vdash B \bullet_j (A \bullet_i C)$
Mixed Associativity	$A \bullet_i (B \bullet_j C) \vdash (A \bullet_i B) \bullet_j C)$

Translated in the labelling format, these structural postulates become rules for manipulating term labels:

Mixed Associativity Rule

$$\frac{X \Rightarrow t : A}{X \Rightarrow t' : A}$$

where t' can be obtained from t by replacing a subterm $r_i(a, t_1, r_j(b, t_2 t_3))$ by a tree term $r_j(a, r_i(c, t_1 t_2) t_3)$, provided that c is fresh.

Permutation Rule

$$\frac{X \Rightarrow t : A}{X \Rightarrow t' : A}$$

where t' can be obtained from t by replacing a subterm $r_i(a, t_1, r_j(b, t_2 t_3))$ by a tree term $r_j(a, t_2 r_i(c, t_1 t_3))$.

For a linguistic application of this type of communication principle, the reader can turn to Moortgat and Oehrle 93, who give a multimodal analysis of head-adjunction phenomena such as can be found in the Germanic Verb-Raising construction. In a sentence such as

dat Jan (een boek (wil lezen))
(that J. a book wants read) i.e.
that John wants to read a book

the verb wil (want) has to be combined semantically with the combination of the main verb lezen (read) and its direct object een boek (a book). But on the syntactic level, wil does not combine with the phrase een boek lezen but just with its head, viz. lezen. Let the main verb combine with its arguments in mode i, and the modal auxiliary wil in mode j, then the Mixed Commutativity rule makes it possible to proceed from the surface syntactic organization to the configuration required for semantic interpretation.

Unary multiplicatives

The labelling method and completeness proof of the previous sections was formulated for families of binary residuated connectives and their ternary accessibility relation. Residuation can of course be generalized to n-ary families of connectives interpreted with respect to n+1-ary accessibility relations, [Dunn 91] for an excellent survey. Our labelling approach can be applied straightforwardly in the generalized residuation setting.

Especially relevant for the linguistic applications is the case of *unary* residuated operators, interpreted with respect to a binary accessibility relation. We have already encountered these operators in Chapter 1 and 2. In Chapter 6 they will play a central role for establishing communication between various categorial systems. The basic residuation pattern for the unary operators assumes the following form:

$$\Diamond A \Rightarrow B \quad \text{iff} \quad A \Rightarrow \Box^{\downarrow} B$$

Semantically, we have the usual truth conditions below:

$$x \models \Diamond A \text{ iff } \exists y(Rxy \;\&\; y \models A)$$
$$x \models \Box^{\downarrow} A \text{ iff } \forall y(Ryx \Rightarrow y \models A)$$

Labelled sequent calculus: unary residuated connectives.
Labelled sequent presentation for unary multiplicatives requires a generalization of the notion of a tree term to include binary tree terms built with the constructor r^2 next to the ternary case we had before: if ξ is a tree terms, $x \in \mathbf{Lab}$, then $r^2(x, \xi)$ is a tree term. The logical rules for the new connectives $\Diamond, \Box^{\downarrow} A$ then assume the following form.

$$\frac{b : A, X \Rightarrow t_1 : B}{a : \Box^{\downarrow} X, \Rightarrow t : B} \qquad \frac{X \Rightarrow r^2(a,t) : A}{X \Rightarrow t : \Box^{\downarrow} A}$$
where $t = t_1[r^2(b,a)/b]$

$$\frac{b : A, X \Rightarrow t_1 : B}{a : \Diamond A, X \Rightarrow t : B} \qquad \frac{X \Rightarrow t : A}{X \Rightarrow r^2(a,t) : \Diamond A}$$
where $t = t_1[a/r^2(a,b)]$

5.3 Digression: Concrete Labelling

The use of labelling in the previous section is still on the conservative side: we have given a uniform presentation for a family of categorial type logics with different resource management properties by introducing a division of labour between the sequent language and the labelling discipline: the sequent language is kept uniform, and the syntactic fine-tuning is taken care of by the labels.

In this section we want to give a very simple illustration of a form of labelling where the label has acquired a greater degree of independence from the formula, i.e. where the label allows one to incorporate *additional* information relevant to the process of linguistic inference. We show how one can decorate simple type assignment with labels capturing *morphophonological sortal* information. The sortal decoration makes it possible to rule out derivations that would go through if we restricted the attention to unlabelled type assignments.

Compare the adjective modifier **very** and the prefix **un-**. On the syntactic level, they are both functors taking adjectives into adjectives. We could assign them the type **a** ← **a** and type **a** to **happy.** But given this type assignment both the grammatical **very unhappy** and the ungrammatical **un-very happy** are derivable according to the scheme

$$a \leftarrow a, a \leftarrow a, a \Rightarrow a$$

Can we refine the type assignment in such a way as to take into account the different combinatory possibilities of affix, words, and phrases? To block the undesired derivation and to keep the desired one we decorate the type formulas with sort labels, characterizing **very** and **happy** as syntactic words and **un-** as an affix. Our assignments in labelling format could take the form

very	word :	$a \leftarrow a$
un-	affix :	$a \leftarrow a$
happy	word :	a

Moreover, we have to impose constraints on the composition relation in order to characterize the well-formed combinations on the morphophonological sort level:

(i) ⟨word, affix word⟩
(ii) ⟨phrase, word phrase⟩
(iii) ⟨phrase, word word⟩

Next, to express the fact that a syntactic word can do the duty as a phrasal expression (but not vice versa, of course) we adopt the following rule:

> Let Ψ be a set of constraints and a triangle $\delta \in \Psi$ has a root –**word**, then there exists $\delta' \in \Psi$ with the same leaves as δ and with the root–**phrase**.

In our case that means that (i) adds one more triangle to the set of constraints:

(i') **⟨word, affix word⟩**

Now using suitable abbreviations and marks to distinguish tree points we can present a derivation of **very unhappy**:

$$\dfrac{w_1 : a \Rightarrow w_1 : a \qquad \dfrac{ph_1 : a \Rightarrow ph_1 : a \qquad ph_2 : a \Rightarrow ph_2 : a}{[\langle ph_2, w_2 ph_1\rangle] w_2 : a \leftarrow a, ph_1 : a \Rightarrow ph_2 : a}}{[\langle ph_2, w_2 ph_1\rangle\langle ph_1, af w_1\rangle]; w_2 : a \leftarrow a, af : a \leftarrow a, w_1 : a \Rightarrow ph_2 : a}$$

On the other hand **un- very unhappy** turns out to be underivable since our constraints on composition relation do not allow to combine **affix** with **phrase.**

Formally passing from the abstract style of labelling to the concrete style exemplified here we have to specify the definition of a model $\langle W, R, V\rangle$ of interpretation by taking W as a set of sorts and imposing frame constraints on R which are indicated above.

Thus on the level of interpretation, filtering out derivations that would be valid in the non-labelled setting is realized by restricting the class of all ternary models to those satisfying the constraints formulated above.

As we remarked at the beginning of this paragraph, our objective here was just to provide a simple illustration of 'autonomous' forms of labelling. For an elaboration of this style of labelling on a much more fundamental and wide-ranging level, we can refer to R.Kempson's work on combining syntactic and semantic inference [Kempson 95].

5.4 Discussion: Labelling and Correspondence Translations

There are some obvious connections between the method of Labelling developed in Chapters 4, 5, and the earlier perspective of Correspondence and translation into fragments of first-order logic (Chapters 1, 2). The aim of this discursive Section is to point out a few analogies, and possible switches. Generally speaking, one can say that labelled systems live somewhere in the gap between pure modal or categorial logics and their translated versions powered by a full first-order engine of deduction.

First Example *Binary Lambek Calculus*

The relational semantics of Chapter 4 may also be viewed as an effective translation taking categorial formulas A to first-order formulas $T_{bin}(A)(x,y)$ having two free state variables x, y. Its recursive clauses transcribe the truth definition:

$$\begin{array}{lcl} T_{bin}(A \bullet B) & = & \exists z(T_{bin}(A)(x,z) \mathbin{\&} T_{bin}(B)(z,y)) \\ T_{bin}(A \to B) & = & \forall z(T_{bin}(A)(z,x) \to T_{bin}(B)(z,y)) \\ T_{bin}(B \leftarrow A) & = & \forall z(T_{bin}(A)(y,z) \to T_{bin}(B)(x,z)) \end{array}$$

The completeness theorem for the associative Lambek Calculus (Chapter 4) may then also be read as a proof-theoretic reduction using Godel's Completeness Theorem for first-order logic:

$A \vdash B$ is L-derivable iff its translation $T_{bin}(A) \to T_{bin}(B)$
is provable in first-order predicate logic.

One can even find analogies at the level of effective comparison between concrete proof steps in both calculi (see below). Note that the predicate-logical formulas employed in this translation display explicit 'labels', being tuples of state variables. They may also encode constraints on these labels, by additional conditions on binary atoms.

Second Example *Ternary Lambek Calculus*
Likewise, the ternary model semantics leads to the following translation from categorial formulas A into first-order formulas $T_{tern}(A)(x)$ having one free state variable x :

$$\begin{array}{lcl} T_{tern}(A \bullet B) & = & \exists y, z(Rx, yz \;\&\; T_{tern}(A)(y) \;\&\; T_{tern}(B)(z)) \\ T_{tern}(A \rightarrow B) & = & \forall y, z(Ry, zxT_{tern}(A)(z) \rightarrow T_{tern}(B)(y)) \\ T_{tern}(B \leftarrow A) & = & \forall y, z(Ry, xz \;\&\; T_{tern}(A)(z) \rightarrow T_{tern}(B)(y)) \end{array}$$

Now, the key Completeness Theorem says

$A \vdash B$ is L-derivable iff its translation $T_{tern}(A) \rightarrow T_{tern}(B)$
is provable in first-order predicate logic.

This result can be extended to other categorial logics which are frame-complete with respect to more restricted first-order frame classes. The associative Lambek Calculus is itself an example: $A \vdash B$ is L-derivable iff its translation $T_{tern}(A) \rightarrow T_{tern}(B)$ is provable in first-order predicate logic from the formulas expressing associativity of the ternary relation R .

Let us now observe some general issues which arise here. First, one can go deeper, and consider combinatorial proof-theoretic aspects of this correspondence. For instance, how do cut-free sequent proofs in the Lambek Calculus compare with cut-free sequent proofs in the usual first-order style for their translations? For our modest purposes, however, some general observations will suffice. We focus on the role played by the structural rules. In categorial derivation with the standard associative Lambek Calculus, it was essential to avoid classical structural rules such as Permutation or Contraction. But on the other hand, in first-order derivations for translated categorial sequents, no such care is needed. (We are allowed unlimited use of every classical first-order facility.) How can this be? The answer here is that, by and large, it does not matter. The first-order representation takes categorial formulas to a fragment of the first-order language which is somewhat insensitive to the precise choice of structural rules. Roughly speaking, the more explicit information is put into our representation of categorial signs, the less important the role of our structural constraints.

Analogies with Modal Logic

It is useful to bring in some lessons here from the related field of Modal Logic (cf.[An.vB.Nem. 95]). Modal languages typically involve restricted quantifiers under their first-order translation and so do their categorial fragments. Thus, the natural first-order representation of modal reasoning is inside so-called 'bounded' or 'restricted' fragments of first-order logic, whose general model-theoretic properties tend to be very much like those of predicate logic as a whole. The reason is that much of classical model theory can be redone with modal bisimulation taking the place of first-order partial isomorphism (cf. also [de Rijke 93]). Incidentally, this analogy may be less straightforward for non-Boolean categorial fragments of modal logics. An example is the Craig Interpolation Theorem. The latter has straightforward proofs for basic modal logic, but its proof for the Lambek Calculus in [Roorda 91] (cf. also [Pentus 92a]) involves delicate new proof-theoretic arguments.

In a sense, first-order translations for modal logics tend to explain 'difficile per difficilius'. Decidable calculi are 'reduced' to a logic which is undecidable, by Church's Theorem. What is going on? The reason is that universal validity in a predicate logic restricted to expressively powerful bounded fragments is often decidable. ([An.vB.Nem. 95] have strong theorems to this effect.) Indeed, the above categorial translation T_{tern} is of this kind, ending up in a decidable bounded fragment of predicate logic (even for the full Boolean version of NL). By contrast, the translation T_{bin} essentially employs unrestricted quantification in its clauses, whence the decidability of the associative Lambek Calculus really requires additional work. (The above ternary reduction with explicit premises for frame associativity does not help either: as we do not know a priori whether the latter axioms have a decidable set of restricted first-order consequences.) A proof-theoretic way of seeing that translated modal or categorial logics are often decidable goes as follows. For instance, in deriving modal or categorial sequents, the structural rule of Contraction is *redundant*. We may use it, but it does not produce new derivable principles. And as is well-known without the latter structural rule, the remainder of predicate-logical deduction in sequent style has a finite proof search space, and hence it is decidable. Again, this stratospheric perspective does not answer every concrete question concerning categorial inference. For instance, the relevant calculus LPC adding Permutation and Contraction to the associative Lambek Calculus is decidable (with high complexity, and by a non-constructive argument). Can this be accounted for by the previous considerations?

What Is Labelling?

On the view outlined here, labelling is a way of operating in the gap between bare categorial systems and their translations in a full predicate-logical language. The idea has been to use essentially the old categorial formulas, with a modicum of 'decoration', using labels (referring to semantic objects) as well as some constraints on these. Apart from the uses of labelling in linguistic applications which we mentioned at the beginning of this chapter, there are also many logical predecessors for this move, before Gabbay made it into a general program. E.g., in

Relational Algebra, the use of explicit pair labels in deduction had already been proposed in [Wadge 75], [Orlowska 92], to overcome some infelicities of the usual algebraic format. Similar motives have driven extended modal formalisms with 'names' or 'distinguished worlds' (cf. [Blackburn 93]). As we have seen, there is a great freedom of operation here: labels will give a fragment of first-order logic, and allow some, though not all first-order 'semantic calculation'. Labelled calculi will show the same phenomena as observed above. For instance, the more explicit the labelling, the less becomes our need for a strict structural regime of deduction. We have worked out this theme of the division of labour between the structural regime and the labelling system in Section 5.2. To make some of these points more concrete for the logical reader, here are some labelled systems of modal logic, in between the standard sequent calculus presentation and a full first-order version.

Example *Labelled Modal Deduction*
We use sequents of the form '$C|\ \Sigma \vdash \Delta$' , where C is a set of atomic conditions of the form Rxy, and Σ, Δ are sets of labelled modal formulas $x : A$. The rules of such systems are very simple. For the propositional base, we transcribe the standard Gentzen rules for left and right introduction of Boolean connectives. E.g.,

$$\frac{C|\ x : A, \Sigma \vdash \Delta \quad C|\ x : B, \Sigma \vdash \Delta}{C|\ x : A \vee B, \Sigma \vdash \Delta} \qquad \frac{C|\ x : A, \Sigma \vdash \Delta, x : A, x : B}{C|\ \Sigma \vdash \Delta, x : A \vee B,}$$

As for structural rules, we have a great variety. For instance, we can allow or forbid Monotonicity in three independent positions (C , Σ and Δ) . Let us assume all standard ones (Contraction is redundant in what follows). Now, say, an existential modality is really a restricted existential quantifier. Thus, we have several ways of designing its rules on standard first-order analogies. Here are two possible forms:

$$\frac{C + Rxy|\ y : A, \Sigma \vdash \Delta}{C|\ x : \Diamond A, \Sigma \vdash \Delta} \qquad \frac{C|\ \Sigma \vdash \Delta, z : A}{C + Rxz|\ \Sigma \vdash \Delta, x : \Diamond A}$$

where y is fresh

It is easy to show that this system is complete for deduction in the minimal modal logic. Moreover, it can easily be adapted to deal with deduction in the basic temporal logic, without a need for any special residuation axioms. This completeness would not be affected if we added further rules, such as a converse to the left introduction rule for the modality $\Diamond$. We conclude with an example of deduction in this intermediate style.

Modal Distribution

$$\dfrac{\dfrac{\dfrac{\dfrac{-|\; \boldsymbol{y} : A \vdash \boldsymbol{y} : A}{-|\; \boldsymbol{y} : A \vdash \boldsymbol{y} : A, \boldsymbol{y} : B} \qquad \dfrac{-|\; \boldsymbol{y} : B \vdash \boldsymbol{y} : B}{-|\; \boldsymbol{y} : B \vdash \boldsymbol{y} : A, \boldsymbol{y} : B}}{-|\; \boldsymbol{y} : A \vee B \vdash \boldsymbol{y} : A, \boldsymbol{y} : B}}{\boldsymbol{Rxy}, \boldsymbol{Rxy}|\boldsymbol{y} : A \vee B \vdash \boldsymbol{x} : \Diamond A, \boldsymbol{x} : \Diamond B}}{-|\; \boldsymbol{x} : \Diamond(A \vee B) \vdash \boldsymbol{x} : \Diamond A, \boldsymbol{x} : \Diamond B}$$

The calculi that we have developed for categorial deduction in the main body of this Chapter can all be viewed as similar designs. They go half-way toward full first-order translation, leaving some standard first-order inferential steps masked all for the purpose of striking some optimum of representation and tractability.

Thus, once again, labelling gives us a spectrum of possible formats for deduction 'in between' pure categorial calculi and their first-order 'completions' arising from making their semantics explicit. And of course, what we have observed is that linguistic convenience may be on this side, as labels acquire independent importance as vehicles for various forms of syntactic, semantic or phonological information.

Part III
Categorial Architecture and Modal Embeddings

We develop a theory of systematic communication between categorial type logics. The communication is two-way: we show how one can fully recover the structural discrimination of a weaker logic from within a stronger one, and how one can reintroduce structural flexibility of stronger categorial logics within weaker ones. We show how unary modal operators can be used to obtain structural relaxation, or to impose structural constraints. From a logical point of view, our contribution consists in some general translation methods, plus a number of embedding theorems connecting the main calculi in the categorial landscape.

