DEGREES OF THE FINITE MODEL PROPERTY: THE ANTIDICHOTOMY THEOREM

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Abstract. A classic result in modal logic, known as the Blok Dichotomy Theorem, states that the degree of incompleteness of a normal extension of the basic modal logic K is 1 or $2^{\aleph_0}$. It is a long-standing open problem whether Blok Dichotomy holds for normal extensions of other prominent modal logics (such as S4 or K4) or for extensions of the intuitionistic propositional calculus IPC (see [11, Prob. 10.5]). In this paper, we introduce the notion of the degree of finite model property (fmp), which is a natural variation of the degree of incompleteness. It is a consequence of Blok Dichotomy Theorem that the degree of fmp of a normal extension of K remains 1 or $2^{\aleph_0}$. In contrast, our main result establishes the following Antidichotomy Theorem for the degree of fmp for extensions of IPC: each nonzero cardinal κ such that $\kappa \leqslant \aleph_0$ or $\kappa = 2^{\aleph_0}$ is realized as the degree of fmp of some extension of IPC. We then use the Blok-Esakia theorem to establish the same Antidichotomy Theorem for normal extensions of S4 and K4. This provides a solution of the reformulation of [11, Prob. 10.5] for the degree of fmp.

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1. Introduction

Since its inception in the late 1950s/early 1960s, Kripke semantics became the most popular tool to study modal and intuitionistic logics. However, examples of Kripke incomplete logics began to emerge already in the 1970s (see, e.g., [11, Ch. 6]). In order to shed light on the phenomenon of Kripke incompleteness, Fine [18] associated with each normal modal logic L a cardinal that measures the degree of incompleteness of L. More precisely, let Fr(L) be the class of Kripke frames validating L. We say that the *degree of incompleteness* of L is the cardinal κ if there are exactly κ logics L′ such that Fr(L′) = Fr(L). Notice that all but one of these L′ are Kripke incomplete.

Blok [9, 10] gave a very unexpected characterization of degrees of incompleteness, which became known as *Blok Dichotomy Theorem*. It states that a normal modal logic L has the degree of

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incompleteness either 1 or $2^{\aleph_0}$; it is 1 iff L is a join-splitting logic (see Section 3 for the definition); otherwise it is $2^{\aleph_0}$. We refer to [34] and [29] for a detailed discussion of Blok Dichotomy and its importance in modal logic.

Blok's result implies that some of the most studied normal modal logics, such as K4 (the logic of transitive Kripke frames) and S4 (the logic of reflexive and transitive Kripke frames), have the degree of incompleteness $2^{\aleph_0}$. However, the logics sharing the Kripke frames with K4 and S4 are not necessarily normal extensions of K4 or S4. Thus, Blok's result does not automatically transfer to normal extensions of K4 or S4 (or, more generally, to normal extensions of a given normal modal logic). There have been several attempts to investigate Blok Dichotomy for normal extensions of K4 and S4. However, this remains an outstanding open problem in modal logic [11, Prob. 10.5].

For a logic L, let $\mathsf{Fin}(\mathsf{L})$ be the class of finite Kripke frames validating L. We recall that L has the *finite model property* (*fmp* for short) if L is complete with respect to $\mathsf{Fin}(\mathsf{L})$. Clearly each logic with the fmp is Kripke complete. Taking inspiration from degrees of incompleteness, it is natural to introduce a similar concept for the fmp. We say that the *degree of fmp* of a logic L is κ provided there exist exactly κ logics L' such that $\mathsf{Fin}(\mathsf{L}') = \mathsf{Fin}(\mathsf{L})$. As with the degree of incompleteness, all but one of such L' lack the fmp. Our main result establishes a complete opposite of Blok Dichotomy theorem for superintuitionistic logics and transitive (normal) modal logics. Namely, we prove that if κ is a nonzero cardinal such that $\kappa \leqslant \aleph_0$ or $\kappa = 2^{\aleph_0}$, then there exists a superintuitionistic logic (or a transitive modal logic) L such that the degree of fmp of L is κ. Under the Continuum Hypothesis (CH) this implies that each nonzero $\kappa \leqslant 2^{\aleph_0}$ is realized as the degree of fmp of some superintuitionistic logic (or some transitive modal logic). For this reason, we refer to these results as the *Antidichotomy Theorems for degrees of fmp* (see Theorems 3.2 and 7.3).

In [29, p. 409] Litak asks "if there is any nontrivial completeness notion for which the Blok Dichotomy does not hold." Our main result provides such a nontrivial and, in our opinion, very natural notion for superintuitionistic logics and transitive modal logics. It also provides a solution of a variant of [11, Prob. 10.5] when the degree of incompleteness is replaced by the degree of fmp.

To give more context, we recall that *superintuitionistic logics* are (axiomatic) extensions of the intuitionistic propositional calculus IPC. They have been studied extensively in the literature (see, e.g., [11]). In particular, there is a close connection between superintuitionistic logics and normal extensions of S4. The *Gödel translation* embeds IPC into S4 fully and faithfully [32]. Thus, each superintuitionistic logic L is embedded into a normal extension of S4, called a *modal companion* of L [11, Sec. 9.6]. Each L has many modal companions, but remarkably each L possesses a largest modal companion. By Esakia's theorem [15, 16], the largest modal companion of IPC is the well-known Grzegorczyk logic Grz. Consequently, the largest modal companion of each superintuitionistic logic is a normal extension of Grz, and there exists an isomorphism between the lattice of superintuitionistic logics and the lattice of normal extensions of Grz (the Blok-Esakia theorem) [8, 15].

Notice that it is a consequence of Blok Dichotomy Theorem that the degree of fmp of a normal extension of the basic modal logic K remains 1 or $2^{\aleph_0}$. Thus, in the lattice of all normal modal logics the dichotomy holds also for the degrees of fmp (see Theorem 7.1). In contrast, it is a consequence of our Modal Antidichotomy Theorem that the situation is drastically different for transitive modal logics (see Corollary 7.4).

We conclude the introduction by discussing how we establish our main results. We first prove the Antidichotomy Theorem for degrees of fmp of superintuitionistic logics. We heavily rely on Esakia duality for Heyting algebras [17], as well as on Fine's completeness theorem for logics of bounded width [19] and the theory of splittings [11, Sec. 10.5]. Our proof is broken into two parts, depending on whether $\kappa \leqslant \aleph_0$ or $\kappa = 2^{\aleph_0}$.

When $\kappa \leqslant \aleph_0$ we work with extensions of the superintuitionistic logic KG, which was introduced by Kuznetsov and Gerčiu [21, 28] and bears their name. The logic KG is the logic of sums of one-generated Heyting algebras, the combinatorics of which allows to construct extensions of KG that lack the fmp [28, 26, 3]. First, we use Fine's completeness theorem to prove that KG is a join-splitting logic over IPC (for a similar result see [26]). Then we develop a method, utilizing a technique of [3], that produces an extension L of KG whose degree of fmp is κ for every nonzero cardinal $\kappa \leqslant \aleph_0$.

To show that there exist superintuitionistic logics whose degree of fmp is $2^{\aleph_0}$ we work with superintuitionistic logics of finite width. Transitive modal logics of finite width were introduced by Fine [19] who showed that each transitive modal logic of finite width has the fmp. The concept was adapted to superintuitionistic logics by Sobolev [37]. For every positive integer n, let BW_n be the least superintuitionistic logic of width n. We prove that if $n > 2$, then the degree of fmp of BW_n is $2^{\aleph_0}$. This is done by a careful analysis of the combinatorics of posets of bounded width.

Under CH our results show that for every nonzero cardinal $\kappa \leqslant 2^{\aleph_0}$ there exists a superintuitionistic logic L whose degree of fmp is κ, thus yielding the Antidichotomy Theorem for degrees of fmp of superintuitionistic logics. Nonetheless, determining the degree of fmp of a given superintuitionistic logic remains an interesting open problem.

Finally, we transfer our results to the setting of modal logics. Following the notation of [11], for a normal modal logic L, let $\mathrm{Next}\,\mathsf{L}$ be the lattice of normal extensions of L. We first use the Blok-Esakia theorem to prove our Antidichotomy Theorem for $\mathrm{Next}\,\mathsf{Grz}$. We next show that for each normal modal logic $\mathsf{L} \subseteq \mathsf{Grz}$ with the fmp, the Antidichotomy Theorem holds for $\mathrm{Next}\,\mathsf{L}$ provided Grz is a join-splitting logic above L. Since $\mathsf{S4}$ and $\mathsf{K4}$ have the fmp and Grz is a join-splitting logic above both, it follows that the Antidichotomy Theorem holds for $\mathrm{Next}\,\mathsf{S4}$ and $\mathrm{Next}\,\mathsf{K4}$. We conclude the paper by listing several open problems and possible future research directions.

2. Superintuitionistic logics

We recall that a *superintuitionistic logic*, or a *si-logic* for short, is a set of formulas L containing IPC and closed under the inference rules of modus ponens and substitution. It is well known (see, e.g., [11, Thm. 4.1]) that consistent si-logics are exactly the logics situated between IPC and the classical propositional calculus CPC. Thus, consistent si-logics are often referred to as *intermediate logics*. Given a set of formulas Σ, we denote by $\mathsf{IPC} + \Sigma$ the si-logic *axiomatized* by Σ; that is, the least si-logic containing Σ.

When ordered by set inclusion, the set of si-logics forms a complete lattice, denoted by $\mathrm{Ext}\,\mathsf{IPC}$, whose bottom and top are IPC and the inconsistent logic, respectively. The meet and join operations in $\mathrm{Ext}\,\mathsf{IPC}$ are defined as

$$\bigwedge_{i \in I} \mathsf{L}_i = \bigcap_{i \in I} \mathsf{L}_i \quad \text{and} \quad \bigvee_{i \in I} \mathsf{L}_i = \text{the si-logic axiomatized by } \bigcup_{i \in I} \mathsf{L}_i.$$

It is a well-known result of Jankov [23] that the cardinality of $\mathrm{Ext}\,\mathsf{IPC}$ is $2^{\aleph_0}$.

Kripke semantics for si-logics is given by partially ordered sets (posets for short). For a poset X, we call $U \subseteq X$ an *upset* (upward closed set) if

$$x \in U \text{ and } x \leqslant y \text{ imply } y \in U.$$

A *valuation* ν on X assigns to each propositional letter p an upset of X. For $x \in X$ and a formula φ we write $x \Vdash_\nu \varphi$ when x satisfies φ under ν. As usual, the satisfaction relation $\Vdash$ is defined by recursion on the construction of formulas:

$$\begin{array}{lll} x \nVdash_\nu \bot & & \\ x \Vdash_\nu p & \text{iff} & x \in \nu(p) \\ x \Vdash_\nu \varphi \wedge \psi & \text{iff} & x \Vdash_\nu \varphi \text{ and } x \Vdash_\nu \psi \\ x \Vdash_\nu \varphi \vee \psi & \text{iff} & x \Vdash_\nu \varphi \text{ or } x \Vdash_\nu \psi \\ x \Vdash_\nu \varphi \to \psi & \text{iff} & \forall y (x \leqslant y \text{ and } y \Vdash_\nu \varphi \text{ imply } y \Vdash_\nu \psi). \end{array}$$

A formula φ is said to be *true* in X under ν if $x \Vdash_\nu \varphi$ for every $x \in X$ and it is said to be *valid* in X if it is true under each valuation, in which case we write $X \vDash \varphi$.

Algebraic semantics for si-logics is given by Heyting algebras. We recall that a *Heyting algebra* $\boldsymbol{A} = \langle A; \wedge, \vee, \to, 0, 1\rangle$ is a bounded distributive lattice such that $\wedge$ has a residual $\to$ given by

$$a \wedge b \leqslant c \iff a \leqslant b \to c$$

for all $a, b, c \in A$.

A valuation ν in a Heyting algebra $\boldsymbol{A}$ assigns to each propositional letter an element of $\boldsymbol{A}$. The logical connectives are then interpreted as the corresponding operations in $\boldsymbol{A}$. A formula φ is *true* in $\boldsymbol{A}$ under ν if $\nu(\varphi) = 1$ and it is *valid* in $\boldsymbol{A}$ if it is true under each valuation, in which case we write $\boldsymbol{A} \vDash \varphi$.

There is a close connection between Kripke and algebraic semantics for si-logics. For a poset X and $U \subseteq X$, let

$$\begin{aligned} {\uparrow}U &= \{x \in X : \exists u \in U \text{ with } u \leqslant x\} \\ {\downarrow}U &= \{x \in X : \exists u \in U \text{ with } x \leqslant u\}. \end{aligned}$$

If $U = \{x\}$, we simply write ${\uparrow}x$ and ${\downarrow}x$ instead of ${\uparrow}\{x\}$ and ${\downarrow}\{x\}$. Let $\mathsf{Up}(X)$ be the set of upsets of X. Then $\mathsf{Up}(X)$ is a Heyting algebra where join and meet are set-theoretic union and intersection, bottom and top are $\varnothing$ and X, and $\to$ is defined by

$$U \to V = X \smallsetminus {\downarrow}(U \smallsetminus V) = \{x \in X : {\uparrow}x \cap U \subseteq V\}.$$

Conversely, for a Heyting algebra $\boldsymbol{A}$, let $X_{\boldsymbol{A}}$ be the poset of prime filters of $\boldsymbol{A}$ ordered by inclusion. Define $\gamma_{\boldsymbol{A}} \colon \boldsymbol{A} \to \mathsf{Up}(X_{\boldsymbol{A}})$ by

$$\gamma_{\boldsymbol{A}}(a) = \{x \in X_{\boldsymbol{A}} : a \in x\}.$$

Then $\gamma_{\boldsymbol{A}}$ is a Heyting algebra embedding. To recognize the image of $\boldsymbol{A}$ in $\mathsf{Up}(X_{\boldsymbol{A}})$, we introduce the topology τ on $X_{\boldsymbol{A}}$ given by the subbasis

$$\{\gamma_{\boldsymbol{A}}(a) : a \in A\} \cup \{X_{\boldsymbol{A}} \smallsetminus \gamma_{\boldsymbol{A}}(a) : a \in A\}.$$

It is well known that τ is a Stone topology on $X_{\boldsymbol{A}}$ (that is, it is compact, Hausdorff, and zero-dimensional). The triple $\boldsymbol{A}_* = \langle X_{\boldsymbol{A}}, \tau, \subseteq\rangle$ is known as the *Esakia space* of $\boldsymbol{A}$. The map $\gamma_{\boldsymbol{A}}$ is an isomorphism from $\boldsymbol{A}$ onto the Heyting algebra of clopen upsets of $\boldsymbol{A}_*$. Thus, each Heyting algebra is represented as the algebra of clopen upsets of an Esakia space.

Esakia spaces are characterized abstractly as triples $X = \langle X, \tau, \leqslant\rangle$ where τ is a Stone topology and $\leqslant$ is a partial order on X that, moreover, is *continuous* in the sense that

(1) ${\uparrow}x$ is closed for all $x \in X$;
(2) $U \subseteq X$ is clopen implies ${\downarrow}U$ is clopen.

We point out that the partial order $\leqslant$ is continuous iff the corresponding map $\rho : X \to \mathcal{V}X$ from X to the Vietoris space $\mathcal{V}X$, given by $\rho(x) = {\uparrow}x$, is a well-defined continuous map [14, 1, 27].

We thus obtain the object level of Esakia duality, namely that there is a one-to-one correspondence between Heyting algebras and Esakia spaces. To extend this correspondence to full duality, we recall that a *p-morphism* (or *bounded morphism*) between two posets X and Y is a map $\alpha \colon X \to Y$ such that ${\uparrow}\alpha(x) = \alpha({\uparrow}x)$ for each $x \in X$.

Let ES be the category of Esakia spaces and continuous p-morphisms between them. Let also HA be the category of Heyting algebras and Heyting homomorphisms between them. The two categories are related as follows [14, 17]:

Theorem 2.1 (**Esakia Duality**). HA *is dually equivalent to* ES*.*

We denote the contravariant functors establishing Esakia duality by $(-)_* : \mathsf{HA} \to \mathsf{ES}$ and $(-)^* : \mathsf{ES} \to \mathsf{HA}$. The functor $(-)_*$ assigns to each Heyting algebra $\boldsymbol{A}$ the Esakia space $\boldsymbol{A}_*$. If $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a Heyting homomorphism, define $f_* \colon \boldsymbol{B}_* \to \boldsymbol{A}_*$ by $f_*(x) = f^{-1}(x)$ for all $x \in B_*$. Then f_* is a continuous p-morphism and $(-)_*$ assigns f_* to f.

The functor $(-)^*$ assigns to an Esakia space X the Heyting algebra X^* of clopen upsets of X. If $\alpha \colon X \to Y$ is a continuous p-morphism, define $\alpha^* \colon Y^* \to X^*$ by $\alpha^*(U) = \alpha^{-1}(U)$ for all $U \in Y^*$. Then α^* is a Heyting homomorphism and $(-)^*$ assigns α^* to α.

The topology of a finite Esakia space is discrete (since it is Hausdorff). Therefore, the full subcategory of ES consisting of finite Esakia spaces is isomorphic to the category of finite posets and p-morphisms between them. Consequently, in the finite case, Esakia duality restricts to the following [14, 17]:

Theorem 2.2 (**Finite Esakia Duality**). *The category of finite Heyting algebras and Heyting homomorphisms is dually equivalent to the category of finite posets and p-morphisms between them.*

In view of Esakia duality, we can define the notion of validity for Esakia spaces as follows. We say that a formula φ is *valid* in an Esakia space X, and write $X \vDash \varphi$, when it is valid in the Heyting algebra X^*. This allows us to associate an si-logic with each class of Esakia spaces (resp. Heyting algebras or posets) as follows.

Definition 2.3. Let K be a class of Esakia spaces (resp. Heyting algebras or posets). The *logic of* K, in symbols $\mathsf{Log}(K)$, is the set of formulas valid in each member of K.

Notice that $\mathsf{Log}(K)$ is always an si-logic. While every si-logic has the form $\mathsf{Log}(K)$ for some class of Esakia spaces (resp. Heyting algebras), the logics of the form $\mathsf{Log}(K)$ for a class K of posets are precisely the Kripke complete ones.

We conclude this preliminary section by a brief dual description of homomorphic images and subalgebras of Heyting algebras. Henceforth, we will freely use these results. To this end, we recall that if $\alpha \colon X \to Y$ is a p-morphism between posets, the map $\alpha^{-1} \colon \mathsf{Up}(Y) \to \mathsf{Up}(X)$ is a Heyting homomorphism that, moreover, is *complete* (i.e., it preserves arbitrary meets and joins). For part (1) of the next result see [17, Lem. 3.3.13(3)], and for part (2) see [12, Thms. 3.4, 3.5, 4.6].

Theorem 2.4. *The following conditions hold.*

(1) *Let X and Y be Esakia spaces, $\alpha \colon X \to Y$ a continuous p-morphism, and $\alpha^{-1} \colon Y^* \to X^*$ the corresponding Heyting homomorphism. Then α^{-1} is one-to-one iff α is onto, and α^{-1} is onto iff α is one-to-one.*

(2) *Let X and Y be posets, $\alpha \colon X \to Y$ a p-morphism, and $\alpha^{-1} \colon \mathsf{Up}(Y) \to \mathsf{Up}(X)$ the corresponding complete Heyting homomorphism. Then α^{-1} is one-to-one iff α is onto, and α^{-1} is onto iff α is one-to-one.*

A closed upset of an Esakia space is an Esakia space (see, e.g., [17, Lem. 3.4.11]). Since one-to-one (continuous) p-morphisms correspond to (closed) upsets, we obtain the following characterization of quotients. For part (1) see [17, Thm. 3.4.16], and for part (2) see [12, Thms. 3.4, 3.5].

Corollary 2.5. *The following conditions hold.*

(1) *For an Esakia space X, the map $U \mapsto U^*$ is a bijection between the closed upsets of X and the quotients of X^*.*
(2) *For a poset X, the map $U \mapsto \mathsf{Up}(U)$ is a bijection between the upsets of X and the complete quotients of $\mathsf{Up}(X)$.*

We next describe the kernels of onto (continuous) p-morphisms. To this end, given a binary relation R on a set X and $U \subseteq X$, we let

$$R(U) = \{x \in X : \langle y, x \rangle \in R \text{ for some } y \in U\}.$$

If R is an equivalence relation, then $R(U) = U$ iff U is a union of equivalence classes of R. In such a case, we say that U is *R-saturated*.

Definition 2.6.

(1) Let X be an Esakia space. An *Esakia partition* (or *E-partition* for short) of X is an equivalence relation R on X satisfying the following conditions:
  (a) If $\langle x, y \rangle \in R$ and $x \leqslant z$, then there is $u \in X$ such that $y \leqslant u$ and $\langle z, u \rangle \in R$;
  (b) If $\langle x, y \rangle \notin R$, then there is an R-saturated clopen upset U such that $x \in U$ and $y \notin U$.
(2) Let X be a poset. An *E-partition* of X is an equivalence relation R on X satisfying Condition (1a) and the following version of Condition (1b):
  (b′) If $\langle x, y \rangle \notin R$, then there is an R-saturated upset U such that $x \in U$ and $y \notin U$.

Let X be an Esakia space or a poset. If R is an E-partition of X, we define a partial order $\leqslant_R$ on X/R as follows for every $x, y \in X$:

$$[x] \leqslant_R [y] \iff \text{ there are } x' \in [x] \text{ and } y' \in [y] \text{ such that } x' \leqslant y'.$$

Since R is an E-partition, the partial order $\leqslant_R$ is well defined and the map $x \mapsto [x]$ is a p-morphism from X to X/R. Furthermore, when X is an Esakia space, the poset X/R endowed with the quotient topology (i.e., the open sets of X/R are the R-saturated open sets of X) is an Esakia space and the map $x \mapsto [x]$ is a continuous p-morphism.

