Topic-relevance and Hyperintensional Belief

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis introduces a novel doxastic logic, denoted as L_{TB}, which addresses the challenges posed by traditional accounts of hyperintensionality. In L_{TB}, we use the notion of topic-relevance to achieve hyperintensionality, modeling belief based on the fragments generated by topics and information pieces.

In this thesis, we show that L_{TB} can deal with different variations of the logical omniscience problem and provide a nuanced interpretation of a model alethic operator within a doxastic context. We present a sound axiomatisation for L_{TB} and we compare L_{TB} with other existing hyperintensional theories.

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Contents

Chapter 1

Introduction

In contemporary logic, the exploration and modeling of knowledge, belief and other epistemic notions is a popular area of research. Tracing back to Hintikka's foundational works [26, 25], epistemic concepts are interpreted in terms of possible world semantics and are framed as modal operators in epistemic modal logic [47]. According to this kind of logic, knowing a proposition means that this proposition is true in all the possible worlds of the epistemic state of the agent, where an epistemic state is usually a set of possible worlds that are considered epistemically indistinguishable to the agent. However, this traditional approach is far from perfect and struggles with challenges, including difficulties in dealing with problems tied to various forms of logical omniscience and the intricacies of alethic modal operators that are used in combination with epistemic attitudes.

While mainly studied in the context of knowledge, the problem of logical omniscience also applies to the doxastic attitude of belief. Belief is typically taken to be non-factive and hence it differs from knowledge in an essential way, as knowledge is assumed to be truthful. Logics for a variety of notions of belief have been studied in the literature and range from the doxastic logic KD45 based to logics based on plausibility models or probabilistic models [4, 48, 28, 9].

The principal aim of this thesis is to confront the challenges posed by standard modal logics for epistemic attitudes, by focusing in the first instance on the doxastic attitude–belief. This thesis adopts a qualitative account of belief, grounded in a relational Kripke-style semantics, while integrating the idea of fragmentationalism. This suggests that an agent's epistemic states are divided into fragments based on subject matter, and it is from these fragments that beliefs arise. Drawing inspiration from recent works of Berto et al. [5, 37, 23], this thesis endeavors to develop a doxastic logic for modelling hyperintensional belief through the use of the concept of topic-relevance.

1.1 Logical omniscience

The traditional Hintikkan style epistemic logics lead to undesired results when modeling the reasoning of human agents. The logical systems demand agents to possess knowledge or beliefs that are unrealistic, given that they only have limited computational, conceptual and reasoning capacities, as well as finite time restrictions [44, 13]. This is called the logical omniscience problem: only omniscient agents know or believe all tautologies and all logical consequences of their knowledge or belief. For instance, if human agents believe or know that 1+1=2, then standard logical systems may require them to believe or know the truth of Fermat's Last Theorem, since the two propositions are logically equivalent. Nonetheless, in reality, agents may believe the former but not the latter if they are not skilled in mathematics.

Although logical omniscience pertains both to knowledge and belief, in this thesis we only focus on the latter, as our aim is to construct a hyperintensional theory for belief. The following list includes some of the most classic logical omniscience principles as applied to belief, following the exposition in [47, 54, 42]:

- Omniscience rule: All truths are believed. Formally, if $\vDash$ then $\phi \vDash B\phi$, where B is the belief operator and $B\phi$ means that ϕ is believed by an implicitly assumed agent. It says that if ϕ is valid, then the agent believes ϕ.

- Closure under disjunction introduction: A disjunction is believed if one disjunct is believed. Formally, it is noted as $B\phi \vDash B(\phi \vee \psi)$. It says that if ϕ is believed then the agent believes $\phi \vee \psi$.

- Closure under material implication: A consequent of a material implication is believed if the antecedent is believed. This is formally noted as $(\psi \rightarrow \phi) \wedge B\psi \vDash B\phi$, where the material implication $\psi \rightarrow \phi$ is defined by $\neg\psi \vee \phi$ classically. It says that if the agent believes ψ and $\psi \rightarrow \phi$ is valid, then the agent believes ϕ.

- Closure under strict implication: A consequent of a strict implication is believed if the antecedent is believed. This is formally noted as $\Box(\psi \rightarrow \phi) \wedge B\psi \vDash B\phi$, where $\Box$ is the S5 modal operator and is interpreted as "necessary". It says that if the agent believes ψ and $\Box(\psi \rightarrow \phi)$ is valid, then the agent believes ϕ.

- Closure under logical consequence: A logical consequence is believed if the logical antecedent is believed. Formally, if $\psi \vDash \phi$, then $B\psi \vDash B\phi$. It says that if ψ logically entails ϕ and the agent believes ϕ, then the agent believes ψ.

- Closure under logical equivalence: A formula is believed if its logically equivalent formula is believed. Formally, if $\psi \dashv\vDash \phi$, then $B\psi \vDash B\phi$. It says that if ψ logically equivalent to ϕ and the agent believes ψ, then the agent believes ϕ.

To address the logical omniscience problem, hyperintensional logics and theories are developed. These approaches are hyperintensional since they allow for distinctions between the propositions that may be true in the same possible worlds but are not equivalent due to some reasons. There are various popular theories aiming to deal with the problem, such as those discussed in [13, 22, 6]. Some theories aim to deal with closures under different kinds of implications, like relevance logic [39] and analytic implication [15], which capture the relevance between antecedents and consequent. There are also approaches for dealing with general logical omniscience, such as syntactical approaches where the knowledge of an agent is represented by a set of formulas. One of the most classic variant of this approach is awareness logic, developed by Halpern and Fagin[14, 21]. In awareness logic an additional function that assigns a set of propositions to each world, indicating which propositions an agent can be aware of. Then an agent can believe a proposition only when he is aware of the proposition [41]. Another important solution to the logical omniscience problem is the impossible worlds approach [38, 35], which relies on the notion of impossible world and posits that the agents may consider possible worlds that are logically inconsistent.

While previous approaches offer some degree of resolution to certain omniscience problems, there is still no universally accepted theory capable of resolving all forms of logical omniscience. Theories based one impossible worlds receives various objections as its notion is both philosophically and logically controversial [50, 51, 45, 6]. Other theories like those based on awareness logic or relevance logic fall short in dealing with some variant versions of omniscient inferences. For example, in awareness logic, if the agent is aware of a proposition and the proposition is a tautology, then the proposition is believed by the agent. But this is counterintuitive. It's easy to imagine a 20th-century mathematician who was aware of Fermat's last theorem yet did not believe it to be true, despite the fact that we currently know that it is equivalent to a tautology.

We denote the condition clause, such as "the agent is aware of the proposition", "the agent is thinking about the proposition" or "the proposition is relevant to the agent" as the hyperintension condition, since they can be used to distinguish the propositions that are true in the same possible worlds. Our aim is not to equate these sentences, but rather to highlight a shared effect: it is noticeable that the omniscience inferences become valid in theories like awareness logic [41] or topic-sensitive logic [37] when they are restricted by the hyperintension condition. But this problem can be avoided in our theory. We introduce the formal form of the hyperintension condition in section 3.5 and we present the proofs in section 4.1 that all of the aformentioned omniscience principles, even when restricted by the hyperintension condition like the following, remain invalid in our logic:

- Restricted omniscience rule: All relevant truths are believed. If ϕ is valid and the agent is aware of or thinking about ϕ, then the agent believes ϕ.
- Restricted closure under disjunction introduction: A relevant disjunction is believed if one disjunct is believed. If ϕ is believed and the agent is aware of or thinking about $\phi \vee \psi$, then the agent believes $\phi \vee \psi$.

- Restricted closure under material implication: A relevant consequent of a material implication is believed if the antecedent is believed. If $\psi \to \phi$ is true and the agent believes ψ and is aware of or thinking about ϕ, then the agent believes ϕ.
- Restricted closure under strict implication: A relevant consequent of a strict implication is believed if the antecedent is believed. It says that if $\Box(\psi \to \phi)$ is valid and the agent believes ψ and is aware of or thinking about ϕ, then the agent believes ϕ.
- Restricted closure under logical consequence: A relevant logical consequence is believed if the logical antecedent is believed. If ψ logically entails ϕ and the agent believes ϕ and is aware of or thinking about ϕ, then the agent believes ψ.
- Restricted closure under logical equivalence: A relevant formula is believed if its logically equivalent formula is believed. If ψ logically equivalent to ϕ and the agent believes ϕ and is aware of or thinking about ϕ, then the agent believes ψ.

1.2 Fragmentation

Traditional epistemic logics and even hyperintensional logics (e.g. awareness logic) assume that an agent has a single coherent belief system or knowledge system that guides their actions at all moments in time. However, it is argued that this is too ideal for an actual human being. Inspired by this concern, the idea of fragmentation was proposed by Lewis[32], Stalnaker [43], Fagin and Halpern [14], and recently developed by Egan [12], Yalcin [55], Berto and Özgün [8]. According to fragmentationalism, the systems of belief that we in fact have are fragmented or compartmentalized. Rather than having a single epistemic state for all of our beliefs, our epistemic state is fragmented into different parts and we can use only one part or several parts in one situation.

Consider the following example from Lewis [32]:

Example 1. I used to think that Nassau Street ran roughly east-west; that the railroad nearby ran roughly north-south; and that the two were roughly parallel.

This belief of Lewis seems inconsistent, since if Nassau Street ran roughly east-west and the railroad nearby ran roughly north-south, then the two could not be roughly parallel. However, this can be a belief for Lewis or any of us in daily life. We can imagine, for example, that a friend of Lewis told him about the orientation of Nassau Street when they were shopping nearby and another friend of Lewis told him the orientation of the railroad when they were drinking on the train and Lewis himself knows that the two are roughly parallel when he checked the map. Then it is possible that Lewis held all of the three inconsistent beliefs since he never put these three situations and beliefs together and considered them as a whole.

Thus, as Lewis writes: "So each sentence in an inconsistent triple was true according to my beliefs, but not everything was true according to my beliefs. Now, what about the blatantly inconsistent conjunction of the three sentences? I say that it was not true according to my beliefs. My system of beliefs was broken into (overlapping) fragments. Different fragments came into action in different situations, and the whole system of beliefs never manifested itself all at once... The inconsistent conjunction of all three did not belong to, was in no way implied by, and was not true according to, any one fragment. That is why it was not true according to my system of beliefs taken as a whole. Once the fragmentation was healed, straightway my beliefs changed: now I think that Nassau Street and the railroad both run roughly northeast-southwest." [32]

From the fragmentationalism point of view, Lewis' example can be explained naturally. Since Lewis' belief system is fragmented, he may use fragment 1 to form the belief that Nassau Street ran roughly east-west and use fragment 2 to form the belief that the railroad nearby ran roughly north-south and use fragment 3 to form the belief that the two were roughly parallel. The three beliefs are formalized reasonable based on the corresponding fragments of the epistemic state. But when Lewis puts the three fragments together and checks them as a whole, he can realize that the three beliefs are inconsistent.

A good motivation for fragmentationalism is that non-omniscient agents only have limited cognitive capacity [54, 55]. We may receive a very large amount of information, but can only process part of the information we get, due to computational, conceptual and time limitation. Given a specific situation, we only need to focus on the information pieces that are relevant to the situation. For example, in a English exam we need not to think about Fermat and in a maths exam we need not to think about Shakespeare, although we have the information of both of them.

The approach of fragmentation blocks at least two kinds of inferences:

- Closure under conjunction: A conjunction is believed if all the conjuncts are believed. Formally, this is noted as $B\phi \wedge B\psi \vDash B(\phi \wedge \psi)$. It says that if the agent believes both ϕ and ψ, then the agent also believes $\phi \wedge \psi$. In short, the agent can always put beliefs together.

- Closure under doxastic implication: A consequence is believed if the implication and the antecedent are believed. Formally, this is noted as $B\psi \wedge B(\psi \to \phi) \vDash B\phi$. It means that if the agent believes both ψ and $\psi \to \phi$, then the agent also believes ϕ. Note that it is different to Closure under material implication, since the latter only requires the implication to be valid but not believed.

Closure under conjunction says that if the two conjuncts are believed, then the conjunction is believed. This is invalid according to fragmentationalism since the two conjuncts can be believed in different fragments and there may be no fragments based on which the conjunction is believed. This is just what Lewis' example indicates. Closure under doxastic implication is also invalid due to the same reason. The agent can use one fragment to believe ψ and use another fragment to believe $\psi \to \phi$, but there may be no fragment based on which the agent believes ϕ.

However, it is also noticeable that the fragmentationalism still remains controversial and has been criticized by Norby and Jago [36, 30]. They pointed out that the fragmentation approach is ad hoc. Although it solves some problems, the theory itself does not explain why the agent's epistemic state should be fragmented in one way rather than another way. In other words, the fragmentation is trivial, since it only says that the agent forms a belief in one circumstance by one fragment but says nothing about why the agent forms the belief in the circumstance by the fragment and nothing about the connections between beliefs, circumstances and fragments. As Norby [36] writes: "The fragmentational theory won't work, then, and it won't work for the simple reason that the theory makes it too easy for a mental state to have the connection to thought and behavior that belief is supposed to have. That's setting the bar too low. It is far from clear that the notion of a 'fragment' serves any purpose except as an aid to imagination, a way to more simply restate the phenomenon that we're trying to understand: that an agent can manifest a belief in one set of circumstances and not another."