Chapter 6

Structural Control

In this chapter[1] we study Lambek systems as *grammar logics*: logics for reasoning about structured linguistic resources. The structural parameters of precedence, dominance and dependency generate a cube of resource-sensitive categorial type logics. From the pure logic of residuation **NL**, one obtains **L**, **NLP** and **LP** in terms of Associativity, Commutativity, and their combination. Each of these systems has a dependency variant, where the product is split up into a left-headed and a right-headed version.

We develop a theory of systematic communication between these systems. The communication is two-way: we show how one can fully recover the structural discrimination of a weaker logic from within a system with a more liberal resource management regime, and how one can reintroduce the structural flexibility of a stronger logic within a system with a more articulate notion of structure-sensitivity.

In executing this programme we follow the standard logical agenda: the categorial formula language is enriched with extra control operators, so-called structural modalities, and on the basis of these control operators, we prove embedding theorems for the two directions of substructural communication. But our results differ from the Linear Logic style of embedding with $S4$-like modalities in that we realize the communication in both directions in terms of a *minimal* pair of structural modalities. The control devices $\Diamond, \Box^{\downarrow}$ used here represent the pure logic of residuation for a family of binary multiplicatives: they do not impose any restrictions on the binary accessibility relation interpreting the unary modalities, unlike the $S4$ operators which require a transitive and reflexive interpretation. With the more delicate control devices we can avoid the model-theoretic and proofthe-

[1] This chapter is based on joint work with Michael Moortgat. It will appear as an independent article under the same title as [Kurton. & Moort. 95]. The notational conventions are slightly different in ways explained in the text.

oretic problems one encounters when importing the Linear Logic modalities in a linguistic setting.

6.1 Logics of structured resources

This paper is concerned with the issue of *communication* between categorial type logics of the Lambek family. Lambek calculi occupy a lively corner in the broader landscape of resource-sensitive systems of inference. We study these systems here as grammar logics. In line with the 'Parsing as Deduction' slogan, we present the key concept in grammatical analysis — *well-formedness* — in logical terms, i.e. grammatical well-formedness amounts to *derivability* in our grammar logic. In the grammatical application, the resources we are talking about are linguistic expressions — multidimensional form-meaning complexes, or *signs* as they have come to be called in current grammar formalisms. These resources are structured in a number of grammatically relevant dimensions. For the sake of concreteness, we concentrate on three types of linguistic structure of central importance: linear order, hierarchical grouping (constituency) and dependency. The structure of the linguistic resources in these dimensions plays a crucial role in determining well-formedness: one cannot generally assume that changes in the structural configuration of the resources will preserve well-formedness. In logical terms, we are interested in structure-sensitive notions of linguistic inference.

Fig 6.1 charts the eight logics that result from the interplay of the structural parameters of precedence, dominance and dependency. The systems lower in the cube exhibit a more fine-grained sense of structure-sensitivity; their neighbours higher up loose discrimination for one of the structural parameters we distinguish here.

Let us present the essentials (syntactically and semantically) of the framework we are assuming before addressing the communication problem. For a fuller treatment of multimodal categorial architecture, the reader can turn to [Moortgat 94, Moort. & Oehrle 94, Moort. & Morrill 91, Moort. & Oehrle 93, Morrill 94]. Consider the standard language of categorial type formulae $\mathcal{F}$ freely generated from a set of atomic formulae $\mathcal{A}$: $\mathcal{F} ::= \mathcal{A} \mid \mathcal{F}/\mathcal{F} \mid \mathcal{F} \bullet \mathcal{F} \mid \mathcal{F}\backslash\mathcal{F}$. The most general interpretation for such a language can be given in terms of Kripke style relational structures — ternary relational structures $\langle W, R^3\rangle$ in the case of the binary connectives (cf. [Došen 92a]). W here is to be understood as the set of linguistic resources (signs) and the accessibility relation R as representing linguistic composition. From a ternary frame we obtain a model by adding a valuation V sending prime formulae to subsets of W and satisfying the clauses

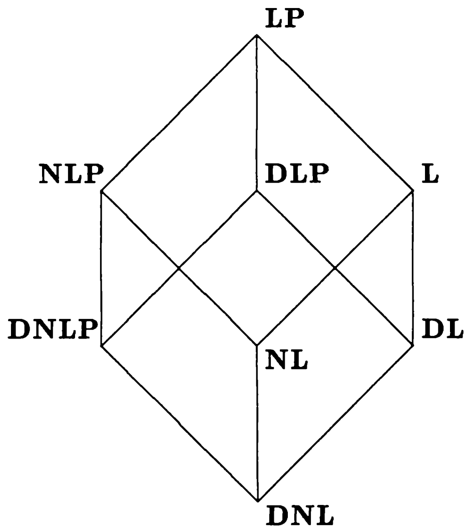


Figure 6.1: Resource-sensitive logics: precedence, dominance, dependency

below for compound formulae.

$$\begin{array}{rcl} V(A \bullet B) & = & \{z \mid \exists x \exists y [Rzxy \ \& \ x \in V(A) \ \& \ y \in V(B)]\} \\ V(C/B) & = & \{x \mid \forall y \forall z [(Rzxy \ \& \ y \in V(B)) \Rightarrow z \in V(C)]\} \\ V(A \backslash C) & = & \{y \mid \forall x \forall z [(Rzxy \ \& \ x \in V(A)) \Rightarrow z \in V(C)]\} \end{array}$$

With no restrictions on R, we obtain the pure logic of residuation known as **NL**.

$$\text{RES}(2) \qquad A \to C/B \iff A \bullet B \to C \iff B \to A \backslash C$$

And with restrictions on the interpretation of R, and corresponding structural postulates, we obtain the systems **NLP**, **L** and **LP**. Below we give the structural postulates of Associativity (A) and Permutation (P) and the corresponding frame conditions $F(A)$ and $F(P)$. Notice that the structural discrimination gets coarser as we impose more constraints on the interpretation of R. In the presence of Permutation, well-formedness is unaffected by changes in the linear order of the linguistic resources. In the presence of Associativity, different groupings of the linguistic resources into hierarchical constituent structures has no influence on derivability.

$$\begin{array}{rc} (A) & A \bullet (B \bullet C) \longleftrightarrow (A \bullet B) \bullet C \\ F(A) & (\forall xyz \in W)\ \exists t.Rxyt \ \& \ Rtzu \Leftrightarrow \exists v.Rvyz \ \& \ Rxvu \\ (P) & A \bullet B \to B \bullet A \\ F(P) & (\forall xyz \in W)\ Rxyz \Leftrightarrow Rxzy \end{array}$$

What we have said so far concerns the upper face of the cube of Fig 6.1. To obtain the systems at the lower face, we split the connective $\bullet$ in left-headed $\bullet_l$ and right-headed $\bullet_r$, taking into account the asymmetry between heads and dependents. It is argued in [Moort. & Morrill 91] that the dependency dimension should be treated as orthogonal in principle to the functor/argument asymmetry. The distinction between left-headed $\bullet_l$ and right-headed $\bullet_r$ (and their residual implications) makes the type language articulate enough to discriminate between head/complement configurations, and modifier/head or specifier/head configurations. A determiner, for example, could be typed as $np/_r n$. Such a declaration naturally accounts for the fact that determiners act semantically as functions from n-type meanings to np-type meanings, whereas in the form dimension they should be treated as dependent on the common noun they are in construction with, so that they can derive their agreement properties from the head noun.

In the Kripke models, the lower plane of Fig 6.1 is obtained by moving from unimodal to multimodal (in this case: bimodal) frames $\langle W, R^3_l, R^3_r\rangle$, with a distinct accessibility relation for each product. Again, we have the pure (bimodal) logic of residuation **DNL**, with an arbitrary interpretation for R^3_l, R^3_r, and its relatives **DNLP**, **DL**, **DLP**, obtained by imposing associativity or (dependency-preserving!) commutativity constraints on the frames. The relevant structural postulates are given below. The distinction between the left-headed and right-headed connectives is destroyed by the postulate (D).

$$\begin{array}{rc}
(A_l) & A \bullet_l (B \bullet_l C) \longleftrightarrow (A \bullet_l B) \bullet_l C \\
(A_r) & A \bullet_r (B \bullet_r C) \longleftrightarrow (A \bullet_r B) \bullet_r C \\
(P_{l,r}) & A \bullet_l B \longleftrightarrow B \bullet_r A \\
(D) & A \bullet_l B \longleftrightarrow A \bullet_r B
\end{array}$$

It will be clear already from the foregoing that in presenting the grammar for a given language, we will in general not be in a position to restrict ourselves to one particular type logic — we want to have access to the combined inferential capacities of the different logics, without destroying their individual characteristics. For this to be possible we need a theory of systematic *communication* between type systems. The structural postulates presented above do not have the required granularity for such a theory of communication: they globally destroy structure sensitivity in one of the relevant dimensions, whereas we would like to have *lexical control* over resource management. Depending on the direction of communication, one can develop two perspectives on controlled resource management. On the one hand, one would like to have control devices to license limited access to a more liberal resource management regime from within a system with a higher sense of structural discrimination. On the other hand, one would like to impose constraints on resource management in systems where such constraints are lacking by default. For discussion of linguistic phenomena motivating these two types of communication, the reader can turn to the papers in

[Barry & Morrill 90] where the licensing perspective was originally introduced, and to [Morrill 94] where apart from licensing of structural relaxation one can also find discussion of constraints with respect to the associativity dimension. We give an illustration for each type of control, drawing on the references just mentioned.

LICENSING STRUCTURAL RELAXATION. For the licensing type of communication, consider type assignment to relative pronouns like *that* in the sentences below.

$$\begin{array}{rl} & \text{the book that John read} \\ & \text{the book that John read yesterday} \\ \mathbf{L} \vdash & r/(s/np), np, (np\backslash s)/np \Rightarrow r \\ \mathbf{L} \not\vdash & r/(s/np), np, (np\backslash s)/np, s\backslash s \Rightarrow r \\ \mathbf{NL} \not\vdash & (r/(s/np), (np, (np\backslash s)/np)) \Rightarrow r \end{array}$$

Suppose first we are dealing with the associative regime of **L**, and assign the relative pronoun the type $r/(s/np)$, abbreviating $n\backslash n$ as r, i.e. the pronoun looks to its right for a relative clause body missing a noun phrase. The first example is derivable[2] (because 'John read np' indeed yields s), the second is not (because the hypothetical np assumption in the subderivation 'John read yesterday np' is not in the required position adjacent to the verb 'read'). We would like to refine the assignment to the relative pronoun to a type $r/(s/np^\sharp)$, where $np^\sharp$ is a noun phrase resource which has access to Permutation in virtue of its $\cdot^\sharp$ decoration. Similarly, if we change the default regime to **NL**, already the first example fails on the assignment $r/(s/np)$ with the indicated constituent bracketing: although the hypothetical np in the subcomputation '((John read) np)' finds itself in the right position with respect to linear order requirements, it cannot satisfy the direct object role for 'read' being outside the clausal boundaries. A refined assignment $r/(s/np^\sharp)$ here could license the marked $np^\sharp$ a controlled access to the structural rule of Associativity which is absent in the **NL** default regime.

IMPOSING STRUCTURAL CONSTRAINTS. For the other direction of communication, we take an example from [Morrill 94] which again concerns relative clause formation, but this time in its interaction with coordination. Assume we are dealing with an associative default regime, and let the conjunction particle 'and' be polymorphically typed as $(X\backslash X)/X$. With the instantiation $X = s/np$ we can derive the first example. But, given Associativity and an instantiation $X = s$, nothing blocks the ungrammatical second example: 'Melville wrote Moby Dick and John read np' derives s, so that withdrawing the np hypothesis indeed gives s/np, the type required for the relative clause body.

[2] The Appendix gives axiomatic and Gentzen style presentation of the logics under discussion.

the book that Melville wrote and John read

$$\mathbf{L} \vdash \; r/(s/np), np, (np\backslash s)/np, (X\backslash X)/X, np, (np\backslash s)/np \Rightarrow r \quad (X = s/np)$$

*the book that Melville wrote Moby Dick and John read

$$\mathbf{L} \vdash \; r/(s/np), np, (np\backslash s)/np, np, (X\backslash X)/X, np, (np\backslash s)/np \Rightarrow r \quad (X = s)$$

To block this violation of the so-called Coordinate Structure Constraint, while allowing Across-the-Board Extraction as exemplified by our first example, we would like to refine the type assignment for the particle 'and' to $(X\backslash X^{\flat})/X$, where the intended interpretation for the marked $X^{\flat}$ now would be the following: after combining with the right and the left conjuncts, the $\cdot^{\flat}$ decoration makes the complete coordination freeze into an island configuration which is inaccessible to extraction under the default associative resource management regime.

MINIMAL STRUCTURAL MODALITIES. Our task in the following pages is to give a logical implementation of the informal idea of decorating formulas with a label $(\cdot)^{\sharp}$ or $(\cdot)^{\flat}$, licensing extra flexibility or imposing a tighter regime for the marked formulae. The original introduction of the licensing type of communication in [Barry & Morrill 90] was inspired by the modalities '!,?' of Linear Logic — unary operators which give marked formulae access to the structural rules of Contraction and Weakening, thus making it possible to recover the full power of Intuitionistic or Classical Logic from within the resource sensitive linear variants. On the proof-theoretic level, the '!,?' operators have the properties of $S4$ modalities. It is not self-evident that $S4$ behaviour is appropriate for substructural systems *weaker* than Linear Logic — indeed [Venema 93b] has criticized an $S4$ '!' in such settings for the fact that the proof rule for '!' has undesired side-effects on the meaning of other operators. On the semantic level it has been shown in [Versmissen 93] that the $S4$ regime is incomplete with respect to the linguistic interpretation which was originally intended for the structural modalities — a subalgebra interpretation in a general groupoid setting, cf. [Morrill 94] for discussion.

Given these model-theoretic and proof-theoretic problems with the use of Linear Logic modalities in linguistic analysis, we will explore a different route and develop an approach attuned to the specific domain of application of our grammar logics — a domain of structured linguistic resources.

[Moortgat 94] proposes an enrichment of the type language of categorial logics with *unary* residuated operators, interpreted in terms of a *binary* relation of accessibility. These operators will be the key devices in our strategy for controlled resource management. If we were talking about temporal organization, $\Diamond$ and $\Box^{\downarrow}$ could be interpreted as future possibility and past necessity, respectively. But in our grammatical application, R^2 just like R^3 is to be interpreted in terms of structural composition. Where a ternary configuration $(xyz) \in R^3$ interpreting the product connective abstractly represents putting together the components y and z into a structured configuration x in the manner indicated by R^3, a binary configuration $(xy) \in R^2$ interpreting the unary $\Diamond$ can be seen as the construction of the sign x out of a structural component y in terms of the building instructions referred to by R^2.

$$\text{RES}(1) \quad \Diamond A \to B \Longleftrightarrow A \to \Box^{\downarrow} B$$

$$V(\Diamond A) = \{x \mid \exists y(R^2 xy \ \wedge \ y \in V(A)\}$$
$$V(\Box^{\downarrow} A) = \{x \mid \forall y(R^2 yx \ \Rightarrow y \in V(A)\}$$

From the residuation laws RES(1) one directly derives the monotonicity laws below and the properties of the compositions of $\Diamond$ and $\Box^{\downarrow}$:

$$A \to B \quad \text{implies} \quad \Diamond A \to \Diamond B \quad \text{and} \quad \Box^{\downarrow} A \to \Box^{\downarrow} B$$

$$\Diamond \Box^{\downarrow} A \to A \qquad A \to \Box^{\downarrow} \Diamond A$$

In the Appendix, we present the sequent logic for these unary operators. It is shown in [Moortgat 94] that the Gentzen presentation is equivalent to the axiomatic presentation, and that it enjoys Cut Elimination. For our examples later on we will use decidable sequent proof search.

Semantically, the pure logic of residuation for $\Diamond, \Box^{\downarrow}$ does not impose any restrictions on the interpretation of R^2. As in the case of the binary connectives, we can add structural postulates for $\Diamond$ and corresponding frame constraints on R^2. With a reflexive and transitive R^2, one obtains an $S4$ system. Our objective here is to show that one can develop a systematic theory of communication, both for the licensing and for the constraining perspective, in terms of the *minimal* structural modalities, i.e. the pure logic of residuation for $\Diamond, \Box^{\downarrow}$.

COMPLETENESS. The communication theorems to be presented in the following sections rely heavily on semantic argumentation. The cornerstone of the approach is the completeness of the logics compared, which guarantees that syntactic derivability $\vdash A \to B$ and semantic inclusion $V(A) \subseteq V(B)$ coincide for the classes of models we are interested in. For the $\mathcal{F}(/, \bullet, \backslash)$ fragment, [Došen 92a] shows that **NL** is complete with respect to the class of all ternary models, and **L**, **NLP**, **LP** with respects to the classes of models satisfying the frame constraints for the relevant packages of structural postulates. The completeness results are obtained on the basis of a simple canonical model construction which directly accommodates bimodal dependency systems with $\mathcal{F}(/_i, \bullet_i, \backslash_i)$ $(i \in \{l, r\})$. And it is shown in [Moortgat 94] that the construction also extends unproblematically to the language enriched with $\Diamond, \Box^{\downarrow}$ as soon as one realizes that $\Diamond$ can be seen as a 'truncated' product and $\Box^{\downarrow}$ its residual implication.

6.1.1. DEFINITION. Define the canonical model for mixed (2,3) frames as $\mathcal{M} = \langle W, R^2, R_i^3 \rangle$, where

> W is the set of formulae $\mathcal{F}(/_i, \bullet_i, \backslash_i, \diamond, \Box^\downarrow)$
> $R_i^3(A,B,C)$ iff $\vdash A \to B \bullet_i C$, $R^2(A,B)$ iff $\vdash A \to \diamond B$
> $A \in V(p)$ iff $\vdash A \to p$.

