Metaphysical Multiversism: from Armchair to Practice

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis assesses whether the ontological commitments of an emerging position within the philosophy of set theory – *metaphysical multiversism* – are justified. Metaphysical multiversism can be characterised as the conjunction of two theses. First, that there is not one unique ultimate set-theoretic universe, but rather a plurality of universes (the 'multiverse'). Second, metaphysical multiversism qualifies as a form of mathematical realism, since it takes these universes to exist 'platonistically'. The present thesis constitutes the first systematic attempt to explore, clarify and evaluate the realist ontology posited by metaphysical multiversism.

I begin by introducing a novel taxonomy: I distinguish between *conservative* and *radical* metaphysical multiversism. According to the former, universes are models of *consistent* set-theoretic theories; according to the latter, universes are models of consistent, or *non-trivial inconsistent* set-theoretic theories. Next, in the central chapters of the thesis, I explore two strategies to justify the ontological commitments of both varieties of metaphysical multiversism. The first strategy relies on what I call the *epistemic* argument, which claims that metaphysical multiversism is the only version of mathematical realism rising to the Benacerraf-Field Challenge, and also on the well-known indispensability argument. The second strategy attempts to justify the realist commitments of metaphysical multiversism on the basis of a *practice-based* argument. This strategy is inspired and informed by Penelope Maddy's Second Philosophy.

My conclusion will be the following: while both strategies can justify the ontology posited by conservative metaphysical multiversism, they do not as straightforwardly justify the ontological commitments of radical metaphysical multiversism. Therefore, only conservative metaphysical multiversism can legitimately justify its realist ontological commitments.

Contents

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Chapter 1

Setting the Stage

1.1 Overview of the Chapter

This chapter introduces the main view that will be discussed in this thesis – *metaphysical multiversism*. I first set the stage by introducing the current debate between universism and multiversism in the philosophy of set theory. Next, I proceed to discuss some general motivations for the multiverse view, and provide a novel taxonomy of metaphysical multiversism. I then raise an important hitherto unsolved problem for this view, namely how to justify its ontological commitments; this will be the driving question of this thesis. I conclude by stating the methodological commitments of the thesis, and with an overview of the remaining chapters.

1.2 Universism vs. Multiversism

Recently, the debate concerning pluralistic interpretations of set theory has taken centre stage within the philosophy of set theory. On the one side of the debate we find the *universist*, who argues that set theory should be interpreted as the theory of a unique and ultimate universe of sets. On the other side, we find the *multiversist*, who claims that set theory should be viewed as a theory about multiple universes: the set-theoretic *multiverse*. Although the present thesis will focus mainly on the latter view, it is worth starting by clarifying the main differences between universism and multiversism. So, let's take a closer look at this debate.

The set-theoretic universe – denoted by V – is the cumulative hierarchy of sets, which is obtained as follows: we start at stage 0 with the empty set and we iterate, along the ordinals, the power-set operation until we reach the transfinite. At limit ordinals, we collect what we had before by the union operation. The standard axioms of set theory, **Z**ermelo-**F**raenkel axioms with the axiom of **C**hoice (ZFC) are however unable to specify many relevant properties of the set-theoretic universe.[1] In

[1] For completeness, the axioms of ZFC are:

fact, set-theorists have developed many different versions of V that are compatible with the axioms through set-theoretic *model construction*.[2] This plurality of set-theoretic models raises a crucial foundational question: is there one unique ultimate set-theoretic universe, or multiple, equally legitimate, versions of it?

There are two possible responses. The orthodox answer is that there is *one* concept of set instantiated in a unique intended model of set theory – *the* set-theoretic universe V – that captures all true properties of sets.[3] This is the universe view (or *universism*). According to the most standard version of universism, any given set-theoretic claim is either true or false, and the chief goal of set theorists is to find out these truths. Universists are of course aware that ZFC-axioms do not offer a fully determinate picture of the set-theoretic universe, but they claim that new axioms must be added in order to achieve a better understanding of the universe, or that we will never reach a complete description of it. The universe view has been famously defended by Gödel (1947) and Woodin (2017). By contrast, the multiverse view (or *multiversism*) claims that there are distinct concepts of set, each instantiated in a different set-theoretic universe. According to this view, there is no ultimate structure of set theory, but rather a collection of set-theoretic *models* (or 'universes') that could all legitimately serve as *the* model of set theory.[4]

- **Extensionality**: $\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$. Informally: two sets are equal if and only if they contain exactly the same elements.
- **Pairing**: $\forall x \forall y \exists z \forall w (w \in z \leftrightarrow w = x \vee w = y)$. Informally: for any sets x, y there exits their pair $z = \{x, y\}$.
- **Union**: $\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \wedge z \in w))$. Informally: for any set there exists the set of all members of its members.
- **Powerset**: $\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x)$. Informally: for any set there exists the set of all its subsets.
- **Separation**: $\forall p_0, ..., p_n \forall x \exists z \forall y (y \in z \leftrightarrow (y \in x \wedge \phi(y, p_0, ..., p_n)))$. Informally: for any set, there is a subset of it consisting of all and only the φs.
- **Infinity**. $\exists x (\emptyset \in x \wedge \forall y \in x (y \cup \{y\} \in x))$. Informally: there is an infinite set.
- **Replacement:** "$\phi(x, y, \bar{p})$ is functional"$\rightarrow \forall d \forall \bar{p} \exists z \forall y (y \in z \leftrightarrow \exists x \in d \phi(x, y, \bar{p}))$. Informally: every function with a domain also has a range.
- **Foundation:** $\forall x (x \neq \emptyset \rightarrow \exists y (y \in x \wedge y \cap x \neq \emptyset))$. Informally: no sets have infinitely descending membership chains.
- **Choice:** $\forall x [(\forall y \in xy \neq \emptyset) \rightarrow f($"$f$ is a function with domain $x'' \rightarrow \forall w \in x (f(w) \in w))]$. Informally: for every collection of non-empty sets, there is a function that chooses a member from each.

[2] I will define set-theoretic models in the next section; at present the details do not matter.

[3] Roughly, a model of a theory is *intended* if it adequately captures our 'intuitions' about the theory. For example, the structure $0, 1, 2...$, used in our everyday life for counting and computing is intended of arithmetic, for it reflects our intuitions of the natural numbers adequately. Thus, the intended model of arithmetic is the sequence of the natural numbers up to isomorphism.

[4] In the 'multiverse' literature it is common to use 'model' and 'universe' interchangeably (see e.g. Antos et al. 2015, Hamkins 2012, Scambler 2020).

Moreover, according to the multiversist, focusing on one single universe is, in fact, deeply problematic, for it ignores set-theorists' experience with alternative models. Antos et al. (2015, 2465) describe this attitude well: "there is no unique universe, nor should there be one". A consequence of this position is that the truth-value of set-theoretic statements can vary across universes. Thus, truth is defined *relative* to a background universe: each universe has its own set-theoretic truths.[5] Versions of this multiverse view have been defended by Linsky and Zalta (1995), Balaguer (1998) and Hamkins (2012), among others.

Thus, we have two competing positions on the table:

- **Universism**: There exists a single 'true' set-theoretic universe in which every set-theoretic question has a definite answer.
- **Multiversism**: There is not a single universe of set theory but rather a multiverse of equally legitimate candidates. Each universe has its own set-theoretic truths.[6]

Before moving on, I would like to make two points of clarification. Firstly, I would like to warn the reader that this reconstruction of the debate oversimplifies many features of both views. To illustrate, not every form of universism accepts that every set-theoretic question has a definite answer. For example, Scambler (2020) has developed a view – which he ascribes originally to Feferman – according to which there is a unique but *indeterminate* universe of sets. Similarly, other proposals, such as Arrigoni and Friedman (2013)'s *hyperuniverse* view, are difficult to categorise in one of these two groups. However, since the aim of this introduction is to provide an initial overview of the debate between universism and multiversism, this rough categorisation is apt for my purpose.

Secondly, it is also important to note that both positions have their own advantages and problems. The universe view is attractive because it provides us with an elegant, concise and unified picture of set theory. However, it faces the difficulty of providing a coherent explanation of why set-theorists work with different set-theoretic models in their practice, and why these should be seen as a mere tool. On the other hand, while the multiverse view can easily capture this, it owes us an account of how all these set-theoretic universes relate to each other, and how it can provide a unified framework for the development of all mathematics.

[5]Balaguer makes the useful distinction between *true simpliciter* (i.e. true in all set-theoretic models) and *true in a structure* (i.e. true in a particular set-theoretic model). For example, statements like the continuum hypothesis are only ever 'true in a structure' , never 'true simpliciter'. What mathematicians standardly mean when they say that a sentence is true is that it is true in the intended structure, or background universe (Balaguer 1998, 13).

[6]Similar characterisations of both positions can be found in Hamkins (2012), Koellner (2013), Antos et al. (2015), Barton (2016, forthcoming).

Given these preliminary points, here's a more detailed plan of the chapter. In the remainder of this chapter I will introduce the multiverse view in detail, with a particular focus on one version of it – *metaphysical multiversism*. I will proceed as follows. In section 1.3 I motivate multiversism on general grounds. In section 1.4, I focus on metaphysical multiversism specifically, and I propose a novel classification for the view. In section 1.5, I outline one of the hardest problem facing metaphysical multiversism; namely: how to justify the metaphysical commitment of the view. Afterward, in section 1.6, I state the methodological commitments of the thesis, and in section 1.4, I layout the structure of the thesis. Section 1.8 concludes the chapter.

1.3 Motivations for Multiversism

The central phenomena that have led scholars to question whether there is a single universe of sets are the so-called *independence results* in set theory: these are natural set-theoretic questions that cannot be answered on the basis of our foundational theory ZFC. According to the multiversist, these results posit an important challenge for universism, for they are proved by constructing different set-theoretic models. The universist view however struggles to make sense of set theorists' mathematical experience with these alternative set-theoretic models, or so multiversists claim. In what follows, I take a look at the most (in)famous of these results: the independence of the Continuous Hypothesis (henceforth: CH).

In order to introduce the independence phenomenon, we can start with the following observation: both the set of real numbers and the set of natural numbers are infinite. Cantor's Theorem however showed that there could be no surjection from the natural numbers to the reals. Put differently: there are more real numbers than natural numbers. A question then comes to mind: is there some size of infinity strictly between the size of the naturals and the real numbers? This is the seemingly innocent question underlying CH, according to which there are *no* sets whose cardinality is strictly greater than the cardinality of the natural numbers, but strictly smaller than the cardinality of the reals. If CH is false, it means that there is a set of real numbers that is bigger than the set of natural numbers but smaller than the set of real numbers. However, perhaps surprisingly, ZFC cannot help with this empasse. In fact:

Theorem. *Suppose ZFC is consistent. Then:*

1. $\mathsf{ZFC} \nvdash \neg CH$ *(Gödel 1938)*
2. $\mathsf{ZFC} \nvdash CH$ *(Cohen 1963)*

Informally: Gödel showed that CH cannot be disproved from ZFC, while Cohen showed that CH cannot be proven from ZFC. In other words, this theorem shows that CH is *independent* from ZFC: it is neither provable nor disprovable from our ZFC axioms.

But, how can we show that a statement, such as CH, is independent from a theory, such as ZFC? The standard way to do this is by the construction of set-theoretic models. We say that statement S is independent of theory $\mathcal{T}$ if and only if there are (at least) two set-theoretic models of $\mathcal{T}$ such that one is a model of S and the other a model of $\neg S$. Thus, we can show that CH is independent of ZFC if and only if (i) we can find at least one set-theoretic model of ZFC that is a model of CH and (ii) we can find another model of ZFC that it is a model of ¬CH.

Before turning to the independence results, I need to first define what set-theoretic models are.[7] Intuitively, a set-theoretic model can be thought of as universe in which all of mathematics takes place (for this reason, in this debate authors tend to use set-thoeretic models and 'universes' interchangeably). More formally, a model of set theory is a pair (M, E), where M is a non-empty set and E is a binary relation on M. For example, a model for ZFC, or simply a ZFC-model, is a pair (M, E) such that all the axioms of ZFC are true when interpreted in (M, E); that is, when the variables that appear in the axioms range over elements of M, and $\in$ is interpreted as E.

Now, back to our theorem. In order to show the independence of CH from ZFC, Gödel (1938) proved that CH was consistent with ZFC; that is, he showed that *adding* CH to these axioms did not result in a contradiction. In order to do this, Gödel carefully constructed a model – called the constructible universe L – in which he was able to prove that the Axioms of Choice and the CH were true. That is, Gödel's constructible universe L is a ZFC+CH-model. By Gödel's Completeness Theorem we know that a theory is consistent if and only if it has a model. Thus, since L is a model for ZFC+CH, ZFC+CH is consistent (relative to ZF, since L was built assuming the consistency of ZF). Gödel's consistency result therefore implies that there cannot be a proof, from the axioms of set theory, that CH is false in ZFC.

To show the independence of CH from ZFC, Cohen hence needed to show the consistency of ZFC + ¬CH. In order to do this, he invented an influential technique for constructing set-theoretic models: the method of *forcing* (Cohen 1963). This technique begins with a *ground model*. In Cohen's case, he started with Gödel's constructible universe L. Very roughly, his strategy was as follows. To the domain of L, he skillfully added new entities whilst ensuring that the resulting model would still be a ZFC-model. This new remaining model, called $L[G]$, is called a *generic* or *forcing* extension of the ground model L. The key result was that CH failed in this generic extension. Thus, using forcing, he constructed a ZFC-model, $L[G]$, in which

[7]A note on terminology, throughout this thesis I will use 'set-theoretic model', 'model of set theory' and 'model' interchangeably.

CH was false. As in Gödel's case, this result implies that there can be no proof, from the axioms of set theory, that CH is true in ZFC. Given these results, one can conclude that CH is independent of ZFC, and consequently, that the question of whether the reals are the infinite set of second-smallest size cannot be settled within ZFC.

But why are these independence results relevant for the multiverse view? First, advocates of the multiverse can give a straightforward explanation of the independence results: since there is a plurality of set-theoretic universes, a statement such as CH can be true in some of these and false in others. That is, the truth-value of set theoretic questions, such as CH, is parametrised to the background model. This is the so-called 'multiverse answer' to CH. The following quote by Balaguer nicely illustrates the multiverse answer to CH:

> ZFC does not describe a unique universe of sets it describes many different universes of sets. For example, it describes some universes in which the continuum hypothesis (CH) is true and others in which it is not true (Balaguer 1998, 59).

Thus, while the universist has to continue searching for new axioms that describe the true universe in a more complete way than the already established ZFC-axioms in order to settle CH, the multiversist has a ready answer. She can claim that CH is already settled. As Hamkins puts it:

> On the multiverse view, consequently, the continuum hypothesis is a settled question; it is *incorrect* to describe CH as an open problem. The answer to CH consists of the expansive, detailed knowledge set theorists have gained about the extent to which it holds and fails in the multiverse, about how to achieve it or its negation in combination with other diverse set-theoretic properties (Hamkins 2012, 429, my emphasis).

This quote is important, as it introduces one key aspect of the multiversist view. In addition to providing a satisfactory solution to the independence results, the multiversist places special emphasis on the *way* in which these results are proved. To underscore this key aspect, Hamkins writes:

> Our situation, after all, is not merely that CH is formally independent and we have no additional knowledge about whether it is true or not. Rather, we have an informed, deep understanding of the CH and ¬CH worlds and of how to build them from each other (Hamkins 2012, 430).

Understanding CH requires understanding the universes in which it holds and doesn't hold, and how to build them through model construction. Current set-theoretic practice is replete with model-theoretic techniques, and models have become essential when proving set-theoretic results. In particular, the forcing method, introduced by Cohen for proving the independence of CH is one of the most productive model-building tools set theorists ever developed. As Hamkins puts is:

> With forcing, we seem to have discovered the existence of other mathematical universes, outside our own universe, and the multiverse view asserts that yes, indeed, this is the case [...] Set theorists have seen via forcing that divergent concepts of set lead to new set-theoretic worlds, extending our previous universe, and many are now busy studying what it would be like to live in them (Hamkins 2012, 425).