Despite the criticism, fragmentationalism remains a valuable tool for solving the logical omniscience problem. The subsequent chapters of this thesis will still adhere to the idea of fragmentation but will also strive to refine it, by linking fragmentation to the agent's belief system. The details of our solution and how it addresses the critics' concerns will be explained and examined in the next chapters.

1.3 Epistemic modality and epistemic contradiction

Epistemic modality pertains to the application of alethic modal operators in epistemic contexts. The epistemic modalized sentences, such as "I believe/know /suppose/think that it is possibly/necessarily/might/may that..." are very special, since they appear to bridge the gap between the epistemic state and the real world, talking about both what the agent supposes to be true and what might be true. This also introduces some unique challenges [49, 53, 24].

In particular, the use of alethic modal operators within a doxastic context can give rise to problems that are similar to, but also different to Moore's Paradox [34, 17]. This happens when one asserts the doxastic possibility of a proposition conjoined with its negation, like "I believe that it's raining and possibly it's not raining". Consider the example from Yalcin [53]:

- It is raining and it might not be raining
- It is raining and possibly it is not raining
- It is not raining and it might be raining

- It is not raining and possibly it is raining

According to Yalcin [53], “all of these sentences are odd, contradictory-sounding, and generally unassertable at a context. They all contain modal operators which, in these sentential contexts, are default interpreted epistemic.” And he calls this phenomenon “epistemic contradiction”.

We use the S5 alethic modal operator $\Box$ and interpret it as “necessary”. We use $\Diamond$ as an abbreviation of $\neg\Box\neg$ and interpret it as “possibly” or “might”. Then all of the above sentences have the form $\phi\wedge\Diamond\neg\phi$. These sentences themselves are satisfiable in a given logical system, but the above example shows that a human agent cannot form beliefs towards the sentences since they are epistemically contradictory. Formally, epistemic contradiction requires $B(\phi \wedge \Diamond\neg\phi)$ to be contradictory. This is also very similar to the form of Moore’s sentence which says that $\phi \wedge \neg K\phi$ cannot be known [34], where K is the knowledge operator and $K\phi$ means that the agent knows ϕ. Thus, it is argued that the reason to accept the principle of epistemic contradiction is the same as the reason to reject knowing a Moore’s sentence, since the former entails the latter: if I believe it is possibly not ϕ, then I don’t know ϕ.

But from another perspective, the issue of epistemic contradiction is different from the Moore paradox. Consider the following sentences:

- Suppose that it is raining and I don’t know that it is raining
- Suppose that it is raining and it might not be raining

The first sentence has no problem and can appear in a daily conversation, for instance, one may say “suppose that it is raining and I don’t know that it is raining, then I would get wet when I step out of the building”. In contrast, the second sentence is not acceptable or not even intelligible, since one cannot suppose something contradictory-sounding. It seems that the Moore’s sentence can be embedded in a supposition context while the epistemic contradiction cannot. This is why people like Yalcin propose to treat epistemic contradiction and Moore’s sentence differently [53].

There are several existing theories that address epistemic modals and the problem of epistemic contradiction. Authors like Yalcin [53], Hawke and Steinert-Threlkeld [24], and Aloni [1] have provided valuable insights and potential solutions. However, there remains substantial room for further exploration of epistemic modalities, particularly regarding the interaction between epistemic modality and other logical omniscience inferences.

In this thesis, we initially focus on the doxastic alethic modality, dealing with the sentences like “I believe that it might be„” or “I believe that it must be...”. The following list includes some features for doxastic alethic modality that will be investigated in this thesis:

- Epistemic contradiction: A belief in an epistemic contradiction leads to contradiction. Formally, this is noted as $B(\phi \wedge \Diamond\neg\phi) \vDash \bot$, where $\bot$ is the abbreviation for “falsum” or “contradiction”.

- Belief to necessity: A belief in n necessity leads to necessity. Formally, this is noted as $B\Box\phi \vDash \Box\phi$. It says that if the agent believes that ϕ is necessary, then ϕ is necessary.
- Apriori Omniscience: All necessities are believed. Formally, this is noted as $\Box\phi \vDash B\Box\phi$. It says that if ϕ is necessary, then the agent believes that ϕ is necessary.

A good theory for doxastic alethic modality should uphold the first principle above, which involves treating the belief towards epistemic contradiction as contradictory itself. The last two principles should be deemed invalid. Due to the non-factive nature of belief, believing a proposition to be necessary does not make the proposition necessary in reality. Also, human agents are not apriori omniscient, implying that they may not hold beliefs in all necessary truths. Notice that these aspects are not discussed or not solved in usual hyperintensional systems, such as in standard awareness semantics [14] or in impossible world semantics [35].

1.4 Overview of results in this thesis

Inspired by recent works of Berto, Özgün and Hawke [5, 7, 37, 23, 8], this thesis aims to develop a logic that achieves hyperintensional belief through the use of the concept of topic-relevance.

The main goal of this thesis is to create a doxastic logic and to partially solve the problems of logical omniscience, fragmentation and epistemic modality in one uniform system. This thesis aims to introduce a novel method for achieving hyperintensional outcomes. The underlying notion is that to believe in a proposition, say A, we only need to concentrate on the information pieces that are relevant to A–the A-relevant parts of our epistemic state. This implies that only specific portions of our epistemic state are utilized, indicating that the epistemic state is fragmented. Hence, topics can segment the belief set into fragments, thereby gaining the anticipated benefits of fragmentation without introducing it as a primitive element in the model.

This approach also allows for a unique yet intuitive interpretation of necessity operators within doxastic contexts, interpreting "Believe that A is necessary" as "Believe A after considering it thoroughly" . While to believe a proposition we only need to check the relevant parts of our epistemic state, to believe that the proposition is necessary we need to check the whole epistemic state, exhausting our epistemic ability. The latter requires that the proposition is topic relevant to all the parts of the epistemic state, which can be modeled by an operator representing full comprehension or thorough understanding. More philosophical discussions will be provided in chapter 2.

Building on these ideas, the expected logic in chapter 3 is weaker than standard awareness logic so that it allows us to avoid some of the criticism on fragmentationalism and to get a more nuanced interpretation of doxastic alethic modality. The main result of the thesis is in chapter 4, where we show the hyperintensional properties of our logic. We provide a sound axiomatisation for our logical system in chapter 5. Connections between this logic and other theories of doxastic attitudes will be investigated in chapter 6.

Chapter 2

From topic-relevance to hyperintensionality

2.1 Topics and the topic-sensitive model

"Topicality" is a concept that has gained significant attention from scholars in various fields, including philosophy, linguistics, and logic (Fine [15]; Humberstone [29]; Roberts [40]; Yablo [52]; Fine [16]; Berto [5]). The majority of scholars identify the topic or aboutness of a sentence as the subject matters it addresses or concerns, and it is widely accepted that this kind of subject matters is the key to deal with the logical omniscience problems.

Recently, a series of hyperintensional systems based on topic have been developed by Berto, Hawke and Özgün [5, 7, 8, 37, 23]. In 2019, Berto [5] originally developed a topic-sensitive model based on a mereology structure of topics. Accoring to Berto, the mereology structure of topics can be defined as:

Definition 1. *A mereology of topic* is a tuple $\mathcal{T} = \langle T, \oplus \rangle$

- T is a non-empty set of topics,
- $\oplus : T \times T \to T$ is a binary operator on topics representing topic fusion, which is idempotent, commutative and associative.

The fusion $x \oplus y$ of x and y is the smallest topic that contains the topic of x and y. With the topic fusion operator $\oplus$, we can further define a binary relation $\leq$ for topic-inclusion on T as: for all $x, y \in T$, $x \leq y$ if and only if $x \oplus y = y$. Then $\langle T, \oplus \rangle$ is a join semilattice and $\langle T, \leq \rangle$ is a poset.

This structure allows topics to possess proper parts, enables distinct topics to share common parts, and permits one topic to be included within another, wherein every part of the former is also a part of the latter. Moreover, there is a topic assignment function in the topic-sensitive system that assigns topics to propositions and epistemic states. Thus, both the topics of propositions and the topic-inclusion relationship between epistemic state and proposition can be modeled. As a direct result, logically or necessarily equivalent propositions can be different since they may be about different topics. Later works of Hawke and Özgün also shows that Berto's model can deal with not only logical omniscience problems [23], but also dynamic hyperintensional updates [37], indicative conditionals [7] and imagination based on fragmentation [8].

Berto and his co-authers have shown that their topic-sensitive system can deal with a large amount of problems caused by hyperintensionality. However, in [23], they also found that mathematical necessity can give rise an difficulty that can hardly be solved in their model. Formally, they found that

$$\Box\phi \wedge B(\phi \vee \neg\phi) \vDash \Box\phi$$

is valid in their system, but this is not a desired result. For example, Fermat's Last Theorem is necessarily true, and anyone that heard the name of the theorem believes that the theorem is either true or false. But one may not believe that the theorem is true if one hasn't heard that the theorem is proved or one doesn't even understand the theorem. We call this the problem of mathematical knowledge.

In this thesis, we build upon the successful approach pioneered by Berto by adopting the topic-sensitive model to formalize doxastic attitudes. The novelty lies in combining their methodology with fragmentation and model-update. The latter is a technique from dynamic epistemic logic [19, 3], and it can be used to constrict a model to a smaller domain. In this way, the logic in this thesis can tackle the unresolved issues, such as the problems of mathematical necessity and epistemic modality. To distinguish my theory from existing ones, I will employ the term "topic-relevant" instead of "topic-sensitive" moving forward.

2.2 Fragmentation generated by topic-relevance

As we discussed in section 1.2, fragmentation is a popular approach for dealing with logical omniscience problems [12, 55]. Fragmentationalism proposes that agents' epistemic sets are divided into fragments and only some of them are used once a time. Given that a non-omniscient agent only has limited computational, conceptual and reasoning capacities, he only considers some of rather than all the information he can access. But then, one may question why the agent chooses to consider some worlds while ignore others and what are the connections between beliefs, circumstances and fragments? This just what philosophers including Norby [36] and Jago [30] are criticising.

Here we want to provide a direct answer to the above question: the agent only considers the relevant information with respect to his epistemic state. And topics can just play the role of determining which information pieces are relevant. Adopting the approach of modelling information and information states from dynamic semantics [11, 20], in this thesis we use sets of possible worlds to represent information pieces. Intuitively, such an information piece means that the agent has information that the actual world is in this set of worlds. Note that the agent may possess different and even inconsistent information pieces and these information pieces may be about different topics.

We propose that the epistemic state of an agent is divided into different parts, which can be regarded as different information pieces. But notice that these information pieces are not fragments yet. With the help of the topic-relevance model and the topic assignment function, we can assign each proposition a topic and each information piece a set of topics. Then we get the relevance relation between propositions and information pieces. To be specific, when an agent is considering a propositional atom p, he just takes the union of all the p-relevant information pieces as a relevant fragment and checks if p is true over this fragment. And the other parts of his epistemic state are just irrelevant and be ignored. As for complex propositions, the agent just takes unions or intersections of information pieces as the corresponding relevant fragment, with respect to the structure of the proposition. For instance, if the relevant fragment of the proposition p is X and the relevant fragment of the proposition is Y, then the relevant fragment of $p \wedge q$ is just the $X \cup Y$, since both of the fragments are relevant to a conjunct of the conjucntion. The details will be explained formally in section 3.3. In short, aiming to take the ideal and technical advantages of the fragmentation approach while avoid the philosophical disadvantages, we no longer treat fragmentation as a primitive element in the model, but as something generated by topics. Fragments here are just unions or intersections of relevant information pieces.

Thus, by tagging information pieces with topics, the agent can fragment his epistemic state naturally. The obtained fragmentation is generated directly by topic-relevance, which means that the connection from beliefs to the fragments that form the beliefs and to the reasons for choosing the fragments is obvious and clear. In this way, we achieve the desired advantages of fragmentation without directly introducing it as primitive element in the model. Unlike traditional fragmentation, the connection between circumstances, fragments and beliefs are well explained by topic-relevance. Hence, the theory can overcome the criticism from Norby [36] and Jago [30].

2.3 Relevance, awareness and comprehension

In the last section, we see that a proposition is relevant to an information piece if and only if the information piece includes the topic of the proposition. In a more general sense, we also say that a proposition is topic-relevant to the agent if there is a relevant fragment generated by the agent's epistemic state. One may feel that this latter notion is very similar to the notion of awareness in awareness logic [14, 21, 41]. Though they have close connections, awareness and relevance do have a radical difference. Awareness is more like a local concept describing whether the agent is aware of something in a world. In contrast, topic-relevance here is about the agent's opinion about a proposition based on his epistemic state, which is global. In short, topic-relevance is about something of the epistemic state while awareness is about something in a world.

It is also very easy to distinguish awareness from relevance in daily conversation. An agent cannot think or know or believe that he himself is not aware of a proposition, since these epistemic attitudes already assume the awareness of the proposition. But one can easily think that the proposition is irrelevant:

Example 2. A student studying cooking enters a wrong room, which is a math classroom, but he does not realize that. Then he asks his teacher in maths class about how to cook lamb shanks.

