Modal Quantifiers

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Natasha Alechina

Chapter 1

Introduction

This thesis contains a study of generalizations of first order quantification. In a sense, the quantifiers studied here stand to $\forall$ and $\exists$ as the modal operators $\Box$ and $\Diamond$ stand to the universal modality.

The idea behind separating this class of quantifiers is based on the following intuition. We understand binding variables as follows: if a 'universal-type' quantifier binds a variable x, then this means that for every object in the range of x, the expression in the scope of the quantifier holds. For the 'existential-type' quantifier, dually, there is an object in the range of x for which the formula under the quantifier holds. For the ordinary quantifiers, the range of a variable is given in advance: it is the domain of the model. For modal quantifiers, the range of a variable depends on the point where the formula is evaluated; a similar definition of 'modal', as depending on the evaluation point, is given in (Blackburn and Seligman 1995). Another property of modal quantifiers, which will be discussed later in this introduction, is that the variables bound by them have internal structure and 'individuality' which ordinary first order variables lack. We will see that the variables bound by modal quantifiers resemble more the variables used in programming languages, such as Pascal.

It turns out that a lot of different quantifiers can be seen as modal quantifiers. In (Venema 1991), (Németi 1992), (van Benthem 1994), (Marx 1995), (Andréka, van Benthem and Németi 1995), a new type of models for predicate logic is studied. In these models, not all assignments of values to variables are possible. $\exists x\varphi$ is satisfied in a model given an assignment s, if there is an assignment $s' =_x s$ which satisfies φ *and which is available in the given model.* (Note that restricting the set of available assignments is one way to restrict the range of x). It is easy to see that an assignment, as a point where a formula is evaluated, plays the same role as a possible world and the relation $=_x$ is an accessibility relation. In this dissertation, one more type of 'assignments models' is introduced, namely partial assignments models, where formulas are evaluated with respect to assignments restricted to their free variables, and, as before, not all partial assignments are available for the quantifier.

Somewhat surpisingly, another example of modal quantifiers are the generalized quantifiers which, intuitively speaking, say that the set of objects satisfying a formula is 'big', like quantifiers 'for almost all', 'for all but countably many', etc. The idea to look at these quantifiers as binding a special kind of variables, whose range is restricted by the values of the free variables of the formula, belongs to Michiel van Lambalgen (cf. (van Lambalgen 1991)). This was studied as an alternative kind of semantics for generalized quantifiers, using a dependency relation between objects in the domain, in (van Benthem and Alechina 1993). Investigating this alternative semantics constitutes the main topic of the dissertation.

But first we investigate in general the idea that the range of a variable can be restricted by the values of some other 'relevant' variables. We introduce a logic corresponding to the following truth definition: $\exists x\varphi(x, y_1, \ldots, y_n)$ is true in a model M under an assignment of values d_i to y_i, if there is an object d, such that $\varphi(d, d_1, \ldots, d_n)$ holds in M and d is a possible value for x given the assignment of values to y_i. Here, only the free variables of the formula $\exists x\varphi$ are considered as relevant for determining the range of x. In 'assignments models' the relevant variables are all the variables of the language. One can also think of intermediate cases, when the relevant variables are at least the free variables, but not necessarily all the variables of the language. However, the results of Chapter 2 suggest that any such logic, given that the number of relevant variables is finite, can be embedded in a logic where the relevant variables are precisely the free variables of the formula by defining a translation function which adds the 'lacking' variables under each quantifier. (At least, this works for the logic of assignments models.) We also consider briefly what happens if the relevant variables are less than the free variables of the formula.

The rest of the Introduction contains some background and motivations for the topics sketched above.

1.1 Generalized quantifiers

A generalized quantifier $\mathbf{Q}$ as defined by Mostowski (1957) is a class of subsets of the universe, so that a model M satisfies $Qx\varphi(x, \bar{d})$ if the set of elements $\{e : M \models \varphi[e, \bar{d}]\}$ is in $\mathbf{Q}$. Examples are: the ordinary existential quantifier (interpreted as the set of all non-empty subsets of the universe); the quantifier 'there are exactly 2'; a filter quantifier (where the only requirement on $\mathbf{Q}$ is that it is a filter), 'there are uncountably many' (where the domain is uncountable, and $\mathbf{Q}$ contains all uncountable subsets), etc. Actually Mostowski required that the generalized quantifier is invariant under permutations of the universe, thus restricting attention to quantifiers related to cardinality. Subsequently, other generalized quantifiers were considered which do not have the property of permutation invariance, such as topological quantifiers or measure quantifiers. An example of the latter is the quantifier 'for almost all', which contains all subsets of measure 1 of [0,1]. For an overview of the subject, one may consult the collection 'Model-Theoretic Logics', edited by Barwise and Feferman (1985), and (Westerståhl 1989).

1.1.1 Generalized quantifiers and modal operators. (How modal quantifiers could have been invented)

There is a certain connection between the theory of generalized quantifiers and modal logic. A set of possible worlds can be viewed as a domain and a modal operator as a generalized quantifier on this domain (which does not have to be a 'modal' generalized quantifier in our sense).

For example, van der Hoek and de Rijke (1991) study in parallel graded modalities and generalized quantifiers 'there are more than n' and 'there are at least as many A's as B's'.

Analogously, modal operators 'the probability of A is at least r' (with probabilistic measure on the set of possible worlds) can be considered as a special case (for a monadic language) of the quantifiers 'the measure of $\{x : A(x)\}$ is at least r'. In particular, the modal operator 'with probability 1' and the quantifier 'for almost all' have clearly related meaning, given that the set of possible worlds and the measure on it present a model for the quantifier.

Let $M = \langle W, P, V \rangle$ be such a model, W an uncountable set of possible worlds, V a valuation and P a probability measure. The truth definition for P_1 is

$$M, w \models P_1\varphi \Leftrightarrow P(\{w' : M, w' \models \varphi\}) = 1.$$

It turns out that the corresponding logic is axiomatized by adding to propositional logic the following simple axioms:

K $P_1(\varphi \to \psi) \to (P_1\varphi \to P_1\psi)$
D $\neg P_1\bot$ where $\bot$ denotes the contradiction
4' $P_1P_1\varphi \equiv P_1\varphi$
5' $P_1\neg P_1\varphi \equiv \neg P_1\varphi$
N $\vdash \varphi \implies \vdash P_1\varphi$

(See, for example, (Alechina 1995b).) But the presence of K and N means that the logic of P_1 has possible worlds models where a binary accessibility relation R between the worlds satisfies the first-order properties corresponding to D, 4' and 5' (since these are Sahlqvist formulas), and P_1 is interpreted as 'true in all accessible worlds':

$$M, w \models P_1\varphi \Leftrightarrow \forall w'(wRw' \implies M, w' \models \varphi)$$

The latter semantics is much simpler and better studied then the semantics involving the probability measure.

A natural question to ask is whether the quantifier 'for almost all' has such alternative models: without a measure but with an additional relation on the domain.

1.1.2 Dependence relation between objects

As we will see, all generalized quantifiers which satisfy the monotonicity axiom

K $\Box_x(\varphi \to \psi) \to (\Box_x\varphi \to \Box_x\psi)$

(even a somewhat weaker property of restricted monotonicity) and necessitation

$\mathbf{N} \vdash \varphi \implies \vdash \Box_x\varphi$,

have an alternative semantics where models are equipped with a family of accessibility relations between objects (for all finite arities). Intuitively, $R^{n+1}(d, e_1, \ldots, e_n)$ will mean 'd is accessible from $e_1, \ldots, e_n$'. We will always assume that $\{e_1, \ldots, e_n\}$ is a *set*, that is, for every permutation π $R^{n+1}(d, \bar{e}) \equiv R^{n+1}(d, \pi(\bar{e}))$ and the repetitions in $\bar{e}$ do not matter. This allows us to replace the family of relations by one binary relation between *elements* and *finite sets of elements*. In such models, $\Box_x\varphi$ is interpreted as '$\varphi(d)$ is true for all objects d dependent on the parameters of $\Box_x\varphi$', more formally,

$$M, s \models \Box_x\varphi(x, \bar{y}) \Leftrightarrow \forall d(R(d, s(\bar{y})) \to M, s^x_d \models \varphi(x, \bar{y}))$$

where $\bar{y}$ are precisely the free variables of $\Box_x\varphi$.

This semantics for the generalized quantifiers and analogies with modal logic are discussed in Chapters 3 and 4.

1.2 Proof-theoretic motivation

1.2.1 Sequent calculus with indexed variables

The idea of interpreting generalized quantifiers by means of a dependence relation between variables comes, however, not from modal logic but from the work on the proof theory for generalized quantifiers in (van Lambalgen 1991). The motivation behind the introduction of the dependence relation is as follows.

For first order logic, it is possible to give a Gentzen-style sequent calculus with left- and right-introduction rules for the quantifiers, where the dependencies between parameters can be managed by means of syntactic side conditions on the rules. Can something similar be done for generalized quantifiers?

This question was answered positively by van Lambalgen (1991), who proposed natural deduction systems for several generalized quantifiers. In further work, (van Lambalgen 1993) and (Alechina and van Lambalgen 1995b) the logics of generalized quantifiers are viewed as substructural logics, where the introduction and elimination rules for the quantifiers remain constant, and the structural rule which deals with dependencies is modified.

In order to be able to talk about dependencies explicitely, the language of first order logic is extended to include *indexed variables* of the form $x_{\bar{y}}$, where $\bar{y}$ may also be indexed variables. The variable $x_{\bar{y}}$ ranges over objects dependent on $\bar{y}$. (For a precise definition, see Chapter 3.)

Below we formulate the left and right introduction rules for the quantifier of the proof system for generalized quantifiers. For the details, and for discussion of the full logic of indexed variables, the reader is referred to (Alechina and van Lambalgen 1995b).

Sequent calculus for generalized quantifiers has the following quantifier introduction and elimination rules:

$$\frac{\Gamma \Longrightarrow \psi(x_{\bar{z}}, \bar{z}), \Delta}{\Gamma \Longrightarrow \Diamond_x \psi(x, \bar{z}), \Delta} \ \Diamond r \qquad \frac{\Gamma, \psi(x_{\bar{z}}, \bar{z}) \Longrightarrow \Delta}{\Gamma, \Diamond_x \psi(x, \bar{z}) \Longrightarrow \Delta} \ \Diamond l$$

where in $\Diamond l$ $x_{\bar{z}}$ does not occur free in Γ and Δ, also not in indices, and in $\Diamond r$ $x_{\bar{z}}$ should occur free in Γ or Δ.

Given the truth definition for $\Diamond$ in the dependence models given above, the indexed variables can be easily interpreted: $x_{\bar{y}}$ ranges of the objects in the relation R to the objects assigned to $\bar{y}$. Under this interpretation, soundness of the rules is obvious. It is clear that $\Diamond_x$ may be introduced only on a variable which depends on the rest of the free variables of a formula. The restriction on $\Diamond l$ looks familiar from classical logic; the restriction on $\Diamond r$ is necessary unless the range of $x_{\bar{z}}$ is assumed to be always non-empty (for every finite set of elements, there is an object dependent on this set).

The corresponding calculus will be called L_{triv}.

1.2.2 Generalized quantification as substructural logic

It is shown in (Alechina and van Lambalgen 1995b) that to make $\Diamond$ in the calculus with indexed variables to behave as the ordinary existential quantifier, one needs to add the following *substitution rule*:

$$\frac{\Gamma \Longrightarrow \psi(t), \Delta}{\Gamma \Longrightarrow \psi(s), \Delta} \ SUB$$

where s and t are any variables; the restriction on SUB is that t does not occur free in Γ and Δ. (Note that

$$\frac{\Gamma, \ \psi(t) \Longrightarrow \Delta}{\Gamma, \ \psi(s) \Longrightarrow \Delta}$$

with the same restriction on t, is derivable from SUB and the rules for negation.)

We call SUB a structural rule since it does not involve any logical connectives.

In between L_{triv} and $L_{triv}+SUB$ a whole class of substructural logics with respect to the substitution rule is situated. The examples below show that modifications of the substitution rule are made possible by the fact that the variables have internal structure.

The weakest system considered in (Alechina and van Lambalgen 1995b) contains the following substitution rule:

$$\frac{\Gamma \Longrightarrow \psi(x_{\bar{z}}), \Delta}{\Gamma \Longrightarrow \psi(x'_{\bar{z}}), \Delta} \ SUB_{av}$$

(given that $x_{\bar{z}}$ does not occur in Γ, Δ).

SUB_{av} corresponds to the principle of renaming bound variables (taking alphabetic variants). $L_{triv} + SUB_{av}$ formalizes the minimal logic of dependence models.

This system is still rather weak. For example,

$$\Diamond_x \varphi(x, \bar{y}) \to \Diamond_x(\varphi(x, \bar{y}) \vee z = z)$$

(monotonicity of $\Diamond$) is not derivable in $L_{triv} + SUB_{av}$.

The first standard generalized quantifier, namely the filter quantifier $\neg\Diamond\neg$ is obtained by strengthening SUB_{av} to

$$\frac{\Gamma \Longrightarrow \psi(x_{\bar{u}\bar{z}}, \bar{z}), \Delta}{\Gamma \Longrightarrow \psi(x'_{\bar{v}\bar{z}}, \bar{z}), \Delta} \quad SUB_{ext}$$

with the customary restrictions on $x_{\bar{u}\bar{z}}$. Observe that this rule allows to prove the monotonicity principle.

The characteristic axiom of the 'for almost all' quantifier

$$\Box_x \Box_y \varphi \to \Box_y \Box_x \varphi$$

corresponds to the following substitution rule:

$$\frac{\Gamma \Longrightarrow \varphi(y_{\bar{z}}, x_{y_{\bar{z}}\bar{z}}, \bar{z}),\ \Delta}{\Gamma \Longrightarrow \varphi(y_{x_{\bar{z}}\bar{z}}, x_{\bar{z}}, \bar{z}),\ \Delta}$$

where both $y_{\bar{z}}$ and $x_{y_{\bar{z}}\bar{z}}$ do not occur free in Γ or Δ.

For determining such substitution rules, and proving their interderivability with the axioms, one can benefit from knowing to what condition on the dependence relation R the axiom corresponds. For example,

$$\Box_x \Box_y \varphi \to \Box_y \Box_x \varphi$$

corresponds to

$$R(y, \bar{z}) \wedge R(x, y\bar{z}) \to R(x, \bar{z}) \wedge R(y, x\bar{z}).$$

The correspondence theory of generalized quantifiers (correspondence between axioms and the properties of R) is studied systematically in Chapter 4.

1.3 Assignments and cylindrifications

Another example of modal quantifiers comes from *cylindric relativized set algebras*, or Crs-algebras (see (Henkin et al. 1981)). These algebras correspond to a new type of models for first order languages (cf. (Németi 1992), (Venema 1991), (Marx 1995), (Andréka, van Benthem and Németi 1995)). One of the main reasons for introducing the models with restricted sets of assignments is that the corresponding logics are decidable and have other pleasant formal properties. The analogy with the treatment of variables in programming languages given in the next section shows that such logics have also a very natural semantics.

We will concentrate mostly on Crs_n-models, which correspond to n-dimensional cylindric relativized set algebras (Crs_n). In such models, one interprets only formulas with at most n first variables $v_0, \ldots, v_{n-1}$. A Crs_n-model consists of a standard model for first order logic (a domain D plus an interpretation function) and a set of assignments $W \subseteq D^n$. The truth definition for the existential quantifier reads $M, s \in W \models \exists v_i \varphi \Leftrightarrow \exists s' \in W(s' =_i s \wedge M, s' \models \varphi)$.

First order logic can be given equivalent formulations using the standard notion of an infinite assignment to individual variables or using assignments restricted to the free variables of a formula. These formulations are equivalent due to the property of locality of first order logic: if two infinite assignments s and z agree on the free variables of φ, then $M, s \models \varphi \Leftrightarrow M, z \models \varphi$. If the set of assignments is restricted, the difference between the two formulations becomes crucial. Crs-models do not satisfy locality. In Chapter 2 we introduce a counterpart of Crs with assignments restricted to the free variables. This logic is local, but the monotonicity property (or distributivity of $\exists$ over $\vee$) is restricted to the formulas with the same free variables; in Crs it holds without restrictions.

There is some connection between logics with restricted sets of assignments and many-sorted logics (see for example (Andréka and Sain 1981)). Namely, many-sorted logics are a special case of models with restricted sets of assignments, since in many-sorted logics only those assignments are possible where variables take values in their associated domains. In 'assignments models', the domain of each variable also does not have to be equal to the domain of the model. It is possible that there is some element d in the domain, such that no assignment s with $s(x) = d$ is available. Then the domain of x is not the whole domain D, but at most $D \backslash \{d\}$. In general, the situation is more complicated than in many-sorted logic: which value a variable may take, depends on the values of other variables. For example, there can be an assignment s with $s(x) = d$, but no assignment s with $s(x) = d$ and $s(y) = e$. If y is assigned e, then x cannot be assigned d, and vice versa: if $s(x) = d$, then the value of y cannot be e.

1.4 Composite variables in programming languages

There is one more property which distinguishes modal quantifiers from classical first order quantifiers. They quantify over variables which have inner structure. In this section we show that such types of variables are well known and used in programming languages (cf. (Watt 1976), (Wirth 1990)).

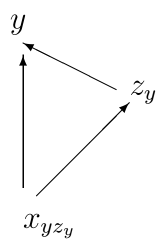
In classical first order logic, variables take values in a domain which consists of points: elements which are not assumed to have internal structure or to have any connections to each other except for the ones denoted by the predicates in the language. On the other hand, in programming languages such as Pascal there are primitive variables (roughly analogous to first order variables) and composite variables. The latter refer to objects having complex structure. Accordingly, composite variables themselves have inner structure: they include other variables as their components. The principal example of a composite variable is a *record variable*.

A record variable is a mapping from an index set to component variables (primitive or composite), each of which can be considered separately. A special case of a record variable is an *array variable* $x = [x[1], \ldots, x[n]]$, where all component variables $x[i]$ are of the same type. The values which such variable can take, are also arrays. For example, think of a variable *date* which is an array of variables *day*, *month*, *year*. The values of *date* are triples of numbers, where the first number is the value of *day*, the second number is the value of *month*, and the third number is the value of *year*. The variable *day* may take values in the set $1, \ldots, 31$; the values of *month* come from $1, \ldots, 12$ and year can be any whole number. Each of these three variables can be updated (or, in our setting, quantified over) separately. It is clear however that if the value of *month* is 2, then the value of *day* cannot be 30; depending on the year, *day* may or may not take the value 29. This is a special case of a Crs_3 setting, with the variable *date* playing the role of assignment. In the generalized Crs_n setting with polyadic quantifiers (cf. Simon and van Lambalgen 1994) it is also possible to quantify over arbitrary subsets of n, including the whole assignment. In other words, generalized Crs_n can be seen as the logic to reason over arrays.

Clearly, the same structure can be described in ordinary first order logic extended with some appropriate functional symbols and predicates. But it is also interesting to try to do without additional predicates and study the theory of composite variables instead. (As we shall see later, this can for example yield a decidable logic, while the logic with explicitly introduced relational symbols is undecidable). Actually, this is what is done in van Lambalgen's approach to generalized quantifiers. Instead of introducing R explicitely in ordinary first order language, he considers a language containing indexed variables $x_{\bar{y}}$. $x_{\bar{y}}$ refers to an object but it also conveys information about the dependence structure: it says that this object must depend on the objects assigned to $\bar{y}$.

Pascal also has means for variables to point to other variables. There is a special type of variables called *pointers*. If x is any variable, the value of a pointer variable $\uparrow x$ is the address of the variable x; $\uparrow x$ points at x.

Consider for example variables x_{yz_y}, z_y, y. Let an arrow (a pointer) from a to b mean that a depends on b. Then there is the following relationship between the variables x_{yz_y}, y and z_y:



Then z_y can be represented as $[z, \uparrow y]$ and x_{yz_y} as $[x, \uparrow y, \uparrow z_y]$. The values which these variables take are objects which stand in the same relations to each other as the corresponding variables. Namely, if y takes a value a, then z_y can only take a value which is in the relation R to a.

The use of pointers is very appropriate in representing indexed variables. Dependencies arising in derivations have dynamic character. Instead of making a derivation in a sequent calculus for generalized quantifiers using indexed variables, one can construct a derivation in a calculus with ordinary variables and simultaneously build a graph where points are variables and arrows correspond to the dependence relation; when a new object is introduced, some new arrows are created. The conditions on applicability of the rules are reformulated in terms of the graph accompanying the derivation; for example, a rule is applicable only if there exists an object to which precisely the variables y, z_y are connected. (Cf. (Fine 1985)). We come back to these ideas in the last section of Chapter 3.

1.5 Overview of the thesis

The first chapter following this introduction contains some facts about various logics which arise from the idea that the range of a variable can be restricted by the values of some 'relevant' variables. We give a Hilbert-style axiomatization of the logic of structured dependence models which we consider as the most basic system under the assumption that 'relevant variables' are precisely the free variables of the formula in the scope of the quantifier.

Also, a Hilbert-style axiomatization for partial assignments and a tableau calculus for the logic of assignments models are given. Further, we show that by a general result of Andréka and Németi (1994) these logics are decidable. We prove some results about the relationships between them.

This chapter owes a lot to discussions with H. Andréka and I. Németi.

Chapter 3 studies model and proof theory of the minimal logic of generalized quantifiers, in particular we prove preservation under bisimulations, decidability and interpolation theorems. It is partly based on (van Benthem and Alechina 1993), 'Modal Quantification over Structured Domains', to appear in M. de Rijke, ed., *Advances in Intensional Logic*, and (Alechina 1995c), 'On A Decidable Generalized Quantifier Logic Corresponding to a Decidable Fragment of First Order Logic', to appear in the *Journal of Logic, Language and Information.*

In Chapter 4 we study the correspondence between quantifier axioms and the properties of R in dependence models and prove a Sahlqvist theorem for correspondence and completeness. It is based on (van Benthem and Alechina 1993), (Alechina and van Lambalgen 1995a): 'Correspondence and Completeness for Generalized Quantifiers', *Bulletin of the IGPL* **3**, 167 – 190, and (Alechina and van Lambalgen 1995b): 'Generalized Quantification as Substructural Logic', to appear in the *Journal of Symbolic Logic.*

In Chapter 5 the approach to unary generalized quantifiers presented in Chapters 3 and 4 is extended to the binary case. It also studies connections between binary generalized quantifiers and conditionals and their applications in formalizing defeasible reasoning. This chapter is based on (Alechina 1993): 'Binary quantifiers and relational semantics', ILLC Report LP-93-13, and (Alechina 1995a): 'For All Typical', in *Symbolic and Quantitative Approach to Reasoning and Uncertainty. Proceedings ECSQARU'95.*

Chapter 2

Various logics of modal quantifiers

As it was mentioned in the Introduction, the main topic of this thesis is the study of dependence models for generalized quantifiers. However, in this chapter we are mostly interested in the family connections of dependence models, first of all, in what is their precise relation to assignments models. To figure it out, we study several new logics which resemble both the logic of dependence models and the logic of assignments models. All these logics arise from the intuition that the range of a variable may depend on the values of other variables. We begin by considering logics where the range of x in $\Diamond_x\varphi$ is restricted by the values of variables including the free variables of $\Diamond_x\varphi$.

First of all, we introduce a very basic logical system, which is weaker than the logic of van Lambalgen's generalized quantifiers and the logic of partial assignments models. We show how to obtain the latter two systems from the basic system by adding additional axioms. The relation between dependence models and partial assignments models becomes more clear. We also show how to embed the logic of assignments models (Crs_n and Crs_n^+) into the logic of partial assignments models. Further, the connections with restricted fragments of first order logic and decidability issues are discussed.

In the last section of this chapter we briefly consider what happens if the set of variables relevant for determining the range of x in $\Diamond_x\varphi$ is less than the set of free variables of $\Diamond_x\varphi$; this clarifies the connections with a generalized version of many-sorted logic.

2.1 Structured dependence models

Consider a language $\mathcal{L}(\Diamond)$, which contains countably many individual variables $v_0, \ldots, v_n, \ldots$, predicate symbols, $\neg$, $\wedge$, $=$ and a quantifier $\Diamond$. A well formed formula of $\mathcal{L}(\Diamond)$ is defined as follows:

1. if $x_1, \ldots, x_n$ are individual variables and P an n-place predicate symbol, then $P(x_1, \ldots, x_n)$ is a w.f.f.; if x, y are individual variables, then $x = y$ is a w.f.f.;

2. if φ_1 and φ_2 are w.f.f.'s, then $\neg\varphi_1$ and $\varphi_1 \wedge \varphi_2$ are w.f.f.'s;
3. if φ is a w.f.f. and x an individual variable, then $\Diamond_x\varphi$ is a w.f.f.;
4. nothing else is a w.f.f.

We define $\Box_x\varphi$ to denote $\neg\Diamond_x\neg\varphi$; $\vee, \rightarrow, \equiv$ are defined as usual.

2.1.1. Definition. A *structured dependence model* is a structure of the form $M = \langle D, R, V\rangle$ where D is a domain, V an interpretation function (a function assigning n-ary predicate symbols subsets of D^n), and R is a relation between a pair $\langle$*variable*, *object*$\rangle$ and a finite set of such pairs. Intuitively, R tells which value a variable can take given the values which other variables have taken. We could have demanded from the start that if $\langle x, d\rangle R\{\langle y_i, d_i\rangle : 1 \leq i \leq n\}$, then $x, y_1, \ldots, y_n$ are all different variables; cf. Lemma 2.2.7.

The relation $M, s \models \varphi$ ('φ is *true* in M under assignment s') is defined as follows:

- $M, s \models P_i^n(x_{j1} \ldots x_{jn}) \Leftrightarrow \langle s(x_{j1}) \ldots s(x_{jn})\rangle \in V(P_i^n)$;
- $M, s \models x = y \Leftrightarrow s(x) = s(y)$;
- $M, s \models \neg\varphi \Leftrightarrow M, s \not\models \varphi$;
- $M, s \models \varphi \wedge \psi \Leftrightarrow M, s \models \varphi$ and $M, s \models \psi$;
- $M, s \models \Diamond_x\psi(x, y_1, \ldots, y_n) \Leftrightarrow$ there exists $s' =_x s$ such that

$$\langle x, s'(x)\rangle \; R \; \{\langle y_1, s'(y_1)\rangle, \ldots, \langle y_n, s'(y_n)\rangle\}$$

and $M, s' \models \psi(x, \bar{y})$, where $\bar{y}$ are precisely the free variables of $\Diamond_x\psi$. (This will be denoted by $FV(\Diamond_x\psi) = \bar{y}$.)

We say that $M \models \varphi$ iff $M, s \models \varphi$ for all variable assignments s. $\Box$

If Γ and Δ are sets of formulas, we say that $\Gamma \models \Delta$ if for every model M and assignment s, if $M, s \models \Gamma$, then $M, s \models \Delta$. The logic corresponding to this notion of consequence is called L_{str}.

Observe that the truth definition for van Lambalgen's generalized quantifiers can be obtained from the truth definition above by erasing the first element in each pair (or dismissing some information about the variables):

$$s'(x) \; R \; \{s'(y_1), \ldots, s'(y_n)\}$$

instead of

$$\langle x, s'(x)\rangle \; R \; \{\langle y_1, s'(y_1)\rangle, \ldots, \langle y_n, s'(y_n)\rangle\}$$

We will see later that this move strengthens the logic by making the axiom $\Diamond_x\varphi \rightarrow \Diamond_y\varphi[x/y]$ valid (given that y is free for x in φ), or, alternatively, making the substitution rule SUB_{av} mentioned in the Introduction, sound.

Another special case of structured dependence models are *partial assignments models* defined in section 2.3.

To finish this section, we prove that structured dependence models share a property which is taken for granted in classical first order logic but, as we shall see later, does not hold for all modal quantifiers.

2.1.2. LEMMA. *(Locality) For every structured dependence model M, pair of assignments s and z and formula φ, if $s\lceil FV(\varphi) = z\lceil FV(\varphi)$, then*

$$M, s \models \varphi \Leftrightarrow M, z \models \varphi.$$

Proof. Completely analogous to the proof for the classical first order logic, the proof goes by induction on the complexity of φ. For atomic formulas the proposition obviously holds, and the steps for $\neg$ and $\wedge$ are also trivial.

Let $\varphi = \Diamond_x\psi(x, \bar{y})$, and the proposition hold for $\psi(x, \bar{y})$. Assume that $M, s \models \Diamond_x\psi(x, \bar{y})$. Then there is an assignment $s' =_x s$ such that the pair $\langle x, s'(x)\rangle$ is in the relation R to the set of pairs $\{\langle y_i, s'(y_i)\rangle : y_i \in FV(\Diamond_x\psi)\}$ and $M, s' \models \psi$. Consider an assignment $z' =_x z$ with $z'(x) = s'(x)$. Obviously, for every variable u in $FV(\psi)$, $z'(u) = s'(u)$; therefore by the inductive hypothesis $M, z' \models \psi$. Also, since $\langle x, s'(x)\rangle = \langle x, z'(x)\rangle$ and $\langle y_i, s'(y_i)\rangle = \langle y_i, z'(y_i)\rangle$ for every $y_i \in FV(\Diamond_x\psi)$, $\langle x, z'(x)\rangle$ is in the relation R to $\{\langle y_i, z'(y_i)\rangle : y_i \in FV(\Diamond_x\psi)\}$, which gives $M, z \models \Diamond_x\psi(x, \bar{y})$. □

2.2 Hilbert-style axiomatization

To give a Hilbert-style axiomatization for the logic of structured dependence models, we need to extend the language by countably many individual constant symbols. The idea is to repeat a standard Henkin argument; however, this extension is less trivial than in case of the first-order logic. The difficulty comes from the fact that the variables have much more 'individuality' than in first order logic. As a result, the constants should also acquire some.

We add to $\mathcal{L}(\Diamond)$ a set of individual constants d^i_j, countably many for each i. We call individual variables and constants *terms*. Intuitively, d^i_j 'behaves approximately as the variable v_i'. The precise meaning of this will become clear below. Formally, we introduce a function $\eta : TERMS \longrightarrow VARIABLES$, such that $\eta(v_i) = v_i$ and $\eta(d^i_j) = v_i$. We call a term t a *n-term*, if $\eta(t) = v_n$.

A well formed formula is defined as usual. A formula without free variables is called a *sentence*.

The interpretation of constants is the same as in first order logic. If $M = \langle D, R, V\rangle$, and d^i_j is a constant, then $V(d^i_j) \in D$.

The meaning of a term t in a model M given an assignment s, $[t]_{M,s}$ is as expected: if $t = v_i$, $[t]_{M,s} = s(v_i)$, and if $t = d^i_j$, $[t]_{M,s} = V(d^i_j)$. The following clauses of the truth definition are extended to the case of formulas containing constants:

- $M, s \models P^n_i(t_1 \ldots t_n) \Leftrightarrow \langle [t_1]_{M,s} \ldots [t_n]_{M,s}\rangle \in V(P^n_i)$;
- $M, s \models \Diamond_x\psi(x, t_1, \ldots, t_n) \Leftrightarrow$ there exists $s' =_x s$ such that

$$\langle x, s'(x)\rangle\ R\ \{\langle \eta(t_1), [t_1]_{M,s'}\rangle, \ldots, \langle \eta(t_n), [t_n]_{M,s'}\rangle\}$$

and $M, s' \models \psi(x, \bar{t})$.

It is in the last clause that the idea that constants are instances of a particular variable becomes clear. For example,

$$M, s \models \Diamond_{v_3} P(d_1^1, v_3) \Leftrightarrow$$

$$\Leftrightarrow \exists d \in D(\langle v_3, d \rangle R\{\langle v_1, V(d_1^1)\rangle\} \wedge M, s_d^3 \models P(d_1^1, v_3))$$

Note that $\Diamond_{v_3} P(d_1^4, v_3)$ can be false in the same model, even if $V(d_1^1) = V(d_1^4)$. Analogously, $\Diamond_{v_3} P(v_1, v_3)$ can be true and $\Diamond_{v_3} P(v_4, v_3)$ false even if $v_1 = v_4$. This corresponds to the failure of the following axiom (even in the language without constants):

$$t_1 = t_2 \wedge \varphi(t_1) \rightarrow \varphi(t_2),$$

given that t_2 is free for t_1 is φ. Instead, two weaker versions of this axiom are valid: the version where φ is an atomic formula and the version where t_1 and t_2 are terms of the same type.

2.2.1. Definition. *L_{str} is the following axiom system:*

A0 *propositional logic;*
A1 $\Box_x(\varphi \rightarrow \psi) \rightarrow (\Box_x\varphi \rightarrow \Box_x\psi)$*, given that* $\Box_x\varphi$ *and* $\Box_x\psi$ *contain the same parameters (free variables and constants);*
A2 $\varphi \rightarrow \Box_x\varphi$*, given that x is not free in φ;*
=1 $t = t$*;*
=2 $t_1 = t_2 \rightarrow t_2 = t_1$*;*
=3 $t_1 = t_2 \wedge t_2 = t_3 \rightarrow t_1 = t_3$*;*
=4 $t_1 = t_2 \wedge \varphi(t_1) \rightarrow \varphi(t_2)$*, given that t_2 is free for t_1 in $\varphi(t_1)$ and either (=4a) φ is atomic, or (=4b) $\eta(t_1) = \eta(t_2)$;*
R1 $\varphi,\ \varphi \rightarrow \psi/\psi$*;*
R2 $\varphi(t^i)/\Box_{v_i}\varphi[t^i/v_i]$*, where t^i is an i-term, i.e. v_i or d_j^i.*

A *derivation* of a formula φ from a set of formulas Γ is a finite sequence of formulas $\varphi_1, \ldots, \varphi_n$, where $\varphi_n = \varphi$ and every φ_i, $i \leq n$, is either an axiom or a member of Γ or is obtained from some φ_j and φ_k with $j, k < i$ by one of the inference rules. A formula is called derivable if there is a derivation of this formula from the empty set of premises. We will write $\Gamma \vdash_{L_{str}} \varphi$ for 'φ is derivable from Γ in L_{str}'. In the sequel, the subscript is omitted when it is clear which system we are talking about.

We say that a formula ψ *depends* on a formula χ in a given derivation, if either $\psi = \chi$, or ψ is obtained by an inference rule with a formula dependent on χ as one of the premises. It is easy to prove that if $\Gamma, \varphi \vdash \psi$ and there is a derivation where ψ does not depend on φ, then $\Gamma \vdash \psi$.

In L_{str} the deduction theorem holds with restrictions (analogous to the restrictions in classical first order logic; cf. for example (Mendelson 1979)).

2.2.2. Theorem. *(Deduction theorem for L_{str}) Let $\Gamma, \varphi \vdash_{L_{str}} \psi$ and there is a derivation of ψ from Γ, φ which does not contain any applications of R2 of the form $\chi(t^i)/\Box_{v_i}\chi(t^i/v_i)$, where v_i or t^i are in φ and χ depends on φ. Then $\Gamma \vdash_{L_{str}} \varphi \rightarrow \psi$.*

Proof. We reason by induction on the length of the derivation.

Let ψ be an axiom or a member of Γ. Then by propositional reasoning $\Gamma \vdash \varphi \to \psi$.

Assume that ψ is obtained by R1 from χ and $\chi \to \psi$, and by the inductive hypothesis $\Gamma \vdash \varphi \to \chi$ and $\Gamma \vdash \varphi \to (\chi \to \psi)$. By propositional reasoning, $\Gamma \vdash \varphi \to \psi$.

Assume that ψ is obtained by R2 from $\chi(t^i)$, that is, $\psi = \Box_{v_i}\chi(t^i/v_i)$. Then either χ does not depend on φ, or v_i and t^i are not in φ. In the former case, $\Gamma \vdash \chi(t^i)$ and by R2 $\Gamma \vdash \psi$, thus $\Gamma \vdash \varphi \to \psi$. In the latter case, by the inductive hypothesis, $\Gamma \vdash \varphi \to \chi(t^i)$. By R2, $\Gamma \vdash \Box_{v_i}(\varphi \to \chi(t^i/v_i))$, and v_i is not free in φ. By A2 and modus ponens, $\Gamma \vdash \varphi \to \psi$. □

The following two weak versions of the deduction theorem are also useful:

2.2.3. CONSEQUENCE. *If $\Gamma, \varphi \vdash_{L_{str}} \psi$ and there is a derivation of ψ from Γ, φ which does not contain any applications of R2 of the form $\chi(t^i)/\Box_{v_i}\chi(t^i/v_i)$, where v_i or t^i are in φ, then $\Gamma \vdash_{L_{str}} \varphi \to \psi$.*

2.2.4. CONSEQUENCE. *If $\Gamma, \varphi \vdash_{L_{str}} \psi$ and in the derivation R2 is applied only to derivable formulas, then $\Gamma \vdash_{L_{str}} \varphi \to \psi$.*

The following formulas are derivable in L_{str}:

A1b $\Box_x(\varphi \to \psi) \to (\varphi \to \Box_x\psi)$, given that $\Box_x\varphi$ and $\Box_x\psi$ contain the same parameters and x is not free in φ. This formula follows immediately from A1 and A2.

T $\Box_x(\Box_x\varphi \to \varphi)$.

Observe that $\Box_x\varphi \to \varphi$ is not valid in L_{str} since it is possible that φ is false on an assignment s, but true on every other assignment which can be obtained from s by changing the value of x to some d, so that $\langle x, d\rangle$ is in the relation R to the assignment of values to the rest of free variables of φ; this just means that $\langle x, s(x)\rangle$ does not satisfy this condition. Prefixing $\Box_x\varphi \to \varphi$ by $\Box_x$ means that we switch to the assignments s with $\langle x, s(x)\rangle$ accessible from the free variables of $\Box_x\varphi \to \varphi$; on such assignments, if $\Box_x\varphi$ is valid, so is φ.

Proof.

1.	$\varphi(x) \to (\Box_x\varphi(x) \to \varphi(x))$	A0
2.	$\Box_x(\varphi(x) \to (\Box_x\varphi(x) \to \varphi(x)))$	1, R2
3.	$\Box_x\varphi(x) \to \Box_x(\Box_x\varphi(x) \to \varphi(x)))$	2, A1, R1
4.	$\neg\Box_x\varphi(x) \to (\Box_x\varphi \to \varphi(x))$	A0
5.	$\Box_x(\neg\Box_x\varphi(x) \to (\Box_x\varphi \to \varphi(x)))$	4, R2
6.	$\neg\Box_x\varphi(x) \to \Box_x(\Box_x\varphi(x) \to \varphi(x))$	5, A1b, R1
7.	$\Box_x(\Box_x\varphi(x) \to \varphi(x))$	3, 6, A0, R1 □

Observe that if ψ_1 and ψ_2 have the same free variables (with the possible exception of x), then from $\psi_1 \to (\psi_2 \to (\psi_1 \wedge \psi_2))$ by two applications of A1 we can derive

$\Box_x\psi_1 \wedge \Box_x\psi_2 \to \Box_x(\psi_1 \wedge \psi_2)$. This implies that if all formulas χ_i have the same free variables, then by T the following formula (used in the proof below) is derivable:

$$\Box_x \bigwedge_i (\Box_x \chi_i \to \chi_i).$$

Now we prove completeness of L_{str} for the original language, not containing constants. We prove that for every set of formulas Γ and formula φ, if Γ and φ do not contain constants,

$$\Gamma \vdash \varphi \Leftrightarrow \Gamma \models \varphi.$$

2.2.5. Theorem. *L_{str} is sound and complete with respect to the class of structured dependence models: $\Gamma \vdash \varphi \Leftrightarrow \Gamma \models \varphi$.*

Proof. The proof of soundness is easy. As usual, we show that axioms are valid and inference rules preserve validity.

It is obvious that A1 is valid. Observe that A2 is valid due to the fact that structured dependence models satisfy locality. We have already given an example showing that unrestricted =4 is not valid. Now we prove that =4a and =4b are valid. Let φ be atomic. Observe that $M, s \models t_1 = t_2 \Leftrightarrow [t_1]_{M,s} = [t_2]_{M,s}$. $M, s \models P(\ldots t_1 \ldots)$ iff $\langle \ldots [t_1]_{M,s} \ldots \rangle \in V(P)$ iff $\langle \ldots [t_2]_{M,s} \ldots \rangle \in V(P) \Leftrightarrow M, s \models P(\ldots t_2 \ldots)$. This shows that =4a is valid. To prove that =4b is valid, assume that $\eta(t_1) = \eta(t_2)$, and reason by induction on complexity of φ. We have already proved the atomic case. The cases for $\neg$ and $\wedge$ are trivial. Finally,

$$M, s \models \Diamond_x \varphi(\ldots t_1 \ldots) \Leftrightarrow \exists d(\langle x, d\rangle \; R \; \{\ldots \langle \eta(t_1), [t_1]_{M,s}\rangle \ldots\} \wedge M, s^x_d \models \varphi);$$

since t_2 is free for t_1 and $\eta(t_1) = \eta(t_2)$, this is equivalent to

$$\exists d(\langle x, d\rangle \; R \; \{\ldots \langle \eta(t_2), [t_2]_{M,s}\rangle \ldots\} \wedge M, s^x_d \models \varphi(t_1/t_2)) \Leftrightarrow M, s \models \Diamond_x \varphi(\ldots t_2 \ldots).$$

To show that R2 preserves validity, assume that $\Box_{v_i}\varphi(v_i)$ is not valid, that is, there exist a model M and an assignment s such that $M, s \not\models \Box_{v_i}\varphi(v_i)$. Then there is a sequence s', which differs from s at most in i, such that $M, s' \not\models \varphi$. If t^i is v_i, we are done: $\varphi(v_i)$ is not valid. If t^i is some d^i, interpret d^i as $s(v_i)$ (observe that d^i is not free in $\varphi(v_i)$!). An argument analogous to the proof of validity of $= 4b$ above shows that $\neg\varphi(d^i)$ is satisfiable, thus $\varphi(d^i)$ is not valid.

For completeness, it is enough to show that every set Δ not containing constants and consistent with L_{str} has a model.

Assume that we have such a set Δ. We are going to construct a canonical model for Δ. The construction is analogous to the proof of completeness theorem for the minimal logic of dependence models given in Chapter 3 (cf. (van Benthem and Alechina 1993)).

We extend the language by adding countably many new constants

$$u_0^0, u_1^0, u_2^0, \ldots, u_n^0, \ldots$$
$$\ldots$$
$$u_0^m, u_1^m, u_2^m, \ldots, u_n^m, \ldots$$
$$\ldots$$

As usual, the domain of the model for Δ will consist of the equivalence classes of constants. Replace each free occurrence of each variable v_i in Δ by u_0^i. The resulting set of sentences is called Δ'. If we construct a model for Δ', then this model and the assignment $s_0 : v_i \mapsto [u_0^i]$ will satisfy Δ.

Make a list of sentences in the new language. Let us call this list Φ.

Let $\Sigma_0 = \Delta$, and $\Sigma_n = \Sigma_{n-1}$ if the nth formula in Φ is not consistent with Σ_{n-1}. If φ_n is consistent with Σ_{n-1}, we have two possibilities. If φ_n is not of the form $\Diamond_{v_i}\psi$, then $\Sigma_n = \Sigma_{n-1} \cup \{\varphi_n\}$.