If these models are taken seriously, then it is just natural for the multiversist to claim that there are a plurality of set-theoretic universes each exhibiting its own set of set-theoretic truths. It is then not the independence results *themselves* that pose a problem for the universist, but rather the abundance of set theoretic extensions that have been developed in order to attempt to settle these results. To illustrate:

> This abundance of set-theoretic possibilities poses a serious difficulty for the universe view, for [...] one must explain or explain away as imaginary all of the alternative universes that set theorists seem to have constructed. This seems a difficult task, for we have a robust experience in those worlds, and they appear fully set theoretic to us. The multiverse view, in contrast, explains this experience by embracing them as real, filling out the vision hinted at in our mathematical experience, that there is an abundance of set-theoretic worlds into which our mathematical tools have allowed us to glimpse (Hamkins 2012, 418).

Summing up: the multiversist has a simple explanation of the independence phenomena. The CH has already been settled: it is true in some universes, and false in others. But even more important than this solution is the *manner* in which this solution is achieved: developing this solution has led to the invention of influential techniques which have made possible the development and investigation of different set-theoretic universes. The way the multiverse view elegantly captures these aspects of set-theoretic practice is the central motivation for the view.

1.4 Varieties of Metaphysical Multiversism

So far, for sake of simplicity, I have assumed that there is just one version of the multiverse view. However, there are many nuanced different versions of this view. In this thesis, I will focus more specifically on one version – *metaphysical* multiversism. But before introducing metaphysical multiversism, I first have to provide a more general taxonomy of the multiverse views.

Following Antos et al. (2015), I will divide the multiverse view in two groups, depending on whether the multiverse is taken to exist objectively or not. That is, the criterion of classification will be their commitment to some form of *realism*. I here follow Clarke-Doane (2020a), who takes mathematical realism to be the combination of five individual theses: aptness, belief, truth, independence and face-value. Roughly, if one is a mathematical realist, then mathematical sentences express beliefs that, when taken at face value, have truth values independent of us, and that some such beliefs are true (Clark-Doane 2020a, Ch. 2; see also Leng forthcoming). Throughout this thesis, I will take set-theoretic realism to be equivalent to set-theoretic *platonism*, so I will use both terms interchangeably.[8]

The first of these groups is the *non-realist* multiverse view. Advocates of this view believe that there are different set-theoretic universes but they are not committed to their 'real' existence. For them, the set-theoretic multiverse is not an independent reality, but only a phenomenon that arises when we need to carry out independence proofs. Within the set theory community, an advocate of this view is Shelah (2002).

Alternatively, a *realist* multiverse advocate claims that the multiverse is a determinate *reality*, consisting of particular entities: the different models, or *universes*, of set theory. Here's Hamkins:

> The [realist multiverse view] is one of higher-order realism — Platonism about universes — and I defend it as a realist position asserting actual existence of the alternative set-theoretic universes into which our mathematical tools have allowed us to glimpse. The multiverse view, therefore, does not reduce via proof to a brand of formalism. (Hamkins 2012, 417).

That is, the multiverse is a view of that Hamkins calls *higher-order realism*: each universe exists 'platonistically' in the very same sense that proponents of the universe view take the 'one true V' to exist. Advocates of a realist multiverse view are Balaguer (1998), Beall (1999) and, more recently, Hamkins (2012). Following Maddy (2019), I call this position **metaphysical multiversism**. It is important to distinguish this position from standard set-theoretic realism, or *metaphysical universism*, which claims that only the "one true V" exists, realistically construed. Hence, metaphysical universism is more ontologically parsimonious than metaphysical multiversism.

[8]It is important to note that other authors define both positions differently. To illustrate, Linnebo (2023) characterises realism to be logically weaker than platonism: according to him, realism leaves out the independence requirement. For different characterisations of mathematical realism and platonism see also Blanchette (1998).

Metaphysical multiversism is the position this thesis will focus on. In the literature, this position tends to be treated as a unified view. However, for sake of clarity, in this thesis I will introduce a novel distinction. Metaphysical multiversism comes in two varieties: conservative and radical metaphysical, depending on what the universes are taken to be. Let's take a closer look at each variety in turn.

1.4.1 Conservative Metaphysical Multiversism

Conservative Metaphysical Multiversism (henceforth in this chapter: CMM) is the most common version of metaphysical multiversism, and it has been defended by Balaguer (1998) and, more recently, by Hamkins (2012), who is its most prominent advocate.[9] It is important to note that both of these positions have been developed separately and on the basis of independent motivations: in fact, both authors name their view differently and Hamkins does not engage with Balaguer's position in his paper.[10] But since both views share similar metaphysical commitments, I here grouped them together for simplicity sake.[11] Let's now outline this position in more detail.

The first feature of Balaguer and Hamkins' position is that universes are tightly related to *concepts of set*.[12] There is no unique absolute background concept of set, instantiated in one uniquely true set-theoretic universe. Rather, there are as many different concepts of set as there are set-theoretic universes: each universe *instantiates* a corresponding concept of set. Here's Hamkins:

> Often, the clearest way to refer to a set concept is to describe the universe of sets in which it is instantiated, and in this article I shall simply identify a set concept with the model of set theory to which it gives rise (Hamkins 2012, 417).

In a similar spirit, Balaguer also writes:

> ZFC describes the universe of sets_1, while ZF+not-C describes sets_2, where sets_1 and sets_2 are different kinds of things (Balaguer 1998, 315).

But, what exactly is a *universe* for CMM? In the literature we find several proposals. A number of authors (e.g. Balaguer 1995, 1998; maybe Hamkins 2012;

[9] Balaguer's original position was called Full Blooded Platonism (Balaguer 1995, 1998).

[10] Probably this is because Hamkins was not familiar with Balaguer's work, since he was not immerse in the philosophy of mathematics at the time his 2012's paper was published.

[11] I take this to be a fairly uncontroversial move. E.g. Antos et al. (2015) and Jonas (forthcoming) are also aware of the connections between both positions and group them together.

[12] Following Incurvati (2020, Ch 1), I take the concept of set to possess three main features: unity, unique decomposition and extensionality. In other words, these features play a key role for the understanding of 'set', as they "determine whether we count somebody as possessing the concept of set" (Incurvati 2020, 13).

Antos et al. 2015) claim that a set-theoretic model of any *consistent* set-theoretic theory gives rise to a universe.[13] As Field puts it: "whenever you have a consistent [formulation of set theory], then there are ... objects that satisfy that theory under a perfectly standard satisfaction relation [...] All the consistent concepts of set ... are instantiated side by side" (Field 1998). Other authors, however, restrict the choice of set-theoretic theories validated in these models. For example, Barton (2016) defines a universe as any ZFC-model. Similarly, Scambler (2020) takes a universe to be a ZFC-model, or a model of some 'similar theory'. However, I will here adopt the most traditional formulation, and take a universe to be a set-theoretic model of a consistent theory of sets. As a consequence, there will be a large number of universes, and this collection of universes conforms 'the multiverse'. Hamkins writes:

> The background idea of the multiverse, of course, is that there should be a large collection of universes, each a model of (some kind of) set theory. There seems to be no reason to restrict inclusion only to ZFC models, as we can include models of weaker theories ZF, ZF^-, KP and so on (Hamkins 2012, 436).

Since the multiverse is so rich then "truth comes cheaply", for consistent set-theoretic models are automatically about the entities of which they are true, and these are so numerous (Clarke-Doane 2020b, 2014). The key claim of conservative metaphysical multiversism is that all we have to do in order to have true set-theoretic beliefs is to have consistent ones. This is summarised in the slogan: 'consistency guarantees existence' (Antos et al. 2015, 2471). However, even if all universes are equal *ontologically* speaking, supporters of CMM claim that there may be good mathematical reasons for considering some universes more interesting or useful than others (Hamkins 2012, 433). I therefore characterise CMM as follows:

> **Conservative Metaphysical Multiversism:** There are many 'platonistically' existing set-theoretic universes, and each instantiates a corresponding concept of set. A universe is simply a set-theoretic model of a consistent theory.

I thus take CMM to be the conjunction of three theses: (1) universes are tightly related to set concepts, (2) universes exist in a 'platonistic' way, and (3) a universe is simply a set-theoretic model of a consistent set-theoretic theory.

[13]Hamkins does not *explicitly* demarcate the range of set-theoretic theories that can make up a universe. In fact, he claims that "there is little need to draw sharp boundaries as to what counts as a universe" (Hamkins 2012, 436). However, I here follow Clarke-Doane (2020a) who takes Hamkins to accept every model of any consistent set theory.

1.4.2 Radical Metaphysical Multiversism

Radical Metaphysical Multiverse (henceforth in this chapter: RMM) has been developed by Beall (1999) as a competitor of Conservative Metaphysical Multiversism.[14] The difference between CMM and RMM is that the only the latter admits *non-trivial inconsistent* set-theoretic theories. This mean that such theories contain some true contradictions or *dialethias*: both A and $\neg A$ obtain, for some A in the language; this is the inconsistency side of such theories. However, the theory should not allow one to prove too much, and in particular should not be trivial; that is, there should be at least one B in the language such that B is not an element of the theory. These two features, Beall writes, are captured nicely by noting that such theories are underwritten by so-called *paraconsistent* logics, a logic in which $A, \neg A \nvdash B$. Thus, Beall's strategy consist in expanding our 'platonic realm' even further to its nontrivial limits. He writes:

> [Conservative Metaphysical Multiversism]'s strategy is ingenious in many ways. But if we really are going to expand platonic heaven [...] then we need to explore the option of expanding heaven to its nontrivial limits" (Beall 1999, 325)

Beall's proposal has not received much attention within the multiverse literature and consequently it has remained greatly underdeveloped. Even though quite a few details are missing, in what follows I will try to reconstruct it further. Since Beall's proposal builds directly on CMM, I'll take the first two feature of RMM to be analogous to those of CMM. That is, I take it that universes are tightly related to *set concepts* and that universes exist 'platonistically'.

However, given than Beall accepts nontrivial inconsistent theories, our previous characterisation of *universe* needs to be expanded. What constitutes a universe for RMM? According to this proposal every set-theoretic model of a *consistent*, or *non-trivial inconsistent* (paraconsistent) set-theoretic theory will give rise to a universe. It is important to point out that Beall does not explicitly discuss which type of paraconsistent formulation of set theory he has in mind. Different strategies for developing paraconsistent set theories have been discussed by Priest (2006). Depending on which strategy we choose, we will obtain different paraconsistent models. Some of these paraconsistent models include NLP models, $\mathsf{NLP}_=$ models and NDLQ models.[15] It is not clear whether Beall would prefer any of these strategies

[14] I should note that Beall's original position was called Really Full Blooded Platonism.

[15] For completeness, NLP stands for Naive Set Theory in Logic of Paradox, $\mathsf{NLP}_=$ for Naive Set Theory in Logic of Paradox with Identity, and NDLQ for Naive Set Theory in logic DLQ, a relevant logic supplemented with the Counterexample Rule $\varphi, \neg\psi \vdash \neg(\varphi \rightarrow \psi)$. See Weber (2022) for an accessible introduction to inconsistent set-theory, and Incurvati (2020, Ch.4) for a comprehensive critique of the strategies proposed by Priest.

in particular, so I will remain neutral on this matter.

A second thing that is missing in Beall's proposal is an explanation of why we should only have *non-trivial* theories. Even if he suggests that we should take paraconsistent mathematics seriously to represent the mathematical landscape accurately, he never provides a positive reason as to why we should leave trivial theories out of this landscape. Estrada-González (2016) mentions three reasons for this.[16] The first reason to avoid triviality (call it *logico-ontological*) is that it might spread and infect all of the platonic realm: "the worst sort of expansion" according to Beall (1999, 323). The idea here seems to be that, once we include a trivial theory, it will trivialise every other theory. This worry seems, as Estrada-González (2016, 289) points out, to be unjustified: if admitting an inconsistent theory does not make automatically every other theory inconsistent, why would admitting a trivial theory make every other theory automatically trivial?

The second reason to exclude trivial theories (call it *practical*) is that some hold that trivial mathematical theories are uninteresting. This is because, if we adopt a trivial theory, we already know all its theorems and properties; namely, *everything* (Estrada-González 2016, 290). However, the fact that a trivial theory is mathematically uninteresting does not seem a sufficient reason to exclude trivial theories from the platonic mathematical realm: if, as mentioned previously, CMM accepts mathematically uninteresting set-theoretic consistent theories, RMM could in principle also accept uninteresting set-theoretic trivial inconsistent theories.

The third reason (call it *epistemic*) states that no knowledge can be obtained from knowledge of triviality (Estrada-González 2016, 290). This is however contentious: even if no knowledge can be obtained from triviality *itself*, surely it is important to learn the conditions under which a theory becomes trivial (in order to, say, avoid triviality).

Thus, RMM must develop a better explanation of why one must exclude set-theoretic trivial inconsistent theories from the mathematical realm. For the purpose of this thesis, I will however assume that this explanation is in place. The following characterisation should capture the position discussed in this subsection:

> **Radical Metaphysical Multiversism:** There are many 'platonistically' existing set-theoretic universes, and each instantiates a corresponding concept of set. A universe is simply a model of a consistent, or non-trivial inconsistent set-theoretic theory.

Hence, I take RMM to be the conjunction of three theses: (1) universes are tightly related to set concepts, (2) universes exist in a 'platonistic' way, and (3) a universe is a model of a consistent, or *non-trivial inconsistent* set-theoretic theory.

[16] Note that he defends that RMM should also accept trivial inconsistent theories.

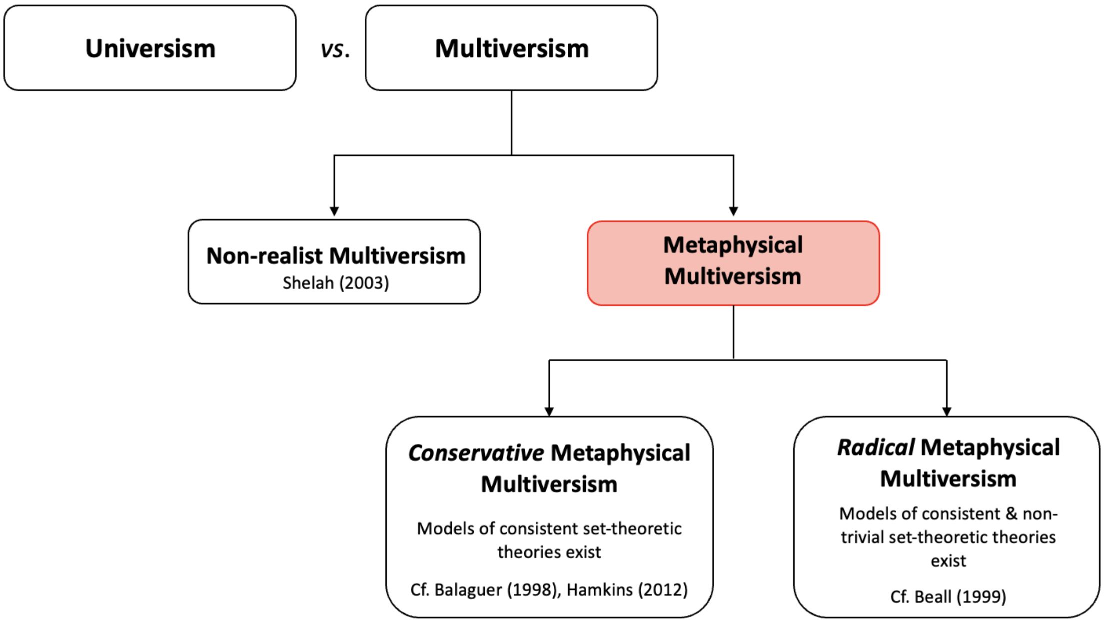

Figure 1.1: Metaphysical multiversism and its varieties: *conservative* and *radical* metaphysical multiversism.

The following simplified diagram (Fig 1.1) illustrates the main positions that have been discussed in this chapter.

1.5 The Central Problem

A crucial question underlies the taxonomy I've provided: which arguments might support a metaphysically weighty view such as metaphysical multiversism? As we have seen, some proponents of the multiverse like Shelah (2002) just hold that that the view should be supported without buying into further metaphysical commitments. However, curiously enough, the most vocal adherents of the view claim that the multiverse exists 'platonistically' (Balaguer 1998, Hamkins 2012).

The goal of this thesis is to investigate this question: I aim to investigate whether the realist component of both varieties of metaphysical multiversism is justified. This assessment matters: while the study the multiverse view is still in its infancy, clarifying its metaphysical foundations is an important theoretical priority. In absence of good arguments for its metaphysical commitments, metaphysical multiversisim (both the conservative and radical versions) would lack a much needed ontological foundation.

