Size Approval Voting Rules

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written by

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Abstract

The field of Voting Theory is concerned with the design and analysis of procedures for collective decision-making, called social choice functions, or voting rules. Many impossibility results having been provided, it is well known that there is no voting rule that will satisfy all kinds of desirable properties that could be asked for. An implication of such results is that the strengths and weaknesses of voting rules need to be assessed carefully so as to determine which rules are appropriate to use for different problems of collective decision-making. For rules in the class of Size Approval Voting rules, voters submit ballots corresponding to those alternatives they approve of, and their ballot is weighted based on the number of alternatives it contains. Despite its inherent simplicity and the prominence of some of its members, this class of voting rules has not been studied in great detail. This thesis studies the class of Size Approval Voting rules from several angles of Computational Social Choice, with the aim of better understanding the properties of the class and its members.

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Contents

1 Introduction **3**

2 Preliminaries **8**

2.1 Basic Framework 8

2.2 Axioms 10

3 Axiomatic Analysis **15**

3.1 Scoring Rules and Size Approval Rules 17

3.1.1 Scoring Rules over Approval Ballots 18

3.1.2 Congruity and Contraction 20

3.2 Classes of Size Approval Voting Rules 28

3.2.1 p-Approval Rules 28

3.2.2 Strictly Decreasing Size Approval Voting Rules 31

3.3 Discussion 38

4 Strategic Behavior **39**

4.1 Underlying Preferences and Sincerity 41

4.2 Complexity of Strategic Manipulation 49

4.3 Manipulation and Partial Information 50

4.4 Discussion 54

5 Computing Possible and Necessary Winners **57**

5.1 Possible and Necessary Winners when New Alternatives are Added . . 59

5.2 Discussion 68

6 Conclusion and Future Work **70**

Chapter 1

Introduction

This thesis studies a class of voting rules called Size Approval Voting rules from several angles of Computational Social Choice, in an effort to get a better understanding of properties of the class and its members.

Voting Theory, a subfield of Social Choice Theory, studies the problem of how best to aggregate the choices of individuals into outcomes that are representative of the preferences of collective. In the typical voting problem, we model choices of individuals as ballots cast over a set of alternatives, and we call a procedure for aggregating the ballots of voters into a collective outcome a voting rule. There is myriad of different ways to make decisions based on the preferences of a group, and throughout the history of voting theory, many different voting rules have been proposed Zwicker [2016], Fleurbaey and Salles [2021]. Out of the many voting rules that have been, and can be, defined, which should we use? The answer seems simple: naturally, we should use the good ones.

What then, makes a voting rule a good voting rule? For one, what normative principles should a voting rule abide by? In the traditional Social Choice Theory, this issue has been studied extensively by use of axioms, formal properties that describe normatively desirable criteria that we would like a voting rule to fulfill, and the analysis of such axioms. As examples of normative desiderata, we might want to require that a voting rule never discriminates between voters or alternatives, or that the voting rule is such that no voter can achieve a more preferred outcome by misrepresenting their preferences instead of voting truthfully. These three characteristics correspond to properties called neutrality, anonymity, and strategyproofness, respectively.

Already early on, several results in Social Choice Theory showed that there is no single voting rule that can simultaneously fulfill all kinds of socially desirable criteria we may want to impose. Notably, Arrow [1951], with a result that is considered to have sparked the beginning of modern Social Choice Theory, proved that dictatorships are the only rules that simultaneously satisfy Independence of Irrelevant Alternatives[1], respects unanimous choices of voters, and always chooses a single alternative as winning, thus showing that a small number of basic desirable properties are inherently incompatible. Gibbard [1973] and Satterthwaite [1975] showed that dictatorships are the only rules that are strategyproof and always choose a unique winnning alternative, a result that was later generalized by Duggan and Schwartz [1999] to show that we cannot guarantee much more when using voting rules that allow for ties between alternatives. We call such results impossibility results, and there are numerous other impossibility results that establish the inherent limitations of voting procedures.

The implication of these impossibility results is not that we should choose voting rules arbitrarily. The fact that there provably exists no perfect voting rule rather motivates studying the properties of different voting rules thoroughly. Since we cannot expect any rule to exhibit all socially desirable properties we could wish for, we need to be able to assess the strengths and weaknesses of each rule, so that we can make informed choices about which rules are appropriate to use, depending on what properties we deem particularly relevant in any given situation.

Over the last few decades, the computational aspects of voting rules, as studied in the area of Computational Social Choice, has gained more attention [Chevaleyre et al., 2007]. The reason is twofold. Firstly, the use of techniques from computer science has provided new insights on the quality of voting rules from perspectives not covered by traditional Voting Theory. For instance, it has been shown that for some voting rules, the problem of finding a successful way to manipulate is computationally hard [Bartholdi et al., 1989a, Xia et al., 2009], suggesting that although no natural voting rule is strategyproof in the general case, some rules may be more robust towards

[1]Independence of Irrelevant Alternatives is a property that requires, roughly, that the collective choice between two options should depend only on the individual preferences between the two options, and not on any other ‘irrelevant’ alternative [Arrow, 1951].

strategic behavior than others. On the other hand, Bartholdi et al. [1989b] showed that for certain voting rules, there mere problem of computing the outcome of an election can be computationally hard, leaving some rules that may be well-regarded in terms of socially desirable characteristics practically infeasible. Secondly, procedures from Voting Theory have many applications in computer science and AI [Chevaleyre et al., 2009], bearing particular relevance to, e.g., multi-agent systems, information retrieval, and crowdsourcing [Endriss, 2014, Brandt et al., 2016]. Many of these use cases require procedures that not only meet a number of normative desiderata, but that also have concise representations, are computationally efficient, and are not too demanding with respect to the information required to be provided by users.

Out of the many voting rules that have been proposed, a notable class is that of scoring rules. Roughly, a scoring rule is a rule that lets each voter assign some number of points to each alternative on their ballot, and that chooses as winning those alternatives with the highest total number of points. This class of rules has been shown to uniquely satisfy a set of very natural and socially desirable properties [Young, 1975, Myerson, 1995, Fishburn, 1979]. In particular, rules in this class never discriminate between voters or alternatives, they always respect the choices of an overwhelming majority, and for any voting situation in which voters are spread across different electorates that mutually favor some alternatives, they always choose those mutually favored candidates as winning when the electorates are assembled. Additionally, scoring rules are among the simplest and most foundational voting procedures, are computationally feasible to use for many problems in collective decision-making, and – perhaps due to their inherent simplicity – are prominent in real-world elections.

In much of the literature on Voting Theory, the ballots of voters are assumed to be linear orders over the set of alternatives. Despite its intuitive appeal with respect to modelling the preferences of voters, the approach has its limitations. For instance, voters may lack the information to compare all alternatives that are voted over, or they may be indifferent between some of the alternatives. In such cases, demanding that voters rank all alternatives is too rigid a requirement. Approval balloting, as an alternative to ranked ballots, has been noticed increasingly in recent years. In this balloting system, the ballot of a voter is simply a subset of the full set of alternatives, intuitively representing those alternatives that the voter approves of. Demanding only that voters are able to determine which alternatives they approve of, approval balloting has been recognized for its simplicity and flexibility.

The central object of study in this thesis is a class of voting rules coined Size Approval Voting rules by Alcalde-Unzu and Vorsatz [2009]. A Size Approval Voting rule takes as input approval ballots from voters, and for each ballot, awards the same non-negative score to each alternative on that ballot, requiring that the score given to alternatives on a ballot is never lower than the score given to alternatives on another ballot of larger size. That is, a Size Approval Voting rule associates a non-negative weight with every ballot size, where weights are assumed to be weakly decreasing in the size of the ballots. The weighted votes are tallied, and those alternatives that have been awarded maximal weighted support are chosen as winning. This class is a class of scoring rules, and contains several natural scoring rules over approval ballots. Most notably, it includes the plurality rule, which can be represented as the Size Approval Voting rule that assigns the same positive weight to all ballots consisting of a single alternative and a weight of zero to all other ballot sizes; Approval Voting, under which all ballot sizes are assigned the same positive weight; and Even and Equal Cumulative Voting, where the weight of each ballot is inversely proportional to the size of the ballot. Plurality voting is perhaps the most prominent rule with respect to real-world usage. Despite its widespread usage, the rule has been widely criticized, among other things for its inclination to elect winning alternatives that have received only a minority of the votes in real-world elections [Zwicker, 2016]. Related to this issue is the problem known as the wasted vote problem, often observed under plurality, where voters have to strategically decide between voting for their favorite alternative and another alternative that is more likely to win [Brams and Fishburn, 1978]. One common argument in support of Approval Voting is that it avoids this drawback of plurality voting [Brams and Fishburn, 1978, Merrill and Nagel, 1987]. Both of these rules have been studied extensively, and for one, it is known that Approval Voting is strategyproof when voters have dichotomous preferences [Brams and Fishburn, 1978].

Despite the prominence of some if its members and the increasing interest in approval balloting, properties of the class of Size Approval Voting rules at large have not been studied in great detail. In particular, there are many questions concerning the computational aspects of the class that remain unsettled. This thesis seeks to analyze the class of Size Approval Rules from various perspectives of Computational Social Choice, in an effort to gain a better understanding of the class at large.

Outline

This thesis is structured as follows. Chapter 2 provides the basic background needed to follow the work in the remainder of the thesis. In Chapter 3, I analyze the class of Size Approval Voting rules from an axiomatic perspective, axiomatically relating the class to scoring rules over approval ballots and exploiting the ideas of Alcalde-Unzu and Vorsatz [2009] to derive the class from the class of scoring rules over approval ballots. I furthermore consider two notable subclasses of the class and provide a characterization for the rule Even and Equal Cumulative Voting. Chapter 4 is dedicated to the issue of strategic manipulation in voting. In this chapter, I study the manipulability of rules in the class and show that many of the rules are particularly vulnerable to the manipulation tactic of bullet voting, under which a voter gives all their support to a single alternative. I furthermore show that it is computationally easy to determine whether there exists a successful manipulation tactic that a voter can use to make a given alternative win, and consider the susceptibility of rules in the class under partial information. In Chapter 5, I study the problem of computing possible and necessary winners in elections with incomplete information about alternatives, under two different assumptions about how voters will respond to new information. I show that the problem of computing necessary winners is computationally easy for all Size Approval Voting rules under both assumptions, and provide some limited results concerning the characterization of possible winners under each of the assumptions. In Chapter 6, I summarize the contents of the thesis, and discuss limitations and future work.

Chapter 2

Preliminaries

2.1 Basic Framework

Let X be a set of alternatives of size $m \geq 2$ and $N \subset \mathbb{N}$ a finite electorate of voters from a universal set of individuals represented by $\mathbb{N}$. For $i \in N$, the ballot of i is a set $B_i \in 2^X$. We call a collection of the ballots of all voters from the electorate N a profile. I will let a profile of ballots cast by a set of voters N be represented as vector $\boldsymbol{B}_N = (B_i)_{i \in N}$, where, with some slight abuse of notation, the index i of a ballot B_i denotes the name of the voter casting that ballot. To give an example, given the set $N = \{3, 4, 6\}$, the ballot profile over the electorate N is $\boldsymbol{B}_N = (B_3, B_4, B_6)$. When no confusion arises from it, I will simply use $\boldsymbol{B}$ in place of $\boldsymbol{B}_N$ to refer to a profile over a given electorate N. Given an electorate N and voter $i \in N$, denote with $\boldsymbol{B}_{-i}$ the partial profile consisting of all ballots aside from B_i. Furthermore, for disjoint electorates $A, A' \subset \mathbb{N}$, let $\boldsymbol{B}_A + \boldsymbol{B}_{A'}$ denote the profile $(B_i)_{i \in A \cup A'}$.

Given an electorate N, a voting rule is a function $f : (2^X)^{|N|} \to 2^X \setminus \{\emptyset\}$, mapping each possible profile over electorate N to some non-empty subset of the set of alternatives X. For example, the plurality rule is the voting rule that maps any profile $\boldsymbol{B}$ over any electorate to the set of those alternatives in X that are on more singleton ballots in $\boldsymbol{B}$ than any other alternative; Approval Voting is the rule that maps profile $\boldsymbol{B}$ to the alternatives that are on the most ballots in $\boldsymbol{B}$; and the trivial rule is the voting rule that maps every profile $\boldsymbol{B}$ to the full set of alternatives X.

I will now introduce some classes of voting rules that are of particular relevance to this thesis, namely the class of (simple) scoring rules, the class of lexicographic scoring rules, and finally, the class of Size Approval Voting rules. Let us first define some notation that will be useful for defining the rules and later discussion.

Given $0 < k \leq m$, call $\mathit{Supp}^k(\boldsymbol{B}, x) = |\{i \in N : x \in B_i \text{ and } |B_i| = k\}|$ the support of size k to alternative x in $\boldsymbol{B}$. That is, the support of size k given to an alternative x is the number of voters with a k-size ballot that includes x. Call $\mathit{Supp}(\boldsymbol{B}, x) = \sum_{k \leq m} \mathit{Supp}^k(\boldsymbol{B}, x)$ the total support of x in profile $\boldsymbol{B}$. Furthermore, given a vector of weights $\boldsymbol{w} = (w_1, ..., w_m)$, let $\mathit{Supp}_{w_k}(\boldsymbol{B}, x) = w_k \cdot \mathit{Supp}^k(\boldsymbol{B}, x)$ be the weighted support of size k to alternative x in profile $\boldsymbol{B}$, and let $\mathit{Supp}_{\boldsymbol{w}} = \sum_{k \leq m} \mathit{Supp}_{w_k}(\boldsymbol{B}, x)$ be the total weighted support to x in profile $\boldsymbol{B}$.

A voting rule f is a (simple) scoring rule if there exists a vector of weights $\boldsymbol{w} = (w_1, ..., w_m)$ of weights $w_k \in \mathbb{R}$ such that, for all profiles $\boldsymbol{B}$,

$$f(\boldsymbol{B}) = \arg\max_{x \in X} \mathit{Supp}_{\boldsymbol{w}}(\boldsymbol{B}, x).$$

In words, a scoring rule is a rule that associates a weight w_k with every ballot size k, and that chooses as winning those alternatives that receive maximal weighted support. For example, the plurality rule is the scoring rule parameterized by the vector $\boldsymbol{w} = (1, 0, ..., 0)$; Approval Voting is the scoring rule parameterized by the vector $\boldsymbol{w} = (1, 1, ..., 1)$; Even and Equal Cumulative Voting is the scoring rule parameterized by $\boldsymbol{w} = (1, \frac{1}{2}, ..., \frac{1}{m})$; anti-plurality is the scoring rule $\boldsymbol{w} = (-1, 0, ..., 0)$, and the trivial rule is the scoring rule $\boldsymbol{w} = (0, 0, ..., 0)$[1].

A voting rule is a lexicographic scoring rule if there exists a sequence of weight vectors $\boldsymbol{w}^1, ..., \boldsymbol{w}^t$ with $t \geq 1$ such that for all profiles $\boldsymbol{B}$ and all $x \in X$, we have that $x \in f(\boldsymbol{B})$ if and only if there are no $y \in X$ and $t' \leq t$ such that $\mathit{Supp}_{\boldsymbol{w}^{t'}}(\boldsymbol{B}, y) > \mathit{Supp}_{\boldsymbol{w}^{t'}}(\boldsymbol{B}, x)$ and $\mathit{Supp}_{\boldsymbol{w}^j}(\boldsymbol{B}, y) = \mathit{Supp}_{\boldsymbol{w}^j}(\boldsymbol{B}, x)$ for all $j < t'$. If for some $x \in X$, there are such $y \in X$ and $t' \leq t$, we will say that y lexicographically dominates x. Thus, a lexicographic scoring rule is a rule that applies a series of scoring rules in lexicographic order and

[1]Note that most scoring rules will have infinitely many representations, i.e., will be induced by infinitely many weight vectors. As such, it is technically not correct to suggest that the rules given are all parameterized by unique weight vectors. However, we will think of the vectors given as the archetypal representations of the rules.

chooses as winning those alternatives that are not lexicographically dominated by any other alternative. As an example of a lexicographic scoring rule, $f_{\boldsymbol{w}^1,\boldsymbol{w}^2}$ with $\boldsymbol{w}^1 = (1, 0, ..., 0)$ and $\boldsymbol{w}^1 = (0, 1, ..., 0)$ is the lexicographic scoring rule which first applies the plurality rule, and, if some alternatives were tied under plurality, chooses as winning those alternatives that are on the maximal number of ballots of size two.

A voting rule f is a Size Approval Voting rule if there exists a vector of weights $\boldsymbol{w} = (w_1, ..., w_m)$, with $w_k \in \mathbb{R}_+$ and $w_k \geq w_{k+1}$ for all $k < m$, such that, for all profiles $\boldsymbol{B} \in (2^X)^n$,

$$f(\boldsymbol{B}) = \arg\max_{x \in X} Supp_{\boldsymbol{w}}(\boldsymbol{B}, x).$$

That is, a Size Approval Voting rule is a scoring rule for which weights are required to be non-negative and weakly decreasing in the size of ballots.

To give some examples of rules that are Size Approval Voting rules, plurality, Approval Voting, Even and Equal Cumulative Voting and the trivial rule are all Size Approval Voting rules. The anti-plurality rule, on the other hand, is an example of a scoring rule that is not a Size Approval Voting rule.

Observe that we did not impose the condition that $w_m \in \mathbb{R}_+$ in the definition of Size Approval Voting rules. The reason is simple: whenever a ballot of size m is cast, all alternatives in the set X are given the same amount of support from the ballot, and as such, a ballot of size m never has any effect on determining the outcome. Therefore, the weight w_m associated with a ballot of size m can be chosen arbitrarily.

2.2 Axioms

In Voting Theory, axioms are formal properties that describe characteristics of rules. This section gives an overview of axioms of particular relevance for the content of this thesis.

I start with the axiom of Anonymity. Anonymity intuitively describes the unbiased treatment of voters. We can consider this property to state that the names of voters should not have an effect on the outcome. We can formally describe this property in the following way.

Axiom (Anonymity)**.** For all profiles $\boldsymbol{B}$ and all permutations $\pi : N \to N$, we have

$$f(\boldsymbol{B}) = f(\pi(\boldsymbol{B})).$$

Neutrality can be considered the counterpart of Anonymity, intuitively describes the unbiased treatment of alternatives that are voted over. While the property of Anonymity requires that the names of voters do not affect the outcome, Neutrality requires that the names of alternatives do not affect the outcome. Given a permutation $\mu : X \to X$, let μ^* be the extension of the permutation μ to a profile $\boldsymbol{B}$ in the natural way. That is, $\mu^*(\boldsymbol{B}) = (\mu(B_1), ..., \mu(B_n))$. We define neutrality in the following way.

Axiom (Neutrality)**.** For all profiles $\boldsymbol{B}$ and all permutations $\mu : X \to X$, we have

$$\mu(f(\boldsymbol{B})) = f(\mu^*(\boldsymbol{B})).$$

Consider now any situation in which an electorate is split into separate districts of voters. The property of reinforcement requires that if two separate districts of voters choose some common alternatives as winning, then those common alternatives should also be the ones that win if the districts are assembled into a single electorate. This property, originally introduced by Young [1974, 1975], is also sometimes called consistency, as it requires some level of consistency between the collective choices of an electorate and the collective choices of subgroups of the electorate.

Axiom (Reinforcement)**.** For all profiles $\boldsymbol{B}$ and all disjoint electorates A, A', we have that $f(\boldsymbol{B}_A) \cap f(\boldsymbol{B}_{A'}) \neq \emptyset$ implies $f(\boldsymbol{B}_A + \boldsymbol{B}_{A'}) = f(\boldsymbol{B}_A) \cap f(\boldsymbol{B}_{A'})$.

Various properties in the literature are so-called continuity properties. From a social perspective, these properties establish some level of influence that an overwhelming majority of voters has on the collective outcome. I mention two. The first one is the continuity-property used by Alcalde-Unzu and Vorsatz [2009]. I will simply call this property continuity. Essentially, the property of continuity states that a group of voters should not be able to dictate the outcome of the election, when massively outnumbered by other voters who together would choose completely different winning alternatives than the group would. We make this precise with the following definition.

Axiom (Continuity)**.** For all profiles $\boldsymbol{B}$, all collections of disjoint electorates $\{N_p\}_{p\in\mathbb{N}}$ such that $f(\boldsymbol{B}_{N_p}) = W$ for all $p \in \mathbb{N}$ and some $W \subseteq X$, and any electorate A for which $A \cap N_p = \emptyset$ for all $p \in \mathbb{N}$ and $f(\boldsymbol{B}_A) \cap W = \emptyset$, there is $k \in \mathbb{N}$ such that

$$f(\boldsymbol{B}_{N_1} + \boldsymbol{B}_{N_2} + \cdots + \boldsymbol{B}_{N_k} + \boldsymbol{B}_A) \cap W \neq \emptyset.$$

Similar, but stronger, is the requirement that if there are alternatives that are uniformly supported by an overwhelming majority of voters, then a voting procedure should limit the set of possible winners to those alternatives. This kind of property is used in Fishburn [1979] under the name continuity, and in Myerson [1995], under the name overwhelming majority. The following axiom is a variation of the property used by Fishburn [1979] and Myerson [1995][2] To distinguish the property from the continuity property given above, I will follow Myerson [1995] and refer to this property as the axiom of Overwhelming Majority. The property is defined as follows.

