Team Semantics for Modal mu-Calculus

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written by

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ABSTRACT. In this thesis, we define a team semantics for modal mu-calculus, show it enjoys the flatness property and aligns well with existing team temporal logics. The approach taken utilises team semantics for modal logic and involves an algebraic study of powerset structures in order to assign a reasonable team semantics to fixed-points.

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Contents

CHAPTER 1

Introduction

The modal μ-calculus, hencefort denoted μML, is an extension of propositional modal logic with two fixed-point operators μ and ν denoting , respectively, the least and the greatest fixed-point of functions induced by formulas with free variables. It was introduced in its current form by Kozen [**15**] but it can be traces back to work by Scott and De Bakker in [**18**]. μML is a very powerful and expressive logic which can capture many temporal operators not expressible in modal logic. One such example is the *until* operator which can be captured by μ. Indeed, μML subsumes Propositional Dynamic Logic (PDL), full Computational Tree Logic (CTL*) and it is thought as *the mother of all temporal logics*.

From the logical point of view, μML is an important extension of modal logic which maintains some desirable properties. Among other properties, μ-calculus enjoys uniform interpolation, finite model property decidability and moreover, its validities have a finitary axiomatization. While modal logic is the bisimilar fragment of first order logic, μML can be shown to be the bisimilar fragment of monadic second order logic [**3**, **6**, **9**].

Modal μ-calculus is also intimately connected with lattice theory, universal algebra, universal coalgebra as well as theoretical computer science through atomata theory and game semantics. It becomes a suitable formalization to study properties about processes and process theory. To date, however, team semantics for μML has not been proposed.

Team semantics can be seen as a generalisation of Tarskian model theoretic semantics. Briefly, while Tarkasian semantics operate on objects alone, team semantics do it on sets of objects. Therefore, if classical semantics for first order logic are defined in terms of interpretations, we can consider team semantics for first order logic defined with respect to sets of interpretations. Likewise, we can consider semantics for modal logic with respect to sets of states in a Kripke model and so on with other logics. In this context, such sets of objects are called *teams*.

Historically, team semantics was first introduced by Hodges for independence friendly logic (IF logic) under the name of compositional semantics [**11**]. Later Väänänen developed team semantics for dependence logic [**19**]. This both logics are extensions of first order logic which formalize the dependence/independence quantifier patterns of first order logic. Such patterns arise because of the possibility of nesting quantifiers in first order logic which is one of the key points of expressibility. The most common example of this pattern can be seen in the definitions of continuity and uniform continuity of a function f expressed, respectively, by the sentences

$$\forall x \forall \varepsilon \exists \delta \forall y(|x-y| < \delta \to |f(x) - f(y)| < \varepsilon)$$
$$\forall \varepsilon \exists \delta \forall x \forall y(|x-y| < \delta \to |f(x) - f(y)| < \varepsilon),$$

where the dependency patterns are as follows. For continuity δ depends on x and ε while for uniform continuity δ only depends on ε. These notions can be syntactically seen because of the order of the quantifiers in the sentences, in the first one the existential is under the scope of both universal quantifiers binding x and ε and in the second one it is only under the scope of ε. However, these notions of *dependency/independency* arise as metanotions which are not formally expressible in FO language. The aforementioned logics (IF logic and dependence logic) introduce syntactical constructs to deal with such patterns. For instance, in IF logic the definition of uniform continuity is equivalent to

$$\forall x \forall \varepsilon \exists (\delta / x) \forall y(|x-y| < \delta \to |f(x) - f(y)| < \varepsilon).$$

where $\exists(\delta/x)$ expresses that the value of δ is not dependent on the value of x while in dependence logic the definition of continuity is equivalent to

$$\forall x \forall \varepsilon \exists \delta \forall y[\mathsf{dep}(x, \varepsilon, \delta) \wedge (|x-y| < \delta \to |f(x) - f(y)| < \varepsilon)]$$

where the atom $\mathsf{dep}(x_0, \ldots, x_n, y)$ means that the value of y is functionally dependant on the values of $x_0, \ldots, x_n$. One can quickly see that standard Tarkasian semantics cannot define the semantics for this two syntactic notions: when only one interpretation is considered there is no possibility of studying the interdependency of variables.

This serves as a very good example of the possibilities that team semantics open. Since the semantical relation is now defined over sets of objects (interpretations in this case), it implicitly carries more information than the classical Tarkasian one. This information allows for the definition of new logical atoms or connectives which expand the expressible power of the logic, e.g., $\mathsf{dep}(x_0, \ldots, x_n, y)$, $\varphi \otimes \psi$, $\dot{\sim} \varphi$, $\exists^1 \varphi, \forall^1 \varphi$, NE (non-emptyness atom) among others in the literature. In particular, it was shown by Väänänen in [**19**] that dependence logic is sentence-wise equivalent to existential second order logic and later by Kontinen and Ville in [**14**] that extending dependence logic with a strong notion of negation makes it equivalent to full second order logic.

Team semantics have also been defined and studied for modal logic, linear temporal logic (`LTL`), full computational tree logic (`CTL`*). Questions of interest include the expressible power of the logics, the relation with classical semantics and the complexity of satisfiability problems, and generally the lifting of well-known theorems for the Tarkasian semantics to the team semantical framework. For instance, the complexity of modal team logic has been studied in [**17**], uniform interpolation was established for the same logic in [**5**] and a van Benthem theorem was shown in [**13**].

In the temporal context, team semantics have been developed to study so called *hyperproperties*. Hyperproperties are properties of sets of traces which in computer science can be thought as processes. Some of these properties arise naturally, for example whether a set of runs of a program reaches every possible final state. In [**16**], Krebs et. al. study how team semantics and a hyper variant of LTL relate for the description of hyperproperties.

Temporal team semantics allows the expression of *synchroncity* which cannot be readily expressed in the classical framework. This notion is one of the reasons which make temporal team semantics an interesting area of study. In particular, team semantics have been defined for LTL, CTL and even CTL* which is a highly expressible fragment of μML. The most recent work in this area investigates their complexity and expressibility [**21**] and studies, from a mathematical point of view, the notion of asynchronicity [**10**].

This thesis contributes by defining team semantics for modal μ-calculus and showing that the classical embedding of CTL can be adapted to the team semantical framework. In summy, we:

(1) Introduce the liftings of some basic algebraic constructions and study their basic properties.
(2) Develop *internal* and *external* team semantics for modal logic giving an algebraic perspective on them.[1]
(3) Introduce a generalization of team semantics for modal logic that we call *general team semantics*.
(4) Build, using the defined general team semantics, fixed-point operators in a team semantical setting and define team μ-calculus and study some basic properties.
(5) Embed CTL with team semantics to team μ-calculus.

We next proceed to summarize the content of the subsequent chapters of this thesis. The reader will find a more detailed explanation at the beginning of each chapter.

In chapter 2, we describe those basic concepts mostly from the fields of order theory, modal μ-calculus, team semantics and powerset structures that will be necessary throughout the thesis.

In chapter 3, we begin by giving a double perspective on team semantics for propositional logic recently developed by Engström and

[1]The term *internal* and *external* logic has recently been introduced in [**8**] to reflect the two semantical layers of propositional team semantics.

Lorimer Olsson [**8**] that we will then extend to modal logic. Once we have developed the notions algebraically, we demostrate how they can be extended to a *general team semantics for modal logic*. Using the just defined semantics we will provide team semantics for $\mu\mathtt{ML}$ and show, among other properties, flatness. Finally, we define the logic of team μ-calculus ($\mathtt{t}\mu\mathtt{ML}$) and a extension of it.

In chapter 4, we briefly introduce the team semantics for LTL and CTL and the important notions of synchronicity and asynchronicity, finally showing that the classical embedding of CTL into $\mu\mathtt{ML}$ can be extended to the team semantical framework.

In chapter 5, we summarize the contents of this thesis highlighting the results obtained and discuss the future open research lines.

CHAPTER 2

Preliminaries

In this chapter we collect those basic facts, mostly without proofs, that we will need in the following chapters to carry out our investigation. Here we fix notation and to give a gentle introduction to team semantics and some of its algebraic considerations. Specifically, in Section 1, we begin by recalling the algebraic basis of fixed-point theory, monotone functions on complete lattices, and state the Knaster-Tarski theorem which guarantees the existence of fixed-points for such a function. Afterwards, we show the equivalent construction by ordinal approximations which we will use throughout the thesis. In Section 2, definition of modal μ-calculus and its standard semantics with respect to Kripke models follows. Some constructions that we will generalize in the subsequent chapters will also be laid out.

In Section 3, we take a closer look at the most known basic team semantics for modal logic and use them as a gentle introduction to team semantics in a more wider sense. We also study some of their properties. Finally, in Section 4, motivated by the constructions of team semantics, we study the structure of double powersets algebras and isolate two different interpretations which we return to in the next chapter. For a full treatment of the topics of this chapter we refer the reader to [**4**, **5**, **13**, **20**].

Our underlying set theory is **ZFC**, Zermelo-Fraenkel set theory with the axiom of choice (also known as **ZFSk**, i.e., Zermelo-Fraenkel-Skolem set theory).

In what follows we use standard concepts and constructions from set theory, see, e.g., [**2**, **7**] and lattice theory, see, e.g., [**1**, **12**]. Nevertheless, regarding set theory, we have adopted the following conventions. An *ordinal* α is a transitive set that is well-ordered by $\in$; thus, $\alpha = \{\beta \mid \beta \in \alpha\}$. The first transfinite ordinal will be denoted by $\mathbb{N}$, which is the set of all *natural numbers*, and, from what we have just said about the ordinals, for every $n \in \mathbb{N}$, $n = \{0, \ldots, n-1\}$. A function from A to B is a subset F of $A \times B$ satisfying the functional condition and a mapping from A to B is an ordered triple $f = (A, F, B)$, denoted by $f : A \to B$, in which F is a function from A to B.

1. Fixed-point theory

In this section we state some basic results about fixed-point theory. We show both perspectives on fixed-points. First showing the purelly algebraic one by stating the Knaster-Tarski theorem and later showing the perspective of ordinal approximations of fixed-points, finally showing that both of them are equivalent. This section is mostly based in the lecture notes of Yde Venema [**20**].

We begin by giving a precise definition of partially-ordered set, complete lattice and monotone function.

DEFINITION 2.1. A *partially-ordered set* is a pair $\mathbf{P} = (P, \leq)$ where P is a set and $\leq$ is a binary relation over P satisfying:

(1) reflexivity: for every $p \in P$, $p \leq p$;
(2) antisymmetry: for every $p, q \in P$, if $p \leq q$ and $q \leq p$, then $p = q$;
(3) transitivity: for every $p, q, r \in P$, if $p \leq q$ and $q \leq r$, then $p \leq r$.

We call $\leq$ a *partial order* over P. If $\leq$ only satisfies reflexivity and transitivity we say that it is a *preorder* and $(P, \leq)$ is a *preordered set*

DEFINITION 2.2. Let $\mathbf{P} = (P, \leq)$ be a partially-ordered set. For a set $A \subseteq P$, an element $p \in P$ is the *supremum* of A, written $\bigvee A$, if

(1) for every $a \in A$, $a \leq p$;
(2) for every $q \in P$, if for every $a \in A$, $a \leq q$, then $p \leq q$.

We define the notion of *infimum* dually and denote it with $\bigwedge A$.

DEFINITION 2.3. A partially-ordered set $\mathbf{P} = (P, \leq)$ is a *complete lattice* if every subset $A \subseteq P$ has supremum and infimum.

REMARK 2.4. Notice that every complete lattice has a maximum and minimum with respect to the order. They are denoted, respectively, $\top := \bigvee P$ and $\bot := \bigwedge P$. In particular, every complete lattice is non-empty.

EXAMPLE 2.5. For every set A, the structure $(\mathcal{P}(A), \subseteq)$ is a complete lattice where, for $\mathcal{S} \subseteq \mathcal{P}(A)$, $\bigvee \mathcal{S} = \bigcup \mathcal{S}$ and $\bigwedge \mathcal{S} = \bigcap \mathcal{S}$. Moreover, its maximum is A and its minimum $\emptyset$.

DEFINITION 2.6. Let $\mathbf{P} = (P, \leq_P)$ and $\mathbf{Q} = (Q, \leq_Q)$ be partially-ordered sets. A function $f : P \to Q$ is said to be *monotone* if, for every $p, q \in P$, $p \leq_P q$ implies $f(p) \leq_Q f(q)$.

DEFINITION 2.7. Let $\mathbf{P} = (P, \leq)$ be a partially-ordered set and $f : P \to P$ a function. We say that $p \in P$ is a *fixed-point (of f)* if $f(x) = x$. The set of fixed-points of f is denoted $\mathrm{FP}(f)$.

The central theorem of this section is the theorem of Knaster-Tarski. It states that the fixed-points of a monotone function on a complete lattice form a complete lattice with the same order, hence being the set of fixed-points non-empty. Particularly, it implies the existence of a least and a greatest fixed-point for which the theorem gives a specific and algebraic description.

THEOREM 2.8 (Knaster-Tarski). *Let* $\mathbf{L} = (L, \leq)$ *be a complete lattice and* $f : L \to L$ *a monotone function. Then,* $\mathrm{FP}(f)$ *is non-empty and* $(\mathrm{FP}(f), \leq)$ *is a complete lattice.*

COROLLARY 2.9. *The following relations follow from Knaster-Tarski theorem:*

(1) *The least fixed-point of* f*, denoted* $\mathrm{lfp}(f)$*, exists and*
$$\mathrm{lfp}(f) = \bigwedge\{x \in L \mid f(x) \leq x\}.$$
(2) *The greatest fixed-point of* f*, denoted* $\mathrm{gfp}(f)$*, exists and*
$$\mathrm{gfp}(f) = \bigvee\{x \in L \mid x \leq f(x)\}.$$

In what follows we will use **ORD** to denote the *class of all ordinals*. As it is well-known, it does not describe a set in **ZFC** that is why it is important to note that we will only use it as notation.

DEFINITION 2.10. Let $(L, \leq)$ be a complete lattice and let $f : L \to L$ be a function. We define by ordinal induction the sequences $(f_\mu^\xi)_{\xi\in\mathbf{ORD}}$ and $(f_\nu^\xi)_{\xi\in\mathbf{ORD}}$ as follows:
$$\begin{cases} f_\mu^0 = \bot \\ f_\mu^{\alpha+1} = f(f_\mu^\alpha) \\ f_\mu^\beta = \bigvee_{\alpha\in\beta} f_\mu^\alpha & \text{for } \beta \text{ limit} \end{cases} \quad \text{and} \quad \begin{cases} f_\nu^0 = \top \\ f_\nu^{\alpha+1} = f(f_\nu^\alpha) \\ f_\nu^\beta = \bigwedge_{\alpha\in\beta} f_\nu^\alpha & \text{for } \beta \text{ limit} \end{cases}$$

PROPOSITION 2.11. *Let* $(L, \leq)$ *be a complete lattice and* $f : L \to L$ *be monotone. If* $\alpha \leq \beta$*, then* $f_\mu^\alpha \leq f_\mu^\beta$ *and* $f_\nu^\beta \leq f_\nu^\alpha$*.*

PROOF. By ordinal induction on β. □

COROLLARY 2.12. *The sequences* $(f_\mu^\xi)_{\xi\in\mathbf{ORD}}$ *and* $(f_\nu^\xi)_{\xi\in\mathbf{ORD}}$ *are both eventually constant. We will denote by* $\lim_{\xi\in\mathbf{ORD}} f_\mu^\xi$ *and* $\lim_{\xi\in\mathbf{ORD}} f_\nu^\xi$ *the respective constant value.*

PROPOSITION 2.13. *The following relations hold*
$$\mathrm{lfp}(f) = \lim\nolimits_{\xi\in\mathbf{ORD}} f_\mu^\xi \quad \textit{and} \quad \mathrm{gfp}(f) = \lim\nolimits_{\xi\in\mathbf{ORD}} f_\nu^\xi$$

PROOF. By a cardinality argument one shows that $\lim_{\xi\in\mathbf{ORD}} f_\mu^\xi$ and $\lim_{\xi\in\mathbf{ORD}} f_\nu^\xi$ are fixed-points and, by ordinal induction on ξ, one shows that they are respectively the least and the greatest ones. □

2. Modal mu-calculus

We now give the classical semantics of modal μ-calculus for Krikpe models. It deserves a special attention how it makes use of the denotation function for its algebraic properties. Further properties of this logic will be studied in Section 1 of Chapter 4. For further details see [**6**] and [**9**].

Fix a countably infinite set Prop of *atomic formulas* from now and for the rest of thesis.

Before formally stating the grammar which defines the formulas of modal logic we remark that throughout this thesis we will be working in *negation normal form*. There are two main reasons for this choice, but since one involves team semantics, for now we will only say that it is to simplify the constructions in modal μ-calculus. Hence, from now on, for $P \in \mathsf{Prop}$, we will write $\overline{P}$ to stand for the negation of P (i.e., $\neg P$) and say that it is a *negative occurrence* of P (in constrast to P being a *positive occurrence* of P).

Definition 2.14. The *formulas of (propositional) modal logic* are given by the following grammar:

$$\varphi ::= \top \mid \bot \mid P \mid \overline{P} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi$$

where P ranges over Prop. The set of all modal formulas will be denoted by $\mathtt{ML}$.

Formulas are denoted by lowercase Greek letter $\varphi, \psi, \chi \ldots$. Moreover, we will say that a formula φ is *positive in* $X \in \mathsf{Prop}$ if every occurrence of X in φ is not negated.

Definition 2.15. A *(Kripke) model* is a truple $\mathbb{S} = (S, R, V)$ where S is a set of *states*, $R \subseteq S \times S$ is an *accessibility relation* and $V : \mathsf{Prop} \to \mathcal{P}(S)$ is a *valuation*. We also say that a pair $\mathbb{F} = (S, R)$ is a *(Kripke) frame*. For a state $s \in S$ we write $R[s] := \{t \in S \mid sRt\}$.

Definition 2.16. *(Classical) semantics (for modal logic)* for a model $\mathbb{S} = (S, R, V)$ and a state $s \in S$ are defined by recursion as follows:

(1) $\mathbb{S}, s \models_{\mathsf{c}} \top$ always.
(2) $\mathbb{S}, s \models_{\mathsf{c}} \bot$ never.
(3) $\mathbb{S}, s \models_{\mathsf{c}} P$ if and only if $s \in V(P)$.
(4) $\mathbb{S}, s \models_{\mathsf{c}} \overline{P}$ if and only if $s \notin V(P)$.
(5) $\mathbb{S}, s \models_{\mathsf{c}} \varphi \wedge \psi$ if and only if $\mathbb{S}, s \models_{\mathsf{c}} \varphi$ and $\mathbb{S}, s \models_{\mathsf{c}} \psi$.
(6) $\mathbb{S}, s \models_{\mathsf{c}} \varphi \vee \psi$ if and only if $\mathbb{S}, s \models_{\mathsf{c}} \varphi$ or $\mathbb{S}, s \models_{\mathsf{c}} \psi$.
(7) $\mathbb{S}, s \models_{\mathsf{c}} \Diamond\varphi$ if and only if there is $t \in S$ such that sRt and $\mathbb{S}, t \models_{\mathsf{c}} \varphi$.
(8) $\mathbb{S}, s \models_{\mathsf{c}} \Box\varphi$ if and only if for every $t \in S$ such that sRt, $\mathbb{S}, t \models_{\mathsf{c}} \varphi$.

The subscript c (not written in the literature) stands for *classical* since later we will define team semantics for the same grammar. The *denotation* of a formula φ is the set $\|\varphi\|^{\mathbb{S}}_{\mathsf{c}} := \{s \in S \mid \mathbb{S}, s \models_{\mathsf{c}} \varphi\}$ and the *denotation function* $\|\cdot\|^{\mathbb{S}}_{\mathsf{c}}$ is a function from $\mathtt{ML}$ to $\mathcal{P}(S)$ which assigns, for every $\varphi \in \mathtt{ML}$, its denotation $\|\varphi\|^{\mathbb{S}}_{\mathsf{c}}$. Notice that the denotation and the denotation function also depend on the model $\mathbb{S}$.

PROPOSITION 2.17. *The recursive definition of the denotation function is as follows:*

$$\begin{aligned}
\|\top\|_{\mathsf{c}}^{\mathbb{S}} &= S \\
\|\bot\|_{\mathsf{c}}^{\mathbb{S}} &= \emptyset \\
\|P\|_{\mathsf{c}}^{\mathbb{S}} &= V(P) \\
\|\overline{P}\|_{\mathsf{c}}^{\mathbb{S}} &= S - V(P) \\
\|\varphi \wedge \psi\|_{\mathsf{c}}^{\mathbb{S}} &= \|\varphi\|_{\mathsf{c}}^{\mathbb{S}} \cap \|\psi\|_{\mathsf{c}}^{\mathbb{S}} \\
\|\varphi \vee \psi\|_{\mathsf{c}}^{\mathbb{S}} &= \|\varphi\|_{\mathsf{c}}^{\mathbb{S}} \cup \|\psi\|_{\mathsf{c}}^{\mathbb{S}} \\
\|\Diamond\varphi\|_{\mathsf{c}}^{\mathbb{S}} &= \langle R\rangle\|\varphi\|_{\mathsf{c}}^{\mathbb{S}} = \{s \in S \mid R[s] \cap \|\varphi\|_{\mathsf{c}}^{\mathbb{S}} \neq \emptyset\} \\
\|\Box\varphi\|_{\mathsf{c}}^{\mathbb{S}} &= [R]\|\varphi\|_{\mathsf{c}}^{\mathbb{S}} = \{s \in S \mid R[s] \subseteq \|\varphi\|_{\mathsf{c}}^{\mathbb{S}}\}
\end{aligned}$$

REMARK 2.18. The just stated proposition entails that a different approach would be to algebraically define the denotation function $\|\cdot\|_{\mathsf{c}}^{\mathbb{S}}$ (as above) and write $\mathbb{S}, s \models_{\mathsf{c}} \varphi$ if $s \in \|\varphi\|_{\mathsf{c}}^{\mathbb{S}}$.

REMARK 2.19. Notice how each logical connective and modality defines (or is related to) an operation on the set $\mathcal{P}(S)$. That is not surprising since, although $\|\cdot\|_{\mathsf{c}}^{\mathbb{S}}$ and $\mathbb{S}, \cdot \models_{\mathsf{c}} \cdot$ are simply two different notations for the semantics of modal logic, each of them has different implicit considerations. On the one hand, $\|\cdot\|_{\mathsf{c}}^{\mathbb{S}}$ can be algebraically seen as the unique homomorphism extending the valuation of $\mathbb{S}$ to the free algebra of formulas. That is, $\|\cdot\|_{\mathsf{c}}^{\mathbb{S}}$ is the unique homomorphism making the diagram

$$\begin{tikzcd}
\mathsf{Prop} \arrow[r, hook, "\iota"] \arrow[dr, "V"'] & \mathtt{ML} \arrow[d, "\|\cdot\|_{\mathsf{c}}^{\mathbb{S}}"] \\
& \mathcal{P}(S)
\end{tikzcd}$$

commute where $\mathcal{P}(S)$ is considered with structure $(\mathcal{P}(S), \cup, \cap, \langle R\rangle, [R])$. On the other hand, the semantic entailment relation $\mathbb{S}, \cdot \models_{\mathsf{c}} \cdot$ carries this algebraic structure in a more obscure way since it is defined as the semantics of a formal grammar. We will be using both notations indistinguisably since modal μ-calculus needs the algebraic counterpart of the semantics.

DEFINITION 2.20. Given a model $\mathbb{S} = (S, R, V)$ and a set $A \subseteq S$, we define the model $\mathbb{S}[X \mapsto A]$ for $X \in \mathsf{Prop}$ as the truple $(S, R, V[X \mapsto A])$ where $V[X \mapsto A]$ is the valuation:

$$V[X \mapsto A](P) := \begin{cases} V(P), & \text{if } P \neq X, \\ A, & \text{if } P = X. \end{cases}$$

The construction of the following function is the meeting point of modal logic and algebraic fixed-point theory for, as we have seen, every powerset can be given a structure of a complete lattice considering the inclusion order. This construction will be one of the main concepts that we will have to generalize in the next chapter.

THEOREM 2.21. *Let $\varphi \in \mathtt{ML}$ and $\mathbb{S}$ be a model. If φ is positive in X, then the function:*

$$\begin{aligned} \varphi_X^{\mathbb{S}} : \mathcal{P}(S) &\to \mathcal{P}(S) \\ A \quad &\mapsto \|\varphi\|_{\mathsf{c}}^{\mathbb{S}[X\mapsto A]} \end{aligned}$$

is monotone over $\subseteq$.

PROOF. By induction on φ. □

COROLLARY 2.22. *For every $\varphi \in \mathtt{ML}$ positive in X and every model $\mathbb{S}$ the function $\varphi_X^{\mathbb{S}}$ has a least and a greatest fixed-point.*

Now, the definition of the μ-calculus is given by syntactically adding two operators μ and ν which will be semantically understand, respectively, as the least and greatest fixed-point of the respective function. The result is a very powerful and expressible temporal logic.

DEFINITION 2.23. The *formulas of modal μ-calculus* are the extension of the modal formulas given by the following grammar:

$$\varphi ::= \top \mid \bot \mid P \mid \overline{P} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mu X.\varphi \mid \nu X.\varphi$$

where P and X range over Prop and φ is positive in X. The set of all modal μ-calculus formulas will be denoted by $\mu\mathtt{ML}$. Notice that $\mathtt{ML} \subseteq \mu\mathtt{ML}$.

DEFINITION 2.24. For a model $\mathbb{S} = (S, R, V)$ we extend the *(classical) denotation map* $\|\cdot\|_{\mathsf{c}}^{\mathbb{S}} : \mu\mathtt{ML} \to \mathcal{P}(S)$ as follows:

(9) $\|\mu X.\varphi\|_{\mathsf{c}}^{\mathbb{S}} := \mathrm{lfp}(\varphi_X^{\mathbb{S}}) = \bigcap\{A \subseteq S \mid \|\varphi\|_{\mathsf{c}}^{\mathbb{S}[X\mapsto A]} \subseteq A\}$;
(10) $\|\nu X.\varphi\|_{\mathsf{c}}^{\mathbb{S}} := \mathrm{gfp}(\varphi_X^{\mathbb{S}}) = \bigcup\{A \subseteq S \mid A \subseteq \|\varphi\|_{\mathsf{c}}^{\mathbb{S}[X\mapsto A]}\}$.

3. Team semantics for modal logic

This section is devoted to provide an introduction to team semantics for modal logic as it is generally defined in the literature and will serve as the basis for defining team semantics for modal μ-calculus in the next chapter. After defining the team semantical relation, we restrict attention to more general constructions common in team semantics. Moreover, we show that the defined semantics enjoy *flatness*, *downwards* and *union closure* and *singleton equivalence*. The main references for the definition and results presented are [**5, 13**].

DEFINITION 2.25. *Team semantics for modal logic* for a model $\mathbb{S} = (S, R, V)$ and a *team* $T \subseteq S$ is defined by recursion as follows:

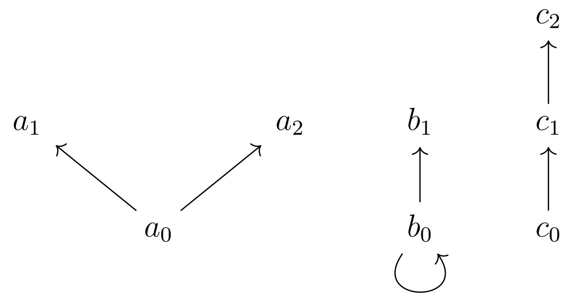
(1) $\mathbb{S}, T \models_{\mathtt{t}}^{\mathtt{ML}} \top$ always.
(2) $\mathbb{S}, T \models_{\mathtt{t}}^{\mathtt{ML}} \bot$ if and only if $T = \emptyset$.
(3) $\mathbb{S}, T \models_{\mathtt{t}}^{\mathtt{ML}} P$ if and only if $T \subseteq V(P)$.

