MODAL LOGICS AND MULTIVERSES

Alexander C. Block and Benedikt Löwe

We provide a general framework to study modal logics of multiverses and survey the results on the modal logic of forcing, the modal logic of grounds, and the modal logic of inner models. We also provide new results on the modal logic of symmetric extensions.

1. Introduction

This paper is a survey on modal logic of multiverses. It summarizes known results by Hamkins, Inamdar, Leibman, and the second author about the modal logics of forcing, grounds, and inner models in a general abstract setting. Most results in this survey come from a series of papers co-authored by the second author [**19**, **20**, **18**, **24**]. Exceptions are the discussion of spiked Boolean algebras and the modal logic of c.c.c. forcing in § 7.1 (these results were obtained by Inamdar in his Master's thesis [**23**] supervised by the second author) and the determination of the modal logic of symmetric extensions in § 7.4 due to the first author.

The study of modal logics of set theoretic constructions started with [**15**] and [**19**]. Various other aspects of the modal logic of forcing are considered in [**29**, **21**, **9**, **10**, **30**, **32**, **8**, **7**, **20**, **18**]. In § 2, we explain the natural (and very general) setting for the study of modal logics of forcing: set theoretic multiverses. After providing some general background in modal logic in § 3 and the abstract definition of modal logics for multiverses in § 4, we then discuss the main result of [**19**] in § 5 and provide the general proof strategy adapted to our general multiverse setting in § 6. Finally, in § 7, we survey various generalisations in the multiverse setting.

2. Set theoretic multiverses

The context in which the investigation of the modal logic of forcing is most naturally seen is the investigation of the *set theoretic multiverse*:

The continuum hypothesis CH had been the most prominent open problem of set theory since Hilbert had listed it as the first problem on his list of problems for the 20th century [**22**]. In 1963, Cohen showed that it can not be solved on the basis of ZFC [**3**]. In our context, the following is the most appropriate formulation of this impossibility result:

Theorem 1. *If $M \models \mathsf{ZFC}$, then there are N, N' such that $M \subseteq N$ and $M \subseteq N'$ and*

$$N \models \mathsf{ZFC} + \mathsf{CH} \text{ and } N' \models \mathsf{ZFC} + \neg\mathsf{CH}.$$

The philosophy of mathematics literature abounds with discussions about what Cohen's result means:[1] Does it mean that the question of the continuum hypothesis is an unsolvable problem? Does it mean that ZFC is a deficient axiom system and we need to search for new axioms?

Recently, a new view, called the *multiverse view*, was proposed by Hamkins [**17**]: on the multiverse view, Cohen's result is not a problem, but rather (part of) the answer. Set theorists gain understanding about the truth values of statements in set theory by understanding their behaviour in the multiverse of models of set theory. The fact that CH can be made true or false in an extension of every model of set theory is one important component of understanding the meaning of CH: it contrasts CH with other statements, e.g., statements that are true in all models of set theory, statements that are false in all models of set theory, and statements that can be false, but once they are true, they remain true in forcing extensions:[2]

> The multiverse view [...] holds that there are diverse distinct concepts of set, each instantiated in a corresponding set-theoretic universe, which exhibit diverse set-theoretic truths. Each such universe exists independently in the same Platonic sense that proponents of the universe view regard their universe to exist. [...] In particular, I shall argue [...] that the question of the continuum hypothesis is settled on the multiverse view by our extensive, detailed knowledge of how it behaves in the multiverse. [**17**, p. 416-417]

Studying the multiverse as a mathematical entity is not restricted to researchers who subscribe to the philosophical position of the multiverse view. In his own work on CH, Woodin (who is not an adherent of the multiverse view) has studied a particular substructure of the multiverse, the *generic multiverse*, which is also the main playing field for the study of modal logics of forcing. In general, *multiverses* are collections of models of set theory, linked by relations that reflect that one model was constructed from another. In full generality, the multiverse should contain all models of set theory: this makes it meta-mathematically difficult to describe since it will become a higher-order object.[3] In contrast with these very general and foundational multiverse concepts, we'll be looking at fragments of the multiverse generated by a fixed collection of construction methods. These fragments will live in second order set theory:

Suppose that we have a meta-universe $\mathbf{V}$ of set theory in which we interpret all statements of our meta-language, like, e.g., "$\mathsf{ZFC} \vdash \varphi$". We shall assume that our meta-universe is rich enough to have (countable) set models of set theory, which will become our main objects of study.[4] Of course, this means that we must work

[1]The literature on the philosophical implication of independence results is vast and we shall not be able to do it justice by giving references.

[2]These latter statements will be called *buttons* later whereas CH will be called a *switch*; cf. Definition 11.

[3]However, cf. [**12**].

[4]Note that we use the slightly ambiguous phrase "models of set theory" rather than "models of ZFC" since we want to consider models of ZF in § 7.4 and because the setting is general enough to consider other base set theories BST as well. For most of this paper, the phrase "M is a model of set theory" can be considered as synonymous to "$M \models \mathsf{ZFC}$".

in a relatively strong meta-theory (e.g., ZFC+"ZFC is consistent"), but we shall not be worried by this.

A *model construction* in our meta-universe $\mathbf{V}$ is a class function $\mathbf{C}$ that gets pairs (M, p) as input where $M \in \mathbf{V}$ is a model of set theory and $p \in \mathbf{V}$ is a parameter. Only for some choice of (M, p) will $\mathbf{C}(M, p)$ be an appropriate object. We shall call those pairs $\mathbf{C}$-*good*. In particular, we shall assume that if (M, p) is $\mathbf{C}$-good, then $\mathbf{C}(M, p)$ is a model of set theory. We may add further restrictions, though. If $V \in \mathbf{V}$ is a model of set theory, we can define the $\mathbf{C}$-multiverse of V by closing under the operation $M \mapsto \mathbf{C}(M, p)$ (of course, this recursion happens in the meta-universe $\mathbf{V}$):

$$\begin{aligned}
\mathrm{Mult}_0^{\mathbf{C}}(V) &:= \{V\}, \\
\mathrm{Mult}_{n+1}^{\mathbf{C}}(V) &:= \mathrm{Mult}_n^{\mathbf{C}}(V) \cup \{\mathbf{C}(M,p)\,;\, M \in \mathrm{Mult}_n^{\mathbf{C}}(V), p \in \mathbf{V} \text{ and} \\
&\qquad\qquad (M,p) \text{ is } \mathbf{C}\text{-good}\}, \text{ and} \\
\mathrm{Mult}^{\mathbf{C}}(V) &:= \bigcup_{n \in \mathbb{N}} \mathrm{Mult}_n^{\mathbf{C}}(V)
\end{aligned}$$

The $\mathbf{C}$-multiverse comes with a natural accessibility relation generated by $\mathbf{C}$ where we say that a model M $\mathbf{C}$-*accesses* another model N (in symbols, $M \leq_{\mathbf{C}} N$) if there is a p such that (M, p) is $\mathbf{C}$-good and $N = \mathbf{C}(M, p)$. Depending on $\mathbf{C}$, it could be that this relation is not transitive. In that case, we could also consider the transitive closure of $\leq_{\mathbf{C}}$ as an accessibility relation.

A natural generalization of this is a multiverse produced by more than one construction method. We shall not deal with these multiverses in this paper, but give the appropriate definitions as a reference. Suppose that $\mathcal{C}$ is a collection of construction methods $\mathbf{C} : (M, p) \mapsto \mathbf{C}(M, p)$ as before. Then

$$\begin{aligned}
\mathrm{Mult}_0^{\mathcal{C}}(V) &:= \{V\}, \\
\mathrm{Mult}_{n+1}^{\mathcal{C}}(V) &:= \mathrm{Mult}_n^{\mathcal{C}}(V) \cup \{\mathbf{C}(M,p)\,;\, \mathbf{C} \in \mathcal{C},\ M \in \mathrm{Mult}_n^{\mathcal{C}}(V), p \in \mathbf{V}, \text{ and} \\
&\qquad\qquad (M,p) \text{ is } \mathbf{C}\text{-good}\}, \text{ and} \\
\mathrm{Mult}^{\mathcal{C}}(V) &:= \bigcup_{n \in \mathbb{N}} \mathrm{Mult}_n^{\mathcal{C}}(V)
\end{aligned}$$

Such a multiverse has several accessibility relations: for each $\mathbf{C} \in \mathcal{C}$, we would have the notion $\leq_{\mathbf{C}}$ of $\mathbf{C}$-accessibility as defined above. In addition to these paths corresponding to a single construction method, we can consider mixed paths: if $\mathcal{C}^* \subseteq \mathcal{C}$ is nonempty, we say that a model M $\mathcal{C}^*$-*accesses* another model N (in symbols, $M \leq_{\mathcal{C}^*} N$) if there is a finite sequence $(M_i\,;\, i \leq n)$ with $M_0 = M$, $M_n = N$, and for each $i < n$ there is some $\mathbf{C} \in \mathcal{C}^*$ such that $M_{i+1} = \mathbf{C}(M_i, p)$ for some p such that (M_i, p) is $\mathbf{C}$-good.[5] These accessibility relations can be studied either individually or with a focus on their interactions (the results in § 7.2, in particular, Theorem 14, can be seen as an example of a bimodal setting with two modalities).