Axiom (Overwhelming Majority)**.** For all profiles $\boldsymbol{B}$, all collections of disjoint electorates $\{N_p\}_{p\in\mathbb{N}}$ such that $f(\boldsymbol{B}_{N_p}) = W$ for all $p \in \mathbb{N}$ and some $W \subseteq X$, and any electorate A for which $A \cap N_p = \emptyset$ for all $p \in \mathbb{N}$ and $f(\boldsymbol{B}_A) \cap W = \emptyset$, there is $k \in \mathbb{N}$ such that

$$f(\boldsymbol{B}_{N_1} + \boldsymbol{B}_{N_2} + \cdots + \boldsymbol{B}_{N_k} + \boldsymbol{B}_A) \subseteq W.$$

Observe that the axiom of Overwhelming Majority only differs from the axiom of Continuity in that it requires the collective outcome over to be a subset of the choices of the majority, whereas Continuity only requires the collective outcome to have a non-empty intersection with the choices of the majority.

Suppose now that some alternative loses an election based on the votes of a given electorate. It seems natural to require that this alternative stays losing if the electorate is extended with a group of voters that uniformly disapprove of the alternative. This

[2] Note that for convenience, the property of Overwhelming Majority is formulated to be as similar as possible to the Continuity axiom introduced by Alcalde-Unzu and Vorsatz [2009]. In this formulation, the property is technically not equivalent to the property of Myerson [1995] and Fishburn [1979] when no other axioms are assumed, but it is straightforward to show that the continuity-property used by both Myerson [1995] and Fishburn [1979] is implied by the axiom of Overwhelming Majority whenever Anonymity, Neutrality and Reinforcement are also required.

property is used by Alcalde-Unzu and Vorsatz [2009] under the name of Congruity, and is formalized as follows.

Axiom (Congruity)**.** For all profiles $\boldsymbol{B}$ and all disjoint electorates A, A', if we have that $x \notin f(\boldsymbol{B}_A)$ and $x \notin B_i$ for all $i \in A'$, then $x \notin f(\boldsymbol{B}_A + \boldsymbol{B}_{A'})$.

The following property, Contraction, also used by Alcalde-Unzu and Vorsatz [2009], may be thought of as a property that grants voters a basic level of decisiveness over the election outcome. Specifically, this property states that if some voter is able to restrict their set of approved alternatives to some smaller set, and that smaller set contains some alternatives that would have been winning before the voter restricted their choice, then those alternatives should stay winning after the voter has restricted their choice. Furthermore, no previously losing alternative should become winning. The property is formulated precisely in the following way.

Axiom (Contraction)**.** For all profiles $\boldsymbol{B}$, $\boldsymbol{B}'$ and all sets of voters $A \subseteq N$ such that $B'_i \subset B_i$ for a single $i \in A$ and $B'_j = B_j$ for all $j \in A \setminus \{i\}$, if $f(\boldsymbol{B}_A) \cap B'_i \neq \emptyset$, we have that

$$B'_i \ \cap \ f(\boldsymbol{B}_A) \ \subseteq \ f(\boldsymbol{B}'_A) \ \subseteq \ f(\boldsymbol{B}_A).$$

Faithfulness is a property that requires a voting rule to exhibit some basic level of faithfulness to the votes of an electorate. Specifically, a voting rule is faithful if, whenever the electorate consists of a single voter, the outcome is exactly the choices of that voter. To the best of my knowledge, this kind of property was first used by Young [1974] in the context of ranked ballot systems, and by Fishburn [1978] in the context of approval-based ballot systems. The property is defined as follows.

Axiom (Faithfulness)**.** For any single-voter profile $\boldsymbol{B} = (B_1)$, we have that $B_1 = f(\boldsymbol{B})$.

Consider now an election with two voters who do not mutually approve of any alternative. The property of Disjoint Equality, first used by Fishburn [1978], states that in such a scenario, the voting procedure should choose the choices of both voters as winning. Formally, we define disjoint equality as follows.

Axiom (Disjoint Equality)**.** For any profile $\boldsymbol{B}$ and $i, j \in N$, $B_i \cap B_j = \emptyset$ implies

$$f(\boldsymbol{B}_{\{i,j\}}) = B_i \cup B_j.$$

Finally, suppose in some election, that all alternatives get the same number of votes. The axiom of Cancellation, used by Fishburn [1979], states that in this case, the collective outcome should correspond to the full set of alternatives.

Axiom (Cancellation)**.** For any profile $\boldsymbol{B}$, whenever $Supp(\boldsymbol{B}, x) = Supp(\boldsymbol{B}, y)$ for all $x, y \in X$, we have

$$f(\boldsymbol{B}) = X.$$

Having defined the formal framework and central axioms, we are now ready to move on Chapter 3, in which I study the class of Size Approval Voting rules from an axiomatic point of view.

Chapter 3

Axiomatic Analysis

In this chapter, I study the class of Size Approval Voting rules from an axiomatic point of view.

The axiomatic method is perhaps the most foundational tool of analysis in Voting Theory. In this chapter, I study the class of Size Approval Voting rules from an axiomatic perspective. In the first part of the chapter, I consider how to axiomatically relate the class of Size Approval Voting rules to the class of scoring rules based on existing characterizations of the classes, and provide different characterizations of the class of Size Approval Voting rules. In the second part of the chapter, I study two of the most notable subclasses of Size Approval Voting rules more closely, and suggest axioms that can be used to derive said subclasses from the class of Size Approval Voting rules. The latter section also includes a possible characterization of the rule of Even and Equal Cumulative Voting.

I will first discuss some work on axiomatic characterizations of voting rules that relates to the contents of the chapter[1]. Most notably, Alcalde-Unzu and Vorsatz [2009] give an axiomatic characterization of the class of Size Approval Voting rules using the axioms of Anonymity, Neutrality, Reinforcement, Contintuity, Congruity, and Con-

[1]Despite the axiomatic method being the most common approach to analyzing the basic properties of voting rules, it is hardly the only one. For instance, in Epistemic Social Choice (see, e.g., [Elkind and Slinko, 2016]), voting rules are interpreted as truth-tracking devices, and their properties are analyzed accordingly. It is worth mentioning that Allouche et al. [2022], in this context, study the truth-tracking abilities of Size Approval Voting rules for which weights decrease strictly in the size of ballots

traction. Furthermore, despite the class itself not having been studied significantly beyond the work of Alcalde-Unzu and Vorsatz [2009], there is a vast body of literature on axiomatic characterizations of classes of voting rules that are related to the class of Size Approval Voting rules. In particular, Young [1975] characterizes the class of positional scoring rules, which are scoring rules defined for ranked balloting systems, using axioms Anonymity, Neutrality, Reinforcement, and a continuity property similar to Overwhelming Majority. Myerson [1995] characterizes the class of generalized scoring rules using Anonymity, Neutrality, Reinforcement and Overwhelming Majority, generalizing the results of Young [1975] to a setting where no assumptions are made about the structure of ballots. Fishburn [1979] uses the same set of axioms to characterize the class of scoring rules over approval ballots, and furthermore characterizes the superclass of lexicographic scoring rules using the axioms Anonymity, Neutrality and Reinforcement.

As mentioned previously, the plurality rule and Approval Voting have both been studied at length. In particular, much of the work from the body of literature on Approval Voting is relevant to this thesis. To name some work of relevance, Fishburn [1978] characterizes Approval Voting using the axioms of Anonymity, Neutrality, Reinforcement, and Disjoint Equality, and Brandl and Peters [2022] provide a series of characterizations of Approval Voting under dichotomous preferences, all of which rely on the axiom of Reinforcement. In particular, they show that Approval Voting can be characterized by Reinforcement, Faithfulness, and Disjoint Equality.

There are also a few works that study classes of rules similar to Size Approval Voting rules in other contexts of Voting Theory and Preference Aggregation. Terzopoulou and Endriss [2019] study the aggregation of incomplete pairwise preferences using weight rules, which are rules that associate with each pairwise preference on a ballot a weight that only depends on the size of said ballot, choosing as winning those sets of preferences that maximize total weight across the electorate. Introducing an axiom called Splitting, which states that election outcome is not affected if a group of voters with disjoint pairwise preferences come together to each report the combination of their preference sets, they show that versions of said axiom can be used to characterize the Even and Equal weight rule, analogous to the rule of Even and Equal Cumulative Voting. Furthermore, Dong and Lederer [2023] study Approval-Based Committee Scoring

(ABCS) rules in the context of Multi-winner Elections[2] and characterize the class of Ballot-Size Weighted Approval Voting rules, a subclass of ABCS rules consisting of rules that associate a non-negative weight with every ballot size, using the axioms of Anonymity, Neutrality, Reinforcement, Overwhelming Majority, Weak Efficiency (disapproved alternatives are never 'better' than disapproved ones), and a new axiom called Choice Set Convexity, particular to the context of Multi-winner Elections.

3.1 Scoring Rules and Size Approval Rules

In this section, I look more closely at the relation between scoring rules and Size Approval Voting rules from an axiomatic perspective. I first consider the class of scoring rules over approval ballots. The Continuity axiom used by Alcalde-Unzu and Vorsatz [2009] in their characterization of Size Approval Voting rules superficially imposes a weaker condition than the continuity properties used in characterizations of scoring rules. Thus, it is not clear whether the axioms of Anonymity, Neutrality, Reinforcement and Continuity alone capture the class of scoring rules, and, as such, whether it is possible to separate the axioms used by Alcalde-Unzu and Vorsatz [2009] to obtain the class of scoring rules over approval ballots. Therefore, I first investigate the relation between the axiom of Continuity and the axiom of Overwhelming Majority, relating characterizations of scoring rules [Fishburn, 1979, Myerson, 1995] to the characterization of Size Approval Voting rules given by Alcalde-Unzu and Vorsatz [2009].

Thereafter, I investigate the axioms of Congruity and Contraction, to clarify the role that these two axioms play in characterizing the class of Size Approval Voting rules. I show that it is possible to exchange Congruity for a weakened version of Faithfulness, and that some aspects of the axiom of Contraction become redundant for the purpose of characterizing Size Approval Voting rules, when combined with other axioms. Finally, I give possible characterizations of Size Approval Rules implied by combinations of results in the section.

[2] See, e.g., [Faliszewski et al., 2017, Lackner and Skowron, 2023].

3.1.1 Scoring Rules over Approval Ballots

The axioms of Anonymity, Neutrality, Reinforcement, and Overwhelming Majority are all extensively used in the literature, and several works use the four axioms in combination to characterize the class of scoring rules over various ballot representations [Young, 1975, Myerson, 1995, Fishburn, 1979]. As noted in Chapter 2, the axiom of Continuity, as defined by Alcalde-Unzu and Vorsatz [2009], superficially appears to be somewhat weaker than the axiom of Overwhelming Majority. Thus, one natural starting point for determining how to derive the class of Size Approval Voting rules from the class of scoring rules, is to consider whether the axioms Anonymity, Neutrality, Reinforcement, and Continuity together are sufficient to characterize said class. The main purpose of this subsection is to explicitly relate the class of Size Approval Voting rules to scoring rules by showing that Overwhelming Majority can be exchanged with Continuity when characterizing the class of scoring rules over approval ballots. To show this, I will rely on two theorems from Fishburn [1979]. The first theorem states that Anonymity, Neutrality and Reinforcement characterize the class of lexicographic scoring rules, defined in Chapter 2. The second theorem states that Anonymity, Neutrality, Reinforcement and Overwhelming Majority together characterize the class of scoring rules over approval ballots. The interested reader may consult Fishburn [1979] for proofs of the theorems.

Theorem 3.1.1 (Fishburn [1979]). A voting rule f satisfies Anonymity, Neutrality and Reinforcement if and only if it is a lexicographic scoring rule.

Theorem 3.1.2 (Fishburn [1979]). A voting rule f satisfies Anonymity, Neutrality, Reinforcement and Overwhelming Majority if and only if it is a (simple) scoring rule.

I will now show that, when we restrict the analysis to lexicographic scoring rules, it turns out that the axiom of Continuity implies the axiom of Overwhelming Majority. The crucial idea is that, whenever f is a lexicographic scoring rule and an overwhelming majority of voters can be guaranteed some representation in the sense that is required by the axiom of Continuity, the overwhelming majority need only increase by a single disjoint electorate with identical collective choice to determine the outcome of the election.

Proposition 3.1.3. Let f be a voting rule that satisfies the axioms of Anonymity, Neutrality, Reinforcement, and Continuity. Then f also satisfies Overwhelming Majority.

Proof. Let f be a voting rule that satisfies Anonymity, Neutrality, Reinforcement and Continuity. I will show that f satisfies Overwhelming Majority. Observe first that by Theorem 3.1.1, f must be a lexicographic voting rule.

Let $\boldsymbol{B}$ be a profile, $\{N_p\}_{p\in\mathbb{N}}$ a collection of disjoint electorates such that $f(\boldsymbol{B}_{N_p}) = W$ for all p and some subset $W \subseteq X$, and let A be a subset of voters such that $A \cap N_p = \emptyset$ for all p and $f(\boldsymbol{B}_A) \cap W = \emptyset$. By Continuity, there is a $k \in \mathbb{N}$ such that $f(\boldsymbol{B}_{N_1} + \cdots + \boldsymbol{B}_{N_k} + \boldsymbol{B}_A) \cap W \neq \emptyset$. Let $\overline{W} = f(\boldsymbol{B}_{N_1} + \cdots + \boldsymbol{B}_{N_k} + \boldsymbol{B}_A) \setminus W$. If $\overline{W} = \emptyset$, then $f(\boldsymbol{B}_{N_1} + \cdots + \boldsymbol{B}_{N_k} + \boldsymbol{B}_A) \subseteq W$, so suppose that $\overline{W} \neq \emptyset$. I will show that $f(\boldsymbol{B}_{N_1} + \cdots + \boldsymbol{B}_{N_{k+1}} + \boldsymbol{B}_A) \subseteq W$.

Since $f(\boldsymbol{B}_{N_1} + \cdots + \boldsymbol{B}_{N_k} + \boldsymbol{B}_A) \cap W \neq \emptyset$, there is some $y \in f(\boldsymbol{B}_{N_1} + \cdots + \boldsymbol{B}_{N_k} + \boldsymbol{B}_A)$ such that, for all $x \in \overline{W}$ and all $p \in \mathbb{N}$, we have that $Supp_{\boldsymbol{w}^{t'}}(\boldsymbol{B}_{N_p}, y) > Supp_{\boldsymbol{w}^{t'}}(\boldsymbol{B}_{N_p}, x)$ for some $t' \in \{1, ..., t\}$ and $Supp_{\boldsymbol{w}^j}(\boldsymbol{B}_{N_p}, y) \geq Supp_{\boldsymbol{w}^j}(\boldsymbol{B}_{N_p}, x)$ for all $j < t'$. That is, for each $x \in \overline{W}$, we have that y lexicographically dominates x in $\boldsymbol{B}_{N_p}$, for all $p \in \mathbb{N}$. From this it follows that y lexicographically dominates every element $x \in \overline{W}$ in the profile $\boldsymbol{B}_{N_1} + \cdots + \boldsymbol{B}_{N_k}$, and that y lexicographically dominates every element $x \in \overline{W}$ in the profile $\boldsymbol{B}_{N_{k+1}}$.

Now, since $y \in f(\boldsymbol{B}_{N_1} + \cdots + \boldsymbol{B}_{N_k} + \boldsymbol{B}_A)$ and $\overline{W} \subseteq f(\boldsymbol{B}_{N_1} + \cdots + \boldsymbol{B}_{N_k} + \boldsymbol{B}_A)$, we must have that $Supp_{\boldsymbol{w}^{t'}}(\boldsymbol{B}_{N_1} + ... + \boldsymbol{B}_{N_k} + \boldsymbol{B}_A, y) = Supp_{\boldsymbol{w}^{t'}}(\boldsymbol{B}_{N_1} + ... + \boldsymbol{B}_{N_k} + \boldsymbol{B}_A, x)$ for all $x \in \overline{W}$ and $t' \in \{1, ..., t\}$. Thus, for all $x \in \overline{W}$, we get that

$$\begin{aligned}
& Supp_{\boldsymbol{w}^{t'}}(\boldsymbol{B}_{N_1} + ... + \boldsymbol{B}_{N_k} + \boldsymbol{B}_{N_{k'}} + \boldsymbol{B}_A, y) \\
= \; & Supp_{\boldsymbol{w}^{t'}}(\boldsymbol{B}_{N_1} + ... + \boldsymbol{B}_{N_k} + \boldsymbol{B}_A, y) + Supp_{w^{t'}}(\boldsymbol{B}_{N_{k'}}, y) \\
> \; & Supp_{\boldsymbol{w}^{t'}}(\boldsymbol{B}_{N_1} + ... + \boldsymbol{B}_{N_k} + \boldsymbol{B}_A, x) + Supp_{w^{t'}}(\boldsymbol{B}_{N_{k'}}, x) \\
= \; & Supp_{\boldsymbol{w}^{t'}}(\boldsymbol{B}_{N_1} + ... + \boldsymbol{B}_{N_k} + \boldsymbol{B}_{N_{k'}} + \boldsymbol{B}_A, x)
\end{aligned}$$

for some $t' \in \{1, ..., t\}$, and furthermore, that

$$\begin{aligned}
&Supp_{\boldsymbol{w}^j}(\boldsymbol{B}_{N_1} + ... + \boldsymbol{B}_{N_k} + \boldsymbol{B}_{N_{k+1}} + \boldsymbol{B}_A, y) \\
&= Supp_{\boldsymbol{w}^j}(\boldsymbol{B}_{N_1} + ... + \boldsymbol{B}_{N_k} + \boldsymbol{B}_A, y) + Supp_{w^j}(\boldsymbol{B}_{N_{k+1}}, y) \\
&\geq Supp_{\boldsymbol{w}^j}(\boldsymbol{B}_{N_1} + ... + \boldsymbol{B}_{N_k} + \boldsymbol{B}_A, x) + Supp_{w^j}(\boldsymbol{B}_{N_{k+1}}, x) \\
&= Supp_{\boldsymbol{w}^j}(\boldsymbol{B}_{N_1} + ... + \boldsymbol{B}_{N_k} + \boldsymbol{B}_{N_{k+1}} + \boldsymbol{B}_A, x)
\end{aligned}$$

for all $j < t'$, so y lexicographically dominates each $x \in \overline{W}$ in the profile $\boldsymbol{B}_{N_1} + ... + \boldsymbol{B}_{N_{k+1}} + \boldsymbol{B}_A$. Consequently, $f(\boldsymbol{B}_{N_1} + ... + \boldsymbol{B}_{N_{k+1}} + \boldsymbol{B}_A) \cap \overline{W} = \emptyset$. Thus, $f(\boldsymbol{B}_{N_1} + ... + \boldsymbol{B}_{N_{k+1}} + \boldsymbol{B}_A) \subseteq W$, so f satisfies Overwhelming Majority.

$\square$

It follows as a corollary of Proposition 3.1.3 that the class of scoring rules over approval ballots can be characterized by the axioms of Anonymity, Neutrality, Reinforcement and Continuity. Note that this strengthens Theorem 3.1.2. Furthermore, this makes it clear that we obtain exactly the scoring rules over approval ballots by excluding the axioms of Congruity and Contraction in the characterization of Alcalde-Unzu and Vorsatz [2009].

Corollary 3.1.4. A voting rule f satisfies Anonymity, Neutrality, Reinforcement and Continuity if and only if f is a (simple) scoring rule.

Let us now move on to study the axioms of Congruity and Contraction, as well as other possible properties that can be used to derive the class of Size Approval Voting rules from the class of scoring rules.

3.1.2 Congruity and Contraction

In contrast to the axioms of Anonymity, Neutrality, Reinforcement and properties of continuity, the axioms of Congruity and Contraction are not commonly used as properties in the literature. Congruity, stating that an alternative x that loses in a given profile $\boldsymbol{B}$ does not become winning when the profile is extended with a set of voters that do not vote for x, is thought to be rather weak, and it mainly excludes a collection of rules in which ballots can be given non-trivial negative weight. I give an example of a rule that would not be allowed by Congruity.

Example 3.1.5. Consider a scoring rule f parameterized by weight vector $\boldsymbol{w} = (-1, 0, ..., 0)$. We can think of f as the veto rule, under which voters are asked to veto a single alternative, and those alternatives that are vetoed by the least number of voters win the election. Consider now the set $X = \{x, y\}$, and let $\boldsymbol{B}$ be a profile with $N = \{1, 2, 3\}$, where $B_1 = \{x\}$ and $B_2 = B_3 = \{y\}$. Then $f(\boldsymbol{B}_{\{1\}}) = \{y\}$, while $f(\boldsymbol{B}) = \{x\}$. Thus, f does not respect Congruity.

Contraction, on the other hand, states that if a single voter restricts their vote to some new vote that contains a previously winning alternative, then that alternative continues to win, and no previously losing alternatives become winning. The main role of the axiom of Contraction is intuitively to force weights to be weakly decreasing in the size of the ballots. However, in the characterization of the class of Size Approval Voting rules provided by Alcalde-Unzu and Vorsatz [2009], this property is also considered to play some role in forcing weights of rules to be non-negative. The following is an example of a rule excluded by Contraction.