The Truth Lemma then states that, for any formula ϕ, $\mathcal{M}, A \models \phi$ iff $\vdash A \to \phi$. Now suppose $V(A) \subseteq V(B)$ but $\not\vdash A \to B$. If $\not\vdash A \to B$ with the canonical valuation on the canonical frame, $A \in V(A)$ but $A \notin V(B)$ so $V(A) \not\subseteq V(B)$. Contradiction.

We have to check the Truth Lemma for the new compound formulae $\diamond A$, $\Box^\downarrow A$. Below the direction that requires a little thinking.

($\diamond$) Assume $A \in V(\diamond B)$. We have to show $\vdash A \to \diamond B$. $A \in V(\diamond B)$ implies $\exists A'$ such that $R^2 AA'$ and $A' \in v(B)$. By inductive hypothesis, $\vdash A' \to B$. By Isotonicity for $\diamond$ this implies $\vdash \diamond A' \to \diamond B$. We have $\vdash A \to \diamond A'$ by (Def R^2) in the canonical frame. By Transitivity, $\vdash A \to \diamond B$.

($\Box^\downarrow$) Assume $A \in V(\Box^\downarrow B)$. We have to show $\vdash A \to \Box^\downarrow B$. $A \in V(\Box^\downarrow B)$ implies that $\forall A'$ such that $R^2 A'A$ we have $A' \in V(B)$. Let A' be $\diamond A$. $R^2 A'A$ holds in the canonical frame since $\vdash \diamond A \to \diamond A$. By inductive hypothesis we have $\vdash A' \to B$, i.e. $\vdash \diamond A \to B$. By Residuation this gives $\vdash A \to \Box^\downarrow B$.

Apart from global structural postulates we will introduce in the remainder of this paper 'modal' versions of such postulates — versions which are relativized to the presence of $\diamond$ control operators. The completeness results extend to these new structural postulates. Syntactically, they consist of formulas built up entirely in terms of the $\bullet$ operator and its truncated one-place variant $\diamond$. This means they have the required shape for a generalized Sahlqvist-van Benthem theorem and frame completeness result which is proved in Chapter 3 of this dissertation:

> If $R_\diamond : A \to B$ is a modal version of a structural postulate, then there exists a first order frame condition effectively obtainable from $R_\diamond$, and any logic $\mathcal{L} + R_\diamond$ is complete if $\mathcal{L}$ is complete.

EMBEDDING THEOREMS: THE METHOD IN GENERAL. In the sections that follow, we consider pairs of logics $\mathcal{L}_0, \mathcal{L}_1$ where $\mathcal{L}_0$ is a 'southern' neighbour of $\mathcal{L}_1$. Let us write $\mathcal{L}\diamond$ for a system $\mathcal{L}$ extended with the unary operators $\diamond, \Box^\downarrow$ with their minimal residuation logic. For the 12 edges of the cube of Fig 6.1, we define embedding translations $(\cdot)^\flat : \mathcal{F}(\mathcal{L}_0) \mapsto \mathcal{F}(\mathcal{L}_1\diamond)$ which impose the structural discrimination of $\mathcal{L}_0$ in $\mathcal{L}_1$ with its more liberal resource management, and $(\cdot)^\sharp : \mathcal{F}(\mathcal{L}_1) \mapsto \mathcal{F}(\mathcal{L}_0\diamond)$ which license relaxation of structure sensitivity in $\mathcal{L}_0$ in such a way that one fully recovers the flexibility of the the coarser $\mathcal{L}_1$.

Our strategy for obtaining the embedding results is quite uniform. It will be helpful to present the recipe first in abstract terms, so that in the following sections we can supply the particular ingredients with reference to the general scheme. The embedding theorems have the format shown below. We call $\mathcal{L}$ the source logic, $\mathcal{L}'$ the target.

$$\mathcal{L} \vdash A \to B \quad \text{iff} \quad \mathcal{L}'\Diamond(+\mathcal{R}_\diamond) \vdash A^\natural \to B^\natural$$

For the constraining perspective, $(\cdot)^\natural$ is $(\cdot)^\flat$ with $\mathcal{L} = \mathcal{L}_0$ and $\mathcal{L}' = \mathcal{L}_1$. For the licensing type of embedding, $(\cdot)^\natural$ is $(\cdot)^\sharp$ with $\mathcal{L} = \mathcal{L}_1$ and $\mathcal{L}' = \mathcal{L}_0$. The embedding translation $(\cdot)^\natural$ decorates critical subformulae in the target logic with the operators $\Diamond, \Box^\downarrow$. The translations are defined on the product $\bullet$ of the source logic: their action on the implicational formulas is fully determined by the residuation laws. A $\bullet$ configuration of the source logic is mapped to the *composition* of $\Diamond$ and the product of the target logic. The elementary compositions are given below (writing $\circ$ for the target product). They mark the product as a whole, or one of the subtypes with the $\Diamond$ control operator.

$$\Diamond(- \circ -) \qquad ((\Diamond -) \circ -) \qquad (- \circ (\Diamond -))$$

Sometimes the modal decoration in itself is enough to obtain the required structural control. We call these cases pure embeddings. In other cases realizing the embedding requires the addition of $\mathcal{R}_\diamond$ — the modalized version of a structural rule package discriminating $\mathcal{L}$ from $\mathcal{L}'$. Typically, this will be the case for communication in the licensing direction: the target logics lack an option for structural manipulation that is present in the source.

The proof of the embedding theorems comes in two parts.

($\Rightarrow$) *Soundness of the embedding.* The ($\Rightarrow$) half is the easy part. Using the Lambek-style axiomatization of 6.5.1 we obtain this direction of the embedding by a straightforward induction on the length of derivations in $\mathcal{L}$.

($\Leftarrow$) *Completeness of the embedding.* For the proofs of the ($\Leftarrow$) part, we reason semantically and rely on the completeness of the logics compared. To show that $\vdash A^\natural \to B^\natural$ in $\mathcal{L}'\Diamond$ implies $\vdash A \to B$ in $\mathcal{L}$ we proceed by contraposition. Suppose $\mathcal{L} \not\vdash A \to B$. By completeness, there is an $\mathcal{L}$ model $\mathcal{M} = \langle F, V \rangle$ falsifying $A \to B$, i.e. there is a point a such that $\mathcal{M}, a \models A$ but $\mathcal{M}, a \not\models B$. We obtain the proof for the ($\Leftarrow$) direction in two steps.

> *Model construction.* From $\mathcal{M}$, we construct an $\mathcal{L}'\Diamond$ model $\mathcal{M}' = \langle F', V' \rangle$. For the valuation, we set $V'(p) = V(p)$. For the frames, we define a mapping between the R^3 configurations in F and corresponding mixed $R^{2'}, R^{3'}$ configurations in F'. We make sure that the mapping reflects the properties of the translation schema, and that it takes into account the different frame conditions for F and F'.

Truth preservation lemma. We prove that for any $a \in W \cap W'$, $\mathcal{M}, a \models A$ iff $\mathcal{M}', a \models A^{\natural}$, i.e. that the construction of $\mathcal{M}'$ is truth preserving.

Now, if $\mathcal{M}$ is a countermodel for $A \to B$, so is $\mathcal{M}'$ for $A^{\natural} \to B^{\natural}$. Soundness then leads us to the conclusion that $\mathcal{L}'\Diamond \not\vdash A^{\natural} \to B^{\natural}$.

With this proof recipe in hand, the reader is prepared to tackle the sections that follow. Recovery of structural discrimination is the subject of §6.2. In §6.3 we turn to licensing of structural relaxation. In §6.4 we reflect on general logical and linguistic features of the proposed architecture, signaling some open questions and directions for future research.

6.2 Imposing structural constraints

Let us first look at the embedding of more discriminating logics within systems with a less fine-grained sense of structure sensitivity. Modal decoration, in this case, serves to block structural manipulation that would be available by default. The section is organized as follows. In §6.2.1, we give a detailed treatment of a representative case for each of the structural dimensions of precedence, dominance and dependency. This covers the edges connected to the pure logic of residuation, **NL**. With minor adaptions the embedding translations of §6.2.1 can be extended to the remaining edges, with the exception of the four associative logics at the right back face of the cube. We present these generalizations in §6.2.2. This time we refrain from fully explicit treatment where extrapolation from §6.2.1 is straightforward. The remaining systems are treated in §6.2.3. They share associative resource management but differ in their sensitivity for linear order or dependency structure. We obtain the desired embeddings in these cases via a tactical manoeuvre which combines the composition of simple translation schemata and the reinstallment of Associativity via *modally controlled* structural postulates.

6.2.1 Simple embeddings

Associativity

Consider first the pair **NL** versus **L**$\Diamond$. Let us subscript the symbols for the connectives in **NL** with 0 and those of **L** with 1. The **L** family $/_1, \bullet_1, \backslash_1$ has an associative resource management. We extend **L** with the operators $\Diamond, \Box^{\downarrow}$ and recover control over associativity by means of the following translation.

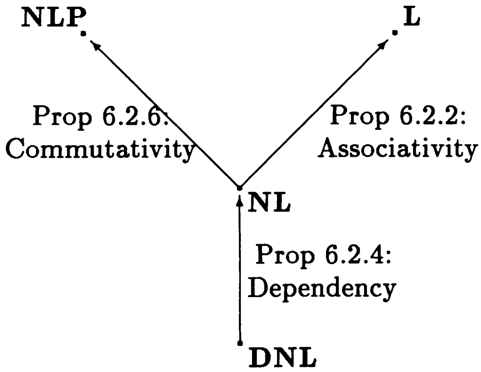


Figure 6.2: Imposing constraints: precedence, dominance, dependency

6.2.1. DEFINITION. Translation $\cdot^\flat : \mathcal{F}(\mathbf{NL}) \mapsto \mathcal{F}(\mathbf{L}\Diamond)$ as below.

$$\begin{aligned} p^\flat &= p \\ (A \bullet_0 B)^\flat &= \Diamond(A^\flat \bullet_1 B^\flat) \\ (A/_0B)^\flat &= \Box^\downarrow A^\flat/_1 B^\flat \\ (B\backslash_0 A)^\flat &= B^\flat\backslash_1 \Box^\downarrow A^\flat \end{aligned}$$

6.2.2. PROPOSITION.

$$\mathbf{NL} \vdash A \to B \quad \text{iff} \quad \mathbf{L}\Diamond \vdash A^\flat \to B^\flat$$

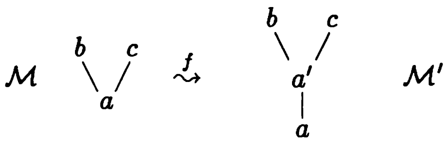
Proof. *($\Rightarrow$) Soundness of the embedding.* For the left-to-right direction we use induction on the length of derivations in **NL** on the basis of the Lambek-style axiomatization given in the Appendix, where apart from the identity axiom and Transitivity, the Residuation rules are the only rules of inference. Assume $A \bullet_0 B \to C$ is derived from $A \to C/_0B$ in **NL**. By inductive hypothesis, $\mathbf{L} \vdash A^\flat \to (C/_1B)^\flat$, i.e. (†) $A^\flat \to \Box^\downarrow C^\flat/_1 B^\flat$. We have to show (‡) $\mathbf{L} \vdash (A \bullet_1 B)^\flat \to C^\flat$, i.e. $\Diamond(A^\flat \bullet_1 B^\flat) \to C^\flat$. By RES(2) we have from (†) $A^\flat \bullet_1 B^\flat \to \Box^\downarrow C^\flat$ which derives (‡) by RES(1). For the other side of the residuation inferences, assume $A \to C/_0B$ is derived from $A \bullet_0 B \to C$. By inductive hypothesis, $\mathbf{L} \vdash (A \bullet_1 B^\flat) \to C^\flat$, i.e. (‡) $\Diamond(A^\flat \bullet_1 B^\flat) \to C^\flat$. We have to show $\mathbf{L} \vdash A^\flat \to C/_1B^\flat$, i.e. (†) $A^\flat \to \Box^\downarrow C^\flat/_1 B^\flat$. By RES(1) we have from (‡) $A^\flat \bullet_1 B^\flat \to \Box^\downarrow C^\flat$ which derives (†) by RES(2). The residual pair $(\bullet_0, \backslash_0)$ is treated in a fully symmetrical way. □

($\Leftarrow$) Completeness of the embedding. We apply the method outlined in §6.1. From a falsifying model $\mathcal{M} = \langle W, R_0^3, V\rangle$ for $A \to B$ in **NL** we construct $\mathcal{M}' = \langle W', R_1^3, R_\Diamond^2, V'\rangle$. We prove that the construction is truth preserving, so that we can conclude from Soundness that $\mathcal{M}'$ falsifies $A^\flat \to B^\flat$ in $\mathbf{L}\Diamond$.

Model construction. Let W_1 be a set such that $W \cap W_1 = \emptyset$ and $f : R_0^3 \mapsto W_1$ a bijection associating each triple $(abc) \in R_0^3$ with a fresh point $f((abc)) \in W_1$. $\mathcal{M}'$ is defined as follows:

$$\begin{array}{rcl} W' & = & W \cup W_1 \\ R_1 & = & \{(a'bc) \mid \exists a.R_0abc \wedge f((abc)) = a'\} \\ R_\diamond & = & \{(aa') \mid \exists bc.R_0abc \wedge f((abc)) = a'\} \\ V'(p) & = & V(p) \end{array}$$

The following picture will help the reader to visualize how the model construction relates to the translation schema.



$$A \bullet_0 B \overset{\flat}{\rightsquigarrow} \Diamond(A^\flat \bullet_1 B^\flat)$$

We have to show that $\mathcal{M}'$ is an appropriate model for $\mathbf{L}$, i.e. that the construction of $\mathcal{M}'$ realizes the frame condition for associativity:

$$F(A) \quad \forall xyzw \in W'(\exists t(R_1wxt \wedge R_1tyz) \Longleftrightarrow \exists t'(R_1wt'z \wedge R_1t'xy))$$

$F(A)$ is satisfied automatically because, by the construction of $\mathcal{M}'$, there are no $x, y, z, w \in W'$ that fulfill the requirements: for every triple $(xyz) \in R_1^3$, the point x is chosen fresh, which implies that no point of W' can be both the root of one triangle and a leaf in another one.

Lemma: Truth Preservation. By induction on the complexity of A we show that for any $a \in W$

$$\mathcal{M}, a \models A \quad \text{iff} \quad \mathcal{M}', a \models A^\flat$$

We prove the biconditional for the product and for one of the residual implications.

($\Rightarrow$). Suppose $\mathcal{M}, a \models A \bullet_0 B$. By the truth conditions for $\bullet_0$, there exist b, c such that (i) R_0abc and (ii) $\mathcal{M}, b \models A$, (iii) $\mathcal{M}, c \models B$. By inductive hypothesis, from (ii) and (iii) we have (ii') $\mathcal{M}', b \models A^\flat$ and (iii') $\mathcal{M}', c \models B^\flat$. By the construction of $\mathcal{M}'$, we conclude from (i) that there is a fresh $a' \in W_1$ such that (iv) $R_\diamond aa'$ and (v) $R_1a'bc$. Then, from (v) and (ii',iii') we have $\mathcal{M}', a' \models A^\flat \bullet_1 B^\flat$ and from (iv) $\mathcal{M}', a \models \Diamond(A^\flat \bullet_1 B^\flat)$.

($\Leftarrow$). Suppose $\mathcal{M}', a \models \Diamond(A^\flat \bullet_1 B^\flat)$. From the truth conditions for $\bullet_1, \Diamond$, we know there are $x, y, z \in W'$ such that (i) $R_\diamond ax$, (ii) R_1xyz and (iii) $\mathcal{M}', y \models A^\flat$ and $\mathcal{M}', z \models B^\flat$. In he construction of $\mathcal{M}'$ the function f is a bijection, so that we can conclude that the configuration (i,ii) has a unique preimage, namely (iv) R_0ayz. By inductive hypothesis, we have from (iii) $\mathcal{M}, y \models A$, and $\mathcal{M}, z \models B$, which then with (iv) gives $\mathcal{M}, a \models A \bullet_0 B$.

($\Rightarrow$). Suppose (i) $\mathcal{M}, a \models A\backslash_0 B$. We have to show $\mathcal{M}', a \models A^\flat\backslash_1\Box^\downarrow B^\flat$. Suppose we have (ii) R_1yxa such that $\mathcal{M}', x \models A^\flat$. It remains to be shown that $\mathcal{M}', y \models \Box^\downarrow B^\flat$. Suppose we have (iii) $R_\diamond zy$. It remains to be shown that $\mathcal{M}', z \models B^\flat$. The configuration (ii,iii) has a unique preimage by the construction of $\mathcal{M}'$, namely R_0zxa. By inductive hypothesis from (ii) we have $\mathcal{M}, x \models A$ which together with (i) leads to $\mathcal{M}, z \models B$ and, again by inductive hypothesis $\mathcal{M}', z \models B^\flat$, as required.