A subalgebra $\boldsymbol{A}$ of a complete Heyting algebra $\boldsymbol{B}$ is called *complete* when $\boldsymbol{A}$ is also a complete sublattice of $\boldsymbol{B}$. Since E-partitions are exactly the kernels of (continuous) p-morphisms, from Theorem 2.4 we deduce:

Corollary 2.7.

(1) *For an Esakia space X, the map $R \mapsto X/R$ is a bijection between the E-partitions of X and the subalgebras of X^*.*
(2) *For a poset X, the map $R \mapsto X/R$ is a bejection between the E-partitions of X and the complete subalgebras of $\mathsf{Up}(X)$.*

For part (1) of the above result see [6, Cor. 2.3.1], and for part (2) see [12, Thm. 4.6].

3. Degrees of the finite model property

We denote the set of posets validating an si-logic L by $\mathsf{Fr}(\mathsf{L})$. The *degree of incompleteness* of L is the number of si-logics L' such that $\mathsf{Fr}(\mathsf{L}) = \mathsf{Fr}(\mathsf{L}')$. In this paper we are concerned with the degree of fmp. Thus, we restrict our attention to finite posets and let $\mathsf{Fin}(\mathsf{L})$ be the set of finite members of $\mathsf{Fr}(\mathsf{L})$.

Definition 3.1. Let L be an si-logic.

(1) The *fmp span* $\mathrm{fmp}(\mathsf{L})$ of L is the set of si-logics L' such that $\mathsf{Fin}(\mathsf{L}') = \mathsf{Fin}(\mathsf{L})$.
(2) The *degree of fmp* $\deg(\mathsf{L})$ of L is the cardinality of $\mathrm{fmp}(\mathsf{L})$.

We call a poset X *rooted* if there is $x \in X$ such that $X = {\uparrow}x$. Such an x is clearly unique and we call it the *root* of X. Given an si-logic L, we denote the class of the rooted members of $\mathsf{Fin}(\mathsf{L})$ by $\mathsf{RFin}(\mathsf{L})$. Notice that, for each pair L and L' of si-logics, we have

$$\mathsf{Fin}(\mathsf{L}) = \mathsf{Fin}(\mathsf{L}') \;\text{ iff }\; \mathsf{RFin}(\mathsf{L}) = \mathsf{RFin}(\mathsf{L}').$$

To see this, suppose that $\mathsf{Fin}(\mathsf{L}) \neq \mathsf{Fin}(\mathsf{L}')$. By symmetry, we may assume that there is a finite poset X validating L and refuting L'. Then there is $x \in X$ such that ${\uparrow}x$ validates L and refutes L'. Consequently, ${\uparrow}x$ is a member of $\mathsf{RFin}(\mathsf{L}) \smallsetminus \mathsf{RFin}(\mathsf{L}')$, as desired. In view of this, the fmp span of an si-logic L is the set of si-logics L' such that $\mathsf{RFin}(\mathsf{L}) = \mathsf{RFin}(\mathsf{L}')$. We will use this fact without further notice.

Since each si-logic L belongs to its own fmp span and there are exactly $2^{\aleph_0}$ si-logics, the obvious lower and upper bounds for $\deg(\mathsf{L})$ are 1 and $2^{\aleph_0}$. The main result of this paper is the Antidichotomy Theorem stating that these restrictions are indeed optimal in that each nonzero cardinal κ such that $\kappa \leqslant \aleph_0$ or $\kappa = 2^{\aleph_0}$ occurs as the degree of fmp of some si-logic. Thus, under CH, every cardinal $1 \leqslant \kappa \leqslant 2^{\aleph_0}$ occurs as the degree of fmp of some si-logic.[1] More precisely, we will prove the following:

Theorem 3.2 (**Antidichotomy Theorem**). *For each nonzero cardinal κ such that $\kappa \leqslant \aleph_0$ or $\kappa = 2^{\aleph_0}$ there is an si-logic L such that $\deg(\mathsf{L}) = \kappa$.*

As we pointed out in the introduction, one of the techniques required to prove this theorem is that of splittings and Jankov formulas. We recall that a pair of elements (a, b) of a lattice L *splits* L if L is the disjoint union of ${\uparrow}a$ and ${\downarrow}b$ [11, Sec. 9.4]. An si-logic L is a *splitting logic* if there is an si-logic M such that the pair (L, M) splits the lattice $\mathrm{Ext}\,\mathsf{IPC}$. An si-logic is *join-splitting* if it is the join in $\mathrm{Ext}\,\mathsf{IPC}$ of a set of splitting si-logics.

Jankov [22] provided an axiomatization of the join-splitting si-logics. We recall that a Heyting algebra $\boldsymbol{A}$ is *subdirectly irreducible* (SI for short) if it has the second largest element (equivalently, the filter $\{1\}$ is completely meet-irreducible in the lattice of filters of $\boldsymbol{A}$). By the Jankov Theorem [22], with each finite SI Heyting algebra $\boldsymbol{A}$ we can associate a formula $\mathcal{J}(\boldsymbol{A})$ (referred to as the *Jankov formula* of $\boldsymbol{A}$) that axiomatizes the least si-logic L such that $\boldsymbol{A} \nvDash \mathsf{L}$:

Theorem 3.3 (**Jankov Theorem**). *An si-logic L is a splitting logic iff there exists a finite SI Heyting algebra $\boldsymbol{A}$ such that $\mathsf{L} = \mathsf{IPC} + \mathcal{J}(\boldsymbol{A})$. Consequently, L is a join-splitting logic iff L is axiomatizable by Jankov formulas.*

The following lemma governs the behavior of Jankov formulas [24]:

Lemma 3.4 (**Jankov Lemma**). *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Heyting algebras with $\boldsymbol{A}$ finite and SI. Then $\boldsymbol{B} \nvDash \mathcal{J}(\boldsymbol{A})$ iff $\boldsymbol{A}$ is a subalgebra of a homomorphic image of $\boldsymbol{B}$.*

It is well known that a Heyting algebra $\boldsymbol{A}$ is SI iff $\boldsymbol{A}_*$ has a root which, moreover, is isolated (see, e.g., [17, Appendix 1.1]). Therefore, the Finite Esakia Duality implies that the finite SI Heyting algebras are those of the form $\mathsf{Up}(X)$ where X is a finite rooted poset. Because of this, given a finite rooted poset X, we denote by $\mathcal{J}(X)$ the Jankov formula of the finite SI Heyting algebra $\mathsf{Up}(X)$. Thus, in view of Theorem 2.4 and Corollary 2.5, the Jankov Lemma can be formulated dually as follows:

Lemma 3.5 (**Dual Jankov Lemma**). *Let X be a finite rooted poset. For every Esakia space Y we have $Y \nvDash \mathcal{J}(X)$ iff X is a continuous p-morphic image of a closed upset of Y.*

Notably, the following variant of the Dual Jankov Lemma for posets holds too [19]:

[1] It is not known whether it is consistent with ZFC that there are si-logics with the degree of fmp κ for $\aleph_0 < \kappa < 2^{\aleph_0}$ (see Problem 1 in the Conclusions).

Lemma 3.6 (**Fine Lemma**). *Let X be a finite rooted poset. For every poset Y we have $Y \nvDash \mathcal{J}(X)$ iff X is a p-morphic image of an upset of Y.*

The next immediate consequence of the Dual Jankov Lemma governs the interaction between Jankov formulas and si-logics.

Corollary 3.7. *For every finite rooted poset X and si-logic L we have $X \vDash \mathsf{L}$ iff $\mathcal{J}(X) \notin \mathsf{L}$.*

We rely on the following folklore result. We provide a full proof of part (2) since we were not able to find one in the literature.

Lemma 3.8. *The following conditions holds.*

1. *Let X be a finite rooted poset and K a class of Esakia spaces. Then $X \vDash \mathsf{Log}(K)$ iff there is $Y \in K$ such that X is a continuous p-morphic image of a closed upset of Y.*
2. *Two si-logics L and L' contain the same Jankov formulas iff $\mathsf{Fin}(\mathsf{L}) = \mathsf{Fin}(\mathsf{L}')$.*

Proof. (1) Immediate from the Dual Jankov Lemma.

(2) First suppose that $\mathsf{Fin}(\mathsf{L}) \neq \mathsf{Fin}(\mathsf{L}')$. Since a poset validates a formula iff each of its principal upsets does, without loss of generality we may assume that there is a finite rooted $X \in \mathsf{Fin}(\mathsf{L}) \smallsetminus \mathsf{Fin}(\mathsf{L}')$. By Corollary 3.7 we have $\mathcal{J}(X) \in \mathsf{L}' \smallsetminus \mathsf{L}$. Conversely, suppose that L and L' do not contain the same Jankov formulas. We may assume without loss of generality that $\mathcal{J}(X) \in \mathsf{L} \smallsetminus \mathsf{L}'$ for a finite rooted poset X. From Corollary 3.7 it follows that $X \in \mathsf{Fin}(\mathsf{L}') \smallsetminus \mathsf{Fin}(\mathsf{L})$. ⊠

In order to describe fmp spans, it is convenient to introduce the following concept.

Definition 3.9. For an si-logic L, define

1. $\mathsf{L}^+ = \mathsf{Log}(\mathsf{Fin}(\mathsf{L}))$;
2. $\mathsf{L}^- = \mathsf{IPC} + \{\mathcal{J}(X) : X \notin \mathsf{Fin}(\mathsf{L})\}$.

Let $[\mathsf{L}^-, \mathsf{L}^+]$ be the interval in the lattice Ext IPC.

Theorem 3.10. *For an si-logic L we have:*

1. $\mathrm{fmp}(\mathsf{L}) = [\mathsf{L}^-, \mathsf{L}^+]$.
2. L^+ *is the only member of* $\mathrm{fmp}(\mathsf{L})$ *that has the fmp.*
3. L^- *is the only member of* $\mathrm{fmp}(\mathsf{L})$ *that is axiomatizable by Jankov formulas.*

Proof. (1) We begin by proving that $\mathsf{L}^- \in \mathrm{fmp}(\mathsf{L})$. By Lemma 3.8(2), it suffices to show that L and L^- contain the same Jankov formulas. In view of Corollary 3.7, $\{\mathcal{J}(X) : X \notin \mathsf{Fin}(\mathsf{L})\}$ is the set of Jankov formulas in L. Since $\mathsf{L}^- = \mathsf{IPC} + \{\mathcal{J}(X) : X \notin \mathsf{Fin}(\mathsf{L})\}$, every Jankov formula in L belongs to L^- and $\mathsf{L}^- \subseteq \mathsf{L}$. The latter implies that every Jankov formula in L^- belongs to L. Thus, $\mathsf{L}^- \in \mathrm{fmp}(\mathsf{L})$, as desired. Since L^- is axiomatized by Jankov formulas, this implies that it is the least element of $\mathrm{fmp}(\mathsf{L})$.

We next prove that L^+ is the greatest logic in $\mathrm{fmp}(\mathsf{L})$. Clearly $\mathsf{Fin}(\mathsf{L}) \subseteq \mathsf{Fin}(\mathsf{L}^+)$ by the definition of L^+. The other inclusion follows from Lemma 3.8(1) and the fact that $\mathsf{Fin}(\mathsf{L})$ is closed under the formation of upsets and p-morphic images. Thus, $\mathsf{Fin}(\mathsf{L}^+) = \mathsf{Fin}(\mathsf{L})$, and so $\mathsf{L}^+ \in \mathrm{fmp}(\mathsf{L})$. Let $\mathsf{L}' \in \mathrm{fmp}(\mathsf{L})$. Then $\mathsf{Fin}(\mathsf{L}') = \mathsf{Fin}(\mathsf{L})$. Since L^+ is the logic of $\mathsf{Fin}(\mathsf{L})$, we conclude that $\mathsf{L}' \subseteq \mathsf{L}^+$. Thus, L^+ is the greatest element of $\mathrm{fmp}(\mathsf{L})$.

It follows from the definition of $\mathrm{fmp}(\mathsf{L})$ that $\mathrm{fmp}(\mathsf{L})$ is an interval in the lattice of si-logics. Together with the fact that L^- and L^+ are the least and greatest elements of $\mathrm{fmp}(\mathsf{L})$, this implies that $\mathrm{fmp}(\mathsf{L}) = [\mathsf{L}^-, \mathsf{L}^+]$.

(2) By definition, L^+ has the fmp. If $\mathsf{L}' \in \mathrm{fmp}(\mathsf{L})$ has the fmp, then L' is the logic of $\mathsf{Fin}(\mathsf{L}')$. But $\mathsf{Fin}(\mathsf{L}') = \mathsf{Fin}(\mathsf{L})$, so $\mathsf{L} = \mathsf{L}^+$. Thus, L^+ is the only member of $\mathrm{fmp}(\mathsf{L})$ with the fmp.

(3) By definition, L^- is axiomatized by Jankov formulas. Let $\mathsf{L}' \in \mathrm{fmp}(\mathsf{L})$ be also axiomatized by Jankov formulas. Since $\mathsf{Fin}(\mathsf{L}') = \mathsf{Fin}(\mathsf{L}) = \mathsf{Fin}(\mathsf{L}^-)$, we can apply Lemma 3.8(2) to obtain

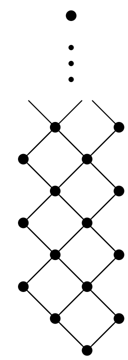

FIGURE 1. The Rieger-Nishimura lattice.

that L' and L^- contain the same Jankov formulas. As both L^- and L' are axiomatized by Jankov formulas, we conclude that $\mathsf{L}^- = \mathsf{L}'$. Thus, L^- is the only member of $\mathrm{fmp}(\mathsf{L})$ axiomatizable by Jankov formulas. ⊠

As a consequence, we obtain a transparent description of the si-logics whose degree of fmp is 1.

Corollary 3.11. *An si-logic* L *has the degree of fmp 1 iff it has the fmp and is axiomatizable by Jankov formulas.*

Proof. First suppose that $\deg(\mathsf{L}) = 1$. Since $\mathsf{L}, \mathsf{L}^-, \mathsf{L}^+ \in \mathrm{fmp}(\mathsf{L})$, this implies that $\mathsf{L} = \mathsf{L}^- = \mathsf{L}^+$. Because L^+ has the fmp and L^- is axiomatizable by Jankov formulas, we conclude that L has the fmp and is axiomatizable by Jankov formulas.

To prove the converse, suppose that L has the fmp and is axiomatizable by Jankov formulas. By Theorem 3.10, the only member of $\mathrm{fmp}(\mathsf{L})$ with the fmp is L^+, and the only member of $\mathrm{fmp}(\mathsf{L})$ that is axiomatizable by Jankov formulas is L^-. Since $\mathsf{L} \in \mathrm{fmp}(\mathsf{L})$, we obtain that $\mathsf{L} = \mathsf{L}^- = \mathsf{L}^+$. Therefore, with an application of Theorem 3.10(1) we conclude that

$$\mathrm{fmp}(\mathsf{L}) = [\mathsf{L}^-, \mathsf{L}^+] = [\mathsf{L}, \mathsf{L}] = \{\mathsf{L}\},$$

and hence $\deg(\mathsf{L}) = 1$. ⊠

Examples of si-logics with the degree of fmp 1 include locally tabular logics. We recall that an si-logic L is *locally tabular* if for each $n < \aleph_0$ the Lindenbaum-Tarski algebra of L in n variables is finite. Clearly each locally tabular logic has the fmp. Moreover, each locally tabular si-logic is axiomatizable by Jankov formulas (see, e.g., [6, Thm. 3.4.24]). Thus, we obtain:

Corollary 3.12. *The degree of fmp of locally tabular si-logics is* 1*.*

Since there are continuum many locally tabular si-logics, the above corollary implies that there are also continuum many si-logics whose degree of fmp is 1. We point out that there are si-logics that are not locally tabular and yet have the degree of fmp 1. For example, IPC is such a logic. More examples will be given in Example 4.11.

4. THE KUZNETSOV-GERCIU LOGIC

In this section we briefly review the si-logic of Kuznetsov and Gerčiu [21, 28]. We start by recalling (see [35, 33]) that the one-generated free Heyting algebra, known as the *Rieger-Nishimura lattice* $\boldsymbol{RN}$, is the Heyting algebra depicted in Figure 1.

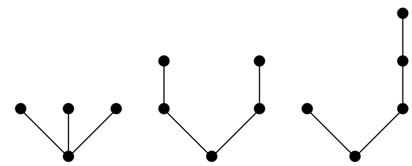

FIGURE 2. The posets P_1, P_2, and P_3.

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Heyting algebras. The *sum* $\boldsymbol{A} + \boldsymbol{B}$ is the Heyting algebra obtained by pasting $\boldsymbol{A}$ *below* $\boldsymbol{B}$ and gluing the top element of $\boldsymbol{A}$ to the bottom element of $\boldsymbol{B}$ [17, Appendix A.9]. As $+$ is clearly associative, there is no ambiguity in writing $\boldsymbol{A}_1 + \cdots + \boldsymbol{A}_n$ for finitely many Heyting algebras $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$, each glued to the next.

Definition 4.1. The *Kuznetsov-Gerčiu* logic KG is the si-logic of all Heyting algebras of the form $\boldsymbol{A}_1 + \cdots + \boldsymbol{A}_n$ where $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$ are one-generated.

We will utilize that KG is a subframe logic. We recall that the theory of subframe modal logics was developed by Fine [20], and that Zakharyaschev [38] studied subframe si-logics. For the present purpose, we concentrate on subframe si-logics.

With each finite rooted poset X we can associate a formula $\beta(X)$ in the language of IPC, called the *subframe formula* of X. Bearing in mind that *frame* and *poset* are synonyms in the context of si-logics, the next result motivates this terminology.

Theorem 4.2 (Fine & Zakharyaschev)**.** *Let X be a finite rooted poset.*

(1) *For every Esakia space Y we have $Y \nvDash \beta(X)$ iff X is a continuous p-morphic image of some clopen $Z \subseteq X$.*
(2) *For every poset Y we have $Y \nvDash \beta(X)$ iff X is a p-morphic image of some $Z \subseteq Y$.*

Proof. See [11, Thm. 9.40(ii)]. Our formulation of the result differs from that of [11, Thm. 9.40(ii)] in that we require Z to be clopen (as opposed to a subframe). The fact that this is harmless follows from the proof of the result (see also [6, Thm. 3.3.16]). ⊠

An si-logic is a *subframe logic* if it is axiomatizable by subframe formulas.

Theorem 4.3 (Fine & Zakharyaschev)**.** *Each subframe si-logic has the fmp.*

Proof. See, e.g., [11, Thm. 11.20]. ⊠

As we pointed out earlier in the section, KG is axiomatizable by subframe formulas (see, e.g., [25] or [6, Thm. 4.3.4]):

Theorem 4.4. KG *is axiomatized by the subframe formulas of the posets in Figure 2.*

For our purposes it is crucial that KG is also axiomatizable by Jankov formulas. For this we first recall the notion of width for posets.

Definition 4.5. Let $1 \leqslant n < \aleph_0$. The *width* of a rooted poset X is n if

$$n = \max\{\kappa : \kappa \text{ is the cardinality of an antichain of } X\}.$$

The *width* of a poset X is n if all principal upsets of X have width $\leqslant n$ and there is a principal upset of width n. The empty poset will be assumed to have width zero.

We next define the notion of width for Heyting algebras.

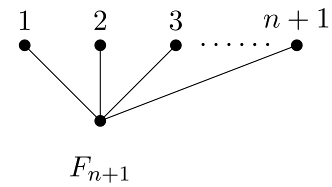

FIGURE 3. The poset F_{n+1}.

Definition 4.6. Let $n < \aleph_0$. A Heyting algebra $\boldsymbol{A}$ has *width* n if $\boldsymbol{A}_*$ has width n. Let

$$\mathsf{W}_n = \{\boldsymbol{A} \in \mathsf{HA} : \boldsymbol{A} \text{ has width } \leqslant n\}.$$

Definition 4.7. For $n < \aleph_0$ let

$$\mathsf{bw}_n = \bigvee_{i=0}^{n} (p_i \to \bigvee_{j \neq i} p_j)$$

and define

$$\mathsf{BW}_n = \mathsf{IPC} + \mathsf{bw}_n.$$

Sobolev [37] proved that a Heyting algebra $\boldsymbol{A}$ validates bw_n iff $\boldsymbol{A} \in \mathsf{W}_n$. Thus, the members of W_n are exactly the algebraic models of BW_n.

Theorem 4.8 (Fine & Zakharyaschev)**.** *Each si-logic* BW_n *is axiomatized by the subframe formula of the poset depicted in Figure 3. Thus, each* BW_n *has the fmp.*

Proof. See [20, 38]. ⊠

We will use the following result of Kracht [26, Prop. 23].