Example 3.1.6. Let f be the scoring rule parameterized by weight vector $\boldsymbol{w} = (-1, 1, 2, ..., m-1)$, where voters are penalized for casting singleton ballots, and where, more generally, the weight of a ballot increases with the size of the ballot. Despite being somewhat strange-looking, this rule bears some relevance to voting situations in which voters are incentivized to be flexible. For example, if we think of the alternatives in X as representing timeslots, and of voters as voting to schedule a meeting, we may consider the rule to require that no voter chooses only a single timeslot, while simultaneously encouraging voters to choose as many of the available slots as possible. Let $X = \{x, y, z\}$, and let $\boldsymbol{B}$ be a profile with $N = \{1, 2, 3\}$, where $B_1 = \{x, y\}$, $B_2 = \{y, z\}$, and $B_3 = \{x, z\}$. Consider the profile $\boldsymbol{B}' = (B_1', B_2, B_3)$, where $B_1' = B_1 \setminus \{y\}$. Then $f(\boldsymbol{B}) = X$ and $f(\boldsymbol{B}) \cap B_1' = \{x\}$, but $f(\boldsymbol{B}') = \{z\}$. Thus, f does not respect Contraction.

In this subsection, I consider the role these two axioms play in capturing the class of Size Approval Voting rules, by looking at the subclasses of scoring rules obtained by requiring each of them separately. I also show that the axioms can be relaxed or exchanged for simpler axioms to obtain the same classes of rules.

Non-negative scoring rules

As noted in the introduction to this section, aside from forcing weights to be weakly decreasing, the axiom of Contraction plays some part in excluding rules with negative weights in the characterization due to Alcalde-Unzu and Vorsatz [2009]. Bearing in mind this dual role played by the axiom in this work, it is worth considering whether, in the context of scoring rules, the axiom of Congruity is sufficient to force weights to be non-negative. For this purpose, it is convenient to define the class of non-negative scoring rules.

Definition 3.1.7 (Non-negative scoring rule)**.** A voting rule f is a non-negative scoring rule if there exists a vector $\boldsymbol{w} = (w_1, ..., w_m)$ of weights, with $w_k \geq 0$ for all $k < m$, such that for all profiles $\boldsymbol{B}$,

$$f(\boldsymbol{B}) = \arg\max_{x \in X} Supp_{\boldsymbol{w}}(\boldsymbol{B}, x)$$

The following proposition states that the scoring rules that satisfy Congruity are exactly the non-negative scoring rules. The proof of the proposition serves as an illustration of how Congruity forces weights to be non-negative. The property requires that no losing alternative can become winning by extending the profile with any set of ballots not including the losing alternative. In other words, no group of voters can make an already losing alternative win by down-voting every other alternative. Now, for any voting rule parameterized by a weight vector containing some non-trivial negative weight, we can construct a voting scenario that defies congruity by defining a sufficiently large set of voters all casting equal-sized negatively valued votes, assigning alternatives to ballots appropriately. This is possible for any case in which there is a negative weight associated with a non-trivial ballot size, and so the property boils down to making it impossible for any ballot to give negative support.

Proposition 3.1.8. A scoring rule f satisfies Congruity if and only if it is a non-negative scoring rule.

Proof. ($\Leftarrow$) Let f be a non-negative scoring rule. Consider an arbitrary profile $\boldsymbol{B}$ and let $A, C \subseteq N$ be disjoint. Suppose that, for some $x \in X$, $x \notin f(\boldsymbol{B}_A)$ and $x \notin B_i$ for $i \in C$. Since $x \notin f(\boldsymbol{B}_A)$, there is some $y \in X$ such that $Supp_{\boldsymbol{w}}(\boldsymbol{B}_A, y) > Supp_{\boldsymbol{w}}(\boldsymbol{B}_A, x)$. Now, since $w_k \geq 0$ for all $k < m$ and $x \notin B_i$

for any $i \in C$, we must have that $Supp_{w_k}(\boldsymbol{B}_C, y) \geq Supp_{w_k}(\boldsymbol{B}_C, x)$ for all $k \leq m$, and so $Supp_{\boldsymbol{w}}(\boldsymbol{B}_C, y) \geq Supp_{\boldsymbol{w}}(\boldsymbol{B}_C, x)$. Thus, $x \notin f(\boldsymbol{B}_A + \boldsymbol{B}_C)$.

($\Rightarrow$) By contraposition. Let X be a set of alternatives with $|X| = m \geq 2$, and let f be a scoring rule for which $w_k < 0$ for some $k < m$. I will show that f does not respect Congruity. The crucial idea is that if all voters vote with ballots of size k, we can make sure that all alternatives are down-voted. Consider a set of voters $N = \{1, ..., n\}$ with $n = \lceil \frac{m}{k} \rceil$, and let $\boldsymbol{B}$ be a profile where $|B_i| = k$ for all $i \in N$. Observe that $\lceil \frac{m}{k} \rceil$ is exactly the number of voters needed for us to be able to place every alternative on some ballot when all ballots are of size k. Now, fix an alternative $x \in X$, and suppose that $x \in B_1$. Since $|B_1| = k$, it follows that $x \notin f(\boldsymbol{B}_{\{1\}})$. Further suppose that $x \notin B_i$, for all remaining voters $i \in N \setminus \{1\}$, and for each alternative $y \in X \setminus \{x\}$, suppose that $y \in B_i$ for some $i \in N$. Then we have that $Supp_{w_k}(\boldsymbol{B}, x) \geq Supp_{w_k}(\boldsymbol{B}, y)$ for all $y \in X$. Since $|B_i| = k$ for all $i \in N$, it follows that $x \in f(\boldsymbol{B}) = f(\boldsymbol{B}_{\{1\}} + \boldsymbol{B}_{N \setminus \{1\}})$. However, since $x \notin f(\boldsymbol{B}_{\{1\}})$ and $x \notin B_i$ for any $i \in N \setminus \{1\}$, we would have by Congruity that $x \notin f(\boldsymbol{B})$. Thus, f does not respect Congruity. $\square$

Could we express a property that excludes the possibility of negative weights in more simple terms? One property of relevance is that of faithfulness. As the name suggests, the faithfulness property requires that a voting rule exhibits some basic level of faithfulness to the votes of an electorate. Specifically, a voting rule is faithful if, in any voting scenario where the electorate consists of a single voter, the voting rule chooses the ballot of the single voter as winning. However basic, this property is still somewhat too strong for the given purposes, as it would exclude any rule that gives value zero to some ballot of size strictly smaller than that of the full set of alternatives. On the other hand, for any electorate consisting of a single voter, we should expect any non-negative scoring rule to produce an output that includes the ballot of the single voter. Conversely, if a scoring rule applied to any single-voter profile does include the ballot of the single voter in its output, no ballot could possibly be valued negatively. Therefore, consider the following weakened version of the faithfulness property.

Axiom (Weak Faithfulness)**.** For any single-voter profile $\boldsymbol{B} = (B_1)$, we have $B_1 \subseteq f(\boldsymbol{B})$.

It is easy to verify that the scoring rules that satisfy the weak faithfulness property are exactly the non-negative scoring rules.

Proposition 3.1.9. A scoring rule f satisfies Weak Faithfulness if and only if it is a non-negative scoring rule.

Proof. ($\Leftarrow$) Let f be a non-negative scoring rule and let $\boldsymbol{B} = (B_1)$. Since $w_k \geq 0$ for all $k < m$, $Supp_{\boldsymbol{w}}(\boldsymbol{B}, x) \geq Supp_{\boldsymbol{w}}(\boldsymbol{B}, y)$ for all $x \in B_1$ and all $y \in X$, and so $B_1 \subseteq f(\boldsymbol{B})$.

($\Rightarrow$) Let f be a scoring rule with $w_k < 0$ for some $k < m$. If $\boldsymbol{B} = (B_1)$ and $|B_1| = k$, then $f(\boldsymbol{B}) = X \setminus B_1$, and so $B_1 \not\subseteq f(\boldsymbol{B})$. Thus, f does not satisfy Weak Faithfulness. □

The reason that we can obtain the class of non-negative scoring rules by imposing such a weak axiom as Weak Faithfulness is the following. Whenever a voting procedure f is neutral and we have a profile $\boldsymbol{B}$ containing only a single individual i, we must have that $f(\boldsymbol{B}) \in \{\{B_i\}, \{X \setminus B_i\}, \{X\}\}$. If we then impose Weak Faithfulness, by definition of the axiom, we cannot have that $f(\boldsymbol{B}) = X \setminus B_i$. The consequence of this fact for the class of scoring rules is that no ballot size can be associated with a negative weight. Note also that if we require Faithfulness in its regular form, we obtain the class of those scoring rules for which all non-trivial weights are positive.

In this subsection, we showed that the scoring rules that satisfy Congruity are exactly those that assign non-negative weights to all ballot sizes, thus establishing that it is sufficient to impose Congruity to exclude rules with negative weights. We also observed that we can obtain the class of non-negative scoring rules by use of the axiom Weak Faithfulness instead of Congruity. We now move on to investigating the axiom of Contraction.

Contraction and weakly decreasing weights

Let us now consider the axiom of Contraction. As was noted previously, this axiom plays a dual role in the characterization by Alcalde-Unzu and Vorsatz [2009]. Consider again the definition of the axiom, as given in Chapter 2:

Axiom (Contraction)**.** For all profiles $\boldsymbol{B}$, $\boldsymbol{B'}$ and all sets of voters $A \subseteq N$ such that $B'_i \subset B_i$ for a single $i \in A$ and $B'_j = B_j$ for all $j \in A \setminus \{i\}$, if $f(\boldsymbol{B}_A) \cap B'_i \neq \emptyset$, we have that

$$B'_i \cap f(\boldsymbol{B}_A) \subseteq f(\boldsymbol{B'}_A) \subseteq f(\boldsymbol{B}_A).$$

To get a better understanding of how the axiom works, it might be convenient to consider the two conditions imposed by the axiom of Contraction, that is, $B'_i \cap f(\boldsymbol{B}_A) \subseteq f(\boldsymbol{B'}_A)$ and $f(\boldsymbol{B'}_A) \subseteq f(\boldsymbol{B}_A)$, separately. I will refer to the properties that we obtain by separating the conditions of Contraction as Contraction 1 and Contraction 2.

Axiom (Contraction 1)**.** For all profiles $\boldsymbol{B}$, $\boldsymbol{B'}$ and all sets of voters $A \subseteq N$ such that $B'_i \subset B_i$ for a single $i \in A$ and $B'_j = B_j$ for all $j \in A \setminus \{i\}$, if $f(\boldsymbol{B}_A) \cap B'_i \neq \emptyset$, we have that

$$B'_i \cap f(\boldsymbol{B}_A) \subseteq f(\boldsymbol{B'}_A).$$

Axiom (Contraction 2)**.** For all profiles $\boldsymbol{B}$, $\boldsymbol{B'}$ and all sets of voters $A \subseteq N$ such that $B'_i \subset B_i$ for a single $i \in A$ and $B'_j = B_j$ for all $j \in A \setminus \{i\}$, if $f(\boldsymbol{B}_A) \cap B'_i \neq \emptyset$, we have that

$$f(\boldsymbol{B'}_A) \subseteq f(\boldsymbol{B}_A).$$

It turns out that if we require a scoring rule to satisfy the property Contraction 1, weights must be weakly decreasing in the size of ballots.

Proposition 3.1.10. A scoring rule f with weight vector $\boldsymbol{w} = (w_1, ..., w_m)$ satisfies Contraction 1 if and only if $w_k \leq w_{k-1}$ for all $1 < k < m$.

Proof. ($\Leftarrow$) Let f be a scoring rule such that $w_k \leq w_{k-1}$ for all $1 < k < m$. Let $\boldsymbol{B}, \boldsymbol{B'}$ be profiles and $A \subseteq N$ a subset of voters such that $B'_i \subset B_i$ for a single $i \in A$ and $B'_j = B_j$ for all $j \in A \setminus \{i\}$, and let $x \in B'_i \cap f(\boldsymbol{B}_A)$. Since $x \in f(\boldsymbol{B}_A)$, we have that $x \in \arg\max_{y \in X} Supp_{\boldsymbol{w}}(\boldsymbol{B}_A, y)$. Now, from $B'_i \subset B_i$, we get that $w_{|B'_i|} \geq w_{|B_i|}$. Thus, since $x \in B'_i$, it follows that $Supp_{\boldsymbol{w}}(\boldsymbol{B'}_A, x) \geq Supp_{\boldsymbol{w}}(\boldsymbol{B}_A, x)$. All else being equal, we must have that $x \in \arg\max_{y \in X} Supp_{\boldsymbol{w}}(\boldsymbol{B'}_A, y)$. Thus, $x \in f(\boldsymbol{B'}_A)$.

($\Rightarrow$) By contraposition. Let X be a set of alternatives with $|X| = m$ and

f a scoring rule with $w_k > w_{k-1}$ for some $k < m$. I will show that we can construct a voting scenario for which f does not respect Contraction 1. Let $|N| = m$ and define $\boldsymbol{B}$ such that $|B_i| = k$ for all $i \in N$ and $Supp_{w_k}(\boldsymbol{B}, x) = k$ for all $x \in X$. By construction $f(\boldsymbol{B}) = X$, and for each $x \in X$ there is some $i \in N$ such that $x \notin B_i$. Let now $\boldsymbol{B}'$ be the profile obtained from $\boldsymbol{B}$ by letting $B'_1 = B_1 \setminus \{x\}$ for some $x \in B_1$, while $B'_j = B_j$ for all $j \in N \setminus \{1\}$. Note that since $k > 1$, we must have that $B'_1 \neq \emptyset$. Fix some alternative $y \in B'_1$. By construction, $y \in f(\boldsymbol{B}) \cap B'_1$. Now, since $|B_1| = k$ and $|B'_1| = k - 1$, we have that $w_{|B'_1|} < w_{|B_1|}$, and so $Supp_{\boldsymbol{w}}(\boldsymbol{B}', y) < Supp_{\boldsymbol{w}}(\boldsymbol{B}, y)$. Now, observe that since $|B_j| = k < m$ for all $j \in N$, we have that $B_j \subset X$ for each $j \in N$, so there is some alternative $z \in X$ such that $Supp_{\boldsymbol{w}}(\boldsymbol{B}', z) = Supp_{\boldsymbol{w}}(\boldsymbol{B}, z) = Supp_{\boldsymbol{w}}(\boldsymbol{B}, y) > Supp_{\boldsymbol{w}}(\boldsymbol{B}', y)$. Thus, $y \notin f(\boldsymbol{B}')$. But $y \in f(\boldsymbol{B}) \cap B'_1$, so $f(\boldsymbol{B}) \cap B'_1 \not\subseteq f(\boldsymbol{B}')$. Hence, f does not respect Contraction 1. □

It furthermore turns out that, in the context of scoring rules, Contraction 1 implies Contraction 2.

Proposition 3.1.11. If f is a scoring rule and f satisfies Contraction 1, then f also satisfies Contraction 2.

Proof. Let f be a scoring rule and suppose that for all $\boldsymbol{B}, \boldsymbol{B}'$ and all $A \subseteq N$ such $B'_i \subset B_i$ for a single $i \in A$, $B'_j = B_j$ for all $j \in A \setminus \{i\}$, and $f(\boldsymbol{B}_A) \cap B'_i \neq \emptyset$, we have $f(\boldsymbol{B}_A) \cap B'_i \subseteq f(\boldsymbol{B}'_A)$. We want to show that $f(\boldsymbol{B}'_A) \subseteq f(\boldsymbol{B}_A)$. Suppose for contradiction that $x \in f(\boldsymbol{B}'_A)$ and $x \notin f(\boldsymbol{B}_A)$ for some $x \in X$. By assumption, $f(\boldsymbol{B}_A) \cap B'_i \neq \emptyset$, and so there is some $y \in B'_i$ such that $Supp_{\boldsymbol{w}}(\boldsymbol{B}_A, y) > Supp_{\boldsymbol{w}}(\boldsymbol{B}_A, x)$. Suppose first that $x \in B'_i$. Then $Supp_{\boldsymbol{w}}(\boldsymbol{B}'_A, x) - Supp_{\boldsymbol{w}}(\boldsymbol{B}_A, x) = Supp_{\boldsymbol{w}}(\boldsymbol{B}'_A, y) - Supp_{\boldsymbol{w}}(\boldsymbol{B}_A, y)$, i.e., y and x must have experienced the same increase in weighted support from i restricting their ballot. Secondly, suppose that $x \notin B'_i$. Then, everything else being equal, $Supp_{\boldsymbol{w}}(\boldsymbol{B}'_A, x) \leq Supp_{\boldsymbol{w}}(\boldsymbol{B}_A, x)$, i.e., x has not gained support in $\boldsymbol{B}'$. In both cases, it follows that $Supp_{\boldsymbol{w}}(\boldsymbol{B}'_A, y) > Supp_{\boldsymbol{w}}(\boldsymbol{B}'_A, x)$, which contradicts the assumption that $x \in f(\boldsymbol{B}'_A)$. Thus, $f(\boldsymbol{B}'_A) \subseteq f(\boldsymbol{B}_A)$. □

For the remainder of the thesis, I will generally not consider voting rules that do not belong to the class of scoring rules over approval ballots, and so it will be sufficient to work with with the property Contraction 1 in place of Contraction in its full form.

Therefore, I will from here on work with the simplified version of Contraction given by Contraction 1.

Axiom (Contraction)**.** For all profiles $\boldsymbol{B}$, $\boldsymbol{B}'$ and all sets of voters $A \subseteq N$ such that $B_i' \subset B_i$ for a single $i \in A$ and $B_j' = B_j$ for all $j \in A \setminus \{i\}$, if $f(\boldsymbol{B}_A) \cap B_i' \neq \emptyset$, we have that

$$B_i' \cap f(\boldsymbol{B}_A) \subseteq f(\boldsymbol{B}_A').$$

I will now summarize the sections so far by giving possible characterizations of the class of Size Approval Voting that follow from the results we have seen.

Size Approval Voting Rules

So far in this section, we have looked at axioms that have previously been used to characterize the class of Size Approval Voting rules, axiomatically related the class to its superclass of scoring rules, and considered whether certain axioms used to characterize Size Approval Voting rules can be simplified or exchanged for other axioms. In this subsection, I conclude the section by giving several characterizations of Size Approval Voting rules that follow directly from results previously given.

Theorem 3.1.12. A voting rule f satisfies each of the following combinations of axioms if and only if f is a Size-Approval Voting rule:

(1) Anonymity, Neutrality, Reinforcement, Overwhelming Majority, Congruity, and Contraction.

(2) Anonymity, Neutrality, Reinforcement, Overwhelming Majority, Weak Faithfulness, and Contraction.

(3) Anonymity, Neutrality, Reinforcement, Continuity, Congruity, and Contraction.

(4) Anonymity, Neutrality, Reinforcement, Continuity, Weak Faithfulness and Contraction.

Proof. I only show (4), the other cases are similar. Let f be a voting rule. By Theorem 3.1.2 due to Fishburn [1979], the rule f satisfies Anonymity, Neutrality, Reinforcement and Overwhelming Majority if and only if f is a scoring rule. By Proposition 3.1.3, it follows that f satisfies Anonymity, Neutrality, Reinforcement and Continuity if and only if f is a scoring rule. By Proposition 3.1.9, a f satisifes Weak Faithfulness if and only if f is a non-negative scoring rule. Finally, by Proposition 3.1.10, a f satisfies Contraction if and only if weights are weakly decreasing in the size of ballots. $\square$

Let us now shift our focus from questions about the area surrounding the class of Size Approval Voting rules, to issues concerning the landscape inside the class. In the next section, I look more closely at subclasses of Size Approval Voting rules and their members.

3.2 Classes of Size Approval Voting Rules

This section is dedicated to a closer inspection of two notable subclasses of Size Approval Voting rules and some important rules in these classes. In particular, I look at how to obtain the subclass of Size Approval Voting rules called p-approval rules, which are those rules that allow voters to approve of any number of candidates up to some fixed value p, and the subclass of Size Approval Voting rules with weights that are strictly decreasing in the size of ballots. These are arguably subclasses that contain the most natural of the Size Approval Voting rules. I mainly consider how to obtain these classes from the class of Size Approval Voting rules. In addition, I give a characterization of the rule Even and Equal Cumulative Voting (EECV), belonging to the subclass of Size Approval Voting rules with strictly decreasing weights.

3.2.1 p-Approval Rules

Let us start by considering the class of p-approval rules. For rules in this class, all ballots of sizes up to some fixed p are awarded some identical non-negative score, and all ballots of size larger than p are discarded. Most notably, this subclass contains the rules Approval Voting and plurality voting, parameterized by the weight vectors $\boldsymbol{w} = (1, 1, ..., 1)$ and $\boldsymbol{w} = (1, 0, ..., 0)$, respectively. These two rules are arguably the most well-known of the Size Approval Voting rules, and have both been studied extensively from many angles. Naturally, the p-approval class also includes all restricted approval rules that allow voters to approve of more than a single alternative, but fewer than all alternatives. I start with giving a formal definition of the class of p-approval rules.

Definition 3.2.1 (p-approval rule)**.** A Size-Approval rule f is a p-approval rule if there exists a $p \leq m$ such that $w_k = w_\ell > 0$ for all $k, \ell \leq p$, and $w_k = 0$ for all $k > p$.

We are primarily interested in determining how these rules are similar. One point worth stressing is that these rules may behave quite differently despite their structural similarity. This point is perhaps best motivated in this context by considering some differences between the plurality rule and Approval Voting. As noted in the introduction, Approval Voting avoids a problem observed in elections under plurality, known as the wasted vote problem, where voter may have to decide between 'wasting' their vote on their most-preferred alternative and voting for another less-preferred alternative that has a better chance of winning the election [Brams and Fishburn, 1978, Merrill and Nagel, 1987]. On the other hand, Approval Voting allows for the possibility of the electorate choosing as winning some alternative, sometimes called the lowest common denominator, which to many voters is not particularly interesting, since additionally voting for said alternative does not affect the support given to other alternatives on the same ballot [Alcalde-Unzu and Vorsatz, 2009].