If φ_n is of the form $\Diamond_{v_i}\psi$, we add a witness for $\Diamond_{v_i}$ as follows. We take the first new constant u_j^i which does not occur in Σ_{n-1} and add to Σ_{n-1} $\psi[v_i/u_j^i]$ and the set $\{\Box_{v_i}\chi \rightarrow \chi(v_i/u_j^i) : \Box_{v_i}\chi$ contains precisely the same constants as $\Diamond_{v_i}\psi\}$.

Let $\Sigma = \bigcup_n \Sigma_n$. Assume that Σ is inconsistent. Then a contradiction is derivable at some finite step n. Let Σ_{n-1} be consistent, and assume that Σ_n is not. Then for some $\Diamond_{v_i}\psi$,

$$\Sigma_{n-1} \cup \{\bigwedge_{\chi}(\Box_{v_i}\chi \rightarrow \chi(u_j^i)) \wedge \Diamond_{v_i}\psi \wedge \psi(u_j^i)\}$$

is inconsistent. Again, a contradiction is derivable from some finite conjunction of formulas in Σ_{n-1} and a finite conjunction of $\Box_{v_i}\chi_k \rightarrow \chi_k$ for $1 \leq k \leq m$.

By R2 we obtain

$$\Sigma_{n-1} \vdash \Box_{v_i} \bigwedge_{k \leq m} (\Box_{v_i}\chi_k \rightarrow \chi_k(v_i)) \rightarrow \Box_{v_i}(\Diamond_{v_i}\psi(v_i) \rightarrow \neg\psi(v_i))$$

Now we apply A1 (all χ_k have the same parameters) and T to derive

$$\Box_{v_i} \bigwedge_{k \leq m} (\Box_{v_i}\chi_k \rightarrow \chi_k).$$

Together with the statement above, this gives

$$\Sigma_{n-1} \vdash \Box_{v_i}(\Diamond_{v_i}\psi(v_i) \rightarrow \neg\psi(v_i)),$$

which by A1b implies

$$\Sigma_{n-1} \vdash \Diamond_{v_i}\psi(v_i) \rightarrow \Box_{v_i}\neg\psi(v_i),$$

that is,

$$\Sigma_{n-1} \vdash \Diamond_{v_i}\psi(v_i) \rightarrow \neg\Diamond_{v_i}\psi(v_i),$$

i.e. contrary to our assumption Σ_{n-1} is inconsistent (note that Σ_{n-1} does not contain u_j^i free, therefore the deduction theorem is applicable).

Now we have established that Σ is a maximal consistent set, i.e. it is consistent, it contains, for every formula $\varphi \in \Phi$, either φ or $\neg\varphi$, it is closed deductively, and it contains $\varphi_1 \wedge \varphi_2$ iff it contains both φ_1 and φ_2.

We define a model M_Σ for Δ using Σ as follows. Let the domain be the set of equivalence classes of constants, i.e. elements of the form $[u^i_j] = \{u : u = u^i_j \in \Sigma\}$.

For the interpretation function, we put

$$\langle [d_1], \ldots, [d_n] \rangle \in V(P) \text{ iff } P(d_1, \ldots, d_n) \in \Sigma.$$

This is well defined by the axiom =4a. The interpretation of constants is quite natural: $V(u) = [u]$.

The definition of R is slightly less straightforward:

$$\langle v_{i_1}, [d_1] \rangle R \{\langle v_{i_2}, [d_2] \rangle, \ldots, \langle v_{i_n}, [d_n] \rangle\}$$

holds in M_Σ iff for some $e_1, \ldots, e_n$, such that $e_j \in [d_j]$ and $\eta(e_j) = i_j$ and for all formulas χ, such that $\Box_{v_{i_1}} \chi$ contains precisely the constants $e_2, \ldots, e_n$,

$$\forall \chi (\Box_{v_{i_1}} \chi(v_{i_1}, e_2, \ldots, e_n) \rightarrow \chi(e_1, e_2, \ldots, e_n) \in \Sigma).$$

Observe that this is equivalent to

$$\forall \chi (\chi(e_1, e_2, \ldots, e_n) \rightarrow \Diamond_{v_{i_1}} \chi(v_{i_1}, e_2, \ldots, e_n) \in \Sigma)$$

Also, observe that by =4b, if

$$\forall \chi (\Box_{v_{i_1}} \chi(v_{i_1}, e_2, \ldots, e_n) \rightarrow \chi(e_1, e_2, \ldots, e_n) \in \Sigma)$$

holds for *some* $e_1, \ldots, e_n$ with $e_j \in [d_j]$ and $\eta(e_j) = i_j$, then it holds for *all* such constants.

2.2.6. Lemma. *(Truth Lemma) For every sentence φ, such that φ contains not more than one term of each type in the scope of each quantifier, $M_\Sigma \models \varphi \Leftrightarrow \varphi \in \Sigma$.*

Proof. The proof of the lemma goes by induction. The inductive hypothesis is somewhat stronger than the statement of the lemma; it says

$$\varphi(d_1, \ldots, d_n) \in \Sigma \Leftrightarrow M_\Sigma, s \models \varphi(v_{i_1}, \ldots, v_{i_m}, d_{m+1}, \ldots, d_n),$$

$0 \leq m \leq n$, where $s(v_{i_j}) = [d_j]$ and $\eta(d_j) = i_j$ for $j \leq m$, for every formula φ which satisfies the condition of the lemma.

Without this condition, the assignment s above is not well defined. Namely, if φ contains two different constants d and e of type i, which are not in the same equivalence class, s would have to assign v_i two different objects $[d]$ and $[e]$.

The clause for atomic formulas follows from the definition of V_Σ. The clauses for negation and conjunction are trivial. Assume that φ is of the form $\Diamond_{v_{i_1}} \psi(v_{i_1}, d_2, \ldots, d_n)$ with $\eta(d_j) = i_j$.

Let $\Diamond_{v_{i_1}}\psi(v_{i_1}, d_2, \ldots, d_n) \in \Sigma$. By the construction of Σ, there is a witness for $\Diamond_{v_{i_1}}$, namely a constant d_1 with $\eta(d_1) = i_1$, such that $\psi(d_1, \ldots, d_n) \in \Sigma$ and

$$\forall\chi(\Box_{v_{i_1}}\chi(v_{i_1}, d_2, \ldots, d_n) \to \chi(d_1, d_2, \ldots, d_n) \in \Sigma).$$

By the definition of R, the latter means

$$\langle v_{i_1}, [d_1]\rangle R\{\langle v_{i_2}, [d_2]\rangle, \ldots, \langle v_{i_n}, [d_n]\rangle\}.$$

By the inductive hypothesis, $M_\Sigma, s \models \psi(v_{i_1}, \ldots, v_{i_m}, d_{m+1}, \ldots, d_n)$, where $s(v_{i_j}) = [d_j]$. Thus $M_\Sigma, s' \models \Diamond_{v_{i_1}}\psi(v_{i_1}, \ldots, v_{i_m}, d_{m+1}, \ldots, d_n)$, where s' is any assignment which differs from s in the i_1st coordinate.

Let $M_\Sigma, s \models \Diamond_{v_{i_1}}\psi(v_{i_1}, \ldots, v_{i_m}, d_{m+1}, \ldots, d_n)$, where $s(v_{i_j}) = [d_j]$ for $1 < j \leq m$, and $\eta(d_j) = i_j$ for $m < j \leq n$. Then for some $[d_1]$

$$\langle v_{i_1}, [d_1]\rangle R\{\langle v_{i_2}, [d_2]\rangle, \ldots, \langle v_{i_n}, [d_n]\rangle\}$$

and $M_\Sigma, s^{i_1}_{[d_1]} \models \psi(v_{i_1}, \ldots, v_{i_m}, d_{m+1}, \ldots, d_n)$. By the definition of R, there are $e_1, \ldots, e_n$, with $e_j \in [d_j]$ and $\eta(e_j) = i_j$, $1 \leq j \leq n$, such that

$$\forall\chi(\chi(e_1, e_2, \ldots, e_n) \to \Diamond_{v_{i_1}}\chi(v_{i_1}, e_2, \ldots, e_n) \in \Sigma).$$

By the inductive hypothesis, $\psi(e_1, \ldots, e_n) \in \Sigma$. Therefore $\Diamond_{v_{i_1}}\psi(v_{i_1}, e_2, \ldots, e_n) \in \Sigma$. By =4b, the same holds for any $e'_2, \ldots, e'_n$ in the eqivalence classes $[d_2], \ldots, [d_n]$ if $\eta(e_j) = i_j$.

This ends the proof of the truth lemma. □

Observe that the set Δ' consists of sentences satisfying the condition of the lemma; so, the lemma gives $M_\Sigma \models \Delta'$.

Coming back to the restriction on the set of formulas for which the truth lemma holds, note that since the extended language contains formulas of the form $\varphi(u^1_0, u^1_1)$, it may happen that the following set of pairs is in R:

$$\langle v_1, u^1_2\rangle R\{\langle v_1, u^1_0\rangle, \langle v_1, u^1_1\rangle\}.$$

This looks quite counterintuitive given that in accordance with the intended meaning of R this means: 'if the variable v_1 takes the value u^1_0 *and* the value u^1_1, u^1_2 is a possible value for v_1'. Later, when we interpret

$$\langle x_1, d_1\rangle R\{\langle x_2, d_2\rangle, \ldots, \langle x_n, d_n\rangle\}$$

as '$\{\langle x_i, d_i\rangle : 1 \leq i \leq n\}$ is a good *assignment*', i.e. a function, this becomes completely unacceptable. So the last step in the construction of the canonical model is to show that we may leave only such decent, or functional sets of pairs in the relation R and throw away all non-functional sets of pairs in R, where some variable occurs with different objects.

2.2.7. Lemma. *Let φ be a formula which contains at most one term of each type in the scope of each quantifier. Then for every structured dependence model M and assignment s, $M, s \models \varphi \Leftrightarrow M', s \models \varphi$, where M' is obtained from M by leaving only functional sets of pairs in the relation R.*

Proof. The proof goes by induction on the complexity of φ. The only non-trivial step is the quantifier clause. Let φ be $\Diamond_{v_i}\psi(v_i, t_1, \ldots, t_n)$. By assumption, $\eta(t_j) \neq v_i$ for $1 \leq j \leq n$, and $\eta(t_j) \neq \eta(t_k)$ for $1 \leq j \neq k \leq n$. But then there is an element d satisfying

$$\langle v_i, d\rangle\ R\ \{\langle \eta(t_1), [t_1]_{M,s}\rangle, \ldots \langle \eta(t_n), [t_n]_{M,s}\rangle\}$$

if and only if there is a corresponding element in M', since the set of pairs above is functional. Therefore

$$M, s \models \Diamond_{v_i}\psi(v_i, t_1, \ldots, t_n) \Leftrightarrow M', s \models \Diamond_{v_i}\psi(v_i, t_1, \ldots, t_n).$$

This finishes the proof of the lemma and of the Theorem 2.2.5. □

2.3 Obtaining stronger logics

2.3.1 In the direction of generalized quantifiers

Every generalized quantifier satisfies the axiom of alphabetic variants: $\Box_x\varphi \to \Box_y\varphi(x/y)$, with y free for x in φ. It turns out that this axiom is the only one which we need to add to L_{str} to obtain the minimal logic of *dependence models*.

2.3.1. Proposition. $L_{str} + \Box_x\varphi \to \Box_y\varphi(x/y)$ *is complete with respect to the class of dependence models.*

Proof. Follows from the completeness theorem for the minimal logic of dependence models, cf. Chapter 3. □

Observe that the *names* of the variables do not matter in dependence models, where R is a relation between an object and a finite set of objects. Not surpisingly, the axiom of renaming bound variables corresponds to a sort of permutation equivalence for variables:

2.3.2. Consequence. $L_{str} + \Box_x\varphi \to \Box_y\varphi(x/y)$ *is complete with respect to the class of structured dependence models satisfying the following property: for any permutation of variables τ,*

$$\langle v_{i_1}, d_{i_1}\rangle R\{\langle v_{i_2}, d_{i_2}\rangle, \ldots, \langle v_{i_n}, d_{i_n}\rangle\} \equiv$$
$$\equiv \langle \tau(v_{i_1}), d_{i_1}\rangle R\{\langle \tau(v_{i_2}), d_{i_2}\rangle, \ldots, \langle \tau(v_{i_n}), d_{i_n}\rangle\} \equiv R(d_{i_1}, \{d_{i_2}, \ldots, d_{i_n}\}).$$

Other examples of strengthening the logic to obtain 'real' generalized quantifiers are given in Chapter 4.

The dependence relation in dependence models has a sort of built-in antisymmetry; in a derivation, relations between parameters can be represented as a directed acyclic graph (cf. (Alechina and van Lambalgen 1995b)). The situation is very different for assignments models.

2.3.2 Circular dependencies, or assignments

In this section we take a somewhat different approach to modal quantifiers, based on the intuition that the truth of a quantified formula depends on availability of a certain assignment (rather than a certain object). First, we consider logics for partial assignments models.

2.3.3. DEFINITION. A *partial assignment* is a finite set of pairs $\langle x, d\rangle$, where x is a variable and d an element of the domain. □

There are two possibilities for defining partial assignments models. Andréka and Németi proposed (for assignments models) to distinguish models where only part of assignments are present (Crs) and models where all assignments are present, but only part of them is available for the quantifier (Crs^+). The first kind of models satisfies $\varphi \rightarrow \Diamond_x \varphi$, and the second kind does not. The latter +-variant is closer to the intuitions underlying the logic of structured dependence models; we shall see that one additional condition on R yields the logic corresponding to the +-variant of partial assignments models. To obtain the variant without +, we'll have to change the truth definition instead of just adding one more condition on R. Most of the proofs below go by way of first proving a certain property for the + variant and then extending the result to the proper (partial) assignments models. This is the reason why we consider both usual and + variants.

2.3.4. DEFINITION. A *partial assignments model* is a structure of the form $M = \langle D, W, V\rangle$, where $\langle D, V\rangle$ is a first order model and W is an arbitrary non-empty set of partial assignments. The relation $M, s \models \varphi$, where $s : FV(\varphi) \longrightarrow D$ is a partial assignment, is defined as follows:

- The clause for atoms and negation is the same as for structured dependence models (with assignments restricted to the free variables of a formula);
- $M, s\lceil FV(\psi_1 \wedge \psi_2) \models \psi_1 \wedge \psi_2 \Leftrightarrow M, s\lceil FV(\psi_1) \models \psi_1 \;\wedge\; M, s\lceil FV(\psi_2) \models \psi_2$;
- $M, s\lceil (y_1, \ldots, y_n) \models \Diamond_x \psi(x, y_1, \ldots, y_n) \Leftrightarrow$
  $\Leftrightarrow\; \exists d \in D(s' = \{\langle x, d\rangle, \langle y_1, s(y_1)\rangle, \ldots, \langle y_n, s(y_n)\rangle\} \in W \;\wedge\; M, s' \models \psi(x, \bar{y}))$

We say that a formula φ is true in a partial assignments model M ($M \models \varphi$) if for all assignments s of the right length *in W* $M, s \models \varphi$. The set of formulas valid in this class of models is called $L_{p.a.}$. For a more familiar notion of validity, namely $M \models \varphi$ if *all* assignments of the right length satisfy φ, we use the notation $L^+_{p.a.}$, following the notation proposed by Andréka and Németi for Crs^+_n. □

This truth definition demands something stronger than that the pair $\langle x, d\rangle$ is accessible from the rest. The fact that

$$\{\langle x, d\rangle, \langle y_1, s(y_1)\rangle, \ldots, \langle y_n, s(y_n)\rangle\}$$

is an admissible partial assignment implies that also $\langle y_1, s(y_1)\rangle$ is accessible from $\{\langle x, d\rangle, \langle y_2, s(y_2)\rangle, \ldots, \langle y_n, s(y_n)\rangle\}$, and so on. The relation '$\langle x, d\rangle$ and the set of pairs D' form a good assignment' is not directed; in a sense, every pair in $\{\langle x, d\rangle\} \cup D'$ depends on the rest.

2.3.5. LEMMA. *The class of structured dependence models satisfying the* circularity property*: if*

$$\langle x_{i_1}, d_{i_1}\rangle R\{\langle x_{i_2}, d_{i_2}\rangle, \ldots, \langle x_{i_n}, d_{i_n}\rangle\}$$

holds, then for every pair $\langle x_{i_j}, d_{i_j}\rangle$,

$$\langle x_{i_j}, d_{i_j}\rangle R\{\langle x_{i_1}, d_{i_1}\rangle, \ldots, \langle x_{i_n}, d_{i_n}\rangle\}\backslash\{\langle x_{i_j}, d_{i_j}\rangle\},$$

defines the same logic as $L^+_{p.a.}$.

Proof. Let φ be any formula, and assume there are a structured dependence model M and an ordinary assignment s, such that $M, s \models \varphi$. Define $M_{p.a.}$ to have the same domain and interpretation function as M and put

$$\{\langle x_1, d_1\rangle, \ldots, \langle x_n, d_n\rangle\} \in W \;\Leftrightarrow\; \langle x_1, d_1\rangle\; R\; \{\langle x_2, d_2\rangle, \ldots, \langle x_n, d_n\rangle\}$$

This is well defined given the circularity property. An easy induction shows that

$$M, s \models \varphi \Leftrightarrow M_{p.a.}, s\lceil FV(\varphi) \models \varphi.$$

Conversely, let M be a partial assignments model, s a partial assignment and $M, s \models \varphi$. We define M_{str} to have the same domain and interpretation, as M, and

$$\langle x, d\rangle\; R\; z \Leftrightarrow \{\langle x, d\rangle\} \cup z \in W,$$

where z is a partial assignment. Let s' agree with s on the free variables of φ, and assign some fixed element of D to the rest of the variables. Again, it is easy to show by induction that $M_{str}, s' \models \varphi$. □

2.3.6. PROPOSITION. $L_{str} + \Box_{v_i}(\varphi \to \Diamond_{v_j}\varphi)$, *where* $v_i, v_j \in FV(\varphi)$ *(A3) is complete with respect to the class of structured dependence models satisfying circularity:*

$$\langle v_i, d_i\rangle R\{\ldots\langle v_j, d_j\rangle\ldots\} \Longrightarrow \langle v_j, d_j\rangle R\{\ldots\langle v_i, d_i\rangle\ldots\}$$

Proof. To show that the axiom is valid in the class of models satisfying circularity, consider an arbitrary model M from this class, an assignment s and a formula φ with $FV(\varphi) = v_i, v_j, \bar{v}$. Note that if $v_i = v_j$, the axiom is valid already in L_{str} (it is equivalent to T). Consider a sequence $s' =_i s$ with $\langle v_i, s'(v_i)\rangle\; R\; s\lceil\bar{v} \cup \{\langle v_j, s(v_j)\rangle\}$, such that $M, s' \models \varphi$. Then by circularity $\langle v_j, s(v_j)\rangle\; R\; s\lceil\bar{v} \cup \{\langle v_i, s'(v_i)\rangle\}$, that is, $M, s' \models \Diamond_{v_j}\varphi$. But then $M, s \models \Box_{v_i}(\varphi \to \Diamond_{v_j}\varphi)$.

To prove completeness, we use the fact that any set of sentences consistent with $L_{str} + A3$ has a canonical model M_Σ as constructed above. Assume that in M_Σ

$$\langle v_i, d_i\rangle R\{\langle v_1, d_1\rangle, \ldots, \langle v_n, d_n\rangle\}\backslash\{\langle v_i, d_i\rangle\}$$

holds, that is, for every formula χ with constants $d_1, \ldots, d_n$, $\chi \to \Diamond_{v_i}\chi \in \Sigma$ (or, equivalently, $\Box_{v_i}\chi \to \chi \in \Sigma$). It suffices to show that

$$\langle v_j, d_j\rangle R\{\langle v_1, d_1\rangle, \ldots, \langle v_n, d_n\rangle\}\backslash\{\langle v_j, d_j\rangle\},$$

that is, that for every formula χ with constants $d_1, \ldots, d_n$, $\chi \to \Diamond_{v_j}\chi \in \Sigma$. Take an arbitrary formula χ with constants $d_1, \ldots, d_n$. Consider the formula $\chi \to \Diamond_{v_j}\chi$. This formula (let us call it θ) contains constants $d_1, \ldots, d_n$. By assumption, $\Box_{v_i}\theta \to \theta \in \Sigma$. But $\Box_{v_i}\theta$ is an axiom, so $\theta \in \Sigma$. □

Due to Lemma 2.3.5, this proves the completeness of L_{str} + A3 wrt $L^+_{p.a.}$.

2.3.7. PROPOSITION. *Adding to L_{str} the axiom $\varphi \to \Diamond_{v_j}\varphi$ (A4) yields a logic complete wrt $L_{p.a.}$.*

Proof. Note that A4 implies A3, so we can use the previous proof to show that there is a partial assignments model M_Σ and an assignment s_0 satisfying a formula φ consistent with A0 – A4, R1, R2. But we do not know yet whether this assignment is in W.

If A4 is true in the canonical model M_Σ, then every assignment

$$\{\langle v_{i_1}, d_{i_1}\rangle, \ldots, \langle v_{i_n}, d_{i_n}\rangle\}$$

is a good assignment given that all d_{i_j} are of type i_j. Note that the assignment s_0 satisfying the original formula φ (which contains only the variables and no constants) is of such type, therefore s_0 is in W. This gives us $M_\Sigma, s_0 \in W \models \Delta$. □

2.4 Crs_n-models

Now we move to classes of models where the notion of assignment stands really central; assignments have a fixed length and do not shrink and extend depending on a formula.

To define assignments models, in particular models with assignments of length α, we need first to define formally a language with at most α variables.

2.4.1. DEFINITION. $\mathcal{L}_\alpha(\Diamond)$ is a first order language containing α many variables v_i, $0 \leq i < \alpha$, $=$, $\neg, \wedge$ and $\Diamond$ and countably many predicate symbols. Well formed formulas (w.f.f.'s) are defined in a standard way. □

In the meta-language we will use $x_1, \ldots, x_n, \ldots$ to denote arbitrary variables of $\mathcal{L}_\alpha(\Diamond)$.

2.4.2. DEFINITION. A *Crs_α-model* for $\mathcal{L}_\alpha(\Diamond)$ is a structure M of the following form: $M = \langle D, W, V\rangle$, where D is a non-empty domain, $W \subseteq D^\alpha$ is non-empty, and V is an interpretation function. Since there are α many variables, an *assignment* of elements of D to the variables is any element of D^α. The relation 'a formula φ is true in M under assignment s ($M, s \models \varphi$)' is defined *only for assignments in W* inductively as follows:

$M, s \models P(x_1, \ldots, x_n) \Leftrightarrow \langle s(x_1), \ldots, s(x_n)\rangle \in V(P)$;
$M, s \models x_1 = x_2 \Leftrightarrow s(x_1) = s(x_2)$;
$M, s \models \neg\psi \Leftrightarrow M, s \not\models \psi$;
$M, s \models \psi_1 \wedge \psi_2 \Leftrightarrow M, s \models \psi_1$ and $M, s \models \psi_2$;
$M, s \models \Diamond_x\psi \Leftrightarrow \exists s' =_x s(s' \in W \wedge M, s' \models \psi)$. □

We write $M \models \varphi$ (φ true in M) for $\forall s \in W(M, s \models \varphi)$.

2.4.3. Definition. A Crs^+_α*-model* for $\mathcal{L}_\alpha(\Diamond)$ is a structure M of the following form: $M = \langle D, W, V\rangle$, where D is a non-empty domain, $W \subseteq D^\alpha$ is non-empty, and V is an interpretation function. The relation 'a formula φ is true in M under assignment s ($M, s \models \varphi$)' is defined inductively as above.

Now $M \models \varphi$ (φ true in M) stands for $\forall s \in D^\alpha(M, s \models \varphi)$. □

A formula is $Crs_\alpha(Crs^+_\alpha)$-valid, if it is true in all models. A formula is $Crs_\alpha(Crs^+_\alpha)$-satisfiable, if its negation is not $Crs_\alpha(Crs^+_\alpha)$-valid.

It is clear that for $\alpha = \omega$ assignments models cannot be reduced to a special case of structured dependence models, since R must become an infinite relation. For $\alpha < \omega$, there is a resemblance between partial assignments models and assignments models. But there is an important property of structured dependence models which holds also for dependence models and partial assignments models (being special cases of structured dependence models), but not for assignments models, namely locality.

To get a feeling why locality fails in assignments models, consider the following example for $\mathcal{L}_2(\Diamond)$. Let M be a Crs_2 (Crs^+_2)-model with domain $D = \{0, 1\}$, $V(P) = \{\langle 1\rangle\}$ and W consisting of the following assignments:

	v_0	v_1
$s =$	0	0
$s' =$	1	0
$s'' =$	0	1

Then $M, s \models \Diamond_{v_1} P(v_1)$ and $M, s' \not\models \Diamond_{v_1} P(v_1)$, although these assignments agree on the free variables of $\Diamond_{v_1} P(v_1)$.

The failure of locality is the reason why the axiom $\varphi \to \Box_x \varphi$, given that x is not free in φ, is not valid in assignments models. Namely, consider the model above; $M, s \models \Diamond_{v_1} P(v_1)$ but $M, s \not\models \Box_{v_0} \Diamond_{v_1} P(v_1)$.

Since we cannot give an axiomatization of assignments models based on L_{str}, we define an *embedding* of Crs^+_n-logic into $L^+_{p.a.}$ and of Crs_n-logic into $L_{p.a.}$. Both embeddings use the following simple idea. Consider the formulas of the n variable language $\mathcal{L}_n$. The difference in the truth conditions for these formulas in assignments models and in partial assignments models is that in the former a quantifier takes into account the assignment of values to all n variables, whether they occur free under the quantifier or not, whereas in the latter only the free variables are important. For the formulas which contain all n variables free under a quantifier, the truth conditions are the same. We take a formula φ of $\mathcal{L}_n$ and add dummy variables under every quantifier in φ. The result is called φ^n. We show that φ^n is equivalent to φ in assignments models, and that φ^n is satisfiable in assignments models if, and only if, it is satisfiable in partial assignments models. This gives the desired embedding.

Observe that formulas which have all n variables under each quantifier satisfy locality in Crs_n:

2.4.4. Lemma. *Let φ be a formula of $\mathcal{L}_n(\Diamond)$ which contains all n variables in the scope of each quantifier. Then for every Crs (Crs^+)-model M and a pair of assignments s, z which agree on the free variables of φ, $M, s \models \varphi \Leftrightarrow M, z \models \varphi$.*

Proof. The proof goes by induction on the complexity of φ; for atomic formulas the claim is obviously true, the clauses for $\neg$ and $\wedge$ use the induction hypothesis in a standard way. Assume $M, s \models \Diamond_{v_i}\psi$. Then there is an assignment $s' =_i s$ in W such that $M, s' \models \psi$. Since all variables except v_i are free in $\Diamond_{v_i}\psi$, and z agrees with s on the free variables of $\Diamond_{v_i}\psi$, $z =_i s'$, which gives $M, z \models \Diamond_{v_i}\psi$. □

This property will be also used in the proof of embedding results below.

Embedding of Crs_n^+ into $L_{p.a.}^+$

Define a translation function ${}^n : \mathcal{L}_n(\Diamond) \longrightarrow \mathcal{L}_n(\Diamond)$ ($\mathcal{L}_n(\Diamond)$ is the language using only the variables $v_0, \ldots, v_{n-1}$) as follows: ^{n} commutes with atomic formulas and propositional connectives, and

$$(\Diamond_{v_i}\psi)^n = \Diamond_{v_i}(\top(v_0, \ldots, v_{n-1}) \wedge \psi).$$

2.4.5. Proposition. $Crs_n^+ \models \varphi \Leftrightarrow L_{p.a.}^+ \models \varphi^n$.

Proof. We will prove $Crs_n^+ \not\models \varphi \Leftrightarrow L_{p.a.}^+ \not\models \varphi$. Assume that $Crs_n^+ \not\models \varphi$. Then there is a Crs_n^+ model for $\neg\varphi$. Since in Crs_n^+ $\neg\varphi$ and $(\neg\varphi)^n$ are equivalent, the same model satisfies $(\neg\varphi)^n$. To prove the proposition, it is enough to show that for every $\theta \in \mathcal{L}_n(\Diamond)$, θ^n is Crs_n^+-satisfiable iff θ^n is $L_{p.a.}^+$-satisfiable.

Assume that θ^n is Crs_n^+-satisfiable, i.e. there is a Crs_n^+-model M and an assignment (of length n) s_0, such that $M, s_0 \models \theta^n$. Define a partial assignments model $M_{p.a.+}$ as follows: $V_{p.a.+} = V$, $D_{p.a.+} = D$, and $W_{p.a.+} = W$.

We prove by induction that for every subformula ψ of θ^n

$$M, s \models \psi \Leftrightarrow M_{p.a.+}, s\lceil FV(\psi) \models \psi.$$

The clauses for atoms and $\neg$ are obvious. The clause for $\wedge$ is also easy:

$$M, s \models \psi_1 \wedge \psi_2 \Leftrightarrow M, s \models \psi_1 \;\wedge\; M, s \models \psi_2 \Leftrightarrow$$

$$\Leftrightarrow M_{p.a.+}, s\lceil FV(\psi_1) \models \psi_1 \;\wedge\; M_{p.a.+}, s\lceil FV(\psi_2) \models \psi_2 \Leftrightarrow$$

$$\Leftrightarrow M_{p.a.+}, s\lceil FV(\psi_1 \wedge \psi_2) \models \psi_1 \wedge \psi_2$$

Finally, $M, s \models \Diamond_{v_i}\psi \Leftrightarrow \exists s' =_i s(s' \in W \wedge M, s' \models \psi)$; since ψ is a subformula of θ in the scope of a quantifier, $FV(\psi) = \{v_0, \ldots, v_{n-1}\}$, and $s'\lceil FV(\psi) = s'$. Therefore the condition above is equivalent to

$$\exists d(s' = s\lceil FV(\Diamond_{v_i}\psi) \cup \{\langle v_i, d\rangle\} \in W_{p.a.+} \;\wedge M_{p.a.+}, s' \models \psi),$$

i.e. $M_{p.a.+}, s\lceil FV(\Diamond_{v_i}\psi) \models \Diamond_{v_i}\psi$.

This shows $M_{p.a.+}, s_0\lceil FV(\theta^n) \models \theta^n$, i.e. if θ^n is Crs_n^+-satisfiable, then it has a $L_{p.a.}^+$-model.

To prove the opposite direction, suppose that θ^n is satisfiable in a partial assignments model M and an assignment s_0. We construct a Crs_n^+-model M' and an assignment $a(s_0)$ with $M', a(s_0) \models \theta^n$. Let M' have the same domain and interpretation function as M, and $W' = W \cap \{s : dom(s) = \{v_0, \ldots, v_{n-1}\}\}$. Of course, all assignments in M' are of length n.

Let a be an arbitrary function which gives a partial assignment s from M an assignment $a(s)$ in M', so that $a(s)\lceil dom(s) = s$.

Now we prove $M, s \models \psi \Leftrightarrow M', a(s) \models \psi$. The clause for atoms and negation is trivial. For the conjunction we have:

$$M, s \models \psi_1 \wedge \psi_2 \Leftrightarrow M, s\lceil FV(\psi_1) \models \psi_1 \;\wedge\; M, s\lceil FV(\psi_2) \models \psi_2 \Leftrightarrow$$

$$\Leftrightarrow M', a(s\lceil FV(\psi_1)) \models \psi_1 \;\wedge\; M', a(s\lceil FV(\psi_2)) \models \psi_2$$

Since ψ_1 and ψ_2 are subformulas of θ^n, and $a(s)\lceil FV(\psi_i) = a(s\lceil FV(\psi_i))\lceil FV(\psi_i)$, by lemma 2.4.4 this is equivalent to

$$M', a(s) \models \psi_1 \;\wedge\; M', a(s) \models \psi_2 \Leftrightarrow M', a(s) \models \psi_1 \wedge \psi_2.$$

The clause for the quantifier is easy. Since the assignments used in the quantifier clause are of length n, we use the fact that such an assignment $s' \in W'$ iff $s' \in W$:

$$M, s \models \Diamond_{v_i}\psi \Leftrightarrow \exists d(s' = s \cup \{\langle v_i, d\rangle\} \in W \;\wedge\; M, s' \models \psi) \Leftrightarrow$$

$$\Leftrightarrow \exists s' =_i a(s) \in W'(M', s' \models \psi) \Leftrightarrow M', a(s) \models \Diamond_{v_i}\psi.$$

This gives $M', a(s_0) \models \theta^n$, that is, if θ^n has a $L^+_{p.a.}$-model, then θ^n has a Crs^+_n-model. □

Embedding of Crs_n into $L_{p.a.}$

Given the result above, it is easy to show that the same embedding works for Crs_n and $L_{p.a.}$.

2.4.6. PROPOSITION. $Crs_n \models \varphi \Leftrightarrow L_{p.a.} \models \varphi$.

Proof. The beginning of the argument is the same as above; the problem is reduced to proving that θ^n is Crs_n-satisfiable iff θ^n is $L_{p.a.}$-satisfiable.

Given the definition of a partial assignments model, it is clear that $L_{p.a}$- and $L_{p.a.+}$-models are the same, and only the notions of satisfiability are different. (We cannot reformulate the definition of partial assignments models so that the satisfaction relation would be defined only for the sequences in W; cf. the clause for conjunction.) Therefore, if φ is satisfiable in an $L^+_{p.a.}$-model and the satisfying sequence is in W, then φ is $L_{p.a.}$-satisfiable.

The corresponding fact for the assignments models is also true, but needs a bit more elaborating. Let M be a Crs_n-model, and M^+ be the corresponding Crs^+_n-model (with the same D, W and V). Then for $s \in W$

$$M, s \models \varphi \Leftrightarrow M^+, s \models \varphi.$$

The proof is an obvious induction.

Consider a formula θ^n from the previous embedding proof. Without loss of generality we may assume that θ^n has all n variables free (in all logics, θ^n is satisfiable iff $\theta^n \wedge \top(v_0, \ldots, v_{n-1})$ is satisfiable).

Assume that θ^n is $L_{p.a.}$ satisfiable, i.e. there is an $L_{p.a.}$ model M and an assignment (of length n) $s \in W$ such that $M, s \models \theta^n$. From the analysis above, M is an $L^+_{p.a.}$ model as well, and the embedding proof above gives a Crs^+_n model M' and assignment $a(s) = s$ which satisfy θ^n. Since s is in W and s is of length n, we have $M', s \in W' \models \theta^n$. But then there is a Crs_n-model satisfying θ^n.

Assume that there is a Crs_n-model M satisfying θ^n. Then there is a Crs^+_n-model M^+ such that $M^+, s \in W \models \theta^n$. By the embedding argument, there is an $L^+_{p.a.}$-model $M_{p.a.+}$ with $W_{p.a.+} = W$, such that $M_{p.a.+}, s\lceil FV(\theta^n) \models \theta^n$. But $s\lceil FV(\theta^n) = s$ and $s \in W$. Therefore, θ^n is $L_{p.a.}$-satisfiable. □

From the following fact (which is well-known, cf. for example (Németi 1992), but which we restate here for the sake of completeness) follows that also Crs_ω is embeddable in $L_{p.a.}$:

2.4.7. PROPOSITION. *For every w.f.f. with at most n first variables φ, $\models_{Crs_n} \varphi$ iff $\models_{Crs_\omega} \varphi$.*

Proof. We prove that $\neg\varphi$ is Crs_n-satisfiable iff $\neg\varphi$ is Crs_ω-satisfiable.

The direction from left to right is easy: one can always transform a Crs_n model M, satisfying $\neg\varphi$, into Crs_ω model, by adding the same infinite 'tail' to all sequences in M.

For the converse, let M be a Crs_ω model, and $M, s \models \neg\varphi$. Just 'cutting the tails' of the sequences in M would not work, since it may make some sequence s' for which $s' =_x s$ does not hold in M, accessible from s in the new model. Let, for example, $n = 3$ (which means that φ contains only the variables v_0, v_1 and v_2). Assume that s' differs from s in the 0th and 5th coordinates. Then $s' =_0 s$ does not hold. But if we consider only the first 3 coordinates of s and s', then $s' =_0 s$.

To make a Crs_n model from M, which satisfies φ, we proced as follows. Let M^- be a submodel of M obtained by leaving out all sequences which differ from s in some coordinate m, $m \geq n$. An easy induction shows that

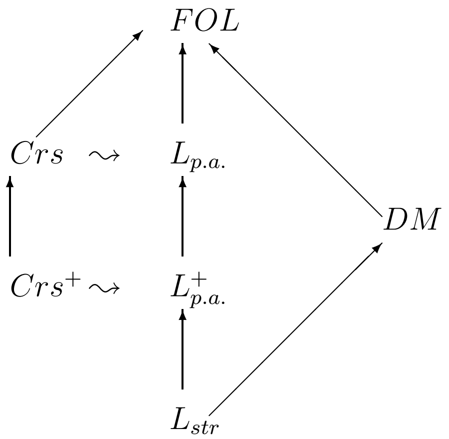
$$M, s \models \varphi \Leftrightarrow M^-, s \models \varphi.$$

(The inductive hypothesis being: for every s' which differs from s in at most n first coordinates, and every formula ψ containing at most n first variables, $M, s' \models \psi \Leftrightarrow M^-, s' \models \psi$).

M^- can be easily transformed into a Crs_n model by 'cutting the tails' of sequences in M^-, making them all of length n. □

2.5 The weakest logic above Crs and dependence models

So far, we have considered the following logics (with inclusions indicated by arrows, DM standing for the minimal logic of dependence models):



There are lots of non-trivial logics between DM and FOL (cf. Chapter 4). But it is interesting that there is no logic which is stronger than both Crs and DM and weaker than first order logic. In this section we are going to show that $Crs + DM = FOL$. This suggests that the intuitions underlying assignments models and dependence models are in a sense orthogonal.

Although we do not give an axiomatization of Crs_n-models, it is clear that at least the following axioms are valid there:

A1' $\Box_x(\varphi \to \psi) \to (\Box_x\varphi \to \Box_x\psi)$ (unrestricted monotonicity);
A4 $\varphi \to \Diamond_x\varphi$.

The following axioms are valid in the class of dependence models:

A0 propositional logic;
A1 $\Box_x(\varphi \to \psi) \to (\Box_x\varphi \to \Box_x\psi)$, given that $\Box_x\varphi$ and $\Box_x\psi$ contain the same parameters (free variables and constants);
A2 $\varphi \to \Box_x\varphi$, given that x is not free in φ;
A5 $\Box_x\varphi \to \Box_y\varphi(x/y)$, given that y is free for x in $\varphi(x)$;
R1 $\varphi,\ \varphi \to \psi/\psi$;
R2 $\varphi(x)/\Box_x\varphi(x)$.

Putting these axioms together and replacing $\Box$ by $\forall$ and $\Diamond$ by $\exists$ we obtain the following system:

Ax0 propositional logic;
Ax1 $\forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi)$, given that x is not free in φ (from unrestricted monotonicity A1' and A2);
Ax2 $\varphi(x) \to \exists y\varphi(x/y)$, with the usual restriction (from A4 and A5);
MP $\varphi,\ \varphi \to \psi/\psi$;
Gen $\varphi(x)/\forall x\varphi(x)$ (from R1).

which axiomatizes first-order logic (cf., for example, (Mendelson 1979)). This proves the following proposition:

2.5.1. Proposition. *The weakest logic which derives all formulas valid in both dependence models and Crs_n-models, is classical first order logic.*

Both the logic of dependence models and Crs_n-models are much weaker than classical first order logic. In particular, they are decidable even in the presence of equality in the language (as we shall see below, this follows from the results of Németi (1992), Andréka and Németi (1994)).

2.6 Restricted fragments of predicate logic

A lot of insight in modal logic comes from considering it as a fragment of first order logic under the *standard translation* (cf. van Benthem (1983)). The same holds for the logics of modal quantifiers.

All logics introduced above can be embedded into fragments of first-order logic with restricted, or bounded, quantification (where every quantifier is restricted by an atomic formula).

In (Andréka, van Benthem and Németi 1995) the following fragment (called Fragment 2) is defined: a fragment of first-order language with equality where each subformula in the scope of $\exists$ is of the form $\exists\bar{x}(R(\bar{x},\bar{y}) \wedge \varphi)$, where R is any relational symbol (not necessarily fixed) and the free variables of φ are among $\bar{x},\bar{y}$. This fragment is shown to be decidable in (Andréka and Németi 1994). In this section, we show that the logics of partial assignments models and the logics of assignments models are embeddable into Fragment 2.

For Crs_n and Crs_n^+ this is a well-known fact (cf., for example, (Németi 1992)). The following translation tr from $\mathcal{L}_n(\Diamond)$ into the language of first-order logic enriched with a predicate R is used to define the embedding: tr commutes with atomic formulas and propositional symbols, and

$$tr(\Diamond_{v_i}\varphi) = \exists v_i(R(v_0,\ldots,v_{n-1}) \wedge tr(\varphi)).$$

Put for Crs_n^+ $\varphi^+ = tr(\varphi)$ and for Crs_n $\varphi^* = R(v_0,\ldots,v_{n-1}) \wedge tr(\varphi)$. Intuitively, $R(d_0,\ldots,d_{n-1})$ means that $\{\langle v_0,d_0\rangle,\ldots,\langle v_{n-1},d_{n-1}\rangle\}$ is an assignment in W. One can easily check that φ is Crs_n (Crs_n^+)-satisfiable iff φ^* (φ^+) is satisfiable in first-order logic.

For the partial assignments models, the idea is completely the same, but we have to introduce countably many relational symbols. The reason is that we must be able to distinguish assignments of the same length but to different variables (coordinates). So we introduce symbols like $R_{i_1,\ldots,i_n}$, $i_1 \leq \ldots \leq i_n$, with $R_{i_1,\ldots,i_n}(d_1,\ldots,d_n)$ meaning that d_j is assigned to the variable v_{i_j}. The translation is, as one would expect, the same as tr except for the quantifiers:

$$tr'(\Diamond_{v_{i_j}}\varphi(v_{i_1},\ldots,v_{i_n})) = \exists v_{i_j}(R_{i_1,\ldots,i_n}(v_{i_1},\ldots,v_{i_n}) \wedge tr'(\varphi))$$

For example, $\Diamond_{v_0}\Diamond_{v_2}P(v_0,v_2)$ is translated a

$$\exists v_0(R_0(v_0) \wedge \exists v_2(R_{0,2}(v_0,v_2) \wedge P(v_0,v_2)))$$

which is equivalent in first order logic to

$$\exists x\exists y(R_0(x) \wedge R_{0,2}(x,y) \wedge P(x,y));$$

$\Diamond_{v_3}\Diamond_{v_1}P(v_3, v_1)$ is translated (up to renaming of the variables) to

$$\exists x\exists y(R_3(x) \wedge R_{1,3}(y, x) \wedge P(x, y)),$$

i.e. to a different formula.