[5]Note that if $\mathcal{C}^* = \{\mathbf{C}\}$, then $\leq_{\mathcal{C}^*}$ is the transitive closure of $\leq_{\mathbf{C}}$.

3. Modal logic and its applications

The propositional language of modal logic consists of $\wedge, \vee, \neg, \bot, \Box$ and $\Diamond$ and propositional variables from a set Prop. Typically we read $\Box\varphi$ as "it is necessary that φ" and $\Diamond\varphi$ as "it is possible that φ". The following semantics for modal logic is called *Kripke semantics* and was introduced by Kripke in [**26**]:

We call any directed graph (K, E) a *Kripke frame*; if $v : \text{Prop} \to \wp(K)$ is a function (called a *valuation function*), we shall call (K, E, v) a *Kripke model*. For any Kripke model (K, E, v) and $x \in K$, we define a satisfaction relation for modal formulas recursively as follows:

$$\begin{aligned}
(K,E,v,x) \models p &\iff x \in v(p),\\
(K,E,v,x) \models \varphi \wedge \psi &\iff (K,E,v,x) \models \varphi \text{ and } (K,E,v,x) \models \psi,\\
(K,E,v,x) \models \varphi \vee \psi &\iff (K,E,v,x) \models \varphi \text{ or } (K,E,v,x) \models \psi,\\
(K,E,v,x) \models \neg\varphi &\iff (K,E,v,x) \not\models \varphi,\\
(K,E,v,x) \models \Box\varphi &\iff \text{for all } y \text{ such that } xEy,\\
&\qquad\quad \text{we have that } (K,E,v,y) \models \varphi,\\
(K,E,v,x) \models \Diamond\varphi &\iff \text{there is a } y \text{ such that } xEy\\
&\qquad\quad \text{and } (K,E,v,y) \models \varphi.
\end{aligned}$$

If φ is a modal formula, we say that it is *valid* in (K, E, v) if $(K, E, v, x) \models \varphi$ for every $x \in K$. We say that it is valid in (K, E) if it is valid in every Kripke model on (K, E). If $\mathcal{K}$ is a class of Kripke frames, we write $\mathbf{ML}(\mathcal{K})$ for the set of modal formulas valid in all frames $(K, E) \in \mathcal{K}$. Clearly, if $\mathcal{K} \subseteq \mathcal{K}'$, then $\mathbf{ML}(\mathcal{K}') \subseteq \mathbf{ML}(\mathcal{K})$.

A set Λ of modal formulas is called a *modal logic* if it contains all propositional tautologies, is closed under modus ponens (i.e., if $\varphi \in \Lambda$ and $\varphi \to \psi \in \Lambda$, then $\psi \in \Lambda$), uniform substitution (of propositional variables in a formula by arbitrary formulas) and necessitation (if $\varphi \in \Lambda$, then $\Box\varphi \in \Lambda$). It is called *normal* if it contains all instances of the scheme

$$\text{(K)} \qquad \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi).$$

It satisfies *duality* if it contains all instances of the scheme

$$\text{(Dual)} \qquad \Diamond\varphi \leftrightarrow \neg\Box\neg\varphi.$$

If is easy to see that for any class $\mathcal{K}$ of Kripke frames, the set $\mathbf{ML}(\mathcal{K})$ is a normal modal logic satisfying duality.

Theorem 2 (Kripke; [**26**]). *We let* $\mathbf{K}$ *be the smallest normal modal logic satisfying duality and* $\mathcal{K}$ *be the class of all Kripke frames. Then* $\mathbf{ML}(\mathcal{K}) = \mathbf{K}$.

The following modal formulas are not theorems of **K**, as can be easily seen using Theorem 2 by providing a Kripke model that invalidates them:

$$\text{(T)} \qquad \Box\varphi \to \varphi$$

$$\text{(4)} \qquad \Box\varphi \to \Box\Box\varphi$$

$$\text{(.2)} \qquad \Diamond\Box\varphi \to \Box\Diamond\varphi$$

$$\text{(5)} \qquad \Diamond\Box\varphi \to \varphi$$

The modal logic **T** is the minimal normal modal logic satisfying duality and including all instances of (T). Similarly, **S4** is the minimal normal modal logic satisfying duality, extending **T** and including all instances of (4), **S4.2** is the minimal normal modal logic satisfying duality, extending **S4** and including all instances of (.2), and **S5** is the minimal normal modal logic satisfying duality, extending **S4.2** and including all instances of (5).

The fundamental idea of Kripke semantics is that properties of the relation E of a Kripke frame (K, E) directly correspond to the validity of modal formulas. This idea is due to Kripke [**26**] and was the beginning of a rich literature of completeness theorems for modal logics using the technique of canonical models (cf. [**2**, § 4.2]). Let us give a number of examples [**2**, Theorems 4.23, 4.28 & 4.29]:

Theorem 3. *Let $\mathcal{K}_{\mathrm{r}}$ be the class of all Kripke frames (K, E) with a reflexive relation E, $\mathcal{K}_{\mathrm{rt}}$ be the class of all Kripke frames (K, E) with a reflexive and transitive relation E (also called* partial pre-orders*), $\mathcal{K}_{\mathrm{rtd}}$ be the class of all Kripke frames (K, E) with a reflexive, transitive and directed relation E, and $\mathcal{K}_{\mathrm{rts}}$ be the class of all Kripke frames (K, E) with a reflexive, transitive and symmetric relation E (also called* equivalence relations*). Then*

(1) $\mathbf{T} = \mathbf{ML}(\mathcal{K}_{\mathrm{r}})$,
(2) $\mathbf{S4} = \mathbf{ML}(\mathcal{K}_{\mathrm{rt}})$,
(3) $\mathbf{S4.2} = \mathbf{ML}(\mathcal{K}_{\mathrm{rtd}})$, *and*
(4) $\mathbf{S5} = \mathbf{ML}(\mathcal{K}_{\mathrm{rts}})$.

It is important to note that Theorem 3 does not mean that every Kripke model of a theory has to have the relational properties of the corresponding frame class: e.g., there can be models $(K, E, v) \models \mathbf{S4}$ where the relation E is not transitive. This is important in our context, since we shall interpret the multiverses of § 2 not simply as Kripke frames, but as Kripke frames with a restriction on the valuations on them we want to consider.[6]

A second crucial property of Kripke semantics is that for many examples of modal logics (in particular, the modal logics **K**, **T**, **S4**, **S4.2**, and **S5** defined above), the class of finite frames with the appropriate property is enough to define the modal logic. Theorems like this are usually proved with the technique of *filtration* (cf. [**13**, pp. 267-268] or [**2**, pp. 77–82]). All of the modal logics listed in Theorem 3 have the finite model property and the finite frame property:[7]

[6] Structures like this are known as *general frames*, cf. [**2**, § 1.4].

[7] Cf. [**2**, Definitions 2.27 & 3.23], respectively. For normal modal logics, the finite model property and the finite frame property are equivalent [**2**, Theorem 3.28].

Theorem 4. *Let $\mathcal{K}_{\mathrm{r}}^{\mathrm{fin}}$ be the class of all finite Kripke frames (K,E) with a reflexive relation E, $\mathcal{K}_{\mathrm{rt}}^{\mathrm{fin}}$ be the class of all finite Kripke frames (K,E) with a reflexive and transitive relation E, $\mathcal{K}_{\mathrm{rtd}}^{\mathrm{fin}}$ be the class of all finite Kripke frames (K,E) with a reflexive, transitive and directed relation E, and $\mathcal{K}_{\mathrm{rts}}^{\mathrm{fin}}$ be the class of all finite Kripke frames (K,E) with a reflexive, transitive and symmetric relation E. Then*

(1) $\mathbf{T} = \mathbf{ML}(\mathcal{K}_{\mathrm{r}}^{\mathrm{fin}})$,
(2) $\mathbf{S4} = \mathbf{ML}(\mathcal{K}_{\mathrm{rt}}^{\mathrm{fin}})$,
(3) $\mathbf{S4.2} = \mathbf{ML}(\mathcal{K}_{\mathrm{rtd}}^{\mathrm{fin}})$, *and*
(4) $\mathbf{S5} = \mathbf{ML}(\mathcal{K}_{\mathrm{rts}}^{\mathrm{fin}})$.