(4) $\mathbb{S}, T \models^{\mathtt{ML}}_{\mathtt{t}} \overline{P}$ if and only if $T \cap V(P) = \emptyset$.
(5) $\mathbb{S}, T \models^{\mathtt{ML}}_{\mathtt{t}} \psi \vee \chi$ if and only if $T = A \cup B$, $\mathbb{S}, A \models^{\mathtt{ML}}_{\mathtt{t}} \psi$ and $\mathbb{S}, B \models^{\mathtt{ML}}_{\mathtt{t}} \chi$.
(6) $\mathbb{S}, T \models^{\mathtt{ML}}_{\mathtt{t}} \psi \wedge \chi$ if and only if $\mathbb{S}, T \models^{\mathtt{ML}}_{\mathtt{t}} \psi$ and $\mathbb{S}, T \models^{\mathtt{ML}}_{\mathtt{t}} \chi$.
(7) $\mathbb{S}, T \models^{\mathtt{ML}}_{\mathtt{t}} \Diamond\psi$ if and only if there is $T' \subseteq S$ such that $\mathbb{S}, T' \models^{\mathtt{ML}}_{\mathtt{t}} \psi$ and
  (a) for every $t \in T$, there is some $t' \in T'$ such that tRt';
  (b) for every $t' \in T'$, there is some $t \in T$ such that tRt'.
(8) $\mathbb{S}, T \models^{\mathtt{ML}}_{\mathtt{t}} \Box\psi$ if and only if $\mathbb{S}, \bigcup_{t \in T} R[t] \models^{\mathtt{ML}}_{\mathtt{t}} \psi$.

In this definition the subscript `t` stands for *team*. Notice that we have also added the superscript `ML` to denote the logic we are referring to. This is just for clarity since we will define team semantics other logics. Team semantics leads to the *(team) denotation* by writing $\|\varphi\|^{\mathbb{S}}_{\mathtt{t}} := \{T \in \mathcal{P}(S) \mid \mathbb{S}, T \models^{\mathtt{ML}}_{\mathtt{t}} \varphi\}$ and the *(team) denotation function* of the form $\|\cdot\|^{\mathbb{S}}_{\mathtt{t}} : \mathtt{ML} \to \mathcal{P}\mathcal{P}(S)$ which maps a formula $\varphi \in \mathtt{ML}$ to $\|\varphi\|^{\mathbb{S}}_{\mathtt{t}}$.

To introduce team semantics let us work out an example and compute some team denotations.

Example 2.26. Consider the frame



with the valuation defined as

$$V(P) := \{a_1, b_1, c_1, c_2\} \quad V(Q) := \{a_2, b_1, c_2\}$$

and let us denote it by $\mathbb{S}$. Let us compute the (team) denotations of $\Box P, \Diamond Q$ and $\Box(P \vee Q)$. Note that, by definition, $\|P\|^{\mathbb{S}}_{\mathtt{t}} = \mathcal{P}(V(P)) = \mathcal{P}(\{a_1, b_1, c_1, c_2\})$ and $\|Q\|^{\mathbb{S}}_{\mathtt{t}} = \mathcal{P}(\{a_2, b_1, c_2\})$.

(1) To compute the denotation of $\Box P$ (and later $\Box(P \vee Q)$), note that the states a_1, a_2, b_1 and c_2 do not play any role since they are endpoints (same as with classical semantics). Formally, if T is any set and s is an endpoint, that is, s is such that $R[s] = \emptyset$, then

$$\mathbb{S}, T \models^{\mathtt{ML}}_{\mathtt{t}} \Box\varphi \text{ if and only if } \mathbb{S}, T \cup \{s\} \models^{\mathtt{ML}}_{\mathtt{t}} \Box\varphi$$

for $\bigcup_{t \in T} R[t] = (\bigcup_{t \in T} R[t]) \cup R[s]$.

Hence, we concentrate ourselves on which subsets $T \subseteq \{a_0, b_0, c_0, c_1\}$ satisfy $\Box P$ and, therefore, the denotation would be all the sets of the form $T \cup A$ for $A \subseteq \{a_1, a_2, b_1, c_2\}$. By definition, we have to compute sets T such that $\bigcup_{t \in T} R[t] \in \|P\|^{\mathbb{S}}_{\mathtt{t}} = \mathcal{P}(\{a_1, b_1, c_1, c_2\})$, i.e., $\bigcup_{t \in T} R[t] \subseteq \{a_1, b_1, c_1, c_2\}$. Note that, for that it suffices to see that, for every $t \in T$, $R[t] \subseteq \{a_1, b_1, c_1, c_2\} = V(P)$, i.e., it suffices to see that, for every $t \in T$, $\mathbb{S}, t \models_{\mathtt{c}} \Box P$. As one can see, such states are c_0 and c_1. Thus,

$$\begin{aligned}\|\Box\varphi\|^{\mathbb{S}}_{\mathtt{t}} &= \{T \cup A \mid T \subseteq \{c_0, c_1\}, A \subseteq \{a_1, a_2, b_1, c_2\}\} \\ &= \mathcal{P}(\{a_1, a_2, b_1, c_0, c_1, c_2\}).\end{aligned}$$

where the last equality is easy to check.

(2) For the denotation of $\Diamond Q$ we have to consider sets which satisfy the forth and back conditions with some set of $\|Q\|^{\mathbb{S}}_{\mathtt{t}} = \mathcal{P}(\{a_2, b_1, c_2\})$. In this particular case, it is quite simple for any state of a_2, b_1 and c_2 only has one predecessor, namely, $a_0 \to a_2$, $b_0 \to b_1$ and $c_1 \to c_2$. Hence, the only possible such sets are the subsets of $\{a_0, b_0, c_1\}$, that is,

$$\|\Diamond Q\|^{\mathbb{S}}_{\mathtt{t}} = \mathcal{P}(\{a_0, b_0, c_1\}).$$

Interestingly enough, note that $\{a_0, b_0, c_1\} = \|\Diamond Q\|^{\mathbb{S}}_{\mathtt{c}}$.

(3) For the denotation of $\Box(P \vee Q)$ we follow the same reasoning as before and conclude that it is the collection of sets $T \cup A$ such that $T \subseteq \{a_0, b_0, c_0, c_1\}$ satisfy $\Box(P \vee Q)$ and $A \subseteq \{a_1, a_2, b_1, c_2\}$.

We begin by computing the denotation of $P \vee Q$. By definition, it is the collection of sets $A \cup B$ such that $\mathbb{S}, A \models^{\mathtt{ML}}_{\mathtt{t}} P$ and $\mathbb{S}, B \models^{\mathtt{ML}}_{\mathtt{t}} Q$, i.e., such that $A \subseteq \{a_1, b_1, c_1, c_2\}$ and $B \subseteq \{a_2, b_1, c_2\}$. As before, it is easy to see that,

$$\{A \cup B \mid A \subseteq \{a_1, b_1, c_1, c_2\}, B \subseteq \{a_2, b_1, c_2\}\} = \mathcal{P}(\{a_1, a_2, b_1, c_1, c_2\})$$

Let us now finish with the example. It is obvious that b_0 is the only state such that $R[t] \nsubseteq \{a_1, a_2, b_1, c_1, c_2\}$. Hence, the sets $T \subseteq \{a_0, b_0, c_0, c_1\}$ which satisfy $\Box(P \vee Q)$ are the sets $T \subseteq \{a_0, c_0, c_1\}$. That is,

$$\begin{aligned}\|\Box(P \vee Q)\|^{\mathbb{S}}_{\mathtt{t}} &= \{T \cup A \mid T \subseteq \{a_0, c_0, c_1\}, A \subseteq \{a_1, a_2, b_1, c_2\}\} \\ &= \mathcal{P}(\{a_0, a_1, a_2, b_1, c_0, c_1, c_2\})\end{aligned}$$

Where the last equality is easy to show. Surprisingly,

$$\|\Box(P \vee Q)\|^{\mathbb{S}}_{\mathtt{c}} = \{a_0, a_1, a_2, b_1, c_0, c_1, c_2\}.$$

The definition of team semantics for modal logic gives an obvious motivation for the study of double powerset structures that we wil carry out in the next chapter. However, our interest now is to see how the classical semantics and team semantics for modal logic, relate. Nonetheless, the restriction to modal logic is not necessary. That is why we will continue with a more general framework and start the study of such relations in general.

For a non-determined set of formulas $\mathtt{Fm}$, we consider the following diagram

$$\begin{tikzcd} \mathtt{Fm} \arrow[r, "\|\cdot\|_{\mathsf{c}}"] \arrow[rr, bend right, "\|\cdot\|_{\mathsf{t}}"'] & \mathcal{P}(X) \arrow[r, "\mathrm{Pow}"] & \mathcal{P}\mathcal{P}(X) \end{tikzcd}$$

which does not necessarily commute and where $\|\cdot\|_{\mathsf{c}}$ (or $\models_{\mathsf{c}}$) will be just refered as classical semantics, $\|\cdot\|_{\mathsf{t}}$ (or $\models_{\mathsf{t}}$) as team semantics and Pow is the *powerset lifting* which maps a set A to its powerset $\mathcal{P}(A)$. Let us say that the set X depends on the chosen logic. For instance:

(1) When considering propositional logic, X would be the set of all truth-assignments for the respective set of propositions.
(2) When considering first order logic, X would be the domain of the chosen model.
(3) When considering modal logic, X would be the set of all states in the chosen Kripke model.
(4) When considering some temporal logic such as linear temporal logic, X would be the set of all traces over the respective propositions.

From now on and for the rest of the thesis we will be usually working with double powersets. Hence, the following remark fixing some notation is necessary.

Remark 2.27. For the sake of redability we adopt the following notational conventions throughout the document. For some set X,

(1) Lowercase Latin letters $a, b, c, \dots$ denote elements of X.
(2) Uppercase Latin letters $A, B, T, U, V \dots$ (and similar) denote subsets of X, that is, elements of $\mathcal{P}(X)$. In the context of team semantics we will call them *teams*.
(3) Uppercase callygraphic Latin letters, $\mathcal{A}, \mathcal{B}, \dots$ denote subsets of teams, that is, elements of $\mathcal{P}\mathcal{P}(X)$. In the context of team semantics we will call them *leagues*.
(4) If neededd, uppercase Gothic Latin letters, $\mathfrak{A}, \mathfrak{B}, \dots$ denote subsets of leagues.

Definition 2.28. For a formula $\varphi \in \mathtt{Fm}$ and teams T and T_i for $i \in I$ we will say that

(1) φ has the *empty team property*, if $\emptyset \models_{\mathsf{t}} \varphi$.
(2) φ has the *singleton property*, if for every $t \in X$, $\{t\} \models_{\mathsf{t}} \varphi$ if and only if $t \models_{\mathsf{c}} \varphi$.
(3) φ is *downwards closed*, if whenver $T \models_{\mathsf{t}} \varphi$ and $U \subseteq T$, then $U \models_{\mathsf{t}} \varphi$.
(4) φ is *union closed*, if whenever $T_i \models_{\mathsf{t}} \varphi$ for all $i \in I$, then $\bigcup_{i \in I} T_i \models_{\mathsf{t}} \varphi$.
(5) φ is *flat*, if $T \models_{\mathsf{t}} \varphi$ if and only if for every $t \in T$, $\{t\} \models_{\mathsf{t}} \varphi$.

We say that $\|\cdot\|_{\mathsf{t}}$ has the empty team property (resp. the singleton property, is downwards closed, union closed or flat) if *every* $\varphi \in \mathtt{Fm}$ has the empty team property (resp. the singleton property, is downwards closed, union closed or flat).

More in general, one can define this notions (except the sigleton property) for any set $\mathcal{A} \in \mathcal{PP}(X)$.

DEFINITION 2.29. For a set $\mathcal{A} \in \mathcal{PP}(X)$ and teams T and T_i for $i \in I$ we will say that

(1) $\mathcal{A}$ has the *empty team property*, if $\emptyset \in \mathcal{A}$.
(2) $\mathcal{A}$ is *downwards closed*, if whenver $T \in \mathcal{A}$ and $U \subseteq T$, then $U \in \mathcal{A}$.
(3) $\mathcal{A}$ is *union closed*, if whenever $T_i \in \mathcal{A}$ for all $i \in I$, then $\bigcup_{i \in I} T_i \in \mathcal{A}$.
(4) φ is *flat*, if $T \in \mathcal{A}$ if and only if for every $t \in T$, $\{t\} \in \mathcal{A}$.

The following are easy to show.

PROPOSITION 2.30. *The following relations hold.*

(1) *If φ is union closed, then φ enjoys the empty team property.*
(2) *φ is flat if and only if φ is downwards and union closed.*

PROOF. (1) It suffices to consider the case where $I = \emptyset$.

(2) Let φ be a flat formula. We want to show that φ is downwards and union closed, i.e., that if $T \models_{\mathsf{t}} \varphi$ and $U \subseteq T$, then $U \models_{\mathsf{t}} \varphi$ and that if, for every $i \in I$, $T_i \models_{\mathsf{t}} \varphi$, then $\bigcup_{i \in I} T_i \models_{\mathsf{t}} \varphi$.

Let T be a set such that $T \models_{\mathsf{t}} \varphi$ and $U \subseteq T$. By flatness, for every $t \in T$, $\{t\} \models_{\mathsf{t}} \varphi$. Hence, for every $u \in U \subseteq T$, $\{u\} \models_{\mathsf{t}} \varphi$ and, by flatness, $U \models_{\mathsf{t}} \varphi$. Moreover, let T_i be such that, for every $i \in I$, $T_i \models_{\mathsf{t}} \varphi$. Thefore, for every $i \in I$ and every $t \in T_i$, $\{t\} \models_{\mathsf{t}} \varphi$. Hence, for every $t \in \bigcup_{i \in I} T_i$, since $t \in T_j$ for some j, $\{t\} \models_{\mathsf{t}} \varphi$ and $\bigcup_{i \in I} T_i \models_{\mathsf{t}} \varphi$ by flatness.

Let φ be downwards and union closed. We want to show that φ is flat, i.e., that for every set T, $T \models_{\mathsf{t}} \varphi$ if and only if for every $t \in T$, $\{t\} \models_{\mathsf{t}} \varphi$.

Let T be such that $T \models_{\mathsf{t}} \varphi$. Since, for every $t \in T$, $\{t\} \subseteq T$, by downwads closure, $\{t\} \models_{\mathsf{t}} \varphi$. Conversely, let T be such that, for every $t \in T$, $\{t\} \models_{\mathsf{t}} \varphi$. By union closure, $\bigcup_{t \in T} \{t\} = T \models_{\mathsf{t}} \varphi$. □

PROPOSITION 2.31. *The following are equivalent.*

(1) *φ is flat and has the singleton property.*
(2) *φ is downwads and union closed and has the singleton property.*
(3) $\|\varphi\|_{\mathsf{t}} = \mathcal{P}(\|\varphi\|_{\mathsf{c}})$.

PROOF. By Proposition 2.30, (1) and (2) are equivalent. We now show that (1) implies (3) and that (3) implies (2).

Let φ be a formula which satisfies condition (1). We want to show that $\|\varphi\|_{\mathsf{t}} = \mathcal{P}(\|\varphi\|_{\mathsf{c}})$. But for that it suffices to notice that condition (1) is equivalent to the condition, $T \models_{\mathsf{t}} \varphi$ if and only for every $t \in T$, $t \models_{\mathsf{c}} \varphi$, i.e., $T \in \|\varphi\|_{\mathsf{t}}$ if and only if $T \subseteq \|\varphi\|_{\mathsf{c}}$.

Let φ be a formula such that $\|\varphi\|_{\mathsf{t}} = \mathcal{P}(\|\varphi\|_{\mathsf{c}})$. We want to show that φ enjoys downwards and union closure and singleton property. Singleton property follows inmediately since $\{t\} \in \|\varphi\|_{\mathsf{t}}$ if and only if $\{t\} \subseteq \|\varphi\|_{\mathsf{c}}$, that is, $t \in \|\varphi\|_{\mathsf{c}}$. Finally, note that every powerset is downwards and union closed. $\square$

Remark 2.32. Note that, if condition (3) of Propostion 2.31 is satisfied for every formula $\varphi \in \mathtt{Fm}$, then the diagram

$$\begin{tikzcd} \mathtt{Fm} \arrow[r, "\|\cdot\|_{\mathsf{c}}"] \arrow[rr, bend right, "\|\cdot\|_{\mathsf{t}}"'] & \mathcal{P}(X) \arrow[r, "\mathrm{Pow}"] & \mathcal{P}\mathcal{P}(X) \end{tikzcd}$$

commutes. This can be seen as a motivation for saying that flatness and singleton property are desirable and natural conditions for a team semantics to have and note that it has consequences which can be seen as *natural*. For instance, it is sensible to think that, if $T \models_{\mathsf{t}} \varphi$, then for any subset $U \subseteq T$, $U \models_{\mathsf{t}} \varphi$ or at least for every non-empty U. Or, as a corollary of union closure, that the emtpy team satisfies every possible formula.

Now, the last point of this section is to show that the defined semantics for modal logic, indeed enjoy flatness and singleton equivalence, thus, all its consequences.

Proposition 2.33 (Flatness and singleton equivalence). *For every model $\mathbb{S}$, every team $T \subseteq S$ and every formula $\varphi \in \mathtt{ML}$, the following are equivalent.*

(1) $\mathbb{S}, T \models^{\mathtt{ML}}_{\mathsf{t}} \varphi$;
(2) *for every* $t \in T$, $\mathbb{S}, \{t\} \models^{\mathtt{ML}}_{\mathsf{t}} \varphi$;
(3) *for every* $t \in T$, $\mathbb{S}, t \models_{\mathsf{c}} \varphi$.

Proof. Let $\mathbb{S}$ be a model. By Proposition 2.31, it suffices to show that, for every formula φ, $T \in \|\varphi\|^{\mathbb{S}}_{\mathsf{t}}$ if and only if $T \subseteq \|\varphi\|^{\mathbb{S}}_{\mathsf{c}}$. We prove this equivalence by induction on φ.

Base case. By definition, $\|\top\|^{\mathbb{S}}_{\mathsf{t}} = \mathcal{P}(S)$, $\|\top\|^{\mathbb{S}}_{\mathsf{c}} = S$, $\|\bot\|^{\mathbb{S}}_{\mathsf{t}} = \emptyset$ and $\|\bot\|^{\mathbb{S}}_{\mathsf{c}} = \emptyset$. Hence, the statement for $\top$ and $\bot$ follows.

For literals, note that $T \in \|\mathsf{P}\|^{\mathbb{S}}_{\mathsf{t}}$ if and only if $T \subseteq V(P) = \|\varphi\|^{\mathbb{S}}_{\mathsf{c}}$ and $T \in \|\overline{P}\|^{\mathbb{S}}_{\mathsf{t}}$ if and only if $T \cap V(P) = \emptyset$, that is, $T \subseteq S - V(P) = \|\overline{P}\|^{\mathbb{S}}_{\mathsf{c}}$.

Boolean cases. Assume that the statement holds for φ, ψ. We restrict ourselves to show the statement for disjuction. The statement for conjunction follows by a similar argument.

Let $T \in \|\varphi \vee \psi\|^{\mathbb{S}}_{\mathsf{t}}$, i.e., $T = A \cup B$ for $A \in \|\varphi\|^{\mathbb{S}}_{\mathsf{t}}$ and $B \in \|\psi\|^{\mathbb{S}}_{\mathsf{t}}$. By induction hypothesis it follows that $A \subseteq \|\varphi\|^{\mathbb{S}}_{\mathsf{c}}$ and $B \subseteq \|\psi\|^{\mathbb{S}}_{\mathsf{c}}$ and $A \cup B \subseteq \|\varphi\|^{\mathbb{S}}_{\mathsf{c}} \cup \|\psi\|^{\mathbb{S}}_{\mathsf{c}} = \|\varphi \vee \psi\|^{\mathbb{S}}_{\mathsf{c}}$. Conversely, let $T \subseteq \|\varphi \vee \psi\|^{\mathbb{S}}_{\mathsf{c}} = \|\psi\|^{\mathbb{S}}_{\mathsf{c}} \cup \|\psi\|^{\mathbb{S}}_{\mathsf{c}}$. It suffices to consider $A := T \cap \|\varphi\|^{\mathbb{S}}_{\mathsf{c}} \subseteq \|\varphi\|^{\mathbb{S}}_{\mathsf{c}}$ and $B := T \cap \|\psi\|^{\mathbb{S}}_{\mathsf{c}} \subseteq \|\psi\|^{\mathbb{S}}_{\mathsf{c}}$ and conclude by induction hypothesis since $A \cup B = T$.

Modal cases. Assume that the statement holds for φ. We show the statement for $\Diamond\varphi$ and $\Box\varphi$.

Case $\Diamond$. Let $T \in \|\Diamond\varphi\|^{\mathbb{S}}_{\mathsf{t}}$, that is, $\mathbb{S}, T \models^{\mathtt{ML}}_{\mathsf{t}} \Diamond\varphi$, i.e., there is a set $U \subseteq S$ such that (1) for every $t \in T$, there is $u \in U$ such that tRu, (2) for every $u \in U$, there is $t \in T$ such that tRu and (3) $\mathbb{S}, U \models^{\mathtt{ML}}_{\mathsf{t}} \varphi$. We want to show that $T \subseteq \|\Diamond\varphi\|^{\mathbb{S}}_{\mathsf{c}}$. Let $t \in T$. By induction hypothesis, $U \subseteq \|\varphi\|^{\mathbb{S}}_{\mathsf{c}}$. Since $TR_{\mathsf{EM}}U$, there is a $u \in U \subseteq \|\varphi\|^{\mathbb{S}}_{\mathsf{c}}$ such that tRu and so $\mathbb{S}, t \models_{\mathsf{c}} \Diamond\varphi$. Conversely, let T be such that $T \subseteq \|\Diamond\varphi\|^{\mathbb{S}}_{\mathsf{c}}$. We want to show that $T \in \|\Diamond\varphi\|^{\mathbb{S}}_{\mathsf{t}}$. By assumption, for every $t \in T$ there is $s_t \in S$ such that tRs_t and $\mathbb{S}, s_t \models_{\mathsf{c}} \varphi$. Consider the set $Z := \{s_t \in S \mid t \in T\} \subseteq \|\varphi\|^{\mathbb{S}}_{\mathsf{c}}$. By construction, Z satisfies the necessary conditions and, by induction hypothesis, $\mathbb{S}, Z \models^{\mathtt{ML}}_{\mathsf{t}} \varphi$.

Case $\Box$. Let $T \in \|\Box\varphi\|^{\mathbb{S}}_{\mathsf{t}}$, that is, $\mathbb{S}, \bigcup_{t\in T} R[t] \models^{\mathtt{ML}}_{\mathsf{t}} \varphi$. We want to show that $T \subseteq \|\Box\varphi\|^{\mathbb{S}}_{\mathsf{c}}$. Let $s \in T$. By induction hypothesis, $R[s] \subseteq \bigcup_{t\in T} \subseteq \|\varphi\|^{\mathbb{S}}_{\mathsf{c}}$ and so $\mathbb{S}, s \models_{\mathsf{c}} \Box\varphi$. Conversely, let T be such that $T \subseteq \|\Box\varphi\|^{\mathbb{S}}_{\mathsf{c}}$, that is, for every $t \in T$, $\mathbb{S}, t \models_{\mathsf{c}} \Box\varphi$. We want to show that $\mathbb{S}, T \models^{\mathtt{ML}}_{\mathsf{t}} \Box\varphi$, i.e., $\mathbb{S}, \bigcup_{t\in T} \models^{\mathtt{ML}}_{\mathsf{t}} \varphi$. By induction hypothesis, it suffices to show that $\bigcup_{t\in T} R[t] \subseteq \|\varphi\|^{\mathbb{S}}_{\mathsf{c}}$ but that follows since, by assumption, for every $t \in T$, $R[t] \subseteq \|\varphi\|^{\mathbb{S}}_{\mathsf{c}}$. $\Box$

Remark 2.34. Notice that, to show that $T \in \|\Diamond\varphi\|^{\mathbb{S}}_{\mathsf{t}}$ if and only if $T \subseteq \|\Diamond\varphi\|^{\mathbb{S}}_{\mathsf{c}}$ the condition that, for every $u \in U$, there is a $t \in T$ such that tRu was never used. That means that the definition of the team semantics for $\Diamond$ can be simplified. We will say more on this later.

Corollary 2.35 (Empty team property). *For every model $\mathbb{S}$ and every formula $\varphi \in \mathtt{ML}$, $\mathbb{S}, \emptyset \models^{\mathtt{ML}}_{\mathsf{t}} \varphi$.*

Corollary 2.36 (Downwards and union closure). *For every model $\mathbb{S}$, every $A, B \subseteq S$ and every formula $\varphi \in \mathtt{ML}$,*

(1) *If $\mathbb{S}, T_i \models^{\mathtt{ML}}_{\mathsf{t}} \varphi$ for $i \in I$, then $\mathbb{S}, \bigcup_{i\in I} T_i \models^{\mathtt{ML}}_{\mathsf{t}} \varphi$.*
(2) *If $\mathbb{S}, T \models^{\mathtt{ML}}_{\mathsf{t}} \varphi$ and $U \subseteq T$, then $\mathbb{S}, U \models^{\mathtt{ML}}_{\mathsf{t}} \varphi$.*

4. Powerset structures

In this section we define, following the study of power structures done in [**4**], power liftings of relations and algebraic operations. In particular, we study the properties of the lifting of the inclusion relation $\subseteq$ and the operations of union $\cup$ and intersection $\cap$. Moreover, we work out the maximum, minimum, supremum and infimums of the order liftings over double powersets.

The work done in this section provides an algebraic glimpse of the ideas fully developed in Chapter 3 about internal and external operations. It also serves as a motivation for the choice of the ordered structure over which we will interpret fixed-point operators. Although the results stated in this section are not strictly necessary for the thesis, we will be using the definitions often.

During this section fix X to be an arbitrary set.

DEFINITION 2.37. Let $R \subseteq X \times X$ be a binary relation. The *lower* and *upper* liftings of R are the relations over $\mathcal{P}(X)$ defined as follows:

$$AR_{\mathsf{L}}B \quad \text{if and only if} \quad \forall a \in A\ \exists b \in B\ (aRb)$$
$$AR_{\mathsf{U}}B \quad \text{if and only if} \quad \forall b \in B\ \exists a \in A\ (aRb)$$

Moreover, the *Egli-Milner lifting* of R is the relation given by

$$AR_{\mathsf{EM}}B \quad \text{if and only if} \quad AR_{\mathsf{L}}B \text{ and } AR_{\mathsf{U}}B$$

We will write R_* to mean any of the three liftings defined above.

DEFINITION 2.38. Let $f : X^n \to X$ be an operation for $n \geq 1$. Its *power operation*, $f^{\mathcal{P}} : \mathcal{P}(X)^n \to \mathcal{P}(X)$ is defined, for every $A_0, \dots A_{n-1} \subseteq X$, as

$$f^{\mathcal{P}}(A_0, \dots, A_{n-1}) := \{f(a_0, \dots a_{n-1}) \mid a_i \in A_i \text{ for } i \in n\}.$$

Moreover, if $f : X^0 \to X$ is a nullary operation (i.e., a constant), its power operation $f^{\mathcal{P}} : \mathcal{P}(X)^0 \to \mathcal{P}(X)$ is defined by $f^{\mathcal{P}}(\emptyset) := \{f(\emptyset)\}$.