Theorem 4.9 (Kracht)**.** *The logic* BW_2 *is axiomatized by the Jankov formulas of the posets in Figure 4.*

Proof. Kracht proved this result in the setting of normal modal logics extending S4. A natural adaptation of the proof yields the analogous result for si-logics. ⊠

We use Theorem 4.9 to prove that KG is also axiomatizable by Jankov formulas:

Theorem 4.10. KG *is axiomatizable by Jankov formulas.*

Proof. Since a similar result was sketched by Kracht in [26, Sec. D] (again in the setting of normal modal logics extending S4) and because full proofs require lengthy combinatorial arguments, they are moved to the Appendix. ⊠

Example 4.11. The above theorem provides further examples of si-logics that are not locally tabular, but have the degree of fmp 1. Let RN be the logic of the Rieger-Nishimura lattice $\boldsymbol{RN}$. It is well known that both KG and RN have the fmp: for KG this follows from Theorems 4.3 and 4.4, while for RN see, e.g., [3, Thm. 5.35]. Moreover, RN is axiomatizable relative to KG by Jankov formulas [3, Thm. 4.33]. Therefore, by Theorem 4.10, both KG and RN are axiomatizable by Jankov formulas. Thus, by Corollary 3.11, both logics have the degree of fmp 1. Clearly neither logic is locally tabular since $\boldsymbol{RN} \vDash \mathsf{RN}, \mathsf{KG}$.

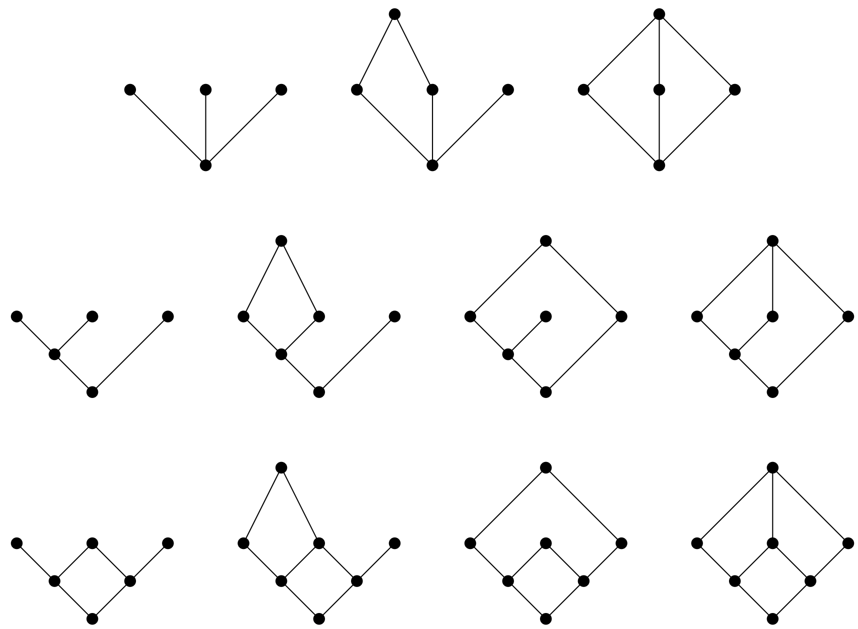

FIGURE 4. The eleven posets whose Jankov formulas axiomatize W_2.

5. The countable case or "anything goes"

In this section we establish the countable case of the Antidichotomy Theorem. We do this by exhibiting for each cardinal $1 \leqslant \kappa \leqslant \aleph_0$, an si-logic L such that $\deg(\mathsf{L}) = \kappa$. As we will see below, L can be chosen to be an extension of KG. More precisely, we will prove the following:

Theorem 5.1. *For each cardinal* $1 \leqslant \kappa \leqslant \aleph_0$ *there exists an si-logic* $\mathsf{L} \supseteq \mathsf{KG}$ *such that* $\deg(\mathsf{L}) = \kappa$.

We will rely on several known facts about the Reieger-Nishimura lattice. We will use [6] as our main reference, but these results can also be found in [3]. The Esakia dual $\mathfrak{L}$ of the Rieger-Nishimura lattice $\boldsymbol{RN}$, often called the *Rieger-Nishimura ladder*, is depicted in Figure 5, where the topology can be described as follows: a subset of $\mathfrak{L}$ is open iff either it misses ω or it is cofinite. In other words, each w_n is an isolated point and ω is the only limit point.

Using the labeling of Figure 5, for each $n \geqslant 0$ let $\mathfrak{L}_n$ be the subspace of $\mathfrak{L}$ whose underlying set is the upset ${\uparrow}w_n$. Let also $\mathbf{1}$ be the one-point Esakia space and $\mathbf{2}$ the Esakia space consisting of two incomparable elements.

For two Esakia spaces X and Y, we denote by $X \oplus Y$ the Esakia space obtained by pasting Y *below* X. If $\boldsymbol{A}_1, \boldsymbol{A}_2$ are Heyting algebras with Esakia duals X_1, X_2, then $X_1 \oplus X_2$ is the dual of the sum $\boldsymbol{A}_2 + \boldsymbol{A}_1$ (see e.g., [6, Thm. 4.1.16]). We will use this construction to produce models of KG.

Definition 5.2. For $m \geqslant 0$ and $n \geqslant 1$, define

$$\mathfrak{C}_m = \underbrace{\mathbf{1} \oplus \cdots \oplus \mathbf{1}}_{m-times} \text{ and } \mathfrak{G}_n = \mathbf{1} \oplus \mathfrak{L} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_n.$$

The poset underlying $\mathfrak{G}_n$ is depicted in Figure 6. Notice that $\mathfrak{G}_n$ is the dual of the sum of Heyting algebras

$$\underbrace{\boldsymbol{B}_2 + \cdots + \boldsymbol{B}_2}_{n-times} + \mathfrak{L}_4^* + \boldsymbol{RN} + \boldsymbol{B}_2,$$

where $\boldsymbol{B}_2$ is the two-element Boolean algebra. Since each of the algebras $\boldsymbol{B}_2, \boldsymbol{RN}$, and $\mathfrak{L}_4^*$ is one-generated, the Heyting algebra in the above display is a model of KG, from which we deduce:

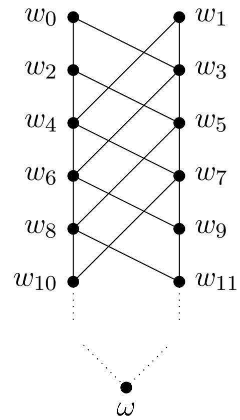

FIGURE 5. The Rieger-Nishimura ladder $\mathfrak{L}$.

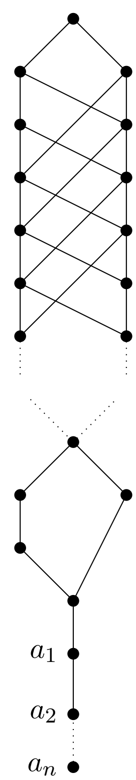

FIGURE 6. The poset underlying $\mathfrak{G}_n$.

Lemma 5.3. *For each $n \geqslant 1$, we have $\mathfrak{G}_n \vDash \mathsf{KG}$.*

We will rely on the following concept:

Definition 5.4. For each $n \geqslant 1$, let $\mathcal{R}_n$ be the class of all finite rooted posets that (when endowed with the discrete topology) are continuous p-morphic images of closed upsets of $\mathfrak{G}_n$.

From Lemma 3.8(1) we deduce:

Lemma 5.5. $\mathcal{R}_n = \mathsf{RFin}(\mathrm{Log}(\mathfrak{G}_n))$.

We will make extensive use of the following class of Esakia spaces:

Definition 5.6. An Esakia space is said to be *simple* if it is a (possibly empty) finite sum of $\mathbf{1}$ and $\mathbf{2}$.

On the other hand, we say that a finite Esakia space X is *complex* if $X \vDash \mathsf{KG}$ and X is not isomorphic to $S \oplus \mathfrak{L}_k$ for any simple Esakia space S and $k \geqslant 0$. The next result is a straightforward adaptation of [6, Thm. 4.5.1]:

Lemma 5.7. *Let X, Y be finite rooted Esakia spaces with Y complex. The following are equivalent.*

(1) *X is a continuous p-morphic image of a closed upset of $\mathbf{1} \oplus \mathfrak{L} \oplus Y$.*
(2) *X is isomorphic to $\mathbf{1} \oplus S \oplus \mathfrak{L}_k$ or it is a continuous p-morphic image of a rooted upset of $\mathbf{1} \oplus S \oplus \mathbf{1} \oplus Y$ for some simple Esakia space S and $k \geqslant 0$.*

As a consequence, we obtain:

Theorem 5.8. *For $n \geqslant 1$, if X is a finite rooted continuous p-morphic image of a closed upset of $\mathfrak{G}_n$, then there are a simple Esakia space S, $k \geqslant 0$, and $m \leqslant n$ such that X is isomorphic to*

$$\mathbf{1} \oplus S \oplus \mathfrak{L}_k \;\; \textit{or} \;\; \mathbf{1} \oplus S \oplus \mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_m.$$

Proof. The Esakia space $Y := \mathfrak{L}_4 \oplus \mathfrak{C}_n$ is not isomorphic to $S \oplus \mathfrak{L}_k$ for any simple Esakia space S and $k \geqslant 0$. On the other hand, since $Y \vDash \mathsf{KG}$, we can apply Lemma 5.7 to obtain that a finite rooted Esakia space X is a continuous p-morphic image of a closed upset of

$$\mathfrak{G}_n = \mathbf{1} \oplus \mathfrak{L} \oplus Y$$

iff it is isomorphic to $\mathbf{1} \oplus S \oplus \mathfrak{L}_k$ or it is a continuous p-morphic image of a rooted upset U of $\mathbf{1} \oplus S \oplus \mathbf{1} \oplus Y$ for some simple Esakia space S and $k \geqslant 0$.

In the former case, there is noting to prove. In the latter case, as U is a rooted upset of $\mathbf{1} \oplus S \oplus \mathbf{1} \oplus Y$, it is of the form

$$\mathbf{1} \oplus S \oplus \mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_m \;\; \text{or} \;\; \mathbf{1} \oplus S' \oplus \mathbf{1}$$

for some $m \leqslant n$ and simple Esakia space S'. Since the continuous p-morphic images of the Esakia spaces in the above display are of the form

$$\mathbf{1} \oplus S'' \oplus \mathbf{1} \;\; \text{or} \;\; \mathbf{1} \oplus S'' \oplus \mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_k$$

for some simple Esakia space S'' and $k \leqslant m \leqslant n$, the result follows. ⊠

As a consequence, we obtain the following characterization of the posets in $\mathcal{R}_n$.

Corollary 5.9. *A finite rooted poset X belongs to $\mathcal{R}_n$ iff X is isomorphic to*

$$\mathbf{1} \oplus S \oplus \mathfrak{L}_k \;\; \textit{or} \;\; \mathbf{1} \oplus S \oplus \mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_m$$

for some simple Esakia space S, $k \geqslant 0$, and $m \leqslant n$.

Definition 5.10. For $n \geqslant 1$ define:

$$\begin{aligned}
\mathsf{L}_0 &= \mathrm{Log}(\mathcal{R}_n);\\
\mathsf{L}_1 &= \mathrm{Log}(\mathcal{R}_n \cup \{\mathfrak{G}_1\});\\
\mathsf{L}_2 &= \mathrm{Log}(\mathcal{R}_n \cup \{\mathfrak{G}_2\});\\
&\;\vdots\\
\mathsf{L}_n &= \mathrm{Log}(\mathcal{R}_n \cup \{\mathfrak{G}_n\}) = \mathrm{Log}(\mathfrak{G}_n),
\end{aligned}$$

where the equality $\mathrm{Log}(\mathcal{R}_n \cup \{\mathfrak{G}_n\}) = \mathrm{Log}(\mathfrak{G}_n)$ holds by Lemma 5.5.

In view of Lemma 5.3 we have:

Lemma 5.11. $\mathsf{L}_0, \dots, \mathsf{L}_n$ *are extensions of* KG.

Our aim is to prove that the fmp span of $\mathrm{Log}(\mathfrak{G}_n)$ is precisely the set $\{\mathsf{L}_0, \dots, \mathsf{L}_n\}$, and hence that the degree of fmp of $\mathrm{Log}(\mathfrak{G}_n)$ is $n+1$. Since $n \geqslant 1$ was arbitrary and we already proved that KG has the degree of fmp 1, this will show that there are extensions of KG with an arbitrary *finite* degree of fmp.

We begin by the following simple observation.

Lemma 5.12. *The following conditions hold.*

(1) *If $m \leqslant n$, then $\mathfrak{G}_m$ is a continuous p-morphic image of $\mathfrak{G}_n$.*
(2) *For each $n \geqslant 1$ the Esakia space $\mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_n$ is a continuous p-morphic image of $\mathfrak{G}_n$.*

Proof. (1) Define $\alpha \colon \mathfrak{G}_n \to \mathfrak{G}_m$ by sending the least $n-m$ points of $\mathfrak{G}_n$ to the root of $\mathfrak{G}_m$ and any other point in $\mathfrak{G}_n$ to its copy in $\mathfrak{G}_m$. It is straightforward to check that α is an onto continuous p-morphism.

(2) Let $X = \mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_n$ and let $\alpha \colon \mathfrak{G}_n \to X$ be the map that sends $\mathbf{1} \oplus \mathfrak{L}$ to the top element of X and is the identity on the rest of $\mathfrak{G}_n$. It is straightforward to check that α is an onto continuous p-morphism. ⊠

We will show that the L_i form a descending chain of logics with the same finite models.

Lemma 5.13. *The following conditions hold.*

(1) $\mathsf{L}_n \subsetneq \cdots \subsetneq \mathsf{L}_0$.
(2) *For all $i \leqslant n$ we have $\mathcal{R}_n = \mathsf{RFin}(\mathsf{L}_i)$.*

Proof. (1) By Lemma 5.12(1) we have the inclusions $\mathsf{L}_n \subseteq \cdots \subseteq \mathsf{L}_0$. To show that these inclusions are proper, for each $0 \leqslant i \leqslant n-1$ consider the subframe formula $\beta(\mathfrak{L}_k)$ for some $k \geqslant 6$ and let

$$\varphi_i \coloneqq \beta(\mathfrak{L}_k) \vee \mathcal{J}(\mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_{i+1}).$$

It is sufficient to show that $\varphi_i \in \mathsf{L}_i \smallsetminus \mathsf{L}_{i+1}$. To see that $\varphi_i \notin \mathsf{L}_{i+1}$, it is enough to show that $\mathfrak{G}_{i+1} \nvDash \varphi_i$. Since $\mathfrak{L}_k$ is a clopen subset of $\mathfrak{G}_{i+1}$, Theorem 4.2(1) implies that $\mathfrak{G}_{i+1} \nvDash \beta(\mathfrak{L}_k)$. Moreover, $\mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_{i+1}$ is a continuous p-morphic image of $\mathfrak{G}_{i+1}$ by Lemma 5.12(2). Thus, $\mathfrak{G}_{i+1} \nvDash \mathcal{J}(\mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_{i+1})$ by the Jankov Lemma. Since a disjunction holds in a rooted Esakia space iff one of the disjuncts does, we conclude that $\mathfrak{G}_{i+1} \nvDash \varphi_i$, and hence $\varphi_i \notin \mathsf{L}_{i+1}$.

To prove that $\varphi_i \in \mathsf{L}_i$, it is sufficient to show that $\mathcal{R}_n \cup \{\mathfrak{G}_i\} \vDash \varphi_i$. From Theorem 5.8 it follows that $\mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_{i+1}$ is not a contionuous p-morphic image of a closed upset of $\mathfrak{G}_i$. Therefore, $\mathfrak{G}_i \vDash \mathcal{J}(\mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_{i+1})$ by the Dual Jankov Lemma, and hence $\mathfrak{G}_i \vDash \varphi_i$. Next, let $X \in \mathcal{R}_n$. If $X \vDash \mathcal{J}(\mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_{i+1})$, then $X \vDash \varphi_i$ as desired. Therefore, we may assume that $X \nvDash \mathcal{J}(\mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_{i+1})$. By the Fine Lemma, $\mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_{i+1}$ is a p-morphic image of an upset of X. Together with Corollary 5.9, this implies that

$$X \cong \mathbf{1} \oplus S \oplus \mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_m,$$

for some simple Esakia space S and $i+1 \leqslant m \leqslant n$ (because the other configuration in Corollary 5.9 cannot have $\mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_{i+1}$ as a p-morphic image of one of its upsets [6, Thm. 4.4.12(1)]). As $k \geqslant 6$, the above display guarantees that $\mathfrak{L}_k$ is not a p-morphic image of a subposet of X. Therefore, $X \vDash \beta(\mathfrak{L}_k)$ by Theorem 4.2(2), and hence $X \vDash \varphi_i$. Thus, $\mathcal{R}_n \cup \{\mathfrak{G}_i\} \vDash \varphi_i$, yielding that the inclusions are proper.

(2) By the definition of L_i we have that $\mathcal{R}_n \subseteq \mathsf{RFin}(\mathsf{L}_i)$. Let $X \in \mathsf{RFin}(\mathsf{L}_i)$. Since $X \nvDash \mathcal{J}(X)$, we have $\mathcal{J}(X) \notin \mathsf{L}_i$. Since $\mathsf{L}_i = \mathrm{Log}(\mathcal{R}_n \cup \{\mathfrak{G}_i\})$, either there exists some $Y \in \mathcal{R}_n$ such that $Y \nvDash \mathcal{J}(X)$ or $\mathfrak{G}_i \nvDash \mathcal{J}(X)$. By the Dual Jankov Lemma, X is a continuous p-morphic image of a closed upset of either some $Y \in \mathcal{R}_n$ or $\mathfrak{G}_i$. In the former case it is clear that $X \in \mathcal{R}_n$. In the latter

case, apply Lemma 5.12(1) to obtain that X is a continuous p-morphic image of a closed upset of a p-morphic image of $\mathfrak{G}_n$. This easily implies that X is also a continuous p-morphic image of a closed upset of $\mathfrak{G}_n$, and hence is a member of $\mathcal{R}_n$. ⊠

Together with Lemma 5.5 this yields that $\mathsf{RFin}(\mathrm{Log}(\mathfrak{G}_n)) = \mathsf{RFin}(\mathsf{L}_i)$ for every $i \leqslant n$. As a consequence, we obtain the following.

Lemma 5.14. $\mathsf{L}_0, \dots, \mathsf{L}_n$ *are* $n+1$ *distinct elements of the fmp span of* $\mathrm{Log}(\mathfrak{G}_n)$.

To show that there are no other logics in the fmp span of $\mathrm{Log}(\mathfrak{G}_n)$, we rely on the following observations.

Lemma 5.15. *Let* X, Y, Z *be Esakia spaces. The following conditions hold.*

(1) $X \oplus \mathfrak{L} \oplus Z$ *is a continuous p-morphic image of* $X \oplus \mathfrak{L} \oplus Y \oplus Z$.
(2) *If* $X, Z \vDash \mathsf{KG}$, *then* $\mathrm{Log}(X \oplus \mathfrak{L} \oplus \mathfrak{L} \oplus Z) = \mathrm{Log}(X \oplus \mathfrak{L} \oplus Z)$.

Proof. (1) The map $\alpha \colon X \oplus \mathfrak{L} \oplus Y \oplus Z \to X \oplus \mathfrak{L} \oplus Z$ that sends the points of Y to the bottom element of $\mathfrak{L}$ and is the identity on the rest of the points is an onto continuous p-morphism.

(2) By (1) we have that $X \oplus \mathfrak{L} \oplus Z$ is a continuous p-morphic image of $X \oplus \mathfrak{L} \oplus \mathfrak{L} \oplus Z$. Thus, $\mathrm{Log}(X \oplus \mathfrak{L} \oplus \mathfrak{L} \oplus Z) \subseteq \mathrm{Log}(X \oplus \mathfrak{L} \oplus Z)$. The reverse inclusion follows from [6, Lem 4.4.9(4)]. ⊠

Consider the si-logic

$$\mathsf{RN.KC} = \mathsf{RN} + (\neg p \vee \neg\neg p).$$

It is well known that a rooted Esakia space X validates RN.KC iff it validates RN and it has a maximum. In other words, $X \vDash \mathsf{RN.KC}$ iff $X \vDash \mathsf{RN}$ and X is of the form $\mathbf{1} \oplus Y$ for some rooted Esakia space Y.

Now, for each si-logic L, let $\mathsf{FGR}(\mathsf{L})$ be the class of Esakia spaces X such that X^* is finitely generated, SI, and $X \vDash \mathsf{L}$. Clearly $\mathsf{FGR}(\mathsf{IPC})$ is the class of all Esakia spaces X with an isolated root such that X^* is finitely generated.

Theorem 5.16. *The following conditions hold for each* $X \in \mathsf{FGR}(\mathsf{IPC})$.

(1) $X \vDash \mathsf{KG}$ *iff* X *is isomorphic to* $X_1 \oplus \cdots \oplus X_n \oplus \mathbf{1}$, *where each* X_i *is isomorphic to* $\mathfrak{L}$ *or to a finite upset of* $\mathfrak{L}$.
(2) $X \vDash \mathsf{RN}$ *iff* X *is isomorphic to* $X_1 \oplus \cdots \oplus X_n \oplus \mathfrak{L}_k$, *where* $k \geqslant 0$ *and each* X_i *is isomorphic to* $\mathbf{1}$, $\mathbf{2}$, *or* $\mathfrak{L}$.
(3) $X \vDash \mathsf{RN.KC}$ *iff* X *is isomorphic to* $\mathbf{1} \oplus X_1 \oplus \cdots \oplus X_n \oplus \mathfrak{L}_k$, *where* $k \geqslant 0$ *and each* X_i *is isomorphic to* $\mathbf{1}$, $\mathbf{2}$, *or* $\mathfrak{L}$
(4) *If* X *is infinite and* $X \vDash \mathsf{RN.KC}$, *then* $\mathrm{Log}(X) = \mathrm{Log}(\mathbf{1} \oplus \mathfrak{L})$ *and* $\mathrm{Log}(X)$ *has the fmp.*

Proof. (1) By [6, Thm. 4.3.9], $X \vDash \mathsf{KG}$ iff X is isomorphic to $X_1 \oplus \cdots \oplus X_n \oplus \mathfrak{L}_k$, where $k \geqslant 0$ and each X_i is isomorphic to $\mathfrak{L}$ or to a finite upset of $\mathfrak{L}$. Next observe that if $k \in \{0, 1\}$, then $\mathfrak{L}_k$ is isomorphic to $\mathbf{1}$; if $k = 2$, then $\mathfrak{L}_k$ is isomorphic to $\mathbf{1} \oplus \mathbf{1}$; and if $k > 2$, then $\mathfrak{L}_k$ is isomorphic to $Y \oplus \mathbf{1}$ where Y is the upset of $\mathfrak{G}$ generated by $\{w_{k-2}, w_{k-3}\}$. Thus, $\mathfrak{L}_k$ is always isomorphic to $Z \oplus \mathbf{1}$ for a (possibly empty) finite upset Z of $\mathfrak{L}$.

