Taming Logics

Szabolcs Mikulás

ILLC Dissertation Series 1995-12

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Institute for Logic, Language and Computation
Universiteit van Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam
phone: +31-20-5256090
fax: +31-20-5255101
e-mail: illc@fwi.uva.nl

Taming Logics

Academisch Proefschrift

ter verkrijging van de graad van doctor aan de
Universiteit van Amsterdam,
op gezag van de Rector Magnificus
Prof.dr P.W.M. de Meijer
in het openbaar te verdedigen in de
Aula der Universiteit
(Oude Lutherse Kerk, ingang Singel 411, hoek Spui)
op vrijdag 8 september 1995 te 11.30 uur

door

Szabolcs Mikulás

geboren te Kecskemét.

Promotors:
Prof.dr. H. Andréka
Prof.dr. J. van Benthem

CIP-GEGEVENS KONINKLIJKE BIBLIOTHEEK, DEN HAAG

Mikulás, Szabolcs

Taming logics / Szabolcs Mikulás. - Amsterdam : Institute
for Logic, Language and Computation, Universiteit van
Amsterdam. - Ill. - (ILLC dissertation series ; 1995-12)
Proefschrift Universiteit van Amsterdam. - Met lit. opg.
ISBN 90-74795-33-1
Trefw.: wiskundige logica / computerlinguïstiek.

On the cover "*Does humor belong in logic?*" by Pál Tóth.
ISBN: 90–74795–33–1

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ACKNOWLEDGMENTS

I am deeply indebted to my supervisors: Hajnal Andréka and Johan van Benthem. Johan, besides being a nice football partner, helped me score a few goals, by stimulating discussions and his intensive care. Hajnal taught me many things: algebra, logic (algebraic and other), persistence, drawing pictures, etc. She, together with István Németi and Ildikó Sain, has been a nice teacher, and they have been my "pótapuka" as well. Thanks to all of you for your moral and scientific support!

I would like to thank my co-authors: Hajnal Andréka, Steven Givant, Maarten Marx, István Németi, and András Simon. Steven Givant, Ágnes Kurucz, Maarten Marx, and Yde Venema read various parts of the dissertation. I found their comments very useful.

I would like to express my gratitude to all my teachers and colleagues at the Symbolic Logic Department of ELTE and at the Mathematical Institute in Budapest, and at the Department of Mathematics and Computer Science of the University of Amsterdam. It has been a great help to work in an inspiring atmosphere and that I could present my (sometimes half-cooked) ideas in the "snelkookpan" sessions in Amsterdam and at the seminars of the Logic Graduate School in Budapest. Thanks are due to everybody who attended these talks and/or showed interest in my work.

I would also like to thank the following people for various reasons: Natasha Alechina, Patrick Blackburn, Frank de Boer, Peter de Jong, Marco Hollenberg, Marianne Kalsbeek, Natasha Kurtonina, Ágnes Kurucz, Anna Madarász, András Máté, the Pijls family, Krisztina Polgárdi, László Pólos, Maarten de Rijke, Jan Rutten, Imre Ruzsa, Jerry Seligman, Daniele Turi, Yde Venema. Special thanks are due to Guszti Eiben and Maarten Marx. Very special thanks are due to Monique for playing darts (which helped me understand the nature of arrows) and other nice games with me.

Thanks are due for financial support to the Hungarian National Foundation for Scientific Research, to TEMPUS and to the Department of Mathematics and Computer Science of the University of Amsterdam.

Finally, I would like to emphasize that, in spite of the title, no creature was harmed by any means in connection with the writing of the dissertation.

Halfway between Amsterdam and Budapest — Szabolcs Mikulás
July, 1995.

INTRO

"What is your conceptual continuity?"
Frank Zappa

This dissertation is about *algebraic logic*, i.e., about *algebras*, *logics*, and their *connection*. In particular, we will investigate *modal logics* with *dynamic* character, *predicate logics*, and the corresponding classes of *algebras of relations*. Further, we will develop a "bridge" connecting logics and algebras, and associating metalogical and algebraic properties.

Let us have a closer look at the connection between logics and algebras. Given a logic L, we may consider the corresponding class $\mathsf{Alg}(\mathsf{L})$ of algebras. Metalogical properties have their algebraic counterparts too, e.g., completeness of a logic corresponds to finite axiomatizability on the algebra side, and the decidability of the set of validities of L is equivalent to the decidability of the equational theory of $\mathsf{Alg}(\mathsf{L})$. Such a bridge between logics and algebras enables us to investigate logics by investigating their algebraic counterparts, and by translating the algebraic result yielding an answer to our (purely) logical question. This approach has the following advantage: the algebraic counterparts of metalogical properties are well-investigated algebraic properties, and we can use the well-developed techniques of (universal) algebra to find the solutions to our problems. On the other hand, logical methods can be used to prove algebraic theorems. Such example is the decidability of the equational theories of certain classes of algebras proved by filtration, cf., e.g., [Ma95].

TAMING. In this dissertation, our main hobbyhorse will be "taming"[1], i.e., finding well-behaved versions of well-investigated logics. We will achieve this goal by applying the powerful machinery of algebraic logic and *universal algcbra*, and by using the bridge described above.

The problem with many logics is that they have some undesirable metalogical properties such as incompleteness, undecidability, or the lack of the Beth definability and the Craig interpolation properties. We will apply several taming strategies to find versions of logics such that these tamed versions behave much nicer than the original logics.

Our main strategy to find tamed versions of logics will be the following. Given a logic L, we will consider its algebraic counterpart $\mathsf{Alg}(\mathsf{L})$. $\mathsf{Alg}(\mathsf{L})$ may have undesirable properties reflecting some ugly feature of L. Using the techniques, e.g., *relativization* (see Section 1.2 and Chapter 3), of algebraic logic, we will define another class K of algebras such that (i) K is rather "close" to $\mathsf{Alg}(\mathsf{L})$ and (ii) K has nicer properties than

[1]Taming is a classical topic in literature, cf. [Sh], but it was hardly ever used in connection with logics.

Alg(L) has. That version of L which corresponds to K has nicer metalogical propertie and its (expressive) power is still large.

ORGANIZATION. The organization of the dissertation reflects the strategy of using th bridge. In Chapter 1, we will introduce the logics we are going to investigate: severa versions of *arrow logic* AL, and (finite variable fragments of) *classical first-order logi* FOL. Then we recall some basic definitions and results of algebras, and work ou the bridge between logics and algebras. In this dissertation, we will concentrate o *completeness* and *decidability*. But the bridge between logics and algebras, and th idea of taming are not restricted to these metalogical properties. See [AKNS] fc developing the bridge for more metalogical properties and [Ma95] and [MMN94] fc taming.

In the other chapters we will develop three taming strategies. These chapter contain a section about logics and another section about algebras. Usually, the re sults of the logic section will follow from the corresponding algebraic results using th bridge described in Chapter 1. In the logic sections, we will state (in)completenes and (un)decidability results for arrow logics and predicate logics. The algebraic sec tions will contain the corresponding (non-)finite axiomatizability and (un)decidabilit theorems for algebras of relations.

In Chapter 2, we will apply the following taming strategy. We will consider *frag ments* of pair arrow logic PAL, and state completeness and decidability results for thes versions, while the full version of pair arrow logic lacks these properties. The idea be hind this approach is that the undecidability and incompleteness of PAL are cause by the interaction of two connectives: composition and disjunction. We will conside fragments in which only one of them (composition) is present. This approach has th advantage that the meanings of the connectives are the original ones. For instance composition remains an *associative* connective. The most important logic of this chap ter is the *Lambek calculus* LC. The main result is the completeness of LC with respec to *relational semantics*. This result provides us with a new perspective on LC: beside its *linguistic* applications, it is a *substructural* logic with a *dynamic* character. Th corresponding algebraic result is the representability of *(semilattice-)ordered residuate semigroups* as algebras of binary relations.

In Chapter 3, we will consider *relativized* versions of arrow logic AL. First, w will widen the class of models for the pair version PAL, and then (re-)introducin connectives that are not definable in the weakened version. The advantage of thi approach is that all of the connectives of the original logic are present. Moreover, w can add connectives that are not definable even in the original version. For instance, w will define complete and decidable versions of PAL in which the *difference operator* an the *graded modalities* are definable. The price we have to pay for these results is tha some of the connectives have slightly different meanings than in the original versior The most important difference is that instead of (full) associativity of compositior only its weakened version holds. The corresponding algebraic results are the finit axiomatizability and decidability of expansions of *weakly associative relation algebra* and other relativized versions of *representable relation algebras*.

In Chapter 4, we will prove completeness results for the original versions of PAL and FOL. That is, in this case, we will not reduce their power. However, this requires to allow inference systems with more complicated rules: we have to make some easily decidable, syntactic restriction on the application of some inference rules. The algebraic counterpart of this kind of completeness results is that those *Boolean monoids*, *relation*, *cylindric*, and *quasi-polyadic* (*equality*) *algebras* which satisfy a certain *density* condition are representable as algebras of relations.

Finally, we will mention some open problems connected to the dissertation, related results, and possible further directions.

1

THE BEASTS AND THE WHIP

The aim of this chapter is to introduce the basic definitions of logics and algebras and to explain the connections between metalogical and algebraic properties. First, we give the definitions of the logics we are going to investigate — the beasts to be tamed —, then recall some basic definitions and results from universal algebra — the whip we will use to tame —, and build a "bridge" between logics and algebras. We will use this bridge to convert algebraic results to theorems about logics. The techniques of algebraic logic and universal algebra will help us solve difficult logical problems. Moreover, we can use these tools to design logical systems with nice metalogical properties and large (expressive) power.

The following chapters will be based on this chapter in the sense that we will use the bridge theorems of Section 1.3 to prove metalogical theorems. To understand the connection between logics and algebras it is *not absolutely* necessary to be familiar with all the technical details of the proofs of these theorems. Although, it may help to get insights about the taming strategies.

1.1 LOGICS

In this section, we give the definition of logic in general, and define several logics that we will investigate later. Then we recall the definitions of some metalogical notions.

1.1.1 LOGIC IN GENERAL

The following definition of logic is slightly narrower than the notion of strongly nice logic in [AKNS]. However, this amount of generality will be sufficient for our purpose, i.e., almost all of the logics investigated later satisfy the conditions of Definition 1.1.1. Some of the logics in Chapter 2 are not logics in the sense of Definition 1.1.1, that is why we develop the algebraization of these logics in Section 2.3.

Most of the logics we will consider extend classical propositional logic, i.e., the connectives of propositional logic are definable, and they have their usual meanings. That is why we use the symbols $\top$ and $\leftrightarrow$ in the following definition. This does not imply that true, and biconditional of classical propositional logic must be definable in a logic. However, in the logics of this section, $\top$ and $\leftrightarrow$ are indeed true and biconditional, respectively. We also note that, instead of the constant $\top$, we could use the formula $\varphi \leftrightarrow \varphi$ (φ any formula).

Definition 1.1.1. (Logic, L) By a *logic* we mean an ordered tuple

$$\mathsf{L} \stackrel{\text{def}}{=} \langle \mathsf{F}_\mathsf{L}, \mathsf{M}_\mathsf{L}, \models_\mathsf{L}, \mathsf{mean}_\mathsf{L} \rangle$$

satisfying the following conditions.

1. There is a set $Cn(\mathsf{L})$, the set of *logical connectives* of L, and every $c \in ($ has a rank $r_c \in \omega$.[1] We will denote the set of logical connectives with ra by $Cn_k(\mathsf{L})$. There is a set P, the set of *propositional variables*, or *param* or *atomic formulas*, such that $\mathsf{F_L}$ is the smallest set satisfying the followin conditions.
   (a) $P \subseteq \mathsf{F_L}$.
   (b) If $c \in Cn_k(\mathsf{L})$ and $\varphi_0, \ldots, \varphi_{k-1} \in \mathsf{F_L}$, then $c(\varphi_0, \ldots, \varphi_{k-1}) \in \mathsf{F_L}$.

   The algebra $\mathfrak{F}_\mathsf{L} \stackrel{\text{def}}{=} \langle \mathsf{F_L}, c \rangle_{c \in Cn(\mathsf{L})}$ is called the *formula algebra* of L.
2. $\mathsf{M_L}$ is some class, called the class of *models*.
3. $\mathsf{mean_L}$ is a function with domain $\mathsf{F_L} \times \mathsf{M_L}$, and, for every $\mathfrak{M} \in \mathsf{M_L}$, the fu $\mathsf{mean}_\mathsf{L}^\mathfrak{M} \stackrel{\text{def}}{=} \langle \mathsf{mean_L}(\varphi, \mathfrak{M}) : \varphi \in \mathsf{F_L} \rangle$ is a homomorphism from $\mathfrak{F}_\mathsf{L}$. Or, in words, the relation $\varphi \sim \psi$ defined by $\mathsf{mean}_\mathsf{L}^\mathfrak{M}(\varphi) = \mathsf{mean}_\mathsf{L}^\mathfrak{M}(\psi)$ is a congr relation on $\mathfrak{F}_\mathsf{L}$.
4. $\models_\mathsf{L}$ is a binary relation, called *truth*, between models and formulas: $\models_\mathsf{L} \subseteq \mathsf{M}$
5. There are a binary connective $\leftrightarrow$ and a zero-ary connective $\top$ such that
   (a) $(\forall \mathfrak{M} \in \mathsf{M_L})(\forall \varphi, \psi \in \mathsf{F_L})(\mathsf{mean_L}(\varphi, \mathfrak{M}) = \mathsf{mean_L}(\psi, \mathfrak{M}) \iff \mathfrak{M} \models_\mathsf{L} \varphi \leftarrow$
   (b) $(\forall \mathfrak{M} \in \mathsf{M_L})(\forall \varphi \in \mathsf{F_L})(\mathfrak{M} \models_\mathsf{L} \varphi \iff \mathfrak{M} \models_\mathsf{L} \varphi \leftrightarrow \top)$.
6. $(\forall \psi, \varphi_0, \ldots, \varphi_k \in \mathsf{F_L})(\forall p_0, \ldots, p_k \in P)((\forall \mathfrak{M} \in \mathsf{M_L})\mathfrak{M} \models_\mathsf{L} \psi(\overline{p}) \Rightarrow (\forall$ $\mathsf{M_L})\mathfrak{M} \models_\mathsf{L} \psi(\overline{p}/\overline{\varphi}))$, where $\psi(\overline{p}/\overline{\varphi})$ denotes the formula given by simultan substituting φ_i for every occurrence of p_i $(i \leq k)$ in ψ.
7. $(\forall \mathfrak{M} \in \mathsf{M_L})(\forall \varphi_0, \ldots, \varphi_k \in \mathsf{F_L})(\forall p_0, \ldots, p_k \in P)(\exists \mathfrak{N} \in \mathsf{M_L})(\forall \psi \in \mathsf{F_L})$

$$\mathsf{mean_L}(\psi, \mathfrak{N}) = \mathsf{mean_L}(\psi(\overline{p}/\overline{\varphi}), \mathfrak{M}).$$

We say that a formula φ is a *semantical consequence* of the set Γ of formulas, in sy $\Gamma \models \varphi$, iff, for every model $\mathfrak{M}$,

$$\mathfrak{M} \models_\mathsf{L} \Gamma \Rightarrow \mathfrak{M} \models_\mathsf{L} \varphi$$

where $\mathfrak{M} \models_\mathsf{L} \Gamma$ abbreviates that, for every $\psi \in \Gamma$, $\mathfrak{M} \models_\mathsf{L} \psi$. A formula φ is *valid* if it is a semantical consequence of the empty set of formulas. ∎

Sometimes, in the definition of a logic, we will omit some of the components ordered tuple above. We can do it without loss of generality, cf. [ANS94]. confusion is likely, we will omit the subscript L form $\mathsf{F_L}$, etc.

1.1.2 DISTINGUISHED LOGICS

Now we give the definitions of several logics. It is easy to check that all of these satisfy the conditions of Definition 1.1.1.

[1] The symbol ω denotes the set of natural numbers, or, equivalently, the set of finite ordir will consider every element n of ω as the set of all natural numbers smaller than n: $n = \{0, 1, ..$ In particular, $0 = \emptyset$.