The starting point of my instigation will be the two main extant arguments in the literature in support metaphysical multiversism:

- **Epistemic Argument:** Balaguer (1998), Beall (1999) and more recently, Clark-Doane (2020a, 2020b) have argued that, unlike metaphysical universism, metaphysical multiversism seems to afford an answer to the Benacerraf-Field epistemic challenge. This has been considered the main philosophical reason to adopt metaphysical multiversism, instead of metaphysical universism.

- **Practice-based Argument:** Antos (forthcoming) has argued that the development of set theory in the last decades has led to models becoming fundamental objects within set theory, and that the multiverse view can nicely capture this set-theoretic fact.

However, these arguments are silent on the ontological commitments of metaphysical multiversism. In fact, even if these arguments seem to offer good reasons for taking metaphysical multiversism seriously, it still remains unclear whether and to what extent they justify the realist ontology of metaphysical multiversism. This observation invites the following question: by building on and implementing these extant arguments, is the realist ontology posited by metaphysical multiversism justified? The central aim of the thesis is to clarify for the first time exactly this issue.

1.6 Two Traditions, Two Methodologies

Finally, some remarks on the methodology I will adopt in this thesis. In the last decades, there have been two important traditions within the philosophy of mathematics; namely the *orthodox* philosophy of mathematics and the *philosophy of mathematical practice* (Mancosu 2008, Introduction). Importantly, both of these programs have focused on different topics and subscribe to different methodologies. On the one hand, orthodox philosophy of mathematics is deeply concerned with long-standing foundational issues in mathematics (i.e. (neo)logicism, Hilbert's program, intuitionism, etc.) and traditional questions within the epistemology and metaphysics of mathematics (i.e. Benacerraf-Field's epistemic challenge, realist/anti-realist debates, etc.). In order to tackle these issues, they make use of the traditional methods within analytic philosophy, such as *a priori* reflection and conceptual analysis. On the other hand, the philosophy of mathematical practice rejects the foundational programs in mathematics, and explore instead other questions. To name but a few: how does mathematics grow? How are informal arguments related to formal arguments? Is there progress, or revolutions, in mathematics? What is the role of diagrams and visualisations in mathematics? Can mathematics be beautiful? What is the role of inductive reasoning in mathematics? In what sense is mathematics a social practice? The methodology used by this tradition to answer these questions is probably the most distinctive feature of this approach: philosophers of mathematical practice claim that "only detailed analysis and reconstruction of large and significant parts of mathematical practice can provide a philosophy of mathematics worth its name" (Mancosu 2008, 5). Thus, according to this tradition, the *a priori* methods used by traditional philosophy of mathematics should be supplemented with a closer attention to *actual* mathematical practice. Put otherwise, philosophers of mathematical practice aim for an engaged investigation of mathematics 'beyond the armchair'.

This thesis combines the methods of both traditions, hence the title of my thesis. When assessing the epistemic argument for metaphysical multiversism (Chapter 2), I will adopt the methods employed within orthodox philosophy of mathematics. At the same time, I will also pay attention to set-theoretic practice (Chapters 3 and 4), when developing a practice-based argument for metaphysical multiversism's ontology. Given the central problem of the thesis, a combination of both methodologies is especially fitting. Here's why: even if this thesis deals with a traditional debated within the philosophy of mathematics – namely, mathematical realism – metaphysical multiversism has been motivated by appeal to set-theoretic practice (as seen in Section 1.3). Therefore, subscribing solely to one methodology would fall short, given the topic.

1.7 Outline of the Thesis

Having introduced the position central to the thesis and its motivations, the research question and methodology, here's an overview of the thesis.

Chapter 2 assesses the *epistemic argument* for metaphysical multiversism developed first by Balaguer (1995) and Beall (1999), and later refined later by Clarke-Doane (2020a, 2020b, 2022). In a nutshell, the argument aims to establish that one ought to adopt metaphysical multiversism instead of metaphysical universism, for only the former can solve the Benacerraf-Field's challenge. The aim of this chapter is two-fold. Firstly, I argue that *only* conservative metaphysical multiversism can overcome this challenge. Secondly, I point out that the epistemic argument *by itself* cannot justify the ontologial commitments of conservative metaphysical multiversism, for it presupposes the realist commitment of the view.

Chapter 3 proposes a turn to set-theoretic practice in order to justify the ontological commitments of the metaphysical multiversism, given the limitations highlighted in Chapter 2. In order to do so, I will present a framework suited to investigate metaphysical questions from a practice-based point of view: Maddy's *Second Philosophical* programme (2007, 2011). Only after having introduced Maddy's Second Philosophical programme in detail, I will be able to assess whether one can provide a practice-based argument for the ontological commitments of metaphysical multiversism.

Chapter 4 develops a novel *practice-based argument* for metaphysical multiversism, by building on Maddy's Second Philosophical programme. I will first introduce Maddy's criticism against conservative metaphysical multiversism's ontology (2016, 2019). I will however argue that, given the fruitfulness of model-building techniques within set theory, the ontology of conservative metaphysical multiversism can be justified from a Second Philosophical point of view. I then argue that whether radical metaphysical multiversism can appeal to a practice-based argument to justify its ontology is uncertain.

Chapter 5 sums up the main points, highlights the novel contributions of the thesis, and suggests directions for further investigation on the topic.

1.8 Chapter Summary

Let's take stock. In this chapter, I introduced the current debate in the philosophy of set theory between the multiversist and the universist. I then provided some general motivations for multiversism, and turned the attention to the view of interest in this thesis – metaphysical multiversism. I proposed a new taxonomy of the varieties of metaphysical multiversism, and I also identified a central problem for this view, which has to do with how to justify its realist metaphysical commitments. I concluded with a brief overview of the upcoming chapters and a note on the methodology I will adopt.

Chapter 2

The Epistemic Argument for Metaphysical Multiversism

2.1 Overview of the Chapter

This chapter takes a close look at the epistemic argument for metaphysical multiversism developed first by Balaguer (1995) and Beall (1999), and later refined by Clarke-Doane (2020a, 2020b, 2022). In a nutshell, the argument aims to establish that one ought to adopt metaphysical multiversism instead of metaphysical universism, for only the former can solve the Benacerraf-Field's challenge. In this chapter, I argue that not every form of metaphysical multiversism can overcome this challenge. In particular, I argue – *contra* Beall (1999) – that his radical metaphysical multiversism cannot. Accordingly, I claim that, on epistemic grounds, conservative metaphysical multiversism seems preferable to radical metaphysical multiversism. I conclude by pointing out at the limitations of the epistemic argument: while it offers good reasons to adopt conservative metaphysical multiversism instead of other forms of realism, it already presupposes the realist commitment of the view. Consequently, the ontology posited by metaphysical multiversism cannot be justified on the basis of the epistemic argument alone.

2.2 The Benacerraf-Field's Challenge

In this section, I will discuss Benacerraf-Field's epistemic challenge against realism, also sometimes called the "Access Problem' or "Reliability Challenge". In a nutshell, the challenge casts doubts on mathematical realism by pointing out that there seems to be no available explanation of why our mathematical beliefs reliably match mathematical facts. The classical version of the challenge goes back to Benacerraf's seminal article *Mathematical Truth*. Crucially, the paper underscores a tension between the "standard" realist interpretation of mathematics and our claim to mathematical knowledge. Benacerraf writes:

> [O]n a realist (i.e., standard) account of mathematical truth our explanation of how we know the basic postulates must be suitably connected with how we interpret the referential apparatus of the theory [. . .] What is missing is precisely [. . .] an account of the link between our cognitive faculties and the objects known .We accept as knowledge only those beliefs which we can appropriately relate to our cognitive faculties (Benacerraf 1973, 674).

Benacerraf himself is skeptical that such account exists because he favours a *causal* account of knowledge which, very roughly, claims that causal interaction with an object is required for knowledge of it (cf. Goldman 1976). However, this causal account of knowledge poses an important challenge to mathematical realism: since realism claims that mathematical objects are abstract (and thus, *causally inert*), a causal connection cannot obtain between the epistemic agent and the object of knowledge. As a result, Benacerraf's causal constraint renders mathematical knowledge impossible if mathematical realism is true. More exactly, Liggins (2010, 68) reconstructs Benacerraf's argument as follows:

P1: If mathematical platonism is true, then we have knowledge of abstract mathematical entities.

P2: If we have knowledge of abstract mathematical entities, then we are causally related to them.

P3: We are not causally related to abstract mathematical entities.

C: Mathematical platonism is not true.

Typically, mathematical realists accept P1 and P3, but reject P2.[1] The standard objection to Benacerraf's formulation is, in fact, that we ought not to presuppose a causal theory of knowledge, for this theory seems unable to capture mathematical knowledge (see e.g. Lewis 1986, Linnebo 2006, Nutting 2022). Moreover, it seems that the causal theory of knowledge faces important counterexamples not limited to mathematics. Potter (2007), for example, considers the following case. We can know the time of tomorrow's sunset by inference, because we have causally interacted with other sunsets. However, we have not yet causally interacted with tomorrow, or with tomorrow's sunset: previous sunsets do not cause tomorrow's. This suggests that the causal theory of knowledge is not only ill-suited for mathematical knowledge,

[1]However, some philosophers of mathematics defend P2. For some recent attempts see e.g. Gail Montero (2022) and Callard (2023).

but also for general truths or with respect to knowledge of truths about spatio-temporally distant events (e.g. future events).[2]

Even if Benacerraf's original formulation had important drawbacks, some philosophers – especially Hartry Field (1988) – took his considerations to be the core of a more devastating objection to mathematical realism. Consequently, he developed an improved formulation of the problem which has become canonical ever since, and also the starting point of many discussions on the topic. Field's remarks are worth quoting in full:

> Benacerraf's formulation of the challenge relied on a causal theory of knowledge which almost no one believes anymore; but I think that he was on to a much deeper difficulty with platonism [...] Benacerraf's challenge [...] is to provide an account of the mechanisms that explain how our beliefs about those remote entities can so well reflect the facts about them. The idea is that *if it appears in principle impossible to explain this*, then that tends to *undermine* the belief in mathematical entities, *despite* whatever reason we might have for believing in them (Field 1988, 26; original italics).

Field avoids appealing to constraints on knowledge in his argument by avoiding any mention of knowledge. Instead, he focuses on the notion of *reliability*. According to him, what the mathematical realist must explain is the seemingly surprising reliability of mathematical beliefs. Field writes:

> The way to understand Benacerraf's challenge, I think, is not as a challenge to our ability to justify our mathematical beliefs, but as a challenge to our ability to explain the *reliability* of those beliefs (Field 1988, 25; my emphasis).

Thus, without a successful interpretation of "explain the reliability", the epistemic justification we have for our mathematical beliefs is lost. Field's contention is that this *reliability claim* (as I will call it from now on) cannot be successfully explained by standard realism (such as metaphysical universism). Consequently, because of the impossibility of answering to the challenge, Field himself endorses nominalism instead. Field's improved formulation of the problem can therefore be reconstructed as follows (adapted from Wirling forthcoming):

P1: All philosophies of mathematics face the challenge to explain the reliability claim.

[2] Causal theories of knowledge face more general counterexamples with some variations of Gettier cases, as Goldman himself noted (Goldman 1976).

P2: Assuming platonism, it is impossible to explain the reliability claim.
C: We have reason to reject platonism.

I will here follow Field (1988) and assume, without further argument, that standard realism cannot respond to the Benacerraf-Field challenge.[3] However, even if standard realism cannot provide a solution to Benacerraf-Field's challenge, some contend that there is one realist position that could avoid it: metaphysical multiversism (see e.g. Balaguer 1998, Beall 1999, Linsky and Zalta 1995, and particularly, Clarke-Doane 2020b for an extensive discussion on this claim). Even Field concedes this point:

> [Metaphysical Multiversism] allows for [...] knowledge in mathematics, and unlike more standard platonist views, they seem to give an intelligible explanation of it (Field 2005, 78).

This is the epistemic argument for metaphysical multiversism: it is epistemic insofar as it identifies a plausible account of how we can reliably have knowledge of mind-independent mathematical objects. But how exactly does metaphysical multiversism afford a successful answer to Benacerraf-Field's challenge? Can both versions of metaphysical multiversism outlined in Chapter 1 solve the problem in an equally satisfactory way? This chapter addresses these questions. In a nutshell, I will argue that that not every form of metaphysical multiversism can overcome the Benacerraf-Field challenge. In particular, I argue – *contra* Beall (1999) – that radical metaphysical multiversism cannot.

2.3 Desiderata for a Multiversist Solution

If there is a solution available to metaphysical multiversism, this seems to depend on the chosen interpretation of the reliability claim, for in Field's argument this is left unspecified. To make progress on this problem, Clarke-Doane (2020b, 2019) argues that if metaphysical multiversism aims to be in a better position to solve the Benacerraf-Field's challenge than standard realist positions, it then needs to seek an interpretation of "explain the reliability" that satisfies the three conditions below:

> **Universism Unreliabiltiy:** It appears impossible to explain the reliability of our set-theoretic beliefs, assuming universism.
>
> **Multiversism Reliability:** It does not appear impossible to explain the reliability of our set-theoretic beliefs, assuming multiversism.

[3] For a review of why the different strategies proposed by standard realism fail to meet the challenge see Balaguer (1998, Ch.2).

> **Undermining Inexplicability:** If it appears impossible to explain the reliability of our set-theoretic beliefs, then this *undermines* those beliefs.

While Clarke-Doane does not motivate these conditions, I would like to point out that they are quite plausible. The plausibility of **Universism Unreliability** and **Multiversism Reliability** is clear enough: if metaphysical multiversism aims to be in a better position to solve the Benacerraf-Field's challenge than metaphysical universism, then it must provide a reading of the reliability claim that is not available to metaphysical universism. **Undermining Inexplicability** however requires further clarification: what does *undermining* mean here? Field writes: "Our belief in a theory should be undermined if the theory requires that it would be a huge coincidence if what we believed about its subject matter were correct" (Field 2005, 77). It follows that when our belief that p is undermined, our belief that p is no longer justified. Thus, **Underming Inexplicability** claims that: if it appears impossible to answer the reliability challenge, then we have a *reason* to dismiss a platonist interpretation of set-theoretic beliefs. Otherwise put, Benacerraf-Field's challenge constitutes a so-called *undercutting defeater*. In general, an undercutting defeater for a belief that p is a reason (in the broad sense) for no longer believing p, not for believing the negation of p (Pollock 1986). The Benacerraf-Field's challenge provides one such reason: it seems impossible to explain the reliability of our mathematical beliefs (given standard realism).[4]

Let's see these conditions in action. For example, if we interpret "explaining the reliability" as positing a 'causal connection' between the agent and the object of knowledge – as Benacerraf originally suggested – then the multiversist is not in a better position to solve the Benacerraf-Field's challenge than the universist. It is straightforward to see why: if there is no causal relation between our set-theoretic beliefs and the "one true V", then there is no causal relation between our set-theoretic beliefs and the multiverse either (Clarke-Doane 2020b, 2020). Thus, according to Benacerraf's own reading of "explaining the reliability", neither metaphysical universism nor metaphysical multiversism can answer the challenge because the second condition – i.e., **Multiversism Reliability** – is not satisfied. For this reason, a different reading of the reliability claim must be offered by metaphysical multiversism.

With these preliminary points on the table, let me turn to the central aim of this chapter. In what follows, I will explain how conservative metaphysical multiversism can solve the Benacerraf-Field's challenge. Afterwards, I will claim that radical metaphysical multiversism cannot solve the challenge by adopting the same strategy of metaphysical multiversism.

[4]Note the contrast with *rebutting defeater*, which provides a reason to believe $\neg p$.

2.4 The Conservative Metaphysical Multiversist's Strategy

Let's begin with conservative metaphysical multiversism (henceforth in this chapter: CMM). In this section, I will explore two possible interpretations of the reliability claim: the first is due to Balaguer, and the second is due to Clarke-Doane. I will argue that only the latter seem to satisfy the above desiderata.

2.4.1 Balaguer's Solution

Balaguer claims that CMM can successfully account for the reliability of our set-theoretic beliefs. However, he does not offer an interpretation of Field's reliability claim. Instead he writes:

> For if [CMM] is correct, then all we have to do in order to attain such knowledge is conceptualize, or think about, or even "dream up" a mathematical object. *Whatever we come up with, so long as it is consistent, we will have formed an accurate representation of some mathematical object*, because, according to [CMM], all possible mathematical objects exist (Balaguer 1995, 305; my emphasis).

