Neglect-Zero Effects on Indicative Conditionals:
*Extending **BSML** and **BiUS** with an implication*

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis researches the effect of the pragmatic "neglect-zero" tendency on indicative conditionals. Aloni (2022a) formalised this pragmatic tendency in *bilateral state-based modal logic* (**BSML**) to account for several *free choice* phenomena. However, this framework lacks an implication, and extending it with a suitable implication is a non-trivial task. To find the most suitable extension, we compare several possibilities on linguistic desiderata by Stalnaker (1976) and Ciardelli (2020), whereby we find that an existential variant of the maximal implication is our best option. Unfortunately, this implication still has some unsatisfactory linguistic and mathematical properties. This gives us the motivation to extend a dynamic variant of **BSML**, as introduced by Aloni (2022b) and called **BiUS**, with an implication. We find that this dynamic framework solves most problems of the static framework by giving us linguistically and mathematically satisfying results. Throughout these considerations we see that the "neglect-zero" tendency can give a simple pragmatic explanation for several linguistic phenomena. Thereby we conclude that this tendency, as formalised by **BSML** and **BiUS**, is likely to be one of the pragmatic tendencies active in natural language concerning indicative conditionals.

Keywords— Neglect-Zero, Indicative Conditionals, Implication, **BSML**, **BiUS**

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Chapter 1

Introduction

In natural language, we sometimes infer a sentence containing "and" from a sentence containing "or". For example, consider the famous example (1) by Kamp (1973). Then we see that (1-a) implies (1-b), a phenomenon that is called a *free choice* (FC) inference.[1]

(1) a. You may go to the beach or to the cinema. (Kamp (1973))
b. $\rightsquigarrow$ You may go to the beach and you may go the cinema.

Formally, we can represent this inference as $\Diamond(\alpha \vee \beta) \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$. However, as Von Wright (1968) pointed out, the following is not a validity in deontic logic.

$$\Diamond(\alpha \vee \beta) \rightarrow \Diamond\alpha$$

In order to account for FC inferences such as (1), we might consider adding this formula, which we will call the FC-principle, as a validity to deontic logic. But then one would be able to infer $\Diamond\beta$ from $\Diamond\alpha$, for any α and β. For consider the following reasoning, as first presented by Kamp (1973).

1.	$\Diamond\alpha$	[assumption]
2.	$\Diamond(\alpha \vee \beta)$	[from 1, by classical reasoning]
3.	$\Diamond\beta$	[from 2, by FC-principle]

But surely, we do not want "You may go to the cinema" to follow from "You may go the beach". In other words, we cannot simply account for the inference in (1) by adding the FC-principle to deontic logic. In the literature this is called the *paradox of free choice.*[2]

One of the possible routes to solving this problem, is to adopt a non-classical logic that can account for (1) without leading to deontic explosion. For this, Aloni

[1]We will follow Aloni (2022a), who in turn follows Kamp (1973), in her presentation of FC inferences and the *paradox of free choice*.
[2]See Aloni (2022a) Section 2 for references and an overview of the literature on this paradox.

(2022a) proposes a *bilateral state-based modal logic* (**BSML**) that can account for this FC inference and similar inferences.

This framework will formally be presented in Chapter 2, but let us shortly look at some of the main features of this logic. First of all, because the logic has separately defined acceptance and rejection clauses, allowing for specific behaviour under negation, we say that **BSML** is *bilateral*.[3] Furthermore, **BSML**-formulas are not evaluated in possible worlds, but rather in sets of possible worlds, called *information states*. And finally, **BSML** is a *modal logic*, i.e., its language has the modality $\Diamond$ that can represent modal words.

The underlying hypothesis that **BSML** formalises is the "neglect-zero" tendency of speakers. This hypothesis relies on the idea that speakers, when interpreting a sentence, create a model that represents a picture of the world. The empirically supported, pragmatic "neglect-zero" tendency claims that speakers avoid models that validate sentences by an empty configuration. As a result, speakers avoid sentences that would be validated by virtue of such an empty configuration.[4] For example, consider the odd, but classically true, sentence (2).

(2) All cows on the moon eat chocolate.

The "neglect-zero" tendency explains the oddity of (2) by the fact that this sentence is validated by an empty configuration. In other words, because there are no cows on the moon we would classically validate this universal quantification, but a natural language speaker would avoid the sentence because of the "emptiness" of the set of cows on the moon. We then say that (2) holds *semantically*, at least if we adopt classical reasoning, but the oddity of this sentence is explained *pragmatically*.

With a pragmatic enrichment function $[\,]^+$, formally modelling the "neglect-zero" tendency using the non-emptiness atom (NE), **BSML** can account for the inference in (1) and other FC-inferences.[5] However, as we will see later, on the NE-free fragment **BSML** simply models classical reasoning. To exemplify this, for NE-free α and β we see that $\Diamond(\alpha \vee \beta) \not\models \Diamond\alpha \wedge \Diamond\beta$. But when we model the "neglect-zero" tendency using the pragmatic enrichment function $[\,]^+$ we find that $[\Diamond(\alpha \vee \beta)]^+ \models \Diamond\alpha \wedge \Diamond\beta$.[6] In other words, **BSML** can pragmatically account for FC-inferences such as (1).

The observant reader might notice that we formulated the FC-principle as $\Diamond(\alpha \vee \beta) \rightarrow \Diamond\alpha$, but in the context of **BSML** discussed the entailment $[\Diamond(\alpha \vee \beta)]^+ \models \Diamond\alpha \wedge \beta$. The reason for this is that **BSML** has notions of negation, disjunction, conjunction and modality, but no notion of implication. And now

[3]Specifically, the bilateralism allows **BSML** to account for the inference from $\neg\Diamond(\alpha \vee \beta)$ to $\neg\Diamond\alpha \wedge \neg\Diamond\beta$. An inference that is called *Dual Prohibition* and is motivated by the cancellation of free choice effects under negation.

[4]Exceptions are sentences such as "Nobody can teleport", of which the logical form demands an empty configuration for its validation. The difference between such cases and (2), is that the latter could also be validated by non-empty configurations.

[5]The NE-atom is only supported by non-empty states, and was first introduced in team semantics by Väänänen (2007), and also appears in more recent work by Yang and Väänänen (2017).

[6]See Fact 5 by Aloni (2022a) for a proof of this.

that we know that **BSML** can account for the FC-inference on an entailment-level, we might also wonder if the FC-principle is a validity in **BSML** extended with an implication.[7]

Furthermore, the "neglect-zero" tendency of speakers is something that might also influence the way we think about conditionals. Specifically, it is interesting to see whether we can explain several phenomena of conditionals using the "neglect-zero" tendency. This does not mean that we are proposing a new and complete theory of conditionals. Rather, we think that there are several pragmatic tendencies at work in natural language, and we will research what role the "neglect-zero" tendency has. And finally, most logics have a notion of implication, either derivable or defined. So from a logico-mathematical perspective it is desirable for **BSML** to also have a well-behaved notion of this operator.

In this thesis we will extend **BSML** with an implication to research the topics mentioned above. A first, and traditional, attempt would be to syntactically define the implication in terms of the negation and disjunction that are available in **BSML**. We will name this implication "the definable $\to_{\neg,\vee}$", and adding this to **BSML** would result in the logic of **BSML** itself.

Definition 1.0.1 (Definable $\to_{\neg,\vee}$)**.**

$$\varphi \to_{\neg,\vee} \psi := \neg\varphi \vee \psi$$

However, we will see in Section 2.2.2 (Fact 2.2.3) that, when pragmatically enriched, this implication is incompatible with its own antecedent. But there is no reason for us to suspect that the "neglect-zero" tendency causes an incompatibility between a conditional and its antecedent. In fact, as we will see later, we would expect it to ensure that the antecedent is a live possibility. Therefore we conclude that the definable $\to_{\neg,\vee}$ is not suitable to extend **BSML**. A straightforward, and equally traditional, second attempt would be to extend **BSML** with a notion of the material implication. This extension has a very strong, and favorable, connection between the implication and entailment, which can be expressed by the Deduction Theorem. However, we will see that there are many linguistic phenomena that cannot be accounted for by **BSML** extended with a material implication. Therefore, we find that extending **BSML** with an implication is a non-trivial task. There are many different other possible candidate implications, which, together with a first review of their suitability, will be introduced in Chapter 2.

Now let us shortly discuss what kind of properties an implication ought to have to be suitable for **BSML**. First of all, we will see later (Fact 2.1.7) that the NE-free fragment of **BSML** determines the same logical notion of consequence as classical modal logic.[8] Only when we introduce pragmatics using the NE-

[7]Similarly, the formulas in (31), (48) and (49) by Aloni (2022a) all contain implications, and thus give a motivation for extending **BSML** with an implication. Note that the latter two also contain quantifiers, so to look at these examples we first need a first-order extension of **BSML**, such as **QBSML**, developed by Aloni and Ormondt (2021). The extension of **QBSML** with an implication will be left for future research.

[8]This means that in **BSML** all classical validities are also valid. Some examples of such classical validities are given in Fact 2.1.3.

atom we get non-classical behaviour. This way we see that **BSML** can make a clear distinction between classical logico-mathematical reasoning and everyday (pragmatic) reasoning of natural language speakers.[9] It would be favorable if this distinction can still be made after we have added an implication to **BSML**. In other words, we would prefer a notion of implication that does not introduce non-classicality, i.e., that preserves the classicality of its antecedent and consequent. In Chapter 5 we will look at this property, and other mathematical properties that we would like our implication to have.

But before we will look at the mathematical properties, we dive into the world of indicative conditionals in Chapters 3 and 4.[10] First we will consider several desiderata that Stalnaker (1976) put forward, whereafter we will consider additional desiderata by Ciardelli (2020). One of the implications, an existential variant of the maximal implication, will be quite satisfactory on many of these linguistic desiderata and mathematical properties. However, some linguistic phenomena cannot be captured by this implication.

For this reason we will consider a dynamic variant of **BSML**, as introduced by Aloni (2022b), in Chapter 6. The extension of this dynamic framework with a dynamic notion of implication will be able to explain all linguistic phenomena that the static framework could not explain. Furthermore, we will see that it is mathematically equally well-behaved, and thus overall gives us a very interesting theory of conditionals that relies on the "neglect-zero" tendency.

Let us now first look at the framework **BSML**, the implications that we can add to it, some of their properties, and the relations between them.

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[9]See Section 4 of Aloni (2022b) for a discussion of this with respect to a dynamic variant of **BSML**.

[10]We will leave the subjunctive, or counterfactual, conditional for future research.

Chapter 2

Preliminaries

As we have seen in the previous chapter, **BSML** models the "neglect-zero" tendency of speakers to explain *free choice inferences*. We will formally introduce **BSML**, and discuss several useful results on this framework, in Section 2.1. As the goal of this thesis is to extend **BSML** with a proper notion of implication, we will introduce and consider several options in Section 2.2.1. We will define the material implication, intuitionistic implication, the maximal implication, an existential variant of the maximal implication and the linear implication, and several "definable" implications using different notions of negation and disjunction. Thereafter we will discuss the suitability of the extension of **BSML** with several of these implications in Section 2.2.2. These discussions leave us with the material implication, intuitionistic implication, maximal implication, the existential variant of the maximal implication and what we will call the **BSML**-implication. Some relations between these implications will be highlighted in Section 2.2.3, whereafter they will be compared on several linguistic desiderata in Chapters 3 and 4, and on mathematical properties in Chapter 5.

2.1 BSML

2.1.1 Framework

Bilateral state-based modal logic (**BSML**) was first introduced by Aloni (2018) and further researched by Aloni (2022a) and Anttila (2021). The following introduction of **BSML** will follow Aloni (2022a) in notation and presentation.

The language of **BSML** extends propositional modal logic with the non-emptiness atom NE.

Definition 2.1.1 (Language). *Let $P = \{p, q, r \dots\}$ be a set of propositional atoms.*

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \text{NE}$$

where $p \in P$.

To interpret this language we use a Kripke model $M = \langle W, R, V\rangle$, where W is a non-empty set of world, R is an accessibility relation on W and V is a valuation function on $W \times P$.

The logic **BSML** is state-based, so the formulas are not interpreted with respect to a single world $w \in W$. Rather, they are interpreted with respect to a state $s \subseteq W$. Both the support condition ($\vDash$) and anti-support conditions ($\Dashv$) are specified below.[11][12] The notions of negation and disjunction that are available in **BSML** are called the *bilateral negation* and *split disjunction*, respectively.

Definition 2.1.2 (Semantic clauses)**.**

$$
\begin{array}{rcl}
M, s \vDash p & \textit{iff} & \forall w \in s : V(w,p) = 1 \\
M, s \Dashv p & \textit{iff} & \forall w \in s : V(w,p) = 0 \\
M, s \vDash \neg\varphi & \textit{iff} & M, s \Dashv \varphi \\
M, s \Dashv \neg\varphi & \textit{iff} & M, s \vDash \varphi \\
M, s \vDash \varphi \vee \psi & \textit{iff} & \exists t, t' : t \cup t' = s \;\&\; M, t \vDash \varphi \;\&\; M, t' \vDash \psi \\
M, s \Dashv \varphi \vee \psi & \textit{iff} & M, s \Dashv \varphi \;\&\; M, s \Dashv \psi \\
M, s \vDash \varphi \wedge \psi & \textit{iff} & M, s \vDash \varphi \;\&\; M, s \vDash \psi \\
M, s \Dashv \varphi \wedge \psi & \textit{iff} & \exists t, t' : t \cup t' = s \;\&\; M, t \Dashv \varphi \;\&\; M, t' \Dashv \psi \\
M, s \vDash \Diamond\varphi & \textit{iff} & \forall w \in s : \exists t \subseteq R[w] : t \neq \emptyset \;\&\; M, t \vDash \varphi \\
M, s \Dashv \Diamond\varphi & \textit{iff} & \forall w \in s : M, R[w] \Dashv \varphi \\
M, s \vDash \text{NE} & \textit{iff} & s \neq \emptyset \\
M, s \Dashv \text{NE} & \textit{iff} & s = \emptyset
\end{array}
$$

where $R[w] = \{v \in W \mid wRv\}$.

We use the abbreviation $\Box\varphi := \neg\Diamond\neg\varphi$, giving rise to the following support and anti-support conditions.

$$
\begin{array}{lcl}
M, s \vDash \Box\varphi & \textit{iff} & \textit{for all } w \in s : M, R[w] \vDash \varphi \\
M, s \Dashv \Box\varphi & \textit{iff} & \textit{for all } w \in s : \exists t \subseteq R[w] : t \neq \emptyset \;\&\; M, t \Dashv \varphi
\end{array}
$$

We say that a formula ψ is a logical consequence of a formula φ, if any model and state supporting φ also support ψ.

Definition 2.1.3 (Logical Consequence)**.** *Let* $\varphi, \psi \in$ **BSML***. Then,*

$$\varphi \vDash \psi \textit{ iff for all } M, s : M, s \vDash \varphi \textit{ implies } M, s \vDash \psi$$

[11]There are strong similarities between **BSML** and dependence logic (and variants thereof) as researched by Väänänen (2007), Yang (2014) and Yang and Väänänen (2017). The key difference is the bilateral nature of **BSML**, allowing for specific behaviour of negated operators.

[12]See Section 3.1 and 4 of Aloni (2022a) for the intended interpretation of these state-based semantics.

We use $\mathbf{BSML}_\emptyset$ to denote the NE-free fragment of **BSML**, and $\mathbf{BSML}_0$ to denote the NE-free and $\Diamond$-free fragment.[13] In other words, $\mathbf{BSML}_\emptyset$ is the classical modal fragment, and $\mathbf{BSML}_0$ is the propositional fragment of **BSML**. We will use φ, ψ, ξ as meta-variables ranging over **BSML**, and $\alpha, \beta, \gamma, \delta$ as meta-variables ranging over $\mathbf{BSML}_\emptyset$ (or $\mathbf{BSML}_0$, in which case it will be made explicit).

Finally, the pragmatic enrichment function $[\,]^+$ is recursively defined on $\mathbf{BSML}_\emptyset$, i.e., the NE-free fragment.

Definition 2.1.4. *Let* $\alpha, \beta \in \mathbf{BSML}_\emptyset$.

$$\begin{array}{rcl} [p]^+ & = & p \wedge \textsc{ne} \\ [\neg\alpha]^+ & = & \neg[\alpha]^+ \wedge \textsc{ne} \\ [\alpha \vee \beta]^+ & = & ([\alpha]^+ \vee [\beta]^+) \wedge \textsc{ne} \\ [\alpha \wedge \beta]^+ & = & ([\alpha]^+ \wedge [\beta]^+) \wedge \textsc{ne} \\ [\Diamond\alpha]^+ & = & \Diamond[\alpha]^+ \wedge \textsc{ne} \end{array}$$

2.1.2 Useful Results on BSML

Throughout this thesis we will be using several known facts about **BSML** on numerous occasions. The first three facts are from Aloni (2022a).[14]

The first fact says that an NE-free formula is a logical consequence of its pragmatically enriched version. Similarly, if its pragmatically enriched version is anti-supported, then so is the formula itself.

Fact 2.1.1. *Let* $\alpha \in \mathbf{BSML}_\emptyset$.

$$\begin{array}{lcl} M, s \vDash [\alpha]^+ & \textit{implies} & M, s \vDash \alpha \\ M, s \,⫤\, [\alpha]^+ & \textit{implies} & M, s \,⫤\, \alpha \end{array}$$

The second fact says that if a formula does not contain a disjunction in a positive environment or a conjunction in a negative environment, then conjoined with the non-emptiness atom it entails its pragmatically enriched version.[15][16]

Fact 2.1.2. *Let* $\alpha \in \mathbf{BSML}_\emptyset$ *be "split-free".*

$$\alpha \wedge \textsc{ne} \vDash_N [\alpha]^+$$

[13]See Table 9 of Aloni (2022a) for these and other fragments of **BSML**.

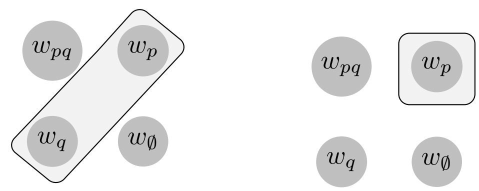
[14]The mapping from these facts to the facts in Aloni (2022a) is as follows: Fact 2.1.1 $\mapsto$ Fact 1 and Fact 2.1.3 $\mapsto$ Fact 6. Fact 2.1.2 occurred in an older version of Aloni (2022a) as Fact 3, but now only occurs in this thesis. It can easily be proven by an double induction on the complexity of α.

[15]The term "split-free" in this context means that any conjunction can only appear in the scope of an even amount of negations, and any disjunction can only appear in an odd amount of negations.

[16]Furthermore, note that we have used the entailment $\vDash_N$, meaning that we restrict ourselves to model-state pairs where $\forall w \in s : R[w]\neg\emptyset$. As this restriction is only relevant for formulas containing the modality $\Diamond$, we will only mention this restriction when needed.

Together with Fact 2.1.1 this means that for classical "split-free" formulas we have $[\alpha]^+ \equiv_N \alpha \wedge \text{NE}$. In other words, pragmatically enriching a "split-free" formula only amounts to requiring a supporting state to be non-empty. However, if α does contain a disjunction, say $\alpha = p \vee q$, then the effect of the pragmatic enrichment is less trivial.

To see this, consider Figure 2.1, where in both models the state s is non-empty.[17] However, as $[p \vee q]^+ = ((p \wedge \text{NE}) \vee (q \wedge \text{NE})) \wedge \text{NE}$ we see that it needs a non-empty substate supporting p and a non-empty substate supporting q. Therefore, $[p \vee q]^+$ is supported in the first model but not in the second. It is exactly this feature of **BSML** that accounts for free choice effects.[18]



(a) $M, s \vDash p \vee q$; and $M, s \vDash [p \vee q]^+$

(b) $M, s \vDash p \vee q$; but $M, s \nvDash [p \vee q]^+$

Figure 2.1: Non-triviality of $[p \vee q]^+$

The third fact by Aloni (2022a) says that the semantics of **BSML** satisfy several classical validities, namely the *De Morgan Laws*, *Double Negation Eliminitation* and *Duality of* $\Box/\Diamond$.

Fact 2.1.3 (Classical validities)**.**

$$
\begin{array}{rcll}
\varphi & \equiv & \neg\neg\varphi & \textit{(Double Negation Elimination)} \\
\neg(\varphi \vee \psi) & \equiv & \neg\varphi \wedge \neg\psi & \textit{(De Morgan Laws)} \\
\neg(\varphi \wedge \psi) & \equiv & \neg\varphi \vee \neg\psi & \\
\neg\Box\varphi & \equiv & \Diamond\neg\varphi & \textit{(Duality } \Box/\Diamond\textit{)} \\
\neg\Diamond\varphi & \equiv & \Box\neg\varphi &
\end{array}
$$

We also have some useful technical facts from Anttila (2021), first of which is an incompatibility result. It says that if one state supports a formula and another supports its negation, then these states are disjoint.[19]

Fact 2.1.4. *Let* $\varphi \in$ **BSML**. *If* $M, s \vDash \varphi$ *and* $M, t \vDash \neg\varphi$*, then* $s \cap t = \emptyset$*. Specifically, if* $t \subseteq s$*, then* $t = \emptyset$*.*

Another useful technical fact is a characterization for the bilateral negation on classical formulas. It states that if all singleton subsets of a state do not support

[17] In our figures we will use w_p to denote a world where p is true, w_q one where q is true etc. And we will use the convention that the shaded area in a model is the state s.

[18] See Section 3.2 and Section 5.1 of Aloni (2022a) for a discussion on the consequences of this non-trivial effect.

[19] This fact is proved in Proposition 3.3.9 of Anttila (2021) and restated as Fact 8 by Aloni (2022a).

α, then the state anti-supports α. This fact is proved in Proposition 2.2.15 by Anttila (2021).

Fact 2.1.5. *Let $\alpha \in \mathbf{BSML}_\emptyset$, then $M, s \vDash \neg\alpha$ if and only if $\forall w \in s : M, \{w\} \nvDash \alpha$.*

Finally, Anttila (2021) presents several state-semantic properties that certain formulas can have, and two useful results concerning these properties.[20]

Definition 2.1.5 (State-Semantic Properties)**.** *Let $\varphi, \psi \in$ **BSML**.*

- *φ is downward closed if for any model M, if $M, s \vDash \varphi$ and $t \subseteq s$, then $M, t \vDash \varphi$;*
- *φ is union closed if for any model M and non-empty set of states S on M, if $M, s \vDash \varphi$ for all $s \in S$, then $M, \bigcup S \vDash \varphi$;*
- *φ has the empty state property if for any model we have $M, \emptyset \vDash \varphi$;*
- *φ is flat if for any model M we have $M, s \vDash \varphi$ if and only if $\forall w \in s$: $M, \{w\} \vDash \varphi$.*

The last of these properties can be characterized by the conjunction of the others, which is proved in Proposition 2.2.2 by Anttila (2021).

Fact 2.1.6. *Let $\varphi \in$ **BSML**. Then φ is flat if and only if it is downward closed, union closed and has the empty state property.*

Using this result Anttila (2021) proved in Proposition 2.2.9 that all classical formulas in **BSML**, i.e., the NE-free formulas, are flat.

Fact 2.1.7. *For any $\alpha \in \mathbf{BSML}_\emptyset$, α is flat.*

So by the characterization of flatness we see that all classical formulas are downward closed, union closed and have the empty state property. Now we can also see that we have rightfully been calling these formulas the "classical" formulas, for Anttila (2021) used this to prove in Proposition 2.2.16 that the NE-free fragment of **BSML** determines the same logical notion of consequence as classical modal logic.

Lastly, in Appendix A we prove that a pragmatically enriched formula $\alpha \in \mathbf{BSML}_0$ is upward closed among states supporting its non-enriched variant.

Fact 2.1.8. *Let $\alpha \in \mathbf{BSML}_0$. Then $M, t \vDash [\alpha]^+$, $t \subseteq s$ and $M, s \vDash \alpha$ imply $M, s \vDash [\alpha]^+$.*

This result is to be expected, as the pragmatic enrichment focuses on non-emptiness, which ought to travel upwards.[21]

[20] These properties also appear in Yang and Väänänen (2017), for example. But Anttila (2021) rephrased them in the context of **BSML**.

[21] To see that this not generalise to $\mathbf{BSML}_\emptyset$, consider the state $s = \{w_q, w_p\}$ with $R[w_q] = \{w_q, w_p\}$ and $R[w_p] = \{w_p\}$ in Figure 5.3(b) (Chapter 5). Then we see that $M, \{w_q\} \vDash [\Diamond(p \vee q)]^+$, $\{w_q\} \subseteq s$ and $M, s \vDash \Diamond(p \vee q)$. However, we also find that $M, s \nvDash [\Diamond(p \vee q)]^+$, showing that Fact 2.1.8 does not hold in general.

2.2 Possible Implications in BSML

Now that we have introduced the framework of **BSML**, we can start with the consideration of possible notions of implication for **BSML**.

We will define several implications in Section 2.2.1, whereafter we will discuss the suitability of these implications for further investigation in Section 2.2.2. Lastly, in Section 2.2.3 we will look at some equivalences that hold between them when we consider certain fragments of **BSML**. But before we continue there are a few minor things that need to be said.

First, when we define an implication, say $\Rightarrow$, then we will call the extension of **BSML** with that implication $\mathbf{BSML}(\Rightarrow)$. Similarly, extensions of $\mathbf{BSML}_0$ and $\mathbf{BSML}_\emptyset$ will be $\mathbf{BSML}_0(\Rightarrow)$ and $\mathbf{BSML}_\emptyset(\Rightarrow)$, respectively.[22]

Secondly, in Definition 2.1.4 we recursively defined the pragmatic enrichment function for NE-free formulas. We will extend this recursive definition for any implication $\Rightarrow$ in the following manner.

$$[\alpha \Rightarrow \beta]^+ = ([\alpha]^+ \Rightarrow [\beta]^+) \wedge \textsc{ne}$$

Note that this definition is in line with the clauses for the other operators.

Lastly, in the following section we will only define the support clauses for our implications. There are several options for the anti-support clauses, and we will discuss these options in Chapter 3, Definition 3.3.1, when they become relevant.

2.2.1 Possible Implications

Remember that in Chapter 1 we introduced the first possible implication, namely the syntactically definable implication $\rightarrow_{\neg,\vee}$ (Definition 1.0.1).

The second implication that we will define is the most discussed implication in the literature, the *material implication* ($\rightarrow$).

Definition 2.2.1 (Material Implication)**.**

$$M, s \vDash \varphi \rightarrow \psi \quad \textit{iff} \quad M, s \vDash \varphi \textit{ implies } M, s \vDash \psi$$

It is easy to see that $\varphi \rightarrow \psi$ is not equivalent to $\neg\varphi \vee \psi$ in **BSML**, but it is equivalent to a formula using alternative notions of negation and disjunction. For this, we need the *Boolean negation* ($\sim$), and the *Boolean disjunction* ($\mathbin{\mathbb{V}}$). The latter is used in inquisitive semantics and therefore we will call it the *inquisitive disjunction* henceforth.[23][24] We follow Anttila (2021) on notation for both notions.

Definition 2.2.2 (Boolean Negation)**.**

$$M, s \vDash {\sim}\varphi \quad \textit{iff} \quad M, s \nvDash \varphi$$

[22]We will also see implications that have strong connections to notions of negation and disjunction that do not occur in **BSML**. Note that the extension of **BSML** with such an implication need not have this negation and disjunction as well. Rather, these notions of negation and disjunction can be seen as inspirations for this particular definition of an implication.

[23]See, for example, Ciardelli, Groenendijk, and Roelofsen (2018).

[24]This disjunction is sometimes also called the *intuitionistic disjunction* or the *global disjunction*. However, we will follow Aloni (2018) in nomenclature.

Definition 2.2.3 (Inquisitive Disjunction)**.**

$$M, s \vDash \varphi \mathbin{\mathbb{W}} \psi \quad \textit{iff} \quad M, s \vDash \varphi \textit{ or } M, s \vDash \psi$$

It then follows directly from the definitions that the material implication is equivalent to an existing formula in **BSML**$(\sim, \mathbb{W})$, i.e., the extension of **BSML** with these operators.[25]

Fact 2.2.1. *$M, s \vDash \varphi \to \psi$ if and only if $M, s \vDash \sim\varphi \mathbin{\mathbb{W}} \psi$*

As mentioned in Chapter 1, the material implication is a traditional option. Unfortunately, in Chapter 5 we will see that the material implication does not satisfy Modus Tollens on the NE-free fragment. In other words, the material implication introduces non-classicality, which also follows from the fact shown in Chapter 5 that it does not preserve flatness. However, we discussed in Chapter 1 that we would prefer an implication that does not introduce non-classicality. On the other hand, in Chapter 5 we will also show that the material implication is the only implication for which the Deduction Theorem holds in general. Therefore we will consider the material implication throughout this thesis, although it faces some serious objections.

In Chapter 1 we discussed that syntactically defining an implication in terms of the *bilateral negation* $(\neg)$ and *split disjunction* $(\vee)$, which we already have in **BSML**, did not work. Specifically, in Section 2.2.2 we will see that under pragmatic enrichment $\to_{\neg,\vee}$ is incompatible with its antecedent. However, we could also semantically define an implication inspired by the bilateral negation and split disjunction. We will call this the **BSML**-implication.

Definition 2.2.4 (**BSML**-implication)**.**

$$M, s \vDash \varphi \Rightarrow_{\neg,\vee} \psi \quad \textit{iff} \quad \exists t, t' : t \cup t' = s \;\&\; M, t \mathrel{=\!\!|} \varphi \;\&\; M, t' \vDash \psi$$

From this definition we can directly see that the **BSML**-implication is equivalent to the disjunction of the negated antecedent and the consequent, although the effect of the pragmatic enrichment function on the two formulas will be different.[26]

Fact 2.2.2. *$M, s \vDash \varphi \Rightarrow_{\neg,\vee} \psi$ if and only if $M, s \vDash \neg\varphi \vee \psi$*

We will consider the **BSML**-implication throughout this thesis, as it would be a very natural implication for **BSML**. As we just saw that this implication is equivalent to a formula in **BSML**, we conclude that **BSML**$(\Rightarrow_{\neg,\vee})$ has the same logical strength as **BSML** itself. And as $\to_{\neg,\vee}$ is an abbreviation of a **BSML**-formula, we see that **BSML**$(\to_{\neg,\vee})$ = **BSML**. However, we saw in Footnote 26

[25] A small note on notation is needed. Formally, we should make explicit what logic our notion of entailment supposes. If we let $\vDash_{\sim,\mathbb{W}}$ be the notion of entailment corresponding to **BSML**$(\sim, \mathbb{W})$, then we would need to say $\varphi \vDash_{\sim,\mathbb{W}} \psi$, for example. However, the notion of entailment will always be clear from context, and therefore we will always use $\vDash$ for entailment.

[26] The difference between this implication and the syntactically defined implication becomes apparent if we pragmatically enrich them. For example, $[p \Rightarrow_{\neg,\vee} q]^+ = ([p]^+ \Rightarrow_{\neg,\vee} [q]^+) \wedge \mathrm{NE} \equiv \neg[p]^+ \vee [q]^+ \not\equiv [\neg p]^+ \vee [q]^+ \equiv [\neg p \vee q]^+ = [p \to_{\neg,\vee} q]^+$ as witnessed by $\{w_{pq}\}$ (for example in Figure 2.1(a)) that supports the former but not the latter.

that the effect of the pragmatic enrichment function was different on the **BSML**-implication and the definable $\to_{\neg,\vee}$. So although extending **BSML** with the **BSML**-implication or the definable $\to_{\neg,\vee}$ leads to logics with the same strength, we see that the pragmatic enrichment function can differentiate between them. The logical strength of **BSML** extended with other implications will be left for future research.

Now, the **BSML**-implication has a strong connection to the intuitionistic implication as discussed by Yang (2014).[27][28][29]

Definition 2.2.5 (Intuitionistic Implication)**.**

$$M, s \vDash \varphi \twoheadrightarrow \psi \quad \textit{iff} \quad \forall t \subseteq s : M, t \vDash \varphi \textit{ implies } M, t \vDash \psi$$

In Section 2.2.3 we will see that for the NE-free fragment $\Rightarrow_{\neg,\vee}$ and $\twoheadrightarrow$ are equivalent. However, we will also see that this does not generalise to all of **BSML**, and thus we will consider both implications throughout this thesis.

A slight alteration of the intuitionistic implication focuses only on maximal subsets of s. We say that $t \subseteq s$ is *maximal* with $M, t \vDash \varphi$ if there is no t' with $t \subset t' \subseteq s$ such that $M, t' \vDash \varphi$. We are inspired by Yang (2014) in notation and definition, and call this the *maximal implication* ($\hookrightarrow_\forall$).[30]

Definition 2.2.6 (Maximal Implication)**.**

$$M, s \vDash \varphi \hookrightarrow_\forall \psi \quad \textit{iff} \quad \textit{for all maximal } t \subseteq s \textit{ with } M, t \vDash \varphi \textit{ we have } M, t \vDash \psi$$

In Section 2.2.3 we will see that the maximal implication is not equivalent to the intuitionistic implication, although they are equivalent on the NE-free fragment.

In the previous definition we saw that the maximal implication has a universal quantification over maximal subsets of s. We could replace this by an existential quantification, giving us the *maximal*$_\exists$ *implication* ($\hookrightarrow_\exists$).[31]

Definition 2.2.7 (Maximal$_\exists$ Implication)**.**

$$M, s \vDash \varphi \hookrightarrow_\exists \psi \quad \textit{iff} \quad \textit{for some maximal } t \subseteq s \textit{ with } M, t \vDash \varphi \textit{ we have } M, t \vDash \psi$$

[27] As we can see in Figure 2.1, **BSML** is not downward-closed. In other words, we can have two teams s and t such that $s \supseteq t$ and $M, s \vDash \varphi$, but $M, t \nvDash \varphi$. This shows that the intuitionistic implication is not equivalent to the material implication.

[28] Note that Yang (2014) uses the symbol $\to$ for the intuitionistic implication. We will use that symbol for the material implication as it is the most common implication.

[29] The intuitionistic implication is sometimes also called the *inquisitive implication*, as it is also used in inquisitive semantics. (See Footnote 23.)

[30] The maximal implication has been introduced in the context of team semantics by Kontinen and Nurmi (2011), but has been thoroughly researched by Yang (2014). See Kolodny and MacFarlane (2010) for a philosophical argument in favor of the maximal implication.

[31] This existential variant is proposed by Aloni and Degano (2022) as an enrichment of their language that possibly can account for negative polarity readings of epistemic indefinites. Their account uses a two-sorted team semantics, so some of the insights from this thesis might be carried over to their framework.

In Chapter 5 we will see that both maximal implications are not transitive, a property that we would expect an implication to have. However, in Chapters 3 and 4 we will see that they behave relatively good from a linguistic point of view. Therefore we will keep both of these implications in consideration throughout this thesis.

Another implication that we will consider is the *linear implication* ($\multimap$). We again follow Yang (2014) for the definition and notation of this implication.

Definition 2.2.8 (Linear Implication)**.**

$$M, s \vDash \varphi \multimap \psi \quad \textit{iff} \quad \textit{for any state } t \textit{ with } M, t \vDash \varphi \textit{ we have } M, s \cup t \vDash \psi$$

Up until now we have considered two implications that are semantically "definable" in terms of negation and disjunction, the material implication and the **BSML**-implication. For this we looked at the *bilateral negation*($\neg$), the *Boolean negation*($\sim$), the *split disjunction*($\vee$) and the *inquisitive disjunction*($\mathbb{V}$). As there are four ways to combine these, we have two more possible "definable" implications to consider.[32]

Definition 2.2.9 (Bilateral Negation and Inquisitive Disjunction)**.**

$$M, s \vDash \varphi \Rightarrow_{\neg,\mathbb{V}} \psi \quad \textit{iff} \quad M, s \dashv \varphi \textit{ or } M, s \vDash \psi$$

Definition 2.2.10 (Boolean Negation and Split Disjunction)**.**

$$M, s \vDash \varphi \Rightarrow_{\sim,\vee} \psi \quad \textit{iff} \quad \exists t, t' : t \cup t' = s \,\&\, M, t \nvDash \varphi \,\&\, M, t' \vDash \psi$$

Again, it follows directly from these definitions that $\varphi \Rightarrow_{\neg,\mathbb{V}} \psi \equiv \neg\varphi \,\mathbb{V}\, \psi$ and $\varphi \Rightarrow_{\sim,\vee} \psi \equiv {\sim}\varphi \vee \psi$. The linguistic and mathematical suitability of these implications, and the implications above, will be further discussed in the following section.

2.2.2 Suitability

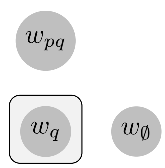
Summing up, in the previous section we have defined the material implication ($\rightarrow$), intuitionistic implication ($\twoheadrightarrow$), **BSML**-implication ($\Rightarrow_{\neg,\vee}$), maximal implication ($\hookrightarrow_{\forall}$), maximal$_{\exists}$ implication ($\hookrightarrow_{\exists}$), linear implication ($\multimap$) and two other "definable" implications ($\Rightarrow_{\neg,\mathbb{V}}$ and $\Rightarrow_{\sim,\vee}$). Furthermore, in Chapter 1 we also defined the syntactically definable $\rightarrow_{\neg,\vee}$.

A first glance at these implications shows that several have bad properties from a linguistic point of view. As the motivation for this research is of a linguistic nature, we will leave out these implications from the rest of this thesis.

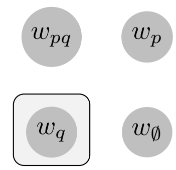
Let us first consider the linear implication ($\multimap$) with respect to the models in Figure 2.2. In the first model, M, we see that only $M, \{w_{pq}\} \vDash p$ and also $M, s \cup \{w_{pq}\} \vDash q$, hence $M, s \vDash p \multimap q$. And in the second model, M', we have

[32] Strictly speaking, only $\rightarrow_{\neg,\vee}$ is definable in **BSML**. However, we will call $\Rightarrow_{\neg,\mathbb{V}}$ and $\Rightarrow_{\sim,\vee}$ "definable" implications throughout this chapter. As they will not be discussed any further in this thesis for reasons that we will see in Section 2.2.2, we will grant ourselves this slight inaccuracy.