Nevertheless, the structural similarity between the p-approval rules suggests some similarity in their behavior. One way to think of the class of p-approval rules is as those rules that behave like, or simulate, Approval Voting up to a certain point. This perspective will motivate the approach I take to obtain the class of p-approval rules from the class of Size Approval Voting rules. Recall the axiom of Disjoint Equality defined in Chapter 2.

Axiom (Disjoint Equality)**.** For any profile $\boldsymbol{B}$ and $i, j \in N$, $B_i \cap B_j = \emptyset$ implies

$$f(\boldsymbol{B}_{\{i,j\}}) = B_i \cup B_j.$$

The axiom of Disjoint Equality has been used in several characterizations of Approval Voting. In particular, Fishburn [1978] shows that Approval Voting is the only voting rule that satisfies Anonymity, Neutrality, Reinforcement and Disjoint Equality, and Brandl and Peters [2022] show that Approval Voting is the only voting rule that satisfies Reinforcement, Faithfulness and Disjoint Equality. Consider now the following property, called Bounded Disjoint Equality, which we can view as a restricted form of Disjoint Equality. The axiom of Bounded Disjoint Equality requires, roughly, that whenever we only consider the ballots of two voters in an election instance and those voters do not have any approved alternatives in common, the voting procedure should elect the choices of both voters as winning, as long as the number of alternatives voted for by each voter is bounded by some fixed $p \leq m$. Additionally, we will require that, whenever a voter submits a ballot of size larger than p, the voting procedure chooses all alternatives as winning.

Axiom (Bounded Disjoint Equality)**.** There exists a $p \leq m$ such that, for any profile $\boldsymbol{B}$ and all $i, j \in N$, $B_i \cap B_j = \emptyset$ and $|B_i|, |B_j| \leq p$ implies $f(\boldsymbol{B}_{\{i,j\}}) = B_i \cup B_j$, and $|B_i| > p$ implies $f(\boldsymbol{B}_{\{i\}}) = X$.

The following proposition show that the Size-Approval rules that satisfy Bounded Disjoint Equality correspond exactly to the subclass of p-approval rules.

Proposition 3.2.2. A Size-Approval rule f satisfies Bounded Disjoint Equality if and only if it is a p-approval rule.

Proof. ($\Leftarrow$) Let f be a p-approval rule. Then there is a $p \leq m$ such that $w_{k-1} = w_k > 0$ for all $k \leq p$ and $w_k = 0$ for all $k > p$. Fix this p. Let $i \in N$ and suppose that $|B_i| = k > p$. Since $k > p$, we have by definition that $w_k = 0$, and so $f(B_i) = X$. Secondly, suppose for some $i, j \in N$ that $|B_i|, |B_j| \leq p$ and $B_i \cap B_j = \emptyset$. Since $w_k = w_l > 0$ for all $k, l \leq p$, $Supp_w(\boldsymbol{B}_{\{i,j\}}, x) = Supp_w(\boldsymbol{B}_{\{i,j\}}, y) > Supp_w(\boldsymbol{B}_{\{i,j\}}, z)$ for all $x, y \in B_i \cup B_j$ and $z \notin B_i \cup B_j$. Thus $f(\boldsymbol{B}_{\{i,j\}}) = B_i \cup B_j$.

($\Rightarrow$) Let f be a Size-Approval rule that satisfies Bounded Disjoint Equality, and fix the p stated to exist by the property. Note that if f is the trivial rule for which $w_k = 0$, for all $k \leq m$, we can easily construct a voting scenario for which f defies Bounded Disjoint Equality, so assume that f is non-trivial. By Bounded Disjoint Equality, for all ballots B_i with $|B_i| > p$, we have that $f(\boldsymbol{B}_{\{i\}}) = X$. Thus, we can let $w_k = 0$ for all $k > p$. It remains to show that $w_k = w_l > 0$ for all $k, l \leq p$. Suppose that there is some $k \leq p$ such that $w_k = 0$. Assume without loss of generality that $k < m$. Since f is not the trivial rule, there is some $l \neq k$ such that $w_l > 0$. Since $w_l > w_k$, we must have $l < k$ by the fact that weights are weakly decreasing. If $l \neq 1$, it follows for the same reason that $w_1 \geq w_l > w_k$, so assume without loss of generality that $l = 1$. Let $i, j \in N$, and let $|B_i| = 1$ and $|B_j| = k$. Then $|B_i|, |B_j| \leq p$, but $f(\boldsymbol{B}_{\{i,j\}}) = B_i$, and so f does not satisfy Bounded Disjoint Equality. $\square$

Are there other approaches we could take to obtain the class of p-approval rules from the class of Size Approval Voting rules? Perhaps. For example, one approach could be to use a limited version of the axiom of Cancellation, saying that the outcome should correspond to the full set of alternatives whenever all alternatives are given the same number of votes and no voter abstains (that is, casts a ballot that awards the same amount of support to every alternative in the set of alternatives). Such a condition would intuitively be trivially fulfilled by the trivial rule, but this issue could be mediated by slightly strengthening the axiom of Weak Faithfulness to require that there exists some profile for which the collective outcome corresponds exactly to the vote of the single-voter electorate. However, such an approach may also rely on properties that turn out to be somewhat technical when formalized.

I will now move on to consider the subclass of Size Approval Voting rules with weights that strictly decrease with the size of ballots. This class will be referred to as the class of Strictly Decreasing Size Approval Voting Rules.

3.2.2 Strictly Decreasing Size Approval Voting Rules

Another natural subclass of Size Approval Voting rules is the subclass containing those rules for which weights are strictly decreasing in the size of ballots. I will call this subclass the class of Strictly Decreasing Size Approval Voting (SDSAV) rules. The most notable of these rules, and perhaps also the most natural, is the rule of Even and Equal Cumulative Voting (EECV), parameterized by the weight vector $\boldsymbol{w} = (w_1, ..., w_m)$ with $w_k = \frac{1}{k}$ for all $k \leq m$. To illustrate the more extreme points of the class, consider the two following rules:

- the rule $\boldsymbol{w} = (w_1, ..., w_m)$, with $w_1 = 1$ and $w_k = w_{k-1} - \epsilon^{k-1}$ for each $k > 1$ and some small $\epsilon > 0$, for which weights decrease infinitesimally with the size of ballots.
- the rule $\boldsymbol{w} = (w_1, ..., w_m)$ with $w_1 = 1$ and $w_k = \epsilon^k$ for each $k > 1$ and some

small $\epsilon > 0$, for which weights drastically decrease with the size of ballots.

In this section, I first discuss how to obtain the class of SDSAV rules from the class of Size Approval Voting rules, and thereafter look more closely at the rule EECV, providing a possible characterization of the rule.

Definition 3.2.3 (Strictly Decreasing Size-Approval Voting rule)**.** A Size Approval Voting rule f is a SDSAV rule if $w_k < w_{k-1}$ for all $1 < k < m$. In words, a SDSAV rule is a SAV rule with weights that are strictly decreasing in the size of ballots.

A simple way to obtain the class of SDSAV rules from the class of Size Approval Voting rules is through strengthening the axiom of Contraction. In this strengthened version of the Contraction axiom, we will require that whenever in some profile a single voter i restricts their ballot to one that includes some alternatives that were already winning, the winners in the profile obtained from i restricting their ballot is exactly the previously winning alternatives on i's new ballot. Intuitively, this property enables voters to sometimes break ties between their approved alternatives and alternatives they do not approve of, describing situations in which the choice of a single voter plays a pivotal role for the election outcome. I will refer to this strengthened version of the Contraction axiom as Pivotal Contraction, formally defined as follows.

Axiom (Pivotal Contraction)**.** For all profiles $\boldsymbol{B}, \boldsymbol{B}'$ and all sets of voters $A \subseteq N$ such that $B_i' \subset B_i$ for a single $i \in A$, $B_j = B_j'$ for all $j \in A \setminus \{i\}$ and $f(\boldsymbol{B}_A) \cap B_i' \neq \emptyset$, we have that $B_i' \cap f(\boldsymbol{B}_A) = f(\boldsymbol{B}_A')$.

Proposition 3.2.4. A SAV rule f satisfies Pivotal Contraction if and only if it is a SDSAV rule.

Proof. ($\Rightarrow$) Let f be a SDSAV rule. By definition, f is a Size Approval Voting rule, and so it is sufficient to show that f satsifies the additional property imposed by the axiom of Pivotal Contraction. Thus, let $\boldsymbol{B}, \boldsymbol{B}'$ be profiles and $A \subseteq N$ be a subset of the electorate such that $B_i' \subset B_i$ for a single $i \in A$ and $B_j' = B_j$ for all $j \in A \setminus \{i\}$, and let $f(\boldsymbol{B}_A) \cap B_i' \neq \emptyset$. The inclusion $f(\boldsymbol{B}_A') \supseteq f(\boldsymbol{B}_A) \cap B_i'$ follows from Conctraction, so I only show that $f(\boldsymbol{B}_A') \subseteq f(\boldsymbol{B}_A) \cap B_i'$. Contrapositively, suppose that $x \notin f(\boldsymbol{B}_A) \cap B_i'$. We want to show that $x \notin f(\boldsymbol{B}'_A)$. Since $f(\boldsymbol{B}_A) \cap B'_i \neq \emptyset$, there is some $y \in B'_i$ such that $Supp_{\boldsymbol{w}}(\boldsymbol{B}_A, y) \geq Supp_{\boldsymbol{w}}(\boldsymbol{B}_A, x')$ for all $x' \in X$. Suppose first that $x \notin f(\boldsymbol{B}_A)$. Then $Supp_{\boldsymbol{w}}(\boldsymbol{B}_A, y) > Supp_{\boldsymbol{w}}(\boldsymbol{B}_A, x)$. Since $y \in B'_i \subset B_i$ and $w_{|B'_i|} > w_{|B_i|}$, it follows that $Supp_{\boldsymbol{w}}(\boldsymbol{B}'_A, y) > Supp_{\boldsymbol{w}}(\boldsymbol{B}'_A, x)$ whether or not $x \in B'_i$, and so $x \notin f(\boldsymbol{B}'_A)$. Suppose on the other hand that $x \in f(\boldsymbol{B}_A)$. Since $x \notin f(\boldsymbol{B}_A) \cap B'_i$, it follows that $x \notin B'_i$. All else being equal, we have that $Supp_{\boldsymbol{w}}(\boldsymbol{B}_A, x) \geq Supp_{\boldsymbol{w}}(\boldsymbol{B}'_A, x)$. Now, $y \in f(\boldsymbol{B}_A) \cap B'_i$ and $B'_i \subset B_i$, from which it follows that $Supp_{\boldsymbol{w}}(\boldsymbol{B}'_A, y) > Supp_{\boldsymbol{w}}(\boldsymbol{B}_A, y) = Supp_{\boldsymbol{w}}(\boldsymbol{B}_A, x) \geq Supp_{\boldsymbol{w}}(\boldsymbol{B}'_A, x)$, and so $x \notin f(\boldsymbol{B}'_A)$. Hence, $f(\boldsymbol{B}'_A) \subseteq f(\boldsymbol{B}_A) \cap B'_i$.

($\Leftarrow$) Let f be a Size Approval Voting rule that satisfies Pivotal Contraction. We want to show that $w_k < w_{k-1}$ for all $1 < k < m$.

By contraposition, suppose that $w_k \geq w_{k-1}$ for some $1 < k < m$. Since f is a Size Approval Voting rule, f satisfies Contraction, and so it cannot be that $w_k > w_{k-1}$. Thus, assume that $w_k = w_{k-1}$. Consider a profile $\boldsymbol{B}$ with $|N| = m$ and $|B_i| = k$ for all $i \in N$. For each $x \in X$, let $Supp(\boldsymbol{B}, x) = k$. That is, let each alternative $x \in X$ be on k votes in $\boldsymbol{B}$. Note that this is possible since $|N| = m$. Since $|B_i| = k$ for all $i \in N$, it follows that $Supp_{\boldsymbol{w}}(\boldsymbol{B}, x) = Supp_{\boldsymbol{w}}(\boldsymbol{B}, y)$ for all $x, y \in X$, and thus $f(\boldsymbol{B}) = X$. Now, let $\boldsymbol{B}'$ be a profile obtained from $\boldsymbol{B}$ by letting a single voter remove a single alternative from their ballot. That is, let $B'_i = B_i \setminus \{y\}$ for some $y \in B_i$ and a single $i \in N$, let $B'_j = B_j$ for all $j \in N \setminus \{i\}$.

Observe that $B'_i \subset B_i$ with $|B'_i| = k-1$, and consequently that $Supp_{\boldsymbol{w}}(\boldsymbol{B}, y) > Supp_{\boldsymbol{w}}(\boldsymbol{B}', y)$. Further, since $f(\boldsymbol{B}) = X$, it follows that $f(\boldsymbol{B}) \cap B'_i = B'_i$. Now, since $w_{|B'_i|} = w_{|B_i|}$ while $B'_j = B_j$ for all $j \in N \setminus \{i\}$, we have that $Supp_{\boldsymbol{w}}(\boldsymbol{B}, x) = Supp_{\boldsymbol{w}}(\boldsymbol{B}', x)$ for all $x \in X$ such that $x \in B'_i$ or $x \notin B_i$. Thus, $f(\boldsymbol{B}') = X \setminus \{y\}$. Now, since $|B_i| = k$ for all $i \in N$, there is some $z \in X$ such that $z \in f(\boldsymbol{B}')$ and $z \notin B'_i = f(\boldsymbol{B}) \cap B'_i$, and so $f(\boldsymbol{B}') \not\subseteq f(\boldsymbol{B}) \cap B'_i$. Thus, f does not satisfy Pivotal Contraction.

$\square$

The axiom of Pivotal Contraction, like Contraction, is perhaps a somewhat technical property. However, the class of SDSAV rules allows for any rule with weights that are strictly decreasing in the size of ballots. Different rules in this class may incentivize very different voting behavior and yield very different outcomes. Compare, for example, the rules $\boldsymbol{w} = (w_1, ..., w_m)$, with $w_1 = 1$ and $w_k = w_{k-1} - \epsilon^{k-1}$ for each $k > 1$ and some small $\epsilon > 0$, for which weights decrease infinitesimally with the size of ballots, and the rules $\boldsymbol{w} = (w_1, ..., w_m)$ with $w_1 = 1$ and $w_k = \epsilon^k$ for each $k > 1$ and some small $\epsilon > 0$, for which weights drastically decrease with the size of ballots, mentioned previously. Under the first kind of rule, the support given to each alternative on a ballot barely reduces as the size of the ballot increases, and as such, voters sacrifice very little in approving of a large number of alternatives. For the second kind of rule, the opposite is true. Under such a rule, the sacrifice that comes with approving of several alternatives is great, as this makes the support given to each alternative miniscule. In this sense, these two kinds of rules are closer to Approval Voting and plurality voting, respectively, than they are to each other. As such, it not a given that there will be many natural properties shared by all of these rules which simultaneously distinguish them from others. A more serious objection is that the axiom of Pivotal Contraction may not be very desirable at all. For one, the property has some relation to bullet voting, an act of manipulation where a voter votes gives all their support to a single alternative that is not necessarily among their favourites, in an effort to bring about some particular election outcome. As we will see in later chapters, many Size Approval Voting rules, and in particular the SDSAV rules, incentivize bullet voting.

Let us now look more closely at the rule Even and Equal Cumulative Voting (EECV). To characterize this rule, I will take another approach than that of building on the characterization of the SDSAV class using Pivotal Contraction, instead using Faithfulness and an axiom more specific to EECV to obtain the rule from the class of Size Approval Voting rules.

Even and Equal Cumulative Voting

The most natural rule in the class of SDSAV rules is probably Even and Equal Cumulative Voting, given by weight vector $\boldsymbol{w} = (1, \frac{1}{2}, ..., \frac{1}{m})$. The rule is a cumulative voting rule[3] that allows voters to distribute one unit of points equally between any number

[3]Under cumulative voting rules, voters are given a fixed amount of points to distribute over some number of chosen candidates. See, e.g., Glasser [1959]

of alternatives. While not as prominent as the plurality rule and Approval Voting, the rule is used in some real-world elections [Alcalde-Unzu and Vorsatz, 2009]. The rule furthermore is the single-winner analogue of the rule Satisfaction Approval Voting originally defined by Brams and Kilgour [2010], used in the context of Multi-winner Elections.

One characteristic feature of EECV is that it can be considered to satisfy the 'one voter, one vote' principle, in the sense that each voter distributes exactly one unit of support across alternatives. That is, under EECV, voters have exactly the same voting power. However, it turns out to not be very straightforward to define a property of equal voting power in a general way that separates the influence of a voter on the election from the actual ballot of the voter.

What other defining characteristics of the rule EECV can be used to characterize the class? I will consider a property motivated by the following scenario. Consider a group of voters who have disjoint but mutually compatible favorite alternatives trying to determine how to cooperatively vote in an election. Under EECV, if each voter favors a single alternative, it should make no difference on the outcome whether each voter in the group votes for their favorite alternative separately or all voters in the group vote for the union of their favorite alternatives. I state this condition formally using the following axiom. This axiom is analogous to a version of the Splitting axiom used by Terzopoulou and Endriss [2019] to characterize the Even and Equal weight rule. I will therefore call the following property Splitting.

Axiom (Splitting)**.** For all $\boldsymbol{B}$ and all $1 < k \leq m$, we have

$$f(\boldsymbol{B} + (\{x_1\}, ..., \{x_k\})) = f(\boldsymbol{B} + (\{x_1, ..., x_k\})^k)$$

I will show that in the context of Size Approval Voting rules, the rule of EECV can be characterized by Faithfulness, and Splitting. Recall that faithfulness requires that, whenever the electorate only has a single voter, the voting procedure used must stay faithful to that voter's choices.

Axiom (Faithfulness)**.** For any single-voter profile $\boldsymbol{B} = (B_1)$, we have that $B_1 = f(\boldsymbol{B})$.

Intuitively, there are only two Size Approval Voting rules that satisfy the Splitting axiom, namely EECV and the trivial rule, represented by $\boldsymbol{w} = (0, ..., 0)$. The following lemma confirms this, showing that the only non-trivial Size Approval Voting rule that satisfies Splitting is EECV. The trivial rule clearly does not satisfy Faithfulness, thus it will follow that the only Size Approval Voting rule satisfying both of these axioms is EECV.

Lemma 3.2.5. If f is a SAV rule that satisfies Splitting, and f is not the trivial rule, then $w_k = \frac{1}{k}$ for all $1 < k < m$.

Proof. Suppose that f is a SAV rule that satisfies Splitting, and that f is not the trivial rule. It is to show that $w_k = \frac{1}{k}$ for all $1 < k < m$. Without loss of generality, we may assume that $w_1 = 1$. Let $\boldsymbol{B}$ be a profile and fix $1 < k < m$. Denote with $\boldsymbol{B}'_1$ the profile $(\{x_1\}, ..., \{x_k\})$ and with $\boldsymbol{B}'_k$ the profiles $(\{x_1, ..., x_k\})^k$, where $x_1, ..., x_k \in X$. I will show that $Supp_{\boldsymbol{w}}(\boldsymbol{B}+\boldsymbol{B}'_1, x) = Supp_{\boldsymbol{w}}(\boldsymbol{B}+\boldsymbol{B}'_2, x)$ for all $x \in X$, from which the claim will follow. Fix some alternative $x \in X$. Observe that if $x \notin \{x_1, ..., x_k\}$, it follows immediately that $Supp_{\boldsymbol{w}}(\boldsymbol{B}+\boldsymbol{B}'_1, x) = Supp_{\boldsymbol{w}}(\boldsymbol{B}+\boldsymbol{B}'_2, x)$. Therefore, suppose that $x \in \{x_1, ..., x_k\}$, and assume for contradiction that $Supp_{\boldsymbol{w}}(\boldsymbol{B} + \boldsymbol{B}'_1, x) \neq Supp_{\boldsymbol{w}}(\boldsymbol{B} + \boldsymbol{B}'_2, x)$. Since f is a Size Approval Rule, $w_k \leq w_{k-1}$ for all $1 < k < m$, so since $Supp_{\boldsymbol{w}}(\boldsymbol{B}, x) = Supp_{\boldsymbol{w}}(\boldsymbol{B}, x)$, it follows that $Supp_{\boldsymbol{w}}(\boldsymbol{B}'_1, x) > Supp_{\boldsymbol{w}}(\boldsymbol{B}'_2, x)$. If $Supp_{\boldsymbol{w}}(\boldsymbol{B}'_2, x') > Supp_{\boldsymbol{w}}(\boldsymbol{B}'_2, x)$ for some $x' \in \{x_1, ..., x_k\}$, let $f(\boldsymbol{B}) = X$. Then, $x \in f(\boldsymbol{B} + \boldsymbol{B}'_1)$, but $x \notin f(\boldsymbol{B} + \boldsymbol{B}'_2)$, since there is $x' \in \{x_1, ..., x_k\}$ such that $Supp_{\boldsymbol{w}}(\boldsymbol{B}'_2, x') > Supp_{\boldsymbol{w}}(\boldsymbol{B}'_2, x)$. Similarly, if there is some $x' \in \{x_1, ..., x_k\}$ such that $Supp_{\boldsymbol{w}}(\boldsymbol{B}'_2, x) > Supp_{\boldsymbol{w}}(\boldsymbol{B}'_2, x')$, then if $f(\boldsymbol{B}) = X$, it follows analogously that $x' \in f(\boldsymbol{B} + \boldsymbol{B}'_1)$ and $x' \notin f(\boldsymbol{B} + \boldsymbol{B}'_2)$. Thus, in both cases, $f(\boldsymbol{B} + \boldsymbol{B}'_1) \neq f(\boldsymbol{B} + \boldsymbol{B}'_2)$, which contradicts Splitting. Thus, suppose that $Supp_{\boldsymbol{w}}(\boldsymbol{B}'_2, x) = Supp_{\boldsymbol{w}}(\boldsymbol{B}'_2, x')$ for all $x' \in \{x_1, ..., x_k\}$. Now, let $f(\boldsymbol{B}) = X \setminus \{x_1, ..., x_k\}$ and $Supp_{\boldsymbol{w}}(\boldsymbol{B}, y) - Supp_{\boldsymbol{w}}(\boldsymbol{B}, x') = 1$ for all $y \in X \setminus \{x_1, ..., x_k\}$ and all $x' \in \{x_1, ..., x_k\}$. It follows that $f(\boldsymbol{B} + \boldsymbol{B}'_1) = X$ and that $f(\boldsymbol{B}) = X \setminus \{x_1, ..., x_k\}$, which contradicts Splitting. Thus, we must have that $Supp_{\boldsymbol{w}}(\boldsymbol{B} + \boldsymbol{B}'_1, x) = Supp_{\boldsymbol{w}}(\boldsymbol{B} + \boldsymbol{B}'_2, x)$ for all $x \in \{x_1, ..., x_k\}$. Consequently, since $Supp_{\boldsymbol{w}}(\boldsymbol{B}, x) = Supp_{\boldsymbol{w}}(\boldsymbol{B}, x)$ for all $x \in X$ and $w_1 = 1$, we must have that $w_k = \frac{1}{k}$. Since $1 < k < m$ was arbitrary, it follows that $w_k = \frac{1}{k}$ for all $1 < k < m$. □

The following proposition states that EECV can be characterized by Faithfulness and Splitting, together with the axioms used to characterize Size Approval Voting rules.