The solution seems to be the following: since according to CMM consistent set-theoretic theories are automatically about the class of objects of which they are true, had our set-theoretic beliefs been different but *still* consistent, they would have been true. Thus, set-theoretic knowledge is obtained *via* knowledge of *consistency*, and since knowledge of consistency is *logical* knowledge, the Benacerraf challenge seems to dissolve. Field agrees. He writes:

> Some philosophers solve the [Benacerraf] problem by articulating views on which though mathematical objects are mind-independent, *any view we had had of them would have been correct* [...]. [T]hese views allow for [...] knowledge in mathematics, and unlike more standard Platonist views, they seem to give an intelligible explanation of it (Field 2005, 78; my emphasis).

Thus, following the above quotations, CMM's interpretation of "explain the reliability" is as follows: had our mathematical beliefs been different, but remained consistent, they still would have been true (Clarke-Doane 2020b). More precisely:

> **Consistent Failsafety**: In order to "explain the reliability" of our set-theoretic beliefs it is necessary to establish, for any one of them, that P, that had we believed $\neg P$, but our set-theoretic beliefs remained consistent, then our belief that $\neg P$ still would have been true (had we used the method that we actually used to determine whether P) (Clarke-Doane 2020b, 2025).

It is important to note that, as a modal proposal cast in terms of subjunctive conditionals, **Consistent Failsafety** requires a relativisation to methods of belief formation.[5] This guarantees that our set-theoretic beliefs are formed via the same method: if the agent forms a false set-theoretic belief in nearby worlds, but the belief-forming method is crucially different, this is not undermining. To illustrate this point, suppose that in the closest possible world in which we form a different set-theoretic belief we employ the following method of belief-formation – a coin flip. These possible worlds are not relevant, as the method of belief-formation is distinct from the one used in the actual world (in the actual world, set-theoretic beliefs are not formed on the basis of a coin flip).[6]

But, is **Consistent Failsafety** a successful interpretation of Field's reliability claim? **Universism Unreliability** is plausibly met: given universism, if our actual set-theoretic beliefs are true, then, had they been different, they would have been false, even if they were still consistent. **Multiversism Reliability** is also plausible: given CMM, if our actual set-theoretic beliefs are true, then, had they been different but consistent, they would have still been true. However, **Undermining Inexplicability** is false: had our set-theoretic beliefs been *very* different, but still consistent, one of them may have been false too. Clarke-Doane illustrates this with the following example concerning perceptual beliefs: had we believed that there were ghosts but our perceptual beliefs remained consistent, then our belief that, for example, we are seeing one would have been false too – since the closest worlds in which these conditions are met are worlds in which we are somehow deluded (2020a, 147). The key insight underlying Clarke-Doane's observation is that the possibility of forming radically different set-theoretic beliefs in close worlds gives us reason to think that we could form a false belief in those same close worlds. Crucially, this gives us evidence that the belief-forming method employed in the actual world is unreliable, and this result is sufficient to undermine our set-theoretic beliefs. Thus, in order for Balaguer's solution to be successful we need an extra condition ensuring that the nearby worlds are similar enough to the actual world. This condition is however lacking.

Consequently, Clarke-Doane's example casts serious doubts on Balaguer's solution: a new promising reading of "explain the reliability" that avoids Benacerraf-Field's challenge must be formulated if metaphysical multiversism aims to overcome the challenge.

[5] The appeal to modal conditions is not surprising: Clark-Doane follows a long-standing tradition in capturing the notion of non-accidentality in purely modal terms (Nozick 1981; Sosa 1999).

[6] See Williamson (2009) for a more extensive discussion of this point.

2.4.2 Clarke-Doane's Solution

Luckily for CMM, Clarke-Doane already suggests a refinement of Balaguer's interpretation for the reliability claim that seems to do the trick: he suggests to avoid the problem with Balaguer's solution by requiring that our set-theoretic beliefs would have been true had they been (i) different (ii) consistent and (iii) "not easily false" (Clarke-Doane 2020b, 2026). Thus, he argues that the Benacerraf-Field's challenge must be understood as a requirement to show that our mathematical beliefs are *safe*. He formulates this interpretation as follows:

> **Safety:** In order to "explain the reliability" of our set-theoretic beliefs it is necessary to establish, for any one of them, that P, that we could not have easily had a false belief as to whether P (using the method that we actually used to determine whether P) (Clarke-Doane 2020b, 2026).

Spelling out the "easily" explicitly in terms of possible worlds, this means that in similar close worlds, the belief that P based on the same method M continues to be true. Let's now test this solution against the desiderata outlined above.

Universism Unreliability is plausibly met because it seems that mathematicians could have *easily* had different set-theoretic beliefs. If we endorse universism, then this implies that we could have easily had *false* beliefs. Clark-Doane argues for this point by noticing that a world in which we rejected standard axioms is not that different or far off: Boolos (1998) does not accept all instances of Replacement, Potter (2004, Sec. 4) is skeptical of Choice, Rieger (2011) rejects the axiom of Foundation, and Jensen (1995) has sympathy for $V = L$. Such disagreement thus suggests that "set-theoretic orthodoxy" is quite contingent: "the mathematical community could have had different set-theoretic beliefs, had history been different" (Clarke-Doane 2020b, 2027).[7]

Multiversism Reliability is also *prima facie* met because it seems that mathematicians couldn't have *easily* had inconsistent set-theoretic beliefs. Thus, we couldn't have *easily* had false set-theoretic beliefs, assuming multiversism, for recall that every consistent theory is true accordign to this view. In order to argue for this point, Clarke-Doane observes that, even if important set-theoretic principles – most famously Comprehension – have turned out to be inconsistent in the development

[7]It is important to note that Clarke-Doane's seems to take our mathematical truths to not be *metaphysically necessary*, but rather *contingent*. This position seems similar to that of Bueno, who takes the truth of mathematical statements be contingent on the underlying mathematical framework in which they are formulated, on the particular formulation of the results in question, and on the relevant logic that is adopted. He writes: "there does not seem to be absolute necessities in mathematics, only relative ones: necessities given a framework, a theory, or a logic. But given that the frameworks, theories and logics in question, in turn, are not necessary either (Bueno 2020, 107)

of mathematics, it is "striking that such inconsistencies have generally been discovered promptly" (2020b, 2027). Thus, it is widely accepted that mathematicians develop the skill to select consistent theories from inconsistent ones early on and, if our mathematical beliefs were inconsistent, then a contradiction would have been already discovered (see Balaguer 1995, 307; Field 1989, Introduction). Clarke-Doane also points out that any good argument against the claim that we could not have easily had inconsistent mathematical beliefs will have to be cumulative. It will have to note not just the mathematicians' failure to discover inconsistencies, but scientists' enormous success in applying mathematical theories to the real world (2020a, Ch. 6). In addition to Clarke-Doane's arguments, I would like to note another way to defend the claim that we could not have easily had inconsistent set-theoretic beliefs: one could point out a rather basic psychological predisposition to detect and remove inconsistent beliefs, and to select only consistent ones. Such predisposition receives relevant empirical support.[8] Be that as it may, whether **Multiversism Reliability** is true depends on the conjecture that our deductive practices are significantly less contingent than our set-theoretic beliefs.

Moreover, unlike Balaguer's **Consistent Failsafety**, **Safety** satisfies **Undermining Inexplicability**. As Clarke-Doane observes, any evidence that the belief-forming method employed in the actual world is unsafe (that is, it could have easily led to form a false belief) is *undermining*, and therefore gives us good reason to give up on our belief.[9] Hence, **Safety**, unlike Balaguer's reading of "explain the reliability", satisfies the three conditions and thus, it is the one that should be adopted by CMM.

At this point one might wonder why we should endorse the safety reading of the reliability claim in the first place. After all, we should accept a reading that is acceptable to every party and not just to CMM. It is important to note that a number of proposals have defended that the reliability claim must be cashed out in modal terms *independently* of CMM (see Clarke-Doane 2017, Field 2005, Topey 2021, Warren 2017). According to these proposals, when we ask for an explanation of the reliability of our set-theoretic beliefs, we are asking for an account according to which the truth of our beliefs isn't completely accidental; that is, isn't the product of an epistemically problematic sort of luck (Topey 2021, 4425). Then, explaining our reliability involves giving an account on which our set-theoretic beliefs are modally robust: we need to show that it would have been difficult for us to have false set-theoretic beliefs. This is exactly what Clark-Doane's safety condition aims to capture. Overall, the safety reading provides a fair common ground for assessing

[8]See Rudnicki (2021) for an overview of psychological studies weighing against the hypothesis that human reasoning tolerates contradictions.

[9]See Clarke-Doane and Baras's (2021) Modal Security Principle for a more precise formulation of the mechanism of this type of defeat.

proposed solutions to the Benacerraf challenge.

Before concluding the section, I would like to point out that, even if yet unchallenged, Clarke-Doane's argument is *not* conclusive. There are two strategies that the universists may pursue to argue, against Clarke-Doane's conclusion, that their set-theoretic beliefs are safe. Firstly, they could claim that mathematical truths could have not easily been false because they are in fact *metaphysically necessary*, and that our belief-forming methods are 'evolutionary designed' to yield true beliefs in such mathematical truths. Secondly, and more plausibly, they could claim that, even if we take our mathematical truths to be *metaphysically contingent*, the "set-theoretic orthodoxy" is not as easily challenged as Clarke-Doane thinks. In other words, the universist may argue that in close worlds the mathematical culture is similar to ours, and therefore mathematicians would still hold true mathematical beliefs using similar methods as in the actual world. However, given that this position has not been substantially challenged yet, for the purposes of this chapter I will accept it.

Let's take stock. In this section I have considered two possible interpretations of Field's reliability claim, and concluded that Clarke-Doane's interpretation, namely **Safety**, unlike Balaguer's, satisfy **Universism Unreliability**, **Multiversism Reliability** and **Undermining Inexplicability**. Hence, CMM is in a better position than metaphysical universism to solve the Benacerraf-Field challenge, understood as the challenge to show that our mathematical beliefs are *safe*.

2.5 Radical Metaphysical Multiversism Exposed

So much for CMM's solution. Let's move on to radical metaphysical multiversism (henceforth in this chapter: RMM): can it also provide an answer to the Benacerraf-Field's challenge? In this section, I argue that RMM does not have a straightforward solution to the challenge. This point is important, as it rejects some hitherto well-established and unquestioned assumptions in the literature on the Benecerraf-Field's challenge. In fact, it is widely accepted that *if* CMM can deal with the challenge, so can RMM (see e.g. Beall 1999, Estrada-González 2016). Hence, one of the key aims of this section is to question, for the first time, Beall's solution to the Benacerraf-Field's challenge.

2.5.1 Beall's Solution

Let's begin with the following observation. When developing his view, Beall (1999) claims that Balaguer is mistaken in claiming that the CMM is the only pluralist position that can rise to the Benacerraf-Field's challenge. As he puts it:

> Balaguer argues for [CMM], and argues that [CMM] alone can solve Benacerraf's epistemic challenge. I have argued that if [CMM] really can solve Benacerraf's epistemic challenge, then [CMM] is not alone in its capacity so to solve; [RMM] can do the trick just as well (Beall 1999, 324-325).

But how exactly does RMM resolve the challenge? Beall's explanation goes as follows:

> After all, [CMM] is supposed to solve the problem by *expanding the platonic heaven to such a degree that one's cognitive faculties can't miss it* (as it were) [...] But, then, since [RMM] simply *expands the heavens* even further, then [RMM] solves the problem if [CMM] does. Hence, Balaguer is wrong to point to [CMM] as a lone contender (Beall 1999, 323; my emphasis).

Beall's interpretation of the reliability claim seems to mirror Balaguer's. Since, according to RMM, consistent and non-trivial inconsistent set-theoretic theories are automatically about the class of objects of which they are true, had our set-theoretic beliefs been different but *still* consistent or non-trivial inconsistent, they would have been true. Put more precisely, and keeping in mind the relativisation to methods of belief-formation, the following gloss captures Beall's interpretation of the reliability claim:

> **Non-trivial Failsafety**: In order to "explain the reliability" of our set-theoretic beliefs it is necessary to establish the following: for any one of them, that P, had we believed $\neg P$ *but our set-theoretic beliefs had remained consistent, or inconsistent non-trivial*, then our belief that $\neg P$ still would have been true (keeping fixed the method that we actually used to determine whether P).

I adapt this from Clarke Doane's **Consistent Failsafety**; note the emphasis on the non-triviality condition, which is meant to capture Beall's RMM.

To be successful, this interpretation needs to meet the desiderata outlined above. However, this proposal faces the same problem as **Consistent Failsafety**: it fails to satisfy **Underminig Inexplicability**. Clarke-Doane's criticism against **Consistent Failsafety** can straightforwardly be adapted to target **Non-trivial Failsafety**: had our set-theoretic beliefs been *radically* different (but still consistent, or non-trivially inconsistent), a given one of them may have been false too. If we grant that Clarke-Doane's criticism works against Balanguer's solution, then by parity of reasoning we should also grant that it works against Beall's solution. The moral of the story seems to be the following: extending our ontology to further limits is not enough to guarantee the reliability of our mathematical beliefs – contrary to what Beall (1999) claims.

2.5.2 Three Alternative Strategies

Thus, Beall's strategy appears to be unsuccessful. What is RMM to do in the face of this objection? There seems to be at least three available strategies:

(a) Interpret the reliability claim as **Safety** once again, or
(b) Adopt a modification of the **Safety** reading: **Paraconsistent Safety**, or
(c) Abandon Clarke-Doane's modal conception of what it would take to explain the reliability altogether, and propose a distinct reading of the reliability claim.

In the remainder of this section, I argue that these strategies are either unsuccessful or too underdeveloped to provide a full solution to the challenge. Let's see each in turn.

A first pass proposal would be to adopt Clark-Doane's **safety** interpretation once again. However, the orthodox reading of the safety condition – which is the one Clark-Doane seems to have in mind – assumes classical logic: as a result, contradictions cannot be safe (a belief that p is safe only if one couldn't have easily believed $\neg p$; so, insofar as one could have easily believed p and $\neg p$ safety is ruled out). But despite this initial difficulty, RMM could still adopt the orthodox reading of safety. For example, supporters of RMM could claim that, even if some theories are non-trivially inconsistent and legitimately describe some part of the set-theoretic realm, the set theory community could have *still* not easily accepted such theories. Put simply: inconsistencies are out there but not within immediate reach. However, I think that this response does not square well with the attitude of many supporters of paraconsistency, who hold that there is an "ongoing *risk* of contradiction *even in orthodox set theory*" (Weber 2022, 21; my emphasis). As a result, these authors are willing to admit the permanent inconsistent status of mathematics. Van Bendegem describes this attitude well:

> [O]ne is willing to admit that the 'normal' situation in mathematics is that at each moment in particular theories a particular contradiction can occur. This consequence tells us that it would be a far better thing to assume that our theories are 'default' inconsistent (van Bendegem 2014, 3071).

This attitude implies accepting that there are inconsistencies easily occurring in the actual world and, as a result, also in nearby worlds. This is in direct tension with adopting the orthodox reading of safety: in fact, as we have seen, the orthodox reading of safety cannot capture any sense in which contradictions can be safe. Thus, strategy (a) doesn't seem to be available for RMM.

Alternatively, RMM could reject Clarke-Doane's orthodox reading of safety and adopt instead a *paraconsistent* safety reading. According to this reading, a belief that p is safe only if one couldn't have easily had a false *only* belief as to whether p. Note that we require that the belief must be false *only* because, according to RMM, some set-theoretic beliefs are *both* true and false. For example, had we believed a set-theoretic theory containing some non-trivial contradiction, our set-theoretic beliefs would have been both true and false. This suggest a new reading of the reliability claim:

> **Paraconsistent Safety:** In order to "explain the reliability" of our set-theoretic beliefs it is necessary to establish the following: for any one of them, that P, *that we could not have easily had a false only belief as to whether P* (using the method that we actually used to determine whether P).

Again, we must test this new reading against our desiderata. **Universism Unreliability** is plausibly met for the same reasons as before: since mathematicians could have *easily* had different set-theoretic beliefs this implies, given universism, that we could have easily had false beliefs. **Undermining Inexplicability** is also satisfied. Much like Clarke-Doane does, RMM could argue that any evidence that the belief-forming method employed in the actual world is paraconsistently unsafe (that is, it leads to triviality) is *undermining*, and therefore it gives us reason to give up on our set-theoretic beliefs.