($\Leftarrow$). Suppose (i) $\mathcal{M}', a \models A^\flat\backslash_1\Box^\downarrow B^\flat$. We have to show $\mathcal{M}, a \models A\backslash_0 B$. Suppose we have (ii) R_0cba such that $\mathcal{M}, b \models A$. To be shown is whether $\mathcal{M}, c \models B$. By the model construction and inductive hypothesis we have $R_\diamond cc'$, $R_1c'ba$ and $\mathcal{M}', b \models A^\flat$. Hence by (i) $\mathcal{M}', c \models \Box^\downarrow B^\flat$ and therefore $\mathcal{M}', c \models B^\flat$. By inductive hypothesis this leads to $\mathcal{M}, c \models B$ as required. $\Box$

ILLUSTRATION: ISLANDS. For a concrete linguistic illustration, we return to the Coordinate Structure Constraint violations of §6.1. The translation schema of Def 6.2.1 was originally proposed by [Morrill 92], who conjectured on the basis of this schema an embedding of **NL** into **L** extended with a pair of unary 'bracket' operators closely related to $\Diamond, \Box^\downarrow$. Whether the conjecture holds for the bracket operators remains open. But it is easy to recast Morrill's analysis of island constraints in terms of $\Diamond, \Box^\downarrow$. We saw above that on an assignment $(X\backslash X)/X$ to the particle 'and', both the grammatical and the illformed examples are **L** derivable. Within **L**$\Diamond$, we can refine the assignment to $(X\backslash\Box^\downarrow X)/X$. The relevant sequent goals now assume the following form (omitting the associative binary structural punctuation, but keeping the crucial $(\cdot)^\diamond$):

(†)	the book that Melville wrote and John read	
$\mathbf{L} \vdash$	$r/(s/np), (np, (np\backslash s)/np, (X\backslash\Box^\downarrow X)/X, np, (np\backslash s)/np)^\diamond \Rightarrow r$	$(X = s/np)$
(‡)	*the book that Melville wrote Moby Dick and John read	
$\mathbf{L}\Diamond \not\vdash$	$r/(s/np), (np, (np\backslash s)/np, np, (X\backslash\Box^\downarrow X)/X, np, (np\backslash s)/np)^\diamond \Rightarrow r$	$(X = s)$

The $(X\backslash\Box^\downarrow X)/X$ assignment allows the particle 'and' to combine with the left and right conjuncts in the associative mode. The resulting coordinate structure is of type $\Box^\downarrow X$. To eliminate the $\Box^\downarrow$ connective, we have to close off the coordinate structure with $\Diamond$ (or the corresponding structural operator $(\cdot)^\diamond$ in the Gentzen presentation) — recall that $\Diamond\Box^\downarrow X \rightarrow X$. The Accross-the-Board case of extraction (†) works out fine, the island violation (‡) fails because the hypothetical gap np assumption finds itself outside the scope of the $(\cdot)^\diamond$ operator.

Dependency

For a second straightforward application of the method, we consider the dependency calculus **DNL** of [Moort. & Morrill 91] and show how it can be emdedded in **NL**. Recall that **DNL** is the pure logic of residuation for a bimodal system with asymmetric products $\bullet_l, \bullet_r$ for left-headed and right-headed composition respectively. The distinction between left- and right-headed products can be recovered within **NL**$\Diamond$, where we have the unary residuated pair $\Diamond, \Box^{\downarrow}$ next to a symmetric product $\bullet$ and its implications. For the embedding translation $(\cdot)^\flat$, we label the head subtype of a product with $\Diamond$. The residuation laws then determine the modal decoration of the implications.

6.2.3. DEFINITION. The embedding translation $(\cdot)^\flat : \mathcal{F}(\mathbf{DNL}) \mapsto \mathcal{F}(\mathbf{NL}\Diamond)$ is defined as follows.

$$\begin{array}{ll} & p^\flat = p \\ (A \bullet_l B)^\flat = \Diamond A^\flat \bullet B^\flat & (A \bullet_r B)^\flat = A^\flat \bullet \Diamond B^\flat \\ (A/_l B)^\flat = \Box^{\downarrow}(A^\flat / B^\flat) & (A/_r B)^\flat = A^\flat / \Diamond B^\flat \\ (B\backslash_l A)^\flat = \Diamond B^\flat \backslash A^\flat & (B\backslash_r A)^\flat = \Box^{\downarrow}(B^\flat \backslash A^\flat) \end{array}$$

6.2.4. PROPOSITION.

$$\mathbf{DNL} \vdash A \to B \quad \text{iff} \quad \mathbf{NL}\Diamond \vdash A^\flat \to B^\flat$$

Proof. ($\Rightarrow$) Soundness of the embedding. The soundness half is proved by induction on the length of the derivation of $A \to B$ in **DNL**. We trace the residuation inferences under the translation mapping for the pair $(\bullet_l, /_l)$. The remaining cases are completely parallel.

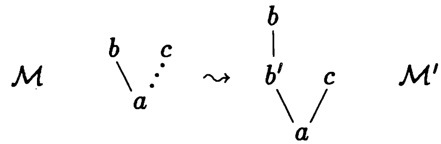
$$\mathbf{DNL} \quad \frac{A \bullet_l B \to C}{A \to C/_l B} \quad \rightsquigarrow \quad \frac{(A \bullet_l B)^\flat \to C^\flat}{A^\flat \to (C/_l B)^\flat} \quad \rightsquigarrow \quad \cfrac{\cfrac{\Diamond A^\flat \bullet B^\flat \to C^\flat}{\Diamond A^\flat \to C^\flat / B^\flat}}{A^\flat \to \Box^{\downarrow}(C^\flat / B^\flat)} \quad \mathbf{NL}$$

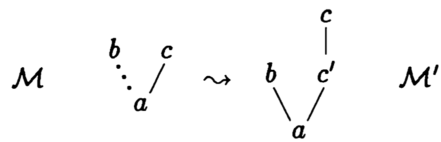
($\Leftarrow$) Completeness of the embedding. Suppose **DNL** $\not\vdash A \to B$. By completeness, there is a model $\mathcal{M} = \langle W, R^3_l, R^3_r, V\rangle$ falsifying $A \to B$. From $\mathcal{M}$, we want to construct a model $\mathcal{M}' = \langle W', R^3_\bullet, R^2_\diamond, V'\rangle$ which falsifies $A^\flat \to B^\flat$. Then from soundness we will be able to conclude **NL**$\Diamond \not\vdash A^\flat \to B^\flat$.

Model construction. Let W, W_l, W_r be disjoint sets and $f : R^3_l \mapsto W_l$ and $g : R^3_r \mapsto W_r$ bijective functions. $\mathcal{M}'$ is defined as follows:

$$\begin{array}{rcl} W' & = & W \cup W_l \cup W_r \\ R_\bullet & = & \{(ab'c) \mid \exists b.R_l abc \wedge f((abc)) = b'\} \quad \cup \\ & & \{(abc') \mid \exists c.R_r abc \wedge g((abc)) = c'\} \\ R_\diamond & = & \{(c'c) \mid \exists ab.R_r abc \wedge g((abc)) = c'\} \quad \cup \\ & & \{(b'b) \mid \exists ac.R_l abc \wedge f((abc)) = b'\} \\ V'(p) & = & V(p) \end{array}$$

We comment on the frames. For every triple $(abc) \in R_l^3$, we introduce a fresh b' and put the worlds $a, b, b', c \in W'$, $(b'b) \in R_\diamond^2$ and $(ab'c) \in R_\bullet^3$. Similarly, for every triple $(abc) \in R_r^3$, we introduce a fresh c' and put the worlds $a, b, c, c' \in W'$, $(c'c) \in R_\diamond^2$ and $(abc') \in R_\bullet^3$. In a picture (with dotted lines for the dependent daughter for R_l, R_r):





Lemma: truth preservation. By induction on the complexity of A, we show that for any $a \in W$, $\mathcal{M}, a \models A$ iff $\mathcal{M}', a \models A^\flat$. We prove the biconditional for the left-headed product. The other connectives are handled in a similar way.

($\Rightarrow$). Suppose $\mathcal{M}, a \models A \bullet_l B$. By the truth conditions for $\bullet_l$, there exist b, c such that (i) $R_l abc$ and (ii) $\mathcal{M}, b \models A$, (iii) $\mathcal{M}, c \models B$. By the construction of $\mathcal{M}'$, we conclude from (i) that there is a fresh $b' \in W'$ such that (iv) $R_\diamond^2 b'b$ and (v) $R_\bullet^3 ab'c$. By inductive hypothesis, from (ii) and (iii) we have $\mathcal{M}', b \models A^\flat$ and $\mathcal{M}', c \models B^\flat$. Then, from (iv) we have $\mathcal{M}', b' \models \Diamond A^\flat$ and from (v), $\mathcal{M}', a \models \Diamond A^\flat \bullet B^\flat$.

($\Leftarrow$). Suppose $\mathcal{M}', a \models \Diamond A^\flat \bullet B^\flat$. From the truth conditions for $\bullet, \Diamond$, we know there are $d', d, e \in W'$ such that (i) $R_\diamond^2 d'd$, (ii) $R_\bullet^3 ad'e$ and (iii) $\mathcal{M}', d \models A^\flat$ and $\mathcal{M}', e \models B^\flat$. From the construction of $\mathcal{M}'$, we may conclude that $d' = b', d = b, e = c$, since every triple $(abc) \in R_l^3$ is keyed to a fresh world $b' \in W'$. So we actually have (i') $R_\diamond^2 b'b$, (ii') $R_\bullet^3 ab'c$ and (iii') $\mathcal{M}', b \models A^\flat$ and $\mathcal{M}', c \models B^\flat$. (i') and (ii') imply $R_l^3 abc$. By inductive hypothesis, we have from (iii') $\mathcal{M}, b \models A$, and $\mathcal{M}, c \models B$. But then $\mathcal{M}, a \models A \bullet_l B$. □

ILLUSTRATION. Below two instances of lifting in **DNL**. The left one is derivable, the right one is not.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^\flat \Rightarrow A^\flat}{(A^\flat)^\diamond \Rightarrow \Diamond A^\flat}\,\Diamond R \qquad B^\flat \Rightarrow B^\flat}{((A^\flat)^\diamond, \Diamond A^\flat\backslash B^\flat)^\bullet \Rightarrow B^\flat}\,\backslash L}{(A^\flat)^\diamond \Rightarrow B^\flat/(\Diamond A^\flat\backslash B^\flat)}\,/R}{A^\flat \Rightarrow \Box^\downarrow(B^\flat/(\Diamond A^\flat\backslash B^\flat))}\,\Box^\downarrow R}{A \Rightarrow B/_l(A\backslash_l B)}\,\cdot^\flat \qquad \dfrac{\dfrac{\dfrac{\dfrac{?}{((A^\flat)^\diamond, \Box^\downarrow(A^\flat\backslash B^\flat))^\bullet \Rightarrow B^\flat}}{(A^\flat)^\diamond \Rightarrow B^\flat/\Box^\downarrow(A^\flat\backslash B^\flat)}\,/R}{A^\flat \Rightarrow \Box^\downarrow(B^\flat/\Box^\downarrow(A^\flat\backslash B^\flat))}\,\Box^\downarrow R}{A \Rightarrow B/_l(A\backslash_r B)}\,\cdot^\flat$$

Commutativity

We can exploit the strategy for modal embedding of the dependency calculus to recover control over Permutation. Here we look at the pure case: the embedding of **NL** into **NLP**$\Diamond$. In §6.2.2 we will generalize the result to the other cases where Permutation is involved. For the embedding, choose one of the (asymmetric) dependency product translations for $\bullet$ in **NL**. Permutation in **NLP** spoils the asymmetry of the product. Whereas one could read the $\Diamond$ label in the cases of Def6.2.3 as a head marker, in the present case $\Diamond$ functions as a marker of the first daughter.

6.2.5. DEFINITION. The embedding translation $\cdot^\flat : \mathcal{F}(\mathbf{NL}) \mapsto \mathcal{F}(\mathbf{NLP}\Diamond)$ is defined as follows.

$$\begin{aligned} p^\flat &= p \\ (A \bullet B)^\flat &= \Diamond A^\flat \otimes B^\flat \\ (A/B)^\flat &= \Box^\downarrow(A^\flat \circ\!\!- B^\flat) \\ (B\backslash A)^\flat &= \Diamond B^\flat -\!\!\circ A^\flat \end{aligned}$$

6.2.6. PROPOSITION.

$$\mathbf{NL} \vdash A \to B \quad \text{iff} \quad \mathbf{NLP}\Diamond \vdash A^\flat \to B^\flat$$

Proof sketch. The ($\Rightarrow$) part again is proved straightforwardly by induction on the length of the derivation of $A \to B$ in **NL**. We leave this to the reader. For the ($\Leftarrow$) direction, suppose $\mathbf{NL} \not\vdash A \to B$. By completeness, there is a model $\mathcal{M} = \langle W, R^3_\bullet, V\rangle$ falsifying $A \to B$. From $\mathcal{M}$, we now have to construct a *commutative* model $\mathcal{M}' = \langle W', R^3_\otimes, R^2_\Diamond, V'\rangle$ which falsifies $A^\flat \to B^\flat$. From soundness we will conclude that $\mathbf{NLP}\Diamond \not\vdash A^\flat \to B^\flat$.

The construction of the frame for $\mathcal{M}'$ in this case proceeds as follows. For every triple $(abc) \in R^3_\bullet$, we introduce a fresh b' and put the worlds $a, b, b', c \in W'$, $(b'b) \in R^2_\Diamond$ and both $(ab'c), (acb') \in R^3_\otimes$. The construction makes the frame for $\mathcal{M}'$ commutative. But because every commutative triple $(ab'c)$ depends on a fresh $b' \in W' - W$, the commutativity of $\mathcal{M}'$ has no influence on $\mathcal{M}$. For the valuation, we set $V'(p) = V(p)$. Now for any $a \in W \cap W'$, we can show by induction on the complexity of A that $\mathcal{M}, a \models A$ iff $\mathcal{M}', a \models A^\flat$ which then leads to the proof of the main proposition in the usual way.

ILLUSTRATION. Below first a theorem of **NL**, followed by a non-theorem. We compare their image under $\cdot^\flat$ in **NLP**$\Diamond$. And we notice that the second example *is* derivable in **NLP**.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B^\flat \Rightarrow B^\flat \quad A^\flat \Rightarrow A^\flat}{(A^\flat \circ\!- B^\flat, B^\flat)^\otimes \Rightarrow A^\flat}\ \circ\!-L}{((\Box^{\downarrow}(A^\flat \circ\!- B^\flat))^\diamond, B^\flat)^\otimes \Rightarrow A^\flat}\ \Box^{\downarrow}L}{(\Box^{\downarrow}(A^\flat \circ\!- B^\flat))^\diamond \Rightarrow A^\flat \circ\!- B^\flat}\ \circ\!-R}{\Box^{\downarrow}(A^\flat \circ\!- B^\flat) \Rightarrow \Box^{\downarrow}(A^\flat \circ\!- B^\flat)}\ \Box^{\downarrow}R}{(\Box^{\downarrow}(A^\flat \circ\!- B^\flat))^\diamond \Rightarrow \Diamond\Box^{\downarrow}(A^\flat \circ\!- B^\flat)}\ \Diamond R \qquad A^\flat \Rightarrow A^\flat}{((\Box^{\downarrow}(A^\flat \circ\!- B^\flat))^\diamond, \Diamond\Box^{\downarrow}(A^\flat \circ\!- B^\flat) -\!\circ A^\flat)^\otimes \Rightarrow A^\flat}\ -\!\circ L}{(\Box^{\downarrow}(A^\flat \circ\!- B^\flat))^\diamond \Rightarrow A^\flat \circ\!- (\Diamond\Box^{\downarrow}(A^\flat \circ\!- B^\flat) -\!\circ A^\flat)}\ \circ\!-R}{\Box^{\downarrow}(A^\flat \circ\!- B^\flat) \Rightarrow \Box^{\downarrow}(A^\flat \circ\!- (\Diamond\Box^{\downarrow}(A^\flat \circ\!- B^\flat) -\!\circ A^\flat))}\ \Box^{\downarrow}R}{\mathbf{NL} \vdash A/B \Rightarrow A/((A/B)\backslash A)}\ \cdot^\flat$$

$$\dfrac{\dfrac{\dfrac{?}{((\Box^{\downarrow}(A^\flat \circ\!- B^\flat))^\diamond, \Box^{\downarrow}(A^\flat \circ\!- (\Diamond B^\flat -\!\circ A^\flat)))^\otimes \Rightarrow A^\flat}}{(\Box^{\downarrow}(A^\flat \circ\!- B^\flat))^\diamond \Rightarrow A^\flat \circ\!- (\Box^{\downarrow}(A^\flat \circ\!- (\Diamond B^\flat -\!\circ A^\flat))}\ \circ\!-R}{\Box^{\downarrow}(A^\flat \circ\!- B^\flat) \Rightarrow \Box^{\downarrow}(A^\flat \circ\!- (\Box^{\downarrow}(A^\flat \circ\!- (\Diamond B^\flat -\!\circ A^\flat)))}\ \Box^{\downarrow}R}{\mathbf{NL} \nvdash A/B \Rightarrow A/(A/(B\backslash A))}\ \cdot^\flat$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B \quad A \Rightarrow A}{(A \circ\!- B, B)^\otimes \Rightarrow A}\ \circ\!-L}{(B, A \circ\!- B)^\otimes \Rightarrow A}\ P}{A \circ\!- B \Rightarrow B -\!\circ A}\ -\!\circ R \qquad A \Rightarrow A}{(A \circ\!- (B -\!\circ A), A \circ\!- B)^\otimes \Rightarrow A}\ \circ\!-L}{(A \circ\!- B, A \circ\!- (B -\!\circ A))^\otimes \Rightarrow A}\ P}{\mathbf{NLP} \vdash A \circ\!- B \Rightarrow A \circ\!- (A \circ\!- (B -\!\circ A))}\ \circ\!-R$$

6.2.2 Generalizations

The results of the previous section can be extended with minor modifications to the five edges that remain when we keep the Associativity face for §6.2.3.