$M', \{w_p\} \vDash p$. But $M', s \cup \{w_p\} \nvDash q$, and thus we find that $M', s \nvDash p \multimap q$. So we see that an extension of the model, outside of the actual state s, can influence the support of the linear implication.[33]



(a) $M, s \vDash p \multimap q$



(b) $M', s \nvDash p \multimap q$

Figure 2.2: Unsuitability of the linear implication

This influence of the extension of the model, outside of the actual information state, on the support of the linear implication is questionable. But matters are even worse, for $M', \{w_p\} \vDash p$ and $M', s \cup \{w_p\} \nvDash p$. Therefore, $M', s \nvDash p \multimap p$ which undoubtedly is problematic from a linguistic point of view. For even if one has full information that it is not raining, one would consider the sentence "If it is raining, then it is raining" a tautology.

Now consider the definable $\rightarrow_{\neg,\vee}$, it is easy to see that its pragmatically enriched version is not compatible with its own antecedent.

Fact 2.2.3. *Let* $\alpha, \beta \in \mathbf{BSML}_\emptyset$. *Then,* $[\alpha \rightarrow_{\neg,\vee} \beta]^+, \alpha \vDash \bot$.[34]

Proof. Suppose that $M, s \vDash [\alpha \rightarrow_{\neg,\vee} \beta]^+$, this means that $M, s \vDash [\neg\alpha]^+ \vee [\beta]^+$. So there are non-empty t, t': $t \cup t' = s$ and $M, t \vDash \neg\alpha$ and $M, t' \vDash \beta$. Now also suppose that $M, s \vDash \alpha$. Then it follows by Fact 2.1.4 that $t = \emptyset$, a contradiction. So anything follows from $[\alpha \rightarrow_{\neg,\vee} \beta]^+$ and α. $\square$

Similarly, we find that $\Rightarrow_{\sim,\vee}$ is incompatible with its own antecedent even when it is not pragmatically enriched.

Fact 2.2.4. *Let* $\alpha, \beta \in \mathbf{BSML}_\emptyset$. *Then,* $\alpha \Rightarrow_{\sim,\vee} \beta, \alpha \vDash \bot$.

Proof. Suppose $M, s \vDash \alpha \Rightarrow_{\sim,\vee} \beta$, then $\exists t, t' : t \cup t' = s$ and $M, t \nvDash \alpha$ and $M, t' \vDash \beta$. Now suppose that $M, s \vDash \alpha$, then by downward closure of α it follows that $M, t \vDash \alpha$.[35] A contradiction, so anything follows from $\alpha \Rightarrow_{\sim,\vee} \beta$ and α. $\square$

Finally, the implication defined using the bilateral negation and inquisitive disjunction ($\Rightarrow_{\neg,\mathbb{W}}$) has several mathematical properties that make it unsuitable to extend **BSML** from a mathematical point of view. Specifically, the Deduction Theorem does not hold and it still fails if we look at the NE-free fragment. As we

[33]Note the similarity with the property of Extensionality from Generalized Quantifier Theory. With some necessary adjustment in the definition of this property, one could say that the linear implication lacks the property of Extensionality.

[34]Let $\bot := \mathrm{NE} \wedge \neg\,\mathrm{NE}$, i.e., a formula that is supported by no state.

[35]The downward closure of α follows from Fact 2.1.7. As this property is used very often, we will not mention its origins in the proofs that follow.

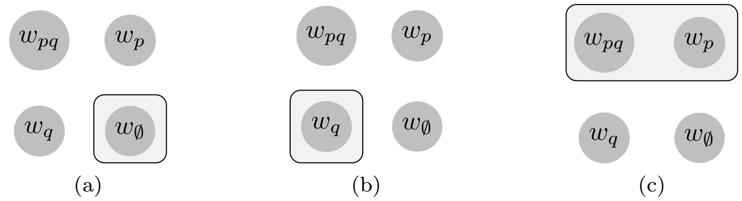


Figure 2.3: Models for non-equivalences of implications

would like to have a strong connection between the notion of entailment and implication, this failure is not preferable. Furthermore, it does not preserve flatness, i.e., adding this implication to the NE-free fragment does not preserve classicality. These results will be proved in Chapter 5, but they are sufficient reason to leave $\Rightarrow_{\neg,\mathbb{V}}$ out of the discussions in Chapters 3 and 4.

These considerations leave us with the material implication ($\rightarrow$), the intuitionistic implication ($\twoheadrightarrow$), the maximal implication ($\hookrightarrow_\forall$), the maximal$_\exists$ implication ($\hookrightarrow_\exists$) and the **BSML**-implication ($\Rightarrow_{\neg,\vee}$). In the following chapter we will compare these on several linguistic desiderata, giving us a better insight in the suitability of these implications for an extension of **BSML**. But before we continue, we will first look at some equivalences between these implications.

2.2.3 Relations between Implications

Let us first consider the relation between the **BSML**-implication and the intuitionistic implication. We can prove that they are equivalent on the NE-free fragment, the proof of this can be found in Appendix A.

Fact 2.2.5. *Let* $\alpha, \beta \in \mathbf{BSML}_\emptyset$. *Then* $\alpha \twoheadrightarrow \beta \equiv \alpha \Rightarrow_{\neg,\vee} \beta$.

To see that this does not generalise, consider Figure 2.3(a). There we see that for all $t \subseteq s$ we have $M, t \not\models [p]^+$, and thus $M, s \models [p]^+ \twoheadrightarrow [q]^+$. However, there is no $t' \subseteq s$ such that $M, t' \models [q]^+$, and thus $M, s \not\models \neg[p]^+ \vee [q]^+$. In other words, $M, s \not\models [p]^+ \Rightarrow_{\neg,\vee} [q]^+$.

Let us shortly think about what this means from a linguistic point of view. The pragmatic enrichment function models the "neglect-zero" tendency, i.e., the tendency to avoid an empty configuration verifying a sentence. Now, in the state $\{w_\emptyset\}$ that we just discussed, p is not a live possibility. So the conditional "if p, then q" would be supported by virtue of an empty configuration. Therefore, we would expect a pragmatically enriched implication not to be supported here. And as the intuitionistic implication is supported while the **BSML**-implication is not, we can conclude that this example is in favor of the latter.[36] However, a similar problem arises for $\Rightarrow_{\neg,\vee}$ if we consider Figure 2.3(b). For then we see that $M, s \models [p \Rightarrow_{\neg,\vee} q]^+$, while p is still not a live possibility. Therefore, we conclude

[36]As the state s is non-empty it follows from the argument above that $M, s \models [p \twoheadrightarrow q]^+$ but $M, s \not\models [p \Rightarrow_{\neg,\vee} q]^+$.

that pragmatic enrichment causes problems for both the intuitionistic implication and the **BSML**-implication.[37]

Now let us look at the relation between the intuitionistic implication and the maximal implication. It is quite straightforward to see that they are equivalent for NE-free formulas.

Fact 2.2.6. *Let $\alpha, \beta \in \mathbf{BSML}_\emptyset$. Then $\alpha \twoheadrightarrow \beta \equiv \alpha \hookrightarrow_\forall \beta$.*

Proof. ($\Rightarrow$): Trivial.

($\Leftarrow$): Suppose $M, s \vDash \alpha \hookrightarrow_\forall \beta$. Now let $t \subseteq s$ be such that $M, t \vDash \alpha$. Then either t is maximal or $t \subset t'$, where t' is maximal with $M, t' \vDash \alpha$. By downward closure of β it follows in both cases from our assumption that $M, t \vDash \beta$. Hence, $M, s \vDash \alpha \twoheadrightarrow \beta$. □

Similarly, if the antecedent is NE-free and the consequent is NE-free and "split-free" we see that the intuitionistic implication and the maximal implication are also equivalent when pragmatically enriched.[38]

Fact 2.2.7. *Let $\alpha, \beta \in \mathbf{BSML}_0$ and β be "split-free". Then $[\alpha \twoheadrightarrow \beta]^+ \equiv_N [\alpha \hookrightarrow_\forall \beta]^+$.*

Proof. ($\Rightarrow$): Trivial.

($\Leftarrow$): Suppose $M, s \vDash [\alpha \hookrightarrow_\forall \beta]^+$. Now let $t \subseteq s$ be such that $M, t \vDash [\alpha]^+$, so $t \neq \emptyset$. If t is maximal, then we are done. So suppose there is a maximal $t \subset t'$ with $M, t' \vDash [\alpha]^+$. Then $M, t' \vDash [\beta]^+$ and thus by Fact 2.1.1, $M, t' \vDash \beta$. Hence, by downward closure of β also $M, t \vDash \beta$. And as $t \neq \emptyset$ and β is "split-free", it follows from Fact 2.1.2 that $M, t \vDash [\beta]^+$. Therefore, $M, s \vDash [\alpha \twoheadrightarrow \beta]^+$. □

To see that this does not hold in general consider $[p \hookrightarrow_\forall (p \vee q)]^+$. This is supported by the state $s = \{w_{pq}, w_p\}$ in Figure 2.3(c), while $[p \twoheadrightarrow (p \vee q)]^+$ is not. This higlights a linguistic problem, for consider $[r \twoheadrightarrow (p \vee q)]^+$. This is only supported by a state that consists of worlds such that if they satisfy r, they must also satisfy p and q. For otherwise there is a singleton satisfying $[r]^+$ that doesn't satisfy $[p]^+$ or doesn't satisfy $[q]^+$. But surely, uttering "if r, then p or q" does not mean the same as "if r, then p and q", not even if we take the "neglect-zero" tendency into account. So this shows that the maximal implication has a linguistic advantage over the intuitionistic implication.

Finally, the maximal implication and its existential variant also have a strong relation. Namely, if the antecedent is NE-free, they are equivalent. The crucial insight for this equivalence is that there cannot be two distinct maximal states for the same formula.

Fact 2.2.8 (Unique maximal state)**.** *Let $\varphi \in \mathbf{BSML}$ and $s \subseteq W$ be a state. Then there is at most one maximal substate $t \subseteq s$ with $M, t \vDash \varphi$.*

[37] In Chapters 3, 4 and 5 we will see that only the maximal$_\exists$ implication correctly interacts with the pragmatic enrichment function by ensuring that the antecedent is a live possibility.

[38] See Footnotes 15 and 16 for the definition of "split-free" and $\equiv_N$.

Proof. Towards contradiction, suppose that there are two distinct maximal states t, t' such that $M, t \vDash \varphi$ and $M, t' \vDash \varphi$. As **BSML** is union closed, it follows that $M, t \cup t' \vDash \varphi$.[39] But as $t, t' \subsetneq t \cup t'$, this is in contradiction with their maximality. $\square$

However, it is also possible that there is no maximal substate supporting a formula, namely when the formula does not have the empty state property. For example, $q \wedge \textsc{ne}$ is not supported by any substate of $s = \{w_\emptyset\}$ in Figure 2.3(a). Combining these insights, we can conclude that the maximal implication and its existential variant are equivalent only when the antecedent is classical.[40]

Fact 2.2.9. *Let* $\alpha \in \mathbf{BSML}_\emptyset$ *and* $\psi \in \mathbf{BSML}$*. Then,* $\alpha \hookrightarrow_\forall \psi \equiv \alpha \hookrightarrow_\exists \psi$

Proof. By the flatness of α (Fact 2.1.7) there is always at least one $t \subseteq s$ that supports α, namely $\emptyset$. Furthermore, by the uniqueness of a maximal state (Fact 2.2.8) we see that there is exactly one such t. Therefore we can conclude the following:

$$\begin{aligned} M, s \vDash \alpha \hookrightarrow_\forall \psi &\text{ iff for all maximal } t \subseteq s \text{ with } M, t \vDash \alpha \text{ also } M, t \vDash \psi \\ &\text{ iff } \exists t \subseteq s \text{ maximal with } M, t \vDash \alpha \text{ such that } M, t \vDash \psi \\ &\text{ iff } M, s \vDash \alpha \hookrightarrow_\exists \psi \end{aligned}$$

$\square$

To conclude, these facts have given us an overview of the relations between the implications that we will consider throughout this thesis. Specifically, we can conclude that the intuitionistic implication, maximal implication, maximal$_\exists$ implication and the **BSML**-implication are all equivalent if their antecedent and consequent are NE-free. However, we have also seen that this is no longer the case if we consider all of **BSML**. These facts will be very helpful for shortening our proofs on these implications in the following chapters.

[39] The union closure of **BSML** can be proved by an easy double induction on the complexity of formulas. Note that other state-based frameworks need not have union closure, and for such systems Fact 2.2.8 no longer holds.

[40] As we saw that $s = \{w_\emptyset\}$ in Figure 2.3(a) has no substate supporting $q \wedge \textsc{ne}$, it follows that $M, s \vDash (q \wedge \textsc{ne}) \hookrightarrow_\forall q$ but $M, s \nvDash (q \wedge \textsc{ne}) \hookrightarrow_\exists q$. So indeed, these implications are not equivalent in general.

Chapter 3

Linguistic Desiderata by Stalnaker

In his seminal work Stalnaker (1976) discusses a non-classical semantic analysis of indicative conditionals.[41] The motivation for this semantic analysis is the failure of the material implication to correctly capture the indicative conditional. Specifically, three entailments and non-entailments are discussed that Stalnaker thinks should be captured by any analysis of implication, but are not all satisfied by the material implication. We will call these *desiderata* and they will be discussed in Section 3.1.[42] As Stalnaker acknowledges, there are two possible routes to overcome the failure of these desiderata for the material implication. The first of these would be à la Grice (1989), who defends the material implication and explains the failure of the desiderata using pragmatic conversational principles discussed by Grice (1975). The second possibility would be to give an alternative analysis of the indicative conditional that is able to satisfy the desiderata, this is the approach of Stalnaker.[43] These two approaches are discussed in more detail in Section 3.2. Finally, we will see where **BSML** extended with an implication lies with respect to these accounts in Section 3.3. We will see that our best option agrees with both accounts on some desiderata, but also differs from both on other desiderata.

[41]This analysis was first developed to account for counterfactual, or subjunctive, conditionals by Stalnaker (1968). But as he explains at the start of Section 3 of Stalnaker (1976), it is intended to capture all different kinds of conditional sentences.

[42]Stalnaker does not list these desiderata explicitly, but argues for them throughout his paper. We list them as desiderata to get a clear overview of the things that we look for in an implication.

[43]As we will see, our approach follows Stalnaker by giving an alternative analysis of conditionals. However, our approach will differ from his account in what is explained by semantics and what is explained using pragmatics.

3.1 The desiderata

The first of these desiderata is what Stalnaker calls the "direct argument". Specifically, it seems to be valid to infer (3-b) from (3-a) in natural language.

(3) a. Either the butler or the gardener did it. (Stalnaker (1976))
b. $\rightsquigarrow$ If the butler didn't do it, the gardener did.

As this inference seems to be valid, we want our logical analysis to account for the direct argument.[44] Now let $\Rightarrow$ be some analysis of the conditional, then the direct argument is captured by the following desideratum.

Desideratum A (Direct argument)**.** *Let* $\alpha, \beta \in \mathbf{BSML}_{\emptyset}$*, then* $\alpha \vee \beta \vDash \neg\alpha \Rightarrow \beta$*.*

As for Stalnaker's second desideratum, assume that the antecedent of some conditional is false. Then Stalnaker argues that it is absurd if this would allow us to infer the truth of the conditional. For example, suppose that the butler and the gardener are under investigation, as well as the driver. Then we have no reason to infer (4-b) from (4-a).

(4) a. The butler did it. (Stalnaker (1976))
b. $\not\rightsquigarrow$ If the butler didn't do it, the gardener did.

In other words, we want the following non-entailment to hold.[45]

Desideratum B (No Trivialization)**.** $\neg p \nvDash p \Rightarrow q$

The third desideratum by Stalnaker has to do with the denial of a conditional. Specifically, it states that the denial of a conditional need not entail the antecedent. Now suppose that I know I did not do it, then it seems very reasonable for me to state (5-a). However, that surely does not put me in a situation where I claim the innocence of the butler. Maybe I have good reason to believe that the butler actually did do it, so the inference from (5-a) to (5-b) is invalid.

(5) (paraphrased from Stalnaker (1976))
a. It is not the case that if the butler didn't do it, then I did it.
b. $\not\rightsquigarrow$ The butler did not do it.

Formally, this means that the negation of an implication should not entail its antecedent.

Desideratum C (Denial of Conditional)**.** $\neg(p \Rightarrow q) \nvDash p$

Note that all entailments above hold for the material implication in classical propositional logic.[46] Therefore, an account that defends the material implication

[44] In the literature potential counterexamples to the direct argument have been proposed ((Adams 1965), (Edgington 1995), (Ciardelli 2020)), which will be discussed in Chapter 4, Desideratum 6.

[45] Note that this is the same desideratum as the first part of Desideratum 1 by Ciardelli (2020), which we will see in Chapter 4. We follow Ciardelli in naming it "No Trivialization".

[46] In Section 3.3 we will see that we have the same result for **BSML** extended with the material implication.

as an analysis of the indicative conditional should give an explanation of the failure of Desideratum B and C. In the following section we will discuss how Grice (1989) approaches this issue.

3.2 Stalnaker vs. Grice

Historically, Stalnaker's paper on indicative conditionals came after Grice's influential lectures on the same subject that have been published later in (Grice 1989). Therefore, we will start with Grice's account and its relation to these desiderata.

Before we can go into Grice's account of the indicative conditional, it is informative to first give a short introduction into Gricean pragmatics as put forward by Grice (1975). The basic idea lies in the assumption that speakers, when in a conversation, tend to cooperate towards a common goal, whatever that goal may be. This tendency is what Grice calls the *Cooperative Principle*, and can be phrased as follows.

> "Make your conversational contribution such as is required [...] by the accepted purpose or direction of the talk exchange in which you are engaged." (Grice (1975), pg. 45)

Grice distinguishes several principles, or maxims, that lead a speaker to act in line with the cooperative principle. These maxims can be divided into the four categories Quantity, Quality, Relation and Manner, the first of which we will shortly discuss. The category, or supermaxim, of Quantity contains the following two maxims:

1. Make your contribution as informative as required. (Grice (1975), pg. 45)
2. Do not make your contribution more informative than required.

For example, if somebody asks whether it is raining one would break the maxim of Quantity by saying that "it is cloudy" or by saying that "it is raining and cats are lovely pets". Now if one does not adhere to this maxim, while we still suppose that one does adhere to the cooperative principle, that might give us new information. For example, if one says "it is cloudy" when asked about the rain, we would infer (pragmatically) that it is not raining. For otherwise the speaker would rather not have flouted the first maxim of Quantity, and just said "it is raining". But surely one can not semantically infer that it is not raining from the statement that it is cloudy. So in Gricean pragmatics we see that there are several inferences that cannot be explained semantically, that can be explained pragmatically using these maxims. We call these inferences *conversational implicatures.*

Now onto Grice's account of the indicative conditional. In his 1975 lectures, as published in Grice (1989), Grice argues that the indicative conditional is best analysed using the material implication.[47] As the direct argument is a validity of classical logic with the material implication, we see that Desideratum A is semantically satisfied by Grice's account. However, $\neg p \vDash p \rightarrow q$ and $\neg(p \rightarrow q) \vDash p$ are

[47]See Gillies (2013) for a discussion on several theoretical arguments for and against this analysis.

also valid for the material implication. Hence, a Gricean account of the indicative conditional needs a pragmatic explanation that can account for desiderata B and C.

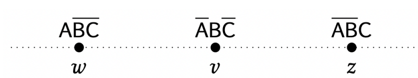
We will give a pragmatic explanation for Desideratum B.[48] For this, consider Grice's view on a conditional with respect to the first maxim of Quantity.[49]

> "To say that $p \supset q$ is to say something logically weaker than to deny that p or to assert that q, and is thus less informative; to make a less informative rather than a more informative statement is to offend against the first maxim of Quantity, provided that the more informative statement, if made, would be of interest." (Grice (1989), pg. 63)

In other words, if $\neg p$ is the case then it would offend against the first maxim of Quantity to utter $p \to q$, as it is less informative. Thereby we see that a Gricean account of the indicative conditional pragmatically explains the non-inference from sentences like (4-a) to sentences like (4-b).

Now let us look at an analysis that rejects the material implication, the *selectional analysis* of the indicative conditional as introduced by Stalnaker (1976). Stalnaker's framework assumes a possible world semantics, where propositions are interpreted as a set of worlds.[50] Furthermore, it defines a similarity function f that takes a proposition A and a possible world i, and gives back the world $f(A, i)$ that is most similar to i and satisfies A. In this framework, a conditional "if A, then B" is true in a possible world i just in case B is true in $f(A, i)$. In other words, the conditional holds in a possible world if the most similar "antecedent-world" makes the consequent true.

This allows us to see how the selectional analysis accounts for Desideratum A.[51] It is straightforward to see that this desideratum is not satisfied semantically. For this we use an example by Cariani (2022), who worked out Stalnaker's formalism very neatly. Suppose that in the actual world (w) Ada alone is guilty (A), but in the closest world in which she is innocent (v), Bo is guilty (B) and Carmen is not (C). This can be visualised by the following picture.[52]



Then we see that w is an $A \vee C$-world, but the closest $\neg A$-world is v. In other words, we have $f(\neg A, w) = v$, but C is not true in v. Hence, we conclude that "if $\neg A$, then C" is not true in w. And thus Desideratum A semantically fails in Stalnaker's account.

[48]For a discussion on Grice's pragmatic explanation of Desideratum C, or *Denial of Conditional*, see pg. 81 of Grice (1989).

[49]Note that $\supset$ is the symbol that Grice uses for the material implication, while we use $\to$.

[50]See Footnote 41 for the origin of this framework.

[51]Similar arguments to the one below can be thought of to show that desiderata B and C are semantically accounted for by Stalnaker. For brevity, we will only discuss Desideratum A.

[52]Both the example and the picture are by Cariani (2022) and have been unchanged.

However, Stalnaker introduces the pragmatic notion of *reasonable inference*. This notion is informally defined as follows.

> "An inference from a sequence of assertions or suppositions (the premisses) to an assertion or hypothetical assertion (the conclusion) is *reasonable* just in case, in every context in which the premisses could appropriately be asserted or supposed, it is impossible for anyone to accept the premisses without committing himself to the conclusion." (Stalnaker (1976), pg. 138)

Let us shortly look at the derivation of the *direct argument* as a reasonable inference. On Stalnaker's account a disjunction $A \vee B$ can only be appropriately asserted if both $\neg A \wedge B$ and $A \wedge \neg B$ are compatible with the context.[53] Similarly, a conditional can only be appropriately made if the antecedent is compatible with the context.[54] Now suppose $A \vee B$ is asserted, then we see that $\neg A$ is compatible with the context. Now, Stalnaker argues that in this context "if $\neg A$, then $A \vee B$" also must be accepted.[55] And as this entails the conditional "if $\neg A$, then B", we see that the direct argument has been derived as a reasonable inference.[56] Hence, we find that, in contrast to Grice, Stalnaker pragmatically accounts for Desideratum A and semantically accounts for Desiderata B and C.

It is interesting to note that the *selectional analysis* invalidates the classically valid principle of *Simplification of Disjunctive Antecedents* (SDA).[57] This principle can be formalised as $(\alpha \vee \beta) \Rightarrow \gamma \vDash (\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma)$. To see that this is not valid in Stalnaker's framework, consider any world w such that γ is true in $f(\alpha, w)$ but not in $f(\beta, w)$, and $f(\alpha, w) = f(\alpha \vee \beta, w)$. Then we see that γ is true in the most similar $\alpha \vee \beta$-world, but not in the most similar β-world. In other words, as the most similar β-world need not be the most similar $\alpha \vee \beta$-world, SDA is not satisfied.[58][59]

To conclude, we can see in Table 3.1 that Stalnaker's account differs from Grice's account on its explanation of all desiderata. In the following section we will see how **BSML** extended with an implication can account for these desiderata.

[53] Notice that this is different from **BSML**'s account of the disjunction. For there we see that a singleton state where both A and B hold supports $[A \vee B]^+$, while this should not be the case on Stalnaker's account.

[54] See Footnote 79 for some discussion on this "antecedent-compatibility".

[55] For this Stalnaker claims that any context in which P is compatible and that entails Q, is a context in which "if P, then Q" is entailed.

[56] For a fully formalised definition and derivation of the direct argument, one can best see Cariani (2022).

[57] See Section 5.2.1 for an example motivating this principle and **BSML**'s account of this principle.

[58] See Alonso-Ovalle (2004) for a broader discussion of this principle in the context of Stalnakerian frameworks.

[59] By considering a context $c = \{w_{AC}, w_B\}$ (where w_{AC} is a world where A and C hold, but B does not), we can see that SDA cannot be accounted for as a reasonable inference. For we see that $\neg A \wedge B$ and $A \wedge \neg B$ are compatible with this context, and thus $A \vee B$ can appropriately be asserted. Hence, the conditional $A \vee B \Rightarrow C$ can also be appropriately asserted. However, if one asserts $A \vee B \Rightarrow C$ we need to consider the updated context $c' = \{w_{AC}\}$. But then B is no longer compatible with the context, and thus "if B, then C" can not be appropriately asserted.

Desideratum	A (*Dir. Arg.*)	B (*No Triv.*)	C (*Denial*)
Grice	Sem.	Prag.	Prag.
Stalnaker	Prag.	Sem.	Sem.
BSML($\Rightarrow$)	?	?	?

Table 3.1: Stalnaker vs. Grice on Stalnaker's desiderata.
Dir. Arg. = Direct Argument, No Triv. = No Trivialization,
Denial = Denial of Conditional

3.3 BSML and Stalnaker

Now that we have seen the approaches by Stalnaker and Grice, it is time to see where **BSML**($\Rightarrow$) falls with respect to Stalnaker's desiderata. We will look at each desideratum, and see whether **BSML** extended with some implication satisfies the desideratum. The proofs from this section that have been omitted from the text can be found in Appendix B.

Now if some desideratum holds for **BSML** extended with some implication $\Rightarrow$, then we say that **BSML**($\Rightarrow$) semantically accounts for that desideratum. However, **BSML** is designed to model a pragmatic tendency, i.e., the "neglect-zero" tendency. Hence, we will also look if the desideratum is satisfied if we pragmatically enrich all the formulas.[60] If that is the case, then we would say that **BSML**($\Rightarrow$) accounts for that desideratum pragmatically. Of course, an explanation is then needed on why the "neglect-zero" tendency can account for the satisfaction of this desideratum, and it will be given where needed.

The implications that we will discuss are the material implication ($\rightarrow$), the intuitionistic implication ($\twoheadrightarrow$), the maximal implication ($\hookrightarrow_\forall$), the maximal$_\exists$ implication ($\hookrightarrow_\exists$) and the **BSML**-implication ($\Rightarrow_{\neg,\vee}$).

Desideratum A Let us start with the *direct argument*, i.e., $\alpha \vee \beta \vDash \neg\alpha \Rightarrow \beta$ for NE-free α and β. Using the facts from Chapter 2 we can see that this desideratum holds for all five implications.[61]

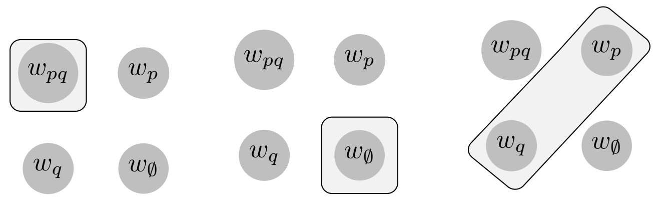
Fact 3.3.1. *Let* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$ *and* $\alpha, \beta \in \mathbf{BSML}_\emptyset$, *then* $\alpha \vee \beta \vDash \neg\alpha \Rightarrow \beta$.

When we add pragmatic enrichment, we see that the material implication still satisfies this desideratum.

Fact 3.3.2. *Let* $\alpha, \beta \in \mathbf{BSML}_\emptyset$, *then* $[\alpha \vee \beta]^+ \vDash [\neg\alpha \rightarrow \beta]^+$.

[60]We have chosen to add the pragmatic enrichment function to both the premises and the conclusion of a(n) (non)-entailment. Another option would be to only enrich the premises as done by Aloni (2022a), sometimes resulting in slightly different results. For example, Fact 5.2.3 would give an entailment for the maximal$_\exists$ implication if we would not enrich the conclusion. However, having the pragmatics on both sides of the entailment seems to be the most complete option, and there are only a few situations where the results would differ.

[61]Additionally, we use the fact that for all formulas φ, ψ we have $\varphi \twoheadrightarrow \psi \vDash \varphi \rightarrow \psi$. This follows directly from the definitions, and will not be made explicit in other proofs. Note that also $[\varphi \twoheadrightarrow \psi]^+ = ([\varphi]^+ \twoheadrightarrow [\psi]^+) \wedge \text{NE} \vDash ([\varphi]^+ \rightarrow [\psi]^+) \wedge \text{NE} = [\varphi \rightarrow \psi]^+$.



(a) (b) (c)

Figure 3.1: Several models to be used as counterexamples.

Similarly, we can see that the intuitionistic implication and the maximal implication still satisfy this desideratum for "split-free" formulas if we add pragmatic enrichment.[62][63]

Fact 3.3.3. *Let* $\Rightarrow \in \{\twoheadrightarrow, \hookrightarrow_\forall\}$ *and "split-free"* $\alpha, \beta \in \mathbf{BSML}_\emptyset$*, then* $[\alpha \vee \beta]^+ \vDash_N [\neg\alpha \Rightarrow \beta]^+$.

However, the same does not hold for the maximal$_\exists$ implication and the **BSML**-implication. For consider the state $s = \{w_{pq}\}$ in Figure 3.1(a), then we see that there is no maximal $t \subseteq s$ with $M, t \vDash [\neg p]^+$. Hence, $M, s \nvDash [\neg p \hookrightarrow_\exists q]^+$ although we do have $M, s \vDash [p \vee q]^+$. Similarly, in Figure 3.1(c) we have $M, s \nvDash \neg[p]^+ \vee [q]^+$. Hence, $M, s \nvDash [\neg p \Rightarrow_{\neg,\vee} q]^+$ although we again do have $M, s \vDash [p \vee q]^+$.

Combining these facts we see that **BSML** extended with any of the implications can account for Desideratum A on a semantic level, in accordance with Grice. However, we saw that the maximal$_\exists$ implication and $\Rightarrow_{\neg,\vee}$ fail this desideratum if we pragmatically enrich the formulas.[64] As the pragmatic enrichment function models the "neglect-zero" tendency of speakers, we would need an explanation of the failure of the direct argument on the basis of this tendency if we want to be able to rival Grice's and Stalnaker's account. This will be discussed at the end of this section, where one can also find a comparison of the different accounts in Table 3.2.

Desideratum B The second desideratum that we will discuss is *No Trivialization*, i.e., $\neg p \nvDash p \Rightarrow q$. It is straightforward to see that this desideratum fails semantically for all implications.

Fact 3.3.4. *For* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$*, we have* $\neg p \vDash p \Rightarrow q$.

[62] To see that this does not generalise to all NE-free formulas, consider $\alpha := p$ and $\beta := p \vee q$. Then it is straightforward to see that state $s = \{w_p, w_q\}$ in Figure 3.1(c) supports $[p \vee (p \vee q)]^+$. However, $\{w_q\}$ is maximal with $M, \{w_q\} \vDash [\neg p]^+$ but $M, \{w_q\} \nvDash [p \vee q]^+$. Hence, s does not support $[\neg p \twoheadrightarrow (p \vee q)]^+$ nor $[\neg p \hookrightarrow_\forall (p \vee q)]^+$.

[63] See Footnotes 15 and 16 for the definition of "split-free" and $\vDash_N$.

[64] The intuitionistic implication and the maximal implication also fail for some formulas containing disjunction. But we will see at the end of this section that their allowance of the direct argument for atomic propositions under the pragmatic enrichment is not preferable. Therefore we will not go into details about the difference between the material implication and these implications with respect to Desideratum A.

And if we pragmatically enrich the formulas we find the same result for the first three implications.

Fact 3.3.5. *For* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall\}$*, we have* $[\neg p]^+ \vDash [p \Rightarrow q]^+$.

In other words, these implications can neither account for Desideratum B on a semantic level nor on a pragmatic level. However, we find that this desideratum is satisfied for the maximal$_\exists$ implication and the **BSML**-implication when we add pragmatic enrichment.

Fact 3.3.6. $[\neg p]^+ \nvDash [p \hookrightarrow_\exists q]^+$ *and* $[\neg p]^+ \nvDash [p \Rightarrow_{\neg,\vee} q]^+$

Proof. Consider Figure 3.1(b), then we see that $M, s \vDash [\neg p]^+$. But there is no $t \subseteq s$ such that $M, t \vDash [p]^+$, hence $M, s \nvDash [p \hookrightarrow_\exists q]^+$. And as there is also no $t' \subseteq s$ such that $M, t' \vDash [q]^+$ we see that $M, s \nvDash [p \Rightarrow_{\neg,\vee} q]^+$. □

Here we see an interesting effect of the pragmatic enrichment function. As Desideratum B did not hold for the non-enriched version we see that **BSML**($\hookrightarrow_\exists$) and **BSML**($\Rightarrow_{\neg,\vee}$) cannot account for it semantically. However, when we model the "neglect-zero" tendency using the pragmatic enrichment function, they do satisfy the desideratum. Hence, **BSML**($\hookrightarrow_\exists$) and **BSML**($\Rightarrow_{\neg,\vee}$) explain *No Trivialization* on a pragmatic level, similar to Grice. At the end of this section we will explain these results given the pragmatic "neglect-zero" tendency.

Desideratum C The final desideratum that we will consider here is the *Denial of Conditional*, i.e., $\neg(p \Rightarrow q) \nvDash p$. As we mentioned in Section 2.2, we have left out the anti-support conditions for the implications until now. For each of these conditions we have chosen to present a version of the anti-support conditions that is most in line with the support conditions of the implication.[65]

Definition 3.3.1 (Anti-support Conditions)**.**

$$\begin{array}{rll}
M, s \mathrel{=\!\!\!|} \varphi \rightarrow \psi & \textit{iff} & M, s \vDash \varphi \textit{ and } M, s \nvDash \psi \\
M, s \mathrel{=\!\!\!|} \varphi \twoheadrightarrow \psi & \textit{iff} & \forall t \subseteq s: \textit{ if } t \neq \emptyset \textit{ and } M, t \vDash \varphi \textit{ then } M, t \nvDash \psi \\
M, s \mathrel{=\!\!\!|} \varphi \hookrightarrow_\forall \psi & \textit{iff} & \exists t \subseteq s: \textit{(maximal with } M, t \vDash \varphi \textit{) and } M, t \mathrel{=\!\!\!|} \psi \\
M, s \mathrel{=\!\!\!|} \varphi \hookrightarrow_\exists \psi & \textit{iff} & M, s \mathrel{=\!\!\!|} \varphi \hookrightarrow_\forall \psi \\
M, s \mathrel{=\!\!\!|} \varphi \Rightarrow_{\neg,\vee} \psi & \textit{iff} & M, s \vDash \varphi \textit{ and } M, s \mathrel{=\!\!\!|} \psi
\end{array}$$

[65] The rejection clause for the *intuitionistic implication* is the same as the negation clause for the implication in inquisitive semantics. (Remember that the intuitionistic implication is the same as the inquisitive implication as discussed by Ciardelli, Groenendijk, and Roelofsen (2018))

The rejection clause for the *maximal implications* says that the consequent should be anti-supported ($M, s \mathrel{=\!\!\!|} \psi$). An alternative option is to say that it should not be supported, so $M, s \nvDash \psi$. For this desideratum both options would give the same result. However, the $\nvDash$-variant would not have the empty state property when the consequent is classical, because the empty state supports and anti-supports any classical formula. On the other hand, we see that the $\mathrel{=\!\!\!|}$ variant will have the empty state property, allowing for the preservation of classicality.

In Chapter 7 we will look at some properties of these different anti-support conditions and point towards future research topics surrounding them.

From this definition it is directly clear that $\neg(p \to q) \vDash p$ and $[\neg(p \to q)]^+ \vDash [p]^+$, i.e., the desideratum fails for the material implication regardless of pragmatic enrichment. Similarly, we see that it fails for $\Rightarrow_{\neg,\vee}$ regardless of pragmatic enrichment. However, this desideratum is satisfied for the other three implications. And this still is the case if we pragmatically enrich the relevant formulas.

Fact 3.3.7. *Let* $\Rightarrow \in \{\twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists\}$*, then* $\neg(p \Rightarrow q) \nvDash p$ *and* $[\neg(p \Rightarrow q)]^+ \nvDash [p]^+$*.*

Proof. Consider the state $s = \{w_p, w_q\}$ in Figure 3.1(c). Then we see that $\{w_p\}$ is the only non-empty substate of s that supports p (and $[p]^+$). And as $M, \{w_p\} \nvDash q$ (and $M, \{w_p\} \nvDash [q]^+$), it follows that $M, s \dashv p \twoheadrightarrow q$ (and $M, s \dashv [p \twoheadrightarrow q]^+$). As $\{w_p\}$ was maximal with $M, \{w_p\} \vDash p$ (and $M, \{w_p\} \vDash [p]^+$), we have the same result for $\hookrightarrow_\forall$ and $\hookrightarrow_\exists$. However, we see that $M, s \nvDash p$ and $M, s \nvDash [p]^+$ and thus the non-entailment holds. □

We can conclude that the intuitionistic implication and both maximal implications can semantically account for this desideratum. And, in contrast to Desideratum A, this result does not change when we add pragmatics to the desideratum.

Stalnaker, Grice and BSML In Section 3.2 we saw that Grice accounted for Desideratum A semantically and for the other desiderata pragmatically, while Stalnaker's account did this the other way around.

We have seen that **BSML** extended with the material implication, intuitionistic implication or maximal implication can not account for *No Trivialization*. And although the **BSML**-implication can pragmatically account for this, it cannot account for *Denial of Conditional*. So we can conclude that these extensions can not rival with Grice's and Stalnaker's account. However, when we extend **BSML** with the maximal$_\exists$ implication we get a more interesting picture.

As we can see in Table 3.2, **BSML**($\hookrightarrow_\exists$) semantically accounts for the direct argument.[66] This is in line with Grice's account, but there seems to be a problem. When we pragmatically enrich the formulas, **BSML**($\hookrightarrow_\exists$) no longer satisfies the desideratum. As mentioned above, this "failure" is in need of an explanation. There are two explanations available, the first discarding this counterexample altogether and the second explaining the failure in this counterexample.