First we recall the definition of a multimodal logic, called arrow logic AL, cf. [vB94], [Ve91], and [Ve92]. Later we will define several versions of AL, e.g., we will consider fragments, or we will expand the set of connectives, or restrict the class of models, etc. For more on arrow logic, we refer to [AKNSS], [Ma95] [MMNS], [MMN94], [Mi92b] and [Si92].

Definition 1.1.2. (Arrow logic, AL) *Arrow logic* AL is defined as

$$\mathsf{AL} \stackrel{\text{def}}{=} \langle \mathsf{F}_{\mathsf{AL}}, \mathsf{M}_{\mathsf{AL}}, \models_{\mathsf{AL}}, \mathsf{mean}_{\mathsf{AL}} \rangle$$

where F_{AL}, M_{AL}, $\models_{\mathsf{AL}}$, and $\mathsf{mean}_{\mathsf{AL}}$ are defined as follows.

1. $Cn(\mathsf{AL}) = \{\wedge, \neg, \bullet, \otimes, \iota\delta\}$ where $\iota\delta$ is a 0-ary connective, $\neg$ and $\otimes$ are unary and $\wedge$, $\bullet$ are binary connectives. That is, F_{AL} is the smallest set satisfying the following four closure conditions:
   - $P \subseteq \mathsf{F}_{\mathsf{AL}}$
   - $\varphi \in \mathsf{F}_{\mathsf{AL}} \Rightarrow \neg\varphi, \otimes\varphi \in \mathsf{F}_{\mathsf{AL}}$
   - $\varphi, \psi \in \mathsf{F}_{\mathsf{AL}} \Rightarrow (\varphi \wedge \psi), (\varphi \bullet \psi) \in \mathsf{F}_{\mathsf{AL}}$
   - $\iota\delta \in \mathsf{F}_{\mathsf{AL}}$.

   We will also use the well-known derived connectives $\rightarrow$, $\leftrightarrow$, $\vee$, and the formulas *false* $(\bot)$ and *true* $(\top)$.
2. An *arrow frame* for AL is a Kripke frame with three accessibility relations corresponding to the extra-Boolean connectives. That is, it is an ordered tuple $\langle W, C, R, I\rangle$, where W, called the set of *arrows*, is a non-empty set, and C, R, and I are ternary, binary, and unary relations on W, respectively; i.e., $C \subseteq W \times W \times W$, $R \subseteq W \times W$, and $I \subseteq W$.
   An *arrow model* for AL is an arrow frame enriched with a valuation v. More precisely, it is an ordered tuple $\langle W, C, R, I, v\rangle$, where $v : P \longrightarrow \mathcal{P}(W)$, i.e., to every parameter, v associates a subset of W. M_{AL} is the class of all arrow models.
3. *(Local) Truth* of a formula φ at an arrow $w \in W$ in a model $\langle W, C, R, I, v\rangle$, denoted as $w \Vdash \varphi$, is defined by recursion as follows.
   - If $p \in P$, then $w \Vdash p \stackrel{\text{def}}{\Longleftrightarrow} w \in v(p)$.
   - $w \Vdash (\varphi \wedge \psi) \stackrel{\text{def}}{\Longleftrightarrow} (w \Vdash \varphi \;\&\; w \Vdash \psi)$.
   - $w \Vdash \neg\varphi \stackrel{\text{def}}{\Longleftrightarrow}$ not $w \Vdash \varphi$ (also denoted as $w \nVdash \varphi$).
   - $w \Vdash (\varphi \bullet \psi) \stackrel{\text{def}}{\Longleftrightarrow} (\exists w_1, w_2 \in W)(Cww_1w_2 \;\&\; w_1 \Vdash \varphi \;\&\; w_2 \Vdash \psi)$.
   - $w \Vdash \otimes\varphi \stackrel{\text{def}}{\Longleftrightarrow} (\exists w_1 \in W)(Rww_1 \;\&\; w_1 \Vdash \varphi)$.
   - $w \Vdash \iota\delta \stackrel{\text{def}}{\Longleftrightarrow} Iw$.

   (Global) Truth in a model and *validity* in a frame are defined in the usual way. That is,
   - $\langle W, C, R, I, v\rangle \models_{\mathsf{AL}} \varphi \stackrel{\text{def}}{\Longleftrightarrow}$ for every arrow $w \in W$, $w \Vdash \varphi$
   - $\langle W, C, R, I\rangle \models \varphi \stackrel{\text{def}}{\Longleftrightarrow}$ for every valuation v, $\langle W, C, R, I, v\rangle \models_{\mathsf{AL}} \varphi$.

4. The function $\mathsf{mean}_{\mathsf{AL}}$ is defined as follows. For every model $\mathfrak{M} = \langle W, C, R,$ M_{AL} and formula $\varphi \in \mathsf{F}_{\mathsf{AL}}$,

$$\mathsf{mean}_{\mathsf{AL}}(\varphi, \mathfrak{M}) \stackrel{\text{def}}{=} \{w \in W : w \Vdash \varphi\}.$$

∎

In the following definition, we restrict the class of models for AL to binary rel and interpret the accessibility relations in a natural way.

Definition 1.1.3. (Pair arrow logics: PAL, PAL_H, and PAL_{sq})

1. *Pair arrow logic* PAL is defined as

$$\mathsf{PAL} \stackrel{\text{def}}{=} \langle \mathsf{F}_{\mathsf{PAL}}, \mathsf{M}_{\mathsf{PAL}}, \models_{\mathsf{PAL}}, \mathsf{mean}_{\mathsf{PAL}} \rangle$$

where $\mathsf{F}_{\mathsf{PAL}}$, $\mathsf{M}_{\mathsf{PAL}}$, $\models_{\mathsf{PAL}}$, and $\mathsf{mean}_{\mathsf{PAL}}$ are defined as follows.
The set of formulas coincide with that of AL: $\mathsf{F}_{\mathsf{PAL}} = \mathsf{F}_{\mathsf{AL}}$.
An *arrow frame* for PAL, also called a *pair frame*, is an ordered tuple $\langle W, C$ where W is a non-empty set of ordered pairs, i.e., W is a binary relatic C, R, and I are *relation composition*, *converse*, and *identity* relativized respectively: that is, for every $\langle x, x'\rangle, \langle y, y'\rangle, \langle z, z'\rangle \in W$,

$$\begin{aligned} C\langle x, x'\rangle\langle y, y'\rangle\langle z, z'\rangle &\stackrel{\text{def}}{\iff} x = y \;\&\; y' = z \;\&\; x' = z' \\ R\langle x, x'\rangle\langle y, y'\rangle &\stackrel{\text{def}}{\iff} x = y' \;\&\; x' = y \\ I\langle x, x'\rangle &\stackrel{\text{def}}{\iff} x = x'. \end{aligned}$$

An *arrow model* for PAL, a *pair model*, is an arrow frame enriched with a va v. The definitions of the other notions are the usual.
Note that it follows that the interpretations of the extra-Boolean connecti as below.

- $\langle x, y\rangle \Vdash (\varphi \bullet \psi) \iff \exists z(\langle x, z\rangle, \langle z, y\rangle \in W \;\&\; \langle x, z\rangle \Vdash \varphi \;\&\; \langle z, y\rangle \Vdash \psi)$.
- $\langle x, y\rangle \Vdash \otimes\varphi \iff (\langle y, x\rangle \in W \;\&\; \langle y, x\rangle \Vdash \varphi)$.
- $\langle x, y\rangle \Vdash \iota\delta \iff x = y$.

2. Let r, s, and t abbreviate 'reflexive', 'symmetric', and 'transitive', respe and let $H \subseteq \{r, s, t\}$.
By PAL_H we mean that version of PAL where the universes of the frames the conditions in H. Thus, e.g., $\mathsf{PAL}_{\{s,r\}}$ denotes that pair arrow logic every frame has a symmetric and reflexive universe.
3. By the *square version* PAL_{sq} we mean that version of PAL where the fran Cartesian spaces, i.e., they have the form $U \times U$.

∎

Next we define several versions of classical first-order logic. We will consider fra with n variables, their restricted versions where the order of the variables in pr symbols is fixed, and their equality-free fragments, cf. [HMT85] 4.3.

Definition 1.1.4. (First-order logics with n variables: $\mathsf{L}_n{}^{=}$, ${}^{r}\mathsf{L}_n{}^{=}$, $\mathsf{L}_n{}^{\neq}$ and ${}^{r}\mathsf{L}_n{}^{\neq}$) Let n be a natural number.

Ordinary first-order logic with n variables with equality is defined as the ordered tuple

$$\mathsf{L}_n{}^{=} \stackrel{\text{def}}{=} \langle \mathsf{F}, \mathsf{M}, \models, \mathsf{mean} \rangle$$

for which the following conditions hold.

1. Let $V \stackrel{\text{def}}{=} \{v_0, \dots, v_{n-1}\}$ be the set of *variables*. Let P denote the set of *atomic formulas*, i.e., $P \stackrel{\text{def}}{=} \{r(v_{j_0}, \dots, v_{j_{n-1}}) : r \in R \;\&\; j_0, \dots, j_{n-1} \in n\}$ for some set R; the symbols r ($r \in R$) are called *relation* or *predicate symbols.* Then the set F of *formulas* is the smallest set H satisfying:
   - $P \subseteq H$
   - $v_i = v_j \in H$ for every $i, j \in n$
   - $\varphi, \psi \in H, i \in n \Rightarrow (\varphi \wedge \psi), \neg\varphi, \exists v_i \varphi \in H.$

   Sometimes we will use the notation $\exists_i \varphi$ instead of $\exists v_i \varphi$, and δ_{ij} instead of $v_i = v_j$. By the set of *connectives* of $\mathsf{L}_n{}^{=}$, $Cn(\mathsf{L}_n{}^{=})$, we mean the set $\{\wedge, \neg, \exists_i, \delta_{ij} : i, j \in n\}$.
2. The class M of *models* is defined by

   $$\mathsf{M} \stackrel{\text{def}}{=} \{\langle M, I \rangle : M \neq \emptyset, I : R \longrightarrow \mathcal{P}({}^nM)\}$$

   where ${}^nM = \{\langle x_0, \dots, x_{n-1} \rangle : x_0, \dots, x_{n-1} \in M\}$. If $\mathfrak{M} = \langle M, I \rangle \in \mathsf{M}$, then M is called the *universe* of $\mathfrak{M}$.
3. Let $k \in {}^nM$. We will consider k as an evaluation of the variables such that, for every $i \in n$, the value given to the variable v_i by k is $k(i)$. We define *local truth* $\langle \mathfrak{M}, k \rangle \Vdash \varphi$ (sometimes also denoted by $\mathfrak{M} \Vdash \varphi[k]$, or by $k \Vdash \varphi$) by induction on the complexity of φ:
   - $\langle \mathfrak{M}, k \rangle \Vdash r(v_{j_0}, \dots, v_{j_{n-1}}) \stackrel{\text{def}}{\iff} \langle k(j_0), \dots, k(j_{n-1}) \rangle \in I(r) \quad (r \in R)$
   - $\langle \mathfrak{M}, k \rangle \Vdash v_i = v_j \stackrel{\text{def}}{\iff} k(i) = k(j) \quad (i, j \in n)$
   - if $\psi_1, \psi_2 \in \mathsf{F}$ and $i \in n$, then

   $$\begin{aligned} \langle \mathfrak{M}, k \rangle \Vdash \neg\psi_1 &\stackrel{\text{def}}{\iff} \text{not } \langle \mathfrak{M}, k \rangle \Vdash \psi_1 \\ \langle \mathfrak{M}, k \rangle \Vdash \psi_1 \wedge \psi_2 &\stackrel{\text{def}}{\iff} \langle \mathfrak{M}, k \rangle \Vdash \psi_1 \;\&\; \langle \mathfrak{M}, k \rangle \Vdash \psi_2 \\ \langle \mathfrak{M}, k \rangle \Vdash \exists v_i \psi_1 &\stackrel{\text{def}}{\iff} (\exists k' \in {}^nM)(\forall j \neq i) k'(j) = k(j) \;\&\; \langle \mathfrak{M}, k' \rangle \Vdash \psi_1. \end{aligned}$$

   If $\langle \mathfrak{M}, k \rangle \Vdash \varphi$, then we say that the evaluation k *satisfies* the formula φ in the model $\mathfrak{M}$. We say that $\mathfrak{M}$ *satisfies*, or *validates* φ, or that φ is *true* in $\mathfrak{M}$, in symbols $\mathfrak{M} \models \varphi$, iff for every $k \in {}^nM$, $\langle \mathfrak{M}, k \rangle \Vdash \varphi$.
4. The *interpretation*, or *meaning*, of a formula φ in a model $\mathfrak{M}$ is defined as

   $$\mathsf{mean}(\varphi, \mathfrak{M}) \stackrel{\text{def}}{=} \{k \in {}^nM : \langle \mathfrak{M}, k \rangle \Vdash \varphi\}.$$

   Instead of $\mathsf{mean}(\varphi, \mathfrak{M})$ sometimes we will use the notation $\varphi^{\mathfrak{M}}$.

Restricted first-order logic with n variables with equality, ${}^r\mathsf{L}_n{}^=$, differs from the corresponding ordinary logic in the following: in restricted logic the order of the variables in atomic formulas $r(v_0, \ldots, v_{n-1})$ is fixed. That is, the set of atomic formulas of ${}^r\mathsf{L}_n{}^=$ is $\{r(v_0, \ldots, r_{n-1}) : r \in R\}$.

$\mathsf{L}_n{}^{\neq}$ and ${}^r\mathsf{L}_n{}^{\neq}$ denote the equality-free fragment of $\mathsf{L}_n{}^=$ and ${}^r\mathsf{L}_n{}^=$, respectively, i.e., in this case, δ_{ij} is not in the set of connectives. ∎

In the above definition we assumed that the language does not contain any constant or function symbol. We assumed also that every relation symbol r has arity n. These are not severe restrictions, a well-known fact.

Later we will define (equivalent) modal versions of first-order logics, cf. Chapter 4. Then the quantifier $\exists_i$ will be treated as a $\Diamond$ type modality and δ_{ij} as a modal constant.

In the following definition, we define extensions of logics with counting and graded modalities, cf. [FC85], [Sa88], [vdH92], [dR93], and [HR93]. In many cases, these extensions are equivalent, cf. Theorem 1.1.6 below. We will assume that local truth $\Vdash$ can be defined, which holds foı the above logics. We note that the graded modalities sometimes are defined using an accessibility relation (i.e., $\langle n \rangle\varphi$ holds at w if there are at least n worlds accessible from w that make φ true), and that Theorem 1.1.6 remains true if we use this definition.

Definition 1.1.5. (Graded and counting logics: ${}^\kappa\mathsf{L}^{grad}$, ${}^\kappa\mathsf{L}^{count}$) Let κ be ω or a natural number greater than 0, i.e., $\kappa \in \{\omega\} \cup \omega \setminus 1$, and let L be a logic where the relation $\Vdash$ is defined.

1. The *graded logic* ${}^\kappa\mathsf{L}^{grad}$ is defined by adding a new unary connective $\langle n \rangle$, for every $n \in \kappa$, to the connectives of L. That is, in the definition of the set of formulas, we further require that $\langle n \rangle\varphi$ is a formula whenever so is φ. The class of models is the same as for L. The interpretation of $\langle n \rangle$ is:

$$w \Vdash \langle n \rangle\varphi \overset{\text{def}}{\Longleftrightarrow} (\exists w_0, \ldots, w_{n-1})|\{w_0, \ldots, w_{n-1}\}| = n \;\&\; (\forall i \in n) w_i \Vdash \varphi.$$

   The definitions of the other notions are the usual.

2. The *counting logic* ${}^\kappa\mathsf{L}^{count}$ is defined by adding a new n-ary connective $\Diamond_n$, for every $n \in \kappa$, to the connectives of L. That is, in the definition of the set of formulas, we further require that $\Diamond_n(\varphi_0, \ldots, \varphi_{n-1})$ is a formula whenever so are $\varphi_0, \ldots, \varphi_{n-1}$. The class of models is the same as that of L. The interpretation of $\Diamond_n$ is:

$$\begin{aligned} w \Vdash \Diamond_n(\varphi_0, \ldots, \varphi_{n-1}) \quad \overset{\text{def}}{\Longleftrightarrow} \quad & (\exists w_0, \ldots, w_{n-1}) \\ & |\{w_0, \ldots, w_{n-1}\}| = n \;\&\; (\forall i \in n) w_i \Vdash \varphi_i. \end{aligned}$$

   The definitions of the other notions are the usual.