4. Modal Logics of Multiverses

As in § 2, we fix a meta-universe $\mathbf{V}$ and a construction method $\mathbf{C}$. For every model of set theory V, we can consider $(\mathrm{Mult}^{\mathbf{C}}(V), \leq_{\mathbf{C}})$ as a Kripke frame. In our set theoretic context, we are not interested in the purely algebraic properties of this frame, but want to restrict our attention to valuations that connect with the set theoretic properties of the multiverse under investigation. A valuation function $v : \mathrm{Prop} \to \wp(\mathrm{Mult}^{\mathbf{C}}(V))$ is called *set theoretic* if there is an assignment $p \mapsto \sigma_p$ assigning a sentence in the language of set theory to any propositional variable in such a way that $v(p) = \{N \in \mathrm{Mult}^{\mathbf{C}}(V)\,;\, N \models \sigma_p\}$. We call a Kripke model $(\mathrm{Mult}^{\mathbf{C}}(V), \leq, v)$ *set theoretic* if v is a set theoretic valuation function. We can now define the *modal logic of the multiverse* $\mathrm{Mult}^{\mathbf{C}}(V)$ by

$$\mathbf{MLC}_V := \{\varphi\,;\, \varphi \text{ is satisfied at } V \text{ in all set theoretic Kripke models on } (\mathrm{Mult}^{\mathbf{C}}(V), \leq_{\mathbf{C}})\}.$$

The following is a syntactic reformulation of the same concept (these definitions are due to Inamdar and the second author [**24**]): Call a function H a *translation* if it assigns to each $p \in \mathrm{Prop}$ a sentence $H(p)$ of the language of set theory. Then for each model of set theory V and every translation H, we define by recursion in the meta-universe $\mathbf{V}$:

$$\begin{aligned}
V \models^H_{\mathbf{C}} p &:\iff V \models H(p),\\
V \models^H_{\mathbf{C}} \varphi \wedge \psi &:\iff V \models^H_{\mathbf{C}} \varphi \text{ and } M \models^H_{\mathbf{C}} \psi,\\
V \models^H_{\mathbf{C}} \varphi \vee \psi &:\iff V \models^H_{\mathbf{C}} \varphi \text{ or } M \models^H_{\mathbf{C}} \psi,\\
V \models^H_{\mathbf{C}} \neg\varphi &:\iff \text{not } V \models^H_{\mathbf{C}} \varphi, \text{ and}\\
V \models^H_{\mathbf{C}} \Box\varphi &:\iff \forall W (V \leq_{\mathbf{C}} W \to W \models^H_{\mathbf{C}} \varphi).
\end{aligned}$$

Proposition 5. *Let $V \in \mathbf{V}$ be a model of set theory and φ be a formula in the language of modal propositional logic. Then the following are equivalent:*

(1) *The formula φ is satisfied at V in every set theoretic Kripke model $(\mathrm{Mult}^{\mathbf{C}}(V), \leq_{\mathbf{C}}, v)$, and*
(2) *for every translation H, we have $V \models^H_{\mathbf{C}} \varphi$.*

Proof. This is proved by induction on the complexity of the formula φ. $\square$

We can now define the *modal logic of the construction method* $\mathbf{C}$ by

$$\mathbf{MLC} := \bigcap\{\mathbf{MLC}_V ; V \models \mathsf{BST} \text{ is a countable model}\}$$

where BST is our basic set theory (i.e., ZFC in almost all cases; cf. Footnote 4).

5. The Modal Logic of Forcing

We now phrase the modal logic of forcing, as investigated by Hamkins and the second author in [**19**] in terms of our general framework. In this section, our base theory is ZFC. As before, we fix a meta-universe $\mathbf{V}$ and let $\mathbf{F}$ be the operation that assigns to a pair (M, G) the generic extension $\mathbf{F}(M, G) := M[G]$ if there is some forcing partial order $\mathbb{P} \in M$ such that G is $\mathbb{P}$-generic over M. Then for every model $V \in \mathbf{V}$, we consider the Kripke frame $(\mathrm{Mult}^{\mathbf{F}}(V), \leq_{\mathbf{F}})$.

5.1. Some words of caution about multiverses as structures. Since we are emphasizing the second order set theory aspect of multiverses in this paper, it is interesting to have a closer look at these forcing multiverses $\mathrm{Mult}^{\mathbf{F}}(V)$ that are concrete structures from the point of view of the meta-universe $\mathbf{V}$. The first observation is that they may trivialize: suppose that the meta-universe is a model of $\mathbf{V}$=$\mathbf{L}$+"κ is an inaccessible cardinal". Let $V := \mathbf{L}_\kappa$. Suppose γ is the statement "there is a Cohen real over $\mathbf{L}$", then this statement is false everywhere in the multiverse $\mathrm{Mult}^{\mathbf{F}}(V)$. This means that the structure $\mathrm{Mult}^{\mathbf{F}}(V)$ does not capture the intuition of what is possible to achieve by forcing very well at all; in fact, in the case of the forcing multiverse, we should restrict ourselves to countable models V in order to have that $\mathrm{Mult}^{\mathbf{F}}(V)$ is a structure that represents the "worlds made possible by forcing" (since in this case, generics for all forcing partial orders in V exist in $\mathbf{V}$ by [**27**, Lemma VII.2.3]).

But even if V is countable, we have to be careful with our structure $\mathrm{Mult}^{\mathbf{F}}(V)$: if $\mathbb{P}$ and $\mathbb{Q}$ are forcing partial orders in some $W \in \mathrm{Mult}^{\mathbf{F}}(V)$ and G and H are $\mathbb{P}$-generic and $\mathbb{Q}$-generic over W, respectively, we cannot assume that $G \times H$ would be $\mathbb{P} \times \mathbb{Q}$-generic. In fact, $G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic if and only if G and H are mutually generic, i.e., G is $\mathbb{P}$-generic over $W[H]$ and H is $\mathbb{Q}$-generic over $W[G]$ [**27**, VIII.1.4]. But by a simple argument due to Woodin, if V is countable and $x \in \omega^\omega$ witnesses the countability of V, then there are two Cohen reals c_1 and c_2 over V such that any model of set theory containing both c_1 and c_2 will also contain x (we call the generic extensions $V[c_1]$ and $V[c_2]$ *non-amalgamable*). Consequently, there can be no forcing extension of V that contains both c_1 and c_2. This means that the preorder $(\mathrm{Mult}^{\mathbf{F}}(V), \leq_{\mathbf{F}})$ is not directed. Since we are restricting our attention to set theoretic valuations, this does not preclude the scheme (.2) from being valid in the modal logic of forcing: suppose that $\mathbb{P}$ forces some statement σ to be necessarily true and $\mathbb{Q}$ forces some other statement τ to be necessarily true, then there are pairs of mutual generics and $\mathbb{P} \times \mathbb{Q}$ forces $\sigma \wedge \tau$ to be necessarily true, even if for some choices of generics G and H, the extensions $V[G]$ and $V[H]$ may be non-amalgamable in the above sense.

As a third example, let us consider a countable non-wellfounded V of set theory. For countable transitive models of set theory, we define the generic extension as the Mostowski collapse of the quotient structure on the set of names, so generic extensions are again transitive countable models of set theory. In particular, if $\mathbb{P}$ in V and $\dot{\mathbb{Q}}$ in $V^{\mathbb{P}}$ is a name for a forcing partial order and if H is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V, then H splits into a $\mathbb{P}$-generic filter G_0 over V and a $\mathbb{Q}$-generic filter G_1 over $V[G_0]$ such that $V[G_0][G_1] = V[H]$.

If V is not wellfounded, then we shall not be able to produce a transitive model corresponding to its generic extension, i.e., we have to work with the quotient structure on the set of names itself. This means that in the above situation, $V[H]$ will be isomorphic to $V[G_0][G_1]$, but will not be the same set. As a consequence, the multiverse is not literally transitive, but only transitive up to isomorphism. A similar observation can be made with respect to reflexivity. For more on forcing over non-wellfounded models, cf. [**4**].