Let us first consider a pragmatic reason for discarding the counterexample, the state $s = \{w_{pq}\}$ from Figure 3.1(a), altogether. When it is the case that it rains and there is a rainbow, it would be odd to say "It rains or there is a rainbow". When we look at Grice's maxims as discussed in Section 3.2, we say that the speaker would flout the first maxim of Quantity. For $p \vee q$ is logically weaker than $p \wedge q$, so the speaker could (and should) have said something more informative. So for this pragmatic reason, one could argue that we need not consider this counterexample. Now note that the other states that support $[p \vee q]^+$ all have

[66]Throughout this thesis we will use colors and fonts in tables to denote what results are **good**, bad or *somewhere in between*. Note that bad means that a framework extended with a certain implication cannot account for a certain phenomenon. Of course, one could still support this implication in combination with an appeal to extra-logical means, such as Gricean pragmatics, to account for certain desiderata.

Desideratum	A (*Dir. Arg.*)	B (*No Triv.*)	C (*Denial*)
Grice	Sem.	Prag.	Prag.
Stalnaker	Prag.	Sem.	Sem.
BSML($\rightarrow$)	*Sem.*	✗	✗
BSML($\twoheadrightarrow$)	*Sem.*	✗	**Sem.**
BSML($\hookrightarrow_\forall$)	*Sem.*	✗	**Sem.**
BSML($\hookrightarrow_\exists$)	**Sem.***	**Prag.**	**Sem.**
BSML($\Rightarrow_{\neg,\vee}$)	*Sem.**	**Prag.**	✗

Table 3.2: BSML and Stalnaker.
Dir. Arg. = Direct Argument, No Triv. = No Trivialization,
Denial = Denial of Conditional
*This desideratum fails when pragmatically enriched.

a subset supporting $[\neg p]^+$ and $[q]^+$, i.e., the same problem does not arise if we only look at pragmatically reasonable models. However, we can also explain the failure of the direct argument in this model using the "neglect-zero" tendency.

Suppose our information state is as in Figure 3.1(a), i.e., we are sure that both p and q hold. Then they are both live possibilities, and therefore we satisfy $[p \vee q]^+$. But then "if $\neg p$, then q" would only be verified by an empty configuration, because $\neg p$ is not a live possibility at all. So it seems to be very reasonable in this case that $[p \vee q]^+$ is supported, but the "neglect-zero" tendency rules out that $[\neg p \hookrightarrow_\exists q]^+$ is supported. In other words, a notion of implication that could account for this would be preferable. Hence, we conclude that this apparent "failure" is actually exactly what we want to have.

Let us shortly also consider the counterexample that we saw for the **BSML**-implication in Figure 3.1(c). There we had the state $s = \{w_p, w_q\}$ that supports $[p \vee q]^+$ but does not support $[\neg p \Rightarrow_{\neg,\vee} q]^+$. However, p, $\neg p$ and q were all live possibilities, so neither an explanation using Gricean pragmatics nor the explanation using the "neglect-zero" tendency that we just saw goes through in this case. Even stronger, we would expect the implication to be supported in this situation, and therefore we see that this "failure" under pragmatic enrichment is a true failure in the case of the **BSML**-implication.

The second desideratum, *No Trivialization*, was not satisfied by **BSML**($\hookrightarrow_\exists$). However, when we add the pragmatic enrichment it does satisfy this desideratum. In other words, **BSML**($\hookrightarrow_\exists$) accounts for this desideratum pragmatically. This makes sense if we think about the "neglect-zero" tendency. If "The butler did it", then we would validate "If the butler didn't do it, then the gardener did" using an empty configuration. But this is exactly what the "neglect-zero" tendency rules out, so we would indeed expect this desideratum to be accounted for pragmatically.

Finally, the third desideratum, *Denial of Conditional*, is satisfied by the maximal$_\exists$ implication regardless of pragmatic enrichment. So similarly to Desideratum A, we would say that **BSML**($\hookrightarrow_\exists$) semantically accounts for this desideratum. At this point we differ from Grice, but agree with Stalnaker.

Before we finish this chapter, there are some observations to be made. First of all, let us shortly compare the Gricean account to **BSML**($\Rightarrow$). Note that the "neglect-zero" hypothesis is compatible with Gricean pragmatics, as Aloni (2022a) acknowledges by stating that the former "connects to a version of Grice's Quality maxim" (pg. 33). In other words, these pragmatic accounts are not competing, but are likely to both be operative in language. Aloni (2022a) further supports this idea by showing that FC-effects cannot be explained as conversational implicatures. Specifically, conversational implicatures have higher processing costs and are acquired later than FC-effects.[67] Hence, another pragmatic explanation, the "neglect-zero" tendency, is given to explain these FC-effects.

Therefore, it might be interesting for future empirical research to look at phenomena such as *No Trivialization*, of which we have seen that both Grice and **BSML**($\hookrightarrow_\exists$) give a pragmatic explanation. This might give us insight in the question whether such phenomena should be explained using Gricean pragmatics or using the "neglect-zero" tendency.[68]

Furthermore, in **BSML** we have made the pragmatic "neglect-zero" tendency formally explicit in the syntax.[69] It would also be interesting to see how Gricean maxims, such as Quantity, can be formalised in **BSML** (or **BSML**($\hookrightarrow_\exists$)) to explain phenomena that the "neglect-zero" tendency alone cannot account for.

Lastly, in this section we have seen that **BSML**($\hookrightarrow_\exists$) can explain, either semantically or pragmatically, the same desiderata as Stalnaker's account. However, in Section 3.2 we saw that the definition of a reasonable inference was rather involved. As the "neglect-zero" tendency is quite simple, this can be seen as an advantage of **BSML**($\hookrightarrow_\exists$) over Stalnaker's account.

To conclude, we have seen that the **BSML**($\hookrightarrow_\exists$)-account is really different from Grice's and Stalnaker's accounts, and is able to explain all desiderata using either semantics or pragmatics. However, we will see in the next section that the maximal$_\exists$ implication has some problems regarding other linguistic desiderata, and in Chapter 5 we will see that it has some unfortunate mathematical properties. Therefore, we will come back to Stalnaker's desiderata in Chapter 6, where we will adopt a dynamic variant of **BSML** to give a more satisfying account on almost all linguistic desiderata with a mathematically better-behaved implication.

[67] See Table 1 by Aloni (2022a) for an overview of these properties for several linguistic phenomena.

[68] Krzyżanowska, Collins, and Hahn (2021) have done empirical research that explicitly goes against the Gricean account. However, their findings are problematic to any analysis of conditionals that does not take the connection between the antecedent and consequent into account. Neither Stalnaker's account nor any of the implications that we are researching for **BSML** takes this connection into account, so it is important to realise that this empirical research causes problems for these accounts as well.

[69] This pragmatic tendency can also be introduced model-theoretically, as done in Section 6.3.1 by Aloni (2022a).

Chapter 4

Linguistic Desiderata by Ciardelli

In the previous chapter we saw that the maximal$_\exists$ implication could account for all of Stalnaker's desiderata using either semantics or pragmatics. In this chapter we will see if the maximal$_\exists$ implication does equally well if we look at other linguistic desiderata. Specifically, we will discuss several desiderata for an analysis of the indicative conditional, as listed by Ciardelli (2020). Ciardelli's account of the indicative conditional combines features of what he calls *information semantics* and *minimal change semantics*.[70] This combination can account for several desiderata that each individual approach traditionally cannot account for. In Section 4.1 we will present and discuss these desiderata. Subsequently, in Section 4.2 we will see which of these are satisfied by **BSML**($\Rightarrow$) for each of our implications.

4.1 The desiderata

The first desideratum discussed by Ciardelli (2020) is the same as Desideratum B (*No Trivialization*) by Stalnaker, but additionally we want an implication to be compatible with the negation of its antecedent. As we argued for in Section 3.1, (4-a) ("If the butler didn't do it, the gardener did.") need not follow from (4-b) ("The butler did it") . But it is not infelicitous to utter both sentences in the same breath, i.e., they should be compatible.

Desideratum 1 (No Trivialization)**.** *(a)* $\neg p \not\models p \Rightarrow q$*;* *(b)* $\neg p, p \Rightarrow q \not\models \bot$

We already have discussed how **BSML**($\Rightarrow$) accounts for Desideratum 1a) in Section 3.3 (as Desideratum B). Specifically, we saw that **BSML**($\hookrightarrow_\exists$) gave a pragmatic account of *No Trivialization*. Therefore, we will not consider the first half of Desideratum 1 again in this chapter. However, it does raise the question

[70] *Information semantics* are accounts like dynamic semantics and **BSML** that evaluate formulas in sets of worlds, rather than in the worlds themselves. *Minimal change semantics* are accounts such as Stalnaker's, as discussed in Section 3.2.

whether Desideratum 1(b) will also be accounted for pragmatically. We will see that none of the implications will give a fully satisfactory account of this, either semantically or pragmatically. This will be one of the motivations for the dynamic approach in Chapter 6.

The second desideratum says that Modus Ponens and Modus Tollens should be satisfied if we restrict ourselves to implication-free propositional sentences. There are arguments against Modus Ponens and Modus Tollens, but these use either modalities or iterated conditionals and will be mentioned for Desiderata 4 & 5. Ciardelli argues in favor of this desideratum because of its intuitive appeal and the lack of counterexamples.

Desideratum 2 (Factual Modus Ponens and Modus Tollens)**.**
For all $\alpha, \beta \in \mathbf{BSML}_0$*: (a)* $\alpha \Rightarrow \beta, \alpha \vDash \beta$*; (b)* $\alpha \Rightarrow \beta, \neg\beta \vDash \neg\alpha$

We will also be interested in whether Modus Ponens also holds under pragmatic enrichment, as the "neglect-zero" tendency does not seem to give us a reason to expect its failure. Hence, the failure of pragmatically enriched propositional Modus Ponens is something that would be undesirable. On the other hand, for Modus Tollens we will only be interested in the non-enriched version. The reason for this is as follows, a pragmatically enriched implication might ensure that its antecedent is a live possibility.[71] However, this need not be compatible with the negation of the antecedent, as is needed for Modus Tollens. Rather than dive into specific instances of the effect of the "neglect-zero" tendency on Modus Tollens, we decide that we will only consider the non-enriched version. The relation between "neglect-zero" and Modus Tollens might be an interesting topic for further research.

The third desideratum concerns the *import/export* principle. It arises from the idea that sentences like (6-a) express the same thing as sentences like (6-b).[72]

(6) a. If Bob is in Paris, then if he's staying in a hotel, he's at the Ritz. (Ciardelli (2020))
b. If Bob is in Paris and he's staying in a hotel, he's at the Ritz.

Formally, we would say that an iterated implication is equivalent to an implication with a conjunction in the antecedent.

Desideratum 3 (Import-export)**.**
For all $\alpha, \beta \in \mathbf{BSML}_0$ *and* $\gamma \in \mathbf{BSML}_\emptyset$*:* $\alpha \Rightarrow (\beta \Rightarrow \gamma) \equiv (\alpha \wedge \beta) \Rightarrow \gamma$

The fourth desideratum is motivated by the famous "counterexample" to Modus Ponens by McGee (1985).[73] For this we consider the context of the 1980 elections in the United States. There are three candidates for presidency: Reagan, a Republican, is leading the polls; Carter, a Democrat, is lagging behind; and Anderson, a Republican running as independent, has virtually no chance at

[71] For Desideratum 7′ we will argue that this is even a desideratum.
[72] See Gibbard (1980), Kremer (1987), Lycan (2001) and Mandelkern (2020) for more discussion on this principle.
[73] We present this example in the wording of Ciardelli (2020).

winning. According to McGee it is very natural to accept (7-a) and (7-b), but to reject (7-c).

(7) a. If a Republican wins, then if Reagan doesn't win, Anderson will. (McGee (1985))
b. A Republican will win.
c. If Reagan doesn't win, Anderson will.

In other words, this desideratum says that we do not want Modus Ponens to hold for an iterated conditional.

Desideratum 4 (No Modus Ponens for iterated conditionals)**.**
$p \Rightarrow (q \Rightarrow r), p \not\models q \Rightarrow r$

However, several accounts have argued that this argument does not hold for information-based semantics, and thus also does not hold for **BSML**. For example, Gillies (2004) argues that in McGee's intuitive assessment of the argument the information of (7-b) seems to be lost as soon as we arrive at (7-c). And the stronger we keep hold of the information from (7-b), the stronger the urge seems to accept (7-c). This can be made more explicit if we consider the following re-wording of the example.[74]

(8) a. If a Republican wins, then if he is not Reagan, he is Anderson.
b. A Republican will win.
c. If he is not Reagan, he is Anderson.

Now, it seems much more reasonable to say that (8-c) follows from (8-a) and (8-b). And as Gillies (2004) argues, the difference between (7) and (8) is merely stylistic, and does not lie in its content. So the only thing that example (8) did, is show that when the information that a Republican will win is more available, it is much more reasonable to accept this instance of Modus Ponens. For an information-based account it seems right to demand that information given earlier, in (7-b), is still available when interpreting (7-c), validating Modus Ponens. And as **BSML** is an information-based account, we do not agree with counting this failure of Modus Ponens as a desideratum.[75] So henceforth, we will only be concerned with the other desiderata by Ciardelli.

The fifth desideratum argues against Modus Tollens when the consequent has an epistemic modality in it. Consider the following example by Ciardelli (2020), a variation on examples by Veltman (1985) and Yalcin (2012). Suppose there is an urn with white and black marbles in it. A marble is randomly extracted from the urn, and its color is not yet revealed. Then we would accept (9-a) and (9-b), but we would reject (9-c).

(9) a. If the marble is not white, it must be black. (Ciardelli (2020))
b. It is not the case that the marble must be black.

[74] Gillies (2004) deals with a similar McGee-like example, and revises it in the same way as we will do.

[75] See Bledin (2015) for another argument against McGee's counterexample to Modus Ponens in the context of information-based semantics.

c. The marble is white.

Formally, this means that we do not want Modus Tollens to hold for implications with an epistemic modal consequent.

Desideratum 5 (No Modus Tollens for modal consequents)**.**
$p \Rightarrow \Box q, \neg\Box q \nvDash \neg p$

The penultimate desideratum by Ciardelli is concerned with the direct argument, or as Ciardelli calls it "*or*-to-*if*". We saw in Section 3.1 that Stalnaker argues that this inference should always be valid. However, Ciardelli (2020) argues that it need not be valid if the disjunct is accepted solely on the acceptance of the first disjunct. We will reformulate this desideratum shortly, but let us first look at Ciardelli's argument.[76] Suppose there were three suspects: the driver, the gardener and the chauffeur.[77] There has been some convincing evidence against the butler, while nothing has been found against the other two. Then the detective will accept (10-a) based on her acceptance of the first disjunct. However, she has no reason at all to accept (10-b), for it might also be the chauffeur who did it.

(10) a. Either the butler or the gardener did it.
b. If the butler didn't do it, the gardener did.

On the other hand, in cases where our acceptance of the disjunction is not merely based on the acceptance of one of the disjuncts, the direct argument should be valid according to Ciardelli. In other words, if the negation of the first disjunct is still a possibility, the direct argument should go through. Ciardelli formulates this as follows.

Desideratum 6 (Cautious *or*-to-*if*)**.**
a) $p \vee q \nvDash \neg p \Rightarrow q$; *b)* $p \vee q, \Diamond\neg p \vDash \neg p \Rightarrow q$

Now, there is an interesting aspect of this line of reasoning that needs to be pointed out. Namely, this argument assumes that acceptance of the first disjunct is sufficient for the acceptance of the disjunction, i.e., $p \vDash p \vee q$. This inference is typical for classical reasoning, and it is exactly such reasoning that the "neglect-zero" tendency is concerned with. So we would expect an account based on this tendency to have something to say about this argument. Specifically, when we take the "neglect-zero" tendency into account, we would not say that in the situation described above (10-a) should be accepted at all. This can formally be seen by the crucial fact that in **BSML** we have $[p \wedge \neg q]^+ \nvDash [p \vee q]^+$.

We agree with Stalnaker in keeping the direct argument as a desideratum and, contrary to Stalnaker, we will account for it semantically. However, when we take the pragmatic "neglect-zero" tendency into account, we agree with Ciardelli by stating that the acceptance of the first disjunct is not sufficient for the acceptance of the direct argument. To make this explicit, we will reformulate the first half of

[76] We will paraphrase Ciardelli's example, but Ciardelli credits Adams (1965) and Edgington (1995).

[77] Note that this example, (10), is the same as example (3), we repeat it here for clarity.

this desideratum as $[p \wedge \neg q]^+ \not\models [\neg p \Rightarrow q]^+$.[78] In other words, we want to explain the situation in example (10) using pragmatics, rather than semantics.

Now for the second half of the sixth desideratum. Remember that in Section 3.3 we saw that under pragmatic enrichment the direct argument no longer holds for the maximal$_\exists$ implication. Specifically, it no longer holds in a situation where the disjunction is supported by a state where in each world both disjuncts are true, e.g., a state such as $s = \{w_{pq}\}$. We argued that this behaviour is desirable if we take the "neglect-zero" tendency into account. For in this situation the antecedent of the implication is not a live possibility, i.e., the implication would be validated by an empty configuration. However, when the antecedent is a live possibility, we expect the direct argument to go through under pragmatic enrichment. Therefore we reformulate the second half of Desideratum 6 so that we are only interested in its pragmatically enriched version. Together with our earlier argument this results in the following reformulated desideratum.

Desideratum 6′. *(*Cautious *or*-to-*if,* reformulated*)*
a) $[p \wedge \neg q]^+ \not\models [\neg p \Rightarrow q]^+$*;* *b)* $[p \vee q]^+, [\Diamond\neg p]^+ \models [\neg p \Rightarrow q]^+$

Finally, the last desideratum deals with the interaction of conditionals and *might*. Consider (11-a), this seems to be very uncontroversial to accept. As we know that Hamlet was written, we might even say "if it was not Shakespeare who wrote it, it *must* have been someone else". However, (11-b) is much more controversial, as it implies that we have been fooled for many centuries.

(11) a. If Hamlet was not written by Shakespeare, it might have been written by someone else. (Ciardelli (2020))
b. It might be that Hamlet was not written by Shakespeare, but by someone else.

This shows that we do not want $p \Rightarrow \Diamond q$ to be equivalent to $\Diamond(p \wedge q)$.[79] Specifically,

[78] Because we have $[p \wedge \neg q]^+ \not\models [p \vee q]^+$ in **BSML**, we see that this is not one of the situations that has been discussed in the pragmatically enriched version of Desideratum A.

[79] Ciardelli mentions that Gillies (2010) takes the opposite view by selecting $p \Rightarrow \Diamond q \equiv \Diamond(p \wedge q)$ as a desideratum. Now note that around Definition 4.1 Gillies (2010) argues that any notion of an indicative conditional should have definedness, i.e., that it should be defined only if the antecedent is compatible with the context. Similarly, Gillies (2020) implicitly assumes $\varphi \to \Diamond\psi \models \Diamond\varphi$ when he states in Figure 1 that in update semantics (similar to the dynamic system we will see in Chapter 6) we have $\varphi \to \Diamond\psi \equiv \Diamond(\varphi \wedge \psi)$.

Ciardelli's account does not assume the same, but Ciardelli discusses this "antecedent compatibility" in Section 5.3. By ensuring $s[\alpha] \neq \emptyset$ for $s \models \alpha \Rightarrow \varphi$, Ciardelli's account would have a notion of implication with definedness. However, he mentions that in that case we would find that $p \Rightarrow \Diamond q \equiv \Diamond(p \wedge q)$, in contradiction to Desideratum 7. Hence, we see that a crucial difference between Ciardelli's and Gillies' accounts is the definedness. It is interesting to note that Stalnaker agrees with Gillies by saying that "it is appropriate to make an indicative conditional statement [...] only in a context which is compatible with its antecedent" (Stalnaker (1976), pg. 146).

As we will see, a pragmatically enriched maximal$_\exists$ implication will have definedness, while this is not the case without pragmatic enrichment. In other words, our pragmatic enrichment function formalises within the maximal$_\exists$ implication what Gillies has (implicitly) assumed. Furthermore, we have a simple pragmatic explanation that can explain why we want to have the definedness. This is an advantage over both Ciardelli's and Gillies' accounts.

this example shows that the former should not entail $\Diamond p$ while the latter should. Therefore, Ciardelli formulates this desideratum as follows.

Desideratum 7 (If-might interaction)**.**
a) $p \Rightarrow \Diamond q \nvDash \Diamond p$*;* *b)* $\Diamond(p \wedge q) \vDash \Diamond p$

However, when we take the "neglect-zero" tendency into account, we no longer think that Desideratum 7a) should be the case. For if somebody says "if p, then it might be that q" without it being an epistemic possibility that p is the case, then the sentence would be validated by an empty configuration. But this is exactly what the "neglect-zero" tendency prevents. In other words, we will reformulate this desideratum to Desideratum 7′, so that the non-entailment will become an entailment under pragmatic enrichment.

Desideratum 7′. *(If-might interaction, reformulated)*

a) $p \Rightarrow \Diamond q \nvDash \Diamond p$ *and* $\Diamond(p \wedge q) \vDash \Diamond p$

b) $[p \Rightarrow \Diamond q]^+ \vDash [\Diamond p]^+$ *and* $[\Diamond(p \wedge q)]^+ \vDash [\Diamond p]^+$

Now that we have discussed all desiderata, we can consider how the different possible implications for **BSML** fare.

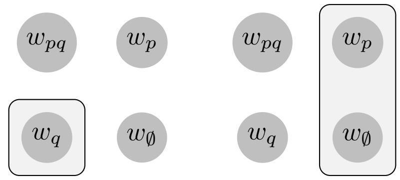
4.2 BSML on Ciardelli's Desiderata

In the previous section we have seen Ciardelli's desiderata for indicative conditionals. As we already discussed Desideratum 1a) as Desideratum B, and we deem the counterarguments to Desideratum 4 valid, we will no longer consider these desiderata. Furthermore, we have reformulated Desideratum 6 and Desideratum 7 to make better use of the pragmatic enrichment function that we have at our disposal. We will now look at **BSML** extended with the material implication ($\rightarrow$), intuitionistic implication ($\twoheadrightarrow$), maximal implication ($\hookrightarrow_\forall$), maximal$_\exists$ implication ($\hookrightarrow_\exists$) or the **BSML**-implication ($\Rightarrow_{\neg,\vee}$), and see if these satisfy the desiderata. Similarly to the desiderata in Section 3.3, we will also see if they are satisfied if we introduce pragmatics with our pragmatic enrichment function. The proofs of this section that are omitted from the text can be found in Appendix C. The proofs that have been included can be insightful for the behaviour of the implications, but need not be thoroughly inspected to understand the discussion that can be found at the end of this section.

Desideratum 1 Let us first look at the second half of Desideratum 1, i.e., $\neg p, p \Rightarrow q \nvDash \bot$. It is quite straightforward to see that all implications satisfy this desideratum.

Fact 4.2.1. *For* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$*, we have* $\neg p, p \Rightarrow q \nvDash \bot$*.*

Proof. Consider the state $s = \{w_q\}$ in Figure 4.1(a). As we have $M, t \vDash q$ for all $t \subseteq s$, we see that indeed $M, s \vDash p \twoheadrightarrow q$. So by Footnote 61, Fact 2.2.5, Fact 2.2.6 and Fact 2.2.9, it follows that $M, s \vDash p \rightarrow q$, $M, s \vDash p \hookrightarrow_\forall q$, $M, s \vDash p \hookrightarrow_\exists q$ and $M, s \vDash p \Rightarrow_{\neg,\vee} q$. Clearly, $M, s \vDash \neg p$ and as $s \neq \emptyset$ we see that $M, s \nvDash \bot$. ☐



(a) (b)

Figure 4.1: Several models to be used as counterexamples.

Similarly, we see that the material implication, intuitionistic implication, maximal implication and the **BSML**-implication still satisfy the desideratum if we add pragmatic enrichment.

Fact 4.2.2. *For* $\Rightarrow \in \{\to, \twoheadrightarrow, \hookrightarrow_\forall, \Rightarrow_{\neg,\vee}\}$*, we have* $[\neg p]^+, [p \Rightarrow q]^+ \nvDash \bot$*.*

Proof. Consider the state $s = \{w_q\}$ in Figure 4.1(a) again. As there is no $t \subseteq s$ with $M, t \vDash [p]^+$ and $s \neq \emptyset$, we see that $M, s \vDash [p \twoheadrightarrow q]^+$. So by Footnote 61 and Fact 2.2.7 we have the same for $\to$ and $\hookrightarrow_\forall$. Now let $t = \emptyset$ and $t' = s$, then clearly $t \cup t' = s$. Furthermore, $M, t \vDash \neg[p]^+$, $M, t' \vDash [q]^+$ and $s \neq \emptyset$, therefore it follows that $M, s \vDash [p \Rightarrow_{\neg,\vee} q]^+$. We can directly see that $M, s \vDash [\neg p]^+$ and as $s \neq \emptyset$ it follows that $M, s \nvDash \bot$. $\square$

However, the same does not hold for the maximal$_\exists$ implication. There we see that the negation of the antecedent is incompatible with the conditional, if we add pragmatic enrichment.

Fact 4.2.3. $[\neg p]^+, [p \hookrightarrow_\exists q]^+ \vDash \bot$

Proof. Suppose $M, s \vDash [\neg p]^+$ and $M, s \vDash [p \hookrightarrow_\exists q]^+$. From the former and Fact 2.1.1 we see that $M, s \vDash \neg p$. And it follows from the latter that there is a $t \subseteq s$ with $M, t \vDash [p]^+$, so $t \neq \emptyset$ and by Fact 2.1.1 also $M, t \vDash p$. Now by Fact 2.1.4 we see that $t = \emptyset$, a contradiction. So anything follows. $\square$

Combining these results in Table 4.1, we see that all implications semantically account for Desideratum 1b). But the maximal$_\exists$ implication does not satisfy it when pragmatically enriched. Similar to the situation with Desideratum A by Stalnaker, this “failure” needs a good explanation if we want to propose the maximal$_\exists$ implication as a suitable implication for **BSML**. At the end of this section we will see that there is something in favor of this “failure” and something against it. Therefore we conclude that none of the implications can give a fully satisfactory account of Desideratum 1b).

Desideratum	1b) (CNA)
BSML($\rightarrow$)	*Sem.*
BSML($\twoheadrightarrow$)	*Sem.*
BSML($\hookrightarrow_\forall$)	*Sem.*
BSML($\hookrightarrow_\exists$)	*Sem.**
BSML($\Rightarrow_{\neg,\vee}$)	*Sem.*

Table 4.1: Desideratum 1 by Ciardelli
CNA = Compatibility with Negated Antecedent
*This desideratum fails when pragmatically enriched.

Desideratum 2 The second desideratum by Ciardelli is about Modus Ponens and Modus Tollens for propositional formulas, i.e., NE-free and $\Diamond$-free formulas. Let us first look at Modus Ponens. It is straightforward to see that this holds for all implications.

Fact 4.2.4. *For $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$ and $\alpha, \beta \in \mathbf{BSML}_0$, we have $\alpha \Rightarrow \beta, \alpha \vDash \beta$.*

Proof. For $\rightarrow$ this follows directly from the definition. And as $s \subseteq s$, it also holds for $\twoheadrightarrow$. So by Fact 2.2.5, Fact 2.2.6 and Fact 2.2.9 it also holds for $\Rightarrow_{\neg,\vee}$, $\hookrightarrow_\forall$ and $\hookrightarrow_\exists$. $\square$

Similarly, Modus Ponens still holds for all implications if we pragmatically enrich the formulas.

Fact 4.2.5. *For $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$ and $\alpha, \beta \in \mathbf{BSML}_0$, we have $[\alpha \Rightarrow \beta]^+, [\alpha]^+ \vDash [\beta]^+$.*

The second part of this desideratum is concerned with Modus Tollens for propositional formulas. This desideratum holds for the intuitionistic implication, maximal implication, maximal$_\exists$ implication and the **BSML**-implication.[80]

Fact 4.2.6. *For $\Rightarrow \in \{\twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$ and $\alpha, \beta \in \mathbf{BSML}_\emptyset$, we have $\alpha \Rightarrow \beta, \neg\beta \vDash \neg\alpha$.*

However, Modus Tollens fails for the material implication. For consider the state $s = \{w_p, w_\emptyset\}$ in Figure 4.1(b), then we see that $M, s \nvDash p$ and thus $M, s \vDash p \rightarrow q$. Also, $M, s \vDash \neg q$ but we do not have $M, s \vDash \neg p$. Note that this failure of Modus Tollens originates in the fact that failure of support for a formula is not the same as anti-support in **BSML**.

[80] Ciardelli's desideratum is only concerned with $\mathbf{BSML}_0$, i.e., the propositional fragment. In Appendix C we have presented a proof for the larger NE-free fragment, $\mathbf{BSML}_\emptyset$, as it will ease our discussion surrounding Desideratum 5.

Desideratum	2a) (MP)	2b) (MT)
BSML($\rightarrow$)	**Sem.**	✗
BSML($\twoheadrightarrow$)	**Sem.**	**Sem.**
BSML($\hookrightarrow_\forall$)	**Sem.**	**Sem.**
BSML($\hookrightarrow_\exists$)	**Sem.**	**Sem.**
BSML($\Rightarrow_{\neg,\vee}$)	**Sem.**	**Sem.**

Table 4.2: Desideratum 2 by Ciardelli
MP = Modus Ponens, MT = Modus Tollens

We combine these results in Table 4.2, where we see that all implications can account for Modus Ponens for propositional formulas. And all implications, except for the material implication, can account for propositional Modus Tollens.[81] We conclude that this result is in favor of all implications but the material implication.

Desideratum 3 The third desideratum that Ciardelli discusses is the *import-export principle*. All implications under discussion can account for this desideratum semantically.

Fact 4.2.7. *Let* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$*,* $\alpha, \beta \in \mathbf{BSML}_0$ *and* $\gamma \in \mathbf{BSML}_\emptyset$*. Then we have* $\alpha \Rightarrow (\beta \Rightarrow \gamma) \equiv (\alpha \wedge \beta) \Rightarrow \gamma$*.*

For the material implication, the maximal$_\exists$ implication and the **BSML**-implication we see that the desideratum is still satisfied if we pragmatically enrich the formulas.

Fact 4.2.8. *Let* $\Rightarrow \in \{\rightarrow, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$*,* $\alpha, \beta \in \mathbf{BSML}_0$ *and* $\gamma \in \mathbf{BSML}_\emptyset$*. Then we have* $[\alpha \Rightarrow (\beta \Rightarrow \gamma)]^+ \equiv [(\alpha \wedge \beta) \Rightarrow \gamma]^+$*.*

However, this desideratum fails under pragmatic enrichment for the intuitionistic implication and the maximal implication.

Fact 4.2.9. *Let* $\Rightarrow \in \{\twoheadrightarrow, \hookrightarrow_\forall\}$*,* $\alpha, \beta \in \mathbf{BSML}_0$ *and* $\gamma \in \mathbf{BSML}_\emptyset$*. Then we have*

$$[\alpha \Rightarrow (\beta \Rightarrow \gamma)]^+ \not\equiv [(\alpha \wedge \beta) \Rightarrow \gamma]^+.$$

Proof. Consider the state $s = \{w_p, w_q\}$ in Figure 4.2(a), then we see that there is no $t \subseteq s$ such that $M, t \vDash [(p \vee q) \wedge p]^+$, and thus $M, s \vDash [((p \vee q) \wedge p) \hookrightarrow_\forall (p \vee q)]^+$. Now we see that $M, s \vDash [p \vee q]^+$ (and s is maximal), and $\{w_p\}$ is maximal with $M, \{w_p\} \vDash p$. However, $M, \{w_p\} \nvDash [p \vee q]^+$ and thus $M, s \nvDash [p \hookrightarrow_\forall (p \vee q)]^+$. Thereby, we conclude that $M, s \nvDash [(p \vee q) \hookrightarrow_\forall (p \hookrightarrow_\forall (p \vee q))]^+$. The same proof works for the intuitionistic implication. $\square$

[81] Remember that we argued in Section 4.1 that we are only interested in the non-enriched Modus Tollens. For the interested reader, the material implication, intuitionistic implication and maximal implication fail Modus Tollens under pragmatic enrichment, as witnessed by $\alpha := p \vee q$ and $\beta := r$.

We summarise these results in Table 4.3. Note that the counterexample from Fact 4.2.9 cannot be explained away using the "neglect-zero" tendency, and thus we conclude that this failure under pragmatic enrichment is unfavorable.[82]

Desideratum	3 (I-E)
BSML($\rightarrow$)	**Sem.**
BSML($\twoheadrightarrow$)	Sem.*
BSML($\hookrightarrow_\forall$)	Sem.*
BSML($\hookrightarrow_\exists$)	**Sem.**
BSML($\Rightarrow_{\neg,\vee}$)	**Sem.**

Table 4.3: Desideratum 3 by Ciardelli
I-E= Import-Export
*This desideratum fails when pragmatically enriched.

Desideratum 5 The next desideratum that we will consider says that we do not want Modus Tollens if we have an epistemic modal consequent, i.e., $p \Rightarrow \Box q, \neg\Box q \not\models \neg p$. This desideratum holds for the material implication, even when we add pragmatic enrichment. Note that this is not very surprising, as the material implication already did not have Modus Tollens for the propositional fragment.

Fact 4.2.10. *$p \rightarrow \Box q, \neg\Box q \not\models \neg p$ and $[p \rightarrow \Box q]^+, [\neg\Box q]^+ \not\models [\neg p]^+$*

Proof. Consider the state $s = \{w_p, w_\emptyset\}$ in Figure 4.2(b). As $s \neq \emptyset$, $M, s \not\models p$ and $M, s \not\models [p]^+$, we see that $M, s \models p \rightarrow \Box q$ and $M, s \models [p \rightarrow \Box q]^+$. And as $\{w_\emptyset\} \subseteq R[w]$ for all $w \in s$ and $M, \{w_\emptyset\} \dashv q$, it follows that $M, s \models \neg\Box q$. Furthermore, $\neg\Box q$ is "split-free", so by Fact 2.1.2 we see that $M, s \models [\neg\Box q]^+$. However, we see that $M, s \not\models \neg p$ and $M, s \not\models [\neg p]^+$. $\Box$

The same desideratum fails for the intuitionistic implication, maximal implication, maximal$_\exists$ implication and the **BSML**-implication. To see this, note that in our discussion of Desideratum 2 in this section we proved that for these notions of implication Modus Tollens holds for all $\alpha, \beta \in \mathbf{BSML}_\emptyset$. And thus it also holds for implications with modal consequents.

And if we pragmatically enrich all formulas, we still see that Modus Tollens holds in this case. Hence, we conclude that these implication also cannot account for this desideratum pragmatically.

Fact 4.2.11. *For $\Rightarrow \in \{\twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists \Rightarrow_{\neg,\vee}\}$, we have $[p \Rightarrow \Box q]^+, [\neg\Box q]^+ \models [\neg p]^+$.*

Combining these results in Table 4.4, we see that the material implication is the only implication that can account for Desideratum 5. It does so semantically, but this result is not surprising as propositional Modus Tollens also failed for the material implication. All in all, we can conclude that none of the implications can give an account of Modus Tollens that takes Desideratum 2b) and 5 both

[82] There is no reason for $(p \vee q) \hookrightarrow_\forall (p \hookrightarrow_\forall (p \vee q))$ to fail when we take the "neglect-zero" tendency into account, for all antecedents and consequents are live possibilities in our state.

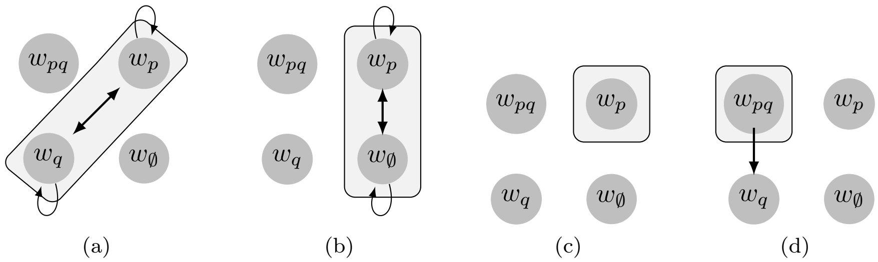


Figure 4.2: Several more models to be used as counterexamples.

into account. In Chapter 6 we will see that a dynamic notion of implication fares much better in this respect.

Desideratum	5 (No MT$^\Diamond$)
BSML($\rightarrow$)	*Sem.*
BSML($\twoheadrightarrow$)	✗
BSML($\hookrightarrow_\forall$)	✗
BSML($\hookrightarrow_\exists$)	✗
BSML($\Rightarrow_{\neg,\vee}$)	✗

Table 4.4: Desideratum 5 by Ciardelli
No MT$^\Diamond$ = No Modus Tollens for modal consequent

Desideratum 6′ As we discussed in Section 4.1, we have reformulated this desideratum so that we want $[p \wedge \neg q]^+ \nvDash [\neg p \Rightarrow q]^+$ and $[p \vee q]^+, [\Diamond \neg p]^+ \vDash [\neg p \Rightarrow q]^+$. The first half of this desideratum is not satisfied by the material implication, intuitionistic implication and maximal implication. As $[p \wedge \neg q]^+ \vDash [p]^+$, this follows directly from a proof analogous to that of Fact 3.3.5.

Fact 4.2.12. *For* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall\}$*, we have* $[p \wedge \neg q]^+ \vDash [\neg p \Rightarrow q]^+$.

However, we find that the maximal$_\exists$ implication and the **BSML**-implication do satisfy Desideratum 6′a).

Fact 4.2.13. *For* $\Rightarrow \in \{\hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$*, we have* $[p \wedge \neg q]^+ \nvDash [\neg p \Rightarrow q]^+$.

Proof. Consider Figure 4.2(c), then we see that $M, s \vDash [p \wedge \neg q]^+$. But there is no $t \subseteq s$ such that $M, t \vDash [\neg p]^+$, and thus $M, s \nvDash [\neg p \hookrightarrow_\exists q]^+$. And as there is no $t \subseteq s$ such that $M, t \vDash [q]^+$, we see that also $M, s \nvDash [\neg p \Rightarrow_{\neg,\vee} q]^+$. □

Now let us turn to the second half of the desideratum. For the material implication, intuitionistic implication and maximal implication we saw in Section 3.3 that the direct argument holds under pragmatic enrichment (Fact 3.3.2 and Fact 3.3.3). Therefore it directly follows that 6′b) is also satisfied by these implications.