∎

Note that the truth of $\langle n \rangle\varphi$ does not depend on the actual choice of w. That is, given a model and two "worlds" w and w', $w \Vdash \langle n \rangle\varphi \Longleftrightarrow w' \Vdash \langle n \rangle\varphi$. The same holds for $\Diamond_n$.

Note that, for $\kappa \geq 3$, ${}^{\kappa}\mathsf{L}^{grad}$ is not a modal logic in the following sense. $\langle n \rangle$ does not distribute over disjunction for $n > 1$, i.e., the following is not a valid formula of ${}^{\kappa}\mathsf{L}^{grad}$:

$$\langle n \rangle(\varphi \vee \psi) \leftrightarrow (\langle n \rangle \varphi \vee \langle n \rangle \psi).$$

On the other hand, ${}^{\kappa}\mathsf{L}^{count}$ is a modal logic, since the following is valid:

$$\Diamond_n(\varphi_0, \ldots, \varphi_i \vee \varphi_i', \ldots, \varphi_{n-1}) \leftrightarrow (\Diamond_n(\varphi_0, \ldots, \varphi_i, \ldots, \varphi_{n-1}) \vee \Diamond_n(\varphi_0, \ldots, \varphi_i', \ldots, \varphi_{n-1}))$$

for every $i \in n$ and $n \in \kappa$. However, as it is shown in [ANS95], the two logics ${}^{\kappa}\mathsf{L}^{grad}$ and ${}^{\kappa}\mathsf{L}^{count}$ are equivalent in the following sense.

Theorem 1.1.6. *Let L be a logic extending propositional logic. Then to each formula φ of ${}^{\kappa}\mathsf{L}^{grad}$, there is a formula φ' of ${}^{\kappa}\mathsf{L}^{count}$ such that truth is preserved: $w \Vdash \varphi$ iff $w \Vdash \varphi'$ for every w. The same holds with ${}^{\kappa}\mathsf{L}^{grad}$ and ${}^{\kappa}\mathsf{L}^{count}$ interchanged. Moreover, the function $\varphi \mapsto \varphi'$ is computable.*

Proof: It is easy to see that $\langle n \rangle \varphi$ can be defined in ${}^{\kappa}\mathsf{L}^{count}$ by the formula $\Diamond_n(\varphi, \ldots, \varphi)$. For the other direction we show that the following formula φ defines $\Diamond_n(\varphi_0, \ldots, \varphi_{n-1})$:

$$\bigwedge\{\langle 1 \rangle \varphi_i : i \in n\} \wedge \bigwedge\{\langle 2 \rangle(\varphi_i \vee \varphi_j) : i, j \in n \;\&\; i \neq j\} \wedge \ldots \wedge \langle n \rangle(\varphi_0 \vee \ldots \vee \varphi_{n-1}).$$

To see this we use a basic combinatorial result, the so-called *Marriage Theorem*[2], cf. [Br77] Theorem 8.1.1. This theorem says that the family $A_0, \ldots, A_{n-1}$ of sets has a system of distinct representatives, i.e., there is a set $\{a_0, \ldots, a_{n-1}\}$ such that

$$|\{a_0, \ldots, a_{n-1}\}| = n \;\&\; (\forall i \in n) a_i \in A_i,$$

iff

$$(\forall k \in n)(\forall 0 \leq i_0 < \ldots < i_k < n)|A_{i_0} \cup \ldots \cup A_{i_k}| > k. \tag{1.1}$$

Now, let $\mathfrak{M}$ be a fixed model, and let $A_i \stackrel{\text{def}}{=} \{w : w \Vdash \varphi_i\}$. Then, for every w, $w \Vdash \varphi$ iff $A_0, \ldots, A_n$ satisfy the formula 1.1 above iff we can find distinct w_i's from the A_i's for each $i \in n$, i.e., iff $w \Vdash \Diamond_n(\varphi_0, \ldots, \varphi_{n-1})$. ∎

In the following definition, we extend logics by adding the difference operator D, cf., e.g., [Se76], [Sa88], [GPT87], and [Ko92]. Then we prove that adding D is equivalent to adding the graded modalities $\langle 1 \rangle$ and $\langle 2 \rangle$.

Definition 1.1.7. (Logic of difference, L^{D}) Let L be a logic where $\Vdash$ is definable. The logic L^{D} is defined by adding a new unary connective D, and interpreting D as:

$$w \Vdash \mathsf{D}\varphi \stackrel{\text{def}}{\Longleftrightarrow} \exists w'(w \neq w' \;\&\; w' \Vdash \varphi).$$

∎

Proposition 1.1.8. *Let L be a logic extending propositional logic. Then the logics L^{D} and ${}^{3}\mathsf{L}^{grad}$ are equivalent in the sense of Theorem 1.1.6.*

Proof: It is easy to see that we can define $\langle 1 \rangle \varphi$ by $\mathsf{D}\varphi \vee \varphi$, and $\langle 2 \rangle \varphi$ by $\mathsf{D}(\mathsf{D}\varphi \wedge \varphi)$. For $\mathsf{D}\varphi$ the following definition works: $((\langle 1 \rangle \varphi \wedge \neg \varphi) \vee \langle 2 \rangle \varphi$. ∎

[2]My girl-friend drew my attention to this theorem with this remarkable name.

1.1.3 Metalogical notions

Now we give definitions of some metalogical notions, and enumerate some metalog properties that we will investigate later.

Definition 1.1.9. (Formula schema) Let L be a logic with the set $Cn(\mathsf{L})$ of log connectives. Fix a countable set $A = \{A_i : i < \omega\}$, called the set of *formula varial* The set Fms_{L} of *formula schemata* of L is the smallest set satisfying the condit below.

(i) $A \subseteq Fms_{\mathsf{L}}$,
(ii) if $c \in Cn_k(\mathsf{L})$ and $\Phi_1, \ldots, \Phi_k \in Fms_{\mathsf{L}}$, then $c(\Phi_1, \ldots, \Phi_k) \in Fms_{\mathsf{L}}$.

An *instance of a formula schema* is given by substituting formulas for the forr variables in it. ∎

Definition 1.1.10. (Inference system, $\vdash$) Let L be a logic. An *inference rul* L is a pair $\langle\langle\langle B_1, \ldots, B_n\rangle, B_0\rangle, C\rangle$, where every B_i $(i \leq n)$ is a formula schema an is some condition. This inference rule will be denoted by

$$\frac{B_1, \ldots, B_n}{B_0} \quad \text{provided } C.$$

An *instance of an inference rule* is given by substituting instances of the forı schemata occurring in the rule such that the formulas satisfy the condition C.

An *inference system*, or *calculus*, for L, denoted by $\vdash$, is a finite set of forı schemata, called *axiom schemata* or *axioms*, together with a finite set of infer rules.

A *Hilbert-style inference system* is a calculus such that the condition C is emp each rule. ∎

Note that every axiom schema is in fact an inference rule such that the hypot $\langle B_1, \ldots, B_n\rangle$ of the rule is empty.

The well-known motivation for calculi is that we would like to mimic the sem cal consequence relation by purely syntactical means, cf. Definition 1.1.11 and 1 below. Besides historical reasons, we distinguished Hilbert-style calculi because o following. Because of their (syntactical) simplicity it is easier to investigate Hil style completeness, cf. Theorem 1.3.7, and to apply them for proving theorems o logic. In addition, sound Hilbert-style calculi are in fact *strongly* sound in many c cf., e.g., classical propositional logic and the modal logic $S5$. Non-Hilbert-style cı sometimes are called *Gabbay-style*[3], cf. [Mi93], or *unorthodox*, cf. [Ve91].

Definition 1.1.11. (Derivability) Let L be a logic, and let $\vdash$ be an inference sy for L. Assume $\Sigma \cup \{\varphi\} \subseteq \mathsf{F}_{\mathsf{L}}$. We say that φ is $\vdash$-*derivable*, or $\vdash$-*provable*, from there is a finite sequence $\langle \varphi_1, \ldots, \varphi_n\rangle$ of formulas, a $\vdash$-*proof* of φ from Σ, such thı is φ, and, for every $1 \leq i \leq n$,

[3]The reason for this name is that [Ga81] initiated completeness investigations using not nece Hilbert-style calculi.

- $\varphi_i \in \Sigma$ or
- φ_i is an instance of an axiom schema (an *axiom*, for short) of $\vdash$ or
- there are $j_1, ..., j_k < i$, and there is an inference rule of $\vdash$ such that $\frac{\varphi_{j_1}, ..., \varphi_{j_k}}{\varphi_i}$ is an instance of this rule.

We write $\Sigma \vdash \varphi$ if φ is $\vdash$-provable from Σ. (We will often identify an inference system $\vdash$ with the corresponding derivability relation.) ∎

Definition 1.1.12. (Completeness and soundness) Let L be a logic, and let $\vdash$ be an inference system for L. Then

- $\vdash$ is *weakly complete* for L iff $\forall\varphi \in \mathsf{F_L}$
$$\models \varphi \Rightarrow \ \vdash \varphi;$$
- $\vdash$ is *finitely complete* for L iff $(\forall\Sigma \subseteq \mathsf{F_L})(\forall\varphi \in \mathsf{F_L})$
$$|\Sigma| < \omega \ \& \ \Sigma \models \varphi \Rightarrow \Sigma \vdash \varphi;$$
- $\vdash$ is *strongly complete* for L iff $(\forall\Sigma \subseteq \mathsf{F_L})(\forall\varphi \in \mathsf{F_L})$
$$\Sigma \models \varphi \Rightarrow \Sigma \vdash \varphi;$$
- $\vdash$ is *weakly sound* for L iff $\forall\varphi \in \mathsf{F_L}$
$$\vdash \varphi \Rightarrow \ \models \varphi;$$
- $\vdash$ is *finitely sound* for L iff $(\forall\Sigma \subseteq \mathsf{F_L})(\forall\varphi \in \mathsf{F_L})$
$$|\Sigma| < \omega \ \& \ \Sigma \vdash \varphi \Rightarrow \Sigma \models \varphi;$$
- $\vdash$ is *strongly sound* for L iff $(\forall\Sigma \subseteq \mathsf{F_L})(\forall\varphi \in \mathsf{F_L})$
$$\Sigma \vdash \varphi \Rightarrow \Sigma \models \varphi.$$

∎

Definition 1.1.13. (Decidability) Let L be a logic. We say that L is *decidable* if the set $\{\varphi : \ \models \varphi\}$ of valid formulas is a decidable set. ∎

1.2 ALGEBRAS

We assume familiarity with the basic definitions and results of universal algebra, cf. [HMT85], [BS81], and [ANS94]. However, we will give some of the most frequently used definitions and results in this section. We will also prove a theorem (Theorem 1.2.6) that we will use several times in the sequel.

Given a class K of algebras of the same similarity type, we will denote the class of isomorphic copies, subalgebras, isomorphic copies of products, homomorphic copies, and ultraproducts of elements of K by $\mathbf{I}\mathsf{K}$, $\mathbf{S}\mathsf{K}$, $\mathbf{P}\mathsf{K}$, $\mathbf{H}\mathsf{K}$, and $\mathbf{Up}\mathsf{K}$, respectively.

Definition 1.2.1. **($\mathbf{P_S}$, Sir, and Sim)** Let $\mathfrak{A}$ be an algebra.

1. By a *subdirect decomposition* of $\mathfrak{A}$ we understand a system $\langle h_i : i \in I\rangle$ of surjective homomorphisms $h_i : \mathfrak{A} \longrightarrow \mathfrak{B}_i$ such that

$$(\forall a, b \in A)(a \neq b \Rightarrow (\exists i \in I) h_i(a) \neq h_i(b)).$$

   Then we say that $\mathfrak{A}$ is a *subdirect product* of the $\mathfrak{B}_i$'s ($i \in I$).
2. A subdirect decomposition is called *non-trivial* if $(\forall i \in I)$ h_i is not an isomorphism.
3. $\mathfrak{A}$ is called *subdirectly irreducible* iff $\mathfrak{A}$ has no non-trivial subdirect decomposition.
4. $\mathfrak{A}$ is called *simple* iff it has only two congruences.

Given a class K of algebras, we denote the class of subdirect products of the elements of K by $\mathbf{P_S}\mathsf{K}$. $\mathbf{Sim}\mathsf{K}$ and $\mathbf{Sir}\mathsf{K}$ denotes the class of simple and subdirectly irreducible members of K, respectively. ∎

Let K and K' be two classes of algebras. We say that an element of K is *representable* (as a member of K') if there is an element of K' such that the two algebras are isomorphic. Usually, K is an axiomatically given class, i.e., $\mathsf{K} = \mathrm{Mod}(\Sigma)$ for some set Σ of formulas in the language of K, while K' is a class of set algebras, i.e., K' consists of algebras whose elements are sets of sets, and the operations are "natural" operations on sets. We will not need the precise definition of set algebras, but we note that the algebraic counterparts $\mathsf{Alg}(\mathsf{L})$ of the logics L of this dissertation are set algebras.

By $\mathrm{Eq}(\mathsf{K})$ we denote the class of equations valid in the elements of K. We recall that a class K of similar algebras is a *variety* if K can be defined by a set of equations, i.e., $\mathsf{K} = \mathrm{Mod}(E)$ for some set E of equations, and that the variety generated by K is the smallest variety containing K. By Birkhoff's theorem, this is is $\mathbf{HSP}\mathsf{K}$. By *quasi-equations* we mean equational implications, i.e., formulas of the form $(\sigma_1 = \tau_1 \;\&\; \ldots \;\&\; \sigma_n = \tau_n) \Rightarrow \sigma_0 = \tau_0$. In other words, a quasi-equation is a Horn clause using only identity atoms. A class K of similar algebras is a *quasi-variety* if K can be defined by a set of quasi-equations, i.e., $\mathsf{K} = \mathrm{Mod}(Q)$ for some set Q of quasi-equations. The quasi-variety generated by K, i.e., the smallest quasi-variety containing K, is $\mathbf{SPUp}\mathsf{K}$. We note that these definability theorems are the algebraic counterparts of preservation theorems of model theory. We recall that every variety V is generated by its subdirectly irreducible members in the following sense (cf. [BS81]):

$$\mathsf{V} = \mathbf{P_S}\mathbf{Sir}\mathsf{V}.$$

We will denote the variety of Boolean algebras by BA. Usually, we will consider BA's as algebras with a unary $(-)$ and a binary $(\cdot)$ operation, corresponding to complement and meet. We can do this, since the other Boolean connectives can be defined by means of $-$ and $\cdot$. Usually, we will denote Boolean join, top, and bottom by $+$, 1, and 0, respectively. However, if we want to emphasize the connection with (propositional) logic, we will denote meet, join, complement, top, and bottom by $\wedge$, $\vee$, $\neg$, $\top$, and $\bot$, respectively.

We say that an algebra $\mathfrak{A}$ has a Boolean reduct if $-$ and $\cdot$ are term definable in $\mathfrak{A}$, and they satisfy the Boolean axioms. The set of atoms of an algebra $\mathfrak{A}$ with a Boolean reduct is denoted by $At(\mathfrak{A})$.

We will use the following technique, called *relativization*, cf. [HMT85] 2.2, rather frequently. Let $\mathfrak{A} = \langle A, \cdot, c_i \rangle_{i \in I}$ be an algebra, and let $a \in A$. Then the function $\mathfrak{Rl}_a$ is defined as follows:

$$\mathfrak{Rl}_a(\mathfrak{A}) \stackrel{\text{def}}{=} \langle \{x \cdot a : x \in A\}, \cdot, c_i^a \rangle_{i \in I}$$

where $c_i^a(a \cdot x_1, \ldots, a \cdot x_n) = a \cdot c_i(x_1, \ldots, x_n)$ for every n-ary connective c_i and $x_1, \ldots, x_n \in A$. Then we say that $\mathfrak{Rl}_a(\mathfrak{A})$ is obtained by *relativizing* $\mathfrak{A}$ with a.