(2) This is [6, Thm. 4.4.12(1)].

(3) This follows from (2) by observing that a rooted Esakia space validates $\neg p \vee \neg\neg p$ iff it has the maximum.

(4) We recall that $\mathrm{Log}(\mathbf{1} \oplus \mathfrak{L}) = \mathsf{RN.KC}$ (see [6, Thm. 4.6.4]). Therefore, from $X \vDash \mathsf{RN.KC}$ it follows that $\mathrm{Log}(\mathbf{1} \oplus \mathfrak{L}) \subseteq \mathrm{Log}(X)$. Because X is infinite, it follows from (3) that X is isomorphic to $\mathbf{1} \oplus Y \oplus \mathfrak{L} \oplus Z \oplus \mathfrak{L}_k$, where $k \geqslant 0$ and Y and Z are possibly empty Esakia spaces. By Lemma 5.15(1), $\mathbf{1} \oplus Y \oplus \mathfrak{L}$ is a continuous p-morphic image of $\mathbf{1} \oplus Y \oplus \mathfrak{L} \oplus Z \oplus \mathfrak{L}_k$. Next identify the points in Y with the maximum to obtain that $\mathbf{1} \oplus \mathfrak{L}$ is a continuous p-morphic image of $\mathbf{1} \oplus Y \oplus \mathfrak{L}$. Thus, $\mathbf{1} \oplus \mathfrak{L}$ is also a continuous p-morphic image of X, and hence $\mathrm{Log}(X) \subseteq \mathrm{Log}(\mathbf{1} \oplus \mathfrak{L})$.

This shows that $\text{Log}(X) = \text{Log}(\mathbf{1} \oplus \mathfrak{L})$. Finally, since every extension of RN has the fmp [6, Thm. 4.4.13], we conclude that $\text{Log}(X)$ has the fmp. ⊠

Lemma 5.17. [6, Cor. 4.2.7] *If S is a simple Esakia space, then $S \oplus \mathfrak{L}$ and $S \oplus \mathbf{1}$ are continuous p-morphic images of $\mathfrak{L}$.*

We are ready for the key result of this section.

Theorem 5.18. *Let L be an extension of KG. If $\mathsf{L} \in \text{fmp}(\text{Log}(\mathfrak{G}_n))$, then $\mathsf{L} = \mathsf{L}_i$ for some $i \leqslant n$.*

Proof. Let L be an extension of KG such that $\mathsf{L} \in \text{fmp}(\text{Log}(\mathfrak{G}_n))$. Then $\mathsf{RFin}(\mathsf{L}) = \mathcal{R}_n$ by Lemma 5.5. Since $\mathsf{L}' = \text{Log}(\mathsf{FGR}(\mathsf{L}'))$ for every si-logic L', this implies that

$$\mathsf{L} = \text{Log}(\mathsf{FGR}(\mathsf{L})) = \bigcap\{\text{Log}(\mathcal{R}_n \cup \{X\}) : X \in \mathsf{FGR}(\mathsf{L})\}.$$

Thus, in order to prove that $\mathsf{L} = \mathsf{L}_i$ for some $i \leqslant n$, it is sufficient to show that for each $X \in \mathsf{FGR}(\mathsf{L})$, there is $j \leqslant n$ with $\text{Log}(\mathcal{R}_n \cup \{X\}) = \mathsf{L}_j$. For in this case, we can take i to be the maximum of the j by Lemma 5.13(1).

Let $X \in \mathsf{FGR}(\mathsf{L})$. First suppose that X is finite. Then $X \in \mathcal{R}_n$, which implies that

$$\text{Log}(\mathcal{R}_n \cup \{X\}) = \text{Log}(\mathcal{R}_n) = \mathsf{L}_0.$$

Next suppose that X is infinite. By Theorem 5.16(1) we may assume that $X = X_1 \oplus \cdots \oplus X_m \oplus \mathbf{1}$, where $m \geqslant 1$ and each X_i is either $\mathfrak{L}$ or a finite upset of $\mathfrak{L}$.

Claim 5.19. $X_1 = \mathbf{1}$.

Proof of the Claim. We first show that X has a maximum. If not, then a finite rooted upset of $\mathfrak{L}$ containing two maximal points is a rooted upset of X, so it is in $\mathcal{R}_n$. But by Corollary 5.9, these do not belong to $\mathcal{R}_n$, a contradiction. Thus, we may assume that X_1 is either $\mathbf{1}$ or the two-element chain $\mathfrak{L}_2$. But $\mathfrak{L}_2$ is isomorphic to $\mathbf{1} \oplus \mathbf{1}$. So without the loss of generality, we may assume that $X_1 = \mathbf{1}$ (otherwise we renumber the summands: the new X_2 becomes $\mathbf{1}$, the new X_3 becomes the old X_2, etc.). ⊠

Since X is infinite, one of the X_k must be $\mathfrak{L}$. Let k be the least such. Then

$$X = \mathbf{1} \oplus S_1 \oplus \mathfrak{L} \oplus X_{k+1} \oplus \cdots \oplus X_m \oplus \mathbf{1},$$

where $S_1 = X_2 \oplus \cdots \oplus X_{k-1}$.

Claim 5.20. *S_1 is a simple Esakia space.*

Proof of the Claim. If not, then $S_1 = X_2 \oplus \cdots \oplus X_{k-1}$ has a finite non-simple upset of $\mathfrak{L}$ as one of its summands. Let $x \in \mathfrak{L}$ be such that $x = w_n$ for $n \geqslant 4$ (see the labelling of Figure 5). By viewing $\mathfrak{L}$ as a subframe of $\mathbf{1} \oplus S_1 \oplus \mathfrak{L}$, we take the upset of X generated by x. This is a finite principal upset of X, hence it belongs to $\mathsf{RFin}(\mathsf{L})$. But it contains two nonsimple finite upsets of $\mathfrak{L}$ as its summands, which contradicts Corollary 5.9. Thus, S_1 is a simple Esakia space. ⊠

Iterating the argument described above, we obtain the following:

Claim 5.21. *One of the following conditions holds.*

(1) *There exist simple Esakia spaces $S_1, \ldots, S_p$ such that*

$$X = \mathbf{1} \oplus S_1 \oplus \mathfrak{L} \oplus S_2 \oplus \cdots \oplus \mathfrak{L} \oplus S_p \oplus \mathbf{1};$$

(2) *There exist simple Esakia spaces $S_1, \ldots, S_p$ and finite upsets $X_i, \ldots, X_m$ of $\mathfrak{L}$ with X_i non-simple such that*

$$X = \mathbf{1} \oplus S_1 \oplus \mathfrak{L} \oplus S_2 \oplus \cdots \oplus \mathfrak{L} \oplus S_p \oplus X_i \oplus \cdots \oplus X_m \oplus \mathbf{1}.$$

Suppose first that Condition (1) of Claim 5.21 holds. Then $X \vDash \mathsf{RN.KC}$ by Theorem 5.16(3). By Condition (4) of the same theorem we obtain that $\mathrm{Log}(X)$ has the fmp. Now, from the assumption that $X \in \mathsf{FGR}(\mathsf{L})$ and $\mathsf{RFin}(\mathsf{L}) = \mathcal{R}_n$ it follows that $\mathsf{RFin}(\mathrm{Log}(X)) \subseteq \mathsf{RFin}(\mathsf{L}) = \mathcal{R}_n$. As $\mathrm{Log}(X)$ has the fmp, this implies that $\mathrm{Log}(\mathcal{R}_n \cup \{X\}) = \mathrm{Log}(\mathcal{R}_n) = \mathsf{L}_0$.

It only remains to consider the case where Condition (2) of Claim 5.21 holds. If there exists $k \geqslant 0$ such that

$$X = \mathbf{1} \oplus S_1 \oplus \mathfrak{L} \oplus S_2 \oplus \cdots \oplus \mathfrak{L} \oplus S_p \oplus \mathfrak{L}_k,$$

then we can repeat the argument detailed above. Therefore, we may assume that there is no $k \geqslant 0$ for which the above holds. Under this assumption, we will prove the following:

Claim 5.22. *There exists $k \leqslant n$ such that*

$$X = \mathbf{1} \oplus S_1 \oplus \mathfrak{L} \oplus S_2 \oplus \cdots \oplus \mathfrak{L} \oplus S_p \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_k.$$

Proof of the Claim. Recall from Condition (2) of Claim 5.21 that

$$X = \mathbf{1} \oplus S_1 \oplus \mathfrak{L} \oplus S_2 \oplus \cdots \oplus \mathfrak{L} \oplus S_p \oplus X_i \oplus \cdots \oplus X_m \oplus \mathbf{1},$$

where each S_j is simple, each X_j is a finite upset of $\mathfrak{L}$, and X_i is not simple. We may also assume that each X_j is nonempty.

We have two cases depending on whether there exists some X_j with $j > i$. First suppose that there is no such j. Then

$$X = \mathbf{1} \oplus S_1 \oplus \mathfrak{L} \oplus S_2 \oplus \cdots \oplus \mathfrak{L} \oplus S_p \oplus X_i \oplus \mathbf{1}.$$

Since we assumed that X is not of the form $\mathbf{1} \oplus S_1 \oplus \mathfrak{L} \oplus S_2 \oplus \cdots \oplus \mathfrak{L} \oplus S_p \oplus \mathfrak{L}_k$ for any $k \geqslant 0$, the non-simple finite upset X_i of $\mathfrak{L}$ must be rooted (otherwise $X_i \oplus \mathbf{1} = \mathfrak{L}_k$ for some $k \geqslant 0$). Therefore, $X_i = \mathfrak{L}_k$ for some $k \geqslant 0$. Because X_i is not simple, we must have $k \geqslant 4$. If $k = 4$, then

$$X = \mathbf{1} \oplus S_1 \oplus \mathfrak{L} \oplus S_2 \oplus \cdots \oplus \mathfrak{L} \oplus S_p \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_1$$

and, since $n \geqslant 1$ by assumption, there is nothing to prove. We will show that the case where $k > 4$ cannot happen. Suppose the contrary. Then

$$X = \mathbf{1} \oplus S_1 \oplus \mathfrak{L} \oplus S_2 \oplus \cdots \oplus \mathfrak{L} \oplus S_p \oplus \mathfrak{L}_k \oplus \mathbf{1}.$$

Therefore, $\mathbf{1} \oplus \mathfrak{L}_k \oplus \mathbf{1}$ is a continuous p-morphic image of X obtained by collapsing the top part of X. Since $X \vDash \mathsf{L}$, we have $\mathbf{1} \oplus \mathfrak{L}_k \oplus \mathbf{1} \in \mathsf{RFin}(\mathsf{L}) = \mathcal{R}_n$. Together with $k \geqslant 5$, this contradicts Corollary 5.9.

It only remains to consider the case where there exists some X_j with $j > i$. We begin by proving that

$$X_{i+1} \oplus \cdots \oplus X_m \oplus \mathbf{1} = \mathfrak{C}_q \tag{1}$$

for some $q \geqslant 2$. Suppose the contrary. Then there exists some $j > i$ which contains two incomparable elements. In this case, one of the following is a continuous p-morphic image of X, depending on whether $i + 1 < j$ or $i + 1 = j$:

$$\mathbf{1} \oplus X_i \oplus \mathbf{1} \oplus X_j \oplus \mathbf{1} \text{ or } \mathbf{1} \oplus X_i \oplus X_j \oplus \mathbf{1}.$$

Now, since X_j is a finite upset of $\mathfrak{L}$ containing two incomparable points, X_j must contain the two maximal elements of $\mathfrak{L}$. It is therefore easy to show that if X_j consists of two disjoint points or of the disjoint union of the two element chain and a point, then $\mathbf{2}$ is a continuous p-morphic image of X_j, and in all other cases $\mathbf{2} \oplus \mathbf{1}$ is a continuous p-morphic image of X_j. By collapsing the summand X_j in this manner in the Esakia spaces in the above display, we obtain that one of the following is a continuous p-morphic image of X:

$$\mathbf{1} \oplus X_i \oplus \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{1} \text{ or } \mathbf{1} \oplus X_i \oplus \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1} \text{ or } \mathbf{1} \oplus X_i \oplus \mathbf{2} \oplus \mathbf{1} \text{ or } \mathbf{1} \oplus X_i \oplus \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1}.$$

Since the finite continuous p-morphic images of X belong to $\mathsf{RFin}(\mathsf{L}) = \mathcal{R}_n$, one of the Esakia spaces above should belong to $\mathcal{R}_n$. As none of them is of the form $\mathbf{1} \oplus S \oplus \mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_k$ for some $k \leqslant n$, by Corollary 5.9 one of them must be of the form $\mathbf{1} \oplus S \oplus \mathfrak{L}_k$ for some $k \geqslant 0$. But this is false because X_i is a not a simple finite upset of $\mathfrak{L}$. This establishes Condition (1).

Consequently, we obtain that

$$X = \mathbf{1} \oplus S_1 \oplus \mathfrak{L} \oplus S_2 \oplus \cdots \oplus \mathfrak{L} \oplus S_p \oplus X_i \oplus \mathfrak{C}_q \tag{2}$$

for some $q \geqslant 2$. From the above display it follows that $\mathbf{1} \oplus X_i \oplus \mathfrak{C}_{q-2} \oplus \mathbf{1} \oplus \mathbf{1}$ is a continuous p-morphic image of X, and so belongs to $\mathcal{R}_n$. Since $\mathbf{1} \oplus X_i \oplus \mathfrak{C}_{q-2} \oplus \mathbf{1} \oplus \mathbf{1}$ is not of the form $\mathbf{1} \oplus S \oplus \mathfrak{L}_k$ for any $k \geqslant 0$ (because X_i is a non-simple finite upset of $\mathfrak{L}$), from Corollary 5.9 it follows that

$$\mathbf{1} \oplus X_i \oplus \mathfrak{C}_q = \mathbf{1} \oplus X_i \oplus \mathfrak{C}_{q-2} \oplus \mathbf{1} \oplus \mathbf{1} = \mathbf{1} \oplus S \oplus \mathbf{1} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_k$$

for some $k \leqslant n$. Since X_i is a non-simple finite upset of $\mathfrak{L}$ we have two cases: either ($X_i = \mathfrak{L}_4$ and $q = k \leqslant n$) or (X_i is the upset $\{w_0, w_1, w_2\}$ of $\mathfrak{L}$ and $q = k + 1$). In both cases, $X_i \oplus \mathfrak{C}_q = \mathfrak{L}_4 \oplus \mathfrak{C}_k$. Together with Condition (2) this implies that $X = \mathbf{1} \oplus S_1 \oplus \mathfrak{L} \oplus S_2 \oplus \cdots \oplus \mathfrak{L} \oplus S_p \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_k$. ⊠

Now, recall from Claim 5.22 that there is $k \leqslant n$ such that

$$X = \mathbf{1} \oplus S_1 \oplus \mathfrak{L} \oplus S_2 \oplus \cdots \oplus \mathfrak{L} \oplus S_p \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_k.$$

Therefore, $\mathbf{1} \oplus \mathfrak{L} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_k$ is a continuous p-morphic image of X obtained by identifying $\mathbf{1} \oplus S_1 \oplus \mathfrak{L} \oplus S_2 \oplus \mathfrak{L} \oplus \cdots \oplus S_{p-1}$ with the maximum of X and S_p with the minimum of the remaining copy of $\mathfrak{L}$. Since $\mathfrak{G}_k = \mathbf{1} \oplus \mathfrak{L} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_k$, this implies that $\mathrm{Log}(X) \subseteq \mathrm{Log}(\mathfrak{G}_k)$. To see that the other inclusion also holds, recall from Lemma 5.17 that S_p and each $S_j \oplus \mathfrak{L}$ are continuous p-morphic images of $\mathfrak{L}$. This implies that X is a continuous p-morphic image of

$$Y \coloneqq \mathbf{1} \oplus \underbrace{\mathfrak{L} \oplus \cdots \oplus \mathfrak{L}}_{p-times} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_k.$$

Therefore, $\mathrm{Log}(Y) \subseteq \mathrm{Log}(X)$. Since at least one copy of $\mathfrak{L}$ appears as a summand of X (because X is infinite), and hence the same holds in the above decomposition of Y, we can apply Theorem 5.15(2) to obtain that

$$\mathrm{Log}(Y) = \mathrm{Log}(\mathbf{1} \oplus \mathfrak{L} \oplus \mathfrak{L}_4 \oplus \mathfrak{C}_k) = \mathrm{Log}(\mathfrak{G}_k).$$

Thus, $\mathrm{Log}(\mathfrak{G}_k) = \mathrm{Log}(Y) \subseteq \mathrm{Log}(X)$, and hence $\mathrm{Log}(X) = \mathrm{Log}(\mathfrak{G}_k)$. Since $k \leqslant n$, we conclude that

$$\mathrm{Log}(\mathcal{R}_n \cup \{X\}) = \mathrm{Log}(\mathcal{R}_n \cup \{\mathfrak{G}_k\}) = \mathsf{L}_k. \qquad ⊠$$

As a consequence, we obtain our desired result:

Theorem 5.23. *The fmp span of* $\mathrm{Log}(\mathfrak{G}_n)$ *is* $\{\mathsf{L}_0, \ldots, \mathsf{L}_n\}$ *and its degree of fmp is* $n + 1$.

Proof. In view of Lemma 5.14, in order to prove that

$$\mathrm{fmp}(\mathrm{Log}(\mathfrak{G}_n)) = \{\mathsf{L}_0, \ldots, \mathsf{L}_n\} \text{ and } \deg(\mathrm{Log}(\mathfrak{G}_n)) = n + 1,$$

it suffices to show that if $\mathsf{L} \in \mathrm{fmp}(\mathrm{Log}(\mathfrak{G}_n))$, then $\mathsf{L} = \mathsf{L}_i$ for some $i \leqslant n$.

Let $\mathsf{L} \in \mathrm{fmp}(\mathrm{Log}(\mathfrak{G}_n))$. We first show that L is an extension of KG. In view of Theorem 4.10, it suffices to prove that L contains all the Jankov formulas in KG. Let X be a finite rooted poset such that $\mathcal{J}(X) \in \mathsf{KG}$. Since $\mathsf{Fin}(\mathsf{L}) = \mathsf{Fin}(\mathrm{Log}(\mathfrak{G}_n))$, from Lemma 5.3 it follows that $\mathsf{Fin}(\mathsf{L}) \vDash \mathsf{KG}$. Because $\mathcal{J}(X) \in \mathsf{KG}$, we obtain $\mathsf{Fin}(\mathsf{L}) \vDash \mathcal{J}(X)$. By the Fine Lemma, $X \notin \mathsf{Fin}(\mathsf{L})$. The application of the Dual Jankov Lemma now yields $\mathcal{J}(X) \in \mathsf{L}$, and hence $\mathsf{KG} \subseteq \mathsf{L}$. Therefore, we can invoke Theorem 5.18 to conclude that $\mathsf{L} = \mathsf{L}_i$ for some $i \leqslant n$. ⊠

As we mentioned earlier, this establishes the following.

Theorem 5.24. *For each* $1 \leqslant n < \aleph_0$, *there exists an si-logic* L *with* $\deg(\mathsf{L}) = n$.

To complete the proof of Theorem 5.1, we require the following.

Theorem 5.25. *There exists an si-logic* L *such that* $\text{deg}(\mathsf{L}) = \aleph_0$.