In this example, the teacher may answer that “I think cooking lamb shanks is irrelevant” and cannot say that “I think I am not aware of cooking lamb shanks”. This is because the teacher is already aware of lamb shanks when the student asks the question. But whether the student asks the question or not, lamb shanks must not be relevant, since it is a maths classroom.

However, the awareness discussed above is just a weak version of awareness in early versions of awareness logic [14]. In some later versions of awareness logic as in [41, 46], for an agent to be explicitly aware of a proposition, we require the agent to believe that he is aware of the proposition, that is to require the agent to be aware of the proposition in all the worlds in his epistemic state. In other words, the agent is not only aware of the proposition in the current world but also aware of the proposition based on the epistemic state. Obviously, the latter notion of awareness is stronger than the former one.

This thesis proposes to treat this strong version of awareness as another epistemic attitude, which I call thorough comprehension. To thoroughly comprehend a proposition is different to just saying that there is one information piece to which the proposition is topic-relevant. While the latter only requires the agent to consider the topic of the proposition, the former indicates a clearer understanding of the proposition. Thorough comprehension requires that all information pieces in the epistemic state are relevant to the proposition, which means that the agent considers the proposition in all the possible worlds he can access, exhausting his epistemic capacity. Thorough comprehension means that the agent has already checked his whole epistemic state to grasp the proposition and cannot have a clearer understanding in the future.

Moreover, this thesis proposes that thorough comprehension is an ideal tool to interpret doxastic necessity. To be specific, we propose: saying that the agent believes that p is necessarily true is the same as saying that the agent comprehends p thoroughly and believes that p is true. This is because as human agents with only limited epistemic capacity, we cannot access any real or ontological necessities. We say we believe that a proposition is necessary only because we are extremely confident that it is true, which only means that we believe that it is true even after we have considered it carefully and have comprehended it thoroughly. In the next chapter, thorough comprehension will be defined formally and be represented by the comprehension operator U.

In summary, this thesis will not use the world-based notion of awareness. We use the term topic-relevance to describe the relationship between propositions and (unions of) information pieces. We treat comprehension as a global notion based on an agent's whole epistemic state.

Chapter 3

Logic of topic-relevant belief L_{TB}

In this chapter, we introduce our logic of topic-relevant belief, which is denoted as L_{TB}. We present and explain all the notions and the formal definitions used in L_{TB}.

3.1 Syntax of L_{TB}

The syntax of logic of topic-relevant belief L_{TB} is an extension of standard epistemic logic that includes an operator U for complete comprehension or thorough understanding. This comprehension operator requires the proposition to be topic-relevant over all parts of the epistemic state, which means that the agent can check the whole epistemic state and exhaust his capacity to form a belief about the proposition. To construct the language, we begin by selecting a countable set $\mathbb{P}$ of proposition letters.

Definition 2 (Language). *The language of* L_{TB} is defined as follows:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \Box\phi \mid B\phi \mid U\phi$$

where $p \in \mathbb{P}$, $B\phi$ means that the agent believes ϕ and $U\phi$ means that the agent comprehends ϕ thoroughly. Other operators are the same as in standard modal logics.

In addition, we will also use the operators for disjunction $\vee$, material implication $\rightarrow$ and possibility $\Diamond$. They are defined by the primitive operators in the classic way: $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$, $\phi \rightarrow \psi := \neg(\phi \wedge \neg\psi)$, $\Diamond\phi := \neg\Box\neg\phi$.

3.2 Model of L_{TB}

Definition 3 (Model). *A model of* L_{TB} is a tuple $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$, such that:

- W is a nonempty set of possible worlds.

- $\mathcal{B} \subseteq \mathcal{P}(W)$ is a finite non-empty set of non-empty sets of possible worlds. $\mathcal{B}$ represents the agent's epistemic state and the elements in $\mathcal{B}$ are information pieces of the agent, from which fragmentation generates. It satisfies the condition that $\mathcal{B}$ is nonempty downward closed under intersection: $\forall f_1, f_2 \in \mathcal{B}$, if $f_1 \cap f_2 \neq \emptyset$ then $f_1 \cap f_2 \in \mathcal{B}$. Moreover, we denote $b = \bigcup \mathcal{B}$, and we call it the epistemic set.

- V is a world-dependent valuation function $V : W \times \mathbb{P} \to \{1, 0\}$.

- $\mathcal{T} = \langle T, \oplus \rangle$ is a topic-sensitive structure. The topic-inclusion relation defined by $\oplus$ is noted as $\leq$.

- $t : Prop \cup \mathcal{B} \to T \cup \mathcal{P}(T)$ is a function assigning topics to each proposition and sets of topics to each information piece, which is similar to the function in [8]. It satisfies the condition that all operators are topic transparent: $t(\phi) = \oplus Var(\phi) = t(p_1) \oplus t(p_2) \oplus ... \oplus t(p_k)$, where $Var(\phi) = \{p_1, ...p_k\}$ is the set of all propositional atoms in ϕ. Additionally, for any information piece f, we denote $\oplus t(f)$ as the fusion of all topics in $t(f)$.

$\mathcal{B}$ represents the agent's epistemic state, which is a finite set of sets of possible worlds. These sets of possible worlds are information pieces in the epistemic state. Considering a specific proposition, the agent only uses relevant information pieces to form a fragment and the agent checks the proposition only on the fragment. Here we require the epistemic state to have the nonempty intersection closure property, which means that the nonempty intersections of information pieces are also information pieces. Intuitively, this indicates that we can always get a new information pieces if we put two different consistent information pieces together. The closure property guarantees proposition 1 in the next section. Moreover, we denote the union of all the information pieces, $\bigcup \mathcal{B}$, as b and call it the epistemic set.

The topic assignment function t assigns topics to propositions and sets of topics to information pieces. With this, we can define the topic-relevance relation between propositions and information pieces. We say a proposition ϕ is relevant to an information piece f if and only if $t(\phi) \leq \oplus t(f)$. In addition, we require all the operators in the logic to be topic transparent, which means that topic-relevance is only generated by the proposition atoms. This is a well motivated requirement that appears in many hyperintensional theories, see [21, 8] for similar requirements and detailed arguments. Notice that we choose to adopt this general transparency condition only for simplicity, more fine-grained requirements such as $t(\phi) \leq t(B\phi)$ are also well-justified and can be adopted in our theory. But since they are not necessary for our logic, we do not discuss them further.

We have already defined topic-relevance and information pieces in this section. But notice that these information pieces are still not the fragments we mentioned in chapter 1 and 2. In the next section, we will see how the agent uses relevant information pieces to form a fragment.

3.3 Relevant domain

To illustrate how to form fragments through information pieces, we need to define a notion called relevant domain. Given a specific proposition, a relevant domain of the proposition is a tuple, consisting of two elements: a positive relevant set and a negative relevant set. The first element is a set of relevant worlds, which we call a positive relevant set, denoting the set of worlds the agent needs to check when he wants to check the proposition. The second element is a set of worlds which we call a negative relevant set, denoting the set of worlds the agent needs to check when he wants to check the negation of the proposition. Here we need both positive and negative relevant domain to achieve doxastic double negation elimination $B\neg\neg\phi \leftrightarrow B\phi$. If we only have positive relevant domain then we get a logic fragment in which doxastic double negation fails, about which we will not discuss in this thesis.

The relevant set of each proposition is just the relevant fragment of the epistemic state of the agent that the agent needs to check, generated by the topic of the proposition. Relevant sets are formed by information pieces recursively as following:

Definition 4. *Given a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$, the relevant domain of a formula ϕ is a binary tuple $D^{\mathbb{M}}[\phi] = \langle D^{\mathbb{M}}[\phi]_1, D^{\mathbb{M}}[\phi]_2\rangle$. When the model is fixed, we usually abbreviate it as $D[\phi] = \langle D[\phi]_1, d[\phi]_2\rangle$ for simplicity. It is defined inductively as follows:*

$$
\begin{aligned}
&D[p] = \langle \bigcup\{f \in \mathcal{B} \mid t(p) \leq \oplus t(f)\}, \bigcup\{f \in \mathcal{B} \mid t(p) \leq \oplus t(f)\}\rangle \\
&D[\neg\phi] = \langle D[\phi]_2, D[\phi]_1\rangle \\
&D[\phi \wedge \psi] = \langle D[\phi]_1 \cup D[\psi]_1, D[\phi]_2 \cap D[\psi]_2\rangle \\
&D[U\phi] = D[B\phi] = D[\phi] \\
&D[\Box\phi] = \langle b, D[\phi]_2\rangle
\end{aligned}
$$

This says that,

- The positive and negative relevant sets of a propositional atom are the same, which is the union of all the information pieces that include the topic of the atom.

- The positive relevant set of a negation $\neg\phi$ is the negative relevant set of ϕ. The negative relevant set of a negation $\neg\phi$ is the positive relevant set of ϕ. This guarantees doxastic double negation elimination $B\neg\neg\phi \leftrightarrow B\phi$.

- The positive relevant set of $\phi \wedge \psi$ is the union of the positive relevant set of the ϕ and the positive relevant set of ψ. The negative relevant set of $\phi \wedge \psi$ is the intersection of the negative relevant set of the ϕ and the negative relevant set of ψ. Briefly, to consider a conjunction we need to consider the union of the fragments of all the conjuncts, and to consider the negation of a conjunction, we need to consider the intersection of the fragments of all the conjuncts.

- The relevant domain of belief $B\phi$ and comprehension $U\phi$ is the relevant domain of ϕ. In this way, we have positive introspection for belief and comprehension in the logic, which means that if the agent believes a proposition then he believes that he believes the proposition and if the agent comprehends the proposition then he comprehends that he comprehends the proposition.

- The positive relevant set of necessity $\Box\phi$ is the whole epistemic set. The negative relevant set of necessity $\Box\phi$ is the negative relevant set of ϕ. This leads to the desired interpretation for necessity operator mentioned in section 2.3. Consequently, several problems can be solved, which will be shown in chapter 4.

In short, the positive relevant sets are the fragments we want. Both positive and negative relevant sets are formed by the information pieces, or to be clearer, they are unions of information pieces. Every relevant set is a union of some elements in $\mathcal{B}$.

Proposition 1. *Given a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$, for any formula ϕ, if $D[\phi]_1$ is nonempty, then there is a nonempty set of information pieces $S_1 \subseteq \mathcal{B}$ such that $\bigcup S_1 = D[\phi]_1$; and if $D[\phi]_2$ is nonempty, then there is a nonempty set of information pieces $S_2 \subseteq \mathcal{B}$ such that $\bigcup S_2 = D[\phi]_2$.*

Proof. We apply induction on the structure of ϕ:

- Base step: $\phi = p$, where p is a propositional atom. Assume the relevant sets are nonempty. Then $D[p]_1 = D[p]_2 = \bigcup\{f \in \mathcal{B} \mid t(p) \leq \oplus t(f)\} \neq \emptyset$. Then let $S = \{f \in \mathcal{B} \mid t(p) \leq \oplus t(f)\}$, we have S is non-empty and $D[p]_1 = D[p]_2 = \bigcup S$.

- Induction step:

  - $\phi = \neg\psi$. Assume the relevant sets are nonempty. By definition 4, We have $D[\phi]_1 = D[\neg\psi]_1 = D[\psi]_2$ and $D[\phi]_2 = D[\neg\psi]_2 = D[\psi]_1$. By the induction hypothesis, there is a nonempty $S_1 \subseteq \mathcal{B}$ such that $\bigcup S_1 = D[\psi]_1$ and a nonempty $S_2 \subseteq \mathcal{B}$ such that $\bigcup S_2 = D[\psi]_2$. Thus there is a nonempty $S_1 \subseteq \mathcal{B}$ such that $\bigcup S_1 = D[\neg\psi]_2$ and a nonempty $S_2 \subseteq \mathcal{B}$ such that $\bigcup S_2 = D[\neg\psi]_1$.
  - $\phi = \psi_1 \wedge \psi_2$. Assume the relevant sets are nonempty. By definition 4, we have $D[\psi_1 \wedge \psi_2]_1 = D[\psi_1]_1 \cup D[\psi_2]_1$ and $D[\psi_1 \wedge \psi_2]_2 = D[\psi_1]_2 \cap D[\psi_2]_2$. By the induction hypothesis, there is a nonempty $S_1 \subseteq \mathcal{B}$ such that $\bigcup S_1 = D[\psi_1]_1$, a nonempty $S_2 \subseteq \mathcal{B}$ such that $\bigcup S_2 = D[\psi_2]_1$, a nonempty $S_3 \subseteq \mathcal{B}$ such that $\bigcup S_3 = D[\psi_1]_2$, a nonempty $S_4 \subseteq \mathcal{B}$ such that $\bigcup S_4 = D[\psi_2]_2$. Thus, we have a nonempty $S_1 \cup S_2 \subseteq \mathcal{B}$ and $\bigcup(S_1 \cup S_2) = D[\psi_1 \wedge \psi_2]_1$. We define $S_0 = \{f_1 \cap f_2 \mid f_1 \in S_3 \;\&\; f_2 \in S_4 \;\&\; f_1 \cap f_2 \neq \emptyset\}$. This set includes all nonempty intersections of information pieces from S_3 and S_4, which by nonempty intersection closure property of $\mathcal{B}$ in definition 3, are also in $\mathcal{B}$. Then we have a nonempty $S_0 \subseteq \mathcal{B}$ and $\bigcup S_0 = D[\psi_1 \wedge \psi_2]_2$.