Proposition 3.2.6. A Size Approval Voting rule f satisfies Faithfulness and Splitting if and only if it is the rule EECV.

Proof. ($\Leftarrow$) First, observe that EECV clearly satisfies Faithfulness: Consider any profile $\boldsymbol{B}$ with $|N| = 1$. Assume without loss of generality that $|B_1| < m$. Since $w_{|B_1|} = \frac{1}{|B_i|}$, $Supp_{\boldsymbol{w}}(\boldsymbol{B}, x) = \frac{1}{|B_1|}$ for all $x \in B_1$, and $Supp_{\boldsymbol{w}}(\boldsymbol{B}, y) = 0$ for all $y \notin B_1$. Thus, $f(\boldsymbol{B}) = B_1$.

Secondly, I show that EECV satisfies Splitting. Let f be such that $\boldsymbol{w} = (1, \frac{1}{2}, ..., \frac{1}{m})$, and consider two profiles $\boldsymbol{B} + (\{x_1\}, ..., \{x_k\})$, $\boldsymbol{B} + (\{x_1, ..., x_k\})^k)$. Suppose that $x \in f(\boldsymbol{B} + (\{x_1\}, ..., \{x_k\}))$. We want to show that $x \in f(\boldsymbol{B} + (\{x_1, ..., x_k\})^k))$. If $x \neq x_i$ for any $i \leq k$, then the claim holds trivially, so suppose that $x = x_i$ for some $i \leq k$. It is sufficient to show that

$$Supp_{\boldsymbol{w}}((\{x_1\}, ..., \{x_k\}), x) = Supp_{\boldsymbol{w}}((\{x_1, ..., x_k\})^k), x).$$

However, by definition of EECV, $\boldsymbol{w} = (1, \frac{1}{2}, ..., \frac{1}{m})$, so then it is clear that $1 = Supp_{\boldsymbol{w}}((\{x_1\}, ..., \{x_k\}), x) = Supp_{\boldsymbol{w}}((\{x_1, ..., x_k\})^k), x) = \frac{1}{k} \cdot k = 1$. Thus, $x \in f(\boldsymbol{B} + (\{x_1, ..., x_k\})^k))$. The converse direction is analogous.

($\Rightarrow$) By Lemma 3.2.5, if f satisfies Splitting and is not the trivial rule, then f must correspond to EECV, so it suffices to show that the trivial rule does not satisfy Faithfulness. This is immediate: let $\boldsymbol{w} = (0, ..., 0)$, and consider any profile $\boldsymbol{B}$ with $|N| = 1$, and suppose that $|B_1| < m$. Since $w_{|B_1|} = 0$ regardless of $|B_1|$, $f(\boldsymbol{B}) = X$. However, by Faithfulness, we should have that $f(\boldsymbol{B}) = B_1 \subset X$. $\square$

We may observe that that any non-trivial Size Approval Voting rule that satisfies the Splitting axiom also satisfies the axiom of Contraction. Since Faithfulness clearly implies Weak Faithfulness, it follows that EECV is the only voting rule satisfying the axioms characterizing scoring rules together with Faithfulness and Splitting.

Corollary 3.2.7. A voting rule f satisfies Anonymity, Neutrality, Reinforcement, Continuity, Faithfulness and Splitting if and only if f is the rule EECV.

3.3 Discussion

In this chapter, I have considered the class of Size Approval Voting rules from an axiomatic perspective. I began by considering axioms that govern the class of Size Approval Voting rules, axiomatically relating the class to its superclass of scoring rules, and studying the role of the axioms of Congruity and Contraction, used by in axiomatic characterization of the Size Approval Voting rules due to Alcalde-Unzu and Vorsatz [2009]. I showed that we can replace Congruity by a weakened form of Faithfulness, and that we can simplify the axiom of Contraction for the purposes of characterizing the class. Subsequently, I looked at two of the most natural subclasses of the class, namely the p-approval and the Strictly Decreasing Size Approval rules. I proposed properties that can be used to obtain these classes from the class of Size Approval Voting rules. Finally, I showed that the rule Even and Equal Cumulative Voting, belonging to the class of Strictly Decreasing Size Approval rules, can be characterized by the axioms of Anonymity, Neutrality, Reinforcement, Continuity, Faithfulness, and Splitting.

As briefly discussed, some axioms suggested in this section, in particular axioms used to capture the two subclasses of Size Approval Rules considered, are somewhat artificial. The primary defense for using these axioms is that both subclasses, while on the surface being natural groups of rules, are composed of rules that may behave quite differently from a normative point of view. As such, properties used to capture such subclasses may end up being more on the technical side. However, there may still be less artificial axioms yet to be considered that could be used to capture the classes we have considered.

Chapter 4

Strategic Behavior

This chapter considers the class of Size Approval Voting rules from the point of view of strategic behavior, studying the vulnerability of Size Approval Voting rules with respect to manipulation by voters.

The issue of manipulability of voting systems is central to Voting Theory. There are many situations in which voters (or other agents with some role in the election) may seek to manipulate so as to produce an outcome that is to their benefit. Manipulation can be detrimental to the decision-making process in several ways. I give some examples of potential adverse consequences of manipulation. For one, the legitimacy of a voting rule that can be manipulated may be questioned, and the use of such a voting rule may reduce voter turnout. Furthermore, if enough voters part-taking in the election attempt to manipulate, we may end up with an outcome that is far from being in line with the preferences of the electorate. Finally, a system that allows for manipulation puts an unnecessarily large burden on voters with respect to deciding how they can best act in their own self-interest: in such a system, voters may have to spend a lot of effort reasoning about the actions of others to be able to determine their own best choice.

The simplest manipulation scenario is that of a single voter submitting a vote that is not in line with their true preferences in an effort to produce an election outcome that is to their benefit. We call this the problem of strategic manipulation, and we say that a rule is strategyproof if it is immune to strategic manipulation [Zwicker, 2016]. Working with rules that are strategyproof is clearly the ideal – if a rule is strategyproof, it will always be in the best interest of voters to vote in line with their true preferences, and so we can avoid the adverse effects of manipulation on the decision-making process. Unfortunately, in the general case, this ideal is out of reach for all natural voting rules, as illustrated by results such as those by Gibbard [1973], Satterthwaite [1975] and Duggan and Schwartz [1999], showing that only dictatorships are strategyproof.

Nevertheless, despite no natural rule being immune to manipulation in the most general case, the are special cases for which some rules become strategyproof. For example, Brams and Fishburn [1978] showed that Approval Voting is strategyproof when voters have true preferences that are dichotomous, and Black [1948] showed that a rule called the median rule is strategyproof when the preferences of voters satisfy a property known as single-peakedness. Furthermore, for some voting rules, the problem of determining whether there exists a successful manipulation tactic is computationally intractable [Bartholdi et al., 1989a, Xia et al., 2009]. In addition, many results concerning the strategic vulnerability of voting rules rely on the assumption that voters have full information about the given election, having access to the ballot of every other voter. It is however rarely the case in practice that voters have access to all information about the choices of other members of the electorate. The problem of manipulability of voting rules under partial information has previously been studied by, e.g., Conitzer et al. [2011], Reijngoud and Endriss [2012], and Endriss et al. [2016], and in particular, some of the results from these works show that it is sometimes possible to guard against manipulation by hiding information from prospective manipulators.

In this chapter, I study the class of Size Approval Voting rules from a strategic point of view. I first describe the problem of relating the underlying preferences of voters to approval ballots and discuss the strategic vulnerability of rules in the class with respect the notion of sincerity defined by Brams and Fishburn [1978] and the notion of strong insincerity, due to Merrill and Nagel [1987]. Subsequently, I consider the computational complexity of strategic manipulation for Size Approval Voting rules, showing that the problem of determining whether there exists a successful manipulation tactic is trivial for all rules in the class. Thereafter, I discuss vulnerability of Size Approval Voting rules to manipulation under partial information, based on a model of partial information developed by Conitzer et al. [2011] and Reijngoud and Endriss [2012].

4.1 Underlying Preferences and Sincerity

To be able to discuss strategic behavior, we first need to represent the underlying preferences of voters. Given a set X of alternatives, let $\succeq_i$ be i's preference order over alternatives in X. That is, $\succeq_i$ is a weak total order on the set X, where $x \succeq_i y$ expresses that i weakly prefers alternative x to alternative y. Let $x \succ_i y$ be shorthand for $x \succeq_i y$ and $y \not\succeq_i x$, expressing that i strictly prefers x to y. Furthermore, let $x \simeq_i y$ be shorthand for $x \succeq_i y$ and $y \succeq_i x$, expressing i is indifferent between x and y.

In the traditional Voting Theory setting, where ballots are represented as rankings over the set of alternatives, we say that the ballot cast by a voter is truthful if the ballot equals the true preferences of the voter. In this case, a voting rule f is said to be strategyproof if there exists no voting scenario in which some voter can benefit from reporting a preference order that is not truthful. Crucially, when working with approval ballots, the above notion of truthfulness, and, as such, of strategyproofness, no longer applies. The reason is the following: in the context of approval balloting, there are 2^m possible ballots that can be cast over a set of alternatives of size m, but a whole $m!$ possible rankings over the set of alternatives. Therefore, we can not obtain underlying preferences of voters from approval ballots. Instead, the standard notion when considering strategic behavior under rules that use approval ballots is sincerity, due to Brams and Fishburn [1978], which roughly requires that voters never approve of some alternative x without also approving of every alternative y preferred to x, and we say that a choice that does not meet this requirement in insincere. In addition to using this notion to evaluate the strategic vulnerability of Size Approval Voting rule, I will also make use of the notion of strong insincerity, due to Merrill and Nagel [1987]. Roughly, the ballot of a voter is strongly insincere if the voter approves of some alternative, but not does not approve of any of their most-preferred alternatives.

In this chapter, it will be convenient to distinguish various Size Approval Voting rules based on which ballots are legal under any given rule. For a Size Approval Voting rule f, ballot B is legal if $w_{|B|} > 0$, and I will say that two rules have the same balloting system if they have the same legal ballots. That is, two Size Approval Voting rules $f_{\boldsymbol{w}}, f_{\boldsymbol{w}'}$ have the same balloting system if $w_k > 0$ whenever $w_{k'} > 0$. For any election instance and any voter $i \in N$, a strategy is any ballot $B_i \in 2^X$. I will consider the set of feasible strategies for any voter to consist of all legal ballots, and of a single abstention ballot if the set of legal ballots does not already include a vote for all alternatives. If the set of legal ballots already includes a vote for all alternatives, we will consider this vote to be an abstention. That is, for any Size Approval Voting rule f, the set of feasible strategies for any given $i \in N$ consists of all ballots B_i for which $w_{|B_i|} > 0$ and a single B_i for which $w_{|B_i|} = 0$, and we require that $w_m = 0$. Note that in general, any two rules with the same balloting system will have the same number of feasible strategies. To illustrate with some examples of feasible strategies under various Size Approval Voting rules, voters have

- $m + 1$ feasible strategies under plurality voting, namely all singleton ballots and abstention;
- $2^m - 1$ feasible strategies under Approval Voting and Even and Equal Cumulative Voting, corresponding to the number of non-empty subsets of the set of alternatives.
- one feasible strategy under the trivial rule $\boldsymbol{w} = (0, ..., 0)$.

In this chapter, I will mainly be concerned with the feasible strategies of voters, and so, when no confusion arises from it, I will simply refer to feasible strategies as strategies. Given a set of (feasible) strategies, it is in the best interest of the voter to choose a strategy that produces the best possible outcome according to the preferences of the voter. In our context, the collective outcome under any voting rule is a non-empty subset of the set of alternatives. In general, it is not possible to determine the preferences a voter i has over subsets of the alternatives based only on the preferences i has over individual alternatives. In particular, the preferences of voter i over subsets of the alternatives will not only depend on their preferences over individual alternatives, but also on i's beliefs about how ties are broken between co-winning alternatives [Endriss, 2013]. However, I will make some mild assumptions about how the the preferences of voters over individual alternatives extend to preferences over subsets of alternatives, imposing the following axiom due to Gärdenfors [1976], commonly referred to as the Gärdenfors principle. Roughly, the axiom states that a subset of the alternatives is strictly preferred to another subset if it can be obtained from the other subset by adding an alternative that is considered better than all the other alternatives in the subset or by removing an alternative that is considered to worse than all other alternatives in the subset. Note that the property is equivalent to the assumptions made about preference extensions by Brams and Fishburn [1978].

Gärdenfors principle. Let $i \in N$, $x \in X$ and $Y \subseteq X$. We then have that

- $Y \cup \{x\} \succ_i Y$ if $x \succ_i y$ for all $y \in Y$, and
- $Y \succ_i Y \cup \{x\}$ if $y \succ_i x$ for all $y \in Y$.

For convenience, it is also worth mentioning that the following axioms are known to be implied by the Gärdenfors principle [Endriss, 2013]. The first axiom, Extension, states that the preferences any voter has over singleton sets directly correspond to the preferences the voter has over individual alternatives. The second axiom states that the singleton set containing the most preferred alternative from a given subset of alternatives is always as least as good as the full subset, and that the full subset is always at least as good as the singleton containing the least preferred alternative from the subset.

Extension. $\{x\} \succ_i \{y\}$ if $x \succ_i y$, for all $i \in N$ and $x, y \in X$.

MaxMin. Let $i \in N$, $Y \subseteq X$ and $x \in X$. Then we have that

- $\{x\} \succeq_i Y$ if $x \succeq_i y$ for all $y \in Y$, and
- $Y \succeq_i \{x\}$ if $y \succeq_i x$ for all $y \in Y$.

For $i \in N$, let $X_1^i, ..., X_\ell^i$ be a partition of the set X of alternatives into subsets such that for each $1 \leq j \leq \ell$, we have $x \simeq_i y$ for all $x, y \in X_j^i$, and for each $j' > j$, we have $x \succ_i y$ for all $x \in X_j^i$ and $y \in X_{j'}^i$. That is, we partition the set of alternatives into subsets of equally preferred alternatives, where all alternatives in X_j^i are strictly preferred to all alternatives in $X_{j'}$ whenever $j' > j$. I will call $X_1^i, ..., X_\ell^i$ voter i's preference partition. If $\ell = 2$, we say that voter i has dichotomous preferences; if $\ell = 3$, we say that voter i has trichotomous preferences; if $\ell \geq 4$, we say that voter i has multichotomous preferences.

Lemma 4.1.1. Let $i \in N$ be a voter with preference partition $X_1^i, ..., X_\ell^i$. For any $X_j^i \in \{X_1^i, ..., X_\ell^i\}$ and any nonempty $Y_j^i \subseteq X_j^i$, we have $Y_j^i \simeq_i X_j^i$.

Proof. Let $i \in N$ be a voter with preference partition $X_1^i, ..., X_\ell^i$, and let $X_j^i \in \{X_1^i, ..., X_\ell^i\}$. Let $x \in X_j^i$. Since $x \in X_j^i$, we have $x \simeq_i y$ for all $y \in X_j^i$, and so $\{x\} \simeq_i X_j^i$. Now, let $y \subseteq X_j$. We want to show that $Y \cup \{x\} \simeq_i X_j^i$. Since $Y \subseteq X_j^i$, $x \simeq_i y$ for all $y \in Y$, so $\{x\} \simeq_i Y$, and so $Y \cup \{x\} \simeq Y$. Thus $Y \cup \{x\} \simeq_i X_j^i$. □

Lemma 4.1.2. Let $i \in N$ and let $X_j^i, X_{j'}^i \in \{X_1^i, ..., X_\ell^i\}$ with $j < j'$. Then $Y_j^i \succ_i Y_{j'}^i$ for all subsets $Y_j^i \subseteq X_j^i$ and $Y_{j'}^i \subseteq X_{j'}^i$.

Proof. By definition of preference partitions, we have that $X_j^i \succ_i X_{j'}^i$. By Lemma 4.1.1, $Y_j^i \simeq_i X_j^i$ for all $Y_j^i \subseteq X_j^i$ and $Y_{j'}^i \simeq X_{j'}^i$. Thus we get that $Y_j^i \simeq_i X_j^i \succ_i X_{j'}^i \simeq Y_{j'}^i$, and so $Y_j^i \succ_i Y_{j'}^i$. □

Let us now discuss the strategies of voters in more detail. Out of the feasible strategies of any given voter, which ones can we expect the voter to use, without making too many assumptions about the strategic attitudes of the voter? Following Brams and Fishburn [1978], I will assume that each $i \in N$ may choose to play any strategy B_i for which there does not exist another unique strategy B_i' such that B_i', according to i's preferences, produces an outcome that is equally good as the outcome produced by B_i in all scenarios, while also producing an outcome that is strictly better in some scenario. If such a strategy B_i' exists for a strategy B_i, we will call B_i dominated. Otherwise, B_i is undominated. Then, the strategies voters may use are exactly those (feasible) strategies that are undominated. I make this precise with the following definition.

Definition 4.1.3 (Domination)**.** Let f be a scoring rule and let $i \in N$. Then, a strategy B_i dominates strategy B_i' if $f(\boldsymbol{B}) \succeq_i f(\boldsymbol{B}_{-i}, B_i')$ for all partial profiles $\boldsymbol{B}_{-i}$ and there is some partial profile $\boldsymbol{B}_{-i}$ such that $f(\boldsymbol{B}) \succ_i f(\boldsymbol{B}_{-i}, B_i')$. A strategy B_i' is undominated if there is no strategy B_i that dominates B_i'.

Let us now define the notions of sincere, insincere and strongly insincere strategies more precisely. We will say that the strategy of a voter i is sincere if whenever i votes for some alternative $x \in X$, i also votes for any alternative $y \in X$ that is strictly preferred to x according to $\succeq_i$, and that a strategy is insincere otherwise. Furthermore, i's strategy is strongly insincere if i does not abstain and does not vote for any of their most preferred alternatives. I state this precisely with the following definition.

Definition 4.1.4 (Sincerity and insincerity)**.** Given a set of alternatives X and voter $i \in N$ with preference order $\succeq_i$, a strategy B_i is sincere if $x \succeq_i y$ for all $x \in B_i$ and all $y \in X \setminus B_i$, and insincere otherwise. Furthermore, a strategy B_i is strongly insincere if $w_{|B_i|} > 0$, while $B_i \cap X_1^i = \emptyset$.

I give an example of sincere strategies and undominated (feasible) strategies under plurality, Approval Voting and Even and Equal Cumulative Voting.

Example 4.1.5 (Sincere and undominated strategies)**.** Let $X = \{x, y, z\}$ and let $i \in N$ be a voter with preference order $x \succ_i y \succ_i z$. The sincere strategies of i are $\{x\}$, $\{x, y\}$ and $\{x, y, z\}$. Under Approval Voting, i has undominated strategies $\{x\}$ and $\{x, y\}$; under the plurality rule, i has undominated strategies $\{x\}$ and $\{y\}$; under EECV, i has undominated strategies $\{x\}$, $\{y\}$, and $\{x, y\}$. Note that the strategy $\{x, y\}$ is undominated under EECV because it is not uniquely dominated by either $\{x\}$ or $\{y\}$. However, it is never better than both strategies simultaneously.