As before, we define $\varphi^+ = tr'(\varphi)$ and $\varphi^* = R_{i_1,\dots,i_n}(v_{i_1}, \dots, v_{i_n}) \wedge tr'(\varphi))$, where $v_{i_1}, \dots, v_{i_n}$ are the free variables of φ. The claim is that φ is $L_{p.a.}(L_{p.a.^+})$-satisfiable iff $\varphi^*(\varphi^+)$ is first-order satisfiable. From this follows

2.6.1. Proposition. *Both logics of partial assignments models are decidable.*

Proof. Follows from the embedding above and the result of Andréka and Németi (1994). □

The same idea applies to L_{str}. Since the difference between L_{str} and $L^+_{p.a.}$ is in the circularity property, we must show which variable was quantified. We add countably many predicates $R^{i_j}_{i_1,\dots,i_n}$, where the superscript stands for the quantified variable. The translation becomes

$$tr''(\Diamond_{v_{i_j}}\varphi(v_{i_1}, \dots, v_{i_n})) = \exists v_{i_j}(R^{i_j}_{i_1,\dots,i_n}(v_{i_1}, \dots, v_{i_n}) \wedge tr''(\varphi))$$

Again, $\varphi^+ = tr''(\varphi)$ and φ is satisfiable in the structured dependence models iff φ^+ is first order satisfiable. We prove that the translation is satisfiability-preserving only for this case; the proofs for partial assignments models are analogous.

2.6.2. Claim. *For any formula φ of $\mathcal{L}(\Diamond)$, φ is satisfiable in a structured dependence model iff $tr''(\varphi)$ is first-order satisfiable.*

Proof. Assume that there is a structured dependence model $M = \langle D, R, V\rangle$ and assignment s which satisfy φ. Let $M' = \langle D, V'\rangle$, where V' agrees with V on all predicates from $\mathcal{L}(\Diamond)$, and

$$\langle d_1, \dots, d_n\rangle \in V'(R^{i_j}_{i_1,\dots,i_n}) \Leftrightarrow \langle v_{i_j}, d_j\rangle\ R\ \{\langle v_{i_1}, d_1\rangle, \dots, \langle v_{i_n}, d_n\rangle\} \setminus \{\langle v_{i_j}, d_j\rangle\}.$$

For example,

$$R^4_{2,4,5}(d_1, d_2, d_3) \Leftrightarrow \langle v_4, d_2\rangle\ R\ \{\langle v_2, d_1\rangle, \langle v_5, d_3\rangle\}.$$

It is easy to prove that for every formula ψ and assignment z,

$$M, z \models \psi \leftrightarrow M', z \models tr''(\psi).$$

The proof goes by induction on the complexity of ψ. The only non-trivial case is of ψ being of the form $\Diamond_{v_{i_j}}\theta(v_{i_1}, \dots, v_{i_n})$. $M, z \models \Diamond_{v_{i_j}}\theta(v_{i_1}, \dots, v_{i_n})$ means that

$$\exists d(\langle v_{i_j}, d\rangle\ R\ \{\langle v_{i_1}, z(v_{i_1})\rangle, \dots, \langle v_{i_n}, z(v_{i_n})\rangle\} \setminus \{\langle v_{i_j}, z(v_{i_j}\rangle\} \wedge M, z^{i_j}_d \models \theta) \Leftrightarrow$$

$$\Leftrightarrow \exists d(R^{i_j}_{i_1,\dots,i_n}(z(v_{i_1}), \dots, d, \dots, z(v_{i_n})) \wedge M', z^{i_j}_d \models tr''(\theta)) \Leftrightarrow$$

$$\Leftrightarrow M', z \models \exists v_{i_j}(R^{i_j}_{i_1,\dots,i_n}(v_{i_1}, \dots, v_{i_n}) \wedge tr''(\theta)).$$

This proves the direction from left to right.

To prove the opposite direction, assume that $tr''(\varphi)$ is first-order satisfiable. We construct a model for φ by keeping the domain and interpretation function the same as in the model for $tr''(\varphi)$ and defining R as follows:

$$\langle v_{i_j}, d_j\rangle \; R \; \{\langle v_{i_1}, d_1\rangle \ldots \langle v_{i_n}, d_n\rangle\} \setminus \{\langle v_{i_j}, d_j\rangle\} \Leftrightarrow \langle d_1, \ldots, d_j, \ldots, d_n\rangle \in V(R^{i_j}_{i_1,\ldots,i_n}).$$

The rest of the argument is the same as above. □

2.6.3. Proposition. *The logic of structured dependence models is decidable.*

Proof. Follows from the embedding above and the result of Andréka and Németi (1994). □

The standard translation for dependence models is given in Chapter 3.

2.7 Tableaux for Crs and Crs^+

To design a proof theory for assignments models, we could have used the same trick with indexed constants as for L_{str}. However there is a much more natural way to do it, which reflects the fact that quantification in Crs has to do with arrays.

In this section, we define analytic tableaux for the logics of assignments models which give their direct formalization independent of the embedding result above. Some well known facts about reduction of $Crs^{(+)}_\omega$- validity to $Crs^{(+)}_n$-validity are also given.

We will use sequences of parameters as labels in analytic tableaux. A labelled formula is an object of the form $\langle d_0, \ldots, d_{n-1}\rangle : \varphi$, where φ is a w.f.f. which uses at most n first variables and $d_0, \ldots, d_{n-1}$ are parameters. In the metalanguage we use $s, s', \ldots$ as metavariables for sequences of parameters, and s_i for the ith element of s.

$$(s' =_i s) =_{df} \bigwedge_j \{s_j = s'_j | j \in \{0, \ldots, n-1\} \setminus \{i\}\}$$

2.7.1. Definition. A *Crs-tableau* for a w.f.f. φ with n variables is a tree, where $\langle d_0, \ldots, d_{n-1}\rangle : \varphi$ is the origin, and the branches are constructed according to the following rules:

$$atom \quad \frac{s : P(x_{i1}, \ldots, x_{ik})}{P(s_{i1}, \ldots, s_{ik})} \qquad \neg atom \quad \frac{s : \neg P(x_{i1}, \ldots, x_{ik})}{\neg P(s_{i1}, \ldots, s_{ik})}$$

$$= \quad \frac{s : x_i = x_j}{s_i = s_j} \qquad \neg = \quad \frac{s : \neg(x_i = x_j)}{s_i \neq s_j}$$

$$\neg\neg \quad \frac{s : \neg\neg\psi}{s : \psi}$$

$$\wedge \quad \frac{s : \varphi_1 \wedge \varphi_2}{s : \varphi_1, s : \varphi_2} \qquad \neg \wedge \quad \frac{s : \neg(\varphi_1 \wedge \varphi_2)}{s : \neg\varphi_1 | s : \neg\varphi_2},$$

where | stands for branching,

$$\Diamond \quad \frac{s : \Diamond_x \psi}{s^x_d : \psi} \qquad \neg\Diamond \quad \frac{s : \neg\Diamond_x \psi}{s' =_x s : \neg\psi}$$

where in the $\Diamond$-rule d is a new element, and in $\neg\Diamond$ $s' =_x s$ is any sequence present in the tableau.

A branch of a tableau is *closed*, if for some P and $\langle d_{i1}, \ldots, d_{ik}\rangle$, both $P(d_{i1}, \ldots, d_{ik})$ and $\neg P(d_{i1}, \ldots, d_{ik})$ occur in this branch; or for some d and e, both $d = e$ and $d \neq e$ occur in this branch; or if one of the axioms below is violated:

I1 $d = d$
I2 $d = e \rightarrow e = d$
I3 $d = c \wedge c = e \rightarrow d = e$
I4 $d_1 = e_1 \wedge \ldots \wedge d_k = e_k \wedge P(d_1, \ldots, d_k) \rightarrow P(e_1, \ldots, e_k)$

A tableau is closed if all its branches are closed. A formula φ is provable if there is a closed tableau for $\neg\varphi$. □

In what follows we call atomic formulas with parameters, like $(\neg)P(d_1, \ldots, d_k)$ or $(\neg)(d_i = d_j)$, also labelled formulas.

2.7.2. Definition. A set Θ of labelled formulas is Crs_n-satisfiable if there is a Crs_n-model M and interpretation of parameters I, with $I(Labels(\Theta)) \subseteq W$, such that for every $s : \psi \in \Theta$, $M, I(s) \models \psi$. □

2.7.3. Theorem. *(Soundness for Crs_n).* For every formula φ which contains at most n first variables, if a tableau for $\neg\varphi$ is closed, then φ is Crs_n-valid.

Proof We first show that for any tableau, if the origin is Crs_n-satisfiable, then there is at least one branch such that the set Θ of labelled formulas on this branch is Crs_n-satisfiable.

The proof goes by induction on the construction of the tableau. We assume that the origin is satisfiable and show that, given a satisfiable set in the premise, any application of a tableau rule yields at least one satisfiable set. The only nontrivial step is for the quantifier rules.

Let $\{\Theta', s : \Diamond_x\psi\}$ be satisfiable. Then there is a model M and an interpretation of parameters I, such that $M, I(s) \models \Diamond_x\psi$. Then there is a sequence $s' =_x I(s)$ in M, such that $M, s' \models \psi$. Take $I(d)$ to be $s'(x)$. Then $M, I(s^x_d) \models \psi$. This implies that the set $\{\Theta', s : \Diamond_x\psi, s^x_d : \psi\}$ is satisfiable.

Let $\{\Theta', s : \neg\Diamond_x\psi\}$ be satisfiable. Obviously, for any sequence $s' =_x s$, $\{\Theta', s : \neg\Diamond_x\psi, s' : \neg\psi\}$ is satisfiable.

Also, if a branch is closed, then the set Θ of formulas on this branch is not Crs_n-satisfiable. Therefore, if a tableau is closed, then the origin is not Crs_n-satisfiable.

But the origin is a labelled formula. We need to show that if $s : \neg\varphi$ is not Crs_n-satisfiable, then $\neg\varphi$ is not Crs_n-satisifable.

Assume that $\neg\varphi$ is Crs_n-satisfiable. Then there is a model M and an assignment s such that $M, s \models \neg\varphi$. But then there is an interpretation of parameters I such that $\langle d_0, \ldots, d_{n-1}\rangle : \neg\varphi$ is satisfiable in M, namely, $I(d_i) = s_i$.

Now we have shown that if a tableau is closed, then $\neg\varphi$ is not Crs_n-satisfiable, that is, φ is Crs_n-valid. □

2.7.4. Definition. A set Θ of labelled formulas is a Hintikka set for Crs_n, if the labels of Θ are of length n and the following conditions hold:

a for no atomic formula with parameters A, $A \in \Theta$ and $\neg A \in \Theta$;
b Θ is consistent with the axioms for identity I1 – I4;
c if $s : \neg\neg\psi \in \Theta$, then $s : \psi \in \Theta$;
d if $s : \psi_1 \wedge \psi_2 \in \Theta$, then $s : \psi_1 \in \Theta$ and $s : \psi_2 \in \Theta$.
e if $s : \neg(\psi_1 \wedge \psi_2) \in \Theta$, then $s : \neg\psi_1 \in \Theta$ or $s : \neg\psi_2 \in \Theta$.
f if $s : \Diamond_x\psi \in \Theta$, then for some sequence $s' =_x s$ $s' : \psi \in \Theta$;
g if $s : \neg\Diamond_x\psi \in \Theta$, then for all sequences $s' \in Labels(\Theta)$, if $s' =_x s$, then $s' : \neg\psi \in \Theta$;
i if $s : (\neg)P(x_{i1}, \ldots, x_{ik}) \in \Theta$, then $(\neg)P(s_{i1}, \ldots, s_{ik}) \in \Theta$;
j if $s : (\neg)(x_i = x_j) \in \Theta$, then $(\neg)(s_i = s_j) \in \Theta$. □

2.7.5. Lemma. *(Hintikka's lemma for Crs_n)* Any Hintikka set Θ for Crs_n is Crs_n-satisfiable.

Proof Take a closure of Θ with respect to identity axioms. Let $[d]$ be the equivalence class of a parameter d. Denote the set $\cup\{range(Labels(\Theta)\}$ (the set of parameters used in Θ) by D. Then the domain of the model for Θ is $D_=$, $I(d) = [d]$, $W = I(Labels(\Theta))$ and V is any function satisfying the following constraints:

$$P(d_1, \ldots, d_k) \in \Theta \Longrightarrow ([d_1], \ldots, [d_k]) \in V(P)$$

$$\neg P(d_1, \ldots, d_k) \in \Theta \Longrightarrow ([d_1], \ldots, [d_k]) \notin V(P)$$

It is easy to check that for any $s : \psi \in \Theta$, $M, I(s) \models \psi$. □

2.7.6. Theorem. *(Completeness for Crs_n)* If φ is Crs_n-valid, then φ is provable, i.e. there exists a closed tableau for $\neg\varphi$.

Proof. Analogously to (Smullyan 1968, p.59), we define a finished systematic tableau as a tableau where every labelled formula has been used as a premise of a rule, and for every labelled formula of the form $s : \neg\Diamond_x\psi$ the rule $\neg\Diamond$ was applied for every label $s' =_x s$ in the tableau. (This can be made constructive by fixing the order of applying the rules by always choosing a formula with the minimal level, i.e. closest to the origin, as a premise, and repeating $s : \neg\Diamond_x\psi$ after introducing all the nodes which can be obtained by applying the $\neg\Diamond$ rule to $s : \neg\Diamond_x\psi$.) A tableau is called finished, when no new nodes can be introduced.

Obviously, every open branch of a finished systematic tableau forms a Hintikka set, and by Hintikka's lemma this set is satisfiable. Since φ is valid, $\neg\varphi$ is not satisfiable and a systematic tableau for $\neg\varphi$ must close. □

2.7.7. Theorem. *(Soundness and completeness for Crs_ω) A formula φ with n variables is Crs_ω valid iff a Crs_n-tableau for φ is closed.*

Proof. Follows from the tableau completeness for Crs_n and Proposition 2.4.7. □

Tableaux for Crs_n^+

A tableau for Crs_n^+ differs from a Crs_n tableau only in the formulation of the quantifier rules:

$$\Diamond^+ \quad \frac{s : \Diamond_x \psi}{s_d^x : \psi \ (d \text{ new}, s_d^x \in W)} \qquad \neg\Diamond^+ \frac{s : \neg\Diamond_x \psi}{s' : \neg\psi \ (s' =_x s, \ s' \in W)}$$

Soundness is proved precisely the same way as for Crs_n. The definition of a Hintikka set for Crs_n^+ differs from the definition for Crs_n in **f** and **g**: add in both $s' \in W$. The Hintikka's lemma is the same but the label of the origin does not have to be in W.

2.7.8. THEOREM. *(Soundness and completeness for Crs_n^+)* Crs_n^+ is sound and complete with respect to the analytic tableaux using the rules $\Diamond^+$ and $\neg\Diamond^+$.

2.8 When there are few relevant variables

So far we considered logics which could be translated into Fragment 2 of Andréka, van Benthem and Németi (1995), namely into the fragment of first order logic where every quantifier is restricted by an atomic formula including all the free variables which occur under the quantifier.

If we change the truth definition of L_{str} to

$$M, s \models \varphi(x, y_1, \ldots, y_n) \Leftrightarrow \exists d(\langle x, d\rangle \ R \ \{\langle z_1, s(z_1)\rangle, \ldots, \langle z_m, s(z_m)\rangle\} \ \wedge \ M, s_d^x \models \varphi),$$

where $\bar{z}$ is some subset of $\bar{y}$ (for example $\emptyset$), the resulting logic cannot be embedded in Fragment 2 anymore. Instead, it ends up in Fragment 3 (where quantifiers are restricted by atomic formulas which include only some free variables occurring under the quantifier). Fragment 3 is shown to be undecidable in (Andréka, van Benthem and Németi 1995).

This does not imply that any logic with such truth definition has to be undecidable. However, a natural logic from the example below turns out to be undecidable.

Let $M, s \models \exists x\varphi(x, y_1, \ldots, y_n) \Leftrightarrow \exists d(\langle x, d\rangle \ R \ \emptyset \ \wedge \ M, s_d^x \models \varphi)$. This can be seen as a special case of structured dependence models, satisfying, for every partial assignment X,

$$\langle x, d\rangle \ R \ X \ \Leftrightarrow \langle x, d\rangle \ R \ \emptyset,$$

or, as initially proposed by Johan van Benthem, as a generalization of many-sorted logic. Let every variable x have its own domain D_x; for different variables, the domains are not necessarily disjoint. Let $\langle x, d\rangle \ R \ \emptyset$ ('d is a possible value for x') stand for $d \in D_x$. Then the truth definition above can be rewritten as

$$M, s \models \exists x\varphi \ \Leftrightarrow \ \exists d \in D_x(M, s_d^x \models \varphi).$$

It turns out that this logic is much stronger than the logics of assignments models and dependence models. In case when the language contains identity, the set of all valid formulas of this logic is undecidable.

2.8.1. THEOREM. *A first order formula φ containing (free and bound) the variables $x_1, \ldots, x_n$, has an ordinary first order model if, and only if,*

$$\varphi \wedge \bigwedge_{i,j} \forall x_i \exists x_j (x_i = x_j),$$

$1 \leq i, j \leq n$, *has a model where each variable has its own domain.*

Proof. One direction is trivial, since every ordinary first order model is a special case of a model where each variable has its own domain, and every ordinary first order model satisfies $\forall x_i \exists x_j (x_i = x_j)$.

Observe that $\forall x_i \exists x_j (x_i = x_j)$ is true in a model where variables have their own domains if, and only if, $D_i = D_j$. It is easy to check that if φ has a model where variables have their own domains, with $D_1 = \ldots = D_n$, and $x_1, \ldots, x_n$ are all the variables occurring in φ, then φ also has a standard model with domain D_1. □

This finishes our investigation of logics of modal quantifiers in general; we turn to our main topic, study of dependence models.

Chapter 3

Minimal logic of dependence models

In this chapter we study some logical properties of the minimal logic of dependence models, where $\Box_x\varphi(x, \bar{e})$ is interpreted as 'for all objects d accessible from $\bar{e}$, $\varphi(d, \bar{e})$ holds'. The motivation for studying such models, as explained in the introduction, is twofold. First, some generalized quantifiers have along with the standard semantics such dependence semantics, and this makes their proof theoretic behaviour tractable. Second, this is an interesting experiment bringing quantification closer to modal logic.

After giving some definitions, we proceed by proving several axiomatic completeness results (which will be used in Chapter 4), preservation theorem, decidability and interpolation. This will serve in the following chapters and hopefully also give a good picture of how the logic behaves.

3.1 Language and models. Informal discussion

The language $\mathcal{L}(\exists\Diamond)$ with a generalized quantifier $\Diamond$ is the ordinary language of first-order predicate logic with equality (without functional symbols) plus an existential generalized quantifier $\Diamond$. The notion of a w.f.f. is extended as follows: if φ is a w.f.f., then so is $\Diamond_x\varphi$. A universal dual of $\Diamond$ is defined as usual: $\Box_x\varphi =_{df} \neg\Diamond_x\neg\varphi$. We shall refer to the sublanguage without ordinary quantifiers as $\mathcal{L}(\Diamond)$. Sometimes we will consider languages without equality, but we always state this explicitely.

We will call the models which we use to interpret these languages, *dependence models*, to distinguish them from the standard models for generalized quantifiers.

3.1.1. Definition. A *dependence model* is a structure of the form $M = \langle D, R, V\rangle$ where D is a domain, V a valuation (a function assigning n-ary predicate symbols subsets of D^n), and R is a relation between elements and finite sets of elements of D, called the dependence relation. We shall write $R(a, b_1, \ldots, b_n)$ or $R(a, \bar{b})$ for $R(a, \{b_1, \ldots, b_n\})$, tacitly assuming that the second argument of R is invariant under permutations and repetitions.

The relation $M, s \models \varphi$ ('φ is *true* in M under assignment s') is defined as in definition 2.1.1, except for

- $M, s \models \exists x\psi(x) \Leftrightarrow$ there exists a variable assignment s' which differs from s at most in its assignment of a value to x ($s' =_x s$) such that $M, s' \models \psi(x)$;
- $M, s \models \Diamond_x \psi(x, y_1, \ldots, y_n) \Leftrightarrow$ there exists $s' =_x s$ such that $R(s'(x), s'(y_1), \ldots, s'(y_n))$ and $M, s' \models \psi(x, \bar{y})$, where $\bar{y}$ are all (and just the) free variables of $\Diamond_x \psi$.
  It is easy to see that
- $M, s \models \Box_x \psi(x, \bar{y}) \Leftrightarrow$ if for all $s' =_x s$: $R(s'(x), s'(\bar{y})) \Rightarrow M, s' \models \psi(x, \bar{y})$.

We say that $M \models \varphi$ iff $M, s \models \varphi$ for all variable assignments s. □

Locality lemma has the same formulation and essentially the same proof as in Chapter 2.

We will call *the minimal logic of dependence models* (or just the minimal logic) the set of all valid formulas in the language of $\mathcal{L}(\exists\Diamond)$.

From the point of view of the theory of generalized quantifiers, the minimal logic has (if any) only technical interest, as a foundation for logics of real generalized quantifiers. Namely, in the minimal logic $\Diamond_x$ need not be a generalized quantifier in Mostovski's sense. Not all formulas satisfiable in dependence models have standard models. This is due to the fact that $\Diamond_x$ is not extensional: if in a model $\{x : \varphi(x)\} = \{x : \psi(x)\}$ holds, it does not follow that $\Diamond_x \varphi$ holds if, and only if, $\Diamond_x \psi$ holds (if φ and ψ have different free variables). If a quantifier is defined as a set of subsets, there is no semantic intuition behind this failure of extensionality.

The failure of extensionality means that the properties which hold for exactly the same objects, are no longer identical. Consider a property P which holds for a single object a. If $s(y) = a$, then $M, s \models \forall x(P(x) \equiv x = y)$. Let $R(a, \emptyset)$ and $\neg R(a, a)$. Then, $M, s \models \Diamond_x P(x)$ and $M, s \not\models \Diamond_x x = y$. This implies that substitution of formulas for predicate letters should also be restricted: only formulas with the same parameters can be substituted.

In fact, the minimal logic can be transformed into a logic with extensional quantifiers: just index quantifiers with finite sets of variables, for example $\Diamond_x^{\bar{y}}$ would replace $\Diamond_x$ in $\Diamond_x \varphi(x, \bar{y})$. These quantifiers will be extensional, but as for their meaning, the question remains the same ...

However strange the minimal logic may be, we will use the metatheorems (about the minimal logic) proved in this chapter, in the following chapters which treat more solid objects. But this is not the only reason for proving these metatheorems. The decidability proof not only gives a decidable fragment of first order logic but analytic tableaux also give an insight in how and why it is decidable: how considering dependencies changes the standard tableau rules for $\forall$ and $\exists$ so that tableau constructions terminate. The interpolation theorem which is proved for the sequent formulation of the minimal logic also demonstrates well how dependencies help to control the derivations in our system: for example, from the form of a formula in a derivation one can immediately determine how many quantifiers can be introduced on the successors of this formula. We believe that this gives the minimal logic some independent interest if not from semantical point of view, then from the proof-theoretic one.

One more reason which can make the minimal logic intuitive is its relation with *modal logic*. We have already mentioned that the truth definition for $\Box_x$ and $\Diamond_x$

resembles the truth definition for modal $\Box$ and $\Diamond$. Now we make the connections between the two logics precise.

The language of propositional modal logic contains propositional variables, boolean connectives (assume for simplicity that we have only $\neg$ and $\wedge$) and a unary modal operator $\Box$ ($\Diamond\varphi = \neg\Box\neg\varphi$). Propositional variables are formulas; if ψ and χ are formulas, so are $\neg\psi$ and $\psi \wedge \chi$. A *model* for this language is a triple $\langle W, R, V\rangle$, where W is a non-empty set of possible worlds, R a binary relation on W and V a functions which assigns propositional variables subsets of W. $M, w \models \varphi$ stands for φ is satisfied in a model M at world w. The clause for modal operators reads $M, w \models \Box\psi \Leftrightarrow \forall v(Rwv \Rightarrow M, v \models \psi)$. The minimal modal logic is called K. φ is valid in K ($\models_K \varphi$) if its negation is not satisfiable.

Let $w_0, \ldots, w_i, \ldots$ be some ordering of the variables of $\mathcal{L}(\Diamond)$. Consider the following translation ^{i} taking modal formulas to formulas of $\mathcal{L}(\Diamond)$ with one free variable w_i:

- $p_n^i = P_n(w_i)$
- commutes with the Booleans
- $(\Box\varphi)^i = \Box_{w_{i+1}}(\top(w_i) \wedge \varphi^{i+1})$

We show that this translation gives an embedding of K into the monadic fragment of the minimal logic of dependence models.

3.1.2. Proposition. *Let φ be a formula of propositional modal logic. Then*

$$\models_K \varphi \Leftrightarrow \models_{DM} \varphi^0.$$

Proof. It is easy to prove that if a model for modal logic M and a world w satisfy $\neg\varphi$, then there is a dependence model M' for $\neg\varphi^0$ and an element a in it, such that $M', w_0/a \models \varphi^0$: just take the domain of M' to be the set of possible worlds in M, put $R'(a, b)$ iff $R(b, a)$ and $a \in V'(P_n)$ iff $a \in V(p_n)$. The same strategy works to convert a model for φ^0 into a model for φ. $\Box$

3.2 Axiomatics and Completeness

3.2.1. Definition. L_{min} is the following system of axiom schemata and inference rules (where φ, ψ are well formed formulas of $\mathcal{L}(\Diamond\exists)$):

C0 all propositional tautologies;
C1 $\Box_x(\varphi \to \psi) \to (\Box_x\varphi \to \Box_x\psi)$, given that $FV(\Box_x\varphi) = FV(\Box_x\psi)$;
C2 $\varphi \to \Box_x\varphi$, given that $x \notin FV(\varphi)$;
C3 $\Box_x\varphi(x) \to \Box_y\varphi(y)$, given that y is free for x in $\varphi(x)$;
C4 $\forall x\varphi(x) \to \varphi(y)$, given that y is free for x in $\varphi(x)$;
C5 $\forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi)$, given that $x \notin FV(\varphi)$;
=1 $x = x$;
=2 $x = y \to y = x$;
=3 $x = y \wedge y = z \to x = z$;
=4 $x = y \wedge \varphi(x) \to \varphi(y)$, given that y is free for x in $\varphi(x)$;

R1 $\varphi, \varphi \to \psi / \psi$;
R2 $\varphi / \Box_x \varphi$;
R3 $\varphi / \forall x \varphi$ □

We will call L^-_{min} the system which is obtained from L_{min} by considering only $\mathcal{L}(\Diamond)$ formulas and leaving out C4, C5 and R3. Observe that C1 is A1 of L_{str}, C2 is A2 and C3 is A5.

For both systems the notions of a derivation and a derivable formula are standard, as well as the formulation of the deduction theorem (cf. Chapter 2).

The proof of the theorem below is not new; this theorem was proved before, for slightly different formulations of the system, by Johan van Benthem and by Cees Doets.

3.2.2. THEOREM. *L_{min} is sound and complete: for every set of formulas Γ and a formula φ,*

$$\Gamma \models \varphi \Leftrightarrow \Gamma \vdash \varphi$$

Proof It is easy to check that the axioms are valid and the rules preserve validity. This proves soundness.

For completeness, we show that if $\Gamma \not\vdash \varphi$, that is, if $\Gamma \cup \{\neg\varphi\}$ is L_{min} – consistent, then there is a model for $\Gamma \cup \{\neg\varphi\}$, that is, $\Gamma \not\models \varphi$.

The proof uses a standard Henkin argument. We extend the language with countably many free variables $u_0, \ldots, u_n, \ldots$. The formulas of the extended language are ordered as follows: $\varphi_0, \ldots, \varphi_n, \ldots$.

Let $\Sigma_0 = \Gamma \cup \{\neg\varphi\}$. Σ_0 is consistent by assumption. Further, let $\Sigma_n = \Sigma_{n-1}$ if $\Sigma_{n-1} \cup \{\varphi_n\}$ is inconsistent. If $\Sigma_{n-1} \cup \{\varphi_n\}$ is consistent, we have three possibilities.

If φ_n is not of the form $\exists x\psi$ or $\Diamond_x\psi$, then $\Sigma_n = \Sigma_{n-1} \cup \{\varphi_n\}$.

If φ_n is of the form $\exists x\psi$, then we choose the first new variable u (a witness for $\exists x\psi$) which does not occur in Σ_{n-1} and $\exists x\psi$ and put $\Sigma_n = \Sigma_{n-1} \cup \{\exists x\psi(x)\} \cup \{\psi[x/u]\}$.

If φ_n is of the form $\Diamond_x\psi$, then we choose the first new variable u which does not occur in Σ_{n-1} and $\Diamond_x\psi$ and put $\Sigma_n = \Sigma_{n-1} \cup \{\Diamond_x\psi(x)\} \cup \{\psi[x/u]\} \cup X$, where $X = \{\Box_x\chi(x) \to \chi[x/u] : FV(\Box_x\chi) = FV(\Diamond_x\psi)\}$. Adding X is necessary since we want the witness for $\Diamond_x\psi$ to satisfy $R(u, FV(\Diamond_x\psi))$.

Let $\Sigma = \cup_n \Sigma_n$.

First of all, we show that Σ is consistent. Assume that it is not. Since every derivation of a contradiction is finite, there is some finite set of formulas used in the derivation. This set must belong to some Σ_n, $n > 0$. Assume that Σ_{n-1} is consistent. Then inconsistency is caused by adding a witness for $\exists x\psi$ or for $\Diamond_x\psi$.

Assume that Σ_n is obtained from Σ_{n-1} by adding $\exists x\psi$ and $\psi(u)$, for a new variable u. Then for a finite conjunction of formulas $\varphi_1, \ldots, \varphi_k$ from Σ_{n-1}

$$\vdash \bigwedge \varphi_i \wedge \exists x\psi \to \neg\psi(u)$$

by R3,

$$\vdash \forall u(\bigwedge \varphi_i \wedge \exists x\psi \to \neg\psi(u))$$

and by C5 (u is not free in $\bigwedge \varphi_i$ and in $\exists x\psi$ by assumption),

$$\vdash \bigwedge \varphi_i \wedge \exists x\psi \to \forall u \neg\psi(u);$$

using C4 an R3, one can show that this is equivalent to

$$\vdash \bigwedge \varphi_i \wedge \exists x\psi \to \forall x \neg\psi(x),$$

which shows that $\bigwedge \varphi_i$ is inconsistent with $\exists x\psi$, contrary to our assumption.

Assume that Σ_n is obtained from Σ_{n-1} by adding $\diamond_x\psi$, $\psi(u)$, and X for a new variable u. Reasoning as before, we obtain

$$\vdash \bigwedge \varphi_i \wedge \diamond_x\psi \to (\bigwedge_j (\Box_x\chi_j \to \chi_j(u)) \to \neg\psi(u))$$

for finitely many formulas χ_j with the same free variables as ψ (except for x, possibly). By R2,

$$\vdash \Box_u(\bigwedge \varphi_i \wedge \diamond_x\psi \to (\bigwedge_j (\Box_x\chi_j \to \chi_j(u)) \to \neg\psi(u)))$$

and by C1 and C2 (u is not free in $\bigwedge \varphi_i$ and in $\diamond_x\psi$ by assumption),

$$\vdash \bigwedge \varphi_i \wedge \diamond_x\psi \to \Box_u(\bigwedge_j (\Box_x\chi_j \to \chi_j(u)) \to \neg\psi(u));$$

by C3, this is equivalent to $\vdash \bigwedge \varphi_i \wedge \diamond_x\psi \to \Box_x(\bigwedge_j(\Box_x\chi_j \to \chi_j(x)) \to \neg\psi(x))$, and since $\Box_x(\bigwedge_j(\Box_x\chi_j \to \chi_j(x))$ is derivable (the proof is given in Chapter 2, page 18), this is equivalent to

$$\vdash \bigwedge \varphi_i \wedge \diamond_x\psi \to \Box_x\neg\psi(x);$$

this implies that $\bigwedge \varphi_i$ is inconsistent with $\diamond_x\psi$, contrary to our assumption.

Now it is easy to show that Σ is a maximal consistent set, i.e. $\psi \in \Sigma$ iff $\neg\psi \notin \Sigma$ and $\psi_1 \wedge \psi_2 \in \Sigma$ iff $\psi_1 \in \Sigma$ and $\psi_2 \in \Sigma$, and Σ is deductively closed.

We construct a model M_Σ using Σ as follows. Let U be the set of variables (old and new ones). The domain D_Σ of our model is $U_{/=}$. We denote by $[y]$ the equivalence class of y ($[y] = \{x : x = y \in \Sigma\}$). $\langle [x_1], \ldots, [x_n] \rangle \in V_\Sigma(P)$ iff $P(x_1, \ldots, x_n) \in \Sigma$ ($= 4$ takes care that this is well defined). Finally,

$$R(u, \{[y_1], \ldots, [y_n]\}) \Leftrightarrow \{\Box_x\chi(x) \to \chi(u) : FV(\Box_x\chi) = \{y_1, \ldots, y_n\}\} \subseteq \Sigma$$

again, we need $= 4$ to show that this is well defined.

Now it is easy to prove the lemma which finishes the proof of the theorem:

3.2.3. Lemma. *(Truth lemma) Let s be an assignment, such that for every variable x $s(x) = [x]$. Then for every formula ψ, $M_\Sigma, s \models \psi \Leftrightarrow \psi \in \Sigma$.*

Proof We need a slightly stronger inductive hypothesis:

$$M_\Sigma, s^{x_1,\ldots,x_n}_{[u_1],\ldots,[u_n]} \models \psi(x_1, \ldots, x_n) \Leftrightarrow M_\Sigma, s \models \psi(u_1, \ldots, u_n)$$

The rest is a simple induction on the complexity of a formula. □

3.2.4. Corollary. *L^-_{min} is sound and complete.*

Proof Inspection of the proof above shows that the same reasoning goes through if the language does not contain ordinary quantifiers and the system does not contain C4 – C5, R3. □

The same holds for both languages without equality (and the systems without equality axioms =1 – =4).

Given these completeness results, we shall sometimes denote the minimal logic of $\mathcal{L}(\exists\Diamond)$ by L_{min} and the minimal logic of $\mathcal{L}(\Diamond)$ by L^-_{min}.

An independent completeness proof for L^-_{min} without equality by means of analytic tableaux is given in section 3.4.

3.3 Preservation

Now, to illustrate the semantical properties of modal quantifiers, we shall consider an analogue to the basic model-theoretic invariance relation of modal logic. In what follows, we talk about the language $\mathcal{L}(\Diamond)$ (without ordinary quantifiers).

3.3.1. Definition. *A* bisimulation *$\mathcal{B}$ between two models $M_1 = \langle D_1, R_1, V_1\rangle$ and $M_2 = \langle D_2, R_2, V_2\rangle$ is a family of partial isomorphisms π with the following properties:*

1 π is a partial bijection with $dom(\pi) \subseteq D_1$ and $ran(\pi) \subseteq D_2$;
2 If $\{d_1, \ldots, d_n\} \subseteq dom(\pi)$, then for all predicate letters

$$\langle d_1, \ldots, d_n\rangle \in V_1(P^n) \;\Leftrightarrow\; \langle \pi(d_1), \ldots, \pi(d_n)\rangle \in V_2(P^n)$$

($d_1, \ldots, d_n$ are not necessarily distinct).

3a If $D \subseteq dom(\pi)$ and $R_1(d, D)$, then there exists an element d' in D_2 such that $R_2(d', \pi[D])$ and $\{\langle d, d'\rangle\} \cup \pi \in \mathcal{B}$.

3b If $D' \subseteq ran(\pi)$, $D' = \pi[D]$, and $R_2(d', D')$, then there exists an element d in D_1 such that $R_1(d, D)$ and $\{\langle d, d'\rangle\} \cup \pi \in \mathcal{B}$. [1] □

3.3.2. Lemma. *(Invariance Lemma) If φ is a formula of $\mathcal{L}(\Diamond)$ with $FV(\varphi) \subseteq \{y_1, \ldots, y_n\}$, M_1 and M_2 are bisimilar models, and for all y_i ($1 \le i \le n$) $s_1(y_i) \in dom(\pi)$ and $s_2(y_i) = \pi(s_1(y_i))$, $\pi \in \mathcal{B}$, then*

$$M_1, s_1 \models \varphi \;\Leftrightarrow\; M_2, s_2 \models \varphi$$

Proof. By induction on the length of φ.

- φ is a k-place predicate letter. - By clause (2) in the definition of bisimulation.
- $\varphi = (x = y)$. $M_1, s_1 \models x = y$ if and only if $s_1(x) = s_1(y)$. Since π is a function, and $s_1(x) \in dom(\pi)$, $\pi(s_1(x)) = \pi(s_1(y))$, that is, $s_2(x) = s_2(y)$ and $M_2, s_2 \models x = y$. Backwards: the same argument, using the fact that π is a bijection.

[1] Alternatively, we could restrict clause 3 to R-successors of the whole domain and range, while adding a further clause closing $\mathcal{B}$ under restrictions.

- $\varphi = \neg\psi$: by a trivial application of the inductive hypothesis;
- $\varphi = \psi_1 \wedge \psi_2$: by a trivial application of the inductive hypothesis;
- $\varphi = \Diamond_x \psi(x, \bar{y})$. Assume $M_1, s_1 \models \Diamond_x \psi(x, \bar{y})$. By the semantic truth definition, there exists an assignment s_1' which differs from s_1 at most in its assignment of value to x, such that $R(s_1'(x), s_1'(\bar{y}))$ and $M_1, s_1' \models \psi(x, \bar{y})$. By assumption, $y_1, \ldots, y_n \in dom(\pi)$. By clause 3a, there is $d' \in D_2$ with $R(d', \pi s_1'(\bar{y}))$, i.e. $R(d', s_2(\bar{y}))$ (since s_1' and s_1 agree on $\bar{y}$), and $\{\langle d, d'\rangle\} \cup \pi \in \mathcal{B}$. Put $s_2' =_x s_2$, $s_2'(x) = d'$. Then, for the $\pi' \in \mathcal{B}$ which consists of π and the pair $\langle d, d'\rangle$, $s_2'(x) = \pi'(s_1'(x))$, and for all y_i, $s_2'(y_i) = \pi(s_1'(y_i))$.
  By the inductive hypothesis, $M_2, s_2' \models \psi(x, \bar{y})$. But then $M_2, s_2 \models \Diamond_x \psi(x, \bar{y})$. The same argument works backwards. □

Continuing the analogy with modal logic, we define a translation of $\mathcal{L}(\Diamond)$ formulas into the appropriate first-order logic, which is our original base language enriched with a dependence predicate R. The *standard translation ST* is defined as follows:

- $ST(P_i^n(t_1 \ldots t_n)) := P_i^n(t_1 \ldots t_n)$;
- $ST(t_1 = t_2) := (t_1 = t_2)$;
- ST commutes with classical connectives;
- $ST(\Diamond_x \varphi(x, \bar{y})) := \exists x(R(x, \bar{y}) \wedge ST(\varphi(x, \bar{y})))$.

3.3.3. Claim. *If φ is a formula of $\mathcal{L}(\Diamond)$, then*

$$M, s \models \varphi \;\Leftrightarrow\; M', s \models ST(\varphi),$$

for the classical model $M' = (D, V')$, where V' extends V to interpret the predicate R as R_M, and R_M satisfies, for every permutation τ,

$$R_M(a, \bar{b}) \Leftrightarrow R(a, \tau(\bar{b}))$$

$$R_M(a, c\bar{b}) \Leftrightarrow R(a, cc\bar{b}))$$

Proof Note that our semantics can be reformulated so that R is a relation between an element and a sequence, not an element and a set, satisfying the two conditions above. □

We use the latter reformulation to state a simple invariance result, which will be used in the sequel.

3.3.4. Claim. *Let φ be a formula of $\mathcal{L}(\Diamond)$ which uses not more that n variables (free or bound). Let the models M and M' have the same domain and interpretation function and agree on R up to arity n (in other words, for every element a and set B with $|B| < n$, $R(a, B) \leftrightarrow R'(a, B)$). Then*

$$M, s \models \varphi \Leftrightarrow M', s \models \varphi$$

Proof By induction on complexity of φ. Note that in the clause for the quantifier, the relation R involved is of arity less or equal to n. □

3.3.5. Definition. *The* modal formulas *(being those formulas which are standard translations of $\mathcal{L}(\Diamond)$ formulas) are the least set X of first-order formulas such that*

- *atomic formulas belong to X,*
- *if ψ_1 and ψ_2 are in X, then so are $\neg\psi_1$ and $\psi_1 \wedge \psi_2$,*
- *if $\varphi(x, \bar{y}) \in X$, then $\exists x(R(x, \bar{y}) \wedge \varphi(x, \bar{y}))$ is in X.* □

The following theorem is proved by standard methods of modal logic: cf. (van Benthem 1983) or (de Rijke 1993).

3.3.6. THEOREM. *A first-order formula φ is equivalent to a modal formula if and only if it is preserved under bisimulation.*

Proof. The direction from left to right follows from Invariance Lemma above. For the converse, let φ be a first-order formula with variables $x_1, \ldots, x_n$, preserved under bisimulation. We want to prove that it is equivalent to a modal formula.

Define the set $CONS_\diamond(\varphi)$ as $\{\alpha : \alpha$ is a modal formula, $\varphi \models \alpha$ and the free variables of α are among $x_1, \ldots, x_n\}$. If we can prove that

$$(*) \qquad CONS_\diamond(\varphi) \models \varphi,$$

then we are done. For, by compactness, there will be some finite subset $\alpha_1, \ldots, \alpha_m$ of $CONS_\diamond(\varphi)$ with $\alpha_1, \ldots, \alpha_m \models \varphi$. By definition, $\varphi \models \alpha_1, \ldots, \alpha_m$. So, then φ is equivalent to $\alpha_1 \wedge \ldots \wedge \alpha_m$, which is a conjunction of standard translations of $\mathcal{L}(\diamond)$ formulas, i.e. a standard translation of the conjunction of those formulas.

Now we start proving $(*)$. Assume that for some model $M, s \models CONS_\diamond(\varphi)$. We show that $M, s \models \varphi$. Let us denote the set of all modal formulas true in M and having free variables among $x_1, \ldots, x_n$ as X_M. This is consistent with φ: for, if it is not, there is a finite set $\psi_1, \ldots, \psi_k$ of formulas from X_M, such that $\bigwedge_i \psi_i \rightarrow \neg\varphi$. Then $\varphi \rightarrow \bigvee_i \neg\psi_i$. But $\bigvee_i \neg\psi_i$ is a modal formula (if every ψ_i is). Since it is a consequence of φ, it must be true in X_M. A contradiction.

Therefore there should be a model N for $\varphi \cup X_M$: say, $N, s' \models \varphi \cup X_M$.