There is an special case of relation and operation liftings, namely, when the original operations can also be considered over the powerset. In this situations the study of the liftings becomes more interesting because we can also study the relation between the original relation/operation and its liftings. In particular in the case of double powerset structures for which we can consider the inclusion order, binary union and binary intersection and their respectives liftings.

Let us consider the liftings of the inclusion relation $\subseteq$ over $\mathcal{PP}(X)$ as $\sqsubseteq_{\mathsf{L}}, \sqsubseteq_{\mathsf{U}}$ and $\sqsubseteq_{\mathsf{EM}}$ and we will write the power operations of binary union and intersection as $\sqcup$ and $\sqcap$. For completeness, let us recall that they are defined as:

$$\mathcal{A} \sqcup \mathcal{B} = \{A \cup B \mid A \in \mathcal{A} \text{ and } B \in \mathcal{B}\} \qquad \text{(Internal union)}$$
$$\mathcal{A} \sqcap \mathcal{B} = \{A \cap B \mid A \in \mathcal{A} \text{ and } B \in \mathcal{B}\} \qquad \text{(Internal intersection)}$$

Hence, we will consider $\mathcal{PP}(X)$ with two different structures:

$$(\mathcal{PP}(X), \subseteq, \cup, \cap) \qquad (\mathcal{PP}(X), \sqsubseteq_*, \sqcup, \sqcap)$$

where $*$ is any of the lower, upper and Egli-Milner liftings. We will refer to the first one as the *external structure* and to the second one as the *internal strcuture*. While in the external structure $\mathcal{PP}(X)$ is regarded as a powerset, hence working with $\mathcal{A} \subseteq \mathcal{B}, \mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \cap \mathcal{B}$, in the internal structure $\mathcal{PP}(X)$ is regarded as a lifting of $\mathcal{P}(X)$, hence working with $\mathcal{A} \sqsubseteq_* \mathcal{B}, \mathcal{A} \sqcup \mathcal{B}$ and $\mathcal{A} \sqcap \mathcal{B}$.

We next proceed to investigate the properties of $\sqsubseteq_*, \sqcup$ and $\sqcap$. Specially we study the lower, upper and Egli-Milner liftings showing they dif and only ifer with respect to the properties of interest and motivate the choice of $\sqsubseteq_{\mathsf{L}}$ as the order to consider in the construction of team semantics for modal μ-calculus.

We begin by investigating the properties of $\sqsubseteq_*$ with respect to $\subseteq$ and the empty set $\emptyset$.

Proposition 2.39. *For $A, B \subseteq X$, the following are equivalent.*

(1) $A \subseteq B$.
(2) $\{A\} \sqsubseteq_{\mathsf{EM}} \{B\}$.
(3) $\{A\} \sqsubseteq_{\mathsf{U}} \{B\}$.
(4) $\{A\} \sqsubseteq_{\mathsf{L}} \{B\}$.

Proof. Clear. $\square$

Proposition 2.40. *For $\mathcal{A}, \mathcal{B} \in \mathcal{PP}(X)$, $\mathcal{A} \subseteq \mathcal{B}$ implies $\mathcal{A} \sqsubseteq_{\mathsf{L}} \mathcal{B}$ and $\mathcal{B} \sqsubseteq_{\mathsf{U}} \mathcal{A}$.*

Proof. Follows from reflexivity of $\subseteq$. $\square$

Remark 2.41. $\sqsubseteq_{\mathsf{EM}}$ does not preserve $\subseteq$ in any form, that is, there are $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$ but $\mathcal{A} \not\sqsubseteq_{\mathsf{EM}} \mathcal{B}$ and $\mathcal{B} \not\sqsubseteq_{\mathsf{EM}} \mathcal{A}$, see [**4**].

Proposition 2.42. *For $\mathcal{A} \in \mathcal{PP}(X)$. The following relations hold.*

(1) $\emptyset \sqsubseteq_{\mathsf{L}} \emptyset, \emptyset \sqsubseteq_{\mathsf{L}} A$ *and* $A \not\sqsubseteq_{\mathsf{L}} \emptyset$.
(2) $\emptyset \sqsubseteq_{\mathsf{U}} \emptyset, \emptyset \not\sqsubseteq_{\mathsf{U}} A$ *and* $A \sqsubseteq_{\mathsf{U}} \emptyset$.
(3) $\emptyset \sqsubseteq_{\mathsf{EM}} \emptyset, \emptyset \not\sqsubseteq_{\mathsf{EM}} A$ *and* $A \not\sqsubseteq_{\mathsf{EM}} \emptyset$.

Proof. (3) is a consequence of (1) and (2) which follows either from the existential quantification or vacuously from the universal quantification in the definition of the relations $\sqsubseteq_*$. $\square$

Proposition 2.43. *The following holds.*

(1) $\sqsubseteq_*$ *are reflexive;*
(2) $\sqsubseteq_*$ *are transitive;*
(3) *None of* $\sqsubseteq_*$ *are antisymmetric.*

Proof. (1) follows from reflexivity of $\subseteq$ and (2) follows from transitivity of $\subseteq$. For (3) we show that $\sqsubseteq_{\mathsf{EM}}$ is not antisymmetric from which the antisymmetry of $\sqsubseteq_{\mathsf{L}}$ and $\sqsubseteq_{\mathsf{U}}$ follow. Consider the set $X = \{0, 1\}$ and we depict the ordered set $(\mathcal{P}(X), \subseteq)$ below.

$$\begin{tikzcd}
& \{0,1\} & \\
\{0\} \arrow[ur] & & \{1\} \arrow[ul] \\
& \emptyset \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

We ommit the description of $(\mathcal{PP}(X), \sqsubseteq_{\mathsf{EM}})$ for simplicity and refer the reader to [**4**] for a complete description.

On the one hand, $\{\{0,1\}, \emptyset\} \sqsubseteq_{\mathsf{EM}} \{\{0,1\}, \{0\}, \{1\}, \emptyset\}$. First,

$$\{\{0,1\}, \emptyset\} \sqsubseteq_{\mathsf{L}} \{\{0,1\}, \{0\}, \{1\}, \emptyset\}$$

because $\{\{0,1\},\emptyset\} \subseteq \{\{0,1\},\{0\},\{1\},\emptyset\}$. Moreover,

$$\{\{0,1\},\emptyset\} \sqsubseteq_{\mathsf{U}} \{\{0,1\},\{0\},\{1\},\emptyset\}$$

because, for $\{0,1\}$ and $\perp$ the condition is obvious and for $\{0\}$ and $\{1\}$ one can simply take $\emptyset$ to witness the existential quantification since $\emptyset \subseteq \{0\}$ and $\emptyset \subseteq \{1\}$.

On the other hand, $\{\{0,1\},\{0\},\{1\},\emptyset\} \sqsubseteq_{\mathsf{EM}} \{\{0,1\},\emptyset\}$. First,

$$\{\{0,1\},\{0\},\{1\},\emptyset\} \sqsubseteq_{\mathsf{U}} \{\{0,1\},\emptyset\}$$

because $\{\{0,1\},\emptyset\} \subseteq \{\{0,1\},\{0\},\{1\},\emptyset\}$. Moreover,

$$\{\{0,1\},\{0\},\{1\},\emptyset\} \sqsubseteq_{\mathsf{L}} \{\{0,1\},\emptyset\}$$

because, for $\{0,1\}$ and $\perp$ the condition is obvious and for $\{0\}$ and $\{1\}$ one can simply take $\{0,1\}$ to witness the existential quantification since $\{0\} \subseteq \{0,1\}$ and $\{1\} \subseteq \{0,1\}$. □

Corollary 2.44. *$(\mathcal{PP}(X), \sqsubseteq_{\mathsf{L}}), (\mathcal{PP}(X), \sqsubseteq_{\mathsf{U}})$ and $(\mathcal{PP}(X), \sqsubseteq_{\mathsf{EM}})$ are preordered sets.*

Remark 2.45. Proposition 2.42 shows the *weird* behavour of $\emptyset$ with respect to $\sqsubseteq_*$ (especially $\sqsubseteq_{\mathsf{EM}}$). Moreover, if we agree that the emtpy team property is desirable, then any possible denotation is never empty. That is, since for every formula φ, $\emptyset \in \|\varphi\|_{\mathsf{t}}$, then $\|\varphi\|_{\mathsf{t}} \neq \emptyset$. Hence, for algebraic and logical reasons, instead of considering $\mathcal{PP}(X)$ we will consider $\mathcal{PP}(X) - \{\emptyset\}$ and we will denote by $\mathcal{P}^*\mathcal{P}(X)$ where we write in general $\mathcal{P}^*(X)$ for $\mathcal{P}(X) - \{\emptyset\}$. Moreover, when talking about the internal structure we will refer to the structure $(\mathcal{P}^*\mathcal{P}(X), \sqsubseteq_*, \sqcup, \sqcap)$.

Remark 2.46. It is necessary to note that antisymetry is necessary in order theory for the uniqueness of many constructions, e.g., maximum, minimum, supremum and infimum. Despite $(\mathcal{P}^*\mathcal{P}(X), \sqsubseteq_*)$ not enjoying antisymmetry, we still think that it is a structure rich enough to carry out our research. That is why many of the constructions will be considered *up to $\equiv_*$ equivalence* where $\mathcal{A} \equiv_* \mathcal{B}$ holds if and only if $\mathcal{A} \sqsubseteq_* \mathcal{B}$ and $\mathcal{B} \sqsubseteq_* \mathcal{A}$. It is easy to see that $\equiv_*$ defines an equivalence relation over $\mathcal{P}^*\mathcal{P}(X)$ and that $\sqsubseteq_*$ defines an order over $\mathcal{P}^*\mathcal{P}(X)/\equiv_*$. As it is standard, when needed, we will write $[\mathcal{A}]_{\equiv_*}$ for the equivalence class of $\mathcal{A}$. However, there are two reasons to not consider the quotient set. (1) To the best of our knowledge, there is not an easy description (in general) of the equivalence clases of $\equiv_*$. Therefore, working in the quotient set is messy and unclear. (2) Being $\equiv_*$ equivalent does not preserve semantical equivalence. That is, two sets $\mathcal{A}$ and $\mathcal{B}$ might be $\equiv_*$ equivalent, but it is not equivalent for a formula φ, $\|\varphi\|_{\mathsf{t}} = \mathcal{A}$ or $\|\varphi\|_{\mathsf{t}} = \mathcal{B}$. Thus, working with $\equiv_*$ equivalence classes seems not to be desirable in team semantics.

Before proceeding any further with the order theoretical study of $\sqsubseteq_*$, let us define the generalizations of internal union and intersection.

Definition 2.47. For a non-empty set $\mathfrak{A} \subseteq \mathcal{P}^*\mathcal{P}(X)$ we define:

$$\begin{aligned}\bigsqcup \mathfrak{A} &:= \{\textstyle\bigcup_{\mathcal{Z}\in\mathfrak{A}} f(\mathcal{Z}) \mid f \in \prod \mathfrak{A}\}\\ \bigsqcap \mathfrak{A} &:= \{\textstyle\bigcap_{\mathcal{Z}\in\mathfrak{A}} f(\mathcal{Z}) \mid f \in \prod \mathfrak{A}\}\end{aligned}$$

where $\prod \mathfrak{A}$ is the usual set-theoretic generalised cartesian product defined as

$$\prod \mathfrak{A} := \{f : \mathfrak{A} \to \bigcup \mathfrak{A} \mid \text{for every } \mathcal{A} \in \mathfrak{A},\ f(\mathcal{A}) \in \mathcal{A}\}.$$

Remark 2.48. The definition of $\bigsqcup$ and $\bigsqcap$ are defined only for non-empty sets for, by definition, $\bigsqcup \emptyset = \emptyset \notin \mathcal{P}^*\mathcal{P}(X)$ and $\bigsqcap \emptyset$ would not be even defined. As we will see we will only use such sets for a non-emtpy $\mathcal{A}$. Although we do not need its definition, in light of the following propositions, we can define $\bigsqcup \emptyset := \{\emptyset\}$ and $\bigsqcap \emptyset := \{X\}$.

It is clear that the general definition generalizes the binary one, that is:

$$\begin{aligned}\bigsqcup\{\mathcal{A}, \mathcal{B}\} &= \mathcal{A} \sqcup \mathcal{B}\\ \bigsqcap\{\mathcal{A}, \mathcal{B}\} &= \mathcal{A} \sqcap \mathcal{B}\end{aligned}$$

Proposition 2.49. *The following hold.*

(1) $[\{X\}]_{\equiv_\mathsf{L}} = \{\mathcal{A} \in \mathcal{P}^*\mathcal{P}(X) \mid X \in \mathcal{A}\}$. *In particular,* $\mathcal{P}(X)$ *is* $\equiv_\mathsf{L}$ *equivalent to* $\{X\}$.
(2) $\{X\}$ *is the maximum of* $\sqsubseteq_\mathsf{L}$ *up to* $\equiv_\mathsf{L}$ *equivalence.*
(3) $[\{\emptyset\}]_{\equiv_\mathsf{L}} = \{\{\emptyset\}\}$.
(4) $\{\emptyset\}$ *is the minimum of* $\sqsubseteq_\mathsf{L}$ *up to* $\equiv_\mathsf{L}$ *equivalence.*
(5) *If* $\mathfrak{A}$ *is non-empty,* $\bigcup \mathfrak{A}$ *is the supremum of* $\mathfrak{A}$ *with respect to* $\sqsubseteq_\mathsf{L}$ *up to* $\equiv_\mathsf{L}$ *equivalence.*
(6) *If* $\mathfrak{A}$ *is non-empty,* $\bigsqcap \mathfrak{A}$ *is the infimum of* $\mathfrak{A}$ *with respect to* $\sqsubseteq_\mathsf{L}$ *up to* $\equiv_\mathsf{L}$ *equivalence.*

Proof. (1) We show both inclusions. Let $\mathcal{A} \in \mathcal{P}\mathcal{P}(X)$ be such that $X \in \mathcal{A}$. We show that $\mathcal{A} \equiv_\mathsf{L} \{X\}$ and so

$$[\{X\}]_{\equiv_\mathsf{L}} \supseteq \{\mathcal{A} \in \mathcal{P}\mathcal{P}(X) \mid X \in \mathcal{A}\}.$$

That $\mathcal{A} \sqsubseteq_\mathsf{L}$ is obvious since, for every $A \in \mathcal{A}$, $A \subseteq X$. Also, for $X \in \{X\}$, since by assumption $X \in \mathcal{A}$ and $X \subseteq X$, $\{X\} \sqsubseteq_\mathsf{L} \mathcal{A}$.

Conversely, suppose that $\mathcal{A} \equiv_\mathsf{L} \{X\}$. In particular, $\{X\} \sqsubseteq_\mathsf{L} \mathcal{A}$ and there should exists $A \in \mathcal{A}$ such that $X \subseteq A$ thus, $A = X \in \mathcal{A}$ and

$$[\{X\}]_{\equiv_\mathsf{L}} \subseteq \{\mathcal{A} \in \mathcal{P}\mathcal{P}(X) \mid X \in \mathcal{A}\}.$$

(2) Take $\mathcal{A} \in \mathcal{P}\mathcal{P}(X)$. Since for every $A \in \mathcal{A}$, $A \subseteq X$, it follows that $\mathcal{A} \sqsubseteq_\mathsf{L} \{X\}$.

(3) It is obvious that

$$[\{\emptyset\}]_{\equiv_\mathsf{L}} \supseteq \{\{\emptyset\}\}$$

by reflexivity of $\sqsubseteq_\mathsf{L}$. Let us show the converse inclusion. Let $\mathcal{A} \in \mathcal{P}\mathcal{P}(X)$ be such that $\mathcal{A} \equiv_\mathsf{L} \{\emptyset\}$. Hence, for every $A \in \mathcal{A}$, $A \subseteq \emptyset$, that is, $A = \emptyset$ and

$$[\{\emptyset\}]_{\equiv_{\mathsf{L}}} \subseteq \{\{\emptyset\}\}.$$

(4) Take $\mathcal{A} \in \mathcal{PP}(X)$. Since, for every $A \in \mathcal{A}$, $\emptyset \subseteq A$, it follows that $\{\emptyset\} \sqsubseteq_{\mathsf{L}} \mathcal{A}$.

(5) We show that $\bigcup \mathfrak{A}$ is an upper bound of $\mathfrak{A}$ and that it is the least such, i.e., that for every $\mathcal{A} \in \mathfrak{A}$, $\mathcal{A} \sqsubseteq_{\mathsf{L}} \bigcup \mathfrak{A}$ and that if $\mathcal{X} \in \mathcal{PP}(X)$ is such that, for every $\mathcal{A} \in \mathfrak{A}$, $\mathcal{A} \sqsubseteq_{\mathsf{L}} \mathcal{X}$, then $\bigcup \mathfrak{A} \sqsubseteq_{\mathsf{L}} \mathcal{X}$.

That, $\bigcup \mathfrak{A}$ is an upper bound of $\mathfrak{A}$ is clear because, for every $\mathcal{A} \in \mathfrak{A}$, $\mathcal{A} \subseteq \bigcup \mathfrak{A}$.

Let $\mathcal{X} \in \mathcal{PP}(S)$ be such that, for every $\mathcal{A} \in \mathfrak{A}$, $\mathcal{A} \sqsubseteq_{\mathsf{L}} \mathcal{X}$. Consider $A \in \bigcup \mathfrak{A}$. Hence, $A \in \mathcal{A}$ for some $\mathcal{A} \in \mathfrak{A}$. Since $\mathcal{A} \sqsubseteq \mathcal{X}$, there is some $X \in \mathcal{X}$ such that $A \subseteq X$.

(6) We show that $\bigsqcap \mathfrak{A}$ is a lower bound of $\mathfrak{A}$ and that it is the greatest such, i.e., that for every $\mathcal{A} \in \mathfrak{A}$, $\bigsqcap \mathfrak{A} \sqsubseteq_{\mathsf{L}} \mathcal{A}$ and that if $\mathcal{X} \in \mathcal{PP}(X)$ is such that, for every $\mathcal{A} \in \mathfrak{A}$, $\mathcal{X} \sqsubseteq_{\mathsf{L}} \mathcal{A}$, then $\mathcal{X} \sqsubseteq_{\mathsf{L}} \bigsqcap \mathfrak{A}$.

Consider $\mathcal{A} \in \mathfrak{A}$, we want to show that $\bigsqcap \mathfrak{A} \sqsubseteq \mathcal{A}$. Let $\bigcap_{\mathcal{Z} \in \mathfrak{A}} f(\mathcal{Z}) \in \bigsqcap \mathfrak{A}$ for some $f \in \prod \mathfrak{A}$. Note that $f(\mathcal{A}) \in \mathcal{A}$ and that $\bigcap_{\mathcal{Z} \in \mathfrak{A}} f(\mathcal{Z}) \subseteq f(\mathcal{A})$. Hence, $\bigsqcap \mathfrak{A} \sqsubseteq_{\mathsf{L}} \mathcal{A}$.

Moreover, let $\mathcal{X} \in \mathcal{PP}(X)$ be such that, for every $\mathcal{A} \in \mathfrak{A}$, $\mathcal{X} \sqsubseteq_{\mathsf{L}} \mathcal{A}$. We want to show that $\mathcal{X} \sqsubseteq_{\mathsf{L}} \bigsqcap \mathfrak{A}$. Let $X \in \mathcal{X}$. We know that, for every $\mathcal{A} \in \mathfrak{A}$, there is a set $f(\mathcal{A}) \in \mathcal{A}$ such that $X \subseteq f(\mathcal{A})$. Hence, consider $\bigcap_{\mathcal{Z} \in \mathfrak{A}} f(\mathcal{Z})$. Note that $\bigcap_{\mathcal{Z} \in \mathfrak{A}} f(\mathcal{Z}) \in \bigsqcap \mathfrak{A}$ and $X \subseteq \bigcap_{\mathcal{Z} \in \mathfrak{A}} f(\mathcal{Z})$. □

The following Proposition states the respective result for $\sqsubseteq_{\mathsf{U}}$ and a consequence we can also write the statement for $\sqsubseteq_{\mathsf{EM}}$.

PROPOSITION 2.50. *The following holds*

(1) $[\{X\}]_{\equiv_{\mathsf{U}}} = \{\{X\}\}$.
(2) $\{X\}$ *is the maximum of* $\sqsubseteq_{\mathsf{U}}$ *up to* $\equiv_{\mathsf{U}}$ *equivalence.*
(3) $[\{\emptyset\}]_{\equiv_{\mathsf{L}}} = \{\mathcal{A} \in \mathcal{P}^*\mathcal{P}(X) \mid \emptyset \in \mathcal{A}\}$. *In particular, any pair of powersets* $\mathcal{P}(A)$ *and* $\mathcal{P}(B)$ *are* $\equiv_{\mathsf{U}}$ *equivalent.*
(4) $\{\emptyset\}$ *is the minimum of* $\sqsubseteq_{\mathsf{U}}$ *up to* $\equiv_{\mathsf{U}}$ *equivalence.*
(5) *For a non-empty* $\mathfrak{A}$, $\bigsqcup \mathfrak{A}$ *is the supremum of* $\mathfrak{A}$ *with respect to* $\sqsubseteq_{\mathsf{U}}$ *up to* $\equiv_{\mathsf{U}}$ *equivalence.*
(6) *For a non-empty* $\mathfrak{A}$, $\bigcup \mathfrak{A}$ *is the infimum of* $\mathfrak{A}$ *with respect to* $\sqsubseteq_{\mathsf{U}}$ *up to* $\equiv_{\mathsf{U}}$ *equivalence.*

PROOF. Similar to Proposition 2.49. □

PROPOSITION 2.51. *The following holds*

(1) $[\{X\}]_{\equiv_{\mathsf{EM}}} = \{\{X\}\}$.
(2) $\{X\}$ *is the maximum of* $\sqsubseteq_{\mathsf{EM}}$ *up to* $\equiv_{\mathsf{EM}}$ *equivalence.*
(3) $[\{\emptyset\}]_{\equiv_{\mathsf{EM}}} = \{\{\emptyset\}\}$.
(4) $\{\emptyset\}$ *is the minimum of* $\sqsubseteq_{\mathsf{EM}}$ *up to* $\equiv_{\mathsf{EM}}$ *equivalence.*

PROOF. Corollary of Propositions 2.49 and 2.50. □

Hence, the study of the internal structures leaves some conclusions to close this Section.

(1) The study of the supremums and infimums with respect to the relation $\sqsubseteq_{\mathsf{EM}}$ does not lead to any conclusion. Even more in general, the Egli-Milner lifting interesting in modal logic because of the notion of bisimulation, seems to not be very well-behaved with respect to orders. As we have said in an earlier remark there are sets $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$ but neither $\mathcal{A} \sqsubseteq_{\mathsf{EM}} \mathcal{B}$ nor $\mathcal{B} \sqsubseteq_{\mathsf{EM}} \mathcal{A}$.

(2) The study of supremums and infimums with respect to the relations $\sqsubseteq_{\mathsf{L}}$ and $\sqsubseteq_{\mathsf{U}}$ leads to the conclusion that, in both cases, while one is internal the other is external. This is interesting because leads to the idea that the differentiation that we did in the beginning between external and internal structure

$$(\mathcal{P}\mathcal{P}(X), \subseteq, \bigcup, \bigcap) \qquad (\mathcal{P}^*\mathcal{P}(X), \sqsubseteq_*, \bigsqcup, \bigsqcap)$$

does not seem to be completely correct when the preorders $\sqsubseteq_*$ play a role. We will say more on this later whe we start our study of the modal μ-calculus.

(3) For the study of maximums and minimums, the relation $\sqsubseteq_{\mathsf{EM}}$ has a strong connection with singletons which in the extreme case of the empty set implies a relation with the powerset for $\{\emptyset\} = \mathcal{P}(\emptyset)$. But that relation is not such for the maximum. This relation with the singletions comes from the definition of power operation when considering constants as operations, recall that it is defined as $\{f(\emptyset)\}$. Taking into account that the maximum and minimum of an order can be undertand as constants for they are definable, this relation is not suprising.

(4) Because of their definition, the last relation about the maximum and minimum is preserved also for $\sqsubseteq_{\mathsf{L}}$ and $\sqsubseteq_{\mathsf{U}}$.

(5) However, the study of $\sqsubseteq_{\mathsf{L}}$ gives more information for $\{X\}$ is not the unique maximum. This fact can give some insight about how the lower lifting works. For more details, see [**4**]. In any case, for the lower lifting we can see that $\mathcal{P}(X)$ is the maximum element up to $\equiv_{\mathsf{L}}$ equivalence, which pointing towards flatness seems desirable.

All the reasons explained above and the relation of $\sqsubseteq_{\mathsf{L}}$ with the $\subseteq$ order should serve motivation for us to say that, when considering fixed-points team semantically, which we will build by ordinal induction, we will consider the structure $(\mathcal{P}^*\mathcal{P}(X), \sqsubseteq_{\mathsf{L}}, \bigcup, \bigsqcap)$.

CHAPTER 3

Team semantics for modal mu-calculus

In this chapter we define the proposed team semantics for modal μ-calculus and study some of its basic properties. For this, we first take a look at the algebraic side of team semantics and find different, but equivalent, definitions for later extending team semantics for modal logic to modal μ-calculus. In particular, in Section 1, we apply the internal and external structures of double powersets to propositional logic and study their relation. We show that external semantics enjoy downwards closure while internal semantics enjoy flatness. Later, in Section 2, we extend this ideas to modal logic defining semantics for $\Diamond$ and $\Box$ both in an external way and in an internal one also showing that internal team semantics for modal logic enjoy flatness. This viewpoint allows us to understand that the internal definition of propositional logic really depends on its context.

Furthermore, in Section 3, we extend the basic semantics for modal logic to what we will call *general team semantics* for modal logic. This will lead to a construction similar to the classical one and to a proposal of team semantics for modal μ-calculus. As we will show, among other properties, they enjoy flatness. Finally, in Section 4, we give some alternative notions considering the singleton lifting from $\mathcal{P}(X)$ to $\mathcal{PP}(X)$ which algebraically completes this chapter.

1. Internal and external definitions of Boolean connectives

In this section, following the ideas of [**8**], we define team semantics for propositional logic in two different ways as was introduced in Section 4 of Chapter 2. In particular, after recalling the classical semantics for propositional logic, we define external and internal semantics for this logic. Later, we relate this semantics with its algebraic conterpart for later studying the relation between them showing that external semantics are downwards closed and internal semantics are flat. Moreover, we prove that with the hypothesis of downwards closure, internal and external conjunction coincide which does not happen with disjunction.

To motivate the work done in this section we should pay attention to the standard team semantics for modal logic defined in 3 in Chapter 2. In particular, it is of great interest the definition of Boolean operators. While the conjunction is defined in a standard (Boolean) way, the disjunction is defined by splitting the team into two subteams. Although, the proof of flatness for that semantics gives an insight about the reason for such definition, it is still surprising because in standard semantics conjunction and disjunction work in a very strong dual pair. With this idea of the *non duality* in the definition of team semantics, we start the algebraic study of them which will be a great part of the thesis.

For completeness, let us first define the syntax of and its classical semantics with respect to truth assignments.

Definition 3.1. The *formulas of propositional logic* are given by the following grammar:

$$\varphi ::= \top \mid \bot \mid P \mid \overline{P} \mid \varphi \vee \varphi \mid \varphi \wedge \varphi$$

where P ranges over Prop. The set of all propositional formulas will be denoted by $\mathtt{PL}$.