Proof. Define

$$\mathcal{R} = \bigcup_{1 \leqslant n} \mathcal{R}_n \text{ and } \mathsf{L}_0^* = \text{Log}(\mathcal{R}).$$

Clearly L_0^* is an si-logic. We show that its degree of fmp is $\aleph_0$. For this, consider the following extensions of KG, where $n \geqslant 1$:

$$\mathsf{L}_n^* = \text{Log}(\mathcal{R} \cup \{\mathfrak{G}_n\});$$
$$\mathsf{L}_\infty^* = \text{Log}(\mathcal{R} \cup \{\mathfrak{G}_n : 1 \leqslant n\}).$$

A proof similar to that of Lemma 5.13 shows that

$$\mathsf{L}_\infty^* \subsetneq \cdots \subsetneq \mathsf{L}_n^* \subsetneq \cdots \subsetneq \mathsf{L}_1^* \subsetneq \mathsf{L}_0^*, \tag{3}$$

and that all the logics in the above display belong to $\text{fmp}(\mathsf{L}_0^*)$. Therefore, in order to prove that $\text{deg}(\mathsf{L}_0^*) = \aleph_0$, it suffices to show that

$$\text{fmp}(\mathsf{L}_0^*) \subseteq \{\mathsf{L}_n^* : n \geqslant 0\} \cup \{\mathsf{L}_\infty^*\}. \tag{4}$$

An argument similar to the one in the proof of Theorem 5.18 shows that for each extension L of KG such that $\mathsf{RFin}(\mathsf{L}) = \mathcal{R}$ and for each $X \in \mathsf{FGR}(\mathsf{L})$, we have $\text{Log}(\mathcal{R} \cup \{X\}) = \mathsf{L}_j^*$ for some $j \geqslant 0$, where j may possibly be ∞. This, by (3), implies that $\mathsf{L} = \mathsf{L}_p^*$ for

$$p \coloneqq \max(\{q \in \{0, 1, 2, \dots, \infty\} : \text{Log}(\mathcal{R} \cup \{X\}) = \mathsf{L}_q^* \text{ for some } X \in \mathsf{FGR}(\mathsf{L})\}).$$

Therefore, every extension of KG in $\text{fmp}(\mathsf{L}_0^*)$ belongs to $\{\mathsf{L}_n^* : n \geqslant 0\} \cup \{\mathsf{L}_\infty^*\}$. As every member of $\text{fmp}(\mathsf{L}_0^*)$ is an extension of KG (which can be shown as in the proof of Theorem 5.23), we conclude that (4) holds. ⊠

6. The continuum case

In order to complete the proof of the antidichotomy theorem, it suffices to exhibit an si-logic whose degree of fmp is $2^{\aleph_0}$. We will do this by proving the following:

Theorem 6.1. *If* $2 < n < \aleph_0$, *then* $\text{deg}(\mathsf{BW}_n) = 2^{\aleph_0}$.

To establish the above result, let $2 < n < \aleph_0$ and let $\mathbb{Z}^+$ be the set of positive integers. For each $m \in \mathbb{Z}^+$, let X_m be the poset in Figure 7 (we point out that ${\uparrow}b_\omega$ is infinite and ${\downarrow}b_\omega$ is finite).[2]

We define a topology on X_m by letting a subset U of X_m be open provided $b_\omega \in U$ implies U is cofinite. Therefore, b_ω is the only limit point of X_m and all the other points are isolated. It is routine to verify that this turns X_m into an Esakia space, which we also denote by X_m.

For each subset M of $\mathbb{Z}^+$ let L_M be the si-logic of the class of Heyting algebras

$$\{X_m^* : m \in M\} \cup \{\boldsymbol{A} \in \mathsf{W}_n : \boldsymbol{A} \text{ is finite}\}.$$

In order to prove Theorem 6.1, it suffices to establish the following.

Proposition 6.2. *The set* $\{\mathsf{L}_M : M \subseteq \mathbb{Z}^+\}$ *has the cardinality* $2^{\aleph_0}$ *and is a subset of* $\text{fmp}(\mathsf{BW}_n)$.

We split the proof of the above proposition in two parts, first showing that the cardinality of $\{\mathsf{L}_M : M \subseteq \mathbb{Z}^+\}$ is $2^{\aleph_0}$ and then that $\{\mathsf{L}_M : M \subseteq \mathbb{Z}^+\}$ is a subset of $\text{fmp}(\mathsf{BW}_n)$.

In order to prove that the cardinality of $\{\mathsf{L}_M : M \subseteq \mathbb{Z}^+\}$ is $2^{\aleph_0}$, recall that F_{n+1} is the poset depicted in Figure 3. We let $Y_m = {\downarrow}d$. It is enough to establish the following result.

[2]Posets similar to the upper part of X_m have been considered in the literature (see, e.g., [11, p. 319]).

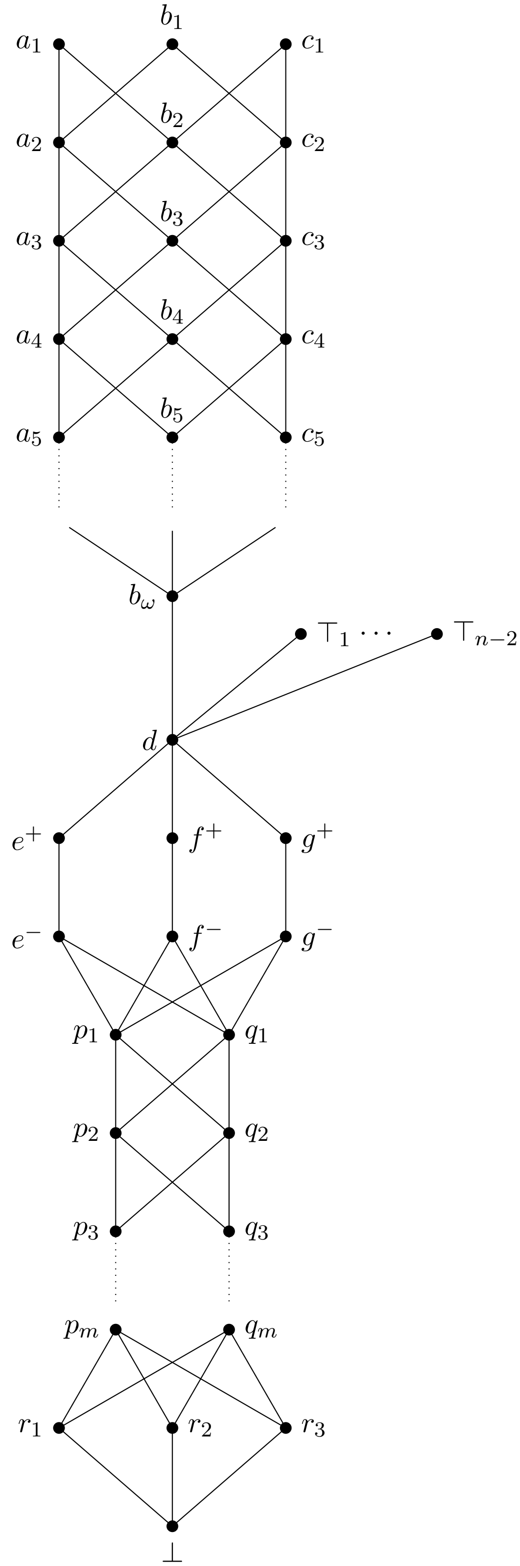

FIGURE 7. The poset X_m.

Lemma 6.3. *Let $M, N \subseteq \mathbb{Z}^+$ and $m \in M \smallsetminus N$. Then $\beta(F_{n+1}) \vee \mathcal{J}(Y_m) \in \mathsf{L}_N \smallsetminus \mathsf{L}_M$.*

Proof. We first show that $\beta(F_{n+1}) \vee \mathcal{J}(Y_m)$ is refuted on X_m. Since X_m is rooted, it suffices to show that neither $\beta(F_{n+1})$ nor $\mathcal{J}(Y_m)$ is valid in X_m. First, as X_m has width $n+1$ and $\beta(F_n)$ axiomatizes the si-logic BW_n of Esakia spaces of width $\leqslant n$, we obtain that $X_m \nvDash \beta(F_{n+1})$. On the other hand, when endowed with the discrete topology, Y_m is a continuous p-morphic image of X_m obtained by collapsing all the elements of $\uparrow y$ to y. By the Dual Jankov Lemma, this yields that $X_m \nvDash \mathcal{J}(Y_m)$. Thus, $\beta(F_{n+1}) \vee \mathcal{J}(Y_m)$ does not belong to L_M.

It remains to prove that $\beta(F_{n+1}) \vee \mathcal{J}(Y_m)$ belongs to L_N. By the definition of L_N, it suffices to show that $\beta(F_{n+1}) \vee \mathcal{J}(Y_m)$ is valid in the class of algebras

$$\{X_k^* : k \in N\} \cup \{\boldsymbol{A} \in \mathsf{W}_n : \boldsymbol{A} \text{ is finite}\}.$$

Since $\beta(F_{n+1})$ axiomatizes BW_n, the disjunction $\beta(F_{n+1}) \vee \mathcal{J}(Y_m)$ holds in the finite members of W_n. Therefore, it only remains to show that $\beta(F_{n+1}) \vee \mathcal{J}(Y_m)$ is valid in $\{X_k : k \in N\}$.

Suppose the contrary, with a view to contradiction. Then there exists $k \in N$ such that $X_k \nvDash \beta(F_{n+1}) \vee \mathcal{J}(Y_m)$. As a consequence, $X_k \nvDash \mathcal{J}(Y_m)$. By the Dual Jankov Lemma there exist a closed upset U of X_k and a surjective continuous p-morphism $\alpha\colon U \to Y_m$ where the poset Y_m is endowed with the discrete topology.

We will show that

$$\alpha^{-1}(\{e^+, e^-, f^+, f^-, g^+, g^-\}) = \{e^+, e^-, f^+, f^-, g^+, g^-\}. \tag{5}$$

To prove the inclusion from left to right, consider $x \in \alpha^{-1}(\{e^+, e^-, f^+, f^-, g^+, g^-\})$. By symmetry, we may assume that $\alpha(x) \in \{e^+, e^-\}$. Since $\alpha\colon U \to Y_m$ is surjective, there are $x_f^+, x_f^-, x_g^+, x_g^- \in U$ such that

$$\alpha(x_f^+) = f^+ \quad \alpha(x_f^-) = f^- \quad \alpha(x_g^+) = g^+ \quad \alpha(x_g^-) = g^-.$$

Furthermore, as α is a p-morphism and e^+, e^-, f^+, f^-, g^+ and g^- are not maximal, the elements $x, x_f^+, x_f^-, x_g^+, x_g^-$ are also not maximal. Now, since α is order preserving, $f(x) \in \{e^+, e^-\}$, and e^+, e^- are incomparable with f^+, f^-, g^+ and g^-, the element x must be incomparable with x_f^+, x_f^-, x_g^+ and x_g^-. By the same token, x_f^+, x_f^- are incomparable with x_g^+, x_g^-. In brief, x is a nonmaximal element that is incomparable with four distinct nonmaximal elements x_f^+, x_f^-, x_g^+ and x_g^- such that x_f^+, x_f^- are incomparable with x_g^+, x_g^-. Examining the pictorial definition of X_m, it is easy to see that

$$x \in \{e^+, e^-, f^+, f^-, g^+, g^-\}.$$

This establishes the inclusion from left to right in (5).

We prove the reverse inclusion by contradiction. By symmetry, we may assume that there exists $x \in \{e^+, e^-, f^+, f^-, g^+, g^-\}$ such that either $x \notin U$ or ($x \in U$ and $\alpha(x) = d$ or $\alpha(x) \leqslant p_1$). By symmetry, we may assume that $x \in \{e^-, e^+\}$. First suppose that $x \in U$ and $\alpha(x) = d$. The monotonicity of α implies that $\alpha(y) = d$ for each $y \in U$ such that $y \geqslant x$. Since $\alpha\colon U \to Y_m$ is surjective, this implies that the restriction $\alpha\colon (U \smallsetminus \uparrow x) \to (Y_m \smallsetminus \{d\})$ is also surjective. But the assumption that $x \in \{e^+, e^-\}$ implies that

$$|U \smallsetminus \uparrow x| \leqslant |U \smallsetminus \uparrow e^+| \leqslant |Y_m| - 2 < |Y_m \smallsetminus \{d\}|,$$

a contradiction to the surjectivity of $\alpha\colon (U \smallsetminus \uparrow x) \to (Y_m \smallsetminus \{d\})$. Therefore, $\alpha(x) \leqslant p_1 \leqslant e^-, f^-$. Since α is a p-morphism and e^-, f^- are non-maximal, there must be two incomparable non-maximal elements $y, z \geqslant x$ such that $\alpha(y) = e^-$ and $\alpha(z) = f^-$. From $x \in \{e^+, e^-\}$ it follows that $a_p \leqslant y, z$ for some $p \in \mathbb{Z}^+$. Moreover, since $\{f^+, f^-\}$ and $\{g^+, g^-\}$ are two-element chains in Y_m that are incomparable with each other and with $\alpha(y) = e^-$, the surjectivity of α implies the existence of two two-element chains in U that are incomparable with each other as well as with y. But the fact that $y \geqslant a_p$ makes this impossible. Therefore, it only remains to consider the case when $x \notin U$. Since U is an upset of X_k, from $x \in \{e^+, e^-\}$ it follows that $U \subseteq {\uparrow}\{e^+, f^-, g^-\}$. Together with the assumption that $\alpha \colon U \to Y_m$ is a surjective p-morphism, this guarantees the existence of some $p \in \mathbb{Z}^+$ and $y \geqslant a_p$ such that $\alpha(y) = e^-$. This allows us to repeat the argument detailed above to obtain the desired contradiction. Thus, the inclusion from right to left in (5) holds.

Now, given a subset V of X_k and $W \subseteq V$, let

$$\uparrow^V W = \{x \in V : x \geqslant y \text{ for some } y \in W\} \text{ and } \downarrow^V W = \{x \in V : x \leqslant y \text{ for some } y \in W\}.$$

Let also $A = \{e^+, e^-, f^+, f^-, g^+, g^-\}$ and notice that $A \subseteq U$ by (5). Moreover, U can be partitioned into the disjoint sets ${\downarrow}^U A$ and ${\uparrow}^U d$. Since $\alpha \colon U \to Y_m$ is order preserving, from (5) it follows that $\alpha({\downarrow}^U A) \subseteq {\downarrow}^{Y_m} A$. Because $d \in Y_m \smallsetminus {\downarrow}^{Y_m} A$ and $\alpha \colon U \to Y_m$ is surjective, we obtain that $d \in \alpha({\uparrow}^U d)$. We show that $\alpha({\uparrow}^U d) \subseteq \{d\}$. Suppose the contrary. As d is the maximum of Y_m and α is order preserving, this implies $\alpha(d) < d$. By the definition of Y_m we obtain that $\alpha(d) \in {\downarrow}^{Y_m} A$. By symmetry, we may assume that $\alpha(d) \leqslant e^+$. Moreover, recall from (5) that $\alpha(A) = A$. Let $x \in A$ be such that $\alpha(x) = f^+$. Since $x \in A$, we have $x \leqslant d$. Because α is order preserving, this implies $f^+ = \alpha(x) \leqslant \alpha(d) \leqslant e^+$, a contradiction. Therefore, we conclude that $\alpha({\uparrow}^U d) \subseteq \{d\}$; that is, $\alpha({\uparrow}^U d) = \{d\}$.

In brief, α sends all elements of ${\downarrow}^U A$ to elements of Y_m that are strictly less than d and all elements of ${\uparrow}^U d$ to d. Since d is an upper bound of A in U, this implies that the restriction $\alpha \colon (\{d\} \cup {\downarrow}^U A) \to Y_m$ is a surjective p-morphism. Because $\{d\} \cup {\downarrow}^U A = Y_k \cap U$, we obtain that the map $\alpha \colon (Y_k \cap U) \to Y_m$ is also a surjective p-morphism. But, inspecting Figure 7, it is easy to see that $m \neq k$ makes this impossible. Hence, we conclude that the disjunction $\beta(F_{n+1}) \vee \mathcal{J}(Y_m)$ is valid in $\{X_k : k \in N\}$. ⊠

The second part of Proposition 6.2 requires to prove that $\{\mathsf{L}_M : M \subseteq \mathbb{Z}^+\}$ is a subset of the fmp span of BW_n, which amounts to the following.

Lemma 6.4. *For each $M \subseteq \mathbb{Z}^+$ and finite poset X,*

$$X \vDash \mathsf{BW}_n \text{ iff } X \vDash \mathsf{L}_M.$$

Proof. Let $M \subseteq \mathbb{Z}^+$. For the left to right implication, if $X \vDash \mathsf{BW}_n$, then X has width $\leqslant n$, and hence $\mathsf{Up}(X)$ is a finite member of W_n. Together with the definition of L_M, this implies that $X \vDash \mathsf{L}_M$.

For the right to left implication, it suffices to prove that if X is a finite poset such that $X \vDash \mathsf{L}_M$, then $X \vDash \mathsf{BW}_n$. Suppose the contrary, with a view to contradiction. Then there exists a finite rooted poset X of width $> n$ such that $X \vDash \mathsf{L}_M$. Since X is finite and rooted, we can consider the Jankov formula $\mathcal{J}(X)$. Now, from $X \vDash \mathsf{L}_M$ it follows that $\mathsf{L}_M \nvDash \mathcal{J}(X)$. The definition of L_M implies that $\mathcal{J}(X)$ fails either in some X_m with $m \in M$ or in some finite member of W_n. Since $\mathsf{W}_n \vDash \mathcal{J}(X)$ because $\mathsf{Up}(X) \notin \mathsf{W}_n$, we conclude that there exists $m \in M$ such that $X_m \nvDash \mathcal{J}(X)$. Therefore, the Dual Jankov Lemma implies that there exist a closed upset U of X_m and an E-partition R of U such that U/R is isomorphic to X. As X is rooted, we may assume that U is also rooted.

Furthermore, as X is not of width $\leqslant n$ and $U/R \cong X$, the set U contains an $(n+1)$-element antichain. An inspection of the pictorial definition of X_m shows that U must contain an antichain of the form $\{\top_1, \dots, \top_{n-2}, a_k, b_k, c_k\}$ for some $k \in \mathbb{Z}^+$. Bearing in mind that U is rooted, this implies that U contains d and, therefore, ${\uparrow}d \subseteq U$ because U is an upset. In brief, U is a rooted upset of X_m such that ${\uparrow}d \subseteq U$ and R is an E-partition of U such that U/R is finite and has an $(n+1)$-element antichain.

Examining again the pictorial definition of X_m, it is easy to see that there must be some $k \in \mathbb{Z}^+$ such that $\{a_k, b_k, c_k\} \subseteq U$ and

$$\{[a_k], [b_k], [c_k], [\top_1], \dots, [\top_{n-2}]\} \tag{6}$$

is an $(n+1)$-element antichain of U/R (notice that $\top_1, \dots, \top_{n-2} \in U$ because ${\uparrow}d \subseteq U$). Consequently, $\{[a_k], [b_k], [c_k]\}$ is a three-element antichain.

Claim 6.5. *There exists the largest $j \in \mathbb{Z}^+$ such that $\{[a_j], [b_j], [c_j]\}$ is a three-element antichain.*

Proof of the Claim. Suppose the contrary and recall that ${\uparrow}b_\omega \subseteq U$. We show that the equivalence class $[b_\omega]$ does not contain any a_i, b_i, or c_i for $i \in \mathbb{Z}^+$. If $[b_\omega]$ contains $x \geqslant b_\omega$, then it also contains the interval $[b_\omega, x]$. In particular, if $[b_\omega]$ contains a_i, b_i or c_i, then $[b_\omega, a_{i+2}] \subseteq [b_\omega]$. This means that for each $t \geqslant i+2$ we have $[a_t] = [b_t] = [c_t] = [b_\omega]$. Hence,

$$p \coloneqq \max\{t \in \mathbb{Z}^+ : t < i+2 \text{ and } \{[a_t], [b_t], [c_t]\} \text{ is a three-element antichain}\}$$

exists (because $k \leqslant i+2$ and $\{[a_k], [b_k], [c_k]\}$ is a three-element antichain) and is the largest positive integer t such that $\{[a_t], [b_t], [c_t]\}$ is a three-element antichain. The obtained contradiction proves that $[b_\omega]$ does not contain any of a_i, b_i or c_i.

As a consequence,

$$[b_\omega] \subseteq \{b_\omega, \top_1, \dots, \top_{n-2}\} \cup {\downarrow}d,$$

whence $[b_\omega]$ is finite. Now, recall that $U/R \cong X$ and that the topology of X is discrete because X is finite. Therefore, $[b_\omega]$ is an isolated point of U/R. Since the map $x \mapsto [x]$ is a continuous p-morphism from U to U/R, it follows that $[b_\omega]$ is a clopen subset of U. But since U is a closed upset of X_m containing ${\uparrow}b_\omega$, the definition of the topology of X_m guarantees that $[b_\omega]$ must contain infinitely many elements of ${\uparrow}b_\omega$. Therefore, $[b_\omega]$ is infinite, a contradiction. ⊠

Let j be the largest positive integer such that $\{[a_j], [b_j], [c_j]\}$ is a three element-antichain, which exists by the Claim. Then $\{[a_{j+1}], [b_{j+1}], [c_{j+1}]\}$ is not a three element antichain. By symmetry, we may assume that

$$[a_{j+1}] \leqslant [b_{j+1}].$$

From $b_{j+1} \leqslant c_j$ it follows that $[b_{j+1}] \leqslant [c_j]$, and so $[a_{j+1}] \leqslant [c_j]$. Therefore, there exist $x \in [a_{j+1}]$ and $y \in [c_j]$ such that $x \leqslant y$. Since $[a_{j+1}] = [x]$ and $x \leqslant y$, the definition of an E-partition guarantees the existence of some $z \in U$ such that $a_{j+1} \leqslant z$ and $[z] = [y]$. Since $[y] = [c_j]$, we obtain

$$a_{j+1} \leqslant z \text{ and } [z] = [c_j].$$

Notice that every element of ${\uparrow}a_{j+1}$ is comparable with a_j or b_j. In particular, z must be comparable with a_j or b_j. Together with $[z] = [c_j]$, this implies that $[c_j]$ is comparable with $[a_j]$ or $[b_j]$. But this contradicts the assumption that $\{[a_j], [b_j], [c_j]\}$ is a three element antichain. ⊠

From Lemmas 6.3 and 6.4 it follows that Proposition 6.2 holds. Therefore, $\deg(\mathsf{BW}_n) = 2^{\aleph_0}$. Since we proved this equality for an arbitrary $2 < n < \aleph_0$, this establishes Theorem 6.1. Together with Theorem 5.1, this concludes the proof of the Antidichotomy Theorem.