A class K of similar algebras is a *discriminator class* if there is a term τ in the language of K such that, in every member of K,

$$\tau(x, y, u, v) = \begin{cases} u & \text{if } x = y \\ v & \text{otherwise.} \end{cases}$$

Such a τ is called a *discriminator term*. A variety V is a *discriminator variety* if there is a discriminator class K such that $\mathsf{V} = \mathbf{HSP}\mathsf{K}$. We note that the existence of a discriminator term corresponds to the deduction theorem on the logic side, cf. [Si92].

Note that, for a discriminator variety V, $\mathbf{Sir}\mathsf{V} = \mathbf{Sim}\mathsf{V}$ ($\supseteq$ always holds, while $\subseteq$ follows from the following: if x and y are different congruent elements, then $\tau(x, y, u, v) = v$ and $\tau(x, x, u, v) = u$ are congruent as well). If K is a class of algebras with a Boolean reduct, then K is a discriminator class iff there is a term $\Diamond x$ in the language of K such that, in every element of K,

$$\Diamond x = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Indeed, $\Diamond x$ can be defined as $\tau(x, 0, 0, 1)$, and $\tau(x, y, u, v)$ can be defined as $(\Diamond(x \oplus y) \cdot v) + (u \cdot -\Diamond(x \oplus y))$, where $\oplus$ denotes symmetric difference.

The following theorem will be useful many times, the proof can be found, e.g., in [Né94].

Theorem 1.2.2. *Let* K *be a discriminator class. Then*

$$\mathbf{SPUp}\mathsf{K} = \mathbf{HSP}\mathsf{K}.$$

By the above theorem, the quasi-varieties generated by discriminator classes are in fact varieties. This also enables us to code quasi-equations as equations.

Next we recall the definition of Boolean algebras with operators (BAO's) from [JT52]. The notion of BAO is important in algebraic logic because the algebraic counterparts of many (modal) logics are classes of BAO's. We also define a special subclass SBAO of BAO's.

Definition 1.2.3. (BAO, SBAO) The class BAO of *Boolean algebras with operators* is defined as follows. An algebra $\mathfrak{A} = \langle A, \cdot, -, f_i \rangle_{i \in I}$ is a BAO if

(i) $\langle A, \cdot, - \rangle$ is a Boolean algebra,

(ii) the operations f_i $(i \in I)$ are *additive* in every coordinate, i.e., if f_i has arity n, then $(\forall j \in n)$

$$f_i(x_0, \ldots, x_j + y_j, \ldots, x_{n-1}) = f_i(x_0, \ldots, x_j, \ldots, x_{n-1}) + f_i(x_0, \ldots, y_j, \ldots, x_{n-1}).$$

An $\mathfrak{A} \in \mathsf{BAO}$ is said to be *normal* if every extra-Boolean operation f_i $(i \in I)$ is normal in each argument, i.e., $(\forall j \in n)$

$$x_j = 0 \Rightarrow f_i(x_0, \ldots, x_j, \ldots, x_{n-1}) = 0.$$

Let $\mathfrak{A} \in \mathsf{BAO}$ be a normal algebra. $\mathfrak{A} \in \mathsf{SBAO}$ if the following holds. There is a term-definable operation $\Diamond$ on A such that

(i) $\Diamond$ is a *complemented closure operator*, i.e.,

$$x \leq \Diamond x \leq \Diamond(x + y) \quad \& \quad \Diamond\Diamond x \leq \Diamond x \quad \& \quad \Diamond - \Diamond x \leq -\Diamond x,$$

(ii) for the extra-Boolean operations f_i $(i \in I)$,

$$f_i(x_0, \ldots, x_{n-1}) \leq \Diamond x_0 \cdot \ldots \cdot \Diamond x_{n-1}.$$

∎

We will use the following fact rather frequently.

Proposition 1.2.4. *If $\mathfrak{A}$ is an algebra with a Boolean reduct, and there is a complemented closure operator $\Diamond$ on $\mathfrak{A}$, then the following holds.*

1. $\Diamond 0 = 0$.
2. $\Diamond(x \cdot \Diamond y) = \Diamond x \cdot \Diamond y$.
3. $\Diamond(x + y) = \Diamond x + \Diamond y$.

Proof: $\Diamond 0 = \Diamond - 1 = \Diamond - \Diamond 1 = -\Diamond 1 = -1 = 0$. Item 2 is proved in [Ve91] Proposition 3.5.6, and 3 easily follows from 2, cf. [HMT85] Theorem 1.2.6. ∎

Definition 1.2.5. (Density) Let K be a class of algebras of the same similarity type with a Boolean reduct, and let $\mathfrak{A} \in \mathsf{K}$. Let τ be any property. We say that $\mathfrak{A}$ is *τ-dense* if

$$(\forall 0 < a \in A)(\exists 0 < b \in A)(b \leq a \;\&\; \tau(b)).$$

We denote the class of τ-dense elements of K by DK. ∎

We recall the following theorem about dense BAO's from [AGMNS]. As we will see, (representability of) dense algebras play an important role in weak completeness of logics, cf. Theorem 1.3.11.

Theorem 1.2.6. *Let K be variety of SBAO's, and let DK be the class of τ-dense elements of K for some property τ preserved under homomorphisms. Assume that $\mathfrak{A} \in \mathsf{DK}$ and $\mathfrak{A}$ is countable: $|A| \leq \omega$. Then there are simple algebras $\mathfrak{A}_a \in \mathbf{Sim}\mathsf{DK}$ $(0 < a \in A)$ such that $\mathfrak{A} \in \mathbf{SP}\{\mathfrak{A}_a : 0 < a \in A\}$.*

Proof: Let $|A| \leq \omega$, and $a \in A$ be an arbitrary non-zero element. We will define a τ-dense simple algebra $\mathfrak{A}_a$ and a homomorphism $f_a : \mathfrak{A} \longrightarrow \mathfrak{A}_a$ such that $f_a(a) \neq 0$.

Let the elements of A be enumerated: $\{a_i : i \in \omega\}$. Since $a \neq 0$, there is a $0 \neq b \leq a$ such that $\tau(b)$ holds. Let $b_0 \stackrel{\text{def}}{=} \Diamond b$.

Now assume that $b_0 \geq \ldots \geq b_n > 0$ are already defined. Take the next element a_n of A. If $a_n \cdot b_n = 0$, then let $b_{n+1} \stackrel{\text{def}}{=} b_n$. If $a_n \cdot b_n \neq 0$, then there is a $0 \neq b'_{n+1} \leq a_n \cdot b_n$ such that $\tau(b'_{n+1})$. Let $b_{n+1} \stackrel{\text{def}}{=} \Diamond b'_{n+1}$. This way we can define a descending chain of $\Diamond$-closed non-zero elements: $b_0 \geq \ldots \geq b_n \geq \ldots$.

We claim that, for every $n \in \omega$, $a \cdot b_n \neq 0$. Indeed, $0 \neq b_n = \Diamond b_n \leq \Diamond b \leq \Diamond a$, whence $0 \neq b_n = b_n \cdot \Diamond a = \Diamond b_n \cdot \Diamond a = \Diamond(\Diamond b_n \cdot a) = \Diamond(b_n \cdot a)$. Thus $b_n \cdot a \neq 0$.

Let I_a be the Boolean ideal generated by $\{-b_n : n \in \omega\}$. Then

$$I_a = \{y : (\exists k, l \in \omega) y \leq -b_{k_0} + \ldots + -b_{k_l}\} = \{y : (\exists n \in \omega) y \leq -b_n\}.$$

Since b_n is closed under $\Diamond$, so is $-b_n$. Then $y \in I_a$ implies $\Diamond y \in I_a$. This implies that $f_i(x_0, \ldots, y, \ldots, x_k) \in I_a$ whenever $y \in I_a$. Then by [Sa82], I_a is an ideal in $\mathfrak{A}$. Since $a \cdot b_n \neq 0$ for every $n \in \omega$, $a \notin I_a$, i.e., I_a is proper.

Let $\mathfrak{A}_a$ be the factor algebra modulo I_a of $\mathfrak{A}$, and let f_a be the canonical homomorphism:

$$\mathfrak{A}_a \stackrel{\text{def}}{=} \mathfrak{A}/I_a \quad \& \quad f_a(x) \stackrel{\text{def}}{=} x/I_a$$

for every $x \in A$. We have to show that $\mathfrak{A}_a$ is a simple element of DK.

Assume that $e \notin I_a$. We show that $-\Diamond e \in I_a$. Since $e \notin I_a$, for every $n \in \omega$, $e \cdot b_n \neq 0$. Let e be the kth element of A. Then $e \cdot b_k \neq 0$, so there is a $0 < b'_{k+1} \leq e \cdot b_k$ such that $\tau(b'_{k+1})$. Then we put $-b_{k+1} = -\Diamond b'_{k+1}$ into the set generating I_a. Since $b_{k+1} \leq \Diamond e$, we have $-\Diamond e \in I_a$. This implies that $\mathfrak{A}_a$ has the following property: $\Diamond x = 1$ whenever $0 < x$. This implies that $\mathfrak{A}_a$ has only two congruences, i.e., $\mathfrak{A}_a$ is simple.

Since f_a is a homomorphism, $\mathfrak{A}_a \in \mathsf{K}$. It remains to show that $\mathfrak{A}_a$ is τ-dense. Let $b' \in A/I_a$ be an arbitrary non-zero element. Then $b' = f_a(b)$ for some $b \notin I_a$. Then, for every $n \in \omega$, $b \not\leq -b_n$, i.e., $b \cdot b_n \neq 0$. Assume that b is the kth element of A. Then $b \cdot b_k \neq 0$ implies that we chose a $0 < b'_{k+1}$ below $b \cdot b_k$ such that $\tau(b'_{k+1})$. Then $b_{k+1} = \Diamond b'_{k+1}$, so $-\Diamond b'_{k+1} \in I_a$. Since τ is preserved under homomorphism, $f_a(b'_{k+1}) = b'_{k+1}/I_a$ has property τ. Clearly, $b'_{k+1}/I_a \leq b/I_a = b'$. Further, since $\Diamond b'_{k+1} \notin I_a$, $0 \neq f_a(\Diamond b'_{k+1}) = \Diamond f_a(b'_{k+1}) = \Diamond(b'_{k+1}/I_a)$. Hence $b'_{k+1}/I_a \neq 0$. Thus, for arbitrary non-zero b', we found a non-zero τ-element below it.

We make the same construction for every $0 < a \in A$. That is, we define simple τ-dense algebras $\mathfrak{A}_a$ $(a \in A)$ such that, for each $0 < a \in A$, $f_a : \mathfrak{A} \longrightarrow \mathfrak{A}_a$ is a homomorphism with $f_a(a) \neq 0$.

Now we embed $\mathfrak{A}$ into $\mathbf{P}\{\mathfrak{A}_a : 0 < a \in A\}$. Let, for every $x \in A$,

$$f(x) \stackrel{\text{def}}{=} \langle f_a(x) : 0 \neq a \in A \rangle.$$

Clearly, f is a homomorphism. Moreover, since $f_x(x) \neq 0$ whenever $0 < x$, f is one-one. Thus $\mathfrak{A}$ is the subdirect product of the $\mathfrak{A}_a$'s. Thus we have proved Theorem 1.2.6. ∎

1.3 Bridge between logics and algebras

In this section, we define the algebraic counterparts of logics, and give algebraic characterizations of metalogical properties. Most of the results of this section are based on [AKNS]. Since less generality is needed for our investigations, we could simplify some of the proofs, and state new theorems as well.

The connection between logics and algebras enables us to prove metalogical theorems using the machinery of algebras. Actually, most of the results about logics in this dissertation are proved in algebraic setting, and then using this bridge we get the answer for our logical problem. We will use these bridge theorems to prove (in)completeness and (un)decidability of logics: these properties follow immediately once we proved that the corresponding classes of algebras have the corresponding (non-)finite axiomatizability and (un)decidability properties.

We note that the bridge between logics and algebras are not restricted to completeness and decidability properties. For instance, several kinds of compactness, Beth definability and Craig interpolation properties can be characterized by algebraic means, cf. [AKNS] for developing these connections, and [Ma95] for applying them.

Definition 1.3.1. (Alg) Let $\mathsf{L} = \langle \mathsf{F_L}, \mathsf{M_L}, \models_\mathsf{L}, \mathsf{mean_L}\rangle$ be a logic in the sense of Definition 1.1.1. Let us recall that the formula algebra $\mathfrak{F}_\mathsf{L}$ of L is defined as:

$$\mathfrak{F}_\mathsf{L} \overset{\text{def}}{=} \langle \mathsf{F_L}, c : c \in Cn(\mathsf{L})\rangle.$$

The algebraic counterpart $\mathsf{Alg}(\mathsf{L})$ of L is defined as:

$$\mathsf{Alg}(\mathsf{L}) \overset{\text{def}}{=} \{\mathsf{mean}_\mathsf{L}^{\mathfrak{M}}{}''\mathfrak{F}_\mathsf{L} : \mathfrak{M} \in \mathsf{M_L}\},$$

i.e., we take the (homomorphic) image of the formula algebra $\mathfrak{F}_\mathsf{L}$ along the meaning function $\mathsf{mean}_\mathsf{L}^{\mathfrak{M}}$, for every $\mathfrak{M} \in \mathsf{M_L}$. ∎

Remark 1.3.2. In [AKNS] another algebraization of logics is defined. Let $\mathsf{Alg}_1(\mathsf{L})$ be the class of the semantical Lindenbaum–Tarski algebras (i.e., we factor out the formula algebra by the semantical equivalence relation). The two kinds of algebraizations are related: $\mathbf{SP}\mathsf{Alg}(\mathsf{L}) = \mathbf{I}\mathsf{Alg}_1(\mathsf{L})$. ∎

Note that $\mathsf{Alg}(\mathsf{L})$ has the same similarity type as $\mathfrak{F}_\mathsf{L}$. Thus the formulas of L are terms of $\mathsf{Alg}(\mathsf{L})$. Our first theorem ensures that semantical consequence in L and validity of quasi-equations in $\mathsf{Alg}(\mathsf{L})$ correspond to each other.

Theorem 1.3.3. *Let $\mathsf{L} = \langle \mathsf{F}, \mathsf{M}, \models, \mathsf{mean}\rangle$ be a logic.*

(i) *For any formulas $\varphi_0, \varphi_1, \ldots, \varphi_k$,*

$$\{\varphi_1, \ldots, \varphi_k\} \models \varphi_0 \quad \Rightarrow \quad \mathsf{Alg}(\mathsf{L}) \models (\varphi_1 = \top \;\&\; \ldots \;\&\; \varphi_k = \top) \Rightarrow \varphi_0 = \top.$$

(ii) *For any quasi-equation q of form $(\tau_1 = \sigma_1 \;\&\; \ldots \;\&\; \tau_k = \sigma_k) \Rightarrow \tau_0 = \sigma_0$,*

$$\mathsf{Alg}(\mathsf{L}) \models q \quad \Rightarrow \quad \{\tau_s \leftrightarrow \sigma_s : 1 \le s \le k\} \models \tau_0 \leftrightarrow \sigma_0.$$

Proof: (i): Assume $p_0, \ldots, p_l$ are the only atomic formulas occurring in $\varphi_0, \ldots, \varphi_k$, and assume that

$$\{\varphi_1(p_0, \ldots, p_l), \ldots, \varphi_k(p_0, \ldots, p_l)\} \models \varphi_0(p_0, \ldots, p_l).$$

Let $\mathfrak{A} \in \mathsf{Alg(L)}$. Then $\mathfrak{A} = \mathsf{mean}^{\mathfrak{M}}{}''\mathfrak{F}$ for some $\mathfrak{M} \in \mathsf{M}$. Let $a \in {}^{P}A$ be arbitrary. For every $i \leq l$, we set $a_i \stackrel{\text{def}}{=} a(p_i)$. Clearly, for every $i \leq l$, $a_i = \mathsf{mean}^{\mathfrak{M}}(\gamma_i)$ for some $\gamma_i \in \mathsf{F}$. For every $s \leq k$,

$$\varphi_s[a_0, \ldots, a_l]^{\mathfrak{A}} = \varphi_s[\mathsf{mean}^{\mathfrak{M}}(\gamma_0), \ldots, \mathsf{mean}^{\mathfrak{M}}(\gamma_l)]^{\mathfrak{A}} = \mathsf{mean}^{\mathfrak{M}}(\varphi_s(\gamma_0, \ldots, \gamma_l)),$$

since $\mathsf{mean}^{\mathfrak{M}}$ is a homomorphism.