– $\phi = B\psi$ or $\phi = U\psi$. Assume the relevant sets are nonempty. Then by definition 4 $D[\phi]_1 = D[\psi]_1$ and $D[\phi]_2 = D[\psi]_2$. By the induction hypothesis, there exist nonempty sets of information pieces $S_1, S_2 \subseteq \mathcal{B}$ such that $\bigcup S_1 = D[\psi]_1$ and $\bigcup S_2 = D[\psi]_2$. It follows that $\bigcup S_1 = D[U\psi]_1 = D[B\psi]_1$ and $\bigcup S_2 = D[U\psi]_2 = D[B\psi]_2$.

– ϕ is $\Box\psi$. Assume the relevant sets are nonempty. Then, by definition 4, $D[\Box\psi]_1 = b$ and $D[\Box\psi]_2 = D[\psi]_2$. By the induction hypothesis, there exist a nonempty set of information pieces $S_2 \subseteq \mathcal{B}$ such that $\bigcup S_2 = D[\psi]_2$. Then we can let $S_1 = \mathcal{B}$, and we have $\bigcup S_1 = b = D[\Box\psi]_1$ and $\bigcup S_2 = D[\Box\psi]_2$.

Therefore, if a relevant set is nonempty, then it is a union of some information pieces. $\Box$

Moreover, we can also see the connection between the topic-inclusion relation and the relevant domain. If a proposition is relevant to an information piece, then the information piece must be a subset of the relevant domain of the proposition.

Proposition 2. *Given a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$, for any formula ϕ and any information piece $f \in \mathcal{B}$, if $t(\phi) \leq \oplus t(f)$, then $f \subseteq D[\phi]_1$ and $f \subseteq D[\phi]_2$.*

Proof. Given a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$ and $f \in mB$. Assume $t(\phi) \leq \oplus t(f)$. We apply induction on the structure of ϕ to show that $f \subseteq D[\phi]_1$ and $f \subseteq D[\phi]_2$:

- Base step: $\phi = p$, where p is a propositional atom. Then $D[p]_1 = D[p]_2 = \bigcup\{f \in \mathcal{B} \mid t(p) \leq \oplus t(f)\}$. Thus, $f \subseteq D[p]_1$ and $f \subseteq D[p]_2$.
- Induction step:
  – $\phi = \neg\psi$. Then $D[\neg\psi]_1 = D[\psi]_2$ and $D[\neg\psi]_2 = D[\psi]_1$. Since logical connectives are topic transparent, by induction hypothesis, we know that $f \subseteq D[\psi]_2$ and $f \subseteq D[\psi]_1$. This means that $f \subseteq D[\phi]_1$ and $f \subseteq D[\phi]_2$.

  – $\phi = \psi_1 \wedge \psi_2$. Then $D[\psi_1 \wedge \psi_2]_1 = D[\psi_1]_1 \cup D[\psi_2]_1$ and $D[\psi_1 \wedge \psi_2]_2 = D[\psi_1]_2 \cap D[\psi_2]_2$. Since logical connectives are topic transparent, by induction hypothesis, we have $f \subseteq D[\psi_1]_1$, $f \subseteq D[\psi_2]_1$, $f \subseteq D[\psi_1]_2$ and $f \subseteq D[\psi_2]_2$. Thus we have $f \subseteq D[\phi]_1$ and $f \subseteq D[\phi]_2$.

  – $\phi = B\psi$ or $\phi = U\psi$. Assume the relevant sets are nonempty. Then $D[\phi]_1 = D[\psi]_1$ and $D[\phi]_2 = D[\psi]_2$. Since logical connectives are topic transparent, by induction hypothesis, we have $f \subseteq D[\psi]_1$ and $f \subseteq D[\psi]_2$. Thus, $f \subseteq D[\phi]_1$ and $f \subseteq D[\phi]_2$.

  – ϕ is $\Box\psi$. Then $D[\Box\psi]_1 = b$ and $D[\Box\psi]_2 = D[\psi]_2$. By the induction hypothesis, $f \subseteq D[\psi]_2$. Also, $f \subseteq b$. Thus, $f \subseteq D[\phi]_1$ and $f \subseteq D[\phi]_2$.

Therefore, if an information piece is relevant to a proposition, then it is a subset of the relevant domain of the proposition. $\Box$

3.4 Semantics of L_{TB}

When the agent is considering a specific proposition, he only needs to check the relevant parts of his epistemic state and the irrelevant parts of the epistemic state are ignored. Then, from the perspective of the agent, the domain of the model shrinks to these relevant parts, which is the positive relevant set. With the help of the technique of model-update from dynamic epistemic logic [48, 3], we can restrict the original model to the relevant set. Under this operation, the domain of the updated model becomes the relevant set while other elements of the model are restricted to the domain. The updated model is just the model that the agent needs to check when he is considering the proposition.

Definition 5. Given a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$ and a proposition ϕ, if $D[\phi]_1 \neq \emptyset$, we define $\mathbb{M}_{D[\phi]} = \langle W_{D[\phi]}, \mathcal{B}_{D[\phi]}, V_{D[\phi]}, \mathcal{T}, t_{D[\phi]}\rangle$, where

- $W_{D[\phi]} = D[\phi]_1$
- $\mathcal{B}_{D[\phi]} = \{f \in \mathcal{B} \mid f \subseteq W_{D[\phi]}\}$
- $V_{D[\phi]}(w, p) = V(w, p)$ for all $w \in W_{D[\phi]}$ and $p \in \mathbb{P}$
- $T_{D[\phi]} = T$
- $t_{D[\phi]}(x) = t(x)$ for all $x \in Prop \cup \mathcal{B}_{D[\phi]}$.

This new model, $\mathbb{M}_{D[\phi]}$, can be regarded as a subjective internal model constructed by the agent. We say it is subjective since the domain of the model is based on a relevant set, and the latter is based on the epistemic state of the agent, which is fully subjective. We say it is internal since the updated model is a submodel of the original model $\mathbb{M}$ and can be constructed only within an existing model. The agent constructs and checks his own model based on his epistemic state when he is considering the proposition. To see whether the agent believes a proposition, we only need to see whether the proposition is globally true in the model of the agent. The idea of working with an agent's subjective and internal model is in line with the ideas and approach developed by Aucher [2] but is here used in the context of a hyperintensional setting.

Through the definition 5 above, we can see that the domain of the updated model is the positive relevant set. The updated epistemic state is a subset of the original epistemic state, and the elements of the updated epistemic state are subsets of the updated domain. The topic-sensitive structure remains the same. The updated truth valuation and the topic assignment function are just the restricted versions of the original ones.

Proposition 3. *Given a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$, for any formula ϕ, if $D[\phi]_1 \neq \emptyset$, then $\mathcal{B}_{D[\phi]} \subseteq \mathcal{B}$ and $\bigcup \mathcal{B}_{D[\phi]} \subseteq W_{D[\phi]} \subseteq b$, where $\mathcal{B}_{D[\phi]}$ is the updated epistemic state.*

Proof. By definition 5, $\mathcal{B}_{D[\phi]} = \{f \in \mathcal{B} \mid f \subseteq W_{D[\phi]}\}$. Then for any $f \in \mathcal{B}_{D[\phi]}$, $f \in \mathcal{B}$ and $f \subseteq W_{D[\phi]}$. Thus $\mathcal{B}_{D[\phi]} \subseteq \mathcal{B}$ and $\bigcup \mathcal{B}_{D[\phi]} \subseteq W_{D[\phi]}$. By proposition 1, there is a $S \subseteq \mathcal{B}$ such that $D[\phi] = \bigcup S$. Since $\bigcup S \subseteq b$, we get $D[\phi] \subseteq b$. $\square$

Most importantly, we can prove that if the positive relevant set is non-empty, then the updated model always exists, since the updated epistemic state is non-empty.

Proposition 4. *Given a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$, for any formula ϕ, if $D[\phi]_1 \neq \emptyset$, then the updated model $\mathbb{M}_{D[\phi]}$ exists.*

Proof. The assumption says that $W_{D[\phi]} = D[\phi]_1 \neq \emptyset$. Given that the topic sensitive model, the valuation function and the topic assignment function always exist, we only need to prove that the updated epistemic state $\mathcal{B}_{D[\phi]}$ exists, that is to prove that it is nonempty.

By proposition 1, we know there is a nonempty $S \subseteq \mathcal{B}$ such that $\bigcup S = D[\phi]_1$. Since S is nonempty, there is at least an $f \in S$. Then $f \in \mathcal{B}$ and $f \subseteq D[\phi]_1$, which also means that $f \subseteq W_{D[\phi]}$. Thus, by definition 5, $f \in \mathcal{B}_{D[\phi]}$ and then $\mathcal{B}_{D[\phi]}$ is nonempty. $\square$

This proves that the nonempty relevant set acts as an effective domain for the updated model and our method for model-update is consistent. We can see that the updated model allows the agent to evaluate a proposition, using only the information pieces which are relevant to the proposition, and ignore all the irrelevant information.

With the notions of relevant domain and model-update, the semantic relation called "support" for the logic of topic-relevance belief L_{TB} can be defined.

Definition 6. Given a world w and a model $\mathbb{M} = \langle W, \mathcal{B}, V_{D[\phi]}, \mathcal{T}, t\rangle$ of L_{TB}, the *support* relation $\vDash$ is defined inductively as follows:

$$\begin{array}{lcc}
\mathbb{M}, w \vDash p & \text{iff} & V(w,p) = 1 \\
\mathbb{M}, w \vDash \neg\phi & \text{iff} & \mathbb{M}, w \nvDash \phi \\
\mathbb{M}, w \vDash \phi \wedge \psi & \text{iff} & \mathbb{M}, w \vDash \phi \text{ and } \mathbb{M}, w \vDash \psi \\
\mathbb{M}, w \vDash U(\phi) & \text{iff} & \forall f \in \mathcal{B},\ t(\phi) \leq \oplus t(f) \\
\mathbb{M}, w \vDash \Box\phi & \text{iff} & \forall x \in W,\ \mathbb{M}, x \vDash \phi \\
\mathbb{M}, w \vDash B\phi & \text{iff} & \forall x \in D[\phi]_1 \neq \emptyset \text{ and } \mathbb{M}_{D[\phi]}, x \vDash \phi \wedge U\phi
\end{array}$$

The semantics clauses for $\neg$, $\wedge$ and $\Box$ are standard. The semantics for U says that the agent can understand or comprehend a proposition thoroughly if and only if the topics of the proposition are included in the topics of all the information pieces of the agent, which means that the agent can employ the whole epistemic state to consider the proposition. The clause for $B\phi$ indicates that the agent believes a proposition if and only if ϕ (1) the positive relevant set of ϕ is nonempty, and (2) the proposition is everywhere true and fully comprehended in the model generated by the relevant domain.

Given proposition 4, (1) means that the updated model generated by the positive relevant set of ϕ exists. (2) means that the agent only needs to check the model to believe ϕ, which leads to the aforementioned interpretation for belief, especially belief towards necessity in section 1.4. This leads to all the desired advantages, including taking the good effect of fragmentation, explaining epistemic modality and invalidating mathematical necessity and all other forms of logical omniscience. All these results will be explored in chapter 4.

Validity and logical consequence of L_{TB} are defined in a standard way:

Definition 7. For any proposition ϕ,

- $\models \phi$ iff for any world w and model $\mathbb{M}$, $\mathbb{M}, w \models \phi$
- $\Sigma \models \phi$ iff for any world w and model $\mathbb{M}$, if $\mathbb{M}, s \models \psi$ for all $\psi \in \Sigma$, then $\mathbb{M}, s \models \phi$.

3.5 Hyperintension condition

In the above sections, we use the notion of topic-relevance to define the operators for belief, comprehension and necessity. But we still need a formula in our logic to express sentences like "the agent is considering the proposition". This kind of sentences is called as the hyperintension condition here, which is firstly introduced in section 1.1. In other words, given a proposition, we need to find a formula to express that the proposition is topic-relevant. For the expected formula to depict the hyperintension condition, it should at least fulfil the following two requirements.

Firstly, it should be compatible with the notion of relevant domain. Given that the positive relevant set of a proposition is the agent's epistemic fragment towards the proposition, it follows that if the proposition holds relevance to the agent, the positive relevant set of the proposition must not be empty. In short, given a proposition ϕ, the expected formula should entail that $D[\phi] \neq \emptyset$. Secondly, the expected formula should serve as a necessary condition for belief. The agent can believe the proposition only when the proposition is relevant to the agent. Thus, given a proposition ϕ, the expected formula should be a consequence of $B\phi$.

In light of the two requirements, we propose that $B(U\phi)$ is the appropriate formula. This formula says that the agent believes that he understands or comprehends the proposition thoroughly. It satisfies the first requirement as such a belief presupposes the non-emptiness of the relevant domain.

