A Program Calculus for Dynamical Systems

MSC THESIS *(Afstudeerscriptie)*

written by

PATRICK WEIGERT
(born May 19, 1998 in Bad Muskau, Germany)

under the supervision of BENNO VAN DEN BERG and LEVIN HORNISCHER, and submitted to the Examinations Board in partial fulfillment of the requirements for the degree of

MSC IN LOGIC

at the *Universiteit van Amsterdam.*

DATE OF THE PUBLIC DEFENSE:
June 12, 2023

MEMBERS OF THE THESIS COMMITTEE:
Benno van den Berg (supervisor)
Nick Bezhanishvili
Malvin Gattinger (chair)
Levin Hornischer (supervisor)
Tobias Kappé

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

We develop a type theory for dynamical systems seen as computational processes. Denotational semantics for dynamical systems are constructed from "finitary approximations" to them, via domain theory. We study the resulting categories of domains, characterise them via certain free constructions, and derive two type theories from them, using techniques from categorical logic. The first theory, obtained from a free construction, is designed to be sound and complete with respect to a finitary fragment of the category; the second is a sound infinitary extension that can express all computable functions. We finish by sketching two applications: a programming language for manipulating dynamical systems, and a logical framework for reasoning about properties of those systems. The work exemplifies a general approach for deriving formal reasoning tools about a computational situation from a suitable category of domains.

A NOTE ON STYLE

A few remarks on the proof style I employ in this thesis seem appropriate. I take inspiration from [20]. Here is an example. First, a lemma.

PROPOSITION 0.0.0. If a divides b and c, then a divides $b - c$. ⟨div-diff⟩

THEOREM 0. There are infinitely many prime numbers. ⟨inf-many-primes⟩

Proof. We show that for any finite list of prime numbers, there exists a prime number unequal to every prime number in that list. It then follows immediately that the set of prime numbers is infinite. Thus let $p_1, \ldots, p_n$ be a finite list of prime numbers. Let $q := p_1 \cdot \ldots \cdot p_n + 1$. By the definitions of prime and composite numbers, we can distinguish three cases: q is prime, or q is composite, or $q = 1$. In each case, we show that there exists a prime number not among $p_1, \ldots, p_n$.

[$q = 1$]

Then $n = 0$ and 2 is a prime number not on the list.

[q is prime]

For all p_i in the list, we have $q > p_i$ by construction, so in particular $q \neq p_i$. Since q is prime by assumption, this proves the claim.

[q is composite]

Then, the set of divisors of q greater than 1 must have a least divisor p. This is a prime number since any divisor p' of p with $1 < p' \leq p$ also divides q and then $p' = p$.

Let p_i be an arbitrary prime from the list. We prove that $p \neq p_i$ by assuming the opposite and showing that p then divides $q-1$ and q. From these claims, it follows by ⟨`div-diff`⟩ that p divides $q-(q-1)=1$, contradicting the fact that p is prime. Thus suppose $p = p_i$.

⟨ p divides $q-1$ ⟩

By definition, $q-1 = p_1 \cdot \ldots \cdot p_n$, in which $p = p_i$ is a factor.

⟨ p divides q ⟩

By construction of p. □

Different kinds of statements, such as propositions, or properties in a definition, may have a label associated to them (like in the margins above), which I can then use to refer to them in a proof. For readers on a screen, the corresponding references are hyperlinked to the place of definition; and readers on paper may find the list of labels in the Index useful.

The layout of a proof visually roughly follows its natural deduction tree using indentation. Substatements are stated inside ⟨ angular brackets ⟩ and followed by an indented proof, while assumptions are stated inside [square brackets], with a vertical line indicating the scope of the assumption.

Structuring the proof in this way makes it—in my opinion—easier to follow, and in addition it allows me to be explicit without boring readers who would prefer a lower level of detail, as they can simply skip more indented passages. An unintended side-effect of the generous use of whitespace associated with this structuring is that the thesis covers more pages than it would under a conventional layout.

Another reason for that lies in my attempt to adhere to established typographic principles. The lines are rather short, to make the dense material easier to digest [6]. To reduce the margins, I then chose a relatively large font size. Since the proportions of the type area should match the proportions of the page [36], the choice of line length also necessitates a shorter height of the text on the page. The hope is that the document is both easy to work with on a screen as well as aesthetically pleasing when printed. The large font size should help maintain readability when printing multiple pages onto one page of paper.

The typeface, Concrete Roman,° with its *slab serifs* evokes the typewriter feel of early foundational works in domain theory such as [33] while still being suitable for body text. The thesis was typeset using LuaTeX in the luatex-plain format.

°designed by Donald Knuth using METAFONT, featuring in *Concrete Mathematics* [10]

LIST OF SYMBOLS

We work with several different kinds of structures, which nevertheless have similar operations defined on them. We keep those corresponding operations apart notationally to aid understanding. Here is an overview.

subsets of X		general poset		category		logic	
subset	$\subseteq$	less-than-or-equal	$\sqsubseteq$	order on morphisms	$\trianglelefteq$	entailment	$\leq$
intersection	$\cap$	meet	$\sqcap$	product	$\times$	conjunction	$\wedge$
union	$\cup$	join	$\sqcup$	coproduct	$+$	disjunction	$\vee$
full set	X	top element	$\top$	terminal object	$\mathbf{1}$	truth	1
empty set	$\emptyset$	bottom element	$\perp$	initial object	$\mathbf{0}$	falsity	0

Throughout the thesis, we frequently commit the following abuses of notation. When there is no danger of confusing an element of a set X with a subset thereof, we drop the curly braces around singleton subsets in the powerset $\mathcal{P}(X)$, pretending that instead of $\mathcal{P}(X)$ we are working with the set $X \cup \{A \subseteq X \mid |A| \neq 1\}$. A relation (or function) R from a set X into a set Y is identified with its image function $\mathcal{P}(X) \to \mathcal{P}(Y)$ on the powersets which sends each $A \subseteq X$ to the set $\{y \in Y \mid \exists x \in A.\, x \mathrel{R} y\}$ of R-successors of *some* $x \in A$. In accordance with the previous remark, $R(x)$ is then used to denote $R(\{x\})$ for $x \in X$. For the set $\{R(x) \mid x \in A\}$, we use the notation $R[A]$, so that $R(A) = \bigcup R[A]$.

Furthermore, we often write pairs (x,y) as $x \times y$, $x \otimes y$, $x \to y$, etc., depending on the context in which they are used.

The following is a list of other notational devices used in this thesis in the order that they appear in.

$f\,;g$	composition of morphisms: $(f\,;g)(x) = g(f(x))$ ("f, then g")	18
$[\![A]\!]$	categorical interpretation of a type (or judgment) A	18
$\mathcal{P}(X)$	powerset of a set X	23
$\max p$	set of maximal elements $m \sqsupseteq p$ in a poset	23
$\uparrow a$	up-set $\{b \in P \mid b \sqsupseteq a\}$ in a poset P	24
$\mathcal{I}(\sigma)$	ideal completion $\{S \subseteq \bigcup \sigma \mid \forall F \subseteq_{\text{fin}} S.\, F \in \sigma\}$	26
$\mathcal{N}(x)$	neighbourhood filter $\{U \in \tau \mid x \in U\}$ of x	29
$\mathcal{C}(\tau)$	set of consistent subsets of τ, i.e. $\{S \subseteq \tau \mid \bigcap S \neq \emptyset\}$	34
$c \cong d$	c is isomorphic to d	35
$\mathrm{At}\, p$	$\{a \sqsubseteq p \mid a \text{ atom}\}$	47
$A \subseteq_{\text{fin}} X$	A is a finite subset of X	48
C^{op}	opposite category of C	60
$[c,d]$	internal hom object of morphisms from c to d	60
$\mathsf{C}(-,c)$	representable presheaf represented by c	61
よ_{C}	Yoneda embedding of C into $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ with $c \mapsto \mathsf{C}(-,c)$	61
$\mathrm{Conc}\,\mathsf{C}^{[+]}_{(\mathrm{fin})}$	category of (finite) [non-empty] concrete presheaves on C	63
$+f_i, \times f_i, \bigotimes f_i$	mediators for (co)cones f_i	69
$\mathrm{el}\, F$	category of elements of presheaf F	69
π^F	diagram of presheaf F	69
$c \leftrightsquigarrow_F d$	c and d are connected via functor F	70
$\mathcal{P}_{\mathcal{C}}(D)$	$\{S \subseteq \mathcal{C}(\mathrm{At}\, D) \mid \bigcup S \in \mathcal{C}(\mathrm{At}\, D)\}$	74
1_c	terminal morphism from c	76
0_c	initial morphism to c	76
$0_{c,d}$	zero morphism from c to d	76
$D \otimes E$	tensor product of domains D and E, i.e. the product in Obs^*	79
1^*	terminal object in Obs^*	81
$\mathsf{C}\,/\,c$	slice category of category C over object c	88

CONTENTS

1 INTRODUCTION

Programming languages are languages for specifying computational processes. These processes can often be described in terms of a machine that modifies its state according to the instructions of the program at hand, e. g. a CPU or more abstractly a Turing machine. Such a machine can be regarded as an instance of a *dynamical system*: a state space with a transformation function acting on it. To what extent can general dynamical systems conversely be regarded as containing computational content? Important differences to more conventional models of computation include that dynamical systems do not necessarily have a well-defined notion of input and output (instead, their evolution follows trajectories that extend infinitely in time), and that the state space is often continuous. What, then, could a program specifying a dynamical system look like?

The objective of this thesis is to develop a ''programming language'', or, to be more precise, a type theory, for describing and manipulating dynamical systems as if they were computer programs. As an application, we also derive a logical calculus for reasoning about these programs.

Here is an outline of the strategy. First, we need a model of what the *meaning* of a dynamical system should be when considered as a computational process. For this, we use ideas from Levin Hornischer's PhD thesis [12], where an interpretation of dynamical systems using *domain theory* is developed in which the meaning of a dynamical system is constructed out of finitary symbolic "approximations" to it. From this interpretation, we obtain a category of mathematical objects called *domains*. Domain theory is usually concerned with assigning meaning to programs of a given programming language. We do the reverse, similar in spirit to [1], and derive a type theory from this category, using the approach of *categorical logic*.

The ambition to derive a type theory from a category of domains representing dynamical systems grew out of Levin Hornischer's thesis. The present work is an attempt to carry out such a programme, but for a simpler domain-theoretic interpretation of dynamical systems than the one given in [12]. This should be seen as a proof-of-concept of a general strategy we pursue for developing a type theory and logic for a given model of computation.

The thesis is structured as follows. The remainder of the introduction serves to detail our approach in order to properly motivate the developments in the subsequent chapters. This includes a (rather informal) introduction to basic concepts from domain theory and categorical logic. Chapter 2 then expounds my view of "approximation through finitary observation", from which we derive a type of domains suitable for modelling the semantics of dynamical systems as intended. This culminates in the description of a category of observation domains, whose properties are studied in Chapter 3, including a characterisation using a *free construction*. The insights gained from this allow us to derive internal type theories from this category, which we do in Chapter 4. There, we also study soundness, completeness, and expressivity of the resulting theories. In Chapter 5, we roughly outline some applications of this framework. In particular, we show how the type theory may be used to actually "program" with dynamical systems, and how it could allow reasoning logically about such programs. To conclude, Chapter 6 discusses the developments in this work in light of its aspirations as well as directions for future research.

1.1 DYNAMICAL SYSTEMS AND COMPUTATION

We shall not be concerned with giving a rigorous definition of what we consider to be a dynamical system here; this is done in Chapter 2. For now, it is enough to think of them as state spaces together with a transformation function assigning to each state its successor at the next time step. That is, our notion of dynamical system is deterministic, time-discrete (and time-invariant), but (possibly) space-continuous.

Straightforward examples of systems satisfying these criteria are given by artificial neural networks. Consider a *Hopfield network* [11], which is a model of associative memory. It may be represented as a set of neurons and weights for the connection between each unordered pair of neurons. Each neuron has a binary activation value (on/off), and activations are propagated to a neuron depending on the activations of the other neurons and the weights of their connections to the neuron in question. This is as a dynamical system, with the state space given by the activation vectors, and the transformation rule determined by an activation function. Starting at any point in the state space, it can be shown that for any Hopfield network, the trajectory obtained by repeatedly applying this transformation eventually converges to a stable state. This makes it easy to accept this as a form of computation: we can choose an initial state as input, then we let the network compute the subsequent states until a stable state is reached, which can then be taken to be its output.

The stable states of such a network depend on the weights of the connections between neurons. It is also possible to train the network, changing its weights so that some given activation patterns are stable. This process can equally be seen as a dynamical system, as described in [12] for learning in neural networks in general: now the state space is given by the set of pairs of weight matrices and sequences of activation vectors (representing the patterns to learn), and the transformation rule changes the weights according to the first pattern in the sequence (e. g. via *Hebbian learning* like in [11]) and shifts the sequence to the succeeding element. Once the end of the sequence is reached, the resulting weight matrix can of course again be interpreted as the output of a computational process, but this interpretation becomes impossible if we allow the sequences to be infinite. In this case, the network continuously learns, but never reaches a "finished" state. It seems plausible to still think of this as a form of computation, since we happily accept any finite initial segment of this process as computation.

Many processes in nature form dynamical systems as well, such as biological neural networks (from individual neurons to the human brain) or the weather. Other systems are more abstract and model interactions between agents, such as ecosystems (e. g. Lotka–Volterra equations for predator–prey relations [22]), social networks, or the financial market. We would usually model such processes in a space-continuous fashion, which is in line with our conception of dynamical systems. However, the assumption of time-discreteness we imposed does not apply to such scenarios. We retain time-discreteness because it allows representing the dynamical system in terms of a function from the current state to a successor state, which is closer to the way conventional models of computation work. Time-continuous systems may be approximated by discretising them using a suitable time interval in order to apply the theory we develop.

While some might reject the notion that such processes perform computation since they do not manipulate explicit discrete symbols and do not have canonical notions of input and output, we would like to posit that they do perform computation, and underpin this claim by showing that we can program with them.

In [12], a step towards this is taken by showing that dynamical systems can be given semantics in which their non-symbolic form of computation is constructed as a limit of increasingly elaborate symbolic models. Roughly, the state space of the system is partitioned into finitely many cells, assuming that we can measure which of these cells the system is in at a given moment in time. Starting in some initial state, we then observe which cells the system visits as time progresses. This way, we obtain symbolic trajectories (in the sense that they are sequences of discrete tokens) forming a "finitary" approximation of the system, and as we increase the fineness of the partition and the observation time, we can, under certain circumstances, get arbitrarily close to approximating the actual, non-symbolic dynamics of the system. This is conceptually related to the practice of *symbolic dynamics* [4], in which dynamical systems are modelled using infinite trajectories of symbols representing regions of the state space.

We use a similar approach, where we also equip dynamical systems with families of subsets of the state space representing "observable" properties, akin to *topological dynamical systems*, except that these families need not form topologies. However, instead of trajectories, we only consider the transitions between these subsets after a single time step since this simplifies the matter while capturing the same amount of information in the limit of the approximation process.

Because there is no way to sensibly assign input and output values to a given trajectory of an arbitrary dynamical system, we simply take the full behaviour of a dynamical system to be the program it computes. Symbolic semantics of this program can be obtained on any desired level of approximation in the fashion just described. Section 1.2 elaborates on this.

1.2 DOMAIN THEORY AND SEMANTICS

Domain theory is often used to tackle the problem of *denotational semantics*: programs should be assigned mathematical objects (e. g. functions) that capture their observable (e. g. input–output) behaviour. That is, two programs should be assigned the same object if and only if they yield the same output when given the same input. This is in contrast to *operational semantics*, which assign a mathematical object to the computation *process* described by a program using a desired model of computation (e. g. a Turing machine). Two programs with different operational semantics may still share a denotation, by computing the same function in two different ways; see the example in the margin.

Two programs that define the same function $\mathbb{N} \to \mathbb{N}$ but may have different machine implementations:

```
f(n) = n
f(n) = n + 1 - 1
```

A naïve approach to denotational semantics might try to interpret a program taking a natural number and returning a natural number as a certain function $f : \mathbb{N} \to \mathbb{N}$, given by some inductive interpretation of the programming language constructs as functions. However, for most languages, such an approach quickly runs into problems. For example, programs may be partial and not actually return a value, due to being stuck in an infinite loop or yielding an error. This may happen in particular when a program is recursively defined, in which case it is not generally guaranteed that the function assigned to a program is well-defined.

Domain theory can solve such problems by interpreting programs as functions between domains instead, which support a richer structure than the set-based approach just described. Domains are certain kinds of ordered sets, where the order provides a formalisation of certain values being more "defined" or more "elaborate" than others. For example, to represent partiality, one may use the domain consisting of the unordered natural numbers together with a bottom element $\bot$. A function on this domain can now be defined to return $\bot$ for the cases where the represented program does not halt.

Usually, the functions one considers between domains have a natural order defined on them, with which they again form a domain, representing the type of functions between these domains. In the example just given, a function would be considered to elaborate another one if its return value on each input elaborates the return value of the other function on the same input.

The typical domain-theoretic endeavour starts with a programming language and designs domains to capture their denotational semantics. In our case, however, we need to do the opposite: we want to *obtain* a programming language from denotational semantics of dynamical systems. Dynamical systems are already mathematical objects, but we envisioned them to be analogous to stateful machines modelling standard computational processes, so they should be considered to represent the operational semantics of a program. As concluded in Section 1.1, we do want to take the whole behaviour of the system into account when determining what it is that the system computes. Therefore, the system may serve as its own denotational semantics.

But there was more to the semantics of a system, namely the aspect of finitary approximations to it. This is what makes domain theory useful in our scenario. A frequent requirement of domains is that all of its elements can be described by "combining" (taking the supremum of) certain "simple", in a sense easily describable elements. Making use of this idea for our interpretation of the semantics of dynamical systems, we can create a domain where each value represents an approximation of a dynamical system, with greater elements corresponding to better approximations. The "simple" elements should be those corresponding to finitary observations.

This way, different dynamical systems will always have different denotational semantics. However, we obtain a useful notion of graded similarity between the behaviours of dynamical systems:

informally, the finer the approximations that simultaneously approximate both systems, the more similar these systems can be considered to be.

Thus, we start with our idea of the semantics of dynamical systems in terms of finitary approximations and turn this into a specific kind of domain. We then accept the resulting notion of domains as the starting point for a category of types of dynamical systems, with each domain interpreted to represent a type, as is standard in domain theory.

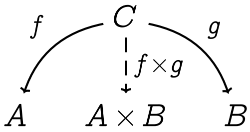
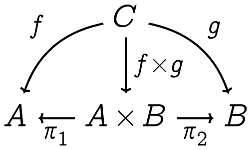
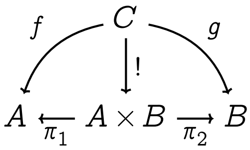
1.3 CATEGORIES AND TYPE THEORIES

We make extensive use of basic concepts from category theory. Some of these are briefly reviewed in the margins when they appear like in the example on the right; but for reference consult [21, 24].

A *category* C consists of: a collection of *objects*; for all pairs of objects $c, d \in \mathsf{C}$ a collection of *morphisms* or *arrows* $\mathsf{C}(c,d)$; for all pairs of arrows $f \in \mathsf{C}(c,d)$ and $g \in \mathsf{C}(d,e)$, a *composite* arrow $f\,;g \in \mathsf{C}(c,e)$; and for all objects $c \in \mathsf{C}$, an *identity* arrow $\mathrm{id}_c \in \mathsf{C}(c,c)$; such that $\mathrm{id}_a\,; f = f = f\,; \mathrm{id}_b$ and $(f\,;g);h = f\,;(g\,;h) =: f\,;g\,;h$ for all arrows for which the composites are defined.

A common approach in categorical logic, treated in detail in [31], is to interpret types A as objects $[\![A]\!]$ in a category C, and a type-theoretic judgment of the form $x : A \vdash N : B$, meaning "if x is of type A, then the term N (which may include x) is of type B", as a morphism from $[\![A]\!]$ to $[\![B]\!]$. We can understand this as meaning that the term N shows how to transform "A-things" x into "B-things" $N(x)$, or a bit more precisely, if x is a "generalised element" of the object interpreting A, i.e. a morphism with codomain $[\![A]\!]$, then N specifies a way to turn x into a generalised element of the object interpreting B (via composition of x with the morphism represented by $x : A \vdash N : B$).

An internal type theory of a category is then given by the constructions that we can carry out on its objects. To derive the rules, we usually start with a universal property we want to capture, which in general consist of five parts, each giving rise to a different kind of rule, as we illustrate for the example of the categorical product:

universal property	type of rule	product	rule for product
existence of an object	type formation	$A \times B$	$\dfrac{\vdash A : \mathsf{C} \quad \vdash B : \mathsf{C}}{\vdash A \times B : \mathsf{C}}$
with morphisms	term elimination/ introduction	$A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$	$\dfrac{x : C \vdash N : A \times B}{x : C \vdash \pi_1 N : A \quad x : C \vdash \pi_2 N : B}$
such that all other candidates have a mediator	term introduction/ elimination		$\dfrac{x : C \vdash N : A \quad x : C \vdash M : B}{x : C \vdash N \times M : A \times B}$
making the diagram commute	β-reduction		$\pi_1(N \times M) = N$ $\pi_2(N \times M) = M$
which is unique with that property	η-conversion		$(\pi_1 N) \times (\pi_2 N) = N$

Thus the expressivity of the type theory depends on what constructions the category allows. If there are (co)products, then the type theory admits product(/sum) types; if there are exponential objects, then it admits function types.

We think of a type theory as being "sound and complete" for a certain category if the type theory can, in a sense made precise in Section 4.1, construct all objects and morphisms of the category and prove all equalities between morphisms, and conversely everything the type theory constructs or shows to be equal should also hold in the category. The key technique for finding a type theory that is both sound and complete for a given category is to exhibit the category via a *free construction*, as we do in Section 3.3.

In addition to the internal type theory, one can think about the internal *logic* of a category. This usually refers to the constructions that can be performed on *subobjects* in the category. Subobjects are a category-theoretic generalisation of the notion of subsets of a set, so we can think of a subobject as containing a ''part'' of the object to which a certain proposition applies. Each proposition about elements of a type is then interpreted as a subobject of the object representing that type. The subobjects of an object form a poset, and if that poset has all meets, then the resulting logic allows the interpretation of conjunction; if it has all joins, we have disjunction, and so on.

Since our focus lies on extracting the internal type theory, we shall not spend too much time on the study of subobjects. However, in Section 5.2, we discuss as an application how we can talk about subobjects and other kinds of propositions *from within* the type theory.

2 A THEORY OF FINITARY OBSERVATION

We aim to capture mathematically the idea that dynamical systems can be approximated via finitary models arising from ‘‘practically feasible observations’’ or ‘‘partial descriptions’’ of these systems. Domain theory is a suitable theory for formalising these notions of approximation and convergence. The purpose of this chapter is to motivate the kind of domains we want to study as semantic models of dynamical systems.

The domains are later going to represent the types of our type theory. The type of a dynamical system is simply going to be the type of endomorphisms on its underlying state space. Thus we would like to have domains corresponding to spaces, and in addition, the endomorphisms on a space should form a ‘‘function space’’ that also constitutes a domain. Observations of both spaces as well as dynamical systems on these spaces should then fit under the same framework of finitary description.

Ultimately, we want to observe a dynamical system by measuring its current state at each point in time. We might not be able to determine its state precisely in a finite amount of time, but we hope that the more observations we make, the more precisely we will know the state. We describe a particular general paradigm of observation of points in a space, and give three different but equivalent views on it: the *spatial* view, in which what we are observing is an element of a state space and measurements correspond to certain subsets of this space, provides a link between observation and the idea of some physical reality that is being observed. The *combinatorial* view, where we forget about the individual states and deal only with sets of properties that may be observed together, enables us to think about observation without the necessity of knowing the underlying reality. Finally, the *ordered* view describes observation very abstractly in terms of how information increases during the observation process.

Each view comes with an associated mathematical structure, and we will show equivalences between these structures. We first develop our theory within the combinatorial view, which provides conceptual justification for the restrictions we impose on the structures we deal with. Showing equivalences is then useful because the spatial view provides a ''physical'' interpretation of the whole theory, connecting to topology and standard notions of dynamical systems, while the abstract ordered view connects to domain theory, enabling the application of the existing body of knowledge on denotational semantics to our situation.

In addition to formalising how to observe points in a space, we will discuss how to integrate the observation of dynamical systems into this framework in Section 2.2, namely by taking dynamical systems to be certain relations on observation spaces. In fact, this treatment will not only apply to dynamical systems but also to suitable functions between potentially different observation spaces. This way, we obtain a (monoidal) closed category of domains admitting a spatial interpretation, which allows types of transformations between dynamical systems in the type theory. Few canonical categories of domains or topological spaces are closed in this way, and much of the developments in this chapter is motivated by the ambition to reconcile closedness with both domains and spaces.

2.1 COMBINATORIAL VIEW: OBSERVATION SYSTEMS

The fundamental setup of our theory is as follows. We observe a state by making some measurements out of a set of basic observable properties (or "atomic description tokens") P. Multiple properties may apply to a single state, and we consider the set σ of sets of properties that can *simultaneously* apply, i. e. that are partial descriptions of at least one state. We call such sets *consistent*, so σ is the set of consistent subsets of P.

From this, it is already clear that the set σ should be closed under taking subsets: if $S \in \sigma$ is a set of properties that apply to an existing state and $S' \subseteq S$, then also the properties in S' all apply to that state. This leads to the following definition.

DEFINITION: SUBSET SYSTEM. A *subset system* is a pair (P, σ) with P a set and $\sigma \subseteq \mathcal{P}(P)$ such that $\langle\subseteq\text{-cl}\rangle$ if $S' \subseteq S \in \sigma$, then $S' \in \sigma$.

For a subset system (P, σ), we assume that any $S \in \sigma$ partially describes a point in some state space. Picking one such point to which all properties in S apply, we can consider the set $S' \in \sigma$ of *all* properties in P applying to that point. Then $S' \supseteq S$, and we will in fact assume that S' is a $\subseteq$-maximal element of σ. In theory, we might imagine that there might be another $S'' \supsetneq S'$ with S'' maximal in σ, describing a point different to the one singled out by S', in that S'' somehow contains additional information that does not apply to the point described by S'. In our interpretation, we will consider this to never be the case, though: instead, S' should then additionally contain the information that some properties applying to the point described by S'' do *not* apply to the point described by S', and then $S' \not\subseteq S''$.

This assumption serves to simplify the situation so that we can think of the maximal elements $\max\sigma$ of σ (seen as a poset) as representing the points of the state space we are observing, and since we are unable to distinguish between points that behave the same under all possible measurements, we furthermore assume that each such maximal element describes a unique such point.

In general, the elements of $\max\sigma$ may be infinite sets, since infinitely many properties may apply to a point. In line with our objective of "description through finitary observations",° we can however get arbitrarily close to singling out any given point by considering increasingly larger finite sets of properties describing that point. This corresponds to making more and more precise observations which "in the limit" uniquely describe the point (arguably at least if P is countable), while only requiring a finite number of observations at each point in time. Indeed, if an infinite set $I \in \sigma$ is a (possibly still partial) description of a point, then also all finite subsets of I are in σ by $\langle\subseteq\text{-cl}\rangle$. Because it is practically infeasible to conversely determine whether an infinite set of properties applies to a single point (there might be infinite $I \subseteq P$ such that all finite subsets of it describe distinct valid points, but there is no point to which all of I applies), we introduce a simple assumption on the consistency of infinite sets of properties in terms of their finite subsets.

°Realistic observable measurements are often considered to be merely *semi-observable*: the result of a measurement may be available after a finite (but unknown) time only if the outcome is positive. In an earlier approach, I tried to take this idea more seriously, imposing certain logical relationships between the observable properties and recording outcomes of semi-observable measurement processes after a finite but unbounded amount of time, and outcomes being "closed" under the logical relationships. This led to a type of domain that was hard to characterise and difficult to work with.

I then made the setup a little less dogmatic, speaking interchangably of "observation" and "description", which is justifiable since in order to construct non-symbolic computation as the "limit" of symbolic computation, it is enough if there is *some* way of describing objects in a finitary fashion, not necessarily via semi-observable measurements.

DEFINITION: OBSERVATION SYSTEM. A pair (P,σ) is an *observation system* if it is a subset system (hence satisfying $\langle\subseteq\text{-cl}\rangle$) such that ⟨fin-compat⟩ for all $I \subseteq P$, if $F \in \sigma$ for all finite $F \subseteq I$, then $I \in \sigma$.

Thus for observation systems, $I \in \sigma$ iff all finite subsets of I are in σ (with $\Rightarrow$ by $\langle\subseteq\text{-cl}\rangle$ and $\Leftarrow$ by ⟨fin-compat⟩), so the infinite elements of σ are completely determined by the finite ones. We furthermore have that any partial description of a point can now be extended to a full description of a point.

PROPOSITION 2.1.1. Let (P,σ) be an observation system and $S \in \sigma$. Then there exists an $S' \in \max\sigma$ such that $S' \supseteq S$. ⟨obs-max-elem⟩

Proof. By Zorn's lemma, it is enough to show that every chain $\{S_i \supseteq S \mid i \in I\}$, totally ordered by $\subseteq$, in the poset $\uparrow S := \{T \in \sigma \mid T \supseteq S\}$ has an upper bound in $\uparrow S$, since it then follows that $\uparrow S$ has a maximal element S', with $S' \supseteq S$ by definition of $\uparrow S$ and $S' \in \max \sigma$ since each $S'' \supseteq S'$ in σ is necessarily a superset of S, hence in $\uparrow S$, so $S'' \subseteq S'$ by maximality of S'.

[Let $\{S_i \mid i \in I\}$ be a chain in $\uparrow S$.]

> We show that the set $\bigcup S_i$, which is clearly an upper bound, is in σ. Then, necessarily $\bigcup S_i \in \uparrow S$. By ⟨fin-compat⟩, it suffices to show that each finite $F \subseteq \bigcup S_i$ is in σ.
>
> [Let $F \subseteq \bigcup S_i$ be finite.]
>
> > We show that $F \in \sigma$. Since F is finite, there exists a finite set $J \subseteq I$ such that $F \subseteq \bigcup\{S_j \mid j \in J\}$. Then, J has an element j^* such that $S_j \subseteq S_{j^*}$ because the S_j are totally ordered and J is finite. Thus $F \subseteq S_{j^*}$. It follows that $F \in \sigma$ by ⟨⊆-cl⟩ since $S_{j^*} \in \uparrow S \subseteq \sigma$. □

We can connect observation systems to several existing concepts. In the presence of ⟨fin-compat⟩, we can simplify ⟨⊆-cl⟩ to only require σ to be closed under taking *finite* subsets.° The resulting definition is identical to the concept of a *family of sets of finite character* [13]. I could not find much in the way of results about these structures, with the exception of *Tukey's lemma*, which is similar to ⟨obs-max-elem⟩ just shown.°

°For, if $S' \subseteq S \in \sigma$, then all finite subsets of S' are finite subsets of S, which are in σ, and then $S' \in \sigma$ by ⟨fin-compat⟩.

°In fact, [13, Theorem 2.1] shows that Tukey's lemma is *equivalent* to the axiom of choice.

A *simplicial complex* is a subset system (P, σ) which additionally satisfies that ⟨all-props-cons⟩ for all $p \in P$, also $\{p\} \in \sigma$ (i.e. that all properties in P are consistent, applying to at least one point), and where all sets in σ are finite and non-empty (of course, $S' \neq \emptyset$ then needs to be demanded in the antecedent of ⟨⊆-cl⟩). Because the elements of an observation system are fully determined by its set of finite non-empty elements, and the finite elements satisfy ⟨fin-compat⟩ trivially, any simplicial complex (P, σ) gives rise to a unique observation system satisfying ⟨all-props-cons⟩

by adding the empty set as well as all subsets of P of which all finite subsets are in σ. Conversely, any observation system (P,σ) satisfying $\langle$all-props-cons$\rangle$ can be turned into a simplicial complex by removing the empty set as well as all infinite sets from σ.

More precisely, given a simplicial complex (P,σ) we turn it into an observation system via a process of *ideal completion*,° which in the case of simplicial complexes reduces to the following.

°For a general poset, this means to take the set of downward-closed directed subsets, called *ideals*.

DEFINITION: IDEAL COMPLETION. Let σ be a family of subsets of a set P satisfying $\langle\subseteq$-cl$\rangle$ and $\langle$all-props-cons$\rangle$. The *ideal completion* of σ is defined as the set $\mathcal{I}(\sigma) := \{S \subseteq \bigcup\sigma = P \mid F \in \sigma \text{ for all finite } F \subseteq S\}$.

Note that the resulting set satisfies $\langle$fin-compat$\rangle$ by construction, and $\sigma \subseteq \mathcal{I}(\sigma)$ by $\langle\subseteq$-cl$\rangle$. Furthermore, observe that the ideal completion may only add infinite sets to σ. This construction will play an important role in some proofs where we construct observation systems. For this reason, we formulated it in terms of $\langle$all-props-cons$\rangle$-subset systems rather than simplicial complexes, which would have required taking into account the exclusion of the empty set.

It is furthermore easily seen that an $\langle$all-props-cons$\rangle$-observation system yields a Scott information system° $(P,\sigma,\vdash)$ with $S \vdash p$ iff $p \in S$. Observation systems are also similar to the concept of *coherence spaces* in the semantics of linear logic [9]. This notion is recovered if in $\langle$fin-compat$\rangle$ we replace "all finite $F \subseteq I$" with "all sets $F \subseteq I$ of cardinality 2". A coherence space is often represented as the reflexive undirected graph of its atoms, with edges indicating coherence. Our concept is instead equivalent to a hypergraph where each edge may connect any finite set of vertices (and edges must be closed under subsets).

°A *Scott information system* is a triple $(P,\sigma,\vdash)$ with $\sigma \subseteq \mathcal{P}(P)$ and $\vdash \subseteq \sigma \setminus \{\emptyset\} \times P$ such that

- if $S \in \sigma$ and $S' \subseteq S$, then $S' \in \sigma$ ($\langle\subseteq$-cl$\rangle$);
- $\{p\} \in \sigma$ for all $p \in P$;
- if $p \in S \in \sigma$, then $S \vdash p$;
- if $S \vdash q$ for all $q \in S'$ and $S' \vdash p$, then $S \vdash p$;
- if $S \vdash p$, then $S \cup \{p\} \in \sigma$.

While $\langle$`all-props-cons`$\rangle$ demands that every property in P actually applies to some point, we may also want to require that no two properties represent the same concept in the sense that they apply to the same set of points.

DEFINITION: SEPARATION. An observation system (P, σ) is called *separated* if $\langle$sep$\rangle$ the function $p \mapsto \max p : P \to \mathcal{P}(\max \sigma)$ on P sending each $p \in P$ to the set of $S \in \max \sigma$ with $p \in S$ is injective.

Note that even in the absense of $\langle$`all-props-cons`$\rangle$, a separated observation system can have at most one $p \in P$ with $\{p\} \notin \sigma$ (then, $\max p = \emptyset$). Both $\langle$`all-props-cons`$\rangle$ and $\langle$sep$\rangle$ are properties that may or may not be applicable to a measurement process. As we will see, they play a role in establishing the equivalences between the different views on observation we describe. In the end, we will only assume $\langle$`all-props-cons`$\rangle$, not $\langle$sep$\rangle$, because this yields the desired equivalence to a notion of domains.

Let us conclude this section by giving some examples of observation systems.

EXAMPLES: OBSERVATION SYSTEMS.

- Any set of properties P gives rise to a "flat" observation system by putting $\sigma := \{\emptyset\} \cup \{\{p\} \mid p \in P\}$. All consistent subsets are trivial (singletons or non-empty), and the maximal elements are given by the singletons from P.
- The powerset of any set: any subset of a subset is a subset, and any union of subsets is a subset, so the definition is clearly satisfied. Here *all* sets of properties are consistent, and there only exists a single maximal element, to which every property applies. Note that this generally does not satisfy $\langle$sep$\rangle$.
- The family of sets of linearly independent vectors in any vector space: satisfaction of $\langle$`fin-compat`$\rangle$ is a consequence of the definition of linear independence depending on *finite* linear

combinations only. Proposition ⟨obs-max-elem⟩ shows in this case that every set of linearly independent vectors can be extended to a maximal such set; in particular, every vector space (which by definition is non-empty, thus has at least a linearly independent singleton set) has a basis.

- The family of consistent sets of first-order sentences: here ⟨fin-compat⟩ is just the compactness theorem. Maximal elements can be seen as equivalence classes of models of first-order theories up to elementary equivalence, and the properties applying to them are just the first-order sentences they satisfy.

2.2 SPATIAL VIEW: OBSERVATION SPACES

We now make precise the way in which observation systems capture the outcomes of a "physical" observation process by giving them a spatial interpretation. The idea is that we are observing points of a state space using the properties from an observation system, and each such property corresponds to a subset of the space containing precisely the points having the property.

A space is just a pair (X,τ) with $\tau \subseteq \mathcal{P}(X)$. Now if (P,σ) is an observation system, and we think of each $p \in P$ as corresponding to some subset of a space where p applies to each point in it, then what is the analogue of a consistent set $S \in \sigma$? Since this is just a set of properties that can all hold together of a single point, it seems appropriate to say that this means that their intersection as sets is non-empty. This leads to the following notion of a space, which we will show to be equivalent to (separated) observation systems.

DEFINITION: OBSERVATION SPACE. A pair (X,τ) is an *observation space* if

⟨T_1⟩ for all $x \neq y \in X$, there exists a $U \in \tau$ with $x \in U \not\ni y$;

⟨`fip-ne`⟩ for each family $I \subseteq \tau$ such that $\bigcap F \neq \emptyset$ for all finite $F \subseteq I$, also $\bigcap I \neq \emptyset$.

We say that a family $I \subseteq \tau$ with $\bigcap F \neq \emptyset$ for each finite $F \subseteq I$ has the *finite intersection property* (FIP). Thus ⟨`fip-ne`⟩ says that each family with the FIP has a *n*on-*e*mpty intersection.

First, we discuss how observation spaces relate to the more familiar topological spaces. Then, we establish the desired equivalence with observation systems. Finally, we describe how the observation of dynamical systems fits into the framework.

CONNECTING TO TOPOLOGICAL SPACES

The assumption of ⟨`fip-ne`⟩ may seem problematic from a topological point of view since topologies do not in general satisfy it.° Let us therefore spend some time to understand what it means for a topological space to be an observation space. In addition to ⟨`fip-ne`⟩, a feature of observation spaces is that points correspond to maximal sets of observable properties with non-empty intersection. What does a family of subsets of a set have to satisfy for this to be the case? We can do something similar to *locale theory* [15], where the points of a *sober* topological space are equivalently the completely prime filters of its opens.

°For example, consider the topological space $(\mathbb{R}, \tau)$ of real numbers together with the standard topology of unions of open intervals (r, s) for $r, s \in \mathbb{R}$. For the family $I := \{(0, 1/n) \mid n \in \mathbb{N}\} \subseteq \tau$, every finite $F \subseteq I$ has $\bigcap F = (0, 1/n)$ for n minimal with $(0, 1/n) \in F$, but $\bigcap I = \emptyset$.

DEFINITION: OBSERVATION FILTERS. Let X be a set and $\tau \subseteq \mathcal{P}(X)$. An *observation filter* on X is a subset $F \subseteq \tau$ such that $\bigcap F \neq \emptyset$. An observation filter F is called *maximal* if each observation filter $F' \supseteq F$ has $F' = F$. An observation filter F is *completely prime* if whenever $\bigcup U_i \supseteq \bigcap F$ for some family of $U_i \in \tau$, then there exists an i such that $U_i \in F$. For each $x \in X$, denote by $\mathcal{N}(x)$ the *neighbourhood filter* $\{U \in \tau \mid x \in U\}$ of x.

Clearly, a neighbourhood filter $\mathcal{N}(x)$ is an observation filter, with the intersection containing x. For sober topological spaces, the notion of a *non-empty* completely prime observation filter coincides with the standard notion of a completely prime filter.° In the following propositions, a *space* (X, τ) refers to any set X with a set of subsets τ (not necessarily forming an observation space). We work towards a one-to-one correspondence between points of the space and maximal observation filters.

°It is proper because its intersection is non-empty, closed under supersets because $U' \supseteq U \supseteq \bigcap F$ if $U \in F$, closed under finite intersections because $\bigcap U_i \supseteq \bigcap F$ with $\bigcap U_i$ open, and completely prime because $\bigcup U_i \supseteq \bigcap F$ if $\bigcup U_i \in F$; conversely, a completely prime filter in a sober space is the neighbourhood filter of a unique point, hence a non-empty observation filter, and if $\bigcup U_i \supseteq \bigcap F = \{x\}$, then $x \in U_i$ for some i, so $U_i \in F$.

PROPOSITION 2.2.1. Every maximal observation filter on a space (X, τ) is of the form $\mathcal{N}(x)$ for some $x \in X$. $\langle$max-$\mathcal{N}\rangle$

Proof. Let F be maximal. By definition, $\bigcap F \neq \emptyset$, say $x \in \bigcap F$. Now $F \subseteq \mathcal{N}(x)$: if $U \in F$, then $x \in U$, so $U \in \mathcal{N}(x)$. Thus the claim follows by maximality of F. □

We also want the converse to hold, though, for which the $\langle \mathsf{T}_1 \rangle$ condition is sufficient.

PROPOSITION 2.2.2. If (X, τ) satisfies $\langle \mathsf{T}_1 \rangle$, then $\bigcap \mathcal{N}(x) = \{x\}$ for all $x \in X$. $\langle \mathsf{T}_1$-$\bigcap\mathcal{N}\rangle$

Proof. Clearly $x \in \bigcap \mathcal{N}(x)$, and if $y \in \bigcap \mathcal{N}(x)$, then $x \in U$ for all $U \ni y$, so $x = y$ by $\langle \mathsf{T}_1 \rangle$. □

PROPOSITION 2.2.3. If (X, τ) satisfies $\langle \mathsf{T}_1 \rangle$, then $\mathcal{N}(x)$ is maximal for every $x \in X$. $\langle \mathsf{T}_1$:$\mathcal{N}$-max$\rangle$

Proof. If $F \supseteq \mathcal{N}(x)$ has $\bigcap F \neq \emptyset$, then $\emptyset \neq \bigcap F \subseteq \bigcap \mathcal{N}(x) = \{x\}$ by $\langle \mathsf{T}_1$-$\bigcap\mathcal{N}\rangle$, so $x \in U$ for all $U \in F$, hence $F \subseteq \mathcal{N}(x)$. □

The notion of a completely prime observation filter is attractive as it offers a more "local" idea of a neighbourhood filter. It is however strictly weaker, but again $\langle \mathsf{T}_1 \rangle$ is enough for equivalence.

PROPOSITION 2.2.4. For each $x \in X$ for (X, τ), the observation filter $\mathcal{N}(x)$ is completely prime. $\langle\mathcal{N}$-cpf$\rangle$

Proof. Let $x \in X$ and $U_i \in \tau$ a family with $\bigcup U_i \supseteq \bigcap \mathcal{N}(x)$. Since $x \in \bigcap \mathcal{N}(x)$, we have $x \in \bigcup U_i$, so $x \in U_i$ for some i. Then $U_i \in \mathcal{N}(x)$. □

PROPOSITION 2.2.5. If (X,τ) satisfies $\langle \mathsf{T}_1 \rangle$, then each completely prime observation filter is of the form $\mathcal{N}(x)$ for some $x \in X$. $\langle \mathsf{T}_1\mathsf{:cpf}\text{-}\mathcal{N} \rangle$

Proof. Let F be completely prime. By $\langle \mathsf{max}\text{-}\mathcal{N} \rangle$, it suffices to show that F is maximal. Suppose $U \notin F$; we show $U \cap \bigcap F = \emptyset$. For each $x \in X \setminus U$ and $y \in U$, by $\langle \mathsf{T}_1 \rangle$ there exists a $V_x^{\neg y}$ such that $x \in V_x^{\neg y} \not\ni y$, since necessarily $x \neq y$. Then for each $y \in U$, we have $\bigcup_{x \in X \setminus U} V_x^{\neg y} \cup U = X \supseteq \bigcap F$, so since $U \notin F$, by complete primeness there exists an $x_y \in X \setminus U$ such that $V_{x_y}^{\neg y} \in F$. Thus $\bigcap F \subseteq \bigcap_{y \in U} V_{x_y}^{\neg y}$. Since $y \notin V_{x_y}^{\neg y}$ for each $y \in U$, we have $U \cap \bigcap_{y \in U} V_{x_y}^{\neg y} = \emptyset$, and so $U \cap \bigcap F = \emptyset$. □

PROPOSITION 2.2.6. If (X,τ) satisfies $\langle \mathsf{T}_1 \rangle$, then a *non-empty* observation filter F on X is completely prime iff for each family of $U_i \in \tau$ such that $\bigcup U_i \supseteq U$ for some $U \in F$, there exists an i such that $U_i \in F$. $\langle \mathsf{T}_1\mathsf{:cpf}\star \rangle$

Proof. The $\Rightarrow$-direction is clear since every $U \in F$ with $\bigcup U_i \supseteq U$ has $U \supseteq \bigcap F$, hence $\bigcup U_i \supseteq \bigcap F$. Now combining $\langle \mathsf{T}_1\mathsf{:cpf}\text{-}\mathcal{N} \rangle$ with $\langle \mathsf{T}_1\mathsf{:}\mathcal{N}\text{-}\mathsf{max} \rangle$, and $\langle \mathsf{max}\text{-}\mathcal{N} \rangle$ with $\langle \mathcal{N}\text{-}\mathsf{cpf} \rangle$, we have that in a $\langle \mathsf{T}_1 \rangle$-space, being completely prime is equivalent to being maximal, so it suffices to show that if F satisfies the stated condition, then it is maximal. This is analogous to the proof of $\langle \mathsf{T}_1\mathsf{:cpf}\text{-}\mathcal{N} \rangle$, except that when applying complete primeness, we note that picking any $W \in F$ (which is non-empty by assumption), we have $X \supseteq W$, and then the stated condition gives us the $x_y \in X \setminus U$ with the required properties. □

That is, the necessity of knowing the intersection of all sets in an observation filter can be dropped (but it must still be non-empty).

PROPOSITION 2.2.7. If (X,τ) is an observation space, then a subset F of τ with $\emptyset \neq F \subsetneq \tau$ is a completely prime observation $\langle \mathsf{obs:cpf}{\star\star} \rangle$

filter iff for each family of $U_i \in \tau$ and each *finite* family of $V_j \in F$ with $\bigcup U_i \supseteq \bigcap V_j$, there exists an i such that $U_i \in F$.

Proof. Again, $\Rightarrow$ is clear because every finite family of sets in F is in particular a family. Now every F satisfying the condition must be a non-empty observation filter: suppose $\bigcap F = \emptyset$, then by ⟨fip-ne⟩ there exists a finite family of $V_j \in F$ with $\bigcap V_j = \emptyset$, and then every $U \in \tau$ has $U \supseteq \bigcap V_j$, hence $U \in F$ and $F = \tau$, contradiction to the assumption that $F \subsetneq \tau$. The result then immediately follows from ⟨T_1:cpf*⟩ by our condition on F since every observation space satisfies ⟨T_1⟩ and every singleton is a finite family. □

The condition given in ⟨obs:cpf**⟩ is now equivalent to the usual notion of a completely prime filter, in the sense that the subsets of a topology satisfying this condition are precisely the completely prime filters, since topologies as well as filters are closed under finite intersections. In particular, this means that if a topological space is an observation space, it is automatically sober: every completely prime filter satisfies the condition in ⟨obs:cpf**⟩, then it is a completely prime observation filter, so by ⟨T_1:cpf-$\mathcal{N}$⟩, it is the neighbourhood filter of a point, and by ⟨T_1-$\bigcap\mathcal{N}$⟩, that point is unique.

Often, we want to be able to recognise the neighbourhood filters by an even more pleasant condition, namely: for each *finite* family $U_i \in \tau$ with $\bigcup U_i \supseteq U$ for some $U \in F$, there exists an i such that $U_i \in F$. We call such filters simply *prime filters*. Clearly every completely prime observation filter is a prime observation filter. For the converse to hold, the following condition is sufficient.

DEFINITION: COMPACTNESS. Let X be a set and $\tau \subseteq \mathcal{P}(X)$. We call (X,τ) *compact* if for every family of $U_i \in \tau$ for $i \in I$ with $\bigcup_{i\in I} U_i = X$, there exists a *finite* $J \subseteq I$ such that $\bigcup_{i\in J} U_i = X$.

PROPOSITION 2.2.8. If (X,τ) is ⟨T_1⟩ and compact, then each prime observation filter is maximal. ⟨T_1+compact:pf-max⟩

Proof. Analogous to $\langle \mathsf{T}_1 : \mathtt{cpf}\text{-}\mathcal{N} \rangle$, except that we use compactness before applying primeness to obtain a finite subcover that still covers X. □

A topological spaces is compact if and only if its family of closed subsets satisfies $\langle \mathtt{fip\text{-}ne} \rangle$ [25, Theorem 26.9]. The proof of that statement shows more generally that any family of sets satisfies the compactness condition if and only if the family of the complements of these sets satisfies $\langle \mathtt{fip\text{-}ne} \rangle$. Thus the family of closed sets of any compact $\langle \mathsf{T}_1 \rangle$ topology forms an observation space.

Furthermore, this implies that a compact family of sets closed under complements automatically satisfies $\langle \mathtt{fip\text{-}ne} \rangle$, which tells us that e. g. the clopens of a Stone space° form an observation space, and the usual Stone duality via prime filters corresponding to points of the space [35] is reflected in the statement just shown. Indeed, [12] works with partitions of the state space using clopens when approximating dynamical systems.

° A *Stone space* is a compact space satisfying that any pair of distinct points can be separated by a clopen set, which means precisely that the clopens satisfy $\langle \mathsf{T}_1 \rangle$.

In particular, the clopens of the *Cantor space* as well as the *Baire space*, which are countably infinite products of the discrete topologies on the set $\{0,1\}$ and $\mathbb{N}$, respectively, form observation spaces. The observable properties, i. e. the basic clopens, of the Cantor space are given by the sets B_f for $f : F \to \{0,1\}$ with F a finite subset of $\mathbb{N}$ of all the binary sequences that coincide with f on F.

In research on dynamical systems, two common assumptions are that the spaces one works with are Polish (i. e. separable and completely metrisable) and zero-dimensional (i. e. having a basis of clopens). These properties also apply to what is called a "measured topological system" in [12], on which topological dynamical systems are considered. Zero-dimensionality is justified there by saying that semi-observable properties as modelled by topological opens may in practice be considered to be fully observable since the points lying on the border of an open are "atypical" (would commonly have measure zero if one were to equip them with a realistic measure).

Now in [17, Theorem 7.8], it is shown that every zero-dimensional separable and metrisable space can be embedded into both the Cantor and the Baire space, and a zero-dimensional Polish space is in fact homeomorphic to a closed subspace of the Baire space. This should demonstrate that our theory can be applied to common situations occurring in the more established theory of topological dynamical systems.

EQUIVALENCE WITH OBSERVATION SYSTEMS

We now establish the equivalence between the concept of separated observation systems and observation spaces. By our guiding intuition, every observation space (X,τ) should give rise to an observation system by taking the set of properties to be τ, and the consistent sets to be those $S \subseteq \tau$ such that $\bigcap S \neq \emptyset$. Denote the set of these subsets by $\mathcal{C}(\tau)$. With the previously introduced terminology, this is also the set of observation filters on τ.

PROPOSITION 2.2.9. For each observation space (X,τ), the pair $(\tau,\mathcal{C}(\tau))$ is a separated observation system. ⟨spc-sepsys⟩

Proof.

⟨⊆-cl⟩

Let $S' \subseteq S \in \mathcal{C}(\tau)$. Then $\bigcap S \neq \emptyset$ by definition, so also $\bigcap S' \neq \emptyset$, hence $S' \in \mathcal{C}(\tau)$.