On the other hand, it is straightforward to see that the same does not hold for the maximal$_\exists$ implication. For consider the state $s = \{w_{pq}\}$ in Figure 4.2(d). Then we see that $M, s \vDash [p \vee q]^+$ and $M, s \vDash [\Diamond\neg p]^+$, but $M, s \nvDash [\neg p \hookrightarrow_\exists q]^+$ as there is no $t \subseteq s$ with $M, t \vDash [\neg p]^+$.

However, Aloni (2022a) introduces some properties that the relation R of a model M can have with respect to a state s. In particular, she defines that R is *state-based* in (M, s) if and only if for all $w \in s : R[w] = s$. As the property of being state-based leads to the satisfaction of the S5-axioms, Aloni argues that for epistemic modalities, such as in our motivation for Desideratum 6′, one only needs to consider models with a state-based R.

But if we restrict ourselves to such models, we see that Desideratum 6′b) actually does hold for the maximal$_\exists$ implication.

Fact 4.2.14. $[p \vee q]^+, [\Diamond\neg p]^+ \vDash [\neg p \hookrightarrow_\exists q]^+$ *(if R is state-based)*

And lastly, for the **BSML**-implication we see that the state $s = \{w_p, w_q\}$ in Figure 4.2(a) supports $[p \vee q]^+$ but not $[\neg p \Rightarrow_{\neg,\vee} q]^+$. And as we also have $M, s \vDash [\Diamond\neg p]^+$, we see that this is a counterexample to Desideratum 6′b). Note that we have a state-based R in this model, and thus we see that the restriction to state-based relations will not make a difference for the **BSML**-implication. Furthermore, this seems to be a model in which we really want our implication to hold under pragmatic enrichment. For $\neg p$ and q are both live possibilities, and thus we cannot give a pragmatic reason for discarding this counterexample.

We combine these results in Table 4.5.[83] We see that the material implication, intuitionistic implication and maximal implication fail to account for Desideratum 6′ by not accounting for Desideratum 6′a). Similarly, the **BSML**-implication does not satisfy Desideratum 6′b), and thus also fails to account for Desideratum 6′. However, if we consider the relevant models, with a state-based R, we see that the maximal$_\exists$ implication satisfies both 6′a) and 6′b), and thus is the only implication that can account for Desideratum 6′.

Desideratum	6′ (COI)
BSML($\rightarrow$)	✗
BSML($\twoheadrightarrow$)	✗
BSML($\hookrightarrow_\forall$)	✗
BSML($\hookrightarrow_\exists$)	✓
BSML($\Rightarrow_{\neg,\vee}$)	✗

Table 4.5: Desideratum 6′ by Ciardelli
COI = Cautious Or-to-If
*The second half of this desideratum fails when pragmatically enriched.

Desideratum 7′ In Section 4.1 we reformulated this desideratum to make better use of the pragmatic enrichment function. So in addition to the old desideratum that says that we want $p \Rightarrow \Diamond q \nvDash \Diamond p$ and $\Diamond(p \wedge q) \vDash \Diamond p$, we also want

[83] We will only show the failure or success of accounting for both Desideratum 6′a) and Desideratum 6′b) at the same time.

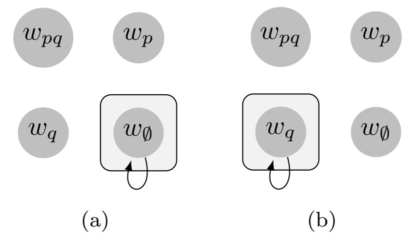


Figure 4.3: Several more models to be used as counterexamples.

to have $[p \Rightarrow \Diamond q]^+ \vDash [\Diamond p]^+$ and $[\Diamond(p \wedge q)]^+ \vDash [\Diamond p]^+$. In other words, when we add pragmatic enrichment we want our implication to entail the possibility of its antecedent, while the rest should remain unchanged.

Now let us first note that part of this desideratum, $\Diamond(p \wedge q) \vDash \Diamond p$, is not concerned with implications. Rather, it is a property that **BSML** satisfies, regardless of pragmatic enrichment.

Fact 4.2.15. $\Diamond(p \wedge q) \vDash \Diamond p$ *and* $[\Diamond(p \wedge q)]^+ \vDash [\Diamond p]^+$

We can now consider the part of this desideratum that does contain an implication, i.e., $p \Rightarrow \Diamond q \vDash \Diamond p$. It is straightforward to see that all implications satisfy the non-enriched non-entailment.

Fact 4.2.16. *For* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$, *we have* $p \Rightarrow \Diamond q \nvDash \Diamond p$.

Proof. Consider the state $s = \{w_\emptyset\}$ in Figure 4.3(a). Then we see that $M, \{w_\emptyset\} \nvDash p$ and $M, \emptyset \vDash \Diamond q$, and thus $M, s \vDash p \twoheadrightarrow \Diamond q$. And by Footnote 61, Fact 2.2.5, Fact 2.2.6 and Fact 2.2.9, the same holds for $\rightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists$ and $\Rightarrow_{\neg,\vee}$. However, there is no non-empty $t \subseteq R[w_\emptyset]$ such that $M, t \vDash p$, so we see that $M, s \nvDash \Diamond p$. □

And when we add pragmatic enrichment, the material implication, intuitionistic implication, maximal implication and **BSML**-implication still have the non-entailment.

Fact 4.2.17. *For* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall, \Rightarrow_{\neg,\vee}\}$, *we have* $[p \Rightarrow \Diamond q]^+ \nvDash [\Diamond p]^+$.

Proof. Consider the state $s = \{w_q\}$ in Figure 4.3(b). As there is no $t \subseteq s$ with $M, t \vDash [p]^+$ we see that $M, s \vDash [p \rightarrow \Diamond q]^+$, and similarly for $\twoheadrightarrow$ and $\hookrightarrow_\forall$. Now let $t = \emptyset$ and $t' = s$, then $M, t \vDash \neg[p]^+$ and $M, t' \vDash [\Diamond q]^+$. Hence, as $t \cup t' = s \neq \emptyset$ we see that $M, s \vDash [p \Rightarrow_{\neg,\vee} \Diamond q]^+$. However, there is no $t \subseteq R[w_q]$ such that $M, t \vDash [p]^+$, and thus $M, s \nvDash [\Diamond p]^+$. □

In other words, for all of these implications we see that pragmatical enrichment ensures nothing about the antecedent being a live possibility. However, for the maximal$_\exists$ implication we see that the pragmatic enrichment does ensure that the antecedent is a live possibility.[84] Similar to the previous desideratum, we will only focus on models with a state-based R because our $\Diamond$ is an epistemic modality.

Fact 4.2.18. $[p \hookrightarrow_{\exists} \Diamond q]^+ \vDash [\Diamond p]^+$ *(if R is state-based)*

We conclude that only the maximal$_{\exists}$ implication can account for this desideratum, as can be seen in Table 4.6.[85]

BSML on Ciardelli's desiderata Now that we have considered all of Ciardelli's desiderata, we can see the overall situation in Table 4.6.[86] We see that the material implication, intuitionistic implication and the maximal implication fail to account for many of the desiderata. So for brevity we will only discuss the behaviour of the maximal$_{\exists}$ implication ($\hookrightarrow_{\exists}$) and the **BSML**-implication ($\Rightarrow_{\neg,\vee}$).

Desideratum	1b) (CNA)	2a) (MP)	2b) (MT)	3 (I-E)	5 (No MT$^{\Diamond}$)	6′ (COI)	7′ (IM)
BSML($\rightarrow$)	*Sem.*	**Sem.**	✗	**Sem.**	*Sem.*	✗	✗
BSML($\twoheadrightarrow$)	*Sem.*	**Sem.**	**Sem.**	Sem.*	✗	✗	✗
BSML($\hookrightarrow_{\forall}$)	*Sem.*	**Sem.**	**Sem.**	Sem.*	✗	✗	✗
BSML($\hookrightarrow_{\exists}$)	*Sem.**	**Sem.**	**Sem.**	**Sem.**	✗	✓	✓
BSML($\Rightarrow_{\neg,\vee}$)	*Sem.*	**Sem.**	**Sem.**	**Sem.**	✗	✗	✗

Table 4.6: Ciardelli's desiderata summarised
CNA = Compatibility with Negated Antecedent, MP = Modus Ponens, MT = Modus Tollens, I-E= Import-Export, No MT$^{\Diamond}$ = No Modus Tollens for modal consequent, COI = Cautious Or-to-If, IM = If-Might interaction
*This desideratum fails when pragmatically enriched.

The first desideratum stated that we do not want an implication to follow from the negation of its antecedent, but it should be compatible with it. The first half of this has been discussed in Chapter 3 as Desideratum B, where we saw that both implications could pragmatically account for this.

The second half of the first desideratum, 1b), can semantically be accounted for by both implications. However, we see that the maximal$_{\exists}$ implication is incompatible with its negated antecedent if pragmatically enriched. Let us consider whether this can be explained with the "neglect-zero" tendency. For this, note

[84] Notice that $[p]^+$ and $[\Diamond p]^+$ are both ways of expressing that p is a live possibility. However, they are not equivalent, not even if we restrict ourselves to state-based relations. For this, consider the state $s = \{w_p, w_\emptyset\}$ in Figure 4.2(b), where we see that $M, s \nvDash [p]^+$ but $M, s \vDash [\Diamond p]^+$. More generally, $[p]^+$ ensures that the state supporting p is non-empty, while $[\Diamond p]^+$ ensures that there is a non-empty substate supporting p. It might be that there are different "levels" of being a live possibility, corresponding to these (non-equivalent) notions. We will leave this for future work, and for now we will use $[p]^+$ and $[\Diamond p]^+$ interchangeably to represent that p is a live possibility.

[85] As the desideratum is concerned with the effect of introducing pragmatics, we cannot say that it is semantically or pragmatically accounted for. Hence, we will just note whether **BSML**($\Rightarrow$) can account for Desideratum 7′ at all. To prevent redundancy, we only show this result in the table summarising all results from this section.

[86] Remember that we use colors and fonts in tables to denote what results are **good**, bad or *somewhere in between*.

that pragmatically enriching the maximal$_\exists$ implication, ensures that both the antecedent and the consequent are live possibilities. But if the negation of the antecedent is already supported, then we see that the antecedent can no longer be a live possibility. So we would indeed expect this to lead to a contradiction.

But now consider the following two sentences. Then (12-a) seems to be okay, while (12-b) is an odd thing to say.[87]

(12) a. If it is raining, the pavement is wet. It is not raining.
b. ??It is not raining. If it is raining, the pavement is wet.

This example shows that the order of these sequences is of importance. However, our static framework either predicts both to be okay or both to be infelicitous. In Chapter 6 we will see that the dynamic variant of **BSML** extended with an implication will be able to account for this difference.

The second desideratum looked at Modus Ponens and Modus Tollens for the propositional fragment, i.e., $\mathbf{BSML}_0$. Both the maximal$_\exists$ implication and the **BSML**-implication can account for these inferences semantically without any problems.

The third desideratum is concerned with the *import-export principle*. We saw that both the maximal$_\exists$ implication and the **BSML**-implication can account for this desideratum semantically. As there is no reason to suspect the "neglect-zero" tendency causing a failure of this principle, we expect this desideratum to also hold under pragmatic enrichment. We found that this indeed is the case, as the desideratum is still satisfied if we add pragmatic enrichment.

The next desideratum, Desideratum 5, argues against Modus Tollens for epistemic modal consequents. Neither of the two implications that we are discussing satisfied this desideratum. This relied on the fact that in Desideratum 2b) we showed that Modus Tollens holds for the whole NE-free fragment when non-enriched. Furthermore, these implications could not pragmatically account for this desideratum either. Luckily, we will see in Chapter 6 that the dynamic notion of implication can semantically account for propositional Modus Tollens and Desideratum 5 simultaneously.

The penultimate desideratum, reformulated as $6'$, is concerned with the direct argument under pragmatic enrichment. Specifically, we do not want the implication $[\neg p \Rightarrow q]^+$ to follow from $[p \wedge \neg q]^+$, while it should follow from $[p \vee q]^+$ together with $[\Diamond\neg p]^+$. The maximal$_\exists$ implication was the only implication that could correctly account for this desideratum.

Similarly, we see that the maximal$_\exists$ implication is the only implication that correctly can account for Desideratum $7'$.

To conclude, in the previous chapter we saw that the maximal$_\exists$ implication could account for all linguistic desiderata by Stalnaker. In this chapter we have seen that the maximal$_\exists$ implication and the **BSML**-implication do a reasonably good job at accounting for Ciardelli's desiderata, with a strong advantage of the former with respect to desiderata $6'$ and $7'$. So again we see that the maximal$_\exists$ implication seems to be the most suitable candidate for **BSML**. However, neither

[87] Note that this is an indicative conditional, rather than a subjunctive conditional. For (12-b) is perfectly fine in a counterfactual situation.

of the implications could fully account for desideratum 1b) nor could they satisfy Desideratum 5. In Chapter 6 we will see that a dynamic implication can overcome these problems while still maintaining good behaviour on the other desiderata. But first we will look at several mathematical properties of all of our implications in Chapter 5.

Chapter 5

Mathematical Properties

In the previous chapters we have compared several notions of implication on linguistic desiderata. As **BSML** is linguistically motivated, we want a notion of implication that can account for many linguistic phenomena. On the other hand, we want to adopt a notion of implication that is mathematically well-behaved. In this chapter we will compare the material implication ($\to$), intuitionistic implication ($\twoheadrightarrow$), maximal implication ($\hookrightarrow_{\forall}$), maximal$_{\exists}$ implication ($\hookrightarrow_{\exists}$), the **BSML**-implication ($\Rightarrow_{\neg,\vee}$) and the "definable" $\Rightarrow_{\neg,\mathbb{W}}$ on several mathematical properties.[88] Specifically, in Section 5.1 we will see if the implications preserve classicality, whether the deduction theorem holds for **BSML** extended with implication and whether the implications are transitive. Thereafter, we will see whether the principle of *Simplification of Disjunctive Antecedent* holds for these implications in Section 5.2. In the same section we will also see whether Fact 2.1.1 and 2.1.2 by Aloni (2022a) still hold if we extend **BSML** with an implication. Specifically, we will see whether, for NE-free and "split-free" α, β, we have $[\alpha \Rightarrow \beta]^+ \equiv (\alpha \Rightarrow \beta) \wedge \textsc{ne}$.

5.1 Logico-Mathematical Properties

5.1.1 Preservation of Classicality

In Chapter 1 we argued that we do not want our notion of implication to introduce non-classicality. At the end of Section 2.1 we saw that $\mathbf{BSML}_{\emptyset}$ determines the same logical notion of consequence as classical modal logic. This result, proven by Anttila (2021), followed from the flatness of the NE-free formulas.[89] So if one of our implications, say $\Rightarrow$, preserves flatness, then we can conclude that $\mathbf{BSML}_{\emptyset}(\Rightarrow)$ still behaves classically. In other words, if the antecedent and consequent are flat,

[88]In Section 2.2.2 we already mentioned that $\Rightarrow_{\neg,\mathbb{W}}$ would not be considered in Chapter 3 and 4 for mathematical reasons. In Section 5.1 we will see that this implication has none of the logico-mathematical properties that we discuss. Hence, we correctly discarded $\Rightarrow_{\neg,\mathbb{W}}$ after Chapter 2.

[89]See Fact 2.1.7.

then a preferable notion of implication would also be flat. However, it might also be the case that this preservation of flatness is not satisfied for some implication $\Rightarrow$. We shall then give an example of a classical validity that does not hold for $\mathbf{BSML}_\emptyset(\Rightarrow)$.

Using the characterization of flatness we see that the intuitionistic implication, maximal implication, maximal$_\exists$ implication and the **BSML**-implication preserve flatness.

Fact 5.1.1. *Let* $\Rightarrow \in \{\twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$ *and let* $\alpha, \beta \in \mathbf{BSML}$ *be flat. Then* $\alpha \Rightarrow \beta$ *is also flat.*

Proof. Remember Fact 2.2.5, Fact 2.2.6 and Fact 2.2.9, stating that the $\twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists$ and $\Rightarrow_{\neg,\vee}$ are equivalent for $\alpha, \beta \in \mathbf{BSML}_\emptyset$. As the same proofs hold for any flat α and β, we see that a proof for the intuitionistic implication ($\twoheadrightarrow$) is sufficient for our current purposes.

- *Downward closed*: Suppose $M, s \vDash \alpha \twoheadrightarrow \beta$ and let $t \subseteq s$. Then for any $t' \subseteq t$ with $M, t' \vDash \alpha$, we also have $t' \subseteq s$ and thus $M, t' \vDash \beta$. In other words, $M, t \vDash \alpha \twoheadrightarrow \beta$ and thus $\alpha \twoheadrightarrow \beta$ is downward closed.
- *Union closed*: Let S be a non-empty set of states such that for all $s \in S$ we have $M, s \vDash \alpha \twoheadrightarrow \beta$. Now let $t \subseteq \bigcup S$ be such that $M, t \vDash \alpha$. Then by downward closure of α we see that $\forall w \in t : M, \{w\} \vDash \alpha$. And as $\{w\} \subseteq s$ for some $s \in S$, we see that $M, \{w\} \vDash \beta$ follows for all $w \in t$. As β is union closed it follows that indeed $M, t \vDash \beta$. So we conclude that $M, \bigcup S \vDash \alpha \twoheadrightarrow \beta$.
- *Empty state property*: Finally, note that $M, \emptyset \vDash \beta$ as β has the empty state property. So clearly $M, \emptyset \vDash \alpha \twoheadrightarrow \beta$.

We conclude by Fact 2.1.6 that $\alpha \twoheadrightarrow \beta$ is flat, and thus by our note the same follows for $\hookrightarrow_\forall, \hookrightarrow_\exists$ and $\Rightarrow_{\neg,\vee}$. $\square$

Now that we have seen that these implication preserve flatness, we can extend Proposition 2.2.16 by Anttila (2021). For this we note that the state-based clauses for singletons of these implications all coincide with the classical clause of the material implication in a world.[90]

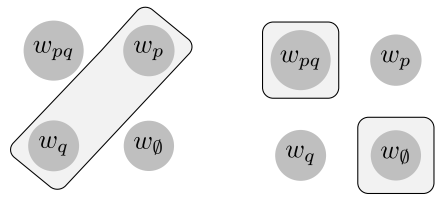
Fact 5.1.2 (Proposition 2.2.16 extended)**.** *For* $\Rightarrow \in \{\twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$*, any model* M *and* $\alpha \in \mathbf{BSML}_\emptyset(\Rightarrow)$*, we have:*

$$M, s \vDash \alpha \quad \textit{iff} \quad M, w \vDash \alpha \textit{ for all } w \in s$$

In particular, for any $w \in W : M, \{w\} \vDash \alpha$ *iff* $M, w \vDash \alpha$*.*

However, we see that the material implication and the definable $\Rightarrow_{\neg,\mathbb{V}}$ do not preserve flatness.

[90] We have phrased the following fact in the words of Antilla to show that it is an extension of Proposition 2.2.16. Note that the $\vDash$ on the left is the support relation of **BSML**, while the $\vDash$ on the right is the truth relation of classical modal logic.



(a) (b)

Figure 5.1: Counterexamples to preservation of flatness

Fact 5.1.3. *Let* $\Rightarrow \in \{\rightarrow, \Rightarrow_{\neg,\mathbb{W}}\}$ *and let* $\alpha, \beta \in$ **BSML** *be flat. Then* $\alpha \Rightarrow \beta$ *need not be flat.*

Proof. Let us first look at $\rightarrow$, and consider Figure 5.1(a). Then we see that $M, s \nvDash p$ and thus $M, s \vDash p \rightarrow q$. However, $M, \{w_p\} \vDash p$ but $M, \{w_p\} \nvDash q$, and thus $M, \{w_p\} \nvDash p \rightarrow q$. So as $\{w_p\} \subseteq s$ we see that $p \rightarrow q$ is not downward closed and thus not flat, while p and q are flat.[91]

For $\Rightarrow_{\neg,\mathbb{W}}$ we consider Figure 5.1(b). As $M, \{w_{pq}\} \vDash q$ we see that $M, \{w_{pq}\} \vDash p \Rightarrow_{\neg,\mathbb{W}} q$. And as $M, \{w_\emptyset\} \vDash \neg p$ we see that $M, \{w_\emptyset\} \vDash p \Rightarrow_{\neg,\mathbb{W}} q$. However, $M, \{w_{pq}, w_\emptyset\} \nvDash \neg p$ and $M, \{w_{pq}, w_\emptyset\} \nvDash q$, so $M, \{w_{pq}, w_\emptyset\} \nvDash p \Rightarrow_{\neg,\mathbb{W}} q$. Hence, $p \Rightarrow_{\neg,\mathbb{W}} q$ is not union closed and thus not flat, while p and q are flat. $\square$

This result shows that we cannot extend Proposition 2.2.16 by Anttila (2021) for $\mathbf{BSML}_\emptyset(\rightarrow)$ and $\mathbf{BSML}_\emptyset(\Rightarrow_{\neg,\mathbb{W}})$ using flatness. Even stronger, we can easily see that the proposition does not hold for these extensions. For in Section 4.2 we saw that Modus Tollens is not valid on $\mathbf{BSML}_\emptyset(\rightarrow)$. And as it might be for some state s that $M, s \nvDash p$ and $M, s \nvDash \neg p$, we see that $p \Rightarrow_{\neg,\mathbb{W}} p$ is not a validity in $\mathbf{BSML}_\emptyset(\Rightarrow_{\neg,\mathbb{W}})$. Both Modus Tollens and $p \Rightarrow p$ are classical validities, so we see that the material implication and $\Rightarrow_{\neg,\mathbb{W}}$ both introduce non-classicality. In Chapter 1 we argued that this is unfavorable, so this result gives us an argument against extending **BSML** with these implications.

5.1.2 Deduction Theorem

Presumably, we want our notion of implication to have a strong connection to the notion of entailment. This connection is normally established by the *Deduction Theorem*, which can be defined for any implication $\Rightarrow$ of **BSML** as follows:[92][93]

[91]Note that the material implication is union closed and preserves the empty state property.

- *Union Closure*: Let S be a non-empty set such that for all $s \in S$, $M, s \vDash \alpha \rightarrow \beta$, and suppose $M, \bigcup S \vDash \alpha$. Then by downward closure of α, $\forall s \in S : M, s \vDash \alpha$, and thus by assumption $M, s \vDash \beta$. So by union closure of β we see that $M, \bigcup S \vDash \beta$.
- *Empty state property*: By the empty state property of β we see that $M, \emptyset \vDash \alpha \rightarrow \beta$.

[92]See Porte (1982) for a historical overview of the Deduction Theorem in logic.

[93]The notion of logical consequence is only defined for single formulas in Section 2.1, but this notion extends to sets of formulas in the obvious way.

Definition (Deduction Theorem)**.** *Let* $\varphi, \psi \in \mathbf{BSML}(\Rightarrow)$ *and* $\Gamma \subseteq \mathbf{BSML}(\Rightarrow)$. *Then,*

$$\Gamma, \varphi \vDash \psi \quad \textit{iff} \quad \Gamma \vDash \varphi \Rightarrow \psi$$

In other words, we want an implication to be entailed by a set of formulas if and only if that set of formulas together with the antecedent entails the consequent of the implication.[94] We will see whether this Theorem holds for the implications under discussion. If it does not hold, we will check whether it holds for the NE-free fragment of **BSML**.

Now, it follows directly from the definition that the Deduction Theorem holds for the material implication.

Fact 5.1.4 (Deduction Theorem $\rightarrow$)**.** *Let* $\varphi, \psi \in \mathbf{BSML}(\rightarrow)$ *and let* $\Gamma \subseteq \mathbf{BSML}(\rightarrow)$. *Then,*

$$\Gamma, \varphi \vDash \psi \quad \textit{iff} \quad \Gamma \vDash \varphi \rightarrow \psi$$

However, the same does not hold for the intuitionistic implication, maximal implication, maximal$_\exists$ implication, the **BSML**-implication and the definable $\Rightarrow_{\neg,\mathbb{V}}$.[95][96]

Fact 5.1.5. *Let* $\Rightarrow \in \{\twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}, \Rightarrow_{\neg,\mathbb{V}}\}$, $\varphi, \psi \in \mathbf{BSML}(\Rightarrow)$ *and let* $\Gamma \subseteq \mathbf{BSML}(\Rightarrow)$. *Then,* $\Gamma, \varphi \vDash \psi$ *does not imply* $\Gamma \vDash \varphi \Rightarrow \psi$.

Proof. We will consider each of the implications separately.

- $\twoheadrightarrow$: Let $\Gamma := \{[p]^+ \vee [q]^+\}$, $\varphi := r$ and $\psi := [p]^+ \vee [r]^+$. Now suppose $M, s \vDash [p]^+ \vee [q]^+$, then there are non-empty $t, t' : t \cup t' = s$ with $M, t \vDash p$ and $M, t' \vDash q$. And also suppose $M, s \vDash r$, then clearly $M, s \vDash [p]^+ \vee [r]^+$ as $t \cup s = s$ and $s \neq \emptyset$. So $[p]^+ \vee [q]^+, r \vDash [p]^+ \vee [r]^+$.

  However, consider the state $s = \{w_p, w_{qr}\}$ in Figure 5.2(a) where $M, s \vDash [p]^+ \vee [q]^+$. Then we see that $\{w_{qr}\} \subseteq s$ and $M, \{w_{qr}\} \vDash r$, but $M, \{w_{qr}\} \nvDash [p]^+ \vee [r]^+$. Hence, $M, s \nvDash r \twoheadrightarrow ([p]^+ \vee [r]^+)$. So $[p]^+ \vee [q]^+ \nvDash r \twoheadrightarrow ([p]^+ \vee [r]^+)$.

- $\hookrightarrow_\forall$: As $\{w_{qr}\}$ is maximal with $M, \{w_{qr}\} \vDash r$, we see that the previous result also follows for the maximal implication.

- $\hookrightarrow_\exists$: Analogous to $\hookrightarrow_\forall$.

- $\Rightarrow_{\neg,\vee}$: Let $\Gamma := \emptyset$, $\varphi := \Diamond([p]^+ \vee [q]^+)$ and $\psi := \Diamond p \wedge \Diamond q$. Now suppose $M, s \vDash \Diamond([p]^+ \vee [q]^+)$, then for all $w \in s$ there are non-empty $t, t' \subseteq R[w]$ such that $M, t \vDash p$ and $M, t' \vDash q$. But then it directly follows that $M, s \vDash \Diamond p \wedge \Diamond q$.

[94] Note that the right-to-left direction is essentially the same as Modus Ponens.

[95] It is straightforward to see that for the intuitionistic implication, maximal implication, maximal$_\exists$ implication and the **BSML**-implication the right-to-left direction of the deduction theorem does hold. While for $\Rightarrow_{\neg,\mathbb{V}}$ we see that $p \vDash \neg p \Rightarrow_{\neg,\mathbb{V}}$ NE but $p, \neg p \nvDash$ NE.

[96] The counterexamples for the intuitionistic implication and the **BSML**-implication are due to Aleksi Antilla.

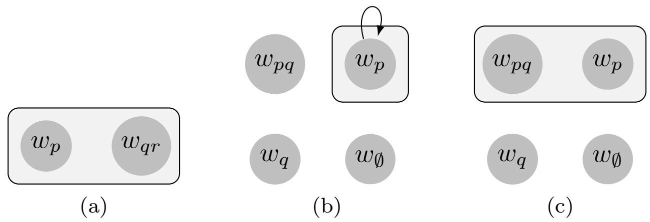


Figure 5.2: Counterexamples to Deduction Theorem and Transitivity

However, consider the state $s = \{w_p\}$ in Figure 5.2(b). The only $t' \subseteq R[w_p]$ such that $M, t' \vDash \Diamond p \wedge \Diamond q$ is $t' = \emptyset$. But $M, R[w_p] \nvDash \neg([p]^+ \vee [q]^+)$, so $M, \{w_p\} \nvDash \neg\Diamond([p]^+ \vee [q]^+)$. And thus we see that there are no $t, t' \subseteq s$ such that $t \cup t' = s$ with $M, t \vDash \neg\Diamond([p]^+ \vee [q]^+)$ and $M, t' \vDash \Diamond p \wedge \Diamond q$. Hence, $M, s \nvDash \Diamond([p]^+ \vee [q]^+) \Rightarrow_{\neg,\vee} (\Diamond p \wedge \Diamond q)$.

- $\Rightarrow_{\neg,\mathbb{W}}$: Let $\Gamma := \emptyset$ and $\varphi = \psi := p$, then clearly $\Gamma, \varphi \vDash \psi$. But again consider Figure 5.2(a), then $M, s \nvDash \neg p$ and $M, s \nvDash p$. So we see that $M, s \nvDash p \Rightarrow_{\neg,\mathbb{W}} p$, i.e., $\emptyset \nvDash p \Rightarrow_{\neg,\mathbb{W}} p$. □

Although the Deduction Theorem does not hold in general for the intuitionistic implication, maximal implication, maximal$_\exists$ implication and the **BSML**-implication, it does hold if we restrict ourselves to NE-free formulas.

Fact 5.1.6 (Deduction Theorem (NE-free))**.** *Let* $\Rightarrow \in \{\twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$, $\alpha, \beta \in$ $\mathbf{BSML}_\emptyset(\Rightarrow)$ *and let* $\Gamma \subseteq \mathbf{BSML}_\emptyset(\Rightarrow)$. *Then*

$$\Gamma, \alpha \vDash \beta \quad \textit{iff} \quad \Gamma \vDash \alpha \Rightarrow \beta$$

Proof. Let us first note that all implications under consideration preserve flatness (Fact 5.1.1), and thus Γ, α and β are all flat.

($\Rightarrow$): Suppose $\Gamma, \alpha \vDash \beta$ and also suppose $M, s \vDash \Gamma$. Now let $t \subseteq s$ be such that $M, t \vDash \alpha$. As all $\xi \in \Gamma$ are NE-free, they are flat. Hence, by downward closure we see that $M, t \vDash \Gamma$. So it follows from our assumption that $M, t \vDash \beta$, and thus $M, s \vDash \alpha \twoheadrightarrow \beta$.

($\Leftarrow$): Suppose $\Gamma \vDash \alpha \twoheadrightarrow \beta$, and also suppose $M, s \vDash \Gamma$ and $M, s \vDash \alpha$. It follows that $M, s \vDash \alpha \twoheadrightarrow \beta$, and as $s \subseteq s$ it directly follows that $M, s \vDash \beta$.

By Fact 2.2.5, Fact 2.2.6 and Fact 2.2.9, the same result also follows for $\Rightarrow_{\neg,\vee}$, $\hookrightarrow_\forall$ and $\hookrightarrow_\exists$. □

But as we saw that $\emptyset, p \vDash p$ but $\emptyset \nvDash p \Rightarrow_{\neg,\mathbb{W}} p$, we see that the Deduction Theorem does not hold for $\Rightarrow_{\neg,\mathbb{W}}$, even when we restrict ourselves to the NE-free fragment.

5.1.3 Transitivity

Another interesting property that implications can have is *transitivity*. This principle states that if we accept (13-a) and (13-b), we should also accept (13-c).[97]

(13) a. If Sal and Dean join the party, it will become a success.
b. If the party is a success, the police will be called.
c. If Sal and Dean join the party, the police will be called.

Formally, we would formulate this in **BSML** as follows.

Definition (Transitivity)**.** *For* $\varphi, \psi, \xi \in \mathbf{BSML}$*, we have* $\varphi \Rightarrow \psi, \psi \Rightarrow \xi \vDash \varphi \Rightarrow \xi$*.*

Directly from the definition we see that the material implication and the intuitionistic implication are transitive. Furthermore, we can prove that the **BSML**-implication is also transitive.

Fact 5.1.7. $\Rightarrow_{\neg,\vee}$ *is transitive.*

Proof. Suppose $M, s \vDash \varphi \Rightarrow_{\neg,\vee} \psi$, then $\exists t, t' : t \cup t' = s$ and $M, t \vDash \neg\varphi$ and $M, t' \vDash \psi$. Now also suppose that $M, s \vDash \psi \Rightarrow_{\neg,\vee} \xi$, then $\exists u, u' : u \cup u' = s$ and $M, u \vDash \neg\psi$ and $M, u' \vDash \xi$. Suppose for contradiction that $t \cup u' \neq s$, then it must be that there is some $w \in t' \cap u$. But by Fact 2.1.4 we know that $t' \cap u = \emptyset$, a contradiction. So we see that $t \cup u' = s$ and thus $M, s \vDash \varphi \Rightarrow_{\neg,\vee} \xi$. □

However, we see that both versions of the maximal implication and the definable $\Rightarrow_{\neg,\mathbb{V}}$ are not transitive.

Fact 5.1.8. *Let* $\Rightarrow \in \{\hookrightarrow_{\forall}, \hookrightarrow_{\exists}, \Rightarrow_{\neg,\mathbb{V}}\}$*, then* $\Rightarrow$ *is not transitive.*

Proof. Consider the state $s = \{w_{pq}, w_p\}$ in Figure 5.2(c), then we see that $\{w_{pq}\}$ is maximal with $M, \{w_{pq}\} \vDash p \wedge q$. And as $M, \{w_{pq}\} \vDash p$ we see that $M, s \vDash p \wedge q \hookrightarrow_{\forall} p$. Furthermore, s is maximal with $M, s \vDash p$ and $M, s \vDash [q \vee \neg q]^+$, and thus $M, s \vDash p \hookrightarrow_{\forall} [q \vee \neg q]^+$. However, $M, \{w_{pq}\} \nvDash [q \vee \neg q]^+$ and thus $M, s \nvDash p \wedge q \hookrightarrow_{\forall} [q \vee \neg q]^+$. The same proof goes through for $\hookrightarrow_{\exists}$.

For $\Rightarrow_{\neg,\mathbb{V}}$ consider any model M and let $s = \emptyset$. Then $M, s \vDash p$ and $M, s \vDash \neg p$, so $M, s \vDash \neg\,\mathrm{NE} \Rightarrow_{\neg,\mathbb{V}} p$ and $M, s \vDash p \Rightarrow_{\neg,\mathbb{V}} \mathrm{NE}$. However, $M, s \nvDash \neg\,\mathrm{NE} \Rightarrow_{\neg,\mathbb{V}} \mathrm{NE}$ as $M, s \nvDash \neg\neg\,\mathrm{NE}$ and $M, s \nvDash \mathrm{NE}$. □

Combining the results of Section 5.1.1, Section 5.1.2 and Section 5.1.3 in Table 5.1[98], we see that none of the implications are fully satisfactory. Even stronger, the definable $\Rightarrow_{\neg,\mathbb{V}}$ is fully unsatisfactory. It introduces non-classicality and fails to satisfy the deduction theorem, even when we restrict our attention to NE-free formulas, and it is not transitive. As we discussed in Section 2.2.2, this has caused

[97] Adams (1965) gives the following counter-example to transitivity: "*If Brown wins the election, Smith will retire to private life. If Smith dies before the election, Brown will win it. Therefore, if Smith dies before the election, then he will retire to private life*". So although transitivity has a strong intuitive appeal, we see that it is not fully uncontroversial.

[98] Remember that we use colors and fonts in tables to denote what results are **good**, bad or *somewhere in between*.

us to leave $\Rightarrow_{\neg,\mathbb{V}}$ out of the consideration in the previous chapters. Now we see that we have rightfully done so.

	Class.	Ded.	Ded. (NE-free)	Trans.
BSML($\rightarrow$)	✗	✓	✓	✓
BSML($\twoheadrightarrow$)	✓	✗	✓	✓
BSML($\hookrightarrow_\forall$)	✓	✗	✓	✗
BSML($\hookrightarrow_\exists$)	✓	✗	✓	✗
BSML($\Rightarrow_{\neg,\vee}$)	✓	✗	✓	✓
BSML($\Rightarrow_{\neg,\mathbb{V}}$)	✗	✗	✗	✗

Table 5.1: Logico-Mathematical Properties
Class. = Preservation of Classicality, Ded. = Deduction Theorem., Trans. = Transitivity

For the other implications, the picture is less one-sided. The material implication introduces non-classicality, but it is the only implication for which the Deduction Theorem holds in general. On the other hand, the intuitionistic implication, maximal implication, maximal$_\exists$ implication and the **BSML**-implication all preserve classicality but only satisfy the Deduction Theorem on the NE-free fragment. These results do not give us a direct reason to discard one of the implications as a possible extension of **BSML**. However, they do show us that there is a trade-off to be made between classicality on the one hand and the connection between implication and entailment on the other. What we can say is that if we choose for the preservation of classicality, the intuitionistic implication and the **BSML**-implication have the preference from a mathematical point of view as they are transitive.[99]

5.2 Linguistic-Mathematical Properties

5.2.1 Simplification of Disjunctive Antecedent

The principle of *Simplification of Disjunctive Antecedent* (SDA) can formally be phrased in **BSML** for some implication $\Rightarrow$ as follows:

Definition (SDA). *Let* $\alpha, \beta, \gamma \in \mathbf{BSML}_\emptyset$*, then* $(\alpha \vee \beta) \Rightarrow \gamma \vDash (\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma)$.

In other words, if the disjunction of α and β implies γ, then both α and β imply γ individually.[100] This principle is motivated by the intuitive appeal to the

[99]Note that one can argue that counterexample to the transitivity of $\hookrightarrow_\forall$ and $\hookrightarrow_\exists$ makes sense from a linguistic point of view. For consider the following example: "*If Paul and Maria are at the office, then Paul is at the office. If Paul is at the office, it is a live possibility both that Maria is at the office or that she is not. If Paul and Maria are at the office, it is a live possibility both that Maria is at the office or that she is not.* In other words, it seems very natural to accept the first two of these sentences in some situation, while the last one is simply invalid. However, we look at the transitivity from a mathematical point of view, and thus deem it undesirable to be intransitive.

[100]We have restricted the principle to the NE-free fragment, because the literature on this principle is concerned with classical logic. However, we will also consider what happens when we pragmatically enrich the formulas.

inference from (14-a) to both (14-b) and (14-c).[101]

(14) a. If you eat sea cucumbers or dung beetles, you'll get a stomachache. (Lassiter (2018))
b. If you eat sea cucumbers, you'll get a stomachache.
c. If you eat dung beetles, you'll get a stomachache.