Let $s(x_1) = d_1, \ldots, s(x_n) = d_n$ in M and $s'(x_1) = d'_1, \ldots, s'(x_n) = d'_n$ in N. Now, take ω-saturated elementary extensions $\mathcal{M}$ and $\mathcal{N}$ of M and N. We define a relation of bisimulation between $\mathcal{M}$ and $\mathcal{N}$ as follows:
(**) $\mathcal{B}$ is the family of partial mappings π such that

$$\pi = \{(e_1, \pi(e_1)), \ldots, (e_n, \pi(e_n))\}$$

if for all modal formulas ψ with at most free variables $x_1, \ldots, x_n$ and any two assignments s, s' with $s(x_i) = e_i$, $s'(x_i) = \pi(e_i)$ $(1 \leq i \leq n)$,

$$\mathcal{M}, s \models \psi \;\Leftrightarrow\; \mathcal{N}, s' \models \psi$$

To prove that $(**)$ indeed defines a bisimulation relation, we must check that the properties (1)–(3) of the definition of bisimulation hold for $\mathcal{B}$. Here, (1) is trivial. Case (2) is immediate, since atomic formulas are also standard translations of (atomic) formulas in $\mathcal{L}(\diamond)$). Next, we check the zigzag clause 3a. Assume that $e_1, \ldots, e_k \in dom(\pi)$ and $R(e, e_1, \ldots, e_k)$. We must prove that there exists e' in $\mathcal{N}$ such that $R(e', \pi(e_1), \ldots, \pi(e_k))$ and $\{\langle e, e' \rangle\} \cup \pi \in \mathcal{B}$. Take the set Ψ of all modal formulas with variables interpreted as $e, e_1, \ldots, e_k$ which are true in $\mathcal{M}$ under variable assignment s. We need an element e' in $\mathcal{N}$ such that all formulas in Ψ are true in $\mathcal{N}$ under s' when e' is assigned to the variable which was assigned e in $\mathcal{M}$. By saturation, it suffices to find such an e' for each finite subset Ψ_0 of Ψ. But these must exist, because the modal formula $ST(\Diamond_x \bigwedge \Psi_0(x, e_1, \ldots, e_k))$ holds in $\mathcal{M}$ and hence $ST(\Diamond_x \bigwedge \Psi_0(x, \pi(e_1), \ldots, \pi(e_k)))$ holds in $\mathcal{N}$. The appropriate check for the converse direction 3b is proved analogously.

Recall that $s(x_i) = d_i$ and $s'(x_i) = d'_i$, $1 \le i \le n$. We must also show that $\{\langle d_1, d'_1\rangle, \ldots, \langle d_n, d'_n\rangle\} \in \mathcal{B}$. But this is so because for all modal formulas ψ with variables interpreted as $d_1, \ldots, d_n$ in M,

$$M, s \models \psi \;\Leftrightarrow\; N, s' \models \psi$$

(by the construction of N), and hence

$$\mathcal{M}, s \models \psi \;\Leftrightarrow\; \mathcal{N}, s' \models \psi.$$

Finally, since φ is invariant under bisimulation and $\{\langle d_1, d'_1\rangle, \ldots, \langle d_n, d'_n\rangle\} \in \mathcal{B}$, $\mathcal{N} \models \varphi(d'_1, \ldots, d'_n)$ will now imply $\mathcal{M} \models \varphi(d_1, \ldots, d_n)$. Since $\mathcal{M}$ is an elementary extension of M, $M \models \varphi(d_1, \ldots, d_n)$, that is, $M, s \models \varphi(x_1, \ldots, x_n)$, and we are done. $\square$

3.4 Tableaux. Decidability

3.4.1 Tableaux for the minimal logic

To check effectively whether a formula of $\mathcal{L}(\Diamond)$ without equality is satisfiable, we introduce analytic tableaux for L^-_{min}.

This time we consider tableaux for *signed formulas*. Every formula is signed by T or F (intuitively standing for truth and falsity). For every connective, there is a pair of rules for decomposing a signed formula with this connective as a principal connective: one rule for a formula signed with T and one rule for a formula signed with F. They correspond to the truth conditions for this connective. If Δ is a set of signed formulas, and A_1 a signed formula, $\dfrac{\Delta, A_1}{\Delta, A_2}$ means: given a node Δ, A_1, create a successor node Δ, A_2, and $\dfrac{\Delta, B}{\Delta, B_1 \mid \Delta, B_2}$ means: given a node Δ, B, create two successors, Δ, B_1 and Δ, B_2. (Of course, we could have omitted T and replaced F by $\neg$ to obtain a system for ordinary formulas. But the present formulation is closer related to the sequent calculus, which we are going to use).

3.4.1. Definition. A tableau for $T(F)\chi$ is a tree where the origin is $T(F)\chi$ and the successors of a node are generated by applying one of the following decompositions rules to this node:

$$T_\neg \quad \frac{\Delta, T\neg\varphi}{\Delta, F\varphi} \qquad F_\neg \quad \frac{\Delta, F\neg\varphi}{\Delta, T\varphi}$$

$$T_\wedge \quad \frac{\Delta, T(\varphi \wedge \psi)}{\Delta, T\varphi, T\psi} \qquad F_\wedge \quad \frac{\Delta, F(\varphi \wedge \psi)}{\Delta, F\varphi \mid \Delta, F\psi}$$

To give rules for $\Diamond$ we need to introduce, in addition to signed formulas, atomic formulas keeping track of the dependence relation, of the form $R(x, \bar{y})$. Now we assume that Δ (also in the rules above) is a union of a set of signed formulas Σ and a set of R-formulas Γ.

$$T_\Diamond \quad \frac{\Delta, T\Diamond_x\varphi(x, \bar{y})}{\Delta, R(z, \bar{y}), T\varphi(z, \bar{y})}$$

where z is a new variable;

$$F_\Diamond \quad \frac{\Delta, F\Diamond_x\varphi(x, \bar{y})}{\Delta, F\varphi(z, \bar{y}), F\Diamond_x\varphi(x, \bar{y})}$$

for every variable z such that $R(z, \bar{y}) \in \Delta$.

Note that the rule $F_\Diamond$ can be used only if previously some variable z with $R(z, \bar{y})$ was introduced by $T_\Diamond$. $F\Diamond_x\varphi(x, \bar{y})$ in the conclusion of the rule means that the application of the rule should be repeated if a new variable u with $R(u, \bar{y})$ is introduced.

A branch is called *closed* if the same formula occurs both under T and under F. A branch is open if this is not the case and no rule can be applied any more. A tableau is closed if all its branches are closed. □

3.4.2. Theorem. *The tableau calculus described above is sound and complete for L^-_{min}: for every formula χ of $\mathcal{L}(\Diamond)$, χ is valid if, and only if, there is a closed tableau for $F\chi$.*

Proof. The proof is analogous to the proof in (Smullyan 1968). To demonstrate soundness, we must show that for every rule, if a set in the premise is satisfiable (i.e. the formulas signed with T are true and the formulas signed with F are false in some model M under some assignment α), then at least one of the successors is satisfiable. For the $T_\Diamond$ rule, notice that if $\Delta \cup \{\Diamond_x\varphi(x, \bar{y})\}$ is satisfiable in M, then there is an element a with $R(a, \alpha(\bar{y}))$ such that for $\alpha' = \alpha[x/a]$ $M \models^{\alpha'} \varphi(x, \bar{y})$. Therefore there is a model and an assignment α' such that the conclusion of the rule is true. To show that the rule $F_\Diamond$ is sound, consider a model where $\Diamond_x\varphi(x, \bar{y})$ is false under an assignment α. This implies that for every object a in the relation R to $\alpha(\bar{y})$, $\varphi(a, \bar{y})$ is false under α; therefore an arbitrary number of formulas of the form $F\varphi(a, \bar{y})$, where $R(a, \bar{y})$ holds, is satisfiable in the same model under the same assignment.

This shows that if the origin is satisfiable, then there is at least one branch, such that the set of all formulas on this branch is satisfiable. But that cannot be a closed branch. Therefore, if a tableau for $F\chi$ is closed, then there is no model and assignment under which χ is false, which means that χ is valid.

To prove completeness, we define a Hintikka set for (a set of variables) D and a dependence relation R between the variables as a set of signed formulas Σ such that

for no formula φ both $T\varphi$ and $F\varphi$ are in Σ;
if $T\neg\varphi \in \Sigma$, then $F\varphi \in \Sigma$, dually for $F\neg\varphi$;
if $T\varphi \wedge \psi \in \Sigma$, then $T\varphi, T\psi \in \Sigma$; if $F\varphi \wedge \psi \in \Sigma$, then either $F\varphi \in \Sigma$ or $F\psi \in \Sigma$;

if $T\Diamond_x\varphi(x,\bar{y}) \in \Sigma$, then $T\varphi(z,\bar{y}) \in \Sigma$ for at least one $z \in D$ with $R(z,\bar{y})$;
if $F\Diamond_x\varphi(x,\bar{y}) \in \Sigma$, then $F\varphi(z,\bar{y}) \in \Sigma$ for all z such that $R(z,\bar{y})$ (may be none such z).

It is easy to check that every Hintikka set for D and R is satisfiable. A satisfying model has domain D, dependence relation R, the assignment is defined by $\alpha(x) = x$ and the valuation is any valuation satifying

$$TP(x_1,\ldots,x_n) \in \Sigma \Rightarrow \langle x_1,\ldots,x_n\rangle \in V(P)$$

and

$$FP(x_1,\ldots,x_n) \in \Sigma \Rightarrow \langle x_1,\ldots,x_n\rangle \notin V(P).$$

Assume that there is no closed tableau for $F\chi$. We are going to show that then there is a tableau for $F\chi$ which has an open branch which constitutes a Hintikka set and therefore is satisfiable. This will imply that χ is not valid.

Let us call a tableau *systematic* if it is constructed in accordance with the following procedure. Given a node Δ, create first all nodes which can be obtained by propositional rules; then all nodes which can be obtained by the $T_\Diamond$-rule; and then all nodes which can be obtained by the $F_\Diamond$-rule. Repeat this procedure as long as the nodes contain signed formulas which can be used to generate new nodes. When there are no such formulas any more, we have a *finished systematic tableau* (note that it can be infinite). It is easy to check that an open branch of a systematic tableau forms a Hintikka set. Therefore, if a systematic tableau for $F\chi$ is open, then χ is not valid. □

A tableau construction need not always terminate after a finite number of steps. Consider a tableau for

$$T(\Diamond_x Q(x) \wedge \neg\Diamond_x\neg(P(x) \wedge \Diamond_z S(z)))$$

$$\begin{array}{lc}
T_{\wedge,\neg} & T\Diamond_x Q(x),\ F\Diamond_x\neg(P(x)\wedge\Diamond_z S(z)) \\
T_\Diamond & R(d),\ TQ(d),\ F\Diamond_x\neg(P(x)\wedge\Diamond_z S(z)) \\
F_\Diamond & R(d),\ TQ(d),\ F\neg(P(d)\wedge\Diamond_z S(z)),\ F\Diamond_x\neg(P(x)\wedge\Diamond_z S(z)) \\
F_\neg & R(d),\ TQ(d),\ T(P(d)\wedge\Diamond_z S(z)),\ F\Diamond_x\neg(P(x)\wedge\Diamond_z S(z)) \\
T_\wedge & R(d),\ TQ(d),\ TP(d),\ T\Diamond_z S(z),\ F\Diamond_x\neg(P(x)\wedge\Diamond_z S(z)) \\
T_\Diamond & R(d),\ TQ(d),\ TP(d),\ R(e),\ TS(e),\ F\Diamond_x\neg(P(x)\wedge\Diamond_z S(z)) \\
F_\Diamond & \ldots R(e),\ F\neg(P(e)\wedge\Diamond_z S(z)),\ F\Diamond_x\neg(P(x)\wedge\Diamond_z S(z)) \\
F_\neg & \ldots T(P(e)\wedge\Diamond_z S(z)),\ F\Diamond_x\neg(P(x)\wedge\Diamond_z S(z)) \\
T_\wedge & \ldots TP(e),\ T\Diamond_z S(z),\ F\Diamond_x\neg(P(x)\wedge\Diamond_z S(z)) \\
T_\Diamond & \ldots R(e_1),\ TS(e_1),\ F\Diamond_x\neg(P(x)\wedge\Diamond_z S(z)) \\
 & \ldots
\end{array}$$

Here the tableau construction starts to loop. Note that $F_\Diamond$ is the rule responsible for this: it does not decrease the complexity of the formula.

We show that a tableau construction for so-called formulas in *normal form* always stops. Then we prove that every formula has an equivalent in normal form.

Let us say that a subformula φ is *immediately in the scope of* a quantifier $\Diamond_x$ if it is in the scope of $\Diamond_x$ and there is no quantifier $\Diamond_y$ 'in between', that is, such that $\Diamond_y$ is in the scope of $\Diamond_x$ and φ is in the scope of $\Diamond_y$.

3.4.3. Definition. *A formula χ is in* normal form *if in χ every subformula immediately in the scope of a quantifier contains the quantified variable of this quantifier free.* □

For example, $\Diamond_y \Diamond_x (P(x, y) \wedge \Diamond_z S(x, z))$ is in normal form, and $\Diamond_x (P(x) \wedge \Diamond_z S(z))$ is not (since $\Diamond_z S(z)$ is immediately in the scope of $\Diamond_x$ and does not have x free).

3.4.4. Lemma. *If a formula is in normal form, then each of its subformulas is.*

Proof. Obvious. □

To prove that a tableau construction always stops for formulas in normal form, we need to define a notion of *dependence between variables*. This notion appears in (Fine 1985). In the present context it has the following meaning:

3.4.5. Definition. *A variable x* depends *on a variable y in a branch of a given tableau, if in this branch either $R(x, \ldots y \ldots)$ holds or (b) there is a variable z such that $R(x, \ldots z \ldots)$ holds and z depends on y.* □

For example, in

$$T\Diamond_x (S(x, y) \wedge \Diamond_z P(x, z))$$

$$\begin{array}{ll} T_\Diamond & R(x, y),\ T(S(x, y) \wedge \Diamond_z P(x, z)) \\ T_\wedge & R(x, y),\ TS(x, y),\ T\Diamond_z P(x, z) \\ T_\Diamond & R(x, y),\ R(z, x),\ TS(x, y),\ TP(x, z) \end{array}$$

x depends on y and z depends on x and y.

Dependency in a tableau has some obvious properties. Transitivity follows from the definition.

3.4.6. Lemma. *If x depends on y on some branch of a tableau, then y appeared free on this branch before x was introduced.*

Proof. Assume that x depends on y on some branch. If $R(x, \ldots y \ldots)$ holds on this branch, then x was introduced by $T_\Diamond$ rule with a premise which had y free. By definition, x is a new variable, therefore there was a stage in the tableau construction (immediately before the rule was applied) when y occurred free on the branch and x did not.

Assume that x depends on y because there are $z_1, \ldots, z_n$ such that $R(x, \ldots z_1 \ldots)$, …, $R(z_n, \ldots y \ldots)$. As before, this means that z_n appeared after y, …, z_1 after z_2, and x after z_1. Thus x appeared after y. □

Lemma 3.4.6 implies that dependence is asymmetric: if x depends on y, then y does not depend on x. It also implies that x cannot depend on a variable which was introduced later on the branch.

If a formula is in normal form, its nested quantifiers are 'hooked' into one another: if $\Diamond_z \varphi$ is immediately in the scope of $\Diamond_y$, then y is free in φ. This yields some important properties of tableaux where the origin is a signed formula in normal form. Before formulating them, some terminology has to be defined.

We shall say that a quantifier $\Diamond_z$ in $\Diamond_z \varphi$ on some branch of a tableau is *instantiated* on a variable z_1, if $\Delta,\ \Diamond_z \varphi(z, \bar{y})$ is a premise of the $T_\Diamond$-rule and the successor of this node is $\Delta,\ R(z', \bar{y}),\ T\varphi[z/z_1]$. Observe that if $\Diamond_z$ is immediately in the scope of $\Diamond_y$, then $\Diamond_z$ will be instantiated on a variable dependent on the variable used to instantiate $\Diamond_y$.

We call a signed formula A a *result of decomposing a signed formula B in accordance with the tableau rules* (on some branch of a tableau) if

(i) either A is obtained by applying one of the tableau rules to B (for example, $A = TP(x)$ is a result of decomposing $B = F\neg P(x)$ in

$$F\neg P(x)$$

$$F_\neg \quad TP(x)$$

(ii) or there are signed formulas $A_1, \ldots, A_n$ on this branch, such that $A = A_1$, A_i is a result of decomposing A_{i+1} $(i < n)$, and $A_n = B$.

Observe that if A is a result of decomposing B, then (with T and F omitted) A is a subformula of B.

3.4.7. Lemma. *Any result of decomposing a formula $\Diamond_x\chi$ in normal form in accordance with the tableau rules will contain a variable dependent on the variable used to instantiate $\Diamond_x$, or this variable itself.*

Proof. Assume that φ is a subformula of $\Diamond_x\chi$ in normal form which is in the scope of quantifiers $\Diamond_x, \Diamond_{x_1}, \ldots, \Diamond_{x_n}$ ($\Diamond_x$ being the outermost). If $n = 0$ (φ is immediately in the scope of $\Diamond_x$), then φ contains x free (since χ is in normal form) and therefore the variable used to instantiate $\Diamond_x$ will be free in φ.

Let $n \geq 1$, and $T(F)\varphi(d_1, \ldots, d_m)$ be the result of decomposing χ according to the tableau rules. We show that at least one of the d_i depends on the variable used to instantiate $\Diamond_x$.

For the sake of simplicity, we assume that $\Diamond_{x_i}$ is instantiated by x_i. By ψ_i we denote the biggest subformula of χ in the scope of $\Diamond_{x_i}$. Note that φ is a subformula of ψ_n, and x_n is free in φ (since χ is in normal form). Since $\Diamond_{x_1}\psi_1$ is immediately in the scope of $\Diamond_x$, x is free in ψ_1, etc. Therefore for the variables used to instantiate the quantifiers holds

$$R(x_1, \ldots x \ldots), R(x_2, \ldots x_1 \ldots), R(x_3, \ldots x_2 \ldots), \ldots, R(x_n, \ldots x_{n-1} \ldots)$$

and by transitivity x_n depends on x. But x_n is free in φ. Thus, $\varphi(d_1, \ldots, d_m)$ contains at least one variable dependent on x. □

3.4.8. Lemma. *Assume that a tableau construction for a formula $T(F)\chi$ in normal form has reached a stage when the only applicable rule is $T_\Diamond$. Let $z_1, \ldots, z_n$ be the list of all variables which occur free on a branch of the tableau at this stage. Any formula which appears later on this branch will contain at least one free variable which is not among $z_1, \ldots, z_n$.*

Proof. If the only applicable rule is $T_\Diamond$, then every formula on the branch at this stage is either atomic, or begins with $T\Diamond_x$ or $F\Diamond_x$. Since χ is in normal form, every formula in the tableau is (Lemma 3.4.4). Any result of decomposing a formula in normal form beginning with $\Diamond_x$ will contain free a variable which depends on the variable used to instantiate $\Diamond_x$, or this variable itself (Lemma 3.4.7). This variable (used to instantiate $\Diamond_x$) is not among $z_1, \ldots, z_n$ since it will be introduced by $T_\Diamond$ rule at a later stage (we have assumed that the $F_\Diamond$ rule was no more applicable, therefore even applications of the $F_\Diamond$ rule to the formulas beginning with $F\Diamond_x$ will use a variable not among $z_1, \ldots z_n$). Since neither of z_i can depend on a variable introduced later on the branch (Lemma 3.4.6), every result of decomposition will contain a variable which is not among $z_1, \ldots, z_n$. □

3.4.9. LEMMA. *Assume that a tableau construction for a formula $T(F)\chi$ in normal form has reached a stage where the only applicable rule is $T_\Diamond$. Let $F\Diamond_x\psi_1, \ldots, F\Diamond_x\psi_n$ be the list of all formulas on a branch of the tableau beginning with $F\Diamond$. For every $F\Diamond_x\psi_i$, the $F_\Diamond$ rule with this formula as a premise will be applied only finitely many times.*

Proof. Let $\bar{z}$ be the free variables of $\Diamond_x\psi_i$. The $F_\Diamond$ rule with $F\Diamond_x\psi_i(x, \bar{z})$ as a premise can be repeated only if a new variable d with $R(d, \bar{z})$ is introduced by the $T_\Diamond$ rule with a formula of the form $T\Diamond_x\alpha(x, \bar{z})$ as a premise. Assume that there are m such formulas present on the branch at this stage. No formula with $\bar{z}$ as its only free variables will appear later on the branch by Lemma 3.4.8. Therefore the $F_\Diamond$ rule with $F\Diamond_x\psi_i(x, \bar{z})$ as a premise will be applied precisely m more times. □

3.4.10. LEMMA. *If χ is in normal form, the tableau construction for $T(F)\chi$ always stops.*

Proof. Take an arbitrary branch of a tableau. Perform all propositional rules and all applications of the $F_\Diamond$ rule with respect to previously introduced variables. One can show that at any stage of constructing a tableau this process stops after a finite number of steps. At some stage we cannot proceed any further without applying the $T_\Diamond$ rule. Assume that at this stage the $F_\Diamond$-formulas (which give rise to repetitions of $F_\Diamond$ rules) are

$$F\Diamond_{x_1}\theta_1, \ldots, F\Diamond_{x_k}\theta_k$$

Due to Lemma 3.4.9, for every $\Diamond_{x_i}\theta_i$ the rule $F_\Diamond$ with this formula as a premise will be repeated only finitely many times.

In a finite number of steps the tableau construction reaches the stage when $F_\Diamond$ will not be repeated with $F\Diamond_x\theta_i$ as a premise, and again only the $T_\Diamond$ rule is applicable. Assume that at this point a branch of the tableau contains atomic formulas, formulas beginning with $T\Diamond_x$, new formulas beginning with $F\Diamond$, $F\Diamond_x\psi_1, \ldots, F\Diamond_x\psi_m$, and $\{F\Diamond_x\theta_i : 1 \leq i \leq k\}$. The complexity of all formulas on the branch except for $\{F\Diamond_x\theta_i : 1 \leq i \leq k\}$ is decreased. For each of the $F\Diamond_x\psi_j$ the $F_\Diamond$ rule will be applied finitely many times. After that the complexity of all formulas on the branch except for the ones which won't be used any more is again decreased. An easy induction shows that the tableau construction will stop after a finite number of steps. □

3.4.11. COROLLARY. *The satisfiability problem for the formulas in normal form is decidable.*

3.4.12. LEMMA. *Every formula has an equivalent in normal form.*

Proof. In (Alechina 1995c) a rather lengthy procedure of reducing a formula to normal form is given. Andréka, van Benthem and Németi (1995) proposed a shorter description of a procedure which gives the same result and uses the fact that if a formula φ occurs immediately in the scope of $\Diamond_x$ and does not contain x free,

$$\Diamond_x(\ldots\varphi(\bar{y})\ldots) \equiv (\varphi(\bar{y}) \wedge \Diamond_x(\ldots\top(x,\bar{y})\ldots)) \vee (\neg\varphi(\bar{y}) \wedge \Diamond_x(\ldots\bot(x,\bar{y})\ldots)).$$

Any formula can be reduced to normal form by applying this equivalence first to the formulas under the maximal number of quantifiers and then moving outwards.

Both procedures (the one from (Alechina 1995) and the one described above) give in the worst case an exponential increase in the length of a formula. □

3.4.13. COROLLARY. *L^-_{min} has the finite model property.*

Proof. Every formula φ of L^-_{min} has an equivalent φ' in normal form, and if φ is satisfiable, then there is a *finite* open branch of a tableau for $F\varphi'$. But then a satisfying model for φ' (and therefore for φ) as constructed in the proof of the Theorem 3.4.2 is finite. □

We can even do better than that and determine the *size* of the satisfying model. To do that, it suffices to count how many new objects are introduced during the proof and to add the free variables of the formula.

Let φ be a formula in normal form with k free variables. Construct a tableau for φ. If this tableau has an open branch, construct a model as in Theorem 3.4.2. The domain of the model contains at least k elements, corresponding to the free variables of φ. In order to count how many new variables are introduced during the construction, recall that any tableau construction for a formula in normal form can be divided into the following stages. At the first stage propositional rules and quantifier rules for the quantifiers of depth 0 are applied. At the second stage propositional rules and quantifier rules for the quantifiers of the depth 1 are applied, and so on. Our decidability proof consisted actually in showing that these stages can be really separated, i.e. after applying all the rules at the stage i no applications of $F_\Diamond$ and subsequently $T_\Diamond$ rules for the formulas at earlier stages become possible. If the maximal number of nested quantifiers in φ is n, the tableau construction tells n stages. At each stage i, not more than m_i new variables are introduced, where m_i is the number of formulas of the form $T\Diamond_x\psi$ at this stage. At the first stage m_1 equals the number of positive occurrences of $\Diamond$. At the second stage, the number of such formulas can be greater than the number of positive occurrences of $\Diamond$ at depth 1 due to repeated applications of the $F_\Diamond$ rule at stage 1. The number of these applications equals to the number of variables introduced at stage 1, that is, m_1. In general, if w_i is the number of positive occurrences of $\Diamond$ at depth $i-1$, then $m_1 = w_1$ and $m_i \leq w_i \times m_{i-1}$ if $i > 1$. The total number of new objects is thus at most $w_1 + w_1 \times w_2 + \ldots + w_1 \times \ldots \times w_n$.

Since every $w_i \leq n$, this number is bound by $n+n^2+\ldots+n^n$, i.e. by an exponent of n; if we count the free variables calculating the length of a formula, the size of a model for φ is bound by an exponent of the length of φ. This proves

3.4.14. Fact. *The satisfiability problem for the formulas in normal form is in EXPTIME.*

3.4.15. Theorem. L^-_{min} *is decidable.*

Proof. The theorem follows immediately from Lemmata 3.4.10 and 3.4.12 and Theorem 3.4.2. □

To calculate the complexity of our algorithm for an arbitrary formula φ, we have to count the complexity of the algorithm used to reduce a formula to normal form. It is at most exponential of the length of the formula. This implies that the upper bound of the complexity of L_{min} without negation is $EXPTIME$.

As for the lower bound, the fact that K is embeddable in L_{min} shows that it is at least $PSPACE$ (cf. (Ladner 1977)).

3.4.2 Related work

The result of Andréka and Németi (1994) on decidability of Fragment 2 implies decidability of L^-_{min} and various extensions of L^-_{min} which are complete with respect to the classes of models defined by first order conditions on R which are *in Fragment 2*. Given a formula φ with n (free and bound) variables, one can show that if it has a model in which the dependence relation R of all arities less or equal to n satisfies a certain condition on R, then it has a model in which this condition is satisfied for all arities. (cf. Claim 3.3.4). This implies that φ is satisfiable in a model with a certain condition on R if, and only if, the conjunction of the standard translation of φ and finitely many formulas defining the condition on R for arity less or equal to n, is first order satisfiable. If this conjunction is in Fragment 2, then by the theorem above, the latter problem is decidable.

There are some important extensions of the minimal logics, for which the result above is not applicable. One such extension is $L^-_{min} + \Diamond_x\varphi \to \Diamond_x(\varphi \vee \Diamond_x\psi)$ without restriction on the free variables. We shall see in Chapter 4 that this axiom does not correspond to a first order condition on R.

3.4.3 Tableaux for extensions of the minimal logic

In this section we look at some extensions of the minimal logic.

Adding $\Diamond_x\top(x,\bar{y})$ as an axiom corresponds to the condition that R is non-empty: $\exists x R(x,\bar{y})$. This can be easily shown to be equivalent to the following modification of the $F_\Diamond$ rule:

$$F_\Diamond \quad \frac{\Delta,\ F\Diamond_x\varphi(x,\bar{y})}{\Delta,\ F\varphi(z,\bar{y}),\ F\Diamond_x\varphi(x,\bar{y})}$$

for every variable z such that $R(z,\bar{y}) \in \Delta$; *if there is no such variable, add a new variable u with $R(u,\bar{y})$ and $F\varphi(u,\bar{y})$.*

The decidability proof is precisely the same, only in the proof of Lemma 3.4.9 the last sentence has to be changed from 'Therefore the $F_\Diamond$ rule with $F\Diamond_x\psi_i(x,\bar{z})$ as a premise will be applied precisely m more times' to 'Therefore the $F_\Diamond$ rule with $F\Diamond_x\psi_i(x,\bar{z})$ as a premise will be applied at most $m+1$ times'.

However, the finite model property for this system does not follow immediately. Although a tableau for a formula in normal form is always finite, a model which is to be built given an open branch of this tableau may be infinite. The construction goes as follows: all free variables which are present on this branch become elements of the domain. R should obey the restrictions which are imposed by the R-formulas in the tableau, plus $\forall\bar{y}\exists x R(x,\bar{y})$. Therefore, to complete the model, for every sequence of elements $\bar{z}$ in the tableau for which no element u with $R(u,\bar{z})$ is present, we should introduce such an element. The modified $F_\diamond$ rule takes care that there is no formula beginning with $F\diamond_x$ such that its free variables are $\bar{z}$, so this will not start an application of the rule. Now the domain has more elements, and there are more possible sequences of elements. For every such sequence introduce a new element in the relation R to it, etc. Every step in this construction is justified by the fact that Γ has an open tableau if and only if $\Gamma \cup \{T\diamond_x \top(x,\bar{u})\}$ does. To be more precise, the construction goes as follows:

$\Gamma_0 = \Gamma$, where Γ is a set of all formulas on an open branch of a tableau with no more rules applicable;

$\vdots$

Γ_n is the result of applying $T_\diamond$ rules to $\Gamma_{n-1} \cup \{T\diamond_x \top(x,\bar{z})$: $\bar{z}$ occurs in Γ_{n-1} but no formula of the form $R(u,\bar{z})$ does $\}$. As we argued above, the $F_\diamond$ rule does not become applicable;

$\vdots$

The union of these sets is still a Hintikka set containing infinitely many variables. The resulting model is infinite. One can prove however that the newly introduced elements may be safely identified, so that a finite model is always possible.

The finite model property for this system also follows from a result of Andréka, van Benthem and Nemeti (1995).

Consider the following extension of L^-_{min}: $L_{min} + \diamond_x\varphi \equiv \Box_x\varphi$. We are going to show that it is complete with respect to the class of models satisfying *functionality*: $\forall\bar{y}\exists^1 x R(x,\bar{y})$. This will be used in Chapter 4 as an example of a first order complete logic which is not axiomatized by weak Sahlqvist formulas. The completeness proof is given here because it essentially uses analytic tableaux.

3.4.16. THEOREM. *$L_{min}+\diamond_x\varphi \equiv \Box_x\varphi$ is complete with respect to the class of models satisfying functionality: $\forall\bar{y}\exists^1 x R(x,\bar{y})$.*

Proof First, we introduce tableaux rules for the logic satisfying functionality. Then we show that this system is equivalent to analytic tableaux for the minimal logic plus $\diamond_x\varphi \equiv \Box_x\varphi$.

To formulate the new rules, we first extend the language by a functional symbol f and a constant $f(\emptyset)$. Terms are defined as usual: individual variables and constants are terms, and if t is a term, then $f(t)$ is a term. The rules are as follows:

$$T^f_\diamond \quad \frac{T\diamond_x \varphi(x, \bar{t})}{T\varphi(f(\bar{t}), \bar{t})} \qquad F^f_\diamond \quad \frac{F\diamond_x \varphi(x, \bar{t})}{F\varphi(f(\bar{t}), \bar{t})}$$

where $\bar{t}$ are all terms in φ which do not contain bound variables. The system is obviously sound with respect to the class of models satisfying functionality: interpret $f(\bar{y})$ as the object in the relation R to $\bar{y}$.

Completeness can be easily proved for *functional models* where R is replaced by a function f from the set of all finite subsets of the domain to the domain, and the truth definition is

$$M \models \diamond_x \varphi(x, \bar{d}) \Leftrightarrow M \models \varphi(f(\bar{d}), \bar{d}).$$

Given an open branch of a tableau for $F\varphi$ (note that any tableau constructed in accordance with the rules above stops after a finite number of steps), a functional model satisfying $\neg\varphi$ is constructed as follows. For every set of variables $\bar{y}$, such that $f(\bar{y})$ occurs on an open branch, introduce a new *indexed variable* $a_{\bar{y}}$ and replace everywhere $f(\bar{y})$ by $a_{\bar{y}}$. Replace all occurrences of $f(\emptyset)$ by $a_\emptyset$. For a new set of formulas on the branch (containing indexed variables), replace every occurrence of $f(\bar{a})$, where $\bar{a}$ is a set of variables (including indexed variables), by a new indexed variable $b_{\bar{a}}$, and so on, until the set of formulas on the branch does not contain f any more. Let us call this set of formulas Σ. The domain of the model is the set of all variables (ordinary and indexed) which occur in Σ plus a new element c. Note that for every set of (indexed) variables A, Σ contains at most one indexed variable with index A. As usual, V is any function satisfying, for every predicate P and variables (ordinary and indexed) $x_1, \ldots, x_n$,

$$TP(x_1, \ldots, x_n) \in \Sigma \Longrightarrow \langle x_1, \ldots, x_n \rangle \in V(P)$$

$$FP(x_1, \ldots, x_n) \in \Sigma \Longrightarrow \langle x_1, \ldots, x_n \rangle \notin V(P)$$

and $f(\bar{x}) = z_{\bar{x}}$, if such variable occurs in Σ, and $f(\bar{x}) = c$ otherwise. One can easily check that for every formula ψ such that $T(F)\psi \in \Sigma$, $T\psi \in \Sigma \Longrightarrow M_\Sigma \models \psi$ and $F\psi \in \Sigma \Longrightarrow M_\Sigma \not\models \psi$.

Completeness for the dependence models satisfying functionality follows immediately.

It is easy to show that $\diamond_x \varphi \equiv \Box_x \varphi$ is provable in the new tableaux system. However, it is less obvious that every formula provable by the new rules is provable in $L^-_{min} + \diamond_x \varphi \equiv \Box_x \varphi$. For example, a tableau for $F\neg(\diamond_x P(x, y) \wedge \diamond_x \neg P(x, y))$ closes in the system with the $T^f_\diamond$ rule, but not in the system with the ordinary $T_\diamond$ rule:

$$\begin{array}{ll}
F_\neg & T\diamond_x P(x, y) \wedge \diamond_x \neg P(x, y) \\
T_\wedge & T\diamond_x P(x, y),\ T\diamond_x \neg P(x, y) \\
T^f_\diamond & TP(f(y), y),\ T\diamond_x \neg P(x, y) \\
T^f_\diamond & TP(f(y), y),\ T\neg P(f(y), y) \\
T_\neg & TP(f(y), y),\ FP(f(y), y)
\end{array}$$

is closed, but

$$\begin{array}{ll} F_\neg & T\Diamond_x P(x,y) \wedge \Diamond_x \neg P(x,y) \\ T_\wedge & T\Diamond_x P(x,y),\ T\Diamond_x \neg P(x,y) \\ T_\Diamond & R(x',y),\ TP(x',y),\ T\Diamond_x \neg P(x,y) \\ T_\Diamond & R(x',y),\ TP(x',y),\ R(x'',y),\ T\neg P(x'',y) \\ T_\neg & R(x',y),\ TP(x',y),\ R(x'',y),\ FP(x'',y) \end{array}$$

is open. However, since the functionality axiom $\Diamond_x\varphi \equiv \Box_x\varphi$ implies

$$\Diamond_x\varphi_1 \wedge \Diamond_x\varphi_2 \to \Diamond_x(\varphi_1 \wedge \varphi_2),$$

given that $\Diamond_x\varphi_1$ and $\Diamond_x\varphi_2$ have the same free variables, $\neg(\Diamond_x P(x,y) \wedge \Diamond_x\neg P(x,y))$ is provable in L_{min} plus functionality. We will use the property of distributivity of $\Diamond$ over conjunction in the general proof below.

Assume that a tableau for $F\varphi$ closes in the system with 'functional' rules. We want to show that φ is provable in L^-_{min} plus functionality. Without lack of generality we may assume that φ is in normal form, that it contains only positive occurrences of $\Diamond$'s and that every quantifier has its own variable (since in $L^-_{min} + \Diamond_x\varphi \equiv \Box_x\varphi$ every formula has an equivalent with these properties).

Since there is a closed tableau for $F\varphi$ constructed in accordance with the functional rules, every branch contains a contradiction: for some formula ψ, $T\psi$ as well as $F\psi$. If ψ does not contain functional symbols, it means that it was not in the scope of any quantifier and the contradiction can be obtained by applying the propositional rules only. Then the same contradiction can be obtained in L^-_{min}, and we are done. Assume that ψ contains functional symbols, that is, $T\psi$ and $F\psi$ are obtained as a result of applying $T^f_\Diamond$ several times. It is clear that the number of iterations of f in ψ (counting $f(\emptyset)$ as an occurrence of f) equals the number of applications of $T^f_\Diamond$ to predecessors of $T\psi$ and $F\psi$. Let this number be n. This means that $T\psi$ is a result of decomposition of some subformula φ_1 of φ, which is not in a scope of a quantifier and contains n iterated quantifiers $\Diamond_{x_1}, \dots, \Diamond_{x_n}$ ($\Diamond_{x_1}$ being the outermost). Let us denote the biggest subformula in the scope of $\Diamond_{x_i}$ ψ_i. $T\psi$ is a result of decomposition of ψ_n. Analogously, $F\psi$ comes from a subformula φ_2 of φ, which contains n iterated quantifiers $\Diamond_{x'_1}, \dots, \Diamond_{x'_n}$; we denote the biggest subformula in the scope of $\Diamond_{x'_i}$ by ψ'_i. It is easy to check that for every i, ψ_i and ψ'_i have the same free variables and terms (otherwise $T\psi$ and $F\psi$ would contain different terms).

Assume now that we are making a tableau for $F\varphi$ using the ordinary rules. Every branch of this standard tableau corresponds to some branch of the closed tableau which was constructed with the use of functional rules. Assume that a branch of the standard tableau is open. We are going to show that the set of formulas which are true on this branch is inconsistent with $\Diamond_x\varphi \equiv \Box_x\varphi$.

From the analysis above it follows that on this branch the rule $T_\Diamond$ was applied n times to the successors of φ_1, introducing new variables $a_1, \dots, a_n$ with $R(a_1, \bar{y}_1)$ (where $\bar{y}_1$ are the free variables of $\Diamond_{x_1}\psi_1$), $R(a_2, a_1, \bar{y}_2)$, where $\bar{y}_2$ are the free variables of ψ_2 except for x_1, x_2, and so on. Note that the free variables of $T\psi$ are among $\bar{a}, \bar{y}_n$. Analogously, the rule $T_\Diamond$ was applied n times to the successors of φ_2, introducing $b_1, \dots, b_n$ with $R(b_1, \bar{y}_1)$, $R(b_2, b_1, \bar{y}_2)$, and so on. The free variables of $F\psi$ are among

$\bar{b}, \bar{y}_n$. The branch is not closed since there are some $a_i \neq b_i$ such that a_i is free in $T\psi$ and b_i in $F\psi$. But the following two formulas are true on this branch:

$$\Diamond_{x_1}(\top(x_1, \bar{y}_1) \wedge \Diamond_{x_2}(\top(x_1, x_2, \bar{y}_2) \wedge \ldots \wedge \Diamond_{x_n}(\top(x_{n-1}, x_n, \bar{y}_n) \wedge \psi(\bar{x}, \bar{y})) \ldots)$$

and

$$\Diamond_{x'_1}(\top(x'_1, \bar{y}_1) \wedge \Diamond_{x'_2}(\top(x'_1, x'_2, \bar{y}_2) \wedge \ldots \wedge \Diamond_{x'_n}(\top(x'_{n-1}, x'_n, \bar{y}_n) \wedge \neg\psi(\bar{x}', \bar{y})) \ldots)$$

(where subformulas in the scope of each quantifier have the same free variables). But then from the set of formulas on this branch a contradiction is derivable by renaming of bound variables and distributivity of $\Diamond$ over conjunction. □

As we have mentioned, decidability of the minimal logic with equality follows from the result of Andréka and Nemeti (1994). The tableaux decidability proof does not work for this system (since Lemma 3.4.6 does not hold any more). Adding one more axiom, namely $\neg\Diamond_x(x = y \wedge \top(\bar{z}))$ (which corresponds to $\neg R(x, \ldots x \ldots)$), yields a decidable system, for which decidability can be shown by tableaux.

There are several ways to introduce equality in analytic tableaux. Beth (1962) adds the following axioms, with all variables universally quantified:

$x = x$;
$x = y \rightarrow y = x$;
$x = y \wedge y = z \rightarrow x = z$;
$x = y \rightarrow (\varphi(x) \rightarrow \varphi(y))$, where φ may contain more free variables.

It is not difficult to show that the minimal logic plus these axioms (for all x, y and z) is equivalent to the system obtained by adding the following rule to the minimal logic (given that x is free for y and y is free for x in all formulas, which can be achieved by using different symbols for bound and free variables):

$T_=$

$$\frac{\Gamma, Tx = y}{\Gamma[x/y], \Gamma[y/x]}$$

where $\Gamma[x/y]$ denotes the result of substituting x for y everywhere in Γ, analogously for $\Gamma[y/x]$,

plus the condition that if for some x $Fx = x$ occurs on a branch, then this branch is closed. (I really don't know where I have seen this formulation, but it's unlikely that I have made it up all myself).

Lemma 3.4.6 does not hold for this system. Consider a tableau for $T\Diamond_x(x = y)$. As a result of $T_\Diamond$ we get $R(u, y)$, $Tu = y$, and as a result of $T_=$ we get $R(y, y)$, i.e. according to the lemma y must have appeared free in the tableau before y was introduced.

Even if a formula is in normal form, it can now have an infinite tableau. Consider a tableau for $F(\neg\Diamond_x\top(x, y) \vee \Diamond_x(x \neq y \vee \neg\Diamond_z P(x, y, z)))$.

At some point we come to

$$R(u, y),\ Tu = y,\ T\Diamond_z P(u, y, z),\ F\Diamond_x(x \neq y \vee \neg\Diamond_z P(x, y, z)),$$

and by $T_\diamond$ to

$$R(u,y),\ Tu=y,\ R(v,yu),\ TP(u,y,v),\ F\diamond_x(x\neq y\vee\neg\diamond_z P(x,y,z)).$$

After applying the $T_=$ rule we get along with other things $R(y,y)$, and the $F_\diamond$ rule can be activated again. The result is

$$Ty=y,\ T\diamond_z P(y,y,z),\ F\diamond_x(x\neq y\vee\neg\diamond_z P(x,y,z))$$

and by the $T_\diamond$ rule applied to $T\diamond_z P(y,y,z)$,

$$R(u_1,y),\ P(y,y,u_1).$$

Now the cycle can start again with u_1 in place of u.

However, the following holds:

3.4.17. THEOREM. *L^-_{min} with equality and additional axiom $\neg\diamond_x(x=y\wedge\top(\bar{z}))$ is decidable.*

Proof. It is easy to check that the axiom implies that every formula φ is equivalent to a formula which does not contain positive occurrences of equality in the scope of $\diamond_x$. (Namely, if the positive occurrence is of the form $x=y$, with x the quantifier variable, this follows from the axiom; if it is of the form $y=z$, this follows from the normal form lemma.) Consider a tableau for such a formula. Since there are no positive occurrences of equality under $\diamond_x$, for no new variable z introduced by the $T_\diamond$ rule $Tz=y$ occurs in the tableau. Note that we have a formulation of our system where the $T_=$ rule is the only rule for equality.