Proposition 5. *Given a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$, a world $w \in W$ and a proposition ϕ, if $\mathbb{M}, w \models B(U\phi)$, then $D[\phi]_1 \neq \emptyset$.*

Proof. From $\mathbb{M}, w \models B(U\phi)$, we know that $D[U\phi]_1 \neq \emptyset$. Since $D[U\phi]_1 = D[\phi]_1$, we get $D[\phi]_1 \neq \emptyset$. $\square$

It also satisfies the second requirement. This is because when an agent believes a proposition, he also believes that he understands or comprehends the proposition. In other words, agents believe that they understand their beliefs.

Proposition 6. *Given a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$, a world $w \in W$ and a proposition ϕ, if $\mathbb{M}, w \models B\phi$, then $\mathbb{M}, w \models B(U\phi)$*

Proof. Assume that $\mathbb{M}, w \vDash B\phi$. Then $D[\phi]_1 \neq \emptyset$ and for any $w' \in W_{D[\phi]}$ $\mathbb{M}_{D[\phi]}, w' \vDash \phi \wedge U\phi$. Given that $D[\phi]_1 = D[U\phi]_1$, we have $D[U\phi]_1 \neq \emptyset$ and $\mathbb{M}_{D[\phi]} = \mathbb{M}_{D[U\phi]}$. Thus, for any $w' \in W_{D[U\phi]}$, $\mathbb{M}_{D[U\phi]}, w' \vDash \phi \wedge U\phi$, which also indicates that $\mathbb{M}_{D[U\phi]}, w' \vDash U\phi \wedge UU\phi$. This means that $\mathbb{M}, w \vDash B(U\phi)$. $\square$

These show that $BU\phi$ successfully meets the two requirements. In forthcoming sections of this thesis, we will use it as the principle of hyperintension condition.

It's worth noting another special formula, $B(\phi \vee \neg\phi)$, which is widely used in the works of Berto et al. [8, 37, 23] as something similar to our hyperintension condition. It is very natural to accept that the proposition is relevant to or is considered by the agent if and only if the agent believes that it is either true or false. However, in our logical system, it is possible that $D[\phi \vee \neg\phi]$ is empty while $D[\phi]$ is nonempty, since $D[\neg\phi]_1$ can be empty. Thus, we cannot deduce $B(\phi \vee \neg\phi)$ from $B\phi$, indicating its non-compliance with the second requirement.

Chapter 4

Hyperintensional properties of L_{TB}

In this chapter, we discuss the hyperintensional properties of L_{TB}. We show why the problems mentioned in section 1.1, section 1.2 and section 1.3 can be solved in L_{TB}. We provide philosophical explanations as well as formal proofs of the results.

4.1 Logical omniscience revisited

The main goal of this thesis to provide a tenable solution for the logical omniscience problems. In this section, we examine all the logical omniscience principles and their restricted versions mentioned in section 1.1. We use $B(U\phi)$ as the hyperintension condition, which can be interpreted as sentences like "the agent is considering about ϕ".

4.1.1 Omniscience rule

The first principle is the omniscience rule $\phi \vDash B\phi$. This principle should be rejected because non-omniscient human agents cannot believe all the truths. Its restricted version $\phi \wedge B(U\phi) \vDash B\phi$ should also be rejected. The reason is that we may not find a proposition to be true even when we are thinking about it. Consider the following example:

Example 3. Tom is a 7 years old little boy. He can recognize the numbers from 1 to 1000 and he can do addition and subtraction very well. But he is not so good at multiplication yet, especially when the numbers are large.

Tom may be not sure about whether $37 \times 5 = 135$ given that he is not good at multiplication. Then, even though we ask him to check the equation and he is also thinking hard about it, he may still not believe the equation is right.

Fact 1. Omniscience rule $\phi \vDash B\phi$ and its restricted version $\phi \wedge BU\phi \vDash B\phi$ are invalid in L_{TB}.

Proof. Counterexample: Construct a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$ such that $W = \{w_1, w_2\}$, $\mathcal{B} = \{f\}$ where $f = \{w_2\}$, $V(p) = \{w_1\}$ and $t(p) \leq \oplus t(f)$. Then, $\mathbb{M}_{D[Up]} \vDash Up$ since $t(p) \leq \oplus t(f)$. Thus, $\mathbb{M}, w_1 \vDash p \wedge BUp$. But $\mathbb{M}, w_1 \nvDash Bp$, since $\mathbb{M}_{D[p]} \nvDash p$. □

4.1.2 Disjunction

The next princple is closure under disjunction introduction $B\phi \vDash B(\phi \vee \psi)$. This principle and its restricted version should be rejected because the agent cannot put the disjuncts together and believe the disjunction if they are irrelevant. In same example of Tom, assume that he believes that snow is white. Even though he is considering whether $37 \times 5 = 135$ is right, it is still very weird for him to have a belief in the disjunction "either snow is white or $37 \times 5 = 135$".

Fact 2. Closure under disjunction introduction $B\phi \vDash B(\phi \vee \psi)$ and its restricted version $B\phi \wedge BU\psi \vDash B(\phi \vee \psi)$ are invalid in L_{TB}.

Proof. Counterexample: Construct a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$ such that $W = \{w_1, w_2\}$, $\mathcal{B} = \{f_1, f_2\}$ where $f_1 = \{w_1\}$ and $f_2 = \{w_2\}$, $V(p) = \{w_1\} = V(q)$, $t(f_1) = \{t(p)\}$ and $t(f_2) = \{t(q)\}$. Then, $\mathbb{M}_{D[Uq]} \vDash Uq$ and $\mathbb{M}_{D[p]} \vDash p \wedge Up$. This means that $\mathbb{M}, w_1 \vDash Bp \wedge BUq$. But $\mathbb{M}, w_1 \nvDash B(p \vee q)$, since $D[p \vee q]_1 = D[p]_2 \cap D[q]_2 = \emptyset$. □

4.1.3 Material implication and strict implication

The next two principles are closure under material implication $B\psi \wedge (\psi \to \phi) \vDash B\phi$ and closure under strict implication $B\psi \wedge \Box(\psi \to \phi) \vDash B\phi$. Since the material implication is weaker than strict implication, if we reject the latter one we also need to reject the former one. Closure under strict implication and its restricted version $B\psi \wedge BU\phi \wedge \Box(\psi \to \phi) \vDash B\phi$ should be rejected, since the agent may not realize that the consequent can be deduced from the antecedent. In example 3, the schoolboy may not be sure whether 136 is bigger than 37×5, even though he believes the obvious proposition 136 is bigger than 135. This is just because he cannot conduct the deduction or perform the calculation $37 \times 5 = 135$.

Fact 3. Closure under strict implication $B\psi \wedge \Box(\psi \to \phi) \vDash B\phi$ and its restricted version $B\psi \wedge BU\phi \wedge \Box(\psi \to \phi) \vDash B\phi$ are invalid in L_{TB}.

Proof. Counterexample: Construct a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$ such that $W = \{w_1, w_2\}$, $\mathcal{B} = \{f_1, f_2\}$ where $f_1 = \{w_1\}$ and $f_2 = \{w_2\}$, $V(p) = V(q) = \{w_1\}$, $t(f_1) = \{t(p)\}$ and $t(f_2) = \{t(q)\}$. Then, we have $\mathbb{M}_{D[Uq]} \vDash Uq$ and $\mathbb{M}_{D[p]} \vDash p \wedge Up$. We also have $\mathbb{M}, w_1 \vDash \Box(p \to q)$. This means that $\mathbb{M}, w_1 \vDash Bp \wedge BUq \wedge \Box(p \to q)$. But $\mathbb{M}, w_1 \nvDash Bq$, since $\mathbb{M}_{D[q]} \nvDash q$. □

The above proof also shows the invalidity of closure under material implication and its restricted version.

Fact 4. Closure under material implication $B\psi \wedge (\psi \to \phi) \vDash B\phi$ and its restricted version $B\psi \wedge BU\phi \wedge (\psi \to \phi) \vDash B\phi$ are invalid in L_{TB}.

Proof. The same counterexample in the proof of fact 3. □

4.1.4 Logical consequence and logical equivalence

The last two principles are closure under logical consequence and closure under logical equivalence. Since a logical equivalence must also be a logical consequence, we can just focus on the latter. Closure under logical equivalence and its restricted version should be rejected. The reason is the same as the reason to reject closure under strict implication: in example 3, the school boy may not realize that the proposition "37×5 is less than 136" is logically equivalent to the proposition "135 is less than 136".

Fact 5. Closure under logical equivalence and its restricted version are invalid in L_{TB}. Formally, $\psi \dashv\vDash \phi$ does not mean that $B\psi \wedge BU\phi \vDash B\phi$.

Proof. Counterexample: Construct a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$ such that $W = \{w_1, w_2\}$, $\mathcal{B} = \{f_1, f_2\}$ where $f_1 = \{w_1\}$ and $f_2 = \{w_2\}$, $V(r) = \{w_1\}$, $t(f_1) = \{t(p), t(r)\}$ and $t(f_2) = \{t(q), t(r)\}$. Let $\phi = (p \to p) \wedge r$ and $\psi = (q \to q) \wedge r$, then we know ϕ and ψ are logically equivalent. We also have $\mathbb{M}_{D[U\psi]} \vDash U\psi$ and $\mathbb{M}_{D[\phi]} \vDash \phi \wedge U\phi$, which means that $\mathbb{M}, w_1 \vDash B\phi \wedge BU\psi$. But $\mathbb{M}, w_1 \nvDash B\psi$, since $\mathbb{M}_{D[\psi]} \nvDash \psi$. □

The above proof also shows the invalidity of closure under logical consequence and its restricted version.

Fact 6. Closure under logical consequence and its restricted version are invalid in L_{TB}. Formally, $\psi \vDash \phi$ does not mean that $B\psi \wedge BU\phi \vDash B\phi$.

Proof. The same counterexample in the proof of fact 5. □

4.2 Fragmentation revisited

Our logic system is built upon the idea of fragmentation. Given a proposition, the agent uses the relevant pieces of information in his epistemic state to form a relevant domain, which is actually a fragment. Then, the agent only needs to check the proposition in this fragment, i.e., to check the updated model generated by the relevant domain. In this way, our theory can leverage the advantages of fragmentation while avoiding some of the philosophical criticisms, as fragments are constructed directly from the epistemic state and are not an initial element in the model.

4.2.1 Conjunction

As a result of fragmentation, our logic invalidates the principle of closure under conjunction introduction $B\phi \wedge B\psi \vDash B(\phi \wedge \psi)$. This states that if the agent believes all the conjuncts, then they should also believe the conjunction. At first glance, this principle seems intuitive, but there are counterexamples in daily life. Recall that in example 1 of section 1.2, Lewis believed that Nassau Street ran roughly east-west, the nearby railroad ran roughly north-south, and the street and the railroad were roughly parallel. Even though these three statements are contradictory, Lewis could still believe them separately. This is because Lewis never combined the three sentences into one belief in one epistemic fragment, and so did not realize the contradiction. This example demonstrates that we may not believe the conjunction even though we believe all the conjuncts.

The lottery paradox also provides a compelling reason to reject closure of belief under conjunction introduction. The lottery paradox was first formulated by Kyburg [31]. Assume there are a million tickets and only one winner in a fair lottery, then the probability of "This ticket is a losing ticket" is very high. Therefore, for any single given ticket, it is reasonable to believe that the ticket will not win. If we label the tickets from number 1 to 1000000, then we believe that "ticket 1 will not win", "ticket 2 will not win"... "ticket 1000000 will not win". However, we do not believe the conjunction of all propositions, i.e., we do not believe that no ticket will win, since there is always a winner. As argued by Foley in [18], the lottery paradox shows that belief is not closed under conjunction. In L_{TB}, it is possible that the agent believes the conjuncts but does not believe the conjunction. This is because different statements about the lottery tickets may have different topics, for example when they are learned or observed in different contexts by the agent, and the agent may have no relevant fragment for the conjunction of all the statements about all the lottery tickets.

L_{TB} invalidates closure under conjunction $B\phi \wedge B\psi \vDash B(\phi \wedge \psi)$. This is because $B\phi \wedge B\psi$ only means that ϕ is everywhere true in $D[\phi]_1$ and ψ is everywhere true in $D[\psi]_1$. But this does not guarantee that $\phi \wedge \psi$ is everywhere true in $D[\phi \wedge \psi]_1 = D[\phi]_1 \cup D[\psi]_1$.

Fact 7. Closure under conjunction introduction $B\phi \wedge B\psi \vDash B(\phi \wedge \psi)$ is invalid in L_{TB}.