⟨fin-compat⟩

Let $I \subseteq \tau$ be such that each finite $F \subseteq I$ is in $\mathcal{C}(\tau)$, i.e. $\bigcap F \neq \emptyset$. Thus I has the FIP, and then $\bigcap I \neq \emptyset$ by ⟨fip-ne⟩, so $I \in \mathcal{C}(\tau)$ by definition.

⟨sep⟩

Let $U \neq V$ in τ, so $U \not\subseteq V$ or $V \not\subseteq U$; w.l.o.g. assume the former. We show that $\max U \neq \max V$. By assumption, there is an $x \in U$

with $x \notin V$. Now $\mathcal{N}(x)$ contains U but not V, and it is maximal in $\mathcal{C}(\tau)$ by ⟨T_1⟩ and ⟨T_1:$\mathcal{N}$-max⟩, so $\max U \ni \mathcal{N}(x) \notin \max V$. □

In the other direction, we have already discussed that we consider the points of the space we are observing to be precisely the maximal elements of σ, and so we can associate with each (P, σ) a pair $(\max\sigma, \max[P])$ with $\max[P]$ consisting of the sets $\max p$ of maximal elements above p, for all $p \in P$.

PROPOSITION 2.2.10. For each observation system (P, σ), the pair $(\max\sigma, \max[P])$ is an observation space. ⟨sys-spc⟩

Proof.

⟨T_1⟩

If $x \neq y$ in $\max\sigma$, then since x, y are both maximal in σ, we have $x \not\subseteq y$, so there exists a $p \in x$ with $p \notin y$. Then $x \in \max p \not\ni y$ as required.

⟨fip-ne⟩

If a family $\max[I]$ for $I \subseteq P$ has the FIP, then $\max[F]$ for each finite $F \subseteq I$ has a non-empty intersection, i.e. there exists an $x_F \in \max\sigma$ with $x_F \in \bigcap \max[F]$, so $F \subseteq x_F$ by definition, and $F \in \sigma$ by ⟨⊆-cl⟩. Then also $I \in \sigma$ by ⟨fin-compat⟩. Let $x \supseteq I$ be maximal in σ by ⟨obs-max-elem⟩. Then $x \in \bigcap \max[I] \neq \emptyset$ as needed. □

To fully formalise what it means for the two notions to be equivalent, we could make use of the categorical concept of equivalence. This requires a definition of morphisms between observation spaces, which at this point would unnecessarily complicate the discussion. We shall merely define the notion of an isomorphism for those structures and prove the "object part" of a categorical equivalence, which for now is a sufficiently strong claim to be made.

Two observation systems $(P, \sigma), (P', \sigma')$ are isomorphic, notation $(P, \sigma) \cong (P', \sigma')$, if there exists a bijection $b : P \to P'$ such that for all $S \in \sigma$, we have $b(S) \in \sigma'$, and conversely $b^{-1}(S') \in \sigma$ for all $S' \in \sigma'$. This is based on the standard notion of a morphism between simplicial complexes. For observation spaces, we use the same definition (which is possible since both structures are pairs of a set and a family of subsets).

PROPOSITION 2.2.11. For all observation spaces (X,τ), we have $(X,\tau) \cong (\max \mathcal{C}(\tau), \max[\tau])$. ⟨spc-sys-spc⟩

Proof. The mapping $x \mapsto \mathcal{N}(x)$ is a well-defined map from X to $\max \mathcal{C}(\tau)$, as we have shown in ⟨T_1:$\mathcal{N}$-max⟩ that $\mathcal{N}(x)$ is maximal in $\mathcal{C}(\tau)$. We show that it is a bijection with the required properties.

⟨surjectivity⟩

If $S \subseteq \tau$ is maximal with $\bigcap S \neq \emptyset$, then by ⟨max-$\mathcal{N}$⟩, we have $S = \mathcal{N}(x)$ for some $x \in X$.

⟨injectivity⟩

If $x \neq y$, then by ⟨T_1⟩, there exists a $U \in \tau$ with $x \in U \not\ni y$, so $\mathcal{N}(x) \neq \mathcal{N}(y)$.

⟨preservation and reflection of consistency⟩

If $U \in \tau$, then $\mathcal{N}(U) = \bigcup\{\mathcal{N}(x) \mid x \in U\} = \bigcup\{\mathcal{N}(x) \mid \mathcal{N}(x) \ni U\} = \max U \in \max[\tau]$ using ⟨T_1:$\mathcal{N}$-max⟩ and ⟨max-$\mathcal{N}$⟩, with the converse holding because $\mathcal{N}(x) \in \max U$ only if $U \in \mathcal{N}(x)$ only if $x \in U$. □

The problem going in the other direction is that generally in an observation system two different properties might actually represent the same subset of a space, namely when they are contained in the same maximal elements. This is prevented by the separation property.

PROPOSITION 2.2.12. For all observation systems (P,σ) satisfying ⟨sep⟩, we have $(P,\sigma) \cong (\max[P], \mathcal{C}(\max[P]))$. ⟨sep:sys-spc-sys⟩

Proof. The mapping $p \mapsto \max p$ from P into $\max[P]$ is clearly well-defined.

$\langle$ surjectivity $\rangle$

By construction.

$\langle$ injectivity $\rangle$

By definition of $\langle \texttt{sep} \rangle$.

$\langle$ preservation of consistency $\rangle$

For all $S \in \sigma$, let $x_S \supseteq S$ be maximal in σ by $\langle \texttt{obs-max-elem} \rangle$, so $x_S \in \max p$ for all $p \in S$. Then $x_S \in \bigcap \max[S] \neq \emptyset$, so $\max[S] \in \mathcal{C}(\max[P])$.

$\langle$ reflection of consistency $\rangle$

If $\max[S] \subseteq \max[P]$ has a non-empty intersection, say $x \in \max \sigma$ is in $\bigcap \max[S]$, then $S \subseteq x$, so $S \in \sigma$ by $\langle \subseteq\texttt{-cl} \rangle$. $\square$

Note that for a similar equivalence, we could also consider general (not necessarily separated) observation systems and allow the set of properties in an observation spaces to be a multiset. We can also separate any observation system by taking the quotient of P by the equivalence relation "$\{p\} \cup S \in \sigma$ iff $\{q\} \cup S \in \sigma$, for all $S \in \sigma$". The resulting observation system is isomorphic to $(\max[P], \mathcal{C}(\max[P]))$.

OBSERVING DYNAMICAL SYSTEMS

Now that we have described a type of space incorporating the idea of finitary observation, how can we describe a dynamical system on such a space? In general, a dynamical system is a state space X with a transformation function $f : X \to X$ acting on it. We will assume that the state space is actually an observation space (X, τ) capturing how we may observe the current state of the system.

Eventually, we want the dynamical systems on a given observation space to themselves be points in a space of dynamical systems, imagining that by observing properties of a dynamical system we can determine which one we are dealing with. Here, we discuss what these properties should look like; that they give rise to an observation space is shown in Section 3.2.

Since we observe the dynamical system by measuring the state at some point in time and then measuring its successor state, the observations we make of dynamical systems should reflect some transition from a basic property applying at some point in time to another applying at the next discrete time step. If $U, V \in \tau$ are basic properties, we let the observation $U \to V$ apply to a dynamical system f if it maps U inside V, i.e. $f(U) \subseteq V$.° Note that this is similar to the subbasic opens of a compact-open topology. Through observation, we thus characterise a dynamical system by noting which properties are mapped inside which other properties.

°Of course, it is not really "finitarily observable" whether a property $U \to V$ applies to a physical system at hand, because it requires observing the successor state for all states in U, which may be infinitely many. This is another reason why one might prefer the term "description" to "observation". I initially pursued a different approach, in which $U \to V$ instead meant that *some* state in U is mapped onto some state in V; this is closer to the implementation in [12]. The theory retained non-finitary aspects even then, since it still made use of observations of transitions for every state of the system.

Furthermore, the category I obtained via this approach did not permit many constructions. In particular, I could not figure out how to define a satisfactory notion of morphism on it that would make the category cartesian or monoidal closed. The main problem appears to be conceptually that lifting the idea of "$\to$ maps a point in U onto some point in V" to a higher order would translate to "this transformation between dynamical systems maps *some* system with a given property onto some system with another property". But a nice notion of transformation would give us a reliable way to transform *any* system that exhibits a certain property into one having another, which is achieved by the approach presented in the main text.

That also means that we cannot distinguish between two dynamical systems that behave identically with respect to the mapping of properties. We need to restrict the dynamical systems we consider accordingly using a condition of continuity.

DEFINITION: CONTINUITY. A function $f : X \to X$ on an observation space (X, τ) is *continuous* with respect to τ if for all $x \in X$ and all $V \ni f(x)$ there exists a $U \ni x$ such that $f[U] \subseteq V$.

For topological spaces, this notion of a continuous function coincides with the standard notion. In that case, the given definition is known to be equivalent to the condition that the preimage of an open set be open. This also means that the category of Stone spaces and continuous functions forms a subcategory of the category of observation spaces and continuous functions. For general observation spaces, however, the two definitions may diverge, essentially because their equivalence rests on the assumption that arbitrary unions of opens are again open.

A continuous dynamical system f on (X, τ) can unambiguously be identified with the collection $\{(U, V) \in \tau \times \tau \mid f[U] \subseteq V\}$. This way, a relation $\rightarrow_f$ on $\tau \times \tau$ can be associated with f, via $U \rightarrow_f V$ iff $f[U] \subseteq V$.

Clearly, not all relations on $\tau \times \tau$ arise as subsets of a relation $\rightarrow_f$ for some continuous f. For example, a property may never be related to two properties describing disjoint regions of the space. More generally, we have the following defining characteristic (see the remark at the List of Symbols before the table of contents for the conventions used in denoting applications of relations).

DEFINITION: CONSISTENCY. A relation $\rightarrow \subseteq \tau \times \tau$ on an observation space (X, τ) is *consistent* if whenever $\bigcap S \neq \emptyset$ for some $S \subseteq \tau$, then also $\bigcap \rightarrow(S) \neq \emptyset$. A consistent relation $\rightarrow$ is called *maximal* if $\rightarrow' = \rightarrow$ for all consistent $\rightarrow' \supseteq \rightarrow$.

Consistent relations should correspond to partial descriptions of dynamical systems, while maximally consistent ones should then be complete descriptions. If $\rightarrow$ is maximally consistent, we associate a relation $f_\rightarrow$ with it via $x\, f_\rightarrow\, y$ iff $y \in \bigcap \rightarrow(\mathcal{N}(x))$. Unfortunately, there appears to be no straightforward one-to-one correspondence between maximally consistent relations and continuous dynamical systems. As it turns out, $f_\rightarrow$ as just defined is not necessarily a function, and even if it is, it may not be continuous. We would still like to use the maximally consistent relations as correspondents for dynamical systems due to the simplicity of the definition. We can do so by expanding the range of what we consider a dynamical system somewhat.

DEFINITION: DYNAMICAL SYSTEM. Let (X, τ) be an observation space. A *dynamical system* on X is a function $f : X \rightarrow \mathcal{P}(X) \setminus \{\emptyset\}$ such that

- ⟨max-det⟩ for all $U, V \in \tau$, if $f(x) \cap V \neq \emptyset$ for all $x \in U$, then $f(U) \subseteq V$; and
- ⟨weak-cont⟩ for all $x \in X$, if $y \in V$ for all $V \supseteq f(U)$ for all $U \ni x$, then $y \in f(x)$.

There are a few modifications to the previous situation here. First, we have replaced the function representing the dynamics by a function assigning to each state a (non-empty) *set* of successor states. We can think of such a dynamical system as a function that may assign a ''partially defined'' successor state to some states. The state of the system could then sometimes be considered to be indeterminate with respect to certain properties. Of course, the previous concept of a dynamical system is still included in this as a function returning a singleton on each input, and we can identify them among the maximally consistent relations $\rightarrow$ via a property of *point-preservation*: for all $x \in X$, there exists a $y \in X$ such that $\bigcap \rightarrow(\mathcal{N}(x)) = \{y\}$. Note that a point-preserving relation is automatically consistent: if $\emptyset \neq \bigcap U_i \ni x$, then $\emptyset \neq \bigcap \rightarrow(\mathcal{N}(x)) \subseteq \bigcap \rightarrow(U_i)$.

The amount of non-determinism is limited by $\langle$max-det$\rangle$: any observable property that could consistently be mapped inside another property is assumed to actually be mapped inside that one. That is, $f(x)$ must be as small as can be determined via the observable properties. Obviously this is satisfied if f is already fully deterministic.

Furthermore, the notion of continuity has been weakened. Let us consider $\langle$weak-cont$\rangle$ for the special case where the dynamics behave like a function $f : X \rightarrow X$. If f is continuous, then $\bigcup_{U \ni x} \{V \mid f(U) \subseteq V\} = \mathcal{N}(f(x))$, so f satisfies $\langle$weak-cont$\rangle$ via $\langle \mathsf{T}_1 : \mathcal{N}\text{-max} \rangle$. In particular, a topological observation space with a topologically continuous function on it constitutes a dynamical system as just defined. Thus, $\langle$weak-cont$\rangle$ is a weakening of the previous continuity requirement: continuity posits that *all* neighbourhoods of $f(x)$ have some neighbourhood of x mapped into them, but we will only need that enough neighbourhoods of $f(x)$ satisfy that property so that the intersection of these neighbourhoods is $\{f(x)\}$.

Now we can show that maximally consistent relations correspond to dynamical systems, as we desired.

THEOREM 1. Let (X,τ) be an observation space. For every dynamical system f on X, the relation $\to_f$, defined as $U \to_f V$ iff $f(U) \subseteq V$, is maximally consistent. Conversely, for any maximally consistent $\to$, the function $f_\to$ mapping each $x \in X$ onto $\bigcap \to(\mathcal{N}(x))$ is a dynamical system. Furthermore, $f_{\to_f} = f$ and $\to_{f_\to} = \to$ for all such functions and relations. ⟨dynsys=max-cons⟩

Proof. Let f be a dynamical system and $\to$ a maximally consistent relation.

⟨$\to_f$ is maximally consistent⟩

⟨$\to_f$ is consistent⟩

Suppose $\bigcap S \neq \emptyset$ for some $S \subseteq \tau$. We show that $\bigcap \to_f (S) \neq \emptyset$. By definition, for each $U \in S$, we have $f(U) \subseteq V$ for each $U \to_f V$, so $f(U) \subseteq \bigcap \to_f(U)$. Let $x \in \bigcap S$, then $f(x) \subseteq f(U) \subseteq \bigcap \to_f(U)$ for all $U \in S$, so $f(x) \subseteq \bigcap \to_f(S)$, and since $f(x) \neq \emptyset$ by definition, this proves the claim.

⟨$\to_f$ is maximal⟩

Suppose $\to' \supseteq \to_f$ is consistent and $U \to' V$. We show that $U \to_f V$, i.e. $f(U) \subseteq V$. By ⟨max-det⟩, it suffices to show that $f(x) \cap V \neq \emptyset$ for all $x \in U$.

Thus let $x \in U$. Then $\bigcap \to'(\mathcal{N}(x)) \subseteq \bigcap \to_f(\mathcal{N}(x)) \cap V$ by assumption, and $\bigcap \to_f(\mathcal{N}(x)) = \bigcap \bigcup_{U' \ni x} \{W \in \tau \mid f(U') \subseteq W\}$ by definition. We first show that the latter is equal to $f(x)$, and then that its intersection with V is non-empty, as required.

⟨$f(x) = \bigcap \bigcup_{U' \ni x} \{W \in \tau \mid f(U') \subseteq W\}$⟩

⟨$\subseteq$⟩

If $y \in f(x)$, then $y \in f(U')$ for all $U' \in \mathcal{N}(x)$, so $y \in W$ for all supersets W of such $f(U')$.

$\langle \supseteq \rangle$

If $y \in V$ for all $W \supseteq f[U]$ for all $U \in \mathcal{N}(x)$, then $y \in f(x)$ by ⟨weak-cont⟩.

$\langle \bigcap \bigcup_{U' \ni x} \{W \in \tau \mid f(U') \subseteq W\} \cap V \neq \emptyset \rangle$

The left-hand side is equal to $\bigcap \rightarrow_f (\mathcal{N}(x)) \cap V$, hence a superset of $\bigcap \rightarrow'(\mathcal{N}(x))$, which is non-empty because $\rightarrow'$ is consistent.

$\langle f_{\rightarrow}$ is a dynamical system $\rangle$

$\langle f_{\rightarrow}(x) \neq \emptyset$ for all $x \in X \rangle$

Clearly $x \in \bigcap \mathcal{N}(x) \neq \emptyset$, so $f_{\rightarrow}(x) = \bigcap \rightarrow(\mathcal{N}(x)) \neq \emptyset$ by consistency of $\rightarrow$.

⟨max-det⟩

Let $U, V \in \tau$. We show that if $f_{\rightarrow}(x) \cap V \neq \emptyset$, for all $x \in U$, then $f_{\rightarrow}(U) \subseteq V$.

$[f_{\rightarrow}(x) \cap V \neq \emptyset$, for all $x \in U]$

Then, $\bigcap \rightarrow(\mathcal{N}(x)) \cap V \neq \emptyset$ for all $x \in U$. Let $\rightarrow' = \rightarrow \cup \{(U, V)\}$. We show that $\rightarrow'$ is consistent; by maximality, then $\rightarrow' = \rightarrow$, hence $U \rightarrow V$ and $V \in \rightarrow(\mathcal{N}(x))$ for all $x \in U$, so $f_{\rightarrow}(U) = \bigcup_{x \in U} \bigcap \rightarrow(\mathcal{N}(x)) \subseteq V$ as needed.

$\langle \rightarrow'$ is consistent $\rangle$

It suffices to check that $\bigcap \rightarrow'(\mathcal{N}(x)) \neq \emptyset$ for all $x \in U$ since every set of consistent sets in τ can be extended to the neighbourhood filter of a point, by ⟨obs-max-elem⟩ "translated" to observation spaces. But $\bigcap \rightarrow'(\mathcal{N}(x)) = \bigcap \rightarrow(\mathcal{N}(x)) \cap V \neq \emptyset$ by assumption.

⟨weak-cont⟩

Let $x, y \in X$. We show that if $y \in V$ for all $V \supseteq f_{\rightarrow}(U)$ and $U \ni x$, then $y \in f_{\rightarrow}(x)$.

[$y \in V$ for all $V \supseteq f_{\to}(U)$ and $U \ni x$]

We show that $y \in V$ for all $V \in \tau$ such that $U \to V$ for some $U \ni x$. It then follows that $y \in \bigcap \to(\mathcal{N}(x)) = f_{\to}(x)$.

[$U \ni x$ and $U \to V$]

If $z \in U$, then $f_{\to}(z) = \bigcap \to(\mathcal{N}(z)) \subseteq \bigcap \to(U) \subseteq V$, so $V \supseteq f_{\to}(U)$, hence $y \in V$ by assumption.

$\langle f_{\to_f} = f \rangle$

We have $f_{\to_f}(x) = \bigcap \to_f(\mathcal{N}(x)) = f(x)$ as shown above in $\langle \to_f$ is maximal$\rangle$.

$\langle \to_{f_\to} = \to \rangle$

We have $U \to_{f_\to} V$ iff $f_{\to}(U) \subseteq V$ iff $\langle\star\rangle$ $\bigcap \to(\mathcal{N}(x)) \subseteq V$ for all $x \in U$. We show that $\langle\star\rangle$ iff $U \to V$.

$\langle U \to V$ implies $\star\rangle$

If $U \to V$, then $\bigcap \to(\mathcal{N}(x)) \subseteq \bigcap \to(U) \subseteq V$ for all $x \in U$.

$\langle \star$ implies $U \to V \rangle$

If $\langle\star\rangle$, let $\to' = \to \cup \{(U, V)\}$. Similarly to $\langle$`max-det`$\rangle$ in $\langle f_{\to}$ is a dynamical system$\rangle$ above, we show that $\to'$ is consistent, from which $U \to V$ again follows: by $\langle\star\rangle$, $\bigcap \to'(\mathcal{N}(x)) = \bigcap \to(\mathcal{N}(x)) \cap V = \bigcap \to(\mathcal{N}(x)) \neq \emptyset$. □

2.3 ORDERED VIEW: OBSERVATION DOMAINS

We now move to the most abstract level, connecting to domain theory. We first repeat the basic definitions of the theory (see [2] for reference), aiming to define *observation domains*, and then prove that they are equivalent to the structures we have seen so far. Lastly, we translate the notion of a consistent relation from Section 2.2, where they corresponded to approximations of dynamical systems, into the domain-theoretic setting. These relations will constitute the morphisms in the category of domains we study in Chapter 3.

TOWARDS A DEFINITION

Domain theory deals with certain types of ordered sets.

DEFINITION: PARTIAL ORDER. A *partially ordered set (poset)* is a pair $(P, \sqsubseteq)$ with P a set and $\sqsubseteq \subseteq P \times P$ a binary relation such that

⟨refl⟩ $p \sqsubseteq p$ for all $p \in P$;

⟨trans⟩ if $p \sqsubseteq q$ and $q \sqsubseteq r$, then $p \sqsubseteq r$; and

⟨antisym⟩ if $p \sqsubseteq q$ and $q \sqsubseteq p$, then $p = q$.

In domain theory, the elements of a poset P are typically interpreted to be partial descriptions of elements of a type, with $p \sqsubset q$ signifying that both p and q are valid partial descriptions of the same element, but q containing strictly more information than p. Often, the elements of the type are identified with the maximal elements of P. A subset of P then represents a set of such descriptions, and if it has a supremum in P, then it can be said that these descriptions can be "unified" under a least specific description extending all of them. That is, there exists an element to which all the descriptions in the set apply, and there is a canonical way of obtaining a description containing precisely the information contained in all the descriptions from the set.

DEFINITION: SUPREMUM. Let $(P, \sqsubseteq)$ be a poset and $A \subseteq P$. An element $p \in P$ is a *supremum* of A, denoted $\bigsqcup A$, if

⟨ub⟩ $a \sqsubseteq p$ for all $a \in A$ (i.e. p is an *upper bound* of A), and

⟨lub⟩ $p \sqsubseteq q$ for all upper bounds q of A.

A central idea is that "complex" descriptions containing an infinite amount of information can be obtained as a "limit" of "finitary" ones. Here, the idea of a "limit" is realised as the supremum of a set of increasingly complex descriptions. "Increasingly complex" means that the set of descriptions contains for each finite subset of itself a description unifying all descriptions in the subset. This is formalised via the notion of a *directed set*.

DEFINITION: DIRECTED SET. Let $(P, \sqsubseteq)$ be a poset. A set $D \subseteq P$ is called *directed* if for all finite $F \subseteq D$, there exists a $d \in D$ such that $p \sqsubseteq d$ for all $p \in F$.

Equivalently, a set is directed if it is non-empty and every pair of elements has an upper bound in the set. Now how can we make the difference between finitary and non-finitary elements precise? In short, a complex element can be obtained as the supremum of an infinite amount of descriptions not already including the element itself. The appropriate technical definition is as follows.

DEFINITION: FINITE ELEMENT. Let $(P, \sqsubseteq)$ be a poset. An element $p \in P$ is called *finite* if for every directed set D that has a supremum $\bigsqcup D$ with $p \sqsubseteq \bigsqcup D$, there is a $d \in D$ such that $p \sqsubseteq d$.

Note that a finite directed set always has a maximum, so for an element to be infinite, it has to be approximated by an infinite directed set not containing it. We want it to be the case that every infinite element is approximable in this way.

DEFINITION: ALGEBRAICITY. A poset $(P, \sqsubseteq)$ is *algebraic* if for all $p \in P$, there is a directed set $D \subseteq P$ of finite elements with supremum p.

For convenience, we also demand the converse to be true: any set of observations of which all finite subsets are consistent (have an upper bound) are actually parts of a description of an existing point.

DEFINITION: DCPO. A poset $(P, \sqsubseteq)$ is called a *directed-complete partial order* (dcpo) if it satisfies ⟨dir-comp⟩ if $D \subseteq P$ is a directed set, then D has a supremum $\bigsqcup D$.

Directed-completeness will turn out to be the domain-theoretic analogue of ⟨fip-ne⟩. Including this property may be seen as pragmatically motivated: what one is interested in is the approximation of "real" elements using finitary elements. Algebraicity is the relevant property here. Directed-completeness is the converse, and demanding it simplifies the technical treatment significantly since the existence of suprema of directed sets can then just be assumed rather than added as a precondition to each statement.

We can also motivate it from the idea of finitary observations: each directed-complete algebraic poset, is fully characterised by its finite elements. The domain can be recovered from the set of finite elements via ideal completion, i. e. identifying each directed down-set of finite elements with an element of a domain. Thus if we want to work with observation spaces not satisfying ⟨fip-ne⟩, we can add "imagined" points to the space this way before applying our theory.

A related criterion of "completeness" of a poset rests on the following idea: a set of descriptions with an upper bound is consistent, so if the poset has "enough" descriptions, it should also contain a least description unifying the bounded set.

DEFINITION: BOUNDED-COMPLETENESS. A poset $(P, \sqsubseteq)$ is called *bounded-complete* if it satisfies ⟨bnd-comp⟩ if $B \subseteq P$ is a bounded set (i. e. there is a $p \in P$ such that $p \sqsupseteq b$ for all $b \in B$), then B has a supremum $\bigsqcup B$.

Now for any observation system (P, σ), the set σ forms a poset under $\subseteq$. We will see that it has all of the aforementioned properties, but it even has some more: crucially, the singletons in σ are atoms, and every element of σ is a union of these atoms in a unique way. We need to translate this into the language of domain theory in order to establish an equivalence between observation systems and certain domains.

DEFINITION: ATOMISTIC ORDER. Let $(P, \sqsubseteq)$ be a partial order. An element $p \in P$ is called *positive* if there exists an $m \in P$ such that $m \sqsubseteq p$ but $m \neq p$. An element $a \in P$ is called an *atom* if a is positive and for all positive $b \in P$ with $b \sqsubseteq a$ also $b = a$. For $p \in P$, denote the set of atoms $a \in P$ with $a \sqsubseteq p$ by $\mathrm{At}\, p$. The poset P is called *atomistic* if $p = \bigsqcup \mathrm{At}\, p$ for every $p \in P$.

In an atomistic order, each description can be "composed" of basic descriptions which cannot be decomposed, namely the atoms. Clearly, the role of the atoms in the poset of an observation system is played by the singleton properties. Atomisticity alone is not enough to characterise the sorts of posets arising from observation systems. It should also be the case that different sets of basic observations are glued together to different unified descriptions. That is, the function mapping each set of atoms with a supremum to their supremum should be injective (note that atomisticity is equivalent to stating that it is surjective with respect to P). We denote the set of sets of atoms with a supremum by $\mathcal{C}(\mathrm{At}\, P)$, analogously to the consistent families in observation spaces.

DEFINITION: NUCLEARITY. A poset $(P, \sqsubseteq)$ is called *nuclear* if the function $\bigsqcup : \mathcal{C}(\mathrm{At}\, P) \to P$ with $A \mapsto \bigsqcup A$ is injective.

Let us clarify the relationship between atoms and finite elements. First, note that in a nuclear atomistic poset, we have $\mathrm{At} \bigsqcup S = \mathrm{At}\, S$ for all $S \subseteq P$ with a supremum: using atomisticity and the associativity of suprema, we have

$$\bigsqcup \mathrm{At} \bigsqcup S = \bigsqcup S = \bigsqcup_{p \in S} p = \bigsqcup_{p \in S} \bigsqcup \mathrm{At}\, p = \bigsqcup \bigcup_{p \in S} \mathrm{At}\, p = \bigsqcup \mathrm{At}\, S,$$

and then the claim follows by nuclearity.

PROPOSITION 2.3.1. In every nuclear atomistic partial order $(P, \sqsubseteq)$, every atom is a finite element. ⟨napo:at-fin⟩

Proof. Let $a \in \operatorname{At} P$ and suppose $D \subseteq P$ is directed with supremum $\bigsqcup D \sqsupseteq a$. Then, $a \in \operatorname{At} \bigsqcup D = \operatorname{At} D = \bigcup_{d \in D} \operatorname{At} d$, so $a \in \operatorname{At} d$ for some $d \in D$, hence $a \sqsubseteq d$. □

PROPOSITION 2.3.2. In every nuclear atomistic partial order $(P, \sqsubseteq)$, if for $p \in P$ the set $\operatorname{At} p$ is finite, then p is a finite element. ⟨napo:finat-fin⟩

Proof. Let $p \in P$ such that $\operatorname{At} p$ is finite. We show that p is a finite element. Thus let D be a directed set with $p \sqsubseteq \bigsqcup D$. For each $a \in \operatorname{At} p$, there must then exist a $d_a \in D$ such that $a \sqsubseteq d_a$ since atoms are finite by ⟨napo:at-fin⟩. Because $\operatorname{At} p$ is finite, $\{d_a \mid a \in \operatorname{At} p\}$ is also finite and thus by directedness has an upper bound $d \in D$. Then $\operatorname{At} p \subseteq \operatorname{At} d$, so $p = \bigsqcup \operatorname{At} p \sqsubseteq \bigsqcup \operatorname{At} d = d$ as required, by atomisticity and monotonicity of the operation sending a set to its supremum. □

PROPOSITION 2.3.3. Every nuclear atomistic bounded-complete partial order $(P, \sqsubseteq)$ is algebraic. ⟨nabcpo-alg⟩

Proof. If $p \in P$, then $\operatorname{At} p$ is bounded, and then so is every subset $A \subseteq \operatorname{At} p$. By bounded-completeness, each finite such A has a supremum $\bigsqcup A$. If $A, A' \subseteq \operatorname{At} p$ are finite, then so is $A \cup A'$, and $\bigsqcup A, \bigsqcup A' \sqsubseteq \bigsqcup (A \cup A')$. Thus the set $P_p := \{\bigsqcup A \mid A \subseteq_{\text{fin}} \operatorname{At} p\}$ is directed. It contains exclusively finite elements by ⟨napo:finat-fin⟩ since $\operatorname{At} \bigsqcup A = A$ by nuclearity, which is finite for all $A \subseteq_{\text{fin}} \operatorname{At} p$. Finally, its supremum is p: it is an upper bound since each $A \subseteq_{\text{fin}} \operatorname{At} p$ has $p \sqsupseteq a$ for all $a \in A$, so $\bigsqcup A \sqsubseteq p$ by ⟨lub⟩ of $\bigsqcup A$, and any upper bound p' of P_p needs to have $p' \sqsupseteq \bigsqcup A$ for all $A \subseteq_{\text{fin}} \operatorname{At} p$ by definition, in particular $p' \sqsupseteq a$ for all $a \in \operatorname{At} p$, so $p' \sqsupseteq \bigsqcup \operatorname{At} p = p$ by atomisticity. □

Note that in a *nuclear* atomistic bounded-complete poset, the finite elements are then *precisely* those $p \in P$ with $\operatorname{At} p$ finite: if $\operatorname{At} p$ is infinite, we can construct a directed set with supremum p as in the statement ⟨nabcpo-alg⟩ just shown, but none of its elements can be equal to p since the set of atoms below each of them is finite, and $\bigsqcup$ is injective.

We are now ready to define the kind of domains we are interested in.

DEFINITION: OBSERVATION DOMAIN. An *observation domain* $(P, \sqsubseteq)$ is a nuclear atomistic bounded-complete directed-complete partial order.

EQUIVALENCE WITH OBSERVATION SYSTEMS

We show how to turn observation systems into observation domains and vice versa.

PROPOSITION 2.3.4. If (P, σ) is a subset system, then $(\sigma, \subseteq)$ is a nuclear atomistic bounded-complete poset. ⟨subs-nabcpo⟩

Proof. First, note that any family of subsets ordered by inclusion is a poset.

⟨bounded-completeness⟩

Let $B \subseteq \sigma$ be bounded with bound $U \in \sigma$. By definition, $\bigcup B \subseteq U$, hence $\bigcup B \in \sigma$ by ⟨$\subseteq$-cl⟩. This is the supremum of B, since it is contained in *any* set containing all sets in B.

For each $S \in \sigma$, also $\{p\} \in \sigma$ for each $p \in S$, by ⟨$\subseteq$-cl⟩. Thus, the singletons $\{p\} \in \sigma$ are precisely the atoms of $(\sigma, \subseteq)$, and $\operatorname{At} S = \{\{p\} \mid p \in S\}$ for each $S \in \sigma$.

⟨nuclearity⟩

If $A, B \subseteq \operatorname{At} \sigma$ are distinct sets of atoms with suprema $\bigcup A, \bigcup B$, say $A \ni \{a\} \notin B$, then $\bigcup A \ni a \notin \bigcup B$, so their suprema are different.

⟨atomisticity⟩

Every $S \in \sigma$ is the union of its singletons, hence $S = \bigcup \operatorname{At} S$. □

Thus $(\sigma, \subseteq)$ is also algebraic by ⟨nabcpo-alg⟩, and then using ⟨napo:finat-fin⟩, it follows that finite elements are precisely those that contain finitely many singletons, i.e. that are finite as sets.

PROPOSITION 2.3.5. For every observation system (P, σ), the poset $(\sigma, \subseteq)$ is an observation domain. ⟨sys-dom⟩

Proof. By ⟨subs-nabcpo⟩, $(\sigma, \subseteq)$ is a nuclear atomistic bounded-complete partial order. It remains to show directed-completeness. Thus let $D \subseteq \sigma$ be directed. It suffices to show that $\bigcup D \in \sigma$. Let $F \subseteq \bigcup D$ be finite. Each $p \in F$ is contained in some $S_p \in D$ (by definition of $\bigcup D$). As $\{S_p \mid p \in F\}$ is finite since F is, by directedness, one obtains an $S \in D$ with $S_p \subseteq S$ for all $p \in F$, hence $F \subseteq \bigcup S_p \subseteq S$. By ⟨$\subseteq$-cl⟩, $F \in \sigma$. Thus each finite subset of $\bigcup D$ is in σ, hence $\bigcup D \in \sigma$ by ⟨fin-compat⟩. □

Now let $(D, \sqsubseteq)$ be an observation domain. Naturally, we would like to consider it as an observation system with the set of properties given by $\mathrm{At}\, D$. The consistent elements are then those $A \subseteq \mathrm{At}\, D$ that have a supremum in D.

PROPOSITION 2.3.6. Let $(D, \sqsubseteq)$ be a bounded-complete dcpo. Then, $(\mathrm{At}\, D, \mathcal{C}(\mathrm{At}\, D))$ is an observation system. ⟨bcdcpo-sys⟩

Proof.

⟨$\subseteq$-cl⟩

If $S' \subseteq S \in \mathcal{C}(\mathrm{At}\, D)$, then by definition $\bigsqcup S = d$ for some $d \in D$. Then $S' \subseteq S$ is a set of atoms bounded by d, so by ⟨bnd-comp⟩ it has a supremum d'. Thus $S' \in \mathcal{C}(\mathrm{At}\, D)$.

⟨fin-compat⟩

If for some $I \subseteq \mathrm{At}\, D$, each finite subset of $F \subseteq I$ is in $\mathcal{C}(\mathrm{At}\, D)$, i.e. $\bigsqcup F = d_F$ for some $d_F \in D$, then the set $\{d_F \mid F \subseteq I\}$ is directed: for finite $F, F' \subseteq I$, also $F \cup F' \subseteq I$ is finite, so there exists a $d_\cup$ such that $d_\cup = \bigsqcup(F \cup F') \sqsupseteq \bigsqcup F, \bigsqcup F' = d_F, d_{F'}$. Thus the set has a supremum d by ⟨dir-comp⟩, and $\mathrm{At}\, d \supseteq \bigcup_{F \subseteq I} \mathrm{At}\, d_F = I$, so $I \in \mathcal{C}(\mathrm{At}\, D)$ by ⟨$\subseteq$-cl⟩ as just shown. □

Say that two domains D, D' are isomorphic if there exists an order-preserving and -reflecting bijection between them, i. e. a bijection $b : D \to D'$ such that $p \sqsubseteq q$ iff $b(p) \sqsubseteq b(q)$.

PROPOSITION 2.3.7. For all observation domains $(D, \sqsubseteq)$, we have $(D, \sqsubseteq) \cong (\mathcal{C}(\mathrm{At}\, D), \subseteq)$. ⟨dom-sys-dom⟩

Proof. Consider the map $d \mapsto \mathrm{At}\, d$ for all $d \in D$. This is well-defined because $\bigsqcup \mathrm{At}\, d = d$ by atomisticity, so $\mathrm{At}\, d \in \mathcal{C}(\mathrm{At}\, D)$.

⟨injectivity⟩

If $\mathrm{At}\, d = \mathrm{At}\, d'$, then by atomisticity $d = \bigsqcup \mathrm{At}\, d = \bigsqcup \mathrm{At}\, d' = d'$.

⟨surjectivity⟩

If $A \subseteq \mathrm{At}\, D$ has a supremum $\bigsqcup A =: d \in D$, then by atomisticity $\bigsqcup \mathrm{At}\, d = d = \bigsqcup A$, and thus by nuclearity $\mathrm{At}\, d = A$.

⟨preservation of order⟩

If $d \sqsubseteq d'$ in D, then $\mathrm{At}\, d \subseteq \mathrm{At}\, d'$ by transitivity of $\sqsubseteq$.

⟨reflection of order⟩

If $A \subseteq A'$ in $\mathcal{C}(\mathrm{At}\, D)$, then $\bigsqcup A \sqsubseteq \bigsqcup A'$ for their preimages since the latter is also an upper bound of A. $\square$

There is again a small obstacle in showing the analogous statement for observation systems: we did not require that all $p \in P$ actually occur in a set in σ, and these disappear in passing to the corresponding observation domain. We can fix this by restricting the equivalence to those (P, σ) with ⟨all-props-cons⟩.

PROPOSITION 2.3.8. For all observation systems (P, σ) satisfying ⟨all-props-cons⟩, we have $(P, \sigma) \cong (\mathrm{At}\, \sigma, \mathcal{C}(\mathrm{At}\, \sigma))$. ⟨apc:dom-sys-dom⟩

Proof. Consider the map $p \mapsto \{p\}$ for all $p \in P$. This well-defined by ⟨all-props-cons⟩.

⟨bijectivity⟩

By ⟨$\subseteq$-cl⟩, the atoms of σ are precisely the singletons, and by ⟨all-props-cons⟩, these are precisely the sets $\{p\}$ for $p \in P$.

⟨ preservation of consistency ⟩

For each $S \in \sigma$, the image of S under the map is the set $\{\{s\} \mid s \in S\}$, which has S itself as a supremum in $(\sigma, \subseteq)$, thus is in $\mathcal{C}(\mathrm{At}\,\sigma)$.

⟨ reflection of consistency ⟩

Every $S' \in \mathcal{C}(\mathrm{At}\,\sigma)$ is a set of singletons from a set $S \subseteq P$, with a supremum in $(\sigma, \subseteq)$, which must then be $\bigcup S' = S$ by $\langle\subseteq\text{-cl}\rangle$, so $S \in \sigma$. □

Notice how the correspondence to observation systems offers another characterisation of observation domains. Namely, for any $d \in D$ in an observation domain, the set $\downarrow d := \{d' \in D \mid d' \sqsubseteq d\}$ is isomorphic to the powerset $\mathcal{P}(\mathrm{At}\,d)$ ordered by inclusion, since it includes a unique join for every subset of atoms. Since powerset lattices are equivalently complete atomic Boolean algebras (CABAs) [37, Theorem 2.4], observation domains are just dcpos where the down-set of each element is a CABA.° In analogy to how *L-domains* are dcpos where each down-set of an element is a complete lattice [16], we could thus refer to observation domains as *CABA-domains*.

°Instead of requiring a dcpo, we could also just require a poset in which each element has a maximal element above it.

TRANSLATING CONSISTENT RELATIONS

In Chapter 3, we want to study the structures presented in this chapter from a categorical perspective. The objects will be observation domains as described here, and as morphisms we generalise the notion of a consistent relation on an observation space to a relation between potentially different observation domains. This way, we obtain a category where the objects correspond to observation spaces/systems/domains, and the approximations to dynamical systems on these spaces are precisely the endomorphisms in this category.

A consistent set of properties on an observation space corresponds to a set of atoms with a supremum in the equivalent domain, so the notion of a maximally consistent relation on an observation space is translated into the domain-theoretic setting as a relation on the atoms of the domain such that the image of any subset of atoms with a supremum in the domain also has a supremum in the codomain.

DEFINITION: DOMAIN MAPPING. Let $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ be observation domains and $\to \subseteq \mathrm{At}\, D_1 \times \mathrm{At}\, D_2$ a relation. Then $\to$ is called a *domain mapping* from D_1 to D_2 if $S \in \mathcal{C}(\mathrm{At}\, D_1)$ implies $\to(S) \in \mathcal{C}(\mathrm{At}\, D_2)$, for all $S \subseteq \mathrm{At}\, D_1$.

We need to take some care when converting between observation spaces and domains: an observation domain is equivalent to an ⟨`all-props-cons`⟩-observation system, while an observation space is equivalent to a separated observation system, so ⟨`all-props-cons`⟩-observation spaces (satisfying $U \neq \emptyset$ for all $U \in \tau$) are equivalent to separated observation domains (satisfying the condition that $\max\colon \mathrm{At}\, D \to \mathcal{P}(\max D)$ is injective).

Let us briefly convince ourselves that we can indeed translate the concept of a maximally consistent relation without problems. A consistent relation on an observation space may not relate any inhabited sets to the empty set, and the empty set may itself be related to any property, so a maximally consistent relation relates the empty set to *everything*. We can therefore safely pass from a maximally consistent relation on an observation space to an equivalent one on an ⟨`all-props-cons`⟩-observation space by ignoring the behaviour with respect to the empty set. On the other side, a maximal domain mapping as just described needs to relate "equivalent" atoms (in the sense that they have identical sets of maximal elements above them) to exactly the same sets of atoms, so we can separate the domain (quotienting by equivalent atoms) without losing information on the behaviour of the original relation. Thus maximal domain mappings correspond to dynamical systems as defined in Section 2.2.

In the remainder of this section, let us look in a bit more detail at some properties of domain mappings. First, some simple examples.

EXAMPLES: DOMAIN MAPPINGS. Let D_1 and D_2 be observation domains.

- The empty relation $\emptyset \subseteq \mathrm{At}\, D_1 \times \mathrm{At}\, D_2$ is a domain mapping if° $D_2 \neq \emptyset$: the image of any set is empty, which has a supremum by ⟨bnd-comp⟩ if D_2 is non-empty.
- Any relation $A \times \{b\} \subseteq \mathrm{At}\, D_1 \times \mathrm{At}\, D_2$ is a domain mapping: a consistent set may or may not have an intersection with A. If it does, then the image is just $\{b\}$, which is consistent like any atom, if it does not, then the image is empty, which is consistent because $b \in D_2$, so $D_2 \neq \emptyset$.
- Let D_1, D_2 be domains and suppose D_2 is a CABA. Then, *every* relation $\rightarrow \subseteq \mathrm{At}\, D_1 \times \mathrm{At}\, D_2$ is a domain mapping, because every set of atoms in D_2 is consistent by completeness.
- Let D_1, D_2 be domains and suppose D_1 is a "flat" domain, i.e. of the form $D \cup \{\bot\}$ for some set D, with the order $\bot \sqsubseteq d$ for all $d \in D$ and all $d, d' \in D$ unordered. Then, a relation $\rightarrow \subseteq \mathrm{At}\, D_1 \times \mathrm{At}\, D_2$ is a domain mapping iff for all atoms $a \in \mathrm{At}\, D_1$, the set $\rightarrow(a)$ is consistent. This is because the only non-empty consistent sets of atoms of D_1 are the singleton atoms.

°and *almost* only if: the exception to the exception is if D_1 is also empty; then the empty relation is again a domain mapping

Every relation $\rightarrow \subseteq \mathrm{At}\, D_1 \times \mathrm{At}\, D_2$ induces an image function $\mathcal{P}(\mathrm{At}\, D_1) \rightarrow \mathcal{P}(\mathrm{At}\, D_2)$ sending each subset of D_1 to its set of successors under $\rightarrow$. It is well-known that the functions $f : \mathcal{P}(\mathrm{At}\, D_1) \rightarrow \mathcal{P}(\mathrm{At}\, D_2)$ arising this way from a relation are precisely those that preserves unions, i.e. $f(\bigcup S_i) = \bigcup f(S_i)$.° Because a domain mapping is a relation that sends consistent sets to consistent sets, its image function is a function $\mathcal{C}(\mathrm{At}\, D_1) \rightarrow \mathcal{C}(\mathrm{At}\, D_2)$ that preserves unions, and conversely every such function comes from a relation sending consistent sets to consistent sets, i.e. a domain mapping.

°It is clear that the image function preserves unions, and conversely, if a function between powersets preserves unions, then we can define a relation relating each element to the value of the function at its singleton, and then the function will be the image function of the relation by preservation of unions.

Note that since every set is a union of its singletons, we can equivalently demand that the function send sets to the union of the images of its singletons, i.e. $f(S) = \bigcup_{s \in S} f(s)$.

Now due to atomisticity and nuclearity, the consistent sets of atoms of an observation domain correspond bijectively to its elements, so a domain mapping $\rightarrow \subseteq \mathrm{At}\, D_1 \times \mathrm{At}\, D_2$ also gives rise to a function $D_1 \rightarrow D_2$ via $x \mapsto \bigsqcup \rightarrow(\mathrm{At}\, x)$. In working with observation domains, we will in fact identify elements with the consistent sets of atoms below them, and thus write $\rightarrow(x)$ instead of $\bigsqcup \rightarrow(\mathrm{At}\, x)$. With the above, a function $f : D_1 \rightarrow D_2$ then comes from a domain mapping iff f preserves existing joins of atoms (and arbitrary existing joins by extension), i.e. $f(x) = \bigsqcup f(\mathrm{At}\, x)$. This motivates the following.

DEFINITION: DOMAIN MAPPING (ALTERNATIVE). Let $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ be observation domains. A function $f : D_1 \rightarrow D_2$ is a *domain mapping* if for all $x \in D_1$, the set $f(\mathrm{At}\, x)$ has a supremum in D_2, and $f(x) = \bigsqcup f(\mathrm{At}\, x)$.

As we have argued, domain mappings in this sense "are" precisely domain mappings in the original sense, and we will thus variously treat domain mappings as relations $\rightarrow$ on atoms or functions f on elements, depending on the desired emphasis. There is another simple condition to recognise functions $D_1 \rightarrow D_2$ that are domain mappings. First, note that every such function is monotone: if $d \sqsubseteq d'$, then $\mathrm{At}\, d \subseteq \mathrm{At}\, d'$, so $\rightarrow(d) \sqsubseteq \rightarrow(d')$.

PROPOSITION 2.3.9. A monotone function $f : D_1 \rightarrow D_2$ between observation domains is a domain mapping if and only if for each $x \in D_1$ and $b \in \mathrm{At}\, f(x)$, there exists an $a \in \mathrm{At}\, x$ such that $b \in \mathrm{At}\, f(a)$. ⟨dm-at-reflect⟩

Proof.

⟨⇒⟩

If f is a domain mapping and $b \in \mathrm{At}\, f(x)$, then since $\mathrm{At}\, f(x) = \mathrm{At} \bigsqcup f(\mathrm{At}\, x) = \mathrm{At}\, f(\mathrm{At}\, x) = \bigcup_{a \in \mathrm{At}\, x} \mathrm{At}\, f(a)$ by nuclearity and atomisticity, there exists an $a \in \mathrm{At}\, x$ with $b \in \mathrm{At}\, f(a)$.

$\langle \Leftarrow \rangle$

If each $b \in \mathrm{At}\, f(x)$ has an $a \in \mathrm{At}\, x$ with $b \in \mathrm{At}\, f(a)$, then $\mathrm{At}\, f(x) \subseteq \bigcup_{a \in \mathrm{At}\, x} \mathrm{At}\, f(a)$, with the reverse inclusion holding by monotonicity, so

$$f(x) = \bigsqcup \mathrm{At}\, f(x) = \bigsqcup_{a \in \mathrm{At}\, x} \mathrm{At}\, f(a) = \bigsqcup f(\mathrm{At}\, x)$$

by atomisticity and nuclearity. □

Let us conclude this chapter by connecting domain mappings to the standard notion of morphisms between domains, given by *Scott-continuous* functions.

DEFINITION: SCOTT-CONTINUITY. For two dcpos $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$, a function $f : D_1 \to D_2$ is called *Scott-continuous* if for each directed $D \subseteq D_1$, the image $f(D)$ is directed (thus has a supremum), and $f(\bigsqcup D) = \bigsqcup f(D)$.

Every domain mapping is Scott-continuous as a function on elements, since it preserves arbitrary existing joins, in particular directed ones.° The converse does not hold in general, since a Scott-continuous function need not preserve all joins. However, since Scott-continuous functions are monotone, ⟨dm-at-reflect⟩ gives a necessary and sufficient condition for a Scott-continuous functions to be a domain mapping. This characterisation is similar to a characterisation of linear functions between coherent spaces in linear logic, except that the latter require the existence of a *unique* $a \in \mathrm{At}\, x$ with $b \in \mathrm{At}\, f(a)$ for all $b \in \mathrm{At}\, f(x)$ [8].

°That the image $f[D]$ is directed follows since for all $f(d_1), f(d_2)$ with $d_1, d_2 \in D$, we have $d_1, d_2 \sqsubseteq d$ for some $d \in D$ by directedness, so $d_1, d_2 \sqcup d = d$, and then $f(d_{1,2}) \sqcup f(d) = f(d)$ follows by preservation of joins, hence $f(d_{1,2}) \sqsubseteq f(d)$.

3 THE CATEGORY OF OBSERVATION DOMAINS

We ultimately want to turn a category of types of dynamical systems into a type theory. In Chapter 2, we have demonstrated how observation domains can be interpreted as structures containing descriptions of points of certain spaces, where each "real" point (a maximal element in the domain) can be seen as the "limit" of increasingly precise finitary descriptions, and how dynamical systems can be seen as relations on these domains. In this chapter, we study the category of observation domains, determining properties that will allow us to straightforwardly derive a type theory from it. First, we fix the definitions of the notions required for forming a category.

DEFINITION: DOMAIN. A *domain* is a *non-empty* bounded-complete, directed-complete partial order $(D, \sqsubseteq)$ satisfying atomisticity and nuclearity: the function from $\mathcal{C}(\mathrm{At}\, D)$ to D sending each $A \subseteq \mathrm{At}\, D$ with a supremum in D to its supremum is bijective.

This is what we previously called an "observation domain", except for the additional requirement that it be non-empty. This amounts to excluding the observation system $(\emptyset, \emptyset)$ (while still allowing $(\emptyset, \{\emptyset\})$), which does not seem to make a big difference for our conceptual setup, but it makes the theory we develop here go more smoothly. It also means that every domain D has a bottom element (by bounded-completeness, with the empty set being bounded since D is non-empty), which we will denote by $\bot_D$ or just $\bot$. The notion of a domain mapping is unaffected by this restriction.

DEFINITION: DOMAIN MAPPING. Let $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ be domains and $\to \subseteq \mathrm{At}\, D_1 \times \mathrm{At}\, D_2$ a relation. Then $\to$ is called a *domain mapping* from D_1 to D_2 if $S \in \mathcal{C}(\mathrm{At}\, D_1)$ implies $\to(S) \in \mathcal{C}(\mathrm{At}\, D_2)$.

Two domain mappings $D_1 \to_f D_2$ and $D_2 \to_g D_3$ can be composed by the usual composition of relations, i.e. $a\, (\to_f ; \to_g)\, b$ iff there exists a $z \in D_2$ such that $a \to_f z \to_g b$. It is easy to see that $\to_f ; \to_g$ is a domain mapping from D_1 to D_3. Furthermore, for each domain D the relation $\to_{\mathrm{id}_D} := \{a \to a \mid a \in \mathrm{At}\, D\}$ is a domain mapping.° Clearly composition is associative and id_D acts as a neutral element for it. Thus domains and domain mappings form a category.