Definition 3.2. A *truth assignment* is a function $v : \mathsf{Prop} \to \{0, 1\}$. The set of all truth assignments is denoted by TA.

Definition 3.3. *Classical semantics for propositional logic* for a truth assignment $v : \mathsf{Prop} \to \{0, 1\}$ are defined by recursion as follows:

(1) $v \models_{\mathsf{c}} \top$ always.
(2) $v \models_{\mathsf{c}} \bot$ never.
(3) $v \models_{\mathsf{c}} P$ if and only if $v(P) = 1$.
(4) $v \models_{\mathsf{c}} \overline{P}$ if and only if $v(P) = 0$.
(5) $v \models_{\mathsf{c}} \varphi \vee \psi$ if and only if $v \models_{\mathsf{c}} \varphi$ or $v \models_{\mathsf{c}} \psi$.
(6) $v \models_{\mathsf{c}} \varphi \wedge \psi$ if and only if $v \models_{\mathsf{c}} \varphi$ and $v \models_{\mathsf{c}} \psi$.

It should be said that the approach taken in this section differs from the one taken in [**8**]. In the cited paper, Engström and Lorimer extend the syntax of $\mathtt{PL}$, which is interpreted naturally (i.e., externally), with new operators that are interpreted internally. However, we define for the standard syntax of $\mathtt{PL}$ two different semantics in line with the two interpretations we gave for double powerset structures. The main reasons to take this approach are (1) to reinforce the semantical point of view of this thesis; and (2) because its algebraic fonaments come from two different interpretations of double powerset structures.

We will define the semantics in a logical way for later giving the algebraic equivalent form.

Definition 3.4. *External team semantics for propositional logic* for a set $T \subseteq \mathsf{TA}$ are defined by recursion as follows:

(1) $T \models_{\mathsf{t}}^{\mathsf{ext}} \top$ always.
(2) $T \models_{\mathsf{t}}^{\mathsf{ext}} \bot$ never.
(3) $T \models_{\mathsf{t}}^{\mathsf{ext}} P$ if and only if for every $v \in T$, $v(P) = 1$.
(4) $T \models_{\mathsf{t}}^{\mathsf{ext}} \overline{P}$ if and only if for every $v \in T$, $v(P) = 0$.
(5) $T \models_{\mathsf{t}}^{\mathsf{ext}} \varphi \vee \psi$ if and only if $T \models_{\mathsf{t}}^{\mathsf{ext}} \varphi$ or $T \models_{\mathsf{t}}^{\mathsf{ext}} \psi$.
(6) $T \models_{\mathsf{t}}^{\mathsf{ext}} \varphi \wedge \psi$ if and only if $T \models_{\mathsf{t}}^{\mathsf{ext}} \varphi$ and $T \models_{\mathsf{t}}^{\mathsf{ext}} \psi$.

Moreover, *internal team semantics for propositional logic* for a set $T \subseteq \mathsf{TA}$ are defined by recursion as follows:

(1) $T \models^{\mathsf{int}}_{\mathsf{t}} \top$ always.
(2) $T \models^{\mathsf{int}}_{\mathsf{t}} \bot$ if and only if $T = \emptyset$.
(3) $T \models^{\mathsf{int}}_{\mathsf{t}} P$ if and only if for every $v \in T$, $v(P) = 1$.
(4) $T \models^{\mathsf{int}}_{\mathsf{t}} \overline{P}$ if and only if for every $v \in T$, $v(P) = 0$.
(5) $T \models^{\mathsf{int}}_{\mathsf{t}} \varphi \vee \psi$ if and only if $T = A \cup B$ such that $A \models^{\mathsf{int}}_{\mathsf{t}} \varphi$ and $B \models^{\mathsf{int}}_{\mathsf{t}} \psi$.
(6) $T \models^{\mathsf{int}}_{\mathsf{t}} \varphi \wedge \psi$ if and only if $T = A \cap B$ such that $A \models^{\mathsf{int}}_{\mathsf{t}} \varphi$ or $B \models^{\mathsf{int}}_{\mathsf{t}} \psi$.

Remark 3.5. The names of external and internal semantics are given because of the algebraic counterpart of the semantics. While for classical semantics we have:

$$\begin{aligned}
\|\top\|_{\mathsf{c}} &= \mathsf{TA} \\
\|\bot\|_{\mathsf{c}} &= \emptyset \\
\|P\|_{\mathsf{c}} &= \{v \in \mathsf{TA} \mid v(P) = 1\} \\
\|\overline{P}\|_{\mathsf{c}} &= \mathsf{TA} - \|P\|_{\mathsf{c}} \\
\|\varphi \vee \psi\|_{\mathsf{c}} &= \|\varphi\|_{\mathsf{c}} \cup \|\psi\|_{\mathsf{c}} \\
\|\varphi \wedge \psi\|_{\mathsf{c}} &= \|\varphi\|_{\mathsf{c}} \cap \|\psi\|_{\mathsf{c}}
\end{aligned}$$

where $\|\cdot\|_{\mathsf{c}} : \mathtt{PL} \to \mathcal{P}(\mathsf{TA})$, for team semantics we have:

$$\begin{aligned}
\|\top\|^{\mathsf{ext}}_{\mathsf{t}} &= \mathcal{P}(\mathsf{TA}) & \|\top\|^{\mathsf{int}}_{\mathsf{t}} &= \mathcal{P}(\mathsf{TA}) \\
\|\bot\|^{\mathsf{ext}}_{\mathsf{t}} &= \emptyset & \|\bot\|^{\mathsf{int}}_{\mathsf{t}} &= \{\emptyset\} \\
\|P\|^{\mathsf{ext}}_{\mathsf{t}} &= \mathcal{P}(\|P\|_{\mathsf{c}}) & \|P\|^{\mathsf{int}}_{\mathsf{t}} &= \mathcal{P}(\|P\|_{\mathsf{c}}) \\
\|\overline{P}\|^{\mathsf{ext}}_{\mathsf{t}} &= \mathcal{P}(\mathsf{TA} - \|P\|_{\mathsf{c}}) & \|\overline{P}\|^{\mathsf{int}}_{\mathsf{t}} &= \mathcal{P}(\mathsf{TA} - \|P\|_{\mathsf{c}}) \\
\|\varphi \vee \psi\|^{\mathsf{ext}}_{\mathsf{t}} &= \|\varphi\|^{\mathsf{ext}}_{\mathsf{t}} \cup \|\psi\|^{\mathsf{ext}}_{\mathsf{t}} & \|\varphi \vee \psi\|^{\mathsf{int}}_{\mathsf{t}} &= \|\varphi\|^{\mathsf{int}}_{\mathsf{t}} \sqcup \|\psi\|^{\mathsf{int}}_{\mathsf{t}} \\
\|\varphi \wedge \psi\|^{\mathsf{ext}}_{\mathsf{t}} &= \|\varphi\|^{\mathsf{ext}}_{\mathsf{t}} \cap \|\psi\|^{\mathsf{ext}}_{\mathsf{t}} & \|\varphi \wedge \psi\|^{\mathsf{int}}_{\mathsf{t}} &= \|\varphi\|^{\mathsf{int}}_{\mathsf{t}} \sqcap \|\psi\|^{\mathsf{int}}_{\mathsf{t}}
\end{aligned}$$

where $\|\cdot\|^{\mathsf{ext}}_{\mathsf{t}}, \|\cdot\|^{\mathsf{int}}_{\mathsf{t}} : \mathtt{PL} \to \mathcal{P}\mathcal{P}(\mathsf{TA})$. Hence, this two interpretations are consequences of the rich structure of $\mathcal{P}\mathcal{P}(X)$ which allows us to define a new set of operations over it.

Remark 3.6. It is worthy noticing that there is not a special description of the literals $\overline{P}$ other than the powerset of the classical denotation. In general, there is not a clear description of the negation of a formula so that it preserves flatness. That is why, we are always working with formulas in negation normal form.

Once all the semantics have been defined we turn our sight on the study of them. We show how internal semantics are flat while external semantics are only downwards closed. Finally, we show that external and internal conjunction, i.e., $\cap$ and $\sqcap$, coincide if the sets are downwards closed which algebraicly explains why team semantics (in general) mix the external and internal perspective when defining conjunction and disjunction.

PROPOSITION 3.7. *The following hold for every sets A and B.*

(1) $\mathcal{P}(A) \sqcup \mathcal{P}(B) = \mathcal{P}(A \cup B)$.
(2) $\mathcal{P}(A) \sqcap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

PROOF. (1) We show both inclusions. Let $T \in \mathcal{P}(A) \sqcup \mathcal{P}(B)$, that is, $T = U \cup V$ where $U \subseteq A$ and $V \subseteq B$. So $T = U \cup V \subseteq A \cup B$ as wanted.

Conversely, suppose that $T \subseteq A \cup B$. Note that $T = (T \cap A) \cup (T \cap B)$ and $T \cap A \subseteq A$ and $T \cap B \subseteq B$. Hence, $T \in \mathcal{P}(A) \sqcup \mathcal{P}(B)$.

(2) We show both inclusion. Let $T \in \mathcal{P}(A) \sqcap \mathcal{P}(B)$, that is, $T = U \cap V$ where $U \subseteq A$ and $V \subseteq B$. So $T = U \cap V \subseteq U \subseteq A$ and $T = U \cap \subseteq V \subseteq V \subseteq B$. Hence, $T \subseteq A \cap B$.

Conversely, suppose that $T \subseteq A \cap B$. Hence $T = T \cap T$, $T \subseteq A$ and $T \subseteq B$, so $T \in \mathcal{P}(A) \sqcap \mathcal{P}(B)$. $\square$

COROLLARY 3.8. *Internal team semantics enjoy flatness and singleton property, that is, they satisfy the following equivalent conditions for every formula $\varphi \in \mathsf{PL}$.*

(1) $\|\varphi\|_{\mathsf{t}}^{\mathsf{int}} = \mathcal{P}(\|\varphi\|_{\mathsf{c}})$.
(2) $\|\varphi\|_{\mathsf{t}}^{\mathsf{int}}$ *is union and downwards closed.*
(3) $T \models_{\mathsf{t}}^{\mathsf{int}} \varphi$ *if and only if, for every $v \in T$, $v \models_{\mathsf{c}} \varphi$.*

PROPOSITION 3.9. $\cup$ *and* $\cap$ *preserve downwards closure, that is, if $\mathcal{A}$ and $\mathcal{B}$ are downwards closed, $\mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \cap \mathcal{B}$ are downwards closed.*

PROOF. Let $\mathcal{A}$ and $\mathcal{B}$ be downwards closed sets. We restrict ourselves to show that $\cup$ preserves downwards closure since $\cap$ follows similarly.

Let $X \in \mathcal{A} \cup \mathcal{B}$ and $Z \subseteq X$. We want to show that $Z \in \mathcal{A} \cup \mathcal{B}$. By assumption, $X \in \mathcal{A}$ or $X \in \mathcal{B}$. By downwards closure, $Z \in \mathcal{A}$ or $Z \in \mathcal{B}$. In either case $Z \in \mathcal{A} \cup \mathcal{B}$. $\square$

COROLLARY 3.10. *External team semantics are downwards closed, that is, for every formula $\varphi \in \mathsf{PL}$, $\|\varphi\|_{\mathsf{t}}^{\mathsf{ext}}$ is downwards closed.*

PROPOSITION 3.11. *The following hold.*

(1) *For every $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \sqcap \mathcal{B}$.*
(2) *If $\mathcal{A}$ and $\mathcal{B}$ are downwards closed, then $\mathcal{A} \cap \mathcal{B} = \mathcal{A} \sqcap \mathcal{B}$.*
(3) *If $\|\varphi\|_{\mathsf{t}}^{\mathsf{int}} = \|\varphi\|_{\mathsf{t}}^{\mathsf{ext}}$ and $\|\psi\|_{\mathsf{t}}^{\mathsf{int}} = \|\psi\|_{\mathsf{t}}^{\mathsf{ext}}$, then $\|\varphi \wedge \psi\|_{\mathsf{t}}^{\mathsf{ext}} = \|\varphi \wedge \psi\|_{\mathsf{t}}^{\mathsf{int}}$.*

PROOF. We restrict ourselves to show (1) and (2) since (3) inmediately follows.

For (1), let $X \in \mathcal{A} \cap \mathcal{B}$. Note that $X = X \cap X$ and, by assumption, $X \in \mathcal{A}$ and $X \in \mathcal{B}$. Hence, $X \in \mathcal{A} \sqcap \mathcal{B}$.

In order to show (2) we show that if both sets are downwards closed, $\mathcal{A} \sqcap \mathcal{B} \subseteq \mathcal{A} \cap \mathcal{B}$. Let $X \in \mathcal{A} \sqcap \mathcal{B}$. We want to show that $X \in \mathcal{A} \cap \mathcal{B}$, that is, $X \in \mathcal{A}$ and $X \in \mathcal{B}$. By assumption, $X = A \cap B$ for $A \in \mathcal{A}$ and

$B \in \mathcal{B}$. Note that $X = A \cap B \subseteq A \in \mathcal{A}$ and $X = A \cap B \subseteq B \subseteq \mathcal{B}$ and we conclude by downwards closure. □

PROPOSITION 3.12. *The following hold.*

(1) *If $\mathcal{A}$ and $\mathcal{B}$ has the empty team property, $\mathcal{A} \cup \mathcal{B} \subseteq \mathcal{A} \sqcup \mathcal{B}$.*
(2) *If $\|\varphi\|_{\mathsf{t}}^{\mathsf{int}} = \|\varphi\|_{\mathsf{t}}^{\mathsf{ext}}$ and $\|\psi\|_{\mathsf{t}}^{\mathsf{int}} = \|\psi\|_{\mathsf{t}}^{\mathsf{ext}}$, then $\|\varphi \vee \psi\|_{\mathsf{t}}^{\mathsf{ext}} \subseteq \|\varphi \vee \psi\|_{\mathsf{t}}^{\mathsf{int}}$.*

PROOF. (2) follows from (1) since downwards closure implies the empty team property and (1) follows from the fact that both have the emtpy team property and $X = X \cup \emptyset = \emptyset \cup X$. □

REMARK 3.13. Condition (1) from the just stated proposition is, in general strict even for powersets (which are downwards and union closed). For example, if we consider two incomparable sets, that is, neither $A \subseteq B$ nor $B \subseteq A$, we have that $A \cup B \in \mathcal{P}(A) \sqcup \mathcal{P}(B)$ but $A \cup B \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

From a team semantical viewpoint it should be interesting to syntactically add an operator to represent external disjunction, for instance $\circledvee$. Hence, we would be expanding the expressible power of the logic. Notice that, if P is some propositional constant,

$$\|P \vee \overline{P}\|_{\mathsf{t}}^{\mathsf{ext}} = \mathcal{P}(\|P\|_{\mathsf{c}}) \cup \mathcal{P}(\mathsf{TA} - \|P\|_{\mathsf{c}})$$

which as we know cannot be written as a powerset because it would be union closed and it is not the case. It suffices to consider a truth-assignment $v : \mathsf{Prop} \to \{0,1\}$ mapping $v(P) = 0$ and a different $v' : \mathsf{Prop} \to \{0,1\}$ mapping $v'(P) = 1$ and we see that

$$\{v, v'\} \notin \|P \vee \overline{P}\|_{\mathsf{t}}^{\mathsf{ext}}$$

while both

$$\{v\}, \{v'\} \in \|P \vee \overline{P}\|_{\mathsf{t}}^{\mathsf{ext}}.$$

We leave this extensions (standard in team semantics) for later since we will study now how this notions of internal and external definitions can be extended to modal logic.

2. Internal and external definitions of modal operators

In this section we expand the ideas of internal and external semantics to the framework of modal logic. In particular, following Section 1 from this chapter, we define the corresponding notion of external team semantics for modal logic and show that they are downwards closed. However, we show that the internal notion used before does not work for the case of modal logic. After defining the correct internal notion for the modalities, we show that such semantics enjoy flatness and singleton property and that both semantics coincide for the modalities.

During this Section, fix a model $\mathbb{S} = (S, R, V)$. Notice that, for every frame $\mathbb{F} = (S, R)$, we can consider its *powerset frame* with the

Egli-Milner lifting of R, that is, $\mathbb{F}^{\mathcal{P}} = (\mathcal{P}(S), R_{\mathsf{EM}})$. Hence, we can define external semantics in the same way that we defined them for propositional logic.

Definition 3.14. *External team semantics for modal logic* for $\mathbb{S}$ and for a team $T \subseteq S$ are defined by recursion as follows:

(1) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathsf{ext}} \top$ always;
(2) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathsf{ext}} \bot$ never;
(3) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathsf{ext}} P$ if and only if $T \subseteq V(P)$;
(4) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathsf{ext}} \overline{P}$ if and only if $T \cap V(P) = \emptyset$;
(5) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathsf{ext}} \varphi \vee \psi$ if and only if $\mathbb{S}, T \models_{\mathsf{t}}^{\mathsf{ext}} \varphi$ or $\mathbb{S}, T \models_{\mathsf{t}}^{\mathsf{ext}} \psi$;
(6) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathsf{ext}} \varphi \wedge \psi$ if and only if $\mathbb{S}, T \models_{\mathsf{t}}^{\mathsf{ext}} \varphi$ and $\mathbb{S}, T \models_{\mathsf{t}}^{\mathsf{ext}} \psi$;
(7) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathsf{ext}} \Diamond\varphi$ if and only if $\mathbb{S}, T' \models_{\mathsf{t}}^{\mathsf{ext}} \varphi$ for some $TR_{\mathsf{EM}}T'$;
(8) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathsf{ext}} \Box\varphi$ if and only if for every $TR_{\mathsf{EM}}T'$, $\mathbb{S}, T' \models_{\mathsf{t}}^{\mathsf{ext}} \varphi$.

Remark 3.15. Unfolding the definition of external team semantics it is easy to see the following characterization:

$$\begin{aligned}
\|\top\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}} &= \mathcal{P}\mathcal{P}(S)\\
\|\bot\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}} &= \emptyset\\
\|P\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}} &= \mathcal{P}(V(P))\\
\|\overline{P}\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}} &= \mathcal{P}(\mathcal{P}(S) - V(P))\\
\|\varphi \vee \psi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}} &= \|\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}} \cup \|\psi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}}\\
\|\varphi \wedge \psi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}} &= \|\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}} \cap \|\psi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}}\\
\|\Diamond\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}} &= \langle R_{\mathsf{EM}}\rangle \|\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}}\\
\|\Box\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}} &= [R_{\mathsf{EM}}] \|\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}}
\end{aligned}$$

where $\langle R_{\mathsf{EM}}\rangle$ and $[R_{\mathsf{EM}}]$ are the unary operations on $\mathcal{P}\mathcal{P}(S)$ defined as for the classical case as,

$$\begin{aligned}
\langle R_{\mathsf{EM}}\rangle \mathcal{X} &= \{T \in \mathcal{P}(S) \mid R_{\mathsf{EM}}[T] \cap \mathcal{X} \neq \emptyset\}\\
[R_{\mathsf{EM}}] \mathcal{X} &= \{T \in \mathcal{P}(S) \mid R_{\mathsf{EM}}[T] \subseteq \mathcal{X}\}
\end{aligned}$$

for $\mathcal{X} \in \mathcal{P}\mathcal{P}(S)$.

Proposition 3.16. *$\langle R_{\mathsf{EM}}\rangle$ preserves downwards closure. That is, if $\mathcal{X} \in \mathcal{P}\mathcal{P}(S)$ is downwards closed, then $\langle R_{\mathsf{EM}}\rangle \mathcal{X}$ is downwards closed.*

Proof. Suppose that $T \in \langle R_{\mathsf{EM}}\rangle \mathcal{X}$ and that $U \subseteq T$. We want to show that $U \in \langle R_{\mathsf{EM}}\rangle \mathcal{X}$.

By assumption, there is $X \in \mathcal{X}$ such that $TR_{\mathsf{EM}}X$. Hence, we know that, for every $t \in T$, there is $x_t \in X$ such that tRx_t. Consider $Y := \{x_u \mid u \in U\}$. By construction, $UR_{\mathsf{EM}}Y$ and, by downwards closure of $\mathcal{X}$, $Y \in \mathcal{X}$ for $Y \subseteq X \in \mathcal{X}$, that is, $U \in \langle R_{\mathsf{EM}}\rangle \mathcal{X}$. $\Box$

Corollary 3.17. *The $\Diamond$ fragment of* `ML` *is downwards closed.*

Remark 3.18. $[R_{\mathsf{EM}}]$ is not downwards closed for there can be elements without successors. Consider the model

$$a_0 \longrightarrow a_1 \longrightarrow a_2 \longrightarrow \dots$$

$$b_0$$

Notice that $R_{\mathsf{EM}}[\{a_0, b_0\}] = \emptyset$ since b_0 does not have any successor and so there is no set T such that $\{a_0, b_0\} R_{\mathsf{EM}} T$. Hence, $\{a_0, b_0\} \in [R_{\mathsf{EM}}]\emptyset$ but it is easy to see that $\{a_0\} \notin [R_{\mathsf{EM}}]\emptyset$ despite being $\emptyset$ downwards closed.

However, if R is serial, that is, every state has a successor, we can see that $[R_{\mathsf{EM}}]$ preserves downwards closure because for every set T, $T R_{\mathsf{EM}} \bigcup_{t \in T} R[t]$.

PROPOSITION 3.19. *If R is serial, $[R_{\mathsf{EM}}]$ preserves downwards closure. That is, if $\mathcal{X} \in \mathcal{PP}(S)$ is downwards closed, then $[R_{\mathsf{EM}}]\mathcal{X}$ is downwards closed.*

PROOF. Similar to the argument for $\langle R_{\mathsf{EM}} \rangle$. $\square$

REMARK 3.20. We can already see that external semantics for modal logic are not as easy to define or study as the ones for porpositional logic. This comes from the fact that they are not automatically definable for a model $\mathbb{S} = (S, R, V)$. Note that in team semantics we cannot work (externally) with the relation R as we did with $\cup$ and $\cap$ but we have to lift the relation to $R_{\mathsf{EM}} \subseteq \mathcal{P}(S) \times \mathcal{P}(S)$. This subtleties will also appear in the definition of internal team semantics.

Mimicking the study of internal team semantics for propositional logic, we have four operations on $\mathcal{PP}(S)$ which can serve as denotations for $\Diamond$ and $\Box$

$$\begin{aligned}
\langle R_{\mathsf{EM}} \rangle \mathcal{X} &= \{T \in \mathcal{P}(S) \mid R_{\mathsf{EM}}[T] \cap \mathcal{X} \neq \emptyset\} \\
[R_{\mathsf{EM}}] \mathcal{X} &= \{T \in \mathcal{P}(S) \mid R_{\mathsf{EM}}[T] \subseteq \mathcal{X}\} \\
\langle R \rangle^{\mathcal{P}} \mathcal{X} &:= \{\langle R \rangle X \mid X \in \mathcal{X}\} \\
[R]^{\mathcal{P}} \mathcal{X} &:= \{[R] X \mid X \in \mathcal{X}\}
\end{aligned}$$

for $\mathcal{X} \in \mathcal{PP}(S)$. Ideally, if $\langle R \rangle^{\mathcal{P}}$ and $[R]^{\mathcal{P}}$ preserved powersets, that is, if

$$\langle R \rangle^{\mathcal{P}} \mathcal{P}(A) = \mathcal{P}(\langle R \rangle A) \qquad [R]^{\mathcal{P}} \mathcal{P}(A) = \mathcal{P}([R] A)$$

for $A \subseteq S$, we would be providing an internal definition of team semantics for $\Diamond$ and $\Box$. Unfortunately (but interestingly) the following example shows that this is not the case.

EXAMPLE 3.21. Consider the frame

$$a \longrightarrow b \longleftarrow c$$

and take $A = \{b\}$. On the one hand

$$\langle R \rangle \{b\} = \{a, c\} \qquad [R]\{b\} = \{a, b, c\}$$

and

$$\mathcal{P}(\langle R\rangle\{b\}) = \{\emptyset, \{a\}, \{c\}, \{a, c\}\} \qquad \mathcal{P}([R]\{b\}) = \mathcal{P}(\{a, b, c\}).$$

While,

$$\langle R\rangle^{\mathcal{P}}\{\emptyset, \{b\}\} = \{\langle R\rangle\emptyset, \langle R\rangle\{b\}\} = \{\emptyset, \{a, c\}\} \neq \mathcal{P}(\{a, c\})$$

and

$$[R]^{\mathcal{P}}\{\emptyset, \{b\}\} = \{[R]\emptyset, [R]\{b\}\} = \{\{b\}, \{a, b, c\}\} \neq \mathcal{P}(\{a, b, c\}).$$

REMARK 3.22. The just stated example gives insight about the internal and external perspectives and it is important to note the following two facts. (1) That the preservation of powersets by internal union and intersection is really dependant on the relation of external union and intersection with respect to powersets. For, as we can see, $\langle R\rangle\{b\} = \{a, c\}$, but because of the nature of the operation $\langle R\rangle$, there is no way of getting singletons $\{a\}$ and $\{c\}$ since, as long as R is involved, the states a and c are "equivalent" (i.e., bisimilar). (2) That the choice of the powerset lifting as the *natural* one is, from the algebraic viewpoint, completely arbitrary while it is forcing us in other aspects. A similar comment was made in [**8**]. We will say more on this in Section 3.

Based on this example, we finally consider what we will call the internal team semantics for modal logic. Before defining them, let us say that we will consider the following two operations as semantics for the modalities

$$\langle\!\langle R\rangle\!\rangle\mathcal{X} := \{T \subseteq \langle R\rangle X \mid X \in \mathcal{X}\}$$
$$[\![R]\!]\mathcal{X} := \{T \subseteq [R]X \mid X \in \mathcal{X}\}$$

based on the idea derived from Example 3.21 that we need to close downwards $\langle R\rangle^{\mathcal{P}}$ and $[R]^{\mathcal{P}}$ to get powersets.

DEFINITION 3.23. *Internal team semantics for modal logic* for $\mathbb{S}$ and for a team $T \subseteq S$ are defined by recursion as follows:

(1) $\mathbb{S}, T \models^{\mathsf{int}}_{\mathsf{t}} \top$ always;
(2) $\mathbb{S}, T \models^{\mathsf{int}}_{\mathsf{t}} \bot$ if and only if $T = \emptyset$;
(3) $\mathbb{S}, T \models^{\mathsf{int}}_{\mathsf{t}} P$ if and only if $T \in \Omega(P)$;
(4) $\mathbb{S}, T \models^{\mathsf{int}}_{\mathsf{t}} \overline{P}$ if and only if for every $X \in \Omega(P)$, $X \cap T = \emptyset$;
(5) $\mathbb{S}, T \models^{\mathsf{int}}_{\mathsf{t}} \varphi \vee \psi$ if and only if $T = A \cap B$ such that $\mathbb{S}, A \models^{\mathsf{int}}_{\mathsf{t}} \varphi$ or $\mathcal{M}, B \models_{\mathsf{t}} \psi$;
(6) $\mathbb{S}, T \models^{\mathsf{int}}_{\mathsf{t}} \varphi \wedge \psi$ if and only if $T = A \cup B$ such that $\mathbb{S}, A \models^{\mathsf{int}}_{\mathsf{t}} \varphi$ and $\mathcal{M}, B \models_{\mathsf{t}} \psi$;
(7) $\mathbb{S}, T \models^{\mathsf{int}}_{\mathsf{t}} \Diamond\varphi$ if and only if $T \subseteq \langle R\rangle X$ for some $\mathbb{S}, X \models^{\mathsf{int}}_{\mathsf{t}} \varphi$;
(8) $\mathbb{S}, T \models^{\mathsf{int}}_{\mathsf{t}} \Box\varphi$ if and only if $T \subseteq [R]X$ for some $\mathbb{S}, X \models^{\mathsf{int}}_{\mathsf{t}} \varphi$.