What we have done in Prop 6.2.4 for the pair **DNL** versus **NL**$\Diamond$ can be adapted straightforwardly to the commutative pair **DNLP** versus **NLP**$\Diamond$. Recall that in **DNLP**, the dependency products satisfy head-preserving commutativity ($P_{l,r}$), whereas in **NLP** we have simple commutativity (P).

$$\begin{array}{rl} P_{l,r}: & A \otimes_l B \longleftrightarrow B \otimes_r A \\ P: & A \otimes B \rightarrow B \otimes A \end{array}$$

Accommodating the commutative products, the embedding translation is that of Prop 6.2.4: $\Diamond$ marks the head subtype.

6.2.7. Definition. Translation $(\cdot)^\flat : \mathcal{F}(\mathbf{DNLP}) \mapsto \mathcal{F}(\mathbf{NLP}\Diamond)$:

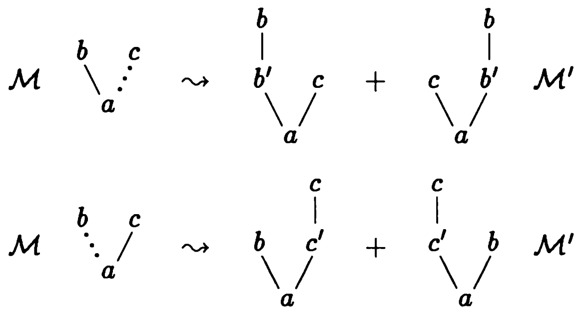
$$p^\flat = p$$

$$\begin{array}{ll}
(A \otimes_l B)^\flat = \Diamond A^\flat \otimes B^\flat & (A \otimes_r B)^\flat = A^\flat \otimes \Diamond B^\flat \\
(A \circ\!-_l B)^\flat = \Box^\downarrow(A^\flat \circ\!- B^\flat) & (A \circ\!-_r B)^\flat = A^\flat \circ\!- \Diamond B^\flat \\
(B -\!\circ_l A)^\flat = \Diamond B^\flat -\!\circ A^\flat & (B -\!\circ_r A)^\flat = \Box^\downarrow(B^\flat -\!\circ A^\flat)
\end{array}$$

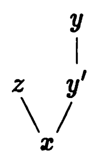
6.2.8. Proposition.

$$\mathbf{DNLP} \vdash A \to B \quad \text{iff} \quad \mathbf{NLP}\Diamond \vdash A^\flat \to B^\flat$$

For the proof of the ($\Leftarrow$) direction, we combine the method of construction of Prop 6.2.4 with that of Prop 6.2.6. For a configuration $R_l^\otimes abc$ in $\mathcal{M}$, we take fresh b' and put the configurations $R_\diamond b'b, R_\otimes ab'c, R_\otimes acb'$ in $\mathcal{M}'$. Similarly, for a configuration $R_r^\otimes abc$ in $\mathcal{M}$, we take fresh c' and put the configurations $R_\diamond c'c, R_\otimes abc', R_\otimes ac'b$ in $\mathcal{M}'$. The commutativity property of $\otimes$ is thus realized by the construction.



Let us check the truth preservation lemma. This time a configuration ($\star$) in $\mathcal{M}'$ does not have a unique pre-image: it can come from $R_l^\otimes xyz$ or $R_r^\otimes xzy$. But because of head-preserving commutativity (DP), these are both in $\mathcal{M}$.

($\star$) 

Similarly, the embedding construction presented in Prop 6.2.6 for the pair **NL** versus **NLP**$\Diamond$ can be generalized directly to the related pair **DNL** versus **DNLP**$\Diamond$. This time, we want the embedding translation to block the structural postulate of *head-preserving* commutativity in **DNLP**. The translation below invalidates the postulate by uniformly decorating with $\Diamond$, say, the left subtype of a product.

6.2.9. Definition. Define $(\cdot)^\flat : \mathcal{F}(\mathbf{DNL}) \mapsto \mathcal{F}(\mathbf{DNLP}\Diamond)$ as follows.

$$\begin{array}{rlrl} & p^\flat = p & & \\ (A \bullet_l B)^\flat &= \Diamond A^\flat \otimes_l B^\flat & (A \bullet_r B)^\flat &= \Diamond A^\flat \otimes_r B^\flat \\ (A/_l B\ ^\flat &= \Box^\downarrow(A^\flat \circ\!-_l B^\flat) & (A/_r B)^\flat &= \Box^\downarrow(A^\flat \circ\!-_r B^\flat) \\ (B\backslash_l A)^\flat &= \Diamond B^\flat -\!\circ_l A^\flat & (B\backslash_r A)^\flat &= \Diamond B^\flat -\!\circ_r A^\flat \end{array}$$

We then have the following proposition. The proof is entirely parallel to that of Prop 6.2.6 before.

6.2.10. Proposition.

$$\mathbf{DNL} \vdash A \to B \quad \text{iff} \quad \mathbf{DNLP}\Diamond \vdash A^\flat \to B^\flat$$

The method of Prop 6.2.2 generalizes to the following cases with some simple changes.

6.2.11. Definition. Translation $(\cdot)^\flat : \mathcal{F}(\mathbf{NLP}) \mapsto \mathcal{F}(\mathbf{LP}\Diamond)$ as below.

$$\begin{aligned} p^\flat &= p \\ (A \otimes B)^\flat &= \Diamond(A^\flat \otimes B^\flat) \\ (A \circ\!- B)^\flat &= \Box^\downarrow A^\flat \circ\!- B^\flat \\ (B -\!\circ A)^\flat &= B^\flat -\!\circ \Box^\downarrow A^\flat \end{aligned}$$

6.2.12. Proposition.

$$\mathbf{NLP} \vdash A \to B \quad \text{iff} \quad \mathbf{LP}\Diamond \vdash A^\flat \to B^\flat$$

The only difference with Prop 6.2.2 is that the product in input and target logic are commutative. Commutativity is realized automatically by the construction of $\mathcal{M}'$.

6.2.13. Proposition.

$$\mathbf{DNL} \vdash A \to B \quad \text{iff} \quad \mathbf{DL}\Diamond \vdash A^\flat \to B^\flat$$

6.2.14. Proposition.

$$\mathbf{DNLP} \vdash A \to B \quad \text{iff} \quad \mathbf{DLP}\Diamond \vdash A^\flat \to B^\flat$$

6.2.3 Composed translations

The remaining cases concern the right back face of the cube, where we find the systems **DL**, **L**, **LP**, and **DLP**. These logics share associative resource management, but they differ with respect to one of the remaining structural parameters — sensitivity for linear order (**L** versus **LP**, **DL** versus **DLP**) or for dependency structure (**DL** versus **L**, and **DLP** versus **LP**). We already know how to handle each of the structural dimensions individually. We use this knowledge to obtain the embeddings for systems with shared Associativity. Our strategy has two components. First we neutralize direct appeal to Associativity by taking the composition of the translation schema blocking Associativity with the schema responsible for control in the structural dimension which discriminates between the source and target logics. This first move does *not* embed the source logic, but its non-associative neighbour. The second move then is to reinstall associativity in terms of $\diamond$ modally controlled versions of the Associativity postulates.

ASSOCIATIVE DEPENDENCY CALCULUS. We work out the 'rear attack' manoeuvre first for the pair **DL** versus **L**. In **DNL** we have no restrictions on the interpretation of $\bullet_l, \bullet_r$. In **DL** we assume $\bullet_l, \bullet_r$ are interpreted on (bimodal) associative frames, and we have structural associativity postulates $A(l), A(r)$ on top of the pure logic of residuation for $\bullet_l, \bullet_r$. In **L** we cannot discriminate between $\bullet_l$ and $\bullet_r$ — there is just one $\bullet$ operator, which shares the associative resource management with its dependency variants. The objective of the embedding is to recover the distinction between left- and right-headed structures in a system which has only one product connective.

$$
\begin{array}{c}
A(l) : (A \bullet_l B) \bullet_l C \longleftrightarrow A \bullet_l (B \bullet_l C) \\
A(r) : A \bullet_r (B \bullet_r C) \longleftrightarrow (A \bullet_r B) \bullet_r C
\end{array}
$$

For the embedding translation, we *compose* the mappings of Def 6.2.3 embedding **DNL** into **NL** and Def 6.2.1 embedding **NL** into **L**.

6.2.15. DEFINITION.

$$
\begin{array}{cc}
\multicolumn{2}{c}{p^\flat = p} \\
(A \bullet_l B)^\flat = \diamond(\diamond A^\flat \bullet B^\flat) & (A \bullet_r B)^\flat = \diamond(A^\flat \bullet \diamond B^\flat) \\
(A/_l B)^\flat = \Box^\downarrow(\Box^\downarrow A^\flat / B^\flat) & (A/_r B)^\flat = \Box^\downarrow A^\flat / \diamond B^\flat \\
(B\backslash_l A)^\flat = \diamond B^\flat \backslash \Box^\downarrow A^\flat & (B\backslash_r A)^\flat = \Box^\downarrow(B^\flat \backslash \Box^\downarrow A^\flat)
\end{array}
$$

From the proof of the embedding of **NL** into **L** we know that $\diamond$ neutralizes the effects of the associativity of $\bullet$ in the target logic **L**: the frame condition for Associativity is satisfied vacuously. To realize the desired embedding of **DL** into **L**, we reinstall modal versions of the associativity postulates.

$$
\begin{array}{c}
A(l)^\diamond : \diamond(\diamond\diamond(\diamond A \bullet B) \bullet C \longleftrightarrow \diamond(\diamond A \bullet \diamond(\diamond B \bullet C)) \\
A(r)^\diamond : \diamond(A \bullet \diamond\diamond(B \bullet \diamond C)) \longleftrightarrow \diamond(\diamond(A \bullet \diamond B) \bullet \diamond C)
\end{array}
$$

Figure 6.3 is a graphical illustration of the interplay between the composed translation schema and the modal structural postulate. f is the translation schema $(\cdot)^{\flat}$ of Def 6.2.1, g that of Def 6.2.3.

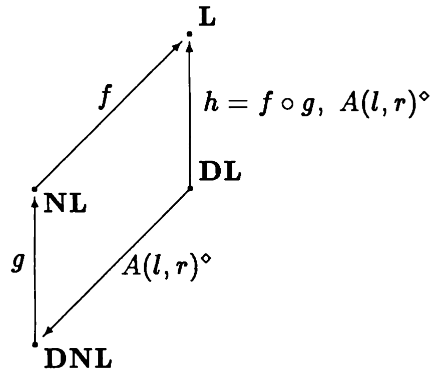


Figure 6.3: Rear Attack Embedding **DL** into **L**.

Modalized structural postulates: frame completeness. The modalized structural postulates $A(l,r)^{\diamond}$ introduce a new element in the discussion. Semantically, these postulates require frame constraints correlating the binary and ternary relations of structural composition. Fortunately we know, from the generalized Sahlqvist-van Benthem Theorem and frame completeness result discussed in §6.1, that from $A(l,r)^{\diamond}$ we can effectively obtain the relevant first order frame conditions, and that completeness of $\mathbf{L}\diamond$ extends to the system augmented with $A(l,r)^{\diamond}$. We check completeness for $A(l)^{\diamond}$ here as an illustration — the situation for $A(r)^{\diamond}$ is entirely similar. Fig 4 gives the frame condition for $A(l)^{\diamond}$.

The models for $\mathbf{L}\diamond$ are structures $\langle W, R^2_{\diamond}, R^3_{\bullet}, V\rangle$. Now consider ($\Rightarrow$) in Figure 4 below. Given the canonical model construction of Def 6.1.1 the following are derivable by the definition of $R^2_{\diamond}, R^3_{\bullet}$:

$$\begin{array}{ll} a \to \diamond b, & e \to \diamond f, \\ b \to c \bullet d, & f \to g \bullet h, \\ c \to \diamond e, & g \to \diamond i. \end{array}$$

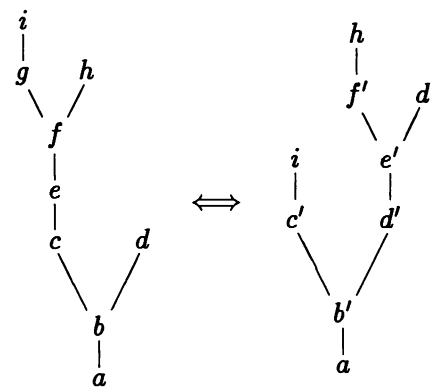
From these we can conclude $\vdash a \to \diamond(\diamond\diamond(\diamond i \bullet h) \bullet d))$, i.e. $a \in V(\diamond(\diamond\diamond(\diamond i \bullet h) \bullet d)))$, given the definition of the canonical valuation ($\star$). For ($\ddagger$) we have to find b', c', d', e', f' such that

$$\begin{array}{ll} a \to \diamond b', & d' \to \diamond e', \\ b' \to c' \bullet d', & e' \to f' \bullet d, \\ c' \to \diamond i, & f' \to \diamond h. \end{array}$$

Let us put

$$
\begin{aligned}
&f' = \Diamond h,\\
&e' = f' \bullet d = \Diamond h \bullet d,\\
&d' = \Diamond e' = \Diamond(\Diamond h \bullet d),\\
&c' = \Diamond i,\\
&b' = c' \bullet d' = \Diamond i \bullet \Diamond(\Diamond h \bullet d).
\end{aligned}
$$

Together they imply $\vdash a \to \Diamond(\Diamond i \bullet \Diamond(\Diamond h \bullet d))$, i.e. $a \in V(\Diamond(\Diamond i \bullet \Diamond(\Diamond h \bullet d)))$ can be shown to follow from $(\star)$. Similarly for the other direction.

(†)  (‡)

$$F(A(l)^\diamond): \quad \exists bcefg(R_\diamond ab \wedge R_\bullet bcd \wedge R_\diamond ce \wedge R_\diamond ef \wedge R_\bullet fgh \wedge R_\diamond gi) \Longleftrightarrow$$
$$\exists b'c'd'e'f'(R_\diamond ab' \wedge R_\bullet b'c'd' \wedge R_\diamond c'i \wedge R_\diamond d'e' \wedge R_\bullet e'f'd \wedge R_\diamond f'h)$$

Figure 6.4: Frame condition for $A(l)^\diamond$

Now for the embedding theorem.

6.2.16. PROPOSITION.

$$\mathbf{DL} \vdash A \to B \quad \text{iff} \quad \mathbf{L}\Diamond + A(l,r)^\diamond \vdash A^\flat \to B^\flat$$

Model construction. Suppose $\mathbf{DL} \nvdash A \to B$. Then there is a model $\mathcal{M} = \langle W, R_l, R_r, V\rangle$ where $A \to B$ fails. From $\mathcal{M}$ we construct $\mathcal{M}'$ as follows. For every triple $(abc) \in R_l$ we take fresh a', b' and put $(aa') \in R_\diamond$, $(a'b'c) \in R_\bullet$, $(b'b) \in R_\diamond$. Similarly, for every triple $(abc) \in R_r$ we take fresh a', c' and put $(aa') \in R_\diamond$, $(a'bc') \in R_\bullet$, $(c'c) \in R_\diamond$.

We have to check whether $\mathcal{M}'$ is an appropriate model for $\mathbf{L}\Diamond + A(l,r)_\diamond$, specifically, whether the frame condition of Fig 4 is satisfied. Suppose (‡) holds, and let us check whether (†). Note that a configuration $R_\diamond ab'$, $R_\bullet b'c'd'$, $R_\diamond c'i$ can only hold in $\mathcal{M}'$ if in $\mathcal{M}$ we had $R_l aid'$ ($\star$). And a configuration $R_\diamond d'e'$, $R_\bullet e'f'd$, $R_\diamond f'h$ can be in $\mathcal{M}'$ only if in $\mathcal{M}$ we had $R_l d'hd$ ($\star\star$). The frame for $\mathcal{M}$ is associative. Therefore, from ($\star, \star\star$) we can conclude $\mathcal{M}$ also contains a configuration $R_l aed, R_l eih$ for some $e \in W$. Applying the $\mathcal{M}'$ construction to that configuration we obtain (†). Similarly for the other direction.

From here on, the proof of Prop 6.2.16 follows the established path.

GENERALIZATION. The rear attack strategy can be generalized to the remaining edges. Below we simply state the embedding theorems with the relevant composed translations and modal structural postulates. We give the salient ingredients for the construction of $\mathcal{M}'$, leaving the elaboration as an exercise to the reader.