In Section 3.2 we saw that Stalnaker's analysis of the indicative conditional does not validate SDA. This originates in the fact that the most similar β-world need not also be the most similar $\alpha \vee \beta$-world. But if we extend **BSML** with any of the implications that we have discussed so far, we see that SDA is valid.[102]

Fact 5.2.1. *Let* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_{\forall}, \hookrightarrow_{\exists}, \Rightarrow_{\neg,\vee}\}$ *and* $\alpha, \beta, \gamma \in \mathbf{BSML}_{\emptyset}$*. Then* $(\alpha \vee \beta) \Rightarrow \gamma \vDash (\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma)$*.*

Proof. Let us start with the material implication and suppose $M, s \vDash (\alpha \vee \beta) \rightarrow \gamma$. Now also suppose $M, s \vDash \alpha$, then by letting $t = s$ and $t' = \emptyset$ we see that $M, s \vDash \alpha \vee \beta$. So it follows that $M, s \vDash \gamma$, and thus $M, s \vDash \alpha \rightarrow \gamma$. Similarly for $\beta \rightarrow \gamma$.

We saw that the intuitionistic implication, maximal implication, maximal$_{\exists}$ implication and the **BSML**-implication all behave classically (Fact 5.1.2) on $\mathbf{BSML}_{\emptyset}$. And as SDA is classically valid, it follows directly from this fact. □

And SDA is still valid for the **BSML**-implication if we add pragmatic enrichment.

Fact 5.2.2. *Let* $\alpha, \beta, \gamma \in \mathbf{BSML}_{\emptyset}$*. Then* $[(\alpha \vee \beta) \Rightarrow_{\neg,\vee} \gamma]^+ \vDash [(\alpha \Rightarrow_{\neg,\vee} \gamma) \wedge (\beta \Rightarrow_{\neg,\vee} \gamma)]^+$*.*

Proof. Suppose $M, s \vDash [(\alpha \vee \beta) \Rightarrow_{\neg,\vee} \gamma]^+$, then we see that $M, s \vDash \neg[\alpha \vee \beta]^+ \vee [\gamma]^+$. So there are $t, t' : t \cup t' = s$ and $M, t \vDash \neg[\alpha \vee \beta]^+$ and $M, t' \vDash [\gamma]^+$. Note that $\neg[\alpha \vee \beta]^+ = \neg(([\alpha]^+ \vee [\beta]^+) \wedge \text{NE}) \equiv \neg([\alpha]^+ \vee [\beta]^+)$, and thus we see that $M, t \vDash \neg[\alpha]^+$ and $M, t \vDash \neg[\beta]^+$. Thereby we can conclude that $M, s \vDash (\neg[\alpha]^+ \vee [\gamma]^+) \wedge (\neg[\beta]^+ \vee [\gamma]^+)$, and as $s \neq \emptyset$ it follows that $M, s \vDash [(\alpha \Rightarrow_{\neg,\vee} \gamma) \wedge (\beta \Rightarrow_{\neg,\vee} \gamma)]^+$. □

However, we see that for the other implications SDA is no longer valid when we add pragmatic enrichment.

Fact 5.2.3. *Let* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_{\forall}, \hookrightarrow_{\exists}\}$ *and* $\alpha, \beta, \gamma \in \mathbf{BSML}_{\emptyset}$*. Then* $[(\alpha \vee \beta) \Rightarrow \gamma]^+ \nvDash [(\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma)]^+$*.*

Proof. Let us start with $\alpha := p$, $\beta := q$ and $\gamma := q$. Consider the state $s = \{w_p\}$ in Figure 5.3(a). Then we see that $M, s \nvDash [p \vee q]^+$ and as $s \neq \emptyset$ it follows that $M, s \vDash [(p \vee q) \rightarrow q]^+$. Similarly, we see that there is no $t \subseteq s$ such that $M, t \vDash [p \vee q]^+$, so the same holds for $\twoheadrightarrow$ and $\hookrightarrow_{\forall}$. However, we see that $M, s \vDash [p]^+$ but $M, s \nvDash [q]^+$ and thus $M, s \nvDash [p \rightarrow q]^+$. As $s \subseteq s$ and is maximal with $M, s \vDash [p]^+$, the same also follows for $\twoheadrightarrow$ and $\hookrightarrow_{\forall}$.

[101] See Lassiter (2018), Khoo (2021) for a broader discussion on SDA for indicative conditionals.
[102] Note that we do not discuss $\Rightarrow_{\neg,\mathbb{W}}$ because of the mathematical problems that we saw at the end of the previous section.

Now let $\alpha := p$, $\beta := q$ and $\gamma := p \vee q$, and consider the state $s = \{w_p, w_{qr}\}$ in Figure 5.3(b). Then we see that s is maximal with $M, s \vDash [p \vee q]^+$ and thus $M, s \vDash [(p \vee q) \hookrightarrow_{\exists} (p \vee q)]^+$. However, $\{w_p\}$ is maximal with $M, \{w_p\} \vDash [p]^+$, but $M, \{w_p\} \nvDash [p \vee q]^+$. Therefore we see that $M, s \nvDash [p \hookrightarrow_{\exists} (p \vee q)]^+$. □

Let us consider whether this non-validity of SDA makes sense if we take the "neglect-zero" tendency into account. Adding pragmatic enrichment to the conditional with a disjunctive antecedent can be seen as stating the following: "if α and β are live possibilities, then it follows that γ". However, if α is supported it does not mean that β is also a live possibility. And therefore it makes sense that it does not follow that α implies γ when we have added the pragmatic enrichment, and similarly for β. We conclude that the non-validity of SDA under pragmatic enrichment is in favor of the material implication, intuitionistic implication, maximal implication and maximal$_{\exists}$ implication.

5.2.2 Extending Fact 2.1.1 & 2.1.2

In Section 2.1.2 we saw that for all NE-free, "split-free" formulas α, we have $[\alpha]^+ \equiv_N \alpha \wedge \text{NE}$.[103] In other words, for these formulas the pragmatic enrichment only adds the requirement that the supporting state is non-empty. This insight was a result of the combination of Fact 2.1.1, stating that $\alpha \in \mathbf{BSML}_{\emptyset}$ is entailed by $[\alpha]^+$, and Fact 2.1.2, stating that "split-free" $\alpha \in \mathbf{BSML}_{\emptyset}$ conjoined with NE entails $[\alpha]^+$ if we restrict model-state pairs (M, s) with $\forall w \in s : R[w] = s$. However, we also saw that for formulas containing a disjunction, like $p \vee q$, the effect of pragmatic enrichment is non-trivial. Similarly, we might wonder whether pragmatically enriching an implication has a non-trivial effect. For this purpose we will look at the connection between $[\alpha \Rightarrow \beta]^+$ and $(\alpha \Rightarrow \beta) \wedge \text{NE}$ for each of our implications. As this thesis researches the effect of the "neglect-zero" tendency on indicative conditionals, we would prefer to find a non-trivial effect.[104]

Let us first note that $[\alpha \Rightarrow \beta]^+ = ([\alpha]^+ \Rightarrow [\beta]^+) \wedge \text{NE}$, and thus the question boils down to whether $[\alpha]^+ \Rightarrow [\beta]^+ \equiv \alpha \Rightarrow \beta$. We will start by looking at the counterpart of Fact 2.1.2, i.e., whether a pragmatically enriched implication follows from its non-enriched variant together with non-emptiness. Similar to Fact 2.1.2 we will focus on "split-free" formulas. Then it is quite straightforward to see that this entailment holds for the material implication, intuitionistic implication and the maximal implication.

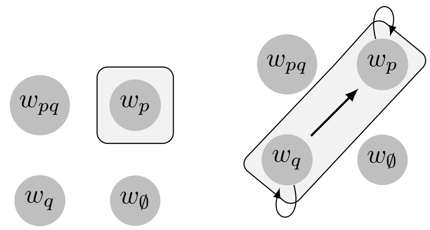
Fact 5.2.4. *For* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_{\forall}\}$ *and "split-free"* $\alpha, \beta \in \mathbf{BSML}_{\emptyset}$*, we have* $(\alpha \Rightarrow \beta) \wedge \text{NE} \vDash_N [\alpha \Rightarrow \beta]^+$.

Proof. Suppose $M, s \vDash \alpha \rightarrow \beta$, and suppose $M, s \vDash [\alpha]^+$. Then by Fact 2.1.1 we see that $M, s \vDash \alpha$ and thus $M, s \vDash \beta$. As β is "split-free", it follows from Fact 2.1.2 that $M, s \vDash [\beta]^+$. Hence, $M, s \vDash [\alpha]^+ \rightarrow [\beta]^+$.

An analogous argument holds for $\twoheadrightarrow$ and $\hookrightarrow_{\forall}$. □

[103] See Footnotes 15 and 16 for the definition of "split-free" and $\equiv_N$.

[104] In the previous chapters we have seen that such a non-trivial effect has allowed the maximal$_{\exists}$ implication and **BSML**-implication to pragmatically account for several linguistic desiderata.



(a) (b)

Figure 5.3: Counterexamples for linguistic-mathematical properties

In other words, for these implications we might have $[\alpha \Rightarrow \beta]^+ \equiv (\alpha \Rightarrow \beta) \wedge \text{NE}$ for α and β NE-free and "split-free". However, this is not the case for the maximal$_\exists$ implication and the **BSML**-implication. For there we see that, even for "split-free" classical antecedent and consequent, non-emptiness is not sufficient for the entailment.

Fact 5.2.5. *For* $\Rightarrow \in \{\hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$ *and "split-free"* $\alpha, \beta \in \mathbf{BSML}_\emptyset$*, we have* $(\alpha \Rightarrow \beta) \wedge \text{NE} \nvDash_N [\alpha \Rightarrow \beta]^+$*.*

Proof. Let us first look at $\hookrightarrow_\exists$, and consider the state $s = \{w_p\}$ in Figure 5.3(a). Then we see that $s \neq \emptyset$ and that $\emptyset$ is maximal with $M, \emptyset \vDash q$. Hence, $M, s \vDash (q \hookrightarrow_\exists q) \wedge \text{NE}$. However, there is no $t \subseteq s$ with $M, t \vDash [q]^+$ and thus $M, s \nvDash [q \hookrightarrow_\exists q]^+$.

For $\Rightarrow_{\neg,\vee}$ we consider $s = \{w_p\}$ in Figure 5.3(a) again. Now notice that $M, s \vDash \neg q$ and $M, \emptyset \vDash q$. And as $s \cup \emptyset = s$ and $s \neq \emptyset$ we get $M, s \vDash (q \Rightarrow_{\neg,\vee} q) \wedge \text{NE}$. However, there is no $t \subseteq s$ with $M, t \vDash [q]^+$ and thus $M, s \nvDash [q \Rightarrow_{\neg,\vee} q]^+$. $\square$

We can conclude for these two implications that the pragmatic enrichment function has a non-trivial effect. Specifically, for the maximal$_\exists$ implication the pragmatic enrichment ensures that both antecedent and consequent are live possibilities. And for the **BSML**-implication the pragmatic enrichment ensures that its consequent is a live possibility.[105]

Now let us turn our attention to the counterpart of Fact 2.1.1, i.e., whether the non-enriched variant of an implication follows from its pragmatically enriched version. It is quite straightforward to see that this is the case for the **BSML**-implication.

Fact 5.2.6. *Let* $\alpha, \beta \in \mathbf{BSML}_\emptyset$*. Then* $[\alpha \Rightarrow_{\neg,\vee} \beta]^+ \vDash (\alpha \Rightarrow_{\neg,\vee} \beta) \wedge \text{NE}$*.*

Proof. Suppose $M, s \vDash [\alpha \Rightarrow_{\neg,\vee} \beta]^+$, then $s \neq \emptyset$ and $M, s \vDash \neg[\alpha]^+ \vee [\beta]^+$. Hence, there are $t, t' : t \cup t' = s$ and $M, t \vDash \neg[\alpha]^+$ and $M, t' \vDash [\beta]^+$. So by Fact 2.1.1 we see that $M, t \vDash \neg\alpha$ and $M, t' \vDash \beta$. So it follows that $M, s \vDash \neg\alpha \vee \beta$, and thus $M, s \vDash (\alpha \Rightarrow_{\neg,\vee} \beta) \wedge \text{NE}$. $\square$

[105] Note that the antecedent need not be a live possibility for the **BSML**-implication as $[\alpha \Rightarrow \beta]^+ \equiv \neg[\alpha]^+ \vee [\beta]^+ \equiv \neg\alpha \vee [\beta]^+$. In other words, under negation the pragmatic enrichment of the antecedent gets canceled.

On the other hand, if we look at the material implication, intuitionistic implication and both maximal implications we see that this entailment does not hold.

Fact 5.2.7. *For* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists\}$ *and* $\alpha, \beta \in \mathbf{BSML}_\emptyset$*, we have* $[\alpha \Rightarrow \beta]^+ \not\models (\alpha \Rightarrow \beta) \wedge \textsc{ne}$.

Proof. Consider the state $s = \{w_p\}$ in Figure 5.3(a), then we see that there is no $t \subseteq s$ such that $M, t \models [p \vee q]^+$. As $s \neq \emptyset$ we see that this means that $M, s \models [(p \vee q) \rightarrow (p \wedge q)]^+$, and similarly for $\twoheadrightarrow$ and $\hookrightarrow_\forall$. However, $M, s \models p \vee q$ (and is maximally so). But as $M, s \not\models p \wedge q$, we see that $M, s \not\models (p \vee q) \rightarrow (p \wedge q)$. Similarly for $\twoheadrightarrow$ and $\hookrightarrow_\forall$.

Now consider Figure 5.3(b), then we see that $M, s \models [p \vee q]^+$. And as $s \subseteq R[w_q]$ and $s \neq \emptyset$, it follows that $M, \{w_q\} \models [\Diamond(p \vee q)]^+$. Because there is no non-empty $t \subseteq R[w_p]$ with $M, t \models [p \vee q]^+$ we see that $\{w_q\}$ is maximal with $M, \{w_q\} \models [\Diamond(p \vee q)]^+$. And as $M, \{w_q\} \models [q]^+$, it follows that $M, s \models [\Diamond(p \vee q) \hookrightarrow_\exists q]^+$. However, s itself is maximal with $M, s \models \Diamond(p \vee q)$ but $M, s \not\models q$. So $M, s \not\models \Diamond(p \vee q) \hookrightarrow_\exists q$. □

At the beginning of this section we saw that for the maximal$_\exists$ implication and the **BSML**-implication, even for "split-free" antecedent and consequent, we do not have $(\alpha \Rightarrow \beta) \wedge \textsc{ne} \models [\alpha \Rightarrow \beta]^+$. So we see that we definitely do not have an equivalence for these implications.

However, for the material implication, intuitionistic implication and the maximal implication we actually do have this equivalence when we focus on "split-free" antecedents and consequents.[106]

Fact 5.2.8. *For* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall\}$ *and "split-free"* $\alpha, \beta \in \mathbf{BSML}_\emptyset$*, we have* $[\alpha \Rightarrow \beta]^+ \equiv_N (\alpha \Rightarrow \beta) \wedge \textsc{ne}$.

Proof. We have already seen that the right-to-left direction holds.

Now suppose $M, s \models [\alpha \rightarrow \beta]^+$, and also suppose $M, s \models \alpha$. Then $s \neq \emptyset$ and by Fact 2.1.2 we see that $M, s \models [\alpha]^+$, and thus it follows that $M, s \models [\beta]^+$. Hence, by Fact 2.1.1 $M, s \models \beta$ and thus indeed $M, s \models (\alpha \rightarrow \beta) \wedge \textsc{ne}$.

The arguments for $\twoheadrightarrow$ and $\hookrightarrow_\forall$ are completely analogous. □

We can conclude that pragmatically enriching the material implication, intuitionistic implication or maximal implication has a trivial effect if its antecedent and consequent are "split-free". Specifically, the only effect of the pragmatic enrichment is the non-emptiness requirement for a supporting state.

Combining the results from Section 5.2 in Table 5.2, we see that all implications satisfy the principle *Simplification of Disjunctive Antecedent*.[107] When we add pragmatic enrichment, all implications except the **BSML**-implication invalidate the principle. However, at the end of Section 5.2.1 we argued that this non-validity

[106] Note that Fact 5.2.7 shows that this equivalence does not generalize to implications with a nested disjunction.

[107] Remember that we use colors and fonts in tables to denote what results are **good**, bad or *somewhere in between*.

makes sense if take the “neglect-zero” tendency into account and actually speaks in favor of these implications.

	SDA	SDA$^+$	Triv.	Triv. (“split-free”)
BSML($\rightarrow$)	✓	✗	✗	$\underline{✓}$
BSML($\twoheadrightarrow$)	✓	✗	✗	$\underline{✓}$
BSML($\hookrightarrow_{\forall}$)	✓	✗	✗	$\underline{✓}$
BSML($\hookrightarrow_{\exists}$)	✓	✗	✗	✗
BSML($\Rightarrow_{\neg,\vee}$)	✓	$\underline{✓}$	✗	✗

Table 5.2: Linguistic-Mathematical Properties
SDA = Simplification of Disjunctive Antecedents, SDA$^+$ = SDA with pragmatic enrichment, Triv. = $[\alpha \Rightarrow \beta]^+ \equiv (\alpha \Rightarrow \beta) \wedge \textsc{ne}$, Triv. (“split-free”) = Triv. for “split-free” α, β.

In Section 5.2.2 we saw that pragmatic enrichment has a non-trivial effect on all implications. However, when we focus on “split-free” antecedents and consequents we notice that this non-trivial effect originated in nested disjunctions for the material implication, intuitionistic implication and maximal implication. For if we have “split-free” formulas, the effect on these implications is only the non-emptiness requirement. On the other hand, for the maximal$_{\exists}$ implication and **BSML**-implication we even have a non-trivial effect with this restriction. Specifically, pragmatic enrichment of the maximal$_{\exists}$ implication ensures that both the antecedent and consequent are live possibilities, while for the **BSML**-implication it ensures only that the consequent is a live possibility. Notice that it was exactly this non-trivial effect that has given these implications its good behaviour in Chapters 3 and 4. Notably, this non-trivial effect was crucial for the satisfaction of Desideratum B for both implications, and Desideratum 6′ and Desideratum 7′ for the maximal$_{\exists}$ implication.

Chapter 6

Dynamics

In the previous chapters we have considered several possible implications for the static system **BSML**. Specifically, we introduced and researched several different possible notions of implication in Chapter 2. In Chapters 3 and 4 we saw that the maximal$_\exists$ implication can best account for several linguistic desiderata by Stalnaker, and the **BSML**-implication also does a reasonably good job on Ciardelli's linguistic desiderata. However, we saw that none of the notions of implication for **BSML** could account for the failure of Modus Tollens with a modal consequent in a non-trivial manner. Furthermore, we saw that the order of utterance matters in Desideratum 1b) when we take the "neglect-zero" tendency into account. In other words, we expect $[p \Rightarrow q]^+$ uttered before $[\neg p]^+$ to be non-problematic, while the reversed order should lead to a contradiction. Our static system could not account for this, so in this chapter we will extend a dynamic variant of **BSML** with an implication. We will see that this dynamic implication can not only account for this "order-problem", but it exceeds our static implications on almost all linguistic desiderata and mathematical properties that we have discussed so far. For example, it can non-trivially account for the failure of Modus Tollens with a modal consequent.

6.1 Preliminaries (BiUS)

Veltman (1996) and Van Der Does, Groeneveld, and Veltman (1997) presented Update Semantics (**US**), which was created to account for the differences between sequences like (15-a) which seem to be okay, and odd sequences like (15-b).[108]

(15) a. Maybe this is Frank Veltman's example. (Aloni (2022b))
It isn't his example!
b. #This is not Frank Veltman's example. Maybe it's his example.

[108]Notice the similarity between this example and example (12), in both cases the oddity seems to come from some non-live possibility. Hence, the "neglect-zero" tendency is a very suitable candidate to explain the difference between (12-a) and (15-a) on the one hand, and on the other (12-b) and (15-b).

Veltman (1996) does not address free choice inferences, but when **US** is extended with the NE-atom it can account for some free choice inferences. However, it is unable to correctly capture the behaviour of enriched disjunction under negation. Therefore, Aloni (2022b) introduced Bilateral Update Semantics (**BiUS**), which is inspired by Veltman's **US**. Similar to **BSML**, **BiUS** formalises the "neglect-zero" tendency, which allows it to account for examples like (15), while also accounting for the same *free choice inferences* as **BSML**. We will first present this framework, whereafter we will introduce the notion of implication that is most natural in this system. This introduction will follow Aloni (2022b) in notation and presentation.

The language of **BiUS** is similar to that of **BSML**, but we do not have the atomic NE anymore. Rather, we have the complex formula φ^{NE} where NE is interpreted as a *post-supposition*. In other words, NE is treated as a constraint that needs to be satisfied after an update.

Definition 6.1.1 (Language)**.** *Let $P = \{p, q, r \dots\}$ be a countable set of propositional atoms.*

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \varphi^{\text{NE}}$$

where $p \in P$.

Similarly to **BSML**, we will use $\mathbf{BiUS}_\emptyset$ and $\mathbf{BiUS}_0$ as the NE-free and propositional fragments of **BiUS**, respectively. Note that the $\Diamond$ can occur in front of every formula, where in **US** it can only occur as the operator with the highest scope. Some properties will hold only for the language of **US** interpreted as in **BiUS** but not for $\mathbf{BiUS}_\emptyset$. In this case we will make this explicit by saying that the property holds for $\mathbf{US}_{BiUS}$.

We interpret the language of **BiUS** on a model $M = \langle W, V \rangle$, and a state $s \subseteq W$, where W is a non-empty set of worlds and V is a valuation function on $W \times P$. Rather than defining support conditions directly, we first define how to update a state s with a formula φ and derive the support conditions from this. Similarly to **BSML**, negation is defined in terms of separately specified rejection updates, allowing the designer of the framework to have more control over the behaviour of operators under negation.

Definition 6.1.2 (Updates)**.**

1. $s[p] = s \cap \{w \in W \mid V(p, w) = 1\}$
2. $s[\varphi \wedge \psi] = s[\varphi] \cap s[\psi]$
3. $s[\varphi \vee \psi] = s[\varphi] \cup s[\psi]$
4. $s[\Diamond\varphi] = s$*, if* $s[\varphi] \neq \emptyset$*;* $\emptyset$*, if* $s[\varphi] = \emptyset$*; undefined (#) otherwise*
5. $s[\varphi^{\text{NE}}] = s[\varphi]$ *if* $s[\varphi] \neq \emptyset$*; undefined (#) otherwise*
6. $s[\neg\varphi] = s[\varphi]^r$

   where $[\varphi]^r$ *is recursively defined as follows:*

   1′. $s[p]^r = s \cap \{w \in W \mid V(p, w) = 0\}$

2′. $s[\varphi \wedge \psi]^r = s[\varphi]^r \cup s[\psi]^r$

3′. $s[\varphi \vee \psi]^r = s[\varphi]^r \cap s[\psi]^r$

4′. $s[\Diamond\varphi]^r = s$, *if* $s[\varphi]^r = s$; $\emptyset$, *if* $s[\varphi]^r \neq s$; *undefined (#) otherwise*

5′. $s[\varphi^{\mathrm{NE}}]^r = s[\varphi]^r$

6′. $s[\neg\varphi]^r = s[\varphi]$

Notice that $s \neq x$ means that s is a state other than x, so this excludes that s is undefined. Furthermore, if either x or y is undefined, then so are $x \cup y$ and $x \cap y$.

We adopt the abbreviation $\Box\varphi := \neg\Diamond\neg\varphi$. Thereby we can derive the following update for the necessity modal:

$$s[\Box\varphi] = s \text{ if } s[\varphi] = s; \emptyset \text{ if } s[\varphi] \neq s;\ \text{undefined } (\#) \text{ otherwise}$$
$$s[\Box\varphi]^r = s \text{ if } s[\varphi]^r \neq \emptyset; \emptyset \text{ if } s[\varphi]^r = \emptyset;\ \text{undefined } (\#) \text{ otherwise}$$

Now that we have defined how we update a state s with a formula φ, we can say when s supports φ. Namely, a state *supports* a formula if and only if the update with that formula results in the same state.[109]

Definition 6.1.3 (Support)**.** $s \vDash \varphi$ *if and only if* $s[\varphi] = s$

Using this notion we say that a formula ψ is the logical consequence of formulas $\varphi_1, \ldots, \varphi_n$ if and only if a state updated with $\varphi_1, \ldots, \varphi_n$, if *defined*, supports ψ.

Definition 6.1.4 (Logical Consequence)**.**
$\varphi_1, \ldots, \varphi_n \vDash \psi$ *iff for all* $s : s[\varphi_1]\ldots[\varphi_n]$ *defined implies* $s[\varphi_1]\ldots[\varphi_n] \vDash \psi$.

Now that we have specified the semantics of **BiUS**, we can also specify the recursively defined pragmatic enrichment function $|\,|^+$.[110]

Definition 6.1.5 (Dynamic Pragmatic Enrichment)**.** *For* NE*-free* α, $|\alpha|^+$ *is defined as follows:*

$$\begin{aligned} |p|^+ &= p^{\mathrm{NE}} \\ |\neg\alpha|^+ &= (\neg|\alpha|^+)^{\mathrm{NE}} \\ |\alpha \vee \beta|^+ &= (|\alpha|^+ \vee |\beta|^+)^{\mathrm{NE}} \\ |\alpha \wedge \beta|^+ &= (|\alpha|^+ \wedge |\beta|^+)^{\mathrm{NE}} \\ |\Diamond\alpha|^+ &= (\Diamond|\alpha|^+)^{\mathrm{NE}} \end{aligned}$$

[109]Note that this is the same as that of **US**, and is a very natural definition of support. For if we get some information and gain no new insights from it, we would indeed say that we already had support for that information.

[110]In Appendix D we have given an overview of the update clauses for pragmatically enriched formulas. These can help with the reading of the proofs in this chapter and in the same appendix.

Finally, we can extend the update clauses of **BiUS** from Definition 6.1.2 with a dynamic implication $\dashrightarrow$.[111][112] We will call the resulting logic **BiUS**($\dashrightarrow$).

Definition 6.1.6 (Dynamic Implication)**.**

7. $s[\varphi \dashrightarrow \psi] = s$ *if* $s[\varphi][\psi] = s[\varphi]$*;* $\emptyset$ *otherwise*[113]

7'. $s[\varphi \dashrightarrow \psi]^r = s$ *if* $s[\varphi][\psi]^r \neq \emptyset$*;* $\emptyset$ *otherwise*

The acceptance update of this definition is equivalent to the *dynamic conditional* as discussed by Gillies (2010) and Gillies (2020).[114] As the former mentions, this definition would pass the Ramsey test and therefore is a natural definition for an implication.[115]

Similarly to **BSML**($\Rightarrow$), we will extend the dynamic pragmatic enrichment function as follows.

$$|\alpha \dashrightarrow \beta|^+ = (|\alpha|^+ \dashrightarrow |\beta|^+)^{\text{NE}}$$

Now that we have introduced **BiUS** and the dynamic implication $\dashrightarrow$, we will give some useful results in the following section.

6.2 Useful Results on BiUS

There are several properties of **BiUS** that we will use throughout this chapter. In this section we will merely state and discuss them, the interested reader can find the proofs in Appendix D.1.[116]

Let us start with two important properties that dynamic semantics can have, *eliminativity* and *distributivity*. If a framework has both properties, then it is

[111] Notice that $\dashrightarrow$ is a test operator, in the terms of Gillies (2020). In other words, it checks whether a state has a given property, and results in the state itself or $\emptyset$ accordingly. This is similar to the $\Diamond$-operator and the NE-post-supposition.

[112] Similarly to **BSML** (Section 2.2.1), we also have the option to syntactically or semantically define an implication in terms of disjunction and negation. Similar to $\rightarrow_{\neg,\vee}$ we could have $\varphi \Rightarrow_1 \psi := \neg\varphi \vee \psi$. But then we would see the following under pragmatic enrichment, $s[|\varphi \Rightarrow_1 \psi|^+] = s[|\neg\varphi \vee \psi|^+] = s[|\neg\varphi|^+] \cup s[|\psi|^+]$. So the pragmatic enrichment ensures that the negation of the antecedent is a live possibility, but says nothing of the antecedent itself.

And similar to $\Rightarrow_{\neg,\vee}$ we could define $s[\varphi \Rightarrow_2 \psi] := s[\varphi]^r \cup s[\psi]$, so that $\varphi \Rightarrow_2 \psi \equiv \neg\varphi \vee \psi$. But then we would see that $s[|\varphi \Rightarrow_2 \psi|^+] = s[|\varphi|^+]^r \cup s[|\psi|^+]$, which does not ensure anything about the antecedent as NE cancels out under the rejection clauses.

In the previous chapters we have repeatedly seen that we expect the pragmatic enrichment function to ensure that the antecedent is a live possibility. So both definitions will not be as suitable as the notion of implication that we will use.

[113] We assume that if a state x is undefined, it is not equal to any state y. In other words, we do not have $\# = \#$.

[114] This implication is also discussed by Dekker (1993)(pg. 202).

[115] Gillies (2010) phrases the Ramsey test as follows: "You accept $(if\, P)(Q)$ in a state B iff Q is accepted in the subordinate state got by taking B and adding the information that P to it." *(pg. 463)*. Gillies concludes that the dynamic conditional, and thus also our dynamic implication, is "very Ramseylike".

[116] Notice that Fact 6.2.1, Fact, 6.2.7, Fact 6.2.8, Fact 6.2.9 and Fact 6.2.10 are all stated in the Appendix of Aloni (2022b), but their proofs are left out. We have included their proofs in Appendix D.1 for completeness.

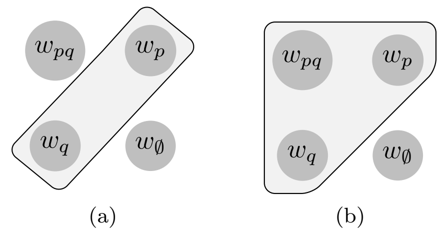


Figure 6.1: Models for useful results on **BiUS**

equivalent to a static framework with a globally defined notion of update.[117] Hence, the failure of one of these properties is seen as a sign of a truly dynamic framework.

First of all, we see that **BiUS** is *eliminative* for all defined updates. In other words, nothing is ever added to a state after succesfully updating with a formula. In particular, for any NE-free formula α, we always have $s[\alpha] \subseteq s$.

Fact 6.2.1 (Eliminativity)**.** *Let* $\varphi \in \mathbf{BiUS}$*, and* s *be any state.*

i) If $s[\varphi]$ *is defined, then* $s[\varphi] \subseteq s$*.*

ii) If $s[\varphi]^r$ *is defined, then* $s[\varphi]^r \subseteq s$*.*

Secondly, we see that $\mathbf{BiUS}_0$ is *distributive* for all defined updates. In other words, if α is $\Diamond$-free and NE-free, then updating a state s with α is the union of all singletons in s that support α.[118]

Fact 6.2.2 (Distributivity)**.** *Let* $\alpha \in \mathbf{BiUS}_0$*, and* s *be any state:*

i) $s[\alpha] = \{w \in s \mid \{w\}[\alpha] = \{w\}\}$

ii) $s[\alpha]^r = \{w \in s \mid \{w\}[\alpha]^r = \{w\}\}$

In other words, we see that for $\alpha \in \mathbf{BiUS}_0$ we have $s \vDash \alpha$ if and only if $\forall w \in s : \{w\} \vDash \alpha$. Notice the strong similarity to the notion of flatness of **BSML**-formulas that we introduced in Definition 2.1.5. For **BSML** we saw that flatness had a strong connection to classicality, and similarly we can now say that $\mathbf{BiUS}_0$ behaves classically. Notice however that this does not hold for **BiUS** in general, and neither for $\mathbf{BiUS}_\emptyset$ nor $\mathbf{US}_{BiUS}$. For an example, consider Figure 6.1(a). Then $s[\Diamond p] = s$, while $\{w \in s \mid \{w\}[\Diamond p] = \{w\}\} = \{w_p\} \neq s$.

From distributivity it follows directly that the order of updates does not matter for $\alpha, \beta \in \mathbf{BiUS}_0$.

Fact 6.2.3. *For* $\alpha, \beta \in \mathbf{BiUS}_0$*:*

[117] See van Benthem (1989) and Groenendijk and Stokhof (1990) for a broader discussion on this.

[118] In the literature *distributivity* is often formulated as "$s[\alpha] = \bigcup_{w \in s}\{w\}[\alpha]$". By eliminativity $\{w\}[\alpha]$ is either $\{w\}$ or $\emptyset$, if defined. So our definition of *distributivity* is equivalent and will make our proofs more readable.

i) $s[\alpha][\beta] = s[\beta][\alpha]$

ii) $s[\alpha][\beta]^r = s[\beta]^r[\alpha]$

iii) $s[\alpha]^r[\beta]^r = s[\beta]^r[\alpha]^r$

Now that we have seen that $\mathbf{BiUS}_0$ is both eliminative and distributive, we can conclude that it is not truly dynamic. As updating with an implication results in either $\emptyset$ or s we see that $\mathbf{BiUS}(\dashrightarrow)$ is still eliminative, and thus similarly for $\mathbf{BiUS}_0(\dashrightarrow)$. However, if we add the dynamic implication to the propositional fragment we see that we no longer have distributivity.

Fact 6.2.4 (Non-distributivity $\mathbf{BiUS}_0(\dashrightarrow)$)**.** *Let $\alpha, \beta \in \mathbf{BiUS}_0$, then we need not have $s[\alpha \dashrightarrow \beta] = \{w \in s \mid \{w\}[\alpha \dashrightarrow \beta] = \{w\}\}$*

Proof. Consider Figure 6.1(b). Then $s[p][q] = \{w_{pq}\} \neq \{w_{pq}, w_p\} = s[p]$ and thus $s[p \dashrightarrow q] = \emptyset$. However, $\{w_{pq}\}[p][q] = \{w_{pq}\} = \{w_{pq}\}[p]$, and thus $\{w_{pq}\}[p \dashrightarrow q] = \{w_{pq}\}$. So we see that $\{w \in s \mid \{w\}[p \dashrightarrow q] = \{w\}\} \neq \emptyset$. □

By the non-distributivity of $\mathbf{BiUS}_0(\dashrightarrow)$, we can conclude that the dynamic implication makes the system truly dynamic. In other words, we can not get an equivalent logic to $\mathbf{BiUS}_0(\dashrightarrow)$ that is static. This is a positive result, for it means that our change to a dynamic setting really goes into new territory that we could not capture in the static **BSML**.[119]

Another useful fact about $\alpha \in \mathbf{BiUS}_0$ is similar to the law of excluded middle. Namely, it is the case that any singleton either supports α or $\neg\alpha$. Remember that $\{w\} \vDash \neg\alpha$ if and only $\{w\}[\alpha]^r = \{w\}$.

Fact 6.2.5. *Let $\alpha \in \mathbf{BiUS}_0$. Then for any $w \in W$ exactly one of the following holds: i) $\{w\}[\alpha] = \{w\}$ or ii) $\{w\}[\alpha]^r = \{w\}$.*

Although we have the law of excluded middle, we do not have bivalence in general. In other words, for an arbitrary state s and formula α it does not always hold that s supports α or $\neg\alpha$. For example, the state $s = \{w_p, w_q\}$ in Figure 6.1(a) neither supports p nor $\neg p$. Notice that this is similar to **BSML** where we also have the law of excluded middle, but no bivalence in general.

Furthermore, we get an analogous result to Fact 2.1.4, stating that updating with a propositional formula and its negation results in the empty set.

Fact 6.2.6. *For $\alpha \in \mathbf{BiUS}_0$, and any state s, we have $s[\alpha][\alpha]^r = \emptyset$.*

Lastly, there are several facts that are stated, but not fully proven, by Aloni (2022b).[120] The first of these states that for NE-free formulas α, updating a state with $|\alpha|^+$ results, if defined, in the same state as updating it with α.

[119]Note that we could also have used a notion of implication that would preserve distributivity. For example, we could define $s[\varphi \Rightarrow_3 \psi] := \{w \in s \mid w \in s[\varphi] \text{ implies } w \in s[\varphi][\psi]\}$ similar to the notion of implication used in the update semantics of Groenendijk, Stokhof, and Veltman (1996). However, the distributivity-preserving of this implication means that $\mathbf{BiUS}_0(\Rightarrow_3)$ will be equivalent to a static framework, and thus we will not be making full use of the advantages of dynamics. For example, we could no longer explain the importance of order such as in Fact 6.3.5. Hence, we see that we prefer to have a truly dynamic implication such as $\dashrightarrow$.

[120]See Footnote 116.

Fact 6.2.7. *For $\alpha \in \mathbf{BiUS}_\emptyset$, and any state s:*

i) If $s[|\alpha|^+]$ is defined, then $s[|\alpha|^+] = s[\alpha]$.

ii) If $s[|\alpha|^+]^r$ is defined, then $s[|\alpha|^+]^r = s[\alpha]^r$.

Furthermore, we find that updating a state with a formula $\alpha \in \mathbf{BiUS}_0$, i.e., α is NE-free and $\Diamond$-free, results in the intersection of that state with $W[\alpha]$. Note that this follows directly from the fact that $\mathbf{BiUS}_0$ is distributive.

Fact 6.2.8. *For $\alpha \in \mathbf{BiUS}_0$ and any state s, $s[\alpha] = s \cap W[\alpha]$.*

From this fact it directly follows that $\mathbf{US}_{BiUS}$ is idempotent. In other words, updating with an NE-free formula that can only have a $\Diamond$ as the outermost operator is the same as updating with that formula twice.

Fact 6.2.9 (Idempotency)**.** *For $\alpha \in \mathbf{US}_{BiUS}$ and any state s, $s[\alpha] = s[\alpha][\alpha]$.*

Finally, it also follows from Fact 6.2.8 that $\mathbf{US}_{BiUS}$ is monotone, i.e., updates preserve subset relations.

Fact 6.2.10 (Monotonicity)**.** *For $\alpha \in \mathbf{US}_{BiUS}$, and any state s and t, $t \subseteq s$ implies $t[\alpha] \subseteq s[\alpha]$.*

Now that we have seen several useful facts concerning **BiUS**, we can look at the behaviour of $\mathbf{BiUS}(\dashrightarrow)$ with respect to the linguistic desiderata and mathematical properties that we have discussed in Chapter 3, Chapter 4 and Chapter 5.

6.3 Linguistic Desiderata

6.3.1 Desiderata by Stalnaker

Let us first look at Stalnaker's desiderata, as introduced in Section 3.1.