5.2. The syntactic description of the modal logic of forcing. The characterisation via translations from Proposition 5 becomes highly relevant in this context since the Forcing Theorem [**27**, Theorem VII.3.6] tells us that for every sentence σ of the language of set theory, we have that σ is true in every generic extension of a countable model M of set theory if and only if

$$M \models \forall\mathbb{P}(\Vdash_{\mathbb{P}} \varphi)$$

where $\mathbb{P}$ ranges over all partial orders and $\Vdash_{\mathbb{P}}$ is the forcing relation recursively defined with parameter $\mathbb{P}$ [**27**, Definition VII.3.3]. This means that the second order definition of $\models^H_{\mathbf{C}}$ from § 4 can be replaced by a purely syntactic first order transformation of formulas:

A function $H \in \mathbf{V}$ that takes a formula φ of the language of modal logic and assigns a sentence of the language of set theory $H(\varphi)$ is called a *Hamkins translation* if

$$\begin{aligned} H(\bot) &= \bot, \\ H(\neg\varphi) &= \neg H(\varphi), \\ H(\varphi \vee \psi) &= H(\varphi) \vee H(\psi), \text{ and} \\ H(\Box\varphi) &= \forall\mathbb{P}(\Vdash_{\mathbb{P}} H(\varphi)). \end{aligned}$$

Proposition 6. *Suppose that V is a countable model of set theory and H is a Hamkins translation. Then $H_0 := H\restriction\mathrm{Prop}$ is a translation in the sense of* § 4 *and furthermore, for all formulas φ in the language of modal propositional logic, the following are equivalent:*

(1) $V \models H(\varphi)$ and
(2) $V \models^{H_0}_{\mathbf{F}} \varphi$.

Proof. Induction on the complexity of formulas, where the only nontrivial step is $\varphi = \Box\psi$:

$$\begin{aligned} V \models^{H_0}_{\mathbf{F}} \varphi &\iff \forall W(V \leq_{\mathbf{F}} W \to W \models^{H_0}_{\mathbf{F}} \psi) \\ &\iff \forall W(V \leq_{\mathbf{F}} W \to W \models H(\psi)) \\ &\iff V \models \forall\mathbb{P}(\Vdash_{\mathbb{P}} H(\psi)) \\ &\iff V \models H(\varphi), \end{aligned}$$

where the second line is the induction hypothesis and for the third equivalence we use the Forcing Theorem and the fact that for any countable model of set theory there are generic filters for every forcing. Furthermore we use that any forcing extension of a countable model is countable again, which follows from the fact that there is a surjection from the class of names in the ground model to the elements of the forcing extension. $\square$

Using Propositions 5 and 6, we obtain that

$$\begin{aligned}\mathbf{MLF} &= \bigcap\{\mathbf{MLF}_V\,;\, V \models \mathsf{ZFC} \text{ is a countable model}\}\\ &= \{\varphi\,;\, V \models^H_{\mathbf{F}} \varphi \text{ for all countable models } V \models \mathsf{ZFC} \text{ and all translations } H\}\\ &= \{\varphi\,;\, \mathsf{ZFC} \vdash H(\varphi) \text{ for all Hamkins translations } H\}.\end{aligned}$$

Theorem 7 (Hamkins)**.** *For every model $V \models \mathsf{ZFC}$, $\mathbf{S4.2} \subseteq \mathbf{MLF}_V$. Consequently, $\mathbf{S4.2} \subseteq \mathbf{MLF}$. There are models W and W' such that $\mathbf{S5} \subseteq \mathbf{MLF}_W$ and $\mathbf{S5} \not\subseteq \mathbf{MLF}_{W'}$.*[8]

The validity of $\mathbf{S4.2}$ is closely connected to closure properties of the multiverses $\mathrm{Mult}^{\mathbf{F}}(V)$; e.g., the validity of (4) related to the *iteration lemma* in the theory of forcing [**27**, VIII.5.5], which essentially expresses that forcing is transitive (up to isomorphism), and the validity of (.2) is related to the *product lemma* [**27**, Theorem VIII.1.4].[9]

Theorem 8 (Hamkins-Löwe)**.** *The modal logic of forcing is exactly* $\mathbf{S4.2}$*.*

In § 6, we shall sketch the general proof strategy for Theorem 8 in our setting. In order to motivate why results like Theorem 8 are relevant, we show that it implies that there can be no pure forcing proof for the mentioned result by Hamkins and Stavi-Väänänen (Theorem 7 and Footnote 8).

Corollary 9. *There can be no pure forcing proof of the existence of a model N such that $\mathbf{S5} \subseteq \mathbf{MLF}_N$.*

Proof. By a pure forcing proof, we mean that ZFC proves that you can force the modal logic of forcing to include $\mathbf{S5}$. This statement itself can be rendered as $\Diamond(\Diamond\Box p \to p) \in \mathbf{MLF}$. But $\Diamond(\Diamond\Box p \to p)$ is not a theorem of $\mathbf{S4.2}$ in contradiction to Theorem 8. $\square$

6. The general proof strategy

Theorem 8 is the prototypical result we are going to discuss in this paper in the multiverse setting. We give a description of the proof strategy for Theorem 8 in such a way that it can be generalized to modal logics of multiverses for any other construction method $\mathbf{C}$:

- **Step 1.:** We first guess a candidate modal logic Λ of which we aim to show that $\mathbf{MLC} = \Lambda$.
- **Step 2.:** For the lower bound, we show $\Lambda \subseteq \mathbf{MLC}$ by showing the validity of all axioms of Λ in every model $V \models \mathsf{BST}$.

[8] The existence of a model W such that $\mathbf{S5} \subseteq \mathbf{MLF}_W$ was proved independently in [**33**].

[9] Cf. [**29**] and [**18**, Theorem 7].

Step 3a.: For the upper bound, we first find an appropriate class $\mathcal{K}$ of Kripke frames such that $\Lambda = \mathbf{ML}(\mathcal{K})$.

Step 3b.: We prove a *transfer lemma* by giving a condition Ξ in terms of so-called *control statements* (cf., e.g., Lemma 12 (3)) under which modal truth from one of the frames in $\mathcal{K}$ can be transferred to a model $V \models \mathsf{BST}$ validating Ξ.

Step 3c.: We find a model $V \models \mathsf{BST}$ validating Ξ.

Concerning the lower bounds (**Step 2**), these are typically connected with the closure properties of the relation $\leq$ in the investigated multiverse. For instance, if $\leq_{\mathbf{C}}$ is reflexive (up to isomorphisms), then $\mathbf{T} \subseteq \mathbf{MLC}$. This was investigated for classes of forcing in [**29**] (cf. also [**18**, Theorem 7]).

Hamkins, Leibman, and the second author also provided the abstract background for the transfer lemmas needed for **Step 3b** [**18**]: Let (K, E) be a Kripke frame with $g \in K$ and $v : \text{Prop} \to \wp(K)$ be a valuation function. Furthermore, let $V \models \mathsf{BST}$ and let H be a translation. We say that that H *transfers truth between* (K, E, v, g) *and* V *with respect to* $\mathbf{C}$ if for every modal formula φ we have that

$$(K, E, v, g) \models \varphi \iff V \models^H_{\mathbf{C}} \varphi.$$

If (K, E) is a finite Kripke frame, V a model of set theory, and $g_0 \in K$. We say that $\{\Phi_g\,;\, g \in K\}$ is a $\mathbf{C}$*-labelling of* (K, E, g_0) *for* V if each of the Φ_g is a sentence in the language of set theory such that

(1) in each model of set theory $W \in \text{Mult}^{\mathbf{C}}(V)$ exactly one Φ_g is true,
(2) $V \models \Phi_{g_0}$
(3) if $h, h' \in K$ and $W \models \Phi_h$ then hEh' if and only if there is some W' with $W \leq_{\mathbf{C}} W'$ and $W' \models \Phi_{h'}$.

We give an example of how this general proof strategy works in terms of the modal logic of forcing given in § 5:

The candidate modal logic for **MLF** was **S4.2** (**Step 1**) and Theorem 7 had established that $\mathbf{S4.2} \subseteq \mathbf{MLF}$ (**Step 2**). Concerning **Step 3a**, Theorem 3 already provides us with a class of Kripke models, viz. the class of finite directed partial pre-orders. However, this class can be slightly refined in order to allow us to prove the transfer lemma.

As usual, pre-orders (reflexive and transitive relations) carry a natural notion of equivalence: If $(K, \leq)$ is a pre-order, let $\equiv$ be defined by $v \equiv w$ if and only if $v \leq w \leq v$. Clearly, $\leq$ is well-defined on the $\equiv$-equivalence classes, so with a slight abuse of notation, we can consider the quotient structure $(K/\!\equiv, \leq)$. For any class $\mathcal{O}$ of ordered structures, we say that a pre-order $(K, \leq)$ is *pre-*$\mathcal{O}$ if $(K/\!\equiv, \leq) \in \mathcal{O}$. E.g., pre-lattices or pre-Boolean algebras are pre-orders such that the natural quotient by $\equiv$ becomes a lattice or a Boolean algebra, respectively.