However, whether **Multiversism Reliability** is satisfied is uncertain. On the one hand, it seems that mathematicians couldn't have easily had trivial inconsistent set-theoretic beliefs. If CMM can argue that mathematicians couldn't have easily had inconsistent set-theoretic beliefs *tout court*, then it looks like RMM can similarly conclude that mathematicians couldn't have easily had trivial inconsistent set-theoretic beliefs.

On the other hand, things are however not so straightforward for RMM, since **Paraconsistent Safety** has a consequence that RMM must address. Adopting **Paraconsistent Safety** implies that it is *safe* to have both consistent and non-trivial inconsistent beliefs; that is, we could have *easily* had both consistent beliefs and non-trivial inconsistent beliefs. It follows that non-trivial inconsistencies *can* obtain in nearby worlds. Otherwise put, non-trivial inconsistencies are "closer" to consistencies than to trivial inconsistencies. However, in order for this to be tenable, RMM needs a somewhat robust criterion to distinguish between 'acceptable' non-trivial inconsistencies and 'non-acceptable' trivial inconsistencies. But this criterion is currently lacking. Absent this criterion we lack a reason to think that our belief forming methods will reliably select non-trivial inconsistencies from trivial ones. This has the potential to undermine our set-theoretic beliefs, since there is no guarantee that we are tracking the inconsistencies of interest.

Of course, supporters of RMM could argue that this reliable method exists. However, even if this is the case, RMM still faces a tricky question: is there any sense in which set-theoretic consistent beliefs can still be more easily believed than non-trivial inconsistent ones? Otherwise put: could a set-theoretic non-trivial inconsistent belief be held more easily than a consistent one? This is a difficult question that has to be settled if RMM opts for **Paraconsistent Safety**. This does not mean that a satisfactory answer to this question cannot be provided; however, such an answer does not seem to be forthcoming in related debates on the metaphysics of modality and modal epistemology. Indeed, views that include both possible and impossible worlds within nearby worlds seem to incur a number of difficulties (Melchior 2020, 727).

Moreover, the claim that we could have easily had non-trivial inconsistent set-theoretic beliefs, but not trivial ones, seems to be in apparent tension with the aforementioned psychological studies showing that we tend to avoid inconsistencies in our daily lives, on the pain of cognitive dissonance (see Rudnicki 2021 for an overview of these studies). Thus, supporters of RMM also owe us an explanation of why we not only easily have but also retain non-trivial inconsistencies, even though in general human reasoning shuns inconsistencies, whether trivial or not. All in all, I think it is fair to say that RMM needs to address a host of serious questions before claiming that **Paraconsistent Safety** satisfies the three proposed desiderata.

Furthermore, adopting **Paraconsistent Safety** seems dialectically problematic. As mentioned before, while CMM could offer independent motivations to endorse orthodox **Safety** as an appropriate reading of the reliability claim, it is dubious whether RMM can provide such independent motivations. **Paraconsistent Safety** seems designed specifically for RMM to solve the Benacerraf-Field challenge: it is unclear that non-paraconsistent mathematicians would endorse a reading of the reliability claim that accepts that our set-theoretic beliefs are still reliable if they turn out to be (non-trivially) inconsistent. As a result, this reading seems to be available only to RMM supporters.

Alternatively, another strategy available to RMM is to reject the modal reading of the Benacerraf-Field challenge altogether. One option could be to develop an *explanationist* reading of Field's reliability claim: instead of cashing out the reliability claim in modal terms, RMM could use an appropriate account of mathematical explanation that could describe the *explanatory* connection between our set-theoretic beliefs and the set-theoretic objects. For example, Faraci (2019) suggests that Lange (2010)'s non-causal account of mathematical explanation may open up new avenues for solving Benacerraf-Field's challenge. However, no one has, to the best of my knowledge, fleshed out the details of Faraci's proposal yet. Moreover, it is not clear whether such suggestion would benefit RMM instead of universism: if there is an appropriate explanatory connection between our set-theoretic beliefs and the 'radical' multiverse, why wouldn't there exist also an explanatory connection between the former and the one single universe? It follows that it is not clear whether such suggestion would satisfy **Universalism Unreliability**. Thus, without an alternative fully developed reading of Field's reliability claim, this strategy lacks a clear enough content.

Let's take stock. In this section, I have argued that RMM does not have a straightforward solution to the Benacerraf-Field's challenge, contrary to what Beall (1999) claims. I first considered Beall's own solution to the Benacerraf-Field's challenge and argued that it is unsuccessful, for it fails to meet one of our proposed desiderata: **Undermining Inexplicability**. Afterwards, I considered three strategies that RMM could adopt to solve the challenge, but I have argued that all of these solution are either unsuccessful or underdeveloped. Thus, the Benacerraf-Field challenge gives us a reason to prefer CMM over RMM.

2.6 Limitations of the Epistemic Argument

So far, this chapter comes with good news for CMM: I have shown that CMM is in a better position to solve the Benacerraf-Field's challenge than the metaphysical universist, understood as the challenge to show that our mathematical beliefs are *safe*. I would however like to conclude on a down note: the epistemic argument *by itself* does not provide solid justification for conservative metaphysical multiversism, for the challenge is met by already presupposing the metaphysical commitments of the view.

This is because, when answering to the challenge from the conservative metaphysical multiversist's perspective, the view must be assumed to be true. As Balaguer (1995, 307) points out, in the context of the Benacerraf-Field challenge conservative metaphysical multiversism is not trying to establish its theory; rather, it is trying to account for a certain fact – i.e., that we have appear to have reliable mathematical beliefs – from *within* the theory. This seems perfectly legitimate: in general, when one is trying to show that a theory $\mathcal{T}$ can account for a certain phenomenon p, it is permissible to assume that $\mathcal{T}$ is true and make use of all of its resources to account for p (Balaguer 1995, 308). *Only* with this assumption in place, conservative metaphysical multiversism can meet the Benacerral-Field challenge. This however, leaves the view in a difficult position. So far, we have shown that *if* one is a mathematical realist, then they should adopt CMM rather than standard realism, for it seems to have a better shot to solve the Benacerraf-Field's challenge (at least under a certain reading). However, we have not provided any *independent* motivations to adopt set-theoretic realism in the first place. Without such independent motivations, conservative metaphysical multiversism could be criticized for merely being an *ad hoc* response to Benacerraf-Field's challenge.[10]

One possible independent motivation for mathematical realism comes from the famous *indispensability argument*. In a nutshell, the indispensability argument for the existence of mathematical entities establishes that we should be ontologically committed only to those parts of a scientific theory that are *explanatory* indispensable, and mathematics is essential in order to explain certain empirical phenomena.[11] Even if the indispensability argument seem to justify only more 'applied' mathematics, some authors (famously, Colyvan 2007) argue for the 'transitivity of indispensability' which aims to enhance traditional indispensability arguments for applied mathematics, so that these can be extended to pure mathematics, all the way up to higher set theory. Thus, according to the transitivity argument, justification of ontological commitment to unapplied mathematics follows abductively from the justication to believe in the existence of all mathematical *explanantia* featuring in confirmed scientific theories.

Thus, CMM could be justified *via* indispensibility as follows:

P1: If mathematical realism is true, then one should adopt CMM instead of standard realism (by the *epistemic argument*)

P2: Mathematical realism is true (by the *indispensability argument*).

C: We should adopt CMM (by P1, P2 and *modus ponens*).

Thus, with the Indispensability Argument at hand, CMM seems available to justify its realist ontology. However, it is important to note that the Indispensibility Argument has not gone unchallenged in the literature. Several authors have questioned the idea that mathematical realism can be supported by way of inference to the best explanation from empirical phenomena (see e.g. Boyce 2018, Heylen and Trump 2019, Hunt 2016). Other authors have claimed that the Indispensibility Argument does not sit well with mathematical practice, for mathematicians do not

[10]This is an important point that has been overlooked by some authors. For example, Jonas mentions the fact that metaphysical multiversism can avoid the Benacerraf-Field challenge as one of the "philosophical reasons" to adopt the view (Jonas forthcoming, 8). At best, the Benacerraf-Field's challenge can be taken as a "philosophical reason" to adopt metaphysical multiversism *instead* of standard realism, but not as a reason to adopt the view *tout court*.

[11]Note that this is not supposed to be a characterisation of the original indispensability arguments proposed by Quine (1981) and Putnam (1975), but rather to more novel "*enhanced*" version of the arguments proposed by Colyvan (2007), Baker (2005; 2009), etc.

regard mathematics as true simply because of its indispensibility for science (see e.g. Leng 2002, Maddy 1992, Sober 1993). Accordingly, these authors claim that we should not take the indispensability of mathematics as providing sufficient grounds for realism. If these arguments are taken seriously, then it is just natural to wonder whether we could justify metaphysical multiversism's ontology following a different strategy. In the upcoming chapters, I explore whether metaphysical multiversisms's ontology can be justified *via* an argument from set-theoretic practice.

2.7 Chapter Summary

In this chapter I have argued for two main claims. Firstly, I have argued – *contra* Beall (1999) – that RMM does not have a straightforward solution to the Benacerraf-Field challenge. Thus, while CMM can solve the challenge – *qua* the challenge to show that our mathematical beliefs are safe – RMM cannot. This gives us a reason to prefer CMM to RMM. Secondly, I have argued that metaphysical multiversism cannot be justified on the basis of this epistemic argument alone. This last point paves the way for the upcoming chapter, which will consider and assess a different motivation for metaphysical multiversism's ontology.

Chapter 3

Interlude: Maddy's Second Philosophy

3.1 Overview of the Chapter

This chapter presents one well-developed framework to philosophically analyse set-theoretic practice – Maddy's *Second Philosophical* programme. The aim of this chapter is to provide the theoretical foundations for Chapter 4: only after having introduced Maddy's Second Philosophical programme in detail, I will be able to assess the feasibility of a practice-based argument for the ontological commitments of metaphysical multiversism. In this sense, this chapter can be seen as an interlude between Chapter 2 and 4.

3.2 A Turn to Practice

In Chapter 2, I argued *via* the indispensability argument and the epistemic argument that the ontological commitments of *conservative* metaphysical multiversism could plausibly be justified. In addition, I also argued that *radical* metaphysical multiversism could not appeal to this strategy. Both of these arguments, however, have been challenged by a number of authors, most famously Maddy (1992, 2011), for being at odds with mathematical practice. If these arguments are successful, then the ontological commitments of conservative metaphysical multiversism would lack a justification which is consistent with mathematical practice.[1]

This invites the following question: rather than being justified by the indispensability and the epistemic argument, could the ontology posited by metaphysical

[1]The term *mathematical practice* is not easy to characterise. Following Van Bendengem and Kerkhove (2014) I take the study of mathematical practice to include *at least* the following seven elements: the community of mathematicians, research programmes, formal languages, proof methods, mathematical concepts, argumentative methods, and proof strategies.

multiversism be justified directly with an argument from mathematical practice?[2] This is the question I will tackle in Chapter 4. But before doing this, in the present Chapter I will introduce the framework that will allow me to develop one such argument from mathematical practice.

The main reason to turn the attention to mathematical practice is simple: if we are talking about mathematics and set-theory, it seems important to pay attention to what mathematicians and set-theorists *actually* do. In fact, if we do not pay attention to practice, some have claimed that the philosophy of mathematics would seem to be using "the wrong philosophical tools for its tasks" (Rittberg 2016, 16), and therefore be in a questionable methodological position. This applies to the ontology of mathematics as well: any attempts to investigate the ontological commitments of a view in complete isolation from mathematical practice seems defective. A second reason to turn to mathematical practice is related to mataphysical multiversism specifically. As we saw in Section 1.3, the multiverse view in set theory has been motivated by appeal to an actual set theoretic phenomenon: the independence results. Accordingly, it is worth exploring whether, and to what extent, mathematical practice favours the ontology posited by metaphysical multiversism.

But, how should we carry out such a task? In recent years, different approaches have been developed to closely investigate mathematical practice. It is important to note that these approaches do not reject traditional methods in philosophy altogether; rather, they *supplement* these traditional methods with a closer attention to actual mathematical practice (see Hamami and Morris 2020, 1117). For instance, one of the main traditional methods of philosophy is *conceptual analysis*, which roughly consists in analyzing a complex concept X by breaking it down into their simpler parts, in order to specify the set of conditions which are necessary and sufficient for something to be or to count as X (see Beaney 2021). This is also a method commonly used to philosophically investigate mathematical practice: one can find in the literature conceptual analyses of what it means for a mathematical proof or theorem to to count as explanatory, beautiful, pure, deep, or fitting (see e.g. Arana 2015, Detlefsen and Arana 2011, Raman-Sundström and Öhman forthcoming, Rota 1997, Steiner 1978). The key difference is that in these investigations one analyses the different values attributed by *mathematicians* themselves to these pieces of mathematics.

Other approaches which are normally used are *case studies* and *empirical methods*. The former consists in "analyzing in detail one or more phenomena in the context of a specific mathematical practice, either past or present" (Hamami and

[2]With *directly* I mean without the aid of any other argument. For example, the epistemic argument cannot justify the ontological commitments directly, as it needs to be paired with the indispensability argument.

Morris 2020, 1118). The latter has been employed to investigate, amongst other things, the peer review process in mathematical practice (e.g. Geist et al. 2010), the way mathematicians write mathematical research papers (Andersen et al. 2019), evaluate the value(s) of mathematical proofs (Inglis and Aberdein 2015) and their beauty (Sa et al. forthcoming), or the kind of explanations mathematicians provide in online mathematical discussions (Pease et al. 2019). Crucially, these methods are not 'mutually exclusive' and can be used all together for a more complete account of mathematical practice (see Rittberg 2016, 23).

In this chapter, I will focus on one well-developed approach to study mathematical practice: Maddy's *Second Philosophical* programme. I choose this approach for two main reasons. Firstly, much of Maddy's work has focused specifically on set theory, and she (and other authors) have already endorsed this approach to investigate ontological debates within set theory (see Maddy 2011 and Rittberg 2016). Secondly, Maddy herself has already engaged with Hamkins' metaphysical multiverse proposal from a Second Philosophical point of view (I will return to this point in more detail in Chapter 4). Overall, Maddy's Second Philosophy seems apt to investigate the ontological commitments of metaphysical multiversism.

The remainder of the chapter is structured as follows. In Section 3.3, I introduce Maddy's Second Philosophy. In Section 3.4, I explain the role that metaphysics plays in Maddy's framework, and the two types of mathematical realisms that she distinguishes: *Robust* and *Thin* Realism. In Section 3.5 I present the notion of mathematical depth, and its relevance for her Thin Realism.

3.3 Maddy's Second Philosophy

Maddy's Second Philosophical programme was firstly developed in her book *Second Philosophy* (2007). Later, in *Defending the Axioms* (2011), she applies her framework to set theory. Since I am here interested in the ontology of set theory, this chapter will follow closely her 2011's exposition.

To get a better purchase on Maddy's programme, let's begin with the following preliminary observation. Second Philosophy is never introduced by means of a sharp definition; instead, Maddy proceeds by portraying the Second Philosopher at work, engaging with a range of philosophical problems and issues in the philosophy of mathematics.[3] The Second Philosopher starts her investigations of set theory

[3]A note on terminology. In 1997, Maddy published *Naturalism in Mathematics*. In this book, Maddy considers her position as a type of naturalism "because it owes so much to Quine" (Maddy 1997, 161). But in another book (Maddy 2007) she coins a new term, *Second Philosophy* because "the term 'naturalism' has acquired so many associations over the years that using it tends to invite indignant responses" (Maddy 2007, 1). Moreover, she wants to "avoid largely irrelevant debates about what 'naturalism' should be" (Maddy 2007, 19). Accordingly, in this chapter I will follow her most recent choice of terminology and use the term 'Second Philosophy'.

with the following question: what are the goals of set theory, and what are the proper methods to achieve such goals? In order to answer this question, the Second Philosopher proceeds "as an active participant, entirely from within [set theory]" (Maddy 2011, 38).

Through the observation of both past and contemporary case studies from set-theoretic practice, the Second Philosopher observes that set-theoretic goals can vary. They include, for example, 'proving a certain theorem', 'developing a new technique' or 'giving an appropriate definition' (Antos forthcoming, 7). However, Maddy concludes that set theory's main goal is *foundational*. But as Maddy herself acknowledges in a more recent paper, it remains underexplored what exactly this 'foundational goal' amounts to (see Maddy 2016, 289). For this reason, she proposes five important set-theoretic goals that are supposed to qualify as foundational. I will briefly survey each of them.