Desideratum A The first desideratum by Stalnaker was concerned with the "direct argument", i.e., $\alpha \vee \beta \vDash \neg\alpha \dashrightarrow \beta$. This desideratum does not hold in general, for consider the state $s = \{w_{pq}, w_\emptyset\}$ in Figure 6.2(a). Then we see that $s[p \vee \Diamond q] = s[p] \cup s[\Diamond q] = \{w_{pq}\} \cup s = s$, and $s[\neg p] = \{w_\emptyset\} \neq \{w_\emptyset\}[\Diamond q] = \emptyset$. In other words, we find that $s[p \vee \Diamond q] \nvDash \neg p \dashrightarrow \Diamond q$.

However, we see that the same desideratum does hold if we restrict ourselves to α and β from the propositional fragment $\mathbf{BiUS}_0$.

Fact 6.3.1. *For $\alpha, \beta \in \mathbf{BiUS}_0$, we have $\alpha \vee \beta \vDash \neg\alpha \dashrightarrow \beta$*

Proof. Suppose $s[\alpha \vee \beta]$ is defined. As order does not matter (Fact 6.2.3), and updating with a formula and its negation results in the empty state (Fact 6.2.6), we see that $s[\alpha \vee \beta][\neg\alpha] = s[\alpha \vee \beta][\alpha]^r = s[\alpha]^r[\alpha \vee \beta] = s[\alpha]^r[\alpha] \cup s[\alpha]^r[\beta] = \emptyset \cup s[\alpha]^r[\beta] = s[\neg\alpha][\beta]$. And thus by idempotency (Fact 6.2.9) we see that $s[\alpha \vee \beta][\neg\alpha][\beta] = s[\neg\alpha][\beta][\beta] = s[\neg\alpha][\beta] = s[\alpha\vee\beta][\neg\alpha]$ and thus $s[\alpha\vee\beta] \vDash \neg\alpha \dashrightarrow \beta$. □

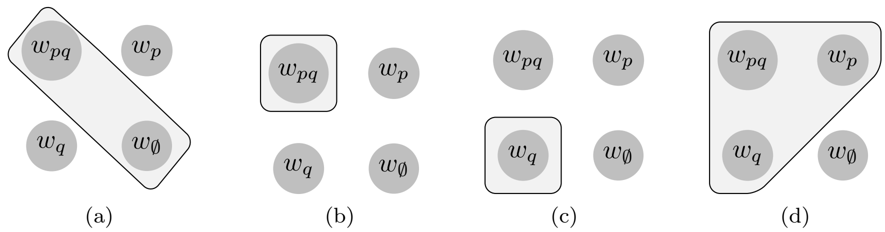


Figure 6.2: Models for **BiUS** and Stalnaker

Similar to the maximal$_\exists$ implication and the **BSML**-implication we see that this result no longer holds if we add pragmatic enrichment. For consider Figure 6.2(b), then we see that $s[|p \vee q|^+] = s$. But $s[|\neg p|^+] = \#$ because $s[\neg p] = \emptyset$, and thus we see that $s[|p \vee q|^+] \not\models |\neg p \dashrightarrow q|^+$. Note that this is the exact same counterexample that we argued against at the end of Section 3.3. So again this failure under pragmatic enrichment is to be expected, and therefore poses no real problem.[121]

To conclude, we see that the dynamic implication can not account for the direct argument in general. On the other hand, it can semantically account for it on the propositional fragment. This is a downside with respect to the behaviour in the static **BSML**, for there we could semantically account for the direct argument over the whole classical fragment.[122] Nevertheless, we will see that there are many other desiderata in favor of the dynamic implication.

Desideratum B The second desideratum by Stalnaker, *No Trivialization*, says that $\neg p$ should not entail the implication $p \dashrightarrow q$. Without semantic enrichment this entailment does hold, as order does not matter (Fact 6.2.3), and updating with a formula and its negation results in the empty state (Fact 6.2.6). For then $s[\neg p][p] = \emptyset = s[\neg p][p][q]$, of which the last step follows from eliminativity. Hence, we see that indeed $s[\neg p] \models p \dashrightarrow q$.

However, if we add pragmatic enrichment we see that the entailment no longer holds.

Fact 6.3.2. $|\neg p|^+ \not\models |p \dashrightarrow q|^+$

[121]In Section 4 of Aloni (2022b) the very similar *disjunctive syllogism* is discussed, i.e., $\alpha \vee \beta, \neg\alpha \models \beta$. There it is shown that this holds under pragmatic enrichment, but we saw that the direct argument does not hold under pragmatic enrichment. Note that the important difference is that for the *disjunctive syllogism* we have $s[|\alpha \vee \beta|^+][|\neg\alpha|^+]$ before the $\models$, and thus we only consider situations where that is defined. But it was exactly a situation where this was not defined that we used as a counterexample to the pragmatically enriched direct argument. Another interesting thing to note, is that the order of the premises matters for the pragmatically enriched disjunctive syllogism. We will see a similar effect occurring in the discussion of Desideratum 1 by Ciardelli (Fact 6.3.5).

[122]Notice that a distributive notion of the modality, using relations such as in **BSML**, would make Fact 6.3.1 hold for the whole NE-free fragment. So we see that the failure of direct argument is a side effect of the non-distributive notion of modality, rather than an effect of our dynamic implication. Nonetheless, the framework that we are working with, **BiUS**, has a non-distributive modality, and thus we can see this failure as a downside of **BiUS**($\dashrightarrow$).

Proof. Consider the state $s = \{w_q\}$ in Figure 6.2(c), then we see that $s[\neg p] = s \neq \emptyset$ and thus $s[|\neg p|^+] = s$. But $s[|p|^+]$ is undefined and therefore we see that $s[|\neg p|^+][|p \dashrightarrow q|^+]$ is also undefined. Hence, $s[|\neg p|^+] \not\models |p \dashrightarrow q|^+$. $\square$

So just like the maximal$_\exists$ implication and the **BSML**-implication, we see that the dynamic implication can account for this desideratum pragmatically.[123] The crucial element to notice is that the pragmatic enrichment ensures that both the antecedent and the consequent are live possibilities, for otherwise updating with them will be undefined. This is very similar to the behaviour of the maximal$_\exists$ implication.[124]

Desideratum C The third desideratum by Stalnaker says that the antecedent should not follow from the denial of a conditional. In other words, we do not want p to follow from $\neg(p \dashrightarrow q)$. The dynamic implication can account for this semantically, and nothing changes when we add pragmatic enrichment.

Fact 6.3.3. *$\neg(p \dashrightarrow q) \not\models p$ and $|\neg(p \dashrightarrow q)|^+ \not\models |p|^+$*

Proof. Consider Figure 6.2(d). Then we see that $s[\neg(p \dashrightarrow q)] = s[p \dashrightarrow q]^r = s$ as $s[p][q]^r \neq \emptyset$. Similarly, we find that $s[|\neg(p \dashrightarrow q)|^+] = s[(|p|^+ \dashrightarrow |q|^+)^{\text{NE}}]^r = s[|p|^+ \dashrightarrow |q|^+]^r = s$ as $s[|p|^+][|q|^+]^r \neq \emptyset$. However, $s[|p|^+] = s[p] = \{w_{pq}, w_p\} \neq s$ and thus we see that $s \not\models p$ and $s \not\models |p|^+$. $\square$

Again this behaviour agrees with **BSML**($\hookrightarrow_\exists$) by semantically accounting for this desideratum. Remember that **BSML**($\Rightarrow_{\neg,\vee}$) could not account for this desideratum in any way.

Stalnaker, Grice, BSML and BiUS We combine these results in Table 6.1, and see that **BiUS**($\dashrightarrow$) behaves similarly to **BSML**($\hookrightarrow_\exists$) with regard to Desiderata B and C.[125] More discussion on this behaviour can be found at the end of Section 3.3.

On Desideratum A we also have similar behaviour, but the dynamic implication fails to account for the direct argument in general. Rather, it only holds when we restrict ourselves to the propositional fragment. Specifically, we saw that $p \vee \Diamond q \not\models \neg p \dashrightarrow \Diamond q$. But if we consider the sentences from (16), then it seems very reasonable to say that, although the sentences are a bit odd, (16-b) follows from (16-a).

(16) a. Either the butler did it or it might be that the gardener did it.
b. If the butler didn't do it, then it might be that the gardener did it.

[123] See the end of Section 3.3 for an explanation of this using the pragmatic "neglect-zero" tendency.
[124] Remember that pragmatic enrichment of the **BSML**-implication only ensures that the consequent is a live possibility, for the enrichment of the antecedent is cancelled out under negation.
[125] Remember that we use colors and fonts in tables to denote what results are **good**, bad or *somewhere in between*.

Desideratum	A (*Dir. Arg.*)	B (*No Triv.*)	C (*Denial*)
Grice	Sem.	Prag.	Prag.
Stalnaker	Prag.	Sem.	Sem.
BSML($\hookrightarrow_\exists$)	**Sem.***	**Prag.**	**Sem.**
BSML($\Rightarrow_{\neg,\vee}$)	*Sem.**	**Prag.**	✗
BiUS($\dashrightarrow$)	*Sem.*^{0*}	**Prag.**	**Sem.**

Table 6.1: BiUS and Stalnaker.
Dir. Arg. = Direct Argument, No Triv. = No Trivialization, Denial = Denial of Conditional
*This desideratum fails when pragmatically enriched.
^{0}This desideratum only holds for $\mathbf{BiUS}_0$.

In other words, we would hope that the direct argument holds for $\mathbf{BiUS}_\emptyset$, not only the propositional fragment. Notice that in state $s = \{w_{pq}, w_\emptyset\}$ in Figure 6.2(a), our counterexample, any p-world is also a q-world. In other words, this is an information state in which either both p and q are the case or neither is the case. Then one could use Grice's first maxim of Quantity to say that this is not a situation in which one would utter $p \vee \Diamond q$. Because this is not informative enough, one should have uttered the more informative $(p \wedge q) \vee (\neg p \wedge \neg q)$ instead. In other words, we could discard this counterexample on pragmatic reasons. However, maybe one does not accept this explanation (as we do not have this pragmatic reasoning build into **BiUS**, for example). Then we cannot use the "neglect-zero" tendency to give a similar explanation of the failure as in Section 3.3, for in this case the failure is without the pragmatic enrichment. Therefore we conclude that with respect to Stalnaker's desiderata, the most suitable account is given by **BSML**($\hookrightarrow_\exists$). However, the dynamic account did not fail altogether and thus this partial failure might be acceptable if it behaves well on other desiderata.

6.3.2 Desiderata by Ciardelli

In Section 4.1 we discussed several desiderata for the indicative conditional as presented by Ciardelli (2020). Remember that we chose to leave out Desideratum 4 and reformulate desiderata 6 and 7. Let us now see how the dynamic implication fares with respect to these desiderata. Note that some of the longer proofs have been left out of the text, these can be found in Appendix D.2.

Desideratum 1 The first half of this desideratum, 1a), is the same as Desideratum B by Stalnaker, of which we have just seen that the dynamic implication can pragmatically account for it. The second half of the desideratum says that an implication should be compatible with the negation of its antecedent, i.e., together they should not entail a contradiction. It is straightforward to see that the dynamic implication satisfies this desideratum.

Fact 6.3.4. $\neg p, p \dashrightarrow q \not\models \bot$

Proof. Consider Figure 6.3(a), then we have $s[\neg p] = s$. And as $s[p] = \emptyset = s[p][q]$, we see that $s[\neg p][p \dashrightarrow q] = s \neq \emptyset$. We conclude that $s[\neg p][p \dashrightarrow q] \not\models \bot$. □

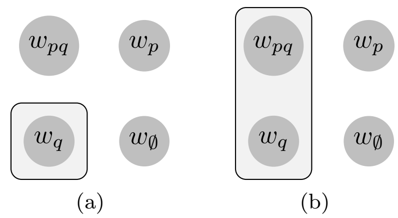


Figure 6.3: Models for **BiUS** and Ciardelli

However, when we pragmatically enrich these formulas we get an interesting result. Namely, the desideratum is only satisfied if we reverse the order of $|\neg p|^+$ and $|p \dashrightarrow q|^+$. This is exactly how we would expect our implication to behave if we consider examples like (12) with the "neglect-zero" tendency in mind.[126]

Fact 6.3.5. $|\neg p|^+, |p \dashrightarrow q|^+ \vDash \bot$, *but* $|p \dashrightarrow q|^+, |\neg p|^+ \nvDash \bot$

Proof. Let s be some state. Then $s[|\neg p|^+][|p|^+]$ is undefined as clearly $s[|\neg p|^+][p] = \emptyset$. Hence, $s[|\neg p|^+][|p \dashrightarrow q|^+]$ is never defined, and the first entailment holds vacuously.

For the reversed order, consider Figure 6.3(b). Then $s \neq \emptyset$ and $s[|p|^+] = \{w_{pq}\} = s[|p|^+][|q|^+]$. Hence, $s[|p \dashrightarrow q|^+] = s$. And as $s[|\neg p|^+] = \{w_q\} \neq \emptyset$ we see that the contradiction does not follow. □

In Section 4.2 we saw that the maximal$_\exists$ implication satisfied the desideratum when pragmatically enriched, while the **BSML**-implication did not. But then example (12) demonstrated that the order of the statements seems to be of importance if we take the "neglect-zero" tendency into account. Therefore we concluded neither of the static accounts was fully satisfactory. As only **BiUS**($\dashrightarrow$) can account for the order, we see this is an advantage of the dynamic account over the static account.

Desideratum 2 The second desideratum by Ciardelli states that Modus Ponens and Modus Tollens should hold for the propositional fragment. We see that regardless of pragmatic enrichment, Modus Ponens holds for the dynamic implication.

Fact 6.3.6. *Let* $\alpha, \beta \in \mathbf{BiUS}_0$. *Then* $\alpha \dashrightarrow \beta, \alpha \vDash \beta$ *and* $|\alpha \dashrightarrow \beta|^+, |\alpha|^+ \vDash |\beta|^+$.

This result is the same as that of **BSML**($\hookrightarrow_\exists$) and **BSML**($\Rightarrow_{\neg,\vee}$). And for the non-enriched version of Modus Tollens we also have the same behaviour, i.e., the dynamic implication satisfies Modus Tollens for the propositional fragment.

[126] We repeat example (12) here for clarity:

(12) a. If it is raining, the pavement is wet. It is not raining.
b. ??It is not raining. If it is raining, the pavement is wet.

Fact 6.3.7. *For $\alpha, \beta \in \mathbf{BiUS}_0$, we have $\alpha \dashrightarrow \beta, \neg\beta \vDash \neg\alpha$.*

We conclude that the dynamic implication satisfies Desideratum 2 semantically.[127]

Desideratum 3 The third desideratum, *import-export*, was semantically accounted for by $\mathbf{BSML}(\hookrightarrow_\exists)$ and $\mathbf{BSML}(\Rightarrow_{\neg,\vee})$. Similarly, we see that the dynamic implication also accounts for this desideratum semantically, and nothing changes if we add pragmatic enrichment.

Fact 6.3.8. *Let $\alpha, \beta \in \mathbf{BiUS}_0$ and $\gamma \in \mathbf{BiUS}_\emptyset$, then we have $\alpha \dashrightarrow (\beta \dashrightarrow \gamma) \equiv (\alpha \wedge \beta) \dashrightarrow \gamma$ and $|\alpha \dashrightarrow (\beta \dashrightarrow \gamma)|^+ \equiv |(\alpha \wedge \beta) \dashrightarrow \gamma|^+$.*

Desideratum 5 This desideratum states that Modus Tollens should fail for an implication with a modal consequent. Remember that in Section 4.2 we saw that only the material implication satisfied this desideratum, but this was unsurprising as the material implication already did not satisfy Modus Tollens on the propositional fragment. On the other hand, we saw that the maximal$_\exists$ implication and the **BSML**-implication did have Modus Tollens on the propositional fragment. But that result could also be extended to the whole classical fragment, and thus Desideratum 5 was not satisfied by these implications.

However, we see that the situation is more interesting for the dynamic implication. For Modus Tollens is satisfied on the propositional fragment, but fails for modal consequents.

Fact 6.3.9. *$p \dashrightarrow \Box q, \neg\Box q \nvDash \neg p$ and $|p \dashrightarrow \Box q|^+, |\neg\Box q|^+ \nvDash |\neg p|^+$*

Proof. Consider state $s = \{w_{pq}, w_\emptyset\}$ in Figure 6.4(a). Then we see that $s[p][q] = s[p]$ and thus it follows that $s[p][\Box q] = s[p]$. Hence, $s[p \dashrightarrow \Box q] = s$ and as $s[q]^r \neq \emptyset$ we see that $s[\neg\Box q] = s[\Box q]^r = s$ as well. And as s, $s[p]$, $s[p][q]$ and $s[q]^r$ are all non-empty, we also find that $s[|p \dashrightarrow \Box q|^+] = s = s[|\neg\Box q|^+]$.

However, $s[\neg p] = s[|\neg p|^+] = \{w_\emptyset\} \neq s$ and thus we see that the non-entailments holds. $\Box$

We conclude that the dynamic implication can semantically account for this desideratum, and nothing changes when we add pragmatic enrichment. The simultaneous satisfaction of Desideratum 2b) and Desideratum 5 is one of the advantages of this dynamic setting.

Desideratum 6′ This desideratum, concerned with the direct argument, says that $[p \wedge \neg q]^+$ should not entail $[\neg p \dashrightarrow q]^+$, but $[p \vee q]^+$ together with $[\Diamond\neg p]^+$ should entail it. It is straightforward to see that dynamic implication satisfies this

[127] Note that the dynamic implication no longer satisfies Modus Tollens when pragmatically enriched. This is witnessed by $\alpha := \neg p \wedge \neg q$ and $\beta := \neg p$, for $\{w_{pq}, w_p, w_\emptyset\}[|(\neg p \wedge \neg q) \dashrightarrow \neg p|^+][|\neg\neg p|^+] = \{w_p\} \nvDash |\neg(\neg p \wedge \neg q)|^+$. However, in Section 4.1 we argued that only the non-enriched Modus Tollens is a desideratum.

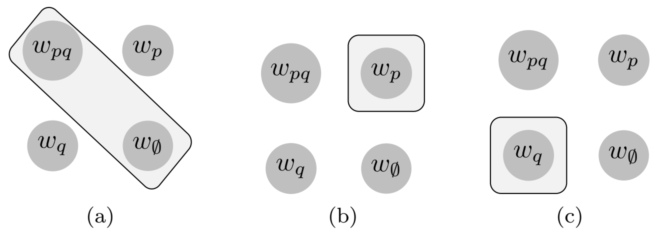


Figure 6.4: More models for **BiUS** and Ciardelli

part of the desideratum, similar to the maximal$_\exists$ implication and the **BSML**-implication.[128]

Fact 6.3.10. $|p \wedge \neg q|^+ \not\vDash |\neg p \dashrightarrow q|^+$

Proof. Consider state $s = \{w_p\}$ in Figure 6.4(b), then we see that $s[|p \wedge \neg q|^+] = s$. However, $s[|\neg p|^+]$ is undefined and thus so is $s[|\neg p \dashrightarrow q|^+]$. Hence, $s[|p \wedge \neg q|^+] \not\vDash |\neg p \dashrightarrow q|^+$. □

The second part of this desideratum states that under pragmatic enrichment, if $\neg p$ is still a live possibility, then $p \vee q$ should entail $\neg p \dashrightarrow q$. In Section 4.2 we saw that only the maximal$_\exists$ implication could account for it, on the condition that we only consider models with a state-based relation R.[129] Now, we see that also the dynamic implication can correctly do so.

Fact 6.3.11. $p \vee q, \Diamond\neg p \vDash \neg p \dashrightarrow q$ *and* $|p \vee q|^+, |\Diamond\neg p|^+ \vDash |\neg p \dashrightarrow q|^+$

We conclude that the dynamic implication does as good as the maximal$_\exists$ implication, but better than the **BSML**-implication, with respect to this desideratum.

Desideratum 7′ The last desideratum says that $p \dashrightarrow \Diamond q$ should not entail $\Diamond p$, while $\Diamond(p \wedge q)$ should entail it. Remember that we reformulated this desideratum so that under pragmatic enrichment we do want the first entailment. In Section 4.2 we saw that the maximal$_\exists$ implication satisfied this desideratum, while the enriched version of the **BSML**-implication still does not entail $\Diamond p$. In the dynamic setting we see that the non-enriched half of the desideratum is satisfied.

Fact 6.3.12. $p \dashrightarrow \Diamond q \not\vDash \Diamond p$ *and* $\Diamond(p \wedge q) \vDash \Diamond p$

Proof. Let us first consider the state $s = \{w_q\}$ in Figure 6.4(c). Then we see that $s[p] = \emptyset = s[p][\Diamond q]$ and thus $s[p \dashrightarrow \Diamond q] = s$. However, as $s[p] = \emptyset$ it follows that $s[\Diamond p] = \emptyset$ and thus $s[p \dashrightarrow \Diamond q] \not\vDash \Diamond p$.

Now suppose $s[\Diamond(p \wedge q)]$ is defined. Then the resulting state is equal to s or it is the empty state. In the latter case we see by eliminativity that $s[\Diamond(p \wedge q)][\Diamond p] =$

[128] See the end of Section 3.3 for an explanation using the "neglect-zero" tendency. There the non-entailment under discussion is $[\neg p]^+ \not\vDash [p \Rightarrow q]^+$, but the same explanation goes for the current situation.

[129] Remember that Aloni (2022a) argues that these are the only relevant models to consider for an epistemic modality, such as the one used in the motivation of Desideratum 6′.

$\emptyset = s[\Diamond(p \wedge q)]$, and thus the entailment holds. Now suppose $s[\Diamond(p \wedge q)] = s$, then it must be that $s[p \wedge q] \neq \emptyset$. Hence, $s[p] \neq \emptyset$ and thus $s[\Diamond(p \wedge q)] \vDash \Diamond p$. $\square$

As we noted in the discussion of Desideratum B, the pragmatic enrichment of the dynamic implication ensures that the antecedent is a live possibility. Hence, under pragmatic enrichment we do have the entailment from $|p \dashrightarrow \Diamond q|^+$ to $|\Diamond p|^+$. Furthermore, we still have $|\Diamond(p \wedge q)|^+ \vDash |\Diamond p|^+$.[130]

Fact 6.3.13. $|p \dashrightarrow \Diamond q|^+ \vDash |\Diamond p|^+$ *and* $|\Diamond(p \wedge q)|^+ \vDash |\Diamond p|^+$

Proof. Suppose $s[|p \dashrightarrow \Diamond q|^+]$ is defined and thus equal to s, then $s \neq \emptyset$ and $s[|p|^+][|\Diamond q|^+] = s[|p|^+]$. And as $s[|p|^+][|\Diamond q|^+]$ must be defined it follows that $s[|p|^+] \neq \emptyset$. Hence, $s[|\Diamond p|^+] = s$ and thus $s[|p \dashrightarrow \Diamond q|^+] \vDash |\Diamond p|^+$. $\square$

So just like the maximal$_\exists$ implication, we see that the dynamic implication fully satisfies Desideratum 7′.

BiUS on Ciardelli's desiderata In Section 4.2 we considered Ciardelli's desiderata with respect to **BSML**($\Rightarrow$) and in this section we did so with respect to **BiUS**($\dashrightarrow$). In Table 6.2 these results are combined, and we see that the dynamic approach has enabled us to correctly account for all revised desiderata.[131]

Desideratum	1b) (CNA)	2a) (MP)	2b) (MT)	3 (I-E)	5 (No MT$^\Diamond$)	6′ (COI)	7′ (IM)
BSML($\hookrightarrow_\exists$)	*Sem.**	**Sem.**	**Sem.**	**Sem.**	✗	✓	✓
BSML($\Rightarrow_{\neg,\vee}$)	*Sem.*	**Sem.**	**Sem.**	**Sem.**	✗	✗	✗
BiUS($\dashrightarrow$)	**Sem.**$^+$	**Sem.**	**Sem.**	**Sem.**	**Sem.**	✓	✓

Table 6.2: Ciardelli's desiderata summarised
CNA = Compatibility with Negated Antecedent, MP = Modus Ponens, MT = Modus Tollens, I-E= Import-Export, No MT$^\Diamond$ = No Modus Tollens for modal consequent, COI = Cautious Or-to-If, IM = If-Might interaction
*This desideratum fails when pragmatically enriched.
$^+$This desideratum's pragmatic enrichment is only satisfied with the premises reversed.

On desiderata 2a) (MP), 2b) (MT) and 3 (I-E) we see that **BiUS**($\dashrightarrow$) behaves the same as the maximal$_\exists$ implication and the **BSML**-implication. And for desiderata 6′ (COI) and 7′ (IM) we see that **BiUS**($\dashrightarrow$) agrees with the maximal$_\exists$ implication, but does better than the **BSML**-implication. However, there are some crucial advantages of the dynamic setting over the static framework.

[130] The proof of $|\Diamond(p \wedge q)|^+ \vDash |\Diamond p|^+$ is essentially the same as the non-enriched version, and thus has been left out. The only difference is that we no longer have the case where updating with the modality results in the empty state, for we quantify over states where $s[|\Diamond(p \wedge q)|^+]$ is defined.

[131] This might raise the question why Ciardelli (2020) needs an ordering on the possible worlds, while we seem to be able to satisfy all desiderata without such an ordering. The crucial detail is that this ordering allowed Ciardelli (2020) to account for Desideratum 4, which is not satisfied by **BiUS**($\dashrightarrow$). However, we argued against this desideratum in 4.1 and so it is the revised desiderata that the dynamic approach can fully account for.

Notably, the dynamic setting allows us to account for the importance of the order of sentences such as in example (12). Specifically, we saw that under pragmatic enrichment the negation of the antecedent only leads to contradiction if it comes before the implication. This is exactly as we would want our implication to behave with the "neglect-zero" tendency in mind.

Another important difference to note is that the dynamic implication simultaneously can account for both Desideratum 2b) (propositional Modus Tollens) and Desideratum 5 (failure of Modus Tollens for modal consequents). As discussed before, in the static setting we found that we could only satisfy either one of them at the same time. Notice that this advantage of $\mathbf{BiUS}(\dashrightarrow)$ stems from the non-classicality of $\Diamond$, while in $\mathbf{BSML}(\Rightarrow)$ the $\Diamond$ is classical.[132]

We can conclude that the move to a dynamic system was not unwarranted. Rather, it allowed us to use a notion of implication that satisfied all of the (revised) desiderata by Ciardelli. And as we discussed in Section 6.1, the dynamic implication gives rise to non-distributivity and thus its introduction makes a framework truly dynamic. Therefore we would not be able to capture a framework equivalent to $\mathbf{BiUS}(\dashrightarrow)$ in a static setting.

6.4 Mathematical Properties

6.4.1 Logico-Mathematical Properties

In Section 6.2 we saw that the notion of distributivity had a similar connection to classicality in the dynamic setting as flatness had in the static setting. As we showed that $\mathbf{BiUS}_0$ is distributive (Fact 6.2.2) , we concluded that $\mathbf{BiUS}_0$ behaves classically. However, the same did not hold for $\mathbf{BiUS}_\emptyset$ and thus we see that the NE-free fragment of **BiUS** already is non-classical. This is what one expects, as the switch to a truly dynamic framework will always give rise to some non-classicality, it might even be the motivation of the dynamic framework. Therefore we will not consider preservation of classicality as one of the favorable mathematical properties that we want to have, in contrast to the situation of Section 5.1.1.

Nonetheless, we are still interested in the Deduction Theorem and transitivity. Similar to all static implications, except the material implication, we see that the Deduction Theorem does not hold in general for $\mathbf{BiUS}(\dashrightarrow)$. Note that the longer proofs of this section can be found in Appendix D.3.

Fact 6.4.1. *Let $\varphi, \psi \in \mathbf{BiUS}(\dashrightarrow)$ and let Γ be a sequence of $\xi_1, \ldots, \xi_n$ such that $\xi_1, \ldots, \xi_n \in \mathbf{BiUS}(\dashrightarrow)$. Then $\Gamma, \varphi \vDash \psi$ does not imply $\Gamma \vDash \varphi \dashrightarrow \psi$.*

Proof. Consider Γ as the empty sequence, $\varphi := p^{\text{NE}}$ and $\varphi := p$, and suppose $s[p^{\text{NE}}]$ is defined. Then we see that $s[p^{\text{NE}}] = s[p] = s[p][p]$, of which the latter equality holds by idempotency. Hence, $p^{\text{NE}} \vDash p$.

[132]Specifically, the flatness of $\Diamond$-formulas in the static setting meant that the proof of Modus Tollens for the propositional fragment directly generalized to the whole classical fragment. But in $\mathbf{BiUS}(\dashrightarrow)$ we saw that Modus Tollens depended on the distributivity that only holds for the propositional fragment.

But if we consider the state $s = \{w_q\}$ in Figure 6.5(a), we see that $s[p^{\text{NE}}]$ is undefined. And thus $s[p^{\text{NE}} \dashrightarrow p]$ is also undefined. In other words, $s \not\models p^{\text{NE}} \dashrightarrow p$. $\square$

However, if we restrict ourselves to NE-free formulas we see that the Deduction Theorem does hold.

Fact 6.4.2. *Let $\alpha, \beta \in \mathbf{BiUS}_{\emptyset}(\dashrightarrow)$ and let Γ be a sequence of $\xi_1, \ldots, \xi_n$ such that $\xi_1, \ldots, \xi_n \in \mathbf{BiUS}_{\emptyset}(\dashrightarrow)$. Then $\Gamma, \alpha \models \beta$ if and only if $\Gamma \models \alpha \dashrightarrow \beta$.*

And similarly to the maximal$_\exists$ implication, we see that the dynamic implication is not transitive.[133][134]

Fact 6.4.3. $\dashrightarrow$ *is not transitive*

Proof. Consider the state $s = \{w_{pq}, w_p\}$ in Figure 6.5(d). Then we see that $s[p \wedge \neg q][p] = s[p \wedge \neg q]$ and $s[p][\Diamond q] = s[p]$. In other words, we find that $s[(p \wedge \neg q) \dashrightarrow p][p \dashrightarrow \Diamond q] = s$. However, $s[p \wedge \neg q][\Diamond q] = \emptyset \neq s[p \wedge \neq q]$ and thus $s \not\models (p \wedge \neg q) \dashrightarrow \Diamond q$. $\square$

Combining these results in Table 6.3, we see that with regard to the Deduction Theorem there is no difference between the maximal$_\exists$ implication, the **BSML**-implication and the dynamic implication. However, we see that just like the maximal$_\exists$ implication, the dynamic implication is also intransitive. Nonetheless, we conclude that, similar to the results from the linguistic sections, these results are quite positive for the dynamic implication.

	Ded.	Ded. (NE-free)	Trans.
$\mathbf{BSML}(\hookrightarrow_\exists)$	✗	✓	✗
$\mathbf{BSML}(\Rightarrow_{\neg,\vee})$	✗	✓	✓
$\mathbf{BiUS}(\dashrightarrow)$	✗	✓	✗

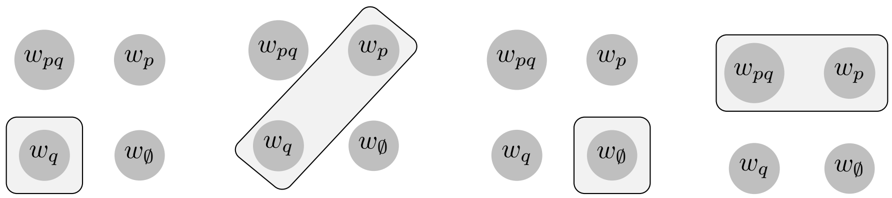
Table 6.3: Logico-Mathematical Properties
Ded. = Deduction Theorem., Trans. = Transitivity

6.4.2 Linguistic-Mathematical Properties

In Section 5.2 we looked at principle of *Simplification of Disjunctive Antecedent* (SDA) in the context of **BSML**. Furthermore, we checked whether the pragmatic enrichment of an implication had a non-trivial effect. In this section we will reconsider this in the context of $\mathbf{BiUS}(\dashrightarrow)$.

[133] Note that this counterexample is very similar to the one discussed in Footnote 99. So from a linguistic point of view one might argue that this failure is not so bad.

[134] Although $\dashrightarrow$ is not transitive in general, it is transitive on the propositional fragment $\mathbf{BiUS}_0$. A proof of this has been added to Appendix D.3.



(a) (b) (c) (d)

Figure 6.5: Models for mathematical properties

Let us first look at SDA, i.e., whether $(\alpha \vee \beta) \dashrightarrow \gamma$ entails $\alpha \dashrightarrow \gamma$ and $\beta \dashrightarrow \gamma$. We see that this principle is satisfied by the dynamic implication on the propositional fragment.[135]

Fact 6.4.4. *Let* $\alpha, \beta, \gamma \in \mathbf{BiUS}_0$. *Then,* $(\alpha \vee \beta) \dashrightarrow \gamma \vDash (\alpha \dashrightarrow \gamma) \wedge (\beta \dashrightarrow \gamma)$.

And similar to the maximal$_\exists$ implication, but contrary to the **BSML**-implication, we see that this principle is no longer satisfied when we add pragmatic enrichment.[136]

Fact 6.4.5. *Let* $\alpha, \beta, \gamma \in \mathbf{BiUS}_\emptyset$. *Then,* $|(\alpha \vee \beta) \dashrightarrow \gamma|^+ \nvDash |(\alpha \dashrightarrow \gamma) \wedge (\beta \dashrightarrow \gamma)|^+$.

Proof. Consider the state $s = \{w_p, w_q\}$ in Figure 6.5(b). Now let $\alpha := p$, $\beta := q$ and $\gamma := p \vee q$. Then we see that $s \neq \emptyset$ and $s[|p \vee q|^+][|p \vee q|^+] = s[|p \vee q|^+]$, i.e., $s[|(p \vee q) \dashrightarrow (p \vee q)|^+] = s$. However, $s[|p|^+] = \{w_p\}$ and $\{w_p\}[|p \vee q|^+]$ is undefined. In other words, $s[|p \dashrightarrow (p \vee q)|^+]$ is undefined and thus we do not have the entailment. $\square$

At the end of Section 5.2.1 we argued that we expect SDA to fail when we consider the "neglect-zero" tendency. Hence, we conclude that the dynamic implication can correctly account for this principle.

Lastly, we mentioned before that pragmatically enriching the dynamic implication ensures that both its antecedent and its consequent are live possibilities. In other words, adding pragmatic enrichment does more than merely ensuring that the supporting state is non-empty, i.e., its effect is non-trivial in that sense. Similarly to the facts proven in Section 5.2.2 we can state this as follows.

Fact 6.4.6. *For "split-free"* $\alpha, \beta \in \mathbf{BiUS}_\emptyset$ *we have* $|\alpha \dashrightarrow \beta|^+ \not\equiv (\alpha \dashrightarrow \beta)^{\mathrm{NE}}$.

Proof. Consider the state $s = \{w_\emptyset\}$ in Figure 6.5(c) and let $\alpha := p$ and $\beta := q$. Then we see that $s \neq \emptyset$, and $s[p][q] = \emptyset = s[p]$ and thus $s[(p \dashrightarrow q)^{\mathrm{NE}}] = s$. However, as $s[|p|^+]$ is undefined we see that $s \nvDash |p \dashrightarrow q|^+$. $\square$

[135] Note that this principle is only satisfied for the propositional fragment $\mathbf{BiUS}_0$. That this doesn't hold in general is witnessed by the state $s = \{w_p, w_q\}$ that supports $p \vee q \dashrightarrow \Diamond q$, but doesn't support $p \dashrightarrow \Diamond q$.

[136] Note that this counterexample is the same as we used for the maximal$_\exists$ implication in Fact 5.2.3.

Combining these results in Table 6.4, we see that the dynamic implication behaves as good as the maximal$_\exists$ implication and better than the **BSML**-implication on the linguistic-mathematical properties.

	SDA	SDA$^+$	Triv.	Triv. ("split-free")
BSML($\hookrightarrow_\exists$)	✓	✗	✗	✗
BSML($\Rightarrow_{\neg,\vee}$)	✓	$\underline{✓}$	✗	✗
BiUS($\dashrightarrow$)	✓	✗	✗	✗

Table 6.4: Linguistic-Mathematical Properties
SDA = Simplification of Disjunctive Antecedents, SDA$^+$ = SDA with pragmatic enrichment, Triv. = Trivial effect of pragmatic enrichment, Triv. ("split-free") = Triv. for "split-free" α, β.

To conclude, in Section 6.3.1 we have seen that **BiUS**($\dashrightarrow$) does almost as good as **BSML**($\hookrightarrow_\exists$) and **BSML**($\Rightarrow_{\neg,\vee}$), while we saw in Sections 6.3.2, 6.4.1 and 6.4.2 that the dynamic setting has several advantages over the static variant. In the final chapter we will make up the final score for both **BSML** an **BiUS**, and discuss several interesting subjects for future research.

Chapter 7

Conclusion

In this thesis we have extended *bilateral stated-based modal logic* (**BSML**) and *bilateral update semantics* (**BiUS**) with different notions of implication.[137] For each of these extensions we have looked at several linguistic desiderata and mathematical properties. In this final chapter we will give an overview of our results, whereafter we will point towards some interesting topics for future research.

Let us first consider the linguistic behaviour of our implication. In Sections 3.3 and 6.3.1 we have looked at three linguistic desiderata as put forward by Stalnaker (1976). An overview of these results can be found in Table 7.1, where we see that **BSML** extended with the maximal$_\exists$ implication can best account for these desiderata.[138]

Desideratum	A (*Dir. Arg.*)	B (*No Triv.*)	C (*Denial*)
BSML($\rightarrow$)	*Sem.*	✗	✗
BSML($\twoheadrightarrow$)	*Sem.*	✗	**Sem.**
BSML($\hookrightarrow_\forall$)	*Sem.*	✗	**Sem.**
BSML($\hookrightarrow_\exists$)	**Sem.***	**Prag.**	**Sem.**
BSML($\Rightarrow_{\neg,\vee}$)	*Sem.**	**Prag.**	✗
BiUS($\dashrightarrow$)	*Sem.*^{0*}	**Prag.**	**Sem.**

Table 7.1: Stalnaker's desiderata

Dir. Arg. = Direct Argument, No Triv. = No Trivialization, Denial = Denial of Conditional

*This desideratum fails when pragmatically enriched.

^{0}This desideratum only holds for $\mathbf{BiUS}_0$.