Theorem 10 (Hamkins-Löwe; [**19**, Theorem 11])**.** *A modal formula is in* **S4.2** *if and only if it is valid in all Kripke frames whose edge relation is a finite pre-Boolean algebra.*

In **Step 3b**, we need to find the right control statements that allow us to prove a transfer theorem for finite pre-Boolean algebras. The notions isolated by Hamkins and the second author are buttons and switches.

Definition 11. A sentence b in the language of set theory is a *button* if it is necessarily possibly necessary;[10] it is *unpushed* in a model V of set theory, if $V \models \neg b$. A sentence s is called a *switch* if both s and $\neg s$ are necessarily possible.

An example button in the case of forcing is $\omega_1^{\mathbf{L}} < \omega_1$ which is unpushed in $\mathbf{L}$, but can be pushed by collapsing $\omega_1^{\mathbf{L}}$ by forcing (note that it remains true in all further forcing extensions; cf. Footnote 2); an example switch is CH which can be forced to be true or false over every model of set theory (cf. Theorem 1).

Suppose that V is a model of set theory. If B is a set of buttons and S is a set of switches, then $B \cup S$ is called an *independent family of buttons and switches over* V if all buttons are unpushed in V and for every V^*, $B' \subseteq B$ and $S' \subseteq S$, if $V \leq_{\mathbf{C}} V^*$ and B^* is the set of buttons pushed in V^*, then there is some V' such that $V^* \leq_{\mathbf{C}} V'$ and $B^* \cup B'$ is the set of buttons pushed in V' and S' is the set of switches true in V'.

Lemma 12 (Transfer Lemma)**.** *(1) If there is a $V \models \mathsf{ZFC}$ such that for every finite pre-Boolean algebra $(K, \leq)$, every valuation function v, and every $g \in K$, there is a Hamkins translation that transfers truth between $(K, \leq, v, g)$ and V with respect to $\mathbf{F}$, then the modal logic of forcing is contained in* **S4.2***.*

(2) Let $(K, \leq)$ be a finite pre-Boolean algebra, V a model of set theory, and $g_0 \in K$ such that there is an $\mathbf{F}$-labelling of $(K, \leq, g_0)$ for V. Then for every valuation function v there is a a Hamkins translation H that transfers truth between $(K, \leq, v, g_0)$ and V with respect to $\mathbf{F}$.

(3) Let $(K, \leq)$ be finite pre-Boolean algebra with $g \in K$ and V a model of set theory with an independent family of buttons and switches of sufficient size[11] *then there is an $\mathbf{F}$-labelling of $(K, \leq, g)$ for V.*

Proof. (1) Suppose ψ is not a theorem of **S4.2**. Then by Theorem 10, there is a finite pre-Boolean algebra $(K, \leq)$, a valuation v, and a $g \in K$ such that

$$(K, \leq, v, g) \models \neg\psi.$$

By the assumption, there is a Hamkins translation H transferring truth between $(K, \leq, v, g)$ and V, so $V \models \neg H(\psi)$. Consequently, $\psi \notin$ **MLF**.

The proofs of (2) and (3) can be found in [**18**, Lemma 9] and [**18**, Theorem 13], respectively. □

Finally, in **Step 3c**, Hamkins and the second author proved that any model satisfying **V**=**L** has an independent family with infinitely many buttons and switches, thus completing the proof [**19**, Lemma 6.1]. It should be mentioned that we do not know how to prove independence of the buttons given in the original proof of [**19**, Lemma 6.1] (this was observed in [**32**]), but numerous other infinite families of buttons have been provided of which independence can be proved (cf. [**18**, § 4]).[12]

[10] I.e., if H is a translation such that $H(p) = b$, then for every model V of set theory, $V \models^H_{\mathbf{C}} \Box\Diamond\Box p$.

[11] It is sufficient that the family has as many buttons as the quotient Boolean algebra $K/{\equiv}$ has atoms and as many switches as the largest $\equiv$-class in K has elements.

[12] The independence of the original list of buttons remains an interesting problem: Inamdar observed that a theorem by Abraham from [**1**] implies a certain fragment of independence [**23**, § 6].

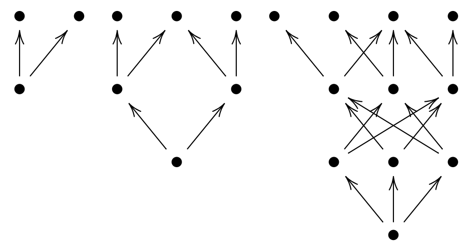

Figure 1. The first three spiked Boolean algebras

7. Generalizations and open questions

7.1. Restricting the class of forcings. The first generalisation that was proposed in the original paper [**19**] was fragments of the forcing multiverse generated by particular types of forcing. Fix any class Γ of forcing partial orders and let $\mathbf{F}_\Gamma$ be the construction that only allows generic extensions with partial order $\mathbb{P} \in \Gamma$. Examples are the classes of proper forcings or c.c.c. forcings.

As mentioned before, the lower bounds in **Step 2** of our proof strategy are very closely related to closure properties of the class Γ [**18**, Theorem 7]. Hamkins, Leibman and the second author have provided a number of other control statements that give rise to proofs of upper bounds and the appropriate transfer theorems, showing that the modal logic of proper forcing and of c.c.c. forcing are contained within a modal logic called **S4.3**, which properly extends **S4.2** [**18**, Corollary 33 (1)].[13]

The most interesting open problem in this direction is to determine the modal logic of c.c.c. forcing $\mathbf{MLF}_{\text{c.c.c.}}$. Here, we know that the axiom (.2) is not valid [**19**, Theorem 34], so the upper bound given by Hamkins, Leibman and the second author cannot be optimal. The currently best known result is due to Inamdar: if $(B, \leq)$ is a finite Boolean algebra with n atoms and n co-atoms $\{c_i\,;\, 1 \leq i \leq n\}$, we define a *spiked Boolean algebra* by adding n additional nodes $\{d_i\,;\, 1 \leq i \leq n\}$ such that for every $b \in B$, we have $b \leq d_i$ if and only if $b \leq c_i$. The spiked Boolean algebras for the Boolean algebras with two, four and eight elements can be seen in Figure 1. We let **S4.sBA** be the set of all modal assertions which are true on all Kripke models whose frame is a finite pre-spiked Boolean algebra. It can be checked that **S4.sBA** is properly contained in **S4.2** [**23**, Theorem 121].

Theorem 13 (Inamdar; [**23**, Theorem 150])**.** $\mathbf{S4} \subseteq \mathbf{MLF}_{\text{c.c.c.}} \subseteq \mathbf{S4.sBA}$.

7.2. Reversing the arrows. The operation **F** corresponds to the relation of "being a generic extension". The inverse of this relation is the relation of "being a ground model". More precisely, the operation **G** is defined on pairs (M, G) such that there is some inner model $N \subseteq M$ such that there is some partial order $\mathbb{P} \in N$ and we have that G is $\mathbb{P}$-generic over N and $M = N[G]$. In that case,

[13] The modal logic **S4.3** is generated from **S4.2** by including all instances of $\Diamond p \wedge \Diamond q \rightarrow \Diamond((p \wedge \Diamond q) \vee (q \wedge \Diamond p))$. It is characterized by the class of finite pre-linear orders [**2**, Exercise 4.33, Theorem 4.96, & Lemma 6.40].

$\mathbf{G}(M,G) := N$. The *modal logic of grounds* $\mathbf{MLG}_V$ is defined by using the general definitions of § 4 applied to $\mathbf{G}$.[14]

The multiverses $\mathrm{Mult}^{\mathbf{G}}(V)$ can look quite different from the multiverses $\mathrm{Mult}^{\mathbf{F}}(V)$: if V is a countable transitive model of set theory, then $\mathrm{Mult}^{\mathbf{F}}(V)$ will not have a $\leq_{\mathbf{F}}$-largest element since for each $M \in \mathrm{Mult}^{\mathbf{F}}(V)$, there are many proper generic extensions; on the other hand, if V is a generic extension of $\mathbf{L}^V$, then $\mathbf{L}^V$ will be a $\leq_{\mathbf{G}}$-largest element in the multiverse $\mathrm{Mult}^{\mathbf{G}}(V)$. In such a $\leq_{\mathbf{G}}$-largest element of the multiverse, all statements that are true are necessarily true, and hence

$$\text{(Top)} \qquad \Diamond((\Box\varphi \leftrightarrow \varphi) \wedge (\Box\neg\varphi \leftrightarrow \neg\varphi))$$

is a valid principle in V. Note that (Top) is not a theorem of $\mathbf{S5}$ and compare this to the result by Hamkins and the second author that for every model V of set theory , $\mathbf{MLF}_V \subseteq \mathbf{S5}$ [**19**, Theorem 15].