The first of these goals is the so-called *Meta-mathematical Corral*, which refers to the role played by set theory in allowing mathematicians to carry out meta-mathematical investigations, such as the search for consistency proofs. The second goal is *Elucidation*, which refers to role that set theory has often held in replacing imprecise mathematical notions with precise, exact, ones. As examples of Elucidation, Maddy mentions the replacement of the imprecise notion of function with the set-theoretic one (see Maddy 2016, 293; and for an overview of the history see Maddy 1997, 118-126). Next, Maddy mentions the goal dubbed as *Risk Assessment*, which refers to the ability to assess the risk or 'danger' that a novel proposed set-theoretic theory presents. In support of this role, she quotes Voevodsky (2014), who claims that "each addition to the language will require the certification by showing that it is at least as consistent as ZFC". The fourth goal is *Shared Standard*, which refers to the role of set theory in setting the standard of what counts as a legitimate construction or proof. Finally, her last goal is *Generous Arena*, which refers to the role played by set theory in allowing all modern mathematics to develop side-by-side.[4]

Similarly, Maddy mentions two roles that have been considered to be foundational by some but should be "dismissed as spurious" (Maddy 2016, 291), as they are ultimately unsupported by mathematical practice. The first of these is *Epistemic Source*, which holds that set theory provides us with a complete account of what mathematical knowledge is. Maddy claims that Epistemic Source is obviously false: "our greatest mathematicians know (and knew!) many theorems without deriving them from the axioms. The observation that our knowledge of mathematics doesn't

[4]It is important to note that in previous work, Maddy claimed that one of the foundational goals of set theory was to provide the a decisive answer to questions of ontology and proof. She famously called this goal *Final Court of Appeal* (see Maddy 1997). However, she now admits that this goal is "something of an exageration" (Maddy 2016, 296). As a result, she proposes to substitute this goal with *Shared Standard* and *Generous Arena*.

flow from the fundamental axioms to the theorems" (Maddy 2016, 291). The second spurious foundational goal is *Metaphysical Insight*. According to this role, set theory reveals the *true* metaphysical identity that objects enjoyed all along. However, Maddy claims that this is also implausible: "Zermelo took the natural numbers to be $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots$ while von Neumann took them to be $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots$ and there's no principled reason to choose one over the other" (Maddy 2016, 192).

In sum, Maddy points out five goals that count as *foundational* in set theory *and* can be traced back to the practice of set theory: Meta-mathematical Corral, Elucidation, Risk Assessment, Shared Standard and Generous Arena. On the other hand, Epistemic Source and Metaphysical Insight should be disregarded as spurious, for they do not bode well with mathematical practice. This is crucial: Second Philosophy only considers as proper those goals that can be backed up with evidence from mathematical practice.

After deriving the foundational goal(s) that set theory plays by closely investigating set-theoretic practice, the Second Philosopher turns to the following *methodological* question: what are the proper methods of set theory? And by the lights of Second Philosophy, the answer to this question is straightforward: the methods that set theorist *actually* use. This is because the methods of set theory are the most effective means to achieve specific set-theoretic goals, and in particular set theory's foundational role. As Maddy herself puts it: "given what set theory is intended to do, relying on [set-theoretic methods] is a perfectly rational way to proceed: embrace effective means toward desired mathematical ends" (Maddy 2011, 52). Thus, the Second Philosopher soon concludes that set-theoretic own methods are both rational and autonomous (see Maddy 2011, 54).

One main upshot of this discussion is that the role of philosophy of mathematics should be *continuous* with the methods of set theory and goals of set theory. In contrast to her Second Philosophy, philosophy undertaken in complete isolation from mathematics is called by Maddy 'First Philosophy', and should be abandoned (Maddy 2011, 40). This does not mean that the philosophy must be silent about mathematics: if the philosopher is interested in a philosophical study related to set theory, then she needs to approach this study from *within* set-theoretical practice. For instance, the philosopher of mathematics can point out a new set-theoretic goal that has emerged within the set-theoretic practice, or suggest that certain methods are not the best means to achieve a certain set-theoretic goal, or compare two set-theoretic frameworks with respect to the foundational goals that they both can play.

But what if a conflict arises between our philosophical account of set theory and actual set-theoretic practice? No matter how reasonable and well-founded our philosophical account is, ultimately "it is the philosophy that must give" (Maddy 1997, 161). Once the Second Philosopher appreciates that mathematics has its own fruitful methods and goals, she is ready to endorse mathematics' effective methods.

Since our aim in Chapter 4 is to investigate whether metaphysical multiversim's ontology can justified with a Second Philosophical, practice-based argument, it is worth exploring the role that metaphysics plays in Maddy's Second Philosophy. I will do so in the next Section.

3.4 Second Philosophy and Metaphysics

Only after the Second Philosopher has established the proper methods of set theory, she can go on to investigate more traditional 'philosophical' questions, such as metaphysical ones: Do sets exist? What are their properties? Under what circumstances are we allowed to expand our ontology? Unsurprisingly, according to Maddy, in order to tackle these metaphysical questions we should rely on the methods that we have accepted as proper: the methods of set theory are *the* reliable guides to (metaphysical) facts about sets (Maddy 2011, 55).

Consequently, metaphysical debates within set theory should also be settled by turning to the own methods of set theory, rather than by appealing to our favourite philosophical views. In fact, Maddy warns us against 'extra' metaphysical talk in mathematics:

> Given the wide range of views mathematicians tend to hold on [philosophical] matters, it seems unlikely that the many analysts, algebraists and set theorists ultimately led to embrace sets would all agree on any single conception of the nature of mathematical objects in general, or of sets in particular; the Second Philosopher concludes that such remarks should be treated as *colourful asides or heuristic aides*, but not as part of the evidential structure of the subject (Maddy 2011, 53; my emphasis).

Hence, a good Second Philosopher does not invoke extra-mathematical reasons to criticize successful mathematical practice. As Leng nicely puts it: "we should cut our metaphysical cloth to fit the practice of mathematicians" (Leng 2016, 825). Thus, if the own methods set theorist actually justify the introduction of sets, then there is no more to be said about them than what set theory says. In fact, Maddy claims that that this is right: the Second Philosopher can conclude from the proper methods "that sets exist, that set theory is a body of truths about them" (Maddy 2011, 61).

Thus, realism about mathematics seems supported by the methods and goals of set theory. However, not every form of realism is compatible with Second Philosophy. Maddy advocates for an alternative form of realism called *Thin Realism*. According to this view, sets are nothing over and above the entities that set theoretic methods describe. Maddy writes:

> Thin Realist begins from her confidence in the authority of set-theoretic methods when it comes to determining what's true and false about sets, and from the observation that her more familiar methods appear irrelevant (Maddy 2011, 63).

Maddy's proposal stands in stark contrast with *Robust Realism*. According to Robust Realism, metaphysics is an *a priori* discipline: the starting point of this view is not set-theoretic methods, but rather a "robust, abstract" ontology supported by philosophical arguments detached from set-theoretic practice (Maddy 2011, 86). Roughly, while Thin Realism shares the realist commitments of Robust Realism; that is, they both aim at describing an "objectively-existing, non-spatiotemoral, acausal reality" (Maddy 2011, 62). However, the main difference between the views is that the second-philosophical Thin Realism is "firmly rooted in the practical details of the actual mathematics" rather than on a metaphysical worldview (Maddy 2011, 86).

Moreover, since Robust Realism's starting point is its robust abstract ontology, its main goal is to provide an explanation of how we can obtain reliable knowledge from this abstract ontology. As Maddy puts it: they must explain the 'great gulf' between the set-theoretic abstracta and the world (see Maddy 2011, 57). This is Benacerraf-Field's challenge discussed in Chapter 2. Thus, in order for Robust Realism to provide a successful epistemology of mathematics, a so-called *robust epistemology* is needed: this consists in a non-trivial account of the the reliability of mathematical methods. However, Maddy thinks this is a mistake: according to her, the Second Philosopher should not expect mathematicians to find external justification for their set-theoretic knowledge. Rather, the Second Philosopher should start from her confidence in the authority of the own methods of set theory and conclude that the call for a non-trivial, external epistemology "that goes beyond what mathematics itself tells us" (Maddy 2011, 117) is misplaced. Maddy concludes:

> The familiar Benacerrafian challenge suggests that the abstract character of the objects of set theory poses a formidable obstacle to the sort of supplementary epistemological account the Robust Realist requires. In contrast, the Second Philosopher's inclination is to think that *no such supplementary account should be required in the first place*: if Robust Realism questions the cogency of apparently sound mathematical reasoning, her guess is that the fault lies with Robust Realism, not the tried-and-true ways of set theory (Maddy 2011, 59; my emphasis).

In short, Robust Relism is deeply problematic from a Second Philosophy point of view. As we have seen Robust Realism begins by positing an abstract ontology and *then* takes set theory to provide us with an account of mathematical knowledge. Otherwise put, Robust Realism takes the goal of set theory to be that of *Epistemic Source*. However, recall that according to Maddy, *Epistemic Source* is a spurious *defective* foundational goal with no support from set-theoretic practice. By contrast, Thin Realism relies on a *thin epistemology* according to which only our set-theoretic methods can provide reliable knowledge of our realist ontology.[5] Because the starting point of Thin Realism is the own methods and goals of set theory, it provides a faithful interpretation of set-theoretic practice. As Maddy puts it: "Thin Realism not only squares with Second Philosopher's austere and hard-nosed scientism, it actually seems to arise naturally from it" (Maddy 2011, 77). Accordingly, Robust Realism must give way to a preferable version of realism – Thin Realism.

Thus, we have two forms of realism within mathematics, but only Thin Realism is acceptable from the point of view of Maddy's Second Philosophy. We can roughly summarise both views as follows:

- **Robust Realism:** defends an *a priori* realist ontology independent of the own goals and methods of set-theoretic practice.
- **Thin Realism:** defends a realist ontology guided by set-theoretic methods (i.e. set-theoretic objects are nothing over and above that which the methods of set theory describe) and coherent with the actual goals of set-theoretic practice.[6]

3.5 Thin Realism and Mathematical Depth

At this point, however, an important question remains unanswered. Recall that Maddy has concluded that sets exist and that they are just the entities that set theoretic methods are suited to tell us about (see Maddy 2011, 55). But, how can we be sure that sets – or any set-theoretic object posited by Thin Realism – *actually* exist? Maddy acknowledges this worry:

> Connecting sets and set-theoretic methods so intimately continues to invite the suspicion that sets aren't fully real, that they're a kind of

[5]See Roland (2007) for some potential problems with Maddy's 'thin' epistemology of mathematics.

[6]It is important to note that Maddy claims that there is another position compatible with Second Philosophy – *Arealism*. Arealism and Thin Realism are indistinguishable at the "level of method": the starting point of both views is set-theoretic methods and goals (Maddy 2011, 76). However, Arealism posits no mathematical objects. Maddy's claim is that whether one supports Thin Realism or Arealism is ultimately a matter of taste: both positions are equally consistent with mathematical practice. In this chapter, I have restricted my discussion to Robust and Thin realism because I am ultimately interested in assessing a realist position – metaphysical multiversism.

> shadow play thrown up by our ways of doing things, by our mathematical decisions (Maddy 2011, 77).

In order to clarify this key issue Maddy introduces the concept of *mathematical depth*. Mathematical depth is taken by Maddy as an umbrella concept that describes several phenomena, such as mathematical fruitfulness, mathematical effectiveness, mathematical importance, or mathematical productivity. These terms could all be used interchangeably to refer to mathematical depth. Maddy holds off from defining the term further because it is "best understood as a catch-all for the various kinds of special virtues we clearly perceive" (Maddy 2011, 81).

According to Maddy, the facts of mathematical depth are *objective* facts about how the world is. To illustrate, the world is in such a way that the Axiom of Choice has a vast array of important implications and plays a fundamental role in mathematics and science. Similarly, the world is in such a way that transfinite ordinals allow to capture facts about the uniqueness of trigonometric representations, or that the algebraic concept 'group' captures important similarities between structures in widely differing areas of mathematics in ways that other concepts do not (see Maddy 2011, 79-80). These facts are, according to Maddy, not "up to us" (Maddy 2011, 80); rather, they are entirely objective facts of how the world is. She thus concludes that these facts share an important feature: they are the kind of mathematics that reveals how the world is – they are *mathematically deep*.

Importantly, Maddy notes that our set theoretic methods are designed to reliably track such facts about mathematical depth: "they are maximally effective trackers of certain strains of mathematical fruitfulness" (Maddy 2011, 82). Maddy is then ready to approach the question opening the section: how can Thin Realism be sure that sets *actually* exist? According to her, Thin Realism can conclude that sets exist because our set theoretic methods are aimed at effectively tracking mathematical depth, and sets themselves *are* "palpable facts" of mathematical depth (Maddy 2011, 137). Thus, mathematical depth is supposed to provide objective grounds for Thin Realism, thereby alleviating the worry that the ontology posited by Thin Realism doesn't genuinely exist.

Crucially, this tight connection between mathematical depth and existence gives Thin Realism the key for deciding whether a set-theoretic posit legitimately exists. According to Maddy, "[we] can't present a set-theoretic posit that does a maximally-efficient job of tracking mathematical fruitfulness and yet doesn't exist" (Maddy 2011, 83). It follows that set-theoretic objects are just those that track mathematical depth. Summarised in a slogan by Leng: "to exist as a mathematical object, then, is to be quantified over in a mathematically deep theory" (Leng 2016, 827). So, once we determine that a set-theoretic posit satisfies the requirement of mathematical depth, there is no further question to be answered about whether the set-theoretic posit really exists.

But when does a set-theoretic posit qualify as mathematically deep? In order to show this, one needs to check the hallmarks of mathematical depth; that is, whether the posit under consideration is mathematically fruitful, effective, productive, important, and so on. Maddy illustrates this with an example:

> The fact of measurable cardinals being mathematically fruitful in ways x, y, z (and these advantages not being outweighed by accompanying disadvantages) is *evidence for their existence*. Why? Because of what [they] are: repositories of mathematical depth (Maddy 2011, 83; my emphasis).

Thus, Maddy's 'recipe for existence' seems to be the following: if Thin Realism shows that a set-theoretic posit is deep (i.e. fruitful, effective, productive, important, etc.) in x, y, z ways (and these advantages are not outweighed by other drawbacks) then the Thin Realist can conclude that it exists, for it constitutes a fact of mathematical depth. Indeed, it *must* exist:

> We can be uncertain whether or not a given set-theoretic posit will pay off, and therefore uncertain about whether or not it exists, but *if it does pay off, there's no longer any room for doubt* (Maddy 2011, 83; my emphasis).

Let's take stock. In this section, I have introduced Maddy's notion of mathematical depth. This concept is crucial for Thin Realism, for it provides the objective ground underlying its ontology: the thin realist can accept that a set-theoretic posit exists *as long as* it is an objective fact of mathematical depth. Moreover, since set-theoretic methods are themselves designed to detect these objective facts of mathematical depth, they're effective and reliable guides to Thin Realism's realist ontology.

It is however important to emphasise that mathematical depth does *not* depend on Thin Realism's ontology. What is really pivotal for Maddy is the notion of mathematical depth, not existence. According to her, one can adopt Thin Realism and assume that set-theoretic existence statements talk about posits that are genuinely 'real'. Alternatively, one can *also* claim that set theory is not "in the business of discovering truth about abstracta" (Maddy 2011, 99). Maddy calls this position *Arealism*. According to Arealism, mathematical practice involves the development of mathematically deep concepts, but incurs no realist commitments. Crucially, both positions are ultimately equally acceptable and grounded in the practice of set theory. As Maddy puts it: "For both positions, the development of set theory responds to an objective reality – and indeed to the very same objective reality: [the facts of mathematical depth]" (Maddy 2011, 100). For these reasons, at the end of the day, whether on chooses Thin Realism or Arealism does not ultimately matter.