This allows us to divide the construction of the tableau in two parts, the first one containing no applications of the quantifier rules and all applications of the $T_=$ rule, and the second one containing no applications of the $T_=$ rule. The construction terminates by Lemma 3.4.10. □

3.5 Sequent calculus

Analytic tableaux described above correspond to a sequent calculus. Informally, a derivation tree in of the sequent calculus can be obtained from a closed tableau by turning the tableau upside down and putting the formulas signed by T to the left of $\Longrightarrow$ and the formulas signed by F to the right. For example,

$$\begin{array}{ll} & F(\varphi\wedge\psi\to\psi)\\ F_\to & T(\varphi\wedge\psi),\ F\psi\\ T_\wedge & T\varphi,\ T\psi,\ F\psi\end{array}$$

becomes

$$\dfrac{\dfrac{\varphi,\ \psi\Longrightarrow\psi}{\varphi\wedge\psi\Longrightarrow\psi}}{\Longrightarrow\varphi\wedge\psi\to\psi}$$

To define the left and right introduction rules for the quantifiers, we introduce a new kind of variables: indexed variables. Thus we avoid explicitely mentioning R in the derivation. Intuitively, $x_{\bar{y}}$, where $\bar{y}$ is a finite set of variables, ranges over the objects dependent on the objects assigned to $\bar{y}$. We use them as a technical device in the proofs.

3.5.1. Definition. Let Var be the set of ordinary variables. Define a new set of variables $IVar$ by recursion, as follows:

$IVar_0 = Var$,
$IVar_{n+1} = IVar_n \cup \{x_{\bar{z}} : x \in Var,\ \bar{z} \subseteq IVar_n \text{ is a finite set}\}$
$IVar = \bigcup_n IVar_n$.

We shall refer to variables in $IVar \backslash Var$ as indexed variables. Elements of $IVar_n$ are called indexed variables of depth n. □

3.5.2. Definition. The sequent calculus for the minimal logic without equality and without ordinary quantifiers is is a formal system including *sequent axioms* of the form $\varphi \Longrightarrow \varphi$, *structural rules*:

$$\textit{Permutation} \quad \frac{\Gamma_1,\ \varphi_1,\ \varphi_2,\ \Gamma_2 \Longrightarrow \Delta}{\Gamma_1,\ \varphi_2,\ \varphi_1,\ \Gamma_2 \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \Delta_1,\ \varphi_1,\ \varphi_2,\ \Delta_2}{\Gamma \Longrightarrow \Delta_1,\ \varphi_2,\ \varphi_1,\ \Delta_2}$$

$$\textit{Weakening} \quad \frac{\Gamma \Longrightarrow \Delta}{\varphi,\ \Gamma \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta,\ \varphi}$$

$$\textit{Contraction} \quad \frac{\varphi,\ \varphi,\ \Gamma \Longrightarrow \Delta}{\varphi,\ \Gamma \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \Delta,\ \varphi,\ \varphi}{\Gamma \Longrightarrow \Delta,\ \varphi}$$

and *left- and right-introduction rules* for connectives:

$$\frac{\Gamma,\ \varphi \Longrightarrow \Delta}{\Gamma \Longrightarrow \neg\varphi,\ \Delta}\ \neg_r \qquad \frac{\Gamma \Longrightarrow \varphi, \Delta}{\Gamma,\ \neg\varphi \Longrightarrow \Delta}\ \neg_l$$

$$\frac{\Gamma,\ \varphi, \psi \Longrightarrow \Delta}{\Gamma,\ \varphi \wedge \psi \Longrightarrow \Delta}\ \neg_l \qquad \frac{\Gamma \Longrightarrow \varphi, \Delta \quad \Gamma \Longrightarrow \psi,\ \Delta}{\Gamma \Longrightarrow \varphi \wedge \psi,\ \Delta}\ \neg_r,$$

dually for $\vee$;

$$\frac{\Gamma, \varphi(u_{\bar{y}}, \bar{y}) \Longrightarrow \Delta}{\Gamma, \Box_x\varphi(x, \bar{y}) \Longrightarrow \Delta}\ \Box_l \qquad \frac{\Gamma \Longrightarrow \varphi(u_{\bar{y}}, \bar{y}), \Delta}{\Gamma \Longrightarrow \Box_x\varphi(x, \bar{y}), \Delta}\ \Box_r$$

(in $\Box_r$ $u_{\bar{y}}$ does not occur in Γ, Δ, also not in the indices; in $\Box_l$ $u_{\bar{y}}$ *does* occur free in Γ or Δ);

$$\frac{\Gamma, \varphi(u_{\bar{y}}, \bar{y}) \Longrightarrow \Delta}{\Gamma, \Diamond_x\varphi(x, \bar{y}) \Longrightarrow \Delta}\ \Diamond_l \qquad \frac{\Gamma \Longrightarrow \varphi(u_{\bar{y}}, \bar{y}), \Delta}{\Gamma \Longrightarrow \Diamond_x\varphi(x, \bar{y}), \Delta}\ \Diamond_r$$

(in $\Diamond_l$ $u_{\bar{y}}$ does not occur in Γ, Δ, also not in the indices; in $\Diamond_r$ $u_{\bar{y}}$ occurs free in Γ or Δ). □

3.5.3. Definition. A *derivation* of a sequent $\Gamma \Longrightarrow \Delta$ (which does not contain indexed variables) in the sequent calculus for the minimal logic without equality and without ordinary quantifiers is a tree where each branch ends with an axiom, successor nodes are obtained from the predecessors by one of the rules, and the root of the tree is $\Gamma \Longrightarrow \Delta$. □

The introduction rules for quantifiers contain a hidden structural rule:

$$\frac{\Gamma, \varphi(u_{\bar{y}}, \bar{y}) \Longrightarrow \Delta}{\Gamma, \varphi(x_{\bar{y}}, \bar{y}) \Longrightarrow \Delta}\ SUB_{av}\ l \qquad \frac{\Gamma \Longrightarrow \varphi(u_{\bar{y}}, \bar{y}), \Delta}{\Gamma \Longrightarrow \varphi(x_{\bar{y}}, \bar{y}), \Delta}\ SUB_{av}\ r$$

both with the restriction that $u_{\bar{y}}$ does not occur in Γ, Δ (also not in the indices).

Given this rule, the introduction rules for quantifiers can be written as

$$\frac{\Gamma, \varphi(x_{\bar{y}}, \bar{y}) \Longrightarrow \Delta}{\Gamma, \Box_x \varphi(x, \bar{y}) \Longrightarrow \Delta}\ \Box_l \qquad \frac{\Gamma \Longrightarrow \varphi(x_{\bar{y}}, \bar{y}), \Delta}{\Gamma \Longrightarrow \Box_x \varphi(x, \bar{y}), \Delta}\ \Box_r$$

with the same restrictions on $x_{\bar{y}}$ as above; analogously for $\Diamond$.

Note that if we do not impose the restriction that $\Box_l$ can be applied to $\varphi(x_{\bar{y}}, \bar{y})$ only if $x_{\bar{y}}$ occurs in some other formula, then it means that we assume the set of objects dependent on $\bar{y}$, for every $\bar{y}$, to be non-empty.

3.5.4. Theorem. *Every closed tableau for $F\varphi$ corresponds to a derivation of $\Longrightarrow \varphi$ in the sequent calculus.*

Proof. The idea of the proof was sketched above. A set of signed formulas at each node of a tableau corresponds to a sequent, where the formulas signed by T are to the left of the turnstyle and the formulas signed by F to the right of the turnstyle. Note that in the process of conversion *sets* of indexed formulas which were nodes in tableaux become *sets* of formulas to the left and to the right of the sequent arrow. The structural rules take care of that.

If a tableau is constructed from the top to the bottom (the most complex formula is at the top, and the decomposition rules create its successors), then constructing the corresponding derivation we work from the bottom to the top: below is the sequent which we want to derive, and we look for the predecessor nodes. The sequent calculus is so devised that every tableau rule has a corresponding sequent rule, and closed branches of a tableau correspond to the branches ending with axioms in a derivation. □

We did not include the Cut rule

$$\frac{\Gamma_1 \Longrightarrow \Delta_1,\ \varphi \qquad \varphi,\ \Gamma_2 \Longrightarrow \Delta_2}{\Gamma_1,\ \Gamma_2 \Longrightarrow \Delta_1,\ \Delta_2}$$

in the definition of the sequent calculus. From the completeness theorem for tableaux and the theorem above follows that every valid formula has a cut-free derivation in the sequent calculus (since every valid formula has a closed tableau).

3.5.5. Consequence. *(Cut elimination) The Cut rule is eliminable in the sequent calculus for the minimal logic.*

Observe that the $F_\Diamond$ rule corresponds to the $\Diamond_r$ rule plus contraction:

$$\begin{array}{ll} & \Sigma,\ F\Diamond_x\varphi(x,\bar{y}) \\ F_\Diamond & \Sigma,\ R(u,\bar{y}),\ F\varphi(u,\bar{y}),\ F\Diamond_x\varphi(x,\bar{y}) \end{array}$$

corresponds to

$$\dfrac{\dfrac{\Gamma \Longrightarrow \varphi(x_{\bar{y}},\bar{y}),\ \Diamond_x\varphi(x,\bar{y}),\ \Delta}{\Gamma \Longrightarrow \Diamond_x\varphi(x,\bar{y}),\ \Diamond_x\varphi(x,\bar{y}),\ \Delta}}{\Gamma \Longrightarrow \Diamond_x\varphi(x,\bar{y}),\ \Delta}$$

and the decidability proof from the previous section can be reformulated as saying that for a formula in normal form, contraction will be performed not more than n^n times, n being the number of quantifiers in the formula. [2]

The definition of a *main formula* of the application of a rule is standard; for example, $\varphi \wedge \psi$ is the main formula of the application of the rule $\wedge_r$ as formulated in definition 3.5.2, and φ and ψ are called main formulas in the premises.

We use a standard notion of a *successor of a formula in the derivation*: if φ is main formula in an application of the rule, it is a successor of the main formulas in the premises, and the successor relation is transitive. Note that each formula in a derivation can have many successors on a given branch, but only one at every node.

If all formulas in the derived sequent are in normal form (every subformula immediately in the scope of the quantifier has the quantifier variable free) and indexed variables do not occur only in the root sequent, some useful properties hold for any formula which occurs in the derivation.

3.5.6. Claim. *Every formula in the derivation is in normal form.*

Proof Follows from Lemma 3.4.4 and Theorem 3.5.4. □

3.5.7. Claim. *For any two indexed variables x and y in one formula, either x occurs in the index of y (we write it $x < y$), or y in the index of x; in other words, the set of indexed variables of one formula is linearly ordered.*

Proof Let $\varphi(x, y)$ occur on some branch of the derivation tree and x does not occur in the index of y, and vice versa. By assumption, the sequent at the root does not contain indexed variables free. Therefore below on the branch quantifier rules must be used to replace x and y by ordinary bound variables. Assume that x is the first variable to be replaced. Then the premise of the rule, which is a successor of $\varphi(x, y)$, $\varphi'(x, y)$, contains y free. But y is not in the index of x, therefore no quantifier rule is applicable to $\varphi'(x, y)$. Similarly, y cannot be the first variable to be replaced. This implies that x and y are free in the derived sequent, contrary to the assumption. □

[2] Which at first sight reminds of n-bounded contraction, cf. (Prijatelj 1994), where only a contraction of n copies of a formula is allowed. However, 'n times 2-contraction' can not be reduced to any form of n-contraction, and vice versa. The formal properties of the systems are also different; for example, in the sequent calculus with n-contraction CUT is not eliminable.

3.5.8. Claim. *In every formula there is an indexed variable in whose index all other indexed variables of the formula occur. It will be called the top indexed variable of the formula.*

Proof Immediately from the previous claim. □

3.5.9. Claim. *The top indexed variable of φ contains in its index all variables which are used to instantiate the quantifiers in whose scope φ occurs in the derived sequent (i.e. quantifiers which are introduced on the successors of φ in the derivation).*

Proof Let x be the top indexed variable of φ. Let φ' be the first successor of φ which is a main premise of a quantifier rule. Since no quantifier rules were applied yet, x is free in φ'. Suppose x is not the variable to be replaced by this rule. Then there is some other variable z in φ' which is to be replaced, and $x < z$. Since x is the top variable of φ, z is not free in φ. Let $Qz_1\varphi'$ ($Q \in \{\Box, \Diamond\}$) be the result of application of the rule. Observe that φ is immediately in the scope of Qz_1 (since no quantifiers were introduced on the successors of φ before), and φ does not contain z free. This means that a derivation contains a formula not in normal form: a contradiction. Therefore, x is the variable to be replaced.

Recall the proof of Lemma 3.4.7. In the tableaux formulation, φ is a result of decomposition of the successor of φ with the maximal number of quantifiers (a formula to the left or to the right of the sequent arrow in the derived sequent). The top indexed variable is the variable which depends on all the variables used to instantiate the quantifiers. Observe that 'x depends on y' is the same, for indexed variables, as 'x contains y in its index'. □

3.5.10. Claim. *Given an arbitrary formula φ in the derivation, one can determine from the top indexed variable of φ how many quantifiers can be introduced on the successors of φ in the derivation.*

Proof It is easy to check that this number is equal to the depth of the top indexed variable plus 1. □

3.5.11. Claim. *Given an arbitrary formula φ in the derivation, one can determine from the top indexed variable of φ what are the free variables of the successor formulas of φ which are main premises of a quantifier introduction rule.*

Proof In the index of the top variable, the variables are also linearly ordered. The index of the top variable of level i are the free variables of the formula which served as a main premise of the ith quantifier introduction rule. □

From the claims, we have

3.5.12. Lemma. *If two formulas φ_1 and φ_2 have the same top indexed variable, then in the derived sequent they are in the scope of the same number of quantifiers $Q_1x_1, \ldots, Q_nx_n$ and $O_1x_1, \ldots, O_nx_n$ (where $Q_i, O_i \in \{\Diamond, \Box\}$), and the biggest subformula in the scope of Q_i has the same free variables as the biggest subformula in the scope of O_i.*

This lemma will be used in the proof of the interpolation theorem below.

3.6 Interpolation

The interpolation theorem is proved here mostly to illustrate the magical properties of indexed variables. However, it is also an interesting thing to know about the minimal logic: is it self-contained enough as a fragment of first-order logic? Since it corresponds to a fragment of first order logic, every derivable sequent has an interpolant in the language containing R and the predicates occurring both in the consequent and antecedent; however, this interpolant does not have to be in the same fragment, i.e. does not have to be a translation of a quantifier formula. We prove that there is always an interpolant in the same fragment.

The proof is more complicated than the proof for the first order logic, although it follows the same strategy. For every rule we assume that the premises have interpolants, and construct an interpolant for the conclusion. For example, consider the following derivation in first order logic:

$$\dfrac{\dfrac{\varphi \Longrightarrow \psi \quad \varphi' \Longrightarrow \psi}{\varphi \vee \varphi' \Longrightarrow \psi}}{\varphi \vee \varphi' \Longrightarrow \exists x \psi}$$

Assume that the interpolants for $\varphi \Longrightarrow \psi$ and $\varphi' \Longrightarrow \psi$ are χ and χ', respectively. Then the interpolant for $\varphi \vee \varphi' \Longrightarrow \psi$ is $\chi \vee \chi'$ and the interpolant for $\varphi \vee \varphi' \Longrightarrow \exists x \psi$ is $\exists x(\chi \vee \chi')$. Since for the quantifier rules in our calculus some additional side conditions hold, the same straightforward strategy does not work. For example, if the free variables of ψ are x_y, y, the free variables of χ - x_y, y and the free variables of χ' - z, then the introduction rule for $\Diamond_x$ is applicable for ψ but not for $\chi \vee \chi'$:

$$\dfrac{\dfrac{\chi(x_y, y) \vee \chi'(z) \Longrightarrow \psi(x_y, y)}{\chi(x_y, y) \vee \chi'(z) \Longrightarrow \Diamond_x \psi(x, y)}}{\ldots}$$

Therefore some more work is needed to construct and interpolant.

3.6.1. THEOREM. *If a sequent $\Gamma \Longrightarrow \Delta$ is provable in the minimal logic without identity and ordinary quantifiers, then there is a formula χ (called an interpolant) such that*

1. $\Gamma \Longrightarrow \chi$ *and* $\chi \Longrightarrow \Delta$ *are provable,*
2. $PRED(\chi) \subseteq PRED(\Gamma) \cap PRED(\Delta)$ *if* $PRED(\Gamma) \cap PRED(\Delta) \neq \emptyset$, *and* $PRED(\chi) = \{\top\}$ *otherwise;*
3. $FV(\chi) \subseteq FV(\Gamma) \cap FV(\Delta)$.

Proof. Without loss of generality we may assume that Γ and Δ are formulas in normal form and that negations are pushed inside to atomic formulas. Then $\Gamma \Longrightarrow \Delta$ has a cut free derivation where the only rules applied are $\Box_{l,r}$, $\Diamond_{l,r}$, $\wedge_{l,r}$ and $\vee_{l,r}$, and axioms are of the form

(a) $\Phi, (\neg)P(\bar{x}) \Longrightarrow (\neg)P(\bar{x}), \Psi$, and P occurs both in Γ and Δ,
(b) $\Phi, P(\bar{x}), \neg P(\bar{x}) \Longrightarrow \Psi$, and P occurs possibly only in one of Γ, Δ.

(c) $\Phi \Longrightarrow P(\bar{x}), \neg P(\bar{x}), \Psi$, and P occurs possibly only in one of Γ, Δ.

We prove the theorem by induction on the length of the derivation of $\Gamma \Longrightarrow \Delta$. First we prove the theorem with condition (2) weakened to $PRED(\chi) \subseteq (PRED(\Gamma) \cap PRED(\Delta)) \cup \{\top\}$.

Assume that $\Gamma \Longrightarrow \Delta$ is an axiom. Then χ equals $P(\bar{x})$ in case (a), $\neg\top(\bar{x})$ in case (b) and $\top(\bar{x})$ in case (c). We shall call the successors of $P(\bar{x})$ and $\neg P(\bar{x})$ 'successors of the axiom'.

The inductive hypothesis says that for the $(n-1)$th step in the derivation the interpolant exists, and the following condition holds. Let φ_1 and φ_2 be successors of $P(\bar{x})$ and $\neg P(\bar{x})$, respectively. The inductive hypothesis says that every time when a quantifier introduction rule is applied to φ_1 or φ_2, $FV(\chi) \subseteq FV(\varphi_1) \cup FV(\varphi_2)$. Note that this additional condition is satisfied for the basis of induction.

Before proceeding further, we need the following

3.6.2. Claim. *Let $\Gamma \Longrightarrow \Delta$ and suppose χ is an interpolant for this sequent. Let $\Gamma' \Longrightarrow \Delta'$ be obtained from $\Gamma \Longrightarrow \Delta$ (and possibly some other sequent) by applying inference rules which do not involve the successors of any axiom as main premises. Then χ is an interpolant for $\Gamma' \Longrightarrow \Delta'$.*

Proof Easy induction. □

In the sequel, we consider only the cases when one of the main premises of the rule is a successor of an axiom.

Assume that $\Gamma \Longrightarrow \Delta$ is obtained by a rule $\wedge_l$ or $\vee_r$ from a sequent which has an interpolant. Then, if χ is an interpolant for $\Gamma', \varphi_i, \psi \Longrightarrow \Delta'$, then it is an interpolant for $\Gamma', \varphi_i \wedge \psi \Longrightarrow \Delta'$, analogously for $\vee_l$.

Consider the rules $\wedge_r$ and $\vee_l$. Let χ_1 be an interpolant for $\Gamma', \varphi_i \Longrightarrow \Delta'$ and χ_2 for $\Gamma', \psi \Longrightarrow \Delta'$. Then $\chi_1 \vee \chi_2$ is an interpolant for $\Gamma', \varphi_i \vee \psi \Longrightarrow \Delta'$. Analogously for $\wedge_r$.

Finally, we consider the quantifier rules. Since $P(\bar{x})$ and $\neg P(\bar{x})$ have the same free variables, they also have the same top indexed variable, which by Lemma 3.5.12 means that both formulas are in the scope of the same number of quantifiers in $\Gamma \Longrightarrow \Delta$ and that if φ_1 is the successor of $P(\bar{x})$ to which the ith quantifier rule is applied, and φ_2 is the successor of $\neg P(\bar{x})$ to which the ith quantifier rule is applied, then φ_1 and φ_2 have the same free variables (including the top indexed variable on which the quantifier is introduced).

From Lemma 3.4.8 it follows that any tableau proof can be arranged so that all decomposition rules involving a variable dependent on a given set of variables $\bar{y}$ can be applied simultaneously, not separated by other rules. For the sequent calculus, that means (by Theorem 3.5.4) that all the rules involving a given indexed variable $x_{\bar{y}}$ can be applied simultaneously. Therefore, without loss of generality we may assume that the ith application of a quantifier rule to the successor formula of $P(\bar{x})$ is immediately followed by an application of a quantifier rule to the successor formula of $\neg P(\bar{x})$, or vice versa.

Let φ_1 be a successor formula of $P(\bar{x})$, φ_2 of $\neg P(\bar{x})$, χ the interpolant for

$$\Gamma', \varphi_1(x_{\bar{y}}, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'$$

and the next rule in the derivation is the ith quantifier rule. By our assumption, φ_1 and φ_2 have the same free variables, and since $FV(\chi) \subseteq FV(\varphi_1) \cup FV(\varphi_2)$, χ has the same free variables or less. We may assume that χ has precisely the same free variables, since we can always add some tautology with the lacking variables conjunctively to χ.

The first rule must be one of the rules $\Box_l$ or $\Diamond_r$, since the variable on which the quantifier is introduced is free in another formula. Let is be $\Box_l$:

$$\frac{\Gamma', \varphi_1(x_{\bar{y}}, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}{\Gamma', \Box_x\varphi_1(x, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}$$

The rule applied to φ_2 can be either $\Box_r$ or $\Diamond_r$.

Assume that it is $\Box_r$:

$$\cfrac{\cfrac{\Gamma', \varphi_1(x_{\bar{y}}, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}{\Gamma', \Box_x\varphi_1(x, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}}{\Gamma', \Box_x\varphi_1(x, \bar{y}) \Longrightarrow \Box_x\varphi_2(x, \bar{y}), \Delta'}$$

The applicability of $\Box_r$ means that $x_{\bar{y}}$ is not free in Γ' and Δ'.

But then we have

$$\cfrac{\cfrac{\Gamma', \varphi_1(x_{\bar{y}}, \bar{y}) \Longrightarrow \chi(x_{\bar{y}}, \bar{y})}{\Gamma', \Box_x\varphi_1(x, \bar{y}) \Longrightarrow \chi(x_{\bar{y}}, \bar{y})}}{\Gamma', \Box_x\varphi_1(x, \bar{y}) \Longrightarrow \Box_x\chi(x, \bar{y})}$$

and

$$\cfrac{\cfrac{\chi(x_{\bar{y}}, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}{\Box_x\chi(x, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}}{\Box_x\chi(x, \bar{y}) \Longrightarrow \Box_x\varphi_2(x, \bar{y}), \Delta'}$$

that is, $\Box_x\chi$ is the interpolant for

$$\Gamma', \Box_x\varphi_1(x, \bar{y}) \Longrightarrow \Box_x\varphi_2(x, \bar{y}), \Delta'.$$

Assume that the rule applied to φ_2 is $\Diamond_r$:

$$\cfrac{\cfrac{\Gamma', \varphi_1(x_{\bar{y}}, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}{\Gamma', \Box_x\varphi_1(x, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}}{\Gamma', \Box_x\varphi_1(x, \bar{y}) \Longrightarrow \Diamond_x\varphi_2(x, \bar{y}), \Delta'}$$

The applicability of $\Diamond_r$ means that $x_{\bar{y}}$ is free in Γ' or Δ' (or both). If $x_{\bar{y}}$ is free in Γ', but not in Δ', we can prove

$$\cfrac{\cfrac{\Gamma', \varphi_1(x_{\bar{y}}, \bar{y}) \Longrightarrow \chi(x_{\bar{y}}, \bar{y})}{\Gamma', \Box_x\varphi_1(x, \bar{y}) \Longrightarrow \chi(x_{\bar{y}}, \bar{y})}}{\Gamma', \Box_x\varphi_1(x, \bar{y}) \Longrightarrow \Diamond_x\chi(x, \bar{y})}$$

(the last step is justified since $x_{\bar{y}}$ is free in Γ'), and

$$\cfrac{\cfrac{\chi(x_{\bar{y}}, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}{\chi(x, \bar{y}) \Longrightarrow \Diamond_x\varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}}{\Diamond_x\chi(x, \bar{y}) \Longrightarrow \Diamond_x\varphi_2(x, \bar{y}), \Delta'}$$

(the last step is justified since $x_{\bar{y}}$ is not free in Δ), that is, $\Diamond_x \chi$ is the interpolant for

$$\Gamma', \Box_x \varphi_1(x, \bar{y}) \Longrightarrow \Diamond_x \varphi_2(x, \bar{y}), \Delta'.$$

Analogously, if $x_{\bar{y}}$ is free only in Δ', the interpolant is $\Box_x \chi$.

If $x_{\bar{y}}$ is free in both Γ' and Δ', we temporarily leave the interpolant to be $\chi(x_{\bar{y}}, \bar{y})$. Note that $FV(\chi) \subseteq FV(\Gamma) \cap FV(\Delta)$, but $FV(\chi) \not\subseteq FV(\Box_x \varphi_1) \cup FV(\Diamond_x \varphi_2)$. To satisfy the induction hypothesis, $x_{\bar{y}}$ should be bound before the next rule is applied to the successors of $\Box_x \varphi_1$ and $\Diamond_x \varphi_2$. By Lemma 3.4.8 we may assume that $x_{\bar{y}}$ is bound before any other quantifier rule is applied to the successors of the axiom (since that rule would involve another variable). The formulas which are the main premises in the applications of quantifier rules on $x_{\bar{y}}$, are not successors of $\Box_x \varphi_1$ and $\Diamond_x \varphi_2$ (otherwise $\Box_x \varphi_1$ and $\Diamond_x \varphi_2$, which do not contain $x_{\bar{y}}$ free, would be immediately in the scope of a quantifier introduced on $x_{\bar{y}}$). We assume therefore that these rules may be performed before any new rule is applied to the successors of $\Box_x \varphi_1$ and $\Diamond_x \varphi_2$, and that by the time that $x_{\bar{y}}$ does not occur free any more in Γ' or in Δ', the interpolant has not changed. Then we can introduce a quantifier on χ as above.

If the first rule applied to

$$\Gamma', \varphi_1(x_{\bar{y}}, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'$$

is $\Diamond_r$, the reasoning is completely analogous.

So far we have proved $PRED(\chi) \subseteq (PRED(\Gamma) \cap PRED(\Delta)) \cup \{\top\}$. To obtain clause (2) of theorem 3.6.1, we reason as follows. If $PRED(\Gamma) \cap PRED(\Delta) \neq \emptyset$, $\top$ is definable for every arity n. Namely, let P^m be both in Γ and Δ; then it is easy to define $\top^n$ for $n \geq m$. For $n < m$, take $\Diamond_{x_1} \ldots \Diamond_{x_k} P(x_1, \ldots, x_k, \ldots) \vee \neg \Diamond_{x_1} \ldots \Diamond_{x_k} P(x_1, \ldots, x_k, \ldots)$. □

Observe that if $\Gamma \Longrightarrow \Delta$ is provable, and $PRED(\Gamma) \cap PRED(\Delta) = \emptyset$, this does not mean that $\Gamma \Longrightarrow$ is provable or $\Longrightarrow \Delta$ is provable. Consider for example $\Box_x P(x) \wedge \Box_x \neg P(x) \Longrightarrow \Box_x (Q(x) \wedge \neg Q(x))$.

If we assume that for any indexed variable the domain is non-empty (which is equivalent to $\forall \bar{y} \exists x R(x, \bar{y})$), the restrictions on $\Box_l$ and $\Diamond_r$ should be dropped. The proof of the interpolation theorem remains the same, but in the step for quantifiers, we should consider one more case: when both rules are weak $\Box_l$ and $\Diamond_r$, and $x_{\bar{y}}$ is not free in Γ and Δ. Then the interpolant can be $\Box_x \chi$ as well as $\Diamond_x \chi$.

For the system without restrictions on $\Box_l$ and $\Diamond_r$, $\Gamma \Longrightarrow \Delta$ and $PRED(\Gamma) \cup PRED(\Delta) = \emptyset$ mean, as in classical predicate logic, that $\Gamma \Longrightarrow$ or $\Longrightarrow \Delta$. The proof of $\Gamma \Longrightarrow$ ($\Longrightarrow \Delta$) is obtained from the proof of $\Gamma \Longrightarrow \Delta$ by omitting the righthand side (lefthand side) of the derivation; now this does not influence the applicability of $\Box_l$ and $\Diamond_r$.

3.7 Using graphs instead of indexed variables

As it was mentioned in the introduction, instead of using indexed variables or explicit statements of the form $R(a, \ldots, b, \ldots)$, we could have used graphs on ordinary variables constructed simultaneously with the construction of a tableau or a sequent derivation.

It is easy to reformulate tableau rules using graphs (and referring to the presence of certain arrows between certain variables in the side conditions of the rules). The graph construction begins with listing all free variables of the formula and a special symbol $\emptyset$. When a new variable is introduced by the $T_\diamond$ rule, it is added to the set of nodes. Instead of adding $R(x, z_1, \ldots, z_n)$ to the set of signed formulas, we add arrows from x to z_i, $1 \leq i \leq n$. This makes new applications of the $F_\diamond$ rule to the formulas of the form $F\diamond_y \psi(y, z_1, \ldots, z_n)$ possible.

In general, such graph can be infinite if the corresponding derivation is. But for the formulas in normal form it is finite, since only finitely many new objects are introduced in the derivation.

Chapter 4

Correspondence and Completeness for Generalized Quantifiers

In the Introduction it was mentioned that the sequent calculus for generalized quantifiers can be extended by substitution rules to yield systems sound and complete with respect to the logics of some well known generalized quantifiers, and that it in order to devise such substitution rules, it is useful to know which properties of the dependence relation correspond to the quantifier axioms.

In this chapter, we prove some general theorems concerning correspondence between properties of the dependence relation R and additional quantifier axioms. First, we study *frame correspondence*, and then *correspondence for completeness*, which allows us to prove general completeness results for some classes of quantifier logics with respect to dependence models.

In order to convince the reader that the minimal logic does have interesting extensions, and to introduce particular axioms as running examples, we state here several axiomatic completeness results (due to Krivine and McAloon (1973), Friedman (cf. (Steinhorn 1985)) and Keisler (1970)).

Free filter quantifier $\Diamond_x$ is axiomatized by adding to L_{min} introduced in definition 3.2.1 the following axioms:

Q1 $\Diamond_x x = x$
Q2 $\neg\Diamond_x x = y$
Q3 $\Box_x\varphi \wedge \Box_x\psi \rightarrow \Box_x(\varphi \wedge \psi)$
Q4 $\Diamond_x\varphi \rightarrow \Diamond_x(\varphi \vee \psi)$

Observe that there are no restrictions on the free variables of φ and ψ in Q3 and Q4. This means that this logic has unrestricted extensionality:

$$\forall x(\varphi \equiv \psi) \rightarrow (\Diamond_x\varphi \equiv \Diamond_x\psi).$$

Adding to Q1 – Q4 above the *Fubini axiom*

Q5 $\Box_x\Box_y\varphi \rightarrow \Box_y\Box_x\varphi$

turns $\Box$ into the quantifier 'for almost all'. We will refer to Q1 – Q5 as *Friedman's axioms.* $\Diamond$ becomes the quantifier 'for uncountably many' if we add to Q1 – Q4

Q6 $\forall x \Box_y \varphi \wedge \Box_x \forall y \varphi \to \Box_y \forall x \varphi$.

Q6 will be referred to as *Keisler's axiom*.

4.1 Frame Correspondence

We define here several standard notions of correspondence theory, completely in parallel with modal logic (cf. (van Benthem 1983, 1984)). The essence of frame correspondence is straightforward reduction of an exotic language (containing modalities or generalized quantifiers) to first order logic.

Recall that the standard translation ST introduced in Chapter 3 leaves atomic fromulas intact, commutes with propositional connectives and ordinary quantifiers and for the generalized quantifiers

$$ST(\Diamond_x \varphi(x, \bar{y})) = \exists x(R(x, \bar{y}) \wedge ST(\varphi(x, \bar{y}))).$$

4.1.1. Definition. A *frame* is a pair $\langle D, R \rangle$ of a domain and dependence relation; it can be identified with a set of dependence models with the same domain and accessibility relation, but all possible interpretation functions. □

If a formula φ of $\mathcal{L}(\exists\Diamond)$ is valid in a frame F (under an assignment s), then classically

$$F, s \models \forall P_i^n \ldots \forall P_l^m ST(\varphi),$$

where $P_i^n, \ldots, P_l^m$ are the predicate letters in φ. If this second-order formula has a first-order equivalent (containing only R and $=$), φ is called *first-order definable*. This means that if φ is true in all models over F, then R has the property defined by φ, and vice versa. Additional quantifier axioms added to the minimal logic will now express special conditions on the relation R in frames.

In order to talk about *axioms* (that is, formulas) and not schemata, we introduce the substitution rule

$$\frac{\vdash \Phi(P(x_1, \ldots, x_n))}{\vdash \Phi(\varphi(x_1, \ldots, x_n))}$$

provided $P(\bar{x})$ and $\varphi(\bar{x})$ have precisely the same free variables. This restriction is necessary due to the fact that in the minimal logic we have only a restricted form of extensionality. As usual, if A is an axiom, then every substitutional instance (in the above sense) of A is an axiom.

4.1.2. Definition. If A is a quantifier axiom, a *frame correspondent* is a first order condition $A^\star$ on R such that $\langle D, R \rangle \models A^\star$ if and only if for *any* interpretation V, $\langle D, R, V \rangle \models A$. □

It is known that the question whether a modal axiom corresponds to a first-order condition on frames is undecidable (Chagrova (1991)). One would expect that the same holds for modal quantifiers. And indeed we have

4.1.3. Theorem. *First-order frame correspondence for $\mathcal{L}(\Diamond)$ formulas is undecidable.*

Proof. The idea is as follows. Let φ be a modal formula. It defines a first-order condition on frames if and only if $\forall P_1 \dots \forall P_n ST^m(\varphi)$ has a first-order equivalent, where $P_1, \dots, P_n$ are all predicate symbols in $ST^m(\varphi)$ and ST^m is the standard translation of modal formulas in the first-order language. Analogously for the generalized quantifier formulas. From Proposition 3.1.2 we know that $\forall P_1 \dots \forall P_n ST^m(\varphi)$ has a first order equivalent if, and only if, $\forall P_1 \dots \forall P_n ST(\varphi^0)$ does, where φ^0 is the translation of φ into the quantifier language. Thus, a modal formula is first-order definable iff its $\mathcal{L}(\Diamond)$-counterpart is, and hence the correspondence problem for generalized quantifiers (in the latter language) is undecidable. □

Theorem 4.1.3 implies that the best we can hope for is to isolate syntactically a *subset* of formulas having a frame correspondent. This is done in the next section.

4.1.1 Sahlqvist theorem for frames

4.1.4. Definition. A formula is called *Sahlqvist formula* if it is of the form

$$\bigwedge_i Qu_1 \dots Qu_k(\varphi \to \psi),$$

where Qu_j is either $\forall u_j$ or $\Box_{u_j}$, and

1. φ is constructed from
   - atomic formulas, possibly prefixed by $\Box_x$, $\forall$;
   - formulas in which predicate letters occur only negatively

   using $\wedge, \vee, \Diamond_x, \exists$
2. in ψ all predicate letters (except =) occur only positively. □

Observe that Q1 – Q6 are Sahlqvist formulas.

4.1.5. Theorem. *Every Sahlqvist formula has a frame correspondent.*

Proof. The proof is analogous to the proof in (van Benthem 1983).

Consider an arbitrary Sahlqvist formula. If every conjunct of a formula is first-order definable, then the whole conjunction is. Therefore without loss of generality we can concentrate on a formula of the form $Qu_1 \dots Qu_k(\varphi \to \psi)$. First we translate it into second-order logic:

$$\forall P_1^n \dots \forall P_l^m \forall u_1 \dots \forall u_k(\mathcal{R} \wedge ST(\varphi) \to ST(\psi)),$$

where $P_1^n \dots P_l^m$ are all the predicates in $\varphi \to \psi$ and $\mathcal{R}$ is a conjunction of R-statements corresponding to the □-quantifiers in the prefix. Then we remove all 'empty' quantifiers (those binding variables not occurring in their scope), and rename bound individual variables in such a way that every quantifier gets its own variable which is distinct from any free variable occurring in the formula. Now it is possible to move all existential quantifiers occurring in positive subformulas of $ST(\varphi)$ to a prefix, using the following equivalences:

$$\exists x A(x) \vee \exists y B(y) \equiv \exists x \exists y (A(x) \vee B(y))$$

$$\exists x A(x) \wedge B \equiv \exists x(A \wedge B)$$

with the usual restrictions. $ST(\varphi)$ has now been rewritten as

$$\exists y_1 \ldots \exists y_m \varphi'.$$

Since ψ does not contain $y_1, \ldots, y_m$ free, $\forall u_1 \ldots \forall u_k(\mathcal{R} \wedge ST(\varphi) \rightarrow ST(\psi))$ is equivalent to

$$\forall x_1 \ldots \forall x_n(\mathcal{R} \wedge \varphi' \rightarrow ST(\psi)),$$

where $x_1, \ldots, x_n$ include $\bar{u}$ and $\bar{y}$.

Next, it would be convenient to get rid of the disjunctions in φ'. Let $\varphi' \equiv \phi_1 \vee \phi_2$.

$$\forall x_1 \ldots \forall x_n((\mathcal{R} \wedge \phi_1) \vee (\mathcal{R} \wedge \phi_2) \rightarrow ST(\psi))$$

is equivalent to

$$\forall x_1 \ldots \forall x_n(\mathcal{R} \wedge \phi_1 \rightarrow ST(\psi)) \wedge \forall x_1 \ldots \forall x_n(\mathcal{R} \wedge \phi_2 \rightarrow ST(\psi)).$$

We can restrict attention to one of these conjuncts (if both components have a first-order equivalent, then so has their conjunction). So, assume that there are no disjunctions in the antecedent. Thus, we have a formula

$$\forall P_1^n \ldots \forall P_l^m \forall x_1 \ldots \forall x_n(\varphi' \rightarrow ST(\psi)),$$

where $P_1^n \ldots P_l^m$ are all the predicates in $\varphi' \rightarrow ST(\psi)$, and φ' is a conjunction of 'blocks' which are of one of the following forms:

1. standard translations of atomic formulas possibly preceded by universal and $\Box$-quantifiers,
2. R-statements,
3. formulas in which all predicate letters occur only negatively.

Next we rewrite the formula so that there are no negative formulas in the antecedent. The point is that $\varphi' \rightarrow ST(\psi)$ can always be rewritten as an implication whose antecedent does not contain negative formulas. Let $\varphi' = \phi_1 \wedge \phi_2$, where ϕ_2 is a negative formula. Then

$$\phi_1 \wedge \phi_2 \rightarrow ST(\psi)$$

is equivalent to

$$\phi_1 \rightarrow \neg\phi_2 \vee ST(\psi),$$

whose consequent contains only positive occurrences of predicate letters.

Let us denote the antecedent obtained (without negative formulas) φ^*. We shall now define the notion of *a minimal substitution* for every predicate letter in φ^*.

A predicate letter P_i^n can occur in φ^* more than once. Consider an occurrence $\bar{P}_i^n$ of P_i^n in φ^*. First we have to classify the variables of this occurrence (this is the only part where the present proof becomes different from the modal case). Let us assume that

- the variables which stand at the places $i_1, \ldots, i_m$ *in this occurrence* are existentially bound or free; let us denote them $x_1, \ldots, x_m$;
- the variables at the places $j_1, \ldots, j_k$ are universally bound by quantifiers which correspond to $\Box$-quantifiers in the original formula; let us call them $z_1, \ldots, z_k$;
- the rest of the variables is bound by ordinary universal quantifiers; let us call them $v_1, \ldots v_l$.

Before defining a minimal substitution we have to define the notion of an 'R-condition' corresponding to the variable z_i:

1. Let $\Box_{z_1}$ be the first (leftmost) generalized quantifier in the sequence of quantifiers preceding $\bar{P}_i^n$, and before $\Box z_1$ the ordinary universal quantifiers $\forall v_1, \ldots, \forall v_s$ occur. Then the R-condition corresponding to z_1 will be $R(z_1, v_1, \ldots, v_s, \bar{x})$,
2. Let $\Box_{z_i}$ be the generalized quantifier following $\Box_{z_{i-1}}$ in our sequence (with some $\forall v_p, \ldots, \forall v_r$ possibly standing in between):

$$\ldots \Box_{z_{i-1}} \forall v_p \ldots \forall v_r \Box_{z_i} \ldots \bar{P}_i^n$$

   If the condition corresponding to z_{i-1} was $R(z_{i-1}, \bar{y})$, then the condition corresponding to z_i is $R(z_i, v_p, \ldots, v_r, z_{i-1}, \bar{y})$.

The minimal substitution $Sb(\bar{P}_i^n)$ for the occurrence of P_i^n in φ^* described above will be:

$P_i^n(u_1, \ldots u_n)$ is the conjunction of

1. $u_{i1} = x_1, \ldots, u_{im} = x_m$;
2. $\top(v_1), \ldots, \top(v_l)$;
3. $R(u_{\alpha_1}, \ldots, u_{\alpha_f})$, where $u_{\alpha_1}, \ldots, u_{\alpha_f}$ are the variables standing at the places $\alpha_1, \ldots, \alpha_f$, and in φ^* for these variables some R-condition (corresponding to one of the variables $z_1, \ldots, z_k$) hold.

Finally, we define

$$Sb(P_i^n, \varphi^*) = \bigvee Sb(\bar{P}_i^n)$$

for all occurrences of P_i^n in φ^*. [1]

The result of substituting $Sb(P_i^n, \varphi^*)$ in $\forall x_1 \ldots \forall x_m(\varphi^* \to \psi')$, which we shall denote as

$$\forall x_1 \ldots \forall x_m(sb(\varphi) \to sb(\psi))$$

is our intended first-order equivalent, which contains no predicate symbols other than R and $=$. It is easy to see that it follows from the original Sahlqvist formula, being an instantiation of a universal second-order formula

$$\forall P_i^n \ldots \forall P_l^m \forall x_1 \ldots \forall x_m(\varphi^* \to \psi').$$

We must prove the other direction to have an equivalence.

[1] Note that we do not need existential quantifiers here to deal with iterations of $\Box$, as in modal logic; instead of $R^n(x, y)$, which is short for $\exists y_1(R(x, y_1) \wedge \ldots \wedge \exists y_{n-1} R(y_{n-1}, y))$, we have, for iterated modalities, $R(y_1, x) \wedge \ldots \wedge R(y, y_{n-1}, \ldots, y_1, x)$.