°Note that we wrote $a \to a$ for the pair (a, a); this kind of notation will be helpful later in this document since we have to deal with many different structures involving pairs of elements.

DEFINITION: CATEGORY OF OBSERVATION DOMAINS. The category Obs is the category of domains with domain mappings between them.

We will frequently consider subcategories of Obs consisting of special types of domain mappings. Call a domain mapping $\to \subseteq \mathrm{At}\, D_1 \times \mathrm{At}\, D_2$ *total* if for all $a \in \mathrm{At}\, D_1$ there exists a $b \in \mathrm{At}\, D_2$ such that $a \to b$, and *univalent* if whenever $a \to b$ and $a \to b'$ for some $a \in \mathrm{At}\, D_1$ and $b, b' \in \mathrm{At}\, D_2$, then $b = b'$. An important class of morphisms is given by those that are both total and univalent.

DEFINITION: ATOM-PRESERVATION. A domain mapping is called *atom-preserving* if it is total and univalent.

Notice that totality is equivalent to "$\to(x) \neq \bot$ for all $x \neq \bot$", and univalence is equivalent to "$\to(a) \in \mathrm{At}\, D_2$ or $\to(a) = \bot$ for all $a \in \mathrm{At}\, D_1$". Thus an atom-preserving domain mapping maps each atom onto an atom, i. e. it is simply a function on the sets of atoms that preserves consistent sets of atoms.

Denote by Obs* the *wide*° subcategory of Obs with domains and atom-preserving domain mappings. This is a category since identities are atom-preserving and so is the composite of atom-preserving maps.

°containing all the objects but possibly only some of the arrows

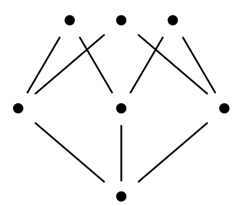
Other categories we work with include the category Set of sets and functions, Rel of sets and binary relations between them, as well as their *full*° subcategories FinSet and FinRel of finite sets, and the full subcategories FinObs and FinObs* of FinObs and FinObs*, respectively, of domains with finitely many elements.

°containing possibly only some of the same objects but all arrows between those objects

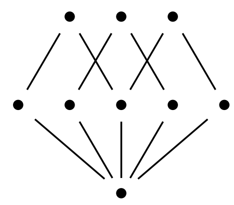
Our categorical investigations will proceed as follows. Section 3.1 performs some preliminary work, representing the categories of domains as categories of certain presheaves, which will help to simplify the further treatment. Then, in Section 3.2, we study the different universal constructions the category admits, including limits and colimits as well as the closed monoidal structure of Obs. Finally, in Section 3.3, we characterise the category FinObs, among others, fully in category-theoretic language by giving a universal property it satisfies, in terms of a *free construction*. This will be central to establishing completeness of a type theory derived from this category in Chapter 4.

3.1 DOMAINS AS PRESHEAVES

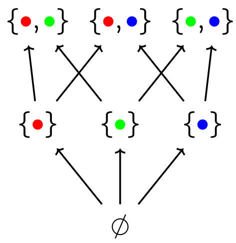
As we have discussed in Section 2.3, a domain is just a (now non-empty) poset where each element has a maximal element above it and the down-set of each element is a complete atomic Boolean algebra. This means that every domain is fully characterised by the CABAs it "contains". Consider the example on the right. It shows



a simple domain with three atoms and three maximal elements, each of which is a two-atom CABA containing a different pair of atoms. But there are other domains that have three two-atom CABAs, like the one in the margin now. There is more relevant information here, namely about which atoms are contained in which CABA.



In the categorical perspective on sets, we cannot "look inside" them to see what the names of their elements are, so sets with the same cardinality are indistinguishable to us. However, we can still specify how different sets relate to each other, via functions. This way, we can talk about the relative containment between the sets of atoms of different CABAs that "fit" into our example domain Consider the diagram on the right. This represents all the information contained in the domain, even though the elements of the sets are "anonymous", because we still know that there are three atoms, three two-atom consistent sets, and the inclusion functions tell us which atom is part of which two-atom set. Note that the diagram looks just like the domain with elements rewritten as sets of atoms like in the equivalent observation system.



Now instead of thinking of the singleton sets in this diagram as a singleton containing an atom of the domain, we can think of each such set as a "way" in which a one-atom CABA is contained in the domain, or more precisely as an atom-preserving map from a one-atom CABA into the domain. Similarly, the sets with two atoms can be seen as atom-preserving maps from a two-atom CABA into the domain. This way, a domain is fully characterised by the sets of atom-preserving maps from all n-atom CABAs into it, together with functions like in the previous diagram that specify how these different ways of mapping CABAs relate to each other. We are now very close to the following concept from category theory.

DEFINITION: PRESHEAF. A *presheaf* on a category C is a functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ from the *opposite category* of C into Set. The *category of presheaves* on C is the functor category $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$,

A *functor* $F : \mathsf{C} \to \mathsf{D}$ is an assignment of objects in C to objects in D and of morphisms $f \in \mathsf{C}(c,d)$ to morphisms $F(f) \in \mathsf{D}(F(c), F(d))$ that preserves identities and composites, i.e. $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$ and $F(f\,;g) = F(f)\,;F(g)$.

with the presheaves on C as objects and natural transformations between them as morphisms.

Any object c of C gives rise to a presheaf $\mathsf{C}(-,c)$ sending $d \in \mathsf{C}$ to the set $\mathsf{C}(d,c)$ of morphisms from d to c and $f : d \to d'$ to its precomposition function $f\,;-$ (which is a function from $\mathsf{C}(d',c)$ to $\mathsf{C}(d,c)$, with $g \mapsto f\,;g$). Such a presheaf captures all the information for how objects in C can be mapped into c, which in fact characterises c up to isomorphism, by the Yoneda lemma [23].

A *natural transformation* $\alpha : F \to G$ between functors is a function assigning to each $c \in \mathsf{C}$ a morphism $\alpha(c) \in \mathsf{D}(F(c), G(c))$ such that for all $f \in \mathsf{C}(c,d)$, the diagram

$$\begin{array}{ccc} F(c) & \xrightarrow{\alpha(c)} & G(c) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(d) & \xrightarrow[\alpha(d)]{} & G(d) \end{array}$$

commutes, i.e. $F(f)\,;\alpha(d) = \alpha(c)\,;G(f)$.

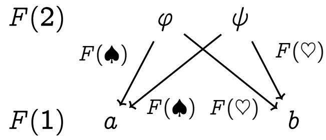
DEFINITION: REPRESENTABLE PRESHEAF. A presheaf is called *representable* if it is naturally isomorphic to $\mathsf{C}(-,c)$ for some $c \in \mathsf{C}$. The functor $\text{よ}_{\mathsf{C}} : \mathsf{C} \to [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ sending a $c \in \mathsf{C}$ to $\mathsf{C}(-,c)$ and $f : c \to d$ to the natural transformation with components $-\,;f$ is called the *Yoneda embedding* of C into its category of presheaves.

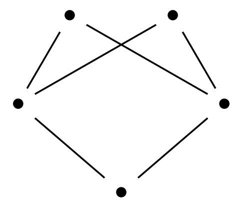
The Yoneda lemma implies that the Yoneda embedding is indeed an embedding, i.e. full and faithful, and so C is equivalent to its image under よ. Now a general presheaf on C can be seen as an "idealised" element of C, because it contains information on how elements of C can be mapped into it, even though there might not be an actual object in C with these specified relationships. More precisely, the Yoneda lemma states that there is a bijection between elements of the set $F(c)$ for F a presheaf on C and $c \in \mathsf{C}$, and the set of natural transformations $\text{よ}(c) \to F$ from the representable presheaf corresponding to c into F (and moreover, this bijection is natural in both F and c).

Let us use presheaves to represent domains by having them keep track of how CABAs can be mapped into them. First, we do this for Obs^*. From what we have discussed, we know that CABAs should correspond to representable presheaves. A morphism between CABAs is just any function between their sets of atoms. This suggests that the subcategory formed by the representables is just the category of sets and functions. In fact, finite sets are enough since every domain is determined by its finite elements, and every atom-preserving domain mapping sends atoms to atoms, so from the point of view of Obs^* it is as though the infinite elements did not exist.

Thus we represent domains in Obs^* as presheaves on the category FinSet of finite sets and functions, with $よ(N)$ for $N \in \mathsf{FinSet}$ representing a CABA with set of atoms given by N. Now a morphism from such a CABA $よ(N)$ into a domain is just a function from N into the set of atoms of the domain such that the image of all of N is consistent. With this, we have that a domain $D \in \mathsf{Obs}^*$ can be represented as a presheaf $\hat{D}$ on FinSet as follows. Send each $N \in \mathsf{FinSet}$ to the set $\{\varphi : N \to \mathrm{At}\, D \mid \varphi(N) \in \mathcal{C}(\mathrm{At}\, D)\}$, containing all consistent $|N|$-tuples of atoms of D, or equivalently all the possible ways in which an $|N|$-element CABA can be mapped into D. Note that in constrast to our example, this includes "degenerate" ways of mapping CABAs into the domains, in the sense that we do not require the functions to be injective; furthermore, the same CABA may hit the same atoms of the domain in different orders, and each of these different orders is a different way of mapping the CABA into the domain. On functions $f : N \to M$, the presheaf needs to turn each $\varphi : M \to \mathrm{At}\, D$ into a function $N \to \mathrm{At}\, D$ with a consistent image so as to specify the relationship between the ways $|M|$-element CABAs can be mapped into D and the ways $|N|$-element CABAs can be mapped into D, so $\hat{D}$ should send f to the precomposition $f\,;-$.

However, not every presheaf on FinSet is of this form. Notice that for a presheaf $\hat{D}$ obtained as just described, the set $\hat{D}(1)$ where 1 denotes the one-element set $\{\bullet\}$ in FinSet contains all functions from the one-element set into $\mathrm{At}\, D$, or equivalently just the set of atoms of D. Given some $\varphi \in \hat{D}(N)$, corresponding to a consistent N-indexed set of atoms of D, we can then evaluate which atom is the n-th atom of φ by computing $\hat{D}(n)(\varphi)$, where we identified n with the function $1 \to N$ given by $\bullet \mapsto n$. This yields an element of $\hat{D}(1)$, namely the one given by the composite $(\bullet \mapsto n)\,;\varphi$, which can be identified with the atom $\varphi(n)$. Now consider the example on the right. It shows part of a presheaf with two "atoms" (the set $F(1)$ has two elements) and two "consistent 2-tuples" (where 2 denotes the two-element set $\{\spadesuit, \heartsuit\}$), but both of them evaluate to the same sequence of atoms. There is nothing preventing a presheaf from doing this, and in this case we cannot interpret it as a presheaf $\hat{D}$ for some domain D. Loosely speaking, if we were to interpret this as a poset, we would get the result on the right, which is evidently not atomistic. We would like to say that the elements $\varphi \in F(N)$ are completely determined by the atoms $F(n)(\varphi)$ they evaluate to, for $n \in N$. Let us make this formal.





DEFINITION: CONCRETE CATEGORY. A category C together with an object $1 \in \mathsf{C}$ is called *concrete* if the functor $\mathsf{C}(1,-)\colon \mathsf{C} \to \mathsf{Set}$ is faithful. That is, for each pair c, d of objects in C and arrows $f, g \in \mathsf{C}(c,d)$, if $x\,;f = x\,;g$ for all $x \in \mathsf{C}(1,c)$, then $f = g$.

Clearly FinSet is concrete: any two functions $f,g\colon N \to M$ with $f(n) = n\,;f = n\,;g = g(n)$ for all $n \in \mathsf{FinSet}(1,N) \cong N$ are identical. In the literature, 1 is often required to be a terminal object, but we will not do so since we would then not be able to apply the theory to Obs.

DEFINITION: CONCRETE PRESHEAF. A presheaf on a concrete category is called *concrete* if for each $c \in \mathsf{C}$, the function sending each $\varphi \in F(c)$ to the map $\mathsf{C}(1,c) \to F(1)$ with $x \mapsto F(x)(\varphi)$, denoted $F(-)(\varphi)$, is injective. The *category of concrete presheaves* $\mathrm{Conc}\,\mathsf{C}$ on C has as objects all concrete presheaves on C and as morphisms all natural transformations between these functors.

A concrete presheaf F thus has a set of "atoms" $F(1)$, and for each $c \in \mathsf{C}$, then $F(c)$ can be seen as a collection of functions mapping global elements of c in C onto elements of $F(1)$, just as we desired.

PROPOSITION 3.1.1. Let C be a concrete category and $F \in \mathrm{Conc}\,\mathsf{C}$ a concrete presheaf on C. Then F is naturally isomorphic ⟨conc-rewrite⟩

to a functor F' with $F'(c) \subseteq \mathsf{Set}(\mathsf{C}(1,c),A)$ for all $c \in \mathsf{C}$ with A a set and $F'(f)(\varphi)(x) = \varphi(x\,;f)$ for all morphisms f in C.

Proof. We can set $A = F(1)$ and

$$F'(c) = \{F(-)(\varphi) \in \mathsf{Set}(\mathsf{C}(1,c),A) \mid \varphi \in F(c)\}.$$

Since $F(x)(\varphi) \in F(1)$ for all $x \in \mathsf{C}(1,c)$ and $\varphi \in F(c)$, this is well-defined. To construct the isomorphism, let $\alpha : F \to F'$ with $\alpha(c)(\varphi) = F(-)(\varphi)$. Each component is surjective by construction of $F'(c)$, and injective by concreteness of F. For $f : c \to d$ and all $\varphi \in F(d)$, we have $\alpha(c)(F(f)(\varphi)) = F(-)(F(f)(\varphi)) = F(-\,;f)(\varphi) = F'(f)(\alpha(d)(\varphi))$, so α is natural. □

Because every CABA is a domain, we need to make sure that the representables are concrete, which is indeed always the case.

PROPOSITION 3.1.2. Every representable presheaf $\mathsf{C}(-,c)$ on a concrete category C is concrete.

Proof. Let $d \in \mathsf{C}$ and $\varphi,\varphi' \in \mathsf{C}(d,c)$ and suppose $\mathsf{C}(x,c)(\varphi) = \mathsf{C}(x,c)(\varphi')$ for all $x : 1 \to d$. That is, $x\,;\varphi = x\,;\varphi'$ for all such x. Then $\varphi = \varphi'$ by concreteness of C. □

Crucially, a morphism between concrete presheaves is determined by its behaviour on 1. In our setup, this translates to the fact that a domain mapping in $\mathsf{Obs}*$ is a certain function between the sets of atoms of the involved domains.

PROPOSITION 3.1.3. Let F,G be concrete presheaves and α,β: $F \to G$ natural transformations. If $\alpha(1) = \beta(1)$, then $\alpha = \beta$. ⟨conc-nat-1⟩

Proof. Let $c \in \mathsf{C}$. Then, for all $\varphi \in F(c)$ and $x \in \mathsf{C}(1,c)$, we have

$$G(x)(\alpha(c)(\varphi)) = \alpha(1)(F(x)(\varphi)) = \beta(1)(F(x)(\varphi)) = G(x)(\beta(c)(\varphi))$$

by naturality, thus $\alpha(c)(\varphi) = \beta(c)(\varphi)$ for all $c \in \mathsf{C}$ and $\varphi \in F(c)$ by concreteness of G, hence $\alpha = \beta$. □

Now we can represent Obs* as a category of concrete presheaves. Call a presheaf $F \in [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ *non-empty* if there is a $c \in \mathsf{C}$ with $F(c) \neq \emptyset$.

PROPOSITION 3.1.4. The category Obs* is equivalent to the category Conc$^+$ FinSet of non-empty concrete presheaves on FinSet.

⟨obs*=conc+finset⟩

An *equivalence of categories* is a functor $F : \mathsf{C} \to \mathsf{D}$ that is full (for each $g \in \mathsf{D}(F(c), F(c'))$ there is an $f \in \mathsf{C}(c, c')$ with $F(f) = g$), faithful(whenever $F(f) = F(g)$ for $f, g \in \mathsf{C}(c, d)$, then $f = g$), and essentially surjective (for each $d \in \mathsf{D}$ there is a $c \in \mathsf{C}$ with $F(c) \cong d$).

Proof. We show that the following assignment constitutes an equivalence of categories from Obs* to Conc$^+$ FinSet. Send $D \in$ Obs* to $\hat{D}$ with $N \mapsto \{\varphi : N \to \mathrm{At}\, D \mid \varphi[N] \in \mathcal{C}(\mathrm{At}\, D)\}$ and $f \mapsto f\,;-$, and $\to : D \to E$ to $\hat{\to} : \hat{D} \to \hat{E}$ with $\hat{\to}(N) = -\,;\to$.

⟨well-defined⟩

Note that if $\varphi : M \to \mathrm{At}\, D$ with $\varphi(M) \in \mathcal{C}(\mathrm{At}\, D)$ and $f : N \to M$, then $f\,;\varphi : N \to \mathrm{At}\, D$ has $\varphi(f(N)) \subseteq \varphi(M)$, so $\varphi(f(N)) \in \mathcal{C}(\mathrm{At}\, D)$ by ⟨⊆-cl⟩ of $(\mathrm{At}\, D, \mathcal{C}(\mathrm{At}\, D))$ (see ⟨bcdcpo-sys⟩), thus $f \mapsto f\,;-$ is well-defined. Functoriality of $\hat{D}$ follows directly from the identity and associativity laws for categories. If $\varphi \in \hat{D}(N)$ and $\to : D \to E$, then $\hat{\to}(N)(\varphi) = \varphi\,;\to : N \to \mathrm{At}\, E$ with $\to(\varphi(N)) \in \mathcal{C}(\mathrm{At}\, E)$ because $\to$ is consistent, so the assignment $\hat{\to}$ is well-defined. Functoriality of $\hat{-}$ follows like for $\hat{D}$.

⟨$\hat{D}$ is a concrete presheaf⟩

Let $N \in$ FinSet and $\varphi, \varphi' \in \hat{D}(N)$ and suppose $\hat{D}(n)(\varphi) = \hat{D}(n)(\varphi')$ for all $n \in N$. That is, $\varphi(n) = \varphi'(n)$ for all n, hence $\varphi = \varphi'$.

⟨$\hat{\to}$ is a natural transformation⟩

We have $\hat{D}(f);\hat{\to}(N) = f;-;\to = \hat{\to}(M);\hat{E}(f)$ for all $f : N \to M$.

⟨essentially surjective⟩

For $P \in$ Conc$^+$ FinSet, we can w.l.o.g. assume that P is of the form given in ⟨conc-rewrite⟩. Let $A := P(1)$ and $\sigma := \bigcup_{N \in \mathsf{FinSet}} \{P(N)(\varphi) \mid \varphi \in P(N)\}$. We show that (A, σ) is a non-empty ($\sigma \neq \emptyset$) ⟨all-props-cons⟩-subset system. It follows that its ideal completion $(\mathcal{I}(\sigma), \subseteq)$ is a domain with $\sigma \subseteq \mathcal{I}(\sigma)$ and all sets in $\mathcal{I}(\sigma) \setminus \sigma$ are infinite. We then show that $\widehat{\mathcal{I}(\sigma)} \cong P$.

$\langle$ (A, σ) is a non-empty $\langle$`all-props-cons`$\rangle$-observation system $\rangle$

$\langle$ non-empty $\rangle$

Since P is non-empty, there is an $N \in \mathsf{FinSet}$ and a $\varphi \in P(N)$, so $P(N)(\varphi) \in \sigma \neq \emptyset$.

$\langle$`all-props-cons`$\rangle$

For all $a \in A = P(1)$, we have $\sigma \ni P(1)(a) = P(\bullet)(a) = \{a\}$.

$\langle$$\subseteq$`-cl`$\rangle$

If $S' \subseteq S \in \sigma$, then there exists an $N \in \mathsf{FinSet}$ such that $\varphi(N) = P(N)(\varphi) = S$ for some $\varphi \in P(N)$. Note that S and then S' are finite. Let $f \in \mathsf{FinSet}(S', N)$ map each $s \in S' \subseteq S$ to an $n \in N$ such that $\varphi(n) = s$. Then necessarily $P(s)(P(f)(\varphi)) = \varphi(f(s)) = s$ for all $s \in S'$, hence $\sigma \ni P(S')(P(f)(\varphi)) = S'$.

$\langle$ $\widehat{\mathcal{I}(\sigma)} \cong P$ $\rangle$

Define $\alpha: \widehat{\mathcal{I}(\sigma)} \to P$ as $\alpha(N)(\varphi)(n) = n$ for all $n \in N \in \mathsf{FinSet}$ and $\varphi \in \widehat{\mathcal{I}(\sigma)}(N)$. Injectivity and naturality are trivial.

$\langle$ well-defined $\rangle$

If $\varphi \in \widehat{\mathcal{I}(\sigma)}(N)$, then $\varphi : N \to \mathrm{At}\,\mathcal{I}(\sigma) = A = P(1)$ with $\varphi(N) \in \mathcal{I}(\sigma)$. Since N is finite, $\varphi(N)$ is finite as well, so $\varphi(N) \in \sigma$. By definition of σ, there exists a $\varphi' \in P(N)$ with $P(N)(\varphi') = \varphi'(N) = \varphi(N)$. That is, there exists a bijection $f : N \to N$ such that $\varphi(n) = \varphi'(f(n))$ for all $n \in N$. Then, $P(N) \ni P(f)(\varphi') = f\,;\varphi' = \varphi$.

$\langle$ surjective $\rangle$

If $\varphi \in P(N)$, then $P(N)(\varphi) \in \sigma \subseteq \mathcal{I}(\sigma)$, so $\varphi(N) \in \mathcal{C}(\mathrm{At}\,\sigma)$ and $\varphi \in \widehat{\mathcal{I}(\sigma)}(N)$.

⟨faithful⟩

If $\hat{\to}_f = \hat{\to}_g : \hat{D} \to \hat{E}$, then in particular $\hat{\to}_f(1) = \hat{\to}_g(1)$, so $\to_f(a) = \to_g(a)$ for all $a \in \hat{D}(1) = \mathrm{At}\,D$, hence $\to_f = \to_g$.

⟨full⟩

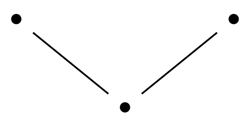
If $\alpha : \hat{D} \to \hat{E}$ is a natural transformation, let $\to \subseteq \mathrm{At}\,D \times \mathrm{At}\,E$ with $f := \alpha(1)$. This is a function $\hat{D}(1) = \mathrm{At}\,D \to \mathrm{At}\,E = \hat{E}(1)$, hence atom-preserving, and a domain mapping since if $S \in \mathcal{C}(\mathrm{At}\,D)$, then $\varphi : N \to \mathrm{At}\,D$ with $\hat{D}(N)(\varphi) = S$ is in $\hat{D}(1)$, and $\to(S) = \alpha(1)(\hat{D}(N)(\varphi)) = \hat{E}(N)(\alpha(N)(\varphi))$ by naturality of α, so $\to(S) \in \mathcal{C}(\mathrm{At}\,E)$. Clearly $\hat{\to}(1)(a) = \to(a) = \alpha(1)(a)$ for all $a \in \hat{D}(1)$, so $\alpha(1) = \hat{\to}(1)$, hence $\alpha = \hat{\to}$ by ⟨conc-nat-1⟩. □

A slight adaptation to this proof shows that Obs^* is also the category $\mathrm{Conc}\,\mathsf{FinSet}^+$ of concrete (possibly empty) presheaves on the category of finite *non-empty* sets. This is also shown in [3, Proposition 27], for the equivalent category of simplicial complexes.

Now following the same approach for Obs, we find that there are now as many morphisms between CABAs as there are relations between them. Thus, we shall consider concrete presheaves on FinRel, the category of finite sets with binary relations, and similarly turn domains D into such presheaves $\hat{D}$ where now $\hat{D}(N)$ contains all the possible consistent *relations* from the CABA with atoms N into D. In particular, $\hat{D}(1)$ is now the set of all consistent subsets of atoms of D. There is an immediate problem with this approach: consider the domain $D := (\mathcal{P}(\mathbb{N}), \subseteq)$. From the one-atom CABA, there are evidently $\mathsf{Rel}(1, \mathrm{At}\,\mathcal{P}(\mathbb{N})) = \mathsf{Rel}(1, \mathbb{N}) \cong \mathcal{P}(\mathbb{N})$ many domain mappings into D, so $\hat{D}(1) \cong \mathcal{P}(\mathbb{N})$ has uncountably many elements. There is however a presheaf F on FinRel that has $F(1) = \{A \mid A \subseteq_{\mathrm{fin}} \mathbb{N}\}$, the set of finite subsets of $\mathbb{N}$, as its set of "consistent sets of atoms", and each $F(N)$ contains all the relations that are subsets of $N \times F(1)$. This is easily seen to be concrete, but what kind of domain should this correspond to? Intuitively, as a poset, this would be the set $(\mathcal{P}_{\mathrm{fin}}(\mathbb{N}), \subseteq)$, which is not directed-complete. I leave the task of finding a way to enforce this directed-completeness in form of a condition on the corresponding concrete presheaves for the future; for our purposes here, this is not necessary.

The reason we might be interested in such a characterisation is that we could use it in Section 3.3 to turn it into a free construction, from which we could then derive a complete type theory in Chapter 4, but as we will see, this ambition is only sensible when restricted to finitary objects anyway, i. e. finite domains in the case of Obs. Since finite posets are trivially directed-complete, we may still try to apply our method to FinObs, the category of finite domains and domain mappings.

Is FinObs the category of non-empty concrete presheaves on FinRel? This is not the case: intuitively, such a presheaf F may have $F(1) = \{\emptyset, \{a\}, \{a,b\}\}$, where we interpret the elements as relations $\mathsf{FinRel}(1, \{a,b\})$ and have F send morphisms in FinRel to their precomposition function as usual. Then some $\varphi \in F(\{\spadesuit, \heartsuit\})$ might have $(\bullet \mapsto \spadesuit) ; \varphi = \{a\}$ and $(\bullet \mapsto \heartsuit) ; \varphi = \{a,b\}$, and then automatically $(\bullet \mapsto \{\spadesuit, \heartsuit\}) ; \varphi = \{a,b\}$, so $\{a,b\}$ does act like a join of $\{a\}$ and $\{a,b\}$. But the only domain with three elements is the one shown in the margin, where no element is a join of two distinct non-bottom elements. We have to somehow enforce that $F(1)$ has a structure that may arise as the set of consistent sets of atoms of a domain.



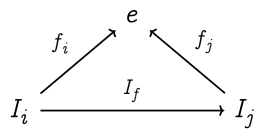
Because FinObs* and FinObs have the "same" objects, we are instead going take the objects from FinObs* and "transfer" them into the category of presheaves on FinRel. Note that going through the proof of ⟨`obs*=conc+finset`⟩, we find that FinObs* is the category $\mathrm{Conc}^{+}_{\mathrm{fin}}$ FinSet of finite° non-empty concrete presheaves on FinSet. The key point that comes to our help is the fact that every presheaf on a category C is a colimit (computed in the category of presheaves on C) of representable presheaves, in a "canonical"

°A presheaf $F \in [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ is called *finite* if for each $c \in \mathsf{C}$, the set $F(c)$ is finite.

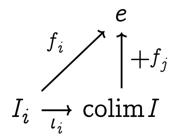
way. We are going to "disassemble" presheaves $F \in \mathrm{Conc}^{+}_{\mathrm{fin}}\,\mathsf{FinSet}$ into a diagram of representable presheaves that it is a colimit of. Then, we embed this diagram into FinRel, and take its colimit in the presheaf category of FinRel. The category of presheaves we obtain this way is equivalent to FinObs.

In the margins, we review the definitions of diagrams and colimits and fix notation and terminology. Note that for diagrams I, in order to reduce clutter we write I_i instead of $I(i)$ to denote functor application. Let us describe how a presheaf can be decomposed into the diagram it is the colimit of.

A *diagram of shape* I in a category C is a functor $I:\mathrm{I}\to\mathsf{C}$. A *cocone from* I is given by a collection of morphisms $f_i : I_i \to e$ for each $i \in \mathrm{I}$ for a fixed object $e \in \mathsf{C}$ such that



commutes for all $f : i \to j$ in I. A *colimit* of I is a cocone given by morphisms $\iota_i : I_i \to \operatorname{colim} I$, called *insertions*, such that for each cocone $f_i : I_i \to e$ from I there exists a *unique* morphism $+f_i : \operatorname{colim} I \to e$, called the *mediator of the cocone* f_i such that



commutes for all $i \in \mathrm{I}$.

DEFINITION: DIAGRAM OF A PRESHEAF. Let $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ be a presheaf. Define the *category of elements* $\mathrm{el}\,F$ of F as the category with objects (c,φ) for all $c \in \mathsf{C}$ and $\varphi \in F(c)$ and morphisms $(c,\varphi) \to (d,\psi)$ being morphisms $f : c \to d$ in C with $F(f)(\psi) = \varphi$. The *diagram of* F is defined as the forgetful functor $\pi^F : \mathrm{el}\,F \to \mathsf{C}$ with $(c,\varphi) \mapsto c$ and $f \mapsto f$.

It is well-known that $F \cong \operatorname{colim}(\pi^F ; \text{よ})$ ("co-Yoneda lemma", see [18, Section 3.4]). In fact, this construction is very similar to what we did at the beginning of the section for an example domain: we looked at maps of CABAs into the domain, and added arrows between them essentially if that arrow commuted with those maps. Now to compute the colimit of such diagrams, note that colimits in categories of presheaves are computed "pointwise". That means that if $I : \mathrm{I} \to \mathsf{C}$ is a diagram, then $\operatorname{colim}(I ; \text{よ})$ is given by the presheaf that sends each $c \in \mathsf{C}$ to the set $\operatorname{colim}(\text{よ}(I(-))(c)) \cong \operatorname{colim} \mathsf{C}(c, I(-))$, which is a colimit computed in Set, and each $f \in \mathsf{C}(c,d)$ to the colimit mediator $+_{i\in \mathrm{I}}\, \text{よ}(I_i)(f) ; \iota_i = +_{i \in \mathrm{I}}\, \iota_i(f ; -)$, which exists because every $g \in \mathrm{I}(i,j)$ has $\text{よ}(I_g)(d) ; \text{よ}(I_j)(f) ; \iota_j = \text{よ}(I_i)(f) ; \text{よ}(I_g)(c) ; \iota_j = \text{よ}(I_i)(f) ; \iota_i$ by naturality of $\text{よ}(I_g)$ and since the ι_i form a cocone.

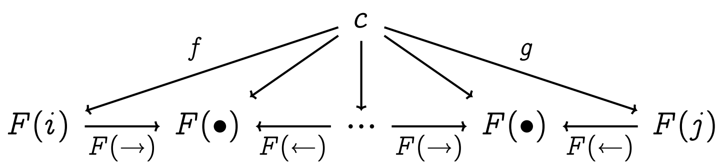
We will take presheaves $F \in \mathrm{Conc}^{+}_{\mathrm{fin}}\,\mathsf{FinSet}$ and send them to

$\operatorname{colim}(\pi^F ; \hookrightarrow ; \text{よ}_{\mathsf{FinRel}})$, computed pointwise in $[\mathsf{FinRel}^{\mathrm{op}}, \mathsf{Set}]$, where $\hookrightarrow$ denotes the inclusion $\mathsf{FinSet} \hookrightarrow \mathsf{FinRel}$. Before we show that the category of presheaves so obtained is indeed equivalent to FinObs, we need a bit more preparation to simplify the calculation of colimits in presheaf categories.

DEFINITION: CONNECTEDNESS. Let C be a category. Two objects $c, d \in \mathsf{C}$ are called *connected*, written $c \leftrightsquigarrow d$, if there is a finite sequence of morphisms

$$c \longrightarrow \bullet \longleftarrow \cdots \longrightarrow \bullet \longleftarrow d$$

in C starting with c and ending with d. For a functor $F : \mathrm{I} \to \mathsf{C}$, two morphisms $f, g : c \to F(i), F(j)$ are *connected via* F, written $f \leftrightsquigarrow_F g$, if they are connected as objects of the comma category $c \,/\, F$, meaning that there is a zig-zag of morphisms in I such that



commutes.

DEFINITION: FINAL FUNCTOR. A functor $F : \mathrm{I} \to \mathsf{J}$ is called *final* if for each $j \in \mathsf{J}$ there exists an $i \in \mathrm{I}$ with an arrow $j \to F(i)$, and any two arrows $j \to F(i), F(i')$ are connected via F.

For final functors, if $G : \mathsf{J} \to \mathsf{C}$ is any functor, then G has a colimit iff $F ; G$ has a colimit, and they are canonically isomorphic as follows [5, Section 2.11]: if $\operatorname{colim} G$ exists, then it is also a colimit of $F ; G$ together with the insertions $\iota^G_{F(i)}$, and if $\operatorname{colim}(F ; G)$ exists, then it is also a colimit of G, with the insertions $f ; \iota^{F;G}_{F(i)}$ where f is any arrow $j \to F(i)$ (which exists and the resulting insertion does not depend on the choice of f, by definition of final functors).

In the example at the beginning of the section, we essentially constructed a domain as a colimit of representables, but only using subset inclusions. The tool of final functors allows us to generalise this to arbitrary domains.

PROPOSITION 3.1.5. For a domain D, let $\mathrm{el}^{\sqsubseteq} D$ denote the category with objects the finite elements $d \in D$ and unique morphisms $d \to d'$ whenever $d \sqsubseteq d'$. Then, there exists a final functor $F : \mathrm{el}^{\sqsubseteq} D \to \mathrm{el}\, \hat{D}$ (from ⟨obs*=conc+finset⟩) sending each $d \in \mathrm{el}^{\sqsubseteq} D$ to $(\mathrm{At}\, d, \subseteq)$, where $\subseteq$ is the inclusion of $\mathrm{At}\, d$ into $\mathrm{At}\, D$, and $d \to d'$ onto the inclusion $\mathrm{At}\, d \subseteq \mathrm{At}\, d'$. ⟨el⊑->el⟩

Proof. Note that $\mathrm{el}^{\sqsubseteq} D$ is a category by reflexivity and transitivity of $\sqsubseteq$. Furthermore, F is well-defined because for each finite $d \in D$, the set $\mathrm{At}\, d$ is finite and $\mathrm{At}\, d \subseteq \mathrm{At}\, D = \hat{D}(1)$, with the inclusion being an element of $\hat{D}(\mathrm{At}\, d)$ because $\mathrm{At}\, d$ is consistent, so the map $a \mapsto a$ for all $a \in \mathrm{At}\, d$ has $\subseteq(\mathrm{At}\, d) \in \mathcal{C}(\mathrm{At}\, D)$. Furthermore, if $d \to d'$ in $\mathrm{el}^{\sqsubseteq}$, then $d \sqsubseteq d'$ by definition, so $\mathrm{At}\, d \subseteq \mathrm{At}\, d'$, which clearly commutes with the inclusions $\mathrm{At}\, d, \mathrm{At}\, d' \subseteq \mathrm{At}\, D$. Functoriality of F is easy to see.

Now let $(N, \varphi) \in \mathrm{el}\, \hat{D}$, so $\varphi : N \to \mathrm{At}\, D$ with $\varphi(N) \in \mathcal{C}(\mathrm{At}\, D)$. Let $d := \bigsqcup \varphi(N)$. Then each $n \in N$ can be mapped onto $\varphi(n) \in \mathrm{At}\, d$, which composed with the inclusion $\mathrm{At}\, d \subseteq \mathrm{At}\, D$ obviously commutes with φ, thus is an arrow $f : (N, \varphi) \to (\mathrm{At}\, d, \subseteq)$. Additionally, every pair of arrows $g, g' : (N, \varphi) \to (\mathrm{At}\, e, \subseteq), (\mathrm{At}\, e', \subseteq)$ must have $g(n) = \varphi(n) = g'(n)$ for all $n \in N$, so $\mathrm{At}\, d \subseteq \mathrm{At}\, e, \mathrm{At}\, e'$, hence $d \sqsubseteq e, e'$, so there are arrows $d \to e, e'$, with $f ; F(d \to e) = \varphi ; (\mathrm{At}\, d \subseteq \mathrm{At}\, e) = g$ and $f ; F(d \to e') = \varphi ; (\mathrm{At}\, d \subseteq \mathrm{At}\, e') = g'$, and then $g \leftrightsquigarrow_F g'$ as needed. □

As a final note before the proof, FinRel with the one-element set 1 is concrete, just like FinSet: if $R, R' \in \mathsf{FinRel}(N, M)$ have $X ; R = X ; R'$ for all $X \in \mathsf{FinRel}(1, N) \cong \mathcal{P}(N)$, then in particular $\bullet\, X\, n\, R\, m$ iff $\bullet\, X\, n\, R'\, m$ for all $n \in N$ and $m \in M$, hence $n\, R\, m$ iff $n\, R'\, m$, so $R = R'$.

PROPOSITION 3.1.6. The category FinObs of finite domains with domain mappings is equivalent to the full subcategory of the category of presheaves $F \in [\mathsf{FinRel}^{\mathrm{op}}, \mathsf{Set}]$ such that $F \cong \mathrm{colim}(\pi^D ; \hookrightarrow ; よ_{\mathsf{FinRel}})$ for some $D \in \mathrm{Conc}^+_{\mathrm{fin}}\,\mathsf{FinSet}$. ⟨finobs-prshf-finrel⟩

Proof. We prove that the following assignment is an equivalence of categories. For each finite domain D, let $\hat{D}(N) := \{\varphi \subseteq N \times \mathrm{At}\, D \mid \varphi(N) \in \mathcal{C}(\mathrm{At}\, D)\}$, and for $R \subseteq N \times M$ let $\hat{D}(R) := R ; -$. For $\rightarrow \subseteq \mathrm{At}\, D \times \mathrm{At}\, E$ let $\hat{\rightarrow} : \hat{D} \rightarrow \hat{E}$ with $\hat{\rightarrow}(N) := - ; \rightarrow$.

⟨well-defined⟩

The fact that $\hat{-}$ is a well-defined functor follows completely analogously to ⟨obs*=conc+finset⟩, except that we need to prove that for each $D \in \mathsf{FinObs}$, we have $\hat{D} \cong \mathrm{colim}(\pi^{D'} ; \hookrightarrow ; よ)$ for some $D' \in \mathrm{Conc}^+_{\mathrm{fin}}\,\mathsf{FinSet}$. Since $D \in \mathsf{FinObs}^*$, we shall in fact show that $\hat{D} \cong \mathrm{colim}(\pi^D ; \hookrightarrow ; よ)$ with π^D the diagram of D seen as a presheaf in $\mathrm{Conc}^+_{\mathrm{fin}}\,\mathsf{FinSet}$.

To this end, let $F : \mathrm{el}^{\sqsubseteq}_D \rightarrow \mathrm{el}\, D$ denote the final functor from ⟨el⊑->el⟩ and let $\pi^{\sqsubseteq} := F ; \pi^D$. Then, it suffices to show that $\hat{D} \cong \mathrm{colim}(\pi^{\sqsubseteq} ; \hookrightarrow ; よ)$ since the latter is then isomorphic to $\mathrm{colim}(\pi^D ; \hookrightarrow ; よ)$ by finality of F.

By the pointwise computation of colimits, it is enough to show that each $\hat{D}(N)$ satisfies the universal property of $\mathrm{colim}(\pi^{\sqsubseteq} ; \hookrightarrow ; よ)(N)$ in Set, and that it maps relations $R \subseteq N \times M$ onto colimit mediators $+_{d \in \mathrm{el}^{\sqsubseteq} D}\, \iota_d(R ; -)$.

Thus let $N \in \mathsf{FinRel}$. For all $d \in \mathrm{el}^{\sqsubseteq} D$, let $\iota_d : よ(\hookrightarrow(\pi^{\sqsubseteq}_d)) = \mathsf{FinRel}(N, \mathrm{At}\, d) \rightarrow \hat{D}$ sending $\varphi \in \mathsf{FinRel}(N, \mathrm{At}\, d)$ to $\varphi \in \hat{D}(N)$. This is well-defined since $\mathrm{At}\, d$ is consistent, and it determines a cocone: for $d \rightarrow d'$, we have $\mathrm{At}\, d \subseteq \mathrm{At}\, d'$, and for all $\varphi \in \mathsf{FinRel}(N, \mathrm{At}\, d)$ then $\iota_d(\varphi) ; \subseteq = \varphi = \iota_{d'}(\varphi) \in \mathsf{FinRel}(N, \mathrm{At}\, d')$.

Now suppose X has a cocone $g_d : \mathsf{FinRel}(N, \mathrm{At}\, d) \rightarrow X$. That is, $g_d(\varphi) ; \subseteq = g_{d'}(\varphi)$ if $d \sqsubseteq d'$. Define $m : \hat{D}(N) \rightarrow X$ as $\varphi \mapsto$

$g_{\bigsqcup\varphi(N)}(\varphi)$. This is well-defined because $\varphi(N)$ is consistent, hence has a supremum which is an element of D, and $\varphi \in \mathsf{FinRel}(N, \mathrm{At}\bigsqcup\varphi(N))$. It commutes with the cocone because for all $\varphi \in \mathsf{FinRel}(N, \mathrm{At}\, d)$, we have

$$m(\iota_d(\varphi)) = m(\varphi) = g_{\bigsqcup\varphi(N)}(\varphi) = g_d(\varphi)$$

as $\varphi(N) \subseteq \mathrm{At}\, d$, so $\bigsqcup\varphi(N) \sqsubseteq \bigsqcup \mathrm{At}\, d = d$. It is unique with that property since any $m'{:}\hat{D}(N) \to X$ with $m'(\iota_d(\varphi)) = g_d(\varphi)$ must have $m'(\varphi) = g_{\bigsqcup\varphi(N)}(\varphi)$ since $\bigsqcup\varphi(N) \sqsubseteq d$.

Finally, for a relation $R \subseteq N \times M$, the colimit mediator sends $\varphi \in \hat{D}(M)$ to $\iota_{\bigsqcup\varphi(M)}(R\,;\varphi) = R\,;\varphi = \hat{D}(R)(\varphi)$, as required.

⟨essentially surjective⟩

If F is of the form $\mathrm{colim}(\pi^D;\hookrightarrow;よ)$ for some $D \in \mathrm{Conc}^+_{\mathrm{fin}}\,\mathsf{FinSet}$, then interpreting D as a domain, we have $\hat{D} \cong \mathrm{colim}(\pi^D\,;\hookrightarrow;\, よ) \cong F$ as shown in ⟨`well-defined`⟩.

⟨faithful⟩

If $\hat{\rightarrow}_1 = \hat{\rightarrow}_2$, then $\varphi\,;\rightarrow_1 = \varphi\,;\rightarrow_2$ for all $\varphi \in \mathsf{FinRel}(1, \mathrm{At}\, D)$ with $\varphi(1) \in \mathcal{C}(\mathrm{At}\, D)$, in particular for all φ selecting an atom, hence $\rightarrow_1 = \rightarrow_2$.

⟨full⟩

Let $\alpha : \hat{D} \to \hat{E}$ and define $\rightarrow \subseteq \mathrm{At}\, D \times \mathrm{At}\, E$ as a function on elements corresponding to consistent subsets, via $\rightarrow = \alpha(1)$. It suffices to show that this function preserves consistent unions. Thus let $S \subseteq \mathcal{C}(\mathrm{At}\, D)$ with $\bigcup S \in \mathcal{C}(\mathrm{At}\, D)$. Note that $S \in \mathsf{FinRel}$ and $\bigcup S \in \hat{D}(1)$. Let $\varphi \in \mathsf{FinRel}(S, \mathrm{At}\, D)$ relate each $s \in S$ to its elements. Clearly $\varphi(S) = \bigcup S \in \mathcal{C}(\mathrm{At}\, D)$, so $\varphi \in \hat{D}(S)$. Because also $S\,;\varphi = \bigcup S$, where we wrote S for the relation $\{\bullet\} \times S \in \mathsf{FinRel}(1, S)$, we then have $\rightarrow(\bigcup S) = \alpha(1)(\bigcup S) = \alpha(1)(S\,;\varphi) = \alpha(1)(\hat{D}(S)(\varphi)) = \hat{E}(S)(\alpha(S)(\varphi)) = S\,;\alpha(S)(\varphi)$ using naturality of α, and similarly $\bigcup_{s\in S} \rightarrow(s) = \bigcup \alpha(1)(s) = \bigcup \alpha(1)(s\,;\varphi) = \bigcup s\,;\alpha(S)(\varphi) = S\,;\alpha(S)(\varphi)$, with the last step following from the definition of the composition of relations.

Like in ⟨obs⋆=conc+finset⟩, we have $\hat{\rightarrow}(1) = \alpha(1)$, hence $\hat{\rightarrow} = \alpha$ since $\hat{D}$ and $\hat{E}$ are concrete (suppose $\hat{D}(X)(\varphi) = \hat{D}(X)(\varphi')$ for all $X \in \mathsf{FinRel}(1, N)$ for some $N \in \mathsf{FinRel}$ and $\varphi, \varphi' \in \hat{D}(N)$, then $X ; \varphi = X ; \varphi'$, hence $\varphi = \varphi'$ since FinRel is concrete). □

3.2 UNIVERSAL CONSTRUCTIONS

By exploring the constructions our category admits, we can determine what constructions can be soundly interpreted in a type theory we create for it. We establish an adjunction between Obs and Obs*, before looking at limits and colimits in the two categories. Then, we show that Obs is monoidal closed, with the tensor product given by the product in Obs*. We will see that Obs* is a *quasitopos* and study the associated structure. Finally, we discuss computationally useful initial and terminal algebras for some functors.

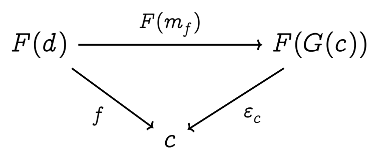
Similarly to how the inclusion $\mathsf{Set} \hookrightarrow \mathsf{Rel}$ has a right adjoint, given by the powerset functor [14], the inclusion $\mathsf{Obs}^* \hookrightarrow \mathsf{Obs}$ also has such an adjoint,° sending each domain to a domain where the atoms are the consistent subsets of the original domain.

°In fact, Obs is the *Kleisli category* of the induced monad, just like Rel is for the powerset monad on Set.

PROPOSITION 3.2.1. The inclusion functor $\mathsf{Obs}^* \hookrightarrow \mathsf{Obs}$ has a right adjoint $\mathcal{P}_{\mathcal{C}} : \mathsf{Obs} \to \mathsf{Obs}^*$ sending each domain $D \in \mathsf{Obs}$ to the domain $(\mathcal{P}_{\mathcal{C}}(D), \subseteq)$ where $\mathcal{P}_{\mathcal{C}}(D) := \{S \subseteq \mathcal{C}(\mathrm{At}\, D) \mid \bigcup S \in \mathcal{C}(\mathrm{At}\, D)\}$, and each domain mapping $\rightarrow \subseteq \mathrm{At}\, D_1 \times \mathrm{At}\, D_2$ to the atom-preserving domain mapping $\mathcal{P}_{\mathcal{C}}(\rightarrow)(\{A\}) = \{\rightarrow(A)\}$.

⟨obs⋆-obs-adj⟩

A functor $F : \mathsf{D} \to \mathsf{C}$ is a *left adjoint* if for each $c \in \mathsf{C}$, there is an object $G(c) \in \mathsf{D}$ with a morphism $\varepsilon_c : F(G(c)) \to c$ such that for every $d \in \mathsf{D}$ and $f : F(d) \to c$ there exists a unique morphism $m_f : d \to G(c)$ such that



commutes. In that case, F has a right adjoint, given by $c \mapsto G(c)$ and $f : c \to c' \mapsto m_{\varepsilon_c ; f}$.

Proof. We first check that $(\mathcal{P}_{\mathcal{C}}(D), \subseteq)$ really is a domain, and then we prove that the required universal property is satisfied. This determines the right adjoint as described in the margins.

⟨$(\mathcal{P}_{\mathcal{C}}(D), \subseteq)$ is a domain⟩

By ⟨sys-dom⟩, it suffices to show that $(\mathcal{C}(\mathrm{At}\, D), \mathcal{P}_{\mathcal{C}}(D))$ is a non-empty observation system.

$\langle$`non-empty`$\rangle$

We have $\emptyset \subseteq \mathcal{C}(\mathrm{At}\,D)$ and $\bigcup\emptyset = \emptyset \in \mathcal{C}(\mathrm{At}\,D)$ since D is non-empty, so $\emptyset \in \mathcal{C}(\mathrm{At}\,D)$.

$\langle\subseteq$`-cl`$\rangle$

If $S' \subseteq S \in \mathcal{P}_{\mathcal{C}}(D)$, then $\bigcup S \in \mathcal{C}(\mathrm{At}\,D)$, so also $\bigcup S' \in \mathcal{C}(\mathrm{At}\,D)$ by $\langle\subseteq$`-cl`$\rangle$ of $(\mathrm{At}\,D, \mathcal{C}(\mathrm{At}\,D))$.

$\langle$`fin-compat`$\rangle$

If $F \in \mathcal{P}_{\mathcal{C}}(D)$, i.e. $\bigcup F \in \mathcal{C}(\mathrm{At}\,D)$, for all finite $F \subseteq I$ for some $I \subseteq \mathcal{C}(\mathrm{At}\,D)$, then $\{\bigsqcup\bigcup F \mid F \subseteq_{\mathrm{fin}} I\}$ is a directed set in D, hence by $\langle$`dir-comp`$\rangle$ has a supremum, with atoms $\bigcup_{F\subseteq_{\mathrm{fin}} I}\bigcup F = \bigcup I$, thus $\bigcup I \in \mathcal{C}(\mathrm{At}\,D)$ and $I \in \mathcal{P}_{\mathcal{C}}(D)$.

Now let $\rightarrow_\varepsilon \subseteq \mathrm{At}\,\mathcal{P}_{\mathcal{C}}(D) \times \mathrm{At}\,D$ with $\{A\} \rightarrow_\varepsilon a$ iff $a \in A$, i.e. $\rightarrow_\varepsilon(\{A\}) = A$.

$\langle \rightarrow_\varepsilon$ is a domain mapping$\rangle$

If $S \in \mathcal{C}(\mathcal{P}_{\mathcal{C}}(D))$, i.e. $\bigcup S \in \mathcal{C}(\mathrm{At}\,D)$, then

$$\rightarrow_\varepsilon(S) = \bigcup_{A\in S} \rightarrow_\varepsilon(\{A\}) = \bigcup_{A\in S} A = \bigcup S \in \mathcal{C}(\mathrm{At}\,D).$$

Suppose $E \in \mathsf{Obs}^*$ and $f : E \to D$ is a domain mapping. Define $m_f : E \to \mathcal{P}_{\mathcal{C}}(D)$ as the atom-preserving domain mapping $e \mapsto \{f(e)\}$.

$\langle m_f$ is an atom-preserving domain mapping$\rangle$

This is well-defined and atom-preserving because f is a domain mapping, so each $f(e)$ is consistent and $\{f(e)\}$ is an atom of $\mathcal{P}_{\mathcal{C}}(D)$. If $A \in \mathcal{C}(\mathrm{At}\,E)$, then $\bigcup m_f(A) = \bigcup\bigcup_{e\in A}\{f(e)\} = \bigcup f[A] = f(A) \in \mathcal{C}(\mathrm{At}\,D)$, so $m_f(A) \in \mathcal{P}_{\mathcal{C}}(D)$.

$\langle m_f$ is a unique mediator$\rangle$

We have $\rightarrow_\varepsilon(m_f(e)) = \rightarrow_\varepsilon(\{f(e)\}) = f(e)$ for all $e \in \mathrm{At}\,E$, and any atom-preserving $m' : E \to \mathcal{P}_{\mathcal{C}}(D)$ with $\rightarrow_\varepsilon(m'(e)) = f(e)$ needs to have $m'(e) = \{f(e)\}$ by definition of $\rightarrow_\varepsilon$.