REMARK 3.24. Unfolding this definition it is easy to shows the algebraic characterization:

$$\begin{aligned}
\|\top\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} &= \mathcal{P}(S)\\
\|\bot\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} &= \{\emptyset\}\\
\|P\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} &= \mathcal{P}(V(P))\\
\|\overline{P}\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} &= \mathcal{P}(\mathcal{P}(S) - V(P))\\
\|\varphi \vee \psi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} &= \|\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} \sqcup \|\psi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}}\\
\|\varphi \wedge \psi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} &= \|\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} \sqcap \|\psi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}}\\
\|\Diamond\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} &= \langle\!\langle R_{\mathsf{EM}}\rangle\!\rangle \|\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}}\\
\|\Box\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} &= [\![R_{\mathsf{EM}}]\!] \|\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}}
\end{aligned}$$

PROPOSITION 3.25. *The following hold for every set A.*

(1) $\langle\!\langle R\rangle\!\rangle\mathcal{P}(A) = \mathcal{P}(\langle R\rangle A)$.
(2) $[\![R]\!]\mathcal{P}(A) = \mathcal{P}([R]A)$.

PROOF. We restrict ourselves to show (1) since (2) follows similarly.
(1) We show both inclusions. Let $T \in \langle\!\langle R\rangle\!\rangle\mathcal{P}(A)$. We want to show that $T \subseteq \langle R\rangle A$. Note that, by assumption, there is $X \subseteq A$ such that $T \subseteq \langle R\rangle X \subseteq \langle R\rangle A$ by monotonicity of $\langle R\rangle$. Conversely, let us assume that $T \subseteq \langle R\rangle A$. We want to show that $T \in \langle\!\langle R\rangle\!\rangle\mathcal{P}(A)$, i.e., there is $X \subseteq A$ such that $T \subseteq \langle R\rangle X$ which is clear by assumption. $\square$

COROLLARY 3.26. *Internal team semantics for modal logic enjoy flatness and singleton property, that is, they satisfy the following equivalent conditions for every formula $\varphi \in$ ML:*

(1) $\|\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} = \mathcal{P}(\|\varphi\|_{\mathsf{c}}^{\mathbb{S}}) = \|\varphi\|_{\mathsf{t}}^{\mathbb{S}}$.
(2) $\|\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}}$ *is union and downwards closed.*
(3) $T \models_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} \varphi$ *if and only if for every $t \in T$, $t \models_{\mathsf{c}} \varphi$.*

REMARK 3.27. Let us remark that, for a model $\mathbb{S}$ and a team $T \subseteq S$, $\mathbb{S}, T \models_{\mathsf{t}}^{\mathsf{int}} \Diamond\varphi$ by definition if $T \subseteq \langle R\rangle X$ for some $X \subseteq S$ such that $\mathbb{S}, X \models_{\mathsf{t}}^{\mathsf{int}} \varphi$ which is equivalent to the condition saying that there is a team $U \subseteq S$ such that, for every $t \in T$, there is $u \in U$ such that tRu and $\mathbb{S}, U \models_{\mathsf{t}}^{\mathsf{int}} \varphi$. And we have show that, considering such definition, which is internally definable, we can shown flatness. This was first stated and considered in the standard proof of flatness for modal logic in Remark 2.34.

The following proposition generalizes the latter one for only the condition of downwards closure of the powersets were needed.

PROPOSITION 3.28. *The following hold.*

(1) *If $\mathcal{A}$ is downwards closed, then $\langle\!\langle R_{\mathsf{EM}}\rangle\!\rangle\mathcal{A} = \langle R_{\mathsf{EM}}\rangle\mathcal{A}$.*
(2) *If $\|\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} = \|\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}}$, then $\|\Diamond\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} = \|\Diamond\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}}$.*

PROOF. We restrict ourselves to show (1) since (2) automatically follows from it.

For (1) we show both inclusions. Let $T \subseteq \langle R\rangle A$ for $A \in \mathcal{A}$. We want to show that $T \in \langle R_{\mathsf{EM}}\rangle\mathcal{A}$, i.e., there is a team X such that $TR_{\mathsf{EM}}X$ and $X \in \mathcal{A}$.

By assumption, for every $t \in T$, $R[t] \cap A \neq \emptyset$. Choose, for every $t \in T$, $x_t \in R[t] \cap A$ and consider $X := \{x_t \in A \mid t \in T\} \subseteq A \in \mathcal{A}$. We conclude by noticing that $TR_{\mathsf{EM}}X$ and, by downwards closure, $X \in \mathcal{A}$.

Conversely, let $T \in \langle R_{\mathsf{EM}}\rangle\mathcal{A}$. We want to show that $T \subseteq \langle R\rangle A$ for some $A \in \mathcal{A}$.

By assumption, for some $A \in \mathcal{A}$, $TR_{\mathsf{EM}}A$. Hence, we conclude by noticing that $T \subseteq \langle R\rangle A$. $\Box$

The following proposition can simply be a corollary of the study done so far to clearly state that for serial models the internal and the external definition of modal operators coincide.

PROPOSITION 3.29. *The following hold if R is serial.*

(1) *If $\mathcal{A}$ is downwards closed, $[\![R_{\mathsf{EM}}]\!]\mathcal{A} = [R_{\mathsf{EM}}]\mathcal{A}$.*
(2) *If $\|\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} = \|\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}}$, then $\|\Box\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} = \|\Box\varphi\|_{\mathsf{t}}^{\mathbb{S},\mathsf{ext}}$.*

PROOF. We restrict ourselves to show (1) since (2) inmediately follows from it.

For (1) let us assume that R is serial, that is, for every $s \in S$, $R[s] \neq \emptyset$. Hence, for every team T, $TR_{\mathsf{EM}} \bigcup_{t\in T} R[t]$. We show both inclusions.

Let $T \in [R_{\mathsf{EM}}]\mathcal{A}$. We want to show that $T \subseteq [R]A$ for some $A \in \mathcal{A}$.

Since $TR_{\mathsf{EM}} \bigcup_{t\in T} R[t]$, $\bigcup_{t\in T} R[t] \in \mathcal{A}$. Noting that $T \subseteq [R] \bigcup_{t\in T} R[t]$ we conclude.

Conversely, let $T \subseteq [R]A$ for $A \in \mathcal{A}$. We want to show that $T \in [R_{\mathsf{EM}}]\mathcal{A}$, i.e., for every X such that $TR_{\mathsf{EM}}X$, $X \in \mathcal{A}$.

It suffices to note that, by definition of R_{EM}, $X \subseteq \bigcup_{t\in T} R[t]$ and that, by assumption, $\bigcup_{t\in T} R[t] \subseteq A \in A$. Hence, we conclude that $X \in \mathcal{A}$. $\Box$

REMARK 3.30. The condition of seriality is not constraining our work for later we will be working with `CTL` in which is usual to consider all models as serial.

We can already see why the study of the internal and external perspective is interesting. While the external definition gives a *natural* interpretation of team semantics, for instance, $T \models_{\mathsf{t}}^{\mathsf{ext}} \Box\varphi$ if and only if for every $TR_{\mathsf{EM}}U$, $U \models_{\mathsf{t}}^{\mathsf{ext}} \varphi$ is a very sensible team semanitcal definition. On the other hand, the internal definition of the operators, carry the structure and preserves powersets, i.e., flatness. The interesting point is when both semantics coincide and we can provide a natural definition which preserves flatness. This was the case of $\wedge$, it is the case of $\Diamond$ and, for a restricted class of models, also of $\Box$.

3. Team mu-calculus

This section is devoted to the definition of team semantics for modal μ-calculus. After redefining team semantics for modal logic in an internal way, we give one more step and define the more abstract *general team semantics* for modal logic. This final notion will allow us to lift the work done in the classical case to the team semantical framework considering two different notions of fixed-points. Finally, we show flatness for a particular fragment of team μ-calculus.

Before proceeding any further, let us motivate the work done in this Section. Recall that, in the classical construction of $\mu\mathtt{ML}$, one considers the function

$$\begin{aligned}\varphi^{\mathbb{S}}_X : \mathcal{P}(S) &\to \mathcal{P}(S)\\ A \ &\mapsto \|\varphi\|_{\mathsf{c}}^{\mathbb{S}[X\mapsto A]}\end{aligned}$$

for a model $\mathbb{S}$ and a formula $\varphi \in \mathtt{ML}$ positive in X. One of the key points of the definition is that any propostion can be interpreted as any set $A \subseteq S$.

However, that is not a property in the team semantics defined so far. That is because the willing of having flatness as a desirable property, restricts the possibilities for interpretations of propositional constants. This is a point done in the paper [**8**] which we try to explain here giving our algebraic perspective. If we consider the classical denotation function as the unique homomorphism extending the valuation of the model, one gets, for a lifting $\mathcal{L} : \mathcal{P}(S) \to \mathcal{PP}(S)$ the following commuting diagram:

$$\begin{tikzcd}
\mathsf{Prop} \arrow[r, hook, "\iota"] \arrow[dr, "V"'] \arrow[ddr, bend right, "\mathcal{L}\circ V"'] & \mathtt{ML} \arrow[d, "\|\cdot\|_{\mathsf{c}}"] \arrow[dd, bend left=60, "\|\cdot\|_{\mathcal{L}}"] \\
 & \mathcal{P}(S) \arrow[d, "\mathcal{L}"] \\
 & \mathcal{PP}(S)
\end{tikzcd}$$

where $\|\cdot\|_{\mathcal{L}}$, the team semantics for the lifting $\mathcal{L}$, is defined as the composition $\mathcal{L} \circ \|\cdot\|_{\mathsf{c}}$. With this diagram, the condition

$$\|\varphi\|_{\mathcal{L}} = \mathcal{L}(\|\varphi\|_{\mathsf{c}})$$

for every formula $\varphi \in \mathtt{ML}$ holds by definition. With this construction in mind, most of the work done in team semantics is to find $\|\cdot\|_{\mathrm{Pow}}$ semantics and characterize them in a natural way for afterwards extend the resultant logic with team semantical atoms.

However, as it is said in the so cited paper, there is a strong drawback in this conception of team semantics, namely, that propositional constants can only represent $\mathcal{L}$-properties, in this case, powersets. Precisely, the restriction that we were stating earlier. Seeing the situation from this side points out that the choice of Pow is not done by any particular algebraic reason, but because it leads to *natural* logical properties. That is why, the algebraic treatment of team semantics that we are doing in this thesis is sometimes messy as with the internal definition of the modalities.

The solution proposed by Engström and Lorimer is to consider the semantic entailment relation $\Gamma \models \varphi$ for a set of formulas Γ and a formula φ if and only if, the relation holds for every possible lifting $\mathcal{L}$. Our treatment in this thesis is to consider *general valuations* as functions $\Omega : \mathsf{Prop} \to \mathcal{P}\mathcal{P}(S)$, i.e., propositional constants represent, in general, properties of legaues, that is, sets of teams.

Once done this instroduction, we begin by defining general models and general team semantics.

Definition 3.31. Given a frame $\mathbb{F} = (S, R)$, a *general (Kripke) model* based on $\mathbb{F}$ is a truple $\mathcal{M} = (S, R, \Omega)$ where $\Omega : \mathsf{Prop} \to \mathcal{P}\mathcal{P}(S)$ is a *general valuation*. General models are noted by callygraphic Latin letters $\mathcal{M}, \mathcal{S}, \ldots$.

Remark 3.32. Notice that every model $\mathbb{S} = (S, R, V)$ canonically defines a general model by considering $(S, R, V^{\mathcal{P}})$ where $V^{\mathcal{P}}$ is the general valuation defined for $P \in \mathsf{Prop}$ as $V^{\mathcal{P}}(P) := \mathcal{P}(V(P))$. As a notational convention we will denote by $\mathcal{S}$ the general model associated with $\mathbb{S}$ in this way.

Definition 3.33. *General team semantics for modal logic* for a general model $\mathcal{M} = (S, R, \Omega)$ and for a team $T \subseteq S$ are defined by recursion as follows:

1. $\mathcal{M}, T \models^{\mathtt{ML}}_{\mathtt{t}} \top$ always.
2. $\mathcal{M}, T \models^{\mathtt{ML}}_{\mathtt{t}} \bot$ if and only if $T = \emptyset$.
3. $\mathcal{M}, T \models^{\mathtt{ML}}_{\mathtt{t}} P$ if and only if $T \in \Omega(P)$.
4. $\mathcal{M}, T \models^{\mathtt{ML}}_{\mathtt{t}} \overline{P}$ if and only if for every $X \in \Omega(P)$, $X \cap T = \emptyset$.
5. $\mathcal{M}, T \models^{\mathtt{ML}}_{\mathtt{t}} \varphi \vee \psi$ if and only if $T = A \cup B$ such that $\mathcal{M}, A \models^{\mathtt{ML}}_{\mathtt{t}} \varphi$ and $\mathcal{M}, B \models^{\mathtt{ML}}_{\mathtt{t}} \psi$.
6. $\mathcal{M}, T \models^{\mathtt{ML}}_{\mathtt{t}} \varphi \wedge \psi$ if and only if $T = A \cap B$ such that $\mathcal{M}, A \models^{\mathtt{ML}}_{\mathtt{t}} \varphi$ and $\mathcal{M}, B \models_{\mathtt{t}} \psi$.
7. $\mathcal{M}, T \models^{\mathtt{ML}}_{\mathtt{t}} \Diamond\varphi$ if and only if $T \subseteq \langle R \rangle X$ for some $\mathcal{M}, X \models^{\mathtt{ML}}_{\mathtt{t}} \varphi$.
8. $\mathcal{M}, T \models^{\mathtt{ML}}_{\mathtt{t}} \Box\varphi$ if and only if $T \subseteq [R] X$ for some $\mathcal{M}, X \models^{\mathtt{ML}}_{\mathtt{t}} \varphi$.

Remark 3.34. Unfolding this definition it is easy to shows the following characterization:

$$\begin{aligned}
\|\top\|^{\mathcal{M}}_{\mathtt{t}} &= \mathcal{P}(S) \\
\|\bot\|^{\mathcal{M}}_{\mathtt{t}} &= \{\emptyset\} \\
\|P\|^{\mathcal{M}}_{\mathtt{t}} &= \Omega(P) \\
\|\overline{P}\|^{\mathcal{M}}_{\mathtt{t}} &= \{T \in \mathcal{P}(S) \mid \text{for every } X \in \Omega(P),\, T \cap X = \emptyset\} \\
\|\varphi \vee \psi\|^{\mathcal{M}}_{\mathtt{t}} &= \|\varphi\|^{\mathcal{M}}_{\mathtt{t}} \sqcup \|\psi\|^{\mathcal{M}}_{\mathtt{t}} \\
\|\varphi \wedge \psi\|^{\mathcal{M}}_{\mathtt{t}} &= \|\varphi\|^{\mathcal{M}}_{\mathtt{t}} \sqcap \|\psi\|^{\mathcal{M}}_{\mathtt{t}}
\end{aligned}$$

$$\|\Diamond\varphi\|_{\mathsf{t}}^{\mathcal{M}} = \langle\!\langle R_{\mathsf{EM}}\rangle\!\rangle \|\varphi\|_{\mathsf{t}}^{\mathcal{M}} = \{T \subseteq \langle R\rangle X \mid X \in \|\varphi\|_{\mathsf{t}}^{\mathcal{M}}\}$$
$$\|\Box\varphi\|_{\mathsf{t}}^{\mathcal{M}} = [\![R_{\mathsf{EM}}]\!] \|\varphi\|_{\mathsf{t}}^{\mathcal{M}} = \{T \subseteq [R] X \mid X \in \|\varphi\|_{\mathsf{t}}^{\mathcal{M}}\}$$

REMARK 3.35. Relating this semantics with our work in Section 2 of this same Chapter, we see that

$$\|\cdot\|_{\mathsf{t}}^{\mathcal{S}} = \|\cdot\|_{\mathsf{t}}^{\mathbb{S},\mathsf{int}} = \mathcal{P}(\|\cdot\|_{\mathsf{c}}^{\mathbb{S}}) = \|\cdot\|_{\mathsf{t}}^{\mathbb{S}}$$

where the last semantics are the basic ones defined in Section 3 of Chapter 2.

The notion of general team semantics allows us to state the following definition.

DEFINITION 3.36. Let $\mathcal{M} = (S, R, \Omega)$ be a general model and a set $\mathcal{A} \in \mathcal{PP}(S)$, we define the general model $\mathcal{M}[X \mapsto \mathcal{A}]$ for $X \in \mathsf{Prop}$ as the truple $(S, R, \Omega[X \mapsto \mathcal{A}])$ where $\Omega[X \mapsto \mathcal{A}]$ is the general valuation:

$$\Omega[X \mapsto \mathcal{A}](P) := \begin{cases} \Omega(P) & \text{if } P \neq X \\ A & \text{if } P = X \end{cases}$$

Now, one of the desired theorems is stated as folows.

THEOREM 3.37. *Let $\varphi \in \mathtt{ML}$ and $\mathcal{M} = (S, R, \Omega)$ a general model. If φ is positive in X, then the function:*

$$\begin{aligned} \varphi_X^{\mathcal{M}} : \mathcal{P}^*\mathcal{P}(S) &\to \mathcal{P}^*\mathcal{P}(S) \\ \mathcal{A} \quad &\mapsto \|\varphi\|_{\mathsf{t}}^{\mathcal{M}[X\mapsto\mathcal{A}]} \end{aligned}$$

is monotone over $\sqsubseteq_{\mathsf{L}}$.

PROOF. By induction on φ.

Base case. We consider several different subcases, namely, $\top, \bot, P, \overline{P}, X$ and $\overline{X}$ for $P \in \mathsf{Prop} - \{X\}$.

Cases $\top$ and $\bot$. Notice that the function $(\top)_X^{\mathcal{M}}$ is constantly equal to $\mathcal{P}(S)$ and that $(\bot)_X^{\mathcal{M}}$ is constantly equal to $\{\emptyset\}$. Hence, trivially monotone (over $\sqsubseteq_{\mathsf{L}}$).

Cases P and $\overline{P}$. Since P and $\overline{P}$ do not depend on X, the functions $(P)_X^{\mathcal{M}}$ and $(\overline{P})_X^{\mathcal{M}}$ are constanly equal to, respectively, $\|P\|_{\mathsf{t}}^{\mathcal{M}}$ and $\|\overline{P}\|_{\mathsf{t}}^{\mathcal{M}}$. Hence, trivially monotone. Hence, trivially monotone (over $\sqsubseteq_{\mathsf{L}}$).

Cases X and $\overline{X}$. The case for X is trivial since $(X)_X^{\mathcal{M}} = \mathrm{id}_{\mathcal{P}^*\mathcal{P}(S)}$ (the identity function on $\mathcal{P}^*\mathcal{P}(S)$) again, trivially monotone (over $\sqsubseteq_{\mathsf{L}}$). In the case $\overline{X}$ there is nothing to prove since it is not positive in X.

Boolean cases. Assume that the statement holds for φ, ψ. We restrict ourselves to show the disjunctive case since the conjunctive case is similar.

Suppose that $\mathcal{A} \sqsubseteq_{\mathsf{L}} \mathcal{B}$. We want to show that

$$\|\varphi \vee \psi\|_{\mathsf{t}}^{\mathcal{M}[X\mapsto\mathcal{A}]} \sqsubseteq_{\mathsf{L}} \|\varphi \vee \psi\|_{\mathsf{t}}^{\mathcal{M}[X\mapsto\mathcal{B}]}.$$

Let $T \in \|\varphi \vee \psi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{A}]} = \|\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{A}]} \sqcup \|\psi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{A}]}$, that is, $T = A \cup B$ where $A \in \|\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{A}]}$ and $B \in \|\psi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{A}]}$. We want to show that there is $T' \in \|\varphi \vee \psi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{B}]}$ such that $T \subseteq T'$.

Note that, if $\varphi \vee \psi$ is positive in X, then both φ, ψ are positive in X. Hence, by induction hypothesis,

$$\|\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{A}]} \sqsubseteq_{\mathsf{L}} \|\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{B}]} \quad \|\psi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{A}]} \sqsubseteq_{\mathsf{L}} \|\psi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{B}]}.$$

Thus, there are $A' \in \|\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{B}]}$ and $B' \in \|\psi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{B}]}$ such that $A \subseteq A'$ and $B \subseteq B'$. Therefore, we can conclude that

$$T = A \cup B \subseteq A' \cup B' =: T' \in \|\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{B}]} \sqcup \|\psi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{B}]} = \|\varphi \vee \psi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{B}]}$$

by monotonicity of $\cup$ showing the statement for $\varphi \vee \psi$.

Modal cases. Assume that the statement holds for φ. We restrict ourselves to show the $\Diamond$ case since the $\Box$ case is similar.

Suppose that $\mathcal{A} \sqsubseteq_{\mathsf{L}} \mathcal{B}$. We want to show that

$$\|\Diamond\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{A}]} \sqsubseteq_{\mathsf{L}} \|\Diamond\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{B}]}.$$

Let $T \in \|\Diamond\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{A}]} = \langle\!\langle R \rangle\!\rangle \|\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{A}]}$, that is, $T \subseteq \langle R \rangle X$ where $X \in \|\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{A}]}$. We want to show that there is $T' \in \|\Diamond\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{B}]}$ such that $T \subseteq T'$.

Note that, if $\Diamond\varphi$ is positive in X, then φ is positive in X. Hence, by induction hypothesis,

$$\|\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{A}]} \sqsubseteq_{\mathsf{L}} \|\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{B}]}$$

Thus, there is $X' \in \|\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{B}]}$ such that $X \subseteq X'$. Therefore, we can conclude that

$$T \subseteq \langle R \rangle X \subseteq \langle R \rangle X' =: T' \in \langle\!\langle R \rangle\!\rangle \|\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{B}]} = \|\Diamond\varphi\|_{\mathsf{t}}^{\mathcal{M}[X \mapsto \mathcal{B}]}$$

by monotonicity of $\langle R \rangle$ showing the statement for $\Diamond\varphi$. $\square$

Once one of the central theorems of modal μ-calculus is shown, we next study how the respective ordinal sequences which define the fixed-points arise in the team semantical framework.

As it was reasoned in Section 4 of Chapter 2, we will work with the structure $(\mathcal{P}^*\mathcal{P}(S), \sqsubseteq_{\mathsf{L}}, \bigcup, \bigsqcap)$ to get the semantics of μ and ν formulas. Let us quickly recall the properties of this structure. The next proposition was stated and proved as Proposition 2.49

Proposition 3.38. *The following hold.*

(1) $[\{X\}]_{\equiv_{\mathsf{L}}} = \{\mathcal{A} \in \mathcal{P}^*\mathcal{P}(X) \mid X \in \mathcal{A}\}$. *In particular,* $\mathcal{P}(X)$ *is* $\equiv_{\mathsf{L}}$ *equivalent to* $\{X\}$.
(2) $\{X\}$ *is the maximum of* $\sqsubseteq_{\mathsf{L}}$ *up to* $\equiv_{\mathsf{L}}$ *equivalence.*
(3) $[\{\emptyset\}]_{\equiv_{\mathsf{L}}} = \{\{\emptyset\}\}$.
(4) $\{\emptyset\}$ *is the minimum of* $\sqsubseteq_{\mathsf{L}}$ *up to* $\equiv_{\mathsf{L}}$ *equivalence.*
(5) *If* $\mathfrak{A}$ *is non-empty,* $\bigcup \mathfrak{A}$ *is the supremum of* $\mathfrak{A}$ *with respect to* $\sqsubseteq_{\mathsf{L}}$ *up to* $\equiv_{\mathsf{L}}$ *equivalence.*

(6) *If $\mathfrak{A}$ is non-empty, $\bigsqcap \mathfrak{A}$ is the infimum of $\mathfrak{A}$ with respect to $\sqsubseteq_{\mathsf{L}}$ up to $\equiv_{\mathsf{L}}$ equivalence.*

One of the last remarks done in Section 4 of Chapter 2 now can be better understand. Until this very point we have been talking about internal and external semantics for modal logic because of the two different interpretations of double powerset structures. Namely, the structures

$$(\mathcal{P}\mathcal{P}(S), \bigcup, \bigcap, \langle R_{\mathsf{EM}}\rangle, [R_{\mathsf{EM}}]) \quad (\mathcal{P}^*\mathcal{P}(S), \bigsqcup, \bigsqcap, \langle\!\langle R\rangle\!\rangle, [\![R]\!]).$$

Our work shows that the external structure is more *natural* while the internal one preserve flatness. However, in light of the just stated proposition, when considering the preorder $\sqsubseteq_{\mathsf{L}}$, the definition of the supremum is external while the definition of the infimum is internal. Hence, we will study team semantics for modal μ-calculus doing the difference between external and internal definitions, but without considering two different defined semantics for the same logic as it was done so far. Instead, we consider the logic of team μ-calculus, $\mathtt{t}\mu\mathtt{ML}$. Theorems 3.50 and 3.58 will later explain this choice.

Now, pointing towards flatness, let us follow the same steps we did for propositional logic in Section 1 of this same chapter.

PROPOSITION 3.39. *If $(A_i)_{i\in I}$ is a family of sets,*

(1) $\bigsqcup_{i\in I} \mathcal{P}(A_i) = \mathcal{P}(\bigcup_{i\in I} A_i)$;
(2) $\bigsqcap_{i\in I} \mathcal{P}(A_i) = \mathcal{P}(\bigcap_{i\in I} A_i)$.

PROOF. Generalization of the binary case. □

Also, the following proposition can also be shown in the infinite case.

PROPOSITION 3.40. *If $(A_i)_{i\in I}$ is a family of sets,*

(1) $\bigcup_{i\in I} \mathcal{P}(A_i) \subseteq \bigsqcup_{i\in I} \mathcal{P}(A_i)$;
(2) $\bigcap_{i\in I} \mathcal{P}(A_i) = \bigsqcap_{i\in I} \mathcal{P}(A_i)$.

PROOF. Generalization of the binary case. □

REMARK 3.41. Notice that condition (1) of the just stated proposition is strict even for families of powersets. For instance, $\bigcup_{n\in\mathbb{N}} \mathcal{P}(n) \subset \mathcal{P}(\mathbb{N})$.

PROPOSITION 3.42. *If $(\mathcal{A}_i)_{i\in I}$ is a family of downwards closed sets, then*

$$\bigcap_{i\in I} \mathcal{A}_i = \bigsqcap_{i\in I} \mathcal{A}_i.$$

PROOF. Generalization of the binary case. □

Thus, note the following facts:

(1) $\bigsqcap$ acts as an infimum and preserves powersets.
(2) $\bigsqcap$ and $\bigcap$ coincide on downwards closed sets.

(3) $\bigsqcup$ preserves powersets.
(4) $\bigcup$ acts as a supremum.
(5) $\bigsqcup$ and $\bigcup$ do no coincide in general.

This properties and obvious dfferences between union and intersection will become crucial for our next definitions and study of $\mathtt{t}\mu\mathtt{ML}$. We begin by defining the respective team semantical ordinal sequences from an internal and an external viewpoint.