Proof. Counterexample: Construct a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$ such that $W = \{w_1, w_2\}$, $\mathcal{B} = \{f_1, f_2\}$ where $f_1 = \{w_1\}$ and $f_2 = \{w_2\}$, $V(p) = \{w_1\}$, $V(q) = \{w_2\}$, $t(f_1) = \{t(p)\}$ and $t(f_2) = \{t(q)\}$. Then, we have $\mathbb{M}_{D[p]} \vDash p \wedge Up$ and $\mathbb{M}_{D[q]} \vDash q \wedge Uq$, which means that $\mathbb{M}, w_1 \vDash Bp \wedge Bq$. However, we also have $\mathbb{M}_{D[p\wedge q]} \nvDash U(p \wedge q)$, since $t(p\wedge q) \not\leq \oplus t(f_1)$ and $t(p \wedge q) \not\leq \oplus t(f_2)$. Consequently, $\mathbb{M}, w_1 \nvDash B(p \wedge q)$. □

However, L_{TB} validates a weaker version of closure under conjunction introduction, formally denoted as $B(\phi \wedge U\psi) \wedge B(\psi \wedge U\phi) \vDash B(\phi \wedge \psi)$. This closure indicates that if the agent believes ϕ when considering ψ and believes ψ when considering ϕ, then the agent believes $\phi \wedge \psi$. This is a desired validity since if the agent already has a large fragment to consider and believes both the conjuncts, then he should, of course, believe the conjunction.

Fact 8. Weak closure under conjunction introduction $B(\phi \wedge U\psi) \wedge B(\psi \wedge U\phi) \vDash B(\phi \wedge \psi)$ is valid in L_{TB}.

Proof. Assume that $\mathbb{M}, w \vDash B(\phi \wedge U\psi) \wedge B(\psi \wedge U\phi)$. Then we have $\mathbb{M}_{D[\phi \wedge U\psi]} \vDash \phi \wedge U(\psi \wedge \phi)$ and $\mathbb{M}_{D[U\phi \wedge \psi]} \vDash \psi \wedge U(\psi \wedge \phi)$. Since $D[\phi \wedge U\psi] = D[U\phi \wedge \psi] = D[\phi \wedge \psi]$, we also have $\mathbb{M}_{D[\phi \wedge \psi]} \vDash \phi \wedge \psi \wedge U(\psi \wedge \phi)$, which means that $\mathbb{M}, w \vDash B(\phi \wedge \psi)$. □

Additionally, the classic principle of conjunction elimination $B(\phi \wedge \psi) \vDash B\phi$ remains valid in L_{TB}. This is also a desirable outcome, given that the agent will believe the conjuncts if he believes the conjunction. This follows from the axiom B_1 in the given axiomatisation section 5.1

Fact 9. Closure under conjunction elimination $B(\phi \wedge \psi) \vDash B\phi$ is valid in L_{TB}.

Proof. Assume that $\mathbb{M}, w \vDash B(\phi \wedge \psi)$. Then we have $\mathbb{M}_{D[\phi \wedge \psi]} \vDash \phi \wedge \psi \wedge U(\psi \wedge \phi)$. This means that for all information pieces $f \in \mathcal{B}_{D[\phi \wedge \psi]}$, $t(\phi) \leq \oplus t(f)$. By proposition 2, we get $f \subseteq D[\phi]_1$ for all $f \in \mathcal{B}_{D[\phi \wedge \psi]}$. Thus, $\bigcup \mathcal{B}_{D[\phi \wedge \psi]} = D[\phi \wedge \psi]_1 \subseteq D[\phi]_1$. By definition 4, $D[\phi]_1 \subseteq D[\phi \wedge \psi]_1$. Thus, we have $D[\phi]_1 = D[\phi \wedge \psi]_1$ and $\mathbb{M}_{D[\phi]} = \mathbb{M}_{D[\phi \wedge \psi]}$. Hence, we get $\mathbb{M}_{D[\phi]} \vDash \phi \wedge U\phi$, which means that $\mathbb{M}, w \vDash B\phi$. □

4.2.2 Doxastic implication

Another result of fragmentation is that L_{TB} invalidates the principle of closure under doxastic implication. This principle indicates that if the agent believes an implication and believes the antecedent of the implication, then he also believes the consequent of the implication. The reason to reject closure under doxastic implication is similar to the reason to reject closure under conjunction introduction. If the agent cannot put the antecedent of the implication and the implication together, then he just cannot identify that the former is the antecedent of the latter, thereby blocking modus ponens deduction. We can replace the "knows" by "believes" in the example of [10] to get a counterexample:

Example 4. Jones knows that Mary lives in New York, that Fred lives in Boston and that Boston is in north of New York. Yet Jones fails to realize the obvious: that Mary will have to travel north to visit Fred.

L_{TB} invalidates closure under doxastic implication $B\psi \wedge B(\psi \to \phi) \vDash B\phi$, because the positive relevant set of $\psi \to \phi$ or the positive relevant set of ψ may be smaller than the positive relevant set of ϕ.

Fact 10. Closure under doxastic implication $B\psi \wedge B(\psi \to \phi) \vDash B\phi$ and its restricted version $B\psi \wedge BU\phi \wedge B(\psi \to \phi) \vDash B\phi$ are invalid in L_{TB}.

Proof. Counterexample: Construct a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$ such that $W = \{w_1, w_2, w_3\}$, $\mathcal{B} = \{f_1, f_2, f_3\}$ where $f_1 = \{w_1\}$, $f_2 = \{w_2\}$ and $f_3 = \{w_3\}$, $V(p) = \{w_1, w_2\}$, $V(q) = \{w_2\}$, $t(f_1) = \{t(p)\}$, $t(f_2) = \{t(p), t(q)\}$ and $t(f_3) = \{t(q)\}$. Then, we have $D[p]_1 = f_1 \cup f_2$, $D[p \to q]_1 = f_2$ and $D[q]_1 = f_2 \cup f_3$. Then we have $\mathbb{M}, w_1 \models Bp \wedge B(p \to q)$, since $\mathbb{M}_{D[p\to q]} \models (p \to q) \wedge U(p \to q)$ and $\mathbb{M}_{D[p]} \models p \wedge Up$. Since $\mathbb{M}_{D[Uq]} \models Uq$, We also have $\mathbb{M}, w_1 \models BUq$. This means that $\mathbb{M}, w_1 \models Bp \wedge BUq \wedge B(p \to q)$. However, $\mathbb{M}, w_1 \not\models Bq$, since $\mathbb{M}_{D[q]} \not\models q$. $\square$

Thus, the modus ponens inference within a doxastic context is blocked in L_{TB}. This means that we achieve the expected results of fragmentation.

4.3 Epistemic modality revisited

4.3.1 Epistemic contradiction

The epistemic contradiction $B(\phi \wedge \Diamond\neg\phi) \models \bot$ should be valid in the L_{TB} logical system, as it is counterintuitive to assume that we can believe or suppose a proposition like "It's raining and it might not be raining". In L_{TB}, this desired outcome can be achieved. If the agent believes a proposition, then this proposition is globally true in the model generated by its relevant domain. Then there can be no world is this model such that the negation of the proposition is true.

Fact 11. Epistemic contradiction $B(\phi \wedge \Diamond\neg\phi) \models \bot$ is valid in L_{TB}.

Proof. Assume that $\mathbb{M}, w \models B(\phi \wedge \Diamond\neg\phi)$. Then $D[\phi \wedge \Diamond\neg\phi]_1 \neq \emptyset$ and $\mathbb{M}_{D[\phi\wedge\Diamond\neg\phi]} \models \phi \wedge \Diamond\neg\phi$. However, $\mathbb{M}_{D[\phi\wedge\Diamond\neg\phi]} \models \phi$ means that for all $w' \in D[\phi \wedge \Diamond\neg\phi]$ we have $\mathbb{M}_{D[\phi\wedge\Diamond\neg\phi]}, w' \models \phi$, while $\mathbb{M}_{D[\phi\wedge\Diamond\neg\phi]} \models \Diamond\neg\phi$ means that there is a $w'' \in D[\phi \wedge \Diamond\neg\phi]$ such that $\mathbb{M}_{D[\phi\wedge\Diamond\neg\phi]}, w'' \models \neg\phi$. Contradiction. $\square$

Thus, believing a proposition and the possibility of the negation of the proposition is contradictory in L_{TB}.

4.3.2 Necessity

Given that beliefs are non-factive, believing a proposition to be necessary does not entail that the proposition is indeed necessary. In L_{TB}, we have this desired outcome, while this kind of doxastic modality is sometimes overlooked in other hyperintensional theories.

L_{TB} invalidates the inference from belief to necessity $B\Box\phi \models \Box\phi$. This is because even though the agent believes that ϕ is necessary, ϕ can still be false in a world outside the epistemic state.

Fact 12. Doxastic necessity to necessity $B\Box\phi \models \Box\phi$ is invalid in L_{TB}.

Proof. Counterexample: Construct a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$ such that $W = \{w_1, w_2\}$, $\mathcal{B} = \{f\}$ where $f = \{w_1\}$, $V(p) = \{w_1\}$ and $t(f_1) = \{t(p)\}$. Then $D[\Box p] = f$ and we have $\mathbb{M}, w_1 \models B\Box p$ since $\mathbb{M}_{D[\Box p]} \models p \wedge Up$. However, $\mathbb{M}, w_1 \not\models \Box p$ since $\mathbb{M}, w_2 \not\models p$. $\square$

In the other direction, not all necessities are believed, as human agents are not a priori omniscient. For instance, even though Fermat's last theorem is a necessary truth, a 20th-century mathematician might not believe it. In L_{TB}, a priori omniscience in the form $\Box\phi \vDash B\phi$ and its weak version in the form $\Box\phi \wedge BU\phi \vDash B\phi$ are invalid. This is because, even though ϕ is necessary and the agent believes ϕ, the agent may not believe $\Box\phi$ as he may not fully comprehend ϕ.

Fact 13. Apriori omniscience $\Box\phi \vDash B\Box\phi$ and its weak version $\Box\phi \wedge BU\phi \vDash B\Box\phi$ are invalid in L_{TB}.

Proof. Counterexample: Construct a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$ such that $W = \{w_1, w_2\}$, $\mathcal{B} = \{f_1, f_2\}$ where $f_1 = \{w_1\}$ and $f_2 = \{w_2\}$, $V(p) = \{w_1, w_2\}$, $t(f_1) = \{t(p)\}$ and $t(f_2) = \emptyset$. Then we have $\mathbb{M}, w_1 \vDash \Box p \wedge Bp$ since $\mathbb{M}_{D[p]} \vDash p \wedge Up$. However, $\mathbb{M}, w_1 \nvDash U\Box p$ since $t(\Box p) = t(p) \not\leq \oplus t(f_2)$. Given that $D[\Box p] = f_1 \cup f_2 = W$, we have $\mathbb{M} = \mathbb{M}_{D[\Box p]}$. Thus, $\mathbb{M}_{D[\Box p]} \nvDash U\Box p$, which means that $\mathbb{M}, w_1 \nvDash B\Box p$. $\Box$

4.3.3 Necessity to comprehension

As discussed in section 2.3, this thesis proposes to interpret the sentences like "the agent believes that the proposition is necessarily true" as "the agent believes that the proposition is true even when he comprehends the proposition thoroughly". Formally, this means that $B\Box\phi \dashv\vDash B\phi \wedge U\phi$. This is valid in L_{TB}, since believing a necessity requires a thorough comprehension in the model generated by the whole epistemic state, which is also the original model.

Fact 14. Doxastic necessity as comprehension: $B\Box\phi \dashv\vDash B\phi \wedge U\phi$ is valid in L_{TB}.

Proof. By definition 6 and definition 4, we have $\mathbb{M}, w \vDash B\Box\phi \Leftrightarrow \mathbb{M}_{D[\Box\phi]} \vDash \Box\phi \wedge U\Box\phi \Leftrightarrow \mathbb{M}_{D[\Box\phi]} \vDash \phi \wedge U\phi$. Thus we only need to prove that $\mathbb{M}_{D[\Box\phi]} \vDash \phi \wedge U\phi \Leftrightarrow \mathbb{M} \vDash B\phi \wedge U\phi$.

From left to right, assume that $\mathbb{M}_{D[\Box\phi]} \vDash \phi \wedge U\phi$. This also means that $t(\phi) \leq \oplus t(f)$ for all $f \in \mathcal{B}_{D[\Box\phi]}$. By proposition 2, we have $f \subseteq D[\phi]_1$ for all $f \in \mathcal{B}_{D[\Box\phi]}$, which means that $D[\Box\phi]_1 = \bigcup \mathcal{B}_{D[\Box\phi]} \subseteq D[\phi]_1$. By definition 4, we have $D[\phi]_1 \subseteq D[\Box\phi]_1$. Thus $D[\phi]_1 = D[\Box\phi]_1$ and $\mathbb{M}_{D[\Box\phi]} = \mathbb{M}_{D[\phi]}$. Then, we have $\mathbb{M}_{D[\phi]} \vDash \phi \wedge U\phi$ which means that $\mathbb{M} \vDash B\phi$. And we also have $\mathbb{M} \vDash U\phi$, since $\mathbb{M}_{D[\Box\phi]} \vDash U\phi$. Thus, we get $\mathbb{M} \vDash B\phi \wedge U\phi$.

From right to left, assume that $\mathbb{M} \vDash B\phi \wedge U\phi$. Then $D[\phi]_1 = D[\Box\phi]_1 = b$, given that $\mathbb{M} \vDash U\phi$ and proposition 2. Thus $\mathbb{M}_{D[\phi]} = \mathbb{M}_{D[\Box\phi]}$. We also have $\mathbb{M}_{D[\phi]} \vDash \phi \wedge U\phi$ since $\mathbb{M} \vDash B\phi$. Thus, we get $\mathbb{M}_{D[\Box\phi]} \vDash \phi \wedge U\phi$. $\Box$

In fact L_{TB}, validates a stronger inference from necessity to comprehension: if the agent believes a conjunction and one of the conjuncts is a necessity, then the agent comprehend all the conjuncts. Formally, $B(\phi \wedge \Box\psi) \vDash U\phi$ is valid in L_{TB}. This is because the relevant domain of the conjunction is the same as the relevant domain of the necessary conjunct.