The phenomenon just discussed is closely related to what Hamkins calls *set theoretic geology* [**16**, **11**]: a $\leq_{\mathbf{G}}$-largest element of the multiverse $\mathrm{Mult}^{\mathbf{G}}(V)$ would be a *bedrock* [**11**, Definition 2]; if $\mathrm{Mult}^{\mathbf{G}}(V)$ contains a bedrock, then (Top) is in $\mathbf{MLG}_V$. Using the so-called *bottomless models* of Reitz, Hamkins and the second author have been able to show that $\mathbf{S4.2}$ is an upper bound for the modal logic of grounds. In fact,

Theorem 14 (Hamkins-Löwe; [**20**, Theorems 7–10])**.** *There are models V, W, and U of set theory such that*

(1) $\mathbf{MLF}_V = \mathbf{S4.2}$ *and* $\mathbf{MLG}_V = \mathbf{S4.2}$*,*
(2) $\mathbf{MLF}_W = \mathbf{S4.2}$ *and* $\mathbf{MLG}_W = \mathbf{S5}$*, and*
(3) $\mathbf{MLF}_U = \mathbf{S5}$ *and* $\mathbf{MLG}_U = \mathbf{S4.2}$*.*

Furthermore there is no *model V of set theory such that* $\mathbf{MLF}_V = \mathbf{S5}$ *and* $\mathbf{MLG}_V = \mathbf{S5}$.

With the upper bound established, we wonder about the lower bound for $\mathbf{MLG}$. It is easy to check that $\mathbf{S4} \subseteq \mathbf{MLG}$, but the scheme (.2) is equivalent to the following set theoretic statement:

(DDG) Let V be a model of set theory, and W and U two grounds of V, i.e., $V = W[G] = U[H]$ for some generic filters G and H. Then there is some model K which is a ground of both W and U, i.e., there are K-generic filters G^* and H^* such that $K[G^*] = W$ and $K[H^*] = U$.

Clearly, if a multiverse has a $\leq_{\mathbf{G}}$-largest element, then (DDG) is trivially true (e.g., in multiverses generated from a generic extension of $\mathbf{L}$ or any other core model). There are multiverses without such a $\leq_{\mathbf{G}}$-largest element (Reitz's bottomless model from [**31**]) where we still know that (DDG) is true (cf. [**20**, Theorem 6]). However, the general status of (DDG) is unknown and is an interesting question in its own right. E.g., is it consistent to have a grounds multiverse with two separate bedrocks?

[14] As in the case of the modal logic of forcing, the modal logic of grounds admits a purely syntactic definition via syntactic translation operations due to the Laver-Woodin theorem [**28**]; cf. also [**11**, Theorem 8].

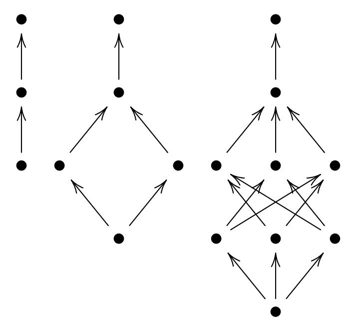

Figure 2. Three inverted lollipops

7.3. Inner models. The following example is similar and yet different to the modal logic of grounds from § 7.2. We consider the operation that takes a model V of set theory, a formula Φ, and some parameters p to form the V-class $X := \{x \in V \,;\, \Phi(x,p)\}$, defined in our meta-universe $\mathbf{V}$. We say that the pair $(V,(\Phi,p))$ is $\mathbf{I}$-good if X is an inner model of V, i.e., a model of set theory having the same ordinals as M. As before, the construction operation $\mathbf{I}$ defines multiverses for every model V of set theory in our meta-universe $\mathbf{V}$.

For a model V, there is a close relationship between $\mathrm{Mult}^{\mathbf{I}}(V)$ and $\mathrm{Mult}^{\mathbf{G}}(V)$. Clearly, all grounds are inner models, but moreover, Grigorieff's theorem tells us more about the relationship between $\leq_{\mathbf{G}}$ and $\leq_{\mathbf{I}}$:

Theorem 15 (Grigorieff; [**14**])**.** *Let V be a transitive model of* ZFC*, W a forcing extension of V and U a transitive model of* ZFC *such that $V \subseteq U \subseteq W$. Then U is a forcing extension of V.*

Consequently, the relation $\leq_{\mathbf{G}}$ is an initial segment of the relation $\leq_{\mathbf{I}}$ in the following sense: if $V \leq_{\mathbf{I}} U$, $U \leq_{\mathbf{I}} W$, and $V \leq_{\mathbf{G}} W$, then $V \leq_{\mathbf{G}} U$.

So, $\mathrm{Mult}^{\mathbf{G}}(V)$ nicely embeds into $\mathrm{Mult}^{\mathbf{I}}(V)$, but in general, the inner model multiverse can be quite different from the grounds multiverse: the inner models multiverse always has a $\leq_{\mathbf{I}}$-largest element, viz. $\mathbf{L}^V$ (of course, $\mathbf{L}^V$ is only in $\mathrm{Mult}^{\mathbf{G}}(V)$ if V is a generic extension of $\mathbf{L}^V$). As discussed in § 7.2, this means that the axiom scheme (Top) is valid in $\mathrm{Mult}^{\mathbf{I}}(V)$ for all models of set theory V.

We let **S4.2Top** be the modal logic obtained from **S4.2** by adding all instances of (Top). The above argument tells us that **S4.2Top** is a lower bound for the modal logic of inner models **MLI**. In order to show that it is also an upper bound, we need to follow **Step 3a**, **Step 3b**, and **Step 3c** of § 6.

Inamdar and the second author did this analysis in [**24**]. A partial pre-order is *topped* if it has a unique largest element; it is called an *inverted lollipop* if it is topped and after removal of the largest element, the remainder is a pre-Boolean algebra (cf. Figure 2 to see three inverted lollipops). If $\mathcal{K}^{\mathrm{fin}}_{\mathrm{IL}}$ is the class of finite inverted lollipops, then $\mathbf{S4.2Top} = \mathbf{ML}(\mathcal{K}^{\mathrm{fin}}_{\mathrm{IL}})$ [**24**, Theorem 6].

Theorem 16 (Inamdar-Löwe; [**24**, Theorem 19])**.** **MLI** $=$ **S4.2Top**.

7.4. Symmetric extensions. In this section, we shall work in ZF as the base theory (cf. Footnote 4). As before, we fix a meta-universe $\mathbf{V}$ of a reasonably strong set theory and remind the reader of the basic definitions of symmetric extensions:[15]

Let Γ be a group. Then a set $\mathcal{F} \subseteq \{\Delta\,;\,\Delta \text{ is a subgroup of } \Gamma\}$ is called a *normal filter over* Γ iff it satisfies the following four properties:[16]

(1) $\Gamma \in \mathcal{F}$;
(2) if $\Delta \in \mathcal{F}$ and Δ' is a subgroup of Γ such that $\Delta \subseteq \Delta'$, then $\Delta' \in \mathcal{F}$;
(3) for any two $\Delta, \Delta' \in \mathcal{F}$, also $\Delta \cap \Delta' \in \mathcal{F}$;
(4) for any $\Delta \in \mathcal{F}$ and $g \in \Gamma$ we have that $g\Delta g^{-1} := \{fdf^{-1}\,;\,d \in \Delta\} \in \mathcal{F}$.

Let V be a transitive model[17] of ZF, $\mathbb{P} \in M$ a forcing partial order, $\Gamma \in V$ a subgroup of the group $\mathrm{Aut}(\mathbb{P})$ of automorphisms of $\mathbb{P}$ and $\mathcal{F} \in V$ a normal filter over Γ. We call a $\mathbb{P}$-name τ *hereditarily $\mathcal{F}$-symmetric* iff $\mathrm{sym}_\Gamma(\tau) := \{g \in \Gamma\,;\, g(\tau) = \tau\} \in \mathcal{F}$ and for every $\langle \sigma, p\rangle \in \tau$ we have that σ is already hereditarily $\mathcal{F}$-symmetric.

Let $G \subseteq \mathbb{P}$ be a generic filter over V. Then we set

$$V_{\mathcal{F}}[G] := \{\tau_G\,;\,\tau \text{ is hereditarily } \mathcal{F}\text{-symmetric}\},$$

where τ^G denotes the interpretation of the name τ in the forcing extension $V[G]$. We call $V_{\mathcal{F}}[G]$ the *symmetric extension of V via $\mathcal{F}$ and G.*

It is not hard to see that in the notation of the above definition $V \subseteq V_{\mathcal{F}}[G] \subseteq V[G]$. Also, $V_{\mathcal{F}}[G] \models \mathsf{ZF}$. Contrary to forcing extensions, symmetric extensions do not necessarily satisfy the axiom of choice whenever the ground model does.