Definition 3.43. For a set S and a map $f : \mathcal{P}^*\mathcal{P}(S) \to \mathcal{P}^*\mathcal{P}(S)$ we define the *external* sequences $(f^{\xi}_{\mu,\cup})_{\xi\in\mathbf{ORD}}$ and $(f^{\xi}_{\nu,\cap})_{\xi\in\mathbf{ORD}}$ as follows:

$$\begin{cases} f^0_{\mu,\cup} = \{\emptyset\} \\ f^{\alpha+1}_{\mu,\cup} = f(f^{\alpha}_{\mu,\cup}) \\ f^{\beta}_{\mu,\cup} = \bigcup_{\alpha\in\beta} f^{\alpha}_{\mu,\cup} \quad \text{for } \beta \text{ limit} \end{cases} \qquad \begin{cases} f^0_{\nu,\cap} = \mathcal{P}(S) \\ f^{\alpha+1}_{\nu,\cap} = f(f^{\alpha}_{\nu,\cap}) \\ f^{\beta}_{\nu,\cap} = \bigcap_{\alpha\in\beta} f^{\alpha}_{\nu,\cap} \quad \text{for } \beta \text{ limit} \end{cases}$$

Definition 3.44. For a set S and a map $f : \mathcal{P}^*\mathcal{P}(S) \to \mathcal{P}^*\mathcal{P}(S)$ we define the *internal* sequences $(f^{\xi}_{\mu,\sqcup})_{\xi\in\mathbf{ORD}}$ and $(f^{\xi}_{\nu,\sqcap})_{\xi\in\mathbf{ORD}}$ as follows:

$$\begin{cases} f^0_{\mu,\sqcup} = \{\emptyset\} \\ f^{\alpha+1}_{\mu,\sqcup} = f(f^{\alpha}_{\mu,\sqcup}) \\ f^{\beta}_{\mu,\sqcup} = \bigsqcup_{\alpha\in\beta} f^{\alpha}_{\mu,\sqcup} \quad \text{for } \beta \text{ limit} \end{cases} \qquad \begin{cases} f^0_{\nu,\sqcap} = \mathcal{P}(S) \\ f^{\alpha+1}_{\nu,\sqcap} = f(f^{\alpha}_{\nu,\sqcap}) \\ f^{\beta}_{\nu,\sqcap} = \bigsqcap_{\alpha\in\beta} f^{\alpha}_{\nu,\sqcap} \quad \text{for } \beta \text{ limit} \end{cases}$$

Proposition 3.45. *Let $f : \mathcal{P}^*\mathcal{P}(S) \to \mathcal{P}^*\mathcal{P}(S)$ be monotone over $\sqsubseteq_{\mathsf{L}}$. If $\alpha \leq \beta$, then*
(1) $f^{\alpha}_{\mu,\cup} \sqsubseteq_{\mathsf{L}} f^{\beta}_{\mu,\cup}$.
(2) $f^{\alpha}_{\mu,\sqcup} \sqsubseteq_{\mathsf{L}} f^{\beta}_{\mu,\sqcup}$.

Proof. We show both statements by ordinal induction on β. We write $f^{\xi}_{\mu,*}$ to mean, wither $f^{\xi}_{\mu,\cup}$ or $f^{\xi}_{\mu,\sqcup}$ since, for the base case and for the successor case, both definitions coincide.

Base case. If $\beta = 0$, then $\alpha = 0$ and the statement is trivial.

Successor step. Suppose that the statement holds for an ordinal β. And let us show the statement for $\beta + 1$.

Notice that, by monotonicity of f, $f^{\beta}_{\mu,*} \sqsubseteq_{\mathsf{L}} f(f^{\beta}_{\mu,*}) = f^{\beta+1}_{\mu,\cup}$.

Let $\alpha \leq \beta + 1$. We show that $f^{\alpha}_{\mu,*} \sqsubseteq_{\mathsf{L}} f^{\beta+1}_{\mu,*}$. By assumption, either $\alpha \leq \beta$ or $\alpha = \beta$. In the second case we conclude by the just stated reasoning. In the first case, by induction hypothesis, $f^{\alpha}_{\mu,*} \sqsubseteq_{\mathsf{L}} f^{\beta}_{\mu,*} \sqsubseteq_{\mathsf{L}} f^{\beta+1}_{\mu,*}$.

Limit step. We now show the statement for the limit case in general. Let $\alpha \leq \beta$ for β limit. On the one hand, $f^{\alpha}_{\mu,\cup} \subseteq f^{\beta}_{\mu,\cup} = \bigcup_{\alpha\in\beta} f^{\alpha}_{\mu,\cup}$ and so $f^{\alpha}_{\mu,\cup} \sqsubseteq_{\mathsf{L}} f^{\beta}_{\mu,\cup}$. On the other hand, noticing that for every $\alpha \in \beta$, $f^{\alpha}_{\mu,\sqcup}$ is non-empty, it follows that $f^{\alpha}_{\mu,\sqcup} \sqsubseteq_{\mathsf{L}} f^{\beta}_{\mu,\sqsubseteq_{\mathsf{L}}}$. □

The similar statement for ν holds following a similar proof.

Proposition 3.46. *Let $f : \mathcal{P}^*\mathcal{P}(S) \to \mathcal{P}^*\mathcal{P}(S)$ be monotone over $\sqsubseteq_{\mathsf{L}}$. If $\alpha \leq \beta$, then*

(1) $f^{\beta}_{\nu,\cap} \sqsubseteq_{\mathsf{L}} f^{\alpha}_{\nu,\cap}$.
(2) $f^{\beta}_{\nu,\sqcap} \sqsubseteq_{\mathsf{L}} f^{\alpha}_{\nu,\sqcap}$.

Remark 3.47. Recall that the relation $\sqsubseteq_{\mathsf{L}}$ is not antisymmetric. Hence, this sequences are eventually constant *up to* $\equiv_{\mathsf{L}}$ *equivalence*. That is, if $(x^{\xi})_{\xi\in\mathbf{ORD}}$ is a sequence of elements of $\mathcal{P}^*\mathcal{P}(S)$ monotone over $\sqsubseteq_{\mathsf{L}}$ there is an ordinal λ such that

$$\{x^{\xi} \mid \xi \in \lambda\} = \{x^{\xi} \mid \xi \in \mathbf{ORD}\}.$$

This allows us to define the following notions.

Definition 3.48. Let $f : \mathcal{P}^*\mathcal{P}(S) \to \mathcal{P}^*\mathcal{P}(S)$ be a monotone function over $\sqsubseteq_{\mathsf{L}}$, then we define

$$\lim_{\xi\in\mathbf{ORD}} f^{\xi}_{\mu,\cup} := \bigcup_{\xi\in\mathbf{ORD}} f^{\xi}_{\mu,\cup} \quad \lim_{\xi\in\mathbf{ORD}} f^{\xi}_{\nu,\cap} := \bigcap_{\xi\in\mathbf{ORD}} f^{\xi}_{\nu,\cap}$$

and

$$\lim_{\xi\in\mathbf{ORD}} f^{\xi}_{\mu,\sqcup} := \bigsqcup_{\xi\in\mathbf{ORD}} f^{\xi}_{\mu,\sqcup} \quad \lim_{\xi\in\mathbf{ORD}} f^{\xi}_{\nu,\sqcap} := \bigsqcap_{\xi\in\mathbf{ORD}} f^{\xi}_{\nu,\sqcap}$$

We know show how this limits relate with the corresponding notion of fixed-point in our framework.

Proposition 3.49. *Let $(\mathcal{A}_i)_{i\in I}$ be a non-empty family. Then, if for every $i, j \in I$, $\mathcal{A}_i \equiv_{\mathsf{L}} \mathcal{A}_j$, then for every $i \in I$, $\bigcup_{i\in I} \mathcal{A}_i \equiv_{\mathsf{L}} \mathcal{A}_i$ and $\bigsqcap_{i\in I} \mathcal{A}_i \equiv_{\mathsf{L}} \mathcal{A}_i$.*

Proof. Let us show first the case of the union. By assumption, it suffices to show that, for some $\mathcal{A}_j$, $\bigcup_{i\in I} \mathcal{A}_i \equiv_{\mathsf{L}} \mathcal{A}_j$. Since $\mathcal{A}_j \subseteq \bigcup_{i\in I} \mathcal{A}_i$, $\mathcal{A}_j \sqsubseteq_{\mathsf{L}} \bigcup_{i\in I} \mathcal{A}_i$. Conversely, let $A \in \bigcup_{i\in I} \mathcal{A}_i$. Hence, for some $k \in I$, $A \in \mathcal{A}_k \equiv_{\mathsf{L}} \mathcal{A}_j$. Therefore, there is $B \in \mathcal{A}_j$, such that $A \subseteq B$, that is, $\bigcup_{i\in I} \mathcal{A}_i \sqsubseteq_{\mathsf{L}} \mathcal{A}_j$.

To show the case of the internal intersection. As before, it suffices to show that, for some $\mathcal{A}_j$, $\bigsqcap_{i\in I} \mathcal{A}_i \equiv_{\mathsf{L}} \mathcal{A}_j$. Since $\bigsqcap_{i\in I} \mathcal{A}_i \subseteq \mathcal{A}_j$, $\bigsqcap_{i\in I} \mathcal{A}_i \sqsubseteq_{\mathsf{L}} \mathcal{A}_j$. Conversely, let $A \in \mathcal{A}_j$. By assumption, for every $k \in I$, $A \in \mathcal{A}_j \equiv_{\mathsf{L}} \mathcal{A}_k$. Therefore, for every $k \in I$, there is $f(\mathcal{A}_k) \in \mathcal{A}_k$ such that $A \subseteq f(\mathcal{A}_k)$. Hence, $A \subseteq \bigcap_{i\in I} f(\mathcal{A}_i) \in \bigsqcap_{i\in I} \mathcal{A}_i$, that is, $\mathcal{A}_j \sqsubseteq_{\mathsf{L}} \bigsqcap_{i\in I} \mathcal{A}_i$. □

This allows us to show that the internal definition of the ν prdinal sequence preserves the intuitive notion of fixed-point while it is the external one which preserves it for the μ case.

Corollary 3.50. *For a function $f : \mathcal{P}^*\mathcal{P}(S) \to \mathcal{P}^*\mathcal{P}(S)$ monotone over $\sqsubseteq_{\mathsf{L}}$,*

(1) $\lim_{\xi\in\mathbf{ORD}} f^{\xi}_{\mu,\cup} \equiv_{\mathsf{L}} f(\lim_{\xi\in\mathbf{ORD}} f^{\xi}_{\mu,\cup})$.
(2) $\lim_{\xi\in\mathbf{ORD}} f^{\xi}_{\nu,\sqcap} \equiv_{\mathsf{L}} f(\lim_{\xi\in\mathbf{ORD}} f^{\xi}_{\nu,\sqcap})$.

Finally, and before formally stating the formulas of team μ-calculus. Let us see, that under the conditions of downwards closure, there is no need to differentiate the internal and the external ν sequences.

PROPOSITION 3.51. *Let $f : \mathcal{P}^*\mathcal{P}(S) \to \mathcal{P}^*\mathcal{P}(S)$ be monotone over $\sqsubseteq_{\mathsf{L}}$. If, for every $\mathcal{X} \in \mathcal{P}^*\mathcal{P}(S)$, $f(\mathcal{X})$ is downwards closed, then*

$$\lim_{\xi \in \mathbf{ORD}} f^{\xi}_{\nu,\sqcap} = \lim_{\xi \in \mathbf{ORD}} f^{\xi}_{\nu,\cap}$$

PROOF. By Proposition 3.42. □

REMARK 3.52. Thanks to the just stated proposition and for the purposes of this thesis, we will consider the grammar of team μ-calculus with two least fixed-point operators (internal and external) and only one greatest fixed-point operator. As well as only talking about the sequence $f^{\xi}_{\nu,\sqcap}$ for both of them coincide. This is because we will always be working under the conditions of downwards closure. However, as we will see later in Section 5, we have to take into account the importance of downwards closure.

DEFINITION 3.53. The *formulas of team modal μ-calculus* are given by the following grammar:

$$\varphi ::= \top \mid \bot \mid P \mid \overline{P} \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mu^{\cup}X.\varphi \mid \mu^{\sqcup}X.\varphi \mid \nu X.\varphi$$

where P and X range over Prop and φ is positive in X. The set of all modal μ-calculus formulas will be denoted by $\mathtt{t}\mu\mathtt{ML}$. Notice that $\mathtt{ML} \subseteq \mathtt{t}\mu\mathtt{ML}$.

DEFINITION 3.54. For a general model $\mathcal{M} = (S, R, \Omega)$ we extend *general team semantics* to team modal μ-calculus formulas as follows:

(9) $\|\mu^{\cup}X.\varphi\|^{\mathcal{M}}_{\mathsf{t}} := \lim_{\xi \in \mathbf{ORD}} (\varphi^{\mathcal{M}}_X)^{\xi}_{\mu,\cup}$.
(10) $\|\mu^{\sqcup}X.\varphi\|^{\mathcal{M}}_{\mathsf{t}} := \lim_{\xi \in \mathbf{ORD}} (\varphi^{\mathcal{M}}_X)^{\xi}_{\mu,\sqcup}$.
(11) $\|\nu X.\varphi\|^{\mathcal{M}}_{\mathsf{t}} := \lim_{\xi \in \mathbf{ORD}} (\varphi^{\mathcal{M}}_X)^{\xi}_{\nu,\sqcap}$.

We will also write, mostly in chapter 4, $\mathcal{M}, T \models^{\mu\mathsf{ML}}_{\mathsf{t}} \varphi$. And, abusing notation $\mathbb{S}, T \models^{\mu\mathsf{ML}}_{\mathsf{t}} \varphi$ for $\mathcal{S}, T \models^{\mu\mathsf{ML}}_{\mathsf{t}} \varphi$.

REMARK 3.55. Let us note that the monotonicity of $\varphi^{\mathcal{M}}_X$ was shown for an arbitrary general model $\mathcal{M}$. In the particular case where we are working with the general model $\mathcal{S}$ associated with $\mathbb{S}$, we can even say more about such function.

It is easy to note that if $\mathcal{B}$ is downwards closed and $\mathcal{A} \sqsubseteq_{\mathsf{L}} \mathcal{B}$, then $\mathcal{A} \subseteq \mathcal{B}$. For, for every $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$ and so $A \in \mathcal{B}$ by downwards closure. Hence, using this fact and the fact that $\bigcup$ also preserves downwards closure as it was the case for the binary union in propositional logic, we can state the following proposition.

PROPOSITION 3.56. *Let $\mathbb{S}$ be a model. The sequences $((\varphi^{\mathcal{S}}_X)^{\xi}_{\mu,\cup})_{\xi \in \mathbf{ORD}}$, $((\varphi^{\mathcal{S}}_X)^{\xi}_{\mu,\sqcup})_{\xi \in \mathbf{ORD}}$ and $((\varphi^{\mathcal{S}}_X)^{\xi}_{\nu,\sqcap})_{\xi \in \mathbf{ORD}}$ are monotone over $\subseteq$.*

COROLLARY 3.57. *Let $\mathbb{S}$ be a model. The sequences $((\varphi^{\mathcal{S}}_X)^{\xi}_{\mu,\cup})_{\xi \in \mathbf{ORD}}$, $((\varphi^{\mathcal{S}}_X)^{\xi}_{\mu,\sqcup})_{\xi \in \mathbf{ORD}}$ and $((\varphi^{\mathcal{S}}_X)^{\xi}_{\nu,\sqcap})_{\xi \in \mathbf{ORD}}$ are eventually constant.*

The following and last result of this Section, shows us that, although it is the external definition of μ which carries the intuitive notion of fixed-point, it is the internal one the construction which preserves flatness. This comment is the same done for modal logic although the framework in μ-calculus do not allow us to consider two different semantics.

Just for completeness, let us recall that in the case of modal logic, we said that it was the internal notion the one which preserved flatness and it was the external definition the one which was more intuitive. However, both definitions coincide for $\Diamond$ and, for a more restricted class of models, for $\Box$. It is important to note, since it will become an important remark later, the external notions which are not equivalent to the internal ones are precisley the notions which expand the expressive power of team semantics.

Now, let us prove flatness for $\mathtt{t}\mu\mathtt{ML}$.

THEOREM 3.58. *Let $\mathbb{S}$ be a model and consider the general model $\mathcal{S} = (S, R, V^{\mathcal{P}})$ where, for every $P \in \mathsf{Prop}$, $V^{\mathcal{P}}(P) := \mathcal{P}(V(P))$. If φ is flat, then*

(1) $\|\mu^{\sqcup} X.\varphi\|_{\mathsf{t}}^{\mathcal{S}} = \mathcal{P}(\|\mu X.\varphi\|_{\mathsf{c}}^{\mathbb{S}})$.
(2) $\|\nu X.\varphi\|_{\mathsf{t}}^{\mathcal{S}} = \mathcal{P}(\|\nu X.\varphi\|_{\mathsf{c}}^{\mathbb{S}})$.

That is, $\mu^{\sqcup}$ and ν preserve flatness.

PROOF. We restrict ourselves to show (1) since (2) is similar.

We show by ordinal induction that, for every ordinal ξ,

$$(\varphi_X^{\mathcal{S}})_{\mu,\sqcup}^{\xi} = \mathcal{P}((\varphi_X^{\mathbb{S}})_{\mu}^{\xi}).$$

Base case. The following equalities hold $(\varphi_X^{\mathcal{S}})_{\mu,\sqcup}^{0} = \{\emptyset\} = \mathcal{P}(\emptyset) = \mathcal{P}((\varphi_X^{\mathbb{S}})_{\mu}^{0})$.

Successor step. Suppose that $(\varphi_X^{\mathcal{S}})_{\mu,\sqcup}^{\alpha} = \mathcal{P}((\varphi_X^{\mathbb{S}})_{\mu}^{\alpha})$

$$\begin{aligned}
(\varphi_X^{\mathcal{S}})_{\mu,\sqcup}^{\alpha+1} &= \varphi_X^{\mathcal{S}}((\varphi_X^{\mathcal{S}})_{\mu,\sqcup}^{\alpha}) && \text{(Def.)}\\
&= \varphi_X^{\mathcal{S}}(\mathcal{P}((\varphi_X^{\mathbb{S}})_{\mu}^{\alpha})) && \text{(I.H.)}\\
&= \|\varphi\|_{\mathsf{t}}^{\mathcal{S}[X \mapsto \mathcal{P}((\varphi_X^{\mathbb{S}})_{\mu}^{\alpha})]} && \text{(Def.)}\\
&= \mathcal{P}(\|\varphi\|_{\mathsf{c}}^{\mathbb{S}[X \mapsto (\varphi_X^{\mathbb{S}})_{\mu}^{\alpha}]}) && \text{(Flatness)}\\
&= \mathcal{P}(\varphi_X^{\mathbb{S}}((\varphi_X^{\mathbb{S}})_{\mu}^{\alpha})) && \text{(Def.)}\\
&= \mathcal{P}((\varphi_X^{\mathbb{S}})_{\mu}^{\alpha+1}) && \text{(Def.)}
\end{aligned}$$

Limit step. Suppose that, for every $\alpha \in \beta$, $(\varphi^{\mathcal{S}}_X)^{\alpha}_{\mu,\sqcup} = \mathcal{P}((\varphi^{\mathbb{S}}_X)^{\alpha}_{\mu})$. Hence, the following chain of equalities hold:

$$\begin{aligned}
(\varphi^{\mathcal{S}}_X)^{\beta}_{\mu,\sqcup} &= \textstyle\bigsqcup_{\alpha\in\beta}(\varphi^{\mathcal{S}}_X)^{\alpha}_{\mu,\sqcup} && \text{(Def.)}\\
&= \textstyle\bigsqcup_{\alpha\in\beta}\mathcal{P}(\varphi^{\mathbb{S}}_X)^{\alpha}_{\mu} && \text{(I.H.)}\\
&= \mathcal{P}(\textstyle\bigcup_{\alpha\in\beta}(\varphi^{\mathbb{S}}_X)^{\alpha}_{\mu}) && \text{(Prop. 3.39)}\\
&= \mathcal{P}((\varphi^{\mathbb{S}}_X)^{\beta}_{\mu}) && \text{(Def.)}
\end{aligned}$$

□

COROLLARY 3.59. *The $\sqcup$ fragment of* $\mathtt{t}\mu\mathtt{ML}$ *is flat.*

4. Singleton lifting

In this last section of the chapter we give a glimpse about the study of another possible lifting. As we said, there is no algebraic preference between the powerset lifting and any other possible lifting from $\mathcal{P}(S)$ to $\mathcal{PP}(S)$. That is why, making use of some of the constructions that we have developed, we give an alternative semantics which are flat with respect to the singleton lifting.

For this section we will consider the singleton lifting $\{\cdot\}$ from $\mathcal{P}(S)$ to $\mathcal{PP}(S)$ which maps a set A to $\{A\}$. We will only give a sketch of how to work with a different kind of lifting mostly without giving the proofs of the theorems. The work is similar to the one we have done because we have chosen a lifting which is preserved under many of the constructions that we have developed.

Let us define the *singleton semantics* for a general model for modal logic.

DEFINITION 3.60. *Singleton semantics* for a general model $\mathcal{M}$ for modal logic are defined by recursion as follows:

(1) $\|\top\|^{\mathcal{M}}_{\{\mathtt{t}\}} = \{S\}$.
(2) $\|\bot\|^{\mathcal{M}}_{\{\mathtt{t}\}} = \{\emptyset\}$.
(3) $\|P\|^{\mathcal{M}}_{\{\mathtt{t}\}} = \Omega(P)$.
(4) $\|\overline{P}\|^{\mathcal{M}}_{\{\mathtt{t}\}} = \{S - X \mid X \in \Omega(P)\}$.
(5) $\|\varphi \vee \psi\|^{\mathcal{M}}_{\{\mathtt{t}\}} = \|\varphi\|^{\mathcal{M}}_{\{\mathtt{t}\}} \sqcup \|\psi\|^{\mathcal{M}}_{\{\mathtt{t}\}}$.
(6) $\|\varphi \wedge \psi\|^{\mathcal{M}}_{\{\mathtt{t}\}} = \|\varphi\|^{\mathcal{M}}_{\{\mathtt{t}\}} \sqcap \|\psi\|^{\mathcal{M}}_{\{\mathtt{t}\}}$.
(7) $\|\Diamond\varphi\|^{\mathcal{M}}_{\{\mathtt{t}\}} = \langle R\rangle^{\mathcal{P}}\|\varphi\|^{\mathcal{M}}_{\{\mathtt{t}\}} = \{\langle R\rangle X \mid X \in \|\varphi\|^{\mathcal{M}}_{\{\mathtt{t}\}}\}$.
(8) $\|\Box\varphi\|^{\mathcal{M}}_{\{\mathtt{t}\}} = [R]^{\mathcal{P}}\|\varphi\|^{\mathcal{M}}_{\{\mathtt{t}\}} = \{[R]X \mid X \in \|\varphi\|^{\mathcal{M}}_{\{\mathtt{t}\}}\}$.

Following the same strategy, we can show the following theorem.

THEOREM 3.61. *Let $\varphi \in$* $\mathtt{ML}$ *and $\mathcal{M}$ a general model. If φ is positive in X, then the function $\varphi^{\mathcal{M}}_X$ is monotone over $\sqsubseteq_{\mathtt{L}}$.*

DEFINITION 3.62. Let us say that a formula $\varphi \in \mathtt{Fm}$ is *singleton-flat* whenever $\|\varphi\|_{\mathtt{t}} = \{\|\varphi\|_{\mathtt{c}}\}$, that is, when the diagram

$$\begin{tikzcd} \mathtt{Fm} \arrow[r, "\|\cdot\|_{\mathsf{c}}"] \arrow[rr, bend right, "\|\cdot\|_{\mathsf{t}}"'] & \mathcal{P}(X) \arrow[r, "\{\cdot\}"] & \mathcal{P}\mathcal{P}(X) \end{tikzcd}$$

commutes.

REMARK 3.63. For a model $\mathbb{S} = (S, R, V)$ we adapt for this section the notation $\mathcal{S}$. Right now and only for this section, $\mathcal{S} = (S, R, V^{\{\cdot\}})$ will denote the general model defined, for every $P \in \mathsf{Prop}$, as $V^{\{\cdot\}}(P) = \{V(P)\}$.

THEOREM 3.64. *For every $\varphi \in \mathtt{ML}$ and every model $\mathbb{S}$,*

$$\|\varphi\|^{\mathcal{S}}_{\{\mathsf{t}\}} = \{\|\varphi\|^{\mathbb{S}}_{\mathsf{c}}\}$$

PROOF. By induction on φ. □

Let us now, consider the respective notion for the ν ordinal sequence. As we will see, we will consdeir the internal definition of the μ ordinal sequence. Note that this is just a particularity of the empty set for $\mathcal{P}(\emptyset) = \{\emptyset\}$.

DEFINITION 3.65. For a function $f : \mathcal{P}^*\mathcal{P}(S) \to \mathcal{P}^*\mathcal{P}(S)$ we define the sequence $(f^{\xi}_{\nu,\{\cdot\}})_{\imath\in\mathbf{ORD}}$ as follows:

$$\begin{cases} f^0_{\nu,\{\cdot\}} = \{S\} \\ f^{\alpha+1}_{\nu,\{\cdot\}} = f(f^{\alpha}_{\nu,\{\cdot\}}) \\ f^{\beta}_{\nu,\{\cdot\}} = \bigsqcap_{\alpha\in\beta} f^{\alpha}_{\nu,\{\cdot\}} \quad \text{for } \beta \text{ limit} \end{cases}$$

The fact that this sequence is monotone over $\sqsubseteq_{\mathsf{L}}$ whenever f is so follows since $\{S\} \equiv_{\mathsf{L}} \mathcal{P}(S)$. As it is seen in the definition, for the singleton lifting everything is computed internally. Hence, we will write

$$\lim_{\xi\in\mathbf{ORD}} f^{\xi}_{\nu,\{\cdot\}} := \bigsqcap_{\xi\in\mathbf{ORD}} f^{\xi}_{\nu,\{\cdot\}}.$$

Thus, let us extend singleton semantics with respective notions.

DEFINITION 3.66. For a general model $\mathcal{M}$ we extend singleton semantics as follows:

(9) $\|\mu X.\varphi\|^{\mathcal{M}}_{\{\mathsf{t}\}} := \lim_{\xi\in\mathbf{ORD}}(\varphi^{\mathcal{M}}_X)^{\xi}_{\mu,\sqcup}$.

(10) $\|\nu X.\varphi\|^{\mathcal{M}}_{\{\mathsf{t}\}} := \lim_{\xi\in\mathbf{ORD}}(\varphi^{\mathcal{M}}_X)^{\xi}_{\nu,\{\cdot\}}$.

Finally showing that this semantics are preserved by fixed-point operators.

THEOREM 3.67. *If φ is singleton-flat and $\mathcal{S} = (S, R, V^{\{\cdot\}})$ is the general model defined, for every $P \in \mathsf{Prop}$, as $V^{\{\cdot\}}(P) = \{V(P)\}$, then*

(1) $\|\mu X.\varphi\|^{\mathcal{S}^{\{\cdot\}}}_{\{\mathsf{t}\}} = \{\|\mu X.\varphi\|^{\mathbb{S}}_{\mathsf{c}}\}$*;*

(2) $\|\nu X.\varphi\|^{\mathcal{S}^{\{\cdot\}}}_{\{\mathsf{t}\}} = \{\|\nu X.\varphi\|^{\mathbb{S}}_{\mathsf{c}}\}$.

5. Extending $\mathtt{t}\mu\mathtt{ML}$

One of the possibilities that team semantics open is the extension of logics with atoms or connectives which are not definable in the classical framework. Although the purpose of this thesis is to relate the defined logic $\mathtt{t}\mu\mathtt{ML}$ with the defined team semantics for $\mathtt{CTL}$, in this section we explore the starting point of this extensions.