Assume that $\forall x_1 \dots \forall x_m(sb(\varphi) \to sb(\psi))$ holds in some frame F under a variable assignment s. Assume, for some interpretation function V, that φ^* holds in $M = \langle F, V\rangle$. To show that ψ' holds in the same model, we need the following two assertions:

4.1.6. Lemma. *For all M, s: $M, s \models \varphi^* \;\Rightarrow\; M, s \models sb(\varphi)$*

4.1.7. Lemma. *Let $M, s \models \varphi^*$, and let $s(x_1) = d_1, \dots, s(x_m) = d_m$. Define $V^*(P_i^n)$ as the set of all n-tuples which satisfy $Sb(P_i^n, \varphi^*)$ under s (that is, with $d_1, \dots, d_m$ assigned to $x_1, \dots, x_m$). Then*

$$V^*(P_i) \subseteq V(P_i).$$

From the first lemma it follows that $sb(\varphi)$ also holds for V and s; and hence $sb(\psi)$ holds. Since ψ' is positive, Lemma 2 (with the Monotonicity Lemma for classical logic) implies that $M, s \models \psi'$, as was to be shown.

Proof of lemma 4.1.6 φ^* has the form $\Psi \wedge \Gamma \wedge \Theta$, where Ψ is a conjunction of R-statements corresponding to the translations of $\Diamond$-quantifiers, Γ is a conjunction of atomic formulas, and Θ a conjunction of universally bound implications. It is easy to check that the two latter conjuncts turn into tautologies after substituting $Sb(P_i, \varphi^*)$ for every P_i in φ^*. It means that $\vdash \; sb(\varphi) \equiv \Psi$, so it follows from any conjunction including Ψ.

Proof of lemma 4.1.7 (a.) Consider the case when the occurrence of P_i is in Γ. Every V which makes the formula true under s should include at least one tuple which satisfies the conditions from Ψ. Then it contains the tuple which satisfies $Sb(\bar{P}_i)$. (b.) Let $\bar{P}_i$ be in Θ. Then it is of the form

$$\forall y_1 \dots \forall y_{k+l}(\mathcal{R}_1 \wedge \dots \wedge \mathcal{R}_k \to P_i(\bar{y}, \bar{x})),$$

where $\mathcal{R}_1 \dots \mathcal{R}_k$ are the R-conditions corresponding to the generalized quantifiers. If φ^* is true under V and s, then this subformula is true, too, which means that $V(P_i)$ includes at least all tuples $\langle d_1, \dots, d_n\rangle$ for which the relation R holds between $\alpha_1, \dots, \alpha_f$th members, for each of the k R-conditions. So, again it contains all tuples which satisfy $Sb(\bar{P}_i, \varphi^*)$. But if for every occurrence of P_i, the set of tuples satisfying $Sb(\bar{P}_i, \varphi^*)$ is a subset of $V(P_i)$, then also their union is in $V(P_i)$. Thus, $V^*(P_i) \subseteq V(P_i)$. □

4.1.8. Example. Here is how the above Sahlqvist algorithm works on some examples:

- *Reflexivity.* Consider $\Diamond_y x = y$. Its standard translation is

$$\exists y(R(y, x) \wedge x = y),$$

which is equivalent to $R(x, x)$.

- *Transitivity.* The standard translation of

$$\Diamond_y(\top(x) \wedge \Diamond_z(\top(y) \wedge P(z))) \to \Diamond_z(\top(x) \wedge P(z))$$

gives us

$$\forall P[\exists y(R(y,x) \wedge \top(x) \wedge \exists z(R(z,y) \wedge \top(y) \wedge P(z))) \to \exists u(R(u,x) \wedge \top(x) \wedge P(u))]$$

which can be rewritten in accordance with the Sahlqvist algorithm as

$$\forall P \forall y \forall z(R(y,x) \wedge R(z,y) \wedge P(z)) \to \exists u(R(u,x) \wedge P(u)))$$

The minimal substitution for $P(u)$ is $u = z$, so we obtain

$$\forall y \forall z(R(y,x) \wedge R(z,y) \wedge z = z \to \exists u(R(u,x) \wedge u = z),$$

which is a first-order equivalent of transitivity:

$$\forall y \forall z(R(y,x) \wedge R(z,y) \to R(z,x))$$

- *Symmetry.* The formula

$$\forall x \Box_y P(x,y) \to \forall y \Box_x P(x,y)$$

is translated as

$$\forall P(\forall x \forall y(R(y,x) \to P(x,y)) \to \forall y \forall x(R(x,y) \to P(x,y)))$$

The minimal substitution for $P(u,v)$ is $\top(u) \wedge R(v,u)$:

$$\forall x \forall y(R(y,x) \to \top(x) \wedge R(y,x)) \to \forall x \forall y(R(x,y) \to \top(x) \wedge R(y,x))$$

The antecedent becomes trivial:

$$\top \to \forall x \forall y(R(x,y) \to R(y,x))$$

which can again be written more elegantly as

$$\forall x \forall y(R(x,y) \to R(y,x)).$$

4.1.2 Limitative Results

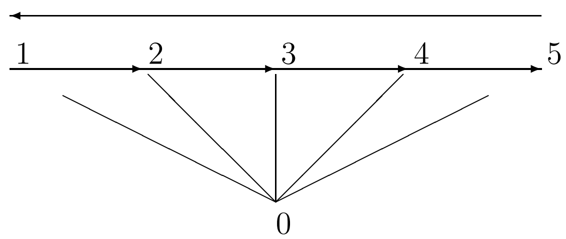
If a formula does not have the form described in our Sahlqvist Theorem, it may lack a first-order equivalent. The proof that a combination $\Box(\ldots \vee \ldots)$ in the antecedent can be fatal, is adapted from the analogous proof for modal logic (see (van Benthem 1983), lemma 10.6).

4.1.9. Lemma. $\Box_x(\Box_y(P(y) \wedge \top(x,z)) \vee P(x)) \to \Diamond_x(\Diamond_y(P(y) \wedge \top(x,z)) \wedge P(x))$ *does not have a frame correspondent.*

Proof. Define a class of frames F_n as follows:

- $D_n = \{0, 1, \ldots, 2n+1\}$;
- $R_n = \{\langle i, 0\rangle : 1 \leq i \leq 2n+1\} \cup \{\langle i+1, i, 0\rangle : 1 \leq i \leq 2n,\} \cup \{\langle 1, 2n+1, 0\rangle\}$.

Here is a picture illustrating this with $R(j, i, 0)$ represented as 'there is a line from 0 to i and an arrow from i to j':



For every n and V,

$$F_n, V, [z/0] \models \Box_x(\Box_y(P(y) \wedge \top(x)) \vee P(x)) \rightarrow \Diamond_x(\Diamond_y(P(y) \wedge \top(x)) \wedge P(x))$$

Indeed, the antecedent is true if

$$\forall x(R(x, z) \rightarrow \forall y(R(y, x, z) \rightarrow P(y)) \vee P(x));$$

that is, if for every i with $R(i, 0)$ $P(i)$ is true or P holds for each j with $R(j, i, 0)$. Each such i has exactly one 'successor' j with $R(j, i, 0)$ and 'predecessor' k with $R(i, k, 0)$. They form a chain which has by definition an odd number of members. That is why, if the antecedent is true, then P should hold for some pair of neighbours in this chain. But then the consequent is also true:

$$\exists x(R(x, z) \wedge \exists y(R(y, x, z) \wedge P(y)) \wedge P(x)).$$

Now, assume that our formula had a frame correspondent. For arbitrary large n, it is consistent with the following set of first-order sentences describing the frames F_n:

$\forall x \forall y(R(x, y) \rightarrow \neg R(y, x))$
$\forall x \forall y \forall z(R(x, y, z) \rightarrow \neg R(y, x, z))$
$\exists^1 z \forall y R(y, z)$
$\forall y(\exists^1 x R(x, y, z) \wedge \exists^1 u R(y, u, z))$
$\neg \exists x_1 \ldots \exists x_{2n} \exists y(R(x_2, x_1 y) \wedge \ldots R(x_{2n}, x_{2n-1}, y) \wedge R(x_1, x_{2n}, y))$.

The latter formula forbids 'loops' of length less than $2n+1$; that is why it is true in F_k for all $k \geq n$.

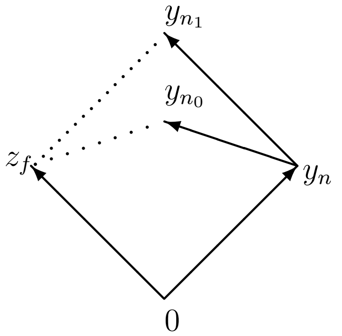
By compactness, since each finite set of these formulas has a model for suitably large n, they also have a countable model simultaneously. But in all countable models with the above properties (which are isomorphic copies of $\mathbf{Z}$ with ternary R interpreted as $R(j, i, 0) := S(j, i)$ and 0 being a fixed element preceding all other elements: $R(i, 0)$ for all $i \neq 0$) the formula can easily be refuted by putting $P(i)$ iff $\neg P(i-1)$ and $\neg P(i+1)$. $\Box$

The same result holds for the combination $\Box_x \ldots \Diamond_y$ in the antecedent (the proof is analogous to the proof of lemma 10.2 in (van Benthem 1983) for McKinsey axiom):

4.1.10. Lemma. $\Box_x\Diamond_y(P(y) \wedge \top(x,z)) \to \Diamond_x\Box_y(P(y) \wedge \top(x,z))$ *does not have a frame correspondent.*

Proof. Consider the following class of models:

$D = \{0\} \cup \{y_n : n \in N\} \cup \{y_{n_i} : n \in N, i \in \{0,1\}\} \cup \{z_f : f : N \to \{0,1\}\}$;
$R = \{\langle y_n, 0\rangle : n \in N\} \cup \{\langle y_{n_i}, y_n, 0\rangle : n \in N, i \in \{0,1\}\} \cup \{\langle z_f, 0\rangle : f : N \to \{0,1\}\} \cup \{\langle y_{n_{f(n)}}, z_f, 0\rangle : n \in N, f : N \to \{0,1\}\}$



(Here an arrow from a to b describes $R(b,a)$, and the combination of arrows from a to b and from b to c - $R(c,b,a)$.)

Any model of this class validates the formula in question: assume

$$M, s = [z/0] \models \Box_x\Diamond_y(P(y) \wedge \top(x,z)).$$

This means that $\forall x(R(x,0) \to \exists y(R(y,x,0) \wedge P(y) \wedge \top(x,0))$ is true, which implies that $\forall n \exists i P(y_{n_i})$ holds. Since for every n either y_{n_0} or y_{n_1} satisfies P, we can choose f such that $P(y_{f(n)})$ for every n. Then the consequent is also true: $\exists x(R(x,0) \wedge \forall y(R(y,x,0) \to (P(y) \wedge \top(x,0)))$ (via $x = z_f$), whence

$$F, s = [z/0] \models \Box_x\Diamond_y(P(y) \wedge \top(x,z)) \to \Diamond_x\Box_y(P(y) \wedge \top(x,z))$$

M is obviously uncountable. Consider any countable elementary submodel M' of F which includes $0, y_n, y_{n_0}, y_{n_1}$ for all n. If our formula had a frame correspondent, it would be true in M' . But it can be refuted there: since M' is countable, it does not contain some z_f. Put $y_{n_i} \in V(P)$ iff $i = f(n)$. Then the antecedent is still true (all elements which had a successor in P, still have it), but the consequent is false. $\Box$

Another limitation to the Sahlqvist theorem for frame correspondence becomes apparent when we try to obtain its natural generalization towards *completeness* of Sahlqvist logics. Here is a striking problem, due to Michiel van Lambalgen.

4.1.11. Example. (Sahlqvist incompleteness). Consider the following three axioms:

Q1. $\Diamond_x x = x$;
Q2. $\neg\Diamond_y x = y$;

Q4. $\Diamond_x \varphi(x, \bar{y}) \to \Diamond_x(\varphi(x, \bar{y}) \vee \psi(x, \bar{z}))$

These properties are consistent (think of an interpretation for $\Diamond$ like 'there exist at least two'). According to the Sahlqvist theorem, these axioms define the following properties of R:

R1. $\exists x R(x)$;
R2. $\neg R(x, x)$;
R3. $R(x, \bar{y}) \to R(x, \bar{y}\bar{z})$;

But together R1–R3 imply $\bot$:

1. $R(x)$ - R1
2. $R(x) \to R(x, x)$ - R3
3. $R(x, x)$ - 1,2
4. $\neg R(x, x)$ R2
5. $\bot$

This example shows that the match between correspondence and completeness is not as good for modal quantifiers as it is for ordinary modal logic. A natural question arises, whether an analogue of the Sahlqvist's theorem can be proved for *correspondence for completeness.*

4.2 Correspondence for completeness

4.2.1. Definition. If A is a quantifier axiom, a *correspondent in the sense of completeness* is a first order condition $A^\dagger$ on R with the following two properties:

i any set of sentences consistent with A in L_{min} has a dependence model where $A^\dagger$ holds;

ii A is satisfied on any dependence model where $A^\dagger$ holds. □

Recall that a modal logic L is called *first order complete* if there is a set Δ of first order sentences in the language $\{R, =\}$ (where R is the accessibility relation) such that

$$\vdash_L \varphi \text{ if and only if for every Kripke model } M\text{: if } M \models \Delta \text{ then } M \models \varphi;$$

equivalently, $\vdash_L \varphi$ if and only if φ is true on any frame which satisfies Δ. This notion can be reformulated for generalized quantifiers as follows

4.2.2. Definition. If L is a generalized quantifier logic, then L is *first order complete* if there is a set Δ of first order sentences in the language $\mathcal{L}(R)$ so that for every quantifier formula φ, $\vdash_L \varphi$ if and only if for every dependence model M: if $M \models \Delta$ then $M \models \varphi$. □

Note that, if L is finitely axiomatizable then by compactness Δ can be taken to be finite.

Definition 4.2.1 can be reformulated as follows: A has a correspondent for completeness $A^\dagger$ if $L_{min} + A$ is first order complete with respect to the class of models satisfying $A^\dagger$.

One can even define a stronger notion of first order completeness using the notion of a canonical model, which plays an important role in modal correspondence theory. For generalized quantifiers, there is no unique canonical model. We define a canonical dependence model as follows:

4.2.3. Definition. A *canonical dependence model* is a model where the dependence relation satisfies

$$R(x,\bar{y}) = \bigwedge_{\varphi(x,\bar{y})\in\mathcal{L}(\exists\Diamond)} \Box_x\varphi(x,\bar{y}) \to \varphi(x,\bar{y}).$$

□

4.2.4. Definition. If L is a generalized quantifier logic, then L is *canonically first order* if L is first order complete and the corresponding set Δ of first order sentences holds in every ω-saturated canonical dependence model of L. □

Since, as we see later, every consistent set of sentences has a canonical ω-saturated model, if $L_{min}+A$ and $L_{min}+B$ are both canonically first order, then $L_{min}+A+B$ is first order complete (recall that inconsistent logics are trivially first order complete). This does not hold in general for any two axioms A and B which have correspondents for completeness; namely, for any set of sentences consistent with $L_{min}+A+B$ there will be a model where $A^\dagger$ holds, a model where $B^\dagger$ holds, but not necessarily a model where $A^\dagger$ and $B^\dagger$ hold simultaneously.

Example 4.1.11 shows that the notions of a frame correspondent and a correspondent for completeness are different; there are formulas which have a frame correspondent, but do not have a correspondent for completeness. However, the following holds:

4.2.5. Proposition. *If A has a correspondent for completeness $A^\dagger$ and a frame correspondent $A^\star$, then $\vdash_{FOL} A^\dagger \to A^\star$.*

Proof. Let $\langle D,R\rangle \models A^\dagger$. Then by Definition 4.2.1 (ii), for every interpretation V, $\langle D,R,V\rangle \models A$. By Definition 4.1.2, $\langle D,R\rangle \models A^\star$. □

4.2.1 Sahlqvist theorem for completeness

We now formulate the completeness part of the Sahlqvist theorem, which describes a class of formulas φ defining first-order conditions on R so that for any logic L in the language of $\mathcal{L}(\exists\Diamond)$ which has a canonical dependence model, $L \cup \{\varphi\}$ as an axiom is complete for the class of models where R has the first order property corresponding to φ. This class is strictly smaller than the class of formulas having a frame correspondent. We shall call these formulas *weak* Sahlqvist formulas. For the formulation of the theorem it is convenient to assume that the language contains special formulas $\top(\bar{z})$ and $\bot(\bar{z})$, which denote a tautology, resp. a contradiction with exactly the free variables $\bar{z}$.

4.2.6. Definition. A formula is called *weak Sahlqvist* if it is of the form $\bigwedge Qz_1 \ldots Qz_n(A \to B)$, where $n \geq 0$, each Q is either $\forall$ or $\Box$, and

1. A is constructed from

a. atomic formulas, possibly with a quantifier prefix $Qx_1 \dots Qx_k$, where each Q is a $\Box$- or $\forall$-quantifier;
b. formulas in which atomic formulas occur only negatively,
c. constant formulas (where the only predicate letters are $\top, \bot$ and $=$),

using $\wedge$ and $\vee$,

2. in B all predicate letters occur only positively,
3. every occurrence of a predicate letter has the same free variables. $\Box$

Every weak Sahqvist formula is a Sahlqvist formula, but not vice versa. Weak Sahlqvist formulas do not contain occurrences of $\exists$ and $\Diamond$ in the antecedent. Observe that Q4 is not a weak Sahlqvist formula.

4.2.7. THEOREM. *(Completeness part of the Sahlqvist theorem) Every weak Sahlqvist formula χ has a correspondent in the sense of completeness; moreover, this correspondent holds in every ω-saturated canonical dependence model of a logic in which χ is provable.*

The idea of the proof of the Theorem 4.2.7 (very similar to the one used in (Sambin and Vaccaro 1989)) can be illustrated by means of the following example.

4.2.8. EXAMPLE. Let $\mathcal{C}$ be a canonical model. We show that if for every P and S

$$\Box_x P(x, \bar{y}) \to \Box_x(P(x, \bar{y}) \vee S(x, \bar{z}))$$

is valid in $\mathcal{C}$, then the accessibility relation in $\mathcal{C}$ has the property $R(x, \bar{y}\bar{z}) \to R(x, \bar{y})$.

Proof. It is easy to see (since the consequent is monotone in S) that the axiom above is equivalent to $\Box_x P(x, \bar{y}) \to \Box_x(P(x, \bar{y}) \vee \bot(x, \bar{z}))$. We translate the validity conditions using second-order quantifiers which range only over *definable* relations of $\mathcal{C}$ (this is the difference with the case of the frame correspondence). To emphasize this difference we use quantifiers $\forall\varphi$. Note that due to the restricted substitution rule, if P is an n-place predicate symbol, then formulas which can be substituted for P must have precisely n variable places. Using this notation the validity condition of the axiom reads as follows:

$$\forall\varphi[\forall x(R(x, \bar{y}) \to \varphi(x, \bar{y})) \to \forall x(R(x, \bar{y}\bar{z}) \to \varphi(x, \bar{y}) \vee \bot(x, \bar{z}))]$$

This is equivalent to $\forall\varphi[\forall x(R(x, \bar{y}) \to \varphi(x, \bar{y})) \to \forall x(R(x, \bar{y}\bar{z}) \to \varphi(x, \bar{y}))]$, and, in turn, to $\bigwedge_{\varphi(x,\bar{y})}\{\forall x(R(x, \bar{y}\bar{z}) \to \varphi(x, \bar{y})) : \forall x(R(x, \bar{y}) \to \varphi(x, \bar{y}))\}$. Moving the conjunction inside (the proof that this can be done for any positive logical function of φ, is given in the Intersection Lemma; in the given case the proof is obvious), we obtain

$$\forall x(R(x, \bar{y}\bar{z}) \to \bigwedge_{\varphi(x,\bar{y})} \{\varphi(x, \bar{y})) : \forall x(R(x, \bar{y}) \to \varphi(x, \bar{y}))\})$$

But in $\mathcal{C}$ $\bigwedge_{\varphi(x,\bar{y})}\{\varphi(x, \bar{y})) : \forall x(R(x, \bar{y}) \to \varphi(x, \bar{y}))\}) \equiv R(x, \bar{y})$. Substituting $R(x, \bar{y})$ instead of the infinite conjunction yields the first-order equivalent $\forall x(R(x, \bar{y}\bar{z}) \to R(x, \bar{y}))$.

It is easy to check that in every model where $\forall x(R(x,\bar{y}\bar{z}) \to R(x,\bar{y}))$ holds, the axiom is valid. $\Box$

The general case is slightly more complicated, because to obtain a correspondent we must sometimes move to a different canonical model, namely, to an ω-saturated canonical model. The existence of such model for every L_{min}-consistent set of sentences is proved in section 4.2.1. For canonical ω-saturated models we have

Intersection Lemma If X is a set of formulas with the same free variables, closed with respect to $\wedge$, and B is a formula where φ occurs positively, then in an ω-saturated model

$$\bigwedge\{B(\varphi) : \varphi \in X\} \equiv B(\bigwedge\{\varphi : \varphi \in X\}).$$

The proof of Theorem 4.2.7 consists of the same three ingredients as those in the example: translation of the validity conditions of an axiom (eventually accompanied by some syntactic transformations), application of the Intersection Lemma, and making use of the fact that for some first-order expression $\mathcal{R}$ with R as the only predicate symbol

$$\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge_{\varphi(\bar{x},\bar{y})} \{\varphi(\bar{x},\bar{y}) : \forall \bar{x}(\mathcal{R}(\bar{x},\bar{y}) \to \varphi(\bar{x},\bar{y}))\}.$$

The class of such expressions will be isolated in the Closure Lemma below. But first we need

4.2.9. Definition. Let M be a canonical model, and A a conjunction of atomic formulas which are prefixed by universal and $\Box$-quantifiers, so that all occurrences of a predicate symbol have the same free variables. Every occurrence of a predicate symbol P in A is therefore of the form $\bar{Q}_i\bar{x}P(\bar{x},\bar{y})$, where $\bar{Q}_i$ is the quantifier prefix of the ith occurrence. P has *a good minimal substitution* in A if $M \models \bigwedge\{\varphi(\bar{x},\bar{y}) : \bigwedge_i \bar{Q}_i\bar{x}\varphi(\bar{x},\bar{y})\} \equiv p$, where p is a first-order formula built using the predicates R, $=$, $\top$ and $\bot$ only. $\Box$

For example, we have seen that if the only occurrence of P in A is of the form $\Box_x P(x,\bar{y})$, then P has a good minimal substitution in A: for every canonical model M,

$$M \models \bigwedge\{\varphi(x,\bar{y}) : \Box_x\varphi(x,\bar{y})\} \equiv R(x,\bar{y}).$$

Before we formulate the Closure Lemma, we shall get rid of two degenerate cases.

(a) Henceforth we assume that *all quantifiers are non-vacuous*, i.e. if we write a formula $\Box_x\varphi$, then x occurs free in φ. This is justified by the assumption that R is always non-empty.

(b) We shall also assume that every quantifier prefix in A contains at least one $\Box$. That this is no loss of generality can be seen as follows.

Let P occur in A with a purely universal prefix, $\forall x_1 \ldots \forall x_n P(\bar{x},\bar{y})$. Then this occurrence implies all other possible occurrences of P in A, and A is equivalent to a conjunction where $\forall x_1 \ldots \forall x_n P(\bar{x},\bar{y})$ is the only occurrence of P.

$$M \models \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall x_1 \ldots \forall x_n\varphi(\bar{x},\bar{y})\} \equiv \top(\bar{x},\bar{y}),$$

so P has a good minimal substitution in A.

Now we can state the Closure Lemma which is proved in section 4.2.1:

Closure Lemma. Let A be a conjunction of atomic formulas prefixed by $\forall$ and $\Box$-quantifiers, so that all occurrences of a predicate symbol in A have the same free variables. Then every atomic formula in A has a good minimal substitution, and this minimal substitution is the same as the one used to obtain the frame correspondent.

ω-saturated models and the Intersection Lemma.

Let $X = \{\varphi_1(x), \varphi_2(x), \ldots\}$ be a finitely realizable type in a model M, that is, for every n there is an element a_n in the domain of M such that $\varphi_1(a_n), \ldots, \varphi_n(a_n)$ is true in M. If M is just an ordinary Henkin model, there does not necessarily exist an element a such that for every φ in X $\varphi(a)$ is true in M. Among other things this implies that $\Diamond_x \bigwedge\{\varphi : \varphi \in X\}$ is not equivalent to $\bigwedge\{\Diamond_x\varphi : \varphi \in X\}$ in M. But in the proof we do need that

$$M \models \Diamond_x \bigwedge\{\varphi : \varphi \in X\} \equiv \bigwedge\{\Diamond_x\varphi : \varphi \in X\}$$

(an analogue of Esakia's lemma). We therefore move from the original Henkin model to a mildly saturated extension.

4.2.10. Theorem. *(van Lambalgen, 1994) Every consistent set of $\mathcal{L}(\exists\Diamond)$ formulas has a model $\mathcal{A}$ which is ω-saturated and canonical, that is*

i *every finitely realizable type which contains finitely many parameters is realized;*
ii $R_{\mathcal{A}}(d, \bar{d}) =_{df} \bigwedge_{\varphi(x,\bar{d})\in\mathcal{L}(\exists\Diamond)} \Box_x\varphi(x, \bar{d}) \rightarrow \varphi(d, \bar{d})$

Proof From the completeness proof for the minimal logic we know that every consistent set of formulas has a canonical model $\mathcal{C}$ where

$$R_{\mathcal{C}}(x, \bar{y}) \equiv \bigwedge_{\varphi(x,\bar{y})\in\mathcal{L}(\exists\Diamond)} \Box_x\varphi(x, \bar{y}) \rightarrow \varphi(x, \bar{y})$$

By the truth definition

$$\mathcal{C} \models \Box_x\varphi(x, \bar{y}) \Leftrightarrow \forall x(R_{\mathcal{C}}(x, \bar{y}) \Rightarrow \mathcal{C} \models \varphi(x, \bar{y}))$$

Therefore there is a first-order model $\mathcal{C}^*$ (with $R = R_{\mathcal{C}}$ just an ordinary predicate) such that if $\psi \in \mathcal{L}(\exists\Diamond)$

$$\mathcal{C} \models \psi \Leftrightarrow \mathcal{C}^* \models ST(\psi).$$

We shall use this fact to build the saturated model which we need, because one can apply the standard procedure of constructing an ω-saturated extension of $\mathcal{C}^*$. (While extending a model for a generalized quantifier is much more difficult.)

Take an ω-saturated elementary extension $\mathcal{A}^*$ of $\mathcal{C}^*$. It is clear that

$$\mathcal{C} \models \psi \Leftrightarrow \mathcal{C}^* \models ST(\psi) \Leftrightarrow \mathcal{A}^* \models ST(\psi),$$

for every sentence ψ of $\mathcal{L}(\exists\Diamond)$.

Every type finitely realizable in $\mathcal{C}$ is finitely realizable in $\mathcal{C}^*$ and is therefore realized in $\mathcal{A}^*$. But $\mathcal{A}^*$ is still a first-order model; to make an $\mathcal{L}(\exists\Diamond)$ model $\mathcal{A}$ out of it, we could take the interpretation of R in $\mathcal{A}^*$ to be the accessibility relation in $\mathcal{A}$, i.e. stipulate

$$\mathcal{A} \models \Box_x \varphi(x,\bar{y}) \Leftrightarrow \forall x(R(x,\bar{y}) \Rightarrow \mathcal{A} \models \varphi(x,\bar{y})).$$

However, it is not obvious that $\mathcal{A}$ is still canonical.

Instead we define the accessibility relation anew in $\mathcal{A}$. $\mathcal{A}$ will be the expansion $\langle \mathcal{A}^*, R_\mathcal{A}\rangle$ of $\mathcal{A}^*$, where $R_\mathcal{A}$ is defined on $\mathcal{A}^*$ as

$$R_\mathcal{A}(x,\bar{y}) = \bigwedge_{ST(\varphi(x,\bar{y})):\varphi(x,\bar{y})\in\mathcal{L}(\exists\Diamond)} \forall x(R(x,\bar{y}) \to ST(\varphi(x,\bar{y}))) \to ST(\varphi(x,\bar{y})).$$

Note that the intersection is only over the formulas $ST(\varphi(x,\bar{y}))$ such that $\varphi(x,\bar{y}) \in \mathcal{L}(\exists\Diamond)$.

We are done if we can show that

4.2.11. LEMMA. *$\mathcal{A} \models \varphi \Leftrightarrow \mathcal{A}^* \models ST(\varphi)$ for all formulas $\varphi \in \mathcal{L}(\exists\Diamond)$.*

Proof By induction on the complexity of φ. The only non-trivial case is $\varphi = \Box_x\psi(x,\bar{y})$.

To prove the direction from right to left, assume that $\mathcal{A}^* \models ST(\Box_x\psi(x,\bar{y}))$, that is, $\mathcal{A}^* \models \forall x(R(x,\bar{y}) \to ST(\psi(x,\bar{y})))$. We want to prove $\mathcal{A} \models \Box_x\psi(x,\bar{y})$, that is $\mathcal{A} \models \forall x(R_\mathcal{A}(x,\bar{y}) \to \psi(x,\bar{y}))$.

Let $R_\mathcal{A}(x,\bar{y})$ hold in $\mathcal{A}$. By the definition of $R_\mathcal{A}$,

$$\mathcal{A}^* \models \forall x(R(x,\bar{y}) \to ST(\psi(x,\bar{y}))) \to ST(\psi(x,\bar{y})).$$

We know that $\mathcal{A}^* \models \forall x(R(x,\bar{y}) \to ST(\psi(x,\bar{y})))$. Therefore $\mathcal{A}^* \models ST(\psi(x,\bar{y}))$ and, by the inductive hypothesis, $\mathcal{A} \models \psi(x,\bar{y})$.

From left to right: let $\mathcal{A} \models \Box_x\psi(x,\bar{y})$, that is $\mathcal{A} \models \forall x(R_\mathcal{A}(x,\bar{y}) \to \psi(x,\bar{y}))$. Let $R(x,\bar{y})$ hold in $\mathcal{A}^*$. We want to show that $\mathcal{A}^* \models ST(\psi(x,\bar{y}))$. It is enough to show that $R(x,\bar{y})$ implies $R_\mathcal{A}(x,\bar{y})$. If this is so, we obtain $\psi(x,\bar{y})$ from $R(x,\bar{y})$ and the fact that $\mathcal{A} \models \forall x(R_\mathcal{A}(x,\bar{y}) \to \psi(x,\bar{y}))$, and hence applying the inductive hypothesis we also get $ST(\psi(x,\bar{y}))$.

Let $R(x,\bar{y})$. Take a formula $ST(\chi)$ such that $\forall x(R(x,\bar{y}) \to ST(\chi(x,\bar{y})))$. Then $ST(\chi(x,\bar{y}))$. This way we prove that for all $ST(\chi)$, $R(x,\bar{y}) \to (\forall x(R(x,\bar{y}) \to ST(\chi(x,\bar{y}))) \to ST(\chi(x,\bar{y})))$.

Therefore $R(x,\bar{y}) \to \bigwedge_{ST(\chi)}(\forall x(R(x,\bar{y}) \to ST(\chi(x,\bar{y}))) \to ST(\chi(x,\bar{y})))$, which means that $R(x,\bar{y})$ implies $R_\mathcal{A}(x,\bar{y})$. □

Comment. $\mathcal{C}$ is an elementary extension of $\mathcal{A}$ with respect to $\mathcal{L}(\exists\Diamond)$ formulas, but not necessarily with respect to $\mathcal{L}(R)$ formulas if R is interpreted as $R_\mathcal{A}$.

Now we are ready to prove that in $\mathcal{A}$ the Intersection Lemma holds.

4.2.12. LEMMA. *(Intersection Lemma) If X is a set of formulas with the same free variables, closed with respect to $\wedge$, and B is a formula where φ occurs positively, then in an ω-saturated model*

$$\bigwedge\{B(\varphi) : \varphi \in X\} \equiv B(\bigwedge\{\varphi : \varphi \in X\})$$

Proof By induction on the complexity of B. The basis and propositional cases are trivial.

- Let $B = \forall x B_1$. $\bigwedge\{\forall x B_1(\varphi) : \varphi \in X\} = \forall x \bigwedge\{B_1(\varphi) : \varphi \in X\}$ (because $\forall$ distributes over $\bigwedge$), and by the inductive hypothesis this is equivalent to $\forall x B_1(\bigwedge\{\varphi : \varphi \in X\})$.
- Let $B = \Box_x B_1$. This case is analogous, but since $\Box$ distributes only over conjunctions of formulas with the same free variables, it is important that all formulas in X (and therefore in $\{B_1(\varphi) : \varphi \in X\}$) have the same free variables.
- Let $B = \Diamond_x B_1$. We have to show $\bigwedge\{\Diamond_x B_1(\varphi) : \varphi \in X\} = \Diamond_x B_1(\bigwedge\{\varphi : \varphi \in X\})$, and here we need the model to be ω-saturated. Since $\Diamond_x B_1(-)$ is monotone, the direction from right to left is immediate. As to the converse, assume $M \models \Diamond_x B_1 \varphi$, for all $\varphi \in X$. Choose $B_1(\varphi_1), \ldots, B_1(\varphi_n)$ with $\varphi_i \in X$. Since X is closed under conjunction, we have by assumption $M \models \Diamond_x B_1(\varphi_1 \wedge \ldots \wedge \varphi_n)$. By monotonicity of $\Diamond_x$ and B_1, $M \models \Diamond_x(B_1(\varphi_1) \wedge \ldots \wedge B_1(\varphi_n))$. This means that there is d_n such that $M \models R(d_n, \bar{e}) \wedge B_1(\varphi_1) \wedge \ldots \wedge B_1(\varphi_n)$. Because M is ω-saturated, there is an element d: $M \models R(u, \bar{e}) \wedge \bigwedge\{B_1(\varphi) : \varphi \in X\}$. Therefore, $M \models \Diamond_x \bigwedge\{B_1(\varphi) : \varphi \in X\}$, as required.
- Let $B = \exists x B_1$: the proof is analogous to the previous case. □

Closure Lemma

Let A be a conjunction as in the condition of the Closure Lemma. We also assume that all quantifiers are non-vacuous and that every quantifier prefix contains at least one $\Box$-quantifier.

Let A' be the subformula of A which contains all and only occurrences of the predicate symbol P. We shall use both the $\mathcal{L}(\exists\Diamond)$-form of A', namely $\bigwedge_i \bar{Q}_i \bar{x} P(\bar{x}, \bar{y})$, and its standard translation $\bigwedge_i \forall \bar{x}(\mathcal{R}_i \to P(\bar{x}, \bar{y}))$, where i runs over the occurrences of P. In the sequel we call the $\mathcal{R}_i$ *R-conditions.* The standard translation of A' is thus equivalent to $\forall \bar{x}(\bigvee_i \mathcal{R}_i \to P(\bar{x}, \bar{y}))$. $P(\bar{x}, \bar{y})$ has a good minimal substitution in A if

$$\bigvee_i \mathcal{R}_i = \bigwedge\{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\bigvee_i \mathcal{R}_i \to \varphi(\bar{x}, \bar{y}))\}.$$

4.2.13. EXAMPLE. The R-condition corresponding to $\Box_x \Box_y P(x, y)$ is $R(x) \wedge R(y, x)$.

4.2.14. EXAMPLE. Let $A' = \forall x \Box_y P(x, y) \wedge \Box_x \forall y P(x, y)$, then

$$A'^* = \forall x \forall y(R(y, x) \to P(x, y)) \wedge \forall x \forall y(R(x) \to P(x, y))$$

which is equivalent to $\forall x \forall y(R(y, x) \vee R(x) \to P(x, y))$. The good minimal substitution for P in A must be therefore $R(y, x) \vee R(x)$.

We are going to prove the existence of good minimal substitutions for all non-vacuous quantifier prefixes containing at least one $\Box$. But first we need several propositions.

4.2.15. PROPOSITION. *Let $\mathcal{R}$ be an R-condition, such that*

$$\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\mathcal{R}(\bar{x},\bar{y}) \to \varphi(\bar{x},\bar{y}))\};$$

then

$$\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\psi(x,\bar{y}\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y}\bar{z}))\},$$

and vice versa: if

$$\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\psi(x,\bar{y}\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y}\bar{z}))\},$$

then

$$\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\mathcal{R}(\bar{x},\bar{y}) \to \varphi(\bar{x},\bar{y}))\}.$$

Proof. Assume $\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\mathcal{R}(\bar{x},\bar{y}) \to \varphi(\bar{x},\bar{y}))\}$. For every $\varphi(\bar{x},\bar{y})$ holds: $\varphi(\bar{x},\bar{y}) \equiv \forall\bar{z}(\varphi(\bar{x},\bar{y}) \wedge \top(\bar{z}))$. Therefore

$$\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\forall\bar{z}(\varphi(\bar{x},\bar{y}) \wedge \top(\bar{z})) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to (\varphi(\bar{x},\bar{y}) \wedge \top(\bar{z}))\}.$$

Since for every φ, $\forall\bar{z}(\varphi(\bar{x},\bar{y}) \wedge \top(\bar{z})) \equiv \varphi(\bar{x},\bar{y}) \wedge \top(\bar{z})$,

$$\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\varphi(\bar{x},\bar{y}) \wedge \top(\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to (\varphi(\bar{x},\bar{y}) \wedge \top(\bar{z}))\}.$$

Now we prove that $\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\psi(\bar{x},\bar{y}\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y}\bar{z}))\}$. Trivially,

$$\bigwedge\{\psi(\bar{x},\bar{y}\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y}\bar{z})\} \to$$

$$\to \bigwedge\{\varphi(\bar{x},\bar{y}) \wedge \top(\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to (\varphi(\bar{x},\bar{y}) \wedge \top(\bar{z}))\}$$

and this implies that $\bigwedge\{\psi(\bar{x},\bar{y}\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y}\bar{z}))\} \to \mathcal{R}(\bar{x},\bar{y})$.

Since $\mathcal{R}(\bar{x},\bar{y}) \to \bigwedge\{\psi(\bar{x},\bar{y}\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y}\bar{z}))\}$, we have $\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\psi(\bar{x},\bar{y}\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y}\bar{z}))\}$.

For the other direction of the proposition, let

$$\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\psi(\bar{x},\bar{y}\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y}\bar{z}))\}.$$

It is easy to check that $\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\forall\bar{z}\psi(\bar{x},\bar{y},\bar{z}) : \forall\bar{x}(\mathcal{R}(\bar{x},\bar{y}) \to \forall\bar{z}\psi(\bar{x},\bar{y},\bar{z}))\}$, and then the reasoning goes as above: the set of φ's with free variables $\bar{x},\bar{y}$ satisfying the same condition is larger than the set of $\forall\bar{z}\psi(\bar{x},\bar{y},\bar{z})$, therefore its conjunction implies the given one; on the other hand, the set of φ's satisfying the condition is implied by $\mathcal{R}$, therefore the two sets are equivalent. $\Box$

To prove the next proposition, we shall use the following tautology of the minimal logic: $L_{min} \vdash \Box_x(\Box_x\theta \to \theta)$ (the proof is given in Chapter 3).

4.2.16. PROPOSITION. $R(z,\bar{x}\bar{y}) \equiv \bigwedge\{\varphi(z,\bar{x},\bar{y}) : \forall z\forall\bar{x}(R(z,\bar{x}\bar{y}) \to \varphi(z,\bar{x},\bar{y}))\}$.

Proof. The direction from left to right is trivial. For the converse direction, we have to prove $\bigwedge\{\varphi(z,\bar{x},\bar{y}) : \forall z\forall\bar{x}(R(z,\bar{x}\bar{y}) \to \varphi(z,\bar{x},\bar{y}))\} \to R(z,\bar{x}\bar{y})$, in other words,

$$\bigwedge_{\varphi}(\forall\bar{x}\Box_z\varphi(z,\bar{x},\bar{y}) \to \varphi(z,\bar{x},\bar{y})) \to \bigwedge_{\psi}(\Box_z\psi(z,\bar{x},\bar{y}) \to \psi(z,\bar{x},\bar{y})).$$

It suffices to derive $\bigwedge_\psi \Box_z\psi(z,\bar{e},\bar{y}) \to \psi(d,\bar{e},\bar{y})$ from $\bigwedge_\varphi \forall\bar{x}\Box_z\varphi(z,\bar{x},\bar{y}) \to \varphi(d,\bar{e},\bar{y})$. Take an arbitrary $\psi(z,\bar{x},\bar{y})$. We substitute this formula for θ in the tautology above:

$$\forall\bar{x}\Box_z(\Box_z\psi(z,\bar{x},\bar{y}) \to \psi(z,\bar{x},\bar{y})).$$

We assume that the conjunction $\bigwedge_\varphi \forall\bar{x}\Box_z\varphi(z,\bar{x},\bar{y}) \to \varphi(d,\bar{e},\bar{y})$ holds. As a special case we obtain

$$\forall\bar{x}\Box_z(\Box_z\psi(z,\bar{x},\bar{y}) \to \psi(z,\bar{x},\bar{y})) \to (\Box_z\psi(z,\bar{e},\bar{y}) \to \psi(d,\bar{e},\bar{y})).$$

Since this holds for every ψ, we can derive $\bigwedge_\psi \Box_z\psi(z,\bar{e},\bar{y}) \to \psi(d,\bar{e},\bar{y})$. $\Box$

4.2.17. PROPOSITION. *If $Qx_1,\ldots,Qx_n$ contains at least one $\Box$-quantifier, and all quantifiers are non-vacuous, and the only occurrence of $P(\bar{x},\bar{y})$ in A is of the form $Qx_1\ldots Qx_nP(\bar{x},\bar{y})$, then P has a good minimal substitution in A.*

Proof. The general form of the prefix described in the condition of this proposition, is

$$\forall(\bar{u})_1\Box_{z_1}\forall(\bar{u})_2\Box_{z_2}\ldots\forall(\bar{u})_k\Box_{z_k}\forall(\bar{u})_{k+1}P(\bar{x},\bar{y}),$$

where $k > 0$ (that is, there is at least one $\Box$ in the prefix), and $\bar{u}\bar{z} = \bar{x}$. The standard translation of this formula is

$$\forall\bar{x}(R(z_1,(\bar{u})_1,\bar{y}) \wedge \ldots \wedge R(z_k,(\bar{u})_k,z_{k-1},(\bar{u})_{k-1},\ldots,z_1,(\bar{u})_1,\bar{y}) \to P(\bar{x},\bar{y}))$$

(since there is at least one $\Box$-quantifier in the prefix).

We have to show that

$$\mathcal{R} \equiv \bigwedge_{i=1}^{i=k} R(z_i,(\vec{uz})_{\leq i},\bar{y}),$$

where $(\vec{uz})_{\leq i}$ are the variables bound by the quantifiers preceding $\Box_{z_i}$, is a good minimal substitution for P.

By propositions 4.2.15 and 4.2.16 (observe that $z_i(\vec{uz})_{\leq i} \subseteq \bar{x}$),

$$R(z_i,(\vec{uz})_{\leq i},\bar{y}) \equiv \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(R(z_i,(\vec{uz})_{\leq i},\bar{y}) \to \varphi(\bar{x},\bar{y}))\}$$

Note that here we essentially use the fact that the z_i occur in $P(\bar{x},\bar{y})$, that is, that $\Box$-quantifiers are non-vacuous.

It is easy to see that $\bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y}) \to \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y}) \to \varphi(\bar{x},\bar{y}))\}$.