Let V be a model of set theory. We say that $(V, (G, \mathcal{F}))$ is $\mathbf{S}$-good if there is a partial order $\mathbb{P} \in M$ for which G is $\mathbb{P}$-generic over V, Γ is a subgroup of $\mathrm{Aut}(\mathbb{P})$, and $\mathcal{F}$ is a normal filter over Γ. We then define the operation $\mathbf{S}(V, (G, \mathcal{F})) := V_{\mathcal{F}}[G]$. The multiverse defined using the operation $\mathbf{S}$ is called the *symmetric extension multiverse*. Note that every forcing extension $V[G]$—where G is $\mathbb{P}$-generic over V—is a symmetric extension by taking the trivial normal filter containing all subgroups of $\mathrm{Aut}(\mathbb{P})$.

As in the case of the forcing multiverse and the grounds multiverse, we can provide a purely syntactic description of the modal logic of symmetric extensions:

By restricting quantifiers to hereditarily symmetric names (cf. [**6**]) we can define for every formula $\varphi(v_1, \ldots, v_n)$ of set theory a formula $\Vdash_{v_{n+1}, v_{n+2}} \varphi(v_1, \ldots, v_n)$ in the language of set theory such that for any transitive model $V \models \mathsf{ZF}$, any forcing $\mathbb{P} \in M$ and any normal filter $\mathcal{F} \in M$ over a subgroup of $\mathrm{Aut}(\mathbb{P})$ and for names $\sigma_1, \ldots, \sigma_n \in M^{\mathbb{P}}$ we have an analogue of the forcing theorem:

Proposition 17. *The following are equivalent:*

(1) $V \models\ \Vdash_{\mathbb{P},\mathcal{F}} \varphi(\sigma_1, \ldots, \sigma_n)$ and
(2) for every generic $G \subseteq \mathbb{P}$: $V_{\mathcal{F}}[G] \models \varphi(\sigma_1^G, \ldots, \sigma_n^G)$.

[15] For more details, cf. [**25**].

[16] We alert the reader to the fact that this normality of filters is entirely different from the normality condition of ultrafilters in the theory of measurable cardinals.

[17] The restriction to transitive models here and in the following only has the purpose to simplify notation. We could work with arbitrary (countable) models in this section, with only the cost of replacing identity of models with identity up to isomorphism in some places.

Using this, we can now say that a function S from the set of modal formulas into the set of sentences of set theory is a *symmetric extension translation* iff

$$\begin{aligned} S(\bot) &= \bot \\ S(\neg\varphi) &= \neg S(\varphi) \\ S(\varphi \vee \psi) &= S(\varphi) \vee S(\psi) \\ S(\Box\varphi) &= \forall\mathbb{P}\forall\mathcal{F}(\Vdash_{\mathbb{P},\mathcal{F}} S(\varphi)), \end{aligned}$$

where $\mathbb{P}$ ranges over partial orders and $\mathcal{F}$ ranges over normal filters over a subgroup of $\mathrm{Aut}(\mathbb{P})$. We then get that

$$\begin{aligned} \mathbf{MLS} &= \bigcap\{\mathbf{MLS}_V\,;\, V \models \mathsf{ZF} \text{ is a countable model}\} \\ &= \{\varphi\,;\, \mathsf{ZF} \vdash S(\varphi) \text{ for all symmetric extension translations } S\}. \end{aligned}$$

Theorem 18 (Block)**.** $\mathbf{MLS} = \mathbf{S4.2}$.

The remainder of this paper will be a proof of Theorem 18. We start with **Step 2**, the lower bound.

Theorem 19 (Grigorieff; [**14**])**.** *If V is a transitive model of* ZF*, $\mathbb{P} \in V$ a forcing, $\mathcal{F} \in V$ a normal filter over a subgroup of* $\mathrm{Aut}(\mathbb{P})$ *and $G \subseteq \mathbb{P}$ a generic filter over V, then $V[G]$ is a forcing extension of $V_{\mathcal{F}}[G]$.*

Lemma 20. $\mathbf{S4.2} \subseteq \mathbf{MLS}$.

Proof. The relation $\leq_{\mathbf{S}}$ is reflexive and transitive (up to isomorphisms), so we get that $\mathbf{S4} \subseteq \mathbf{MLS}$ (cf. [**18**, Theorem 7]). In order to show (.2), we fix a sentence σ and suppose that in some countable model V, σ is possibly necessary, i.e., there is some $\mathbb{P}$ and some filter $\mathcal{F}$ over a subgroup of $\mathrm{Aut}(\mathbb{P})$ such that $\Vdash_{\mathbb{P},\mathcal{F}} \Box\sigma$.[18] Towards a contradiction, we suppose that σ is also possibly necessarily false, i.e., there is some $\mathbb{Q}$ and some filter $\mathcal{G}$ over a subgroup of $\mathrm{Aut}(\mathbb{Q})$ such that $\Vdash_{\mathbb{Q},\mathcal{G}} \Box\neg\sigma$. Then consider the forcing $\mathbb{P} \times \mathbb{Q}$; since V was countable, we find a $\mathbb{P} \times \mathbb{Q}$-generic $G \times H$ such that G is $\mathbb{P}$-generic over $V[H]$ and H is $\mathbb{Q}$-generic over $V[G]$. By our assumption, $V_{\mathcal{F}}[G] \models \Box\sigma$ and $V_{\mathcal{G}}[H] \models \Box\neg\sigma$, but by Theorem 19, we have that $V_{\mathcal{F}}[G] \leq_{\mathbf{S}} V[G \times H]$ and $V_{\mathcal{G}}[H] \leq_{\mathbf{S}} V[G \times H]$. Contradiction. □

Now, for the upper bound we shall give an independent families of infinitely many switches and buttons with respect to symmetric extensions over any model of $\mathbf{V}{=}\mathbf{L}$. Here and in the following, we use the notation $\mathbf{L}$ for one such model, arbitrarily chosen. As in [**18**, Theorem 15], we can construct independent switches from a uniform independent family of buttons, indexed by the ordinals.

Let

$$b(\alpha) := \aleph_{\alpha+1}^{\mathbf{L}} \text{ is not a cardinal or } T_\alpha^{\mathbf{L}} \text{ is not an } \aleph_{\alpha+1}^{\mathbf{L}}\text{-Suslin tree},$$

where $T_\alpha^{\mathbf{L}}$ denotes the $\mathbf{L}$-least well-pruned $\aleph_{\alpha+1}^{\mathbf{L}}$-Suslin tree[19] . These statements are due to Friedman, Fuchino, and Sakai who proved in ZFC that they form an independent family of buttons with respect to forcing [**8**]. If we live in an inner

[18]Once more, by $\Box\sigma$, we mean $S(\Box p)$ for any symmetric translation S such that $S(p) = \sigma$.

[19]For definitions, cf. [**27**, Chapter II].

model of a set forcing extension of $\mathbf{L}$, then a proper class of the buttons $b(\alpha)$ are unpushed. Let λ_b be the least limit ordinal such $b(\lambda_b)$ is not pushed. For $n \in \omega$, we now define buttons $b_n := b(n)$ and switches $s_n := b(\lambda_b + n + 1)$. Clearly, the b_n are buttons with respect to symmetric extensions and are unpushed in $\mathbf{L}$. Also we have the following:

Lemma 21. *Let M be a symmetric extension of $\mathbf{L}$. Working in M, let $A := \{\alpha < \lambda_b\,;\, b(\alpha)\}$ and let $B \supset A$ be an arbitrary superset of ordinals. Then there is a forcing $\mathbb{P}(B)$ such that $\Vdash_{\mathbb{P}(A,B)} (\check{B} = \{\alpha\,;\, b(\alpha)\})$.*

Proof. We work in M. Let $\mathbb{P}(B) := \prod^{\text{fin}}_{\alpha \in B \setminus A} T^{\mathbf{L}}_{\alpha}$, where $\prod^{\text{fin}}$ denotes the finite support product. Then for any $\alpha \in B$ we have that $\Vdash_{\mathbb{P}(B)} b(\check{\alpha})$.