As we saw in the propositional case, union, that is, external semantics for $\vee$, already extended the power of the logic since $\bigcup$ does not preserve flatness. Once having proved flatness, we can consider external disjunction or, as we will refer to from now on, *Boolean disjunction* as one more connective in our language to have a richer and a more expressive logic. That is, we will consider the following extension of $\mathtt{t}\mu\mathtt{ML}$.

Definition 3.68. The *formulas of extended team modal μ-calculus* are given by the following grammar:

$$\varphi ::= \top \mid \bot \mid P \mid \overline{P} \mid \varphi\vee\varphi \mid \varphi\wedge\varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mu^{\cup}X.\varphi \mid \mu^{\sqcup}X.\varphi \mid \nu X.\varphi \mid \varphi \circledvee \varphi$$

where P and X range over Prop and φ is positive in X. The set of all extended modal μ-calculus formulas will be denoted by $\mathtt{t}\mu\mathtt{ML}^*$.

And we extend the semantics in the intuitive way

Definition 3.69. For a general model $\mathcal{M} = (S, R, \Omega)$ we extend *general team semantics* to extended team modal μ-calculus formulas by writing

$$\|\varphi \circledvee \psi\|^{\mathcal{M}}_{\mathsf{t}} := \|\varphi\|^{\mathcal{M}}_{\mathsf{t}} \cup \|\psi\|^{\mathcal{M}}_{\mathsf{t}}.$$

Remark 3.70. It is of great importance to note that this extension is well-defined because union is monotone over $\subseteq$ and so monotone over $\sqsubseteq_{\mathsf{L}}$. Hence, all the work done to define the semantics for fixed-points do also hold for this extension. Note also that we have preserved downwards closure.

The fact that we are, indeed, extending the expressible power was already remarked on our work of propositional logic.

Lastly, let us say that we could have extended the logic with many other atoms or connectives usual in team semantics, as for example, dependence atoms $\mathsf{dep}(x_0, \ldots, x_n, y)$, independence atoms $x_0, \ldots, x_n \perp_{z_0,\ldots,z_k} y_0, \ldots, y_m$, inclusion atoms $\vec{x} \subseteq \vec{y}$, strong negation $\sim \varphi$ or the non-emptyness atom NE (first checking whether or not they are monotone over $\sqsubseteq_{\mathsf{L}}$). However, some of those, such as NE or $\sim$, break downwards closure of the semantics which, as we have seen, was crucial for most of the work. That is not in a strict sense a problem but then one should start differentiating Boolean conjunction $\circledwedge$ and internal conjunction $\wedge$ or internal and external definitions of ν since they will not coincide in general.

CHAPTER 4

Temporal team semantics

In this chapter we extend the classical embedding of computational tree logic (CTL) into modal μ-calculus to the framework of team semantics. Temporal team semantics, that is, team semantics for linear temporal logic (LTL) and CTL become really interesting since the notion of synchronicity plays an important role. In this chapter is where both definitions of μ fixed-points and the difference between union and internal union comes into play. In particular, in Section 1, we recall the proof of the embedding of CTL into (classical) modal μ-calculus. We do it from an ordinal perspective since we will later generalize it. Later, in Section 2, we give team semantics for LTL and CTL. Specially studying synchronicity which is only definable in a team semantical setting. Finally, in section 3, we prove the translation between CTL with team semantics and extended team μ-calculus.

As it is customary when working with CTL, we will assume that all the models in this chapter are *serial*, that is, every state has a successor. Recall that, thanks to Proposition 3.29 the internal and external team definitions of $\Box$ coincide.

1. Embedding of CTL into μML

In this section after defining classical semantics for CTL, we study the proof of its embedding into modal μ-calculus specially studuying the case of *until* formulas.

DEFINITION 4.1. The *formulas of linear temporal logic* are given by the following grammar:

$$\varphi ::= \top \mid \bot \mid P \mid \overline{P} \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \mathsf{X}\,\varphi \mid \varphi \mathsf{U} \varphi \mid \varphi\,\mathsf{R}\,\varphi$$

where P ranges over Prop. The set of all formulas will be denoted by LTL.

DEFINITION 4.2. A *trace* π is a function $(\pi(i))_{i\in\mathbb{N}}$ from $\mathbb{N}$ to $\mathcal{P}(\mathsf{Prop})$.

DEFINITION 4.3. Let π be a trace. For $k \in \mathbb{N}$, its j-*shifted* trace is $\pi[j] = (\pi_{i+j})_{i\in\mathbb{N}}$.

DEFINITION 4.4. *(Classical) semantics (for linear temporal logic)* for a trace π and $i \in \mathbb{N}$ are defined by recursion as follows:

(1) $\pi \models_{\mathsf{c}} \top$ always.
(2) $\pi \models_{\mathsf{c}} \bot$ never.

(3) $\pi \models_{\mathsf{c}} P$ if and only if $P \in \pi(0)$.
(4) $\pi \models_{\mathsf{c}} \overline{P}$ if and only if $P \notin \pi(0)$.
(5) $\pi \models_{\mathsf{c}} \varphi \vee \psi$ if and only if $\pi \models_{\mathsf{c}} \varphi$ or $\pi \models_{\mathsf{c}} \psi$.
(6) $\pi \models_{\mathsf{c}} \varphi \wedge \psi$ if and only if $\pi \models_{\mathsf{c}} \varphi$ and $\pi \models_{\mathsf{c}} \psi$.
(7) $\pi \models_{\mathsf{c}} \mathsf{X}\,\varphi$ if and only if $\pi(1) \models_{\mathsf{c}} \varphi$.
(8) $\pi \models_{\mathsf{c}} \varphi \mathsf{U} \psi$ if and only if there is some $j \in \mathbb{N}$ such that
   (a) $\pi(j) \models_{\mathsf{c}} \psi$;
   (b) for every $0 \leq k < j$, $\pi(k) \models_{\mathsf{c}} \varphi$.
(9) $\pi \models_{\mathsf{c}} \varphi\,\mathsf{R}\,\psi$ if and only if for every $k \in \mathbb{N}$, $\pi(k) \models_{\mathsf{c}} \psi$ or there is some $j \in \mathbb{N}$ such that
   (a) $\pi(j) \models_{\mathsf{c}} \varphi, \psi$;
   (b) for every $0 \leq k < j$, $\pi(k) \models_{\mathsf{c}} \psi$.

DEFINITION 4.5. The *formulas of computational tree logic* are given by the following grammar:

$$\varphi ::= \top \mid \bot \mid P \mid \overline{P} \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists \mathsf{X}\,\varphi \mid \exists(\varphi \mathsf{U} \varphi) \mid \exists(\varphi\,\mathsf{R}\,\varphi) \mid \\ \forall \mathsf{X}\,\varphi \mid \forall(\varphi \mathsf{U} \varphi) \mid \forall(\varphi\,\mathsf{R}\,\varphi)$$

where P ranges over Prop. The set of all formulas will be denoted by $\mathtt{CTL}$.

DEFINITION 4.6. Let $\mathbb{S}$ be a model and $s \in S$. We define its *set of traces*as follows:

$$\mathfrak{Tr}_{\mathbb{S}}(s) := \{(V(s_i))_{i\in\omega} \mid s_0 = s \text{ and, for every } i \in \omega, s_i R s_{i+1}\}$$

and for $T \subseteq S$ we write $\mathfrak{Tr}_{\mathbb{S}}(T) = \bigcup_{t \in T} \mathfrak{Tr}_{\mathbb{S}}(t)$.

DEFINITION 4.7. *(Classical) semantics (for computational tree logic)* for a model $\mathbb{S} = (S, R, V)$ are defined by recursion as follows:

(1) $\mathbb{S}, s \models_{\mathsf{c}} \top$ always.
(2) $\mathbb{S}, s \models_{\mathsf{c}} \bot$ never.
(3) $\mathbb{S}, s \models_{\mathsf{c}} P$ if and only if $s \in V(P)$.
(4) $\mathbb{S}, s \models_{\mathsf{c}} \overline{P}$ if and only if $s \notin V(P)$.
(5) $S, s \models_{\mathsf{c}} \varphi \vee \psi$ if and only if $\mathbb{S}, s \models_{\mathsf{c}} \varphi$ or $\mathbb{S}, s \models_{\mathsf{c}} \psi$.
(6) $\mathbb{S}, s \models_{\mathsf{c}} \varphi \wedge \psi$ if and only if $\mathbb{S}, s \models_{\mathsf{c}} \varphi$ and $\mathbb{S}, s \models_{\mathsf{c}} \psi$.
(7) $\mathbb{S}, c \models_{\mathsf{c}} \exists \psi$ if and only if there is a trace $\pi \in \mathfrak{Tr}_{\mathbb{S}}(s)$ such that $\pi \models_{\mathsf{c}} \psi$.
(8) $\mathbb{S}, c \models_{\mathsf{c}} \forall \psi$ if and only if for every trace $\pi \in \mathfrak{Tr}_{\mathbb{S}}(s)$, $\pi \models_{\mathsf{c}} \psi$.

REMARK 4.8. Note that we have stated all the cases of $\exists \mathsf{X}\,\varphi$, $\forall \mathsf{X}\,\varphi$, $\exists(\varphi \mathsf{U} \psi)$, $\forall(\varphi \mathsf{U} \psi)$, $\exists(\varphi\,\mathsf{R}\,\psi)$ and $\forall(\varphi\,\mathsf{R}\,\psi)$ at once but only this formulas are allowed under the scope of a quantifier. That is, we are *not* working with full $\mathtt{CTL}$ but we express the condition as shown for simplicity.

Before proceeding with the proof of the translation, let us state it completely.

$\varphi \in \mathtt{CTL}$	$\mathtt{tr}(\varphi) \in \mu\mathtt{ML}$
$\top$	$\top$
$\bot$	$\bot$
P	P
$\overline{P}$	$\overline{P}$
$\varphi \vee \psi$	$\mathtt{tr}(\varphi) \vee \mathtt{tr}(\psi)$
$\varphi \wedge \psi$	$\mathtt{tr}(\varphi) \wedge \mathtt{tr}(\psi)$
$\exists \mathsf{X}\, \varphi$	$\Diamond \mathtt{tr}(\varphi)$
$\forall \mathsf{X}\, \varphi$	$\Box \mathtt{tr}(\varphi)$
$\exists(\varphi \mathsf{U} \psi)$	$\mu X.\mathtt{tr}(\psi) \vee (\mathtt{tr}(\varphi) \wedge \Diamond X)$
$\forall(\varphi \mathsf{U} \psi)$	$\mu X.\mathtt{tr}(\psi) \vee (\mathtt{tr}(\varphi) \wedge \Box X)$
$\exists(\varphi\, \mathsf{R}\, \psi)$	$\nu X.\mathtt{tr}(\psi) \wedge (\mathtt{tr}(\varphi) \vee \Diamond X)$
$\forall(\varphi\, \mathsf{R}\, \psi)$	$\nu X.\mathtt{tr}(\psi) \wedge (\mathtt{tr}(\varphi) \vee \Box X)$

where the variable X in the last four cases is a fresh new variable.

THEOREM 4.9. *For every formula $\varphi \in \mathtt{CTL}$ and every model $\mathbb{S}$,*

$$\mathbb{S}, s \models_{\mathsf{c}}^{\mathtt{CTL}} \varphi \textit{ if and only if } \mathbb{S}, s \models_{\mathsf{c}}^{\mu\mathtt{ML}} \mathtt{tr}(phi).$$

That is, $\|\varphi\|_{\mathsf{c}}^{\mathbb{S}} = \|\mathtt{tr}(\varphi)\|_{\mathsf{c}}^{\mathbb{S}}$.

The proof of the theorem goes by induction on φ. The Boolean and modal cases are trivial. Among the temporal cases, we restrict attention to the case of an *until* formula $\exists(\varphi \mathsf{U} \psi)$. From now and for the rest of the proof fix a model $\mathbb{S}$. We first show the following equivalence in CTL:

$$\exists(\varphi \mathsf{U} \psi) \equiv_{\mathtt{CTL}} \psi \vee (\varphi \wedge \exists \mathsf{X}\, \exists(\varphi \mathsf{U} \psi))$$

PROPOSITION 4.10. $\|\exists(\varphi \mathsf{U} \psi)\|_{\mathsf{c}}^{\mathbb{S}} \subseteq \|\psi \vee (\varphi \wedge \exists \mathsf{X}\, \exists(\varphi \mathsf{U} \psi))\|_{\mathsf{c}}^{\mathbb{S}}$.

PROOF. Suppose that $\mathbb{S}, s \models_{\mathsf{c}}^{\mathtt{CTL}} \exists(\varphi \mathsf{U} \psi)$. Then, there is a path π starting from s (i.e., $\pi(0) = s$), such that $\pi \models_{\mathsf{c}}^{\mathtt{CTL}} \varphi \mathsf{U} \psi$, that is, for some $j \in \omega$,

$$\pi(j) \models_{\mathsf{c}}^{\mathtt{LTL}} \psi \text{ and } \pi(0), \dots, \pi(j-1) \models_{\mathsf{c}}^{\mathtt{LTL}} \varphi.$$

If $j = 0$, then $\mathbb{S}, s \models_{\mathsf{c}}^{\mathtt{CTL}} \psi$ and so $\mathbb{S}, s \models_{\mathsf{c}}^{\mathtt{CTL}} \psi \vee (\varphi \wedge \exists \mathsf{X}\, \exists(\varphi \mathsf{U} \psi))$ as wanted.

If $j > 0$, then $\mathbb{S}, \pi(0) \models_{\mathsf{c}}^{\mathtt{CTL}} \varphi$, $\pi(0) R \pi(1)$ and $\mathbb{S}, \pi(1) \models_{\mathsf{c}}^{\mathtt{CTL}} \exists(\varphi \mathsf{U} \psi)$. Hence, $\mathbb{S}, s \models_{\mathsf{c}}^{\mathtt{CTL}} \varphi \wedge \exists \mathsf{X}(\varphi \mathsf{U} \psi)$ and $\mathcal{S}, s \models_{\mathsf{c}}^{\mathtt{CTL}} \psi \vee (\varphi \wedge \exists \mathsf{X}\, \exists(\varphi \mathsf{U} \psi))$. $\Box$

PROPOSITION 4.11. $\|\psi \vee (\varphi \wedge \exists \mathsf{X}\, \exists(\varphi \mathsf{U} \psi))\|_{\mathsf{c}}^{\mathbb{S}} \subseteq \|\exists(\varphi \mathsf{U} \psi)\|_{\mathsf{c}}^{\mathbb{S}}$.

PROOF. Suppose that $\mathbb{S}, s \models_{\mathsf{c}}^{\mathtt{CTL}} \psi \vee (\varphi \wedge \exists\ (\varphi \mathsf{U} \psi))$.

If $\mathbb{S}, s \models_{\mathsf{c}}^{\mathtt{CTL}} \psi$, then $\mathbb{S}, s \models_{\mathsf{c}}^{\mathtt{CTL}} \exists(\varphi \mathsf{U} \psi)$ trivially.

Otherwise, if $\mathbb{S}, s \models_{\mathsf{c}}^{\mathtt{CTL}} \varphi \wedge \exists \mathsf{X}\, \exists(\varphi \mathsf{U} \psi)$ let sRs' such that $\mathbb{S}, s' \models_{\mathsf{c}}^{\mathtt{CTL}} \exists(\varphi \mathsf{U} \psi)$. Considering the path π starting from s' which withnesses $\mathbb{S}, s' \models_{\mathsf{c}}^{\mathtt{CTL}} \exists(\varphi \mathsf{U} \psi)$, it is easy to see that $s \curlywedge \pi$ (the concatenation) is a witness for $\mathbb{S}, s \models_{\mathsf{c}}^{\mathtt{CTL}} \exists(\varphi \mathsf{U} \psi)$. $\Box$

We now show that $\exists(\varphi \mathsf{U} \psi)$ is equivalent to $\mu X.\mathtt{tr}(\psi) \vee (\mathtt{tr}(\varphi) \wedge \Diamond X)$ assuming that φ is equivalent to $\mathtt{tr}(\varphi)$ and that ψ is equivalent to $\mathtt{tr}(\psi)$ (induction hypothesis). Let us denote

$$\chi := \mathtt{tr}(\psi) \vee (\mathtt{tr}(\varphi) \wedge \Diamond X).$$

We show that $\|\exists(\varphi \mathsf{U} \psi)\|^{\mathbb{S}}_{\mathsf{c}} = \lim_{\xi \in \mathbf{ORD}} (\chi^{\mathbb{S}}_X)^{\xi}_{\mu}$ by showing both inclusions.

PROPOSITION 4.12. $\lim_{\xi \in \mathbf{ORD}} (\chi^{\mathbb{S}}_X)^{\xi}_{\mu} \subseteq \|\exists(\varphi \mathsf{U} \psi)\|^{\mathbb{S}}_{\mathsf{c}}$.

PROOF. By ordinal induction.

Base case. Obvious since $(\chi^{\mathbb{S}}_X)^{0}_{\mu} = \emptyset \subseteq \|\exists(\varphi \mathsf{U} \psi)\|^{\mathbb{S}}_{\mathsf{c}}$

Successor step. Suppose that $(\chi^{\mathbb{S}}_X)^{\alpha}_{\mu} \subseteq \|\exists(\varphi \mathsf{U} \psi)\|^{\mathbb{S}}_{\mathsf{c}}$. We want to show that $(\chi^{\mathbb{S}}_X)^{\alpha+1}_{\mu} \subseteq \|\exists(\varphi \mathsf{U} \psi)\|^{\mathbb{S}}_{\mathsf{c}}$. Let $s \in (\chi^{\mathbb{S}}_X)^{\alpha+1}_{\mu} = \|\chi\|^{\mathbb{S}[X \mapsto (\chi^{\mathbb{S}}_X)^{\alpha}_{\mu}]}_{\mathsf{c}}$. Hence,

$$\mathbb{S}[X \mapsto (\chi^{\mathbb{S}}_X)^{\alpha}_{\mu}], s \models^{\mu\mathsf{ML}}_{\mathsf{c}} \mathtt{tr}(\psi) \vee (\mathtt{tr}(\varphi) \wedge \Diamond X)$$

Suppose $\mathbb{S}[X \mapsto (\chi^{\mathbb{S}}_X)^{\alpha}_{\mu}], s \models^{\mu\mathsf{ML}}_{\mathsf{c}} \mathtt{tr}(\psi)$. Hence $\mathbb{S}, s \models^{\mu\mathsf{ML}}_{\mathsf{c}} \mathtt{tr}(\psi)$ since $\mathtt{tr}(\psi)$ does not contain X. Therefore, $\mathbb{S}, s \models^{\mathsf{CTL}}_{\mathsf{c}} \varphi$ by induction hypothesis, and

$$\mathbb{S}, s \models^{\mathsf{CTL}}_{\mathsf{c}} \psi \vee (\varphi \wedge \exists \mathsf{X} \exists(\varphi \mathsf{U} \psi)).$$

Hence, by the just proven equivalence, $\mathbb{S}, s \models^{\mathsf{CTL}}_{\mathsf{c}} \exists(\varphi \mathsf{U} \psi)$.

Suppose $\mathbb{S}[X \mapsto (\chi^{\mathbb{S}}_X)^{\alpha}_{\mu}], s \models^{\mu\mathsf{ML}}_{\mathsf{c}} \mathtt{tr}(\varphi) \wedge \Diamond X$, i.e., $\mathbb{S}[X \mapsto (\chi^{\mathbb{S}}_X)^{\alpha}_{\mu}], s \models^{\mu\mathsf{ML}}_{\mathsf{c}} \mathtt{tr}(\varphi)$ and, for some sRt, $t \in (\chi^{\mathbb{S}}_X)^{\alpha}_{\mu}$. On the one hand, as before, $\mathbb{S}, s \models^{\mu\mathsf{ML}}_{\mathsf{c}} \mathtt{tr}(\varphi)$ since $\mathtt{tr}(\varphi)$ does not contain X and so, by induction hypothesis, $\mathbb{S}, s \models^{\mathsf{CTL}}_{\mathsf{c}} \varphi$. On the other hand, by induction hypothesis, $\mathbb{S}, t \models^{\mathsf{CTL}}_{\mathsf{c}} \exists(\varphi \mathsf{U} \psi)$. That is, $\mathbb{S}, s \models^{\mathsf{CTL}}_{\mathsf{c}} \varphi \wedge \exists \mathsf{X} \exists(\varphi \mathsf{U} \psi)$. Therefore,

$$\mathbb{S}, s \models^{\mathsf{CTL}}_{\mathsf{c}} \psi \vee (\varphi \wedge \exists \mathsf{X} \exists(\varphi \mathsf{U} \psi))$$

and, by the just proven equivalence, $\mathbb{S}, s \models^{\mathsf{CTL}}_{\mathsf{c}} \exists(\varphi \mathsf{U} \psi)$.

Limit step. If, for every $\alpha \in \beta$, $(\chi^{\mathbb{S}}_X)^{\alpha}_{\mu} \subseteq \|\exists(\varphi \mathsf{U} \psi)\|^{\mathbb{S}}_{\mathsf{c}}$, then

$$(\chi^{\mathbb{S}}_X)^{\beta}_{\mu} = \bigcup_{\alpha \in \beta} (\chi^{\mathbb{S}}_X)^{\alpha}_{\mu} \subseteq \|\exists(\varphi \mathsf{U} \psi)\|^{\mathbb{S}}_{\mathsf{c}}.$$

□

PROPOSITION 4.13. $\|\exists(\varphi \mathsf{U} \psi)\|^{\mathbb{S}}_{\mathsf{c}} \subseteq \lim_{\xi \in \mathbf{ORD}} (\chi^{\mathbb{S}}_X)^{\xi}_{\mu}$.

PROOF. Let $\mathbb{S}, s \models^{\mathsf{CTL}}_{\mathsf{c}} \exists(\varphi \mathsf{U} \psi)$. Then, there is a path π (starting from s) such that, $\pi \models^{\mathsf{LTL}}_{\mathsf{c}} \varphi \mathsf{U} \psi$, that is, for some $\pi(j) \models^{\mathsf{LTL}}_{\mathsf{c}} \psi$ and, for every $0 \leq i < j$, $\pi(i) \models^{\mathsf{LTL}}_{\mathsf{c}} \varphi$. Note that it suffices to show that $s \in (\chi^{\mathbb{S}}_X)^{j+1}_{\mu}$. We show that

$$\pi(j - i) \in (\chi^{\mathbb{S}}_X)^{i+1}_{\mu}$$

by induction on i and the proof ends by setting $i = j$.

Base case. Notice that:

$$\pi(j-0) \in \|\psi\|^{\mathsf{CTL}}_{\mathsf{c}} \subseteq \|\psi\|^{\mathsf{CTL}}_{\mathsf{c}} \cup (\|\varphi\|^{\mathsf{CTL}}_{\mathsf{c}} \cap \langle R \rangle \emptyset) = \|\psi\|^{\mathsf{CTL}}_{\mathsf{c}} = \|\mathtt{tr}(\psi)\|^{\mu\mathsf{ML}}_{\mathsf{c}} = (\chi^{\mathbb{S}}_X)^{1}_{\mu}.$$

Inductive step. Suppose that $\pi(j-i) \in (\chi_X^{\mathbb{S}})_\mu^{i+1}$. We want to show that

$$\pi(j-i-1) \in (\chi_X^{\mathbb{S}})_\mu^{i+2} = \|\mathtt{tr}(\psi)\|_{\mathsf{c}}^{\mu\mathtt{ML}} \cup (\|\mathtt{tr}(\varphi)\|_{\mathsf{c}}^{\mu\mathtt{ML}} \cap \langle R\rangle (\chi_X^{\mathbb{S}})_\mu^{i+1})$$

Note that $j-i-1 < j$, hence $\pi(j-i-1) \in \|\varphi\|_{\mathsf{c}}^{\mathtt{CTL}} = \|\mathtt{tr}(\varphi)\|_{\mathsf{c}}^{\mu\mathtt{ML}}$. Moreover note that $\pi(j-i-1)R\pi(j-i) \in (\chi_X^{\mathbb{S}})_\mu^{i+1}$ and so

$$\pi(j-i-1) \in \|\varphi\|_{\mathsf{c}}^{\mathtt{CTL}} \cap \langle R\rangle (\chi_X^{\mathbb{S}})_\mu^{i+1} = \|\mathtt{tr}(\varphi)\|_{\mathsf{c}}^{\mu\mathtt{ML}} \cap \langle R\rangle (\chi_X^{\mathbb{S}})_\mu^{i+1}.$$

□

2. Team semantics for LTL and CTL

In this section we give the definition of team semantics for temporal logics, namely, LTL and CTL. For the first one, we follow Lück in [**17**]. Afterwards, we bring his ideas to computational tree logic mimicking the classical construction.

Before defining the formulas of team LTL let us clarify that we are not considering the R operator because ot deserves a special treatment. We will say more on this on the next section.

Definition 4.14. The *formulas of team linear temporal logic* are defined by the following grammar

$$\varphi ::= \top \mid \bot \mid P \mid \overline{P} \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \mathsf{X}\,\varphi \mid \varphi \mathsf{U}^{\mathsf{s}} \varphi \mid \varphi \mathsf{U}^{\mathsf{a}} \varphi$$

The set of all formulas will be denoted by tCTL.

As convention, sets of traces will be denoted by uppercase Greek letters Π, Δ, Γ...

Definition 4.15. Let Π be a set of traces and $f : \Pi \to \mathbb{N}$ a function. We write $\Pi[f] := \{\pi[f(\pi)] \mid \pi \in \Pi\}$. If $f, f' : \Pi \to \mathbb{N}$, we write $f < f'$ if, for every $\pi \in \Pi$, $f(\pi) < f'(\pi)$. We will also write $\Pi[j] := \{\pi[j] \mid \pi \in \Pi\}$ for $j \in \mathbb{N}$.

Definition 4.16. *Team semantics for* tLTL for a set of traces Π are defined by recursion as follows:

(1) $\Pi \models_{\mathsf{t}}^{\mathtt{LTL}} \top$ if and only if always.
(2) $\Pi \models_{\mathsf{t}}^{\mathtt{LTL}} \bot$ if and only if $\Pi = \emptyset$.
(3) $\Pi \models_{\mathsf{t}}^{\mathtt{LTL}} P$ if and only if for every $\pi \in \Pi$, $P \in \pi(0)$.
(4) $\Pi \models_{\mathsf{t}}^{\mathtt{LTL}} \overline{P}$ if and only if for every $\pi \in \Pi$, $P \notin \pi(0)$.
(5) $\Pi \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi \vee \psi$ if and only if $\Pi = \Delta \cup \Gamma$, $\Delta \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi$ and $\Gamma \models_{\mathsf{t}}^{\mathtt{LTL}} \psi$.
(6) $\Pi \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi \wedge \psi$ if and only if $\Pi \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi$ and $\Pi \models_{\mathsf{t}}^{\mathtt{LTL}} \psi$.
(7) $\Pi \models_{\mathsf{t}}^{\mathtt{LTL}} \mathsf{X}\,\varphi$ if and only if $\Pi[1] \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi$.
(8) $\Pi \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi \mathsf{U}^{\mathsf{s}} \psi$ if and only if there is $j \geq 0$ such that
   (a) $\Pi[j] \models_{\mathsf{t}}^{\mathtt{LTL}} \psi$;
   (b) for every $0 \leq k < j$, $\Pi[k] \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi$.
(9) $\Pi \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi \mathsf{U}^{\mathsf{a}} \psi$ if and only if there is $f : \Pi \to \mathbb{N}$ such that
   (a) $\Pi[f] \models_{\mathsf{t}}^{\mathtt{LTL}} \psi$;

(b) for every $f' : \Pi \to \mathbb{N}$, if $f' < f$, then $\Pi[f'] \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi$.