To prove the other direction, namely

$$\bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y}) \to \varphi(\bar{x},\bar{y}))\} \to \bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y}),$$

we argue as follows:

$$\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(R(z_i,(\vec{uz})_{\leq i},\bar{y}) \to \varphi(\bar{x},\bar{y}))\} \subseteq$$
$$\subseteq \{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y}) \to \varphi(\bar{x},\bar{y}))\}$$

therefore

$$\bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y}) \to \varphi(\bar{x},\bar{y}))\} \to$$
$$\to \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(R(z_i,(\vec{uz})_{\leq i},\bar{y}) \to \varphi(\bar{x},\bar{y}))\}$$

and this means that $\bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y}) \to \varphi(\bar{x},\bar{y}))\} \to R(z_i,(\vec{uz})_{\leq i},\bar{y})$. Since this holds for every i,

$$\bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y}) \to \varphi(\bar{x},\bar{y}))\} \to \bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y}),$$

that is, $\mathcal{R}$ is a good minimal substitution. □

4.2.18. PROPOSITION. *A disjunction of good minimal substitutions is a good minimal substitution, i.e. if for every i, $1 \leq i \leq n$,*

$$\mathcal{R}_i(\bar{z}_i,\bar{y}) = \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\},$$

where $\bar{z}_i \subseteq \bar{x}$, then

$$\bigvee_i \mathcal{R}_i(\bar{z}_i,\bar{y}) = \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\}.$$

Proof. Since for every $\mathcal{R}_i$

$$\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\} \subseteq \{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\},$$

$$\bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\} \to \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\},$$

that is, for every $\mathcal{R}_i$, $\mathcal{R}_i \to \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\}$, and this implies $\bigvee_i \mathcal{R}_i \to \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\}$.

Now we prove that the implication holds also in the other direction. From Proposition 4.2.15 follows that if $\mathcal{R}_i(\bar{z}_i,\bar{y}) = \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\}$, $\bar{z}_i \subseteq \bar{x}$, then $\mathcal{R}_i(\bar{z}_i,\bar{y}) = \bigwedge\{\psi(\bar{z}_i,\bar{y}) : \forall\bar{z}_i(\mathcal{R}_i(\bar{z}_i,\bar{y}) \to \psi(\bar{z}_i,\bar{y}))\}$.

Now, assume that $\bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\}$ holds and none of the $\mathcal{R}_i(\bar{z}_i,\bar{y})$ holds. Then as we have just seen, there are formulas $\psi_1,\ldots,\psi_n$, such that for every i, $\forall\bar{z}_i(\mathcal{R}_i(\bar{z}_i,\bar{y}) \to \psi_i(\bar{z}_i,\bar{y}))$ and $\neg\psi_i(\bar{z}_i,\bar{y})$. Take the disjunction of these formulas, $\bigvee_i \psi_i(\bar{z}_i,\bar{y})$. The resulting formula is also false. An equivalent formula with free variables $\bar{x},\bar{y}$, namely, $\bigvee_i \psi_i(\bar{z}_i,\bar{y}) \vee \bot(\bar{x})$, belongs to the set $\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\}$; but this formula is false, a contradiction. □

4.2.19. LEMMA. *(Closure Lemma.) Let A be a conjunction of atomic formulas prefixed by $\forall$ and $\Box$-quantifiers, so that all occurrences of a predicate symbol in A have the same free variables. Then every atomic formula in A has a good minimal substitution, and this minimal substitution is the same as the one used to obtain a frame correspondent.*

Proof. The lemma follows from the four propositions proved above. □

Syntactic transformations

Syntactic transformations which reduce the task of finding a correspondent for an axiom χ to simple applications of the Intersection and Closure Lemmas are the same as in the proof of Theorem 4.1.5.

Assume that we have a formula

$$(*) \quad \forall z_1 \dots \forall z_m(\Gamma' \to \forall\varphi_1 \dots \forall\varphi_n(A' \to B')),$$

where Γ' contains only predicate R, B' is a positive first-order formula, and A' is a conjunction of formulas $ST(Qx_1 \dots Qx_k\varphi_i)$, $Q \in \{\forall, \Box\}$, where all quantifiers are non-vacuous.

Assume that there is only one predicate letter P in χ. Then the reasoning goes as follows: P occurs in A' in subformulas of the form $ST(Qx_1 \dots Qx_kP(\bar{x}, \bar{y}))$, where the $\bar{x}$ are bound and the $\bar{y}$ free (rename the bound variables if necessary). The condition (*) can be rewritten as $\forall z_1 \dots \forall z_m(\Gamma' \to \bigwedge\{B'(\varphi(\bar{x},\bar{y})) : \bigwedge_i ST(\bar{Q}_i\bar{x}\varphi(\bar{x},\bar{y}))\})$.

Applying the Intersection Lemma,

$$\forall z_1 \dots \forall z_m(\Gamma' \to B'(\bigwedge\{\varphi(\bar{x},\bar{y}) : \bigwedge_i ST(\bar{Q}_i\bar{x}\varphi(\bar{x},\bar{y}))\})),$$

and by the Closure Lemma (P has a good minimal substitution in A', say p), this is equivalent to $\forall z_1 \dots \forall z_m(\Gamma' \to B'(p))$, which is a first-order statement in R.

Now we consider the general case, when there is more than one predicate symbol. Then we eliminate the second-order quantifiers one by one in the following way. Split A' in two parts, A_1 and A_2, so that A_2 contains all and only occurrences of P_n:

$$\forall z_1 \dots \forall z_m(\Gamma' \to \forall\varphi_1 \dots \forall\varphi_n(A' \to B'))$$

is equivalent to $\forall z_1 \dots \forall z_m(\Gamma' \to \forall\varphi_1 \dots \forall\varphi_n(A_1 \wedge A_2 \to B'))$, and this in turn to $\forall z_1 \dots \forall z_m(\Gamma' \to \forall\varphi_1 \dots \forall\varphi_{n-1}(A_1 \to \forall\varphi_n(A_2 \to B')))$. We now apply the Intersection Lemma and the Closure Lemma to $\forall\varphi_n(A_2 \to B')$.

This way all second-order quantifiers which bind predicate symbols occurring both in the antecedent and in the consequent can be eliminated.

If B contains predicate symbols which are not in the antecedent, these can be replaced by a fixed contradiction having the same parameters as the original atomic formula; since B is positive, and therefore monotone, the resulting formula is equivalent to the original one. Analogously, a predicate symbol occurring only in the antecedent can be replaced by a tautology.

Assume that A contains a predicate symbol which does not have a quantifier prefix, that is, $A \to B$ can be written as $A' \wedge P(\bar{x}) \to B(P(\bar{x}))$. By assumption, the $\bar{x}$ are free in B. Since B is positive, $B(P(\bar{x}))$ can be equivalent to $B' \wedge P(\bar{x})$ or $B' \vee P(\bar{x})$.

$$A' \wedge P(\bar{x}) \to B' \vee P(\bar{x})$$

obviously corresponds to a first order condition, namely a trivial one, and

$$A' \wedge P(\bar{x}) \to B' \wedge P(\bar{x})$$

is equivalent to $A' \wedge P(\bar{x}) \to B'$, the case which we treated above.

Let $\chi^\dagger$ be the result of applying this algorithm to χ. We proved that if χ is an axiom, then in a canonical ω-saturated model $\chi^\dagger$ holds.

The proof of the converse, namely that for *every* model M, if $M \models \chi^\dagger$, then $M \models \chi$, is standard (cf. (Sambin and Vaccaro 1989)). From the proof of Theorem 4.1.5 and the minimal substitutions we used follows that $\vdash \chi^\dagger \equiv \forall P_1 \dots \forall P_n \chi$, where $P_1, \dots P_n$ are all predicate letters occurring in χ And this implies that if $M \models \chi^\dagger$, then M makes every instance of χ true. This finishes the proof of Theorem 4.2.7. □

4.2.20. Example. Q1 – Q3 have correspondents for completeness.

Proof.

Q1 $\Diamond_x x = x$ has a correspondent in every model, namely $\exists x(R(x) \wedge x = x)$, which is equivalent to $\exists x R(x)$.

Q2 $\Box_x x \neq y$ has a correspondent in every model, namely, $\forall x(R(x,y) \to x \neq y)$, which is equivalent to $\forall x \neg R(x,x)$.

Q3 $\Box_x \varphi \wedge \Box_x \psi \to \Box_x(\varphi \wedge \psi)$ is equivalent to the formula treated in the example 4.2.8; its correspondent is $\forall x \forall \bar{y} \forall \bar{z}(R(x, \bar{y}\bar{z}) \to R(x, \bar{y}))$.

4.2.21. Example. The characteristic axiom of the 'for almost all' quantifier

Q5 $\Box_x \Box_y P(x,y,\bar{z}) \to \Box_y \Box_x P(x,y,\bar{z})$

corresponds to the following condition on R:

$$R(y,\bar{z}) \wedge R(x, y\bar{z}) \to R(x,\bar{z}) \wedge R(y, x\bar{z}).$$

Proof. Rewriting the validity conditions of the axiom gives

$$\forall \varphi(\forall x \forall y(R(x,\bar{z}) \wedge R(y,x\bar{z}) \to \varphi(x,y,\bar{z})) \to \forall y \forall x(R(y,\bar{z}) \wedge R(x,y\bar{z}) \to \varphi(x,y,\bar{z}))$$

which is equivalent to

$$\bigwedge\{\forall y \forall x(R(y,\bar{z}) \wedge R(x,y\bar{z}) \to \varphi(x,y,\bar{z})) : \forall x \forall y(R(x,\bar{z}) \wedge R(y,x\bar{z}) \to \varphi(x,y,\bar{z}))\}.$$

By the Intersection Lemma,

$$\forall y \forall x(R(y,\bar{z}) \wedge R(x,y\bar{z}) \to \bigwedge\{\varphi(x,y,\bar{z}) : \forall x \forall y(R(x,\bar{z}) \wedge R(y,x\bar{z}) \to \varphi(x,y,\bar{z}))\}),$$

while by the Closure Lemma

$$\bigwedge\{\varphi(x,y,\bar{z})) : \forall x \forall y(R(x,\bar{z}) \wedge R(y,x\bar{z}) \to \varphi(x,y,\bar{z}))\} \equiv R(x,\bar{z}) \wedge R(y,x\bar{z})$$

in every canonical model. Thus we obtain the correspondent

$$\forall y \forall x(R(y,\bar{z}) \wedge R(x,y\bar{z}) \to R(x,\bar{z}) \wedge R(y,x\bar{z})).$$

□

4.2.22. Example. The characteristic axiom of the 'co-countably many' quantifier (Keisler's axiom):

Q6 $\forall x \Box_y P(x, y, \bar{z}) \wedge \Box_x \forall y P(x, y, \bar{z}) \rightarrow \Box_y \forall x P(x, y, \bar{z})$,
corresponds to

$$\forall x \forall y (R(y, \bar{z}) \rightarrow R(y, x\bar{z}) \vee R(x, \bar{z})).$$

Proof. The axiom is valid iff

$$\forall \varphi (\forall x \forall y (R(y, x\bar{z}) \rightarrow \varphi(x, y, \bar{z})) \wedge \forall x \forall y (R(x, \bar{z}) \rightarrow \varphi(x, y, \bar{z})) \rightarrow$$

$$\rightarrow \forall x \forall y (R(y, \bar{z}) \rightarrow \varphi(x, y, \bar{z})),$$

namely, $\forall \varphi (\forall x \forall y (R(y, x\bar{z}) \vee R(x, \bar{z})) \rightarrow \varphi(x, y, \bar{z})) \rightarrow \forall x \forall y (R(y, \bar{z}) \rightarrow \varphi(x, y, \bar{z}))$. This can be rewritten as

$$\bigwedge \{\forall x \forall y (R(y, \bar{z}) \rightarrow \varphi(x, y, \bar{z})) : \forall x \forall y (R(y, x\bar{z}) \vee R(x, \bar{z})) \rightarrow \varphi(x, y, \bar{z}))\};$$

by the Intersection Lemma,

$$\forall x \forall y (R(y, \bar{z}) \rightarrow \bigwedge \{\varphi(x, y, \bar{z}) : \forall x \forall y (R(y, x\bar{z}) \vee R(x, \bar{z})) \rightarrow \varphi(x, y, \bar{z}))\},$$

and by the Closure Lemma, $\forall x \forall y (R(y, \bar{z}) \rightarrow R(y, x\bar{z}) \vee R(x, \bar{z}))$. □

4.2.2 Non-existence of correspondents for completeness

Restrictions imposed in the Sahlqvist theorem

In this section we show that not all formulas have a correspondent for completeness. First we prove this for the truth definition we are working with; later the result is generalized.

The following theorem also shows why occurrences of $\Diamond_x$ in the antecedent are forbidden by theorem 4.2.7.

4.2.23. THEOREM. *$\Diamond_x \varphi \rightarrow \Diamond_x (\varphi \vee \psi)$ does not have a correspondent in the sense of completeness.*

Proof. Although Q4 does have a frame correspondent, namely $\forall x \forall \bar{y} \forall \bar{z} (R(x, \bar{y}) \rightarrow R(x, \bar{y}\bar{z}))$, we show that it does not have a correspondent for completeness. Assume that such a correspondent $A^\dagger$ exists. Q1 and Q2 are consistent with Q4. Hence, together with Q4 they have a model where $A^\dagger$ holds. From Proposition 4.2.5 follows that $A^\dagger$ implies $\forall x \forall \bar{y} \forall \bar{z} (R(x, \bar{y}) \rightarrow R(x, \bar{y}\bar{z}))$. Since the correspondents of the other two axioms (being their standard translations) hold in every model, we have a model where $\forall x \forall \bar{y} \forall \bar{z} (R(x, \bar{y}) \rightarrow R(x, \bar{y}\bar{z}))$, $\exists x R(x)$ and $\forall x \neg R(x, x)$ hold simultaneously. But this is impossible, as we have seen in Example 4.1.11. □

An immediate consequence of the theorem is that '$\Box$ over $\wedge$'-combination in the antecedent cannot be allowed, since the axiom considered above can be written equivalently as

$$\Box_x (\varphi \wedge \psi) \rightarrow \Box_x \varphi.$$

Also, we have

4.2.24. Corollary. *Extensionality, that is $\forall x(\varphi \equiv \psi) \to (\Box_x\varphi \equiv \Box_x\psi)$ does not have a correspondent in the sense of completeness.*

Proof. One can check that extensionality is consistent with $\Diamond_x x = x$ and $\neg\Diamond_y x = y$, and that extensionality implies $\Diamond_x\varphi \to \Diamond_x(\varphi \vee \psi)$. Hence if extensionality would have a correspondent for completeness, it would imply the frame correspondent of $\Diamond_x\varphi \to \Diamond_x(\varphi \vee \psi)$. The rest of the argument is the same as above. □

The clause of Theorem 4.2.7 forbidding occurrences of the same predicate letter with different free variables is also necessary. Suppose that there is a variable in the antecedent not occurring in the consequent, as in $\Box_y(\Box_x P(x,y) \to \Box_z P(x,z))$, is equivalent to $\Diamond_y(\Box_x P(x,y) \wedge \top(x)) \to \Box_z P(x,z)$, and
$\forall y(\Box_x P(x,y) \to \Box_z P(x,z))$ is equivalent to $\exists y \Box_x P(x,y) \to \Box_z P(x,z)$.

This shows that the class of weak Sahlqvist formulas is strictly smaller than the class of all Sahlqvist formulas and that none of the conditions of the Theorem 4.2.7 can be dropped.

Definability of singletons

Reflection on the proofs of Theorems 4.1.5 and 4.2.7 suggests a closer look at the behaviour of singleton sets, which are used as minimal substitutions in the proof of the Theorem 4.1.5 (and in modal logic), but do not occur as good minimal substitutions in the proof of the Theorem 4.2.7. Note that singletons as minimal substitutions are used precisely in the cases ruled out in the Theorem 4.2.7. The semantical correlate of 'singletons as minimal substitutions' is distinguishability. A model is called *distinguishable* if every element is uniquely determined by the set of formulas in one free variable which are true for this element. For example, canonical models for modal logic are distinguishable. But in general, our models will not be distinguishable. Suppose an ω-saturated canonical model satisfies $\Box_x x \neq y$ and extensionality for $\Box_x$, and d is the unique element satisfying $\bigwedge\{\varphi(x) : \varphi(d)\}$. We show that $\forall x(R(x) \to x \neq d)$. In a distinguishable model this would hold for every element, hence R would be empty.

Suppose $\exists x(R(x) \wedge x = d)$, then for all φ such that $\varphi(d)$, $\exists x(R(x) \wedge \varphi(x))$. Extensionality implies for every formula ψ

$$\exists x(R(x) \wedge \psi(x)) \to \exists x(R(x,d) \wedge \psi(x)).$$

Hence $\exists x(R(x,d) \wedge \varphi(x))$, and by ω-saturation $\exists x(R(x,d) \wedge \bigwedge \varphi(x))$. It follows that $R(d,d)$, a contradiction. □

A first order complete logic which is not axiomatized by weak Sahlqvist formulas

Recall the logic $L^-_{min} + \Diamond_x\varphi \equiv \Box_x\varphi$ introduced in Chapter 3. This logic is first-order complete. In Chapter 3 we proved that it is complete with respect to the class of models defined by the functionality condition on R: $\forall\bar{y}\exists^1 x R(x,\bar{y})$. From the proposition below follows that $\Diamond_x\varphi \equiv \Box_x\varphi$ is not weak Sahlqvist (and that

$\Diamond_x\varphi \to \Box_x\varphi$ is not weak Sahlqvist, otherwise in conjunction with the weak Sahlqvist formula $\Box_x\varphi \to \Diamond_x\varphi$ it would give a weak Sahlqvist formula).

4.2.25. Proposition. *$L_{min} + \Diamond_x\varphi \equiv \Box_x\varphi$ is not canonically first order.*

Proof. First, we show that $\Diamond_x\varphi \equiv \Box_x\varphi$ is consistent with Q1 – Q3. Then we will see that there is no canonical model where Q1 – Q3 and $\Diamond_x\varphi \equiv \Box_x\varphi$ are valid together.

Observe that $\Box_x\varphi \to \Diamond_x\varphi$ follows from Q1, hence it is enough to prove that Q1 – Q3 are consistent with $\Diamond_x\varphi \to \Box_x\varphi$.

Consider a model M where D is infinite, $R(x,\bar{y}) =_{df} \bigwedge_i x \neq y_i$, and for each predicate P^n, $V(P^n) = D^n$. In this model Q1 – Q3 are trivially valid. The proof that for every formula φ, $M \models \Diamond_x\varphi \to \Box_x\varphi$ goes by induction on the complexity of φ. We may assume that x is free in φ since the axiom is trivially true otherwise. We want to show that for any formula φ and any variable x which is free in φ,

$$\exists x(\bigwedge x \neq y_i \wedge \varphi(x,\bar{y})) \to \forall x(\bigwedge x \neq y_i \to \varphi(x,\bar{y})),$$

where y_i are the free variables of φ different from x.

- let $\varphi = P(x,\bar{y})$; given the interpretation function,

$$\exists x(\bigwedge x \neq y_i \wedge P(x,\bar{y})) \to \forall x(\bigwedge x \neq y_i \to P(x,\bar{y}))$$

- let $\varphi = (x = y)$. Then $\exists x(x \neq y \wedge x = y) \to \forall x(x \neq y \to x = y)$ holds trivially;
- let $\varphi = \neg\psi$ and $\exists x(\bigwedge x \neq y_i \wedge \psi(x,\bar{y})) \to \forall x(\bigwedge x \neq y_i \to \psi(x,\bar{y}))$ holds. Assume that $\exists x(\bigwedge x \neq y_i \wedge \neg\psi)$ holds. Then $\neg\forall x(\bigwedge x \neq y_i \to \psi)$. By the inductive hypothesis, $\neg\exists x(\bigwedge x \neq y_i \wedge \psi)$, that is, $\forall x(\bigwedge x \neq y_i \to \neg\psi)$.
- for $\varphi = \psi_1 \wedge \psi_2$, the claim holds trivially;
- let $\varphi = \exists z\psi(x,z,\bar{y})$, and the inductive hypothesis holds for $\psi(x,z,\bar{y})$ and for any free variable of this formula. Assume $\exists x(\bigwedge x \neq y_i \wedge \exists z\psi(x,z,\bar{y}))$. This is equivalent to
  $\exists x(\bigwedge x \neq y_i \wedge$
  $\wedge[\exists z((z = x \vee \bigvee z = y_i) \wedge \psi(x,z,\bar{y})) \vee \exists z(z \neq x \wedge \bigwedge z \neq y_i \wedge \psi(x,z,\bar{y}))])$ After some calculations involving the inductive hypothesis, we obtain $\forall x(\bigwedge x \neq y_i \to \exists z\psi(x,z,\bar{y}))$.
  This shows that Q1 – Q3 are consistent with $\Diamond_x\varphi \to \Box_x\varphi$.
  Assume that $\Diamond_x\varphi \equiv \Box_x\varphi$ is canonically first order. Then there is a canonical model where Q1 – Q3 and $\Diamond_x\varphi \equiv \Box_x\varphi$ are true and the correspondent for completeness of the latter holds. Note that the correspondents for completeness of Q1 – Q3 hold in every canonical model where Q1 – Q3 are valid (since in Q3 there are no existential quantifiers in the consequent).
  Then by Q1, in this model for every element a there is an element b, such that $R(b,a)$; by Q2, this element is not equal to a, and by the additional axiom, it is unique. By Q1, for any pair of elements a, b there is an element c with $R(c,ab)$, which is again unique. By Q3, $R(c,ab) \to R(c,a)$ and $R(c,ab) \to R(c,b)$, that is, $R(c,a)$ holds in the model, which means that $c = b$. But then $R(b,b)$ holds, which contradicts our assumption. □

4.2.26. COROLLARY. *$L_{min}+\exists x(x \neq y \wedge \varphi(x,y)) \rightarrow \Diamond_x(\varphi(x,y) \vee \psi)$ is not canonically first order.*

Proof. Analogous. □

4.2.3 Other truth definitions

The fact that Q4 (or extensionality) does not have correspondents for completeness, is rather disappointing. A natural question is, whether a better truth definition involving a dependence relation can be found, under which extensionality does have a correspondent for completeness.

For example, the following truth definition studied in (Jervell 1975), (Mijajlovic 1985) and (Krynicki 1990) trivially yields extensionality:

$$\Box_x \varphi(x) \text{ if and only if } \exists y \forall x (R(x,y) \rightarrow \varphi(x)),$$

where R is a new binary predicate and y does not occur free in φ. But in this case Keisler's axiom for 'co-countably many'

$$\forall x \Box_y \varphi \wedge \Box_x \forall y \varphi \rightarrow \Box_y \forall x \varphi$$

corresponds to a schema, not to a first order condition as above.

This is not accidental. For instance, for the quantifier 'almost all' it can be shown that any truth definition, however complex, involving an accessibility relation R, will make at least one axiom correspond to a schema.

Consider even more general truth definitions involving a relation $R(x,Y)$, where Y is a finite *subset* of the domain. Consideration of this relation is natural, because one might argue that dependence really is a relation between objects and finite sets of objects (compare algebraic or linear dependence). Such truth definitions, where quantifiers over finite sets are allowed, will be called *weak second order*. The truth definition that we employed up till now can be expressed in the new language as follows:

$$\Box_x \varphi(x, y_1, \ldots, y_n) \;\Leftrightarrow$$

$$\Leftrightarrow\; \exists Y[\forall x(R(x,Y) \rightarrow \varphi(x, y_1, \ldots, y_n)) \wedge \bigwedge_i y_i \in Y \wedge \forall z \in Y(\bigvee_i z = y_i)]$$

For any given weak second order definition, we may now ask whether quantifier axioms have correspondents for completeness in the language $\{R, \in, =\}$. The following theorem by van Lambalgen shows that the negative result still holds:

4.2.27. THEOREM. *(van Lambalgen, 1994) For any weak second order truth definition, the conjunction of the Friedman axioms does not have a correspondent for completeness.*

Proof. The proof, using Levy - Shoenfield Absoluteness Lemma (cf. (Jech 1978), is given in (Alechina and van Lambalgen 1995b). □

4.2.4 Undecidability of the correspondence problem

In the section on frame correspondence we proved that the problem whether a formula has a first order frame correspondent is undecidable. We finish this chapter by stating the result by van Lambalgen on correspondence for completeness.

4.2.28. Theorem. *(van Lambalgen, 1994) For any weak second order truth definition, it is undecidable whether a formula in the language with generalized quantifiers has a correspondent for completeness.*

Proof. It is undecidable whether a formula without ordinary quantifiers is satisfiable in Friedman's logic (cf. (Steinhorn 1985)). Let F be Friedman's logic; F can be written as $F_0 + EXT$, where EXT is extensionality, and F_0 is a conjunction of weak Sahlqvist formulas.

Let A be any formula with $\Box$'s but no first order quantifiers.

- If $F_0 + EXT + A$ is satisfiable, then it does not have a correspondent for completeness by the argument in **L**.
- If $F_0 + EXT + A$ is not satisfiable, then $\bot$ is a correspondent for completeness.

Hence if we could decide whether a schema has a correspondent for completeness, we could also decide satisfiability in Friedman's logic. $\Box$

Chapter 5

Binary quantifiers

In this chapter we move from unary to binary quantifiers. Standardly, a binary quantifier is a relation $\prod$ on the powerset of the domain, such that $\Pi x(\varphi, \psi)$ is true in a model if the pair $\langle\{x : \varphi(x)\}, \{x : \psi(x)\}\rangle$ is in $\prod$. Like in the previous chapters we have given a non-standard semantics for unary quantifiers, here we give a non-standard semantics for binary quantifiers, which is a natural generalization of the unary case.

Unary quantifiers which admit dependence models may be intuitively understood as quantifying over 'almost all', 'all but a few' objects. A natural step is to consider binary quantifiers talking about 'almost all objects with property φ', 'all typical φ's'. While $\Box_x \varphi(x, \bar{y})$ can be translated into first order logic as $\forall x(R(x, \bar{y}) \to \varphi(x, \bar{y}))$, the binary quantifier Π which we are going to study in this chapter is translated as follows:

$$ST(\Pi x(\varphi(x, \bar{y}), \psi(x, \bar{z})) = \forall x(R_{\varphi(\underline{x}, \bar{y})}(x, \bar{y}\bar{z}) \to ST(\psi(x, \bar{z}))).$$

This continues the analogy with modal logic which was developed in the previous chapters. Π resembles the binary modality 'if... then' from conditional logic. In conditional logic, $\varphi > \psi$ (to be read 'normally, if ... then ... '), is true in a world w if all worlds which are normal φ-worlds from the point of view of w, satisfy ψ [1]. As in conditional logic, φ-accessibility of x from $\bar{y}\bar{z}$ ($R_{\varphi(\underline{x}, \bar{y})}(x, \bar{y}\bar{z})$) will have the intuitive meaning of being a normal or typical $\varphi(x)$ with respect to $\bar{y}\bar{z}$.

The reason for considering such quantifiers except for a purely technical interest of generalizing the previous approach to the binary case is that a quantifier $\Pi x(\varphi, \psi)$ with the meaning of 'typically, φ's are ψ's' can be used for representing defaults, or commonsense generalizations.

There are several kinds of commonsense generalizations. Some have clearly statistical meaning and can be better represented by a binary quantifier or operator with probabilistic meaning, like 'the chance that a randomly chosen object has the property ψ, given that it has the property φ, is greater than 1/2'. Some,

[1] This is just one of the numerous semantics for conditional logic; see (Lewis 1973) and (Veltman 1985) for an overview.

like 'Birds fly' or 'The Dutch dine at 6' refer not to the majority of individuals, but rather to what is to be expected from a typical individual under normal circumstances. The most well known approach towards formalizing the latter type of defaults is the theory of circumscription (see for example (McCarthy 1980)). In this theory, defaults are represented as quantifying over normal objects, normality being related to some aspect. For example, 'Birds fly' is represented by '$\forall x(Bird(x) \wedge \neg abnormal(x, aspect\ i) \rightarrow Fly(x))$.

In predicate conditional logic (cf. (Delgrande 1991), (Morreau 1992)), defaults are represented by expressions of the form $\forall x(\varphi > \psi)$, quantifying over all individuals and φ-normal worlds; however, the intuitive meaning of defaults in (Morreau 1992) is also explained as quantification over typical objects. We show to what extend the representation of defaults in conditional logic and our approach give the same results and where they differ.

Badaloni and Zanardo (1990, 1991) propose a representation of defaults by means of a binary generalized quantifier. Although their semantics is different from the one proposed here (typical objects are not introduced explicitely in semantics; instead, a binary 'high plausibility' relation between two sets is considered), their work substantially influenced the research below.

5.1 Semantics and axiomatizations

5.1.1 Minimal logic

Consider a first-order language $\mathcal{L}(\forall\Pi)$ with a binary generalized quantifier Π. A well-formed formula is defined as usual; if φ and ψ are well-formed formulas, so is $\Pi x(\varphi, \psi)$. As before, we denote the fragment of $\mathcal{L}(\forall\Pi)$ without ordinary quantifiers by $\mathcal{L}(\Pi)$.

Let D be a domain, φ a well formed formula of $\mathcal{L}(\forall\Pi)$ and x a variable (which may be free or not free in φ). Replacing all the free variables in φ, except for x, by parameters from D defines a unary property $\varphi(-, \bar{d})$, where $-$ stands on the place of x. For every such property we introduce a relation $R_{\varphi(-,\bar{d})}$ between an element from D and a finite set of such elements, analogously to the dependence relation used to interpret unary quantifiers. We shall write $R_{\varphi(\underline{x},\bar{d})}$ for $R_{\varphi(-,\bar{d})}$ if $\varphi(-, \bar{d})$ is defined as described above. The set of all $R_{\varphi(-,\bar{d})}$, for every unary property definable with parameters from D, is denoted by $\mathcal{R}$.

A model for $\mathcal{L}(\forall\Pi)$ is a triple $M = \langle D, \{R_{\varphi(-)} : R_{\varphi(-)} \in \mathcal{R}\}, V\rangle$. The truth definition for Π reads as follows:

$M, s \models \Pi x(\varphi(x, \bar{y}), \psi(x, \bar{z})) \Leftrightarrow \forall d(R_{\varphi(\underline{x},s(\bar{y}))}(d, s(\bar{y}\bar{z})) \Rightarrow M, s^x_d \models \psi)$, where $\bar{y}$ and possibly x are the free variables of φ and $\bar{z}$ and possibly x are the free variables of ψ.

Observe that this makes the standard translation of Π to be as intended:

$$ST(\Pi x(\varphi(x, \bar{y}), \psi(x, \bar{z})) = \forall x(R_{\varphi(\underline{x},\bar{y})}(x, \bar{y}\bar{z}) \rightarrow ST(\psi(x, \bar{z}))).$$

Since x in $R_{\varphi(\underline{x})}$ only indicates the place, $R_{\varphi(\underline{x})}$ and $R_{\varphi(\underline{y})}$ are the same if $\varphi(x)$ and $\varphi(y)$ define the same logical function, i.e. are similar in the sense of Mendelson (1979, p.65): $\varphi(y) = \varphi(x/y)$, y is free for x in $\varphi(x)$ and y does not occur free in $\varphi(x)$. In particular, if neither x nor y are free in φ, for whatever parameters $\bar{d}$ replace the free variables of φ, $R_{\varphi(\underline{x})}$ and $R_{\varphi(\underline{y})}$ are the same.

The semantics above corresponds to an axiomatic system which is very similar to the minimal logic of unary quantifiers. Here we give an axiomatization for the minimal logic of $\mathcal{L}(\Pi)$, without ordinary quantifiers. Observe that $\Pi x(\varphi \to \ldots)$ is a L_{min}-modality, i.e. it satisfies axioms C1 – C3 for $\Box_x$ from Definition 3.2.1.

5.1.1. Definition. The *minimal logic for binary quantifiers B_{min}* is the least set of formulas closed under propositional derivability and

B1 $\Pi x(\varphi, \psi_1 \to \psi_2) \to (\Pi x(\varphi, \psi_1) \to \Pi x(\varphi, \psi_2))$, given that $\Pi x(\varphi, \psi_1)$ and $\Pi x(\varphi, \psi_2)$ have the same free variables;
B2 $\psi \to \Pi x(\varphi, \psi)$, given that x is not free in ψ;
B3 $\Pi x(\varphi, \psi) \leftrightarrow \Pi y(\varphi, \psi)$, where $\varphi(x)$ and $\varphi(y)$ are similar;
R $\psi / \Pi x(\varphi, \psi)$ □

The deduction theorem is analogous to the one for L_{min}.

5.1.2. Theorem. *B_{min} is sound and complete with respect to the semantics described above.*

Proof. The soundness part is easy. The proof of completeness is similar to the proof for L_{min}. Instead of $\Box_x(\Box_x \varphi \to \varphi)$ we use the following formula:

Iteration $\vdash \Pi x(\varphi(x), \Pi x(\varphi(x), \psi(x)) \to \psi(x))$

1. $\Pi x(\varphi, \psi \to (\Pi x(\varphi(x), \psi(x)) \to \psi(x))$ classical logic and R;
2. $\Pi x(\varphi(x), \psi(x)) \to \Pi x(\varphi(x), \Pi x(\varphi(x), \psi(x)) \to \psi(x))$ from 1, B1;
3. $\Pi x(\varphi, \neg\Pi x(\varphi(x), \psi(x)) \to (\Pi x(\varphi, \psi) \to \psi))$ classical logic, R;
4. $\Pi x(\varphi, \neg\Pi x(\varphi, \psi)) \to \Pi x(\varphi, \Pi x(\varphi, \psi) \to \psi))$ 3,B1;
5. $\neg\Pi x(\varphi, \psi) \to \Pi x(\varphi, \neg\Pi x(\varphi, \psi))$ B2;
6. $\neg\Pi x(\varphi, \psi) \to \Pi x(\varphi, \Pi x(\varphi(x), \psi(x)) \to \psi)$ from 4, 5;
7. $\Pi x(\varphi, \Pi x(\varphi, \psi) \to \psi)$ 2,6.

Now we a going to show that every set of formulas consistent with B_{min} has a model. We construct a maximally consistent set as usual [2], and for every formula $\neg\Pi x(\varphi, \psi)$ consistent with Σ_n add a new variable x' such that

1. $\neg\psi(x/x') \in \Sigma_{n+1}$
2. $\Pi x(\varphi, \chi) \to \chi(x/x') \in \Sigma_{n+1}$ for all formulas χ such that $\Pi x(\varphi, \chi)$ has the same free variables as $\Pi x(\varphi, \psi)$.

[2]The only difference with the standard setup is that we add new variables to the language instead of new constants to keep the semantic definitions simpler. But a reader can substitute the word "constants" or "parameters" everywhere instead of "free variables".

Assume that the above algorithm gives rise to inconsistencies. Then

$$\Sigma_n \vdash \bigwedge_i (\Pi x(\varphi, \chi_i) \to \chi_i(x')) \to \psi(x')$$

and, by R and B1,

$$\Sigma_n \vdash \Pi z(\varphi(z), \bigwedge_i (\Pi x(\varphi, \chi_i) \to \chi_i(z))) \to \Pi z(\varphi(z), \psi(z));$$

by B3, this is equivalent to

$$\Sigma_n \vdash \Pi z(\varphi(z), \bigwedge_i (\Pi x(\varphi, \chi_i) \to \chi_i(z))) \to \Pi x(\varphi(x), \psi(x))$$

that is, if

$$\Pi z(\varphi, \bigwedge_i (\Pi x(\varphi, \chi_i) \to \chi_i(z)))$$

is derivable from Σ_n, then Σ_n is contradictory.

But

$$\Pi z(\varphi(z), \Pi x(\varphi(x), \chi_i(x)) \to \chi_i(z)))$$

is derivable by Iteration and B3 for every χ_i, and then we can apply B1.

The model is based on the set $\Sigma = \cup_{n<\omega} \Sigma_n$.

$R_{\varphi(\underline{x},\bar{y})}(u, \bar{y}\bar{z})$ holds in Σ iff for all formulas χ such that $FV(\Pi x(\varphi(x, \bar{y}, \chi)) = \bar{y}\bar{z}$,

$$\Pi x(\varphi, \chi) \in \Sigma \Rightarrow \chi(x/u) \in \Sigma.$$

It is easy to prove now that for any formula ϕ, $\phi \in \Sigma \Leftrightarrow \Sigma \models \phi$. □

The question whether B_{min} is decidable is open. We have the following partial result.

Consider the fragment of $\mathcal{L}(\Pi)$ where $\Pi x(\varphi, \psi)$ is a well formed formula only if φ has at most one free variable x. Let us call this fragment B. We can prove that B is decidable by embedding it in L_{min}.

For every formula φ with at most one free variable introduce a new constant a_φ in $\mathcal{L}(\Pi)$. Define the translation $* : \mathcal{L}(\Pi) \longrightarrow \mathcal{L}(\Box)$ by induction as follows: $*$ commutes with atomic formulas and propositional connectives, and

$$(\Pi x(\varphi, \psi))^* = \Box_x(\psi^* \wedge \top(a_\varphi)).$$

5.1.3. Proposition. *For every $\varphi \in B$, $B_{min} \vdash \varphi \Leftrightarrow L_{min} \vdash \varphi^*$.*

Proof. The direction from left to right can be proved easily by induction on the length of the derivation.

To prove the opposite direction, assume that there is a model for binary quantifiers satisfying $\neg\varphi$. The following transformation gives a dependence model for $\neg\varphi^*$.

Let $M, s \models \neg\varphi$, and $M = \langle D, \{R_\varphi : R_\varphi \in \mathcal{R}\}, V\rangle$. Without loss of generality, we may assume that $\mathcal{R}$ contains only relations corresponding to formulas with at most one free variable. The dependence model M' will have a domain $D' = D \cup A$, where A is the set of objects interpreting the new constants; somewhat sloppily, we denote the interpretation of a_φ in M' by a_φ. For predicate symbols, V' consides with V on D and is arbitrary otherwise. Finally, $R'(d, \bar{e}, a_\varphi)$ holds in M' iff $d, \bar{e} \in D$ and $R_\varphi(d, \bar{e})$ holds in M; for all other cases R is empty. It is easy to check that for all subformulas ψ of $\neg\varphi$, and any assignment $z : Var \longrightarrow D$,

$$M, z \models \psi \Leftrightarrow M', z \models \psi^*.$$

Let us check the quantifier case:
$M, z \models \Pi x(\theta(x), \chi(x, \bar{y})) \Leftrightarrow \forall d(R_\theta(d, z(\bar{y})) \rightarrow M, z^x_d \models \chi) \Leftrightarrow \forall d(R'(d, \bar{e}, a_\theta) \rightarrow M', z^x_d \models \chi) \Leftrightarrow M', z \models \Box_x(\chi \wedge \top(a_\theta))$. □

5.1.4. Corollary. *B is decidable.*

5.1.2 Definability

A natural question to ask is whether Π is definable via unary quantifiers. We cannot hope for semantic definability, since the meanings of R and R_φ are completely independent. But the following question remains open: is there a translation * from $\mathcal{L}(\Pi)$ into $\mathcal{L}(\Diamond)$ which preserves validity, namely

$$\vdash_{B_{min}} \varphi \Leftrightarrow \vdash_{L_{min}} \varphi^*?$$

One can however show that there is no unary quantifier $\Box$ satisfying L_{min} for which holds

$$\vdash_{B_{min}} \Pi x(\varphi, \psi) \Leftrightarrow \vdash_{L_{min}} \Box_x(\varphi^* \rightarrow \psi^*).$$

Namely, for any such quantifier holds

$$\vdash_{L_{min}} \Box_x(\varphi^* \rightarrow \psi^*) \equiv \Box_x(\neg\psi^* \rightarrow \neg\varphi^*)$$

(at least if * commutes with the booleans), and in B_{min} it is not provable that

$$\Pi x(\varphi, \psi) \equiv \Pi x(\neg\psi, \neg\varphi).$$

Eventually, Π becomes syntactically definable if the following axiom is added: $\Pi x(\varphi, \psi) \equiv \Pi x(\top, \varphi \rightarrow \psi)$. Then $\Pi x(\top, \ldots)$ can be identified with $\Box_x$.

5.1.3 Other truth definitions. Correspondence.

In principle the same questions about correspondence and completeness as for the unary quantifiers can be asked about the logic above. Note, however, that since R is indexed by formulas, one cannot speak about pure first order correspondence in the sense of the previous chapters.

Here are some examples of correspondence in the sense of completeness, given in parallel with a discussion of possible other truth definitions for Π.

Reflexivity, or alternative truth definition

We could have given the truth definition for Π so that the standard translation becomes

$$ST(\Pi x(\varphi(x,\bar{y}),\psi(x,\bar{z}))) = \forall x(R_{x,\varphi(x,\bar{y})}(x,\bar{y}\bar{z}) \wedge ST(\varphi(x,\bar{y})) \rightarrow ST(\psi(x,\bar{z})))$$

(cf. (Alechina 1993), (Fernando 1995)).

However, it is easy to check that the same logic can be obtained by imposing an additional condition on R in the original semantics, namely

$$R_{\varphi(\underline{x},\bar{y})}(x,\bar{y}\bar{z}) \rightarrow \varphi(x,\bar{y}).$$

This corresponds to adding one more axiom to our minimal logic:

5.1.5. PROPOSITION. *B_{min} with additional* reflexivity axiom $\Pi x(\varphi,\varphi\wedge\top(\bar{z}))$ *is complete with respect to the class of models satisfying*

$$R_{\varphi(\underline{x},\bar{y})}(x,\bar{y}\bar{z}) \rightarrow \varphi(x,\bar{y}).$$

Proof. Soundness is immediate. To prove completeness, consider a canonical model for B_{min}. If $R_{\varphi(\underline{x},\bar{y})}(x,\bar{y}\bar{z})$, then for all formulas χ with $FV(\Pi x(\varphi(x,\bar{y}),\chi)) = \bar{y}\bar{z}$, $\Pi x(\varphi,\chi) \rightarrow \chi \in \Sigma$. In particular, $\Pi x(\varphi,\varphi\wedge\top(\bar{z})) \rightarrow \varphi \in \Sigma$. Since now

$$\Pi x(\varphi,\varphi\wedge\top(\bar{z})) \in \Sigma,$$

$\varphi \in \Sigma$, and by the truth lemma $\Sigma \models \varphi$. $\Box$.

Relaxing restrictions on the free variables

Another truth definition considered in (Alechina 1993) gives

$$ST(\Pi x(\varphi(x,\bar{y}),\psi(x,\bar{z}))) = \forall x(R_{\varphi(\underline{x},\bar{y})}(x,\bar{y}) \rightarrow ST(\psi(x,\bar{z}))),$$

i.e. only the free variables of the antecedent matter; this is an intuitive truth definition, which better corresponds to the reading 'x is a typical $\varphi(x,\bar{y})$'. Recall that our original truth definition requires in addition typicality with respect to parameters not free in φ.

Observe that this truth definition can be rewritten as

$$ST(\Pi x(\varphi(x,\bar{y}),\psi(x,\bar{z}))) = \forall x(R_{\varphi(\underline{x},\bar{y})}(x) \rightarrow ST(\psi(x,\bar{z}))),$$

since it actually defines a property of x.