We close this section with an observation about the logics BW_n.

Proposition 6.6. *For each $n < \aleph_0$, the logic BW_n can be axiomatized by Jankov formulas iff $n \leqslant 2$.*

Proof. Since BW_0 is the trivial variety, it is axiomatizable by the Jankov formula of the two-element Boolean algebra. Also, since BW_1 is the Gödel-Dummett logic [13], it is well known that BW_1 is axiomatizable by the Jankov formulas of the posets in Figure 8 (see, e.g., [2, Thm. 4.23(4)]).

If $n = 2$, then it follows from Theorem 4.9 that BW_2 is also axiomatizable by Jankov formulas. Finally, let $n > 2$. By Theorem 4.8, BW_n has the fmp. Since BW_n has the degree of fmp $2^{\aleph_0}$ by

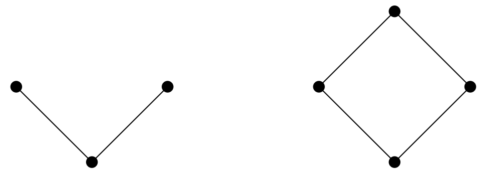

FIGURE 8. The two posets whose Jankov formulas axiomatize BW_1.

Theorem 6.1, we can use Corollary 3.11 to deduce that BW_n cannot be axiomatized by Jankov formulas. ⊠

In view of the Antidichotomy Theorem, every nonzero cardinal κ such that $\kappa \leqslant \aleph_0$ or $\kappa = 2^{\aleph_0}$ may occur as the degree of fmp of some si-logic. An opposite scenario appears if we restrict our attention to the logics BW_n.

Theorem 6.7 (**Width Dichotomy Theorem**). *For each $n < \aleph_0$, we have*

$$\deg(\mathsf{BW}_n) = \begin{cases} 1 & \text{if } n \leqslant 2; \\ 2^{\aleph_0} & \text{if } n > 2. \end{cases}$$

Proof. By Theorem 4.8, each BW_n has the fmp. Therefore, from Proposition 6.6 and Corollary 3.11 it follows that if $n \leqslant 2$, then $\deg(\mathsf{BW}_n) = 1$. The case where $n > 2$ is a consequence of Theorem 6.1. ⊠

7. Degrees of fmp for modal logics

In this section we investigate the degree of fmp for normal extensions of some prominent modal logics. To this end, we denote the class of Kripke frames (resp. finite Kripke frames) validating a normal modal logic L by $\mathsf{Fr}(\mathsf{L})$ (resp. $\mathsf{Fin}(\mathsf{L})$). The *degree of incompleteness* (resp. *the degree of fmp*) of L is the number of normal modal logics L' such that $\mathsf{Fr}(\mathsf{L}) = \mathsf{Fr}(\mathsf{L}')$ (resp. $\mathsf{Fin}(\mathsf{L}) = \mathsf{Fin}(\mathsf{L}')$).

As a consequence of the Blok Dichotomy Theorem, we obtain a dichotomy theorem for the degree of fmp of normal extensions of the basic modal logic K.

Theorem 7.1 (**FMP Dichotomy Theorem**). *The degree of the fmp of a normal modal logic L is either 1 or $2^{\aleph_0}$.*

Proof. Let L be a normal modal logic. By Blok Dichotomy Theorem, its degree of incompleteness is either 1 or $2^{\aleph_0}$. First suppose that the degree of incompleteness of L is $2^{\aleph_0}$. Then there are $2^{\aleph_0}$ normal modal logics L' such that $\mathsf{Fr}(\mathsf{L}') = \mathsf{Fr}(\mathsf{L})$. Since $\mathsf{Fin}(\mathsf{L}) \subseteq \mathsf{Fr}(\mathsf{L})$, it follows that $\mathsf{Fin}(\mathsf{L}') = \mathsf{Fin}(\mathsf{L})$. Thus, the degree of fmp of L is also $2^{\aleph_0}$.

Next suppose that the degree of incompleteness of L is 1. Then L is a join-splitting logic (see, e.g., [11, Thm. 10.59]), and hence L has the fmp (see, e.g., [11, Thm. 10.54]).

Recall that a Kripke frame X is said to be *cycle free* if there is no path of length $n > 0$ from a point of X to itself. For each finite rooted cycle-free Kripke frame X we denote by $\mathcal{J}(X)$ an analogue of the Jankov formula in the language of modal logic [11, p. 362]. The join-splitting normal modal logics are precisely those axiomatized by formulas of the form $\mathcal{J}(X)$ where X is a finite rooted cycle free Kripke frame (see, e.g., [11, Thm. 10.53]). In particular, since L is a join-splitting logic, $\mathsf{L} = \mathsf{K} + \{\mathcal{J}(X_i) : i \in I\}$ for some set $\{X_i : i \in I\}$ of finite rooted cycle free Kripke frames.

Let L' be a normal modal logic such that $\mathsf{Fin}(\mathsf{L}) = \mathsf{Fin}(\mathsf{L}')$. We show that $\mathsf{L} = \mathsf{L}'$. Since L has the fmp, from $\mathsf{Fin}(\mathsf{L}) = \mathsf{Fin}(\mathsf{L}')$ it follows that $\mathsf{L}' \subseteq \mathsf{L}$. On the other hand, the modal analogue of

Lemma 3.8(2) yields that $\mathcal{J}(X_i) \in \mathsf{L}'$ for each $i \in I$. Therefore, $\mathsf{L} \subseteq \mathsf{L}'$, and hence $\mathsf{L} = \mathsf{L}'$. Thus, the dichotomy theorem holds for degrees of fmp. ⊠

However, the situation changes dramatically if we relativize the notion of the degree of fmp to stronger normal modal logics. Following the terminology of [11], given a normal modal logic L, let $\mathrm{Next}\,\mathsf{L}$ be the lattice of normal extensions of L.

Definition 7.2. For a normal extension L of a normal modal logic M, let

$$\mathrm{fmp}_{\mathsf{M}}(\mathsf{L}) = \{\mathsf{L}' \in \mathrm{Next}\,\mathsf{M} : \mathsf{Fin}(\mathsf{L}') = \mathsf{Fin}(\mathsf{L})\};$$
$$\deg_{\mathsf{M}}(\mathsf{L}) = |\mathrm{fmp}_{\mathsf{M}}(\mathsf{L})|.$$

Recall that the *Grzegorczyk logic* Grz is the normal extension of $\mathsf{S4}$ by the formula

$$\Box(\Box(p \to \Box p) \to p) \to \Box p)$$

(see, e.g., [11, pp. 74 and 93]).

Theorem 7.3 (**Modal Antidichotomy Theorem**). *Let $\mathsf{M} \subseteq \mathsf{Grz}$ be a normal modal logic with the fmp such that Grz is a join-splitting in $\mathrm{Next}\,\mathsf{M}$. For each nonzero cardinal κ such that $\kappa \leqslant \aleph_0$ or $\kappa = 2^{\aleph_0}$ there is a normal extension L of M with $\deg_{\mathsf{M}}(\mathsf{L}) = \kappa$.*

Before proving the Modal Antidichotomy Theorem, we point out that it holds for $\mathsf{S4}$ and $\mathsf{K4}$. For recall that Grz is a join-splitting in $\mathrm{Next}\,\mathsf{S4}$ [39, Exmp. 1.11] and that $\mathsf{S4}$ is a join-splitting in $\mathrm{Next}\,\mathsf{K4}$ [11, Exmp. 10.48]. Consequently, Grz is also a join-splitting in $\mathrm{Next}\,\mathsf{K4}$. Since both $\mathsf{S4}$ and $\mathsf{K4}$ have the fmp, we obtain that the modal antidichotomy theorem holds in both $\mathrm{Next}\,\mathsf{S4}$ and $\mathrm{Next}\,\mathsf{K4}$:

Corollary 7.4. *For each nonzero cardinal κ such that $\kappa \leqslant \aleph_0$ or $\kappa = 2^{\aleph_0}$ there is $\mathsf{L} \in \mathrm{Next}\,\mathsf{S4}$ with*

$$\deg_{\mathsf{K4}}(\mathsf{L}) = \deg_{\mathsf{S4}}(\mathsf{L}) = \kappa.$$

In order to prove the Modal Antidichotomy Theorem, we recall that the *Gödel translation*, associating with each intuitionistic formula φ the modal formula φ^t, is defined recursively as follows:

$$\begin{aligned}
p^t &= \Box p \text{ for each propositional variable } p\\
\bot^t &= \bot\\
(\chi \wedge \psi)^t &= \chi^t \wedge \psi^t\\
(\chi \vee \psi)^t &= \chi^t \vee \psi^t\\
(\chi \to \psi)^t &= \Box(\chi^t \to \psi^t).
\end{aligned}$$

By [32], for each intuitionistic formula φ, we have

$$\varphi \in \mathsf{IPC} \;\text{ iff }\; \varphi^t \in \mathsf{S4}.$$

Let L be an si-logic and M a normal extension of $\mathsf{S4}$. Following the standard terminology (see, e.g., [11, Sec. 9.6]), we say that M is a *modal companion* of L provided for each intuitionistic formula φ, we have

$$\varphi \in \mathsf{L} \;\text{ iff }\; \varphi^t \in \mathsf{M}.$$

Notably, each si-logic L has the least and greatest modal companions, denoted by $\tau(\mathsf{L})$ and $\sigma(\mathsf{L})$. For example, $\tau(\mathsf{IPC}) = \mathsf{S4}$ and $\sigma(\mathsf{IPC}) = \mathsf{Grz}$. More generally, $\tau(\mathsf{L}) = \mathsf{S4} + \{\varphi^t : \varphi \in \mathsf{L}\}$ and $\sigma(\mathsf{L}) = \mathsf{Grz} + \{\varphi^t : \varphi \in \mathsf{L}\}$ (see, e.g., [11, Sec. 9.6]). The latter is a consequence of an important result in modal logic, known as the Blok-Esakia theorem.

Theorem 7.5 (**Blok-Esakia Theorem**). *The map $\sigma \colon \mathrm{Ext}\,\mathsf{IPC} \to \mathrm{Next}\,\mathsf{Grz}$ is an isomorphism.*

Proof. See Blok [8] and Esakia [15, 16]. ⊠

When dealing with the degree of fmp, the following observation will also be useful.

Proposition 7.6. *For every si-logic* L,

$$\mathsf{Fin}(\mathsf{L}) = \mathsf{Fin}(\sigma(\mathsf{L})).$$

Proof. We recall from [17, Cor. 3.5.10] that $\mathsf{Fin}(\mathsf{Grz})$ is the class of all finite posets. This yields the result together with the fact that a finite poset validates an intuitionistic formula φ iff the same poset, when viewed as a Kripke frame, validates the modal formula φ^t. ⊠

Proposition 7.7. *For each nonzero cardinal* κ *such that* $\kappa \leqslant \aleph_0$ *or* $\kappa = 2^{\aleph_0}$ *there is a normal extension* L *of* Grz *with* $\deg_{\mathsf{Grz}}(\mathsf{L}) = \kappa$.

Proof. Consider a nonzero cardinal κ such that $\kappa \leqslant \aleph_0$ or $\kappa = 2^{\aleph_0}$. By the Antidichotomy Theorem, there is an si-logic L such that $\deg(\mathsf{L}) = \kappa$. To complete the proof, it suffices to show that $\deg_{\mathsf{Grz}}(\sigma(\mathsf{L})) = \kappa$. Since $\sigma\colon \mathrm{Ext}\,\mathsf{IPC} \to \mathrm{Next}\,\mathsf{Grz}$ is an isomorphism, it is enough to prove that

$$\mathrm{fmp}_{\mathsf{Grz}}(\sigma(\mathsf{L})) = \{\sigma(\mathsf{L}') : \mathsf{L}' \in \mathrm{fmp}(\mathsf{L})\}.$$

The inclusion from right to left is an immediate consequence of Proposition 7.6. Indeed, if L and L' share the class of finite posets, then $\sigma(\mathsf{L})$ and $\sigma(\mathsf{L}')$ also share the same class of finite Kripke frames by Proposition 7.6.

To prove the other inclusion, let $\mathsf{S} \in \mathrm{fmp}_{\mathsf{Grz}}(\sigma(\mathsf{L}))$. By the Blok-Esakia Theorem, there is a unique $\mathsf{L}' \in \mathrm{Ext}\,\mathsf{IPC}$ such that $\mathsf{S} = \sigma(\mathsf{L}')$. Moreover, $\mathsf{Fin}(\mathsf{S}) = \mathsf{Fin}(\mathsf{L}')$ by Proposition 7.6. Therefore,

$$\mathsf{Fin}(\mathsf{L}) = \mathsf{Fin}(\sigma(\mathsf{L})) = \mathsf{Fin}(\mathsf{S}) = \mathsf{Fin}(\mathsf{L}').$$

Thus, $\mathsf{L}' \in \mathrm{fmp}(\mathsf{L})$, which together with $\sigma(\mathsf{L}') = \mathsf{S}$ yields that $\mathsf{S} \in \{\sigma(\mathsf{L}') : \mathsf{L}' \in \mathrm{fmp}(\mathsf{L})\}$. ⊠

We are now ready to prove the Modal Antidichotomy Theorem.

Proof. Consider a nonzero cardinal κ such that $\kappa \leqslant \aleph_0$ or $\kappa = 2^{\aleph_0}$. By Proposition 7.7, there exists a normal extension L of Grz such that $\deg_{\mathsf{Grz}}(\mathsf{L}) = \kappa$. Therefore, to conclude the proof it suffices to show that

$$\mathrm{fmp}_{\mathsf{M}}(\mathsf{L}) = \mathrm{fmp}_{\mathsf{Grz}}(\mathsf{L}).$$

The inclusion from right to left is obvious because $\mathsf{M} \subseteq \mathsf{Grz}$ by assumption. To prove the other inclusion, consider $\mathsf{L}' \in \mathrm{fmp}_{\mathsf{M}}(\mathsf{L})$. Since $\mathsf{Fin}(\mathsf{L}') = \mathsf{Fin}(\mathsf{L})$, in order to prove that $\mathsf{L}' \in \mathrm{fmp}_{\mathsf{Grz}}(\mathsf{L})$ it remains to show that L' extends Grz.

Since M has the fmp and Grz is a join-splitting in $\mathrm{Next}\,\mathsf{M}$, it follows from a general result of McKenzie [30, Thm. 4.3] that there exists a set K of finite rooted Kripke frames validating M such that Grz is the least normal extension N of M with $\mathsf{Fin}(\mathsf{N}) \cap K = \varnothing$. In particular, since L is an extension of Grz, we have $\mathsf{Fin}(\mathsf{L}) \cap K = \varnothing$. Together with the assumption that $\mathsf{Fin}(\mathsf{L}) = \mathsf{Fin}(\mathsf{L}')$, this yields that $\mathsf{Fin}(\mathsf{L}') \cap K = \varnothing$. As L' is a normal extension of M, we conclude that $\mathsf{Grz} \subseteq \mathsf{L}'$ as desired. ⊠

8. Conclusions

In this paper, we introduced the notion of the degree of fmp for superintuionistic and modal logics in analogy with the classic notion of the degree of incompleteness for these logics. We proved the Antidichitomy Theorem for the degree of fmp for superintuionistic and transitive modal logics. Namely, for every nonzero cardinal κ such that $\kappa \leqslant \aleph_0$ or $\kappa = 2^{\aleph_0}$ there is a superintuitionistic or transitive modal logic L such that the degree of fmp of L is κ. We conclude by discussing possible future research directions that could originate from this work.

(1) By assuming the Continuum Hypothesis (CH), our results show that every nonzero $\kappa \leqslant 2^{\aleph_0}$ is the degree of fmp of some si-logic or transitive modal logic. However, for proving this result the power of CH may not be necessary. We leave it as an open problem whether the assumption of CH can be avoided.
(2) In this paper we determined what cardinalities can be realized as degrees of fmp for superintuitionistic and modal logics. However, it still remains an open problem to characterize the degree of fmp of a given si-logic or an extension of S4 or K4. Note that for an extension L of K, the degree of fmp, as well as the degree of incompleteness of L, is determined by the Blok Dichotomy Theorem: If L is join-splitting, then it is 1; otherwise it is $2^{\aleph_0}$. In analogy with this, we showed that if a logic L has the fmp and is join-splitting, then its degree of fmp is 1. But if these conditions are not met, it is still unclear what is the degree of fmp of L.
(3) The first step to answer (2) would be to determine the degree of fmp of a given extension of KG. In particular, it is still unclear whether the continuum degree of fmp can occur above KG.
(4) We also find it interesting to study the degree of fmp for other prominent deductive systems such as bi-intuionistic logic, tense and temporal logics, and fixpoint logics such as PDL and the modal μ-calculus. In fact, one can define and investigate the degree of fmp for any logic (or a variety of algebras thereof) that has finite models.

More generally, one can apply this perspective to other logically interesting properties. For a given logic L, let $\mathcal{S}$ be a semantics of L. In other words, let $\mathcal{S}$ be a class of models of L (relational, topological, algebraic, etc.). Also, let P be a property of $\mathcal{S}$-models. Then the *P-degree of the $\mathcal{S}$-semantics* is the cardinality of the set of logics L′ such that L and L′ share the same class of $\mathcal{S}$-models satisfying property P. The degree of fmp is then the P-degree of the $\mathcal{S}$-semantics when the $\mathcal{S}$-semantics is Kripke semantics and P is the property of being finite. Note that being finite can be replaced by other properties; for example, by being countable, etc.

Since every si-logic or modal logic L is complete with respect to its algebraic semantics, the P-degree of the $\mathcal{S}$-semantics of each L is always 1 when the $\mathcal{S}$-semantics is the algebraic semantics and P is any property true in each algebraic model. Indeed, in this case two logics have the same P-degree if they have the same class of algebraic models. Every such logic is complete with respect to its algebraic models. Hence, every logic has the P-degree 1. However, if $\mathcal{S}$-semantics is the topological semantics, then the situation changes drastically since it is well known that there exist topologically incomplete modal logics (see, e.g., [36]) and it remains an outstanding open problem whether there exist topologically incomplete si-logics. In a recent paper [5], it was shown that there exist (continuum many) extensions of the bi-intuitionistic logic that are topologically incomplete.

In topological semantics of modal logic, it is customary to interpret $\Diamond$ as topological closure. Under such interpretation, S4 is the least topologically complete modal logic, and the degree of topological fmp coincides with the degree of fmp in NExt S4 since finite topological spaces are in one-to-one correspondence with finite S4-frames. On the other hand, if we interpret $\Diamond$ as topological derivative [31, Appendix I] (the so-called *d-semantics*; see [4]), then it makes sense to investigate the degree of topological fmp (which modal logics have the same class of finite topological models).

In our opinion, the study of P-degrees of $\mathcal{S}$-semantics for non-classical logics is a promising direction for future research.

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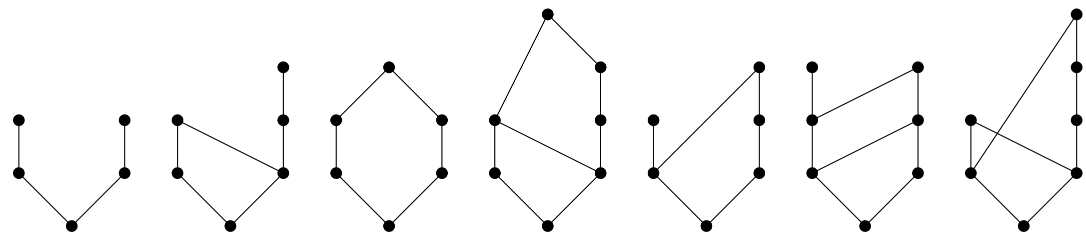

FIGURE 9. The posets $K_1, \ldots, K_7$.

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APPENDIX

The aim of the Appendix is to prove Theorem 4.10 that KG is axiomatizable by Jankov formulas. For this we utilize the following classic result of Fine [19]. For $n \geqslant 1$ we say that an si-logic L is of *width* $\leqslant n$ if each Esakia space X validating L is of width $\leqslant n$ (see Definition 4.5 for the definition of the width of a poset). We call L of *finite width* if there is n such that L is of width $\leqslant n$. Clearly L is of finite width provided $\mathsf{BW}_n \subseteq \mathsf{L}$ for some n.

We recall that a poset is *Noetherian* if it has no infinite strictly ascending chains. We then have (see, e.g., [11, Thm. 10.45]):

Theorem 8.1 (**Fine Completeness Theorem**). *If L is an si-logic of width $\leqslant n$, then there is a class K of rooted Noetherian posets of width $\leqslant n$ such that $\mathsf{L} = \mathrm{Log}(K)$.*

The proof of Theorem 4.10 is based on the following two combinatorial observations.

Lemma 8.2. *A rooted Noetherian poset X of width $\leqslant 2$ validates $\beta(P_2)$ iff it validates the Jankov formulas of the posets in Figure 9.*

Lemma 8.3. *Let X be a rooted Noetherian poset validating $\beta(P_2)$. Then X validates $\beta(P_3)$ iff it validates the Jankov formulas of the posets in Figure 10.*

We point out that the posets K_3 and K_4 are obtained by adding a new top to K_1 and K_2, respectively. Moreover, G_3 and G_4 are obtained in a similar manner from G_2 and G_1.