THEOREM 4.17. *The asynchronized fragment of* tLTL *is flat. In particular,*

(1) $\Pi \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi \mathsf{U}^{\mathsf{a}} \psi$ *if and only if for every* $\pi \in \Pi$, $\pi \models_{\mathsf{c}}^{\mathtt{LTL}} \varphi \mathsf{U} \psi$.

PROOF. The proof follows a similar argument to the one used to show flatness of CTL which we will prove later. □

DEFINITION 4.18. The *formulas of team computational tree logic* are defined by the following grammar

$$\varphi ::= \top \mid \bot \mid P \mid \overline{P} \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists \mathsf{X}\, \varphi \mid \exists(\varphi \mathsf{U}^{\mathsf{s}} \varphi) \mid \exists(\varphi \mathsf{U}^{\mathsf{a}} \varphi) \mid \\ \forall \mathsf{X}\, \varphi \mid \forall(\varphi \mathsf{U}^{\mathsf{s}} \varphi) \mid \forall(\varphi \mathsf{U}^{\mathsf{a}} \varphi)$$

The set of all formulas will be denoted by tCTL.

DEFINITION 4.19. *Team semantics for* tCTL are defined for a model $\mathbb{S}$ and $T \subseteq S$ by recursion as follows:

(1) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathtt{CTL}} \top$ if and only if always;
(2) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathtt{CTL}} \bot$ if and only if $T = \emptyset$;
(3) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathtt{CTL}} P$ if and only if $T \subseteq V(P)$;
(4) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathtt{CTL}} \overline{P}$ if and only if $T \cap V(P) = \emptyset$;
(5) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathtt{CTL}} \varphi \vee \psi$ if and only if $T = U \cup V$, $\mathbb{S}, U \models_{\mathsf{t}}^{\mathtt{CTL}} \varphi$ and $\mathbb{S}, V \models_{\mathsf{t}}^{\mathtt{CTL}} \psi$;
(6) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathtt{CTL}} \varphi \wedge \psi$ if and only if $\mathbb{S}, T \models_{\mathsf{t}}^{\mathtt{CTL}} \psi$ and $\mathbb{S}, T \models_{\mathsf{t}}^{\mathtt{CTL}} \theta$;
(7) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathtt{CTL}} \exists \psi$ if and only if for every $t \in T$ there is $\emptyset \neq \Pi_t \subseteq \mathfrak{Tr}_{\mathbb{S}}(t)$ such that $\bigcup_{t \in T} \Pi_t \models_{\mathsf{t}}^{\mathtt{LTL}} \psi$;
(8) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathtt{CTL}} \forall \psi$ if and only if $\mathfrak{Tr}_{\mathbb{S}}(T) \models_{\mathsf{t}}^{\mathtt{LTL}} \psi$.

THEOREM 4.20. *The asynchronized fragment of* tCTL *is flat. In particular,*

(1) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathtt{CTL}} \exists(\varphi \mathsf{U}^{\mathsf{a}} \psi)$ *if and only if for every* $t \in T$, $\mathbb{S}, t \models_{\mathsf{c}}^{\mathtt{CTL}} \exists(\varphi \mathsf{U} \psi)$*;*
(2) $\mathbb{S}, T \models_{\mathsf{t}}^{\mathtt{CTL}} \forall(\varphi \mathsf{U}^{\mathsf{a}} \psi)$ *if and only if for every* $t \in T$, $\mathbb{S}, t \models_{\mathsf{c}}^{\mathtt{CTL}} \forall(\varphi \mathsf{U} \psi)$*;*

PROOF. By induction on φ. We restrict ourselves to show the case $\exists(\varphi \mathsf{U}^{\mathsf{a}} \psi)$ for φ, ψ flat.

Suppose that $\mathbb{S}, T \models_{\mathsf{t}}^{\mathtt{CTL}} \exists(\varphi \mathsf{U}^{\mathsf{a}} \psi)$ for $T \subseteq S$. We want to show that, for every $t \in T$, $\mathbb{S}, t \models_{\mathsf{c}}^{\mathtt{CTL}} \exists(\varphi \mathsf{U} \psi)$.

By assumption, there is some $\Pi = \bigcup_{t \in T} \Pi_t$ where $\emptyset \neq \Pi_t \subseteq \mathfrak{Tr}_{\mathbb{S}}(t)$ such that $\Pi \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi \mathsf{U}^{\mathsf{a}} \psi$. That is, for some $f : \Pi \to \omega$, $\Pi[f] \models_{\mathsf{t}}^{\mathtt{LTL}} \psi$ and, for every $f' : \Pi \to \omega$ such that $f' < f$, $\Pi[f'] \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi$. Let $t \in T$, we want to show that there is a path π starting from t such that $\pi \models_{\mathsf{c}}^{\mathtt{LTL}} \varphi \mathsf{U} \psi$. Take any $\pi \in \Pi_t$ since it is non-empty. Note that, by flatness of φ and ψ, it follows that $\pi(f(\pi)) \models_{\mathsf{c}}^{\mathtt{LTL}} \psi$ and, for every $0 \leq i < f(\pi)$, $\pi(i) \models_{\mathsf{c}}^{\mathtt{LTL}} \varphi$.

Conversely, suppose that, for every $t \in T$, $\mathbb{S}, t \models^{\mathtt{CTL}}_{\mathsf{c}} \exists(\varphi \mathsf{U} \psi)$. We want to show that $\mathbb{S}, T \models^{\mathtt{CTL}}_{\mathsf{t}} \exists(\varphi \mathsf{U}^{\mathsf{a}} \psi)$.

Consider the sets $\Pi_t = \{\pi \in \mathfrak{Tr}_{\mathbb{S}}(t) \mid \pi \models^{\mathtt{LTL}}_{\mathsf{c}} \varphi \mathsf{U} \psi\}$ which, by assumption are non-empty. We claim that $\Pi = \bigcup_{t \in T} \Pi_t \models^{\mathtt{LTL}}_{\mathsf{t}} \varphi \mathsf{U}^{\mathsf{a}} \psi$. Consider the set

$$\{j_\pi \in \omega \mid \pi \in S\}$$

where, for $\pi \in S$, j_π is the witness of $\varphi \mathsf{U} \psi$. That is, for $\pi \in S$,

(1) $\pi(j_\pi) \models^{\mathtt{LTL}}_{\mathsf{c}} \psi$
(2) $\pi(0), \dots \pi(j_\pi - 1) \models^{\mathtt{LTL}}_{\mathsf{c}} \varphi, \neg\psi$

Consider the function

$$\begin{aligned} f : \Pi &\to \omega \\ \pi &\mapsto j_\pi \end{aligned}$$

Hence, by flatness of ψ, $\Pi[f] \models^{\mathtt{LTL}}_{\mathsf{t}} \psi$ and by flatness of φ, for every $f' : S \to \omega$ such that $f' < f$, $\Pi[f'] \models^{\mathtt{LTL}}_{\mathsf{t}} \varphi$.

The rest of the cases are similar. □

3. Embeddding of tCTL into tμML*

In this section we bring together all the research we have done to lift the classical embedding of CTL to team semantics. Basically, we will follow the same proof scketch making the necessary changes. We next state the full translation of tCTL.

$\varphi \in$ tCTL	$\mathtt{tr}(\varphi) \in$ tμML*
$\top$	$\top$
$\bot$	$\bot$
P	P
$\overline{P}$	$\overline{P}$
$\varphi \vee \psi$	$\mathtt{tr}(\varphi) \vee \mathtt{tr}(\psi$
$\varphi \wedge \psi$	$\mathtt{tr}(\varphi) \wedge \mathtt{tr}(\psi)$
$\exists \mathsf{X} \varphi$	$\Diamond \mathtt{tr}(\varphi)$
$\forall \mathsf{X} \varphi$	$\Box \mathtt{tr}(\varphi)$
$\exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)$	$\mu^{\cup} X.\mathtt{tr}(\psi) \circledvee (\mathtt{tr}(\varphi) \wedge \Diamond X)$
$\forall(\varphi \mathsf{U}^{\mathsf{s}} \psi)$	$\mu^{\cup} X.\mathtt{tr}(\psi) \circledvee (\mathtt{tr}(\varphi) \wedge \Box X$
$\exists(\varphi \mathsf{U}^{\mathsf{a}} \psi)$	$\mu^{\sqcup} X.\mathtt{tr}(\psi) \vee (\mathtt{tr}(\varphi) \wedge \Diamond X)$
$\forall(\varphi \mathsf{U}^{\mathsf{a}} \psi)$	$\mu^{\sqcup} X.\mathtt{tr}(\psi) \vee (\mathtt{tr}(\varphi) \wedge \Box X)$

The translation of the asyncronized fragment is trivial thanks to flatness of the constructions.

PROPOSITION 4.21. *Let $\varphi \in$ tCTL be a formula from the asynchronized fragment and let $\mathbb{S}$ be a model, then*

$$\mathbb{S}, T \models^{\mathtt{CTL}}_{\mathsf{t}} \varphi \text{ if and only if } \mathbb{S}, T \models^{\mu\mathtt{ML}}_{\mathsf{t}} \mathtt{tr}(\varphi)$$

PROOF. It follows from flatness of the asynchronized fragment of tCTL and μML. $\mathbb{S}, T \models^{\mathtt{CTL}}_{\mathsf{t}} \varphi$ if and only if, for every $t \in T$, $\mathbb{S}, t \models^{\mathtt{CTL}}_{\mathsf{c}} \varphi$

if and only if, for every $t \in T$, $\mathbb{S}, t \models_{\mathsf{c}}^{\mu\mathtt{ML}} \mathtt{tr}(\varphi)$ if and only if $\mathbb{S}, T \models_{\mathsf{t}}^{\mu\mathtt{ML}} \mathtt{tr}(\varphi)$. □

However, the synchronized translation is more difficult to show. As we will see, we will follow the same scketch as in the classical semantics since in serial models, the description of team $\Diamond$ and $\Box$ are the *natural* ones given by the external definition. From now and for the rest of the proof fix a model $\mathbb{S}$. We first show the following equivalence in $\mathtt{tCTL}$:

$$\exists(\varphi \mathsf{U}^{\mathsf{s}} \psi) \equiv_{\mathtt{tCTL}} \psi \otimes (\varphi \wedge \exists \mathsf{X} \exists (\varphi \mathsf{U}^{\mathsf{s}} \psi))$$

PROPOSITION 4.22. $\|\exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)\|_{\mathsf{t}}^{\mathbb{S}} \subseteq \|\psi \otimes (\varphi \wedge \exists \mathsf{X} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi))\|_{\mathsf{t}}^{\mathbb{S}}$.

PROOF. Suppose that $\mathbb{S}, T \models_{\mathsf{t}}^{\mathtt{CTL}} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)$. Then, there is is some $\Pi = \bigcup_{t \in T} \Pi_t$ where $\emptyset \neq \Pi_t \subseteq \mathfrak{Tr}_{\mathbb{S}}(t)$, such that $\Pi \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi \mathsf{U}^{\mathsf{s}} \psi$. That is, for some $j \in \omega$,

$$\Pi[j] \models_{\mathsf{t}}^{\mathtt{LTL}} \psi \text{ and } \Pi[0], \ldots, \Pi[j-1] \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi, \neg\psi.$$

If $j = 0$, then $\Pi[0] \models_{\mathsf{t}}^{\mathtt{CTL}} \psi$. Noticing that $\Pi[0] = T$, by definition of $\otimes$, $T \models_{\mathsf{t}}^{\mathtt{CTL}} \psi \otimes (\varphi \wedge \exists \mathsf{X} \exists(\varphi \mathsf{U} \psi))$ as wanted.

If $j > 0$, then $\Pi[0] = T \models_{\mathsf{t}}^{\mathtt{CTL}} \varphi$, $\Pi[0] R_{\mathsf{EM}} \Pi[1]$ and $\Pi[1] \models_{\mathsf{t}}^{\mathtt{CTL}} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)$. Hence, $T \models_{\mathsf{t}}^{\mathtt{CTL}} \varphi \wedge \exists \mathsf{X} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)$ and $T \models_{\mathsf{t}}^{\mathtt{CTL}} \psi \otimes (\varphi \wedge \exists \mathsf{X} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi))$. □

PROPOSITION 4.23. $\|\psi \otimes (\varphi \wedge \exists \mathsf{X} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi))\|_{\mathsf{t}} \subseteq \|\exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)\|_{\mathsf{t}}$.

PROOF. Suppose that $T \models_{\mathsf{t}}^{\mathtt{CTL}} \psi \otimes (\varphi \wedge \exists \mathsf{X} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi))$.

If $T \models_{\mathsf{t}}^{\mathtt{CTL}} \psi$, then $T \models_{\mathsf{t}}^{\mathtt{CTL}} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)$ trivially choosing $j = 0$.

Otherwise, if $T \models_{\mathsf{t}}^{\mathtt{CTL}} \varphi \wedge \exists \mathsf{X} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)$, then $T \models_{\mathsf{t}}^{\mathtt{CTL}} \varphi$ and for some $\Pi = \bigcup_{t \in T} \Pi_t$ where $\emptyset \neq \Pi_t \subseteq \mathfrak{Tr}_{\mathbb{S}}(t)$, $\Pi[1] \models_{\mathsf{t}}^{\mathtt{CTL}} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)$. Hence, consider now $\Delta = \bigcup_{w \in \Pi[1]} \Delta_w$ where $\emptyset \neq \Delta_w \subseteq \mathfrak{Tr}_{\mathbb{S}}(w)$ such that $\Delta \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi \mathsf{U}^{\mathsf{s}} \psi$. Notice that $T = \Pi[0] R_{\mathsf{EM}} \Pi[1]$. Consider

$$\Delta' := \bigcup_{t \in T} \{ t \curlywedge \pi \mid t \to w, w \in \Pi[1], \pi \in \Delta_w \}.$$

Notice, that since all the Δ_w are non-empty, all the Δ'_t are non-empty. Moreover, it is easy to see that, by construction, $\Delta'_t \subseteq \mathfrak{Tr}_{\mathbb{S}}(t)$.

Now, since $\Delta \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi \mathsf{U}^{\mathsf{s}} \psi$, [by unfolding definitions] $\Delta' \models_{\mathsf{t}}^{\mathtt{LTL}} \varphi \mathsf{U}^{\mathsf{s}} \psi$ □

We now show that $\exists(\varphi \mathsf{U}^{\mathsf{s}} \psi) \equiv \mu^{\cup} X.\mathtt{tr}(\psi) \otimes (\mathtt{tr}(\varphi) \wedge \Diamond X)$ assuming that φ is equivalent to $\mathtt{tr}(\varphi)$ and that ψ is equivalent to $\mathtt{tr}(\psi)$ (induction hypothesis). Let us denote

$$\chi := \mathtt{tr}(\psi) \otimes (\mathtt{tr}(\varphi) \wedge \Diamond X).$$

We show that $\|\exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)\|_{\mathsf{t}} = \lim_{\xi \in \mathbf{ORD}}((\chi_X^{\mathcal{S}})_{\mu,\cup}^{\xi})$ by showing both inclusions.

PROPOSITION 4.24. $\lim_{\xi \in \mathbf{ORD}}(\chi_X^{\mathcal{S}})_{\mu,\cup}^{\xi} \subseteq \|\exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)\|_{\mathsf{c}}^{\mathbb{S}}$.

PROOF. By ordinal induction.

Base case. Obvious since $(\chi_X^{\mathcal{S}})^0_{\mu,\cup} = \{\emptyset\} \subseteq \|\exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)\|^{\mathbb{S}}_{\mathsf{t}}$

Successor step. Suppose that $(\chi_X^{\mathcal{S}})^\alpha_{\mu,\cup} \subseteq \|\exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)\|^{\mathbb{S}}_{\mathsf{t}}$. We want to show that $(\chi_X^{\mathcal{S}})^{\alpha+1}_{\mu,\cup} \subseteq \|\exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)\|^{\mathbb{S}}_{\mathsf{t}}$. Let $T \in (\chi_X^{\mathcal{S}})^{\alpha+1}_{\mu,\cup} = \|\chi\|_{\mathsf{t}}^{\mathcal{S}[X \mapsto (\chi_X^{\mathcal{S}})^\alpha_{\mu,\cup}]}$. Hence,

$$\mathcal{S}[X \mapsto (\chi_X^{\mathcal{S}})^\alpha_{\mu,\cup}], T \models^{\mu\mathsf{ML}}_{\mathsf{t}} \mathtt{tr}(\psi) \vee (\mathtt{tr}(\varphi) \wedge \Diamond X)$$

Suppose $\mathcal{S}[X \mapsto (\chi_X^{\mathcal{S}})^\alpha_{\mu,\cup}], T \models^{\mu\mathsf{ML}}_{\mathsf{t}} \mathtt{tr}(\psi)$. Hence $\mathbb{S}, T \models^{\mu\mathsf{ML}}_{\mathsf{t}} \mathtt{tr}(\psi)$ since $\mathtt{tr}(\psi)$ does not contain X. Therefore, $\mathbb{S}, T \models^{\mathsf{CTL}}_{\mathsf{t}} \varphi$ by induction hypothesis, and

$$\mathbb{S}, T \models^{\mathsf{CTL}}_{\mathsf{t}} \psi \otimes (\varphi \wedge \exists \mathsf{X} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)).$$

Hence, by the just proven equivalence, $\mathbb{S}, T \models^{\mathsf{CTL}}_{\mathsf{t}} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)$.

Suppose $\mathcal{S}[X \mapsto (\chi_X^{\mathcal{S}})^\alpha_{\mu,\cup}], T \models^{\mu\mathsf{ML}}_{\mathsf{t}} \mathtt{tr}(\varphi) \wedge \Diamond X$, i.e.,

$$\mathcal{S}[X \mapsto (\chi_X^{\mathcal{S}})^\alpha_{\mu,\cup}], T \models^{\mu\mathsf{ML}}_{\mathsf{t}} \mathtt{tr}(\varphi) \text{ and, for some } T R_{\mathsf{EM}} U,\ U \in (\chi_X^{\mathcal{S}})^\alpha_{\mu,\cup}.$$

On the one hand, as before, $\mathbb{S}, T \models^{\mu\mathsf{ML}}_{\mathsf{t}} \mathtt{tr}(\varphi)$ since $\mathtt{tr}(\varphi)$ does not contain X and so, by induction hypothesis, $\mathbb{S}, T \models^{\mathsf{CTL}}_{\mathsf{t}} \varphi$. On the other hand, by induction hypothesis, $\mathbb{S}, U \models^{\mathsf{CTL}}_{\mathsf{t}} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)$. That is, $\mathbb{S}, T \models^{\mathsf{CTL}}_{\mathsf{t}} \varphi \wedge \exists \mathsf{X} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)$. Therefore,

$$\mathbb{S}, T \models^{\mathsf{CTL}}_{\mathsf{t}} \psi \otimes (\varphi \wedge \exists \mathsf{X} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi))$$

and, by the just proven equivalence, $\mathbb{S}, T \models^{\mathsf{CTL}}_{\mathsf{t}} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)$.

Limit step. If, for every $\alpha \in \beta$, $(\chi_X^{\mathcal{S}})^\alpha_{\mu,\cup} \subseteq \|\exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)\|^{\mathbb{S}}_{\mathsf{t}}$, then

$$(\chi_X^{\mathcal{S}})^\beta_{\mu,\cup} = \bigcup_{\alpha \in \beta} (\chi_X^{\mathcal{S}})^\alpha_{\mu,\cup} \subseteq \|\exists(\varphi \mathsf{U} \psi)\|^{\mathbb{S}}_{\mathsf{t}}.$$

□

PROPOSITION 4.25. $\|\exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)\|^{\mathbb{S}}_{\mathsf{t}} \subseteq \lim_{\xi \in \mathbf{ORD}} (\chi_X^{\mathcal{S}})^\xi_{\mu,\cup}$.

PROOF. Let $\mathbb{S}, T \models^{\mathsf{CTL}}_{\mathsf{t}} \exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)$. Then, there is a path Π (starting from T) such that, $\Pi \models^{\mathsf{LTL}}_{\mathsf{t}} \varphi \mathsf{U}^{\mathsf{s}} \psi$, that is, for some $\Pi(j) \models^{\mathsf{LTL}}_{\mathsf{t}} \psi$ and, for every $0 \leq i < j$, $\Pi(i) \models^{\mathsf{LTL}}_{\mathsf{t}} \varphi$. Note that it suffices to show that $T \in (\chi_X^{\mathcal{S}})^{j+1}_{\mu,\cup}$. We show that

$$\Pi(j - i) \in (\chi_X^{\mathcal{S}})^{i+1}_{\mu,\cup}$$

by induction on i and the proof ends by setting $i = j$.

Base case. Notice that:

$$\Pi(j-0) \in \|\psi\|^{\mathsf{CTL}}_{\mathsf{t}} \subseteq \|\psi\|^{\mathsf{CTL}}_{\mathsf{t}} \cup (\|\varphi\|^{\mathsf{CTL}}_{\mathsf{c}} \cap \langle R_{\mathsf{EM}} \rangle \emptyset) = \|\psi\|^{\mathsf{CTL}}_{\mathsf{t}} = \|\mathtt{tr}(\psi)\|^{\mathcal{S}}_{\mathsf{t}} = (\chi_X^{\mathcal{S}})^1_{\mu,\cup}.$$

Inductive step. Suppose that $\Pi(j - i) \in (\chi_X^{\mathcal{S}})^{i+1}_{\mu,\cup}$. We want to show that

$$\Pi(j - i - 1) \in (\chi_X^{\mathcal{S}})^{i+2}_{\mu,\cup} = \|\mathtt{tr}(\psi)\|^{\mathcal{S}}_{\mathsf{t}} \cup (\|\mathtt{tr}(\varphi)\|^{\mathcal{S}}_{\mathsf{t}} \cap \langle R_{\mathsf{EM}} \rangle (\chi_X^{\mathcal{S}})^{i+1}_{\mu,\cup})$$

Note that $j - i - 1 < j$, hence $\Pi(j - i - 1) \in \|\varphi\|^{\mathsf{CTL}}_{\mathsf{t}} = \|\mathtt{tr}(\varphi)\|^{\mathcal{S}}_{\mathsf{t}}$. Moreover note that $\Pi(j - i - 1) R_{\mathsf{EM}} \Pi(j - i) \in (\chi_X^{\mathcal{S}})^{i+1}_{\mu,\cup}$ and so

$$\Pi(j - i - 1) \in \|\varphi\|^{\mathsf{CTL}}_{\mathsf{t}} \cap \langle R_{\mathsf{EM}} \rangle (\chi_X^{\mathcal{S}})^{i+1}_{\mu,\cup} = \|\mathtt{tr}(\varphi)\|^{\mathcal{S}}_{\mathsf{t}} \cap \langle R_{\mathsf{EM}} \rangle (\chi_X^{\mathcal{S}})^{i+1}_{\mu,\cup}.$$

□

REMARK 4.26. In this last proof we have ommited minor details such as if $\mathcal{S}[X \mapsto (\chi_X^{\mathcal{S}})^{\alpha}_{\mu,\cup}], T \models_{\mathtt{t}}^{\mu\mathsf{ML}} \mathtt{tr}(\psi)$, then $\mathbb{S}, T \models_{\mathtt{t}}^{\mu\mathsf{ML}} \mathtt{tr}(\psi)$ since $\mathtt{tr}(\psi)$ does not contain X in order to reduce verbosity. Note how, the key steps in the proof are given because of the *natural* description of the team semantical $\Diamond$ and $\Box$ in serial models. And the limit steps, because of considering the union, i.e., the internal definition.

REMARK 4.27. Notice that we have only showed the translation for $\exists(\varphi \mathsf{U}^{\mathsf{s}} \psi)$. The team semantical translation of $\forall(\varphi \mathsf{U}^{\mathsf{s}} \psi)$ follows in a similar way, because, thanks to seriality of $\mathbb{S}$, we can work with the external definition of $\Box$.

Conjecture. We have only proved the team semantical translation of synchronized until operators since, to our understanding, they represent in an understandable way the notion of synchronicity and asynchronicity. The respective, synchronized or asynchronized notions of the release operator R are more obscure and, in the literarure, often ommited. However, as we have remarked before, the core of the proof relies on the external definitions of $\Diamond$ and $\Box$ and the respective notion of fixed-point. Hence, it should be possible to lift the translation of R operator. Nonetheless, some problems may appear because of the relation

$$\varphi \,\mathsf{R}\, \psi \equiv_{\mathrm{LTL}} \neg(\neg\varphi \mathsf{U} \neg\psi)$$

since negation does not behave well with team semantics and downwards closure. In conclusion, the R case deserves a special study, but, to our understanding, it should be possible to translate.

CHAPTER 5

Conclusion and further work

In this thesis, we defined a team semantics for modal μ-calculus and we showed how team CTL embeds into team μ-calculus. Following the ideas from [**8**] and [**4**], we introduced an algebraic approach to team semantics for modal logic, first by means of relational powerset liftings and later giving an internal definition of the modalities. Using these approach, we developed the classical construction of modal μ-calculus in a team semantical framework using the notions of general model and general team semantics for modal logic, finding the need of defining two least fixed-point operators. Later, we studied the defined semantics showing, among other properties, flatness for a fragment of it and extended the logic thanks to the rich team semantical relation. Finally, extending a classical result, we showed how the two least fixed-point operators algebraically encode the notions of synchronizity and asynchronocity known in temporal team semantics.

The work presented in this thesis goes in the recent line of bringing an algebraic perspective to team semantics done in [**17**] in 2020 and in [**8**] in 2023. In particular, Lück presents a universal algebraic view of team semantics and Engström et. al. study powerset structures of Boolean algebras to develop team semantics for propositional logic. By our side, we study the relations of powerset structures with ordered structures and Kripke models to define team semantics for modal μ-calculus. The work leaves many natural continuations for this research. We highlight some of these below.

We did not fully elaborate the algebraic study of fixed-points in double powerset structures. The study of how orders can be lifted to powerset structures and the study of fixed-point theory in such structures would be a natural continuation of the work presented here. We expect that some of the properties and results can be lifted, at least, up to the respective equivalence relation. However, there are still questions about how this notions relate with the internal/external perspective studied in this thesis.

Related to the generalization of classical constructions, it is worth to say that game-theoretic semantics have been defined for dependence logic. Thus, can we adapt those semantics to extend the game-theoretic semantics for classical μ-calculus. Again, some questions arise when thinking about how this semantics would relate with the internal/external interpretations.

Although this lines go by extending classical constructions of the modal μ-calculus, the definition of the external box modality leaves many open questions as for example what is its relation with team semantics for modal logic or what is its expressible power. In this line we expect that the methods from [**8**] can be generalised to define, for instance, a proof system for the external-internal team semantics of modal logic.

One of the reasons for working with team semantics is to increase the expressive power of a logic from the semantical side. As it was shown by Kontinen and Ville in [**14**], dependence logic with a (Boolean) negation operator is equivalent to full second order logic. As it is known, modal μ-calculus is equivalent to the bisimilar invariant fragment of monadic second order logic. Hence, some questions arise as for example, to find the expressive power of some extensions of team μ-calculus or if it is possible to find a translation between dependence logic with negation operator and team μ-calculus.

Finally, team semantics in temporal logics can be used to the study of hyperproperties, that is, properties of sets of traces. Hyperproperties have been identified as a key concept in the verification of information flow properties and some logics have been defined to formalize this concepts as HyperLTL. Moreover, in [**16**] the authors study the relation between HyperLTL and team semantics for LTL. However, the complexity of modal μ-calculus makes the relation of Hyper modal μ-calculus (defined in the literature) and team semantics defined here a long term question.

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