Consider first embedding of **L** into **LP**. The discriminating structural parameter is Commutativity. For the translation schema, we compose the translations of Def 6.2.11 and Def 6.2.5. Associativity is reinstalled in terms of the structural postulate $A^\diamond_\otimes$.

$$A^\diamond_\otimes: \quad \diamond(\diamond\diamond(\diamond A \otimes B) \otimes C \longleftrightarrow \diamond(\diamond A \otimes \diamond(\diamond B \otimes C))$$

6.2.17. DEFINITION. Embedding translation $(\cdot)^\flat : \mathcal{F}(\mathbf{L}) \mapsto \mathcal{F}(\mathbf{LP}\diamond)$.

$$\begin{aligned} p^\flat &= p \\ (A \bullet B)^\flat &= \diamond(\diamond A^\flat \otimes B^\flat) \\ (A/B)^\flat &= \Box^\downarrow(\Box^\downarrow A^\flat \mathbin{\circ\!-} B^\flat) \\ (B\backslash A)^\flat &= \diamond B^\flat \multimap \Box^\downarrow A^\flat \end{aligned}$$

6.2.18. PROPOSITION.

$$\mathbf{L} \vdash A \to B \quad \text{iff} \quad \mathbf{LP}\diamond + A^\diamond_\otimes \vdash A^\flat \to B^\flat$$

Semantically, the commutativity of $R_\otimes$ is realized via the construction of $\mathcal{M}'$, as in the case of Prop 6.2.6:

$$\mathcal{M} \quad \begin{array}{ccc} b & & c \\ & \diagdown\,\diagup & \\ & a & \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} b & & \\ | & & \\ b' & & c \\ & \diagdown\,\diagup & \\ & a' & \\ & | & \\ & a & \end{array} \quad + \quad \begin{array}{ccc} & & b \\ & & | \\ c & & b' \\ & \diagdown\,\diagup & \\ & a' & \\ & | & \\ & a & \end{array} \quad \mathcal{M}'$$

For the pair **DL** versus **DLP**, again Commutativity is the discriminating structural parameter, but now in a bimodal setting. We compose the translations for the embedding of **DNLP** into **DLP** and **DNL** into **DNLP** The structural postulates $A^{\diamond}_{\otimes_l}$ and $A^{\diamond}_{\otimes_r}$ are the dependency variants of $A^{\diamond}_{\otimes}$ above.

$$\begin{array}{ll} A^{\diamond}_{\otimes_l}: & \diamond(\diamond\diamond(\diamond A \otimes_l B) \otimes_l C \longleftrightarrow \diamond(\diamond A \otimes_l \diamond(\diamond B \otimes_l C)) \\ A^{\diamond}_{\otimes_r}: & \diamond(\diamond\diamond(\diamond A \otimes_r B) \otimes_r C \longleftrightarrow \diamond(\diamond A \otimes_r \diamond(\diamond B \otimes_r C)) \end{array}$$

6.2.19. DEFINITION. Embedding translation $(\cdot)^\flat : \mathcal{F}(\mathbf{DL}) \mapsto \mathcal{F}(\mathbf{DLP}\diamond)$.

$$\begin{array}{rr} & p^\flat = p \\ (A \bullet_l B)^\flat = \diamond(\diamond A^\flat \otimes_l B^\flat) & (A \bullet_r B)^\flat = \diamond(\diamond A^\flat \otimes_r B^\flat) \\ (A/_l B)^\flat = \Box^\downarrow(\Box^\downarrow A^\flat \circ\!-_l B^\flat) & (A/_r B)^\flat = \Box^\downarrow(\Box^\downarrow A^\flat \circ\!-_r B^\flat) \\ (B\backslash_l A)^\flat = \diamond B^\flat -\!\circ_l \Box^\downarrow A^\flat & (B\backslash_r A)^\flat = \diamond B^\flat -\!\circ_r \Box^\downarrow A^\flat \end{array}$$

6.2.20. PROPOSITION.

$$\mathbf{DL} \vdash A \to B \quad \text{iff} \quad \mathbf{DLP}\diamond + (A^{\diamond}_{\otimes_l}, A^{\diamond}_{\otimes_r}) \vdash A^\flat \to B^\flat$$

Finally, for the pair **DLP** versus **LP**, the objective of the embedding is to recapture the dependency distinctions. We compose the translations of Def 6.2.11 and Def 6.2.7. The modal structural postulates $A(l,r)^{\diamond}_{\otimes}$ are obtained from $A(l,r)^{\diamond}$ by replacing $\bullet$ by $\otimes$.

6.2.21. DEFINITION. Embedding translation $\cdot^\flat : \mathcal{F}(\mathbf{DLP}) \mapsto \mathcal{F}(\mathbf{LP}\diamond)$.

$$\begin{array}{ll} & p^\flat = p \\ (A \otimes_l B)^\flat = \diamond(\diamond A^\flat \otimes B^\flat) & (A \otimes_r B)^\flat = \diamond(A^\flat \otimes \diamond B^\flat) \\ (A \circ\!-_l B)^\flat = \Box^\downarrow(\Box^\downarrow A^\flat \circ\!- B^\flat) & (A \circ\!-_r B)^\flat = \Box^\downarrow A^\flat \circ\!- \diamond B^\flat \\ (B -\!\circ_l A)^\flat = \diamond B^\flat -\!\circ \Box^\downarrow A^\flat & (B -\!\circ_r A)^\flat = \Box^\downarrow(B^\flat -\!\circ \Box^\downarrow A^\flat) \end{array}$$

6.2.22. PROPOSITION.

$$\mathbf{DLP} \vdash A \to B \quad \text{iff} \quad \mathbf{LP}\diamond + A(l,r)^{\diamond}_{\otimes} \vdash A^\flat \to B^\flat$$

6.2.4 Constraining embeddings: summary

We have completed the tour of the landscape and shown that the connectives $\diamond$, $\Box^\downarrow$ can systematically reintroduce structural discrimination in logics where on the level of the binary multiplicatives such discrimination is destroyed by global structural postulates. In Fig 6.5 we label the edges of the cube with the numbers of the embedding theorems.

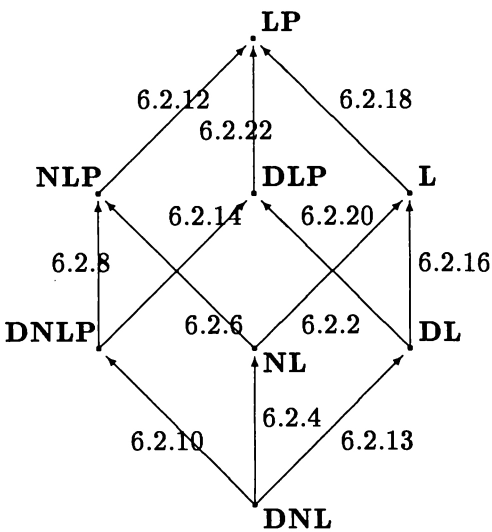


Figure 6.5: Embedding translations: recovering resource control

6.3 Licensing structural relaxation

In the present section we shift the perspective: instead of using modal decorations to *block* structural options for resource management, we now take the more discriminating logic as the starting point and use the modal operators to recover the flexibility of a neighbouring logic with a more liberal resource management regime from within a system with a more rigid notion of structure-sensitivity.

Licensing of structural relaxation has traditionally been addressed (both in logic [Došen 88,89] and in linguistics [Morrill 94]) in terms of a single universal $\Box$ modality with $S4$ type resource management. Here we stick to the minimalistic principles set out at the beginning of this paper, and realize also the licensing embeddings in terms of the pure logic of residuation for the pair $\Diamond, \Box^{\downarrow}$ plus modally controlled structural postulates. In §6.3.1 we present an external strategy for modal decoration: in the scope of the $\Diamond$ operator, products of the more discriminating logics gain access to structural rules that are inaccessible in the non-modal part of the logic. In §6.3.2 we develop a complementary strategy for internal modal decoration, where modal versions of the structural rules are accessible provided one or all of the immediate substructures are labelled with $\Diamond$. We present linguistic considerations that will affect the choice for the external or internal approach.

6.3.1 Modal labelling: external perspective

Licensing structural relaxation is simpler than recovering structural control: the target logics for the embeddings in this section *lack* an option for structural manipulation which can be reinstalled straightforwardly in terms of a modal version of the relevant structural postulate. We do not have to design specific translation strategies for the individual pairs of logics, but can do with one general translation schema.

6.3.1. DEFINITION. General translation schema $(\cdot)^\sharp : \mathcal{F}(\mathcal{L}_1) \mapsto \mathcal{F}(\mathcal{L}_0\Diamond)$ embedding a stronger logic $\mathcal{L}_1$ into a weaker logic $\mathcal{L}_0$ extended with $\Diamond, \Box^\downarrow$.

$$\begin{aligned} p^\sharp &= p \\ (A \bullet_1 B)^\sharp &= \Diamond(A^\sharp \bullet_0 B^\sharp) \\ (A/_1B)^\sharp &= \Box^\downarrow A^\sharp/_0 B^\sharp \\ (B\backslash_1 A)^\sharp &= B^\sharp\backslash_0\Box^\downarrow A^\sharp \end{aligned}$$

The embedding theorems we are interested in now have the general format shown below, where $\mathcal{R}_\diamond$ is (a package of) the modal translation(s) $A^\sharp \to B^\sharp$ of the structural rule(s) $A \to B$ which differentiate(s) $\mathcal{L}_1$ from $\mathcal{L}_0$.

$$\mathcal{L}_1 \vdash A \to B \quad \text{iff} \quad \mathcal{L}_0\Diamond + \mathcal{R}_\diamond \vdash A^\sharp \to B^\sharp$$

We look at the dimensions of dependency, precedence and dominance in general terms first, discussing the relevant aspects of the model construction. Then we comment on individual embedding theorems.

RELAXATION OF DEPENDENCY SENSITIVITY. For a start let us look at a pair of logics $\mathcal{L}_0$,$\mathcal{L}_1$, where $\mathcal{L}_0$ makes a dependency distinction between a left-dominant and a right-dominant product, whereas $\mathcal{L}_1$ cannot discriminate these two. There is two ways of setting up the coarser logic $\mathcal{L}_1$. Either we present $\mathcal{L}_1$ as a bimodal system where the distinction between right-dominant $\bullet_r$ and left-dominant $\bullet_l$ collapses as a result of the structural postulate (D).

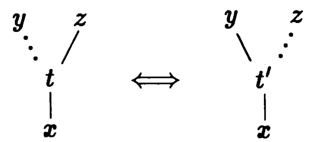
$$\mathcal{L}_1: \quad A \bullet_r B \longleftrightarrow A \bullet_l B \quad (D)$$

Or we have a unimodal presentation for $\mathcal{L}_1$ and pick an arbitrary choice of the dependency operators for the embedding translation. We take the second option here, and realize the embedding translation as indicated below.

$$\begin{aligned} p^\sharp &= p \\ (A \bullet B)^\sharp &= \Diamond(A^\sharp \bullet_r B^\sharp) \\ (A/B)^\sharp &= \Box^\downarrow A^\sharp/_r B^\sharp \\ (B\backslash A)^\sharp &= B^\sharp\backslash_r\Box^\downarrow A^\sharp \end{aligned}$$

Relaxation of dependency sensitivity is obtained by means of a modally controlled version of (D). Corresponding to the structural postulate $(D_\diamond)$ we have the frame condition $F(D_\diamond)$ as a restriction on models for the more discriminating logic.

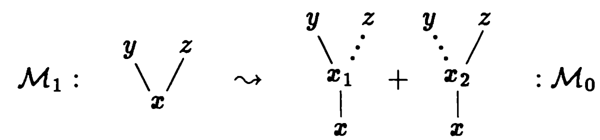
$$\mathcal{L}_0 : \quad \diamond(A \bullet_r B) \longleftrightarrow \diamond(A \bullet_l B) \quad (D_\diamond)$$



$$F(D_\diamond) : \quad (\forall xyz \in W_0)\ \exists t(R_\diamond xt \wedge R_r tyz) \Leftrightarrow \exists t'(R_\diamond xt' \wedge R_r t'yz)$$

Model construction. To construct an $\mathcal{L}_0$ model $\langle W_0, R^2_\diamond, R^3_l, R^3_r, V_0\rangle$ from a model

$\langle W_1, R^3_1, V_1\rangle$ for $\mathcal{L}_1$ we proceed as follows. For every triple $(xyz) \in R_1$ we take fresh points x_1, x_2, put x, x_1, x_2, y, z in W_0 with $(xx_1) \in R_\diamond$, $(x_1yz) \in R_l$ and $(xx_2) \in R_\diamond$, $(x_2yz) \in R_r$.



To show that the generated model $\mathcal{M}_0$ satisfies the required frame condition $F(D_\diamond)$, assume there exists $b \in W_0$ such that $R_\diamond ab$ and $R_r bcd$. Such a configuration has a unique preimage in $\mathcal{M}_1$ namely $R_1 acd$. By virtue of the construction of $\mathcal{M}_0$ this means there exists $b' \in W_0$ such that $R_\diamond ab'$ and $R_l b'cd$, as required for $F(D_\diamond)$.

Truth preservation of the model construction is unproblematic. The proof of the following proposition then is routine.

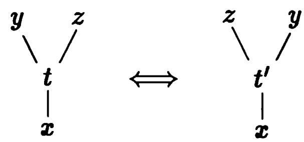
6.3.2. PROPOSITION.

$$\mathbf{NL} \vdash A \to B \quad \text{iff} \quad \mathbf{DNL}\diamond + D_\diamond \vdash A^\natural \to B^\natural$$

RELAXATION OF ORDER SENSITIVITY. Here we compare logics $\mathcal{L}_1$ and $\mathcal{L}_0$ where the structural rule of Permutation is included in the resource management package for $\mathcal{L}_1$, but not in that of $\mathcal{L}_0$. Controlled Permutation is reintroduced in $\mathcal{L}_0$ in the form of the modal postulate $(P_\diamond)$. The corresponding frame condition on $\mathcal{L}_0$ models $\mathcal{M}_0$ is given as $F(P_\diamond)$.

$$\mathcal{L}_1: \quad A \bullet_1 B \longleftrightarrow B \bullet_1 A \quad (P)$$

$$\mathcal{L}_0: \quad \Diamond(A \bullet_0 B) \longleftrightarrow \Diamond(B \bullet_0 A) \quad (P_\Diamond)$$



$$F(P_\Diamond): \quad (\forall xyz \in W_0)\ \exists t(R_\Diamond xt \wedge R_0 tyz) \Rightarrow \exists t'(R_\Diamond xt' \wedge R_0 t'zy)$$

To generate the required model $\mathcal{M}_0$ from $\mathcal{M}_1$ we proceed as follows. If $(xyz) \in R_1$ we take fresh x_1, x_2 and put both $(xx_1) \in R_\Diamond$ and $(x_1yz) \in R_0$ and $(xx_2) \in R_\Diamond$ and $(x_2zy) \in R_0$.

We have to show that the generated model $\mathcal{M}_0$ satisfies $F(P_\Diamond)$. Assume there exists $b \in W_0$ such that $R_\Diamond ab$ and R_0bcd. Because of the presence of Permutation in $\mathcal{L}_1$ this configuration has two preimages, R_1acd and R_1adc. By virtue of the construction algorithm for $\mathcal{M}_0$ each of these guarantees there exists $b' \in W_0$ such that $R_\Diamond ab'$ and R_0xdc.

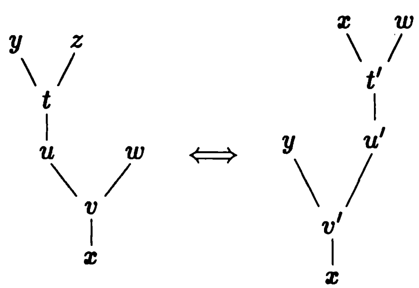
6.3.3. Proposition.

$$\mathbf{NLP} \vdash A \to B \quad \text{iff} \quad \mathbf{NL}\Diamond + P_\Diamond \vdash A^\natural \to B^\natural$$

Relaxation of constituent sensitivity. Next compare a logic $\mathcal{L}_1$ where Associativity obtains with a more discriminating logic without global Associativity. We realize the embedding by introducing a modally controlled form of Associativity $(A_\Diamond)$ with its corresponding frame condition $F(A_\Diamond)$.

$$\mathcal{L}_1: \quad A \bullet_1 (B \bullet_1 C) \longleftrightarrow (A \bullet_1 B) \bullet_1 C \quad (A)$$

$$\mathcal{L}_0: \quad \Diamond(A \bullet_0 \Diamond(B \bullet_0 C)) \longleftrightarrow \Diamond(\Diamond(A \bullet_0 B) \bullet_0 C) \quad (A_\Diamond)$$



$$F(A_\diamond): \quad (\forall xyzw \in W_0)$$

$$\exists tuv(R_\diamond xv \wedge R_0 vuw \wedge R_\diamond ut \wedge R_0 tyz) \Leftrightarrow \exists t'u'v'(R_\diamond xv' \wedge R_0 v'yu' \wedge R_\diamond u't' \wedge R_0 t'zw)$$

The $\mathcal{M}_0$ model is generated from $\mathcal{M}_1$ in the familiar way. For every triple $(xyz) \in R_1$, we take a fresh point x', and put $x, x', y, z \in W_0$, with $(xx') \in R_\diamond$ and $(x'yz) \in R_0$.

We have to show that the frame condition $F(A_\diamond)$ holds in the generated model. Suppose (†) $R_\diamond ab$ and $R_0 bcd$ and (‡) $R_\diamond ce$ and $R_0 efg$. We have to show that there are $x, y, z \in W_0$ such that $R_\diamond ax$ and $R_0 xfy$ and $R_\diamond yz$ and $R_0 zgd$. Observe that the configurations (†) and (‡) both have unique preimages in $\mathcal{M}_1$, $R_1 acd$ and $R_1 cfg$ respectively. Because R_1 is associative, there exists $y \in W_1$ such that $R_1 afy$ and $R_1 ygd$. But then, by the construction of $\mathcal{M}_0$, also $y \in W_0$ and there exist $x, z \in W_0$ such that $R_\diamond ax$, $R_0 xfy$, $R_\diamond yz$ and $R_0 zgd$, as required.

6.3.4. PROPOSITION.

$$\mathbf{L} \vdash A \to B \quad \text{iff} \quad \mathbf{NL}\diamond + A_\diamond \vdash A^\natural \to B^\natural$$

GENERALIZATIONS. The preceding discussion covers the individual dimensions of structural organization. Generalizing the approach to the remaining edges of Fig 6.1 does not present significant new problems. Here are some suggestions to assist the tenacious reader who wants to work out the full details.

The embeddings for the lower plane of Fig 6.1 are obtained from the parallel embeddings in the upper plane by doubling the construction from a unimodal product setting to the bimodal situation with two dependency products.

Embeddings between logics sharing associative management, but differing with respect to order or dependency sensitivity require modal associativity $A_\diamond$ in addition to $P_\diamond$ or $D_\diamond$ for the more discriminating logic: as we have seen in §6.2, the external $\diamond$ decoration on product configurations pre-empts the conditions of application for the non-modal associativity postulate. We have already come across this interplay between the translation schema and modal structural postulates in §6.2.3. For the licensing type of embedding, concrete instances are the embedding of **LP** into $\mathbf{L}\diamond + A_\diamond + P_\diamond$, and the embedding of **L** into $\mathbf{DL}\diamond + A_\diamond + D_\diamond$.