Fact 15. Doxastic necessity to comprehension: $B(\phi \wedge \Box\psi) \vDash U\phi$ is valid in L_{TB}.

Proof. Assume that $\mathbb{M}, w \vDash B(\phi \wedge \Box\psi)$. Then $\mathbb{M}_{D[\phi\wedge\Box\psi]} \vDash \phi \wedge \Box\psi \wedge U(\phi \wedge \Box\psi)$. Since $D[\phi \wedge \Box\psi]$ is the epistemic set b, we also have $\mathbb{M}, w \vDash U(\phi \wedge \Box\psi)$, which means that $\mathbb{M}, w \vDash U\phi$. $\Box$

This validity may seem weird at first glance, but it can be explained naturally. Assume that the agent believes $\phi \wedge \Box\psi$, then he should search over his whole epistemic state to see if ψ is true. But during this process, he is not only considering ϕ, but also considering the whole conjunction. Thus, if the agent believes the conjunction, then he also exhausts his whole epistemic state to believe ϕ. This means that the agent comprehends ϕ thoroughly.

4.4 The problem of mathematical knowledge

Most importantly, L_{TB} can address the problem of mathematical knowledge mentioned in [23]. Many hyperintensional theories that do not introduce impossible worlds validate the principle $\Box\phi \wedge B(\phi \vee \neg\phi) \vDash B\phi$. This suggests that if a proposition is necessarily true and the agent is considering whether it is true, then the agent believes the proposition. If this were true, a mathematician would believe every mathematical knowledge under consideration, given that mathematical knowledges are necessary and mathematical propositions are either true or false. As a result, any mathematician in the last century would have believed Fermat's Last Theorem, provided they had thought about it. However, this is obviously not the case, indicating that this principle should be rejected.

Unlike most other possible world semantics, L_{TB} handles this issue very well. Firstly, L_{TB} invalidates the variant of a priori omniscience $\Box\phi \wedge B(\Box\phi \vee \neg\Box\phi) \vDash B\Box\phi$, as the antecedent does not imply $U(\phi)$, suggesting that the agent might not fully comprehend the proposition and thus may not believe it. Consequently, L_{TB} successfully invalidates $\Box\phi \wedge B(\phi \vee \neg\phi) \vDash B\phi$. We can easily get a counterexample when $\phi = \Box p$. For instance, suppose $\Box p$ is necessarily true, and the agent believes that $\Box p$ is either true or false, indicating the positive relevant set of $\Box p$ is non-empty. However, the agent might still not fully comprehend $\Box p$, which means that the agent might not believe $\Box p$.

Fact 16. Mathematical knowledge $\Box\phi \wedge B(\phi \vee \neg\phi) \vDash B\phi$ and its variation $\Box\phi \wedge BU\phi \vDash B\phi$ are invalid in L_{TB}.

Proof. Counterexample: Construct a model $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$ such that $W = \{w_1, w_2\}$, $\mathcal{B} = \{f_1, f_2\}$ where $f_1 = \{w_1\}$ and $f_2 = \{w_2\}$, $V(p) = \{w_1, w_2\}$, $t(f_1) = \{t(p)\}$ and $t(f_2) = \emptyset$. Then $D[p]_1 = D[\Box p]_2 = D[\Box p \vee \neg\Box p]_1 = f_1$, we have $\mathbb{M}, w_1 \vDash \Box\Box p$ and $\mathbb{M}, w_1 \vDash B(\Box p \vee \neg\Box p) \wedge BUp$, since $\mathbb{M}_{D[\Box p\vee\neg\Box p]} \vDash U\Box p \wedge (\Box p \vee \neg\Box p)$. However, $\mathbb{M}, w_1 \nvDash U\Box p$ since $t(\Box p) = t(p) \not\lesssim \oplus t(f_2)$. Given that $D[\Box p]_1 = f_1 \cup f_2 = W$, we have $\mathbb{M} = \mathbb{M}_{D[\Box p]}$. Thus, $\mathbb{M}_{D[\Box p]} \nvDash U\Box p$, which means that $\mathbb{M}, w_1 \nvDash B\Box p$. $\Box$

Our solution can also be naturally explained: all mathematical propositions best be interpreted by $\Box$, meaning that we need to fully comprehend them before believing them. Even though we believe Fermat's Last Theorem to be either true or false and we cannot find a world in which it is false, we do not believe it as long as we fail to comprehend it thoroughly and check it across the whole epistemic state. The reason for why a mathematical knowledge is not believed by us is precisely because we do not fully comprehend it.

4.5 Overview

The following table summarizes the hyperintensional properties of L_{TB} that we proved in this chapter.

Properties	**Validities**
Omniscience rule: $\phi \vDash B\phi$	Invalid
Restricted omniscience: $\phi \wedge BU\phi \vDash B\phi$	Invalid
Closure under disjunction: $B\phi \vDash B(\phi \vee \psi)$	Invalid
Restricted closure under disjunction: $B\phi \wedge BU\psi \vDash B(\phi \vee \psi)$	Invalid
Closure under material implication: $B\psi \wedge (\psi \to \phi) \vDash B\phi$	Invalid
Restricted closure under material implication: $B\psi \wedge BU\phi \wedge (\psi \to \phi) \vDash B\phi$	Invalid
Closure under strict implication: $B\psi \wedge \Box(\psi \to \phi) \vDash B\phi$	Invalid
Restricted closure under strict implication: $B\psi \wedge BU\phi \wedge \Box(\psi \to \phi) \vDash B\phi$	Invalid
Closure under logical consequence: $\psi \vDash \phi$ then $B\psi \vDash B\phi$	Invalid
Restricted closure under logical consequence: $\psi \vDash \phi$ then $B\psi \wedge BU\phi \vDash B\phi$	Invalid
Closure under logical equivalence: $\psi \Dashv\vDash \phi$ then $B\psi \vDash B\phi$	Invalid
Restricted closure under logical equivalence: $\psi \Dashv\vDash \phi$ then $B\psi \wedge BU\phi \vDash B\phi$	Invalid
Closure under conjunction introduction: $B\phi \wedge B\psi \vDash B(\phi \wedge \psi)$	Invalid
Weak closure under conjunction introduction: $B(\phi \wedge U\psi) \wedge B(\psi \wedge U\phi) \vDash B(\phi \wedge \psi)$	Valid
Closure under conjunction elimination: $B(\phi \wedge \psi) \vDash B\phi$	Valid
Closure under doxastic implication: $B\psi \wedge B(\psi \to \phi) \vDash B\phi$	Invalid
Restricted closure under doxastic implication: $B\psi \wedge BU\phi \wedge B(\psi \to \phi) \vDash B\phi$	Invalid
Epistemic contradiction: $B(\phi \wedge \Diamond\neg\phi) \vDash \bot$	Valid
Doxastic necessity to necessity: $B\Box\phi \vDash \Box\phi$	Invalid
Apriori omniscience: $\Box\phi \vDash B\Box\phi$	Invalid
Restricted apriori omniscience: $\Box\phi \wedge B\phi \vDash B\Box\phi$	Invalid
Doxastic necessity as comprehension: $B\Box\phi \Dashv\vDash B\phi \wedge U\phi$	Valid
Doxastic necessity to comprehension: $B(\phi \wedge \Box\psi) \vDash U\phi$	Valid
mathematical knowledge 1: $\Box\phi \wedge B(\phi \vee \neg\phi) \vDash B\phi$	Invalid
mathematical knowledge 2: $\Box\phi \wedge BU\phi \vDash B\phi$	Invalid

Chapter 5

Axiomatisation of L_{TB}

5.1 Axiomatisation

The following table provides a sound axiomatisation TB for the logic of topic-relevant belief L_{TB}. The notion of derivation, denoted by $\vdash$, in TB is defined as usual. Thus, $\vdash \phi$ means ϕ is a theorem in TB.

Axiomatisation TB

CPL	classic propositional tautologies and Modus Ponens
$S5_{\Box}$	$S5$ axioms and rules for $\Box$
	Axioms for U:
U_1	$U(\phi \wedge \psi) \leftrightarrow (U(\phi) \wedge U(\psi))$
U_2	$U(\phi) \leftrightarrow U(\neg\phi)$
U_3	$U(\phi) \leftrightarrow U(U(\phi))$
U_4	$U(\phi) \leftrightarrow U(B\phi)$
U_5	$U(\phi) \leftrightarrow U(\Box\phi)$
	Axioms for B:
B_1	$B(\phi \wedge \psi) \rightarrow B\phi \wedge B\psi$
B_2	$B(\phi \wedge U\psi) \wedge B(\psi \wedge U\phi) \rightarrow B(\phi \wedge \psi)$
B_3	$B\phi \rightarrow \neg B\neg\phi$
B_4	$B\phi \rightarrow BU\phi$
	Axioms connecting B, U and $\Box$:
C_1	$B\phi \rightarrow \Box B\phi$
C_2	$U\phi \rightarrow \Box U\phi$
C_3	$U\phi \rightarrow BU\phi$
C_4	$\Box(\psi \rightarrow \phi) \wedge B(\psi \wedge U\phi) \rightarrow B\phi$
C_5	$B\Box\phi \leftrightarrow B\phi \wedge U\phi$
C_6	$B(\phi \wedge \Box\psi) \rightarrow U\phi$
C_7	$B\neg\Box\phi \rightarrow \neg B\phi$

5.2 Soundness

We prove that TB is a sound axiomatisation for L_{TB}.

Theorem 1. The logic TB is sound with respect to the models of L_{TB} as given above in the semantics in section 3.4: for every formula ϕ, $\vdash \phi \Rightarrow \vDash \phi$.

Proof. Soundness is proved as usual by showing the validity of the axioms and the preservation of soundness under the inference rules. We skip the trivial proofs for the validity of the axioms and inference rules of classical propositional logic. Also, the proofs for validity of the S5 axioms and rules for $\Box$ are standard. For the proofs of other cases, let $\mathbb{M} = \langle W, \mathcal{B}, V, \mathcal{T}, t\rangle$ be an arbitrary model and $w \in W$.

- Axioms for U:
  - Validity of U_1 $U(\phi \wedge \psi) \leftrightarrow (U(\phi) \wedge U(\psi))$:
    $\mathbb{M}, w \vDash U(\phi \wedge \psi)$ if and only if $t(\phi \wedge \psi) \leq \oplus t(f)$ for all the $f \in \mathcal{B}$. The latter is equivalent to the statement that $t(\phi) \leq \oplus t(f)$ and $t(\psi) \leq \oplus t(f)$ for all the $f \in \mathcal{B}$. This is also equivalent to $\mathbb{M}, w \vDash U\psi \wedge U\phi$.
  - Validity of U_2 $U(\phi) \leftrightarrow U(\neg\phi)$:
    $\mathbb{M}, w \vDash U(\phi)$ if and only if $t(\phi) \leq \oplus t(f)$ for all the $f \in \mathcal{B}$. The latter is equivalent to the statement that $t(\neg\phi) \leq \oplus t(f)$ for all the $f \in \mathcal{B}$. This is also equivalent to $\mathbb{M}, w \vDash U\neg\phi$.
  - Validity of U_3 $U(\phi) \leftrightarrow U(U\phi)$:
    $\mathbb{M}, w \vDash U(\phi)$ if and only if $t(\phi) \leq \oplus t(f)$ for all the $f \in \mathcal{B}$. The latter is equivalent to the statement that $t(U\phi) \leq \oplus t(f)$ for all the $f \in \mathcal{B}$. This is also equivalent to $\mathbb{M}, w \vDash UU\phi$.
  - Validity of U_4 $U(\phi) \leftrightarrow U(B\phi)$:
    $\mathbb{M}, w \vDash U(\phi)$ if and only if $t(\phi) \leq \oplus t(f)$ for all the $f \in \mathcal{B}$. The latter is equivalent to the statement that $t(B\phi) \leq \oplus t(f)$ for all the $f \in \mathcal{B}$. This is also equivalent to $\mathbb{M}, w \vDash UB\phi$.
  - Validity of U_5 $U(\phi) \leftrightarrow U(\Box\phi)$:
    $\mathbb{M}, w \vDash U(\phi)$ if and only if $t(\phi) \leq \oplus t(f)$ for all the $f \in \mathcal{B}$. The latter is equivalent to the statement that $t(\Box\phi) \leq \oplus t(f)$ for all the $f \in \mathcal{B}$. This is also equivalent to $\mathbb{M}, w \vDash U\Box\phi$.
- Axioms for B:
  - Validity of B_1 $B(\phi \wedge \psi) \rightarrow B\phi \wedge B\psi$:
    The same proof of fact 9.
  - Validity of B_2 $B(\phi \wedge U\phi) \wedge B(\psi \wedge U\phi) \rightarrow B(\phi \wedge \psi)$:
    The same proof of fact 8.
  - Validity of B_3 $B\phi \rightarrow \neg B\neg\phi$:
    Assume that $\mathbb{M}, w \vDash B\phi$, $\mathbb{M}_{D[\phi]} \vDash \phi$. Then assume towards contradiction that $\mathbb{M}, w \vDash B\neg\phi$, then $\mathbb{M}_{D[\phi]} \vDash \neg\phi$, contradiction. Thus, $\mathbb{M}, w \nvDash B\neg\phi$, which means that $\mathbb{M}, w \vDash \neg B\neg\phi$.