In order to shorten the proofs of Lemmas 8.2 and 8.3, we use the following equivalent formulation of Condition (1a) of Definition 2.6 (see [7, Rem. 3.1]):

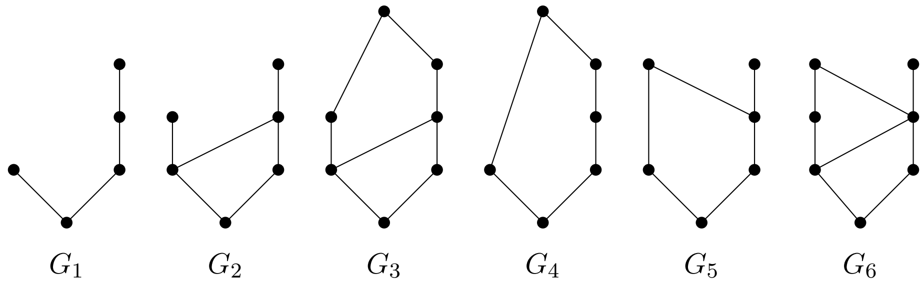

FIGURE 10. The posets $G_1, \dots, G_6$.

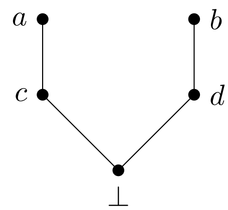

FIGURE 11. The poset P_2 viewed as a subposet of X.

(1a′) Suppose that $\langle x, y\rangle \in R$ and $x, y \in X$ are distinct. If there is $z \geqslant x$ such that $y \nleqslant z$ and $\langle x, z\rangle \notin R$, then there is $u \in X$ such that $y \leqslant u$ and $\langle z, u\rangle \in R$.

We will also use repeatedly that for every poset X and upset U, identifying U into a point is an E-partition on X.

Proof of Lemma 8.2. Let X be a rooted Noetherian poset of width $\leqslant 2$. Suppose first that $X \vDash \beta(P_2)$. Consider a nonnegative integer $i \leqslant 7$. By the Fine Lemma, to show that $X \vDash \mathcal{J}(K_i)$, it suffices to prove that K_i is not a p-morphic image of any upset of X. Suppose the contrary. Then there exists an upset U of X and a surjective p-morphism $\alpha\colon U \to K_i$. As a consequence, K_i validates all the formulas valid in X and, in particular, $\beta(P_2)$. But in view of Theorem 4.2(2) this is false because P_2 is isomorphic to a subposet of K_i, as it can be checked by inspecting the posets in Figure 9.

To prove the converse, assume that X validates the Jankov formulas of the posets $K_1, \dots, K_7$ in Figure 9 and suppose, with a view to contradiction, that $X \nvDash \beta(P_2)$. By Theorem 4.2(2) this implies that P_2 is isomorphic to a p-morphic image of a subposet of X. The definition of a p-morphism and the structure of P_2 imply that actually P_2 is isomorphic to a subposet of X. We name the elements of this subposet as in Figure 11.

As X is Noetherian and every element in the interval $[c, a]$ is incomprable with b and d, we may assume that c is an immediate predecessor of a (otherwise we replace c by a maximal element in $[c, a)$). Similarly, we may assume that d is an immediate predecessor of b. By the same token, we may assume that $\bot$ is maximal in ${\downarrow}c \cap {\downarrow}d$, whence we obtain that for every $x \in X$,

$$(\text{if } \bot < x \leqslant c, \text{ then } x \nleqslant d) \text{ and } (\text{if } \bot < x \leqslant d, \text{ then } x \nleqslant c). \tag{7}$$

Lastly, as the upset of X generated by $\bot$ validates all the formulas valid in X, we may also assume that $\bot$ is the minimum of X (otherwise we replace X by ${\uparrow}\bot$).

Since $({\downarrow}\{a, b\})^c$ is an upset, the following relation is an E-partition of X:

$$R \coloneqq \{\langle x, y\rangle \in X \times X : x = y \text{ or } x, y \notin {\downarrow}\{a, b\}\}.$$

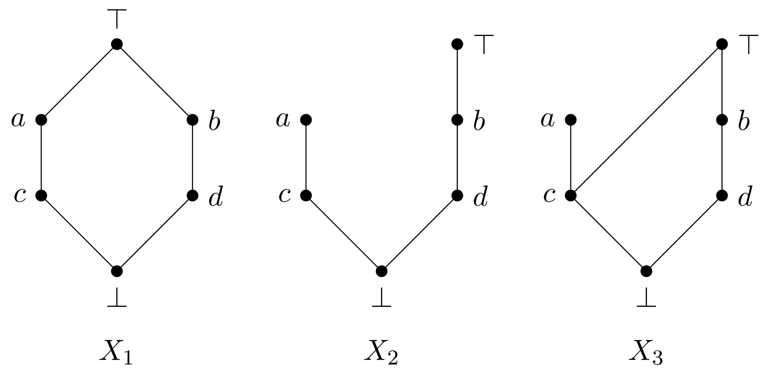

FIGURE 12. The posets X_1, X_2 and X_3.

Notice that the subposet of X/R with the universe $\{[a], [b], [c], [d], [\bot]\}$ is isomorphic to the subposet of X with the universe $\{a, b, c, d, \bot\}$. Therefore, since X/R validates all the formulas valid in X, we may assume that R is the identity relation (otherwise we replace X by X/R). Consequently,

$$\text{either } ({\downarrow}\{a, b\})^c = \varnothing \;\text{ or }\; ({\downarrow}\{a, b\})^c = \{\top\}$$

for some element $\top \in X$.

Notice that if $({\downarrow}\{a, b\})^c = \{\top\}$, then obviously $\top \nleqslant a$ and $\top \nleqslant b$. As X is a rooted poset of width $\leqslant 2$ and a and b are incomparable, by symmetry we may assume that b and $\top$ are comparable which, together with $\top \nleqslant b$, yields $b < \top$. Therefore, one of the following Conditions holds.

(1) $({\downarrow}\{a, b\})^c = \varnothing$;
(2) $a, b < \top$;
(3) a and $\top$ are incomparable and $b < \top$.

If $({\downarrow}\{a, b\})^c = \{\top\}$, the subposet of X with the universe $\{\bot, a, b, c, d, \top\}$ is one of the three X_i depicted in Figure 12. Thus,

$$X = {\downarrow}\{a, b\} \;\text{ or }\; (X = \{\top\} \cup {\downarrow}\{a, b\} \text{ and the poset } \{\bot, a, b, c, d, \top\} \text{ is one of the } X_i). \tag{8}$$

As a and b are incomparable, this implies that for every $x \in X$,

$$\text{if } a < x \text{ or } b < x, \text{ then } x = \top > b. \tag{9}$$

Given a pair y_1, y_2 of elements of a poset Y, we denote by (y_1, y_2) the open interval $\{z \in Y : y_1 < z < y_2\}$. We will prove that

$$X = \begin{cases} \{\bot, a, b, c, d\} \cup (\bot, c) \cup (\bot, d) & \text{if } \top \text{ does not exist} \\ \{\bot, a, b, c, d, \top\} \cup (\bot, c) \cup (\bot, d) & \text{if } \top \text{ exists.} \end{cases} \tag{10}$$

The inclusion from right to left is obvious. To prove the other inclusion, consider some $x \in X$ other than $\top$. In view of Condition (8), we have $x \leqslant a$ or $x \leqslant b$. If $x \in \{a, b\}$, we are done. Therefore, we may assume that

$$\text{either } x < a \;\text{ or }\; x < b.$$

Now, if $x \in {\downarrow}c \cup {\downarrow}d$, we are done because ${\downarrow}c \cup {\downarrow}d \subseteq \{\bot, c, d\} \cup (\bot, c) \cup (\bot, d)$. Consequently, we may assume that $x \notin {\downarrow}\{c, d\}$. Since c and d are incomparable and X is a rooted poset of width $\leqslant 2$, this yields

$$c < x \;\text{ or }\; d < x.$$

As a and c are incomparable with b and d, the two displays above imply that

$$\text{either } c < x < a \ \text{ or } \ d < x < b.$$

But this contradicts the assumption that c (resp. d) is an immediate predecessor of a (resp. b). Hence, Condition (10) holds as desired.

Now, we consider the sets

$$\begin{aligned} Y_1 &= (\bot, c] \smallsetminus ({\downarrow}\top \cup {\downarrow}b); \\ Y_2 &= ((\bot, c] \cap {\downarrow}\top) \smallsetminus {\downarrow}b; \\ Y_3 &= (\bot, c] \cap {\downarrow}b; \\ Y_4 &= (\bot, d] \smallsetminus {\downarrow}a; \\ Y_5 &= (\bot, d] \cap {\downarrow}a. \end{aligned}$$

If $\top$ does not exist, the expression ${\downarrow}\top$ in the above definition should be interpreted as denoting the empty set.

Claim 8.4. *The following relation is an E-partition of X:*

$$S = \{\langle x, y\rangle \in X \times X : x = y \text{ or } x, y \in Y_i \text{ for some } i \leqslant 5\}.$$

Proof of the Claim. Since the various Y_i are pairwise disjoint, S is an equivalence relation on X. To prove that it is also an E-partition, it suffices to show that there are no distinct $x, y \in X$ such that $\langle x, y\rangle \in S$ and there exists $z \in X$ such that $x \leqslant z$ and $y \nleqslant z$ and $\langle x, z\rangle \notin S$ (see Condition (1a′) if necessary). Suppose the contrary, with a view to contradiction. Since x and y are distinct and related by S, we have

$$\bot < x, y \leqslant c \ \text{ or } \ \bot < x, y \leqslant d.$$

Suppose first that $\bot < x, y \leqslant c$. As $d \nleqslant c$, we have $d \nleqslant x, y$. Furthermore, from Condition (7) it follows that $x, y \nleqslant d$. Thus, x and y are incomparable with d. As X is a rooted poset of width $\leqslant 2$, this implies that x and y are comparable. Since by assumption $x \leqslant z$ and $y \nleqslant z$, we conclude that $x < y$. Now, by applying the assumption that X has width $\leqslant 2$ to the fact that y and d are incomparable, we obtain that z is comparable with either y or d. We will prove that z is incomparable with y. On the one hand, by assumption $y \nleqslant z$. On the other hand, if $z \leqslant y$, then we would have $x \leqslant z \leqslant y$, because $x \leqslant z$ by assumption. Since $\langle x, y\rangle \in S$, the equivalence class $[x]$ contains the interval (x, y). In particular, $\langle x, z\rangle \in S$, a contradiction. Thus, we conclude that $z \nleqslant y$. Consequently, y and z are incomparable, which in turn means that d and z are comparable. Since x and d and incomparable and $x \leqslant z$, this means that $d < z$. A similar argument shows that if $\bot < x, y \leqslant d$, then $c < z$. Thus, we obtain that

$$(\bot < x, y \leqslant c \text{ and } d < z) \ \text{ or } \ (\bot < x, y \leqslant d \text{ and } c < z).$$

We need to prove that both cases lead to a contradiction. First suppose that $\bot < x, y \leqslant c$ and $d < z$. Since $d < z$, by Condition (10) we obtain that $z \in \{b, \top\}$. Recall that $x, y \in Y_i$ for some $i \leqslant 5$ because x and y are different and related by S. Furthermore, as $\bot < x \leqslant c$, Condition (7) implies that Y_i is Y_1, Y_2, or Y_3. We have two cases: either $z = b$ or $z = \top$. If $z = \top$, then $Y_i \neq Y_1$ because $x \in Y_i$ and $x \leqslant z = \top$ and $Y_1 \subseteq ({\downarrow}\top)^c$. Therefore, Y_i is Y_2 or Y_3. Since $Y_2 \cup Y_3 \subseteq {\downarrow}\top = {\downarrow}z$ and $y \in Y_i$, we obtain $y \leqslant \top = z$, a contradiction. If $z = b$, then $Y_i = Y_3$ because $x \in Y_i$, $x \leqslant z = b$, and $Y_1 \cup Y_2 \subseteq ({\downarrow}b)^c$. As a consequence, $y \in Y_i = Y_3$. Since $Y_3 \subseteq {\downarrow}b = {\downarrow}z$, this implies that $y \leqslant z$, a contradiction.

Next we consider the case where $\bot < x, y \leqslant d$, and $c < z$. Since $c < z$, Condition (10) implies that $z = \top$ or $z = a$. If $z = \top$, then $z \geqslant d \geqslant y$, a contradiction. Suppose that $z = a$. Recall that $x, y \in Y_i$ for some $i \leqslant 5$ because x and y are distinct and related by S. As $\bot < x \leqslant d$, Condition (7) implies that Y_i is Y_4 or Y_5. But, as $x \in Y_i$, $x \leqslant z = a$, and $Y_4 \subseteq ({\downarrow}a)^c$, we must have

$Y_i = Y_5$. Consequently, $y \in Y_i = Y_5 \subseteq {\downarrow}a$. This yields that $y \leqslant a = z$, a contradiction. Hence, we conclude that S is an E-partition of X. ⊠

Claim 8.5. *For every $x, y \in \{\bot, a, b, c, d, \top\}$,*

$$x \leqslant y \Longleftrightarrow [x] \leqslant [y].$$

Proof of the Claim. The implication from left to right is obvious. To prove the other implication, suppose that $[x] \leqslant [y]$. The definition of S guarantees that

$$[\bot] = \{\bot\} \qquad [\top] = \{\top\} \qquad [a] = \{a\} \qquad [b] = \{b\} \qquad [d] = (\bot, d] \smallsetminus {\downarrow}a$$

and

$$[c] = \begin{cases} (\bot, c] \smallsetminus {\downarrow}b & \text{if } \top \text{ does not exist or it exists and } c \leqslant \top \\ (\bot, c] \smallsetminus {\downarrow}\top & \text{if } \top \text{ exists and } c \nleqslant \top. \end{cases}$$

Consequently, y is the maximum of the equivalence class $[y]$. Therefore, the assumption that $[x] \leqslant [y]$ guarantees the existence of some $x' \in [x]$ such that $x' \leqslant y$. If the equivalence class $[x]$ is a singleton, then $x \leqslant y$ and we are done. Otherwise, in view of the above displays, x is either c or d. As $\bot$ is maximal in ${\downarrow}c \cap {\downarrow}d$, the above displays guarantee that $[c]$ and $[d]$ are incomparable. Then we may assume that $x \in \{c, d\}$ and $y \in \{\bot, \top, a, b\}$.

We begin by the case where $x = d$. If $y \in \{b, \top\}$, then clearly $x \leqslant y$. Therefore, we consider the case where $y \in \{\bot, a\}$. As the set $[x] = [d] = (\bot, d] \smallsetminus {\downarrow}a$ does not contain any element below $\bot$ or a, this case never happens and we are done.

Then we turn our attention to the case where $x = c$. If $y = a$, then clearly $x \leqslant y$. Moreover, $[x] = [c] \subseteq (\bot, c]$, and hence $[x]$ does not contain any element below $\bot$ or b. Together with $[\bot] = \{\bot\}$ and $[b] = \{b\}$, this yields that $y \notin \{b, \bot\}$. It only remains to consider the case where $y = \top$. But the above display guarantees that if $[c]$ contains an element below $\top$, then $c \leqslant \top$, and hence $x \leqslant y$ as desired. ⊠

Together with the fact that X/S validates all the formulas valid in X, Claim 8.5 allows us to assume that S is the identity relation (otherwise we replace X by X/S).

Claim 8.6. *One of the following conditions holds.*

(M1) $(\bot, c) = \varnothing$;
(M2) $(\bot, c) = \{x\}$ *for some x such that* ${\uparrow}x = \{x, a, c\} \cup {\uparrow}b$;
(M3) $\top$ *exists*, $c \nleqslant \top$, *and* $(\bot, c) = \{x\}$ *for some x such that* ${\uparrow}x = \{x, a, c, \top\}$;
(M4) $\top$ *exists*, $c \nleqslant \top$, *and* $(\bot, c) = \{x, y\}$ *for x and y such that* ${\uparrow}y = \{y, a, c, \top\}$ *and* ${\uparrow}x = \{x, y, a, b, c, \top\}$.

Proof of the Claim. First, if $(\bot, c)$ is empty, Condition (M1) holds. Suppose $(\bot, c) \neq \varnothing$. The definition of the various Y_i guarantees that $(\bot, c] \subseteq Y_1 \cup Y_2 \cup Y_3$. Bearing in mind that each Y_i is either empty or a singleton (since S is the identity relation), this implies that $(\bot, c]$ has at most three elements, which in turn means that $(\bot, c)$ has at most two. Furthermore,

$$\{\{x\} : x \in (\bot, c]\} \subseteq \{Y_1, Y_2, Y_3\}. \tag{11}$$

First suppose that $(\bot, c)$ has precisely two elements x and y. By Condition (7), x and y are incomparable with d. As X is a rooted poset of width $\leqslant 2$, this yields that both x and y must be comparable. Without loss of generality we may assume that $\bot < x < y < c$. From Condition (11) it follows that

$$\{\{x\}, \{y\}, \{c\}\} = \{Y_1, Y_2, Y_3\}.$$

Together with $x < y < c$ and the definition of the various Y_i, this implies that

$$Y_1 = \{c\} \qquad Y_2 = \{y\} \qquad Y_3 = \{x\}.$$

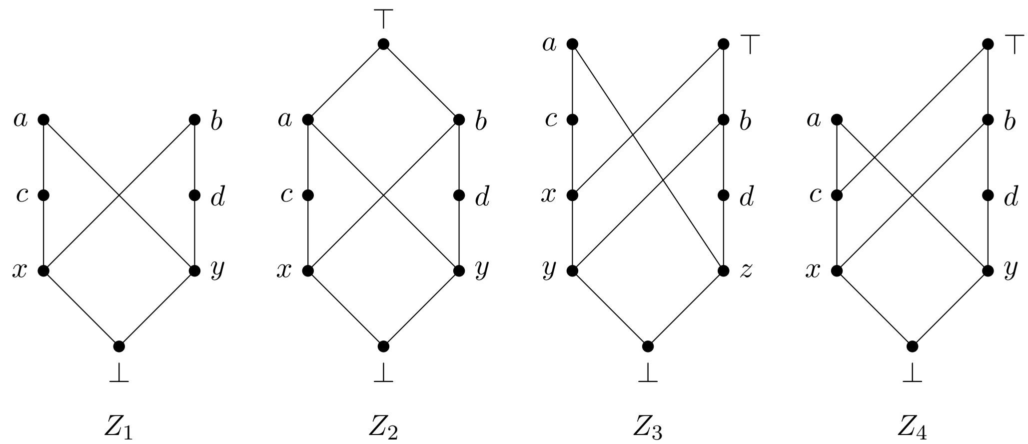

FIGURE 13. The posets Z_1, Z_2, Z_3, and Z_4.

This, in turn, guarantees that $\top$ exists and that

$$x \leqslant b \qquad y \leqslant \top \qquad y \nleqslant b \qquad c \nleqslant \top.$$

Together with Condition (10) and the facts that $(\bot, c) = \{x, y\}$, $\bot < x < y < c$, and $x, y, c \nleqslant d$, this implies that

$$\uparrow x = \{x, y, a, b, c, \top\} \qquad \uparrow y = \{y, a, c, \top\}.$$

Therefore, Condition (M4) holds.

It only remains to consider the case where $(\bot, c) = \{x\}$ for some $x \in X$. First suppose that $x \leqslant b$. As before, Condition (7) implies that $x \nleqslant d$. Together with Condition (10) and the assumption that $(\bot, c) = \{x\}$, this implies that $\uparrow x = \{x, a, c\} \cup \uparrow b$. Hence, Condition (M2) holds. Next suppose that $x \nleqslant b$. Since $x, c \nleqslant b$, both $[c]$ and $[x]$ are different from Y_3. By Condition (11) and the fact that $c \neq x$, this implies that $\{\{c\}, \{x\}\} = \{Y_1, Y_2\}$. As $x \leqslant c$, the definition of the various Y_i guarantees that $c \in Y_1$ and $x \in Y_2$. Consequently, $\top$ exists and

$$c \nleqslant \top \qquad x \leqslant \top \qquad x \nleqslant b.$$

Bearing in mind that $x \nleqslant d$ by Condition (7), we conclude that $\uparrow x = \{x, a, c, \top\}$. Therefore, Condition (M3) holds as desired. ⊠

A similar (but shorter) argument yields the following:

Claim 8.7. *One of the following conditions holds.*

(N1) $(\bot, d) = \varnothing$;
(N2) $(\bot, d) = \{x\}$ *for some* x *such that* $\uparrow x = \{x, a\} \cup \uparrow d$.

At last, we are ready to give a more concrete description of the poset X. First, the order structure of the subposet $\{\bot, a, b, c, d\}$ of X is that of Figure 11 and, if $\top$ exists, the subposet $\{\bot, a, b, c, d, \top\}$ is one of those depicted in Figure 12. By Condition (10), the elements of X other than $a, b, c, d, \top$, and $\bot$ lie in $(\bot, c) \cup (\bot, d)$. But recall from Claim 8.6 that one of Conditions (M1)–(M4) holds and that each of them gives a complete description of the interval $(\bot, c)$. Similarly, one of (N1) or (N2) holds by Claim 8.7 and each of them gives a complete description of the interval $(\bot, d)$.

As a consequence, we obtain that X is a subposet of one of the rooted posets in Figure 13. Moreover, X contains $\bot, a, b, c, d$ plus $\top$ if $\top$ appears in the corresponding picture.