Now we note that for any $\alpha \in \mathbf{Ord} \setminus A$, the forcing $T^{\mathbf{L}}_{\alpha}$ is $< \aleph_{\alpha+1}$-closed and $\aleph_{\alpha+2}$-Knaster (using the canonical well-ordering of $T^{\mathbf{L}}_{\alpha}$ in $\mathbf{L}$). Now let $\beta \in \mathbf{Ord} \setminus B$. Then we get that $\prod_{\alpha \in B \setminus A, \alpha < \beta} T^{\mathbf{L}}_{\alpha}$ is $\aleph_{\beta+1}$-Knaster and $\prod_{\alpha \in B \setminus A, \alpha > \beta} T^{\mathbf{L}}_{\alpha}$ is $< \aleph_{\beta+1}$-closed. Therefore $\mathbb{P}(B)$ preserves the Suslinness of β by [**5**, Proposition 5], where we use a well-ordering of $T^{\mathbf{L}}_{\beta}$. Hence we have for any $\beta \in \mathbf{Ord} \setminus B$ that $\Vdash_{\mathbb{P}(B)} \neg b(\beta)$, which shows the claim. □

As a consequence we get that the s_n are indeed switches over $\mathbf{L}$ with respect to symmetric forcing. Moreover we get the following:

Lemma 22. *The set $\{b_n\,;\, n \in \omega\} \cup \{s_n \;\; n \in \omega\}$ is an independent set of buttons and switches over $\mathbf{L}$ with respect to symmetric extensions.*

Proof. That $\{b_n; n \in \omega\}$ is an independent family of buttons over $\mathbf{L}$ with respect to symmetric extensions follows directly from Lemma 21. Now we work in a symmetric extension M of $\mathbf{L}$ and let $S \subseteq \omega$. Then let μ be the least limit ordinal such that for all $\alpha \geq \mu$, $b(\alpha)$ is not pushed, and let $B := \{\alpha\,;\, \omega \leq \alpha < \mu\} \cup \{\mu + n + 1\,;\, n \in S\}$. Then by Lemma 21 in any forcing extension of M via $\mathbb{P}(B)$, we have $\lambda_b = \mu$ and the switches in $\{s_n\,;\, n \in S\}$ hold and the switches in $\{s_n\,;\, n \in \omega \setminus S\}$ do not hold. □

This lemma together with Lemma 12 for symmetric forcing and Lemma 20 establishes Theorem 18.

References

1. Uri Abraham, *On forcing without the continuum hypothesis*, Journal of Symbolic Logic **48** (1983), no. 3, 658–661.
2. Patrick Blackburn, Maarten de Rijke, and Yde Venema, *Modal logic*, Cambridge Tracts in Theoretical Computer Science, vol. 53, Cambridge University Press, Cambridge, 2001.
3. Paul Cohen, *The independence of the continuum hypothesis*, Proceedings of the National Academy of Sciences of the United States of America **50** (1963), 1143–1148.
4. Paul Corazza, *Forcing with non-wellfounded models*, Australasian Journal of Logic **5** (2007), 20–57.
5. René David, *Some results on higher suslin trees*, Journal of Symbolic Logic **55** (1990), no. 2, 526–536.
6. Ioanna M. Dimitriou, *Symmetric models, singular cardinal patterns, and indiscernibles*, Ph.D. thesis, Rheinische Friedrich-Wilhelms-Universität Bonn, 2012.

7. Leo Esakia and Benedikt Löwe, *Fatal Heyting algebras and forcing persistent sentences*, Studia Logica **100** (2012), no. 1-2, 163–173.
8. Sy Friedman, Sakaé Fuchino, and Hiroshi Sakai, *On the set-generic multiverse*, 2012, Submitted.
9. Gunter Fuchs, *Closed maximality principles: implications, separations and combinations*, Journal of Symbolic Logic **73** (2008), no. 1, 276–308.
10. ______, *Combined maximality principles up to large cardinals*, Journal of Symbolic Logic **74** (2009), no. 3, 1015–1046.
11. Gunter Fuchs, Joel David Hamkins, and Jonas Reitz, *Set-theoretic geology*, Annals of Pure and Applied Logic **166** (2015), no. 4, 464–501.
12. Victoria Gitman and Joel David Hamkins, *A natural model of the multiverse axioms*, Notre Dame Journal of Formal Logic **51** (2010), no. 4, 475–484.
13. Valentin Goranko and Martin Otto, *Model theory of modal logic*, Handbook of Modal Logic (Patrick Blackburn, Johan F. A. K. van Benthem, and Frank Wolter, eds.), Studies in Logic and Practical Reasoning, vol. 3, Elsevier, 2007, pp. 249–330.
14. Serge Grigorieff, *Intermediate submodels and generic extensions in set theory*, Annals of Mathematics **101** (1975), no. 3, 447–490.
15. Joel David Hamkins, *A simple maximality principle*, Journal of Symbolic Logic **68** (2003), no. 2, 527–550.
16. ______, *Some second order set theory*, Logic and its applications. Proceedings of the Third Indian Conference, ICLA 2009, held in Chennai, January 7–11, 2009 (R. Ramanujam and Sundar Sarukkai, eds.), Lecture Notes in Computer Science, vol. 5378, Springer-Verlag, 2009, pp. 36–50.
17. ______, *The set-theoretic multiverse*, Review of Symbolic Logic **5** (2012), 416–449.
18. Joel David Hamkins, George Leibman, and Benedikt Löwe, *Structural connections between a forcing class and its modal logic*, Israel Journal of Mathematics (2015), in press.
19. Joel David Hamkins and Benedikt Löwe, *The modal logic of forcing*, Transactions of the American Mathematical Society **360** (2008), no. 4, 1793–1817.
20. ______, *Moving up and down in the generic multiverse*, Logic and Its Applications, 5th International Conference, ICLA 2013, Chennai, India, January 10-12, 2013, Proceedings (Kamal Lodaya, ed.), Lecture Notes in Computer Science, vol. 7750, Springer-Verlag, 2013, pp. 139–147.
21. Joel David Hamkins and W. Hugh Woodin, *The necessary maximality principle for c.c.c. forcing is equiconsistent with a weakly compact cardinal*, Mathematical Logic Quarterly **51** (2005), no. 5, 493–498.
22. David Hilbert, *Mathematische Probleme. Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900*, Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-physikalische Klasse **3** (1900), 253–?297.
23. Tanmay C. Inamdar, *On the modal logics of some set-theoretic constructions*, Master's thesis, Universiteit van Amsterdam, 2013, ILLC Publications MoL-2013-07.
24. Tanmay C. Inamdar and Benedikt Löwe, *The modal logic of inner models*, submitted.
25. Thomas J. Jech, *The axiom of choice*, Studies in Logic and the Foundations of Mathematics, vol. 75, North-Holland, 1973.
26. Saul A. Kripke, *A completeness theorem in modal logic*, Journal of Symbolic Logic **24** (1959), 1–14.
27. Kenneth Kunen, *Set theory. an introduction to independence proofs*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland, 1980.
28. Richard Laver, *Certain very large cardinals are not created in small forcing extensions*, Annals of Pure and Applied Logic **149** (2007), no. 1–3, 1–6.

29. George Leibman, *Consistency strengths of modified maximality principles*, Ph.D. thesis, City University of New York, 2004.

30. ———, *The consistency strength of* $\mathrm{MP}_{\mathrm{CCC}}(\mathbb{R})$, Notre Dame Journal of Formal Logic **51** (2010), no. 2, 181–193.

31. Jonas Reitz, *The Ground Axiom*, Ph.D. thesis, City University of New York, September 2006.

32. Colin Jakob Rittberg, *The modal logic of forcing*, Master's thesis, Westfälische Wilhelms-Universität Münster, 2010.

33. Jonathan Stavi and Jouko Väänänen, *Reflection principles for the continuum*, Logic and algebra, Contemporary Mathematics, vol. 302, American Mathematical Society, Providence, RI, 2002, pp. 59–84.

(A.C.B. & B.L.) Fachbereich Mathematik, Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany
E-mail address: alexander.block@uni-hamburg.de

(B.L.) Institute for Logic, Language and Computation, Universiteit van Amsterdam, Postbus 94242, 1090 GE Amsterdam, The Netherlands
E-mail address: b.loewe@uva.nl

(B.L.) Corpus Christi College, University of Cambridge, Trumpington Street, Cambridge CB2 1RH, United Kingdom

Acknowledgements. This survey paper is the written version of the second author's tripartite tutorial entitled "The Modal Logic of Forcing" at the workshop *Infinitary Combinatorics in Set Theory and its Applications* held at the *Research Institute for Mathematical Sciences* in Kyōto that was given on 11, 12, and 13 November 2014. The second author would like to thank the organizers of this workshop for the invitation. Both authors would like to thank Tanmay Inamdar for reading an earlier version of the paper and providing comments as well as Nathan Bowler for spotting an incorrect lemma in an earlier version of this paper.