Assume that, for every $1 \leq s \leq k$, $\mathfrak{A} \models \varphi_s = \top[a]$. Then

$$\mathsf{mean}^{\mathfrak{M}}(\varphi_s(\gamma_0, \ldots, \gamma_l)) = \mathsf{mean}^{\mathfrak{M}}(\top) \quad (1 \leq s \leq k).$$

By Definition 1.1.1, there exists an $\mathfrak{N} \in \mathsf{M}$ such that

$$\mathsf{mean}^{\mathfrak{N}}(\varphi_s(p_0, \ldots, p_l)) = \mathsf{mean}^{\mathfrak{M}}(\varphi_s(\gamma_0, \ldots, \gamma_l)) \quad (0 \leq s \leq k),$$

whence

$$\mathsf{mean}^{\mathfrak{N}}(\varphi_s(p_0, \ldots, p_l)) = \mathsf{mean}^{\mathfrak{N}}(\top) \quad (1 \leq s \leq k).$$

Hence

$$\mathfrak{N} \models \varphi_s(p_0, \ldots, p_l) \quad (1 \leq s \leq k),$$

and then, by assumption,

$$\mathfrak{N} \models \varphi_0(p_0, \ldots, p_l).$$

Thus

$$\mathsf{mean}^{\mathfrak{N}}(\varphi_0(p_0, \ldots, p_l)) = \mathsf{mean}^{\mathfrak{N}}(\top),$$

whence

$$\mathsf{mean}^{\mathfrak{M}}(\varphi_0(\gamma_0, \ldots, \gamma_l)) = \mathsf{mean}^{\mathfrak{M}}(\top),$$

that is,

$$\mathfrak{A} \models \varphi_0 = \top[a],$$

proving Theorem 1.3.3(i), since a was chosen arbitrarily.

(ii): Assume that, for every $\mathfrak{A} \in \mathsf{Alg(L)}$, and, for every valuation $a \in {}^{P}A$,

$$\mathfrak{A} \models q[a].$$

Let $\mathfrak{M} \in \mathsf{M}$ such that $\mathfrak{M} \models \{\tau_s \leftrightarrow \sigma_s : 1 \leq s \leq k\}$. Then $\mathsf{mean}^{\mathfrak{M}}(\tau_s) = \mathsf{mean}^{\mathfrak{M}}(\sigma_s)$ for each $1 \leq s \leq k$. Now let $\mathfrak{A} \stackrel{\text{def}}{=} \mathsf{mean}^{\mathfrak{M}}{}''\mathfrak{F}$, and let $a \in {}^{P}A$ be such that, for each $p \in P$, $a(p) \stackrel{\text{def}}{=} \mathsf{mean}^{\mathfrak{M}}(p)$. Then

$$\mathfrak{A} \models (\tau_1 = \sigma_1 \;\&\; \ldots \;\&\; \tau_k = \sigma_k)[a],$$

which implies, by our assumption, that $\mathfrak{A} \models (\tau_0 = \sigma_0)[a]$. This is the same as $\mathsf{mean}^{\mathfrak{M}}(\tau_0) = \mathsf{mean}^{\mathfrak{M}}(\sigma_0)$, thus, $\mathfrak{M} \models \tau_0 \leftrightarrow \sigma_0$, which proves Theorem 1.3.3(ii). ∎

It is worth stating the following special case of the above theorem.

Corollary 1.3.4. *Let* L *be a logic.*

(i) *For any formula* φ,

$$\models \varphi \quad \Rightarrow \quad \mathsf{Alg}(\mathsf{L}) \models \varphi = \top.$$

(ii) *For any equation* $\tau = \sigma$,

$$\mathsf{Alg}(\mathsf{L}) \models \tau = \sigma \quad \Rightarrow \quad \models \tau \leftrightarrow \sigma.$$

Proof: It is a straightforward consequence of Theorem 1.3.3. ∎

1.3.1 HILBERT-STYLE COMPLETENESS

We turn to investigating algebraic characterizations of several kinds of completeness for logics. Roughly speaking, completeness of a logic is equivalent to the finite axiomatizability of its algebraic counterpart. We can build up a hierarchy according to how much of the semantical consequence relation we would like to mimic by purely syntactical means. That is, we can consider necessary and sufficient conditions for weak, finite, and strong completeness and soundness. On the other hand, we can characterize inference systems by their forms as well. We already distinguished Hilbert-style inference systems. We can make a finer distinction by considering the type of the Hilbert-style rules occurring in a calculus. For instance, in Definition 1.3.6(i) below, if Ax is a set of equations, then the corresponding calculus $\vdash_{Ax}$ contains only some simple rules ensuring that $\leftrightarrow$ is a congruence relation on the formula algebra[4], and that φ is provably equivalent to $\varphi \leftrightarrow \top$. If Ax contains quasi-equations, then we have to add more complex rules to the calculus, cf. below.

First we define a translation between quasi-equations in the language of $\mathsf{Alg}(\mathsf{L})$ and Hilbert-style inference rules for L.

[4]These rules correspond to the rules of equational logic.

Definition 1.3.5. Let $\overline{x} = \langle x_0, \ldots, x_n \rangle$ be a sequence of (algebraic) variables, and let $\overline{\Psi} = \langle \Psi_0, \ldots, \Psi_n \rangle$ be a sequence of formula variables.

(i) Let q be

$$(\tau_1(\overline{x}) = \sigma_1(\overline{x}) \ \& \ \ldots \ \& \ \tau_k(\overline{x}) = \sigma_k(\overline{x})) \Rightarrow \tau_0(\overline{x}) = \sigma_0(\overline{x})$$

a quasi-equation in the language of $\mathsf{Alg}(\mathsf{L})$. The Hilbert-style inference rule r_q corresponding to q is

$$\frac{\tau_1(\overline{\Psi}) \leftrightarrow \sigma_1(\overline{\Psi}), \ldots, \tau_k(\overline{\Psi}) \leftrightarrow \sigma_k(\overline{\Psi})}{\tau_0(\overline{\Psi}) \leftrightarrow \sigma_0(\overline{\Psi})}.$$

(ii) Let r be

$$\frac{\tau_1(\overline{\Psi}), \ldots, \tau_k(\overline{\Psi})}{\tau_0(\overline{\Psi})}$$

a Hilbert-style inference rule for L. The corresponding quasi-equation q_r is defined as

$$(\tau_1(\overline{x}) = \top \ \& \ \ldots \ \& \ \tau_k(\overline{x}) = \top) \Rightarrow \tau_0(\overline{x}) = \top.$$

∎

Now we extend the above translation to sets of quasi-equations and Hilbert-style inference systems.

Definition 1.3.6. (i) Let Ax be a finite set of quasi-equations in the language of $\mathsf{Alg}(\mathsf{L})$. The corresponding Hilbert-style inference system $\vdash_{Ax}$ is defined as follows.

AXIOM SCHEMATA: $\Phi_0 \leftrightarrow \Phi_0$, and, for every equation e of Ax, the axiom r_e corresponding to e.

INFERENCE RULES: For every quasi-equation q of Ax, the rule r_q corresponding to q. Other rules are: rules corresponding to equational logic, cf. [BS81],

$$\frac{\Phi_0 \leftrightarrow \Phi_1, \ \ \Phi_1 \leftrightarrow \Phi_2}{\Phi_0 \leftrightarrow \Phi_2},$$

$$\frac{\Phi_0 \leftrightarrow \Phi_1}{\Phi_1 \leftrightarrow \Phi_0},$$

$$(\forall c \in Cn_l(\mathsf{L})) \quad \frac{\Phi_1 \leftrightarrow \Psi_1, \ldots, \Phi_l \leftrightarrow \Psi_l}{c(\Phi_1, \ldots, \Phi_l) \leftrightarrow c(\Psi_1, \ldots, \Psi_l)},$$

and rules ensuring that φ and $\varphi \leftrightarrow \top$ are provably equivalent

$$\frac{\Phi_0 \leftrightarrow \top}{\Phi_0},$$

$$\frac{\Phi_0}{\Phi_0 \leftrightarrow \top}.$$

(ii) Let $\vdash$ be a Hilbert-style inference system for L. The set $Ax_\vdash$ of quasi-equations is defined as follows. For every axiom (and inference rule) r of $\vdash$, let the corresponding (quasi-)equation q_r belong to $Ax_\vdash$. $Ax_\vdash$ also contains the following two quasi-equations: $(x_0 = x_1) \Rightarrow (x_0 \leftrightarrow x_1) = \top$ and $(x_0 \leftrightarrow x_1) = \top \Rightarrow (x_0 = x_1)$.

∎

Now we are in the position to state equivalence theorems for Hilbert-style completeness.

Theorem 1.3.7. *Assume* L *is a logic and* $Cn(\mathsf{L})$ *is finite.*[5]

(I) *There is a finite set of quasi-equations* Ax *such that* $\mathsf{Alg}(\mathsf{L}) \subseteq \mathrm{Mod}(Ax) \subseteq \mathbf{HSP}\mathsf{Alg}(\mathsf{L})$ *iff there is a strongly sound and weakly complete Hilbert-style calculus for* L. *In more detail:*

(i) *if* $\mathsf{Alg}(\mathsf{L}) \subseteq \mathrm{Mod}(Ax) \subseteq \mathbf{HSP}\mathsf{Alg}(\mathsf{L})$ *for a finite set* Ax *of quasi-equations, then the Hilbert-style calculus* $\vdash_{Ax}$ *is strongly sound and weakly complete for* L;

(ii) *given* $\vdash$, $\mathsf{Alg}(\mathsf{L}) \subseteq \mathrm{Mod}(Ax_\vdash) \subseteq \mathbf{HSP}\mathsf{Alg}(\mathsf{L})$.

(II) $\mathbf{SPUp}\mathsf{Alg}(\mathsf{L})$ *is a finitely axiomatizable quasi-variety iff there is a strongly sound and finitely complete Hilbert-style calculus* $\vdash$ *for* L. *In more detail:*

(i) *if* $\mathbf{SPUp}\mathsf{Alg}(\mathsf{L}) = \mathrm{Mod}(Ax)$ *for a finite set* Ax *of quasi-equations, then the Hilbert-style calculus* $\vdash_{Ax}$ *is strongly sound and finitely complete for* L;

(ii) *given* $\vdash$, $\mathbf{SPUp}\mathsf{Alg}(\mathsf{L}) = \mathrm{Mod}(Ax_\vdash)$.

(III) *Assume that* $\mathbf{SP}\mathsf{Alg}(\mathsf{L})$ *is a quasi-variety. Then* $\mathbf{SP}\mathsf{Alg}(\mathsf{L})$ *is finitely axiomatizable iff there exists a Hilbert-style calculus* $\vdash$ *such that* $\vdash$ *is strongly complete and strongly sound for* L. *In more detail:*

(i) *if* $\mathbf{SP}\mathsf{Alg}(\mathsf{L}) = \mathrm{Mod}(Ax)$ *for a finite set* Ax *of quasi-equations, then the Hilbert-style calculus* $\vdash_{Ax}$ *is strongly sound and strongly complete for* L;

(ii) *given* $\vdash$, $\mathbf{SPUp}\mathsf{Alg}(\mathsf{L}) = \mathrm{Mod}(Ax_\vdash)$.

In this dissertation, we only use Theorem 1.3.7(III), that is why we will only sketch the proofs for the other items. These proofs consist of three essential steps: (a) Theorem 1.3.3 above, (b) that the derivability relation (in equational logic) determined by Ax (and by $Ax_\vdash$) and $\vdash_{Ax}$ (and $\vdash$) correspond to each other, and (c) completeness of equational logic.

Proof of Theorem 1.3.7: Let $\Phi_0, \Phi_1, \ldots$ denote formula variables, $\tau_0, \tau_1, \ldots$ denote formula schemata, $\overline{\Phi}$ denote sequence of formula variables, and $\overline{x}$ denote sequence of variables.

The proof of items (i) goes, mutatis mutandis, as follows. Let Ax be the set of quasi-equations satisfying the conditions of the theorem, and let $\vdash_{Ax}$ be the corresponding calculus. We have to prove that $\vdash_{Ax}$ is sound and complete w.r.t. L.

SOUNDNESS: The soundness of $\vdash_{Ax}$ can be proved by induction on the length of the $\vdash_{Ax}$-proof of φ_0 (from Σ). We only show one part of the induction step, namely the case

[5] One can eliminate the assumption of $Cn(\mathsf{L})$ being finite. Then the finitary character of a Hilbert-style inference system has to be ensured in a more subtle way. Also, 'finitely axiomatizable' must be replaced by 'finite schema axiomatizable', cf., e.g., [Mo69], [Né94], and [SG95].

when φ_0 is obtained by one of the inference rules corresponding to a quasi-equation $q \in Ax$. Say q has the form

$$(\tau_1(\overline{x}) = \tau_1'(\overline{x})\ \&\ \ldots\ \&\ \tau_r(\overline{x}) = \tau_r'(\overline{x})) \Rightarrow \tau_0(\overline{x}) = \tau_0'(\overline{x}),$$

where $\overline{x} = \langle x_1, \ldots, x_z\rangle$. Then the corresponding inference rule is

$$\frac{\tau_1(\overline{\Phi}) \leftrightarrow \tau_1'(\overline{\Phi}), \ldots, \tau_r(\overline{\Phi}) \leftrightarrow \tau_r'(\overline{\Phi})}{\tau_0(\overline{\Phi}) \leftrightarrow \tau_0'(\overline{\Phi})}.$$

Assume that φ_0 is obtained with the help of this rule by substituting the members of the sequence $\overline{\gamma} = \langle\gamma_1, \ldots, \gamma_z\rangle$ of formulas for the members of the sequence $\overline{\Phi} = \langle\Phi_1, \ldots, \Phi_z\rangle$ of formula variables, i.e., φ_0 has the form $\tau_0(\overline{\gamma}) \leftrightarrow \tau_0'(\overline{\gamma})$.

Now fix a model $\mathfrak{M}$, and assume that

$$\mathfrak{M} \models \tau_1(\overline{\gamma}) \leftrightarrow \tau_1'(\overline{\gamma}), \ldots, \mathfrak{M} \models \tau_r(\overline{\gamma}) \leftrightarrow \tau_r'(\overline{\gamma}).$$

We have to show that $\mathfrak{M} \models \tau_0(\overline{\gamma}) \leftrightarrow \tau_0'(\overline{\gamma})$.

Let $\mathfrak{A} \stackrel{\text{def}}{=} \mathsf{mean}^{\mathfrak{M}}{}''\mathfrak{F} \in \mathsf{Alg}(\mathsf{L})$, and let a be a valuation into A such that, for every $1 \le v \le z$, $a(x_v) \stackrel{\text{def}}{=} \mathsf{mean}^{\mathfrak{M}}(\gamma_v)$. Since

$$(\forall 1 \le j \le r)\mathsf{mean}^{\mathfrak{M}}(\tau_j(\overline{\gamma})) = \mathsf{mean}^{\mathfrak{M}}(\tau_j'(\overline{\gamma})),$$

we have

$$\mathfrak{A} \models (\tau_1(\overline{x}) = \tau_1'(\overline{x})\ \&\ \ldots\ \&\ \tau_r(\overline{x}) = \tau_r'(\overline{x}))[a].$$

Then, by $\mathsf{Alg}(\mathsf{L}) \models Ax$,

$$\mathfrak{A} \models (\tau_0(\overline{x}) = \tau_0'(\overline{x}))[a],$$

whence

$$\mathfrak{M} \models \tau_0(\overline{\gamma}) \leftrightarrow \tau_0'(\overline{\gamma}).$$

This finishes the proof of the soundness of $\vdash_{Ax}$.