Before concluding, I would like to end the section by briefly clarifying the relationship between mathematical depth and our set-theoretic goals (even though, admittedly, Maddy herself does not expand too much on this issue). As we have seen, according to Maddy set-theoretic methods are effective means to achieve our goals (including our foundational goal), and also effective trackers of mathematical depth. This seems to suggest that our set-theoretic goals are closely connected to the notion of depth. But how exactly? According to Maddy, mathematical goals are responsive to the facts of mathematical depth. This means that mathematical depth is not an instrumental concept with respect to the goals of set theory, as mathematical depth should not be characterised as that which helps us meet our set-theoretic goals. Instead, the set-theoretic goals are "only proper" as long as they are shaped by the mathematical depth (Maddy 2011, 82). She writes:

> The key here is that mathematical [depth] isn't defined as 'that which allows us to meet our goals', irrespective of what these might be. [...] [T]he goals are answerable to the facts of mathematical depth, not the other way 'round. (Maddy 2011, 82).

Thus, since the correct application of our set-theoretic methods reliably tracks the facts of mathematical depth, and since these facts shape our set-theoretic goals, we can expect our set-theoretic methods to be effective means to achieve our set-theoretic goals. This gives us the desired connection between mathematical depth and set-theoretic goals and methods.

3.6 Chapter Summary

In this chapter, I have introduced Maddy's Second Philosophy, and the role that ontology may play in her framework. Crucially for our purposes, we have concluded that a realist position can legitimately posit set-theoretic objects insofar as they are taken to be facts of mathematical depth. With this framework on the table, we can now move on: in Chapter 4, I will assess whether on can provide a practice-based, Second Philosophical argument for the ontological commitments of metaphysical multiversism.

Chapter 4

The Practice-based Argument for Metaphysical Multiversism

4.1 Overview of the Chapter

A number of prominent authors have claimed that the ontology posited by conservative metaphysical multiversism cannot be justified from a Second Philosophical point of view (see Antos forthcoming; Maddy 2016, 2019). In this chapter, I challenge this claim by developing a novel practice-based Second Philosophical argument in support of conservative metaphysical multiversism's ontology. I will argue that, given the fruitfulness of model-building techniques within set-theoretic practice in the last decades, set-theoretic models can legitimately be seen as facts of *mathematical depth*. I will conclude that the rich ontology posited by conservative metaphysical multiversism can be justified from a Second Philosophical point of view. Afterwards, I will explore whether radical metaphysical multiversism can appeal to a similar argument to justify its even richer ontology.

4.2 Maddy's Criticism of Conservative Metaphysical Multiversism

In two recent papers, Maddy (2016, 2019) has expressed some concerns regarding the ontology of conservative metaphysical multiversism. In a nutshell, according to her, metaphysics plays too much of a role in the characterisation of metaphysical multiversism: accordingly, she considers it a form of Robust Realism. In order to support this claim, she appeals to some relevant textual evidence and quotes the following Hamkins' remarks.

Firstly, she is concerned with what she calls Hamkins' "ontological flight" (Maddy 2016, 315). She refers to Hamkins' much discussed quote:

> Each [...] universe exists independently in the same Platonic sense that proponents of the universe view regard their universe to exist. [...] The multiverse view is one of higher-order realism – Platonism about universes – and I defend it as a realist position asserting actual existence of the alternative set-theoretic universes (Hamkins 2012, 416–417).

Secondly, she expresses some discomfort with the role attributed to set theory by Hamkins' conservative metaphysical multiversism:

> The multiverse view does not abandon the goal of using set theory as an *epistemological and ontological foundation* for mathematics, for we expect to find all our familiar mathematical objects [...] inside any one of the universes of the multiverse (Hamkins 2012, 419; my emphasis).

The reason underlying Maddy's discomfort has to do with the alleged foundational goals attributed by Hamkins to set theory. Hamkins' 'epistemological' and 'metaphysical' foundational goals, which are heavily reminiscent of Maddy's *Epistemic Source* and *Metaphysical Insight*, are problematic because they cannot be traced back to set-theoretic practice (Maddy 2016, 296). Accordingly, following Maddy's terminology, Hamkins' foundational goals should be taken as 'spurious'. Maddy thus concludes that Hamkins' proposal is "high metaphysics" (Maddy 2016, 312).

Taken together, the previous quotes suggest that Hamkins' conservative metaphysical multiversism is the pluralist variety of Robust Realism, which – as we saw in the previous chapter – is a form of realism in tension with Maddy's Second Philosophy. As a result, Maddy concludes that Hamkins' conservative metaphysical multiversism must also be at odds with her Second Philosophical framework. Notably, she is not the only one claiming that conservative metaphysical multiversism clashes with Second Philosophy. Antos has also noted that Hamkins' ontological conclusions cannot be justified from a naturalistic, Second Philosophy point of view, writing that "when following Second Philosophy, a Robust Realism of the [Hamkins'] kind cannot be supported" (Antos forthcoming, 24).

As a result of these criticisms, some authors have defended that metaphysical multiversism would be better off if it deflated its ontology (see Ternullo 2019) or if it abandoned its metaphysical addenda altogether (see Maddy 2016). In this chapter, I argue that this needn't be the case, and show that conservative metaphysical multiversism and its ontology can be compatible with Second Philosophy.

However, before proceeding, I would like to make two points of clarifications. Firstly I would like to clarify the scope of my objection. I'm overall sympathetic to Maddy's Second Philosophical programme, and I agree that metaphysics should play a generally limited role. That is, I agree that a naturalisation is desirable: my aim in this chapter is not to challenge Maddy's efforts towards a metaphysics continuous with mathematical practice. Similarly, I should note that I am *not* arguing against Maddy's and Antos' interpretation of Hamkins' quotes. In fact, I take their criticism to Hamkins' to be justified: insofar as conservative metaphysical multiversism admits new posits in their ontology (i.e. set-theoretic models of consistent theories of sets) independently of set-theoretic practice, and takes the foundational goals of set theory to be the 'epistemological and metaphysical foundations of mathematics', then it must be considered a form of Robust Realism. Instead, my aim is just to show that conservative metaphysical multiversism need not be seen as a form of Robust Realism – once we go beyond the letter of Hamkins' quotes.

Moreover, it is important to note that Maddy focuses solely on Hamkins' *conservative* metaphysical multiversism and not on Beall's *radical* metaphysical multiversism.[1] Thus, since I am addressing Maddy's criticism, I will first focus on whether conservative metaphysical multiversism is compatible with Maddy's Second Philosophy. Afterwards, I will explore whether Maddy's ciriticism can be applied to Beall's radical metaphysical multiversim. My argument proceeds as follows. In Section 4.3, after recalling *what* needs to be shown for conservative metaphysical multiversism to be compatible with Maddy's Second Philosophy, I argue that conservative metaphysical multiversism's ontology can actually be justified within a Second Philosophical framework. Afterwards, In Section 4.4. I argue that it is inconclusive whether radical metaphysical multiversism can be justified following the same strategy. In Section 4.5, I conclude the chapter.

4.3 Defending Conservative Metaphysical Multiversism

In this section, I will argue that conservative metaphysical multiversism is actually compatible with Maddy's Second Philosophy. However, before laying out the argument, it is worth recalling *what* needs to be shown in order for conservative metaphysical multiversism to be considered a realist position compatible with Maddy's Second Philosophical framework (a form of Thin Realism). This is no easy task, as Maddy herself thinks that Second Philosophy "has no theory", and that its methods will remain "without any definitive way of characterizing exactly what that term entails" (Maddy 2007, 1-2). Similarly, Maddy does not provide a procedural way to check whether a position counts as a form of Thin Realism. However, based on the previous exposition of Maddy's work in Chapter 3, there are at least two fea-

[1] I should also note that Maddy does not engage with Balaguer's work (which I have discussed in the previous chapters). Since I take conservative metaphysical multiversism to include the views of *both* Hamkins and Balaguer, the practice-based argument developed in this chapter is also available to Balaguer.

tures that we should check if we want to conclude that conservative metaphysical multiversism is compatible with Second Philosophy:

1. **Mathematical Depth:** recall that insofar as Thin Realism can show that a set-theoretic posit is deep (i.e. fruitful, effective, productive, important, etc.), then it can conclude that it exists (see Maddy 2011, 83). Accordingly, conservative metaphysical multiversism can legitimately posit that models of consistent set-theoretic theories exist as long as it can be showed that they are facts of mathematical depth.

2. **Foundational Goal:** recall that Thin Realism must take the main set-theoretic goal to be *foundational*, in the five specific ways that Maddy favours, and that this foundational goal must be traced back to the facts of mathematical depth. In particular, Thin Realism must ensure that the foundational goal is shaped after the facts of mathematical depth (see Maddy 2011, 82). Accordingly, conservative metaphysical multiversism must be shown to respect the foundational goal of set theory, while at the same time taking this goal to be shaped by the facts of mathematical depth.

Hence, my defense of conservative metaphysical multiversism will proceed in two steps. First, I will argue that set-theoretic models of consistent theories of sets are facts of mathematical depth, and consequently, conservative metaphysical multiversism can legitimately posit models as part of its ontology. Second, I will argue that conservative metaphysical multiversism is compatible with the goals that Maddy assigns to set theory. My conclusion will be that conservative metaphysical multiversism can be seen as a form of Thin Realism, and thus it is compatible with Maddy's framework.

4.3.1 Models and Mathematical Depth

Let's begin with the first argument. Recall that for Maddy, the Second Philosopher can claim that a set-theoretic object exists *as long as* it is a "palpable" fact of mathematical depth (Maddy 2011, 137). This is one key difference between Thin and Robust Realism: while Robust Realism's starts positing its abstract ontology purely on *a priori* grounds, Thin Realism defends a realist ontology guided by set-theoretic practice, in particular, *mathematically deep* practice.

In order to justify that set-theoretic models of consistent theories are facts of mathematical depth, I will follow Maddy's proposal and highlight the mathematical fruitfulness, effectiveness, productiveness and importance of models within set theory. In order to do so, I will: (i) highlight the historical importance of models for set theory and (ii) consider the relevance of models for contemporary set-theoretic practice and the set-theoretic community.[2]

Luckily for conservative metaphysical multiversism, many authors have already highlighted the importance of set-theoretic models for the development of set theory. For instance, in a recent paper, Antos has provided a full argument from set-theoretic practice supporting the fact that set-theoretic models of consistent theories can be taken, together with sets, to be *fundamental* entities in set theory (Antos forthcoming). Even though the aims of her argument are different, her points help to bring out the importance of set-theoretic models in the practice of set theory. In what follows, I will carefully reconstruct her discussion on this topic.

It's worth starting by highlighting the historical importance of models within set theory. This discussion will be reminiscent of the points I already made in Chapter 1 (Section 1.3), where I explained that set-theoretic models were originally developed for the purpose of investigating the independence results in set theory. As Antos observes, up until the late 1950s model-building was carried out at a *local* level; that is, while specific models were constructed for specific purposes, the construction method was not yet generalised (Antos forthcoming, 12). In this way, some prominent models were developed: Gödel's L (1938), von Neumann's model of well-founded sets (1925), and models with urelements (Zarmelo 1930). Albeit important, these models only allowed to prove independence result *partially*, in the sense that only one direction of the independence proof could be shown. For example, a model was constructed for ZF+CH but not for $\mathsf{ZF} + \neg$CH.

However, the construction of set-theoretic models changed drastically when Cohen introduced his flexible and general *forcing* technique. With forcing, one first starts from some model of ZFC (the so-called *ground* model) and then produces a different model that satisfies additional statements. This is a *general* model building technique, as it allows to construct different models by 'adjoining' new elements that are missing from the original ground model. This gives us an enormous flexibility to create models with different desired properties (e.g. ones that satisfy CH, and others that do not). *Only* with these techniques on the table, *full* independence results were achieved, and a wide range of statements were shown to be independent. Crucially for our purposes, some authors have argued that the development of the forcing technique couldn't have been possible without the use of set-theoretic models. For instance, Kanamori writes: "models evidently played a central role in the discovery of forcing, and the simplifications and intuitive underpinnings afforded by them were crucial factors in the development of forcing as a general method" (Kanamori 2008, 364). The historical importance of models should now be clear:

[2] Because I am talking about conservative metaphysical multiversism, in this section I will use "models" to refer to set-theoretic models of *consistent* theories of set.

models were crucial for the development of forcing and this technique has enabled us to prove and better understand a number of independence results.

With these historical points in play, let's now move on to contemporary set theory. As Antos points out, the construction of models had important repercussions for the *contemporary* development of set theory. Independence results are nowadays one of the most discussed topics in set theory. Furthermore, the relevance of the independent results is not restricted only to set theory: in algebra, Shelah (1974) showed that Whitehead's Problem in group theory is undecidable, and in analysis Solovay (1976) proved that Kaplansky's conjecture is also undecidable (see Antos forthcoming, 13).

The forcing technique has also been further developed. Nowadays, two main approaches are available: the *Countable Transtive Model* approach (CTM) and the *Boolean Valued Models* approach (BVM). Both approaches rely on the use of models and both approaches arrive at the same mathematical results; that is, they are *mathematically equivalent*. Crucially for conservative metaphysical multiversism, the forcing practice itself is not nowadays limited only to models of ZF or ZFC; sometimes models of weaker theories such as ZF^- are used, or instead ZFC plus additional axioms, such as the large cardinal axiom. Further current research on forcing includes the modal logic of forcing and the set-theoretic genealogy (which is a study of the structure of ground models). Also, forcing has turned out to have different applications such as, for example, a tool for theorem-proving (see Antos forthcoming, 14). Thus, the use of models continues to be deeply ingrained in contemporary set theory.

Finally, it is also worth paying attention to some important sociological factors highlighted by Rittberg (2016). It bears noting that model-centred approaches to set theory have attracted the attention of important mathematicians. Not only Hamkins, but also Löwe (e.g. Hamkins and Löwe 2008), Leibman (e.g. Hamkins et al. 2015), Gitman (e.g. Gitman and Hamkins, 2010) and many more. These are all relevant and influential figures within they set-theoretic community, and thus we should not take their choice of topic as a "quirk and weird option" (Rittberg 2016, 86). Of course, appealing to the members of the set-theoretic community might look like a weak kind of argument. However, philosophers that take mathematical practice seriously often argue that in order to show that a concept is mathematically relevant, one must show that it has been discussed and spread through sufficiently large parts of the community (see Antos forthcoming, 9; Maddy 1997, 197; Rittberg 2016, 100-101). Accordingly, the fact that many influential mathematicians are engaging with model-centred approaches to set-theory is compelling evidence of the fruitfulness of these approaches.

Given these observations regarding historical and contemporary mathematical practice, it is clear that the use of models have had fruitful and important consequences. To make this observation clearer, here's a telling quote from Antos: "some parts of set-theoretic reasoning would simply not be possible without the consideration of models [...] reasoning with models is not only *fruitful and relevant*, it is *essential* for set-theoretic practice" (Antos forthcoming, 22; my emphasis).

Let's take stock. In this subsection I have argued that not only set-theoretic models have played an historical important role within set theory, but also that nowadays they are *still* one of the main tools used within set-theoretic practice. Furthermore, I have argued that an important number of set theorists are sympathetic to the use of models. Thus, I have showed that set-theoretic models (and model-building techniques such as forcing) have been proven *objectively* fruitful, effective and mathematically important. Consequently, they accord with Maddy's criterion of mathematical depth, and we can take models to be facts of mathematical depth. The ramifications of this argument are important for conservative metaphysical multiversism. Recall that, *as long as* we take set-theoretic models of consistent theories to be facts of mathematical depth, which is plausible given their fruitful consequences within set theory and mathematics as a whole, conservative metaphysical multiversism can claim that these exist – in fact, their mathematical depth is *evidence* for their existence (see Maddy 2011, 82-83). It follows that conservative metaphysical multiversism can take its ontology to be supported by the facts of mathematical depth.[3]

However, this conclusion has one consequence that is worth discussing. In this section, I have argued that the there are facts of mathematical depth underlying conservative metaphysical multiversism. However, adherents of the universe view (such as Maddy) could claim that their view is also backed up by *further* facts of mathematical depth. For example they could point out to the remarkable linearity pattern in the hierarchy of large cardinal consistency strength, and take this as evidence for the fact that ZFC might be the ultimate theory of sets (see e.g. Maddy 2011, 80-81).[4] This suggests that there could be different equally legitimate paths to mathematical depth. Accordingly, the Thin Realist could hold, with the universist, that there exist a single 'true' set-theoretic universe, while simultaneously holding, with the conservative metaphysical multiversist, that there is a multiverse of legitimate candidates. This is however, incoherent, as universism and multiversism are incompatible views. This seems to imply that the notion of mathematical depth

[3]In her paper, Antos (forthcoming) argues that models are fundamental entities within set theory. Nevertheless, she also claims that Hamkins' ontological conclusion cannot be justified from a Second Philosophical point of view. On this point, I disagree: I suspect that Antos' makes this claim because she doesn't take into account Maddy's notion of mathematical depth and its connections with ontological commitments.