EXTERNAL DECORATION: APPLICATIONS. Linguistic application for the external strategy of modal licensing will be found in areas where one wants to induce structural relaxation in a configuration from the outside. The complementary view, where a subconfiguration induces structural relaxation in its context, is explored in §6.3.2 below. For the outside perspective, consider a non-commutative

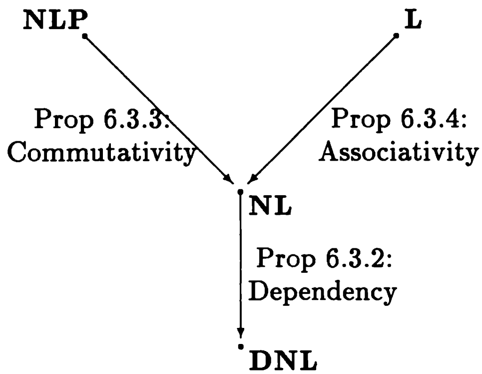


Figure 6.6: Licensing structural relaxation: precedence, dominance, dependency

default regime with $P_\Diamond$ for the modal extension. Collapse of the directional implications is underivable, $\nvdash A/B \longleftrightarrow B\backslash A$, but the modal variant below is. In general terms: a lexical assignment $A/\Box^\downarrow\Diamond B$ will induce commutativity for the argument subtype.

$$\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B \quad (A)^\circ \Rightarrow \Diamond A}{((A/B, B)^\bullet)^\circ \Rightarrow \Diamond A}\,/L}{((B, A/B)^\bullet)^\circ \Rightarrow \Diamond A}\,P_\Diamond}{(B, A/B)^\bullet \Rightarrow \Box^\downarrow\Diamond A}\,\Box^\downarrow R}{A/B \Rightarrow B\backslash\Box^\downarrow\Diamond A}\,\backslash R$$

Similarly, in the context of a non-associative default regime with $A_\Diamond$ for the modal extension, one finds the following modal variant of the Geach rule, which remains underivable without the modal decoration.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \Rightarrow C \quad (\Box^\downarrow B)^\circ \Rightarrow B}{(\Box^\downarrow B/C, C)^\bullet)^\circ \Rightarrow B}\,/L \quad (A)^\circ \Rightarrow \Diamond A}{((A/B, ((\Box^\downarrow B/C, C)^\bullet)^\circ)^\bullet)^\circ \Rightarrow \Diamond A}\,/L}{((((A/B, \Box^\downarrow B/C)^\bullet)^\circ, C)^\bullet)^\circ \Rightarrow \Diamond A}\,A_\Diamond}{(((A/B, \Box^\downarrow B/C)^\bullet)^\circ, C)^\bullet \Rightarrow \Box^\downarrow\Diamond A}\,\Box^\downarrow R}{((A/B, \Box^\downarrow B/C)^\bullet)^\circ \Rightarrow \Box^\downarrow\Diamond A/C}\,/R}{(A/B, \Box^\downarrow B/C)^\bullet \Rightarrow \Box^\downarrow(\Box^\downarrow\Diamond A/C)}\,\Box^\downarrow R}{A/B \Rightarrow \Box^\downarrow(\Box^\downarrow\Diamond A/C)/(\Box^\downarrow B/C)}\,/R$$

6.3.2 Modal labelling: the internal perspective

The embeddings discussed in the previous section license special structural behaviour by *external* decoration of product configurations: in the scope of the $\Diamond$

operator the product gains access to a structural rule which is unavailable in the default resource management of the logic in question. In view of the intended linguistic applications of structural modalities we would like to complement the external modalization strategy by an *internal* one where a structural rule is applicable to a product configuration provided one of its subtypes is modally decorated. In fact, the examples of modally controlled constraints we gave at the beginning of this paper were of this form. For the internal perspective, the modalized versions of Permutation and Associativity take the form shown below.

$$\begin{array}{ll} (P'_\diamond) & \diamond A \bullet B \longleftrightarrow B \bullet \diamond A \\ (A'_\diamond) & A_1 \bullet (A_2 \bullet A_3) \longleftrightarrow (A_1 \bullet A_2) \bullet A_3 \quad (\text{provided } A_i = \diamond A, 1 \leq i \leq 3) \end{array}$$

We prove embedding theorems for internal modal decoration in terms of the following translation mapping, which labels positive (proper) subformulae with the modal prefix $\diamond\Box^{\downarrow}$ and leaves negative subformulae undecorated.

6.3.5. DEFINITION. Embedding translations $(\cdot)^+, (\cdot)^- : \mathcal{F}(\mathcal{L}_1) \mapsto \mathcal{F}(\mathcal{L}_0\diamond)$ for positive and negative formula occurrences.

$$\begin{array}{rcc rcc} (p)^+ & = & p & (p)^- & = & p \\ (A \bullet_1 B)^+ & = & \diamond\Box^{\downarrow}(A)^+ \bullet_0 \diamond\Box^{\downarrow}(B)^+ & (A \bullet_1 B)^- & = & (A)^- \bullet_0 (B)^- \\ (A/_1 B)^+ & = & \diamond\Box^{\downarrow}(A)^+/_0(B)^- & (A/_1 B)^- & = & (A)^-/_0\diamond\Box^{\downarrow}(B)^+ \\ (B\backslash_1 A)^+ & = & (B)^-\backslash_0\diamond\Box^{\downarrow}(A)^+ & (B\backslash_1 A)^- & = & \diamond\Box^{\downarrow}(B)^+\backslash_0(A)^- \end{array}$$

The theorems embedding a stronger logic $\mathcal{L}_1$ into a more discriminating system $\mathcal{L}_0$ now assume the following general form, where $\mathcal{R}'_\diamond$ is the modal version of the structural rule package discriminating between $\mathcal{L}_1$ and $\mathcal{L}_0$.

6.3.6. PROPOSITION.

$$\mathcal{L}_1 \vdash A \rightarrow B \quad \text{iff} \quad \mathcal{L}_0\diamond + \mathcal{R}'_\diamond \vdash A^+ \rightarrow B^-$$

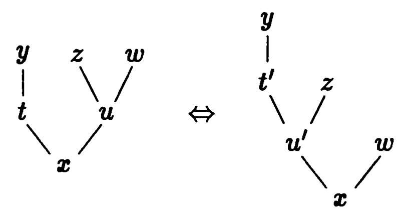
As an illustration we consider the embedding of **L** into **NL**$\diamond$ which involves licensing of Associativity in terms of the postulate $(A'_\diamond)$. The frame construction method we employ is completely general: it can be used unchanged for the other cases of licensing embedding one may want to consider.

The proof of the $(\Rightarrow)$ direction of Prop 6.3.6 is by easy induction. We present a Gentzen derivation of the Geach rule as an example. The type responsible for licensing $A'_\diamond$ in this case is $\diamond\Box^{\downarrow}(B/C)^+$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C^+ \Rightarrow C^-}{(\Box^{\downarrow}C^+)^{\diamond} \Rightarrow C^-}\,L\Box^{\downarrow}}{\Diamond\Box^{\downarrow}C^+ \Rightarrow C^-}\,L\Diamond \qquad \dfrac{\cdots}{\Diamond\Box^{\downarrow}B^+ \Rightarrow B^-}}{\Diamond\Box^{\downarrow}B^+/C^-, \Diamond\Box^{\downarrow}C^+ \Rightarrow B^-}\,L/}{((\Box^{\downarrow}(\Diamond\Box^{\downarrow}B^+/C^-))^{\diamond}, \Diamond\Box^{\downarrow}C^+) \Rightarrow B^-}\,L\Box^{\downarrow} \qquad \dfrac{\cdots}{\Diamond\Box^{\downarrow}A^+ \Rightarrow A^-}}{(\Diamond\Box^{\downarrow}A^+/B^-, ((\Box^{\downarrow}(\Diamond\Box^{\downarrow}B^+/C^-))^{\diamond}, \Diamond\Box^{\downarrow}C^+)) \Rightarrow A^-}\,L/}{((\Diamond\Box^{\downarrow}A^+/B^-, (\Box^{\downarrow}(\Diamond\Box^{\downarrow}B^+/C^-))^{\diamond}), \Diamond\Box^{\downarrow}C^+) \Rightarrow A^-}\,A'_{\diamond}}{((\Diamond\Box^{\downarrow}A^+/B^-, \Diamond\Box^{\downarrow}(\Diamond\Box^{\downarrow}B^+/C^-)), \Diamond\Box^{\downarrow}C^+) \Rightarrow A^-}\,L\Diamond}{\Diamond\Box^{\downarrow}A^+/B^- \Rightarrow (A^-/\Diamond\Box^{\downarrow}C^+)/\Diamond\Box^{\downarrow}(\Diamond\Box^{\downarrow}B^+/C^-)}\,R/, R/}{(A/B)^+ \Rightarrow ((A/C)/(B/C))^-}\,(\cdot)^+, (\cdot)^-
$$

For the ($\Leftarrow$) direction, we proceed by contraposition. Suppose $\mathbf{L} \not\vdash A \to B$. Completeness tells us there exists an $\mathbf{L}$ model $\mathcal{M}_1 = \langle W_1, R_1, V_1\rangle$ with a point $a \in W$ such that $\mathcal{M}_1, a \models A$ but $\mathcal{M}_1, a \not\models B$. From $\mathcal{M}_1$ we want to construct an $\mathbf{NL}\Diamond + A'_{\diamond}$ model $\mathcal{M}_0 = \langle W_0, R_{\diamond}, R_0, V_0\rangle$ such that $A^+ \to B^-$ fails. Recall that R_0 has to satisfy the frame conditions for the modal versions $A'_{\diamond}$ of the Associativity postulate. We give one instantiation below.

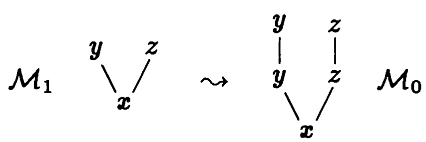
$$(A'_{\diamond}) \quad \Diamond A \bullet_0 (B \bullet_0 C) \longleftrightarrow (\Diamond A \bullet_0 B) \bullet_0 C$$



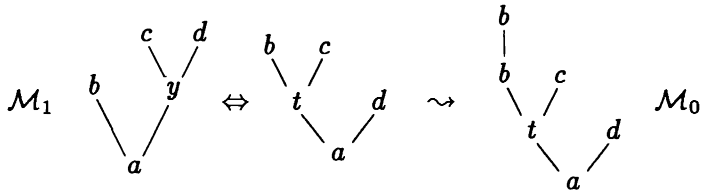
$$(\dagger) \quad (\forall xyzw \in W_0)$$

$$\exists tu(R_0xtu \wedge R_{\diamond}ty \wedge R_0uzw) \Leftrightarrow \exists t'u'(R_0xu'w \wedge R_0u't'z \wedge R_{\diamond}t'y)$$

The model construction proceeds as follows. We put the falsifying point $a \in W_0$, and for every triple $(xyz) \in R_1$ we put $x, y, z \in W_0$ and $(xyz) \in R_0$, $(yy) \in R_{\diamond}$, $(zz) \in R_{\diamond}$.



We have to show that the model construction realizes the frame condition ($\dagger$) (and its relatives) in $\mathcal{M}_0$. Suppose $\exists xy(R_0axy \wedge R_{\diamond}xb \wedge R_0ycd)$. By the model construction, $x = b$, so R_0aby which has the pre-image R_1aby. The pre-image of R_0ycd is R_1ycd. The combination of these two R_1 triangles satisfies the Associativity frame condition of **L**, so that we have a point t such that $R_1atd \wedge R_1tbc$. Again by the model construction, this means in $\mathcal{M}_0$ we have $\exists z, t(R_0tzc \wedge R_\diamond zb \wedge R_0atd)$, as required.

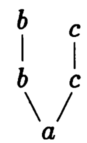


The central Truth Preservation Lemma now is that for any $a \in W_1 \cap W_0$,

$$\mathcal{M}_1, a \models A \quad \text{iff} \quad \mathcal{M}_0, a \models A^+ \quad \text{iff} \quad \mathcal{M}_0, a \models A^-$$

We concentrate on the $(\cdot)^+$ case — the $(\cdot)^-$ case is straightforward.

($\Rightarrow$) Suppose $\mathcal{M}_1, a \models A \bullet_1 B$. We have to show that $\mathcal{M}_0, a \models \diamond\Box^\downarrow A^+ \bullet_0 \diamond\Box^\downarrow B^+$. By assumption, there exist b, c such that R_1abc, and $\mathcal{M}_1, b \models A$, $\mathcal{M}_1, c \models B$. By inductive hypothesis and the model construction algorithm, we have in $\mathcal{M}_0$



$$\mathcal{M}_0, b \models A^+ \quad \mathcal{M}_0, c \models B^+$$

Observe that if x is the only point accessible from x via $R_\diamond$ (as is the case in $\mathcal{M}_0$), then for any formula ϕ, $x \models \phi$ iff $x \models \diamond\phi$ iff $x \models \Box^\downarrow\phi$ iff $x \models \diamond\Box^\downarrow\phi$. Therefore, from the above we can conclude $\mathcal{M}_0, b \models \diamond\Box^\downarrow A^+$ and $\mathcal{M}_0, c \models \diamond\Box^\downarrow B^+$, hence $\mathcal{M}_0, a \models \diamond\Box^\downarrow A^+ \bullet_0 \diamond\Box^\downarrow B^+$.

($\Leftarrow$) Suppose $\mathcal{M}_0, a \models \diamond\Box^\downarrow A^+ \bullet_0 \diamond\Box^\downarrow B^+$. We show that $\mathcal{M}_1, a \models A \bullet_1 B$. By assumption, there exist b, c such that R_0abc, and $\mathcal{M}_0, b \models \diamond\Box^\downarrow A^+$, $\mathcal{M}_0, c \models \diamond\Box^\downarrow B^+$. In $\mathcal{M}_0$ all triangles are such that the daughters have themselves and only themselves accessible via $R_\diamond$. Using our observation again, we conclude that $\mathcal{M}_0, b \models A^+$, $\mathcal{M}_0, c \models B^+$, and by inductive assumption $\mathcal{M}_1, a \models A \bullet B$.

We leave the implicational formulas to the reader.

COMMENT: FULL INTERNAL LABELING. Licensing of structural relaxation is implemented in the above proposal via modal versions of the structural postulates requiring at least *one* of the internal subtypes to be $\diamond$ decorated. It makes good sense to consider a variant of internal licensing, where one requires *all* relevant subtypes of a structural configuration to be modally decorated — depending on the application one has in mind, one could choose one or the other. Embeddings with this property have been studied for algebraic models by [Venema 93b, Versmissen 93]. In the terms of our minimalistic setting, modal structural postulates with full internal labeling would assume the following form.

$$\begin{array}{ll} (P''_\diamond) & \diamond A \bullet \diamond B \longleftrightarrow \diamond B \bullet \diamond A \\ (A''_\diamond) & \diamond A \bullet (\diamond B \bullet \diamond C) \longleftrightarrow (\diamond A \bullet \diamond B) \bullet \diamond C \end{array}$$

One obtains the variant of the embedding theorems for full internal labeling on the basis of the modified translation $(\cdot)^{++}$ which marks *all* positive subformulae with the modal prefix $\diamond\Box^\downarrow$. (Below we abbreviate $\diamond\Box^\downarrow$ to μ.) In the model construction, one puts $(xx) \in R_\diamond$ (and nothing more) for every point x that has to be put in W_0.

$$\begin{array}{lcclcc} (p)^{++} & = & \mu p & (p)^- & = & p \\ (A \bullet_1 B)^{++} & = & \mu(\mu(A)^{++} \bullet_0 \mu(B)^{++}) & (A \bullet_1 B)^- & = & (A)^- \bullet_0 (B)^- \\ (A/_1B)^{++} & = & \mu(\mu(A)^{++}/_0(B)^-) & (A/_1B)^- & = & (A)^-/_0\mu(B)^{++} \\ (B\backslash_1 A)^{++} & = & \mu((B)^-\backslash_0\mu(A)^{++}) & (B\backslash_1 A)^- & = & \mu(B)^{++}\backslash_0(A)^- \end{array}$$

6.3.7. PROPOSITION.

$$\mathcal{L}_1 \vdash A \rightarrow B \quad \text{iff} \quad \mathcal{L}_0\diamond + \mathcal{R}''_\diamond \vdash A^{++} \rightarrow B^-$$

ILLUSTRATION: EXTRACTION. For a concrete linguistic illustration of $\diamond\Box^\downarrow$ labeling licensing structural relaxation, we return to the example of extraction from non-peripheral positions in relative clauses. The example below becomes derivable in $\mathbf{NL}\diamond + (A'_\diamond, P'_\diamond)$ given a modally decorated type assignment $r/(s/\diamond\Box^\downarrow np)$ to the relative pronoun, which allows the hypothetical $\diamond\Box^\downarrow np$ assumption to find its appropriate location in the relative clause body via controlled Associativity and Permutation. We give the relevant part of the Gentzen derivation, abbreviating $(np\backslash s)/np$ as tv.

$$\begin{array}{ll} & \ldots\text{that ((John read) yesterday)} \\ \mathbf{NL}\diamond + (A'_\diamond, P'_\diamond) \vdash & (r/(s/\diamond\Box^\downarrow np), ((np, (np\backslash s)/np), s\backslash s)) \Rightarrow r \end{array}$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\cdots}{((np,(tv,np)),s\backslash s) \Rightarrow s}}{((np,(tv,(\Box^{\downarrow}np)^{\diamond})),s\backslash s) \Rightarrow s}\,L\Box^{\downarrow}}{(((np,tv),(\Box^{\downarrow}np)^{\diamond}),s\backslash s) \Rightarrow s}\,A'_{\diamond}}{((np,tv),((\Box^{\downarrow}np)^{\diamond},s\backslash s)) \Rightarrow s}\,A'_{\diamond}}{((np,tv),(s\backslash s,(\Box^{\downarrow}np)^{\diamond})) \Rightarrow s}\,P'_{\diamond}}{(((np,tv),s\backslash s),(\Box^{\downarrow}np)^{\diamond}) \Rightarrow s}\,A'_{\diamond}}{(((np,tv),s\backslash s),\Diamond\Box^{\downarrow}np) \Rightarrow s}\,L\Diamond}{((np,tv),s\backslash s) \Rightarrow s/\Diamond\Box^{\downarrow}np}\,R/
$$

Comparing this form of licensing modal decoration with the treatment in terms of a universal $\Box$ operator with $S4$ structural postulates, one observes that on the proof-theoretic level, the $\Diamond\Box^{\downarrow}$ prefix is able to mimick the behaviour of the $S4$ $\Box$ modality, whereas on the semantic level, we are not forced to impose transitivity and reflexivity constraints on the interpretation of $R_{\diamond}$. With a translation $(\Box A)^{\sim} = \Diamond\Box^{\downarrow}(A)^{\sim}$, the characteristic T and 4 postulates for $\Box$ become valid type transitions in the pure residuation system for $\Diamond$, $\Box^{\downarrow}$, as the reader can check.

$$
\begin{array}{llll}
T: & \Box A \rightarrow A & \rightsquigarrow & \Diamond\Box^{\downarrow}A \rightarrow A \\
4: & \Box A \rightarrow \Box\Box A & \rightsquigarrow & \Diamond\Box^{\downarrow}A \rightarrow \Diamond\Box^{\downarrow}\Diamond\Box^{\downarrow}A
\end{array}
$$

6.4 Discussion

In this final section, we reflect on some general logical and linguistic aspects of the proposed architecture, and raise a number of questions for future research.