COMPLETENESS: To prove completeness, we need a claim. For any set Σ of formulas, we define

$$\psi \sim_\Sigma \psi' \stackrel{\text{def}}{\Longleftrightarrow} \Sigma \vdash_{Ax} \psi \leftrightarrow \psi'.$$

Note that, by the definition of $\vdash_{Ax}$ and by the definition of derivability, $\sim_\Sigma$ is a congruence relation on the formula algebra $\mathfrak{F}$ for any Σ. Thus, we can define the factor algebra $\mathfrak{F}/\sim_\Sigma$, called the *(syntactical) Lindenbaum–Tarski algebra* of L corresponding to Σ (defined by $\vdash_{Ax}$).

Claim 1.3.8. *For any* $\Sigma \subseteq \mathsf{F}$,

(i) $(\forall\varphi \in \Sigma)(\mathfrak{F}/\sim_\Sigma) \models \varphi = \top$,
(ii) $(\mathfrak{F}/\sim_\Sigma) \models Ax$.

Proof: (i): For every $\varphi \in \Sigma$, the following holds: $\Sigma \vdash_{Ax} \varphi$, whence $\Sigma \vdash_{Ax} \varphi \leftrightarrow \top$, i.e., $\varphi \sim_\Sigma \top$.

(ii): Let $q \in Ax$, and assume that q is of the form

$$(\tau_1(\overline{x}) = \tau_1'(\overline{x}) \ \& \ \dots \ \& \ \tau_k(\overline{x}) = \tau_k'(\overline{x})) \Rightarrow \tau_0(\overline{x}) = \tau_0'(\overline{x}).$$

Let $\mathfrak{A} \stackrel{\text{def}}{=} (\mathfrak{F}/{\sim_\Sigma})$. We want to provr that, for every valuation a of the variables into A, $\mathfrak{A} \models q[a]$.

So let a be an arbitrary valuation into A. Then, for every i, $a(x_i) = \varphi_i/{\sim_\Sigma}$ for some $\varphi_i \in \mathsf{F}$. Assume that

$$\mathfrak{A} \models \tau_1[\overline{\varphi/{\sim_\Sigma}}] = \tau_1'[\overline{\varphi/{\sim_\Sigma}}] \ \& \ \dots \ \& \ \tau_k[\overline{\varphi/{\sim_\Sigma}}] = \tau_k'[\overline{\varphi/{\sim_\Sigma}}].$$

Then

$$(\tau_1(\overline{\varphi}))/{\sim_\Sigma} = (\tau_1'(\overline{\varphi}))/{\sim_\Sigma}, \dots, (\tau_k(\overline{\varphi}))/{\sim_\Sigma} = (\tau_k'(\overline{\varphi}))/{\sim_\Sigma},$$

since $\sim_\Sigma$ is a congruence on $\mathfrak{F}$. Then

$$\tau_1(\overline{\varphi}) \sim_\Sigma \tau_1'(\overline{\varphi}), \dots, \tau_k(\overline{\varphi}) \sim_\Sigma \tau_k'(\overline{\varphi}),$$

that is,

$$\Sigma \vdash_{Ax} \{\tau_j(\overline{\varphi}) \leftrightarrow \tau_j'(\overline{\varphi}) : 1 \leq j \leq k\},$$

by the definition of $\sim_\Sigma$. In $\vdash_{Ax}$, we have the following rule (corresponding to the quasi-equation q):

$$\frac{\tau_1(\overline{\Phi}) \leftrightarrow \tau_1'(\overline{\Phi}), \dots, \tau_k(\overline{\Phi}) \leftrightarrow \tau_k'(\overline{\Phi})}{\tau_0(\overline{\Phi}) \leftrightarrow \tau_0'(\overline{\Phi})}.$$

By this rule, we get that $\Sigma \vdash_{Ax} \tau_0(\overline{\varphi}) \leftrightarrow \tau_0'(\overline{\varphi})$. Then $\tau_0(\overline{\varphi}) \sim_\Sigma \tau_0'(\overline{\varphi})$, whence $(\tau_0(\overline{\varphi}))/{\sim_\Sigma} = (\tau_0'(\overline{\varphi}))/{\sim_\Sigma}$, that is, $\mathfrak{A} \models \tau_0[\overline{\varphi/{\sim_\Sigma}}] = \tau_0'[\overline{\varphi/{\sim_\Sigma}}]$ which implies $\mathfrak{A} \models (\tau_0(\overline{x}) = \tau_0'(\overline{x}))[a]$. By this we proved Claim 1.3.8. ∎

Now completeness in cases (I), (II) can be proved as follows. In case (III), since we have to prove *strong* completeness, i.e., we consider $\Sigma \models \varphi$ for *infinite* Σ's, we need a modified version of the argument below. By Theorem 1.3.3,

$$\begin{array}{rcl} \Sigma \models \varphi & \Rightarrow & \mathsf{Alg(L)} \models \&\{\sigma = \top : \sigma \in \Sigma\} \Rightarrow \varphi = \top \iff \\ & \iff & Ax \models \&\{\sigma = \top : \sigma \in \Sigma\} \Rightarrow \varphi = \top \Rightarrow \\ \text{[by Claim 1.3.8]} & \Rightarrow & \mathfrak{F}/{\sim_\Sigma} \models \varphi = \top \iff \\ & \iff & \Sigma \vdash_{Ax} \varphi. \end{array}$$

In case (III), the following proof works. Assume $\Sigma \models \varphi$. We want $\Sigma \vdash_{Ax} \varphi$.

First we prove that if $\Sigma \models \varphi$, then the corresponding "infinitary" quasi-equation is valid in $\mathbf{SP}\mathsf{Alg(L)}$. Let $\mathfrak{A} \in \mathbf{SP}\mathsf{Alg(L)}$, and let k be an arbitrary valuation into A. Assume that, for every $\psi \in \Sigma$, $\mathfrak{A} \models \psi = \top[k]$. Let I and $\mathfrak{A}_i$ $(i \in I)$ be such that $\mathfrak{A} \subseteq \mathbf{P}\{\mathfrak{A}_i : i \in I \ \& \ \mathfrak{A}_i \in \mathsf{Alg(L)}\}$. Then, for every $i \in I$, $\mathfrak{A}_i \models \psi = \top[k_i]$, where $k_i(x) = k(x)(i)$ for every variable x. We know that, for every $i \in I$, $\mathfrak{A}_i$ is the $\mathsf{mean}^{\mathfrak{M}_i}$-image of $\mathfrak{F}$ (for some $\mathfrak{M}_i \in \mathsf{M}$), and that there are $\psi_{i_0}, \dots, \psi_{i_l}$ such that

$\mathsf{mean}^{\mathfrak{M}_i}(\psi_{i_j}) = k_i(x_j)$ $(j \leq l)$. Moreover, $\mathfrak{M}_i \models \psi(\overline{p}/\overline{\psi_i})$, by $\mathfrak{A}_i \models \psi = \top[k_i]$. Let, for every $i \in I$, $\mathfrak{N}_i \in \mathsf{M}$ such that

$$\mathsf{mean}^{\mathfrak{N}_i}(\psi(\overline{p})) = \mathsf{mean}^{\mathfrak{M}_i}(\psi(\overline{p}/\overline{\psi_i})).$$

Then, for every $i \in I$, and for every $\psi \in \Sigma$, $\mathfrak{N}_i \models \psi(\overline{p})$. Hence, for every $i \in I$, $\mathfrak{N}_i \models \varphi(\overline{p})$. That is, $\mathfrak{M}_i \models \varphi(\overline{p}/\overline{\psi_i})$. Hence, $\mathfrak{A}_i \models \varphi = \top[k_i]$, thus $\mathfrak{A} \models \varphi = \top[k]$.

By Claim 1.3.8(ii), $\mathfrak{F}/\sim_\Sigma \models Ax$, whence $\mathfrak{F}/\sim_\Sigma \in \mathbf{SPUp}\mathsf{Alg}(\mathsf{L}) = \mathbf{SP}\mathsf{Alg}(\mathsf{L})$. By Claim 1.3.8(i), $\mathfrak{F}/\sim_\Sigma \models \psi = \top$ for every $\psi \in \Sigma$. Then, by the previous paragraph, $\mathfrak{F}/\sim_\Sigma \models \varphi = \top$, i.e., $\Sigma \vdash_{Ax} \varphi$.

To prove items (ii) of Theorem 1.3.7, assume that $\vdash$ is a complete and sound Hilbert-style inference system for the logic L, and let $Ax_\vdash$ be the corresponding set of quasi-equations. The following claim ensures that $Ax_\vdash$ is valid in $\mathsf{Alg}(\mathsf{L})$.

Claim 1.3.9. $\mathsf{Alg}(\mathsf{L}) \models Ax_\vdash$.

Proof: It is easy to see that the quasi-equations $(x_0 = x_1) \Rightarrow (x_0 \leftrightarrow x_1) = \top$ and $(x_0 \leftrightarrow x_1) = \top \Rightarrow (x_0 = x_1)$ are valid in $\mathsf{Alg}(\mathsf{L})$.

Let $\&\{\tau_s(\overline{x}) = \top : 1 \leq s \leq k\} \Rightarrow \tau_0(\overline{x}) = \top \in Ax_\vdash$ be a quasi-equation corresponding to a rule of $\vdash$. Let $\mathfrak{A} \in \mathsf{Alg}(\mathsf{L})$, and let a be an arbitrary valuation of the variables into A. Let $\mathfrak{M}$ be such that $\mathfrak{A} = \mathsf{mean}^{\mathfrak{M}''}\mathfrak{F}$. Then, for every i, $a(x_i) = \mathsf{mean}^{\mathfrak{M}}(\varphi_i)$ for some $\varphi_i \in \mathsf{F}$. Assume that

$$\mathfrak{A} \models \&\{\tau_s(\overline{x}) = \top : 1 \leq s \leq k\}[a].$$

Then

$$\mathfrak{M} \models \tau_s(x_1/\varphi_1, \ldots, x_z/\varphi_z) \quad \text{(for each } 1 \leq s \leq k).$$

$\frac{\tau_1(\overline{\Phi}),\ldots,\tau_k(\overline{\Phi})}{\tau_0(\overline{\Phi})}$ is an inference rule of $\vdash$, therefore $\{\tau_1(\overline{\varphi}), \ldots, \tau_k(\overline{\varphi})\} \vdash \tau_0(\overline{\varphi})$. This implies, by the strong soundness of $\vdash$, that $\{\tau_1(\overline{\varphi}), \ldots, \tau_k(\overline{\varphi})\} \models \tau_0(\overline{\varphi})$. Then $\mathfrak{M} \models \tau_0(\overline{\varphi})$, hence $\mathfrak{A} \models \tau_0(\overline{x}) = \top[a]$, as desired.

Validity of equations can be proved similarly. ∎

It remains to prove that every (quasi-)equation valid in $\mathsf{Alg}(\mathsf{L})$ is a consequence of $Ax_\vdash$. In case (I), this easily follows from the fact that we only have to consider equations. In cases (II), (III), the following claim will help.

Claim 1.3.10. *For any formulas* $\varphi_0, \varphi_1, \ldots, \varphi_k$,

$$\{\varphi_1, \ldots, \varphi_k\} \vdash \varphi_0 \quad \Rightarrow \quad Ax_\vdash \models (\varphi_1 = \top \;\&\; \ldots \;\&\; \varphi_k = \top) \Rightarrow (\varphi_0 = \top).$$

Proof: The proof is by induction on the length of the $\vdash$-proof of φ_0 from $\{\varphi_1, \ldots, \varphi_k\}$. We only show one part of the induction step, namely the case when φ_0 is obtained by an inference rule $\frac{\tau_1(\overline{\Phi}),\ldots,\tau_r(\overline{\Phi})}{\tau_0(\overline{\Phi})}$, where $\overline{\Phi} = \langle \Phi_1, \ldots, \Phi_z \rangle$. Then there are formulas $\gamma_1, \ldots, \gamma_z$ such that φ_0 is $\tau_0(\gamma_1, \ldots, \gamma_z)$, and, for every $1 \leq l \leq r$, $\{\varphi_1, \ldots, \varphi_k\} \vdash \tau_l(\overline{\gamma})$. By the induction hypothesis,

$$Ax_\vdash \models \&\{\varphi_s = \top : 1 \leq s \leq k\} \Rightarrow \tau_l(\overline{\gamma}) = \top \quad \text{(for each } 1 \leq l \leq r).$$

By the definition of $Ax_\vdash$, the following quasi-equation belongs to $Ax_\vdash$:

$$\&\{\tau_l(\overline{x}) = \top : 1 \leq l \leq r\} \Rightarrow \tau_0(\overline{x}) = \top.$$

Let $\mathfrak{B}$ be an algebra such that $\mathfrak{B} \models Ax_\vdash$, and let b be any valuation of the variables into B. Now we can define a valuation b' with $b'(x_v) \stackrel{\text{def}}{=} \gamma_v[b]^{\mathfrak{B}}$ $(1 \leq v \leq z)$. Then, for every $0 \leq l \leq r$, $\tau_l(\overline{x})[b']^{\mathfrak{B}} = \tau_l(\overline{\gamma})[b]^{\mathfrak{B}}$. Thus

$$\mathfrak{B} \models (\&\{\varphi_s = \top : 1 \leq s \leq k\} \Rightarrow \tau_0(\overline{\gamma}) = \top)[b],$$

as desired. ▮

Now we can prove that each (quasi-)equation which holds in $\mathsf{Alg}(\mathsf{L})$ is a consequence of $Ax_\vdash$. Assume that

$$\mathsf{Alg}(\mathsf{L}) \models (\tau_1 = \tau_1' \;\&\; \ldots \;\&\; \tau_k = \tau_k') \Rightarrow \tau_0 = \tau_0'.$$

Then, by Theorem 1.3.3(ii),

$$\{\tau_s \leftrightarrow \tau_s' : 1 \leq s \leq k\} \models \tau_0 \leftrightarrow \tau_0',$$

whence, by (strong) completeness,

$$\{\tau_s \leftrightarrow \tau_s' : 1 \leq s \leq k\} \vdash \tau_0 \leftrightarrow \tau_0'.$$

By Claim 1.3.10,

$$Ax_\vdash \models \&\{(\tau_s \leftrightarrow \tau_s') = \top : 1 \leq s \leq k\} \Rightarrow (\tau_0 \leftrightarrow \tau_0') = \top.$$

Since we added the quasi-equations $(x_0 = x_1) \Rightarrow (x_0 \leftrightarrow x_1) = \top$ and $(x_0 \leftrightarrow x_1) = \top \Rightarrow (x_0 = x_1)$ to $Ax_\vdash$, we have

$$Ax_\vdash \models \&\{\tau_s = \tau_s' : 1 \leq s \leq k\} \Rightarrow \tau_0 = \tau_0'$$

completing the proof of Theorem 1.3.7. ▮

1.3.2 Weak completeness

Now we turn to investigating weak completeness of logics by not necessarily Hilbert-style calculi. This kind of calculi are frequently used in (modal) logic, especially when Hilbert-style completeness is impossible. See, e.g., [Ga81] and [Ve91]. Below we will give a sufficient condition in terms of algebras for weak soundness and completeness by calculi that contain only one non-Hilbert-style rule. Moreover, the condition C in this rule is that a certain atomic formula does not occur in the conclusion of the rule, an easily decidable syntactic condition.

The theorem below states that this kind of completeness can be obtained if the **SP**-closure of $\mathsf{Alg}(\mathsf{L})$ coincide with the **SP**-closure of the subclass K of dense elements of a finitely axiomatizable discriminator variety K'.

Let us recall that density is defined as follows, cf. Definition 1.2.5. Let R be a property, and let an algebra $\mathfrak{A}$ be R-dense if

$$\forall a(0 < a \Rightarrow (\exists b \leq a)R(b) \;\&\; b \neq 0).$$

This is equivalent to

$$\forall a(\forall b \leq a)(b = 0 \text{ or not } R(b)) \Rightarrow -a = 1.$$

We can write this in a rule format:

$$\frac{-(b \cdot a) = 1 \text{ or not } R(b \cdot a)}{a = 0}$$

The corresponding logical rule (using p and $\neg\varphi$ instead of b and a, respectively) is

$$\frac{\vdash \neg(p \wedge \neg\varphi) \vee \neg R(p \wedge \neg\varphi)}{\vdash \varphi}.$$

Indeed, we will add this rule with the condition that p does not occur in φ to the Hilbert-style calculus defined by the discriminator variety K' above, and prove completeness.