Thus the inclusion is a left adjoint, with the right adjoint given on objects by $\mathcal{P}_C$, and sending morphisms $\to$ to maps $\{A\} \mapsto m_{\to_\varepsilon;\to}(\{A\}) = \{\to(\to_\varepsilon(\{A\}))\} = \{\to(A)\}$, as claimed. □

Similarly, we can prove that there is a right adjoint to $\mathsf{FinObs}^* \hookrightarrow \mathsf{FinObs}$ with the same definition and proof. This adjunction can probably be derived from the analogous adjunction $\mathsf{FinSet} \hookrightarrow \mathsf{FinRel}$, as FinObs^* is the category of finite non-empty concrete presheaves on FinSet and to obtain FinObs, we decompose presheaves into their elements and apply $\mathsf{FinSet} \hookrightarrow \mathsf{FinRel}$ before recomposing them. In fact, many of the phenomena discussed here may have a simpler explanation in terms of such adjunctions, for example the monoidal closed structure of Obs arises in parallel to how it arises for (Fin)Rel from (Fin)Set.

LIMITS AND COLIMITS

Let us begin with simple structures in Obs.

PROPOSITION 3.2.2. The set $\{\bot\}$ with the reflexive relation is a zero object in Obs, denoted by $0 \cong 1$. The unique morphisms $1_D : D \to 1$ and $0_D : 0 \to D$ are given by the empty relation.

⟨obs-zero⟩

A *terminal object* is an object 1 such that for each c there exists a unique morphism $c \to 1$. An *initial object* is an object 0 such that for each c there exists a unique morphism $0 \to c$. A *zero object* is an object that is both initial and terminal, i.e. $0 \cong 1$.

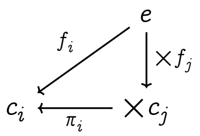
Proof. We first note that $0 \in \mathsf{Obs}$ since it is just the zero-atom CABA. Now let $D \in \mathsf{Obs}$. We need to show that there exist a unique morphisms $D \to 1$ and $0 \to D$. Clearly the empty relation is a domain mapping. Furthermore any domain mapping out of and into $0 \cong 1$ must be empty since 0 has no atoms. □

As a function on elements, the empty domain mapping sends every element in its domain to $\bot$. The presence of a zero object makes Obs a *pointed* category. In a pointed category, any pair c, d of objects has a morphism $0_{c,d} : c \to d$ given by the composite $1_c ; 0_d$. In our case this just means that the empty relation is a domain mapping between any pair of objects.

PROPOSITION 3.2.3. For any collection of domains $(D_i, \sqsubseteq_i) \in$ Obs, their product $\times(D_i, \sqsubseteq_i)$ in Obs exists and is given by the cartesian product $D_\times := \times D_i$ together with the relation $\times x_i \sqsubseteq_\times \times y_i$ iff $x_i \sqsubseteq_i y_i$ for all i and projections $\pi_i : D_\times \to D_i$ with $\pi_i(\times x_j) = x_i$. For any $E \in$ Obs with morphisms $f_i : E \to D_i$ for each i, the mediator $\times f_i : E \to D_\times$ is given by $x \mapsto \times(f_i(x))$.

⟨obs-prod⟩

A *product* of a collection c_i of objects is an object $\times c_i$ with morphisms $\pi_i : \times c_j \to c_i$ for all i, called *projections*, such that for each e with morphisms $f_i : e \to c_i$ for all i there exists a unique mediator $\times f_i : e \to \times c_i$ such that



commutes for all i.

Proof.

⟨$(D_\times, \sqsubseteq_\times) \in$ Obs⟩

We perform an equivalent construction on the corresponding observation systems. Recall that each $(\mathrm{At}\, D_i, \mathcal{C}(\mathrm{At}\, D_i))$ is an observation system by ⟨bcdcpo-sys⟩. Let $P := \{(a,i) \mid a \in \mathrm{At}\, D_i\}$ denote the disjoint union of all $\mathrm{At}\, D_i$ and $\sigma := \{S \subseteq P \mid \forall i. \{a \mid (a,i) \in S\} \in \mathcal{C}(\mathrm{At}\, D_i)\}$ the set of subsets of P such that each "component" of elements coming from the same domain is consistent in that domain. We show that (P, σ) is a non-empty observation system, and that $(\sigma, \subseteq) \cong (D_\times, \sqsubseteq_\times)$.° Clearly σ is non-empty since the D_i are.

°Recall that we defined two domains to be isomorphic if there is a bijection on their sets of atoms that preserves and reflects consistent sets of atoms. It is easy to see that this indeed determines an isomorphism in Obs, which justifies using it here. Later in this section, we give a more formal argument for this.

⟨(P, σ) is an observation system⟩

⟨$\subseteq$-cl⟩

If $S' \subseteq S \in \sigma$, then for all i, we have $\{a \mid (a,i) \in S'\} \subseteq \{a \mid (a,i) \in S\} \in \mathcal{C}(\mathrm{At}\, D_i)$, so $\{a \mid (a,i) \in S'\} \in \mathcal{C}(\mathrm{At}\, D_i)$ by ⟨$\subseteq$-cl⟩ of $(\mathrm{At}\, D_i, \mathcal{C}(\mathrm{At}\, D_i))$, and $S' \in \sigma$ by definition of σ.

⟨fin-compat⟩

If $F \in \sigma$ for all finite $F \subseteq I$ for some $I \subseteq P$, then for each i, in particular any finite $F \subseteq \{(a,i) \mid (a,i) \in I\}$ is in σ, hence $\{a \mid (a,i) \in F\} \in \mathcal{C}(\mathrm{At}\, D_i)$, so $\{a \mid (a,i) \in I\} \in \mathcal{C}(\mathrm{At}\, D_i)$ by ⟨fin-compat⟩ of $(\mathrm{At}\, D_i, \mathcal{C}(\mathrm{At}\, D_i))$, and then $I \in \sigma$ by definition of σ.

⟨$(\sigma, \subseteq) \cong (D_\times, \sqsubseteq_\times)$⟩

Consider $b : \sigma \to D_\times$ with $b(S) := \times_i \bigsqcup \{a \mid (a,i) \in S\}$. This is well-defined since each $\{a \mid (a,i) \in S\} \in \mathcal{C}(\mathrm{At}\, D_i)$ by construction. It is injective and surjective because each D_i is nuclear and atomistic, respectively. Finally, $S \subseteq S'$ iff for all i, we have $\{a \mid (a,i) \in S\} \subseteq \{a \mid (a,i) \in S'\}$, which is the case iff $\bigsqcup\{a \mid (a,i) \in S\} \sqsubseteq \bigsqcup\{a \mid (a,i) \in S'\}$ for all i, iff $b(S) \sqsubseteq_\times b(S')$.

The bijection just constructed shows that the atoms of $D_\times$, given by the images of singletons of $(a,i) \in P$, are precisely the tuples $\times a_i$ with $a_i \neq \bot_i$ for a unique i and $a_i \in \mathrm{At}\, D_i$ for that i, and that a set $A := \{\times_i a_i^j \mid j \in J\}$ of such atoms is consistent iff for each i, the set $\{a_i^j \in A \mid a_i^j \neq \bot_i\}$ is consistent in D_i.

⟨each π_i is a domain mapping⟩

First, π_i is monotone since $\times x_j \sqsubseteq_\times \times y_j$ implies $\pi_i(\times x_j) = x_i \sqsubseteq_i y_i = \pi_i(\times y_j)$ by definition of $\sqsubseteq_\times$. Furthermore, for each $\times x_j \in D_\times$, any $b \in \mathrm{At}\, \pi_i(\times x_j) = \mathrm{At}\, x_i$ has that $\times a_j$ with $a_j = \bot_j$ for all $j \neq i$ and $a_i = b$ is an atom of $D_\times$, and clearly $\mathrm{At}\, \pi_i(\times a_j) = \mathrm{At}\, b \ni b$. The claim follows by ⟨`dm-at-reflect`⟩.

⟨$\times f_i$ is a domain mapping⟩

We again use ⟨`dm-at-reflect`⟩. Clearly $\times f_i$ is monotone, by monotonicity of each f_i and the definition $\sqsubseteq_\times$. For each $x \in E$ and $\times b_i \in \mathrm{At} \times (f_i(x))$, we have that $b_i \neq \bot_i$ for a unique i and $b_i \in \mathrm{At}\, D_i$. Then, $b_i \sqsubseteq f_i(x)$, so since f_i is a domain mapping, by ⟨`dm-at-reflect`⟩ there exists an $a \in \mathrm{At}\, E$ with $b_i \in \mathrm{At}\, f_i(a)$, as needed.

⟨$\times f_i$ is a unique mediator⟩

That $\times f_i$ makes the diagram commute is obvious: for all i and $x \in E$, we have $\pi_i(\times(f_j(x))) = f_i(x)$. Now any $m{:}E \to D_\times$ with $\pi_i(m(x)) = f_i(x)$ for all i and $x \in E$ needs to have $m(x) = \times y_i$ for some collection of $y_i \in D_i$, and since $f_i(x) = \pi_i(m(x)) = y_i$ for all i, we have that $m(x) = \times(f_i(x))$, so $m = \times f_i$. □

We could have given a simpler direct proof instead of constructing equivalent observation systems. However, with this proof, we have gained a new perspective on products in Obs: the set of atoms of the product can be seen as the disjoint union of the individual sets of atoms, and a set of such atoms is consistent iff each subset of atoms coming from the same domain is consistent in that domain. Converting this into observation spaces, we have that the points of a product space are tuples of points of the individual spaces (since points are maximal elements, which in the product are tuples of maximal elements), and the basic observables applying to each point are just the properties from each individual space applying to the corresponding component of the tuple. With our understanding of atoms, we can also reformulate the projections $\pi_i : \times D_j \to D_i$ as relations on atoms, with $(a,i) \to_{\pi_i} a$ for all $a \in \mathrm{At}\, D_i$, and mediators $\times f_i : E \to \times D_i$ as $e \to_{\times f_j} f_i(e) \times \{i\}$ for all i.

The remark after the proof of ⟨obs*=conc+finset⟩ pointed out that Obs^* is equivalent to $\mathrm{Conc}\,\mathsf{FinSet}^+$. As shown in [3], $\mathrm{Conc}\,\mathsf{C}$ forms a quasitopos for each concrete category C. In particular, this means that Obs^* has all (small) limits and colimits.

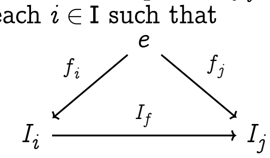
PROPOSITION 3.2.4. The category Obs^* has limits of all small diagrams. For I a small category and $I : \mathrm{I} \to \mathsf{Obs}^*$ a diagram, the limit $\lim I$ is given by the domain $(\sigma, \subseteq)$ for (P, σ) the following observation system. Let

$$P := \{ \bigotimes_{i \in \mathrm{I}} a_i \in \bigotimes_{i \in \mathrm{I}} \mathrm{At}\, I_i \mid I_f(a_i) = a_j \text{ for all } f : i \to j \text{ in } \mathrm{I} \}$$

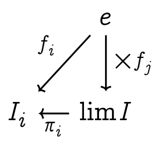
be the set of tuples in the cartesian product of the sets of atoms of all domains in the diagram such that each domain mapping in the diagram maps the atom coming from its domain to the atom coming from its codomain. Here we denote the cartesian product for atoms using $\otimes$ to distinguish it from the product in Obs. Let $\sigma := \{ S \subseteq P \mid \forall i. \{ a_i \mid \bigotimes a_j \in S \} \in \mathcal{C}(\mathrm{At}\, D_i) \}$. The projections π_i^* are given by the domain mappings $\pi_i^*(\bigotimes a_j) = a_i$ for all atoms of

⟨obs*-lim⟩

A *cone* to a diagram $I : \mathrm{I} \to \mathsf{C}$ is a collection of morphisms $f_i : e \to I_i$ for each $i \in \mathrm{I}$ such that



commutes for all $f : i \to j$ in I. A *limit* of I is a cone $\pi_i : \lim I \to I_i$ of morphisms called *projections*, such that for each cone $f_i : e \to I_i$ to I there exists a unique mediator $\times f_i : e \to \lim I$ such that



commutes for all $i \in \mathrm{I}$.

σ, and for any collection of $f_i : E \to I_i$ in Obs^* with $f_i \mathbin{;} I_f = f_j$ for all $f : i \to j$, the mediator $\bigotimes f_i$ is given by $a \mapsto \bigotimes(f_i(a))$ for all $a \in \operatorname{At} E$.

Proof. Limits in the category $[\mathsf{FinRel}^{\mathrm{op}}, \mathsf{Set}]$ of presheaves are computed pointwise, and Obs^* is a full subcategory of this category by ⟨`obs*=conc+finset`⟩. As is easy to see from inspecting the definition of limits, full subcategory inclusions reflect them, so if the pointwise limit happens to be in $\mathrm{Conc}^+_{\mathrm{fin}} \mathsf{FinSet}$, it is also the limit in Obs^*. A pointwise limit of concrete presheaves is always concrete [3, Proposition 39].

Furthermore, a pointwise limit of non-empty concrete presheaves is always non-empty: if I_i is non-empty, say $\varphi \in I_i(N)$, then $I_i(0_N)(\varphi) \in I_i(0)$, where $0_N : 0 \to N$ is the empty map. Any two $\varphi, \varphi' \in I_i(0)$ vacuously have $I_i(n)(\varphi) = I_i(n)(\varphi')$ for all $n \in \mathsf{FinSet}(1, 0) \cong \emptyset$, so $\varphi = \varphi'$ by concreteness. Thus each $I_i(0)$ has a unique element, and every natural transformation $\alpha : I_i \to I_j$ must map the unique $\bullet \in I_i(0)$ onto $\bullet \in I_j(0)$. By the way limits are computed in Set, it follows that $\lim(I(-)(0))$ is a singleton, so $\lim I$ is non-empty.

Thus the pointwise limit of I in $[\mathsf{FinRel}^{\mathrm{op}}, \mathsf{Set}]$ is in $\mathrm{Conc}^+ \mathsf{FinSet}$. Explicitly, for each $N \in \mathsf{FinSet}$ we have that $\lim I(N)$ is the set of tuples $\bigotimes \varphi_i$ with $\varphi_i \in I_i(N)$ for all $i \in \mathrm{I}$ and $I_f(N)(\varphi_i) = \varphi_j$ for all $f : i \to j$ in I. Then $I_j(n)(\varphi_j) = I_j(n)(I_f(N)(\varphi_i)) = I_f(1)(I_i(n)(\varphi_i))$ for all $n \in N$ by naturality, so by concreteness we can equivalently consider $\bigotimes \varphi_i$ as an N-indexed tuple of atoms $\bigotimes a_i \in \bigotimes I_i(1)$ such that for each i the component $\{a_i \mid \bigotimes a_j \in \bigotimes \varphi_i(N)\}$ is consistent (i.e. the image of some φ_i under $I_i(N)$), with $I_f(1)(a_i) = a_j$ for all $f : i \to j$.

Using the construction given in the proof of the essential surjectivity in ⟨`obs*=conc+finset`⟩, we can then determine what observation system this corresponds to. First, we have that $\lim I(1) \cong \lim(I(-)(1))$ in Set is simply given by P as stated above. By the above, the set σ of consistent subsets is given $\lim I(N)(\varphi)$ for each $\varphi \in \lim I(N)$ for each $N \in \mathsf{FinSet}$ contains precisely the sets of tuples of atoms from P such that each component is consistent. Thus σ is as stated in the proposition. The equivalent domain is then given by the ideal completion of σ, but in this case this has no effect because (P, σ) already satisfies $\langle$`fin-compat`$\rangle$: suppose $F \in \sigma$ for all $F \subseteq_{\text{fin}} S \subseteq P$, then for all i, we have that each finite $F' \subseteq \{a_i \mid \bigotimes a_j \in S\}$ must be in $\mathcal{C}(\mathrm{At}\, D_i)$ by definition, hence $\{a_i \mid \bigotimes a_j \in S\} \in \mathcal{C}(\mathrm{At}\, D_i)$ by $\langle$`fin-compat`$\rangle$ of $(\mathrm{At}\, D_i, \mathcal{C}(\mathrm{At}\, D_i))$.

The definition of the projections and mediators directly follows from the corresponding definitions for $\lim(I(-)(\mathbf{1}))$ in Set. $\square$

As a corollary, we in particular have a terminal object $\mathbf{1}^*$ (the limit of the empty diagram), given by the one-atom CABA $\{\bot, \top\}$ with a top element $\top$. The unique atom-preserving domain mapping $\mathbf{1}^*_D : D \to \mathbf{1}^*$ has $a \mapsto \top$ for all $a \in \mathrm{At}\, D$.

Furthermore, Obs^* has all (small) products, since products are just limits of discrete (containing only identity arrows) diagrams. We readily compute the product $\bigotimes D_i$ in Obs^* of a collection of domains to have atoms $\bigotimes \mathrm{At}\, D_i$, and a set $\{\bigotimes a_i^j \mid j \in J\}$ is consistent iff each $\{a_i^j \mid j \in J\}$ is consistent in D_i. There does not seem to be a simple characterisation of this construction in domain-theoretic terms without resorting to the language of observation systems, except if the domains are of the form $\mathcal{P}_{\mathcal{C}}(D_i)$: then, $\bigotimes \mathcal{P}_{\mathcal{C}}(D_i) \cong \mathcal{P}_{\mathcal{C}}(\bigtimes D_i)$ because $\mathcal{P}_{\mathcal{C}}$ is a right adjoint by $\langle$`obs⋆-obs-adj`$\rangle$, and right adjoints preserve limits. We can still consider observation spaces equivalent to products in Obs^*: the points of such spaces are maximal consistent sets, which are clearly given by $\{\bigotimes a_i^j \mid j \in J\}$ such that each $\{a_i^j \mid j \in J\}$ is maximal in D_i, i.e. they are again tuples of maximal elements of D_i, just like for the product in Obs. However, in contrast to the product in Obs, the observable properties are now given by tuples of observable properties, each applying to the respective component.

PROPOSITION 3.2.5. The category Obs* has colimits of all small diagrams. For I a small category and $I : \mathrm{I} \to \mathsf{Obs}^*$ a diagram, the colimit colim I is given by $(\mathcal{I}(\sigma), \subseteq)$, for (P, σ) the following observation system. Let P be the disjoint union $\bigcup_{i \in \mathrm{I}} \{(a, i) \mid a \in \mathrm{At}\, I_i\}$ quotiented by the equivalence relation $\sim$ generated by "$(a,i) \sim (b,j)$ iff $I_f(a) = b$ for some $f : i \to j$ in I", and let σ contain precisely the sets of the form $\{[(a,i)] \mid a \in A\}$ for some $i \in \mathrm{I}$ and $A \in \mathcal{C}(\mathrm{At}\, I_i)$. The insertions ι_i are given by $a \mapsto [(a,i)]$, and for any collection of atom-preserving $f_i : I_i \to E$ with $I_f \,;\, f_j = f_i$ for all $f : i \to j$ in I, the mediator $+f_i$ is given by $[(a,i)] \mapsto f_i(a)$. ⟨obs*-colim⟩

Proof. By [3, Proposition 50], colimits in categories of concrete presheaves can be computed by first computing pointwise and then applying a "concretisation" functor which simply quotients each $F(c)$ under the equivalence relation $F(-)(\varphi) = F(-)(\varphi')$. This functor has a right adjoint [3, Lemma 47], given by the inclusion $\mathrm{Conc}\,\mathsf{C} \hookrightarrow [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$, and the adjunction clearly restricts to an adjunction between $\mathrm{Conc}^+\,\mathsf{C}$ and $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ since the concretisation of a presheaf F just takes quotients of each $F(c)$, which is non-empty if $F(c)$ is non-empty, so the concretisation of a non-empty presheaf is non-empty.

Thus colimits exist in Obs* and we can compute them in the same manner. Again, we use ⟨obs*=conc+finset⟩ to find the equivalent observation system. Its set of properties is given by $\mathrm{colim}(I(-)(\mathbf{1}))$ in Set, which is just P. Now each $[\varphi] \in \mathrm{colim}(I(-)(N))$ is the equivalence class of some $\varphi \in I_i(N)$, under the equivalence relation generated by $I_f(N)(\psi) = \psi'$ for some $f : j \to k$ in I, which like in ⟨obs*-lim⟩ just means that there is an arrow f such that $I_f(\mathbf{1})(I_j(n)(\psi)) = I_k(n)(\psi')$ for all $n \in N$, so that after applying concretisation we can think of such a $[\varphi]$ as corresponding to the set of equivalence classes of atoms of φ. It directly follows that the consistent sets are given by σ as stated. This time, ideal completion is not automatic° and needs to be applied, as we do in the statement of the proposition. Insertions and mediators again follow from their definitions for $\mathrm{colim}(I(-)(\mathbf{1}))$ in Set. □

°For example, consider the chain

$$1 \subseteq 2 \subseteq 3 \subseteq \ldots$$

containing a set of each cardinality in FinSet together with inclusions, embedded into Obs*. The pointwise colimit of this diagram only specifies finite consistent sets, but the colimit in Obs* is $(\mathcal{P}(\mathbb{N}), \subseteq)$, which also contains infinite sets.

The colimit can also be described as the (order-theoretic) ideal completion of the quotient of the disjoint union of the domains I_i under the equivalence relation generated by $I_f(x) = y$ for some f and identifying bottom elements of all I_i, together with the order $[x] \sqsubseteq [y]$ if $x \sqsubseteq_i y$ in D_i, extended to the ideal completion, with insertions $\iota_i(x) = [(x,i)]$ and mediators $+f_i$ defined on *finite* elements by $[(x,i)] \mapsto f_i(x)$ and then uniquely extended to the ideal completion.

Left adjoints preserve° colimits, Obs^* is cocomplete (has all colimits) by ⟨obs⋆-colim⟩, and the inclusion $\mathsf{Obs}^* \hookrightarrow \mathsf{Obs}$ is a left adjoint by ⟨obs⋆-obs-adj⟩. As a corollary, the initial object in Obs^* (the colimit of the empty diagram) is sent by the inclusion to an initial object in Obs, which as we know from ⟨obs-zero⟩ is given by the zero-atom CABA $\mathbf{0}$, together with empty domain mappings (which are atom-preserving when going out of $\mathbf{0}$), so $\mathbf{0}$ must also be initial in Obs^*.

°We say that a functor $F : \mathsf{C} \to \mathsf{D}$ *preserves* the colimit of a functor $I : \mathrm{I} \to \mathsf{C}$ if $F(\operatorname{colim} I)$ is a colimit of $\operatorname{colim}(I\,;F)$, together with the insertions $F(\iota_i)$ for $i \in \mathrm{I}$.

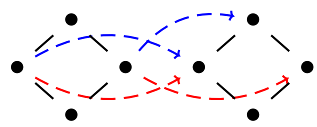
Coproducts in Obs^*, which are colimits of discrete diagrams, are then simply given by disjoint unions of domains, but identifying bottom elements. As is easy to see, the disjoint union automatically satisfies ⟨dir-comp⟩, so ideal completion need not be applied in this case. We then also have the following.

PROPOSITION 3.2.6. The category Obs has all (small) coproducts, computed like in Obs^*. ⟨obs-coprod⟩

Proof. A coproduct is a colimit of a discrete diagram, and discrete diagrams automatically contain only atom-preserving maps, so the diagram is in Obs^*. Now Obs^* is cocomplete by ⟨obs⋆-colim⟩, and the inclusion $\mathsf{Obs}^* \hookrightarrow \mathsf{Obs}$ is a left adjoint by ⟨obs⋆-obs-adj⟩, hence preserves colimits. It follows that the coproduct in Obs^* exists and is also the coproduct in Obs. □

The observation spaces corresponding to coproducts are simply disjoint unions of the spaces, and each observable property only applies to the points of the original space it belongs to.

More generally, it follows that Obs has colimits of all diagrams factoring through the inclusion $\mathsf{FinObs}^* \hookrightarrow \mathsf{FinObs}$. However, Obs is otherwise neither complete nor cocomplete. The diagram in the margin does not have an equaliser, nor a coequaliser, which follows from the fact that a (co)equaliser would imply a splitting of the idempotent relation $\{(\spadesuit,\spadesuit),(\heartsuit,\spadesuit),(\heartsuit,\heartsuit)\}$ [5, Proposition 6.5.4], which is the blue arrow in the diagram, but this cannot be split.



MONOIDAL CLOSEDNESS

Intuitively speaking, a category is closed if it has an object $[c,d]$ for each pair of objects that is in a sense an "object of morphisms" from c to d. In our case, this is desirable because we want to have a function space between domains that is itself a domain, so that e.g. the dynamical systems on a space are (maximal) elements of the function space on the corresponding domain. Often, one cares about categories that are *cartesian closed*, which can be defined succinctly by saying that the category has finite products and the induced functor $-\times c$ has a right adjoint $[c,-]$ for every c. This is motivated by analogy to the situation in Set, where functions going out of a product $X \times Y \to Z$ are equivalently functions $X \to (Y \to Z)$ going into a function space ("currying").

The category Obs is not cartesian closed: every cartesian closed category with a zero object is trivial.° This is not a significant limitation, though, because it instead comes with a monoidal structure with respect to which it is closed. Pleasantly, this structure is provided by the cartesian monoidal structure of Obs^*.

°For, we would have $\mathsf{C}(c,d) \cong \mathsf{C}(c\times 1,d) \cong \mathsf{C}(c,[1,d]) \cong \mathsf{C}(c,[0,d]) \cong \mathsf{C}(c\times 0,d) \cong \mathsf{C}(0,d) \cong \{\bullet\}$ for all pairs of objects $c,d \in \mathsf{C}$.

DEFINITION: TENSOR PRODUCT. For every collection of domains $D_i \in \mathsf{Obs}$, their *tensor product* is defined as their product in Obs^* and denoted $\bigotimes D_i$.

PROPOSITION 3.2.7. The category Obs forms a symmetric monoidal category together with $\otimes$.

A *symmetric monoidal category* is a category C with a functor $\otimes : \mathsf{C}\times\mathsf{C} \to \mathsf{C}$, an object called the *tensor unit*, several natural isomorphisms, namely the *associator*, the *braiding*, and the *left* and *right unitor* satisfying various identities, namely the triangle, pentagon, and hexagon identities, and a symmetry condition for the braiding; see [5, Section 6.1].

Proof. First, we need to check that the tensor product $\otimes$ can be extended to a functor from $\mathsf{Obs} \times \mathsf{Obs}$ into Obs. For objects, this is clear since this is just the product in Obs^*, which is a wide subcategory of Obs, so the tensor product is defined for all objects. For domain mappings $\rightarrow_f, \rightarrow_g : D \rightarrow D_1, D_2$, define $a\ (\rightarrow_f \otimes \rightarrow_g)\ (b_1 \otimes b_2)$ iff $a \rightarrow_f b_1$ and $a \rightarrow_g b_2$, and let the tensor product functor map $(\rightarrow_f, \rightarrow_g) : (D_1, D_2) \rightarrow (D_1', D_2')$ to $\rightarrow_{f,g} := (\pi_1^*; \rightarrow_f) \otimes (\pi_2^*; \rightarrow_g)$, which has the property that $(a_1 \otimes a_2) \rightarrow_{f,g} (b_1 \otimes b_2)$ iff $a_1 \rightarrow_f b_1$ and $a_2 \rightarrow_g b_2$.°

°Note that the action of this functor on morphisms is often denoted $\rightarrow_f \otimes \rightarrow_g$, but we avoid this here because we use this notation for mediators, and the extension we made to our notation coincides with mediators if both maps are atom-preserving.

⟨ $\otimes$ is a functor ⟩

⟨ $\otimes$ is well-defined ⟩

We need to show that $\rightarrow_{f,g}$ is a domain mapping. Let $A \in \mathcal{C}(\mathrm{At}(D_1 \otimes D_2))$. Then, $\rightarrow_{f,g}(A) = \{a' \otimes b' \mid \exists (a \otimes b) \in A.\, a \rightarrow_f a', b \rightarrow_g b'\}$. Here $\{a' \in \mathrm{At}\, D_1' \mid \exists (a \otimes b) \in A, b' \in \mathrm{At}\, D_2'.\, a \rightarrow_f a', b \rightarrow_g b'\} \subseteq \rightarrow_f(A) \in \mathcal{C}(\mathrm{At}\, D_1')$ since $\rightarrow_f$ is a domain mapping, so the subset is consistent by ⟨$\subseteq$-cl⟩ of $(\mathrm{At}\, D_1', \mathcal{C}(\mathrm{At}\, D_1'))$, and analogously for the subset of the $b' \in \mathrm{At}\, D_2'$, so $\rightarrow_{f,g}(A) \in \mathcal{C}(\mathrm{At}(D_1 \otimes D_2))$ by definition of consistency in the product in Obs^*.

⟨ $\otimes$ preserves identities ⟩

We have $(a \otimes b) \rightarrow_{\mathrm{id},\mathrm{id}} (c \otimes d)$ iff $a \rightarrow_{\mathrm{id}} c$ and $b \rightarrow_{\mathrm{id}} d$ iff $a = c$ and $b = d$.

⟨ $\otimes$ preserves composites ⟩

For pairs $(\rightarrow_f, \rightarrow_g) : (D_1, D_2) \rightarrow (D_1', D_2')$ and $(\rightarrow_{f'}, \rightarrow_{g'}) : (D_1', D_2') \rightarrow (D_1'', D_2'')$, we have $(a \otimes b) \rightarrow_{f;f',g;g'} (a'' \otimes b'')$ iff $a \rightarrow_{f;f'} a''$ and $b \rightarrow_{g;g'} b''$, iff there are a', b' such that $a \rightarrow_f a' \rightarrow_{f'} a''$ and $b \rightarrow_g b' \rightarrow_{g'} b''$, iff there is an $a' \otimes b' \in \mathrm{At}(D_1' \otimes D_2')$ such that $(a \otimes b) \rightarrow_{f,g} (a' \otimes b')$ and $(a' \otimes b') \rightarrow_{f',g'} (a'' \otimes b'')$, iff $(a \otimes b)\ (\rightarrow_{f,g}; \rightarrow_{f';g'})\ (a' \otimes b')$, as required.

As the tensor unit, we designate the object $\mathbf{1}^*$. The associator is given as a domain mapping by $((a \otimes b) \otimes c) \rightarrow_\alpha (a \otimes (b \otimes c))$, the left and right unitors are $\top \otimes a \rightarrow_\lambda a$ and $a \otimes \top \rightarrow_\varrho a$, respectively, and the braiding is $a \otimes b \rightarrow_\beta b \otimes a$.

The triangle, pentagon, and hexagon identities, as well as symmetry of the braiding are satisfied because the structure we have specified restricted to Obs^* is the cartesian monoidal structure on Obs^* coming with any category with finite products, and any such structure is automatically symmetric monoidal, hence satisfying all these identities, which suffices since Obs^* is a wide subcategory of Obs.

However, we still need to show that all of these are natural isomorphisms. The isomorphism part is obvious (just transpose all the relations in the definitions to get the inverse), so it remains to show that naturality extends to arbitary morphisms in Obs.

⟨naturality of α⟩

Let $((\rightarrow_f, \rightarrow_g), \rightarrow_h)$ be a morphism in $(\mathsf{Obs} \times \mathsf{Obs}) \times \mathsf{Obs}$. Then, there exists an $(a' \otimes b') \otimes c')$ with $((a \otimes b) \otimes c) \rightarrow_{\rightarrow_{f,g}, \rightarrow_h} ((a' \otimes b') \otimes c') \rightarrow_\alpha (a' \otimes (b' \otimes c'))$ iff there exist a', b', c' with $a \rightarrow_f a'$, $b \rightarrow_g b'$, and $c \rightarrow_h c'$, iff $((a \otimes b) \otimes c) \rightarrow_\alpha (a \otimes (b \otimes c)) \rightarrow_{\rightarrow_f, \rightarrow_{g,h}} (a' \otimes (b' \otimes c'))$.

⟨naturality of λ and ϱ⟩

Let $\rightarrow_f$ be a morphism in Obs. Then, there exists a b with $\top \otimes a \rightarrow_{\mathrm{id}, f} \top \otimes b \rightarrow_\lambda b$ iff there exists a b with $a \rightarrow_f b$, iff $\top \otimes a \rightarrow_\lambda a \rightarrow_f b$. The proof for ϱ is analogous.

⟨naturality of β⟩

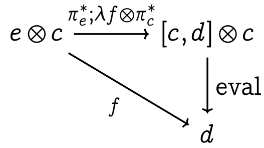
Let $(\rightarrow_f, \rightarrow_g)$ be a morphism in $\mathsf{Obs} \times \mathsf{Obs}$. Then, there exists an $a' \otimes b'$ with $(a \otimes b) \rightarrow_{f,g} (a' \otimes b') \rightarrow_\beta (b' \otimes a')$ iff there exist a', b' with $a \rightarrow_f a'$, $b \rightarrow_g b'$, iff $(a \otimes b) \rightarrow_\beta (b \otimes a) \rightarrow_{g,f} (b' \otimes a')$.

□

PROPOSITION 3.2.8. The category Obs with $\otimes$ is symmetric monoidal closed. For two domains $(D_1, \sqsubseteq_1), (D_2, \sqsubseteq_2) \in \mathsf{Obs}$, the ⟨obs-closed⟩

internal hom $[D_1, D_2]$ is given by the set of domain mappings with domain D_1 and codomain D_2, ordered by inclusion of relations. The morphism eval is given by $(a \to b) \otimes a \to_{\text{eval}} b$, and for any $\to_f$ from $E \otimes D_1$ to D_2, the transpose λf is given by $e \to_{\lambda f} (a \to b)$ iff $e \otimes a \to_f b$.

In a monoidal category, an *internal hom* of c and d is an object $[c,d]$ together with a morphism eval: $([c,d] \otimes c) \to d$ such that for any object e with a morphism $f : (e \otimes c) \to d$, there is a unique morphism $\lambda f : e \to [c,d]$ such that



commutes.

Proof.

$\langle [D_1, D_2]$ is a domain$\rangle$

It suffices to show that $(\mathrm{At}\, D_1 \times \mathrm{At}\, D_2, [D_1, D_2])$ is an observation system since $[D_1, D_2]$ is ordered by inclusion of sets.

$\langle$⊆-cl$\rangle$

If $\to \in [D_1, D_2]$ and $\to' \subseteq \to$, and $S \in \mathcal{C}(\mathrm{At}\, D_1)$, then we have $\to(S) \in \mathcal{C}(\mathrm{At}\, D_2)$ and $\to'(S) \subseteq \to(S)$, so $\to'(S) \in \mathcal{C}(\mathrm{At}\, D_2)$ by $\langle$⊆-cl$\rangle$ of $(\mathrm{At}\, D_2, \mathcal{C}(\mathrm{At}\, D_2))$, so $\to'$ is a domain mapping, hence in $[D_1, D_2]$.

$\langle$fin-compat$\rangle$

If for some relation $\to \subseteq \mathrm{At}\, D_1 \times \mathrm{At}\, D_2$, every finite $\to_F \subseteq \to$ is in $[D_1, D_2]$, and $S \in \mathcal{C}(\mathrm{At}\, D_1)$, let $F \subseteq \to(S)$ be finite. For each $a \in F$, there is an $s_a \in S$ such that $s_a \to a$. Let $\to_F := \{s_a \to a \mid a \in F\}$, then $\to_F$ is clearly finite and a subset of $\to$, so it must be a domain mapping, and then $F = \to_F(S) \in \mathcal{C}(\mathrm{At}\, D_2)$. Thus $\to(S) \in \mathcal{C}(\mathrm{At}\, D_2)$ by $\langle$fin-compat$\rangle$ of $(\mathrm{At}\, D_2, \mathcal{C}(\mathrm{At}\, D_2))$, and $\to \in [D_1, D_2]$.

$\langle$eval is a domain mapping$\rangle$

If $\{(a_i \to b_i) \otimes c_i \mid i \in I\}$ is consistent, then by definition of the tensor product, $\{a_i \to b_i \mid i \in I\}$ and $\{c_i \mid i \in I\}$ are consistent. Then also $\{c_i \mid i \in I, a_i = c_i\} = \mathrm{eval}(\{(a_i \to b_i) \otimes c_i \mid i \in I\})$ is consistent, as required.

$\langle \lambda f$ is a domain mapping$\rangle$

If $\{e_i \mid i \in I\}$ is consistent, then $\to_{\lambda f}(\{e_i \mid i \in I\}) = \{a \to b \mid e_i \otimes a \to_f b\}$, which is consistent since if a set $A \subseteq \mathrm{At}\, D_1$ is consistent, then its image under the relation given by $\{b \mid \exists i \in$

$I, a \in A. e_i \otimes a \rightarrow_f b\} = \rightarrow_f(\{e_i \otimes a \mid i \in I, a \in A\})$ is consistent because the latter set is consistent in $E \otimes D_1$, and then so must be its image under $\rightarrow_f$ since $\rightarrow_f$ is a domain mapping.

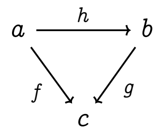
$\langle \lambda f$ is a unique mediator$\rangle$

We have that $e \otimes a \rightarrow_f b$ iff $e \rightarrow_{\lambda f} (a \to b)$ iff $e \otimes a \rightarrow_{\lambda f, \mathrm{id}} (a \to b) \otimes a \rightarrow_{\mathrm{eval}} b$. Now if $\rightarrow_m$ satisfies this property, then $e \otimes a \rightarrow_f b$ iff there exists an $(a' \to b')$ such that $e \otimes a \rightarrow_{m,\mathrm{id}} (a' \to b') \otimes a \rightarrow_{\mathrm{eval}} b$, which by definition of $\rightarrow_{\mathrm{eval}}$ is the case iff $a' = a$ and $b' = b$, so iff $e \rightarrow_m (a \to b)$. □

THE QUASITOPOS Obs*

As mentioned in the previous subsection, Obs* is a *quasitopos*, which is a finitely complete, cocomplete, locally cartesian closed category with a classifier for strong subobjects. We have already seen that Obs* is in fact complete and cocomplete (not just finitely so).

The *slice category* of C over an object c, denoted C/c, has as objects all morphisms in C with codomain c and as morphisms between f and g all commutative triangles



A category is *locally cartesian closed* if all its slice categories are cartesian closed.

The category Obs* is locally cartesian closed. For a domain $D \in \mathsf{Obs}^*$ and $f, g \in \mathsf{Obs}^*/D$, the exponential $[f, g]$ is given by the object $(\sigma, \subseteq)$, where P is the set of pairs (d, h) for $d \in \mathrm{At}\, D$ and $h \in \mathsf{Set}(f^{-1}(d), g^{-1}(d))$, and σ has all subsets $\{(d_i, h_i) \mid i \in I\} \subseteq P$ such that $\{d_i \mid i \in I\} \in \mathcal{C}(\mathrm{At}\, D)$ and if $\bigcup\{A_i \subseteq f^{-1}(d_i) \mid i \in I\}$ is consistent, then so is $\bigcup\{h_i(A_i) \mid i \in I\}$; together with the morphism $p_{[f,g]} : [f, g] \to D$ with $(d, h) \mapsto d$. The map $\mathrm{eval} : p_{[f,g]} \otimes_D f \to g$ is given by $(d, h) \otimes d_1 \mapsto h(d_1)$. For $p_E \in \mathsf{Obs}^*/D$ with $m : (p_E \otimes_D f) \to g$, the adjunct λm is given by $e \mapsto (p_E(e), m(e \otimes -))$.

The locally cartesian closed structure is never used in this thesis, so we will not discuss it. The interested reader can find a definition of the local internal hom in the margins.

For the strong-subobject classifier, we need a bit of preparation on the various kinds of monomorphisms in Obs and Obs*; this is also relevant for the developments in Section ‘`sec:cat-char`’. Recall that a *monomorphism* is a morphism m such that for every diagram

$$b \underset{g}{\overset{f}{\rightrightarrows}} c \xrightarrow{m} d$$

if $f ; m = g ; m$, then $f = g$, while an *epimorphism* is a morphism e such that for every diagram

$$c \xrightarrow{e} d \underset{g}{\overset{f}{\rightrightarrows}} b$$

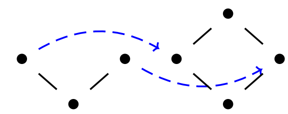
if $e;f = e;g$, then $f = g$. As [3, Proposition 31] shows, a morphism $\alpha : F \to G$ in a category of concrete presheaves is a monomorphism iff $\alpha(1)$ is an injective function, and an epimorphism iff $\alpha(1)$ is surjective. This means that a domain mapping in Obs^* is a mono/epi in Obs^* iff it is an injective/surjective function on atoms.

Now for Obs, we first note that the functor $\mathcal{P}_{\mathcal{C}} : \mathsf{Obs} \to \mathsf{Obs}^*$ is a right adjoint and faithful, which implies that it preserves and reflects monomorphisms, respectively. This means that a morphism $\to\, \subseteq \mathrm{At}\,D \times \mathrm{At}\,E$ in Obs is a monomorphism iff $\mathcal{P}_{\mathcal{C}}(f)$ is a morphism in Obs^*, i. e. iff it is injective as a function $D \to E$ on elements (via the correspondence of consistent subsets and elements).°

°The condition for a morphism in Obs to be an epimorphism, however, is surjectivity on atoms, i. e. every atom in the codomain must be the image of an atom in the domain.

While a morphism in Obs^* that is both a monomorphism and an epimorphism is thus always a bijection on atoms, in contrast to categories like Set, it is not necessarily an isomorphism, as the example in the margin shows: the domain mapping does not have an inverse.

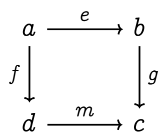
epic monomorphism does not imply isomorphism:



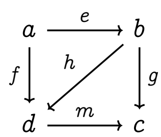
For Obs^*, an isomorphism is an atom-preserving domain mapping with an atom-preserving inverse. Since every inverse to a morphism f is also an inverse to f seen as a function between sets, it follows that f is an isomorphism iff it is a bijection on atoms such that its (unique) inverse is also a domain mapping. This is evidently the case iff it is a bijection that not only preserves, but also reflects consistent sets of atoms, i. e. if $S \in \mathcal{C}(\mathrm{At}\,D_2)$, then $f^{-1}(S) \in \mathcal{C}(\mathrm{At}\,D_1)$, which clearly fails in the example just shown.

As it turns out, reflection of consistent subsets is also precisely the condition that distinguishes *strong* monomorphisms in Obs^* from general monomorphisms [3, Proposition 34].

A *strong monomorphism* is a monomorphism m such that for every commutative square



with e an epimorphism, there exists a unique h such that

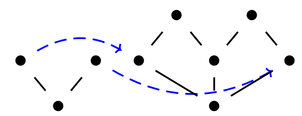


commutes.

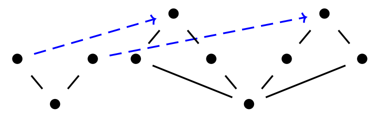
It is easy to see that every morphism in Obs that is an injective function on atoms and reflects consistent subsets has a left inverse, given by its transpose relation. By definition, this makes such morphisms *split monomorphisms*, and conversely every atom-preserving split monomorphism in Obs is a monomorphism in Obs* that reflects consistent subsets, because left-inverses of injective total univalent relations are necessarily given by their transposes, and the transpose of a domain mapping is a domain mapping iff the domain mapping reflects consistent subsets, as is clear from the definition.

Thus strong monomorphisms in Obs* are precisely the atom-preserving split monomorphisms in Obs. In fact, in Obs, these are also the *embeddings* from domain theory, namely morphisms f that have a left-inverse f^{T} such that $f^{\mathrm{T}} ; f \subseteq \mathrm{id}$. Note, however, that a strong monomorphism in Obs* is not necessarily split in Obs* (see the margins) and a split monomorphism in Obs is not necessarily atom-preserving, not even when it is an injective relation, i.e. no atom is hit twice (moreover, the notions of strong and split monomorphisms in Obs coincide).

strong monomorphism does not imply split in Obs*:



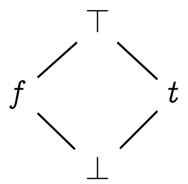
injective split monomorphism does not imply atom-preserving in Obs:



Now we can work out what strong subobjects in Obs* look like. Subobjects in any category are obtained as follows. The collection of monomorphisms $i : D_i \hookrightarrow D$ with codomain D forms a preorder under the relation "$i \trianglelefteq j$ iff there exists a $k : D_i \hookrightarrow D_j$ such that $i = k ; j$". The poset of subobjects is obtained from this preorder by turning it into a partial order in the usual way. Thus subobjects of D are equivalence classes of monomorphisms under the equivalence relation $i \trianglelefteq j \trianglelefteq i$.

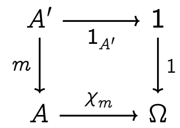
If two monomorphisms are part of the same equivalence class, then their domains are isomorphic, and because monomorphisms in Obs* are injective, there is a canonical representative for each subobject of D that is given by a subset of D together with the induced order and the inclusion into D. Now a strong subobject is an equivalence class of a strong monomorphism, so it is represented by a subset of the atoms of D together with all joins of those atoms that D admits. Thus a strong subobject is completely determined by the subset of atoms it contains, and every such subset gives rise to a unique strong subobject.

In a topos, subobjects are "classified" by a *subobject classifier*. However, Obs^* is not a topos, but just a quasitopos, and it only has a classifier for strong subobjects. The classifier Ω^* for strong subobjects in Obs^* is given by the domain



together with the domain mapping $1 : \mathbf{1}^* \to \Omega^*$ with $\top \to_1 t$, and for each strong subobject $E \hookrightarrow D$ the mapping $\chi_E : D \to \Omega^*$ is defined as $\chi_E(a) = t$ if $a \in \mathrm{At}\, E$ and $\chi_E(a) = f$ otherwise. This only works for strong subobjects: all the morphism χ_E can do is map each atom of D to either f or t, and the pullback along χ_E will always include all joins of the atoms mapped to t that D admits. Observe that Ω^* is the product $\mathbf{1}^* \times \mathbf{1}^*$ in Obs, and the arrow 1 is given by the product mediator $\mathrm{id}_{1^*} \times \mathbf{0}_{1^*,1^*}$.

A *subobject classifier* is an object Ω together with a morphism $1 : \mathbf{1} \to \Omega$ from the terminal object such that for each monomorphism $m : A' \to A$, there exists a unique $\chi_m : A \to \Omega$ such that



is a pullback square (i.e. A' is a limit of the rest of the diagram).

As a final note, every quasitopos contains a topos as a full subcategory. The objects of this subcategory are those objects for which epic monomorphisms are "seen as if they were" isomorphisms. These objects are called *coarse* objects. For Obs^*, coarse objects can be seen to be precisely the domains that are CABAs, so the topos inside Obs^* is in fact just Set.

An object c is called *coarse* if for every epic monomorphism $f : d \to d'$ and every morphism $g : d \to c$ there is a unique $h : d' \to c$ such that $g = f\,; h$.

ALGEBRAS OF ENDOFUNCTORS

There are many techniques for solving recursive domain equations in categories of domains in the literature. We will not pursue this in any complexity; instead, we just give two objects that are useful as computational types.

PROPOSITION 3.2.9. The category Obs has an inductive natural numbers object, i.e. an object with morphisms ⟨obs-nno⟩

$$1^* \xrightarrow{z} N \xrightarrow{s} N$$

such that for each

$$1^* \xrightarrow{o} M \xrightarrow{f} M$$

there exists a unique $u : N \to M$ such that

$$\begin{array}{ccccc} 1^* & \xrightarrow{z} & N & \xrightarrow{s} & N \\ & \searrow^{o} & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} \\ & & M & \xrightarrow{f} & M \end{array}$$

commutes.

Proof. Let $\mathbb{N}_\perp$ denote the domain consisting of a bottom element and elements n for each $n \in \mathbb{N}$, with the order $\perp \sqsubseteq n$ for all $n \in \mathbb{N}$ but each pair of $n \neq m \in \mathbb{N}$ unordered. Define z as the map $\top \mapsto 0$ and s as $n \mapsto n+1$. Both are clearly domain mappings.

For any object M as stated, let $u(0) = o(\top)$ and for each n, let $u(n+1) = f(u(n))$. This defines u by the recursion principle for the natural numbers. That this is a domain mapping can be shown by induction: $u(0) = o(\top)$ is consistent because o is a domain mapping, and for $n \in \mathbb{N}$, we have that $u(n+1) = f(u(n))$ is consistent because $u(n)$ is consistent by the induction hypothesis, and then $f(u(n))$ is consistent because f is a domain mapping. This suffices since $\mathbb{N}_\perp$ does not have any non-trivial consistent sets.

Clearly $u(z(\top)) = u(0) = o(\top)$ and $u(s(n)) = u(n+1) = f(u(n))$. Every u' with these properties needs $u'(0) = u'(z(\top)) = o(\top)$ and $u'(n+1) = u'(s(n)) = f(u'(n))$, so $u' = u$ by the uniqueness property of recursively defined functions. □

This also shows that $\mathbb{N}$ is the initial algebra for the endofunctor $\mathbf{1}^* + -$, and it is equivalently the coproduct $+_{n\in\mathbb{N}}\mathbf{1}^*$. We also have something close to a dual, namely a terminal coalgebra for the functor $\mathbf{1}^* \times -$, given by the object $\times_{n\in\mathbb{N}}\mathbf{1}^*$. This object will be useful for expressing μ-recursion in the type theory. Note that $\mathbf{1}^*$ is not terminal in Obs, and $\times$ is not a (co)product in Obs^*, so this is not precisely a dual construction to the above (however, the material in Section 3.3 implies that $\mathbf{1}^*$ is terminal and $\times$ a *co*product when restricted to the full subcategory Set inside Obs^*).

PROPOSITION 3.2.10. The category Obs has a coinductive natural numbers object, i.e. an object with morphisms ⟨obs-conno⟩

$$\mathbf{1}^* \xleftarrow{\;z\;} \overline{\mathbb{N}} \xleftarrow{\;p\;} \overline{\mathbb{N}}$$

such that for each

$$\mathbf{1}^* \xleftarrow{\;o\;} M \xleftarrow{\;f\;} M$$

there exists a unique $u : M \to \overline{\mathbb{N}}$ such that

$$\begin{array}{ccccc}
\mathbf{1}^* & \xleftarrow{\;z\;} & \overline{\mathbb{N}} & \xleftarrow{\;p\;} & \overline{\mathbb{N}} \\
& \nwarrow^{o} & \uparrow{\scriptstyle u} & & \uparrow{\scriptstyle u} \\
& & M & \xleftarrow{\;f\;} & M
\end{array}$$

commutes.

Proof. Let $\overline{\mathbb{N}} := (\mathcal{P}(\mathbb{N}), \subseteq)$. Define z as the relation with $0 \to_z \top$ and p as $n+1 \to_p n$ for all $n \in \mathbb{N}$. For any object M as stated, let u be the map $m \mapsto \{n \mid o(f^n(m)) = \top\}$. This is trivially a domain mapping because $\overline{\mathbb{N}}$ is a CABA.

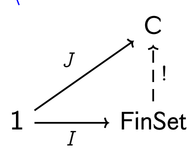
Now $z(u(m)) = \top$ iff $0 \in u(m)$ iff $o(f^0(m)) = \top$, i.e. $o(m) = \top$, so $u\,;z = o$. Furthermore, $m \to_{u;p} n$ iff $m \to_u n+1$, which is the case iff $o(f^{n+1}(m)) = \top$. On the other hand, $m \to_{f;u} n$ iff $o(f^n(f(m))) = o(f^{n+1}(m)) = \top$, so $u\,;p = f\,;u$.