Linear Logic and the sublinear landscape. In order to obtain controlled access to Contraction and Weakening, Linear Logic extends the formula language with operators which on the proof-theoretic level are governed by an $S4$-like regime. The 'sublinear' grammar logics we have studied show a higher degree of structural organization: not only the multiplicity of the resources matters, but also the way they are put together into structured configurations. These more discriminating logics suggest more delicate instruments for obtaining structural control. We have presented embedding theorems for the licensing and for the constraining perspective on substructural communication in terms of the pure logic of residuation for a set of unary multiplicatives $\Diamond$, $\Box^{\downarrow}$. In the frame semantics setting, these operators make more fine-grained structural distinctions than their $S4$ relatives which are interpreted with respect to a transitive and reflexive accessibility relation. But they are expressive enough to obtain full control over grammatical resource management. Our minimalistic stance is motivated by linguistic considerations. For reasons quite different from ours, and for different types of models, a number of recent proposals in the field of Linear Logic proper have argued for a decomposition of the '!,?' modalities into more elementary operators. For comparison we refer the reader to [Bucalo 94] , [Girard 95].

Multiplicatives versus Booleans.

The price of diamonds. We have compared logics with a 'standard' language of binary multiplicatives with systems where the formula language is extended with the unary logical constants $\diamond, \Box^{\downarrow}$. The unary operators, one could say, are the price one has to pay to gain structural control. Do we really have to pay this price, or could one faithfully embed the systems of Fig 6.1 *as they stand*? For answers in a number of specific cases, one can turn to [van Benthem 91b].

A question related to the above point is the following. Our embeddings compare the logics of Fig 6.1 pairwise, adding a modal control operator for each translation. This means that self-embeddings, from $\mathcal{L}$ to $\mathcal{L}'$ and back, end up two modal levels higher, a process which reaches equilibrium only in languages with infinitely many $\diamond, \Box^{\downarrow}$ control operators. Can one stay within some finite modal repertoire? We conjecture the answer is positive, but a definitive result would require a deeper study of the residuation properties of the $\diamond, \Box^{\downarrow}$ family.

Pure embeddings versus modal structural rules. The embedding results presented here are globally of two types. One type — what we have called the pure embeddings — obtains structural control solely in terms of the modal decoration added in the translation mapping. The other type adds a relativized structural postulate which can be accessed in virtue of the modal decoration of the translation. For the licensing type of communication, the second type of embedding is fully natural. The target logic, in these cases, does not allow a form of structural manipulation which is available in the source logic: in a *controlled* form, we want to regain this flexibility. But the distinction between the two types of embedding does not coincide with the shift from licensing to constraining communication. We have seen in §6.2.3 that imposing structural constraints for logics sharing associative resource management requires modalized structural postulates, in addition to the modal decoration of the translation mapping. In these cases, the $\diamond$ decoration has accidentally damaged the potential for associative rebracketing: the modalized associativity postulates repair this damage. We leave it as an open question whether one could realize pure embeddings for some of the logics of §6.2.3. A related question can be raised for the same family of logics under the licensing perspective: in these cases, we find not just the modal structural postulate for the parameter which discriminates between the logics, but in addition modal associativity, again because the translation schema has impaired the normal rebracketing.

Uniform versus customized translations. Another asymmetry that may be noted here is our implementation of the licensing type of communication in terms of a uniform translation schema, versus the constraining type of embeddings where the translations are specifically tailored towards the particular structural dimension one wants to control. Could one treat the constraining embeddings of §6.2 also in terms of a uniform translation scheme? And if so, would such a scheme be cheaper or more costly than the individual schemes in the text?

Complexity. A final set of questions relates to issues of computational complexity. For many of the individual logics in the sublinear cube complexity results (pleasant or unpleasant) are known. Do the embeddings allow transfer of such results to systems where we still face embarrassing open questions (such as: the issue of polynomial complexity for **L**)? In other words: what is the computational cost of the translations and modal structural postulates proposed? We conjecture that modalized versions of structural rules have the same computational cost as corresponding structural rules themselves.

Notice furthermore that all given reductions can be extended from the pure categorial language to its Boolean completion. By boolean completion we mean the system which can be obtained from the corresponding Lambek system by adding booleans and translating slashes into their product versions. Our semantic arguments can be easily extended with routine clauses for booleans due to the completeness proof given in Chapter 3. Here we have one confirmation of the above conjecture. Full boolean categorial logic with associative product now can be embedded into the modal version of non-associative Lambek Calculus with booleans and a modal associativity postulate. Since the first logic is undecidable (Andréka, Mikulás), the modal associativity postulate should bring undecidability to the second logic.

Embeddings: linguistic relevance. We close with a remark for the reader with a linguistics background. The embedding results presented in this paper may seem somewhat removed from the daily concerns of the working grammarian. Let us try to point out how our results can contribute to the foundations of grammar development work. In the literature of the past five years, a great variety of 'structural modalities' has been introduced, with different proof-theoretic behaviour and different intended semantics. It has been argued that the defects of particular type systems (either in the sense of overgeneration, or of undergeneration) can be overcome by refining type assignment in terms of these structural modalities. The accounts proposed for individual linguistic phenomena are often ingenious, but one may legitimately ask what the level of generality of the proposals is. The embedding results of this paper show that the operators $\diamond$, $\Box^{\downarrow}$ provide a general logic of constraints in the dimensions of order, dominance and dependency.

6.5 Appendix. Axiomatic and Gentzen presentation

In this Appendix we juxtapose the axiomatic presentations and the Gentzen formulation of the logics under discussion. The Lambek and Došen style axiomatic presentations are two equivalent ways of characterizing $\Diamond, \Box^{\downarrow}$, $\bullet, /$ and $\bullet, \backslash$ as residuated pairs of operators. For the equivalence between the axiomatic and the Gentzen presentations, we refer to [Moortgat 94]. This paper also establishes a Cut Elimination result for the language extended with $\Diamond, \Box^{\downarrow}$.

6.5.1. Definition. Lambek-style axiomatic presentation.

$$A \to A \qquad \frac{A \to B \quad B \to C}{A \to C}$$

$$\Diamond A \to B \Longleftrightarrow A \to \Box^{\downarrow} B$$

$$A \to C/B \quad \Longleftrightarrow \quad A \bullet B \to C \quad \Longleftrightarrow \quad B \to A\backslash C$$

6.5.2. Definition. Došen style axiomatization.

$$A \to A \qquad \frac{A \to B \quad B \to C}{A \to C}$$

$$\begin{array}{cc} \Diamond\Box^{\downarrow} A \to A & A \to \Box^{\downarrow}\Diamond A \\ A/B \bullet B \to A & A \to (A \bullet B)/B \\ B \bullet B\backslash A \to A & A \to B\backslash(B \bullet A) \end{array}$$

$$\frac{A \to B}{\Diamond A \to \Diamond B} \qquad \frac{A \to B}{\Box^{\downarrow} A \to \Box^{\downarrow} B}$$

$$\frac{A \to B \quad C \to D}{A \bullet C \to B \bullet D}$$

$$\frac{A \to B \quad C \to D}{A/D \to B/C} \qquad \frac{A \to B \quad C \to D}{D\backslash A \to C\backslash B}$$

The formulations of Def 6.5.1 and Def 6.5.2 give the pure residuation logic for the unary and binary families. The logics of Fig 6.1 are then obtained by adding different packages of structural postulates, as discussed in §6.1.

6.5.3. Definition. Gentzen presentation. Sequents $\Gamma \Rightarrow A$ with Γ a structured database of linguistic resources, A a formula. Structured databases are inductively defined as terms $\mathcal{T} ::= \mathcal{F} \mid (\mathcal{T}, \mathcal{T})^m \mid (\mathcal{T})^{\diamond}$, with binary $(\cdot, \cdot)^m$ or unary $(\cdot)^{\diamond}$ structural connectives corresponding to the (binary, unary) logical connectives.

We add resource management mode indexing for logical and structural connectives to keep families with different resource management properties apart. This strategy goes back to [Belnap 82] and has been applied to *modal* display logics in [Kracht 93a], [Wansing 92], two papers which are related in a number of respects to our own efforts.

$$[\mathrm{Ax}]\frac{}{A \Rightarrow A} \qquad \frac{\Gamma \Rightarrow A \quad \Delta[A] \Rightarrow C}{\Delta[\Gamma] \Rightarrow C}[\mathrm{Cut}]$$

$$[\mathrm{R}\Diamond]\frac{\Gamma \Rightarrow A}{(\Gamma)^{\diamond} \Rightarrow \Diamond A} \qquad \frac{\Gamma[(A)^{\diamond}] \Rightarrow B}{\Gamma[\Diamond A] \Rightarrow B}[\mathrm{L}\Diamond]$$

$$[\mathrm{R}\Box^{\downarrow}]\frac{(\Gamma)^{\diamond} \Rightarrow A}{\Gamma \Rightarrow \Box^{\downarrow} A} \qquad \frac{\Gamma[A] \Rightarrow B}{\Gamma[(\Box^{\downarrow} A)^{\diamond}] \Rightarrow B}[\mathrm{L}\Box^{\downarrow}]$$

$$[\mathrm{R}/_m]\frac{(\Gamma, B)^m \Rightarrow A}{\Gamma \Rightarrow A/_m B} \qquad \frac{\Gamma \Rightarrow B \qquad \Delta[A] \Rightarrow C}{\Delta[(A/_m B, \Gamma)^m] \Rightarrow C}[\mathrm{L}/_m]$$

$$[\mathrm{R}\backslash_m]\frac{(B, \Gamma)^m \Rightarrow A}{\Gamma \Rightarrow B\backslash_m A} \qquad \frac{\Gamma \Rightarrow B \qquad \Delta[A] \Rightarrow C}{\Delta[(\Gamma, B\backslash_m A)^m] \Rightarrow C}[\mathrm{L}\backslash_m]$$

$$[\mathrm{L}\bullet_m]\frac{\Gamma[(A, B)^m] \Rightarrow C}{\Gamma[A \bullet_m B] \Rightarrow C} \qquad \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{(\Gamma, \Delta)^m \Rightarrow A \bullet_m B}[\mathrm{R}\bullet_m]$$

Structural postulates, in the axiomatic presentation, have been presented as transitions $A \to B$ where A and B are constructed out of formula variables $p_1, \ldots, p_n$ and logical connectives $\bullet_m$, $\Diamond$. For structure variables $\Delta_1, \ldots, \Delta_n$ and structural connectives $(\cdot, \cdot)^m$, $(\cdot)^{\diamond}$, define the structural equivalent $\sigma(A)$ of a formula A as indicated below (cf [Kracht 93a]):

$$\sigma(p_i) = \Delta_i \quad \sigma(A \bullet_m B) = (\sigma(A), \sigma(B))^m \quad \sigma(\Diamond A) = (\sigma(A))^{\diamond}$$

The transformation of structural postulates into Gentzen rules allowing Cut Elimination then is straightforward: a postulate $A \to B$ translates as the Gentzen rule

$$\frac{\Gamma[\sigma(B)] \Rightarrow C}{\Gamma[\sigma(A)] \Rightarrow C}$$

In the cut elimination algorithm, one shows that if a structural rule precedes a Cut inference, the order of application of the inferences can be permuted, pushing the Cut upwards. See [Došen 88,89] for the case of global structural rules, [Moortgat 94] for the $\Diamond$ cases.

In the multimodal setting, structural rules are relativized to the appropriate resource management modes, as indicated by the mode index. An example is given below (for k a commutative and l an associative regime). Where no confusion is likely to arise, in the text we use the conventional symbols for different families of operators, rather than the official mode indexing on one generic set of symbols.

$$\frac{\Gamma[(\Delta_2, \Delta_1)^k] \Rightarrow A}{\Gamma[(\Delta_1, \Delta_2)^k] \Rightarrow A}[P] \qquad \frac{\Gamma[((\Delta_1, \Delta_2)^l, \Delta_3)^l] \Rightarrow A}{\Gamma[(\Delta_1, (\Delta_2, \Delta_3)^l)^l] \Rightarrow A}[A]$$

$$cf\ A \bullet_k B \rightarrow B \bullet_k A \qquad cf\ A \bullet_l (B \bullet_l C) \rightarrow (A \bullet_l B) \bullet_l C$$

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Samenvatting

Categoriale afleidingsbegrippen zijn in de literatuur om vele redenen en vanuit verschillende achtergronden van taalwetenschap en filosofie tot logica en informatica bestudeerd. In de laatste tien jaar hebben Categoriale Grammatica's hun plaats gekregen in de bredere context van zg *resource*-gevoelige substructurele afleidingssystemen. In deze dissertatie wordt een modaal-semantische basis geconstrueerd voor de studie van categoriale afleidingsbegrippen die ons in staat stelt de structurele regels te representeren als beperkingen opgelegd aan informatie-structuren. In de volgende hoofdstukken ontmoeten drie onderzoekslijnen elkaar: categoriale type-systemen, modale logica and gelabelde deductie.

Deel I introduceert de ternaire frame-semantiek en geeft een basis voor een modeltheorie en een correspondentietheorie voor categoriale talen. Hoofdstuk 1 behandelt de logische, filosofische and linguistische achtergronden van de ternaire frame-semantiek en de beweegredenen voor de keuzes van de talen over ternaire modellen. Een simpele modeltheorie wordt ontwikkeld met het begrip bisimulatie als fundament. Daarna worden ternaire frame-constructies besproken die nuttig kunnen zijn bij het bewijzen van de categoriale ondefinieerbaarheid van bepaalde eerste-orde principes. Hoofdstuk 2 gaat over Correspondentietheorie voor categoriale principes. We bewijzen een Sahlqvist-van Benthemstelling voor categoriale talen en geven een aanzet tot een algemene definieerbaarheidstheorie. Als toepassing verkrijgen we een semantische karakterisering van structurele regels vanuit het perspectief van de correspondentietheorie. We stellen twee methodes voor om te bewijzen dat categoriale principes niet eerste-orde definieerbaar zijn. De eerste is gebaseerd op het vertalen van categoriale formules in een niet eerste-orde definieerbare standaard modale formule. De tweede methode is directer: het niet eerste-orde zijn van categoriale formulas uit zich in het niet opgaan van de Lowenheim-Skolemstelling.

In Deel II we stappen we over van de studie van zuiver semantische uitdrukkingskracht naar de combinatoriek van categoriale deductie. Hoofdstuk 3 stelt een analyse voor van de volledigheidsstellingen voor de categoriale axiomasystemen in het perspectief van filterrepresentatie. Bovendien bewijzen we een onvolledigheidsstelling m.b.t. frames en onderscheiden we een wel volledige klasse van categoriale Sahlqvist-formules. Vervolgens beschouwen we gelabelde deductie. Daarin kunnen sequenten dragers zijn van informatie over linguistische tekens. In Hoofdstuk 4 gebruiken we de methode van het labelen bij het verkrijgen van een vrij simpel volledigheidsbewijs voor de Lambek Calculus met betrekking tot de binaire relatie-semantiek door gebruik te maken van passende labelparen. In Hoofdstuk 5 tenslotte, geven we een meer algemene labelmethode voor ternaire frame-semantiek tezamen met het leggen van een verband tussen deze methode en het eerdere correspondentie-perspectief van het vertalen naar fragmenten van de eerste-orde predicatenlogica.

Het laatste deel, Deel III, houdt zich bezig met de beheersing en besturing van de talige 'resources' in categoriale systemen. We ontwikkelen een theorie van systematische communicatie tussen deze systemen. De communicatie is tweezijdig: we laten zien hoe men de structurele onderscheidingen van een zwakkere logica binnen een sterkere kan terughalen en hoe men de structurele flexibiliteit van sterkere categoriale logica's in zwakkere kan herintroduceren. Verder laten we zien hoe unaire modale operatoren gebruikt kunnen worden om structuurgevoeligheid af te zwakken of juist op te leggen. Vanuit logisch standpunt bestaat onze bijdrage uit een aantal algemene vertaalmethoden plus een aantal inbeddingsstellingen die een verband leggen tussen de belangrijkste formele systemen in het categoriale landschap.

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