Let $\mathsf{L} = \langle \mathsf{F}, \mathsf{M}, \models, \mathsf{mean}\rangle$ be a logic. Assume that the Boolean connectives are definable in L, and that the universal modality $\Diamond$ is expressible in L, that is, for every model $\mathfrak{M}$ and formula φ, if $\mathfrak{M} \not\models \neg\varphi$, then $\mathfrak{M} \models \Diamond\varphi$. Then we say that L extends the modal logic $S5$. Below $\oplus$ stands for Boolean symmetric difference, i.e., $x \oplus y$ abbreviates $(x \wedge \neg y) \vee (y \wedge \neg x)$. Let τ be a term, and let an algebra be $\Diamond\tau$-dense if $(\forall a \neq 0)(\exists b \neq 0) b \leq a \;\&\; \Diamond\tau(b) = 1$.

Theorem 1.3.11. *Let L be a logic extending $S5$ such that there are infinitely many atomic formulas. Let K' be a finitely axiomatizable discriminator variety such that $\mathsf{K}' \supseteq$ $\mathbf{SP}\mathsf{Alg}(\mathsf{L})$, and the discriminator term is defined as $(\Diamond(x \oplus y) \wedge v) \vee (u \wedge \neg\Diamond(x \oplus y))$. Let K be the $\Diamond\tau$-dense members of K'. Assume that $\Diamond\tau(\bot) = \top$ and that $\mathbf{SP}\mathsf{K} = \mathbf{SP}\mathsf{Alg}(\mathsf{L})$. Then there is a weakly sound and complete calculus $\vdash$ for L.*

We will use the above theorem for proving completeness results for the finite variable fragments of classical first-order logic and for the square version of arrow logic, cf. Chapter 4.

Usually, the density condition can be expressed by a universal-existential equation: $\forall x \exists y \varphi$ for some equation φ. We conjecture that representability of algebras satisfying such conditions can be used to prove more completeness results. It is an intriguing open question whether these representation theorems are necessary as well, i.e., whether this kind of completeness results imply algebraic representation theorems.

Proof of Theorem 1.3.11: Let the inference system $\vdash$ be defined as follows. Let Ax be the finite set of equations axiomatizing K'. Then the axiom schemata and rules of inference of $\vdash$ are those of $\vdash_{Ax}$ in Definition 1.3.6 plus the following rule

$$\frac{\vdash \neg(p \wedge \neg\varphi) \vee \neg\Diamond\tau(p \wedge \neg\varphi)}{\vdash \varphi} \qquad \text{provided } p \notin \varphi$$

where $p \notin \varphi$ denotes that p is an atomic formula not occurring in φ.

SOUNDNESS: This amounts to prove that $\not\models \varphi$ implies $\not\models \neg(p \wedge \neg\varphi) \vee \neg\Diamond\tau(p \wedge \neg\varphi)$ whenever $p \notin \varphi$.

Assume $\not\models \varphi$. Then, by Corollary 1.3.4, $\mathsf{Alg}(\mathsf{L}) \not\models \varphi = \top$, i.e., $\mathsf{Alg}(\mathsf{L}) \not\models \neg\varphi = \bot$. By $\mathbf{SP}\mathsf{Alg}(\mathsf{L}) = \mathbf{SP}\mathsf{K}$, there is a $\mathfrak{B} \in \mathsf{K}$ such that $\mathfrak{B} \not\models \neg\varphi = \bot$. Then, for some valuation k, $\mathfrak{B} \models \neg\varphi \neq \bot[k]$. Since $\mathfrak{B}$ is $\Diamond\tau$-dense, $\mathfrak{B} \models (\bot < x \leq \neg\varphi \;\&\; \Diamond\tau(x) = \top)[k]$ for some variable x. By $\mathfrak{B} \in \mathsf{K} \subseteq \mathbf{SP}\mathsf{Alg}(\mathsf{L})$, there are algebras $\mathfrak{A}_i \in \mathsf{Alg}(\mathsf{L})$ $(i \in I)$ such that $\mathfrak{B} \subseteq \mathbf{P}_{i\in I}\mathfrak{A}_i$. Then, for every $i \in I$, $\mathfrak{A}_i \models (x \leq \neg\varphi \;\&\; \Diamond\tau(x) = \top)[k_i]$, where $k_i(y) = k(y)(i)$ for every variable y. Further, by the definition of direct product, for some $j \in I$, $\mathfrak{A}_j \models x \neq \bot[k_j]$. Thus, we found an algebra $\mathfrak{A}_j \in \mathsf{Alg}(\mathsf{L})$ and a valuation k_j such that $\mathfrak{A}_j \models (\bot < x \leq \neg\varphi \;\&\; \Diamond\tau(x) = \top)[k_j]$. Then, by the same argument as in the proof of Theorem 1.3.3(i), there is a model $\mathfrak{M}$ such that $\mathfrak{M} \models (\psi \to \neg\varphi) \wedge \Diamond\tau(\psi)$ for some formula ψ. Let p be an atomic formula not occurring in φ, and let p be evaluated to $\psi^{\mathfrak{M}}$: $\mathsf{mean}^{\mathfrak{M}}(p) = \mathsf{mean}^{\mathfrak{M}}(\psi)$. Note that $\mathfrak{M} \not\models \neg p$, by $\mathsf{mean}^{\mathfrak{M}}(p) = \mathsf{mean}^{\mathfrak{M}}(\psi) = x[k_j]^{\mathfrak{A}} \neq \bot[k_j]^{\mathfrak{A}} = \mathsf{mean}^{\mathfrak{M}}(\bot)$.

Now, if we assume that $\mathfrak{M} \models \neg(p \wedge \neg\varphi) \vee \neg\Diamond\tau(p \wedge \neg\varphi)$, then, by $\mathfrak{M} \models (p \to \neg\varphi) \wedge \Diamond\tau(p)$ and propositional logic, we get $\mathfrak{M} \models \neg p$. This is a contradiction.

COMPLETENESS: First we show that the syntactical Lindenbaum–Tarski algebra $\mathfrak{A}$ of L is in K. Clearly $\mathfrak{A} \in \mathsf{K}'$, by Claim 1.3.8(ii). It remains to show that $\mathfrak{A}$ is $\Diamond\tau$-dense. Assume $\varphi \neq \bot$ in $\mathfrak{A}$. This implies $\not\vdash \neg\varphi$. Then $\not\vdash \neg(p\wedge\varphi)\vee\neg\Diamond\tau(p\wedge\varphi)$ whenever $p \notin \varphi$, i.e., $\mathfrak{A} \not\models (p \wedge \varphi) \wedge \Diamond\tau(p \wedge \varphi) = \bot$. Clearly $(p \wedge \varphi) \wedge \Diamond\tau(p \wedge \varphi) \leq \varphi$. We have to show that $(p \wedge \varphi) \wedge \Diamond\tau(p \wedge \varphi)$ has property $\Diamond\tau$. We will show that $\mathsf{K}' \models \Diamond\tau(x \wedge \Diamond\tau(x)) = \top$ for any variable x. By assumption K' is a variety, so it suffices to show that in the subdirectly irreducible algebras $\Diamond\tau(x\wedge\Diamond\tau(x)) = \top$. Since K' is a discriminator variety, in $\mathbf{Sir}\mathsf{K}'$, $\Diamond y \in \{\bot, \top\}$, whence the above equation follows (here we use that $\bot$ has property $\Diamond\tau$, i.e., $\Diamond\tau(\bot) = \top$). Thus $\mathfrak{A}$ is $\Diamond\tau$-dense, whence $\mathfrak{A} \in \mathsf{K}$.

Now assume $\not\vdash \varphi$. Then $\mathfrak{A} \not\models \varphi = \top$. Since by the previous paragraph $\mathfrak{A} \in \mathsf{K} \subseteq \mathbf{SP}\mathsf{K}$, and by assumption $\mathbf{SP}\mathsf{K} = \mathbf{SP}\mathsf{Alg}(\mathsf{L})$, we have $\mathbf{SP}\mathsf{Alg}(\mathsf{L}) \not\models \varphi = \top$. Then $\mathsf{Alg}(\mathsf{L}) \not\models \varphi = \top$. Thus, by Corollary 1.3.4, φ is not valid. ∎

1.3.3 DECIDABILITY

Finally, we state that decidability of a logic is equivalent to the decidability of the equational theory of its algebraic counterpart.

Theorem 1.3.12. *Let L be a logic. The set of valid formulas is a decidable set iff the equational theory* $\mathrm{Eq}(\mathsf{Alg}(\mathsf{L}))$ *is decidable.*

Proof: By Corollary 1.3.4. ∎

2

The Lambek calculus

In this chapter[1], our main concern is (several versions of) the Lambek calculus, LC, cf. [La58], on the logic side, and ordered residuated semigroups on the algebra side. We will prove several finite axiomatizability theorems in Section 2.2 which yield completeness results for LC w.r.t. the so-called relational semantics, cf. Section 2.1. The logics of this chapter are not in the scope of Definition 1.1.1, that is why we build a bridge between the logics and the algebras of this chapter in Section 2.3.

LC gives us a good opportunity to introduce our first strategy for taming logics. This strategy, probably the most obvious one, amounts to finding well-behaved *fragments* of interesting logics with nasty behavior. The situation can be described as follows.

Let us consider the transitive version PAL_H $(t \in H)$ of pair arrow logic, cf. Definition 1.1.3. PAL_H is undecidable by [AKNSS], and incomplete, cf. [Mo69], [An88] and Theorem 2.1.10, Theorem 2.2.5, and Remark 2.2.6. Roughly speaking, the reason for this ugly behavior of PAL_H is that composition is an associative connective distributing over disjunction. Thus, one natural try to find nice versions of PAL_H is to consider such fragments in which (a) composition has its original meaning, e.g., it is associative, and (b) disjunction is not a connective. Completeness of LC w.r.t. transitive (or relativized) and square relational semantics, Theorem 2.1.5 and Theorem 2.1.13, provides us decidable[2] and complete fragments of PAL_H. The connectives of these fragments are composition with its two residuals[3], and implication: $\bullet, \backslash, /, \rightarrow$. It is a natural question whether we can strengthen these logics by adding more connectives without losing the nice properties. We will show (Theorem 2.1.9) that adding conjunction does not ruin the nice behavior, while disjunction cannot be added without losing completeness and decidability. An intriguing open problem is whether identity can be added while preserving completeness and decidability.

2.1 Completeness of the Lambek calculus

The Lambek calculus was introduced in [La58] with both linguistic and logical motivations. It has been intensively investigated since then, e.g., because of its connections to categorial grammar and context-free languages, and because it is an example of substructural logics. The book [vB91] gives a good picture on these investigations. See also [Bu86], [Do92], [Ro91], [Ga92], and [Pe93].

[1]This chapter is based on [AM94], and the results are joint with Hajnal Andréka.

[2]The decidability of these logics follows from the cut-elimination proof in [La58].

[3]The residuals are not among the basic connectives of arrow logic, but they are term definable in frames with symmetric and transitive universes, cf. below.

The so-called relational semantics for the Lambek calculus first appeared in print in [Or88] and [vB89], where plenty of motivation is given for this semantics. Clearly, relational semantics is strongly motivated by dynamic semantics for natural languages. We quote the "slogan" behind dynamic semantics from [vB91]: "Natural language is a programming language for effecting cognitive transitions between information states of its users." It is proved in [vB91] that the Lambek calculus is sound w.r.t. relational semantics, and it was asked whether it was also complete. In this section we prove (Theorem 2.1.5) that the Lambek calculus is indeed complete w.r.t. relational semantics.

In order to prove this completeness, we had to allow relativized relational models, i.e., models with transitive (and not necessarily symmetric or reflexive) universes, because the original Lambek calculus is not complete w.r.t. "unrelativized" relational models, cf. Definition 2.1.3 and Definition 2.1.11. The question naturally arises: what strengthening of the Lambek calculus would be complete w.r.t. the more natural unrelativized, or square relational semantics? It turns out that the Lambek calculus can be modified in two very natural ways, both modifications making it complete w.r.t. the stronger relational semantics (see Theorem 2.1.13).

We also investigate connection with another kind of semantics for the Lambek calculus, the so-called language-model semantics. This semantics reflects the original (syntactic) motivation behind LC. We will concentrate on language models where we admit languages *with* the empty word. We show that the Lambek calculus is not weakly complete, and that there is no strengthening of the Lambek calculus which is sound w.r.t. relational semantics and would be strongly complete w.r.t. the language models of the above kind. Weak completeness of the Lambek calculus w.r.t. language models *without* the empty word had long been an intriguing open problem. Interesting results in this line can be found, e.g., in [Bu86] and in [Ga92]. [Pę93] gave a positive solution.

We also investigate what happens if we introduce new connectives in the Lambek calculus, moving towards the language of linear logic. We find that structural (or additive, or static, or Boolean) conjunction does not cause any problem, however, structural disjunction makes a strongly complete strengthening impossible. (Weak completeness is still possible.)

Now we recall the definition of the Lambek calculus from [La58].

Definition 2.1.1. (Lambek calculus, LC) We define the language of the Lambek calculus, LC, as follows. Given a denumerable set P of *primitive symbols*, we let the set $\mathrm{Form}_{\mathrm{LC}}$ of *formulas* be the smallest set containing every primitive symbol and closed under $\backslash$, $/$, and $\bullet$, i.e., if $A, B \in \mathrm{Form}_{\mathrm{LC}}$, then $A\backslash B, A/B, A \bullet B \in \mathrm{Form}_{\mathrm{LC}}$. The set of *sequents* is the set of all expressions of the form $A_1, \ldots, A_n \to A_0$ where n is a positive integer and $A_i \in \mathrm{Form}_{\mathrm{LC}}$ for each $i \leq n$.

LC is given by the following axiom and rules of inference, where A, B, C stand for formulas and x, y, z stand for finite sequences of formulas including the empty sequence unless the contrary is asserted.

AXIOM:

$$(LC0) \quad A \to A.$$

RULES OF INFERENCE:

$$(LC1) \quad \frac{x \to A \quad A \to B}{x \to B} \quad x \text{ non-empty}$$

$$(LC \bullet r) \quad \frac{x \to A \quad y \to B}{x, y \to A \bullet B} \quad x, y \text{ non-empty}$$

$$(LC \bullet l) \quad \frac{x, A, B, y \to C}{x, A \bullet B, y \to C}$$

$$(LC\backslash r) \quad \frac{A, x \to B}{x \to A\backslash B} \quad x \text{ non-empty}$$

$$(LC\backslash l) \quad \frac{x \to A \quad y, B, z \to C}{y, x, A\backslash B, z \to C} \quad x \text{ non-empty}$$

$$(LC/r) \quad \frac{x, A \to B}{x \to B/A} \quad x \text{ non-empty}$$

$$(LC/l) \quad \frac{x \to A \quad y, B, z \to C}{y, B/A, x, z \to C} \quad x \text{ non-empty.}$$

A *theorem* of LC is a sequent deducible in LC ($\vdash_{\text{LC}}$), i.e., by the usual recursive definition, a sequent is a theorem iff it is an instance of $(LC0)$, or it is given by some rule of inference from some theorem(s). More generally, let Γ be a set of sequents and φ be a sequent. We say that φ is LC-*deducible from* Γ, $\Gamma \vdash_{\text{LC}} \varphi$, iff

1. $\varphi \in \Gamma$ or
2. φ is an instance of $(LC0)$ or
3. there is a set Δ of sequents each of whose elements is LC-deducible from Γ, and there is an inference rule such that $\frac{\Delta}{\varphi}$ is an instance of this rule.