Suppose u' satisfies the same property. We show by induction on $n \in \mathbb{N}$ that for all $m \in M$, we have $n \in u'(m)$ iff $o(f^n(m)) = \top$; it then immediately follows that $u' = u$. First, $o(f^0(m)) = o(m) = \top$ iff $z(u'(m)) = \top$, by commutativity of the left triangle, and the latter is the case iff $0 \in u'(m)$. Now if the claim holds for some n, then $n+1 \in u'(m)$ iff $n \in p(u'(m)) = u'(f(m))$ by commutativity of the square, which by the induction hypothesis is the case iff $o(f^n(f(m))) = o(f^{n+1}(m)) = \top$. □

3.3 CHARACTERISATION AS A FREE CATEGORY

In order to capture the properties of our categories fully in a type theory, it is beneficial to characterise them in terms of a *free construction*. Intuitively, this is because the types in a type theory are "freely generated" from the rules of the theory, which in turn capture universal properties, so if we can describe how to construct a category via certain universal properties, then we can automatically derive a type theory from that description which matches the category perfectly. Since a uniform definition of free constructions runs into size issues, let us instead see with a well-known example how this works. This is just for illustration; we give more detailed proofs for the other categories.

PROPOSITION 3.3.1. The category FinSet is the free finitely cocartesian category generated by the trivial category $\mathbf{1}$ with a single object and a single morphism. That is, FinSet has finite coproducts and there is a functor $I : \mathbf{1} \to \mathsf{FinSet}$ such that for each $J : \mathbf{1} \to \mathsf{C}$ with C having all finite coproducts, there exists a finite-coproduct-preserving functor $F : \mathsf{FinSet} \to \mathsf{C}$, unique up to natural isomorphism, with $J \cong I\,; F$ naturally. ⟨finset-free⟩



Proof. Clearly FinSet has finite coproducts, given by disjoint unions, with the initial object (the empty coproduct) given by the empty set. Take $I : 1 \to \mathsf{FinSet}$ with $I(\bullet) = 1$ and let $J : 1 \to \mathsf{C}$ with C finitely cocartesian. Let $F : \mathsf{FinSet} \to \mathsf{C}$ map $N \in \mathsf{FinSet}$ to $+_{n\in N} J(\bullet)$ and $f : N \to M$ to $+_{n\in N} \iota_{f(n)} : +_{n\in N} J(\bullet) \to +_{m\in M} J(\bullet)$. This arrow exists by the mediating property of the coproduct in C. Notice that $F(f)$ is unique with the property that $\iota_n ; F(f) = \iota_{f(n)}$ for all $n \in N$. This defines a functor since $\iota_n ; F(\mathrm{id}_N) = \iota_{\mathrm{id}_N(n)} = \iota_n$ for all $n \in N$, so $F(\mathrm{id}_N) = \mathrm{id}_{F(N)}$ by uniqueness of mediators, and $\iota_n ; F(f) ; F(g) = \iota_{f(n)} ; F(g) = \iota_{g(f(n))} = \iota_n ; F(f ; g)$, so $F(f ; g) = F(f) ; F(g)$. It preserves finite coproducts: on objects, this is clear, and an insertion $\iota_N : +_N 1 \to +_{M\in S} +_{m\in M} 1$ is mapped to $+_{n\in N} \iota_n^{\mathsf{C}} = \iota_N^{\mathsf{C}}$ in C. Clearly $F(I(\bullet)) = F(1) = +_{n\in 1} J(\bullet) \cong J(\bullet)$ and any finite-coproduct-preserving functor F' satisfying this must have $F'(N) \cong F'(+_N 1) \cong +_N F'(1) \cong +_N J(\bullet) \cong F(\bullet)$, with naturality following by uniqueness of mediators. □

This means that we can generate FinSet by taking a trivial object and "freely adding" finite coproducts, in the sense that we add (nothing but) fresh objects for every coproduct and all the morphisms that are required to satisfy the universal property, only identifying those morphisms that are *necessarily* equal by category-theoretic laws. We now do the same for FinRel. Then, we show how subcategories of categories of presheaves can be obtained as free constructions, including a general recipe for concrete presheaves. The latter is mainly for future reference; for the purposes of this thesis, we will need a different construction to build FinObs out of FinRel, which is the final task we will complete in this chapter.

THE CATEGORY FinRel AS A FREE CATEGORY

Recall that a *pointed* category is a category C with a zero object $\mathsf{0}$, which is both initial and terminal. The unique composite $\mathsf{0}_{c,d} : c \to \mathsf{0} \to d$ between any pair of objects is termed the *zero* *morphism*; a zero morphism composed with any morphism yields a zero morphism by uniqueness. Note that FinRel is pointed, with the zero object given by the empty set and the zero morphisms given by empty relations.

Suppose a pointed category C has the coproduct $+c_i$ of a collection of objects c_i. Notice that then, each c_i also has a canonical "projection" $\pi_i : +c_j \to c_i$ defined by $\iota_i ; \pi_i = \mathrm{id}$ and $\iota_j ; \pi_i = 0_{j,i}$ if $j \neq i$. This defines π_i uniquely by the uniqueness property of the mediator from the coproduct. Writing $(i = j)^? : c_i \to c_j$ for id if $i = j$ and $0_{i,j}$ otherwise, we can thus define $\pi_i = +_j (j = i)^?$. We then also get a cone from $+c_i$ given by the collection of the π_i.

This cone need not make $+c_i$ into a product of the c_i, but if it does, we call it a *biproduct* of the c_i and denote it $\ast\, c_i$. Note that a morphism $f : +c_i \to \times d_j$ is uniquely determined by its collection of $\iota_i ; f ; \pi_j$. Namely, if $\iota_i ; f ; \pi_j = \iota_i ; g ; \pi_j$ for all i, j, then each $\iota_i ; g$ is, by uniqueness of the mediator of the product $\times d_i$, equal to $\times_j (\iota_i ; g ; \pi_j)$. At the same time, g is, by uniqueness of the mediator of the product $+c_i$, equal to $+_i (\iota_i ; g) = +_i \times_j (\iota_i ; g ; \pi_j) = +_i \times_j (\iota_i ; f ; \pi_j) = f$. In particular, this holds for any morphism between biproducts.

Call a category C *finitely bicartesian* if it has biproducts of all finite discrete diagrams. The category FinRel is finitely bicartesian, with biproducts given by the disjoint union of sets together with the obvious projections and insertions. Mediators $\times R_i$ and $+R_i$ are simply given by the union of relations (appropriately composed with biproduct projections and insertions).

In well-behaved cases, including FinRel, we can compose mediators of arrows $f_i : b \to c_i$ and $g_i : c_i \to d$ in a component-wise fashion: each $f_i ; g_i$ gives an arrow $b \to d$, and we have the composite $\times f_i ; + g_i$, but not much can be derived about the calculation of this composite in terms of the components $f_i ; g_i$. In FinRel, we can just take the union of the individual composites, and without making additional assumptions, we cannot construct FinRel as a free finitely bicartesian category. One possibility is to enrich the category one works with in commutative monoids, so that its morphisms already come with a predefined addition operation. Indeed, FinRel is known to be the free "semi-additive" bicompletion of the Boolean ring with two elements [7, Example 2.4]. This makes the treatment easier, but less amenable to our type-theoretic aspirations. There is a simple condition on the composition of mediators we can impose instead.

DEFINITION: IDEMPOTENCE. Let C be a finitely bicartesian category. Then C is called *idempotent* if for all $f_{1,2} : b \to c_{1,2}$ and $g_{1,2} : c_{1,2} \to d$, if $f_1 ; g_1 = f_2 ; g_2$, then $(f_1 \times f_2) ; (g_1 + g_2) = f_1 ; g_1$.

From the canonical isomorphisms $c * d \cong d * c$, $c * (d * e) \cong (c * d) * e$, and $c * 0 \cong c$, we can furthermore derive that this operation (seen as an operation on the morphisms $f_i ; g_i$) is commutative and associative, and that zero morphisms are neutral for it, i.e. if $f ; g = 0_{b,d}$, then $(f \times f') ; (g + g') = f' ; g'$ (see [29, Section 4.1], which shows this for *additive* categories, but only using properties that generally apply in categories with biproducts).

PROPOSITION 3.3.2. FinRel is the free idempotent finitely bicartesian category on the trivial category. That is, FinRel has finite biproducts, there is a functor $I : \mathbf{1} \to$ FinRel, and for each $J : \mathbf{1} \to$ C with C idempotent finitely bicartesian there exists an F : FinRel $\to$ C unique up to isomorphism such that F preserves finite biproducts and $I ; F \cong J$ naturally. ⟨finrel-free⟩

Proof. We have already discussed that FinRel is finitely bicartesian. Furthermore, for any relations $R_{1,2} : N \to M_{1,2}$ and $R'_{1,2} : M_{1,2} \to L$ we have $n\ (R_1 \times R_2 ; R'_1 + R'_2)\ l$ iff there exists an i and an $m \in M_i$ such that $n\ R_i\ m\ R'_i\ l$. Now if $R_1 ; R'_1 = R_2 ; R'_2$, then this is clearly the case iff $n\ (R_1 ; R'_1)\ l$. Thus FinRel is idempotent. We set $I : \mathbf{1} \to$ FinRel as $I(\bullet) = 1$.

Let C be an idempotent finitely bicartesian category and $J : \mathbf{1} \to \mathsf{C}$. Define $F : \mathsf{FinRel} \to \mathsf{C}$ sending $N \in \mathsf{FinRel}$ to $\ast_{n \in N} J(\bullet)$ and relations $R : N \to M$ to $+_{n \in N} \times_{m \in M} (n\,R\,m)^?$, with the property that $\iota_n\,;F(R)\,;\pi_m = \mathrm{id}$ if $n\,R\,m$ and $0_{J(\bullet),J(\bullet)}$ otherwise.

⟨ F is functorial ⟩

⟨ well-defined ⟩

By assumption, C has finite biproducts and all sets in FinSet are finite, and thus the given mediators must exist in C.

⟨ preservation of identities ⟩

Identities $\mathrm{id} : N \to N$ are mapped onto morphisms with $\iota_n\,;F(\mathrm{id});\pi_m = (n = m)^?$. But id also satisfies this by definition of the π_m, so $F(\mathrm{id}) = \mathrm{id}$ by the biproduct mediator property discussed at the beginning of this subsection.

⟨ preservation of composites ⟩

For $R : N \to M$ and $R' : M \to L$, we check that $F(R\,;R') = F(R)\,;F(R')$ by showing that $F(R)\,;F(R')$ fulfills the defining mediating property. Note that for all $n \in N$ and $l \in L$, we have $\iota_n\,;F(R)\,;F(R')\,;\pi_l = \times_m (n\,R\,m)^?\,;+_m (m\,R'\,l)^?$.° Clearly each $(n\,R\,m)^?\,;(m\,R'\,l)^?$ is id if $n\,R\,m\,R'\,l$ and $0_{J(\bullet),J(\bullet)}$ otherwise. Since each composite is either id or a zero morphism, by repeatedly applying idempotence, neutrality of zero morphisms, as well as associativity and commutativity, the composite $\times_m (n\,R\,m)^?\,;+_m (m\,R'\,l)^?$ reduces to id if $(n\,R\,m)^?\,;(m\,R'\,l)^? = \mathrm{id}$ for *some* $m \in M$, i.e. $n\,R\,m\,R'\,l$, and to $0_{J(\bullet),J(\bullet)}$ otherwise, i.e. to $(n\,(R\,;R')\,l)^?$. But this defines precisely $F(R\,;R')$.

°Here we used the fact that $+_i \times_j f_{i,j} = \times_j +_i f_{i,j}$, which follows by observing that $\iota_k\,;+_i \times_j f_{i,j}\,;\pi_l = f_{k,l} = \iota_k\,;\times_j +_i f_{i,j}\,;\pi_l$ by associativity of composition.

⟨ F preserves finite biproducts ⟩

Biproducts in any bicartesian category are associative: if each I_s for $s \in S$ is a set of indices and each x_i^s for $s \in I_s$ for $s \in S$ an object, then the biproduct $\ast_{s \in S} \ast_{i \in I_s} x_i^s$ is also a biproduct $\ast_{s \in S, i \in I_s} x_i^s$, namely via insertions $\iota_{s,i} := \iota_i^s;\iota_s$ and projections

$\pi_{s,i} := \pi_s ; \pi^s_i$. Thus each insertion $\iota_s : \ast_{i \in I_s} x^s_i \to \ast_{t \in S, i \in I_t} x^t_i$ uniquely has the property that $\iota^s_i ; \iota_s ; \pi_{t,j} = \iota^s_i ; \iota_s ; \pi_t ; \pi^t_j = \mathrm{id}$ iff $s = t$ and $i = j$, and $0_{(s,i),(t,j)}$ otherwise.

Now F maps biproducts $\ast_{s \in S} N_s \cong \ast_{s \in S, n \in N_s} \mathbf{1}$ onto biproducts $\ast_{s \in S, n \in N_s} J(\bullet) \cong \ast_{s \in S} \ast_{n \in N_s} J(\bullet) \cong \ast_{s \in S} F(N_s)$, and it maps insertions $\iota_s : \ast_{m \in N_s} \mathbf{1} \to \ast_{t \in S, n \in N_t} \mathbf{1}$ with $m \, \iota_s \, (t, n)$ iff $s = t$ and $m = n$ to the morphism $+_{m \in N_s} \times_{t \in S, n \in N_t} (s = t \wedge m = n)^?$, which defines precisely the appropriate insertion, as just discussed. The argument for the projections is analogous.

$\langle I ; F \cong J \rangle$

Clearly $F(I(\bullet)) = F(\mathbf{1}) = \ast_{n \in \mathbf{1}} J(\bullet) \cong J(\bullet)$, which is necessarily natural since F and J are functors and $\mathbf{1}$ is trivial.

$\langle F$ is unique$\rangle$

If $I ; F' \cong J$ naturally for F' preserving finite biproducts, denote by $\cong_1$ the isomorphism $F(\mathbf{1}) = F(I(\bullet)) = J(\bullet) \cong F'(I(\bullet)) = F'(\mathbf{1})$. Then for all $N \in \mathsf{FinRel}$, we have $F'(N) \cong F'(\ast_{n \in N} \mathbf{1}) \cong \ast_{n \in N} F'(\mathbf{1}) \cong \ast_{n \in N} F(\mathbf{1}) \cong F(N)$ by preservation of finite biproducts, and so $\cong_N := +_{n \in N} (\cong_1 ; \iota'_n) : F(N) \to F'(N)$ is an isomorphism since the $\cong_1 ; \iota'_n$ make $F'(N)$ into a coproduct $+_{n \in N} F(\mathbf{1})$. Notice that also $\cong_N = \times_{n \in N} (\pi_n ; \cong_1)$ since $\iota_n ; +_{n \in N} (\cong_1 ; \iota'_n) ; \pi'_m = \cong_1 ; \iota'_n ; \pi'_m = \cong_1$ if $n = m$ and $0_{F(\mathbf{1}), F'(\mathbf{1})}$ otherwise, which is the same for $\iota_n ; \times_{n \in N} (\pi_n ; \cong_1) ; \pi'_m = \iota_n ; \pi_m ; \cong_1$.

We show that this isomorphism defines a natural transformation. Let $R : N \to M$. Then for all $n \in N$, we have $\iota_n ; F(R) ; \cong_M ; \pi'_m = \times_{m \in M} (n \, R \, m)^? ; \pi_m ; \cong_1 = (n \, R \, m)^? ; \cong_1$, and $\iota_n ; \cong_N ; F'(R) ; \pi'_m = \cong_1 ; \iota'_n ; F'(R) ; \pi'_m = \cong_1 ; F'(\iota^{\mathsf{FinRel}}_n ; R ; \pi^{\mathsf{FinRel}}_m)$ because F' preserves the biproduct insertions and projections, and this is $\cong_1$ if $n \, R \, m$ since then $\iota^{\mathsf{FinRel}}_n ; R ; \pi^{\mathsf{FinRel}}_m = \mathrm{id}$ and $F'(\mathrm{id}) = \mathrm{id}$, and otherwise $0_{F(\mathbf{1}), F'(\mathbf{1})}$ since $\iota^{\mathsf{FinRel}}_n ; R ;$

$\pi_m^{\mathsf{FinRel}} = 0_{1,1}$ and $F'(0_{1,1}) = 0_{F'(1),F'(1)}$ because F' preserves the zero object and composites, hence zero morphisms. Thus $F(R)\,;\cong_M = \cong_N; F'(R)$ as required. □

CATEGORIES OF CONCRETE PRESHEAVES AS FREE CATEGORIES

As we have seen, Obs^* and FinObs are subcategories of categories of presheaves. The category of *all* presheaves on a category C is its *free cocompletion*, i.e. the free completion of C under taking arbitrary (small) colimits (at least if C is small) rather than just coproducts as for FinSet. A subcategory of the category of presheaves can then be obtained as the free completion under *some* colimits only. The following lemma should be well-known, but I cannot find it in the literature, so let us prove it here.

PROPOSITION 3.3.3. Let $\mathbb{D}$ be a class of diagrams in a category C that includes all functors $\mathbf{1} \to \mathsf{C}$. Then, the full subcategory $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]_{\mathbb{D}}$ of the category of presheaves $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ on C given by functors $F \cong \mathrm{colim}(I\,;\,よ)$ where $I \in \mathbb{D}$, together with $よ : \mathsf{C} \to [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]_{\mathbb{D}}$, the corestriction of the Yoneda embedding, is the free $\mathbb{D}$-cocompletion of C. That is, $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]_{\mathbb{D}}$ has all colimits of diagrams $I\,;\,よ$ for $I \in \mathbb{D}$, and for each category D with a functor $J : \mathsf{C} \to \mathsf{D}$ that has all colimits of diagrams $I\,;J$ for $I \in \mathbb{D}$, there exists, up to natural isomorphism, a unique functor $M : [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}] \to \mathsf{D}$ such that M preserves colimits of diagrams $I\,;\,よ$ for $I \in \mathbb{D}$, and $よ\,; M \cong J$ naturally. ⟨prshf-free⟩

To prepare for the proof, we need another lemma: colimits of diagrams computed pointwise after embedding them into the presheaf category of their codomain are "free" in the sense that if two diagrams have the same pointwise colimit in the presheaf category, then they have the same colimit "everywhere", i.e. even if they are mapped via some functor into another category.

PROPOSITION 3.3.4. For all $I : \mathrm{I} \to \mathsf{C}$ and $I' : \mathrm{I}' \to \mathsf{C}$, if $\mathrm{colim}(I\,;\,よ) \cong \mathrm{colim}(J\,;\,よ)$ naturally, then for all $J : \mathsf{C} \to \mathsf{D}$, the colimit ⟨prshf-colim-absolute⟩

colim($I\,;J$) exists iff colim($I'\,;J$) exists, and if they exist, they are isomorphic as objects.

Proof. This is claimed in [30, Proposition 3.9] and shown in [28, Theorem 3.2]. One constructs for all diagrams $I : \mathrm{I} \to \mathsf{C}$ a final functor $F : \mathrm{I} \to \mathrm{el}\,\mathrm{colim}(I\,;よ)$ into the category of elements of $\mathrm{colim}(I\,;よ)$ such that $F\,;\pi^{\mathrm{colim}(I;よ)} \cong I$. Then, if $\mathrm{colim}(I\,;よ) \cong \mathrm{colim}(I'\,;よ)$, the categories of elements of these two functors are equivalent and $\pi^{\mathrm{colim}(I;よ)} \cong \pi^{\mathrm{colim}(I';よ)}$, and so $\mathrm{colim}(I\,;J) \cong \mathrm{colim}(F\,;\pi^{\mathrm{colim}(I;よ)}\,;J) \cong \mathrm{colim}(\pi^{\mathrm{colim}(I;よ)}\,;J) \cong \mathrm{colim}(\pi^{\mathrm{colim}(I';よ)}\,;J) \cong \mathrm{colim}(F'\,;\pi^{\mathrm{colim}(I';よ)}\,;J) \cong \mathrm{colim}(I'\,;J)$ if either colimit exists, by finality of F and F'.

The functor F maps i onto $(I_i, \iota_i(I_i)(\mathrm{id}))$, and $f : i \to j$ onto I_f, where $\iota_i : よ(I_i) \to \mathrm{colim}(I\,;よ)$ is the insertion. From the pointwise computation of colimits of presheaves in Set, each $(c,\varphi) \in \mathrm{el}\,\mathrm{colim}(I\,;よ)$ is the equivalence class of some element in $よ(I_i)(c)$, which gives an arrow into $(I_i, \iota_i(I_i)(\mathrm{id}))$, and any two such arrows yield two elements with the same equivalence class, so they must be connected via a zig-zag in I that translates to a zig-zag connecting the two in the category of elements; see [30, Proposition 3.7] for details. □

Now we can prove ⟨`prshf-free`⟩.

Proof. It is clear that $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]_{\mathbb{D}}$ has all $\mathbb{D}$-colimits since it is a full subcategory of the category of presheaves and full subcategory inclusions reflect colimits. Furthermore, $よ : \mathsf{C} \to [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]_{\mathbb{D}}$ is well-defined because $\mathbb{D}$ contains all trivial diagrams in C, so the image of the Yoneda embedding of C is a full subcategory of $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]_{\mathbb{D}}$.

Thus suppose $J : \mathsf{C} \to \mathsf{D}$ is a functor such that D has all colimits of diagrams $I\,;J$ for $I \in \mathbb{D}$. Construct a functor $M : [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]_{\mathbb{D}} \to \mathsf{D}$ as follows. Each $F \in [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]_{\mathbb{D}}$ by definition is of the form $\mathrm{colim}(I\,;よ)$ for $I \in \mathbb{D}$. Let M map F to $\mathrm{colim}(\pi^F\,;J)$, which by ⟨`prshf-colim-absolute`⟩ is isomorphic to $\mathrm{colim}(I\,;J)$ because

$\mathrm{colim}(I\,;\,よ) \cong F \cong \mathrm{colim}(\pi^F\,;\,よ)$, hence exists by assumption on D.

A natural transformation $\alpha : F \to G$ between presheaves induces a functor $\mathrm{el}F \to \mathrm{el}G$ mapping (c,φ) to $(c,\alpha(c)(\varphi))$ and $f:(c,\varphi) \to (d,\psi)$ to $f:(c,\alpha(c)(\varphi)) \to (d,\alpha(d)(\psi))$ which is in $\mathrm{el}G$ since

$$G(f)(\alpha(d)(\psi)) = \alpha(c)(F(f)(\psi)) = \alpha(c)(\varphi)$$

by naturality of α. For each $(c,\varphi) \in \mathrm{el}F$, we have that $J(\mathrm{id}_c) : J(\pi^F(c,\varphi)) \to J(\pi^G(c,\alpha(c)(\varphi)))$, giving an arrow

$$\iota_{(c,\alpha(c)(\varphi))} : J(\pi^F(c,\varphi)) \to \mathrm{colim}(\pi^G\,;J).$$

This constitutes a cocone from $\pi^F\,;J$ to $\mathrm{colim}(\pi^G\,;J)$ since each $f:(c,\varphi) \to (d,\psi)$ has that

$$J(\pi^F(f))\,;\iota_{(c,\alpha(c)(\varphi))} = J(\pi^G(f))\,;\iota_{(c,\alpha(c)(\varphi))} = \iota_{(d,\alpha(d)(\psi))}$$

by the cocone property of $\mathrm{colim}(\pi^G\,;J)$. Thus let M map α to $+_{(c,\varphi)\in \mathrm{el}F}\,\iota_{(c,\alpha(c)(\varphi))}$. Then $M(\alpha)$ is unique with the property that $\iota_{(c,\varphi)}\,;M(\alpha) = \iota_{(c,\alpha(c)(\varphi))}$ for all $c \in \mathsf{C}$ and $\varphi \in F(c)$.

$\langle$ functoriality of M $\rangle$

Just like for FinSet: we have $\iota_{(c,\varphi)}\,;M(\mathrm{id}) = \iota_{(c,\mathrm{id}(\varphi))} = \iota_{(c,\varphi)}$ and $\iota_{(c,\varphi)};M(\alpha);M(\beta) = \iota_{(c,\alpha(c)(\varphi))};M(\beta) = \iota_{(c,\beta(c)(\alpha(c)(\varphi)))} = \iota_{(c,(\alpha;\beta)(c)(\varphi)}$.

$\langle$ $よ\,;M \cong J$ $\rangle$

We have $M(よ(c)) = \mathrm{colim}(\pi^{よ(c)};J) = J(c)$ since $\mathrm{el}\,よ(c)$ has a terminal object (c,id_c),° and the colimit of a diagram with a terminal object in the index category is just the image of that object (a corollary to the relevant property of final functors). Now for any $f : c \to d$ in C, we have $M(よ(f)) = +_{(e,\varphi)\in \mathrm{el}\,よ(c)}\,\iota_{(e,\varphi;f)}$, which reduces to just $\iota_{(c,\mathrm{id}_c;f)} = \iota_{(c,f)} = J(f)$ since the insertions into the colimit of a diagram with a terminal object are

°For all $(d,f) \in \mathrm{el}\,よ(c)$, we have $よ(c)(f)(\mathrm{id}_c) = f\,;\mathrm{id}_c = f$, so f is an arrow $(d,f) \to (c,\mathrm{id}_c)$, and it is the unique choice for g such that $g\,;\mathrm{id}_c = f$.

clearly given by the unique morphisms into that object, and f is a morphism $(c,f) \to (d,\mathrm{id}_d)$ in $\mathrm{el}\,\text{よ}(d)$. Naturality is trivial since we can even choose the colimits in constructing M so that commutativity is strict.

⟨preservation of colimits⟩

Let $I : \mathsf{I} \to \mathsf{C}$ be a diagram in $\mathbb{D}$. We have $M(\mathrm{colim}(I\,;\text{よ})) \cong \mathrm{colim}(\pi^{\mathrm{colim}(I;\text{よ})};J) \cong \mathrm{colim}(I;J)$ by ⟨prshf-colim-absolute⟩, and $\mathrm{colim}(I;J) \cong \mathrm{colim}(I\,;\text{よ}\,;M)$ by commutativity just shown. Insertions $\iota_i : \text{よ}(I_i) \to \mathrm{colim}(I\,;\text{よ})$ are mapped onto morphisms $+_{(c,\varphi)\in\mathrm{el}\,\text{よ}(I_i)} \iota^{\mathsf{D}}_{(c,\iota_i(c)(\varphi))} = \iota^{\mathsf{D}}_{(I_i,\iota_i(I_i)(\mathrm{id}))} = \iota^{\mathsf{D}}_i$, as required.°

°Again, the category $\mathrm{el}\,\text{よ}(I_i)$ has a terminal object (I_i, id), and mediators of cocones from colimits of such diagrams are clearly given by the unique arrow in the cocone that comes from the terminal object.

⟨uniqueness⟩

If $\text{よ}\,;M \cong J' \cong \text{よ}\,;M'$ naturally, then also $M(F) \cong M(\mathrm{colim}(I\,;\text{よ})) \cong \mathrm{colim}(I\,;\text{よ}\,;M) \cong \mathrm{colim}(I\,;\text{よ}\,;M') \cong M'(\mathrm{colim}(I\,;\text{よ})) \cong M'(F)$ for some $I \in \mathbb{D}$ for all $F \in [\mathsf{C}^{\mathrm{op}},\mathsf{Set}]_{\mathbb{D}}$ since M' must preserve colimits of diagrams in $\mathbb{D}$. Let δ_F denote the unique isomorphisms $M(F) \cong M'(F)$ commuting with the colimit insertions. We show that this is a natural transformation. Let $\alpha \in [\mathsf{C}^{\mathrm{op}},\mathsf{Set}]_{\mathbb{D}}(F,G)$. Then, for all $(c,\varphi) \in \mathrm{el}\,F$, we have $\iota^{M}_{(c,\varphi)}\,;M(\alpha)\,;\delta_G = \iota^{M}_{(c,\alpha(c)(\varphi))}\,;\delta_G = \iota^{M'}_{(c,\alpha(c)(\varphi))}$ and $\iota^{M}_{(c,\varphi)}\,;\delta_F\,; M'(\alpha) = \iota^{M'}_{(c,\varphi)};M'(\alpha) = M'(\varphi);M'(\alpha) = M'(\varphi;\alpha) = \iota^{M'}_{(c,\alpha(c)(\varphi))}$, where the last steps follow because M' preserves the colimit of $\pi^F\,;\text{よ}$ (as this is isomorphic to a colimit of $I\,;\text{よ}$ for $I \in \mathbb{D}$), hence the insertions of $\text{よ}(\pi^F(c,\varphi))$ into $\mathrm{colim}(\pi^F\,;\text{よ})$, given by φ.° Thus $M(\alpha);\delta_G = \delta_F;M'(\alpha)$ by uniqueness of mediators, as required. □

°The property of final functors gives a canonical correspondence of colimiting cocones, and then the proof of ⟨prshf-colim-absolute⟩ yields a canonical correspondence of colimiting cocones of the two functors given here, from which it can straightforwardly be shown that insertions are preserved.

To exhibit Obs^* as a free cocompletion, we need to identify which diagrams of representables have concrete presheaves as their colimit. We provide a general condition that can be applied to any category of concrete presheaves. A concise way of defining a diagram $I : \mathsf{I} \to \mathsf{C}$ such that $\mathrm{colim}(I\,;\text{よ})$ is concrete would be to say that whenever $c \in \mathsf{C}$ and $\varphi \in \mathsf{C}(c,d)$ and $\varphi' \in \mathsf{C}(c,d')$ have

$x\,;\varphi \leftrightsquigarrow_I x\,;\varphi'$ for all $x \in \mathsf{C}(1,c)$, then $\varphi \leftrightsquigarrow_I \varphi'$. We will give a definition that is a bit more involved, but it avoids having to talk about zig-zags in the diagram. The definition is not equivalent to the previous one: there are diagrams that have concrete colimits without being concrete in the sense of the following definition.

DEFINITION: CONCRETE DIAGRAM. A diagram $I : \mathsf{I} \to \mathsf{C}$ in a concrete category C is called *concrete* if

⟨conc-1⟩ for each commuting square

$$\begin{array}{ccccc} & & I_j & & \\ & \nearrow^{I_f} & & \nwarrow^{I_{f'}} & \\ I_i & & & & I_{i'} \\ & \nwarrow_{x} & & \nearrow_{y} & \\ & & 1 & & \end{array}$$

in C, there exists a diagram

$$i \xleftarrow{\;g\;} k \xrightarrow{\;g'\;} i'$$

in I and an arrow $z : 1 \to I_k$ in C such that

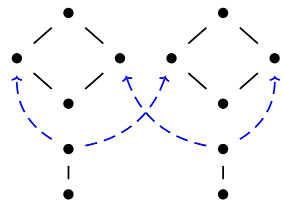
$$\begin{array}{ccccc} I_i & \xleftarrow{\;I_g\;} & I_k & \xrightarrow{\;I_{g'}\;} & I_{i'} \\ & \nwarrow_{x} & \uparrow{\scriptstyle z} & \nearrow_{y} & \\ & & 1 & & \end{array}$$

commutes, and

⟨conc-2⟩ for each pair $i, i' \in \mathsf{I}$ and every diagram

$$I_i \xleftarrow{\;f\;} c \xrightarrow{\;f'\;} I_{i'}$$

The situation we need to prevent, illustrated in terms of Obs, is that all the atoms of two CABAs are glued together without their joins being glued together, which happens e.g. in the pointwise colimit of the diagram



which is not a concrete presheaf.

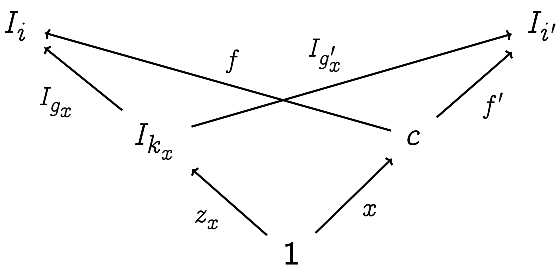
The condition ⟨conc-1⟩ simplifies reasoning about the diagrams, because it implies that if two atoms in the CABAs coming from I_i and $I_{i'}$ are identified in I_j, then there is already an object in the diagram that accounts for this.

Condition ⟨conc-2⟩ then says that if there are is a CABA (given by よ(c)) that fits into two objects of the diagram such that all the corresponding atoms hit by each CABA are identified (which is detected by the presence of objects directly identifying each pair of atoms like in ⟨conc-1⟩), then there must be a CABA in the diagram that directly identifies these pairs of atoms, which means that it also identifies their joins.

in C: if for each $x \in \mathsf{C}(\mathbf{1}, c)$ there exists a diagram

$$i \xleftarrow{g_x} k_x \xrightarrow{g'_x} i'$$

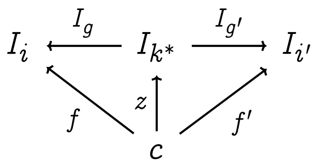
in I and an arrow $z_x : \mathbf{1} \to I_{k_x}$ in C such that



commutes, then there exists a diagram

$$i \xleftarrow{g} k^* \xrightarrow{g'} i'$$

in I and an arrow $z : c \to I_{k^*}$ such that



commutes.

PROPOSITION 3.3.5. Let C be a small concrete category. The ⟨conc-free⟩
category Conc C together with よ : C → Conc C is the free cocompletion of C under small concrete diagrams in C.

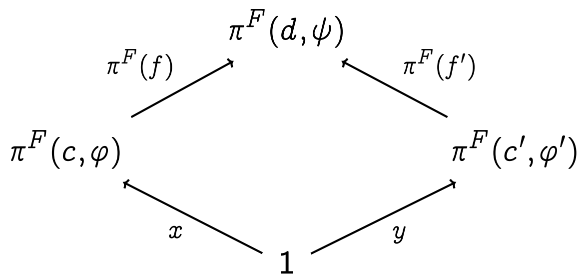
Proof. By ⟨prshf-free⟩, we just need to show that every concrete presheaf is isomorphic to colim(I ; よ) for some concrete diagram I, and conversely that every such colimit is a concrete presheaf.

⟨concrete presheaves are colimits of concrete diagrams⟩

Let $F \in \mathrm{Conc}\,\mathsf{C}$. We simply show that π^F is concrete; that $\mathrm{colim}(\pi^F\,;\,よ) \cong F$ is the co-Yoneda lemma.

⟨conc-1⟩

Suppose we have a commuting square



in C. By definition of $\mathrm{el}\,F$, then $F(f)(\psi) = \varphi$ and $F(f')(\psi) = \varphi'$, and so $F(x)(\varphi) = F(x)(F(f)(\psi)) = F(x\,;f)(\psi) = F(y\,;f')(\psi) = F(y)(F(f')(\psi)) = F(y)(\varphi')$ by assumption of commutativity. We then get arrows

$$(1, F(x)(\varphi)) = (1, F(y)(\varphi')) \to (c,\varphi), (c',\varphi')$$

in $\mathrm{el}\,F$ via x and y, and with $\mathrm{id}_1 : 1 \to \pi^F(1, F(x)(\varphi))$ we have $\mathrm{id}_1\,;\pi^F(x) = x$ and $\mathrm{id}_1\,;\pi^F(y) = y$ as needed.

⟨conc-2⟩

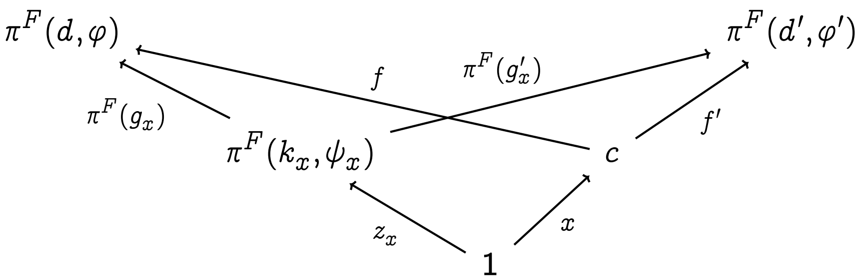
Suppose that for $(d,\varphi),(d',\varphi') \in \mathrm{el}\,F$ we have

$$\pi^F(d,\varphi) \xleftarrow{\;f\;} c \xrightarrow{\;f'\;} \pi^F(d',\varphi')$$

in C such that for each $x \in \mathsf{C}(1,c)$ there exist

$$(d,\varphi) \xleftarrow{\;g_x\;} (k_x,\psi_x) \xrightarrow{\;g'_x\;} (d',\varphi')$$

in $\mathrm{el}\,F$ and an arrow $z_x : \mathbf{1} \to \pi^F(k_x, \psi_x)$ in C such that



commutes. For all $x \in \mathsf{C}(\mathbf{1}, c)$, we then have $F(g_x)(\varphi) = \psi_x = F(g'_x)(\varphi')$. Now $z_x \,;\, g_x = x \,;\, f$, so

$$F(z_x)(\psi_x) = F(z_x)(F(g_x)(\varphi)) = F(z_x \,;\, g_x)(\varphi) = F(x \,;\, f)(\varphi) = F(x)(F(f)(\varphi)),$$

and with $z_x \,;\, g'_x = x \,;\, f'$ we analogously derive $F(z_x)(\psi_x) = F(x)(F(f')(\varphi'))$, hence $F(x)(F(f)(\varphi)) = F(x)(F(f')(\varphi'))$. By concreteness of F, then $F(f)(\varphi) = F(f')(\varphi')$. Thus f, f' induce arrows $(c, F(f)(\varphi)) \to (d,\varphi), (d',\varphi')$ and $\mathrm{id}_c : c \to \pi^F(c, F(f)(\varphi))$, with $\mathrm{id}_c \,;\, f = f$ and $\mathrm{id}_c \,;\, f' = f'$.

⟨concrete colimits are concrete⟩

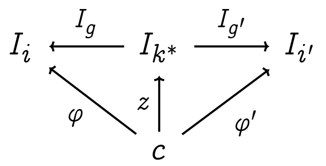
Let I be a concrete diagram in C. We show that the pointwise colimit of $I' := I \,;\, \text{よ}$ in the category of presheaves is concrete. The claim directly follows because full subcategories reflect all colimits.

The pointwise colimit of $I \,;\, \text{よ}$ is the functor sending $c \in \mathsf{C}$ to the disjoint union of all $I'_i(c)$ quotiented by the equivalence relation generated by "$[\varphi] \sim [\varphi']$ iff there exists an $f : i \to j$ such that $I'_f(c)(\varphi) = \varphi'$", and morphisms f to $+(I'_i(f) \,;\, \iota_i)$.

Now let $c \in \mathsf{C}$, $[\varphi], [\varphi'] \in \mathrm{colim}\, I'(c)$ for $\varphi \in I'_i(c)$ and $\varphi' \in I'_{i'}(c)$, and suppose $\mathrm{colim}\, I'(x)([\varphi]) = \mathrm{colim}\, I'(x)([\varphi'])$ for all $x \in \mathsf{C}(\mathbf{1}, c)$. We want to show that $[\varphi] = [\varphi']$, i.e. $\varphi \sim \varphi'$, by using ⟨conc-2⟩ to obtain

$$i \xleftarrow{\;g\;} k^* \xrightarrow{\;g'\;} i'$$

in I such that $I'_g(c)(\psi) = \varphi$ and $I'_{g'}(c)(\psi) = \varphi'$ for some $\psi \in I'_{k^*}(c)$, which would prove $\varphi \sim \varphi'$. By the Yoneda lemma, there is a bijection between $I'_i(c) = \text{よ}(I_i(c))$ and natural transformations from $\text{よ}(c)$ to $\text{よ}(I_i)$, which correspond to morphisms $c \to I_i$ in C, so let us identify $\varphi : c \to I_i$ and $\varphi' : c \to I_{i'}$ with their morphisms in C. By definition of the Yoneda embedding, it then suffices if there is an arrow $z : c \to I_{k^*}$ such that



commutes, like in the consequent of ⟨conc-2⟩.

Thus we need to check that the requirements of the antecedent are satisfied. Let $x \in \mathsf{C}(1, c)$. Now because $\operatorname{colim} I'(x)([\varphi]) = \operatorname{colim} I'(x)([\varphi'])$, by definition of the equivalence relation, there exists a finite sequence of elements i_k in I of length n and arrows $f_k : i_k \to i_{k+1}$ or $f_k : i_{k+1} \to i_k$ and elements $a_k \in I'_{i_k}(1)$ such that $a_1 = I'_i(x)(\varphi)$, $a_n = I'_{i'}(x)(\varphi')$, and $I'_{f_k}(1)(a_k) = a_{k+1}$ or $I'_{f_k}(1)(a_{k+1}) = a_k$ for all k. Here we used that $\operatorname{colim} I'(x)([\varphi]) = +(I'_j(x)\,;\iota_j)([\varphi]) = \iota_i(I'_i(x)(\varphi))$.

We show by induction on n that there exists a k_x in I with a $z_x \in I'_{k_x}(1)$ and arrows $g_x, g'_x : k_x \to i, i'$ such that $I'_{g_x}(1)(z_x) = a_1 = I'_i(x)(\varphi)$ and $I'_{g'_x}(1)(z_x) = a_n = I'_{i'}(x)(\varphi')$. This is clearly the required commutativity in ⟨conc-2⟩.

⟨$n = 1$⟩

We can choose $k_x = i$ and $z_x = a_1$ with $g_x, g'_x = \mathrm{id}$.

⟨$n=2$⟩

W.l.o.g. $f_1{:}i \to i'$ and let $k_x = i$, $z_x = a_1$, $g_x = \mathrm{id}$, and $g'_x = f_1$. Then $I'_{g_x}(1)(z_x) = a_1$ and $I'_{g'_x}(1)(z_x) = I'_{f_1}(1)(a_1) = a_2$ by assumption.

⟨$n>2$⟩

By the induction hypothesis, there exists a k_x^- in I with a $z_x^- \in I'_{k_x^-}(1)$ and arrows $g_x^-, g_x^{-\prime} : k_x^- \to i_2, i'$ such that $I'_{g_x^-}(1)(z_x^-) = a_2$ and $I'_{g'_x}(1)(z_x^-) = a_n$.

[$f_1 : i_2 \to i$]

Let $k_x = k_x^-$, $z_x = z_x^-$, $g_x = g_x^-; f_1$, and $g'_x = g_x^{-\prime}$. Then, we have $I'_{g_x}(1)(z_x) = I'_{f_1}(I'_{g_x^-}(1)(z_x^-)) = I'_{f_1}(a_2) = a_1$ as well as $I'_{g'_x}(1)(z_x) = I'_{g_x^{-\prime}}(1)(z_x^-) = a_n$ by assumption.

[$f_1 : i \to i_2$]

Then $I'_{f_1}(1)(a_1) = a_2 = I'_{g_x^-}(1)(z_x^-)$, and so by ⟨conc-1⟩, there exists a $k_x \in$ I and $g_x, g_x^+ : k_x \to i, k_x^-$ and an arrow $z_x : 1 \to I'_{k_x} \in I'_{k_x}(1)$ such that $I'_{g_x}(1)(z_x) = a_1$ and $I'_{g_x^+;g'_x}(1)(z_x) = I'_{g_x^-}(1)(z_x^-) = a_n$, as required. □

We then have the following corollaries.

PROPOSITION 3.3.6. The category Obs* is the free cocompletion of FinSet$^+$ under colimits of concrete diagrams as well as the free cocompletion of FinSet under colimits of concrete non-empty diagrams. ⟨obs*-free⟩

Proof. As remarked before, an adaptation to the proof of the statement ⟨obs*=conc+finset⟩ shows that Obs* is equivalent to the category Conc FinSet$^+$, which is the free cocompletion under concrete diagrams by ⟨conc-free⟩. A presheaf is non-empty iff it is the pointwise colimit of a non-empty diagram by the way pointwise colimits are computed, since unions of non-empty sets are non-empty. Since Obs* is equivalent to Conc$^+$ FinSet and ⟨conc-free⟩

shows that concrete presheaves are precisely those of the form colim(I ; よ) for I concrete, together we obtain that Obs^* is the category of presheaves on FinSet with objects colim(I ; よ) for I a non-empty concrete diagram, hence the free cocompletion under such diagrams by ⟨prshf-free⟩. □

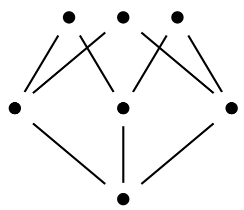
PROPOSITION 3.3.7. The category FinObs is the free cocompletion of FinRel under colimits of diagrams I;↪;よ factoring through the inclusion FinSet ↪ FinRel such that colim(I;よ) $\in \mathrm{Conc}^+_{\mathrm{fin}}$ FinSet, or equivalently of diagrams I;↪;よ such that I is finite, non-empty, and concrete in FinSet. ⟨finobs-prshf-free⟩

Proof. The first part of the claim follows directly from the statements ⟨finobs-prshf-finrel⟩ and ⟨prshf-free⟩ since the presheaves colim(I ; ↪ ; よ) for colim(I ; よ) $\in \mathrm{Conc}^+_{\mathrm{fin}}$ FinSet are precisely those of the form colim(π^F ; ↪ ; よ) for $F \in \mathrm{Conc}^+_{\mathrm{fin}}$ FinSet by ⟨prshf-colim-absolute⟩ because colim($\pi^{\mathrm{colim}(I;よ)}$; よ) ≅ colim($I$; よ) for all $I : \mathbf{I} \to$ FinSet. The second part follows from that and ⟨obs⋆-free⟩ while noting that finite presheaves are precisely those that are colimits of finite diagrams of representables. □

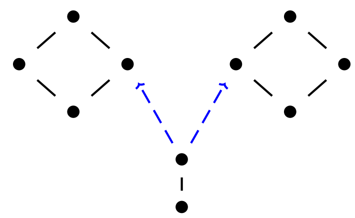
For the full category Obs, things are more complicated; colimits do not seem to be enough. Since we cannot capture infinitary objects fully in a type theory, there would not be any benefit to describing the whole category Obs as a free category for our purposes, but for future reference, let me conjecture that Obs can be obtained from FinRel by performing a different kind of construction on the same diagrams as for FinObs (except that they may be infinite), namely taking colimits which additionally satisfy the property that any object with a *cone* to the "transposed" diagram (transposing all relations in it) of a directed subset of such a diagram has a unique mediator *to* the colimit commuting with the cone and projections given by the transposes of the colimit insertions. That is, the object should be a colimit but also canonically behave like a limit to each directed subset of its diagram.

The characterisation of FinObs via colimits of concrete diagrams seems to be of limited use for creating a type theory out of it, because it is difficult to describe concrete diagrams using type-theoretic notation. Future research may determine a generally applicable way for doing that. For the case of FinObs, we will instead give an alternative representation as a free category of presheaves constructed from repeated pushouts of a certain shape. See Section 3.2, subsection on the quasitopos Obs^*, for a discussion of split and strong monomorphisms in Obs and Obs^*, which straightforwardly transfers to FinObs and FinObs^*.

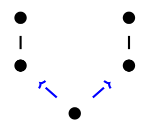
The example domain from the beginning of this chapter



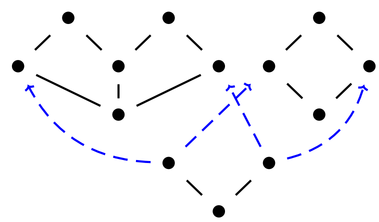
can be constructed using repeated split pushouts by first taking



then constructing the intersection with the third maximal element



and then putting everything together



DEFINITION: SPLIT PUSHOUT. Let C be a category and $J : \mathsf{FinRel} \to \mathsf{C}$ a functor. A *split pushout diagram* is a diagram in C that is either of the form $I;J$ for some $I : \mathbf{1} \to \mathsf{FinRel}$, or of the form $I : \mathsf{P} \to \mathsf{C}$ where P is the category $\bullet \leftarrow \bullet \to \bullet$ and the image of I is

$$A \xleftarrow{\ r\ } C \xrightarrow{\ s\ } B$$

where A, B, and C are colimits of split pushout diagrams, r is a split monomorphism, and both r and s are functional. A morphism in C is called *functional* if it is one of the following:

- a morphism $J(\hookrightarrow(f)) : J(\hookrightarrow(N)) \to J(\hookrightarrow(M))$ for some $f \in \mathsf{FinSet}(N, M)$,
- an identity morphism,
- a composite of functional morphisms,
- an insertion into $\operatorname{colim} I$ for I a split pushout diagram,
- a mediator of functional morphisms forming a cocone from a split pushout diagram.

The pair (C, J) is called *split-pushout-cocomplete* if it has colimits of all split pushout diagrams. A functor $F : (\mathsf{C}, J) \to (\mathsf{D}, J')$ between split-pushout-cocomplete categories is called *split-pushout-cocontinuous* if it preserves colimits of split pushout diagrams.

We are going to use split pushouts in the type theory to construct all the objects of FinObs. This is more feasible than using the characterisation as strong monomorphisms in FinObs^* because it will be easy to define split monomorphisms in the type theory, and furthermore every functor preserves split monomorphisms, which will play a central role in proving the claim we want to establish. However, it is often easier to work with the category FinObs^* in the proofs that follow, so we shall also define a corresponding construction for FinObs^*.

DEFINITION: STRONG PUSHOUT. A *strong pushout diagram* is a diagram in FinObs^* that is either of the form $I\,;\,よ$ for $I : \mathsf{I} \to \mathsf{FinSet}$ or of the form $I : \mathsf{P} \to \mathsf{C}$ where P is the category $\bullet \leftarrow \bullet \to \bullet$ and the image of I is

$$A \xleftarrow{\;r\;} C \xrightarrow{\;s\;} B$$

such that A, B, C are colimits of strong pushout diagrams and r is a strong monomorphism.

We refer to colimits of split/strong pushout diagrams simply as *split/strong pushouts.*

PROPOSITION 3.3.8. Every object $D \in \mathsf{FinObs}^*$ is a strong pushout. ⟨finobs*-strong⟩

Proof. By induction on $|\max D|$ (which is at least 1 since D is non-empty). If $|\max D| = 1$, then D is necessarily a CABA, hence isomorphic to $よ(\mathrm{At}\, D)$, with $\mathrm{At}\, D \in \mathsf{FinSet}$ because D is finite.

Now suppose $|\max D| > 1$ and all D' with $|\max D'| < |\max D|$ are strong pushouts. Let $x \in \max D$ and consider the set $E := {\downarrow}(\max D \setminus \{x\})$ of elements in D that are below some maximal element other than x. Together with the induced order from D, this is a domain: it is non-empty because D has at least two maximal elements by assumption, it is nuclear and atomistic, and bounded-complete because it is a down-set in a domain,° and

°That is, it includes all atoms below elements it contains, and all joins of these atoms, which are in bijection with consistent sets of atoms because D is a domain.

it is trivially directed-complete because it is finite. In addition, let $E' := E \cap {\downarrow} x$. This is also a domain (like any intersection of two domains, in fact): non-emptiness follows because $\bot \in E$ and $\bot \in {\downarrow} x$ and the rest follows like for E since E' is a down-set in D.

By ⟨obs⋆-colim⟩, it is clear that D is the colimit of the diagram

$$E \hookleftarrow E' \hookrightarrow \text{よ}(\mathrm{At}\, x)$$

where $E' \hookrightarrow E$ and $E' \hookrightarrow \text{よ}(\mathrm{At}\, x) \cong {\downarrow} x$ are the obvious inclusions (which are clearly atom-preserving domain mappings): naming the elements of E, E', and $\text{よ}(\mathrm{At}\, x)$ in accordance with the names in D, of which all these domains are subsets, the inclusions identify all elements with the same name, so the computation of the colimit amounts to taking the union $E \cup E' \cup \text{よ}(\mathrm{At}\, x)$, which is D because every element of D is below x or a maximal element of E.

Thus it only remains to show that the diagram satisfies the conditions from the definition of strong pushouts. First, E' is a strong subobject of E: whenever $S \subseteq \mathrm{At}\, E'$ has a join in E, then because $S \subseteq \mathrm{At}\, {\downarrow} x$ and ${\downarrow} x$ is a CABA, that join is also in ${\downarrow} x$, hence in E'. Thus the inclusion $E' \hookrightarrow E$ is a strong monomorphism in FinObs^*.