We say that a voting rule f is sincere if, for each voter i and any voting situation, all (feasible) undominated strategies of i are sincere. Furthermore, a voting rule f is strategyproof if in each profile, each voter has a unique undominated strategy, and that unique undominated strategy is sincere.

Defining voting rules only with respect to balloting systems, and working with assumptions on preference extensions that are equivalent to the Gärdenfors principle, Brams and Fishburn [1978] show that all voting rules are sincere and that Approval Voting is strategyproof when voters have dichotomous preferences. In this case, under Approval Voting, the unique undominated (sincere) strategy for each voter is to vote for all of their most-preferred alternatives. Furthermore, it is shown that Approval Voting is the only sincere voting rule when voters have trichotomous preferences, and that no voting rule is sincere when voters have multichotomous preferences. These results directly apply to all p-approval rules in the class of Size Approval Voting rules. In particular, we may observe that any p-approval rule simulates Approval Voting as long as every voter has at most p most-preferred alternatives. An implication of this is that any p-approval rule is strategyproof in the case where voters have preferences that are dichotomous and sets of most-preferred alternatives of size bounded by p. The result, however, is not applicable to the class of Size Approval Voting rules in general. This is due to the fact that Brams and Fishburn [1978] only define voting rules with respect to balloting systems. Since several different Size Approval Voting rules may have the same balloting system, this definition does not enable us to distinguish sufficiently between rules in the class, and as such, the class of Size Approval Voting rules does not fit the definition of approval-based systems used by Brams and Fishburn [1978]. For example, under this definition, any Size Approval Voting rules for which $w_k > 0$ for all $k < m$ are equivalent to Approval Voting. The implication is of this is that all properties of Approval Voting with respect to sincerity would also hold for such rules. This is clearly not the case, as illustrated by Example 4.1.5.

The following proposition shows that whenever voters have dichotomous preferences, any Size Approval Voting rule is sincere.

Proposition 4.1.6. Every Size Approval Voting rule f is sincere when voters have dichotomous preferences.

Proof. Let f be a Size Approval Voting rule, and let $i \in N$ be a voter with preference partition X_1^i, X_2^i. We want to show that i only has undominated strategies that are sincere. Let B_i^* be an insincere strategy. I will show that there exists a sincere strategy B_i' that dominates B_i^*. Observe that since B_i^* is insincere, we have that $X_1^i \not\subseteq B_i^*$ and $B_i^* \not\subseteq X_1^i$. We can generally distinguish two cases: $B_i^* \cap X_1^i = \emptyset$, and $B_i^* \cap X_1^i \neq \emptyset$.

If $B_i^* \cap X_1^i = \emptyset$, let $B_i' = \{x\}$ for some $x \in X_1^i$, and if $B_i^* \cap X_1^i \neq \emptyset$, let $B_i' = B_i^* \cap X_1^i$. Observe that in both cases, B_i' is a sincere strategy. The proof that B_i' is a dominating strategy in the second case is analogous to the proof that B_i' is a dominating strategy in the first case, so I only show the first case. Thus, suppose that $B_i^* \cap X_1^i = \emptyset$ and let $B_i' = \{x\}$ for some $x \in X_1^i$.

Claim: *B_i' dominates B_i^*.*

I will show that $f(\boldsymbol{B}_{-i}.B_i') \succeq_i f(\boldsymbol{B}_{-i}, B_i^*)$ for all partial profiles $\boldsymbol{B}_{-i}$, and $f(\boldsymbol{B}_{-i}.B_i') \succ_i f(\boldsymbol{B}_{-i}, B_i^*)$ for some partial profile $\boldsymbol{B}_{-i}$. First, to see that $f(\boldsymbol{B}_{-i}, B_i') \succ_i f(\boldsymbol{B}_{-i}, B_i^*)$ for some partial profile $\boldsymbol{B}_{-i}$, let $\boldsymbol{B}_{-i}$ be a partial profile such that $f(\boldsymbol{B}_{-i}) = X$. Since $B_i^* \cap X_1^i = \emptyset$, $B_i^* \subseteq X_2^i$, and so

$f(\boldsymbol{B}_{-i}, B_i^*) \subseteq X_2^i$. Further, $f(\boldsymbol{B}_{-i}, B_i') = \{x\}$. Since $x \in X_1^i$, $x \succ_i y$ for all $y \in X_2^i$, and so it follows by the Gärdenfors principle that $f(\boldsymbol{B}, B_i') \succ_i f(\boldsymbol{B}, B_i^*)$.

It remains to show that $f(\boldsymbol{B}_{-i}.B_i') \succeq_i f(\boldsymbol{B}_{-i}, B_i^*)$ for all partial profiles $\boldsymbol{B}_{-i}$. Note that $f(\boldsymbol{B}_{-i}.B_i') \simeq_i f(\boldsymbol{B}_{-i}, B_i^*)$ whenever $f(\boldsymbol{B}_{-i}.B_i') = f(\boldsymbol{B}_{-i}, B_i^*)$, so assume that $f(\boldsymbol{B}_{-i}.B_i') \neq f(\boldsymbol{B}_{-i}, B_i^*)$. Distinguish three cases: $f(\boldsymbol{B}_{-i}, B_i^*) \subseteq X_1^i$, $f(\boldsymbol{B}_{-i}, B_i^*) \subseteq X_2^i$, and $\{y, z\} \subseteq f(\boldsymbol{B}_{-i}, B_i^*)$ for some $y \in X_1^i$ and $z \in X_2^i$.

Suppose first that $f(\boldsymbol{B}_{-i}, B_i^*) \subseteq X_1^i$. Since $B_i^* \cap X_1^i = \emptyset$ and $f(\boldsymbol{B}_{-i}, B_i^*) \subseteq X_1^i$, it follows that $f(\boldsymbol{B}_{-i}, B_i') \subseteq X_1^i$, since no alternative $y \in X_1^i$ has lost support and no alternative $z \in X_2^i$ has gained support. It follows by Lemma 4.1.1 that $f(\boldsymbol{B}_{-i}, B_i') \simeq_i f(\boldsymbol{B}_{-i}, B_i^*)$.

Secondly, suppose that $f(\boldsymbol{B}_{-i}, B_i^*) \subseteq X_2^i$. Since $B_i^* \cap X_1^i = \emptyset$, we have that $Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, B_i^*, y) = Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, B_i', y)$ for all $y \in X_1^i \setminus \{x\}$. Thus, $f(\boldsymbol{B}_{-i}, B_i') \subseteq X_2^i \cup \{x\}$. If $x \in f(\boldsymbol{B}_{-i}, B_i')$, it follows by the Gärdenfors principle that $f(\boldsymbol{B}_{-i}, B_i') \succ_i f(\boldsymbol{B}_{-i}, B_i^*)$. If $x \notin f(\boldsymbol{B}_{-i}, B_i')$, it follows by Lemma 4.1.1 that $f(\boldsymbol{B}_{-i}, B_i') \simeq_i f(\boldsymbol{B}_{-i}, B_i^*)$.

Finally, suppose that $\{y, z\} \subseteq f(\boldsymbol{B}_{-i}, B_i^*)$ for some $y \in X_1^i$ and $z \in X_2^i$. Since $B_i^* \cap X_1^i = \emptyset$, we must have that $Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, B_i', y) \geq Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, B_i^*, y)$ for all $y \in X_1^i$ and $Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, B_i', z) \leq Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, B_i^*, z)$ for all $z \in X_2^i$, and as such, $f(\boldsymbol{B}_{-i}, B_i') \cap X_1^i \neq \emptyset$. Now, if $f(\boldsymbol{B}_{-i}, B_i') \cap X_2^i = \emptyset$, it follows by the Gärdenfors principle and Lemma 4.1.1 that $f(\boldsymbol{B}_{-i}, B_i') \succ_i f(\boldsymbol{B}_{-i}, B_i^*)$. Suppose therefore that $f(\boldsymbol{B}_{-i}, B_i') \cap X_2^i \neq \emptyset$. By the fact that $Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, B_i', y) \geq Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, B_i^*, y)$ for all $y \in X_1^i$ and $Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, B_i', z) \leq Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, B_i^*, z)$ for all $z \in X_2^i$, it follows that $f(\boldsymbol{B}_{-i}, B_i') \cap X_2^i \subseteq f(\boldsymbol{B}_{-i}, B_i^*) \cap X_2^i$. Consequently, by the fact that $f(\boldsymbol{B}_{-i}, B_i') \neq f(\boldsymbol{B}_{-i}, B_i^*)$, we get that $f(\boldsymbol{B}_{-i}, B_i') \cap X_2^i \subset f(\boldsymbol{B}_{-i}, B_i^*) \cap X_2^i$ or that there is some $y^* \in X_1^i$ such that $y^* \in f(\boldsymbol{B}_{-i}, B_i')$ and $y^* \notin f(\boldsymbol{B}_{-i}, B_i^*)$. Since $Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, B_i', y) = Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, B_i^*, y)$ for all $y \in X_1^i \setminus \{x\}$, we must have that $y^* = x$. In each case, it follows that $f(\boldsymbol{B}_{-i}, B_i') \succ_i f(\boldsymbol{B}_{-i}, B_i^*)$.

Thus, we have that $f(\boldsymbol{B}_{-i}.B_i') \succeq_i f(\boldsymbol{B}_{-i}, B_i^*)$ for all partial profiles $\boldsymbol{B}_{-i}$ and

$f(\boldsymbol{B}, B_i') \succ_i f(\boldsymbol{B}, B_i^*)$ for some partial profile $\boldsymbol{B}_{-i}$, so $B_i' = \{x\}$ dominates B_i^*. □

The above result does not extend to those cases in which the preferences of voters are trichotomous or multichotomous. In fact, in these cases, we can show that any voter will have an undominated and strongly insincere strategy whenever f is a Size Approval Voting rule for which $w_1 > w_2$. In particular, for any voter with trichotomous and multichotomous preferences, there will always be some partial ballot $\boldsymbol{B}_{-i}$ in which a strategy of bullet voting for some alternative that is not among the most-preferred alternatives of the voter will be strictly better than any other strategy.

Proposition 4.1.7. Let f be a Size Approval Voting rule with $w_1 > w_2$ and let $i \in N$. If i has trichotomous or multichotomous preferences, then i has an undominated strategy that is strongly insincere.

Proof. Let f be a Size Approval Voting rule with $\boldsymbol{w} = (w_1, ..., w_m)$ such that $w_1 > w_2$. Assume without loss of generality that $w_1 = 1$. Let $i \in N$ be a voter with trichotomous or multichotomous preferences, and let $X_1^i, ..., X_\ell^i$ be i's preference partition. Since i has trichotomous or multichotomous preferences, $\ell \geq 3$ and $|X| \geq 3$. We want to show that i has an undominated, strongly insincere strategy. Let $B_i = \{x\}$ for some $x \in X_2^i$. Since $B_i \neq \emptyset$ and $B_i \cap X_1^i = \emptyset$, B_i is strongly insincere. I will show that there is no B_i' such that $f(\boldsymbol{B}_{-i}, B_i') \succeq_i f(\boldsymbol{B}_{-i}, B_i')$ for all partial profile $\boldsymbol{B}_{-i}$ and $f(\boldsymbol{B}_{-i}, B_i') \succ_i f(\boldsymbol{B}_{-i}, B_i)$ for some partial profile $\boldsymbol{B}_{-i}$. It is sufficient to show that there is some partial profile $\boldsymbol{B}_{-i}$ such that $f(\boldsymbol{B}_{-i}, B_i) \succ_i f(\boldsymbol{B}_{-i}, B_i')$ for all strategies $B_i' \neq B_i$.

Let $\boldsymbol{B}_{-i}$ be a partial profile such that $f(\boldsymbol{B}_{-i}) = X_\ell^i$ and $Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, y) - Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, x) = 1$ and $Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, y) - Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, z) > 1$ for all $y \in X_\ell^i$ and all $z \in X \setminus (X_\ell^i \cup \{x\})$. Observe that since $w_1 = 1$ and $w_k \geq w_{k+1}$ for all $k < m$, there is no strategy B_i' such that $z \in f(\boldsymbol{B}_{-i}, B_i')$ for any $z \in X \setminus (X_\ell^i \cup \{x\})$. Thus, $f(\boldsymbol{B}_{-i}, B_i') \subseteq X_\ell^i \cup \{x\}$ for all B_i'. Further, since $x \in B_i$ and $w_{|B_i|} = 1$, it follows that $f(\boldsymbol{B}_{-i}, B_i) = X_\ell^i \cup \{x\}$. Since $x \in X_2^i$ and $\ell \geq 3$, $x \succ_i y$ for all $y \in X_\ell^i$. It follows by the Gärdenfors principle that $X_\ell^i \cup \{x\} \succ_i X_\ell^i$, and by Lemma 4.1.1 that $X_\ell^i \cup \{x\} \succ_i Y_\ell^i$ for any

$Y^i_\ell \subseteq X^i_\ell$. Thus, $f(\boldsymbol{B}_{-i}, B_i) = X^i_\ell \cup \{x\} \succ_i f(\boldsymbol{B}_{-i}, B'_i)$ for all B'_i, and so B_i is undominated. □

It follows, in particular, that under all SDSAV rules, all voters with trichotomous or multichotomous preferences will have some undominated and strongly insincere strategy. Note how this is reflected by the axiom of Pivotal Contraction introduced in Chapter 3.

In this section, I have considered properties of Size Approval Voting rules with respect to sincerity and manipulation. We have seen that all Size Approval Voting rules are sincere when voters have dichotomous preferences. On the other hand, for many rules in the class, voters will have strongly insincere undominated strategies whenever their preferences are multichotomous, due to the fact that the structure of these rules incentivize bullet voting. I will now move on to consider the problem of strategic manipulation from the perspective of computational complexity, where the incentives of voters to bullet vote will turn out to be highly relevant.

4.2 Complexity of Strategic Manipulation

As noted in the chapter introduction, the computational complexity[1] of the problem of determining whether there exists a successful manipulation tactic has been shown to be $\mathcal{NP}$-complete for certain voting rules [Bartholdi et al., 1989a, Xia et al., 2009], and one interpretation of such results is that said rules are resistant to manipulation, since the problem of computing a successful manipulation tactic is computationally intractable in the worst case [2]. In this section, I consider the computational complexity of strategic manipulation under Size Approval Voting rules. Consider the following decision problem, Manipulability(f).

[1]See, e.g., Arora and Barak [2009] for basic background on computational complexity.

[2]Note, however, that since $\mathcal{NP}$-hardness relates to worst-case complexity, this interpretation can be contested, since it is not a given that the instances would-be manipulators would need to compute manipulation tactics for are among the hard instances. It has therefore been suggested that determining whether manipulation is 'usually' hard is more informative. See, e.g., Conitzer and Walsh [2016], section 6.5, for more on this.

MANIPULABILITY(f)

Input: Partial profile $\boldsymbol{B}_{-i}$, voter $i \in N$ alternative $x^* \in X$

Question: Does there exists a ballot B_i such that $x^* \in f(\boldsymbol{B}_{-i}, B_i)$?

As we saw in the previous section, many of the Size Approval Voting rules incentivize manipulation by bullet voting, under which a would-be manipulator gives all their support to a single alternative, which is not necessarily among the most-preferred alternatives of that voter. For rules in the class in general, the best tactic of a given voter i who wants to make an alternative x^* win will always be to bullet vote for x^*, as no other ballot of i will give the alternative more weighted support. From this, it is easy to see that the problem of determining a successful manipulation tactic is trivial: suppose $\boldsymbol{B}_{-i}$ is a partial profile containing all ballots except that of i, and that i seeks to vote so as to make x^* win. Observe that if $Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, y) - Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, x^*) > w_1$ for some $y \in X$, no single vote can make x^* win. If, however, $Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, y) - Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, x^*) \leq w_1$ for all $y \in X$, the vote $\{x^*\}$ is guaranteed to make x^* win.

Thus, to solve $Manipulability(f)$ for any Size Approval Voting rule f, we can just compute the collective outcome under f of the given partial profile $\boldsymbol{B}_{-i}$ together with the ballot $\{x^*\}$. If $x^* \in f(\boldsymbol{B}_{-i}, \{x^*\})$, there exists a successful manipulation tactic. Otherwise, such a tactic does not exist. It follows that MANIPULABILITY(f) (trivially) is in $\mathcal{P}$ for all Size Approval Voting rules f.

Conversely, we may consider a destructive version of $Manipulability(f)$, and ask whether there exists a ballot a voter i can submit so as to make sure a given alternative x^* does not win. This problem is also easily shown to be in $\mathcal{P}$. If there is some y for which $Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, x^*) - Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-i}, y) < w_1$, then i can submit the singleton ballot $\{y\}$ to make x^* lose. If no such y exists, there is nothing i can do to stop x^* from winning.

4.3 Manipulation and Partial Information

When considering manipulability in the previous subsections, we have assumed that the would-be manipulator has full information about the election sought to be manipulated. It is however rarely the case in practice that voters have access to all information about the choices of other members of the electorate. The impact of informational restrictions on the manipulability of voting rules has previously been studied by, among others, Conitzer et al. [2011], Reijngoud and Endriss [2012], and Endriss et al. [2016], who provide results about the vulnerability of voting rules with respect to manipulation under various cases of of incomplete information. In these works, the ballots of voters are rankings, and the limited information of a voter i in a given profile $\boldsymbol{B}$ is represented by use of information sets composed of all profiles that i cannot distinguish from $\boldsymbol{B}$ based on the information i has. In this section, I make some observations about the manipulability of Size Approval Voting rules based on the model of partial information used by Conitzer et al. [2011], Reijngoud and Endriss [2012] and Endriss et al. [2016].

Given a voter i, let $\pi : \boldsymbol{B} \to \mathcal{I}$ be a function mapping a profile $\boldsymbol{B}$ to the information $\mathcal{I}$ that i has access to. We call π an information function. Let $\mathcal{W}_i^{\pi(\boldsymbol{B})}$ be the set of profiles that i cannot distinguish from $\boldsymbol{B}$ having access only to information $\pi(\boldsymbol{B})$. Formally, we let

$$\mathcal{W}_i^{\pi(\boldsymbol{B})} = \{\boldsymbol{B}' \in (2^X)^n \mid \pi(\boldsymbol{B}') = \pi(\boldsymbol{B}) \text{ and } B_i = B_i'\}.$$

Reijngoud and Endriss [2012] consider a series of possible information functions. Not all of the information functions considered in their work are applicable to our context, but I will consider the following:

- the zero information function $\pi(\boldsymbol{B}) = \bot$, under which a voter has no information about the election instance. In this case, we have that

$$\mathcal{W}_i^{0(\boldsymbol{B})} = \{\boldsymbol{B}' \in (2^X)^n \mid B_i = B_i'\}.$$

- the winner information function $\pi(\boldsymbol{B}) = f(\boldsymbol{B})$, under which the voter i has access to the outcome of the given profile. The set of indistinguishable profiles for i is then

$$\mathcal{W}_i^{f(\boldsymbol{B})} = \{\boldsymbol{B}' \in (2^X)^n \mid f(\boldsymbol{B}) = f(\boldsymbol{B}') \text{ and } B_i = B_i'\}.$$

- the score information function $w(\boldsymbol{B})$, under which a voter has information about the weighted support of every alternative voted over in the profile $\boldsymbol{B}$. We then have set of indistinguishable profiles

$$\mathcal{W}_i^{w(\boldsymbol{B})} = \{\boldsymbol{B'} \in (2^X)^n \mid Supp_{\boldsymbol{w}}(\boldsymbol{B'}, x) = Supp_{\boldsymbol{w}}(\boldsymbol{B}, x) \text{ for all } x \in X, \text{ and } B_i = B'_i\}.$$

- the full information function $\pi(\boldsymbol{B}) = \boldsymbol{B}$ that maps a profile to itself. Then

$$\mathcal{W}_i^{\pi(\boldsymbol{B})} = \{\boldsymbol{B}\}.$$

Furthermore, in Conitzer et al. [2011], Reijngoud and Endriss [2012], Endriss et al. [2016], it is assumed that a voter i only has incentive to manipulate in a truthful profile $\boldsymbol{B}$ if the manipulation is equally good as the truthful strategy B_i in all profiles indistinguishable from $\boldsymbol{B}$ with respect to the information available, while also being strictly better in some profile indistinguishable from $\boldsymbol{B}$. Intuitively, this is exactly when the manipulation is 'safe' for i, based on the limited information i has. Followiing this approach, let us assume that given a profile $\boldsymbol{B}$ that includes a sincere strategy B_i, and some insincere strategy B^*, i will only use B_i^* instead of B_i if $f(\boldsymbol{B'}_{-i}, B_i^*) \succeq_i f(\boldsymbol{B'}_{-i}, B_i^*)$ for all $\boldsymbol{B'}_{-i}$ with $(\boldsymbol{B'}_{-i}, B_i) \in \mathcal{W}_i^{g(\boldsymbol{B})}$ and $f(\boldsymbol{B'}_{-i}, B_i^*) \succ_i f(\boldsymbol{B'}_{-i}, B_i)$ for some $\boldsymbol{B'}_{-i}$ with $(\boldsymbol{B'}_{-i}, B_i) \in \mathcal{W}_i^{g(\boldsymbol{B})}$.

This is a stronger assumption than the one we used, following Brams and Fishburn [1978], in Section 4.1: under this assumption, if a voter i has full information in a given profile $\boldsymbol{B}$, then an insincere strategy B_i^* would only be used by the voter if $f(\boldsymbol{B}_{-i}, B_i^*) \succ_i f(\boldsymbol{B}_{-i}, B'_i)$ for any sincere strategy B'_i available to i. That is, in this case, a voter will only use an insincere strategy if it is better than any sincere strategy.