[4]For a comprehensive account of this linearity phenomenon see Grotenhuis (2022), and for alleged counterexamples to this phenomenon see Hamkins (2021).

is not as robust or discriminating as Maddy originally hoped for, as she seems to claim in this quote: "A mathematician may blanch and stammer, unsure of himself, when confronted with questions of truth and existence, but on judgements of mathematical importance and depth he brims with conviction" (Maddy 2011, 117). By contrast, this subsection has shown that mathematicians do not have a shared confident judgment about the matters of mathematical depth, since both universists and multiversists disagree on which mathematical practices are actually mathematically deep.

Of course, Maddy could try to resist my conclusion. For instance, she could claim that the kind of mathematical depth underlying universism is preferable to the mathematical depth underlying conservative metaphysical multiversism. However, to do so, more needs to be said about the notion of mathematical depth. This point has been acknowledged by other commentators: for example, as Incurvati and Smith put it, when it comes to comparing the depth displayed by forcing with the depth displayed by the large cardinal hierarchy, "nothing in the notion of depth itself seems available to do the work" (Incurvati and Smith 2012, 198). Alternatively, she could claim that it is *more* mathematically fruitful to not give up the quest for an answer to the CH. But, as Maddy herself acknowledges, "this is inconclusive" (see Maddy 2016, 318). Without an argument of this kind on the table, and any independent criterion to evaluate mathematical depth, Maddy's claim that there is a unique path to mathematical depth does not bear scrutiny.

Given my aim in this chapter, let's bracket these important difficulties with Maddy's notion of mathematical depth, and move on to the relation between conservative metaphysical multiversism and set theory's foundational goal.

4.3.2 Conservative Metaphysical Multiversism and the Foundational Goal

Recall that a form of Thin Realism must shape our foundational goal(s) after the facts of mathematical depth (see Maddy 2011, 82), while at the same time it should respect the foundational goals played by set theory. This is a second key difference between Robust and Thin Realism.

Since we have already concluded that models of consistent set-theoretic theories are facts of mathematical depth, it seems legitimate for conservative metaphysical multiversism to take this as evidence that the set-theoretic foundational goal is best met by taking set theory to be about a variety of universes, instead of one unique absolute background universe. This is in line with Maddy's Second Philosophy, as it considers the foundational goal of set theory to be responsive to the facts of mathematical depth (see Maddy 2011, 82).

But can conservative metaphysical multiversism continue to play all the five foundational goals that Maddy proposes? Traditionally, Maddy has defended that, currently, ZFC, provides the simplest theory that can satisfy the foundational goals (see Maddy 2016, 316). However, recently she has acknowledged that, in principle, conservative metaphysical multiversism can also carry out these foundational goals. This is how Maddy suggests that the foundational goals of set theory may be carried out in a conservative metaphysical multiverse framework:

> *Shared Standard* and *Generous Arena* in practice are treated much as the universe advocate would, with ZFC as the default Shared Standard and V the default Generous Arena. [...] *Elucidation* seems unproblematic, as it, too, can be carried out within any universe of the multiverse. For *Risk Assessment*, presumably we call on a universe with large cardinals. For purposes of *Meta-mathematical Corral* [...] perhaps we're to turn to the theory of the multiverse itself (Maddy 2016, 312).

Here, I am not suggesting that this is the best way to carry out these foundational goals: it may well be that another proposal turns out to be better. The important thing to note is that, in principle, conservative metaphysical multiversism seems actually compatible with the foundational goals that Maddy proposed.

I would like to conclude the subsection by emphasizing one final important point. It may well be that universism will *ultimately* play the foundational goals better than conservative metaphysical multiversism. However, I want to emphasize that this would still not mean that set-theoretic models are any less deep. Recall that, for Maddy, depth is not defined as that which allows us to meet our foundational goals (see Maddy 2011, 82). Thus, the depth of set-theoretic models stands, and universism will need to find a way to accommodate this result. However, this seems no easy task, and the fact that set-theoretic models can be taken as facts of mathematical depth plausibly challenges the idea that universism is the simplest theory that can satisfy the foundational goals (see Antos forthcoming, 25).

4.4 Radical Metaphysical Multiversism: a New Hope?

Let's now turn our attention to the ontology of the other variety of metaphysical multiversism, Beall's radical metaphysical multiversism: can it also be justified *via* a practice-based, second philosophical argument?

It is first important to note that if an argument from set theoretic practice *in favour* of radical metaphysical multiversism were to be provided, this would be seen as *independent* from the argument I put forth *against it* in the previous Chapter 2. As we have seen, some authors have challenged the indispensability argument for being at odds with set-theoretic practice (see Leng 2002, Maddy 1992, Sober 1993). Moreover, Maddy takes the Benacerraf challenge to be misguided: the philosopher of mathematics should not suppose that perfectly ordinary mathematical reasoning should stand in need of any philosophical supplementation (see Maddy 2011, Ch.2). Thus, even if I argued that radical metaphysical multiversism cannot solve the Benacerraf-Field challenge, some may argue that it does not have to so in the first place. If this is the case, then a practice-based argument would be more desirable.

But can such an argument be provided? As I mentioned before, Maddy's has never engaged with Beall's position, but I would expect Maddy to find Beall's motivations troubling. First, Beall's main motivation for the development of radical metaphysical multiversism is to show that conservative metaphysical multiversism is not the only position that solve Benacerraf-Field's challenge. Recall the previous quote:

> Balaguer argues for [CMM], and argues that [RMM] alone can solve Benacerraf's epistemic challenge. I have argued that if [CMM] really can solve Benacerraf's epistemic challenge, then [CMM] is not alone in its capacity so to solve; [RMM] can do the trick just as well (Beall 1999, 324-325)

Moreover, his decision to expand the platonic realm even further, including non-trivial inconsistent models, is not justified by paying attention to mathematical practice. Rather, his aim is to guarantee that the Benacerraf-Field challenge is properly met. In his own words:

> If we really are going to expand platonic heaven *in an effort to ensure our epistemic footing*, then we need to explore the option of expanding heaven to its nontrivial limits. If this option is to be rejected, then we need good reason for rejecting it. For now, no such reason seems to exist (Beall 1999, 325; my emphasis).

Plausibly, and given our discussion so far, Maddy would find Beall's reason to expand the 'platonic heaven' to be in direct tension with her Second Philosophy: this is even *higher* metaphysics completely detached from mathematical practice.

However, despite these quotes and as in the case of conservative metaphysical multiversism, we could also attempt to provide an argument from set-theoretic practice by highlighting the fruitfulness, effectiveness and importance of models of non-trivial inconsistent (i.e. paraconsistent) set-theoretic theories. If we show that these models are facts of mathematical depth then, following Maddy, we should take these facts to be evidence for their existence. Let's now try to see how this could be attempted.

Non-classical set theories whose underlying logic is *paraconsistent* have been studied by many authors (see e.g. Brady 1971; Brady and Routley 1989; Restall

2003; Libert 2005; Weber 2010, 2013). These accounts start from the observation that ZF was developed to avoid the contradiction that obtains from the axiom scheme *Comprehension* – $\exists x \forall y(y \in x \leftrightarrow \varphi(y))$ – *via* Russell's paradox. These proposals reintroduce *Comprehension* by arguing that it does not give rise to devastating (i.e. trivial) contradictions within paraconsistent setting. It is however unclear whether this strategy can be used to develop a successful paraconsistent set theory (see Incurvati 2020, Chapter 4 for a number of important challenges).

However, and interestingly for my focus on mathematical practice, in recent years a number of set theorists have proposed to develop paraconsistent set theories *via* model-building techniques (see e.g. Löwe and Tarafder 2015, Jockwich Martinez and Venturi 2020, Tarafder and Venturi 2021, Tarafder 2022). The starting point of their investigation is Boolean-valued models. The theory of Boolean-valued models was first introduced by Scott, Solovay and Vopenka in the 1960s (for an introduction see Jech 2003 and Bell 2005). If $\mathbb{B}$ is a Boolean algebra and V is a model of set theory, we can construct by transfinite recursion a new *Boolean-valued model of set theory*, $V^{\mathbb{B}}$ that verifies all axioms of ZFC. The paraconsistent models investigated by these authors follow very closely the Boolean-valued construction, but instead they take as starting point models of (fragments of) ZF whose internal logic is non-classical and algebras that are non-Boolean. The first steps in the construction of these models were undertaken by Löwe and Tarafder (2015), who constructed an algebra-valued model of paraconsistent set theory, $V^{(PS_3)}$, by taking a negation free fragment of ZF and the algebra PS_3 associated to paraconsistent logic. This work has been recently extended by Jockwich Martinez et al. (In Press) who showed, by discovering several algebras for which the corresponding new algebra-valued models are paraconsistent and satisfy all ZF-axioms, that there are actually many different models of ZF-like paraconsistent set theories.

We can already find several applications in the literature of these algebra-valued models of paraconsistent set theories. The first application is the investigation of the independence phenomena in this novel paraconsistent context. This has been carried out by Tarafder and Venturi (2021). Their paper brings about two important points. Firstly, they show that all the independence results obtained from classical ZFC can be imported into this context (2021, Theorem 5.35). For instance, they prove that CH is independent from the paraconsistent algebra-valued model $V^{(PS_3)}$ (2021, Theorem 5.45). Secondly, they show that new instances of the independence results arise *only* within paraconsistent set theory (2021, Theorem 5.48).

A second application is the development of the basic foundations of cardinal numbers in paraconsistent set theories. This investigation has been carried out by Tarafder (2022). Crucially, his paper shows that algebra-valued models of paraconsistent set theory can be expressive enough to have all the properties of the natural numbers, the cardinal numbers, and cardinal arithmetic (i.e. cardinal addition, cardinal multiplication, and cardinal exponentiation). Surprisingly, these properties are similar to those in classical set theory.

However, even if we have some pioneering applications of models of paraconsistent set theory, it is important to note that the study of paraconsistent set theories is still rare in the literature (see Tarafder 2022, 348), and much remains to be done. For instance, it is important to observe that it is still an open question whether one can extend the forcing method to paraconsistent contexts (see Tarafder and Venturi 2021, 30). Furthermore, the community of set theorists developing this line of research seems still too small and at its earliest stages. As a result, it may well be possible to give a practice-based argument in favour of radical metaphysical multiversism, but since its practices are still in infancy, it just seems to early to tell. This is an emerging literature and it is difficult to assess the relevance of model-building techniques within paraconsistent set theory: only time will tell if these models turn our to be fruitful, effective, and relevant.

4.5 Chapter Summary

In this chapter I have developed a practice-based argument for conservative metaphysical multiversism following Maddy's Second Philosophical framework. Crucially, I argued that given the fruitfulness of model-building techniques within set-theoretic practice in the last decades, set-theoretic models can be seen as facts of *mathematical depth*. The final upshot has been the following: while conservative metaphysical multiversism can legitimately posit models of consistent theories as part of their ontology, matters are more complicated for radical metaphysical multiversism, as it's too early to tell whether an effective practice-based argument for the view can be developed.

Chapter 5

Conclusion

5.1 Summary

This thesis has investigated whether the ontological commitments of two varieties of metaphysical multiversism — *conservative* and *radical* metaphysical multiversism – are justified. In order to do so, I have assessed and developed two different strategies. The first strategy relied on what I called the *epistemic* argument for metaphysical multiversism (which claims that metaphysical multiversism is the only version of mathematical realism rising to the Benacerraf-Field Challenge), and also on the well-known indispensability argument. The second strategy attempted to justify the realist commitments of metaphysical multiversism on the basis of a practice-based argument inspired and informed by Penelope Maddy's Second Philosophy. My final conclusion has been the following: while both strategies can justify the ontology posited by conservative metaphysical multiversism, they do not as straightforwardly justify the ontological commitments of radical metaphysical multiversism. Therefore, only conservative metaphysical multiversism can legitimately justify its ontological commitments.

5.2 Novelty

I will now highlight the main novel contributions of this thesis:

- This thesis is, to the best of my knowledge, the first attempt to provide a systematic discussion of whether the ontological commitments of metaphysical multiversism can be justified, and if so, how they can be justified.

- This thesis provides a discussion of metaphysical multiversism while distinguishing for the first time between Balaguer and Hamkins' *conservative* metaphysical multiversism and Beall's *radical* metaphysical multiversism. This approach is worth emphasising: prior to the present thesis, the literature on metaphysical multiversism had focused on Balaguer and Hamkins' work, mostly neglecting (or at best briefly mentioning) Beall's proposal (see e.g. Antos et al. 2015, Clarke-Doane 2020b, Jonas forthcoming, Ternullo 2019). In this thesis, I aimed to reverse this trend by devoting more attention to Beall's version of metaphysical multiversism.

- The argument put forth in Chapter 2 defends a novel conclusion: it argues, *contra* Beall (1999), that his radical metaphysical multiversism cannot solve the Benacerraf-Field's challenge following the same strategy as conservative metaphysical multiversism.

- Similarly, the argument developed in Chapter 4 is also novel: even though some authors have defended the multiverse view by appeal to Maddy's Second Philosophical framework (e.g. Antos forthcoming, Rittberg 2016), this chapter develops the first argument for the ontology of conservative metaphysical multiversism within a Second Philosophical framework. Moreover, it is also the first explicit attempt to provide an argument from practice in support of radical metaphysical multiversism.

5.3 Further Work

Even if the thesis has made significant progress on the metaphysics of the multiverse view, there are many questions open for future investigation. I here briefly discuss three of them.

Formulating Conservative and Radical Metaphysical Multiversism

As pointed out in Chapter 1, there is still considerable debate on how to best formulate *conservative* metaphysical multiversism, and what this view should take a universe to be. On this matter, responses vary. Antos et al. (2015), Balaguer (1995, 1998), and (maybe) Hamkins (2012) take the multiverse view to be about *consistency*: they seem to accept that a set-theoretic model of any consistent theory should count as a universe. By contrast, other authors propose a more restrictive approach. For example, Barton (2016) defines a universe as a ZFC-model. Similarly, Scambler (2020) takes a universe to be a ZFC-model, or a model of some similar theory. Analogous questions can be raised for *radical* metaphysical multiversism: should the view take a universe to be a set-theoretic model of any consistent or paraconsistent set-theoretic theory? Or should we somehow restrict the theories of sets validated in these models? These are important questions that need to be answered if we want the multiverse view to be a clear proposal. Moreover, absent an answer to this question, we cannot tell with sufficient precision which models constitute the ontology of both views.

The Referential Challenge for Mathematical Realism

In this thesis, I have argued that (at least some version of) metaphysical multiversism can justify its ontological commitments *via* two different strategies. However, these strategies do not solve further potential problems for metaphysical multiversism. One traditional challenge for mathematical realism is the so-called *referential challenge*, which claims that mathematical realism must offer an account of how we are able to refer to mathematical objects given the gap between the epistemic agent and mathematical *abstracta* (see Cole 2010 for an overview of the referential challenge). In a recent paper, Barton (2016) has argued that metaphysical multiversism cannot secure successful reference to any particular universe within the multiverse. If the referential challenge poses a genuine problem for mathematical realism, then Barton's challenge promises to be relevant for metaphysical multiversism. Crucially, his challenge has not been met yet.

Maddy's Mathematical Depth

Maddy's notion of mathematical depth is elusive, and difficult to analyse in a principled manner. Moreover, it does not seem to be sufficient to settle some ontological debates: as we saw in Chapter 4, the notion cannot help to conclusively settle the metaphysical disagreement between the universist and the multiversist. This opens up interesting venues for future research, which I will only sketch here in the form of an open question. Is mathematical depth our best (and only) resource to do metaphysics continuous with mathematical practice? For instance, in the metaphysics of science literature, Amanda Bryant (2018) has suggested that there might be several ways for metaphysics to engage conscientiously with science. She claims that having a single requirement (such as mathematical depth) unrealistically implies that engaging with practice is an all or nothing matter. To do better, Bryant advocates for a *plurality* of criteria to assess conscientious engagement of metaphysics with science. In a similar spirit, it would be interesting to see whether a novel *pluralist* naturalistic proposal could be developed to settle the key ontological debates within mathematics. In general, enriching the notion of mathematical depth might have noteworthy consequences for a number of debates in the philosophy of mathematics.

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