Furthermore, E has strictly fewer maximal elements than D since $\max E = \max D \setminus \{x\} \subsetneq \max D$ because x is maximal. The domain E' also has fewer maximal elements, which we show via an injection $\max E' \to \max E$: send every maximal element in E' to a maximal element of E above it, which exists because E is a domain. If two $x, y \in \max E'$ have $x, y \sqsubseteq z \in \max E$, then $x \sqcup y$ exists by ⟨bnd-comp⟩, and $\mathrm{At}(x \sqcup y) \in \mathcal{C}(\mathrm{At}\, E)$ implies $\mathrm{At}(x \sqcup y) \in \mathcal{C}(\mathrm{At}\, E')$ by strongness, so $x \sqsubseteq x \sqcup y = y$ by maximality of y and then $x = y$ by maximality of x. Clearly $|\max \text{よ}(\mathrm{At}\, x)| = |\{x\}| = 1 < |\max D|$ by assumption. By the induction hypothesis, it follows that E, E', and $\text{よ}(\mathrm{At}\, x)$ are strong pushouts, concluding the proof. □

Because FinObs^* is cocomplete, the converse holds, too: every strong pushout diagram has a colimit in FinObs^*. However, to show that they are "free", we need to make sure that strong pushouts can be computed pointwise, in the category of all presheaves on FinSet.

PROPOSITION 3.3.9. The pointwise colimit of every strong pushout diagram in FinObs^*, calculated in $[\mathsf{FinSet}^{\mathrm{op}}, \mathsf{Set}]$, is in FinObs^*. ⟨pointwise-strong-finobs*⟩

Proof. If the diagram is trivial, this is clear because $\mathrm{Conc}^{+}_{\mathrm{fin}}\,\mathsf{FinSet}$ contains all the representables. Thus let

$$A \xleftarrow{\;r\;} C \xrightarrow{\;s\;} B$$

be a strong pushout diagram with pointwise colimit P. Note that P is non-empty and finite since the diagram is non-empty and finite and so are the presheaves in the diagram. We show that P is concrete, from which the claim immediately follows. Thus let $N \in \mathsf{FinSet}$ and $[\varphi], [\varphi'] \in P(N)$ such that $P(n)([\varphi]) = P(n)([\varphi'])$ for all $n \in N$. We show that $[\varphi] = [\varphi']$. The only non-trivial case is where w.l.o.g. $\varphi \in A(N)$ and $\varphi' \in B(N)$. This means that for all $n \in N$, there exists an $a_n \in C(1)$ such that $r(1)(a_n) = A(n)(\varphi)$ and $s(1)(a_n) = B(n)(\varphi')$. Then, $r(1)(\{a_n \mid n \in N\}) = A(N)(\varphi)$, so because m is strong, there exists a $\psi \in C(N)$ with $r(N)(\psi) = \varphi$ (like in [3, Proposition 34]), so $r(1)(C(n)(\psi)) = A(n)(r(N)(\psi)) = A(n)(\varphi) = r(1)(a_n)$ for all $n \in N$, hence $C(n)(\psi) = a_n$ because $r(1)$ is injective. It follows that $B(n)(s(N)(\psi)) = s(1)(C(n)(\psi)) = s(1)(a_n) = B(n)(\varphi')$ for all $n \in N$, so $s(N)(\psi) = \varphi'$ by concreteness of B. Thus $r(N)(\psi) = \varphi$ and $s(N)(\psi) = \varphi'$, hence $[\varphi] = [\varphi']$. □

We transfer split pushouts in FinObs^* to strong pushouts in FinObs. We can see directly using the equivalences in ⟨obs*=conc+finset⟩ and ⟨finobs-prshf-finrel⟩ that the inclusion $\mathsf{FinObs}^* \hookrightarrow \mathsf{FinObs}$ sends representable presheaves $よ(N) \in \mathsf{FinObs}^*$ to representables $よ(\hookrightarrow(N)) \in \mathsf{FinObs}$ and morphisms $よ(f)$ to $よ(\hookrightarrow(f))$. It then follows that FinObs has colimits of all finite diagrams $I \,;\, \hookrightarrow \,;\, よ$ for $\mathrm{colim}(I \,;\, よ) \in \mathsf{FinObs}$ since the inclusion preserves colimits and FinObs^* is finitely cocomplete. By ⟨prshf-free⟩, it follows that there is a unique functor $\mathsf{FinObs}^* \to \mathsf{FinObs}$ commuting with よ and よ ; $\hookrightarrow$ and preserving colimits of finite diagrams I ; よ with colim(I ; よ) $\in \mathsf{FinObs}$. But the inclusion $\mathsf{FinObs}^* \hookrightarrow \mathsf{FinObs}$ does precisely that, so we know that it is of the form given in the proof of ⟨prshf-free⟩. Note furthermore that there exists a similar colimit-preserving inclusion $[\mathsf{FinSet}^{\mathrm{op}}, \mathsf{Set}] \hookrightarrow [\mathsf{FinRel}^{\mathrm{op}}, \mathsf{Set}]$ since $[\mathsf{FinRel}^{\mathrm{op}}, \mathsf{Set}]$ is cocomplete and thus in particular has all colimits of diagrams factoring through $\hookrightarrow$; よ, and inspecting its definition in ⟨prshf-free⟩, we find that it restricts to the inclusion $\mathsf{FinObs}^* \hookrightarrow \mathsf{FinObs}$ on those full subcategories.

PROPOSITION 3.3.10. Every split pushout diagram in FinObs factors through the inclusion $\mathsf{FinObs}^* \hookrightarrow \mathsf{FinObs}$ via a strong pushout diagram in FinObs^*. ⟨split-strong⟩

Proof. For a diagram I ; よ$_{\mathsf{FinRel}}$ where I selects a single object $N \in \mathsf{FinRel}$, this is clear since then $N \in \mathsf{FinSet}$ and the inclusion sends よ$_{\mathsf{FinSet}}(N)$ to よ$_{\mathsf{FinRel}}(\hookrightarrow(N))$. Thus let

$$A \xleftarrow{\ r\ } C \xrightarrow{\ s\ } B$$

be a split pushout diagram in FinObs. Clearly $A, B, C \in \mathsf{FinObs}^*$, and by ⟨finobs*-strong⟩, they are strong pushouts. Furthermore, as we discussed in Section 3.2, every split monomorphism in FinObs that happens to be atom-preserving is a strong monomorphism in FinObs^*, so we only need to check that r and s are atom-preserving.

We show that every functional morphism is atom-preserving by induction on the definition of functional morphisms. A morphism of the form よ($\hookrightarrow(f)$) : よ($\hookrightarrow(N)$) $\to J(\hookrightarrow(M))$ for $f \in \mathsf{FinSet}(N, M)$ has that よ(f) : よ(N) $\to$ よ(M) is a morphism in FinObs^* and the inclusion sends よ(f) to よ($\hookrightarrow(f)$). Identities and composites of atom-preserving maps are atom-preserving, and insertions into split pushouts that by induction hypothesis factor through $\mathsf{FinObs}^* \hookrightarrow \mathsf{FinObs}$ are the image of atom-preserving maps because FinObs^* is finitely cocomplete, hence the split pushout has a colimit, and the inclusion preserves colimits, hence colimit insertions.

Finally, a mediator of atom-preserving maps for a pushout factoring through the inclusion is equally atom-preserving in FinObs*, again by finite cocompleteness of FinObs, and so the image of the corresponding mediator under the inclusion is that same mediator since the inclusion preserves colimits and composites. □

Eventually, we want to use ⟨prshf-free⟩ to show the claim. Note that ⟨prshf-free⟩ only makes a claim about the category of presheaves that are colimits of diagrams $I\,;\,よ$ with $I : \mathsf{I} \to \mathsf{C}$ a specified type of diagram. However, we want to use these freely constructed pushouts from objects that may already be presheaves. There may be an obvious abstract argument that solves this issue, but the only solution I found was to "unfold" a pushout diagram in the category of presheaves recursively into a diagram in the underlying category, so this is what we shall do.

DEFINITION: RECURSIVE PUSHOUT. A *recursive pushout shape* is a category I such that either $\mathsf{I} \cong \mathbf{1}$ or I consists of subcategories I_A, I_B, and I_C, such that each of these categories is a recursive pushout shape, and for each $c \in \mathsf{I}_C$ there exists an $a \in \mathsf{I}_A$ with an arrow $c \to a$ and a $b \in \mathsf{I}_B$ with an arrow $c \to b$ in I and any two such arrows $f, f' : c \to a, a' \in \mathsf{I}_{A,B}$ have $f \leftrightsquigarrow_{\mathrm{id}} g$. A *recursive pushout diagram* is a diagram $I : \mathsf{I} \to \mathsf{C}$ of recursive pushout shape.

Given a recursive pushout diagram, which we think of as a representation of repeated applications of pushouts, we can reassemble it under certain conditions into a pushout diagram.

PROPOSITION 3.3.11. Let $I : \mathsf{I} \to \mathsf{C}$ be a recursive pushout diagram. If $\mathsf{I} \cong \mathbf{1}$, then I evidently has a colimit $\operatorname{colim} I \cong I(\bullet)$. Otherwise, if $I_{A,B,C}$° have a colimit and the diagram ⟨rec-colim⟩

$$\operatorname{colim} I_A \xleftarrow{\ r\ } \operatorname{colim} I_C \xrightarrow{\ s\ } \operatorname{colim} I_B$$

with arrows r, s given as the mediators $+_{i:c\to a\in \mathsf{I}_A} I(i)\,;\,\iota_a$ and $+_{j:c\to b\in \mathsf{I}_B} I(j); \iota_b$ has a pushout, then I has a colimit, given by that

°From here on, we denote diagram functor application by $I(i)$, because we use $I_{A,B,C}$ to denote the restrictions of recursive pushout diagrams I to their constituent categories.

pushout together with the natural insertions $\iota_x ; \iota_X$ arising from composition of the insertions into $\operatorname{colim} I_{A,B,C}$ with the arrows of those colimits into the pushout.

Proof. First, note that the respective mediators exist because by definition of recursive pushout shapes, there exists an arrow $i : c \to a$ for each $c \in \mathrm{I}_C$, hence an arrow $i ; \iota_a : c \to \operatorname{colim} I_A$, and any two such arrows i, i' have $i \leftrightsquigarrow i'$, hence also $i ; \iota_a = i' ; \iota_{a'}$ because the ι form a cocone which commutes with the zig-zag; analogously for I_B, so the collections of arrows form cocones into the respective colimits.

Let P denote the pushout of the given diagram, with insertions $\iota_{A,B}$. We show that it satisfies the universal property of a colimit of I. First, we establish a cocone from each element of I. For $a \in \mathrm{I}_A$, let $f_a = \iota_a ; \iota_A$, for $b \in \mathrm{I}_B$, let $f_b = \iota_b ; \iota_B$, and for $c \in \mathrm{I}_C$, let $f_c = \iota_c ; r ; \iota_A = \iota_c ; s ; \iota_B$ by the cocone property of the pushout. To show that this is a cocone, let f be an arrow in I and distinguish three cases.

[$f : a \to a'$ for a, a' both in I_A or I_B]

W.l.o.g. assume both are in I_A. Then $I(f) ; f_{a'} = I(f) ; \iota_{a'} ; \iota_A = \iota_a ; \iota_A = f_a$ by the cocone property of $\operatorname{colim} I_A$.

[$f : c \to c'$ for c, c' both in I_C]

Then $I(f) ; f_{c'} = I(f) ; \iota_{c'} ; r ; \iota_A = \iota_c ; r ; \iota_A = f_c$ by the cocone property of $\operatorname{colim} I_C$.

[$f : c \to a$ for $c \in \mathrm{I}_C$ and $a \notin \mathrm{I}_C$]

W.l.o.g. assume $a \in \mathrm{I}_A$. Then $I(f) ; f_a = I(f) ; \iota_a ; \iota_A = \iota_c ; r ; \iota_A = f_c$ by the mediating property of r.

Now suppose Q has a cocone g_i from I. We construct an arrow $P \to Q$ as follows. Since Q has a cocone from I, in particular, it has cocones from I_A and I_B, hence colimit mediators $m_{A,B} : A, B \to Q$. Then for all $c \in \mathrm{I}_C$, we have $\iota_c ; r ; m_A = I(i) ; m_A = g_c$ for some $i : c \to a$ by the mediating properties of r and m_A, and similarly $\iota_c ; s ; m_B = I(j) ; m_B = g_c$, so $r ; m_A = s ; m_B$ by uniqueness of mediators. Thus there exists a mediator $m{:}P \to Q$ with $\iota_A;m = m_A$ and $\iota_B\,;m = m_B$.

We show that m is a unique mediator of the cocone g_i. First, commutativity. Distinguish three cases.

[g_a for $a \in \mathrm{I}_{A,B}$]

W.l.o.g. $a \in \mathrm{I}_A$. Then, $f_a\,;m = \iota_a\,;\iota_A\,;m = \iota_a\,;m_A = g_a$ by the mediating properties of m and m_A.

[g_c for $c \in \mathrm{I}_C$]

Then, $f_c\,;m = \iota_c\,;r\,;\iota_A\,;m = I(i)\,;\iota_a\,;m_A = I(i)\,;g_a = g_c$ for some $i : c \to a$ since the g_j form a cocone.

Finally, we show that every mediator $m' : P \to Q$ with $f_i\,;m' = g_i$ also has $\iota_A\,;m' = m_A$ and $\iota_B\,;m' = m_B$, from which uniqueness follows by uniqueness of the pushout mediator. We have $\iota_a;\iota_A;m' = f_a\,;m' = g_a$ and $\iota_a\,;m_A = g_a$ for all $a \in \mathrm{I}_A$, so $\iota_A\,;m' = m_A$ by uniqueness of the colimit mediator for I_A, and analogously for I_B.

□

Now let us see how a recursive pushout diagram can be created from a pushout diagram.

DEFINITION: UNFOLDING. For a strong pushout diagram in FinObs^*, define its *unfolding*, if it exists, recursively as a diagram $I : \mathrm{I} \to \mathsf{FinSet}$ as follows. If the diagram is trivial with value よ(N), let $\mathrm{I} = 1$ and I map $\bullet$ onto N. Otherwise, it is of the form

$$A \xleftarrow{\;r\;} C \xrightarrow{\;s\;} B$$

where we suppose that A, B, C are colimits of strong pushout diagrams with unfoldings $I_{A,B,C} : \mathrm{I}_{A,B,C} \to \mathsf{FinSet}$ with colimits $\mathrm{colim}(I_{A,B,C}\,;$ よ$) \cong A, B, C$ in FinObs^*. Then, let I be the disjoint union of $\mathrm{I}_{A,B,C}$ together with arrows $c \to a, b$ whenever there is an $f : I_C(c) \to I_A(a), I_B(b)$ in FinSet such that $\iota_c\,;r =$ よ$(f)\,;\iota_{a,b}$, and let I map each such arrow onto the corresponding arrow f, and otherwise coincide with $I_{A,B,C}$.

PROPOSITION 3.3.12. Every unfolding is a recursive pushout diagram. ⟨unfold-rec⟩

Proof. By structural induction on the strong pushout diagram. Suppose it is of the form よ(N) for some $N \in \mathsf{FinSet}$. Then $\mathrm{I} = 1$ and $I : \bullet \mapsto N$, which is a recursive pushout diagram. Now suppose the diagram is of the form

$$A \xleftarrow{r} C \xrightarrow{s} B$$

with unfoldings $I_{A,B,C} : \mathrm{I}_{A,B,C} \to \mathsf{FinSet}$ with $\mathrm{colim}(I_{A,B,C} ; よ) \cong A, B, C$ and assume each of these unfoldings is a recursive pushout diagram. We show that I as in the definition of unfoldings is a recursive pushout.

For that, we only need to check that for each $c \in \mathrm{I}_C$, there exists an $a \in \mathrm{I}_A$ with an arrow $i : c \to a$ in I, and any two such i, i' have $i \leftrightsquigarrow i'$; the case for $b \in \mathrm{I}_B$ will be analogous. Thus let $c \in \mathrm{I}_C$. We first show that there exists an $f : I_C(c) \to I_A(a)$ in FinSet such that $\iota_c ; r = よ(f) ; \iota_a$. Because $\iota_c ; r$ maps $よ(I_C(c))$ into $\mathrm{colim}\, I_A$, it must map all of the CABA $よ(I_C(c))$ onto a consistent subset of $\mathrm{colim}\, I_A$, which by computation of colimits is contained entirely in some $よ(I_A(a))$, so $\iota_c ; r$ factors through ι_a via a map between representables, which is a function in FinSet. If it factors through two such insertions, the corresponding consistent sets must be mapped onto the same set in the colimit, hence connected by a zig-zag in I_A. □

PROPOSITION 3.3.13. Every strong pushout diagram $I : \mathrm{I} \to \mathsf{FinSet}$ has an unfolding I such that the diagram for $I ; よ_{\mathsf{FinSet}}$ in ⟨rec-colim⟩ is a strong pushout diagram with the same colimit as $I ; よ$. ⟨strong-unfold⟩

Proof. By induction on the diagram. If the diagram is trivial, this is clear. Otherwise, it is of the form

$$A \xleftarrow{r} C \xrightarrow{s} B$$

where we suppose that A,B,C are colimits of strong pushout diagrams that have unfoldings with $\mathrm{colim}(I_{A,B,C}\,;\,\text{よ}) \cong A,B,C$. Construct I like in the definition of unfoldings. By ⟨unfold-rec⟩, this is a recursive pushout, and then the pushout of

$$\mathrm{colim}(I_A\,;\,\text{よ}) \xleftarrow{r'} \mathrm{colim}(I_C\,;\,\text{よ}) \xrightarrow{s'} \mathrm{colim}(I_B\,;\,\text{よ})$$

with r' given as $+_{i:c\to a\in I_A} \text{よ}(I(i));\iota_a = +_{i:c\to a\in I_A}\iota_c;r = r$ by definition of the unfolding, and s' analogously, is simply the original strong pushout, which has a colimit in FinObs^* by finite cocompleteness. Then by ⟨rec-colim⟩, the colimit of $I\,;\,\text{よ}$ exists and is given by the pushout. □

Note that this also shows that for *every* unfolding of a strong pushout, the colimit can be computed like in ⟨rec-colim⟩, which reduces it to the computation of the original pushout. Using ⟨pointwise-strong-finobs*⟩, colimits of strong pushout diagrams can be computed pointwise, then with ⟨strong-unfold⟩ we can see inductively that colimits of $I\,;\,\text{よ}$ where I is an unfolding can also be computed pointwise.

To establish the free construction, we will unfold strong pushout diagrams and then reassemble them in a split-pushout-cocomplete category. We have to make sure that the unfoldings do reassemble into a split pushout diagram. The crucial point is that split monomorphisms are preserved by this process.

Define the *canonical unfolding* of a domain $D \in \mathsf{FinObs}^*$ as the unfolding arising from the construction in ⟨finobs*-strong⟩.° That is, write each domain D as the colimit of a trivial diagram if $|\max D| = 1$ and otherwise write it as a strong pushout like in ⟨finobs*-strong⟩, create the canonical unfoldings recursively for the domains in that diagram, and compose them into a recursive pushout by adding arrows like in the definition of unfoldings. We first show some facts about these canonical unfoldings.

°Note that the canonical unfolding is not unique, but it can be made unique by fixing an enumeration of the elements of the domain.

PROPOSITION 3.3.14. Every recursive pushout shape I has a *weakly terminal set* $T \subseteq \mathbf{I}$. That is, for each $i \in \mathbf{I}$ there exists a $t \in T$ such that there exists an arrow $i \to t$. ⟨rec-term⟩

Proof. By structural induction on I. For $\mathbf{I} \cong \mathbf{1}$, this is trivial. Thus suppose the statement holds for $\mathbf{I}_{A,B,C}$. We show that $T_A \cup T_B$ is a terminal set. Let $i \in \mathbf{I}$ and distinguish two cases.

$[i \in \mathbf{I}_{A,B}]$

By the induction hypothesis, there exists a $t \in T_{A,B}$ with an arrow $i \to t$.

$[i \in \mathbf{I}_C]$

By definition, there exists an $a \in \mathbf{I}_A$ with an arrow $i \to a$, which by the induction hypothesis has an arrow $a \to t \in T_A$, so the composite $i \to a \to t$ gives an arrow into $T_A \cup T_B$. □

PROPOSITION 3.3.15. For each domain $D \in \mathsf{FinObs}^*$, its canonical unfolding $I : \mathbf{I} \to \mathsf{FinSet}$ has that for each terminal $t \in \mathbf{I}$, the insertion $\iota_t : よ(I(t)) \to \mathrm{colim}(I\,;よ) \cong D$ is a monomorphism, and the image of the maximal element of $よ(I(t))$ under ι_t is a maximal element of D. ⟨canon-inj-max⟩

Proof. By induction on I. If I is trivial, this is clear. Otherwise, by ⟨strong-unfold⟩ the colimit $I\,;よ$ is given as the pushout from ⟨rec-colim⟩. Let $t \in \mathbf{I}$ be terminal, w.l.o.g. $t \in \mathbf{I}_A$. Then, $\iota_t : よ(I(t)) \to \mathrm{colim}(I_A\,;よ)$ is a monomorphism by the inductive hypothesis, and $\iota_A : \mathrm{colim}(I_A\,;よ) \to \mathrm{colim}(I\,;よ)$, the insertion into the pushout, is clearly a monomorphism as computation of colimits in FinObs^* by ⟨obs⋆-colim⟩ shows. Thus $\iota_t ; \iota_A$ is a monomorphism as the composite of monomorphisms.

Furthermore, the canonical unfolding has that $\mathrm{colim}(I_A\,;よ) \cong {\downarrow}(\max D \setminus \{x\})$ and $\mathrm{colim}(I_B\,;よ) \cong {\downarrow} x$, so every terminal element t (which is either in $\mathbf{I}_A$ or $\mathbf{I}_B$) has that the image of $よ(I(t))$ under ι_t hits a maximal element in ${\downarrow}(\max D \setminus \{x\})$ or in ${\downarrow} x$ by the inductive hypothesis. But ${\downarrow} x$ only has one maximal element, which is maximal in D by construction of the canonical unfolding, and every maximal element in $\downarrow(\max D \setminus \{x\})$ is necessarily maximal in D. Thus the image under $\iota_t ; \iota_{A,B}$ is maximal in D. □

PROPOSITION 3.3.16. Let (C, J) be split-pushout-cocomplete and $I : \mathrm{I} \to \mathsf{FinSet}$ an unfolding of a strong pushout diagram in FinObs^*. Then $I ; \hookrightarrow ; J$ has a colimit given as a split pushout like in ⟨rec-colim⟩. ⟨unfold-colim⟩

Proof. By structural induction on I. The case where $\mathrm{I} \cong \mathbf{1}$ is trivial. Thus suppose that $I_{A,B,C}$ are unfoldings and the diagrams $I_{A,B,C} ; \hookrightarrow ; J$ have colimits given as split pushouts like in ⟨rec-colim⟩. We can restrict our attention to canonical unfoldings of A, B, C: let I_A be an unfolding and I'_A the canonical unfolding of the colimit of $I_A ; よ$ in FinObs^*. By ⟨pointwise-strong-finobs*⟩ and ⟨strong-unfold⟩, $\mathrm{colim}(I_A ; よ) \cong \mathrm{colim}(I'_A ; よ)$ in the presheaf category, so by ⟨prshf-colim-absolute⟩, also $\mathrm{colim}(I_A ; \hookrightarrow ; J) \cong \mathrm{colim}(I'_A ; \hookrightarrow ; J)$.

Since I is a recursive pushout by ⟨unfold-rec⟩, $I;\hookrightarrow;J$ is a recursive pushout as well. By ⟨rec-colim⟩, it suffices to show that the pushout

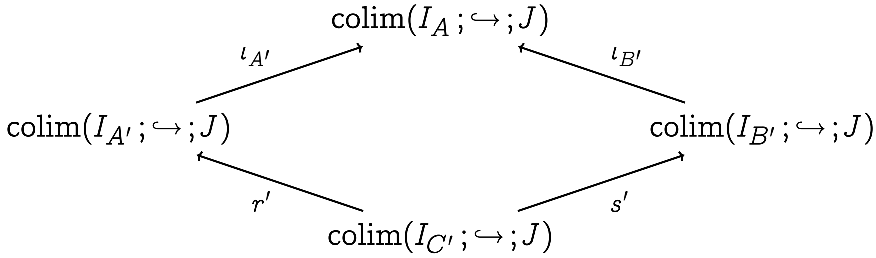
$$\mathrm{colim}(I_A ; \hookrightarrow ; J) \xleftarrow{\ r\ } \mathrm{colim}(I_C ; \hookrightarrow ; J) \xrightarrow{\ s\ } \mathrm{colim}(I_B ; \hookrightarrow ; J)$$

with $r = +_{i:c \to a \in \mathrm{I}_A} J(\hookrightarrow(I(i))) ; \iota_a$ and $s = +_{j:c \to b \in \mathrm{I}_B} J(\hookrightarrow(I(i)))$; ι_b exists. Since C is split-pushout-cocomplete, we simply show that r and s are functional and that r is a split monomorphism. This also shows that the pushout is split since each $\mathrm{colim}(I_{A,B,C} ; \hookrightarrow ; よ)$ is a split pushout by the induction hypothesis.

⟨r and s are functional⟩

We first show that each ι_a is functional, by structural induction on I_A. If $\mathrm{I}_A \cong \mathbf{1}$, then $\iota_\bullet = \mathrm{id}$, which is functional. Otherwise,

I_A consists of $\mathrm{I}_{A',B',C'}$, and $\mathrm{colim}(I_A\,;\hookrightarrow;J)$ is the split pushout



with each insertion ι_x given by $\iota_x\,;\iota_X$ for $x \in \mathrm{I}_X$ and $X \in \{A',B',C'\}$. Then $\iota_{a'}$ is functional by hypothesis, $\iota_{A'}$ is functional since it is a split pushout insertion, and so ι_a is functional as the composite of functional morphisms.

Now we show that $r = +_{i:c\to a\in \mathrm{I}_A} J(\hookrightarrow(I(i)))\,;\iota_a$ is functional, by structural induction on I_C. If $\mathrm{I}_C \cong \mathbf{1}$, then r is just a composite $J(\hookrightarrow(I(i)));\iota_a$ for some $a \in \mathrm{I}_A$, which is functional since it is the composite of $J(\hookrightarrow(I(i)))$ with $I(i)$ a morphism in FinSet, and ι_a, which is functional as just shown. Otherwise, decompose I_C as $\mathrm{I}_{A',B',C'}$, and then $\mathrm{colim}(I_C\,;\hookrightarrow;J)$ is the split pushout of a diagram like the above. Then r can by uniqueness of mediators be written as $(+_{i:a'\to a\in \mathrm{I}_A} J(\hookrightarrow(I(i)))\,;\iota_a) + (+_{i:b'\to a\in \mathrm{I}_A} J(\hookrightarrow(I(i)))\,;\iota_a)$, which is a split pushout mediator of by hypothesis functional morphisms, hence functional. The argument for s is analogous.

$\langle\, r$ is a split mono$\,\rangle$

We explicitly construct the retraction of r, as follows. Because the inclusion $\mathsf{FinObs}^* \hookrightarrow \mathsf{FinObs}$ preserves colimits and by ⟨`strong-unfold`⟩, the strong pushout diagram is sent by the inclusion to a diagram in FinObs with the same colimit as $I\,;\hookrightarrow;\,\text{よ}_{\mathsf{FinRel}}$ which under ⟨`rec-colim`⟩ reduces to that diagram. Because strong monomorphisms in FinObs^* are split in FinObs, the mediator $R := +_{i:c\to a}\,\text{よ}_{\mathsf{FinRel}}(\hookrightarrow(I(i)))\,;\iota_a$ from ⟨`rec-colim`⟩ is a split monomorphism, so it has a retraction R', which is a mediator $+\,\text{よ}(r'_a)\,;\iota_c$ for all morphisms $r'_a \in$

$\mathsf{FinRel}(\hookrightarrow(I(a)), \hookrightarrow(I(c)))$ such that $\iota_a ; R' = よ(r'_a) ; \iota_c$, by an argument like in the proof of ⟨strong-unfold⟩.

For each $c \in \mathrm{I}_C$, by ⟨rec-term⟩, there is a terminal element $t \in \mathrm{I}_C$ with an arrow $t_c : c \to t$, and $\iota_c = よ(\hookrightarrow(t_c)) ; \iota_t$ since the insertions form a cocone. Since $\iota_t ; R$ factors through some ι_a for $a \in \mathrm{I}_A$, which has an arrow $t'_a : a \to t'$ with t' terminal in I_A, it also factors through $\iota_{t'}$ by composition with $よ(\hookrightarrow(t'_a))$. Thus there exists an arrow $f_t : t \to t'$ such that $\iota_t ; R = よ(\hookrightarrow(I(f_t))) ; \iota_{t'}$, so $\iota_t = \iota_t ; R ; R' = よ(\hookrightarrow(I(f_t))) ; \iota_{t'} ; R'$. By ⟨canon-inj-max⟩, ι_t is an injective function on atoms, and then $よ(\hookrightarrow(I(f_t)))$ must also be injective, and on the image of $よ(\hookrightarrow(I(f_t)))$, the morphism $\iota_{t'} ; R'$ also needs to be an injective function on atoms. Now the image of $よ(\hookrightarrow(I(t')))$ under $\iota_{t'} ; R'$ needs to be a consistent set of atoms of $\mathrm{colim}(I_C ; \hookrightarrow ; よ)$, and it is necessarily a superset of the atoms in $よ(\hookrightarrow(I(t)))$. By maximality of $よ(\hookrightarrow(I(t)))$, it follows that the image of $よ(\hookrightarrow(I(t')))$ under $\iota_{t'} ; R'$ is precisely $よ(\hookrightarrow(I(t)))$, so combined with injectivity we have that $\iota_{t'} ; R'$ factors through ι_t via the transpose $よ(\hookrightarrow(I(f_t))^{\mathrm{T}})$. Note that $\hookrightarrow(I(f_t)) ; \hookrightarrow(I(f_t))^{\mathrm{T}} = \mathrm{id}$.

Now let $r' := \dotplus J(r'_a) ; \iota^{\mathsf{C}}_c$. Because the r'_a were obtained from a mediator like in the definition of unfoldings, an argument like in the proof of ⟨rec-colim⟩ shows that the mediator r' exists. Then, for all $c \in \mathrm{I}_C$, we have $\iota^{\mathsf{C}}_c ; r ; r' = J(\hookrightarrow(I(t_c ; f_t))) ; \iota^{\mathsf{C}}_{t'} ; r'$ for $t_c : c \to t$ with t terminal, and then this is $J(\hookrightarrow(I(t_c ; f_t))) ; J(\hookrightarrow(I(f_t))^{\mathrm{T}}) ; \iota^{\mathsf{C}}_t = J(\hookrightarrow(I(t_c))) ; J(\hookrightarrow(I_{f_t}) ; \hookrightarrow(I_{f_t})^{\mathrm{T}}) ; \iota^{\mathsf{C}}_t = J(\hookrightarrow(I(t_c))) ; \mathrm{id} ; \iota^{\mathsf{C}}_t = J(\hookrightarrow(I(t_c))) ; \iota^{\mathsf{C}}_t = \iota^{\mathsf{C}}_c$, so $r ; r' = \mathrm{id}$ and r is a split monomorphism. □

Now we can finally put all the pieces together.

THEOREM 2. The category FinObs together with the inclusion FinRel $\hookrightarrow$ FinObs is the free split-pushout-cocomplete category. That is, it is split-pushout-cocomplete, and for any category that is ⟨finobs-free⟩

split-pushout-cocomplete, there exists a unique-up-to-isomorphism split-pushout-cocontinuous functor from FinObs into it.

Proof. By ⟨split-strong⟩, every split pushout diagram in FinObs factors through the inclusion $\mathsf{FinObs}^* \hookrightarrow \mathsf{FinObs}$ and FinObs^* is finitely cocomplete, with the inclusion preserving colimits, so FinObs is split-pushout-cocomplete. Thus let (C, J) split-pushout-cocomplete. We show that ⟨unfold-finobs⟩ the pointwise colimit of every diagram $I\,;\hookrightarrow;\,よ$ such that I is an unfolding of a strong pushout is in FinObs, and that ⟨finobs-unfold⟩ every presheaf in FinObs is the pointwise colimit of a diagram $I\,;\hookrightarrow;\,よ$ for I the unfolding of a strong pushout. Thus FinObs is precisely the full category of presheaves $F \in [\mathsf{FinRel}^{\mathrm{op}}, \mathsf{Set}]$ such that $F \cong \mathrm{colim}(I;\hookrightarrow;よ)$ for I some unfolding. By ⟨prshf-free⟩, it follows that FinObs is the free cocompletion of FinRel for diagrams $I\,;\hookrightarrow$ with I the unfolding of a strong pushout in FinObs. By ⟨unfold-colim⟩, C has colimits of all diagrams $I;\hookrightarrow;J$ with I an unfolding. By ⟨prshf-free⟩, there then exists a functor $F : \mathsf{FinObs} \to \mathsf{C}$, unique up to isomorphism with the property that F preserves colimits of diagrams $I\,;\hookrightarrow;\,よ$ for I an unfolding and $よ\,;F \cong J$. Finally, we show that F preserves split pushouts, and then that every split-pushout-preserving functor $F' : \mathsf{FinObs} \to \mathsf{C}$ with $よ\,;F' \cong J$ preserves colimits of diagrams $I\,;\hookrightarrow;\,よ$ with I an unfolding, so that $F' \cong F$ by the defining property of F. It then follows that there exists a unique-up-to-isomorphism split-pushout-cocontinuous functor $\mathsf{FinObs} \to \mathsf{C}$, as claimed.

⟨unfold-finobs⟩

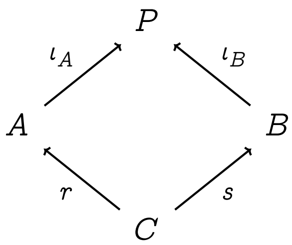
Let I be the unfolding of a strong pushout. The colimit of $I\,;\,よ$ in FinObs^* can be computed pointwise by the remark after ⟨strong-unfold⟩. Because the functor $[\mathsf{FinSet}^{\mathrm{op}}, \mathsf{Set}] \hookrightarrow [\mathsf{FinRel}^{\mathrm{op}}, \mathsf{Set}]$ given by $\mathrm{colim}(-\,;\hookrightarrow;\,よ)$ as well as its restriction $\mathsf{FinObs}^* \hookrightarrow \mathsf{FinObs}$ preserve colimits, it follows that $\mathrm{colim}(I\,;\hookrightarrow;\,よ)$ is both the colimit in the presheaf category $[\mathsf{FinRel}^{\mathrm{op}}, \mathsf{Set}]$ as well as in FinObs, hence computed pointwise.

⟨`finobs-unfold`⟩

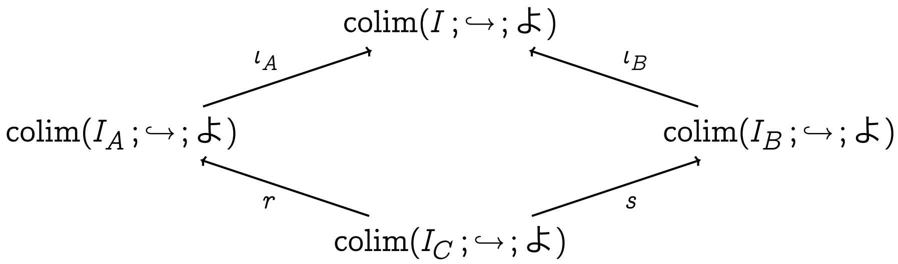
Let $D \in \mathsf{FinObs}$. Since $D \in \mathsf{FinObs}^*$, by ⟨`finobs*-strong`⟩, it is a strong pushout, and by ⟨`strong-unfold`⟩, its diagram has an unfolding I with $\mathrm{colim}(I\,;\text{よ}) \cong D$. Then, because the inclusion $\mathsf{FinObs}^* \hookrightarrow \mathsf{FinObs}$ preserves colimits, $\mathrm{colim}(I\,;\hookrightarrow;\text{よ}) \cong D \in \mathsf{FinObs}$.

⟨F preserves split pushouts⟩

For split pushout diagrams of the form $\text{よ}(\hookrightarrow(N))$ for $N \in \mathsf{FinSet}$, the functor F preserves the colimit trivially. Thus let



be a split pushout in FinObs. Using ⟨`split-strong`⟩ as well as ⟨`strong-unfold`⟩, there is an unfolding I with $\mathrm{colim}(I;\hookrightarrow;\text{よ}) \cong P$ so that we can rewrite the above diagram as



with $\mathrm{colim}(I_{A,B,C};\hookrightarrow;\text{よ}) \cong A,B,C$ and $r = +_{i:c\to a}\text{よ}(\hookrightarrow I(i));\iota_a$ and $s = +_{j:c\to b}\text{よ}(\hookrightarrow(I(j)));\iota_b$.

We show that $F(\mathrm{colim}(I\,;\hookrightarrow;\text{よ}))$ together with $F(\iota_{A,B})$ is the split pushout of

$$F(\mathrm{colim}(I_A\,;\hookrightarrow;\text{よ}))\xleftarrow{F(r)}F(\mathrm{colim}(I_C\,;\hookrightarrow;\text{よ}))\xrightarrow{F(s)}F(\mathrm{colim}(I_B\,;\hookrightarrow;\text{よ}))$$

as required. By preservation of colimits of unfoldings, F maps $\mathrm{colim}(I;\hookrightarrow;\text{よ})$ onto $\mathrm{colim}(I;\hookrightarrow;\text{よ};F)\cong\mathrm{colim}(I;\hookrightarrow;J)$, which by ⟨`unfold-colim`⟩ is given by the split pushout of

$$\mathrm{colim}(I_A\,;\hookrightarrow;J)\xleftarrow{r'}\mathrm{colim}(I_C\,;\hookrightarrow;J)\xrightarrow{s'}\mathrm{colim}(I_B\,;\hookrightarrow;J)$$

with $r'=+_{i:c\to a}J(\hookrightarrow(I(i)));\iota^{\mathsf{C}}_a$ and $s'=+_{j:c\to b}J(\hookrightarrow(I(j)));\iota^{\mathsf{C}}_b$, together with insertions $\iota^{\mathsf{C}}_x\,;\iota^{\mathsf{C}}_X$. Since $\text{よ}\,;F\cong J$, this is the pushout of

$$\mathrm{colim}(I_A\,;\hookrightarrow;\text{よ}\,;F)\xleftarrow{r'}\mathrm{colim}(I_C\,;\hookrightarrow;\text{よ}\,;F)\xrightarrow{s'}\mathrm{colim}(I_B\,;\hookrightarrow;\text{よ}\,;F)$$

where we have $r'=+_{i:c\to a}F(\text{よ}(\hookrightarrow(I(i))));\iota^{\mathsf{C}}_a$ as well as $s'=+_{j:c\to b}F(\text{よ}(\hookrightarrow(I(j))));\iota^{\mathsf{C}}_b$. Preservation of colimits of unfoldings also implies that insertions and composites, hence mediators, are mapped appropriately, so that $F(r)=r'$ and $F(s)=s'$ since $\iota^{\mathsf{C}}_c\,;r'=F(\text{よ}(\hookrightarrow(I(i))));\iota^{\mathsf{C}}_a=F(\iota_c\,;r)=F(\iota_c)\,;F(r)=\iota^{\mathsf{C}}_c\,;F(r)$ and mediators are unique. Again by preservation of colimits of unfoldings, this is the pushout of

$$F(\mathrm{colim}(I_A\,;\hookrightarrow;\text{よ}))\xleftarrow{F(r)}F(\mathrm{colim}(I_C\,;\hookrightarrow;\text{よ}))\xrightarrow{F(s)}F(\mathrm{colim}(I_B\,;\hookrightarrow;\text{よ}))$$

where we have that $\iota^{\mathsf{C}}_a\,;F(\iota_A)=F(\iota_a\,;\iota_A)$ is the insertion of $\text{よ}(\hookrightarrow(I_A(a)))$ into $\mathrm{colim}(I;\hookrightarrow;\text{よ};F)$ by preservation of colimits of unfoldings, just like $\iota^{\mathsf{C}}_a\,;\iota^{\mathsf{C}}_A$, so $F(\iota_A)=\iota^{\mathsf{C}}_A$ by uniqueness of the colimit mediator from $\mathrm{colim}(I_A;\hookrightarrow;\text{よ};F)$, and analogously for ι_B, as claimed.

⟨F' preserves colimits $I\,;\hookrightarrow;\text{よ}$ for I unfolding⟩

By structural induction on I. If $\mathrm{I}\cong\mathbf{1}$, this is trivial. Otherwise, by ⟨`unfold-colim`⟩, we have that $\mathrm{colim}(I\,;\hookrightarrow;\text{よ})$ is given by the split pushout

$$\text{colim}(I_A\,;\hookrightarrow;\text{よ}) \xleftarrow{\;r\;} \text{colim}(I_C\,;\hookrightarrow;\text{よ}) \xrightarrow{\;s\;} \text{colim}(I_B\,;\hookrightarrow;\text{よ})$$

for $r = +_{i:c\to a}\text{よ}(\hookrightarrow(I(i)))\,;\iota_a$ and $s = +_{j:c\to b}\text{よ}(\hookrightarrow(I(j)))\,;\iota_b$ together with the insertions $\iota_x\,;\iota_X$ for $x \in \mathrm{I}_X$ for $X \in \{A,B,C\}$. The induction hypothesis is that F' preserves colimits of $I_{A,B,C}\,;\hookrightarrow;\text{よ}$. Then $F'(\text{colim}(I\,;\hookrightarrow;\text{よ}))$ is the split pushout of

$$F'(\text{colim}(I_A\,;\hookrightarrow;\text{よ})) \xleftarrow{F'(r)} F'(\text{colim}(I_C\,;\hookrightarrow;\text{よ})) \xrightarrow{F'(s)} F'(\text{colim}(I_B\,;\hookrightarrow;\text{よ}))$$

by preservation of split pushouts. By the induction hypothesis, this is also the pushout of

$$\text{colim}(I_A\,;\hookrightarrow;\text{よ}\,;F') \xleftarrow{F'(r)} \text{colim}(I_C\,;\hookrightarrow;\text{よ}\,;F') \xrightarrow{F'(s)} \text{colim}(I_B;\hookrightarrow;\text{よ}\,;F')$$

which since each $I_{A,B,C}\,;\hookrightarrow;\text{よ}\,;F'$ is a recursive pushout with $F'(r) = +_{i:c\to a}F'(\text{よ}(\hookrightarrow(I(i))))\,;\iota_a^{\mathsf{C}}$ and analogously for s is the colimit of $I\,;\hookrightarrow;\text{よ}\,;F$ by ⟨`rec-colim`⟩, with insertions $\iota_x^{\mathsf{C}}\,;\iota_X^{\mathsf{C}} = F'(\iota_x)\,;F'(\iota_X) = F'(\iota_x\,;\iota_X)$ by preservation of the insertions ι_x by the induction hypothesis and preservation of ι_X by preservation of split pushouts. □

4 TYPE THEORIES FOR DYNAMICAL SYSTEMS

We finally turn to the derivation of the type theories we set out to develop. Because Obs^* is locally cartesian closed, we could actually write down an internal dependent type theory for it. However, this is not necessary for us in order to create a type theory that is complete for the finitary fragment of Obs, and not sufficient for the infinitary case. Since the aim is to show how the derivation of a type theory can be done *in principle*, we will instead keep it as simple as possible.

The building blocks of any type theory are *terms* and *types*, as well as *judgments* stating that a term (usually containing a free variable of a given type) has a certain type. The valid judgments are generated via *derivation rules*. We will also use *kinds*, which are types of types, to model multiple categories within a single theory. This way, we can mimick the construction of categories as free categories on top of another category in the type theory.

DEFINITION: TYPE THEORY. Let V be a set, called the set of *variables*. A *judgment* is an expression of one of the four following forms:

- $K : \square$, where K is a *kind*;

- $A : K$, where K is a kind and A a *type*;
- $x : A \vdash N : B : K$, where $x \in V$, K is a kind, A and B are types, and N is a *term*;
- $x : A \vdash N = M : B : K$, where $x \in V$, K is a kind, A and B are types, and N and M are terms.

A *type theory* is a set of judgments closed under a set of *derivation rules*.

Categories will be turned into kinds, and objects in the category become types of the corresponding kind. Judgments $x : A \vdash N : B : K$ where A and B are types representing objects and N is a term should correspond to morphisms in the category from the object represented by A to the object represented by B. Judgments $x : A \vdash N = M : B : K$ indicate that the morphisms represented by $x : A \vdash N : B : K$ and $x : A \vdash M : B : K$ are equal. This is all we need to capture the structure of a category in a type theory.

Note that the definition leaves unspecified what a kind, a type, or a term is. This is because the sets of these things are generated inductively via the derivation rules. A judgment $x : A \vdash N : B : K$ in the conclusion of a derivation rule thus actually expresses two things: that N in context $x : A$ is a well-formed term, and that it is of type B. Since derivation rules are in turn defined in terms of judgments, this means that the formal definitions of judgments and type theories should be understood as being mutually recursive.

Note also that in contrast to common type theories, our contexts (the left-hand side of a judgment) contain exactly one variable. This is to keep interpretations simple, and it suffices because multiple variables can still be modelled using product types. In fact, product types are necessary for this, but some of our categories do not even have products. We are not trying to model a particular type theory categorically, but we are trying to do the reverse, so we shall tailor the type theories to fit the demands of the categories we want to describe.

For a judgment $x:A \vdash N:B:K$, the term N will usually contain occurrences of the variable x. We define *substitution* $N[M\,/\,x]$ where M is another term as the term N where x is replaced with M. However, we may only replace *free* occurrences of x in N. There may be subterms of N of the form $x.N'$, indicating that x is *bound* by the binder $x.$ if it occurs in N', and then x inside N' must not be replaced with M. By the principle of *α-equivalence*, we do not distinguish between terms that only differ in the names of bound variables. We will not specify the details of bound and free variables and substitution and rely on the reader's intuition to handle these correctly; a proper discussion of this for the λ-calculus is found in [26, Sections 1.3–1.5].

Derivation rules will be added step by step in this chapter. They are of the form

$$\frac{\Gamma_1 \qquad \Gamma_2 \qquad \dots \qquad \Gamma_n}{\Delta_1 \qquad \Delta_2 \qquad \dots \qquad \Delta_n}$$

stating that if all judgments Γ_i are included in a type theory, then all judgments Δ_i are included in it, too. The judgments in a derivation rule usually contain variables for terms and types and are thus not technically judgments but merely stand for judgments of the given shapes.°

°Variables, including term and type variables, occurring in judgments are implicitly universally quantified. With this, the rules ⟨tt-var⟩ and ⟨tt-subs⟩ presented soon allow changing the names of free variables in judgments as well, which we will often do implicitly.

We often simplify notation using the following conventions. We drop the left-hand side of a judgment along with the turnstile $\vdash$ if the variable declared in it is common to all judgments and not explicitly used, and we drop the upper part of a derivation rule if it is empty. We may leave out antecedents that are obviously necessary to soundly interpret a rule and rely on the reader's benevolence in using the rules. We usually leave out kind annotations in type and equality judgments if it is obvious what kind we are talking about, and we may similarly leave out type annotations from equality judgments.

There are a few derivation rules we will assume *any* type theory to be closed under. First, there are rules corresponding to the axioms of category theory:

$$\langle \texttt{tt-var} \rangle \; x:A \vdash x:A \qquad \langle \texttt{tt-subs} \rangle \; \frac{x:A \vdash N:B \qquad y:B \vdash M:C}{x:A \vdash M[N\,/\,y]:C}$$

For equality, we need

$$\langle \texttt{tt-eq-refl} \rangle \; N = N \qquad \langle \texttt{tt-tt-eq-sym} \rangle \; \frac{N = M}{M = N} \qquad \langle \texttt{tt-eq-trans} \rangle \; \frac{N = M \qquad M = L}{N = L}$$

as well as a rule

$$\langle \texttt{tt-eq-subs} \rangle \; \frac{x:A \vdash N = N':B \qquad y:B \vdash M = M':C}{x:A \vdash M[N\,/\,y] = M'[N'\,/\,y]:C}$$

stating that we can replace the free variable in a term with equal terms and obtain equal terms. Now, we are ready to look at the categorical interpretation of type theories, which we should discuss before giving the type theories to make it clear where the rules are coming from. Then, we will first give a sound and complete type theory for finite observation domains along with relevant subcategories of FinObs, before giving a sound and incomplete, but more expressive, type theory that can deal with all computable functions.

4.1 CATEGORICAL SEMANTICS OF TYPE THEORIES

We first define how to interpret a type theory in a category. Note that a type theory may speak about multiple categories, so we will just pick one of them. For a type theory Θ and $K : \square \in \Theta$, write Λ_K for the *language* of K, defined as $\{A{:}K \mid A{:}K \in \Theta_K\} \cup \{x{:}A \vdash N{:}B{:}K \in \Theta \mid A{:}K, B{:}K \in \Theta\}$, and then define Θ_K as $\Lambda_K \cup \{x{:}A \vdash N = M : B : K \in \Theta \mid x:A \vdash N:B:K \in \Lambda_K, x:A \vdash M:B:K \in \Lambda_K\}$, containing all judgments of Θ that are about K.

DEFINITION: INTERPRETATION. Let Θ be a type theory and $K:\Box\in\Theta$. An *interpretation* of Θ_K is given by a category C along with an assignment $[\![\cdot]\!]_{\mathsf{C}}:\Lambda_K\to\mathsf{C}$ such that

⟨intp-obj⟩ $[\![A]\!]_{\mathsf{C}}\in\mathsf{C}$ for $A:K\in\Lambda_K$,

⟨intp-arr⟩ $[\![x{:}A\vdash N{:}B]\!]_{\mathsf{C}}\in\mathsf{C}([\![A]\!]_{\mathsf{C}},[\![B]\!]_{\mathsf{C}})$ for $x{:}A\vdash N{:}B\in\Lambda_K$,

⟨intp-id⟩ $[\![x:A\vdash x:A]\!]_{\mathsf{C}}=\mathrm{id}_{[\![A]\!]_{\mathsf{C}}}$ for $x:A\vdash x:A\in\Lambda_K$,

⟨intp-comp⟩ $[\![x{:}A\vdash N{:}B]\!]_{\mathsf{C}};[\![x{:}B\vdash M{:}C]\!]_{\mathsf{C}}=[\![x{:}A\vdash M[N/x]{:}C]\!]_{\mathsf{C}}$ for $x:A\vdash N:B\in\Lambda_K$ and $x:B\vdash M:C\in\Lambda_K$.

DEFINITION: CATEGORY OF INTERPRETATIONS. Let Θ be a type theory and $K:\Box\in\Theta$. The *category of interpretations* $\mathtt{I}(\Lambda_K)$ of Λ_K has as objects all pairs $(\mathsf{C},[\![\cdot]\!]_{\mathsf{C}})$ where C is a small category and $[\![\cdot]\!]_{\mathsf{C}}:\Lambda_K\to\mathsf{C}$ an interpretation of Λ_K in C. Morphisms from $(\mathsf{C},[\![\cdot]\!]_{\mathsf{C}})$ to $(\mathsf{D},[\![\cdot]\!]_{\mathsf{D}})$ are given by all functors $F:\mathsf{C}\to\mathsf{D}$ such that

⟨intp-iso⟩ $F([\![A]\!]_{\mathsf{C}})\cong[\![A]\!]_{\mathsf{D}}$ for all $A:K\in\Lambda_K$, and

⟨intp-nat⟩ $F([\![x:A\vdash N:B]\!]_{\mathsf{C}});\cong_B=\cong_A;[\![x:A\vdash N:B]\!]_{\mathsf{D}}$ for all $x:A\vdash N:B\in\Lambda_K$, where $\cong_C:F([\![C]\!]_{\mathsf{C}})\to[\![C]\!]_{\mathsf{D}}$ is the isomorphism from ⟨intp-iso⟩.

Note that interpretations behave like functors from a category built out of Λ_K, and morphisms of interpretations are required to commute with those functor-like interpretations "naturally".

PROPOSITION 4.1.1. Let Θ be a type theory and $K:\Box\in\Theta$. Then $\mathtt{I}(\Lambda_K)$, with composition defined as the usual composition of functors, is a category.