I will first make some comments about manipulation when voters have full information or information about the score of alternatives. First of all, note that for all Size Approval Voting rules, having access to the scores given to each alternative provides voters with all relevant information about the election, i.e., for any rule in the class, having information about the scores of alternatives coincides with having full information about the election. Furthermore, it is easy to see that there will be rules in the class for which there are profiles $\boldsymbol{B}$ in which a voter i has an insincere strategy B_i^* such that $f(\boldsymbol{B}_{-i}, B_i^*) \succ_i f(\boldsymbol{B}_{-i}, B'_i)$ for any sincere strategy B'_i available to i. I illustrate with an example under the rule of EECV.

Example 4.3.1. Let f be the EECV rule, and let $X = \{x, y, z, w\}$ and $N = \{1, 2, 3, 4\}$. Suppose that $x \succ_1 y \succ_1 z \succ_1 w$, and consider the profile $\boldsymbol{B}$ in which $B_1 = \{x, y\}$, $B_2 = \{y\}$, $B_3 = \{w\}$, and $B_4 = \{z, w\}$. Then $f(\boldsymbol{B}) = \{y, w\}$. Consider now the insincere strategy $B_1^* = \{y\}$. Since $f(\boldsymbol{B}) = \{y, w\}$ and $B_1^* \subset B_1$, it follows that $f(\boldsymbol{B}_{-1}, B_1^*) = \{y\}$. Now, since $y \succ_1 w$, we have that $f(\boldsymbol{B}_{-1}, B_1^*) \succ_1 f(\boldsymbol{B})$. Furthermore, observe that there is exists no sincere strategy B_1' such that $\{x\} \subseteq f(\boldsymbol{B}_{-1}, B_1')$, and that $Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-1}, B_1^*, y) > Supp_{\boldsymbol{w}}(\boldsymbol{B}_{-1}, B_1', y)$ for all sincere strategies B_1'. Thus, the insincere strategy B_1^*

Let us now discuss how voters will behave under zero information. In this case, since $\mathcal{W}_i^{0(\boldsymbol{B})} = \{\boldsymbol{B'} \in (2^X)^n \mid B_i = B_i'\}$, it is clear that for any rule f in the class, every voter has some undominated sincere strategy in any profile, namely one in which the voter votes for some feasible subset of their most preferred alternatives. To see that this is the case, it is sufficient to observe that, given a profile $\boldsymbol{B}$ and a voter i who votes with a (feasible) ballot $B_i \subseteq X_1^i$, there is some $\boldsymbol{B'} \in \mathcal{W}_i^{0(\boldsymbol{B})}$ such that $f(\boldsymbol{B'}) = B_i'$. Since $f(\boldsymbol{B'}$ is composed only of alternatives that are among i's most preferred alternatives, it follows that this outcome is as good as any other outcome. Thus, there is no Size Approval Voting rule f for which voters have incentive to manipulate under zero information.

Finally, consider the case where voters have access (only) to the set of winning alternatives $f(\boldsymbol{B})$ in a profile $\boldsymbol{B}$. In this case, a strategy B_i^* of a voter i dominates B_i if $f(\boldsymbol{B}, B_i^*) \succ_i f(\boldsymbol{B'})$ for some $\boldsymbol{B'}$ such that $f(\boldsymbol{B'}) = f(\boldsymbol{B})$ and $B_i' = B_i$, and $f(\boldsymbol{B}, B_i^*) \succeq_i f(\boldsymbol{B'})$ for all $\boldsymbol{B'}$ such that $f(\boldsymbol{B'}) = f(\boldsymbol{B})$ and $B_i' = B_i$. Now, the crucial thing to observe is that, when a voter i only has access to $f(\boldsymbol{B})$, it is not possible for i to determine whether an alternative $x \notin f(\boldsymbol{B})$ can become winning if i changes their strategy. Thus, for every $x \notin f(\boldsymbol{B})$, i must consider possible some profile $\boldsymbol{B'}$ such that $x \in f(\boldsymbol{B'}_{-i}, B_i')$ for some strategy $B_i' \neq B_i$. For some rules in the class, in partcular those rules for which $w_1 > w_2 > 0$, there will in this case be situations in which the outcomes $f(\boldsymbol{B'}), f(\boldsymbol{B''})$ of profiles $\boldsymbol{B'}, \boldsymbol{B''}$ that only differ with respect to the strategy chosen by i cannot be compared based on the assumptions made in Section 4.1 about how the preferences of voters over individual alternatives extend to preferences over sets. This is perhaps best illustrated with an example.

Example 4.3.2. Suppose that f is a Size Approval Voting rule for which $w_1 > w_2 > 0$. Let $X = \{x, y, z\}$ and i a voter with preference order $x \succ_i y \succ_i z$. In this case, i has (feasible) sincere strategies $\{x\}$ and $\{x, y\}$. Let $\boldsymbol{B}$ be a profile for which $f(\boldsymbol{B}) = \{y, z\}$, and $B_i = \{x, y\}$. Since $f(\boldsymbol{B}) = \{y, z\}$ and $B_i = \{x, y\}$, the insincere strategy $B_i^* = \{y\}$ dominates B_i. Consider now the other sincere strategy $B_i' = \{x\}$ of i. Observe that there is a profile $\boldsymbol{B}'$ such that $f(\boldsymbol{B}'') = f(\boldsymbol{B})$ and $B''_i = B_i'$, where $x \in f(\boldsymbol{B}''_{-i}, B')$. However, since $y \in B_i$, $w_{|B_i|} = w_2 > 0$ and $y \notin B_i'$, it follows by the fact that $f(\boldsymbol{B}) = \{y, z\}$ that $y \notin f(\boldsymbol{B}'')$. On the other hand, since the support of alternative z is not affected by i's change of strategy, it is possible that $z \in f(\boldsymbol{B}''_{-i}, B')$. Thus, suppose that $f(\boldsymbol{B}''_{-i}, B') = \{x, z\}$, and observe that we cannot determine whether or not $\{x, z\} \succeq_i \{y\}$ based only on the assumptions made in Section 4.1 about how preferences over individual alternatives extend to sets of alternatives.

4.4 Discussion

In this chapter, I have considered the properties of the class of Size Approval Voting rules with respect to manipulability. Relying on the notion of sincerity from Brams and Fishburn [1978], I showed that all rules in the class are sincere when all voters have dichotomous preferences, and that many of the rules in the class can be seen as to incentivize strongly insincere behavior under less strict assumptions about the preferences of voterrs. In particular, we saw that many rules in the class are vulnerable to the manipulation act of bullet voting whenever preferences are trichotomous or multichotomous. These results can be interpreted as showing that members of the class are not vulnerable to manipulative behavior when the approval balloting system is in direct correspondence with the preferences of voters, and furthermore that we lose this robustness towards manipulation for many of the rules in the class whenever the preferences of voters are too multifaceted to be captured by mere approval and disapproval. I subsequently discussed the complexity of determining whether a given voter has a successful manipulation tactic to make a given alternative win (resp. lose). Under any rule in the class, the best strategy of a voter who wishes to make a given alternative win is the strategy of bullet voting for that alternative. An implication of this property is that the problem of determining whether a voter has a successful manipulation tactic to make a given alternative win (resp. lose) is computationally easy. Finally, I discussed some basic observations about the susceptibility of various

Size Approval Voting rules to manipulation under different cases of partial information.

In Section 4.1, following Brams and Fishburn [1978], we assumed that a voter will consider using any strategy for which there does not exist another strategy that, according to the preferences of the voters, produces an equally good outcome in every profile and a strictly better outcome in some profile. This is a relatively weak assumption that leaves open the possibility that a voter will use an insincere strategy in situations where there is some equally good sincere strategy. It is perhaps equally reasonable to assume that a voter will choose among their sincere strategies whenever there is no insincere strategy that dominates all their sincere strategies. However, as we saw when considering the incentives of voters to manipulate under full information in Section 4.3, certain rules in the class are still susceptible to manipulation by insincere strategies when we assume that voters will only use an insincere strategy when it strictly dominates their sincere strategies. There are also other assumptions that could have been made as to how the preferences of voters over individual alternatives extend to sets. For instance, the result of Duggan and Schwartz [1999] relies on notions of optimistic and pessimistic voters[3]. We say that a voter is optimistic if the voter prefers one set to another whenever the favorite alternative of the voter in the first set is preferred to the favorite alternative in the second set, and that a voter is pessimistic if the voter prefers one set to another whenever the least favorite alternative of the voter in the first set is preferred to the least favorite alternative in the second set [Endriss, 2013]. Note that, in Example 4.3.2, there is a dominant sincere strategy if we assume that voters are optimistic, and a dominant insincere strategy if we assume voters are pessimistic.

The issues of manipulability discussed in the chapter were restricted to the setting in which a single voter attempts to manipulate the outcome of a given election. In practice, however, a single voter may not have much impact on the election alone. Therefore, it may be equally relevant to consider the problem of coalitional manipulation, in which a group of voters coordinate their votes in an attempt to affect the outcome [Conitzer and Walsh, 2016]. The computational complexity of coalitional manipulation has been studied by, e.g., Betzler et al. [2011], Xia et al. [2010]. For the rules in our class, we may observe that the best strategy for a group of voters collaboratively

[3]Duggan and Schwartz [1999] refer to voters that are optimistic and pessimistic as taking the Heroic Approach and the Maximin approach, respectively.

attempting to make an alternative win, will be the one in which each member of the group bullet votes for that alternative, for reasons analogous to the case of a single manipulator. Similarly, to determine whether a coalition of voters has a successful strategy to make a given alternative x lose, we can simply compute whether there is another alternative y such that the weighted support of x minus the weighted support of y is less than $\ell \cdot w_1$, where ℓ is the size of the coalition. If so, all members bullet voting for y is a successful strategy; if not, no successful manipulation tactic exists for the coalition.

Furthermore, there are various other categories of manipulation problems that have not been considered, such as electoral control and bribery [Faliszewski and Rothe, 2016]. In problems of electoral control, some agent, often thought of as the chair, attempts to control the outcome of the election by, e.g., adding or removing alternatives or voters. Note that the special case of the problem of electoral control in which the chair attempts to control the election by adding alternatives is closely related to the problem of determining possible and necessary winners when some number of alternatives is missing, studied in Chapter 5. This problem has been studied by Bartholdi et al. [1992], Hemaspaandra et al. [2009], who provide complexity results for problems electoral control under several voting rules. In particular, Bartholdi et al. [1992] studies the complexity of electoral control by adding alternatives when there is no limitation on the number of alternatives that can be added, and Hemaspaandra et al. [2009] studies the complexity of the same problem where the number of alternatives that can be added is bounded by some fixed ℓ. In bribery problems, some external entity attempts to bribe voters in an election to vote in a way preferred by the entity. Such problems were first studied by Faliszewski et al. [2009], and shortly after by Elkind et al. [2009], who both study the complexity of various bribery problems. Note that certain versions of bribery problems also closely relate to the problem of determining possible winners [Faliszewski and Rothe, 2016].

Chapter 5

Computing Possible and Necessary Winners

In this chapter, I consider some basic issues concerning the problem of determining possible and necessary winners given incomplete profiles with missing alternatives under two different assumptions about how profiles can be completed.

Suppose, in some election, that we only have partial information about the ballots of voters. This can be the case in many situations. For instance, if an electorate consists of subelectorates that vote at different times, or if an election allows voters to vote by, e.g., post, some voter ballots may become available later than others. We may also have that alternatives become available at different times – think, for example, of political elections in which some candidates join the race late, or of job applicants submitting applications for a given position at different times.

In such situations, we will often not be able to decide whether an alternative will be a winner of the election in its complete form. However, we may still be interested in determining whether an alternative can come out as winning, based on the information we already have. This can be due to concerns of resource use with respect to computing the outcome of the election, or relate to the interest of voters and/or alternatives. Suppose, for instance, that we can determine, in some incomplete election, that some alternatives cannot win regardless of how the election is made complete. Then, for the sake of computing the outcome of the election, it will usually be more efficient to restrict the analysis to those alternatives that still stand a chance of winning. Furthermore, if the alternatives voted over are, e.g., job applicants in a hiring process, it will be in the interest of applicants who do not stand a chance of being hired to know as soon as possible, so that they can look for other opportunities[1].

The problem of determining whether it is possible for an alternative to come out as winning, based only on incomplete information about the ballots of voters, is called the Possible Winner Problem. Similarly, we call the problem of determining whether an alternative is a necessary winner, given incomplete information, the Necessary Winner Problem. These problems were first studied by Konczak and Lang [2005], who study the computational complexity of the Possible Winner Problem and the Necessary Winner Problem under various voting rules in the context of ranked ballots. Besides being relevant to the situations described above, the Possible Winner and Necessary Winner problems are closely related to the problem of Coalitional Manipulation when only voters, and not alternatives, are missing [Boutilier and Rosenschein, 2016]: if an alternative is a possible winner in an incomplete profile missing only voters, then a coalition can successfully manipulate the election to make the alternative win. On the other hand, if the alternative is not a possible winner, there is no way the coalition can vote to make the alternative win. The problem of determining possible and necessary winners when only alternatives are missing is, on the other hand, related to the problem of Electoral Control by Adding Candidates [Faliszewski and Rothe, 2016], in which the chair of the election attempts to control the outcome of the election by introducing some amount of new alternatives. This problem was first studied by [Bartholdi et al., 1992], who study a version of the problem where the chair can add an unlimited (unbounded) number of new alternatives, and later by [Hemaspaandra et al., 2009], who consider a version of the problem where the number of alternatives that can be added by the chair is bounded by some number ℓ. Furthermore, the problem of determining possible and necessary winners is of relevance to procedures of preference elicitation [2]: if, when eliciting preferences, we keep track of the possible and necessary winners, then we know that we have elicited preferences sufficiently when the sets of possible and necessary winners coincide.

In this chapter, I study basic issues related to the problem of determining possible and

[1] This example is similar to a motivating example used by Xia et al. [2011], in which committees vote over research proposals and some proposals may arrive late.

[2] See, e.g., [Boutilier and Rosenschein, 2016], section 10.5 and [Conitzer and Sandholm, 2002].

necessary winners of elections with missing alternatives for the class of Size Approval Voting rules. The problem of determining possible and necessary winners in profiles with (only) missing alternatives has been studied by Chevaleyre et al. [2012] and Xia et al. [2011]. Chevaleyre et al. [2012] study the complexity of computing possible winners in profiles with missing alternatives under various scoring rules over ranked ballots. Xia et al. [2011] study the problem of determining possible winners when alternatives are missing for various voting rules not considered by Chevaleyre et al. [2012], and in particular, they consider the computational complexity of determining possible winners with missing alternatives for Approval Voting, under three different assumptions about how voters may change their ballots when gaining access to new information. Furthermore, Barrot et al. [2013] study the Possible Winner problem under Approval Voting in the context of ranked-ballot systems. I follow the approach of Xia et al. [2011], and consider some issues related to determining possible and necessary winners in profiles with missing information under two of the three assumptions about how voters respond to new information used in Xia et al. [2011].

5.1 Possible and Necessary Winners when New Alternatives are Added

Before discussing the problem of determining possible and necessary winners under Size Approval Voting rules, we first need to define what it means for a ballot to be missing information. To avoid confusion with the notion of a partial ballot, as defined in Chapter 2, I will call a ballot that possibly is missing some information about the election an incomplete ballot. As usual, let X denote the full set of alternatives and N the full set of voters, and let $|X| = m$ and $|N| = n$. We will assume that X can be divided into two disjoint subsets representing the known alternatives and unknown alternatives. Let K denote the set of known alternatives of X, and U the set of unknown alternatives of X, from which some subset $U' \subseteq U$ of new alternatives can be added. I will in some cases assume that the number of new alternatives that can be added is unrestricted, that is, that the number of new alternatives that can be added is not restricted by some fixed number ℓ. However, in general, the number of alternatives that can be added may also be restricted[3]. Furthermore, I will require that $X = K \cup U$

[3]Note that this should not be confused the case in which the number of alternatives that will be added is fixed.

and that $K \cap U = \emptyset$. Note that, if all alternatives are known, we have $X = K \cup U = K$. An incomplete profile is defined as follows.

Definition 5.1.1 (Incomplete profile)**.** Given a set of alternatives $X = K \cup U$ and a set of voters $N = \{1, ..., n\}$, an incomplete profile is an element $\boldsymbol{B}^{inc} = (B_1^{inc}, ..., B_{n'}^{inc}) \in (2^K)^{n'}$, where $n' \leq n$.

In words, an incomplete profile is a profile of the ballots cast by some (not necessarily strict) subset of the electorate over the set of known alternatives. As discussed in the chapter introduction, I will only be concerned with the problem of determining possible and necessary winner for the case in which only alternatives are missing. Therefore, unless stated otherwise, we will assume that an incomplete profile contains a ballot for each member of the electorate. However, this definition of incomplete profiles also allows us to consider the problem of determining possible and necessary winners with some number of voters missing.

To be able to consider the problem of determining whether an alternative is a possible or necessary winner in an incomplete $\boldsymbol{B}^{inc}$ under some voting rule f, we need to be able to pinpoint how an incomplete profile can be made complete. This is less than straightforward. If for some voting problem, all alternatives are known but some voters are missing, an incomplete profile may be completed by any combination of possible ballots for remaining voters without issue. However, if some alternatives are missing, we need to account for the fact that voters may change their ballots at the arrival of new alternatives not previously known or considered. In this case, voters may not only expand or restrict their ballot, but may also exchange some alternatives on their ballot for newly arrived alternatives. For example, consider the following scenario. A group of voters are voting over three alternatives, where the winner is chosen by plurality vote and the third alternative joins the election later than the first two. Suppose now that there is some voter who would vote for the first alternative over the second when only the two first alternatives are considered. Further, suppose that when the third alternative joins, this voter prefers the first alternative to the third alternative to the second alternative, and that the voter believes the third alternative to have a better chance of winning against the second alternative than the first alternative has. Under these circumstances, it is reasonable to think that the voter will change their vote so as to vote for the third alternative instead of the first. On the other hand, if the voter prefers alternative one to two to three, but believes alternative two to have a better chance of winning than three, the voter may even exchange the first alternative for the second.

Let us call a ballot that completes another ballot an extension of the first ballot, and say that a profile $\boldsymbol{B}$ extends an incomplete profile $\boldsymbol{B}^{inc}$ if each voter's ballot in $\boldsymbol{B}$ is an extension of their ballot in $\boldsymbol{B}^{inc}$. If we allow voters to make any conceivable change to their ballot (that is, expanding, restricting, exchanging alternatives for old and new ones), then any ballot extends the old ballot. In this case, it follows that for any incomplete profile $\boldsymbol{B}^{inc}$, any alternative is a possible winner. On the other hand, no alternative is a necessary winner. This is not very informative. Therefore, we need to restrict the possible ways voters can extend their ballots. I make use of two different definitions of ballot extensions due to Xia et al. [2011][4].

The first definition requires that voters only possibly extend their ballot with new alternatives, while never removing some already approved alternative or approving of some previously disapproved alternative. Note that this definition of ballot extension is quite restrictive. In particular, since this definition does not allow voters to re-evaluate their approval of old alternatives when new alternatives arrive, a consequence of using the definition is that the problem of determining possible and necessary winners under plurality voting becomes trivial.

Definition 5.1.2 (Extension 1)**.** A ballot $B' \subseteq X = K \cup U$ is an extension of a ballot $B \subseteq K$ if $B' \cap K = B$. Furthermore, a profile $\boldsymbol{B}$ is an extension of an incomplete profile $\boldsymbol{B}^{inc}$ if B_i extends B_i^{inc} for all $i \in N$.

Under the second definition of ballot extension, voters can either keep their ballot unchanged or exchange some number of old alternatives for some number of new alternatives.

Definition 5.1.3 (Extension 2)**.** $B' \subseteq X = K \cup U$ is an extension of $B \subseteq K$ if one of the following conditions holds

[4]Xia et al. [2011] also consider a third definition of ballot extension, under which voters may come to approve of previously disapproved alternatives when new information arrives. Under such a definition of ballot extensions, every alternative will a possible winner, so I will not consider it.

- $B = B'$
- $B' \cap U \neq \emptyset$ and $B' \cap K \subseteq B$.

I will first discuss the problem of determining possible and necessary winners for various Size Approval Voting rules under Extension 1. In this case, I will assume that no voter will extend their ballot in such a way that a ballot that was non-trivial becomes trivial. That is, given an incomplete profile $\boldsymbol{B}^{inc}$, I will only consider extensions $\boldsymbol{B}'$ of $\boldsymbol{B}^{inc}$ for which it holds, for each $i \in N$, that $w_{|B_i'|} > 0$ whenever $w_{|B_i^{inc}|} > 0$. If we exclude this assumption, the problem of computing possible winners becomes for many rules in the class: if all voters extend their ballots to ballots that have zero weight, all alternatives become winning. Thus, every alternative is a possible winner.

We can formulate the problem of determining whether an alternative is a possible winner as the following decision problem.

PWE1(f)
Input: Incomplete profile $\boldsymbol{B}^{inc}$, alternative $x^* \in K$
Question: Is x^* a possible winner with under voting rule f with Extension 1?

Xia et al. [2011] show that under Approval Voting, an alternative $x \in K$ is a possible winner in an incomplete profile $\boldsymbol{B}^{inc}$ if and only if the alternative is a winner in the incomplete profile $\boldsymbol{B}^{inc}$. This result extends easily to any other p-approval rule, under the assumption made above.