The dynamic framework **BiUS** extended with an implication gives almost equally good results, although Desideratum A (the *direct argument*) is only satisfied on the propositional fragment $\mathbf{BiUS}_0$. As we would like to be able to account for

[137] Remember that throughout this thesis we have discussed the material implication ($\rightarrow$), intuitionistic implication ($\twoheadrightarrow$), maximal implication ($\hookrightarrow_\forall$), maximal$_\exists$ implication ($\hookrightarrow_\exists$), **BSML**-implication ($\Rightarrow_{\neg,\vee}$) and the dynamic implication ($\dashrightarrow$).

[138] Remember that we use colors and fonts in tables to denote what results are **good**, bad or *somewhere in between*.

the direct argument with modalities as well, we see that this is a downside of the dynamic setting. One solution would be to switch to a distributive notion of modality as discussed in Footnote 122. However, as hinted at in Footnote 132, we would risk losing the good behaviour of the dynamic setting regarding Desiderata 2b) (MT) and Desideratum 5 (No MT for Modal Consequent). Therefore we see that solving this problem is non-trivial, which highlights the possibility of future research into different notions of modality for $\mathbf{BiUS}(\dashrightarrow)$.

In Sections 4.2 and 6.3.2 we discussed several other desiderata for the indicative conditional by Ciardelli (2020). An overview of these results can be found in Table 7.2, where we see the advantages of the dynamic setting of **BiUS**.

Desideratum	1b) (CNA)	2a) (MP)	2b) (MT)	3 (I-E)	5 (No MT$^\diamond$)	6′ (COI)	7′ (IM)
$\mathbf{BSML}(\to)$	*Sem.*	**Sem.**	✗	**Sem.**	*Sem.*	✗	✗
$\mathbf{BSML}(\twoheadrightarrow)$	*Sem.*	**Sem.**	**Sem.**	Sem.*	✗	✗	✗
$\mathbf{BSML}(\hookrightarrow_\forall)$	*Sem.*	**Sem.**	**Sem.**	Sem.*	✗	✗	✗
$\mathbf{BSML}(\hookrightarrow_\exists)$	*Sem.**	**Sem.**	**Sem.**	**Sem.**	✗	✓	✓
$\mathbf{BSML}(\Rightarrow_{\neg,\vee})$	*Sem.*	**Sem.**	**Sem.**	**Sem.**	✗	✗	✗
$\mathbf{BiUS}(\dashrightarrow)$	**Sem.**$^+$	**Sem.**	**Sem.**	**Sem.**	**Sem.**	✓	✓

Table 7.2: Ciardelli's desiderata summarised
CNA = Compatibility with Negated Antecedent, MP = Modus Ponens, MT = Modus Tollens, I-E= Import-Export, No MT$^\diamond$ = No Modus Tollens for modal consequent, COI = Cautious Or-to-If, IM = If-Might interaction
*This desideratum fails when pragmatically enriched.
$^+$This desideratum's pragmatic enrichment is only satisfied with the premises reversed.

The static **BSML** extended with the maximal$_\exists$ implication can account for quite a few of these desiderata, but $\mathbf{BiUS}(\dashrightarrow)$ solves the remaining problems that the static framework has. Specifically, the dynamic setting allowed us to take the order in Desideratum 1b) (CNA) into account, and we were able to satisfy Desideratum 2b) (MT) and Desideratum 5 (No MT$^\diamond$) simultaneously.

Now let us consider the mathematical side of things. In Chapter 5 and Section 6.4 we looked at several mathematical properties of our logics extended with an implication. An overview of these results can be found in Table 7.3, where we see that none of the implications satisfies all mathematical properties that we would like to have.

We see that the material implication is the only implication that satisfies the Deduction Theorem in general. However, this came at the cost of non-classicality and the inability to account for many of the desiderata. The intuitionistic implication and the maximal implication do preserve classicality and satisfy the Deduction Theorem on the NE-free fragment. But unfortunately we see that they cannot account for quite a few of the linguistic desiderata. This originates in the trivial effect the pragmatic enrichment function has on these implications, i.e., it only ensures that the state of evaluation is non-empty.[139]

[139] Actually, this effect is non-trivial if the antecedent and consequent can contain disjunctions, but if we restrict ourselves to "split-free" antecedent and consequent the non-triviality disap-

	Class.	Ded.	Ded.$^{\emptyset}$	Tran.	SDA	SDA^{+}	Triv.	Triv.$'$
BSML($\rightarrow$)	$\underline{✗}$	✓	✓	✓	✓	✗	✗	$\underline{✓}$
BSML($\twoheadrightarrow$)	✓	$\underline{✗}$	✓	✓	✓	✗	✗	$\underline{✓}$
BSML($\hookrightarrow_{\forall}$)	✓	$\underline{✗}$	✓	$\underline{✗}$	✓	✗	✗	$\underline{✓}$
BSML($\hookrightarrow_{\exists}$)	✓	$\underline{✗}$	✓	$\underline{✗}$	✓	✗	✗	✗
BSML($\Rightarrow_{\neg,\vee}$)	✓	$\underline{✗}$	✓	✓	✓	$\underline{✓}$	✗	✗
BiUS($\dashrightarrow$)	-	$\underline{✗}$	✓	$\underline{✗}$	✓	✗	✗	✗

Table 7.3: Mathematical Properties
Class. = Preservation of Classicality, Ded. = Deduction Theorem.,
Ded.$^{\emptyset}$ = Deduction Theorem NE-free, Tran. = Transitivity, SDA = Simplification of Disjunctive Antecedents, SDA^{+} = SDA with pragmatic enrichment,
Triv. = $[\alpha \Rightarrow \beta]^{+} \equiv (\alpha \Rightarrow \beta) \wedge \text{NE}$, Triv.$'$ = Triv. for "split-free" α, β.

The maximal$_{\exists}$ implication and the **BSML**-implication both also preserve classicality and satisfy the Deduction Theorem on the NE-free fragment. But on top of that the pragmatic enrichment function has a non-trivial effect that we will discuss in a moment. On the other hand, the downside of the maximal$_{\exists}$ implication is its intransitivity, and the downside of the **BSML**-implication is that it validates SDA under pragmatic enrichment. Similar to the maximal$_{\exists}$ implication, the dynamic implication satisfies the Deduction Theorem on the NE-fragment but is intransitive.

Now let us shortly review the non-trivial effect that the pragmatic enrichment function has on the maximal$_{\exists}$ implication, the **BSML**-implication and the dynamic implication. By pragmatically enriching a maximal$_{\exists}$ implication or a dynamic implication, we ensure that both the antecedent and the consequent are live possibilities. When we take the "neglect-zero" tendency into account, we see that we would expect this. For a conditional with an antecedent that is not a live possibility would only be validated by an empty configuration. In other words, the "neglect-zero" tendency gives us a simple pragmatic explanation of this non-trivial effect.[140] It is exactly this effect that allows these implications to correctly account for Desideratum A, Desideratum B, Desideratum 6$'$ (COI) and Desiderataum 7$'$ (IM). On the other hand, the **BSML**-implication only ensures that the consequent is a live-possibility. It is this weaker non-trivial effect that causes the **BSML**-implication to be less well-behaved throughout all linguistic desiderata.

Now if we consider the formula $[\Diamond(\alpha \vee \beta) \Rightarrow \Diamond\alpha]^{+}$ that we discussed in Chapter 1, we see that we do not expect this to be a valid conditional if we take the "neglect-zero" tendency into account. For there is no reason to suspect that $\Diamond(\alpha \vee \beta)$ is always a live possibility, and thus its non-validity in **BSML**($\hookrightarrow_{\exists}$) and **BiUS**($\dashrightarrow$) is a positive result.[141]

pears. In other words, we see that it was not an effect of the implications but rather of the nested disjunctions.

[140] Remember that this non-trivial effect is an explicit formalisation of the "definedness" or "antecedent compatibility", as discussed in Footnote 79.

[141] Notice that the formula is valid for the material implication, intuitionistic implication and the maximal implication. So if one would prefer $[\Diamond(\alpha \vee \beta) \Rightarrow \Diamond\alpha]^{+}$ to be a validity, one could

Now let us take up the final score. When we take all linguistic and mathematical considerations into account, we conclude that the static **BSML** can best be extended with the maximal$_\exists$ implication. However, the dynamic framework **BiUS** extended with a dynamic implication gives the best linguistic behaviour while improving the mathematical behaviour.[142] It is important to realise that this thesis has researched the linguistic behaviour of **BSML** and **BiUS** extended with an implication and the effect of the "neglect-zero" tendency in this. This does not mean that we put **BSML**($\hookrightarrow_\exists$) or **BiUS**($\dashrightarrow$) forward as a theory of conditionals that can account for all relevant linguistic phenomena.[143] Rather, we think that there are many different pragmatic phenomena operative in language that explain the behaviour of indicative conditionals. This research has shown that the "neglect-zero" tendency has earned its place, by giving a simple pragmatic explanation for several phenomena concerning the indicative conditional.

Now that we have seen the effect of the "neglect-zero" tendency on indicative conditionals as modeled by **BSML**($\hookrightarrow_\exists$) and **BiUS**($\dashrightarrow$), we can look at several interesting topics for future research. One of the first, and most obvious, topics would be to extend the first-order variant of **BSML**, called **QBSML** and developed by Aloni and Ormondt (2021), with an implication. This would allow one to analyse sentences like (48) and (49) from Aloni (2022a) that contain both implications and quantifiers.

The more mathematically-oriented reader might also be interested in the logical strength and axiomatization of **BSML**($\hookrightarrow_\exists$) and **BiUS**($\dashrightarrow$). This will be left for future research, which might be inspired by the work of Anttila (2021) on the axiomatization of state-based logics.

Another interesting topic would be the formalisation of some of Grice's maxims, such as the maxims of Quantity, in **BSML** or **BiUS**. For example, in our current frameworks we see that the state $\{w_{pq}\}$ supports the disjunction $p \vee q$. However, it is odd to say that "it rains or the wind is blowing" when actually it is both raining and the wind is blowing. This oddity can be explained using Grice's first maxim of Quantity, which states that a contribution should be as informative as required by the current conversation. So we see that a formalisation of Gricean pragmatics should makes sure that $\{w_{pq}\}$ no longer supports $p \vee q$ but does support $p \wedge q$.[144]

extend **BSML** with one of these implications. However, this comes at the cost of significantly less interesting linguistic behaviour on the desiderata that we have discussed.

[142] It is not surprising that the maximal$_\exists$ implication gives similar results to the dynamic implication. Remember that in Footnote 115 we mentioned that the dynamic implication is very "Ramseylike". And similarly, for the maximal$_\exists$ implication we consider the (greatest) subordinate state that satisfies the antecedent and look if the consequent holds in that state. So we see that this implication is also very "Ramseylike".

[143] For this it is also important to realise that there is an empirical objection, as has been discussed in Footnote 68, against a theory of conditionals that does not take the logical connection between the antecedent and consequent into account. This highlights the possibility of future research extending **BSML** or **BiUS** with an implication that does take this connection into account.

[144] The difficulty of this lies in the vagueness of what is "required", i.e., maybe only the disjunction is relevant while the conjunction is not. Similarly, it is odd to say "it is raining and the

As Grice's account of the indicative conditional and both $\mathbf{BSML}(\hookrightarrow_\exists)$ and $\mathbf{BiUS}(\dashrightarrow)$ explain Desideratum B (*No Trivialization*) pragmatically, it might be interesting to empirically research what pragmatic account gives the most plausible explanation.[145]

We saw that Stalnaker argued in Desideratum C that the denial of a conditional should not entail its antecedent. But what is it that we actually mean when we deny an indicative conditional? Grice (1989) points out that "it is by no means always clear what a speaker who says "it is not the case that if p, q" is committing himself to" (pg. 80). One of the possible interpretations of a denied conditional that Grice discusses is the *counterconditional*. So by denying "if p, then q" we mean to say "if p, then $\neg q$". If we look at the anti-support clauses for our conditionals, we see that $\neg(p \hookrightarrow_\exists q) \equiv p \hookrightarrow_\exists \neg q$ but $\neg(p \dashrightarrow q) \not\equiv p \dashrightarrow \neg q$.[146] As it is certain that we, sometimes, but not always interpret the denial of a conditional as a counterconditional, this does not speak in favor of either of the accounts. However, it does highlight the open topic of further research into the anti-support clauses for the implications in **BSML** and **BiUS**.

Lastly, we have looked at several desiderata for indicative conditionals, but there are many phenomena that have been discussed in the literature that we did not discuss in this thesis. One of these is the failure of monotonicity, motivated by the so-called Sobel sequences and argued for by Von Fintel (2001). This work is concerned with counterfactuals, but there are similar examples that go against monotonicity for the indicative conditional.[147] For example, "if you strike a match, then a fire will start" should not imply "if you strike a match and the match is wet, then a fire will start". In other words, it might be that an indicative conditional holds in general, but it no longer holds if we strengthen the antecedent. Neither $\mathbf{BSML}(\hookrightarrow_\exists)$ nor $\mathbf{BiUS}(\dashrightarrow)$ seems to be capable to capture such behaviour. A *selectional analysis*, such as Stalnaker's account, will be able to explain this, as the most similar "strike-a-match"-world need not be the most similar "strike-a-wet-match"-world. Therefore, it might be interesting to consider a variant of $\mathbf{BSML}(\hookrightarrow_\exists)$ or $\mathbf{BiUS}(\dashrightarrow)$ with a similarity ordering of the possible worlds, wherein we can combine the pragmatic "neglect-zero" tendency with a *selectional analysis* of the indicative conditional.[148]

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wind is blowing" when asked only about the rain, i.e., sometimes a statement is required that is not as informative as possible.

[145]Remember that we shortly discussed this at the end of Section 3.3.

[146]The former follows directly from the definition of the maximal$_\exists$ implication. The latter is witnessed by the state $s = \{w_{pq}, w_p\}$, where we see that $s[\neg(p \dashrightarrow q)] = s$ but $s[p \dashrightarrow \neg q] = \emptyset$.

[147]See Williams (2008), Klecha (2015) and Willer (2017) for more on indicative Sobel sequences.

[148]Similarly, Von Fintel (2001) argues for a dynamic approach by noting that sequences like (i-a) seem to be okay, while sequences like (i-b) are odd.

(i) a. If you strike a match, then a fire will start. But if you strike a wet match, then no fire will start.
b. # If you strike a wet match, then no fire will start. But if you strike a match, a fire will start.

It might be interesting to see whether a $\mathbf{BiUS}(\dashrightarrow)$ with a similarity ordering can account for this. Again, Von Fintel (2001) is concerned with counterfactuals, but (i-a) and (i-b) are indicative variants of Von Fintel's (10) and (11).

To conclude, in this thesis we have looked at several implications that can extend **BSML** and **BiUS**. We found that the static **BSML** can best be extended with the maximal$_\exists$ implication, but the best linguistic and mathematical account is given by the dynamic **BiUS** extended with a dynamic implication. Furthermore, we have seen that the "neglect-zero" tendency can give a simple explanation for certain phenomena surrounding the indicative conditional. Several questions have been left open, of which some hopefully will be answered in the future.

Appendix

A Proofs of Chapter 2

Fact (2.1.8). *Let $\alpha \in$ **BSML**$_0$. Then $M, t \vDash [\alpha]^+$, $t \subseteq s$ and $M, s \vDash \alpha$ imply $M, s \vDash [\alpha]^+$.*

Proof. We prove this by a double induction on the complexity of α. For this we will simultaneously prove the fact above and the following: $M, t \mathrel{=\!\!|} [\alpha]^+$, $t \subseteq s$ and $M, s \mathrel{=\!\!|} \alpha$ imply $M, s \mathrel{=\!\!|} [\alpha]^+$.

1. $\alpha = p$:
   - If $M, t \vDash [p]^+$, then $t \neq \emptyset$ and thus $s \neq \emptyset$. So from $M, s \vDash p$ it follows directly that $M, s \vDash [p]^+$.
   - Analogously for $\mathrel{=\!\!|}$.
2. $\alpha = \gamma \vee \delta$:
   - If $M, s \vDash \gamma \vee \delta$, there are $s', s'' : s' \cup s'' = s$ and $M, s' \vDash \gamma$ and $M, s'' \vDash \delta$. Similarly, from $M, t \vDash [\gamma \vee \delta]^+$ we know there are $t', t'' : t' \cup t'' = t$ and $M, t' \vDash [\gamma]^+$ and $M, t'' \vDash [\delta]^+$. Now we know that $t' \subseteq t' \cup s'$ and $t'' \subseteq t'' \cup s''$ and by Fact 2.1.1 and union closure of **BSML**, $M, t' \cup s' \vDash \gamma$ and $M, t'' \cup s'' \vDash \delta$. Then by the induction hypothesis $M, t' \cup s' \vDash [\gamma]^+$ and $M, t'' \cup s'' \vDash [\delta]^+$. Hence, as $(t' \cup s') \cup (t'' \cup s'') = s$ we see that $M, s \vDash [\gamma \vee \delta]^+$.
   - If $M, s \mathrel{=\!\!|} \gamma \vee \delta$, then $M, s \mathrel{=\!\!|} \gamma$ and $M, s \mathrel{=\!\!|} \delta$. And if $M, t \mathrel{=\!\!|} [\gamma \vee \delta]^+$, then $t \neq \emptyset$, $M, t \mathrel{=\!\!|} [\gamma]^+$ and $M, t \mathrel{=\!\!|} [\delta]^+$. So by our induction hypothesis it follows that $M, s \mathrel{=\!\!|} [\gamma]^+$ and $M, s \mathrel{=\!\!|} [\delta]^+$. We conclude that $M, s \mathrel{=\!\!|} [\gamma \vee \delta]^+$ as $t \neq \emptyset$ implies $s \neq \emptyset$.
3. $\alpha = \gamma \wedge \delta$:
   - $\vDash$ is analogous to the $\mathrel{=\!\!|}$-case of the disjunction.
   - $\mathrel{=\!\!|}$ is analogous to the $\vDash$-case of the disjunction.
4. $\alpha = \neg\gamma$:
   - If $M, s \vDash \neg\gamma$, then $M, s \mathrel{=\!\!|} \gamma$. And if $M, t \vDash [\neg\gamma]^+$, then $t \neq \emptyset$ and $M, t \mathrel{=\!\!|} [\gamma]^+$. So by the induction hypothesis we see that $M, s \mathrel{=\!\!|} [\gamma]^+$, i.e., $M, s \vDash \neg[\gamma]^+$. And as $t \neq \emptyset$ implies $s \neq \emptyset$, we see that $M, s \vDash [\neg\gamma]^+$.

- If $M, s \dashv \neg\gamma$, then $M, s \vDash \gamma$. And if $M, t \dashv [\neg\gamma]^+$, then $M, t \dashv \neg[\gamma]^+$ and thus $M, t \vDash [\gamma]^+$. So by the induction hypothesis we see that $M, s \vDash [\gamma]^+$, i.e., $M, s \dashv \neg[\gamma]^+$ and thus $M, s \dashv [\neg\gamma]^+$.

□

Fact (2.2.5). *Let* $\alpha, \beta \in \mathbf{BSML}_\emptyset$. *Then* $\alpha \twoheadrightarrow \beta \equiv \alpha \Rightarrow_{\neg,\vee} \beta$.

Proof. ($\Rightarrow$): Suppose $M, s \vDash \alpha \twoheadrightarrow \beta$. Now let $t = \{w \in s \mid M, \{w\} \nvDash \beta\}$ and $t' = s \backslash t$, then clearly $t \cup t' = s$. And by our assumption it follows that $\forall w \in t$: $M, \{w\} \nvDash \alpha$, so by Fact 2.1.5 we see that $M, t \vDash \neg\alpha$. If $t' = \emptyset$ it follows from the empty state property of β that $M, t' \vDash \beta$. Otherwise, as β is union closed and $\forall w \in t' : M, \{w\} \vDash \beta$, we see that $M, t' \vDash \beta$. Hence, $M, s \vDash \neg\alpha \vee \beta$, i.e., $M, s \vDash \alpha \Rightarrow_{\neg,\vee} \beta$.

($\Leftarrow$): Suppose $M, s \vDash \alpha \Rightarrow_{\neg,\vee} \beta$, then $\exists t, t' : t \cup t' = s$ and $M, t \vDash \neg\alpha$ and $M, t' \vDash \beta$. Now let $u \subseteq s$ be such that $M, u \vDash \alpha$, then by Fact 2.1.4 we see that $t \cap u = \emptyset$. Hence, $u \subseteq t'$ and thus by downward closure of β it follows that $M, u \vDash \beta$. So indeed $M, s \vDash \alpha \twoheadrightarrow \beta$. □

B Proofs of Chapter 3

Fact (3.3.1). *For* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$, *we have* $\alpha \vee \beta \vDash \neg\alpha \Rightarrow \beta$.

Proof. Suppose $M, s \vDash \alpha \vee \beta$, then there are $t, t' : t \cup t' = s$ and $M, t \vDash \alpha$ and $M, t' \vDash \beta$. Now suppose for some $u \subseteq s$ that $M, u \vDash \neg\alpha$, then we see by Fact 2.1.4 that $u \cap t = \emptyset$ and thus $u \subseteq t'$. So by downward closure of β it follows that $M, u \vDash \beta$, and thus $M, s \vDash \neg\alpha \twoheadrightarrow \beta$. Now by Footnote 61 we also have $M, s \vDash \neg\alpha \rightarrow \beta$, and by Fact 2.2.5, Fact 2.2.6 and Fact 2.2.9 we find that the same holds for $\Rightarrow_{\neg,\vee}, \hookrightarrow_\forall$ and $\hookrightarrow_\exists$. □

Fact (3.3.2). *For* $\alpha, \beta \in \mathbf{BSML}_\emptyset$, *we have* $[\alpha \vee \beta]^+ \vDash [\neg\alpha \rightarrow \beta]^+$.

Proof. Suppose $M, s \vDash [\alpha \vee \beta]^+$, then $s \neq \emptyset$ and there are $t, t' : t \cup t' = s$ and $M, t \vDash [\alpha]^+$ and $M, t' \vDash [\beta]^+$. Now suppose $M, s \vDash [\neg\alpha]^+$, then by definition we see that $M, s \vDash \neg[\alpha]^+$. So by Fact 2.1.4 it follows that $t = \emptyset$, i.e., $s = t'$. We conclude that $M, s \vDash [\beta]^+$ and thus indeed $M, s \vDash [\neg\alpha \rightarrow \beta]^+$. □

Fact (3.3.3). *For* $\Rightarrow \in \{\twoheadrightarrow, \hookrightarrow_\forall\}$ *and "split-free"* $\alpha, \beta \in \mathbf{BSML}_\emptyset$, *we have* $[\alpha \vee \beta]^+ \vDash_S [\neg\alpha \Rightarrow \beta]^+$.

Proof. Suppose $M, s \vDash [\alpha \vee \beta]^+$, then $s \neq \emptyset$ and there are non-empty $t, t' : t \cup t' = s$ and $M, t \vDash \alpha$ and $M, t' \vDash \beta$. Let $u \subseteq s$ be such that $M, u \vDash [\neg\alpha]^+$, then $u \neq \emptyset$ and by Fact 2.1.1 we get $M, u \vDash \neg\alpha$. So by Fact 2.1.4 it must be that $u \subseteq t'$, and thus by downward closure $M, u \vDash \beta$. As $u \neq \emptyset$ and β is "split-free", it follows from Fact 2.1.2 that $M, u \vDash [\beta]^+$, and thus $M, s \vDash [\neg\alpha \twoheadrightarrow \beta]^+$. As $\neg\alpha$ is "split-free", we see by Fact 2.2.7 that $M, s \vDash [\neg\alpha \hookrightarrow_\forall \beta]^+$. □

Fact (3.3.4). *For* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$, *we have* $\neg p \vDash p \Rightarrow q$.

Proof. Suppose $M, s \vDash \neg p$. Now let $t \subseteq s$ be such that $M, t \vDash p$, then by Fact 2.1.4 we know that $t = \emptyset$. Hence, by the empty state property of q we find that $M, t \vDash q$ and thus $M, s \vDash p \twoheadrightarrow q$. By Footnote 61, Fact 2.2.5, Fact 2.2.6 and Fact 2.2.9, it follows that $M, s \vDash p \rightarrow q$, $M, s \vDash p \hookrightarrow_\forall q$, $M, s \vDash p \hookrightarrow_\exists q$ and $M, s \vDash p \Rightarrow_{\neg,\vee} q$. $\square$

Fact (3.3.5). *For* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall\}$*, we have* $[\neg p]^+ \vDash [p \Rightarrow q]^+$.

Proof. Suppose $M, s \vDash [\neg p]^+$, then $s \neq \emptyset$ and by Fact 2.1.1 we see that $M, \vDash \neg p$. So by Fact 2.1.4 it follows that any $t \subseteq s$ with $M, t \vDash p$ is empty. So for all $t \subseteq s$ we find that $M, t \nvDash [p]^+$. Hence, $M, s \vDash [p]^+ \twoheadrightarrow [q]^+$ and as s is non-empty also $M, s \vDash [p \twoheadrightarrow q]^+$. Similarly, we find that $M, s \vDash [p \rightarrow q]^+$ and $M, s \vDash [p \hookrightarrow_\forall q]^+$. $\square$

C Proofs of Chapter 4

Fact (4.2.5). *For* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$ *and* $\alpha, \beta \in \mathbf{BSML}_0$*, we have* $[\alpha \Rightarrow \beta]^+, [\alpha]^+ \vDash [\beta]^+$.

Proof. For $\rightarrow$ this follows directly from the definition. For $\twoheadrightarrow$, $\hookrightarrow_\forall$ and $\hookrightarrow_\exists$ it follows because $s \subseteq s$ and s must be maximal with $M, s \vDash [\alpha]^+$. Now suppose $M, s \vDash [\alpha \Rightarrow_{\neg,\vee} \beta]^+$, then we see that $M, s \vDash \neg[\alpha]^+ \vee [\beta]^+$. In other words, $\exists t, t' : t \cup t' = s$ and $M, t \vDash \neg[\alpha]^+$ and $M, t' \vDash [\beta]^+$. Now also suppose $M, s \vDash [\alpha]^+$, then by Fact 2.1.4 we see that $t = \emptyset$ and thus $t' = s$. Hence, we indeed see that $M, s \vDash [\beta]^+$. $\square$

Fact (4.2.6). *For* $\Rightarrow \in \{\twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$ *and* $\alpha, \beta \in \mathbf{BSML}_\emptyset$*, we have* $\alpha \Rightarrow \beta, \neg\beta \vDash \neg\alpha$.

Proof. Suppose $M, s \vDash \alpha \twoheadrightarrow \beta$, then for all $t \subseteq s$ we have that $M, t \vDash \alpha$ implies $M, t \vDash \beta$. Now suppose $M, s \vDash \neg\beta$, then by Fact 2.1.4 it must be that for all $w \in s : M, \{w\} \nvDash \beta$. For otherwise, it would follow that $\{w\} = \emptyset$, a contradiction. Hence, for all $w \in s$, we see that $M, \{w\} \nvDash \alpha$ and thus by Fact 2.1.5 it follows that $M, s \vDash \neg\alpha$. By Fact 2.2.5, Fact 2.2.6 and Fact 2.2.9 have the same result for $\Rightarrow_{\neg,\vee}, \hookrightarrow_\forall$ and $\hookrightarrow_\exists$. $\square$

Fact (4.2.7). *Let* $\Rightarrow \in \{\rightarrow, \twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$, $\alpha, \beta \in \mathbf{BSML}_0$ *and* $\gamma \in \mathbf{BSML}_\emptyset$. *Then we have* $\alpha \Rightarrow (\beta \Rightarrow \gamma) \equiv (\alpha \wedge \beta) \Rightarrow \gamma$.

Proof. $(\Rightarrow)$: Let us first shortly consider the material implication. Suppose $M, s \vDash \alpha \rightarrow (\beta \rightarrow \gamma)$ and suppose $M, s \vDash \alpha \wedge \beta$. Then by Modus Ponens applied twice we see that $M, s \vDash \gamma$. So indeed, $M, s \vDash (\alpha \wedge \beta) \rightarrow \gamma$.

Similarly, suppose that $M, s \vDash \alpha \twoheadrightarrow (\beta \twoheadrightarrow \gamma)$. Now suppose for some $t \subseteq s$ that $M, t \vDash \alpha \wedge \beta$, then by our assumption and Modus Ponens applied twice we see that $M, t \vDash \gamma$. Hence, $M, s \vDash (\alpha \wedge \beta) \twoheadrightarrow \gamma$.

$(\Leftarrow)$: Again, we first look at the material implication. Suppose, $M, s \vDash (\alpha \wedge \beta) \rightarrow \gamma$. Now suppose $M, s \vDash \alpha$ and also suppose $M, s \vDash \beta$, then it follows that $M, s \vDash \gamma$. So $M, s \vDash \beta \rightarrow \gamma$ and thus $M, s \vDash \alpha \rightarrow (\beta \rightarrow \gamma)$.

Similarly, suppose $M, s \vDash (\alpha \wedge \beta) \twoheadrightarrow \gamma$, then for all $t \subseteq s$: $M, t \vDash \alpha \wedge \beta$ implies $M, t \vDash \gamma$. Now suppose for some $t \subseteq s$ that $M, t \vDash \alpha$, and suppose for some $t' \subseteq t$

that $M, t' \vDash \beta$. By downward closure of α we see that $M, t' \vDash \alpha \wedge \beta$, and thus it follows from our first assumption that $M, t' \vDash \gamma$. Hence, $M, t \vDash \beta \twoheadrightarrow \gamma$ and thus $M, s \vDash \alpha \twoheadrightarrow (\beta \twoheadrightarrow \gamma)$.

By Fact 2.2.5, Fact 2.2.6 and Fact 2.2.9 everything also follows for $\Rightarrow_{\neg,\vee}$, $\hookrightarrow_{\forall}$ and $\hookrightarrow_{\exists}$. $\square$

Fact (4.2.8). *Let* $\Rightarrow \in \{\rightarrow, \hookrightarrow_{\exists}, \Rightarrow_{\neg,\vee}\}$, $\alpha, \beta \in \mathbf{BSML}_0$ *and* $\gamma \in \mathbf{BSML}_{\emptyset}$. *Then we have* $[\alpha \Rightarrow (\beta \Rightarrow \gamma)]^+ \equiv [(\alpha \wedge \beta) \Rightarrow \gamma]^+$.

Proof. ($\Rightarrow$):

- $\rightarrow$: Suppose $M, s \vDash [\alpha \rightarrow (\beta \rightarrow \gamma)]^+$ and also suppose $M, s \vDash [\alpha \wedge \beta]^+$. Then it follows that $M, s \vDash [\alpha]^+$ and thus by our assumption $M, s \vDash [\beta \rightarrow \gamma]^+$. But from our second assumption we also see that $M, s \vDash [\beta]^+$ and thus indeed $M, s \vDash [\gamma]^+$. We conclude that $M, s \vDash [(\alpha \wedge \beta) \rightarrow \gamma]^+$.

- $\hookrightarrow_{\exists}$: Suppose $M, s \vDash [\alpha \hookrightarrow_{\exists} (\beta \hookrightarrow_{\exists} \gamma)]^+$, then $\exists t \subseteq s$ that is maximal with $M, t \vDash [\alpha]^+$ and $M, t \vDash [\beta \hookrightarrow_{\exists} \gamma]^+$. Hence, $\exists t' \subseteq t$ that is maximal in t with $M, t' \vDash [\beta]^+$ and $M, t' \vDash [\gamma]^+$. Now suppose for contradiction that t' is not maximal in s with $M, t' \vDash [\alpha \wedge \beta]^+$. Then there is t'' with $t' \subset t'' \subseteq s$ such that $M, t'' \vDash [\alpha \wedge \beta]^+$. In fact, it must be that $t'' \subseteq t$, for otherwise we see by union closure of **BSML** that $M, t \cup t'' \vDash [\alpha]^+$ which is in contradiction to the maximality of t with respect to $[\alpha]^+$. But this is in contradiction to the maximality of t' with respect to $[\beta]^+$, and thus we see that t' is maximal with $M, t' \vDash [\alpha \wedge \beta]^+$. Hence, $M, s \vDash [(\alpha \wedge \beta) \hookrightarrow_{\exists} \gamma]^+$.

- $\Rightarrow_{\neg,\vee}$: Suppose, $M, s \vDash [\alpha \Rightarrow_{\neg,\vee} (\beta \Rightarrow_{\neg,\vee} \gamma)]^+$. Then we see that $M, s \vDash \neg[\alpha]^+ \vee [\beta \Rightarrow_{\neg,\vee} \gamma]^+$, i.e., $\exists t, t' : t \cup t' = s$ and $M, t \vDash \neg[\alpha]^+$ and $M, t' \vDash [\beta \Rightarrow_{\neg,\vee} \gamma]^+$. So $M, t' \vDash \neg[\beta]^+ \vee [\gamma]^+$, and thus there are $t'', t''' : t'' \cup t''' = t'$ such that $M, t'' \vDash \neg[\beta]^+$ and $M, t''' \vDash [\gamma]^+$. Now $M, t \cup t'' \vDash \neg[\alpha \wedge \beta]^+$ as $M, t \vDash \neg[\alpha]^+$, $M, t'' \vDash \neg[\beta]^+$ and $t \neq \emptyset \neq t''$.[149] And as $(t \cup t'') \cup t''' = s$, we see that it follows that $M, s \vDash [(\alpha \wedge \beta) \Rightarrow_{\neg,\vee} \gamma]^+$.

($\Leftarrow$):

- $\rightarrow$: Suppose $M, s \vDash [(\alpha \wedge \beta) \rightarrow \gamma]^+$, then $s \neq \emptyset$. Now suppose $M, s \vDash [\alpha]^+$ and also suppose $M, s \vDash [\beta]^+$. Then we see that $M, s \vDash [\alpha \wedge \beta]^+$ by the non-emptiness of s, and thus by our assumption $M, s \vDash [\gamma]^+$. Hence, $M, s \vDash [\beta \rightarrow \gamma]^+$ and thus $M, s \vDash [\alpha \rightarrow (\beta \rightarrow \gamma)]^+$.

- $\hookrightarrow_{\exists}$: Suppose $M, s \vDash [(\alpha \wedge \beta) \hookrightarrow_{\exists} \gamma]^+$. Then $\exists t \subseteq s$ with $M, t \vDash [\alpha \wedge \beta]^+$ and $M, t \vDash [\gamma]^+$. Without loss of generality, let $t'' \supseteq t$ be maximal with $M, t'' \vDash [\alpha]^+$. Now suppose for contradiction that t is not maximal in t'' with $M, t \vDash [\beta]^+$. Then there is some t' with $t \subset t' \subseteq t''$ such that $M, t' \vDash [\beta]^+$. As $M, t'' \vDash [\alpha]^+$ we see by Fact 2.1.1 that $M, t' \vDash \alpha$, and thus by downward closure $M, t' \vDash \alpha$. But then by Fact 2.1.8 it follows that $M, t' \vDash [\alpha]^+ \wedge [\beta]^+$, i.e., $M, t' \vDash [\alpha \wedge \beta]^+$. But this is in contradiction to the

[149] Here we use the fact that $\neg[\alpha \wedge \beta]^+ = \neg(([\alpha]^+ \wedge [\beta]^+) \wedge \text{NE}) \equiv \neg([\alpha]^+ \wedge [\beta]^+) \equiv \neg[\alpha]^+ \vee \neg[\beta]^+$.

maximality of t, and thus it must be that t is maximal with $M, t \vDash [\beta]^+$. Hence, $M, t'' \vDash [\beta \hookrightarrow_\exists \gamma]^+$ and thus $M, s \vDash [\alpha \hookrightarrow_\exists (\beta \hookrightarrow_\exists \gamma)]^+$.

- $\Rightarrow_{\neg,\vee}$: Suppose $M, s \vDash [(\alpha \wedge \beta) \Rightarrow_{\neg,\vee} \gamma]^+$, then $M, s \vDash \neg[\alpha \wedge \beta]^+ \vee [\gamma]^+$. In other words, there are $t, t' : t \cup t' = s$ and $M, t \vDash \neg[\alpha \wedge \beta]^+$ and $M, t' \vDash [\gamma]^+$. Then there are $t'', t''' : t'' \cup t''' = t$ such that $M, t'' \vDash \neg[\alpha]^+$ and $M, t''' \vDash \neg[\beta]^+$. So we see that $M, t''' \cup t' \vDash [\beta \Rightarrow_{\neg,\vee} \gamma]^+$ and thus as $t'' \cup (t''' \cup t') = s$ it follows that $M, s \vDash [\alpha \Rightarrow_{\neg,\vee} (\beta \Rightarrow_{\neg,\vee} \gamma]^+$. $\square$

Fact (4.2.11). *For* $\Rightarrow \in \{\twoheadrightarrow, \hookrightarrow_\forall, \hookrightarrow_\exists, \Rightarrow_{\neg,\vee}\}$*, we have* $[p \Rightarrow \Box q]^+, [\neg\Box q]^+ \vDash [\neg p]^+$.

Proof. Suppose $M, s \vDash [p \twoheadrightarrow \Box q]^+$ and $M, s \vDash [\neg\Box q]^+$, then $s \neq \emptyset$ and $M, s \vDash \neg[\Box q]^+$. Now suppose towards contradiction that $M, s \nvDash [\neg p]^+$. As $s \neq \emptyset$, it must be that $M, s \nvDash \neg p$. So then there must be a non-empty $t \subseteq s$ such that $M, t \vDash p$, and thus $M, t \vDash [p]^+$. But then it follows from our assumption that $M, t \vDash [\Box q]^+$, so by Fact 2.1.4 we see that $t = \emptyset$. A contradiction, so it must be that $M, s \vDash [\neg p]^+$. By Fact 2.2.7 the same holds for $\hookrightarrow_\forall$.

Suppose $M, s \vDash [p \hookrightarrow_\exists \Box q]^+$, then $\exists t \subseteq s$ with $M, t \vDash [p]^+$ and $M, t \vDash [\Box q]^+$. Hence, $t \neq \emptyset$ and by Fact 2.1.1 we see that $M, t \vDash \Box q$. Now also suppose that $M, s \vDash [\neg\Box q]^+$, then again from Fact 2.1.1 it follows that $M, s \vDash \neg\Box q$. But then by Fact 2.1.4 we see that $t = \emptyset$, a contradiction. So this cannot happen, and anything follows.