Proof. The composite of functors is a functor, and preservation of interpretations along composition of $F:(\mathsf{C},[\![\cdot]\!]_{\mathsf{C}})\to(\mathsf{D},[\![\cdot]\!]_{\mathsf{D}})$ and $G:(\mathsf{D},[\![\cdot]\!]_{\mathsf{D}})\to(\mathsf{E},[\![\cdot]\!]_{\mathsf{E}})$ is seen in as folllows.

⟨intp-iso⟩

Since $\cong^{\mathsf{D}}_A:F([\![A]\!]_{\mathsf{C}})\to[\![A]\!]_{\mathsf{D}}$ is an isomorphism by ⟨intp-iso⟩ of F, then so is $G(\cong^{\mathsf{D}}_A):G(F([\![A]\!]_{\mathsf{C}}))\to G([\![A]\!]_{\mathsf{D}})$ as functors preserve isomorphisms, and then so is $G(\cong^{\mathsf{D}}_{A});\cong^{\mathsf{E}}_{A} : (F;G)(\llbracket A\rrbracket_{\mathsf{C}}) \to \llbracket A\rrbracket_{\mathsf{E}}$ by ⟨intp-iso⟩ of G and because the composite of isomorphisms is an isomorphism.

⟨intp-nat⟩

We have a diagram

$$\begin{array}{ccccc} G(F(\llbracket A\rrbracket_{\mathsf{C}})) & \xrightarrow{G(\cong^{\mathsf{D}}_{A})} & G(\llbracket A\rrbracket_{\mathsf{D}}) & \xrightarrow{\cong^{\mathsf{E}}_{A}} & \llbracket A\rrbracket_{\mathsf{E}} \\ {\scriptstyle G(F(\llbracket x:A\vdash N:B\rrbracket_{\mathsf{C}}))}\downarrow & & {\scriptstyle G(\llbracket x:A\vdash N:B\rrbracket_{\mathsf{D}})}\downarrow & & \downarrow{\scriptstyle \llbracket x:A\vdash N:B\rrbracket_{\mathsf{E}}} \\ G(F(\llbracket B\rrbracket_{\mathsf{C}})) & \xrightarrow[G(\cong^{\mathsf{D}}_{B})]{} & G(\llbracket B\rrbracket_{\mathsf{D}}) & \xrightarrow[\cong^{\mathsf{E}}_{B}]{} & \llbracket B\rrbracket_{\mathsf{E}} \end{array}$$

where the left side is the image of ⟨intp-nat⟩ of F under G, which commutes since functors preserve commuting squares, and the right side is ⟨intp-nat⟩ of G, so the whole rectangle commutes.

The identity functor trivially preserves interpretations and is the neutral element for composition of functors. □

Our aim is to prove soundness and completeness for certain type theories and classes of interpretations. We can now define what that means precisely.

DEFINITION: SOUNDNESS AND COMPLETENESS. Let Θ be a type theory, $K : \square \in \Theta$, and $\mathbf{I} \hookrightarrow \mathbf{I}(\Lambda_K)$ a full subcategory of the category of interpretations of Λ_K. Then Θ_K is called *sound* for $\mathbf{I}$ if for all $(\mathsf{C}, \llbracket\cdot\rrbracket_{\mathsf{C}}) \in \mathbf{I}$ and $x : A \vdash N = M : B \in \Theta_K$, we have that $\llbracket x : A \vdash N : B\rrbracket_{\mathsf{C}} = \llbracket x : A \vdash M : B\rrbracket_{\mathsf{C}}$ in $\mathsf{C}(\llbracket A\rrbracket_{\mathsf{C}}, \llbracket B\rrbracket_{\mathsf{C}})$. Conversely, Θ_K is called *complete* for $\mathbf{I}$ if for each pair $x : A \vdash N : B \in \Lambda_K$ and $x : A \vdash M : B \in \Lambda_K$ with $\llbracket x : A \vdash N : B\rrbracket_{\mathsf{C}} = \llbracket x : A \vdash M : B\rrbracket_{\mathsf{C}}$ for all $(\mathsf{C}, \llbracket\cdot\rrbracket_{\mathsf{C}}) \in \mathbf{I}$, we have $x : A \vdash N = M : B \in \Theta_K$.

We can easily define a canonical subcategory of the category of interpretations for each theory for which that theory is both sound and complete, so the equalities between morphisms constructed by the theory are *precisely* those that are provable (i. e. contained) in the theory.

DEFINITION: MODELS. Let Θ be a type theory and $K : \square \in \Theta$. A *model* of Θ_K is an object $(\mathsf{C}, [\![\cdot]\!]_{\mathsf{C}}) \in \mathrm{I}(\Lambda_K)$ such that for each $x : A \vdash N = M : B \in \Theta_K$, we have $[\![x : A \vdash N : B]\!]_{\mathsf{C}} = [\![x : A \vdash M : B]\!]_{\mathsf{C}}$. The *category of models* $\mathsf{M}(\Theta_K)$ is the full subcategory of $\mathrm{I}(\Lambda_K)$ containing precisely the models of Θ_K.

Clearly Θ_K is sound for $\mathsf{M}(\Theta_K)$. To show completeness, we first need to construct a generic model of the theory, which we call its *syntactic category*.

DEFINITION: SYNTACTIC CATEGORY. Let Θ be a type theory and $K : \square \in \Theta$. The *syntactic category* $\mathrm{Syn}(\Theta_K)$ of Θ_K is defined as follows. Its objects are types A for $A : K \in \Lambda_K$, and its morphisms from A to B with $A : K, B : K \in \Lambda_K$ are equivalence classes of judgments $x : A \vdash N : B \in \Lambda_K$ under the equivalence relation "$(x : A \vdash N : B) \sim (x : A \vdash M : B)$ iff $x : A \vdash N = M : B \in \Theta_K$". Composition is defined as $[x{:}A \vdash N{:}B];[x{:}B \vdash M{:}C] = [x{:}A \vdash M[N/x]]$, with the identity on A given by $[x : A \vdash x : A]$.

PROPOSITION 4.1.2. Let Θ be a type theory and $K : \square \in \Theta$. The syntactic category $\mathrm{Syn}(\Theta_K)$ is a category.

Proof. First, note that $\sim$ is really an equivalence relation, due to Θ being closed under the rules ⟨`tt-eq-refl`⟩, ⟨`tt-eq-sym`⟩, and ⟨`tt-eq-trans`⟩. Then, composition is well-defined since if $x : A \vdash N : B \in \Theta_K$ and $x : B \vdash M : C \in \Theta_K$, then by ⟨`tt-subs`⟩, $x : A \vdash M[N\,/\,x] : C \in \Theta_K$, and if $[x : A \vdash N : B] = [x : A \vdash M' : B]$ and $[x : B \vdash N : C] = [x : B \vdash N' : C]$, then $x : A \vdash N = N' : B \in \Theta_K$ and $x : B \vdash M = M' : C \in \Theta_K$, so $x : A \vdash M[N\,/\,x] = M'[N'\,/\,x] : C$ by ⟨`tt-eq-subs`⟩ and thus $[x : A \vdash M[N\,/\,x]] = [x : A \vdash M'[N'\,/\,x]]$ by definition, i. e. composition is defined independently of the choice of representatives.

Composition is associative due to the purely syntactic fact that $(L[M/x])[N/x]$ yields the same expression as $L[M[N/x]/x]$. Now for each object A, we have $A:K \in \Theta_K$, thus by ⟨tt-var⟩, $x:A \vdash x:A \in \Theta_K$. This is the identity with respect to composition since $N[x/x]$ is just N. □

PROPOSITION 4.1.3. Let Θ be a type theory and $K:\square \in \Theta$. The syntactic category $\mathrm{Syn}(\Theta_K)$ has an interpretation $[\![\cdot]\!]_{\mathrm{Syn}}$ of Λ_K given by $[\![A]\!]_{\mathrm{Syn}} = A$ and $[\![x:A \vdash N:B]\!]_{\mathrm{Syn}} = [x:A \vdash N:B]$, and this interpretation makes $(\mathrm{Syn}(\Theta_K), [\![\cdot]\!]_{\mathrm{Syn}})$ a model of Θ_K. ⟨syn-mod⟩

Proof.

⟨ $[\![\cdot]\!]_{\mathrm{Syn}}$ is an interpretation ⟩

⟨intp-obj⟩

For each $A:K \in \Lambda_K$, we have that $[\![A]\!]_{\mathrm{Syn}} = A$ is an object in $\mathrm{Syn}(\Theta_K)$.

⟨intp-arr⟩

If $x:A \vdash N:B \in \Lambda_K$, then by definition of Λ_K we have $A:K, B:K \in \Theta_K$, so A and B are objects in $\mathrm{Syn}(\Theta_K)$, and thus $[x:A \vdash N:B]$ is a morphism from $[\![A]\!]_{\mathrm{Syn}} = A$ to $[\![B]\!]_{\mathrm{Syn}} = B$.

⟨intp-id⟩

Clearly $[\![x:A \vdash x:A]\!]_{\mathrm{Syn}} = [x:A \vdash x:A] = \mathrm{id}_A = \mathrm{id}_{[\![A]\!]_{\mathrm{Syn}}}$.

⟨intp-comp⟩

We have $[\![x{:}A \vdash N{:}B]\!]_{\mathrm{Syn}} ; [\![x{:}B \vdash M{:}C]\!]_{\mathrm{Syn}} = [x{:}A \vdash N{:}B] ; [x{:}B \vdash M{:}C] = [x{:}A \vdash M[N/x]{:}C] = [\![x{:}A \vdash M[N/x]{:}C]\!]_{\mathrm{Syn}}$.

⟨ $(\mathrm{Syn}(\Theta_K), [\![\cdot]\!]_{\mathrm{Syn}})$ is a model of Θ_K ⟩

If $x:A \vdash N = M:B \in \Theta_K^{\mathrm{Syn}}$, then by definition of Θ_K, we have $x:A \vdash N:B \in \Theta_K$ and $x:A \vdash M:B \in \Theta_K$, and by definition of $\sim$ for $\mathrm{Syn}(\Theta_K)$, then $[x:A \vdash N:B] = [x:A \vdash M:B]$. □

PROPOSITION 4.1.4. Let Θ be a type theory and $K : \square \in \Theta$. Then Θ_K is complete for its category of models $\mathsf{M}(\Theta_K)$.

Proof. Suppose that for two judgments $x : A \vdash N : B \in \Lambda_K$ and $x : A \vdash M : B \in \Lambda_K$ we have $[\![x : A \vdash N : B]\!]_{\mathsf{C}} = [\![x : A \vdash M : B]\!]_{\mathsf{C}}$ for all $(\mathsf{C}, [\![\cdot]\!]_{\mathsf{C}}) \in \mathsf{M}(\Theta_K)$. By ⟨syn-mod⟩, $\mathrm{Syn}(\Theta_K) \in \mathsf{M}(\Theta_K)$, so in particular, $[\![x : A \vdash N : B]\!]_{\mathrm{Syn}} = [\![x : A \vdash M : B]\!]_{\mathrm{Syn}}$. By definition of $\mathrm{Syn}(\Theta_K)$, then $[x : A \vdash N : B] = [x : A \vdash M : B]$, which is the case only if $x : A \vdash N = M : B \in \Theta_K$. □

In fact, there is an even closer relationship between a type theory, its category of models, and its syntactic category, encapsulated by the following.

PROPOSITION 4.1.5. Let Θ be a type theory and $K : \square \in \Theta$. Then $\mathrm{Syn}(\Theta_K)$ is an initial object in $\mathsf{M}(\Theta_K)$.

⟨syn-free⟩

An initial object in a 2-category $\mathbb{C}$ of categories and (some) functors is given by a category $\mathsf{0} \in \mathbb{C}$ such that for each category $\mathsf{C} \in \mathbb{C}$ there exists a functor $F \in \mathbb{C}(\mathsf{0}, \mathsf{C})$ and for any functor $F' \in \mathbb{C}(\mathsf{0}, \mathsf{C})$ there exists a natural isomorphism $F \cong F'$.

Initial objects in such 2-categories are unique up to equivalence of categories: if $\mathsf{0}, \mathsf{0}'$ are initial objects, then there are functors $F \in \mathbb{C}(\mathsf{0}, \mathsf{0}')$ and $F' \in \mathbb{C}(\mathsf{0}', \mathsf{0})$, and then $F ; F' \cong \mathrm{id}_{\mathsf{0}}$ and $F' ; F \cong \mathrm{id}_{\mathsf{0}'}$ since all functors in $\mathbb{C}(\mathsf{0}, \mathsf{0})$ and $\mathbb{C}(\mathsf{0}', \mathsf{0}')$ are naturally isomorphic to each other, which includes the identity functors and composites $F;F'$ since $\mathbb{C}$ is a category.

Proof. Let $(\mathsf{C}, [\![\cdot]\!]_{\mathsf{C}}) \in \mathsf{M}(\Theta_K)$. There is an obvious functor $F : (\mathrm{Syn}(\Theta_K), [\![\cdot]\!]_{\mathrm{Syn}}) \to (\mathsf{C}, [\![\cdot]\!]_{\mathsf{C}})$ given by $F(A) = [\![A]\!]_{\mathsf{C}}$ and $F([x : A \vdash N : B]) = [\![x : A \vdash N : B]\!]_{\mathsf{C}}$.

⟨ F is a morphism in $\mathsf{M}(\Theta_K)$ ⟩

⟨ well-defined ⟩

By ⟨intp-obj⟩ and ⟨intp-arr⟩, and since if $[x : A \vdash N : B] = [x : A \vdash N' : B]$, then $x : A \vdash N = M \in \Theta_K$ and thus $[\![x : A \vdash N : B]\!]_{\mathsf{C}} = [\![x : A \vdash N' : B]\!]_{\mathsf{C}}$ as C is a model.

⟨ functorial ⟩

It preserves identities and composites since $F([x : A \vdash x : A]) = [\![x : A \vdash x : A]\!]_{\mathsf{C}} = \mathrm{id}_{[\![x]\!]_{\mathsf{C}}}$ by ⟨intp-id⟩, and $F([x : A \vdash N : B]; [x : B \vdash M : C]) = [\![x : A \vdash N : B]\!]_{\mathsf{C}}; [\![x : B \vdash M : C]\!]_{\mathsf{C}} = [\![x : A \vdash M[N/x] : C]\!]_{\mathsf{C}} = F([x : A \vdash M[N/x] : C])$ by ⟨intp-comp⟩.

⟨ commutes with interpretation ⟩

By construction, $F([\![A]\!]_{\mathrm{Syn}}) = F(A) = [\![A]\!]_{\mathsf{C}}$, so the identity constitutes an isomorphism $F(A) \cong [\![A]\!]_{\mathsf{C}}$, from which

⟨intp-nat⟩ trivially follows since $F(\llbracket x : A \vdash N : B \rrbracket_{\mathrm{Syn}}) = F([x : A \vdash N : B]) = \llbracket x : A \vdash N : B \rrbracket_{\mathsf{C}}$.

⟨F is unique⟩

Let $F' \colon (\mathrm{Syn}(\Theta_K), \llbracket \cdot \rrbracket_{\mathrm{Syn}}) \to (\mathsf{C}, \llbracket \cdot \rrbracket_{\mathsf{C}})$. Then necessarily $F'(A) = F'(\llbracket A \rrbracket_{\mathrm{Syn}}) \cong \llbracket A \rrbracket_{\mathsf{C}} = F(A)$ by ⟨intp-iso⟩ of F', with an isomorphism $\cong_A : F'(A) \to F(A)$ for all $A \in \mathrm{Syn}(\Theta_K)$, and for all $[x{:}A \vdash N{:}B] \in \mathrm{Syn}(\Theta_K)(A, B)$, we have $F'([x{:}A \vdash N{:}B]);\cong_B = F'(\llbracket x{:}A \vdash N{:}B \rrbracket_{\mathrm{Syn}});\cong_B = \cong_A;\llbracket x{:}A \vdash N{:}B \rrbracket_{\mathsf{C}} = \cong_A;F([x{:}A \vdash N : B])$, by ⟨intp-nat⟩ of F', so the isomorphism is natural. □

We have thus basically characterised the syntactic category of a theory Θ_K via a free construction: it is the free model of Θ_K. This will be useful for proving a strong correspondence between the categories we have constructed and the type theories we are using to describe them. We will do so by proving that $\mathrm{Syn}(\Theta_K)$ satisfies the universal property of the category we are trying to describe, as established in Section 3.3.° It then follows that $\mathrm{Syn}(\Theta_K)$ is equivalent to that category. Such an equivalence has many useful consequences.

°A bit more elegant would be an argument to show that the category of models of the theory is equivalent to the category of categories of which our category is the "free" category, which implies that their initial objects are equivalent. However, this is out of reach with our setup, because e. g. models of a type theory with coproduct ⟨syn-equiv⟩ types need not have finite coproducts, but they only have to contain a (not even necessarily full) subcategory restricted to which there are finite coproducts, and then we would have had to prove e. g. that FinSet is the free category that has a subcategory with all coproducts.

DEFINITION: STRONG SOUNDNESS AND COMPLETENESS. Let Θ be a type theory and $K : \square \in \Theta$. Then Θ_K is called *strongly sound and complete* for a category C if $\mathrm{Syn}(\Theta_K) \simeq \mathsf{C}$.

PROPOSITION 4.1.6. Let Θ be a type theory, $K : \square \in \Theta$, and suppose Θ_K is strongly sound and complete for a category C. Let $F \colon \mathrm{Syn}(\Theta_K) \to \mathsf{C}$ denote the equivalence. Then $\llbracket \cdot \rrbracket_{\mathsf{C}} \colon \Lambda_K^{\mathsf{C}} \to \mathsf{C}$ given by $A{:}K \mapsto F(A)$, $x{:}A \vdash N{:}B \mapsto F([x{:}A \vdash N{:}B])$ has the following properties:

- ⟨syn-obj-sound⟩ for each $A : K \in \Lambda_K$, we have that $\llbracket A \rrbracket_{\mathsf{C}}$ is an object in C,
- ⟨syn-obj-compl⟩ conversely, for each object $c \in \mathsf{C}$, there is an $A : K \in \Lambda_K$ such that $\llbracket A \rrbracket_{\mathsf{C}} \cong c$;

⟨syn-arr-sound⟩ for each $x:A \vdash N:B \in \Lambda_K$, we have that $[\![x:A \vdash N:B]\!]_{\mathsf{C}}$ is a morphism in C from $[\![A]\!]_{\mathsf{C}}$ to $[\![B]\!]_{\mathsf{C}}$,

⟨syn-arr-compl⟩ conversely, for each morphism f in C from $[\![A]\!]_{\mathsf{C}}$ to $[\![B]\!]_{\mathsf{C}}$, there is an $x:A \vdash N:B \in \Lambda_K$ such that $[\![x:A \vdash N:B]\!]_{\mathsf{C}} = f$;

⟨syn-eq-sound⟩ for each $x:A \vdash N = M:B \in \Theta_K$, we have that $[\![x:A \vdash N:B]\!]_{\mathsf{C}} = [\![x:A \vdash M:B]\!]_{\mathsf{C}}$, and

⟨syn-eq-compl⟩ conversely, if for some pair $x:A \vdash N:B \in \Lambda_K$ and $x:A \vdash M:B \in \Lambda_K$ we have $[\![x:A \vdash N:B]\!]_{\mathsf{C}} = [\![x:A \vdash M:B]\!]_{\mathsf{C}}$, then $x:A \vdash N = M:B \in \Theta_K$.

Proof.

⟨syn-obj-sound⟩

Because F is a functor.

⟨syn-obj-compl⟩

Since F is an equivalence, it is essentially surjective, so for each $c \in \mathsf{C}$ there is an $A \in \mathrm{Syn}(\Theta_K)$ with $c \cong F(A) = [\![A]\!]_{\mathsf{C}}$.

⟨syn-arr-sound⟩

Because F is a functor and by construction of $\mathrm{Syn}(\Theta_K)$.

⟨syn-arr-compl⟩

Since F is an equivalence, it is full, so for each morphism f in C from $[\![A]\!]_{\mathsf{C}} = F(A)$ to $[\![B]\!]_{\mathsf{C}} = F(B)$ there is an $[x:A \vdash N:B] \in \mathrm{Syn}(\Theta_K)(A,B)$, i.e. $x:A \vdash N:B \in \Lambda_K$, such that $[\![x:A \vdash N:B]\!]_{\mathsf{C}} = F([x:A \vdash N:B]) = f$.

⟨syn-eq-sound⟩

If $x:A \vdash N = M:B \in \Theta_K$, then $[x:A \vdash N:B] = [x:A \vdash M:B]$ by construction of $\mathrm{Syn}(\Theta_K)$.

⟨syn-eq-compl⟩

Since F is an equivalence, it is faithful, so if $F([x:A \vdash N:B]) = [\![x:A \vdash N:B]\!]_{\mathsf{C}} = [\![x:A \vdash M:B]\!]_{\mathsf{C}} = F([x:A \vdash M:B])$, then

$[x:A \vdash N:B] = [x:A \vdash M:B]$, hence $x:A \vdash N = M:B \in \Theta_K$ by definition of $\mathrm{Syn}(\Theta_K)$. □

Thus, a category C equivalent to $\mathrm{Syn}(\Theta_K)$ is "essentially" completely described by Θ_K: every object can be constructed in the type theory (up to isomorphism), every morphism between constructible objects can be constructed, and every equality between constructible morphisms between constructible objects can be shown. The converse holds, too, making Θ_K an arguably perfect match as *the* type theory of C. Note that in particular, ⟨syn-equiv⟩ implies that Θ_K is sound and complete for the one-object subcategory of $\mathrm{I}(\Lambda_K)$ given by $(\mathsf{C}, [\![\cdot]\!]_{\mathsf{C}})$.

4.2 A COMPLETE TYPE THEORY FOR FINITE DOMAINS

We construct the type theory in several stages. First, we create a type theory for FinSet, then for FinRel, and finally for FinObs. We let Θ denote the smallest type theory which is closed under the rules presented in this section.

TYPE THEORY I: FinSet

As shown in ⟨finset-free⟩, FinSet is the free cocompletion of the trivial category under finite coproducts. First, we introduce a new kind and make sure that the trivial object is included.

⟨tt-set-kind⟩ SET : ▢ ⟨tt-unit-form⟩ Unit : SET

For a category to have finite coproducts is equivalent to having an initial object and binary coproducts. This formulation is better suited for type-theoretic syntax. The initial object becomes an Empty type. The universal property states that there should be an arrow to every object, and any two such arrows should be equal. With the material from Section 4.1, it should be clear how the following rules capture precisely this property; we will give slightly more detail for the case of coproducts.

$$\langle\texttt{tt-empty-form}\rangle\ \mathsf{Empty} : \mathsf{SET}$$

$$\langle\texttt{tt-empty-elim}\rangle\ x : \mathsf{Empty} \vdash \mathsf{bot}\ x : A \qquad \langle\texttt{tt-empty-eta}\rangle\ x : \mathsf{Empty} \vdash \mathsf{bot}\ x = N : A$$

Then, we just need to describe the universal property of a coproduct. First, we include a type formation rule that says that if we have two types, we can create their coproduct or Sum type.

$$\langle\texttt{tt-sum-form}\rangle\ \frac{A : \mathsf{SET} \qquad B : \mathsf{SET}}{\mathsf{Sum}\ A\ B : \mathsf{SET}}$$

Next, we specify how to create terms of that type, namely via the insertions into the coproduct.

$$\langle\texttt{tt-sum-intro}\rangle\ x : A \vdash \mathsf{inl}\ x : \mathsf{Sum}\ A\ B \qquad x : B \vdash \mathsf{inr}\ x : \mathsf{Sum}\ A\ B$$

A term elimination rule is given by the coproduct mediators.

$$\langle\texttt{tt-sum-elim}\rangle\ \frac{y : A \vdash N : C \qquad y : B \vdash M : C}{x : \mathsf{Sum}\ A\ B \vdash \mathsf{case}\ x\ y.N\ y.M : C}$$

Coproduct mediators commute with cocones, which is expressed by *β-reduction* rules.

$$\langle\texttt{tt-sum-beta}\rangle\ \mathsf{case}\ (\mathsf{inl}\ x)\ y.N\ y.M = N[x\,/\,y] \qquad \mathsf{case}\ (\mathsf{inr}\ x)\ y.N\ y.M = M[x\,/\,y]$$

Finally, mediators are unique, leading to an *η-conversion* rule.°

$$\langle\texttt{tt-sum-eta}\rangle\ \mathsf{case}\ x\ (y.N[\mathsf{inl}\ y\,/\,z])\ (y.N[\mathsf{inr}\ y\,/\,z]) = N[x\,/\,z]$$

°The rule expresses that $(\iota_1 ; f) + (\iota_2 ; f) = f$, which is true by uniqueness of mediators, and conversely it implies uniqueness: if m, m' have $\iota_{1,2} ; m = \iota_{1,2} ; m'$, then by this rule $m = (\iota_1 ; m) + (\iota_2 ; m) = (\iota_1 ; m') + (\iota_2 ; m') = m'$.

We now show soundness and completeness for this theory to illustrate the general approach. Intuitively, we want to say that the rules above say precisely that a category has an object 1, an initial object, and binary coproducts; and FinSet is the free such category. We do this in detail for FinSet to check that the setup works since this is the simplest type theory we discuss.

Let us formulate more precisely what universal property FinSet satisfies. Call a pair (C, I) *finitely cocartesian* if C is a small° category with finite coproducts and $I : \mathbf{1} \to \mathsf{C}$ a functor. Call a functor $F : (\mathsf{C}, I) \to (\mathsf{D}, J)$ *finitely cocartesian* if F is a functor from C to D that preserves finite coproducts and $I ; F \cong J$ naturally. Then, ⟨finset-free⟩ implies that the skeleton of the category FinSet together with the functor selecting $\mathbf{1}$ is an initial object in the category of small finitely cocartesian categories with finitely cocartesian functors. It then suffices to show that $\mathrm{Syn}(\Theta_{\mathsf{SET}})$ is also an initial object in that category to prove that it is equivalent to the skeleton of FinSet, and by extension to FinSet.

°The category FinSet is not small, because there is no set of all finite sets. To avoid size issues, we prefer dealing with 2-categories of small categories only. This is not a problem because every category is equivalent to its skeleton, which is obtained as the quotient under isomorphisms, i.e. picking a single set of each cardinality in the case of FinSet. The proof of ⟨finset-free⟩ straightforwardly applies to the skeleton as well.

PROPOSITION 4.2.1. The theory Θ_{SET} is strongly sound and complete for the category FinSet. ⟨set-sound-complete⟩

Proof. We show that $\mathrm{Syn}(\Theta_{\mathsf{SET}})$ together with the functor $I : \mathbf{1} \to \mathrm{Syn}(\Theta_{\mathsf{SET}})$ with $I(\bullet) = \mathtt{Unit}$ is the free finitely cocartesian category. It follows that $\mathrm{Syn}(\Theta_{\mathsf{SET}})$ and FinSet are equivalent by ⟨finset-free⟩.

⟨ $\mathrm{Syn}(\Theta_{\mathsf{SET}})$ has finite coproducts ⟩

We show that $\mathrm{Syn}(\Theta_{\mathsf{SET}})$ has both an initial object and binary coproducts, which implies that it has all finite coproducts.

For the initial object, we have that $\mathtt{Empty} \in \mathrm{Syn}(\Theta_{\mathsf{SET}})$ by ⟨tt-empty-form⟩. Let $A \in \mathrm{Syn}(\Theta_{\mathsf{SET}})$. Then, using the rule ⟨tt-empty-elim⟩, there is an arrow $[x : \mathtt{Empty} \vdash \mathtt{bot}\ x : A] : \mathtt{Empty} \to A$ in $\mathrm{Syn}(\Theta_{\mathsf{SET}})$. Now if $[x : \mathtt{Empty} \vdash N : A] : \mathtt{Empty} \to A$ is an arrow, then $x : \mathtt{Empty} \vdash N : A \in \Theta_{\mathsf{SET}}$, so by ⟨tt-empty-eta⟩, we have $x : \mathtt{Empty} \vdash \mathtt{bot}\ x = N : A$, hence $[x : \mathtt{Empty} \vdash \mathtt{bot}\ x : A] = [x : \mathtt{Empty} \vdash N : A]$, as required.

Now let $A, B \in \mathrm{Syn}(\Theta_{\mathsf{SET}})$, i.e. $A : \mathsf{SET}, B : \mathsf{SET} \in \Theta_{\mathsf{SET}}$. Then, by ⟨tt-sum-form⟩, we have $\mathtt{Sum}\ A\ B \in \Theta_{\mathsf{SET}}$, hence $\mathtt{Sum}\ A\ B \in \mathrm{Syn}(\Theta_{\mathsf{SET}})$. We show that this is a coproduct of A and B. First, by ⟨tt-sum-intro⟩, there are morphisms $[x : A \vdash \mathtt{inl}\ x : \mathtt{Sum}\ A\ B] : A \to \mathtt{Sum}\ A\ B$ and $[x : B \vdash \mathtt{inr}\ x : \mathtt{Sum}\ A\ B] : B \to \mathtt{Sum}\ A\ B$. Now suppose C has morphisms $[y : A \vdash N : C] : A \to C$

and $[y:B \vdash M:C]:B \to C$. Then, $y:A \vdash N:C \in \Theta_{\mathsf{SET}}$ and $y:B \vdash M:C \in \Theta_{\mathsf{SET}}$, so by ⟨tt-sum-elim⟩, $x:\mathtt{Sum}\ A\ B \vdash \mathsf{case}\ x\ y.N\ y.M:C \in \Theta_{\mathsf{SET}}$, i.e $[x:\mathtt{Sum}\ A\ B \vdash \mathsf{case}\ x\ y.N\ y.M:C]:\mathtt{Sum}\ A\ B \to C$. For commutativity, we have $[x:A \vdash \mathtt{inl}\ x:\mathtt{Sum}\ A\ B];[x:\mathtt{Sum}\ A\ B \vdash \mathsf{case}\ x\ y.N\ y.M:C]$, which is equal to $[x:A \vdash \mathsf{case}\ (\mathtt{inl}\ x)\ y.N\ y.M:C] = [x:A \vdash N[x/y]:C] = [y:A \vdash N:C]$ by definition of substitution in $\mathrm{Syn}(\Theta_{\mathsf{SET}})$ and due to ⟨tt-sum-beta⟩, with inr analogous. For uniqueness, suppose $[x:\mathtt{Sum}\ A\ B \vdash X:C]$ is a mediator, i.e. $[y:A \vdash X[\mathtt{inl}\ y/x]:C] = [y:A \vdash N:C]$ and $[y:B \vdash X[\mathtt{inr}\ y/x]:C] = [y:A \vdash M:C]$. Then, $y:A \vdash X[\mathtt{inl}\ y/x] = N:C \in \Theta_{\mathsf{SET}}$ and $y:B \vdash X[\mathtt{inr}\ y/x] = M:C \in \Theta_{\mathsf{SET}}$. We then have $x:\mathtt{Sum}\ A\ B \vdash X = \mathsf{case}\ x\ (y.X[\mathtt{inl}\ y/x])\ (y.X[\mathtt{inr}\ y/x]) = \mathsf{case}\ x\ y.N\ y.M:C$ by ⟨tt-sum-eta⟩ and ⟨tt-eq-subs⟩.

Now let (C, J) be finitely cocartesian. We want to use ⟨syn-free⟩ to obtain a functor $\mathrm{Syn}(\Theta_{\mathsf{SET}}) \to \mathsf{C}$. To this end, we need to show that C is a model of Θ_{SET}. We equip it with the following interpretation:

$$[\![\mathtt{Unit}]\!] = I(\bullet) \qquad [\![\mathsf{Empty}]\!] = \mathsf{0} \qquad [\![\mathtt{Sum}\ A\ B]\!] = [\![A]\!] + [\![B]\!]$$

$$[\![x:A \vdash x:A]\!] = \mathrm{id}_{[\![A]\!]} \qquad [\![x:\mathsf{Empty} \vdash \mathsf{bot}\ x:A]\!] = \mathsf{0}_{[\![A]\!]}$$

$$[\![x:A \vdash \mathtt{inl}\ x:\mathtt{Sum}\ A\ B]\!] = \iota_1 : [\![A]\!] \to [\![A]\!] + [\![B]\!] \quad [\![x:B \vdash \mathtt{inr}\ x:\mathtt{Sum}\ A\ B]\!] = \iota_2 : [\![B]\!] \to [\![A]\!] + [\![B]\!]$$

$$[\![x:\mathtt{Sum}\ A\ B \vdash \mathsf{case}\ x\ y.N\ y.M:C]\!] = [\![y:A \vdash N:C]\!] + [\![y:B \vdash M:C]\!] : [\![A]\!] + [\![B]\!] \to [\![C]\!]$$

⟨ $(\mathsf{C}, [\![\cdot]\!]) \in \mathsf{M}(\Theta_{\mathsf{SET}})$ ⟩

⟨ $[\![\cdot]\!]$ is an interpretation ⟩

⟨intp-obj⟩

Every $A:\mathsf{SET} \in \Lambda_{\mathsf{SET}}$ has $A = \mathtt{Unit}$, $A = \mathsf{Empty}$, or $A = \mathtt{Sum}\ B\ B'$ with $B:\mathsf{SET}, B':\mathsf{SET} \in \Lambda_{\mathsf{SET}}$. Furthermore $I(\bullet)$ is an object in C and C has an initial object as well as all binary coproducts.

⟨intp-arr⟩

Every $x:A \vdash N:B \in \Lambda_{\mathsf{SET}}$ is a consequence of the rule ⟨tt-var⟩, ⟨tt-subs⟩, ⟨tt-empty-elim⟩, ⟨tt-sum-intro⟩, or ⟨`tt-sum-elim`⟩. We prove by structural induction on derivations that in each case, the assigned morphism exists in $\mathsf{C}([\![A]\!], [\![B]\!])$.

⟨`tt-var`⟩

If $x : A \vdash x : A \in \Lambda_{\mathsf{SET}}$ as a result of this rule, then $A : \mathsf{SET} \in \Lambda_{\mathsf{SET}}$ since the antecedent must be satisfied. Thus $[\![A]\!]$ is an object in C by ⟨`intp-obj`⟩, and it has an identity morphism by definition of categories.

⟨`tt-subs`⟩

It can be shown by induction that every judgment that is a consequence of ⟨`tt-subs`⟩ can be derived using the other rules, i. e. the rule itself is derivable by structural induction, as is usually the case in type theory [32, A.2.2].

⟨`tt-empty-elim`⟩

Since `Empty` is interpreted as the initial object, every object has an initial morphism from $\mathsf{0}$.

⟨`tt-sum-intro`⟩

If $x : A \vdash \texttt{inl}\ x : \mathsf{Sum}\ A\ B \in \Lambda_{\mathsf{SET}}$, then $A, \mathsf{Sum}\ A\ B \in \Lambda_{\mathsf{SET}}$ by definition of Λ_{SET}. These objects are interpreted as $[\![A]\!]$ and $[\![A]\!] + [\![B]\!]$, and so the coproduct insertion exists. The case for `inr` is analogous.

⟨`tt-sum-elim`⟩

If $x : \mathsf{Sum}\ A\ B \vdash \mathsf{case}\ x\ (y.N)\ (y.M) : C \in \Lambda_{\mathsf{SET}}$, then $y : A \vdash N : C \in \Lambda_{\mathsf{SET}}$ and $y : B \vdash M : C \in \Lambda_{\mathsf{SET}}$. By the inductive hypothesis, these are interpreted as morphisms $[\![A]\!] \to [\![C]\!]$ and $[\![B]\!] \to [\![C]\!]$, thus there exists a mediator from the coproduct $[\![A]\!] + [\![B]\!]$ to $[\![C]\!]$.

⟨`intp-id`⟩

By construction.

⟨intp-comp⟩

If $x : A \vdash N : B \in \Lambda_{\mathsf{SET}}$ and $x : B \vdash M : C \in \Lambda_{\mathsf{SET}}$, then a tedious double structural induction over the derivations of both judgments shows that the composite of their interpretations is the interpretation of $x{:}A \vdash M[N/x]{:}C$.

⟨ $[\![\cdot]\!]$ makes C a model⟩

We use structural induction on the derivations of equality judgments to show that if $x : A \vdash N = M : B \in \Theta_{\mathsf{SET}}$, then $[\![x : A \vdash N : B]\!] = [\![x : A \vdash M : B]\!]$ in C. For ⟨tt-eq-refl⟩, ⟨tt-eq-sym⟩, and ⟨tt-eq-trans⟩, this is clear since equality of morphisms is reflexive, symmetric, and transitive. For ⟨tt-eq-subs⟩, this follows from ⟨intp-comp⟩ and the fact that equality of morphisms is preserved by composition.

⟨tt-empty-eta⟩

If $x : \mathsf{Empty} \vdash \mathtt{bot}\ x = N : A \in \Theta_{\mathsf{SET}}$, then $[\![x : \mathsf{Empty} \vdash N : A]\!] \in \mathsf{C}(0, [\![A]\!])$, hence $[\![x : \mathsf{Empty} \vdash N : A]\!] = 0_{[\![A]\!]} = [\![x : \mathsf{Empty} \vdash \mathtt{bot}\ x : A]\!]$ by the universal property of the initial object.

⟨tt-sum-beta⟩

If $x{:}A \vdash \mathsf{case}\ (\mathtt{inl}\ x)\ y.N\ y.M = N{:}C \in \Theta_{\mathsf{SET}}$, then we calculate $[\![x : A \vdash \mathsf{case}\ (\mathtt{inl}\ x)\ y.N\ y.M : C]\!] = [\![x : A \vdash \mathtt{inl}\ x : \mathsf{Sum}\ A\ B]\!] \,;[\![x : \mathsf{Sum}\ A\ B \vdash \mathsf{case}\ x\ y.N\ y.M : C]\!] = \iota_1\,; ([\![y : A \vdash N : C]\!] + [\![y : B \vdash M : C]\!])$ using ⟨tt-subs⟩, which is equal to $[\![y{:}A \vdash N{:}C]\!] = [\![x{:}A \vdash N[x/y]{:}A]\!]$ by the mediating property of the coproduct, and the case for inl is analogous.

⟨tt-sum-eta⟩

If $x{:}\mathsf{Sum}\ A\ B \vdash \mathsf{case}\ x\ (y.\,\mathtt{inl}\ y)\ (y.\,\mathtt{inr}\ y) = x{:}\mathsf{Sum}\ A\ B \in \Theta_{\mathsf{SET}}$, then $[\![x{:}\mathsf{Sum}\ A\ B \vdash \mathsf{case}\ x\ (y.\,\mathtt{inl}\ y)\ (y.\,\mathtt{inr}\ y)]\!] = [\![y : A \vdash \mathtt{inl}\ y : \mathsf{Sum}\ A\ B]\!] + [\![y : B \vdash \mathtt{inr}\ y : \mathsf{Sum}\ A\ B]\!] = \iota_1 + \iota_2$, which is equal to $\mathrm{id}_{[\![A]\!]+[\![B]\!]} = \mathrm{id}_{[\![\mathsf{Sum}\ A\ B]\!]} = [\![x :$

$\texttt{Sum}\ A\ B \vdash x : \texttt{Sum}\ A\ B]\!]$ by uniqueness of coproduct mediators.

Thus by $\langle\texttt{syn-free}\rangle$, there exists a functor $F : \mathrm{Syn}(\Theta_{\mathsf{SET}}) \to \mathsf{C}$, unique up to isomorphism such that F commutes with the interpretation. We show that this functor has $I\,;F \cong J$ and preserves finite coproducts, and that conversely every such functor commutes with the interpretation. It immediately follows that there is a unique-up-to-isomorphism functor $\mathrm{Syn}(\Theta_{\mathsf{SET}}) \to \mathsf{C}$ commuting with I and J and preserving finite coproducts, as required.

$\langle\, I\,;F \cong J$ and F preserves finite coproducts$\,\rangle$

We have $F(I(\bullet)) = F(\texttt{Unit}) = F([\![\texttt{Unit}]\!]_{\mathrm{Syn}}) \cong [\![\texttt{Unit}]\!] = J(\bullet)$ by $\langle\texttt{intp-iso}\rangle$, and since the category $\mathbf{1}$ is trivial, $I\,;F \cong J$ naturally.

For the preservation of finite coproducts, it suffices to show the preservation of the initial object and of binary coproducts. For the initial object, this is clear by construction.

Now $F(A{+}B) = F(\texttt{Sum}\,A\,B) = F([\![\texttt{Sum}\,A\,B]\!]_{\mathrm{Syn}}) \cong [\![\texttt{Sum}\,A\,B]\!] = [\![A]\!] + [\![B]\!] \cong F([\![A]\!]_{\mathrm{Syn}}) + F([B]\!]_{\mathrm{Syn}}) = F(A) + F(B)$ by property $\langle\texttt{intp-iso}\rangle$, and $F(\iota_1);\cong_{\texttt{Sum}\ A\ B} = F([x{:}A \vdash \texttt{inl}\ x{:}\texttt{Sum}\,A\,B]);\cong_{\texttt{Sum}\ A\ B} = F([\![x{:}A \vdash \texttt{inl}\ x{:}\texttt{Sum}\,A\,B]\!]_{\mathrm{Syn}});\cong_{\texttt{Sum}\ A\ B} = \cong_A; [\![x{:}A \vdash \texttt{inl}\ x : \texttt{Sum}\ A\ B]\!] = \cong_A; \iota_1$ by $\langle\texttt{intp-nat}\rangle$, and analogously for ι_2, so because $\cong_A; \iota_1$ and $\cong_B; \iota_2$ make $[\![A]\!] + [\![B]\!]$ a coproduct of $F(A)$ and $F(B)$, it follows that $F(\iota_1)$ and $F(\iota_2)$ make $F(A+B)$ a coproduct of these, too.

Now let F' be a finite-coproduct-preserving functor $\mathrm{Syn}(\Theta_{\mathsf{SET}}) \to \mathsf{C}$ with $I\,;F' \cong J$.

$\langle\, F'$ commutes with interpretations$\,\rangle$

$\langle\texttt{intp-iso}\rangle$

We have that $F'([\![\texttt{Unit}]\!]_{\mathrm{Syn}}) = F'(\texttt{Unit}) = F'(I(\bullet)) \cong J(\bullet) = [\![\texttt{Unit}]\!]$ by commutativity, that $F'([\![\texttt{Empty}]\!] = F'(\texttt{Empty}) \cong \mathsf{0} = [\![\texttt{Empty}]\!]$ by preservation of initial objects, and that $F'([\![\texttt{Sum}\ A\ B]\!]_{\mathrm{Syn}}) = F'(\texttt{Sum}\ A\ B) \cong F'(A+B) \cong F'(A) +$

$F'(B) \cong \llbracket A \rrbracket + \llbracket B \rrbracket = \llbracket A + B \rrbracket$ inductively by preservation of coproducts, the last isomorphism being the mediator $\iota_{F'(A)} ; \cong_A + \iota_{F'(B)} ; \cong_B$.

$\langle$`intp-nat`$\rangle$

We have

$$\begin{array}{ccc} F'(\llbracket \mathtt{Empty} \rrbracket_{\mathrm{Syn}}) & \xrightarrow{\cong} & \mathsf{0} \\ {\scriptstyle F'(\llbracket x{:}\mathtt{Empty} \vdash \mathtt{bot}\ x{:}A \rrbracket_{\mathrm{Syn}})} \downarrow & & \downarrow {\scriptstyle \mathsf{0}_{\llbracket A \rrbracket}} \\ F'(\llbracket A \rrbracket_{\mathrm{Syn}}) & \xrightarrow{\cong} & \llbracket A \rrbracket \end{array}$$

for the initial morphisms by their uniqueness property, and

$$\begin{array}{ccccc} F'(\llbracket A \rrbracket_{\mathrm{Syn}}) & \xrightarrow{\mathrm{id}} & F'(\llbracket A \rrbracket_{\mathrm{Syn}}) & \xrightarrow{\cong} & \llbracket A \rrbracket \\ {\scriptstyle F'(\llbracket x{:}A \vdash \mathtt{inl}\ x{:}\mathtt{Sum}\ A\ B \rrbracket_{\mathrm{Syn}})} \downarrow & & \downarrow {\scriptstyle \iota_{F'(\llbracket A \rrbracket_{\mathrm{Syn}})}} & & \downarrow {\scriptstyle \iota_{\llbracket A \rrbracket}} \\ F'(\llbracket \mathtt{Sum}\ A\ B \rrbracket_{\mathrm{Syn}}) & \xrightarrow{\cong} & F'(\llbracket A \rrbracket_{\mathrm{Syn}}) + F'(\llbracket B \rrbracket_{\mathrm{Syn}}) & \xrightarrow{\cong} & \llbracket A \rrbracket + \llbracket B \rrbracket \end{array}$$

for insertions, with the left side commuting due to preservation of coproduct cocones, and the right side due to the mediating property of the isomorphism as defined above. For $x : \mathtt{Sum}\ A\ B \vdash \mathtt{case}\ x\ (y.N)\ (y.M) : C$, we have

$$\begin{array}{ccccc} F'(\llbracket \mathtt{Sum}\ A\ B \rrbracket_{\mathrm{Syn}}) & \xrightarrow{\cong} & F'(\llbracket A \rrbracket_{\mathrm{Syn}}) + F'(\llbracket B \rrbracket_{\mathrm{Syn}}) & \xrightarrow{\cong} & \llbracket A \rrbracket + \llbracket B \rrbracket \\ \downarrow {\scriptstyle F'(\llbracket x{:}\mathtt{Sum}\ A\ B \vdash \mathtt{case}\ x\ (y.N)\ (y.M){:}C \rrbracket)} & & \downarrow {\scriptstyle F'(\llbracket y{:}A \vdash N{:}C \rrbracket_{\mathrm{Syn}}) + F'(\llbracket y{:}B \vdash M{:}C \rrbracket_{\mathrm{Syn}})} & & \downarrow {\scriptstyle \llbracket y{:}A \vdash N{:}C \rrbracket + \llbracket y{:}B \vdash M{:}C \rrbracket} \\ F'(\llbracket C \rrbracket_{\mathrm{Syn}}) & \xrightarrow[\mathrm{id}]{} & F'(\llbracket C \rrbracket_{\mathrm{Syn}}) & \xrightarrow[\cong]{} & \llbracket C \rrbracket \end{array}$$

where the left side commutes since the top isomorphism is the mediator of the insertions into the coproduct in the middle, which exists because $F'([\![\mathsf{Sum}\ A\ B]\!]_{\mathrm{Syn}})$ is a coproduct of the same objects by preservation of coproducts, and then commutativity follows because the composite arrows satisfy the same mediating property, and the argument for the right side is similar, except that we also need to apply ⟨intp-nat⟩ for A, B, and C by induction. □

That was a lot of entirely trivial work, but we had to do it to verify that the categorical semantics we defined generally work. In the future, we shall omit these proofs, and instead trust our ability to read off from the type theory the universal property its syntactic category satisfies. This is of course an informal way of argument, but as we have just seen, the actual proofs are not very insightful, and any mistakes we make this way are probably easy to fix.

TYPE THEORY II: FinRel

The category FinRel is the free idempotent finitely bicartesian category on the trivial category. Like for FinSet, we simply write down the definition of the universal properties involved.

$$\langle\texttt{tt-rel-kind}\rangle\ \mathsf{REL} : \square$$

The existence of finite biproducts is equivalent to the existence of a zero object and binary biproducts. We import all the rules from Θ_{SET} while replacing the (implicit) kind annotations $:\mathsf{SET}$ with $:\mathsf{REL}$. Then, we modify the imported types, adding new rules to them. First, we turn the initial object into a zero object.

$$\langle\texttt{tt-empty-intro}\rangle\ x : A \vdash \texttt{bot}\ x : \texttt{Empty} \qquad \langle\texttt{tt-empty-beta}\rangle\ x : A \vdash \texttt{bot}\ x = N : \texttt{Empty}$$

We turn coproducts into biproducts.

$$\langle\texttt{tt-bi-elim}\rangle\ x : \mathsf{Sum}\ A\ B \vdash \texttt{outl}\ x : A \qquad x : \mathsf{Sum}\ A\ B \vdash \texttt{outr}\ x : B$$

$$\langle\texttt{tt-bi-intro}\rangle\ \frac{x : C \vdash N : A \quad x : C \vdash M : B}{x : C \vdash \texttt{pair}\ N\ M : \mathsf{Sum}\ A\ B}$$

⟨tt-bi-beta⟩ $\mathtt{outl}\ (\mathtt{pair}\ N\ M) = N : A \qquad \mathtt{outr}\ (\mathtt{pair}\ N\ M) = M$

⟨tt-bi-eta⟩ $\mathtt{pair}\ (\mathtt{outl}\ x)\ (\mathtt{outr}\ x) = x$

We also need to specify the relationship between the insertions and projections.

⟨tt-bi-in-out-id⟩ $\mathtt{outl}\ (\mathtt{inl}\ x) = x \qquad \mathtt{outr}\ (\mathtt{inr}\ x) = x$

⟨tt-bi-in-out-bot⟩ $\mathtt{outl}\ (\mathtt{inr}\ x) = \mathsf{bot}\ x \qquad \mathtt{outr}\ (\mathtt{inl}\ x) = \mathsf{bot}\ x$

Finally, biproducts need to be idempotent.

$$\langle\texttt{tt-bi-comp}\rangle\ \frac{M[N\,/\,y] = M'[N'\,/\,y]}{\mathtt{case}\ (\mathtt{pair}\ N\ N')\ y.M\ y.M' = M[N\,/\,y]}$$

Analogously to FinSet, we can now show the following.

PROPOSITION 4.2.2. The type theory Θ_{REL} is strongly sound and complete for the category FinRel. ⟨rel-sound-complete⟩

Proof. Like ⟨set-sound-complete⟩, by noticing that the rules capture the definition of finite idempotent biproducts via binary biproducts and a zero object, and FinRel is the free idempotent finitely bicartesian category on a trivial object by ⟨finrel-free⟩. We defined a biproduct to be a coproduct such that the mediators $+_j (j = i)^?$ form a product cone; it is clear that then, the object is both a product and a biproduct, and $\iota_j\,;\pi_i = \mathsf{id}$ if $j = i$ and $0_{j,i}$ otherwise, just as prescribed by ⟨tt-bi-in-out-id⟩ and ⟨tt-bi-in-out-bot⟩, but also the converse holds: if an object has coproduct insertions and product projections and it satisfies the conditions ⟨tt-bi-in-out-id⟩ and ⟨tt-bi-in-out-bot⟩, then the projections have $\iota_j;\pi_i = (j = i)^?$, so $\pi_i = +_j (j = i)^?$ by uniqueness of coproduct mediators, so the object is a biproduct. □

TYPE THEORY III: FinObs

We add split pushouts to REL. In order to keep track of the functional—i. e. atom-preserving—morphisms, we define a type theory for FinObs^* in parallel.

$$\langle\texttt{tt-obs-kind}\rangle\ \texttt{OBS}:\square \qquad \langle\texttt{tt-obs*-kind}\rangle\ \texttt{OBS*}:\square$$

We import FinRel into FinObs.

$$\langle\texttt{tt-rel-obs}\rangle\ \frac{A:\texttt{REL}\quad B:\texttt{REL}\quad x:A\vdash N:B:\texttt{REL}\quad x:A\vdash M:B:\texttt{REL}\quad x:A\vdash N=M:B:\texttt{REL}}{A:\texttt{OBS}\quad B:\texttt{OBS}\quad x:A\vdash N:B:\texttt{OBS}\quad x:A\vdash M:B:\texttt{OBS}\quad x:A\vdash N=M:B:\texttt{OBS}}$$

In the same fashion, we import FinSet into FinObs^* via a rule ⟨tt-set-obs*⟩ that is like ⟨tt-rel-obs⟩ where REL is replaced with SET and OBS with OBS*. This kind of import is different to the way we imported rules from SET into REL in that those rules are now "fossilised": new judgments cannot be constructed in OBS* and OBS from these rules; in particular, the sum types will not be coproducts anymore in the new syntactic categories.