It is easy to check that the logic with such truth definition is equivalent to the original logic with the following additional condition on R:

$$R_{\varphi(\underline{x},\bar{y})}(x,\bar{y}) \equiv R_{\varphi(\underline{x},\bar{y})}(x,\bar{y}\bar{z}).$$

Again, this class of models can be axiomatized by adding to B_{min} two more axioms:

5.1.6. Proposition. *B_{min} with additional axioms of* unrestricted monotonicity *B1 and* exchange axiom

$$\forall\bar{z}\Pi x(\varphi,\psi)\to\Pi x(\varphi,\forall\bar{z}\psi)\ \text{ given that } \bar{z} \text{ not free in } \varphi$$

is complete with respect to the class of models satisfying

$$R_{\varphi(\underline{x},\bar{y})}(x,\bar{y})\equiv R_{\varphi(\underline{x},\bar{y})}(x,\bar{y}\bar{z}).$$

Proof. We are going to show that if in a canonical model unrestricted B1 and exchange hold, then

$$\forall\bar{z}(R_{\varphi(\underline{x},\bar{y})}(x,\bar{y})\equiv R_{\varphi(\underline{x},\bar{y})}(x,\bar{y}\bar{z}))$$

First, assume that $R_{\varphi(\underline{x},\bar{y})}(x,\bar{y}\bar{z})$ holds. This means that for every formula χ, such that $FV(\Pi x(\varphi,\chi))=\bar{y}\bar{z}$, $\Pi x(\varphi,\chi)\to\chi(x)\in\Sigma$. To prove that $R_{\varphi(\underline{x},\bar{y})}(x,\bar{y})$ holds, assume that for some formula ψ such that $FV(\Pi x(\varphi,\psi))=\bar{y}$, $\Pi x(\varphi,\psi)\in\Sigma$. Then by unrestricted B1 $\Pi x(\varphi,\psi\wedge\top(\bar{z}))\in\Sigma$. Since this formula has as its free variables $\bar{y}\bar{z}$, $\psi(x)\wedge\top(\bar{z})\in\Sigma$, that is, $\psi(x)\in\Sigma$. Since this reasoning can be applied to any formula ψ, we have $R_{\varphi(\underline{x},\bar{y})}(x,\bar{y})$.

Assume that $R_{\varphi(\underline{x},\bar{y})}(x,\bar{y})$ holds. Suppose $\Pi x(\varphi,\chi)\in\Sigma$, where $FV(\Pi x(\varphi,\chi)=\bar{y}\bar{z}$ and $\bar{z}$ are not free in φ. The following example of Iteration is derivable:

$$\forall\bar{z}\Pi x(\varphi,\Pi x(\varphi,\chi)\to\chi).$$

By exchange, $\Pi x(\varphi,\forall\bar{z}(\Pi x(\varphi,\chi)\to\chi))$ is also derivable and therefore in Σ. Since this formula has only $\bar{y}$ as its free variables, $\forall\bar{z}(\Pi x(\varphi,\chi)\to\chi)\in\Sigma$. Therefore $\chi\in\Sigma$. □

Full extensionality

So far we have indexed R with *formulas* and not with *sets*; in other words, the first argument of Π remained intensional.

The following axiom (extensionality of Π)

$$\forall x(\varphi\equiv\psi)\to(\Pi x(\varphi,\chi)\equiv\Pi x(\psi,\chi)),$$

corresponds in the original semantics to

$$\forall x(\varphi(x,\bar{y})\equiv\psi(x,\bar{u}))\to(R_{\varphi(\underline{x},\bar{y})}(x,\bar{y}\bar{z})\equiv R_{\psi(\underline{x},\bar{u})}(x,\bar{u}\bar{z})).$$

Here is an example of a quantifier satisfying extensionality. It has a clear interpretation as a quantifier over typical objects.

5.1.7. Proposition. *A quantifier which satisfies the following set of axioms:*

A0 *first order logic*

B1' $\Pi x(\varphi,\psi_1\to\psi_2)\to(\Pi x(\varphi,\psi_1)\to\Pi x(\varphi,\psi_2))$, *without restrictions on the free variables;*

B2 $\psi\to\Pi x(\varphi,\psi)$, *given that x is not free in ψ;*

B3 $\Pi x(\varphi,\psi) \leftrightarrow \Pi y(\varphi,\psi)$, *where* $\varphi(x)$ *and* $\varphi(y)$ *are similar;*
B4 $\forall\bar{z}\Pi x(\varphi,\psi) \rightarrow \Pi x(\varphi,\forall\bar{z}\psi)$, *given that* $\bar{z}$ *is not free in* φ *(exchange);*
B5 $\forall x(\varphi \equiv \psi) \rightarrow (\Pi x(\varphi,\chi) \equiv \Pi x(\psi,\chi))$ *(extensionality)*

(note that in the presence of first order logic R becomes derivable), admits a semantics for Π *where models are equipped with a function* T *on the powerset of the domain, which gives every set* Q *the set of* typical objects *corresponding to this set,* Q_T. *Then*

$$M,s \models \Pi x(\varphi,\psi) \Leftrightarrow \{d : M,s \models \varphi(x/d)\}_T \subseteq \{d : M,s \models \psi(x/d)\}.$$

Such quantifier is explicitely quantifying over typical objects. In (Alechina 1995a) a logic containing one more axiom $\Pi x(\varphi,\varphi)$ (reflexivity), which under this truth definition corresponds to $Q_T \subseteq Q$, is called *Bin* and proposed as a suitable system for representing defaults.

Proof of Proposition 5.1.7. From Proposition 5.1.6 above follows that the axioms B1' and B4 correspond to the following property of R:

$$R_{\varphi(\underline{x},\bar{y})}(x,\bar{y}) \equiv R_{\varphi(\underline{x},\bar{y})}(x,\bar{y}\bar{z}).$$

It is completely straightforward that B5 corresponds to

$$\forall x(\varphi(x,\bar{y}) \equiv \psi(x,\bar{u})) \rightarrow (R_{\varphi(\underline{x},\bar{y})}(x,\bar{y}\bar{z}) \equiv R_{\psi(\underline{x},\bar{u})}(x,\bar{u}\bar{z})).$$

This allows to define T as follows:

$$x \in \{x' : \varphi(x',\bar{y}) \in \Sigma\}_T \Leftrightarrow R_{\varphi(\underline{x'},\bar{y})}(x,\bar{y}).$$

By Proposition 5.1.5, if $\Pi x(\varphi,\varphi)$ is an axiom, then any canonical model satisfies

$$R_{\varphi(\underline{x},\bar{y})}(x,\bar{y}) \Longrightarrow \varphi(x,\bar{y}) \in \Sigma.$$

This forces

$$\{x : \varphi(x,\bar{y}) \in \Sigma\}_T \subseteq \{x : \varphi(x,\bar{y}) \in \Sigma\}.$$

□

Some more examples of correspondence

Here we give two more examples of correspondence; the proofs use the same methods as analogous results for unary quantifiers.

5.1.8. Example. $\exists x\varphi \rightarrow \neg\Pi x(\varphi,\bot)$ corresponds to the condition $\exists x\varphi(x,\bar{y}) \rightarrow \exists x R_{\varphi(\underline{x},\bar{y})}(x,\bar{y}\bar{z})$.

Proof. Obvious. □

5.1.9. Example. $\Pi x(\varphi,\Pi y(\psi,\chi)) \rightarrow \Pi y(\psi,\Pi x(\varphi,\chi))$ corresponds to the following condition in semantics:

$$R_{\varphi(\underline{x})}(x,\bar{z}) \wedge R_{\psi(\underline{y})}(y,x\bar{z}) \rightarrow R_{\psi(\underline{y})}(y,\bar{z}) \wedge R_{\varphi(\underline{x})}(x,y\bar{z}),$$

where $x,y \notin \bar{z}$ and $FV(\varphi), FV(\psi) \in \{x,y,\bar{z}\}$.

Proof. Assume that $\Pi x(\varphi, \Pi y(\psi, \chi)) \to \Pi y(\psi, \Pi x(\varphi, \chi))$ is valid, and assume that $R_{\varphi(x)}(x, \bar{z})$ and $R_{\psi(y)}(y, x\bar{z})$ hold. We want to show that $R_{\psi(y)}(y, \bar{z})$ and $R_{\varphi(x)}(x, y\bar{z})$ hold. (Where $x, y \notin \bar{z}$ and $R_{\psi(y)}$ stands for $R_{\psi(\underline{y})}$.)

Note that if x is free in ψ, there is no such formula θ that $FV(\Pi y(\psi, \theta)) = \bar{z}$ and, by the definition of R, $R_{\psi(y)}(y, \bar{z})$.

Assume that x is not free in ψ and $\neg R_{\psi(y)}(y, \bar{z})$. Then $\neg(\Pi y(\psi, \theta) \to \theta) \in \Sigma$ for some θ such that $FV(\Pi x(\psi, \theta)) = \bar{z}$. Also, $\neg(\top(x) \wedge \Pi y(\psi, \theta) \to \theta) \in \Sigma$. This formula has as its free variables $x, y, \bar{z}$. Assume that

$$\Pi y(\psi(y), \top(x) \wedge \Pi y(\psi, \theta) \to \theta) \in \Sigma$$

Since $R_{\psi(y)}(y, x\bar{z})$, it will mean that $\top(x) \wedge \Pi y(\psi, \theta) \to \theta \in \Sigma$. This contradicts the assumption, thus

$$\neg\Pi y(\psi(y), \top(x) \wedge \Pi y(\psi, \theta) \to \theta) \in \Sigma$$

This formula has as its free variables x and $\bar{z}$. Applying the same reasoning and the fact that $R_{\varphi(x)}(x, \bar{z})$, we obtain

$$\neg\Pi x(\varphi(x), \Pi y(\psi(y), \top(x) \wedge \Pi y(\psi, \theta) \to \theta)).$$

Permutation gives $\neg\Pi y(\psi(y), \Pi x(\varphi(x), \top(x) \wedge \Pi y(\psi, \theta) \to \theta))$. Now it is possible to get rid of $\top(x)$. By B1, $\neg\Pi y(\psi(y), \Pi x(\varphi(x), \Pi y(\psi, \theta) \to \theta))$; now Permutation gives $\neg\Pi x(\varphi(x), \Pi y(\psi(y), \Pi y(\psi, \theta) \to \theta))$. Since the following formula as an example of Iteration is derivable: $\Pi y(\psi(y), \Pi y(\psi, \theta) \to \theta)$, then

$$\Pi x(\varphi(x), \Pi y(\psi(y), \Pi y(\psi, \theta) \to \theta))$$

is also derivable. A contradiction, thus $R_{\psi(y)}(y, \bar{z})$.

Assume $\neg R_{\varphi(x)}(x, y\bar{z})$. As before, for some θ such that $FV(\Pi x(\varphi, \theta) = y\bar{z}$,

$$\neg(\Pi x(\varphi(x), \theta) \to \theta) \in \Sigma.$$

From $R_{\psi(y)}(y, \bar{z})$, $\neg\Pi y(\psi(y), \Pi x(\varphi(x), \theta) \to \theta) \in \Sigma$. The free variables of the above formula are $\bar{z}$. From $R_{\varphi(x)}(x, \bar{z})$

$$\neg\Pi x(\varphi(x), \Pi y(\psi(y), \Pi x(\varphi(x), \theta) \to \theta)) \in \Sigma$$

By Permutation, $\neg\Pi y(\psi(y), \Pi x(\varphi(x), \Pi x(\varphi(x), \theta) \to \theta)) \in \Sigma$ - a contradiction again. So, $R_{\varphi(x)}(x, y\bar{z})$. □

5.2 Conditionals

As it was mentioned in the introduction to this chapter, there is a certain parallelism between our semantics for Π and semantics for conditional logic using a selection function. In this section we are going to study this connection more systematically.

5.2.1 Propositional conditional logic

In Chapter 3 we defined an embedding of modal logic (system K) into L_{min} (cf. Proposition 3.1.2). The same can be done with the following basic system of propositional conditional logic L.

The language $\mathcal{L}(>)$ of L contains propositional letters, boolean connectives and a binary modality $>$. The notion of a well formed formula is standard; if φ and ψ are w.f.f.'s, then so is $\varphi > \psi$. The set of all w.f.f.'s is denoted by Φ.

A model for $\mathcal{L}(>)$ is $M = \langle W, S, V\rangle$, where W is a set of possible worlds, V an interpretation and S a selection function $W \times \Phi \longrightarrow \mathcal{P}(W)$. Intuitively, $S(w,\varphi)$ gives a set of worlds which are φ-normal from the point of view of w. The truth definition for $>$ reads as follows:

$$M, w \models \varphi > \psi \Leftrightarrow \forall w'(w' \in S(w,\varphi) \Longrightarrow M, w' \models \psi).$$

Let $w_0, \dots, w_n, \dots$ be some fixed ordering of the variables of $\mathcal{L}(\Pi)$. Then ${}^{*i} : \mathcal{L}(>) \longrightarrow \mathcal{L}(\Pi)$ (with ${}^{*0} = {}^{*}$) is defined as follows:

$(p)^{*i} = P(w_i)$;
^{*i} commutes with the propositional connectives;
$(\varphi > \psi)^{*i} = \Pi w_{i+1}(\varphi^{*i+1}, \psi^{*i+1} \wedge \top(w_i))$, where $\top$ is any fixed tautology in one free variable.

This translation looks very similar to the translation used in Proposition 3.1.2. It also yields an analogous result:

5.2.1. Proposition. *Let φ be a formula of $\mathcal{L}(>)$. Then*

$$\models_L \varphi \Leftrightarrow \models_{B_{min}} \varphi^0.$$

Proof. Let $M, w \models \neg\varphi$, where M is a model for conditional logic. Define a model M' for B_{min} as follows: $D' = W$; $d \in V'(P_n)$ iff $d \in V(p_n)$ (note that φ^0 contains only monadic predicates); $R_{\psi^{*i}(\underline{w_i})}(a,b)$ iff $a \in S(b,\psi)$, for the rest of the properties C let R_C be arbitrary. It is easy to check that $M, v \models \chi \Leftrightarrow M', w_i/v \models \chi^{*i}$, therefore $M', w_0/w \models \neg\varphi^{*0}$.

For the reverse, assume that there is a model for B_{min}, M, such that $M, w_0/a \models \neg\varphi^{*0}$. Construct a model M' for L as follows: let $W' = D$, $d \in V'(p_n) \Leftrightarrow d \in V(P_n)$, and $d \in S(c,\psi) \Leftrightarrow R_{\psi^{*i}(\underline{w_i})}(d,c)$. The latter is well defined, since $\psi^{*i}(\underline{w_i})$ and $\psi^{*k}(\underline{w_k})$ define the same property, for all i and k. Again, it is easy to see that for every formula χ, $M', v \models \chi \Leftrightarrow M, w_i/v \models \chi^{*i}$, therefore $M', a \models \neg\varphi$. □

This is an easy result, analogous to the one from Chapter 2. To compare the ways of representing defaults by means of conditionals and by means of generalized quantifiers seriously, it is necessary to move to stronger systems, at least to consider predicate conditional logic. In the rest of this section we compare the system of predicate conditional logic *Cond* which is the weakest system proposed to represent defaults in (Morreau 1992), and a logic with a binary quantifier.

5.2.2 Predicate conditional logic

Let $\mathcal{L}(\forall >)$ denote the language of first-order conditional logic (the language of the first-order predicate logic plus binary modal operator $>$). M is a model for $\mathcal{L}(\forall >)$ if $M =< D, W, S, V >$, where D is a non-empty universe (the same for all possible worlds), W is a non-empty set of possible worlds, $S : W \times \mathcal{P}(W) \to \mathcal{P}(W)$ is a selection function (given a world and a proposition, S provides a set of worlds which are normal with respect to this proposition from the point of view of this world; note that this time S is extensional), and V is an interpretation function. The truth definition for conditionals is as follows:

$$M, s, w \models \varphi > \psi \Leftrightarrow S(w, [\varphi]_{M,s}) \subseteq [\psi]_{M,s},$$

where $[\varphi]_{M,s} = \{w' : M, s, w' \models \varphi\}$. The selection function satisfies the following constraint:

ID $S(w, [\varphi]_{M,s}) \subseteq [\varphi]_{M,s}$

5.2.2. Definition. *Cond is the least set of formulas closed under first-order derivability and the axioms and rules below:*

CI $\varphi > \varphi$;
CC $(\varphi > \psi_1) \wedge (\varphi > \psi_2) \to (\varphi > \psi_1 \wedge \psi_2)$;
E $\forall x(\varphi > \psi) \to (\varphi > \forall x\psi)$, *if x is not free in φ;*
RCEA $\vdash \varphi \leftrightarrow \psi \Longrightarrow \vdash (\varphi > \chi) \leftrightarrow (\psi > \chi)$
RCM $\vdash \varphi \to \psi \Longrightarrow \vdash (\chi > \varphi) \to (\chi > \psi)$

Cond is complete (see (Morreau 1992)).

The resemblance between *Bin* from Proposition 5.1.7, where $\Pi x(\varphi, \psi)$ is interpreted as 'all typical φ's are ψ's', and *Cond* is rather remarkable, and one can ask oneself if the two representations of defaults are not identical.

In (Morreau 1992) generics are indeed informally understood as quantifying over normal (typical) individuals[3]. However, representing the sentence 'Normally, φ's are ψ's' by $\forall x(\varphi(x) > \psi(x))$ involves quantifying both over individuals and worlds and it is not obvious that this can be reduced to quantifying only over individuals. $\Pi x(\varphi, \psi)$ and $\forall x(\varphi > \psi)$ behave similarly only if conditionals (quantifiers) are not iterated:

5.2.3. Theorem. *If φ and ψ are first-order formulas, then*

$$Bin \vdash \Pi x(\varphi, \psi) \;\Leftrightarrow\; Cond \vdash \forall x(\varphi > \psi).$$

[3]We assume for the present purposes that 'normal' and 'typical' have more or less the same meaning: a normal (typical) φ is an object which has all the properties one would expect from a φ-object without having any specific information about this very object. In particular, we are not going to make a distinction between 'normal' as 'average' and 'typical' as 'having all the features of its kind in the most condensed form'.

Proof. The theorem easily follows from the two facts:
Fact 1. If φ, ψ are first-order formulas, $Cond \vdash \forall x(\varphi > \psi) \;\Leftrightarrow\; FOL \vdash \forall x(\varphi \to \psi)$;
Fact 2. If φ, ψ are first-order formulas, $Bin \vdash \Pi x(\varphi, \psi) \;\Leftrightarrow\; FOL \vdash \forall x(\varphi \to \psi)$.
In both cases, the direction from right to left is obvious. Assume that $FOL \not\vdash \forall x(\varphi \to \psi)$. Then there is a first-order model M and an object d such that $M \models \varphi[d]$ and $M \not\models \psi[d]$. Let M' be a model for $Cond$ consisting of just one world w, corresponding to M, and $S(w, [\varphi[d]]) = \{w\}$. Then $M', w \not\models \forall x(\varphi(x) > \psi(x))$. Therefore $Cond \not\vdash \forall x(\varphi(x) > \psi(x))$. Analogously, let M" be a model for Bin with the same domain and interpretation function as that of M, and $\{a : M\text{"} \models \varphi(x/a)\}_T = \{a : M\text{"} \models \varphi(x/a)\}$. Then $M\text{"} \not\models \Pi x(\varphi, \psi)$ and $Bin \not\vdash \Pi x(\varphi, \psi)$. □

If φ or ψ contain conditionals / generalized quantifiers the statement does not hold: for example,

$$Bin \vdash \Pi x(\varphi(x), \Pi y(\psi(y), \varphi(x))),$$

but

$$Cond \not\vdash \forall x(\varphi(x) > \forall y(\psi(y) > \varphi(x))).$$

It is however possible to find a translation from $Cond$ into a two-sorted binary quantifier language, where formulas of the form $\forall x(\varphi(x) > \psi(x))[w]$ are translated as $\forall x \Pi w'(\varphi(x, w'), \psi(x, w') \wedge \top(w))$. To make the embedding faithful, the extensionality axiom of Bin has to be restricted to the formulas with the same world variables; therefore the semantics should again use the relations R_φ instead of T.

Logic SBin. The language of $SBin$ is the same as $\mathcal{L}(\forall\Pi)$, but has two sorts of individual variables: sort 1 (objects): $x_1, x_2, \ldots$ and sort 2 (worlds): w_1, w_2, ... The definition of w.f.f. is standard: if u is a sort 1 or sort 2 variable, and φ and ψ are w.f.f.'s, so are $\forall u\varphi$ and $\Pi u(\varphi, \psi)$.

A model for $SBin$ $M =< O \cup W, \{R_{\varphi(-)} : R_{\varphi(-)} \in \mathcal{R}\}, V >$, where $O \cap W = \emptyset$, O and W are nonempty, O is a domain for the first sort, and W is a domain for the second sort, and V is a valuation. The truth definition for Π is the same as in the minimal logic.

We impose the following conditions on R:

1. $R_{\varphi(\underline{u}\bar{w}\bar{d})}(d, \bar{w}\bar{d}) \Leftrightarrow R_{\varphi(\underline{u}\bar{w}\bar{d})}(d, \bar{w}\bar{d}\bar{e})$, where $\bar{w} \in W$ and $\bar{d}, \bar{e} \in O$ (cf. Proposition 5.1.6);
2. if φ and ψ are formulas with the same world parameters, and
$$\{d : M \models \varphi(u/d)\} = \{d : M \models \psi(u/d)\},$$
then
$$R_{\varphi(\underline{u})}(u, \bar{w}, \bar{d}) \Leftrightarrow R_{\psi(\underline{u})}(u, \bar{w}, \bar{e}),$$
where $\bar{w}$ are world parameters and $\bar{d}, \bar{e} \in O$ (cf. Proposition 5.1.7);
3. for every formula with parameters φ,
$$R_{\varphi(\underline{x}\bar{w}\bar{e})}(d, \bar{w}\bar{e}) \Longrightarrow \varphi(d, \bar{w}\bar{e})$$
(cf. Proposition 5.1.5).

5.2.4. Definition. *The logic SBin is the least set of formulas closed under first-order derivability and the following axioms (where u is a variable of an arbitrary sort):*

B1" $\Pi u(\varphi, \psi_1 \to \psi_2) \to (\Pi u(\varphi, \psi_1) \to \Pi u(\varphi, \psi_2))$, *where* $\Pi u(\varphi, \psi_1)$ *and* $\Pi u(\varphi, \psi_2)$ *have the same free variables of sort 2;*

B2 $\psi \to \Pi u(\varphi, \psi)$, *given that u is not free in* ψ*;*

B3' $\Pi u(\varphi, \psi) \leftrightarrow \Pi y(\varphi, \psi)$, *where* $\varphi(u)$ *and* $\varphi(u/y)$ *are similar and u and y are variables of the same sort;*

B4' $\forall \bar{z} \Pi u(\varphi, \psi) \to \Pi u(\varphi, \forall \bar{z} \psi)$, *given that* $\bar{z}$ *are sort 1 variables which are not free in* φ*;*

B5' $\forall u(\varphi \equiv \psi) \to (\Pi u(\varphi, \chi) \equiv \Pi u(\psi, \chi))$, *where* $\Pi u(\varphi, \chi)$ *and* $\Pi u(\psi, \chi)$ *have the same free variables of sort 2;*

B6 $\Pi u(\varphi, \varphi)$. □

5.2.5. Theorem. *SBin is sound and complete for the semantics described above.*

Proof. What we actually need to prove the embedding result is soundness of *SBin*. We have the monotonicity property restricted with respect to sort 2 variables and unrestricted with respect to sort 1 variables; this axiom (B'') is valid due to the first condition on R. $B2$ and $B3'$ are as in the minimal logic. $B4'$ is valid due to the first condition on R. $B5'$ is valid due to the second condition on R; this property is also restricted with respect to sort 2 variables and unrestricted with respect to sort 1 variables. $B6$ is valid due to the third property of R.

To prove completeness, we translate the language of *SBin* into $\mathcal{L}(\forall\Pi)$ with an additional predicate O (for object sort) in the following way: every atomic formula $P(u_1, \ldots, u_n)$ of *SBin* gets translated as a conjunction of $P(u_1, \ldots, u_n)$ and, for every u_i $(1 \leq i \leq n)$, either $O(u_i)$ or $\neg O(u_i)$, depending on whether u_i was a sort 1 or a sort 2 variable. The translation commutes with logical connectives. It is easy to check that this translation gives rise to an equivalent system. We construct a canonical model for this system in the same way as for B_{min}. The proof that R-conditions 1 – 3 hold is analogous to the proof of Propositions 5.1.6–5.1.7. □

Now we define a translation function * from $L_{\forall >}$ to $L_{\forall\Pi}$ (two- sorted), such that

$$Cond \vdash \varphi \Leftrightarrow SBin \vdash \varphi^*.$$

5.2.6. Definition. $^* : L\forall > \to L_{\forall\Pi}$ is the following function:

- for every n-place predicate symbol P of $L_{\forall >}$, let P^* be an $n+1$ predicate symbol of $L_{\forall\Pi}$; $(P(x_1, \ldots, x_n))^{*i} = P^*(x_1, \ldots, x_n, \mathrm{w}_i)$;
- ^{*i} commutes with Boolean connectives and ordinary quantifiers;
- $(\varphi > \psi)^{*i} = \Pi \mathrm{w}_{\mathrm{i+1}}(\varphi^{*i+1}, \psi^{*i+1} \wedge \top(\mathrm{w}_i))$.
- $^* = ^{*0}$.

Observe that the only free variable of sort 2 of φ^{*i} is w_i, and $\varphi^{*i}(\underline{\mathrm{w}_i})$ and $\varphi^{*j}(\underline{\mathrm{w}_j})$ define the same logical function.

5.2.7. Theorem. $SBin \vdash \varphi^{*0} \Rightarrow Cond \vdash \varphi$

Proof We shall show that there is a transformation mapping an $L_{\forall >}$-model M, a world w_0 and a variable assignment s to an $L_{\forall\Pi}$ model M^* and a variable assignment s^* such that

$$M, s, w_0 \models \varphi \;\Leftrightarrow\; M^* \models^{s^*} (\varphi)^{*0}$$

(this will show that if a formula is consistent with *Cond*, then its translation is consistent with *SBin*).

Given an $L_>$ model $M =< D, W, S, I >$, s and $w_0 \in W$, construct M^* and s^* as follows:

- $O^* = D$;
- $W^* = W$;
- $V^* : \langle d_1, \ldots, d_n, w\rangle \in V^*(P^*) \;\Leftrightarrow\; \langle d_1, \ldots, d_n\rangle \in V_w(P)$;
- s^* agrees with s on O^* and assigns w_0 to w_0;
- $R_{\varphi^{*i}(\underline{\mathrm{w}_i})}(w', w\bar{d})$ *iff* $w' \in S(w, [\varphi(\bar{d})])$; for the rest of the formulas ψ and variables x let $R_{\psi(\underline{x})}$ be empty unless there is at least for one φ^{*i}, $\psi(\underline{x})$ and $\varphi^{*i}(\underline{\mathrm{w}_i})$ are equivalent; then chose such $\varphi^{*i}(\underline{\mathrm{w}_i})$ and put $R_{\psi(\underline{x})} = R_{\varphi^{*i}(\underline{\mathrm{w}_i})}$.

5.2.8. LEMMA.

$$M, w \models \varphi[\bar{d}] \Leftrightarrow M^* \models (\varphi)^{*i}[\mathrm{w}_i/w, \bar{d}]$$

The proof goes by induction on the complexity of φ.

(i) $M, w \models P(d_1, \ldots, d_n) \Leftrightarrow \langle d_1, \ldots, d_n\rangle \in V_w(P) \Leftrightarrow \langle d_1, \ldots, d_n, w\rangle \in V^*(P^*) \Leftrightarrow M^* \models P^*(d_1, \ldots, d_n, \mathrm{w}_i)[w]$

(ii) - (iii) $\neg, \wedge, \forall$: easy (note that in φ and in φ^{*i} ordinary quantifiers bind only sort 1 variables);

(iv) $M, w \models \chi(\bar{d}) > \psi(\bar{e}) \Leftrightarrow \forall w'(w' \in S(w, [\chi(\bar{d})]) \to M, w' \models \psi(\bar{e})) \Leftrightarrow$
$\Leftrightarrow \forall w'(R_{\chi^{*i+1}(\underline{\mathrm{w}_{i+1}})}(w', w, \bar{d}\bar{e}) \to M^* \models \psi(\bar{e}))^{*i+1}[\mathrm{w}_{i+1}/w']) \Leftrightarrow$
$\Leftrightarrow M^* \models \Pi\mathrm{w}_{i+1}(\chi^{*i+1}, \psi^{*i+1})$ □

It remains to show that in this model the R-conditions hold (note that we did not use them in the proof of the lemma above!).

1. holds by the definition of R;
2. From the lemma above follows that $\{w : M^* \models \varphi^{*i}[\bar{d}, \mathrm{w}_i/w]\} = [\varphi(\bar{d})]_M$. Therefore if $\{w : M^* \models \varphi^{*i}(\bar{d}, \mathrm{w}_i/w)\} = \{w : M^* \models \psi^{*k}(\bar{d}, \mathrm{w}_k/w)\}$, then $[\varphi(\bar{d})]_M = [\psi(\bar{e})]_M$, and $w' \in S(w, [\varphi(\bar{d})]) \Leftrightarrow w' \in S(w, [\psi(\bar{e})])$, therefore $R_{\varphi^{*i}(\underline{\mathrm{w}_i})}(w', w\bar{d}) \Leftrightarrow R_{\psi^{*k}(\underline{\mathrm{w}_k})}(w', w\bar{e})$.
   Note that it is here that we need the condition that the world parameters of the two formulas are the same; namely, even if $[\varphi] = [\psi]$, it does not follow that $w' \in S(w, [\varphi]) \Leftrightarrow w' \in S(w'', [\psi])$.
3. Let $R_{\varphi^{*i}(\underline{\mathrm{w}_i})}(w', w\bar{d})$. This means $w' \in S(w, [\varphi(\bar{d})])$. By ID, $M, w' \models \varphi$; by the lemma above, $M^* \models \varphi[\mathrm{w}_i/w']$. □

5.2.9. THEOREM. $Cond \vdash \varphi \Rightarrow SBin \vdash \varphi^{*0}$.

Proof. The proof of the theorem goes by induction on the length of the derivation of φ. The case when φ is an axiom is trivial (it is easy to check that translations of the axioms of *Cond* are theorems of *SBin*). In the inductive step, we show that for every inference rule of *Cond*, if the translation of the premise is provable in *SBin*, then so is the translation of the conclusion.

RCEA Let $\vdash \varphi^{*i} \leftrightarrow \psi^{*i}$. By *Gen*, $\vdash \forall \mathrm{w}_i(\varphi^{*i} \leftrightarrow \psi^{*i})$. Observe that this formula does not contain any world variables free and probably the variables $\mathrm{w}_{i+1}, \ldots, \mathrm{w}_k$ bound (if $\varphi > \psi$ contains k nested conditionals). Applying B3' several times, we can rename the variables as follows: $\mathrm{w}_k \mapsto \mathrm{w}_{k+1}, \ldots, \mathrm{w}_i \mapsto \mathrm{w}_{i+1}$. This will give $\vdash \forall \mathrm{w}_{i+1}(\varphi^{*i+1} \leftrightarrow \psi^{*i+1})$. Observe that $\Pi \mathrm{w}_{i+1}(\varphi^{*i+1}, \chi^{*i+1} \wedge \top(\mathrm{w}_i))$ and $\Pi \mathrm{w}_{i+1}(\psi^{*i+1}, \chi^{*i+1} \wedge \top(\mathrm{w}_i))$ have the same free variables of sort 2, namely w_i. Now axiom B5' gives

$$\vdash \Pi w_{i+1}(\varphi^{*i+1}, \chi^{*i+1} \wedge \top(\mathrm{w}_i)) \leftrightarrow \Pi w_{i+1}(\psi^{*i+1}, \chi^{*i+1} \wedge \top(\mathrm{w}_i)),$$

that is,

$$\vdash (\varphi > \chi)^{*i} \leftrightarrow (\psi > \chi)^{*i}.$$

RCM (analogously). □

The question concerning the existence of a backward translation is open. It is easy to show that a very natural translation of $\Pi x(\varphi, \psi)$ as $\forall x(\varphi > \psi)$ *does not* work. A counterexample (not involving ordinary quantifiers) is:

$$SBin \vdash \Pi x(Q(x), \Pi y(P(y), Q(x)))$$

but

$$Cond \nvdash \forall x(Q(x) > \forall y(P(y) > Q(x)))$$

(by a straightforward semantic argument). If we somehow manage to translate the above formula into propositional conditional logic, still

$$Cond \nvdash q > (p > q).$$

The most essential difference between formalizing defaults by means of binary quantifiers and by means of conditional logic is that in the first one the formulas

$$\forall x(\varphi \rightarrow \psi) \wedge \Pi x(\chi, \varphi) \rightarrow \Pi x(\chi, \psi)$$

and

$$\forall x(\varphi \leftrightarrow \psi) \wedge \Pi x(\chi, \varphi) \rightarrow \Pi x(\chi, \psi),$$

are valid and in the second one the corresponding properties

$$\forall x(\chi > \varphi) \wedge \forall x(\varphi \rightarrow \psi) \rightarrow \forall x(\chi > \psi)$$

and

$$\forall x(\varphi > \chi) \wedge \forall x(\varphi \leftrightarrow \psi) \rightarrow \forall x(\psi > \chi)$$

are not (since $\forall x(\varphi \to (\leftrightarrow)\psi)$ does not imply $[\varphi]_M \subseteq (=)[\psi]_M$). Therefore the first approach can be called (cf. (Morreau 1992)) extensional, and the second one intensional.

A special case of

$$\forall x(\varphi \to \psi) \to \Pi x(\varphi, \psi),$$

the statement that if there are no φ's, then anything is plausible with respect to something being a typical φ:

$$\neg\exists x\varphi \to \Pi x(\varphi, \psi)$$

caused the strongest objections towards the extensional approach in (Morreau 1992).

However, in the two-sorted language

$$\forall w(\varphi \to \psi) \to \Pi w(\varphi, \psi)$$

is not an extensional principle, but a true statement, not expressible in the language of conditional logic. A possible way to make this statement expressible is to add to the conditional language a universal modality $\Box$: $M, w \models \Box\varphi \Leftrightarrow \forall w' M, w' \models \varphi$. Then this principle would become $\Box(\varphi \to \psi) \to (\varphi > \psi)$.

5.3 Defeasible reasoning for *Bin*

It is important not only to reason *about* statements of the form 'typically, A's are B's', but also to be able to make defeasible inferences from them. The simplest example of such inference would be

Typically, A's are B's
a is A
It can be assumed that a is B

(cf. the Tweety example). Usually, such inference is defined semantically, using some notion of minimal models (cf. (McCarthy 1980)).

Badaloni and Zanardo (1990, 1991) argued that being able to make such inferences inside the system without referring explicitely to models would be much better, and proposed a system where this is possible. Badaloni and Zanardo formalize defeasible reasoning using a three-valued logic. In their system constants refer not to single objects but to sets of objects (possible denotations of the constant). $A(a)$ is neither true nor false if some of the objects referred to by a have the property A, and some do not. To express the fact that "*all* we know about a is A", one can say that an object is a possible denotation of a if and only if it has the property A:

$$\forall x(x \stackrel{\diamond}{=} a \leftrightarrow A(x))$$

The following valid inference corresponds in their system to the example of defeasible inference given above:

$$\frac{\begin{array}{c}\Pi x(A(x), B(x)) \\ \forall x(x \stackrel{\diamond}{=} a \leftrightarrow A(x))\end{array}}{\Pi x(x \stackrel{\diamond}{=} a, B(x)).}$$

Below, a simpler definition of defeasible inference in a language with a generalize quantifier is proposed.

The problem of defining defeasible inference resembles the so-called direct inference problem in probabilistic logic (from the knowledge of objective probabilities to subjective degrees of belief in individual events).

In (Bacchus 1990) the latter problem was solved as follows. Let Γ be a finite knowledge base, and $\varphi(a)$ a statement concerning an object a [4]. The subjective degree of belief in $\varphi(a)$ given Γ, $prob(\varphi(a)|\Gamma)$, is computed from the objective probability $P(\{x : \varphi(a/x)\}|\{x : \Gamma(a/x)\})$.

Our definition of defeasible inference is analogous.

5.3.1. DEFINITION. *Let* Γ *be a finite set of formulas, and* $\varphi(x)$ *a formula with one free variable* x. *Then* $\varphi(a)$ *defeasibly follows from* Γ *in Bin (in symbols* $\Gamma \rhd \varphi(a)$*), if* $\Gamma \vdash_{Bin} \Pi x(\Gamma(a/x), \varphi(a/x))$.

This definition is very simple, but it makes the following inferences (often used to test adequacy of systems for non-monotonic reasoning) valid:

Typically, A's are B's
a is A
$\rhd$ a is B

Typically, A's are B's
All C's are A's
Typically, C's are non-B's
a is C
$\rhd$ a is not B

Following a tradition, we call these principle defeasible modus ponens and specificity, respectively.

5.3.2. THEOREM. *In Bin, defeasible modus ponens and specificity are valid, i.e.*
$\Pi x(B(x), F(x)), B(a) \rhd F(a)$;
$\Pi x(B(x), F(x)), \Pi x(P(x), \neg F(x)), \forall x(P(x) \to B(x)), B(a), P(a) \rhd \neg F(a)$.

Proof. The first statement is trivial. As for specificity, we need to show that

$$\Gamma \vdash \Pi x(\Gamma(a/x), \neg F(a/x)),$$

where $\Gamma(a/x)$ is

$$\Pi x(B(x), F(x)) \wedge \Pi x(P(x), \neg F(x)) \wedge \forall x(P(x) \to F(x)) \wedge B(x) \wedge P(x).$$

Clearly,

[4] We assume here that φ is a monadic property. For dealing with polyadic properties one can introduce a polyadic quantifier in an obvious way: quantifying over typical tuples of objects.

$$\Gamma \vdash \forall x(\Pi x(B(x), F(x)) \wedge \Pi x(P(x), \neg F(x)) \wedge \forall x(P(x) \to F(x)) \wedge B(x) \wedge P(x) \leftrightarrow$$
$$\leftrightarrow P(x)).$$

By extensionality and

$$\Gamma \vdash \Pi x(P(x), \neg F(x)),$$

we have

$$\Gamma \vdash \Pi x(\Gamma(a/x), \neg F(a/x)).$$

□

Note that above we essentially made use of the fact that

$$\forall x(P(x) \to B(x)) \vdash \forall x(B(x) \wedge P(x) \leftrightarrow P(x)).$$

Adding to Γ one more formula containing a, for example $Y(a)$, would block the inference, since

$$\Gamma \wedge Y(a) \not\vdash \forall x(\Gamma(a/x) \wedge Y(x) \leftrightarrow P(x)).$$

In general, learning some new information about a, even a completely irrelevant one, blocks the reasoning. This is a disadvantage of syntactic approach in comparison with the theory of circumscription or minimal entailment for conditional logic.

However, there is one important special case of dealing with irrelevant information which presents a difficulty for the minimal models approach but not for this logic. Consider the following example:

Normally, birds fly
There is a non-flying bird (or: Sam does not fly)
Tweety is a bird
It can be assumed, that Tweety flies.

This is not a correct inference if correctness is checked by considering the models where the cardinality of the set of non-flying birds is minimal. There are minimal models with domains consisting only of Tweety, and there it has to be the nonflying bird. (This can be repaired by making the notion of a suitable minimal model more sophisticated.)

Under our definition of defeasible inference, the example above is correct. Namely, if $\Gamma \rhd \varphi(a)$ and we add to Γ some irrelevant information Δ which does not contain a free, $\Gamma \cup \Delta \rhd \varphi(a)$, since

$$\Gamma, \Delta \vdash \forall x(\Gamma(a/x) \wedge \Delta(a/x) \leftrightarrow \Gamma(a/x)).$$

Therefore if irrelevant information is of the form 'There is a non-flying bird' ($\exists x \neg F(x)$) or of the form 'Sam does not fly' ($\neg F(b)$), the reasoning goes thorough: we still are able to derive that Tweety flies.

5.4 Conclusion

This chapter shows that the approach towards unary quantifiers studied in this thesis, can be naturally extended to the binary case. The basic step seems to be just adding one more argument - a formula - into the dependence relation. Here again we meet quantifiers which have a very natural meaning, in particular can be applied for modelling commonsense reasoning about laws with exceptions.

Most of the technical results of the previous chapters can be transferred to the binary case, with conditional logic playing the same role as modal logic for unary quantifiers.

Samenvatting

Dit proefschrift gaat over een quantificatietheorie waar de variabelen van elkaar afhankelijk kunnen zijn: de waarden die een variabele kan nemen, hangen af van de waarden van andere variabelen. De quantoren in zo'n theorie noem ik modale quantoren, omdat ze zich in veel opzichten gedragen als modale operatoren. Namelijk, de betekenis van een quantor is pas gegeven als wij weten in welke punt (bedeling) de formule geevalueerd wordt.

In de inleiding geef ik voorbeelden van modale quantoren: 'gewone' $\forall$ en $\exists$ (geen afhankelijkheid tussen variabelen), gegeneraliseerde quantoren als 'voor bijna alle', 'voor overaftelbaar veel', quantoren in de logica's met beperkte verzamelingen bedelingen, etc. Gegeven zo veel voorbeelden uit verschillende gebieden, lijkt het nuttig de algemene theorie van dergelijke quantoren te bestuderen en de analogie met modale logica te gebruiken om nieuwe stellingen te bewijzen.

In hoofdstuk 2 probeer ik een algemene definitie van modale quantoren te geven en onderzoek ik de relaties tussen verschillende voorbeelden van modale quantoren. Ik presenteer een axiomatisering van een 'basislogica' en geef aan hoe andere quantorlogica's zich verhouden tot deze basislogica.

In hoofdstukken 3 en 4 bestudeer ik gegeneraliseerde quantoren die de volgende semantiek hebben: de quantor loopt over alle objecten die afhankelijk zijn van de parameters van de formule onder de quantor.

In hoofdstuk 3 geef ik een axiomatisering van de minimale logica van dergelijke quantoren en bewijs dat deze beslisbaar is en de interpolatie eigenschap heeft.

In hoofdstuk 4 bestudeer ik extensies van de minimale logica, vooral de correspondentie tussen quantoraxioma's en eigenschappen van de afhankelijkheidsrelatie.

In hoofdstuk 5 breid ik deze technieken uit naar het geval van binaire gegeneraliseerde quantoren, waar de binaire quantor $\Pi x(\varphi, \psi)$ de intuitive betekenis heeft 'voor alle x's die φ-typisch zijn, ψ geldt'. Ik laat zien dat met behulp van deze quantoren het redeneren met plausibele generalisaties geformaliseerd kan worden.

Dit proefschrift moet aantonen dat modale quantoren in veel gebieden op natuurlijke wijze ontstaan en interessante technische eigenschappen hebben, en dat men kan verwachten dat er in veel gebieden ook toepassingen van modale quantoren gevonden kunnen worden.

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