Suppose $M, s \vDash [p \Rightarrow_{\neg,\vee} \Box q]^+$, then we see that $s \neq \emptyset$ and $M, s \vDash \neg[p]^+ \vee [\Box q]^+$. In other words, $\exists t, t' : t \cup t' = s$ and $M, t \vDash \neg[p]^+$ and $M, t' \vDash [\Box q]^+$. Now also suppose that $M, s \vDash [\neg\Box q]^+$, then by Fact 2.1.1 we see that $M, s \vDash \neg\Box q$ and $M, t' \vDash \Box q$. So by Fact 2.1.4 it follows that $t' = \emptyset$ and thus $t = s$. Therefore, $M, s \vDash \neg[p]^+$ and as $s \neq \emptyset$ it follows that $M, s \vDash [\neg p]^+$.[150] $\square$

Fact (4.2.14). $[p \vee q]^+, [\Diamond\neg p]^+ \vDash [\neg p \hookrightarrow_\exists q]^+$ *(if R is state-based)*

Proof. Suppose $M, s \vDash [p \vee q]^+$, then $s \neq \emptyset$ and there are non-empty $t, t' : t \cup t' = s$ and $M, t \vDash p$ and $M, t' \vDash q$. Now suppose $M, s \vDash [\Diamond\neg p]^+$. As R is state-based, we see that there is a $u \subseteq s$ with $M, u \vDash [\neg p]^+$ and without loss of generality we can say that u is maximally so. So then clearly $u \neq \emptyset$ and $u \subseteq t'$ and thus by downward closure of q we find that $M, u \vDash [q]^+$. Subsequently, we conclude that $M, s \vDash [\neg p \hookrightarrow_\exists q]^+$. $\square$

Fact (4.2.15). $\Diamond(p \wedge q) \vDash \Diamond p$ *and* $[\Diamond(p \wedge q)]^+ \vDash [\Diamond p]^+$

Proof. Let us start with the non-enriched version, and suppose $M, s \vDash \Diamond(p \wedge q)$. Then we know that for all $w \in s$, there is a non-empty $t \subseteq R[w]$ with $M, t \vDash p \wedge q$. So then it directly follows that $M, s \vDash \Diamond p$.

Similarly, suppose $M, s \vDash [\Diamond(p \wedge q)]^+$, then $s \neq \emptyset$ and by Fact 2.1.1 we see that $M, s \vDash \Diamond(p \wedge q)$. And thus for all $w \in s$, there is a non-empty $t \subseteq R[w]$ with $M, t \vDash p$ by the previous argument. But then we see that $M, t \vDash [p]^+$ as it is non-empty. Hence, $M, s \vDash (\Diamond[p]^+) \wedge \textsc{ne}$ and thus $M, s \vDash [\Diamond p]^+$. $\square$

[150]Note that we use $\neg[\alpha]^+ \wedge \textsc{ne} = [\neg\alpha]^+$, for NE-free α.

Fact (4.2.18). $[p \hookrightarrow_{\exists} \Diamond q]^+ \vDash [\Diamond p]^+$ *(if R is state-based)*

Proof. Suppose $M, s \vDash [p \hookrightarrow_{\exists} \Diamond q]^+$, then $s \neq \emptyset$ and there is a non-empty $t \subseteq s$ with $M, t \vDash [p]^+$. As R is state-based, it follows that for each $w \in s$, there is a non-empty $t \subseteq R[w]$ with $M, t \vDash [p]^+$. Hence, $M, s \vDash (\Diamond [p]^+) \wedge \text{NE}$ and thus $M, s \vDash [\Diamond p]^+$. $\square$

D Proofs of Chapter 6

Fact (Update Clauses for Pragmatically Enriched formulas). *Let $\alpha, \beta \in$ **BiUS**.*

$$s[|p|^+] = s[p] \textit{ if } s[p] \neq \emptyset; \ \# \textit{ otherwise}$$
$$s[|p|^+]^r = s[p]^r$$
$$s[|\neg\alpha|^+] = s[|\alpha|^+]^r \textit{ if } s[|\alpha|^+]^r \neq \emptyset; \ \# \textit{ otherwise}$$
$$s[|\neg\alpha|^+]^r = s[|\alpha|^+]$$
$$s[|\alpha \wedge \beta|^+] = s[|\alpha|^+] \cap s[|\beta|^+] \textit{ if } s[|\alpha|^+] \neq \emptyset \neq s[|\beta|^+]; \ \# \textit{ otherwise}$$
$$s[|\alpha \wedge \beta|^+]^r = s[|\alpha|^+]^r \cup s[|\beta|^+]^r$$
$$s[|\alpha \vee \beta|^+] = s[|\alpha|^+] \cup s[|\beta|^+] \textit{ if } s[|\alpha|^+] \neq \emptyset \neq s[|\beta|^+]; \ \# \textit{ otherwise}$$
$$s[|\alpha \vee \beta|^+]^r = s[|\alpha|^+]^r \cap s[|\beta|^+]^r$$
$$s[|\Diamond\alpha|^+] = s \textit{ if } s[|\alpha|^+] \neq \emptyset; \ \# \textit{ otherwise}$$
$$s[|\Diamond\alpha|^+]^r = s \textit{ if } s[|\alpha|^+]^r = s; \ \emptyset \textit{ if } s[|\alpha|^+]^r \neq s; \ \# \textit{ otherwise}$$
$$s[|\Box\alpha|^+] = s \textit{ if } s \neq \emptyset \textit{ and } s[|\alpha|^+] = s; \ \# \textit{ otherwise}$$
$$s[|\Box\alpha|^+]^r = s \textit{ if } s[|\alpha|^+]^r \neq \emptyset; \ \# \textit{ otherwise}$$
$$s[|\alpha \dashrightarrow \beta|^+] = s \textit{ if } s \neq \emptyset \textit{ and } s[|\alpha|^+][|\beta|^+] = s[|\alpha|^+]; \ \# \textit{ otherwise}$$
$$s[|\alpha \dashrightarrow \beta|^+]^r = s \textit{ if } s[|\alpha|^+][|\beta|^+] \neq \emptyset; \ \# \textit{ otherwise}$$

D.1 Proofs of Section 6.2

Fact (6.2.1 (Eliminativity)**).** *Let $\varphi \in$ **BiUS**, and s be any state.*

i) If $s[\varphi]$ is defined, then $s[\varphi] \subseteq s$.

ii) If $s[\varphi]^r$ is defined, then $s[\varphi]^r \subseteq s$.

Proof. By double induction on the complexity of φ.

1. $\varphi = p$:
   - $s[p] = s \cap \{w \in W \mid V(p, w) = 1\} \subseteq s$.
   - $s[p]^r = s \cap \{w \in W \mid V(p, w) = 0\} \subseteq s$.

2. $\varphi = \psi \wedge \xi$:

- If defined, $s[\psi \wedge \xi] = s[\psi] \cap s[\xi]$, and $s[\psi]$ and $s[\xi]$ must be defined. So by the induction hypothesis both are subsets of s, hence $s[\psi] \cap s[\xi] \subseteq s$.
- If defined, $s[\psi \wedge \xi]^r = s[\psi]^r \cup s[\xi]^r$, and $s[\psi]^r$ and $s[\xi]^r$ must be defined. So by the induction hypothesis both are subsets of s, hence $s[\psi]^r \cup s[\xi]^r \subseteq s$.

3. $\varphi = \psi \vee \xi$: This case is dual to the $\wedge$-case.
4. $\varphi = \Diamond\psi$: If defined, $s[\Diamond\psi] = s$ or $s[\Diamond\psi] = \emptyset$. In both cases it is a subset of s. Analogously for $s[\Diamond\psi]^r$.
5. $\varphi = \neg\psi$:
   - If defined, $s[\neg\psi] = s[\psi]^r \overset{\text{IH}}{\subseteq} s$.
   - If defined, $s[\neg\psi]^r = s[\psi] \overset{\text{IH}}{\subseteq} s$.
6. $\varphi = \psi^{\text{NE}}$: If defined, $s[\psi^{\text{NE}}] = s[\psi] \overset{\text{IH}}{\subseteq} s$. Similarly for $s[\psi^{\text{NE}}]^r$.

□

Fact (**6.2.2** (Distributivity))**.** *Let $\alpha \in$ **BiUS** be* NE*-free and $\Diamond$-free, and s be any state:*

i) $s[\alpha] = \{w \in s \mid \{w\}[\alpha] = \{w\}\}$

ii) $s[\alpha]^r = \{w \in s \mid \{w\}[\alpha]^r = \{w\}\}$

Proof. By double induction on the complexity of α.

1. $\alpha = p$:
   - Note that $\{w\}[p] = \{w\}$ if and only if $V(p, w) = 1$. So $s[p] = s \cap \{w \in W \mid V(p, w) = 1\} = \{w \in s \mid \{w\}[p] = \{w\}\}$.
   - Analogous for $s[p]^r$.
2. $\alpha = \gamma \wedge \delta$:

$$\begin{aligned} s[\gamma \wedge \delta] =& s[\gamma] \cap s[\delta] \\ \overset{\text{IH}}{=}& \big\{w \in s \mid \{w\}[\gamma] = \{w\}\big\} \cap \big\{w \in s \mid \{w\}[\delta] = \{w\}\big\} \\ =& \big\{w \in s \mid \{w\}[\gamma] = \{w\} \text{ and } \{w\}[\delta] = \{w\}\big\} \\ =& \big\{w \in s \mid \{w\}[\gamma] \cap \{w\}[\delta] = \{w\}\big\} \\ =& \big\{w \in s \mid \{w\}[\gamma \wedge \delta] = \{w\}\big\} \end{aligned}$$

$$\begin{aligned} s[\gamma \wedge \delta]^r =& s[\gamma]^r \cup s[\delta]^r \\ \overset{\text{IH}}{=}& \big\{w \in s \mid \{w\}[\gamma]^r = \{w\}\big\} \cup \big\{w \in s \mid \{w\}[\delta]^r = \{w\}\big\} \\ =& \big\{w \in s \mid \{w\}[\gamma]^r = \{w\} \text{ or } \{w\}[\delta]^r = \{w\}\big\} \\ =& \big\{w \in s \mid \{w\}[\gamma]^r \cup \{w\}[\delta]^r = \{w\}\big\} \\ =& \big\{w \in s \mid \{w\}[\gamma \wedge \delta]^r = \{w\}\big\} \end{aligned}$$

3. $\alpha = \gamma \vee \delta$: This case is dual to the $\wedge$-case.
4. $\alpha = \neg\gamma$:

   - $s[\neg\gamma] = s[\gamma]^r \stackrel{\text{IH}}{=} \{w \in s \mid \{w\}[\gamma]^r = \{w\}\} = \{w \in s \mid \{w\}[\neg\gamma] = \{w\}\}$
   - $s[\neg\gamma]^r = s[\gamma] \stackrel{\text{IH}}{=} \{w \in s \mid \{w\}[\gamma] = \{w\}\} = \{w \in s \mid \{w\}[\neg\gamma]^r = \{w\}\}$

$\square$

Fact (6.2.3). *For $\alpha, \beta \in \mathbf{BiUS}_0$:*

i) $s[\alpha][\beta] = s[\beta][\alpha]$

ii) $s[\alpha][\beta]^r = s[\beta]^r[\alpha]$

iii) $s[\alpha]^r[\beta]^r = s[\beta]^r[\alpha]^r$

Proof. We will only prove *i)*, for *ii)* and *iii)* are completely analogous. From distributivity (Fact 6.2.2) it follows that,

$$\begin{aligned} s[\alpha][\beta] &= \{w \in s[\alpha] \mid \{w\}[\beta] = \{w\}\} \\ &= \{w \in s \mid \{w\}[\alpha] = \{w\} \text{ and } \{w\}[\beta] = \{w\}\} \\ &= \{w \in s[\beta] \mid \{w\}[\alpha] = \{w\}\} \\ &= s[\beta][\alpha] \end{aligned}$$

$\square$

Fact (6.2.5). *Let $\alpha \in \mathbf{BiUS}_0$. Then for any $w \in W$ exactly one of the following holds: i) $\{w\}[\alpha] = \{w\}$ or ii) $\{w\}[\alpha]^r = \{w\}$.*

Proof. By induction on the complexity of α.

1. $\alpha = p$: Either $V(p, w) = 1$ or $V(p, w) = 0$. In the former case we have $\{w\}[p] = \{w\}$ and in the latter $\{w\}[p]^r = \{w\}$.
2. $\alpha = \gamma \wedge \delta$: Then there are four possibilities arising from the induction hypothesis.

   - $\{w\}[\gamma] = \{w\} = \{w\}[\delta]$: Then $\{w\}[\gamma \wedge \delta] = \{w\}[\gamma] \cap \{w\}[\delta] \stackrel{\text{IH}}{=} \{w\} \cap \{w\} = \{w\}$. And by the induction hypothesis $\{w\}[\gamma]^r = \emptyset = \{w\}[\delta]^r$ and thus $\{w\}[\gamma \wedge \delta]^r \neq \{w\}$.
   - $\{w\}[\gamma]^r = \{w\} = \{w\}[\delta]$: Then $\{w\}[\gamma \wedge \delta]^r = \{w\}[\gamma]^r \cup \{w\}[\delta]^r \stackrel{\text{IH}}{=} \{w\} \cup \emptyset = \{w\}$. And by the induction hypothesis we see that $\{w\}[\gamma] = \emptyset$ and thus $\{w\}[\gamma \wedge \delta\} \neq \{w\}$.
   - $\{w\}[\gamma] = \{w\} = \{w\}[\delta]^r$: Analogous to the previous case.
   - $\{w\}[\gamma]^r = \{w\} = \{w\}[\delta]^r$: Analogous to the first case.

3. $\alpha = \gamma \vee \delta$: Analogous to the $\wedge$-case.
4. $\alpha = \neg\gamma$: Then there are two possibilities arising from the induction hypothesis.

   - $\{w\}[\gamma] = \{w\}$: Then $\{w\}[\neg\gamma]^r = \{w\}[\gamma] \stackrel{\text{IH}}{=} \{w\}$. And as $\{w\}[\gamma]^r = \emptyset$ by the induction hypothesis, we see that $\{w\}[\neg\gamma] = \{w\}[\gamma]^r \neq \{w\}$.

- $\{w\}[\gamma]^r = \{w\}$: Then $\{w\}[\neg\gamma] = \{w\}[\gamma]^r \overset{\text{IH}}{=} \{w\}$. And as $\{w\}[\gamma] = \emptyset$ by the induction hypothesis, we see that $\{w\}[\neg\gamma]^r = \{w\}[\gamma] \neq \{w\}$.

$\square$

Fact (6.2.6). *For $\alpha \in \mathbf{BiUS}_0$, and any state s, we have $s[\alpha][\alpha]^r = \emptyset$.*

Proof. By induction on the complexity of α.

1. $\alpha = p$: We see that $s[p][p]^r = s \cap \{w \in W \mid V(p,w) = 1\} \cap \{w \in W \mid V(p,w) = 0\} = \emptyset$, as we never have $V(p,w) = 1$ and $V(p,w) = 0$ at the same time.
2. $\alpha = \gamma \wedge \beta$: Using the fact that order does not matter (Fact 6.2.3) multiple times, we see the following:

$$\begin{aligned} s[\gamma \wedge \delta][\gamma \wedge \delta]^r &= s[\gamma \wedge \delta][\gamma]^r \cup s[\gamma \wedge \delta][\delta]^r \\ &\overset{Fact}{=} s[\gamma]^r[\gamma \wedge \delta] \cup s[\delta]^r[\gamma \wedge \delta] \\ &= (s[\gamma]^r[\gamma] \cap s[\gamma]^r[\delta]) \cup (s[\delta]^r[\gamma] \cap s[\delta]^r[\delta]) \\ &\overset{Fact}{=} (s[\gamma][\gamma]^r \cap s[\gamma]^r[\delta]) \cup (s[\delta]^r[\gamma] \cap s[\delta][\delta]^r) \\ &\overset{\text{IH}}{=} (\emptyset \cap s[\gamma]^r[\delta]) \cup (s[\delta]^r[\gamma] \cap \emptyset) \\ &= \emptyset \end{aligned}$$

3. $\alpha = \gamma \vee \delta$: Again, we use Fact 6.2.3 to see the following:

$$\begin{aligned} s[\gamma \vee \delta][\gamma \vee \delta]^r &= s[\gamma \vee \delta][\gamma]^r \cap s[\gamma \vee \delta][\delta]^r \\ &\overset{Fact}{=} s[\gamma]^r[\gamma \vee \delta] \cap s[\delta]^r[\gamma \vee \delta] \\ &= (s[\gamma]^r[\gamma] \cup s[\gamma]^r[\delta]) \cap (s[\delta]^r[\gamma] \cup s[\delta]^r[\delta]) \\ &\overset{Fact}{=} (s[\gamma][\gamma]^r \cup s[\gamma]^r[\delta]) \cap (s[\delta]^r[\gamma] \cup s[\delta][\delta]^r) \\ &\overset{\text{IH}}{=} (\emptyset \cup s[\gamma]^r[\delta]) \cap (s[\delta]^r[\gamma] \cup \emptyset) \\ &= s[\gamma]^r[\delta] \cap s[\delta]^r[\gamma] \end{aligned}$$

   And by distributivity (Fact 6.2.2) we see that this is equal to the set $\big\{w \in s \mid \{w\}[\gamma]^r = \{w\}$ and $\ldots$ and $\{w\}[\gamma] = \{w\}\big\}$, but by the law of excluded middle (Fact 6.2.5) we see that this is the empty set.
4. $\alpha = \neg\gamma$: As order does not matter (Fact 6.2.3), we see the following:

$$\begin{aligned} s[\neg\gamma][\neg\gamma]^r &= s[\gamma]^r[\gamma] \\ &\overset{Fact}{=} s[\gamma][\gamma]^r \\ &\overset{\text{IH}}{=} \emptyset \end{aligned}$$

$\square$

Fact (**6.2.7**)**.** *For* $\alpha \in \mathbf{BiUS}_\emptyset$*, and any state* s*:*

i) If $s[|\alpha|^+]$ *is defined, then* $s[|\alpha|^+] = s[\alpha]$*.*

ii) If $s[|\alpha|^+]^r$ *is defined, then* $s[|\alpha|^+]^r = s[\alpha]^r$*.*

Proof. By double induction on the complexity of α. The overview of update clauses for pragmatically enriched formulas at the beginning of this appendix can be helpful for this proof.

1. $\alpha = p$
   - If defined, $s[|p|^+] = s[p]$.
   - If defined, then $s[|p|^+]^r = s[p]^r$.
2. $\alpha = \gamma \wedge \delta$:
   - If defined, $s[|\gamma \wedge \delta|^+] = s[|\gamma|^+] \cap s[|\delta|^+] \overset{\text{IH}}{=} s[\gamma] \cap s[\delta] = s[\gamma \wedge \delta]$.
   - If defined, $s[|\gamma \wedge \delta|^+]^r = s[|\gamma|^+]^r \cup s[|\delta|^+]^r \overset{\text{IH}}{=} s[\gamma]^r \cup s[\delta]^r = s[\gamma \wedge \delta]^r$.
3. $\alpha = \gamma \vee \delta$: Analogous to the $\wedge$-case.
4. $\alpha = \Diamond\gamma$:
   - If defined, $s[|\Diamond\gamma|^+] = s$ and $s[|\gamma|^+] \neq \emptyset$. So then by the induction hypothesis, $s[\gamma] \neq \emptyset$ and thus $s[\Diamond\gamma] = s$ as well.
   - If defined, there are two cases for the rejection clause.
     - $s[|\gamma|^+]^r = s$: Then we see that $s[|\Diamond\gamma|^+]^r = s$ and by the induction hypothesis $s[\gamma]^r = s$. Hence, we also have $s[\Diamond\gamma]^r = s$.
     - $s[|\gamma|^+]^r \neq s$: Then $s[|\Diamond\gamma|^+]^r = \emptyset$. By the induction hypothesis we see that $s[\gamma]^r \neq s$ and thus also $s[\Diamond\gamma]^r = \emptyset$.
5. $\alpha = \neg\gamma$:
   - If defined, $s[|\neg\gamma|^+] = s[|\gamma|^+]^r \overset{\text{IH}}{=} s[\gamma]^r = s[\neg\gamma]$.
   - If defined, $s[|\neg\gamma|^+]^r = s[|\gamma|^+] \overset{\text{IH}}{=} s[\gamma] = s[\neg\gamma]^r$.

□

Fact (**6.2.8**)**.** *For* $\alpha \in \mathbf{BiUS}_0$ *and any state* s*,* $s[\alpha] = s \cap W[\alpha]$*.*

Proof. By distributivity (Fact 6.2.2) we see the following.

$$\begin{aligned} s[\alpha] &= \big\{w \in s \mid \{w\}[\alpha] = \{w\}\big\} \\ &= s \cap \big\{w \in W \mid \{w\}[\alpha] = \{w\}\big\} \\ &= s \cap W[\alpha] \end{aligned}$$

□

Fact (**6.2.9** (Idempotency))**.** *For* $\alpha \in \mathbf{US}_{BiUS}$ *and any state* s*,* $s[\alpha] = s[\alpha][\alpha]$*.*

Proof. Note that if $\alpha \in \mathbf{US}_{BiUS}$, then either α is also in $\mathbf{BiUS}_0$ or $\alpha = \Diamond\beta$ for $\beta \in \mathbf{BiUS}_0$. In the former case we see by Fact 6.2.8 that $s[\alpha] = s \cap W[\alpha] = s \cap W[\alpha] \cap W[\alpha] = s[\alpha] \cap W[\alpha] = s[\alpha][\alpha]$.

If $\alpha = \Diamond\beta$ for $\beta \in \mathbf{BiUS}_0$, then $s[\Diamond\beta] = s$ or $s[\Diamond\beta] = \emptyset$. In the former case we directly see that $s[\Diamond\beta][\Diamond\beta] = s[\Diamond\beta]$ and in the latter it follows from eliminativity that $s[\Diamond\beta][\Diamond\beta] = \emptyset[\Diamond\beta] = \emptyset = s[\Diamond\beta]$. □

Fact (6.2.10 (Monotonicity)**).** *For $\alpha \in \mathbf{US}_{BiUS}$, and any state s and t, $t \subseteq s$ implies $t[\alpha] \subseteq s[\alpha]$.*

Proof. Again we see that $\alpha \in \mathbf{BiUS}_0$ or is the form $\alpha = \Diamond\beta$ for $\beta \in \mathbf{BiUS}_0$. In the former case we see by Fact 6.2.8 that $t[\alpha] = t \cap W[\alpha] \subseteq s \cap W[\alpha] = s[\alpha]$.

Now suppose $\alpha = \Diamond\beta$ for $\beta \in \mathbf{BiUS}_0$, then we know that $t[\Diamond\beta]$ and $s[\Diamond\beta]$ are defined. Either $t[\beta] \neq \emptyset$ or $t[\beta] = \emptyset$, in the latter of which it directly follows that $t[\Diamond\beta] = \emptyset \subseteq s[\Diamond\beta]$. And if $t[\beta] \neq \emptyset$ then it follows from the argument above that $s[\beta] \neq \emptyset$. Hence, $t[\Diamond\beta] = t \subseteq s = s[\Diamond\beta]$. □

D.2 Proofs of Section 6.3

Fact (6.3.6). *Let $\alpha, \beta \in \mathbf{BiUS}_0$. Then $\alpha \dashrightarrow \beta, \alpha \vDash \beta$ and $|\alpha \dashrightarrow \beta|^+, |\alpha|^+ \vDash |\beta|^+$.*

Proof. We start with the non-enriched version, so suppose $s[\alpha \dashrightarrow \beta][\alpha]$ is defined. There are two cases.

- $s[\alpha][\beta] = s[\alpha]$: Then we see that $s[\alpha \dashrightarrow \beta][\alpha] = s[\alpha] = s[\alpha][\beta] = s[\alpha \dashrightarrow \beta][\alpha][\beta]$ by our assumption and the definition of $\dashrightarrow$. Hence, $s[\alpha \dashrightarrow \beta][\alpha] \vDash \beta$.
- $s[\alpha][\beta] \neq s[\alpha]$: Then $s[\alpha \dashrightarrow \beta] = \emptyset$ and by eliminativity we see that $s[\alpha \dashrightarrow \beta][\alpha][\beta] = \emptyset = s[\alpha \dashrightarrow \beta][\alpha]$. So again $s[\alpha \dashrightarrow \beta][\alpha] \vDash \beta$.

Now let us consider the pragmatically enriched version, and suppose $s[|\alpha \dashrightarrow \beta|^+][|\alpha|^+]$ is defined. Then it must be that $s \neq \emptyset$ and $s[|\alpha|^+][|\beta|^+] = s[|\alpha|^+]$, and thus $s[|\alpha \dashrightarrow \beta|^+] = s$. Hence, $s[|\alpha \dashrightarrow \beta|^+][|\alpha|^+][|\beta|^+] = s[|\alpha|^+][|\beta|^+] = s[|\alpha|^+] = s[|\alpha \dashrightarrow \beta|^+][|\alpha|^+]$. So we see that indeed $s[|\alpha \dashrightarrow \beta|^+][|\alpha|]^+ \vDash |\beta|^+$. □

Fact (6.3.7). *For $\alpha, \beta \in \mathbf{BiUS}_0$, we have $\alpha \dashrightarrow \beta, \neg\beta \vDash \neg\alpha$.*

Proof. Suppose $s[\alpha \dashrightarrow \beta][\neg\beta]$ is defined. There are two cases.

- $s[\alpha][\beta] = s[\alpha]$: Then by distributivity (Fact 6.2.2) we know that $\forall w \in s : \{w\}[\alpha] = \{w\}$ implies $\{w\}[\beta] = \{w\}$. Furthermore, $s[\alpha \dashrightarrow \beta] = s$ and thus $s[\alpha \dashrightarrow \beta][\neg\beta][\neg\alpha] = s[\neg\beta][\neg\alpha]$. Now by eliminativity we see that $s[\neg\beta][\neg\alpha] \subseteq s[\neg\beta]$. For the other direction, let $w \in s[\neg\beta]$ and thus by distributivity $\{w\}[\beta]^r = \{w\}$. Then by the law of excluded middle (Fact 6.2.5) it follows that $\{w\}[\beta] \neq \{w\}$ and thus from our assumption we see that $\{w\}[\alpha] \neq \{w\}$. So by the same fact it follows that $\{w\}[\alpha]^r = \{w\}$ and thus by distributivity $w \in s[\beta]^r[\alpha]^r = s[\neg\beta][\neg\alpha]$. Hence, $s[\neg\beta] \subseteq s[\neg\beta][\neg\alpha]$ and thus $s[\neg\beta][\neg\alpha] = s[\neg\beta]$. We conclude that $s[\alpha \dashrightarrow \beta][\neg\beta] \vDash \neg\alpha$.

- $s[\alpha][\beta] \neq s[\alpha]$: Then we see that $s[\alpha \dashrightarrow \beta] = \emptyset$ and by eliminativity $s[\alpha \dashrightarrow \beta][\neg\beta][\neg\alpha] = \emptyset = s[\alpha \dashrightarrow \beta][\neg\beta]$. Hence, we again see that $s[\alpha \dashrightarrow \beta][\neg\beta] \vDash \neg\alpha$.

$\square$

Fact (6.3.8). *Let* $\alpha, \beta \in \mathbf{BiUS}_0$ *and* $\gamma \in \mathbf{BiUS}_\emptyset$*, then we have* $\alpha \dashrightarrow (\beta \dashrightarrow \gamma) \equiv (\alpha \wedge \beta) \dashrightarrow \gamma$ *and* $|\alpha \dashrightarrow (\beta \dashrightarrow \gamma)|^+ \equiv |(\alpha \wedge \beta) \dashrightarrow \gamma|^+$.

Proof. Let us first look at the non-enriched version.
($\Rightarrow$): Suppose $s[\alpha \dashrightarrow (\beta \dashrightarrow \gamma)]$ is defined. Then there are two cases.

- $s[\alpha][\beta][\gamma] = s[\alpha][\beta]$: Then we see that $s[\alpha][(\beta \dashrightarrow \gamma)] = s[\alpha]$ and thus $s[\alpha \dashrightarrow (\beta \dashrightarrow \gamma)] = s$. By distributivity (Fact 6.2.2) we know that $s[\alpha][\beta] = s[\alpha \wedge \beta]$ and thus in this case we see that $s[\alpha \wedge \beta][\gamma] = s[\alpha \wedge \beta]$, i.e., $s[\alpha \dashrightarrow (\beta \dashrightarrow \gamma)] \vDash (\alpha \wedge \beta) \dashrightarrow \gamma$.
- $s[\alpha][\beta][\gamma] \neq s[\alpha][\beta]$: Then we see that $s[\alpha][\beta \dashrightarrow \gamma] = \emptyset$. Again there are two cases:
  - $s[\alpha] = \emptyset$: It must be that $s[\alpha][\beta][\gamma]$ is defined, so by eliminativity we see that actually $s[\alpha][\beta][\gamma] = \emptyset = s[\alpha][\beta]$. A contradiction, so this cannot happen.
  - $s[\alpha] \neq \emptyset$: By our assumption $s[\alpha \dashrightarrow (\beta \dashrightarrow \gamma)] = \emptyset$. And by eliminativity, $\emptyset[\alpha \wedge \beta] = \emptyset = \emptyset[\alpha \wedge \beta][\gamma]$. So then we also conclude that $s[\alpha \dashrightarrow (\beta \dashrightarrow \gamma)] \vDash (\alpha \wedge \beta) \dashrightarrow \gamma$.

($\Leftarrow$): Suppose $s[(\alpha \wedge \beta) \dashrightarrow \gamma]$ is defined. Then there are two cases.

- $s[\alpha \wedge \beta][\gamma] = s[\alpha \wedge \beta]$: Then we see that $s[(\alpha \wedge \beta) \dashrightarrow \gamma] = s$. And by distributivity we see that $s[\alpha][\beta][\gamma] = s[\alpha][\beta]$, and thus $s[\alpha][\beta \dashrightarrow \gamma] = s[\alpha]$. Hence, $s[\alpha \dashrightarrow (\beta \dashrightarrow \gamma)] = s$ and we can conclude that $s[(\alpha \wedge \beta) \dashrightarrow \gamma] \vDash \alpha \dashrightarrow (\beta \dashrightarrow \gamma)$.
- $s[\alpha \wedge \beta][\gamma] \neq s[\alpha \wedge \beta]$: Then $s[(\alpha \wedge \beta) \dashrightarrow \gamma] = \emptyset$. And thus by eliminativity it follows that $\emptyset[\alpha][\beta][\gamma] = \emptyset[\alpha][\beta] = \emptyset[\alpha] = \emptyset$ and thus $s[(\alpha \wedge \beta) \dashrightarrow \gamma] \vDash \alpha \dashrightarrow (\beta \dashrightarrow \gamma)$.

The enriched version is simpler.
($\Rightarrow$): Suppose $s[|\alpha \dashrightarrow (\beta \dashrightarrow \gamma)|^+]$ is defined and thus is equal to s. Then we see that $s[|\alpha|^+][|\beta|^+][|\gamma|^+] = s[|\alpha|^+][|\beta|^+]$. Because a defined update with pragmatic enrichment is the same as without pragmatic enrichment (Fact 6.2.7) and by distributivity (Fact 6.2.2) we see that $s[|\alpha|^+][|\beta|^+] = s[|\alpha|^+] \cap s[|\beta|^+] = s[|\alpha \wedge \beta|^+]$ as $s[|\alpha|^+] \neq \emptyset \neq s[|\beta|^+]$. So by our assumption, $s[|\alpha \wedge \beta|^+][|\gamma|^+] = s[|\alpha \wedge \beta|^+]$. Hence, $s[|\alpha \dashrightarrow (\beta \dashrightarrow \gamma)|^+] \vDash |(\alpha \wedge \beta) \dashrightarrow \gamma|^+$.
($\Leftarrow$): This is analogous to the previous case. $\square$

Fact (6.3.11). $p \vee q, \Diamond\neg p \vDash \neg p \dashrightarrow q$ *and* $|p \vee q|^+, |\Diamond\neg p|^+ \vDash |\neg p \dashrightarrow q|^+$

Proof. Let us start with the non-enriched version. Suppose $s[p \vee q][\Diamond\neg p]$ is defined, then there are two cases.

- $s[p \vee q][p]^r \neq \emptyset$: Then $s[p \vee q][\Diamond\neg p] = s[p \vee q]$, and by the direct argument (Fact 6.3.1) we see that $s[p \vee q] \vDash \neg p \dashrightarrow q$.
- $s[p \vee q][p]^r = \emptyset$: Then we see that $s[p \vee q][\Diamond\neg p] = \emptyset$, and thus by eliminativity also $s[p \vee q][\Diamond\neg p][\neg p \dashrightarrow q] = \emptyset$.

So in both cases we see that $s[p \vee q][\Diamond\neg p] \vDash \neg p \dashrightarrow q$.

Now consider the pragmatically enriched version and suppose $s[|p \vee q|^+][|\Diamond\neg p|^+]$ is defined, then it must be that $s[|p \vee q|^+] \neq \emptyset \neq s[|p \vee q|^+][|\neg p|^+]$. And as a defined update with pragmatic enrichment is the same as without pragmatic enrichment (Fact 6.2.7), we see that $s[|p \vee q|^+][|\neg p|^+] = s[p \vee q][\neg p]$. Now by distributivity and the law of excluded middle (Fact 6.2.5) it follows that $s[p \vee q][\neg p] = \{w \in s \mid \{w\}[p]^r = \{w\} \text{ and } \{w\}[q] = \{w\}\} = s[p \vee q][\neg p][q]$. And as this is non-empty we can finally conclude that $s[|p \vee q|^+][|\neg p|^+] = s[|p \vee q|^+][|\neg p|^+][|q|^+]$, i.e., $s[|p \vee q|^+][|\Diamond\neg p|^+] \vDash |\neg p \dashrightarrow q|^+$. $\square$

D.3 Proofs of Section 6.4

Fact (6.4.2). *Let $\alpha, \beta \in \mathbf{BiUS}_\emptyset(\dashrightarrow)$ and let Γ be a sequence of $\xi_1, \ldots, \xi_n$ such that $\xi_1, \ldots, \xi_n \in \mathbf{BiUS}_\emptyset(\dashrightarrow)$. Then $\Gamma, \alpha \vDash \beta$ if and only if $\Gamma \vDash \alpha \dashrightarrow \beta$.*

Proof. ($\Rightarrow$): Suppose $\Gamma, \alpha \vDash \beta$, and also suppose $s[\Gamma]$ is defined.[151] As α is NE-free, we see $s[\Gamma][\alpha]$ is always defined. And thus by our first assumption it follows that $s[\Gamma][\alpha][\beta] = s[\Gamma][\alpha]$. But this directly shows that $s[\Gamma] \vDash \alpha \dashrightarrow \beta$.

($\Leftarrow$): Suppose $\Gamma \vDash \alpha \dashrightarrow \beta$. Now also suppose that $s[\Gamma][\alpha]$ is defined, then it must be that $s[\Gamma]$ is defined. So by our assumption we see that $s[\Gamma][\alpha \dashrightarrow \beta] = s[\Gamma]$, i.e., $s[\Gamma][\alpha][\beta] = s[\Gamma][\alpha]$. In other words, $s[\Gamma][\alpha] \vDash \beta$. $\square$

Fact (Transitivity $\mathbf{BiUS}_0(\dashrightarrow)$)**.** *Let $\alpha, \beta, \gamma \in \mathbf{BiUS}_0$, then $\alpha \dashrightarrow \beta, \beta \dashrightarrow \gamma \vDash \alpha \dashrightarrow \gamma$.*

Proof. Suppose $s[\alpha \dashrightarrow \beta][\beta \dashrightarrow \gamma]$ is defined. Then there are two cases:

- $s[\alpha][\beta] = s[\alpha]$: Then $s[\alpha \dashrightarrow \beta] = s$ and again there are two cases.
  - $s[\beta][\gamma] = s[\beta]$: Then it follows from the fact that order doesn't matter (Fact 6.2.3) and these two assumptions that $s[\alpha][\gamma] = s[\alpha][\beta][\gamma] = s[\beta][\gamma][\alpha] = s[\beta][\alpha] = s[\alpha][\beta] = s[\alpha]$. And thus $s \vDash \alpha \dashrightarrow \gamma$.
  - $s[\beta][\gamma] \neq s[\beta]$: Then we know that $s[\alpha \dashrightarrow \beta][\beta \dashrightarrow \gamma] = \emptyset$ and by eliminativity we find that $s \vDash \alpha \dashrightarrow \gamma$.
- $s[\alpha][\beta] \neq s[\alpha]$: Analogous to the previous case.

$\square$

Fact (6.4.4). *Let $\alpha, \beta, \gamma \in \mathbf{BiUS}_0$. Then, $(\alpha \vee \beta) \dashrightarrow \gamma \vDash (\alpha \dashrightarrow \gamma) \wedge (\beta \dashrightarrow \gamma)$.*

Proof. Suppose $s[(\alpha \vee \beta) \dashrightarrow \gamma]$ is defined, then there are two cases.

[151] By this we mean that $s[\xi_i] \ldots [\xi_n]$ is defined.

- $s[\alpha \vee \beta][\gamma] = s[\alpha \vee \beta]$: Then we see that $s[(\alpha \vee \beta) \dashrightarrow \gamma] = s$. By distributivity (Fact 6.2.2) we see that this means that $\forall w \in s$: $\big(\{w\}[\alpha] = \{w\}$ or $\{w\}[\beta] = \{w\}\big)$ implies $\{w\}[\gamma] = \{w\}$. Hence, $\big\{w \in s \mid \{w\}[\alpha] = \{w\}$ and $\{w\}[\gamma] = \{w\}\big\} = \big\{w \in s \mid \{w\}[\alpha] = \{w\}\big\}$, i.e., $s[\alpha][\gamma] = s[\alpha]$. Hence, $s[(\alpha \vee \beta) \dashrightarrow \gamma] \vDash \alpha \dashrightarrow \gamma$. Similarly for $\beta \dashrightarrow \gamma$.
- $s[\alpha \vee \beta][\gamma] \neq s[\alpha \vee \beta]$: Then we see that $s[(\alpha \vee \beta) \dashrightarrow \gamma] = \emptyset$. And by eliminativity we see that it $\emptyset[\alpha][\gamma] = \emptyset = \emptyset[\alpha]$, i.e., $s[(\alpha \vee \beta) \dashrightarrow \gamma] \vDash \alpha \dashrightarrow \gamma$. Similarly for $\beta \dashrightarrow \gamma$. $\square$

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