Proposition 5.1.4. Let f be a p-approval rule, $\boldsymbol{B}^{inc}$ an incomplete profile. Then $x \in K$ is a possible winner in $\boldsymbol{B}^{inc}$ with Extension 1 if and only if

$$x \in f(\boldsymbol{B}^{inc}).$$

Proof. Let f be a p-approval rule, $\boldsymbol{B}^{inc}$ an incomplete profile, and $x \in K$. If $\boldsymbol{B}'$ is an extension of $\boldsymbol{B}^{inc}$ for which it holds that $w_{|B_i'|} > 0$ for all $i \in N$ such that $w_{|B_i^{inc}|} > 0$, we must have that $Supp_{\boldsymbol{w}}(\boldsymbol{B}', x) = Supp_{\boldsymbol{w}}(\boldsymbol{B}^{inc}, x)$ for all $x \in K$. Thus, it follows that if $x \notin f(\boldsymbol{B}^{inc})$, there is no extension $\boldsymbol{B}'$ of $\boldsymbol{B}^{inc}$ for which $x \in f(\boldsymbol{B}')$. Suppose now that $x \in f(\boldsymbol{B}^{inc})$, and consider the extension $\boldsymbol{B}' = \boldsymbol{B}^{inc}$, where no voter extends their ballot with any new alternative. It is immediate that $x \in f(\boldsymbol{B}')$. □

It follows that PWE1(f) is in $\mathcal{P}$ for all p-approval rules f.

For those rules for which the (non-zero) weights of ballot sizes differ, the problem of determining possible winners in incomplete profiles becomes more complicated. Under such rules, an alternative may come to win because another alternative that previously had strictly more weighted support has lost a significant amount of that weighted support through voters extending their ballots to include new alternatives. Furthermore, whether it is possible for such a situation to occur under a given rule is dependent not only on the weights that parameterize the rule, but also on the number of new alternatives that can be added. Thus, there is not an immediate clear cut way to characterize the possible winners in a given profiles for such rules. However, if the number of alternatives is sufficiently large and f is a rule with weights that decrease 'sufficiently' with the size of ballots, we can characterize the possible winners in an incomplete profile $\boldsymbol{B}^{inc}$ under Extension 1 as follows: the possible winners are those known alternatives x for which there exists no other known alternative y such that the set of ballots in support of y is a strict superset of the ballots in support of x. The idea is that, in such cases, we can construct an extension $\boldsymbol{B}'$ of $\boldsymbol{B}^{inc}$ by 'overloading´ the ballots not in support of x with new alternatives. If the set of new alternatives is sufficiently large and weights decrease sufficiently with the size of ballots, we can make sure that the weighted support given to x in the profile $\boldsymbol{B}$ exceeds both that of the new alternatives and that of old alternatives that previously had more weighted support than x. In particular, this is possible for many of the SDSAV rules. As a concrete example, I show that this is the case for the rule of EECV.

Proposition 5.1.5. If the number of new alternatives to be added is unrestricted, $x \in K$ is a possible winner under EECV with Extension 1 if and only if there is no $y \in K$ such that $y \in B_i$ for all $i \in N$ with $x \in B_i$ and $x \notin B_i$ for some $i \in N$ with $x \in B_i$.

Proof sketch.
($\Rightarrow$) Let $x \in K$ and suppose there is a $y \in K$ such that $y \in B_i^{inc}$ for all $i \in N$ with $x \in B_i^{inc}$ and $x \notin B_i^{inc}$ for some $i \in N$ with $x \in B_i^{inc}$. Then $Supp_{\boldsymbol{w}}(\boldsymbol{B}^{inc}, y) > Supp_{\boldsymbol{w}}(\boldsymbol{B}^{inc}, x)$. Furthermore, because $y \in B_i^{inc}$ for all

$i \in N$ with $x \in B_i^{inc}$, it follows that if y experiences some reduction in support by some voter i with $y \in B_i$ extending their ballot, then x will experience the same reduction. Thus. $Supp_{\boldsymbol{w}}(\boldsymbol{B'}, y) > Supp_{\boldsymbol{w}}(\boldsymbol{B'}, x)$, for all profiles $\boldsymbol{B'}$ that extend $\boldsymbol{B}^{inc}$, so x is not a possible winner in $\boldsymbol{B}^{inc}$.

($\Leftarrow$) Suppose there is no $y \in K$ such that $y \in B_i^{inc}$ for all $i \in N$ with $x \in B_i^{inc}$ and $x \notin B_i^{inc}$ for some $i \in N$ with $x \in B_i^{inc}$. Observe that since the number of new alternatives that can be added is not restricted to some fixed ℓ, we can keep adding new alternatives from U to each ballot B_i^{inc} such that $x \notin B_i^{inc}$ until we have an extension $\boldsymbol{B'}$ of $\boldsymbol{B}^{inc}$ in which $Supp_{\boldsymbol{w}}(\boldsymbol{B'}, x) \geq Supp_{\boldsymbol{w}}(\boldsymbol{B'}, y)$ for all $y \in K \cup U$. Thus, x is a possible winner in $\boldsymbol{B}^{inc}$. $\square$

I give an example of a SDSAV rule for which the above characterization does not hold.

Example 5.1.6. Consider the SDSAV rule given by $\boldsymbol{w} = (w_1, ..., w_m)$, with $w_1 = 1$ and $w_k = w_{k-1} - \epsilon^{k-1}$ for each $k > 1$ and some small $\epsilon > 0$, briefly mentioned in 3.2.2. Suppose that $\epsilon = 0.1$. Since $w_k = w_{k-1} - 0.1^{k-1}$ for any $k > 1$, we have, for a given ballot B, that $w_{|B|}$ tends to 0.9 as $|B|$ goes towards ∞. Consider now a known set of alternatives $K = \{x, y\}$ and voters $N = \{1, 2, 3\}$, and suppose that $B_1 = \{x\}$ and $B_2 = B_3 = \{y\}$. Obviously, the set ballots in support of alternative y is not a superset of the ballots in support of x. However, by the above discussion, there is no way to extend the ballots B_2 and B_3 so that alternative x wins.

Consider now the problem of determining necessary winners under Size Approval Voting rules, given Extension 1, formulated as the following decision problem.

NWE1(f)

Input:	Incomplete profile $\boldsymbol{B}^{inc}$, alternative $x^* \in K$
Question:	Is x^* a necessary winner under voting rule f with Extension 1?

Under Extension 1, each voter may extend their ballot to include any new alternative, and in particular, all voters may extend their ballot to include the same new alternative. Thus, an alternative $x \in K$ can only be a necessary winner in an incomplete profile $\boldsymbol{B}^{inc}$ under Approval Voting if the alternative x has unanimous support in $\boldsymbol{B}^{inc}$. The same holds for any Size Approval Voting rule for which $w_k > 0$ for all $k < m$. It is however not generally true for rules in the class. For example, under the trivial rule, all alternatives are both possible and necessary winners. More crucially, under the assumption that voters will not extend a ballot B_i^{inc} with associated non-zero weight to one for which the associated weight is zero, there are several rules for which it may not be possible for some voter i to extend their ballot B_i^{inc} with new alternatives, because $w_{B_i'} = 0$ for all ballots B_i' of size larger than B_i. In such cases, an alternative could be a necessary winner in an incomplete profile $\boldsymbol{B}^{inc}$ without having unanimous support in said profile. For example, as mentioned earlier, under the plurality rule no voter will ever be able to include new alternatives on their ballot. In this case, the necessary winners of an incomplete profile $\boldsymbol{B}^{inc}$ correspond to the set $f(\boldsymbol{B})$.

Consider now the following algorithm for determining whether an alternative is a necessary winner in incomplete profile $\boldsymbol{B}^{inc}$ under Size Approval Voting rule f and Extension 1. The algorithm works as follows: given an incomplete profile $\boldsymbol{B}^{inc}$ and some alternative $x^* \in K$ as input, we let the set of alternatives X be composed of K and a single new alternative d. If x^* is dominated in $\boldsymbol{B}^{inc}$ by some other alternative $y \in K$, the algorithm rejects, as x^* cannot be a necessary winner. Otherwise, we construct an extension $\boldsymbol{B}'$ of $\boldsymbol{B}^{inc}$ by adding the alternative d to every ballot that can be extended with a new alternative. If the total weighted support given to alternative d in the profile $\boldsymbol{B}'$ is greater than that given to x^*, the algorithm accepts. Otherwise, it rejects.

Algorithm 1: NWE1(f)

```
Input: B^inc = (B_1, ..., B_n), x* ∈ K
X ← K ∪ {d}
if x ∉ f(B^inc) then
 | Reject.
else
 | for B_i ∈ B^inc do
 |  | if w_{|B_i|+1} > 0 then
 |  |  | B'_i ← B_i ∪ {d}
 |  | else
 |  |  | B'_i ← B_i
 | B' ← (B'_1, ..., B'_n)
 | if x* ∈ f(B') then
 |  | Accept.
 | else
 |  | Reject.
```

Observe that the algorithm must correctly determine whether an alternative $x^* \in K$ is a necessary winner in a profile $\boldsymbol{B}^{inc}$ under any Size Approval Voting rule f that is not the trivial rule. By definition, the alternative x^* is a necessary winner in $\boldsymbol{B}^{inc}$ if and only if it is not dominated by any other (old or new) alternative in any extension $\boldsymbol{B}'$ of $\boldsymbol{B}^{inc}$. If the alternative x^* is dominated by an alternative $y \in K$, then x^* is also dominated by the alternative y in the extension $\boldsymbol{B}' = \boldsymbol{B}^{inc}$, so x^* is not a necessary winner. On the other hand, since no alternative $y \in K$ can gain support by ballots being extended under Extension 1, it follows that if x^* is not dominated by y in $\boldsymbol{B}^{inc}$, then $\boldsymbol{B}^{inc}$ is not dominated by y in any extension $\boldsymbol{B}'$ of $\boldsymbol{B}^{inc}$. Furthermore, if a new alternative $d \in U'$ does not dominate the alternative x^* in the extension $\boldsymbol{B}'$ where d is the only new alternative added to those ballots that can be extended with new alternatives, then d does not dominate x^* in any extension $\boldsymbol{B}'' \neq \boldsymbol{B}'$. On the other hand, if the alternative d does dominate x^* in such an extension $\boldsymbol{B}'$, x^* is clearly not a necessary winner.

It follows that the problem of determining necessary winners in an incomplete profile under Extension 1 is in $\mathcal{P}$ for all Size Approval Voting rules.

Proposition 5.1.7. NWE1(f) is in $\mathcal{P}$ for all Size Approval Voting rules f.

Proof. Let f be a Size Approval Voting rule. First, if f is the trivial rule, it is immediate that all alternatives are necessary winners. Therefore, suppose f is not the trivial rule. Consider a problem instance consisting of some incomplete profile $\boldsymbol{B}^{inc}$ and alternative $x^* \in K$. By the discussion above, Algorithm 1 accepts the problem instance if and only if the alternative x^* is a possible winner in the incomplete profile $\boldsymbol{B}^{inc}$. Furthemore, it is easy to see that the algorithm runs in time polynomial in the size of the input. □

Let us now consider the problem of determining possible and necessary winners under Size Approval Voting rules f, assuming that voters extend their ballots according to Extension 2.

PWE2(f)
Input: Incomplete profile $\boldsymbol{B}^{inc}$, alternative $x^* \in K$
Question: Is x^* a possible winner with Extension 2 under voting rule f?

I again discuss the special case of determining possible winners under the assumption that the number of new alternatives that can be added is unrestricted. Let us assume additionally that for any voter i, we have $B_i^{inc} \cap B_i' = \emptyset$ for any ballot B_i^{inc} with associated weight $w_{|B_i^{inc}|} = 0$ and extension B_i' with $w_{|B_i^{inc}} > 0$. In words, assume that a voter who abstained from giving strictly positive support to any known alternative will not change their mind about this once new alternatives arrive. Then it holds for any Size Approval Voting rule f, modulo the trivial rule, that an alternative is a possible winner in a profile $\boldsymbol{B}^{inc}$ if and only if it is on some ballot that has strictly positive weight.

Proposition 5.1.8. Let f be a Size Approval Voting rule, and suppose that f is not the trivial rule. If no voter who abstained from giving strictly positive support to any known alternative changes their mind about the known alternatives when new alternatives arrive, then an alternative $x \in K$ is a possible winner in incomplete profile $\boldsymbol{B}^{inc}$ if and only if $x \in B_i^{inc}$ for some B_i^{inc} with $w_{|B_i^{inc}|} > 0$.

Proof. Observe first that by the assumption made above, a known alternative that received zero weighted support in $\boldsymbol{B}^{inc}$ cannot be a winner in any extension $\boldsymbol{B}'$ of $\boldsymbol{B}^{inc}$. On the other hand, suppose for $x \in K$ that $x \in B_i^{inc}$ for some B_i^{inc} with $w_{|B_i^{inc}|} > 0$. Consider a profile $\boldsymbol{B}'$ with $B_j' \cap B_{j'}' = \emptyset$ for all $j, j' \in N$, obtained from $\boldsymbol{B}^{inc}$ by replacing the ballot B_j of every $j \in N \setminus \{i\}$ with a ballot B_j' containing $|B_i|$ new alternatives. This profile is a valid extension of $\boldsymbol{B}^{inc}$ under Extension 2. Furthermore, since there is no restriction on the number of alternatives that can be added, we can clearly construct it. Observe now that since $B_j' \cap B_{j'}' = \emptyset$ for all $j, j' \in N$, each new alternative added only occurs on a single ballot. Furthermore, no known alternative that is not on the ballot B_i^{inc} receives any support in B_i'. It follows that the alternative x must have maximal weighted support in $\boldsymbol{B}'$. □

Determining possible winners under Extension 2 in the more general case where the number of new alternatives that can be added may be bounded by some ℓ is less straightforward. Xia et al. [2011] shows that the Possible Winner Problem under Extension 2 is $\mathcal{NP}$-complete for Approval Voting, and it is reasonable to think that this is the case for several other rules in the class. However, I have not been able to provide results to confirm this.

Finally, consider the problem of determining necessary winners in an incomplete profile $\boldsymbol{B}^{inc}$ under Extension 2.

NWE2(f)
Input: Incomplete profile $\boldsymbol{B}^{inc}$, alternative $x^* \in K$
Question: Is x^* a necessary winner with Extension 2 under voting rule f?

It is easy to see that under this definition of ballot extensions, there is no alternative that is a necessary winner, for any Size Approval Voting rule f that is not the trivial rule.

Proposition 5.1.9. No alternative is a necessary winner under Extension 2 for any incomplete profile $\boldsymbol{B}^{inc}$ or Size Approval Voting rule f.

Proof. Let f be any Size Approval Voting rule, let $\boldsymbol{B}^{inc}$ be an incomplete profile and let $x \in K$. Consider any extension $\boldsymbol{B}'$ of $\boldsymbol{B}^{inc}$ in which $B'_i \cap K = \emptyset$ for all $i \in N$. Clearly, $x \notin f(\boldsymbol{B}')$, and so x is not a necessary winner in $\boldsymbol{B}^{inc}$. □

5.2 Discussion

This chapter has been dedicated to issues concerning the problem of determining possible and necessary winners under Size Approval Voting rules when some alternatives are missing. Under two different assumptions, due to Xia et al. [2011], about how voters will respond when new alternatives arrive, I studied the problem of characterizing possible and necessary winners under various Size Approval Voting rules. I showed that, under each of the assumptions, the Necessary Winner problem is in $\mathcal{P}$ for all Size Approval Voting rules, being trivial under one of the assumptions. Furthermore, under each of the assumptions, I characterized possible winners under various rules in the class for the special case where there is no fixed limit on the number of new alternatives that can be added. In particular, I characterized possible winners under the rule of EECV when the number of new alternatives that can be added is not limited by some fixed number ℓ. It is worth mentioning that these results suggest the problem of electoral control by adding an unlimited number of new alternatives, as studied by Bartholdi et al. [1992], is in $\mathcal{P}$ for EECV.

The results obtained in this chapter were quite limited. In particular, I have not provided many general results regarding the complexity of the Possible Winner Problem for rules in the class. Such results should be provided for more rules in the class. In particular, it would be beneficial to determine the complexity of the Possible Winner Problem for the rule of EECV, under each of the assumptions about voters' response to new information that were considered.

Chapter 6

Conclusion and Future Work

In an attempt to better understand the class of Size Approval Voting rules and its members, I have analyzed the class from different angles of Computational Social Choice. In this section, I discuss the work that has been done. I will first summarize results obtained, reiterate limitations discussed, and make some general observations about the class, and thereafter outline some directions for future work.

In Chapter 3, I considered the class of Size Approval Voting rules from an axiomatic point of view. I axiomatically related the class to the class of scoring rules, simplified a previous characterization of the class, and gave some alternative axiomatizations of the class based on its relation to scoring rules and said simplification. I further suggested axioms that can be used to obtain two important subclasses of the class, and gave a characterization of the rule EECV. As was pointed out in Chapter 3, some axioms suggested in this section, in particular axioms used to capture the subclasses of Size Approval Rules considered, were somewhat artificial. I defended the use of these axioms by pointing out that the classes themselves can be seen as somewhat artificial. However, it should still be considered whether these subclasses can be captured using axioms that are less technical than the ones suggested in this thesis.

Subsequently, in Chapter 4, I studied the properties of the class with respect to manipulability. Relying on the notion of sincerity from Brams and Fishburn [1978], I showed that all rules in the class are sincere when all voters have dichotomous preferences, and that many of the rules in the class can be seen as to incentivize strongly insincere behavior under less strict assumptions about the preferences of voterrs. As mentioned, these results can be thought of as showing that no rule in the class is vulnerable to manipulative behavior when the approval balloting system directly reflects the preferences of voters, and that many rules are vulnerable to manipulation when this does not hold. I furthermore showed that the decision problem of strategic manipulation is trivial for all Size Approval Voting rules. Additionally, I discussed the problem of manipulation under partial information. Under the assumption that a voter will only use an insincere strategy if it dominates the sincere strategies of the voter, I showed that voters are disincentivized from using insincere strategies under zero information. We also saw that, under the (lack of) assumptions made about how the preferences of voters extend to sets of alternatives, it is not possible to determine whether voters are incentivized to use insincere strategies when knowing (only) the winners of an election under certain rules in the class. Some of the results concerning the (in)sincerity of rules in the class could be made more precise, e.g., by characterizing more exactly and exhaustively the conditions under which rules in the class admit the use of insincere strategies. The issues considered in Chapter 4 were limited to ones that concern the vulnerability of the rules with respect to manipulation by individual voters. However, as noted in the discussion, there are several other kinds of manipulation problems in Voting Theory, such as the problem of coalitional manipulation [Conitzer and Walsh, 2016], bribery, and electoral control [Faliszewski and Rothe, 2016]. Although we briefly discussed the properties of Size Approval Voting rules with respect to the problem of coalitional manipulation, the properties of the class with respect to the latter two problems should be studied further.

Lastly, in Chapter 5, I considered the problem of computing possible and necessary winners in incomplete elections with missing alternatives, under two different assumptions about how voters complete their votes when given access to new information. I showed that the problem of determining necessary winners is computationally easy for all rules in the class under each of the assumptions considered. I furthermore provided results characterizing possible winners under various rules in the class when the number of new alternatives that can be introduced is restricted. The results concerning the problem of determining possible winners were limited, and more general results should be provided. In particular, it would be interesting to determine the complexity of both versions of the Possible Winner Problem for more natural rules in the class, such as the rule of EECV.

Overall, many of the results obtained in this thesis suggest that properties shared by rules in the class are limited. The fact that rules in the class exhibit quite different properties with respect to several of the problems considered could suggest that the class has some degree of artificiality. On the other hand, being quite general and imposing relatively few restrictions, it is hardly surprising that the class allows for rules that can behave very differently. The class perhaps also serves as an example of the limitations to defining scoring rules over approval ballots. It contains some of the most natural scoring rules that we can define in an approval-based system, but there aren't many such rules relative to the size of the class.

I now briefly discuss future work beyond the open problems discussed above. There are many problems studied in Voting Theory that go beyond the scope of this thesis. In particular, it could be of interest to study the properties of the class of Size Approval Voting rules with respect to several problems not considered that, like the problems of determining possible and necessary winners based on incomplete information, concern the informational and communicational burdens imposed by voting rules. I mention two. Firstly, it is sometimes advantageous to use voting rules that minimize the communication burden put on voters with respect to computing the outcome of an election. The number of bits required from voters to be able to compute the outcome of an election under a given voting rule f is called the communication complexity of f. This problem has been studied by Conitzer and Sandholm [2005], who, for one, derive tight bounds on the communcation complexity of Approval Voting. It is easy to see that, for n voters and m alternatives, the communication complexity of rules in the class of Size Approval Voting is upper bounded by nm, as each voter can submit one bit corresponding to approval or disapproval for each of the m alternatives. However, lower bounds on the communication complexity of rules in the class should also be derived. A related problem is that of determining the minimal amount of information from an intermediary election needed to be stored for later purposes. For a given voting rule f, the number of bits required to store the information of an intermediary election is called the compilation complexity of f. Chevaleyre et al. [2009] study the compilation complexity of several voting rules, among others the plurality rule, and derive tight bounds on the compilation complexity of the rules using a technique that relies on counting the number of profile equivalence classes under a given voting rule.

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