Now we can define the pushout type. In the following rules, whenever we leave out kind annotations, it means that the rules are valid irrespective of whether the kind in question is OBS or OBS*.

$$\langle\texttt{tt-push-form}\rangle\ \frac{y:C\vdash N:A:\texttt{OBS*}\quad y:C\vdash M:B:\texttt{OBS*}\quad z:A\vdash N':C\quad y:C\vdash N'[N/z]=y:C}{\mathsf{Push}\ C\ y.N:A\ y.M:B}$$

$$\langle\texttt{tt-push-intro}\rangle\ x:A\vdash \mathsf{pshl}\ x:\mathsf{Push}\ C\ y.N:A\ y.M:B$$

$$x:B\vdash \mathsf{pshr}\ x:\mathsf{Push}\ C\ y.N:A\ y.M:B$$

$$y:C\vdash \mathsf{pshl}\ N=\mathsf{pshr}\ M:\mathsf{Push}\ C\ y.N:A\ y.M:B$$

$$\langle\texttt{tt-push-elim}\rangle\ \frac{z:A\vdash S:D\quad z:B\vdash T:D\quad y:C\vdash S[N/z]=T[M/z]:D}{x:\mathsf{Push}\ C\ y.N:A\ y.M:B\vdash \mathsf{pshcase}\ x\ z.S\ z.T:D:\texttt{OBS}}$$

$$\langle\texttt{tt-push-elim*}\rangle\ \frac{z:A\vdash S:D:\texttt{OBS*}\quad z:B\vdash T:D:\texttt{OBS*}\quad y:C\vdash S[N/z]=T[M/z]:D}{x:\mathsf{Push}\ C\ y.N:A\ y.M:B\vdash \mathsf{pshcase}\ x\ z.S\ z.T:D:\texttt{OBS*}}$$

$$\langle\texttt{tt-push-beta}\rangle\ \mathsf{pshcase}\ (\mathsf{pshl}\ x)\ z.S\ z.T=S[x/z]\qquad \mathsf{pshcase}\ (\mathsf{pshr}\ x)\ z.S\ z.T=T[x/z]$$

$$\langle\texttt{tt-push-eta}\rangle\ \mathsf{pshcase}\ x\ (z.X[\mathsf{pshl}\ z/y])\ (z.X[\mathsf{pshr}\ z/y])=X[x/y]$$

THEOREM 3. The theory Θ_{OBS} is sound and complete for the category FinObs. ⟨obs-sound-complete⟩

Proof. (Sketch.) By ⟨finobs-free⟩, FinObs is the free cocompletion of FinRel under split pushouts, and the ⟨tt-push-*⟩-rules formulate the universal property of such a pushout, while the rule ⟨tt-rel-obs⟩ makes sure that the theory of FinRel is included. It is clear that the rules applying to OBS* capture the definition of a functional morphism, namely a morphism either imported from SET, or an identity or composite of functional morphisms (via ⟨tt-var⟩ and ⟨tt-subs⟩, which also applies to OBS*), or an insertion into a split pushout, or a mediator of split pushouts via ⟨tt-push-elim*⟩, and the antecedent of ⟨tt-push-form⟩ requires both morphisms to be functional and one of them to be a split monomorphism. Finally, every type in OBS is either the result of a split pushout or a type imported from REL, so the definition of split pushouts is satisfied. □

4.3 AN INFINITARY EXTENSION

While the theory $\Theta_{\mathtt{OBS}}$ is complete for FinObs, it is not very expressive. Obviously, we cannot construct infinite domains with it, but there are more limitations: for example, the Empty type is a zero object not only with respect to the objects coming from REL, but for all of FinObs. However, there is no rule in the type theory that makes this apparent. We can prove in the meta-logic that all terms $x : \mathtt{Empty} \vdash N : A : \mathtt{OBS}$ and $x : A \vdash N : \mathtt{Empty} : \mathtt{OBS}$ are equal and that at least one such term exists, by induction on the structure of types, using pushout insertions and mediators from and to Empty, but we cannot show it using a derivation inside the type theory alone. Similarly, the type theory is silent on the existence of product types in FinObs, and if we added them, we would somehow have to ensure that they are not "freely" added in the syntactic category but are in some specified relationship to the existing types by adding equality rules that allow constructing the relevant isomorphisms.

This is rather complicated, so we shall instead give up on completeness (which seems even more difficult, if not impossible, to achieve in the infinitary case) and just write down a type theory that allows us to construct as many objects and morphisms in Obs as possible while at least being soundly interpretable in Obs. As we shall see, with the type theory we give here, we can in a sense construct all computable domain mappings.

In the following, we start with a "clean" type theory Θ that only includes the necessary rules as stated in the beginning of this chapter. We add two kinds, one for Obs and one for Obs*.

⟨tt-obsx-kind⟩ `OBSX`: ▢   ⟨tt-obsx*-kind⟩ `OBSX*`

Unless the kind is explicitly specified, it is assumed that the rules apply to both; in particular, we will leave out the kind in type formation rules. Furthermore, we make sure that everything constructed in `OBSX*` can be imported into `OBSX` via a rule ⟨tt-obsx*-obsx⟩ analogous to ⟨tt-rel-obs⟩.

TERMINAL AND INITIAL OBJECTS

Recall from ⟨obs-zero⟩, ⟨obs*-colim⟩, and ⟨obs*-lim⟩ that Obs and Obs* have isomorphic initial objects, and that in Obs this is also a terminal object, while the category Obs* has a different terminal object.

⟨tt-empty-form⟩ `Empty`

⟨tt-empty-intro⟩ $x : A \vdash$ bot x : `Empty` : `OBSX`

⟨tt-empty-elim⟩ x : `Empty` $\vdash$ bot $x : A$

⟨tt-empty-eta⟩ $x : A \vdash N =$ bot x : `Empty` x : `Empty` $\vdash N =$ bot $x : A$

⟨tt-unit-form⟩ `Unit`

⟨tt-unit-intro⟩ $x : A \vdash$ top x : `Unit`

⟨tt-unit-eta⟩ $x : A \vdash N =$ top x : `Unit` : `OBSX*`

We have products in Obs and Obs* by ⟨obs-prod⟩ and ⟨obs*-lim⟩, as well as coproducts by ⟨obs*-colim⟩ which coincide as shown in ⟨obs-coprod⟩. The tensor product, which is the product in Obs* is cast into a Smash type.

⟨tt-prod-form⟩ $$\frac{A \quad B}{\mathsf{Prod}\ A\ B}$$

⟨tt-prod-elim⟩ $x : \mathsf{Prod}\ A\ B \vdash \mathsf{outl}\ x : A : \mathsf{OBSX} \qquad x : \mathsf{Prod}\ A\ B \vdash \mathsf{outr}\ x : B : \mathsf{OBSX}$

⟨tt-prod-intro⟩ $$\frac{x : C \vdash N : A \quad x : C \vdash M : B}{x : C \vdash \mathsf{pair}\ N\ M : \mathsf{Prod}\ A\ B : \mathsf{OBSX}}$$

⟨tt-prod-beta⟩ $\mathsf{outl}\ (\mathsf{pair}\ N\ M) = N \qquad \mathsf{outr}\ (\mathsf{pair}\ N\ M) = M$

⟨tt-prod-eta⟩ $\mathsf{pair}\ (\mathsf{outl}\ x)\ (\mathsf{outr}\ x) = x$

⟨tt-sum-form⟩ $$\frac{A \quad B}{\mathsf{Sum}\ A\ B}$$

⟨tt-sum-intro⟩ $x : A \vdash \mathsf{inl}\ x : \mathsf{Sum}\ A\ B \quad x : B \vdash \mathsf{inr}\ x : \mathsf{Sum}\ A\ B$

⟨tt-sum-elim⟩ $$\frac{y : A \vdash N : C \qquad y : B \vdash M : C}{x : \mathsf{Sum}\ A\ B \vdash \mathsf{case}\ x\ y.N\ y.M : C : \mathsf{OBSX}}$$

⟨tt-sum-elim*⟩ $$\frac{y : A \vdash N : C : \mathsf{OBSX*} \qquad y : B \vdash M : C : \mathsf{OBSX*}}{x : \mathsf{Sum}\ A\ B \vdash \mathsf{case}\ x\ y.N\ y.M : C}$$

⟨tt-sum-beta⟩ $\mathsf{case}\ (\mathsf{inl}\ x)\ y.N\ y.M = N[x\,/\,y] \qquad \mathsf{case}\ (\mathsf{inr}\ x)\ y.N\ y.M = M[x\,/\,y]$

⟨tt-sum-eta⟩ $\mathsf{case}\ x\ (y.N[\mathsf{inl}\ y\,/\,z])\ (y.N[\mathsf{inr}\ y\,/\,z]) = N[x\,/\,z]$

⟨tt-smash-form⟩ $$\frac{A \quad B}{\mathsf{Smash}\ A\ B}$$

⟨tt-smash-elim⟩ $x : \mathsf{Smash}\ A\ B \vdash \mathsf{atl}\ x \qquad x : \mathsf{Smash}\ A\ B \vdash \mathsf{atr}\ x$

⟨tt-smash-intro⟩ $$\frac{x : C \vdash N : A \quad x : C \vdash M : B}{x : C \vdash \mathsf{atpair}\ N\ M : \mathsf{Smash}\ A\ B}$$

⟨tt-smash-beta⟩ $\mathsf{atl}\ (\mathsf{atpair}\ N\ M) = N : A : \mathsf{OBSX*} \quad \mathsf{atr}\ (\mathsf{atpair}\ N\ M) = M : B : \mathsf{OBSX*}$

⟨tt-smash-eta⟩ $\mathsf{atpair}\ (\mathsf{atl}\ x)\ (\mathsf{atr}\ x) = x : \mathsf{Smash}\ A\ B : \mathsf{OBSX*}$

MONOIDAL HOM

There is a function type, which uses the tensor product induced by the product from Obs^*, see ⟨obs-closed⟩. To extend the tensor product to a functor on Obs, we need to allow tensor product "mediators" of non-atom-preserving maps, which we did by leaving out the kind specification in the rule ⟨tt-smash-intro⟩.

⟨tt-fun-form⟩ $\dfrac{A \quad B}{\mathsf{Fun}\ A\ B}$

⟨tt-fun-elim⟩ $x : \mathsf{Smash}\ (\mathsf{Fun}\ A\ B)\ A \vdash \mathtt{eval}\ x : B : \mathtt{OBSX}$

⟨tt-fun-intro⟩ $\dfrac{z : \mathsf{Smash}\ C\ \ A \vdash N : B}{y : C \vdash \mathtt{lambda}\ x\ N[\mathtt{atpair}\ y\ x\,/\,z] : \mathsf{Fun}\ A\ B : \mathtt{OBSX}}$

⟨tt-fun-beta⟩ $\mathtt{eval}\ (\mathtt{atpair}\ (\mathtt{lambda}\ y\ N[\mathtt{atpair}\ (\mathtt{atl}\ x)\ y\,/\,z])\ (\mathtt{atr}\ x)) = N[x\,/\,z]$

⟨tt-fun-eta⟩ $\mathtt{lambda}\ x\ (\mathtt{eval}\ (\mathtt{atpair}\ f\ x)) = f$

ADJUNCTION BETWEEN Obs AND Obs^*

The inclusion $\mathsf{Obs}^* \hookrightarrow \mathsf{Obs}$ is left adjoint to the power domain operation by ⟨obs⋆-obs-adj⟩. We only formulate the universal property of the left adjoint. The right adjoint operations can be derived. This way, we avoid having to define the whole powerset functor.

⟨tt-pow-form⟩ $\dfrac{A}{\mathsf{Pow}\ A}$

⟨tt-pow-elim⟩ $x : \mathsf{Pow}\ A \vdash \mathtt{join}\ x : A : \mathtt{OBSX}$

⟨tt-pow-intro⟩ $\dfrac{x : A \vdash N : B}{x : A \vdash \mathtt{atoms}\ N : \mathsf{Pow}\ B}$

⟨tt-pow-beta⟩ $\mathtt{join}\ (\mathtt{atoms}\ x) = x$

⟨tt-pow-eta⟩ $\mathtt{atoms}\ (\mathtt{join}\ x) = x$

QUASITOPOS STRUCTURE

We will not describe the whole structure, but we shall add pullbacks from the strong-subobject classifier (see Section 3.2 on the quasitopos Obs*) as well as quotient types, which in combination with sum types should allow computing many types of colimits. See ⟨obs*-lim⟩ and ⟨obs*-colim⟩. We shall write Prop as an abbreviation for Prod Unit Unit and true x for pair (top x) (bot x). The rules for the type Sub $x.P$ A formulate the universal property of a pullback of the arrow $\mathbf{1}^* \to \Omega^* = [\![\mathtt{Prop}]\!]$ with $\top \mapsto \top \times \bot$ along the arrow from $[\![A]\!]$ into Ω^* represented by $x.P$.

⟨tt-sub-form⟩ $$\frac{x : A \vdash P : \mathtt{Prop} : \mathtt{OBSX*}}{\mathtt{Sub}\ x.P\ A}$$

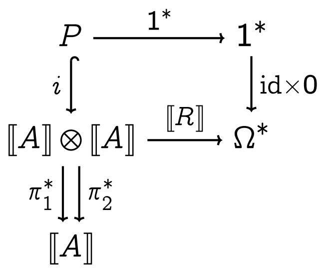
⟨tt-sub-elim⟩ $x : \mathtt{Sub}\ y.P\ A \vdash \mathtt{incl}\ x : A \qquad P[\mathtt{incl}\ x\,/\,y] = \mathtt{true}\ x$

⟨tt-sub-intro⟩ $$\frac{x : C \vdash N : A : \mathtt{OBSX*} \quad P[N\,/\,y] = \mathtt{true}\ x}{x : C \vdash \mathtt{sub}\ N : A}$$

⟨tt-sub-beta⟩ $\mathtt{incl}\ (\mathtt{sub}\ N) = N$

⟨tt-sub-eta⟩ $\mathtt{sub}\ (\mathtt{incl}\ x) = x$

The Quot A $x.R$ type below formulates the universal property of the following coequaliser. The term R is intended to represent an equivalence relation on the atoms of $[\![A]\!]$ stating which atoms we would like to identify. This can be realised as a coequaliser of the arrows $i\,;\pi_1^*$ and $i\,;\pi_2^*$ in the diagram in the margins, where P is the strong subobject of $[\![A]\!] \otimes [\![A]\!]$ corresponding to R. Note that R need not actually be an equivalence relation; the resulting domain will have all atoms identified that are related by the equivalence relation *generated* by R.



⟨tt-quot-form⟩ $$\frac{x : \mathtt{Smash}\ A\ A \vdash R : \mathtt{Prop} : \mathtt{OBSX*}}{\mathtt{Quot}\ A\ x.R}$$

⟨tt-quot-intro⟩ $x : A \vdash \mathtt{in}\ x : \mathtt{Quot}\ A\ x.R$

$\mathtt{in}\ (\mathtt{atl}\ (\mathtt{incl}\ x)) = \mathtt{in}\ (\mathtt{atr}\ (\mathtt{incl}\ x))$

⟨tt-quot-elim⟩ $$\frac{y : A \vdash N : C : \mathtt{OBSX*} \quad N[\mathtt{atl}\ (\mathtt{incl}\ x)\,/\,y] = N[\mathtt{atr}\ (\mathtt{incl}\ x)\,/\,y]}{x : \mathtt{Quot}\ A\ y.R \vdash \mathtt{match}\ x\ y.N}$$

⟨tt-quot-beta⟩ $\mathtt{match}\ (\mathtt{in}\ x)\ y.N = N[x\,/\,y]$

⟨tt-quot-eta⟩ $\mathtt{match}\ x\ (y.N[\mathtt{in}\ y\,/\,z]) = N[x\,/\,z]$

Finally, we add the computational types from ⟨obs-nno⟩ and ⟨obs-conno⟩.

⟨tt-nat-form⟩ Nat

⟨tt-nat-intro⟩ $x : \mathsf{Unit} \vdash \mathsf{zero}\ x : \mathsf{Nat} \qquad x : \mathsf{Nat} \vdash \mathsf{succ}\ x : \mathsf{Nat}$

⟨tt-nat-elim⟩ $$\frac{y : \mathsf{Unit} \vdash Z : A \quad y : A \vdash S : A}{x : \mathsf{Nat} \vdash \mathsf{rec}\ x\ y.Z\ y.S : A : \mathsf{OBSX}}$$

⟨tt-nat-beta⟩ $\mathsf{rec}\ (\mathsf{zero}\ x)\ y.Z\ y.S = Z \qquad \mathsf{rec}\ (\mathsf{succ}\ n)\ y.Z\ y.S = S[\mathsf{rec}\ n\ y.Z\ y.S\,/\,y]$

⟨tt-nat-eta⟩ $\mathsf{rec}\ x\ (\mathsf{zero}\ y)\ (y.\mathsf{succ}\ y) = x$

⟨tt-conat-form⟩ Conat

⟨tt-conat-intro⟩ $x : \mathsf{Conat} \vdash \mathsf{iszero}\ x : \mathsf{Unit} : \mathsf{OBSX} \qquad x : \mathsf{Conat} \vdash \mathsf{pred}\ n : \mathsf{Conat} : \mathsf{OBSX}$

⟨tt-conat-elim⟩ $$\frac{y : A \vdash Z : \mathsf{Unit} \quad y : A \vdash P : A}{x : A \vdash \mathsf{corec}\ x\ y.Z\ y.P : \mathsf{Conat} : \mathsf{OBSX}}$$

⟨tt-conat-beta⟩ $\mathsf{iszero}\ (\mathsf{corec}\ x\ y.Z\ y.P) = Z[x\,/\,y]$

$\mathsf{pred}\ (\mathsf{corec}\ x\ y.Z\ y.P) = \mathsf{corec}\ x\ y.Z\ y.P[P\,/\,y]$

⟨tt-conat-eta⟩ $\mathsf{corec}\ x\ (y.\mathsf{iszero}\ y)\ (y.\mathsf{pred}\ y) = x$

This concludes the definition of the type theory. We should now have the following.

PROPOSITION 4.3.1. The theories Θ_{OBSX} and $\Theta_{\mathsf{OBSX*}}$ are sound for Obs and Obs*, respectively.

Proof. (Sketch.) Before giving the rules describing each individual universal property, we gave references to the statements showing that the respective construction exists in Obs or Obs*. It should then be clear how the categories can be equipped with an interpretation that makes them models of the type theories. □

As discussed at the beginning of this section, this theory cannot be expected to be complete. However, we want to make a smaller claim, namely that it is in a sense "Turing-complete", i.e. it can express all computable functions. This requires a small trick in defining the interpretation of computable functions. We will use *general recursive functions* $f : \mathbb{N}^k \rightharpoonup \mathbb{N}$, which are partial functions, and define their interpretation $\hat{f}$ to be the univalent domain mapping $\bigotimes_{i=1}^{k} \mathbb{N} \to \overline{\mathbb{N}}$ from the tensor product of the natural numbers object into the *coinductive* natural numbers object, defined as $\hat{f}(\bigotimes a_i) = f(\times a_i)$ if f is defined on $\times a_i$, and $\hat{f}(\bigotimes a_i) = \bot$ otherwise. Notice that this always defines a univalent domain mapping because $\overline{\mathbb{N}}$ is a CABA and f outputs a natural number, which can be interpreted as an atom of $\overline{\mathbb{N}}$, or nothing, which is interpreted as $\bot$. This is clearly a faithful interpretation of f.

We trust that it is clear how to interpret the judgments in Θ_{OBSX} as objects and morphisms in Obs, so we will not justify uses of such interpretations. Write $\mathtt{Nat}^k$ for Smash Nat (Smash Nat (Smash Nat ...)) with k instances of Nat. This is interpreted as the domain $\bigotimes_k \mathbb{N}$, with atoms all k-tuples of natural numbers and no non-trivial consistent sets of atoms, see ⟨obs-nno⟩ and ⟨obs⋆-lim⟩. Thus we will give a construction that turns general recursive functions $f : \mathbb{N}^k \to \mathbb{N}$ into judgments $x : \mathtt{Nat}^k \vdash F : \mathtt{Conat}$ such that $[\![x : \mathtt{Nat}^k \vdash F : \mathtt{Conat}]\!] = \hat{f}$.

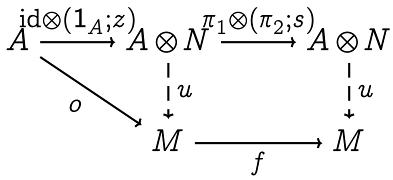
Let us see what terms of type Nat and Conat actually look like. For Nat, the term introduction rules give us terms zero and succ x for $x : \mathtt{Nat}$. Thus every term is an n-fold application of succ to zero, corresponding to an atom $n \in \mathbb{N}$, as was to be expected. For Conat, we cannot explicitly construct terms out of nothing, but given a term, we can check whether it is 0, and if not, we can do the same for its predecessor. Of course, if the term is bot, then we will never be able to tell. To interpret a term of type Conat as a natural number (extended by $\bot$), we would like to first check if the term evaluates to bot, and if not, count how many times we need to apply pred to the outcome until iszero yields top. This can of course not be an effective procedure, chiefly because checking if a term of type Conat is $\perp$ cannot be implemented effectively—otherwise, the theorem we are about to prove would imply that the halting problem is decidable. It directly follows that term equality is not decidable in the theory.

THEOREM 4. The theory Θ_{OBSX} is Turing-complete. That is, for every general recursive function $f : \mathbb{N}^k \to \mathbb{N}$, there exists a judgment $x : \mathtt{Nat}^k \vdash F : \mathtt{Conat}$ such that $[\![x : \mathtt{Nat}^k \vdash F : \mathtt{Conat}]\!] = \hat{f}$, where $[\![\cdot]\!]$ is the interpretation of Θ_{OBSX} in Obs. ⟨obsx-turing⟩

Proof. (Sketch.) Every general recursive function can be written as an application of the μ-operator to a primitive recursive function. We first show that all primitive recursive functions $\mathbb{N}^k \to \mathbb{N}$ can be constructed. We interpret such functions as atom-preserving domain mappings $\mathbb{N} \to \mathbb{N}$, i. e. the codomain is the natural numbers object, not the coinductive one.

Every category with a *parametrised* natural numbers object can "express" all primitive recursive functions. While Obs does not have such an object, it has a "tensorial" parametrised natural numbers object; see the margin for the relevant diagram. This follows from the fact that for each $A \in \mathsf{Obs}$, the functor $A \otimes -$ is a left adjoint, with the right adjoint given by the internal hom $[A, -]$, as ⟨obs-closed⟩ shows. This implies that $A \otimes -$ preserves initial algebras [27, Theorem 7.2], in particular the natural numbers object, from which the diagram on the right directly follows. The proof that categories with parametrised natural numbers objects express all primitive recursive functions as morphisms $\mathbb{N} \to \mathbb{N}$, given in [19, Theorem 2.4], also shows that the same applies to a category with a tensorial parametrised natural numbers object, and then a corresponding judgment can be derived in Θ_{OBSX} because it contains all the rules necessary to prove the existence of such an object in the syntactic category.

For all A:



In short, the type Nat provides constant zero and a successor function by ⟨tt-nat-intro⟩, and primitive recursion via ⟨tt-nat-elim⟩ together with the Smash type to model the parametrised version of the natural numbers object, which is necessary to account for the fact that in a primitive recursive definition $\mathsf{rec}\ x\ (y.Z)\ (y.S)$, the term S may not just depend on the result of itself applied to the predecessor of the argument x, but also on the *value* of the predecessor. Without this, e.g. the predecessor function is not definable. Functions with arities greater than 1 can be handled using ⟨tt-smash-intro⟩, ⟨tt-smash-elim⟩, and ⟨tt-subs⟩. Now for general recursive functions, for simplicity we ignore these cases of arities greater than 1.

Thus let $f:\mathbb{N}\to\mathbb{N}$ be a general recursive function of the form $f(n)=\mu m.p(m,n)$ for p primitive recursive. Create a new function $p':\mathbb{N}^2\to\mathbb{N}$ as follows. On input (m,n), if $p(i,n)=0$ for some $i<m$, return 1. Otherwise, return $p(m,n)$. Now p' is primitive recursive because p is and bounded quantification and case distinction are primitive recursive operations. By the above, there then is a judgment $x:\mathsf{Smash\ Nat\ Nat}\vdash P':\mathsf{Nat}$ in Θ_{OBSX} interpreted as p' seen as an atom-preserving domain mapping $\mathbb{N}\otimes\mathbb{N}\to\mathbb{N}$.

We derive a term that represents f.

⟨1⟩ $x:\mathsf{Smash\ Nat\ Nat}\vdash \mathsf{atl}\ x:\mathsf{Nat}$ ⟨tt-smash-elim⟩

⟨2⟩ $x:\mathsf{Smash\ Nat\ Nat}\vdash \mathsf{atr}\ x:\mathsf{Nat}$ ⟨tt-smash-elim⟩

⟨3⟩ $x:\mathsf{Nat}\vdash \mathsf{succ}\ x:\mathsf{Nat}$ ⟨tt-nat-intro⟩

⟨4⟩ $x:\mathsf{Smash\ Nat\ Nat}\vdash \mathsf{succ}\ (\mathsf{atl}\ x):\mathsf{Nat}$ ⟨tt-subs⟩ @ ⟨1⟩,⟨3⟩

⟨5⟩ $x:\mathsf{Smash\ Nat\ Nat}\vdash \mathsf{atpair}\ (\mathsf{succ}\ (\mathsf{atl}\ x))\ (\mathsf{atr}\ x):\mathsf{Smash\ Nat\ Nat}$ ⟨tt-smash-intro⟩ @ ⟨4⟩,⟨2⟩

Denote the term just derived by Q. We find that ⟨5⟩ is interpreted as a function $q:\mathbb{N}\otimes\mathbb{N}\to\mathbb{N}\otimes\mathbb{N}$ with $m\otimes n\mapsto(m+1)\otimes n$.

⟨6⟩ $y:\mathsf{Unit}\vdash y:\mathsf{Unit}$ ⟨tt-var⟩

⟨7⟩ $y:\mathsf{Unit}\vdash \mathsf{bot}\ y:\mathsf{Empty}$ ⟨tt-empty-intro⟩

⟨8⟩ $y:\mathsf{Empty}\vdash \mathsf{bot}\ y:\mathsf{Unit}$ ⟨tt-empty-elim⟩

⟨9⟩ $y:\mathsf{Unit}\vdash \mathsf{bot}\ y:\mathsf{Unit}$ ⟨tt-subs⟩ @ ⟨7⟩,⟨8⟩

⟨A⟩ $x:\mathsf{Nat}\vdash \mathsf{rec}\ x\ y.y\ y.\mathsf{bot}\ y:\mathsf{Unit}$ ⟨tt-nat-elim⟩ @ ⟨6⟩,⟨9⟩

⟨B⟩ $x:\mathsf{Smash\ Nat\ Nat} \vdash P' : \mathsf{Nat}$ *by assumption*

⟨C⟩ $x:\mathsf{Smash\ Nat\ Nat} \vdash \mathsf{rec}\ P'\ y.y\ y.\mathsf{bot}\ y : \mathsf{Unit}$ ⟨tt-subs⟩ @ ⟨B⟩,⟨A⟩

Denote the term in ⟨C⟩ by Z. This gets interpreted as a function $z : \mathbb{N} \otimes \mathbb{N} \to \mathbf{1}^*$ that sends $m \otimes n$ to $\top$ if $p'(m,n) = 0$ and otherwise to $\bot$, which we also denote $(p'(m,n) = 0)^?$.

⟨D⟩ $y:\mathsf{Nat} \vdash \mathsf{zero}\ y : \mathsf{Nat}$ ⟨tt-nat-intro⟩

⟨E⟩ $y:\mathsf{Nat} \vdash y : \mathsf{Nat}$ ⟨tt-var⟩

⟨F⟩ $y:\mathsf{Nat} \vdash \mathsf{atpair}\ (\mathsf{zero}\ y)\ y : \mathsf{Smash\ Nat\ Nat}$ ⟨tt-smash-intro⟩ @ ⟨D⟩,⟨E⟩

⟨G⟩ $y:\mathsf{Smash\ Nat\ Nat} \vdash \mathsf{corec}\ y\ x.Z\ x.Q : \mathsf{Conat}$ ⟨tt-conat-elim⟩ @ ⟨C⟩,⟨5⟩

⟨H⟩ $y:\mathsf{Nat} \vdash \mathsf{corec}\ (\mathsf{atpair}\ (\mathsf{zero}\ y)\ y)\ x.Z\ x.Q : \mathsf{Conat}$ ⟨tt-subs⟩ @ ⟨F⟩,⟨G⟩

Denote the interpretation of ⟨H⟩ by $f' : \mathbb{N} \to \overline{\mathbb{N}}$. I claim that $\hat{f} = f'$. On input n, we have that f' returns the set of all m such that $z(q^m(0,n)) = \top$. Since $q(m',n') = (m'+1,n')$ for all m',n', we find that $q^m(0,n) = (m,n)$ for all m, and then $z(q^m(0,n)) = z(m,n) = (p'(m,n) = 0)^?$. By construction of p', we have that $p'(m,n) = 0$ iff m is least such that $p(m,n) = 0$. Of course, there is at most one such m, and so this returns $\{m\}$ if $p(m,n) = 0$ and $p(i,n) \neq 0$ for all $i < m$, or $\bot$ if no such m exists, which is precisely the definition of the interpretation of $\mu m.\, p(m,n) = f(n)$. □

5 APPLICATIONS

We briefly sketch some ideas for how the framework we have developed may be applied in practice to dynamical systems, first within the realm of programming, then within the realm of logical reasoning.

5.1 PROGRAMMING WITH DYNAMICAL SYSTEMS

Let us consider the Cantor space, ubiquitous in the theory of dynamical systems, as an observation space. This is the set of countably infinite binary sequences $(a_n)_{n\in\mathbb{N}}$, equipped with the countable product of the discrete topology on $\{0,1\}$. We will use a set of subbasic clopens as observable properties, specified by 0_n and 1_n for each $n\in\mathbb{N}$, where $b_n\in\{0_n,1_n\}$ represents the set of all binary sequences $(a_i)_{i\in\mathbb{N}}$ such that $a_n = b_n$. The maximally consistent sets of properties are the points of the Cantor space, i. e. collections of properties that include either 0_n or 1_n for each $n\in\mathbb{N}$, and it follows that the consistent sets are precisely those that do not include both 0_n and 1_n for any n. We can see from $\langle$obs-prod$\rangle$ that the domain $\times_{n\in\mathbb{N}}(\mathbf{1}^* + \mathbf{1}^*)$ satisfies precisely this property.

We did not include the possibility of recursive type definitions in the theory (with which we could define a type such as $\mathtt{Cantor} :=$ $\mathtt{Prod}$ ($\mathtt{Sum}$ $\mathtt{Unit}$ $\mathtt{Unit}$) $\mathtt{Cantor}$), but there is an isomorphic domain that we can construct, namely the function space $[\mathbb{N}, \mathbf{1}^* + \mathbf{1}^*]$. Let $\mathtt{State} := \mathtt{Sum\ Unit\ Unit}$ and $\mathtt{Cantor} := \mathtt{Fun\ Nat\ State}$. We a shorthand $\mathtt{Dyn}\ A$ for $\mathtt{Fun}\ A\ A$, the type of dynamical systems on the space represented by the type A. In symbolic dynamics, one often considers *shift operators*, which we can use here to immediately obtain a term of type $\mathtt{Dyn\ Cantor}$. The shift operator sends the sequence $(a_n)_{n\in\mathbb{N}}$ to $(a_{n+1})_{n\in\mathbb{N}}$. We can interpret this by saying that each point in the Cantor space represents a history of events 0 and 1, and the shift operator updates a history, sending it one step into the future. We straightforwardly construct this as $\mathtt{shift}\ x := \mathtt{lambda}\ y\ (\mathtt{eval}\ (\mathtt{atpair}\ x\ (\mathtt{succ}\ y)))$, for which the judgment $x : \mathtt{Cantor} \vdash \mathtt{shift}\ x : \mathtt{Cantor}$ can be derived in Θ_{OBSX}, and then we can obtain $y : \mathtt{Unit} \vdash \mathtt{lambda}\ x\ (\mathtt{shift}\ x) : \mathtt{Dyn\ Cantor}$. Contrast this with the definition of the shift operator on the domain $[\mathbb{N}, \mathbf{1}^* + \mathbf{1}^*]$ as a relation: it sends every history to which property $(n+1) \to b$ applies to one where $n \to b$ applies, and the properties $0 \to b$ to $\bot$. It can easily be seen that this relation is indeed a maximal element of $[[\mathbb{N}, \mathbf{1}^* + \mathbf{1}^*], [\mathbb{N}, \mathbf{1}^* + \mathbf{1}^*]]$, hence corresponds to a dynamical system via ⟨dynsys=max-cons⟩.

Since we can form arbitrary function spaces, we can also consider transformations between dynamical systems. Let us construct such a transformation taking a dynamical system on the Cantor space and perturbing it by inverting its output. First, we define $\mathtt{flip}\ x :=$ $\mathtt{case}\ x\ (y.\mathtt{inr}\ y)\ (y.\mathtt{inl}\ y)$, with $x : \mathtt{State} \vdash \mathtt{flip}\ x : \mathtt{State}$. Then, we have $x : \mathtt{Cantor} \vdash \mathtt{lambda}\ y\ (\mathtt{eval}\ (\mathtt{atpair}\ x\ (\mathtt{flip}\ y))) : \mathtt{Cantor}$, which gives us

$$x : \mathtt{Dyn\ Cantor} \vdash \mathtt{lambda}\ y\ (\mathtt{lambda}\ z\ (\mathtt{flip}\ (\mathtt{eval}\ (\mathtt{atpair}\ x\ (\mathtt{eval}\ (\mathtt{atpair}\ y\ z)))))) : \mathtt{Dyn\ Cantor}.$$

Applying this to the shift operator, the resulting dynamical system flips the history on every second time step.

One of the more unusual features of the type theory is the presence of different interlinking kinds. We have a theory $\Theta_{\mathsf{OBSX}*}$, which can be interpreted to make judgments about *atoms*, i.e. observable properties, and Θ_{OBSX}, which makes judgments about the *elements*, i.e. (partial descriptions of) points of the space. For example, we have two kinds of product types (see ⟨tt-prod-form⟩ and ⟨tt-smash-form⟩), one being universal for atom-preserving maps only, and the other one being universal for general domain mappings. A term atpair $x\ y$: Smash $A\ B$ should be thought of as a pair of atoms, while pair $x\ y$: Prod $A\ B$ is an arbitrary pair of elements of the domain.

The interaction between these constructions and types of dynamical systems seems worth exploring further; we will have to leave this to the future. For example, the types Dyn (Prod $A\ B$), Dyn (Smash $A\ B$), Prod (Dyn A) (Dyn B), and Smash (Dyn A) (Dyn B) all provide different kinds of terms that may be considered to be pairs of dynamical systems. As we have mentioned, tensor product and cartesian product spaces both have the same maximal elements, but they are observed using different properties: pairs of properties in the first case, and individual properties from either space in the second case. An element of a cartesian product space of dynamical systems is then specified via a collection of properties that either dynamical system satisfies, while for the tensor product space, we need to simultaneously specify two properties, one satisfied by each dynamical system.

The coinductive natural numbers type from ⟨tt-conat-form⟩ is not only useful to model general recursion: ⟨obs-conno⟩ suggests that we can feed it with a dynamical system $f : A \to A$ as well as a domain mapping $o : A \to \mathbf{1}^*$ that makes a selection of observable properties of A that interest us, and then corecursion returns a function that takes a point of A and returns a set of time steps such that the state of the dynamical system is in the region specified by o at that time step. This suits our conception of dynamical systems as functions for which time progresses through self-application very well.

5.2 REASONING ABOUT DYNAMICAL SYSTEMS

There are many different levels on which logical reasoning may be implemented in our framework. First, we have the equality rules in the type theory, which for some theories fully capture the equalities in the categories they describe. All of the following approaches rely on those equalities, which means that completeness of the resulting logic depends on the completeness of the type theory we use, and also that these logics will generally not be decidable.

Any quasitopos comes with an internal logic, which is very similar to the intuitionistic predicate logic that can be developed inside a topos. See [38, Chapter 3] for an in-depth exposition of this approach. We did not include local internal homs or related constructions in the type theory, which implies that we cannot talk about quantification. However, we did include pullbacks from the strong-subobject classifier. The Sub types from ⟨tt-sub-form⟩ can be used to talk about atoms of a domain satisfying a proposition. Namely, if $x{:}\mathtt{Unit} \vdash X{:}A{:}\mathtt{OBSX*}$ is a term, which gets interpreted as an atom-preserving domain mapping $\mathbf{1}^* \to [\![A]\!]$ selecting an atom of $[\![A]\!]$, and $y:A \vdash P:\mathtt{Prop}$ represents an atom-preserving domain mapping into the strong-subobject classifier, then we could say that $X \vDash P{:}A$ if $x{:}\mathtt{Unit} \vdash P[X/y] = \mathtt{true}\ x{:}\mathtt{Prop}$, which is the case precisely if the element selected by the term X is in the strong subobject generated by pullback along P (see the quasitopos subsection in Section 3.2). This way, we can determine whether the atom we selected is in a selected set of atoms, which means that we can interpret P as a proposition about atoms of A. We can define logical operations on the terms, for example conjunction of $x:A \vdash P,Q:\mathtt{Prop}$ as $P \wedge Q$ constructed from composing the pair $\mathtt{atpair}\ P\ Q:\mathtt{Smash\ Prop\ Prop}$ with a term corresponding to the domain mapping $\Omega^* \otimes \Omega^* \to \Omega^*$ with $(\top \times \bot) \otimes (\top \times \bot) \mapsto (\top \times \bot)$ and $x \otimes y \mapsto (\bot \times \top)$ otherwise. However, notice that this is just the logic of subsets of atoms, which completely ignores the other elements of the domains and is thus not very interesting.

We would like to talk about subsets other than strong subobjects. The natural choice in domain theory are certain up-sets of domains, because they more closely correspond to actual logical properties. For example, for an atom $a \in \mathrm{At}\,D$, the set $\uparrow a$ contains all the points of the observation space to which a applies. Unfortunately, up-sets of atoms generally do not form domains, but we can still talk about them in the type theory. Namely, we can select a set of atoms via a term $x{:}A \vdash P{:}\mathtt{Unit}$, sending the atoms we want to select to top. Then, also all elements above the selected atoms will be sent to top. We can again whether $X \vDash P{:}A$ for any global element $x:\mathtt{Unit} \vdash X:A$ by checking if $x:\mathtt{Unit} \vdash P[X\,/\,x] = \mathtt{top}\ x:\mathtt{Unit}$. For this, we can define truth 1 as the term $x:A \vdash \mathtt{top}:\mathtt{Unit}$ given by ⟨tt-unit-intro⟩, falsity 0 as $x{:}A \vdash \mathtt{bot}\ x{:}\mathtt{Unit}$ obtained from ⟨tt-empty-intro⟩ and ⟨tt-empty-elim⟩ via ⟨tt-subs⟩, and disjunction $P \vee Q$ as the composite $x:A \vdash \mathtt{top}\ (\mathtt{pair}\ P\ Q){:}\mathtt{Unit}$. Conjunction $P \wedge Q$ cannot be defined, however: there is no way to map the product $\mathbf{1}^* \times \mathbf{1}^*$ to $\mathbf{1}^*$ such that only the top element of $\mathbf{1}^* \times \mathbf{1}^*$ is sent to $\top$ and the rest to $\bot$, by preservation of joins. Indeed, intersections of up-sets of atoms are usually not up-sets of atoms.

An approach somewhere between the two approaches just discussed would be to consider general subobjects in Obs*, rather than just strong ones.° This means that we need to select a set of atoms, and then we can specify a down-set of joins of these atoms we want to include. This includes some interesting sets such as $\downarrow\uparrow a$ for a an atom, which contains the same maximal elements of $\uparrow a$ but has the advantage of forming a domain. This set includes precisely the descriptions that are consistent with a, i.e. those that may apply together with a to a point in the space. Because the Sub types can only create strong subobjects, we would have to assemble these types "by hand". We can however define these subobjects by defining what a monomorphism is in Obs*: namely, $x{:}A' \vdash M{:}A$ is a monomorphism if whenever $y{:}B \vdash M[N/x] = M[N'/x]{:}A \in \Theta_{\mathsf{OBSX}*}$ for some B, N, N', then $y:B \vdash N = N' : A' \in \Theta_{\mathsf{OBSX}*}$. Note that we cannot make this definition into a derivation rule of the usual

°Subobjects in Obs, however, are too general to admit a useful interpretation as logical properties.

form, but we can only prove that something is a monomorphism "from outside", by proving things *about* the theory. Since the type theory is most likely not complete, it cannot be expected that all monomorphisms can be identified this way. Now, if $x:\texttt{Unit} \vdash X:A$ and $y:A' \vdash M:A$ is a monomorphism, we can define $X \vDash M:A$ as the existence of an $x:\texttt{Unit} \vdash N:A'$ such that $x:\texttt{Unit} \vdash M[N/y] = X:A$.

If we allow such meta-inferences about the type theory, we can in fact recover the order relation of the domains in the theory, as follows. In any pointed category C, we can define an order $\trianglelefteq$ on the morphisms by saying that $f \trianglelefteq g$ iff for all appropriate h, h', we have $h\,;g\,;h' = \mathsf{0}$ implies $h\,;f\,;h' = \mathsf{0}$. This is always reflexive and transitive.° Observe that in the case of Obs (and Rel), this precisely means that $f \subseteq g$ as a relation: if $H\,;G\,;H' = \mathsf{0}$, then there are no u, v, w, x with $u\,H\,v\,G\,w\,H'\,x$, and then there can be no such u, v, w, x for $F \subseteq G$ either; conversely, if the implication holds and $v\,F\,w$, then $\bullet \to v\,F\,w \to \bullet$ with arrows from and into $\mathsf{1}$ relating to nothing but v and w makes the composite non-empty, and the only way for the composite with G to be non-empty is then if $v\,G\,w$. We can now talk about the order on global elements $x:\texttt{Unit} \vdash X:A$ and $x:\texttt{Unit} \vdash Y:A$ by saying $X \leq Y$ iff $x:\texttt{Unit} \vdash N[Y/y] = \texttt{bot}\ x:B$ implies $x:\texttt{Unit} \vdash N[X/y] = \texttt{bot}\ x:B$ for all $y:A \vdash N:B$. We can interpret global elements variously as propositions selecting a consistent set of atoms, or as elements of a domain, and we could take $X \vDash P:A$ to just mean $P \leq X$, so logical entailment reduces to the order on the domain. We can now also define what an atom is in Θ_{OBSX}, namely a global element $x:\texttt{Unit} \vdash X:A$ that is an atom under $\leq$, and then an atom-preserving map is just a term that composed with every atom yields an atom, which would allow us to include another rule into the theory stating that if we have proved a term to be atom-preserving in this way, we may also add it to $\Theta_{\mathsf{OBSX}*}$.

°This means that pointed categories are not just automatically enriched in pointed sets, but even in preordered sets.

6 CONCLUSION

We have derived a type theory from a model of denotational semantics for dynamical systems within a paradigm of "finitary observation" or "description". Here is a summary of the individual steps we took.

1. In Chapter 2, we developed *observation systems* to implement the concept of finitary observation via observable properties. We showed that they have a spatial interpretation, given by *observation spaces*, which connect to topology and topological dynamical systems (Theorem ⟨dynsys=max-cons⟩), see Section 2.2, and a domain-theoretic interpretation, Section 2.3. Morphisms between observation domains called *domain mappings* can be seen as partial descriptions of dynamical systems.

2. We studied the category of domains and domain mappings and found that it is monoidal closed, implying that dynamical systems on a domain form a domain on their own, that it has products and coproducts in addition to a tensor product, and that it contains a subcategory which is a quasitopos, in Section 3.2. We characterised several relevant categories using *free constructions* in Section 3.3, in particular the category of finite domains in Theorem ⟨obs-free⟩.

3. We developed a framework of categorical semantics for type

theories in Section 4.1 and showed how characterisations of free constructions can be used to obtain sound and complete type theories for individual categories. In particular, we argued that we have a complete type theory for finite observation domains in Theorem ⟨obs-sound-complete⟩, but we did not prove this in detail. Then, we gave a sound theory that is Turing-complete, Theorem ⟨obsx-turing⟩.°

°We have also briefly discussed that complete logics for (finite) dynamical systems may be developed on the basis of the complete type theory, Section 5.2, while the Turing-complete type theory may serve as a programming language for dynamical systems, Section 5.1.

Other notable results include a characterisation of the category of finite sets and relations as a free construction in Proposition ⟨finrel-free⟩, as well as a generally applicable characterisation of categories of concrete presheaves as certain free cocompletions. In dealing with denotational semantics of a particular model of computation via domain theory, one often works with a category of domains representing types, and it might then be of great interest to obtain a sound and complete theory for this particular category. We have demonstrated that this can be done (at least for the case of the category of finite sets and functions, in ⟨set-sound-complete⟩) by characterising the category using a free construction. This is a somewhat unusual approach in categorical semantics because one typically only considers completeness of a theory for a *class* of categories (e. g. locally cartesian closed categories for dependent type theories) [34].

While we did obtain a type theory that arguably speaks about dynamical systems, there still remains a lot more work to be done. First and foremost, applications of the theories to the study of dynamical systems remain mostly unexplored, so it is yet to be determined how useful it is in practice. In particular, the difference between tensor and product types calls for an interpretation in terms of constructions involving dynamical systems. Furthermore, if we want to be certain about the completeness of the type theory for finite domains, we may still want to work out a full proof of Theorem ⟨obs-sound-complete⟩. We deliberately kept the type theories simple because we focused on the overall approach we tried to demonstrate, so the theories are not optimised for usability

e. g. as a programming language. For increased usefulness, the Turing-complete theory may also be extended with recursive type definitions, as is common in domain theory [2]. An even more ambitious project would be to obtain a complete type theory for suitably defined "computable" countable domains and computable domain mappings.

On a more general level, we may wonder about the applicability of our conceptual setup of "finitary observation" to the theory of dynamical systems. While we have argued for the connection of the definitions we made to the standard notions of dynamical systems, the relationship between these needs to be elucidated in more detail. What examples of observation spaces exist, besides the straightforward ones such as the Cantor space? What properties does a "weakly continuous" (see ⟨weak-cont⟩) function between topological spaces have compared to a (standard) continuous one? What are the implications of the condition ⟨max-det⟩ in the definition of dynamical systems? How can what we did be integrated into the study of measure-theoretic dynamical systems? Can we generalise the method sufficiently and apply it to the category of dynamical domains in [12]?

Also on the category-theoretic side, some open questions remain. The relationship between Obs and Obs^* is intriguingly similar to the relationship between Set and Rel, which also becomes apparent from the way we can represent the two as categories of certain presheaves. How can this be made precise? Finally, how can the concrete diagrams from ⟨concrete-free⟩ be formalised in a type theory? Doing so would allow us to automatically derive sound and complete type theories for categories of concrete presheaves, which have been explored as "convenient" categories of spaces since they always form quasitopoi [3]. In this way, I hope to have demonstrated the feasibility of a general method that may be used in the future to derive type theories for a variety of computational processes from their semantics.

REFERENCES

[1] Samson Abramsky. Domain theory in logical form. *Annals of pure and applied logic*, 51(1-2):1–77, 1991.

[2] Samson Abramsky and Achim Jung. Domain theory. 1994.

[3] John Baez and Alexander Hoffnung. Convenient categories of smooth spaces. *Transactions of the American Mathematical Society*, 363(11):5789–5825, 2011.

[4] Valérie Berthé. Symbolic dynamics and representations. *Les cours du CIRM*, 5(1):1–16, 2017.

[5] Francis Borceux. *Handbook of categorical algebra: volume 1, Basic category theory*, volume 1. Cambridge University Press, 1994.

[6] Robert Bringhurst. *The elements of typographic style*. Hartley & Marks Vancouver, 2004.

[7] Bob Coecke, John Selby, and Sean Tull. Two roads to classicality. *arXiv preprint arXiv:1701.07400*, 2017.

[8] Jean-Yves Girard. The system f of variable types, fifteen years later. *Theoretical computer science*, 45:159–192, 1986.

[9] Jean-Yves Girard. Linear logic: its syntax and semantics. *London Mathematical Society Lecture Note Series*, pages 1–42, 1995.

[10] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete Mathematics: A Foundation for Computer Science*. Addison-Wesley, Reading, 1989.

[11] John J Hopfield. Neural networks and physical systems with emergent collective computational abilities. *Proceedings of the national academy of sciences*, 79(8):2554–2558, 1982.

[12] Levin Hornischer. *Dynamical systems via domains: Toward a unified foundation of symbolic and non-symbolic computation*. PhD thesis, University of Amsterdam, 2021.

[13] Thomas J Jech. *The axiom of choice*. Courier Corporation, 2008.

[14] Anna Jenčová and Gejza Jenča. On monoids in the category of sets and relations. *International Journal of Theoretical Physics*, 56:3757–3769, 2017.

[15] Peter T Johnstone. The point of pointless topology. *Bulletin of the American Mathematical Society*, 8(1):41–53, 1983.

[16] Achim Jung. *Cartesian closed categories of domains*, volume 66. Citeseer, 1989.

[17] Alexander Kechris. *Classical descriptive set theory*, volume 156. Springer Science & Business Media, 2012.

[18] Max Kelly. *Basic concepts of enriched category theory*, volume 64. CUP Archive, 1982.

[19] Joachim Lambek and Philip J Scott. *Introduction to higher-order categorical logic*, volume 7. Cambridge University Press, 1988.

[20] Leslie Lamport. How to write a 21st century proof. *Journal of fixed point theory and applications*, 11(1):43–63, 2012.

[21] Tom Leinster. *Basic category theory*, volume 143. Cambridge University Press, 2014.

[22] Alfred J Lotka. Contribution to the theory of periodic reactions. *The Journal of Physical Chemistry*, 14(3):271–274, 2002.

[23] Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 2013.

[24] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012.

[25] James R Munkres. Topology (2nd edn), 2000.

[26] Rob Nederpelt and Herman Geuvers. *Type theory and formal proof: an introduction*. Cambridge University Press, 2014.

[27] Fernando Lucatelli Nunes and Matthijs Vákár. Chad for expressive total languages. *arXiv preprint arXiv:2110.00446*, 2021.

[28] Robert Paré. Connected components and colimits. *Journal of Pure and Applied Algebra*, 3(1):21–42, 1973.

[29] Bodo Pareigis. *Kategorien und Funktoren*. Springer-Verlag, 2013.

[30] Paolo Perrone and Walter Tholen. Kan extensions are partial colimits. *Applied Categorical Structures*, 30(4):685–753, 2022.

[31] Andrew M Pitts. Categorical logic. *Handbook of logic in computer science*, 5:39–128, 2001.

[32] The Univalent Foundations Program. Homotopy type theory: Univalent foundations of mathematics. *arXiv preprint arXiv:1308.0729*, 2013.

[33] Dana Scott. *Outline of a mathematical theory of computation*. Oxford University Computing Laboratory, Programming Research Group Oxford, 1970.

[34] Robert AG Seely. Locally cartesian closed categories and type theory. In *Mathematical proceedings of the Cambridge philosophical society*, volume 95, pages 33–48. Cambridge University Press, 1984.

[35] Marshall H Stone. The theory of representation for boolean algebras. *Transactions of the American Mathematical Society*, 40(1):37–111, 1936.

[36] Jan Tschichold. *Willkürfreie Maßverhältnisse der Buchseite und des Satzspiegels*. Springer, 1975.

[37] Jaap van Oosten and BRICS Lecture Series LS. Basic category theory. 1995.

[38] Oswald Wyler. *Lecture notes on topoi and quasitopoi*. World Scientific, 1991.

INDEX