Now, recall that X validates the Jankov formulas of $K_1, \dots, K_7$. Therefore, none of these posets is a p-morphic image of an upset of X. Bearing this in mind, we begin by considering the case where X is a subposet of Z_1 containing $\bot, a, b, c$, and d. Notice that $X \neq \{\bot, a, b, c, d\}$, otherwise we would obtain $X \cong K_1$, a contradiction. Then X contains x or y. If X does not contain both, it is isomorphic to K_2 against the assumptions. Therefore, we conclude that X contains both x and y. But this is also impossible as in this case K_3 is a p-morphic image of X.

The case where X is a subposet of Z_2 containing $\bot, a, b, c, d$, and $\top$ leads to a contradiction in a similar way (where K_3 takes the role of K_1 and K_4 that of K_2).

Next we consider the case where X is a subposet of Z_3 containing $\bot, a, b, c, d$, and $\top$. Notice that X contains one of x, y, and z (otherwise K_1 is a p-morphic of X, which is impossible). If x and z or y and z belong to X, then K_3 is a p-morphic image of X, which is also false. More precisely, when $x, z \in X$, we collapse $\{a, b, \top\}$ if $y \notin X$ and we collapse $\{a, b, c, \top\}$ if $y \in X$. In both cases, we obtain a p-morphic image of X isomorphic to K_3. On the other hand, when $y, z \in X$ but $x \notin X$, we collapse $\{a, b, \top\}$, thus obtaining a p-morphic image of X isomorphic to K_3. Therefore, we may assume that the universe of X is the union of $A \coloneqq \{\bot, a, b, c, d, \top\}$ with $\{x\}$ or $\{y\}$ or $\{x, y\}$ or $\{z\}$. We will show that each of these cases leads to a contradiction.

If $X = A \cup \{x\}$ or $X = A \cup \{y\}$, then K_2 is a p-morphic image of X obtained by collapsing $\{b, d, \top\}$. Moreover, if $X = A \cup \{x, y\}$, then K_6 is a p-morphic image of X obtained by collapsing $\{a, c\}$. Lastly, if $X = A \cup \{z\}$, then K_2 is a p-morphic image of X obtained by collapsing $\{b, \top\}$.

It remains to consider the case where X is a subposet of Z_4 containing $\bot, a, b, c, d$, and $\top$. Observe that $X \neq \{\bot, a, b, c, d, \top\}$ (otherwise $X \cong K_5$, which is false). Therefore, x or y belong to X. If both x and y belong to X, then K_3 is a p-morphic image of X obtained by collapsing $\{a, b, \top\}$, against the assumptions. Thus, X is A together with x or y. If $X = A \cup \{x\}$, then $X \cong K_6$, which is false. On the other hand, if $X = A \cup \{y\}$, then $X \cong K_7$, which is also false. Hence, we reach the desired contradiction. ⊠

Proof of Lemma 8.3. Let X be a rooted Noetherian poset of width $\leqslant 2$ validating $\beta(P_2)$. By Theorem 4.2(2), from $X \vDash \beta(P_2)$ it follows that P_2 is not a p-morphic image of any subposet of X. This fact will be used repeatedly in the proof.

First suppose that $X \vDash \beta(P_3)$ and consider a nonnegative integer $i \leqslant 6$. By the Fine Lemma, to show that X validates $\mathcal{J}(G_i)$, it suffices to prove that G_i is not a p-morphic image of any upset of X. Suppose the contrary. Then there exist an upset U of X and a surjective p-morphism $\alpha \colon U \to G_i$. As a consequence, G_i validates all the formulas valid in X and, in particular, $\beta(P_3)$. But in view of Theorem 4.2(2) this is false because P_3 is isomorphic to a subposet of G_i, as it can be checked by inspecting the posets in Figure 10.

To prove the converse, assume that X validates the Jankov formulas of the posets $G_1, \dots, G_6$ in Figure 10 and suppose, with a view to contradiction, that $X \nvDash \beta(P_3)$. By Theorem 4.2(2) this implies that P_3 is isomorphic to a p-morphic image of a subposet of X. The definition of a p-morphism and the structure of P_3 imply that actually P_3 is isomorphic to a subposet of X. We name the elements of this subposet as in Figure 14.

We may assume that $\bot$ is the minimum of X (otherwise we replace X by $\uparrow\bot$). In addition, since X is Noetherian, we may assume that

$$\bot \text{ is maximal in } {\downarrow} a \cap {\downarrow} d \text{ and } b \text{ (resp. } c\text{) is an immediate successor of } c \text{ (resp. } d\text{).} \tag{12}$$

Claim 8.8. *We may assume, without loss of generality, that either $X = {\downarrow}\{a, b\}$ or $X = \{\top\} \cup {\downarrow}\{a, b\}$ for some $\top \in X$ such that the subposet of X with the universe $\{\bot, a, b, c, d, \top\}$ is one of the posets X_i depicted in Figure 15.*

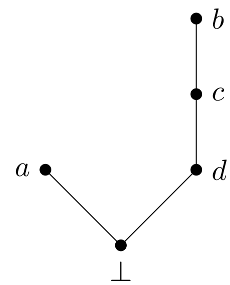

FIGURE 14. The poset P_3 viewed as a subposet of X.

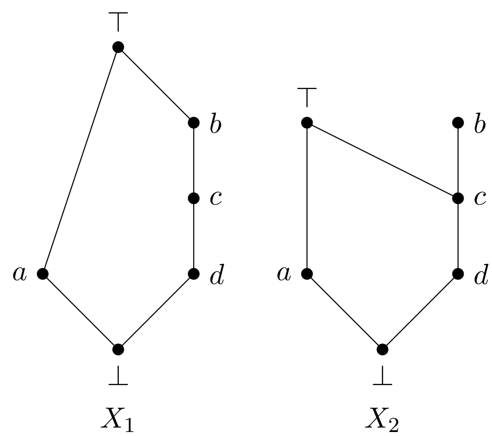

FIGURE 15. The posets X_1 and X_2.

Proof of the Claim. If $X = {\downarrow}\{a, b\}$, we are done. Suppose $X \neq {\downarrow}\{a, b\}$. Since $({\downarrow}\{a, b\})^c$ is an upset, the following relation is an E-partition of X:

$$R = \{\langle x, y\rangle \in X \times X : x = y \text{ or } x, y \notin {\downarrow}\{a, b\}\}.$$

Notice that the subposet of X/R with the universe $\{[a], [b], [c], [d], [\bot]\}$ is isomorphic to the subposet of X with the universe $\{a, b, c, d, \bot\}$. Therefore, since X/R validates all the formulas valid in X, we may assume that R is the identity relation (otherwise we replace X by X/R). Consequently, being nonempty by assumption, the set $({\downarrow}\{a, b\})^c$ is an equivalence class of the identity relation R. Therefore, $({\downarrow}\{a, b\})^c = \{\top\}$ for some $\top \in X$. This implies that $X = \{\top\} \cup {\downarrow}\{a, b\}$.

It only remains to prove that the subposet of X with the universe $\{\bot, a, b, c, d, \top\}$ is one of the posets X_i. First, recall that $\top \nleqslant a, b$. Since X is a rooted poset of width $\leqslant 2$, this yields that either $a < \top$ or $b < \top$. If $a, b < \top$, then the subposet $\{\bot, a, b, c, d, \top\}$ is isomorphic to X_1 and we are done. Suppose $a \nleqslant \top$ or $b \nleqslant \top$. Since $\top \nleqslant a, b$, we have that

$$(a \text{ and } \top \text{ are incomparable and } b < \top) \text{ or } (b \text{ and } \top \text{ are incomparable and } a < \top).$$

First suppose that a and $\top$ are incomparable and $b < \top$. Since ${\uparrow}b$ is an upset, the following relation is an E-partition of X:

$$S = \{\langle x, y\rangle \in X \times X : x = y \text{ or } b \leqslant x, y\}.$$

Notice that S does not alter the order relation between $\bot, a, b, c, d$. Together with the fact that X/S validates all the formulas valid in X, this means that X/S is a poset of width $\leqslant 2$ that validates $\beta(P_2)$ and the various $\mathcal{J}(G_i)$. Moreover, X/S contains a subposet isomorphic to P_3, namely $\{[\bot], [a], [b], [c], [d]\}$. Therefore, in our proof we may replace X by X/S and each element $\bot, a, b, c, d$ by its equivalence class. Furthermore, since $b < \top$, the definition of S ensures that $[\top] = [b]$. Bearing in mind that $X = \{\top\} \cup {\downarrow}\{a, b\}$, this means that $X/S = {\downarrow}\{[a], [b]\}$. Because of this, by replacing X by X/S, we may assume that $X = {\downarrow}\{a, b\}$ as desired.

Therefore, it only remains to consider the case where b and $\top$ are incomparable and $a < \top$. We will prove that $c \leqslant \top$. Suppose the contrary. Then c and $\top$ are incomparable because by assumption $c \leqslant b$ and $\top \nleqslant b$. Therefore, both c and b are incomparable with $\top$. By assumption, they are also incomparable with a. Together with the fact that $\bot < a < \top$ and $\bot < c < b$, this implies that $\{\bot, a, b, c, \top\}$ is a subposet of X isomorphic to P_2. But this contradicts the assumption that X validates $\beta(P_2)$. Hence, we conclude that $c \leqslant \top$ as desired. Bearing in mind that b and $\top$ are incomparable, that $a < \top$, and that the structure of the poset $\{\bot, a, b, c, d\}$ is as in Figure 14, this implies that the subposet of X with universe $\{\bot, a, b, c, d, \top\}$ is X_2. ⊠

Claim 8.9. *We have that*

$$X = \begin{cases} \{\bot, a, b, c, d\} \cup ({\downarrow}c \cap (\bot, a)) \cup (\bot, d) & \text{if } \top \text{ does not exist} \\ \{\bot, a, b, c, d, \top\} \cup ({\downarrow}c \cap (\bot, a)) \cup (\bot, d) & \text{if } \top \text{ exists.} \end{cases}$$

Proof of the Claim. The inclusion from right to left is obvious. To prove the other inclusion, consider some $x \in X \smallsetminus \{\bot, a, b, c, d\}$ other than $\top$. In view of the Claim 8.8, either $X = {\downarrow}\{a, b\}$ or $X = \{\top\} \cup {\downarrow}\{a, b\}$. Since x is different from $a, b, \top$ and from the minimum $\bot$, we have two cases: $x \in (\bot, a)$ or $x \in (\bot, b)$.

First suppose that $x \in (\bot, a)$. To prove that x belongs to the set in the right hand side of the statement, it suffices to show that $x \leqslant c$. Suppose the contrary, with a view to contradiction. Since $x \leqslant a$ and $c \nleqslant a$, this means that x and c are incomparable. Furthermore, since $\bot$ is maximal in ${\downarrow}a \cap {\downarrow}d$ by Condition (12) and $\bot < x \leqslant a$, we obtain $x \nleqslant d$. In addition, $d \nleqslant x$ because $x \leqslant a$ and $d \nleqslant a$. Thus, x is also incomparable with d. Therefore, $\bot < x < a$ and $\bot < d < c$ and x, a are incomparable with d, c. Consequently, $\{\bot, a, b, c, d\}$ is a subposet of X isomorphic to P_2. But this contradicts the assumption that $X \vDash \beta(P_2)$.

Next suppose that $x \in (\bot, b)$. We may assume that $x \nless a$ (otherwise $x \in (\bot, a)$ and we repeat the argument of the previous case). Consequently, in order to prove that a and x are incomparable, it suffices to show that $a \nleqslant x$. But this is clear because by assumption $x < b$ and $a \nleqslant b$. Furthermore, by assumption, a is incomparable with c and d. Together with the facts that a is incomparable with x and that X has width $\leqslant 2$, this implies that x is comparable with both c and d. If $x < d$, then $x \in (\bot, d)$, and hence x belongs to the right hand side of the statement. Suppose $x \nless d$. Since x and d are comparable and distinct, this means that $d < x$. Together with the assumption in Condition (12) that c is an immediate successor of d and the fact that $x \neq c$, this implies that $x \nleqslant c$. Since x and c are comparable, we obtain that $c < x$. But then we have $c < x < b$, a contradiction to the assumption that b is an immediate successor of c (see Condition (12)). ⊠

Now, we consider the relation

$$T = \{\langle x, y\rangle \in X \times X : x = y \text{ or } x, y \in (\bot, a) \cap {\downarrow}c \text{ or } x, y \in (\bot, d]\},$$

where $(\bot, d]$ stands for $\{x \in X : \bot < x \leqslant d\}$.

Claim 8.10. *The relation T is an E-partition of X.*

Proof of the Claim. Since $\bot$ maximal in ${\downarrow}a \cap {\downarrow}d$ by Condition (12), the sets $(\bot, a) \cap {\downarrow}c$ and $(\bot, d]$ are disjoint, and hence T is an equivalence relation on X. We will prove that it is also an E-partition.

To this end, it suffices to show that there are no distinct $x, y \in X$ such that $\langle x, y\rangle \in T$ and for which there exists an element $z \in X$ such that $x \leqslant z$ and $y \nleqslant z$ and $\langle x, z\rangle \notin T$. Suppose the contrary, with a view to contradiction. Since x and y are distinct and related by T, we have that

$$x, y \in (\bot, a) \cap {\downarrow} c \text{ or } x, y \in (\bot, d].$$

First suppose that $x, y \in (\bot, a) \cap {\downarrow} c$. From Claim 8.9 it follows that $z \in \{\bot, a, b, c, d, \top\} \cup ((\bot, a) \cap {\downarrow} c) \cup (\bot, d)$. Clearly, $z \notin (\bot, a) \cap {\downarrow} c$ (otherwise $\langle x, z \rangle \notin T$ contradicting the assumption). Moreover, $z \notin (\bot, d]$ because otherwise $\bot < x \leqslant z \leqslant d$ and by assumption $\bot < x \leqslant a$, contradicting the maximality of $\bot$ in ${\downarrow} a \cap {\downarrow} d$ (see Condition (12)). Therefore, $z \in \{\bot, a, b, c, \top\}$. Since by assumption $y \nleqslant z$ and $y \in (\bot, a) \cap {\downarrow} c$, we obtain that $z \notin \{a, c, b, \top\}$. Consequently, $z = \bot$. But this contradicts the assumption that $\bot < x \leqslant z$.

Next suppose that $x, y \in (\bot, d]$. Since $y \nleqslant z$, this implies that $d \nleqslant z$. Furthermore, $z \nleqslant d$ (otherwise $\bot < x \leqslant z \leqslant d$, and hence $\langle x, z \rangle \in T$, a contradiction). Therefore, z and d are incomparable. Since a and d are also incomparable and X is a rooted poset of width $\leqslant 2$, we conclude that z and a are comparable. As $\bot$ is maximal in ${\downarrow} a \cap {\downarrow} d$ by Condition (12) and $\bot < x \leqslant d$, we obtain that $x \nleqslant a$. Together with $x \leqslant z$, this implies that $z \nleqslant a$. Thus, since z and a are comparable, we must have $a < z$. As $\{\bot, a, b, c, d, \top\}$ is one of the posets depicted in Figure 15, we conclude that $z = \top$. But since $y \leqslant d \leqslant \top$, this implies that $y \leqslant z$, a contradiction. ⊠

Lastly, we will make use of the following.

Claim 8.11. *For every $x, y \in \{\bot, a, b, c, d, \top\}$,*

$$x \leqslant y \iff [x] \leqslant [y].$$

Proof of the Claim. The implication from left to right is obvious. To prove the other one, suppose that $[x] \leqslant [y]$. The definition of T guarantees that

$$[\bot] = \{\bot\} \qquad [\top] = \{\top\} \qquad [a] = \{a\} \qquad [b] = \{b\} \qquad [c] = \{c\} \qquad [d] = (\bot, d].$$

In view of the above display, if $x, y \neq d$, then $[x] = \{x\}$ and $[y] = \{y\}$, whence $[x] \leqslant [y]$ implies $x \leqslant y$ as desired. Therefore, we consider the case where either $x = d$ or $y = d$. First suppose that $x = d$. If $y \in \{b, c, d, \top\}$, then $x = d \leqslant y$ and we are done. Thus, it suffices to show that $y \notin \{\bot, a\}$. Since there is no element in $[y] = [d] = (\bot, d]$ below $\bot$ or a (the latter, by the maximality of $\bot$ in ${\downarrow} a \cap {\downarrow} d$; see Condition (12)), the fact that $[\bot] = \{\bot\}$ and $[a] = \{a\}$ implies that $[y] \nleqslant [\bot], [a]$, thus preventing y from being $\bot$ or a as desired. Next suppose that $y = d$. If $x = d$, we are done. Therefore, we suppose that $x \in \{\bot, a, b, c, \top\}$. In this case, $[x] = \{x\}$, thus the assumption that $\{x\} = [x] \leqslant [y] = [d] = (\bot, d]$ implies that $x \leqslant d = y$. ⊠

Together with the fact that X/T validates all the formulas valid in X, this allows us to assume that T is the identity relation (otherwise we replace X by X/T). Consequently, Claim 8.9 specializes to the following:

$$X = \begin{cases} \{\bot, a, b, c, d\} \cup ({\downarrow} c \cap (\bot, a)) & \text{if } \top \text{ does not exist} \\ \{\bot, a, b, c, d, \top\} \cup ({\downarrow} c \cap (\bot, a)) & \text{if } \top \text{ exists,} \end{cases} \tag{13}$$

where ${\downarrow} c \cap (\bot, a)$ is either empty or a singleton. Bearing in mind that if $\top$ exists, then the subposet of X with the universe $\{\bot, a, b, c, d, \top\}$ is one of the posets depicted in Figure 15, we conclude that X is a subposet of one of the posets depicted in Figure 16 containing $\bot, a, b, c$, and d. Furthermore, when we identify X with a subposet of Z_2 or Z_3 we assume that it contains $\top$, otherwise we identify it with a subposet of Z_1.

Now, recall that X validates the Jankov formulas of $G_1, \dots, G_6$, and hence none of these posets is a p-morphic image of an upset of X by the Fine Lemma. Bearing this in mind, we begin by considering the case where X is a subposet of Z_1 containing $\bot, a, b, c$, and d. In this case, X is isomorphic to either G_1 or G_2, a contradiction. Next we consider the case where X is a subposet of Z_2 (resp. Z_3) containing $\bot, a, b, c, d$, and $\top$. In this case, X is isomorphic to either G_3 or G_4 (resp. G_5 or G_6), which is also false. Hence, we reach the desired contradiction. ⊠

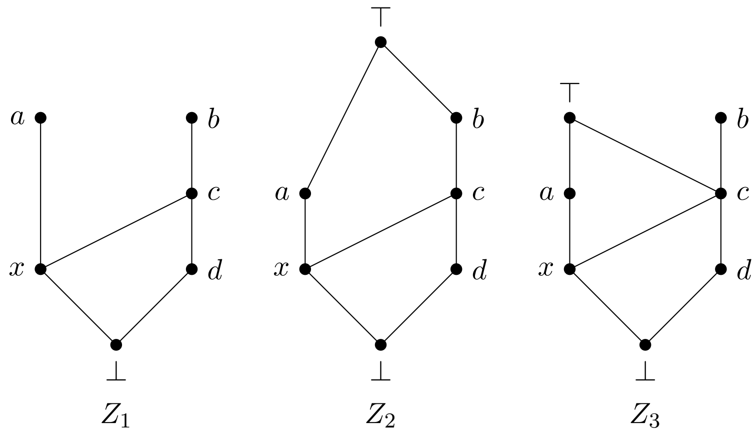

FIGURE 16. The posets Z_1, Z_2, and Z_3.

We are now ready to prove that KG can be axiomatized by Jankov formulas.

Proof of Theorem 4.10. Let Σ be the union of the set of Jankov formulas that axiomatize BW_2 and the set of Jankov formulas of the posets in Figures 9 and 10. We will prove that Σ axiomatizes KG.

Suppose the contrary. Since sums of one-generated Heyting algebras have width $\leqslant 2$, we have that $\mathsf{BW}_2 \subseteq \mathsf{KG}$, and hence the Jankov formulas axiomatizing BW_2 belong to KG. Furthermore, observe that the posets in Figures 9 and 10 are not models of KG (because each of them contains one of the posets P_1, P_2, P_3 in Figure 2 as a subposet and KG is axiomatized by $\beta(P_1), \beta(P_2), \beta(P_3)$ by Theorem 4.4). Therefore, in view of the Dual Jankov Lemma, the Jankov formulas of these posets belong to KG. As a consequence, we obtain that $\Sigma \subseteq \mathsf{KG}$. Since by assumption Σ does not axiomatize KG, this yields that the si-logic L axiomatized by Σ is strictly contained in KG.

From Theorem 4.9 it follows that $\mathsf{BW}_2 \subseteq \mathsf{L}$. By Fine Completeness Theorem, there is a class K of rooted Noetherian posets of width $\leqslant 2$ such that $\mathsf{L} = \mathrm{Log}(K)$. Since $\mathsf{KG} \nsubseteq \mathsf{L}$, by Theorem 4.4 there is a poset $X \in K$ refuting $\beta(P_i)$ for some $i \leqslant 3$. Because X has width $\leqslant 2$, we have that $X \vDash \beta(P_1)$. Therefore, either $X \nvDash \beta(P_2)$ or $X \nvDash \beta(P_3)$. By Lemmas 8.2 and 8.3, there is a poset Y in Figure 9 or 10 such that $X \nvDash \mathcal{J}(Y)$. Since $X \in K$ and $\mathsf{L} = \mathrm{Log}(K)$, we obtain that $\mathcal{J}(Y)$ does not belong to L. The obtained contradiction proves that $\mathsf{KG} = \mathsf{IPC} + \Sigma$. ⊠

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