- Validity of B_4 $B\phi \to BU\phi$:
  The same proof of proposition 6.

- Axioms connecting B, U and $\Box$:

  - Validity of C_1 $B\phi \to \Box B\phi$:
    Assume that $\mathbb{M}, w \vDash B\phi$, then $\mathbb{M}_{D[\phi]} \vDash \phi$. This means that $\mathbb{M}, w' \vDash B\phi$ for any $w' \in W$. Thus, $\mathbb{M}, w \vDash \Box B\phi$.
  - Validity of C_2 $U\phi \to \Box U\phi$:
    Assume that $\mathbb{M}, w \vDash U\phi$, then $t(\phi) \leq \oplus t(f)$ for any $f \in \mathcal{B}$. This means that $\mathbb{M}, w' \vDash U\phi$ for any $w' \in W$. Thus, $\mathbb{M}, w \vDash \Box U\phi$.
  - Validity of C_3 $U\phi \to BU\phi$:
    Assume that $\mathbb{M}, w \vDash U\phi$, then $t(\phi) \leq \oplus t(f)$ for any $f \in \mathcal{B}$. This also means that $\mathbb{M}_{D[\phi]} \vDash U\phi$. Thus, $\mathbb{M}, w \vDash BU\phi$.
  - Validity of C_4 $\Box(\psi \to \phi) \land B(\psi \land U\phi) \to B\phi$:
    Assume that $\mathbb{M}, w \vDash \Box(\psi \to \phi) \land B(\psi \land U\phi)$, then $\mathbb{M}_{D[\psi \land U\phi]} \vDash \psi \land U(\psi \land U\phi) \land \Box(\psi \to \phi)$. This means that $\mathbb{M}_{D[\psi \land U\phi]} \vDash \phi \land U\phi$. This indicates that $t(\phi) \leq \oplus t(f)$ for all $f \in \mathcal{B}_{D[\psi \land U\phi]}$. By proposition 2, $f \subseteq D[\phi]_1$ for all $f \in \mathcal{B}_{D[\psi \land U\phi]}$, which also means that $D[\psi \land U\phi]_1 = \bigcup \mathcal{B}_{D[\psi \land U\phi]} \subseteq D[\phi]_1$. By definition 4, we have $D[\phi]_1 \subseteq D[\psi \land U\phi]_1$. Thus, we get $D[\phi]_1 = D[\psi \land U\phi]_1$ and $\mathbb{M}_{D[\phi]} = \mathbb{M}_{D[\psi \land U\phi]}$. Hence, we have $\mathbb{M}_{D[\phi]} \vDash \phi \land U\phi$, which means that $\mathbb{M}, w \vDash B\phi$.
  - Validity C_5 $B\Box\phi \leftrightarrow B\phi \land U\phi$:
    The same proof of fact 14.
  - Validity C_6 $B(\phi \land \Box\psi) \to U\phi$:
    The same proof of fact 15.
  - Validity C_7 $B\neg\Box\phi \to \neg B\phi$:
    Assume that $\mathbb{M}, w \vDash B\neg\Box\phi$. Assume towards contradiction that $\mathbb{M}, w \vDash B\phi$. This means that $\mathbb{M}, w \vDash B(\Diamond\neg\phi \land U\phi)$ and $\mathbb{M}, w \vDash B(\phi \land U(\Diamond\neg\phi))$. By the validity of B_2, we have $\mathbb{M}, w \vDash B(\neg\Box\phi \land \phi)$. However, the proof of fact 11 shows that this is contradictory. Thus, $\mathbb{M}, w \nvDash B\phi$, which means that $\mathbb{M}, w \vDash \neg B\phi$.

This concludes the soundness proof of TB. $\Box$

Chapter 6

Comparison

In this chapter, we compare our logic L_{TB} with other pre-existing theories. In the following sections, we briefly introduce the awareness logic approach developed by Fagin and Halpern [14], the question-sensitive approach from Lewis [33] and Yalcin [54], and the topic-sensitive approach of Berto, Hawke and Özgün [5, 37, 23]. We explain the connections between our theory and the previous ones and we show the advantages of working with L_{TB} in comparison to these works.

6.1 Awareness logic approaches

Traditional epistemic logic for addressing knowledge or belief leads to the logical omniscience problem. Awareness logic, introduced by Fagin and Halpern [14], offers an approach to solve the logical omniscience problem by modelling not only what agents believe but also what they are aware of. This framework addresses the problem of logical omniscience by differentiating between implicit and explicit belief. Implicit belief is just the classical Hintikkan concept of belief. In contrast, explicit belief consists of two elements: a statement ϕ is explicitly believed if and only if (1) ϕ is implicitly believed, and (2) the agent is aware of ϕ. This distinction ensures that logical omniscience is a characteristic of implicit belief, but not of explicit belief.

Awareness in this system is modeled by a set of propositions called the awareness set. This set contains all the propositions that the agent is aware of, and it captures the hyperintensional aspects of belief in a syntactic manner. The structure of an awareness set can vary greatly depending on the particular approach and it might consist of any arbitrary collection of propositions. Specific constraints may be imposed, for example, in many awareness logics we require that if a formula is in the awareness set then all its subformulas should also be in the set.

However, the awareness logic approach encounters several challenges. The first one is determining the appropriate constraints for the awareness set, such that it can validate the desired epistemic closure principles while invalidating the undesired ones. Since the awareness set is defined purely syntactically, it is difficult to find a perfect justification for why should the agent be aware of these propositions but not the others [42]. Moreover, as previously mentioned in section 1.1, standard awareness logic cannot handle some variant versions of logical omniscience, such as the principles restricted by the hyperintension condition. In the example of the Fermat's Last Theorem, a 20-century mathematician might not believe the theorem even when he was aware of it. But in awareness logic, awareness of the theorem leads to the omniscient belief of it, which is problematic.

In contrast, these obstacles are circumvented in L_{TB}. We adopt the topic-sensitive model in L_{TB} so that we do not need a syntactical awareness set to achieve hyperintensionality. Moreover, we demonstrated in section 4.1 that L_{TB} is weaker than the standard awareness logic [41], as the logical omniscience principles restricted by the hyperintension condition are still invalid in L_{TB}. This means that the hyperintensional senarios like the Fermat example can be handled well in L_{TB}.

6.2 Question-sensitive approaches

In awareness logics, the notion of awareness is utilized to achieve the hyperintensional results while in L_{TB} we use topic-relevance to do the same job. However, alternative conceptions might be employed instead of topicality or awareness. It has also been proposed that questions are the essential reason for hyperintensionality. This approach was initiated by Lewis [33], with more recent developments by Yalcin [54] and Hoek [27].

The question-sensitive approaches propose that the meaning of a proposition is shaped by a context in which it is evaluated and so that a belief can be formed only as an answer towards a question. Thus, beliefs are question-sensitive. For example, in [54], Yalcin proposes the concept of resolution of a logical space as a way to understand how beliefs are influenced by the questions being asked and the context in which they are evaluated. The resolutions of a logical space are partitions over the logical space, representing agents' cognitive abilities and computational limitations. A resolution divides the space into exclusive cells, such that the worlds within a cell are indistinguishable to the agent. This means that agents may not have the ability to distinguish the true world from others. In Yalcin's theory, he defines foreground and background propositions based on the notion of resolution: With respect to a resolution, foreground propositions are those that align with the resolution, and are constructed entirely from (unions of) cells of the resolution, which means that they do not cut through the cells [54]. The remaining propositions are background propositions, which with respect to the resolution, cannot be believed explicitly. As a result, the similar conceptions of explicit and implicit belief as in awareness logics can be defined, but in terms of partitions and questions.

The question-sensitive approaches can successfully achieve hyperintensionality non-syntactically. However, being partition-based, these approaches cannot handle some kinds of logical omniscience. For instance, they cannot invalidate closure under logical equivalence, as logically equivalent formulas are true in the same possible worlds and thus always cut cross the same partition cells. Consequently, confining content to possible worlds, these approaches cannot distinguish the logically equivalent sentences. In [54], Yalcin also admits that fragmentation should be introduced to solve the problem.

In contrast, this problem is dealt with well in L_{TB}. As shown in section 4.1, L_{TB} invalidates both closure under logical equivalence and its restricted version. And this invalidity itself does not require fragmentation, since we use topic-relevance to distinguish two logical equivalent propositions.

6.3 Topic-sensitive approaches

In 2019, Berto [5] originally developed a topic-sensitive model based on a mereology structure of topics. Building on Berto's initial concept, recent collaborations by Berto, Hawke, and Özgün [7, 8, 37, 23] have led to a series of hyperintensional systems based on the topic-sensitive model, which can solve various kinds of logical omniscience problems.

Our topic-relevance logic L_{TB} can be regarded as a close variant of the topic-sensitive logic that originated from Berto's work, since we also adopt the topic-sensitive structure and assign topics to propositions. As already shown in [37, 7], the topic-sensitive approaches can not only deal with logical omniscience problems but also handle hyperintensional belief revision and indicative conditionals.

Despite these achievements, we notice that epistemic modality –the issue pertaining to the modal operators in an epistemic context, does not receive enough attention from the authors following the topic-sensitive approaches. This oversight is also found in other hyperintensional theories such as awareness logics and question-sensitive approaches. As a result, the topics of the sentences like "the proposition might be true", along with the modeling of beliefs towards such sentences remain relatively unexplored.

Additionally, as Berto, Hawke and Özgün noted in [23], mathematical knowledge can give rise to a problem in their logic. Formally, their topic-sensitive logics validate $\Box\phi \wedge B(\phi \vee \neg\phi) \vDash B\phi$, which is not a desired result. Recall the example in section 2.1, Fermat's Last Theorem is necessarily true, and anyone that heard the name of the theorem believes that the theorem is either true or false. But one may not believe that the theorem is true if one hasn't heard that the theorem has been proved or one doesn't even understand the theorem.

This observation leads us to a significant innovation in our theory L_{TB}. We introduce an operator for comprehension and we interpret sentences like "believing the proposition is necessarily true" as "believing the proposition even after comprehending it thoroughly". As a result, the problem of mathematical knowledge is resolved in L_{TB}, as already shown in section 4.4.

Chapter 7

Conclusion

This thesis introduces L_{TB}, a novel hyperintensional doxastic logic based on the idea of topic-relevance and fragmentation. By modeling belief based on fragments generated by topics and information pieces, L_{TB} overcomes some of the challenges faced by traditional epistemic logic in handling hyperintensional scenarios.

In chapter 2 of the thesis, we explain the philosophical ideas behind our logic. The formal semantics are laid out in chapter 3. The primary results of our research, namely the hyperintensional properties of L_{TB}, are presented in chapter 4.

We have demonstrated that a variety of omniscience principles, including the omniscience rule, closure under disjunction, closure under material implication, closure under strict implication, closure under logical consequence, and closure under logical equivalence, as well as their variations restricted by the hyperintension condition, are invalid in L_{TB}. Since our framework is built upon the notion of fragmentation, we have also proved the invalidity of closure under conjunction, supported by arguments from sources such as [42, 31, 23], while a weaker version remains valid and is motivated by taking beliefs tied to topicality into account. As another result of fragmentation, closure under doxastic implication and its restricted version are invalid in L_{TB}. Moreover, we have established the validity of the epistemic contradiction, whereas the inferences from doxastic necessity to necessity, a priori omniscience inferences, and their restricted versions are found to be invalid in L_{TB}. Most importantly, our research has led to a compelling interpretation of doxastic necessity as comprehension, contributing a possible solution to the problem of mathematical knowledge.

In chapter 5 we introduce a provisional logical axiomatisation for L_{TB}, and its soundness has been proved. Finally, chapter 6 draws comparisons between our logic and other pre-existing theories. L_{TB} not only addresses more versions of logical omniscience than the awareness logic approaches and the question-sensitive approaches, it also proposes a novel and intuitive interpretation for the alethic operator within a doxastic context. Thus, L_{TB} provides a potential solution to the unresolved issue of mathematical knowledge, a problem that persists in the topic-sensitive approaches.

While L_{TB} offers a promising framework for hyperintensional doxastic reasoning, there are still open areas and opportunities for further exploration and refinement. One area of exploration is extending L_{TB} with the operators for dynamic belief revision and belief update. Incorporating operators that can cause changes of information or topics in L_{TB} would allow for more sophisticated modeling of agents' evolving epistemic states. Exploring a multi-agent framework or adding operators for more attitudes in a multi-modal framework are similarly intriguing directions. Another important direction for investigation concerns the meta-logical theory involving our newly proposed axiomatisation. Given that in this thesis we only have a sound axiomatisation, the pursuit of a sound and complete axiomatisation for our logic remains an essential and stimulating challenge ahead.

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