∎

Note that if we consider only derivations from the empty set, then, in the definition of LC, $(LC1)$ is superfluous, a result of [La58]. $(LC1)$ is really the cut-rule, and the result of Lambek is a cut-elimination theorem. As an immediate consequence, LC is decidable. On the other hand, if we want to have strong completeness, i.e., we are dealing with derivations of the form $\Gamma \vdash_{\text{LC}} \varphi$ (Γ arbitrary set of sequents), then $(LC1)$ is needed. (Indeed, let $A, B, C \in P$ and $\{A \to B,\ B \to C\} = \Gamma$. Then, as we will see soon, $A \to C$ is a semantical consequence of Γ. Since each rule but $(LC1)$ introduces a new connective in the sequent to be derived, $\Gamma \vdash_{\text{LC}} A \to C$ uses $(LC1)$.)

The rules $(LC \bullet r)$ and $(LC \bullet l)$ ensures that composition $\bullet$ is an associative connective. There are non-associative versions of LC as well, cf. [Ka88].

Remark 2.1.2. If the set of primitive symbols is the set of basic types, then the formulas are types and, roughly speaking, $\to$ of LC corresponds to the derivability relation of a version of *categorial grammar*. At the same time, if P is considered as a

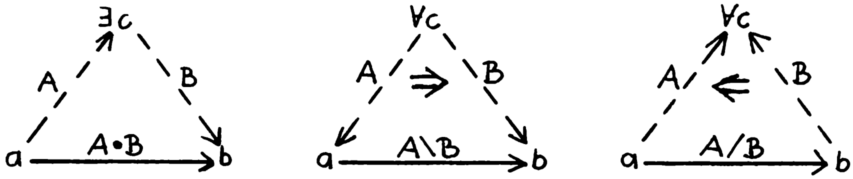

Figure 2.1: RelSem

set of propositional variables, then LC is a Gentzen type inference system, and hence it is a fragment of *linear logic*. ∎

Now we recall the intended dynamic semantics for LC from [vB91].

Definition 2.1.3. (Relational semantics, RelSem) By a (*relativized*) *relational model* for LC we mean an ordered tuple $\langle W, C, v\rangle$ such that W is a transitive binary relation, $C \subseteq {}^3W$ is relational composition, i.e.,

$$C\langle x_0, x_1\rangle\langle y_0, y_1\rangle\langle z_0, z_1\rangle \iff x_0 = y_0 \;\&\; x_1 = z_1 \;\&\; y_1 = z_0,$$

and v is a mapping of the set P of primitive symbols into the powerset $\mathcal{P}(W)$ of W.

Next we define *local truth*, or the *satisfaction relation*. Let $\mathfrak{W} = \langle W, C, v\rangle$ be a relational model for LC, and let $A, B, A_0, A_1, \ldots, A_n \in \mathrm{Form}_{\mathrm{LC}}$ for any $n > 0$. Let $x \in W$ be arbitrary. We define $x \Vdash \varphi$ for formulas and sequents φ inductively.

$$\begin{array}{rll}
x \Vdash p & \text{iff} & x \in v(p), \quad \text{for } p \in P,\\
x \Vdash A \bullet B & \text{iff} & \text{there are } y, z \in W \text{ such that } Cxyz \text{ and}\\
& & y \Vdash A,\ z \Vdash B,\\
x \Vdash A \backslash B & \text{iff} & \text{for all } y, z \in W \text{ such that } Czyx \text{ and } y \Vdash A,\\
& & \text{we have } z \Vdash B,\\
x \Vdash A/B & \text{iff} & \text{for all } y, z \in W \text{ such that } Cyxz \text{ and } z \Vdash B,\\
& & \text{we have } y \Vdash A,
\end{array}$$

$$x \Vdash (A_1, \ldots, A_n \to A_0) \quad \text{iff} \quad (x \Vdash ((A_1 \bullet A_2) \ldots \bullet A_n) \text{ implies } x \Vdash A_0).$$

We say that a sequent φ of LC is *true* in a model $\mathfrak{W}$, in symbols $\mathfrak{W} \models \varphi$, iff $x \Vdash \varphi$ for all $x \in W$. A sequent is *valid* with respect to RelSem iff it is true in every relational model. We denote this by $\models_{\mathsf{R}} \varphi$. We say that φ is a (RelSem) *consequence* of Γ, in symbols $\Gamma \models_{\mathsf{R}} \varphi$, iff, for every relational model $\mathfrak{W}$ of LC, $\mathfrak{W} \models \Gamma$ implies $\mathfrak{W} \models \varphi$ ($\mathfrak{W} \models \Gamma$ abbreviates that, for every $\psi \in \Gamma$, $\mathfrak{W} \models \psi$). ∎

We note that, like in the case of arrow logic, there are more abstract relational type semantics for LC, cf. [Ku95] for abstract ternary frame semantics. She proved completeness of LC w.r.t. this abstract semantics using Henkin-style methods with witnesses, cf. also [Ku94].

The following remark may be skipped for the first reading.

Remark 2.1.4. **(Duals and conjugates)** The residuals \ and / are a kind of conjugates of duals of $\bullet$. Indeed, if we fix one argument, then the modality \ is related to the modality $\bullet$ in a similar fashion as the temporal modality *always-in-the-past*, denoted as $[P]$, is related to *sometime-in-the-future*, denoted as $\langle F\rangle$, cf. [ANS91] and [Go87]. In [ANS91], $\langle P\rangle$ is called the *conjugate* of $\langle F\rangle$ and $[P]$ is the *dual* of $\langle P\rangle$. Then \ is a conjugate of a dual of $\bullet$, cf. below. It is instructive to meditate over the two steps leading to \ from $\bullet$.

We obtain a conjugate modality in temporal logic by reversing the accessibility relation, this corresponds to permuting the arguments of the ternary accessibility relation C, and we obtain a dual modality by passing from an existential quantifier to a universal one (more precisely, by replacing the arguments with their negations and then negating the whole expression). So the obvious dual of $\bullet$ would be $\boxdot$ defined as

$$x \Vdash A \boxdot B \quad \text{iff} \quad \begin{array}{l}\text{for all } y, z \text{ such that } Cxyz,\\ \text{either } y \Vdash A \text{ or } z \Vdash B.\end{array}$$

We can get another dual $\boxtimes$ of $\bullet$ by fixing (i.e., not negating) the first argument as

$$x \Vdash A \boxtimes B \quad \text{iff} \quad \begin{array}{l}\text{for all } y, z \text{ such that } Cxyz,\\ y \Vdash A \text{ implies } z \Vdash B.\end{array}$$

We then get a conjugate $\vec{\boxtimes}$ by interchanging the first and third arguments of C obtaining

$$x \Vdash A \vec{\boxtimes} B \quad \text{iff} \quad \begin{array}{l}\text{for all } y, z \text{ such that } Czyx,\\ y \Vdash A \text{ implies } z \Vdash B.\end{array}$$

Then one can see that $\vec{\boxtimes}$ is just \, i.e., $x \Vdash A\backslash B$ iff $x \Vdash A\vec{\boxtimes}B$. This is what we meant by saying that the slashes \, / are certain conjugates of duals of $\bullet$. ∎

See [Ro91] for a multimodal logic extending the Lambek calculus and containing the existential versions of the residuals. See also [Mi92b] and [Ma95].

We also note that the residuals are term-definable in the symmetric and transitive version of pair arrow logic: $x \Vdash A\backslash B$ iff $x \Vdash \neg(\otimes A \bullet \neg B)$ and $x \Vdash A/B$ iff $x \Vdash \neg(\neg A \bullet \otimes B)$.

2.1.1 COMPLETENESS W.R.T. RELATIVIZED RELATIONAL SEMANTICS

Now we are ready to formulate the strong completeness of LC w.r.t. its relational semantics.

Theorem 2.1.5. *For any set Γ of sequents, and for any sequent φ,*

$$\Gamma \vdash_{\mathsf{LC}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathsf{R}} \varphi.$$

The proof of Theorem 2.1.5 will be based on an algebraic representation theorem (Theorem 2.2.3) which ensures that the Lindenbaum–Tarski algebra of LC (reflecting $\vdash_{\mathsf{LC}}$) is isomorphic to an algebra of binary relations (reflecting $\models_{\mathsf{R}}$).

First we define the Lindenbaum–Tarski algebras of LC. Let $\mathfrak{F} = \langle \mathrm{Form}_{\mathrm{LC}}, \bullet, \backslash, / \rangle$ be the formula algebra, where $\bullet, \backslash, /$ are the natural operations on $\mathrm{Form}_{\mathrm{LC}}$. Let Γ be any set of sequents. We define the relations $\leq_\Gamma$ and $\equiv_\Gamma$ on $\mathrm{Form}_{\mathrm{LC}}$ as follows. For any $A, B \in \mathrm{Form}_{\mathrm{LC}}$,

$$\begin{array}{lll} A \leq_\Gamma B & \text{iff} & \Gamma \vdash_{\mathrm{LC}} A \to B \\ A \equiv_\Gamma B & \text{iff} & (A \leq_\Gamma B \text{ and } B \leq_\Gamma A). \end{array}$$

Lemma 2.1.6. *For any set Γ of sequents,*

(i) *$\equiv_\Gamma$ is a congruence relation on $\mathfrak{F}$ and*
(ii) *for any A, B, A', B' such that $A \equiv_\Gamma A'$ and $B \equiv_\Gamma B'$, we have*

$$A \leq_\Gamma B \quad \textit{iff} \quad A' \leq_\Gamma B'.$$

Proof: (i): $\leq_\Gamma$ is reflexive and transitive by $(LC0)$ and $(LC1)$, so $\equiv_\Gamma$ is an equivalence relation. To show congruence, assume that $A \equiv_\Gamma A'$, $B \equiv_\Gamma B'$. First we want to show $A \bullet B \equiv_\Gamma A' \bullet B'$. By $\Gamma \vdash_{\mathrm{LC}} A \to A'$, $\Gamma \vdash_{\mathrm{LC}} B \to B'$ and $(LC \bullet r)$, we obtain $\Gamma \vdash_{\mathrm{LC}} A, B \to A' \bullet B'$, from which we obtain $\Gamma \vdash_{\mathrm{LC}} A \bullet B \to A' \bullet B'$ by using $(LC \bullet l)$, i.e., $A \bullet B \leq_\Gamma A' \bullet B'$. We obtain $A' \bullet B' \leq_\Gamma A \bullet B$ similarly, so $A \bullet B \equiv_\Gamma A' \bullet B'$ as desired. The proofs for $\backslash, /$ are completely analogous, therefore we omit them. So $\equiv_\Gamma$ is a congruence relation.

(ii): Assume now further that $A \leq_\Gamma B$. Then $A' \leq_\Gamma A$ and $B \leq_\Gamma B'$, by $A \equiv_\Gamma A'$ and $B \equiv_\Gamma B'$, so by transitivity of $\leq_\Gamma$ we obtain $A' \leq_\Gamma B'$. ∎

For $A \in \mathrm{Form}_{\mathrm{LC}}$, $A/{\equiv_\Gamma}$ denotes the equivalence class of $\equiv_\Gamma$ A is in.

Definition 2.1.7. (Lindenbaum–Tarski algebra of LC) Fix Γ. The *Lindenbaum–Tarski algebra*[4] $\mathfrak{L}_\Gamma$ of LC is defined as

$$\mathfrak{L}_\Gamma = \langle L, \bullet, \backslash, /, \leq \rangle,$$

where $\langle L, \bullet, \backslash, / \rangle$ is the factor algebra $\mathfrak{F}/{\equiv_\Gamma}$, and $\leq$ is the image of $\leq_\Gamma$, i.e.,

$$L \stackrel{\text{def}}{=} \mathrm{Form}_{\mathrm{LC}}/{\equiv_\Gamma} = \{A/{\equiv_\Gamma} : A \in \mathrm{Form}_{\mathrm{LC}}\},$$

and

$$(A/{\equiv_\Gamma}) \bullet (B/{\equiv_\Gamma}) \stackrel{\text{def}}{=} (A \bullet B)/{\equiv_\Gamma},$$

and similarly for $/$, $\backslash$, and

$$(A/{\equiv_\Gamma}) \leq (B/{\equiv_\Gamma}) \quad \text{iff} \quad A \leq_\Gamma B.$$

∎

Proof of Theorem 2.1.5: Theorem 2.1.5 will follow from Lemma 2.3.2 and Theorem 2.2.3 in the following way.

In the next section, we will define two classes, ORS and RRS, of (ordered) algebras. As we shall see, ORS reflects very closely the syntactic derivations of LC, (actually,

[4] The Lindenbaum–Tarski algebra of LC is in fact an *ordered* algebra.

we will prove that, for any Γ, $\mathfrak{L}_\Gamma \in$ ORS) while RRS reflects very closely relational semantics of LC. Completeness of LC w.r.t. relational semantics will then be based on the algebraic representation theorem saying that ORS and RRS coincide, up to isomorphisms (see Theorem 2.2.3).

To make the above ideas more concrete, for any subclass K of ORS, we will define a semantics $\models_{\mathsf{K}}$ for LC (this semantics will be invariant under isomorphism, i.e, K and **I**K define the same consequence relation). Then the overall idea of our completeness proof will be the following. For any set Γ of sequents, and for any sequent φ, we prove

(1) $\quad \Gamma \vdash_{\mathsf{LC}} \varphi$ iff $\Gamma \models_{\mathsf{ORS}} \varphi$,

(2) $\quad \mathsf{ORS} = \mathbf{I}\mathsf{RRS}$,

(3) $\quad \Gamma \models_{\mathsf{RRS}} \varphi$ iff $\Gamma \models_{\mathsf{R}} \varphi$.

In the above, (1), (3) are more or less trivial (because ORS is "very close" to the definition of LC, while RRS is "very close" to the definition of relational semantics), cf. Lemma 2.3.2. The hard part will be step (2), cf. Theorem 2.2.3. ∎

Remark 2.1.8. **(GS-semantics)** In [Bu86], a semantics called GS-semantics, is introduced and completeness of LC w.r.t. GS-semantics is proved. Here we show that Theorem 2.1.5 is a strengthening of this theorem. Namely, we show that RelSem is a kind of "subsemantics" of GS-semantics.

Let W be a transitive relation. We define a semigroup as follows. Let u be a new element, not a pair and not in W, and let $W^+ = W \cup \{u\}$. We define the binary operation $\bullet$ on W^+ as follows: for any $x, y \in W^+$,

$$x \bullet y = \begin{cases} \langle a, c\rangle & \text{if } x = \langle a, b\rangle, y = \langle b, c\rangle \text{ for some } a, b, c \\ u & \text{otherwise.} \end{cases}$$

For any $R \subseteq W$ let us define $h(R) = R \cup \{u\}$. Then it is easy to check that h is an isomorphic embedding of the RRS $\langle \mathcal{P}(W), \circ, \backslash_W, /_W, \subseteq\rangle$ (cf. Definition 2.2.1) into the r. semigroup spread over $\langle W^+, =, \bullet\rangle$. This shows that completeness w.r.t. GS-semantics follows from completeness w.r.t. RelSem. Since h also preserves intersection, completeness w.r.t. IGS-semantics in §4.1 of [Bu86] also follows from our Theorem 2.1.9 (which we shall state later). ∎

2.1.2 EXTENSIONS OF THE LAMBEK CALCULUS

We will investigate what happens if we add static conjunction $\wedge$, and static disjunction $\vee$ to LC.

Let LCC denote the Lambek calculus enriched with *static conjunction.* This means the following: in the language of LCC we have one more binary connective $\wedge$, i.e., $A \wedge B \in \mathrm{Form}_{\mathrm{LCC}}$ whenever $A, B \in \mathrm{Form}_{\mathrm{LCC}}$. Otherwise, $\mathrm{Form}_{\mathrm{LCC}}$ and the sequents are defined as above. The models for LCC are those for LC. Let $\langle W, C, v\rangle$ be a relational model for LC, $x \in W$, and let $A, B \in \mathrm{Form}_{\mathrm{LCC}}$. Then

$$x \Vdash A \wedge B \quad \text{iff} \quad (x \Vdash A \;\&\; x \Vdash B).$$

This conjunction is sometimes called structural, or Boolean (e.g., in [vB91]), or additive (e.g., in [Ro91]) versus multiplicative. We adopted the term 'static' (versus 'dynamic') from [Pr92].

The axioms and rules of LCC are those of LC together with the following two axioms, and rule:

$$(\wedge l) \qquad A \wedge B \to A \qquad A \wedge B \to B$$

$$(\wedge r) \qquad \frac{x \to A \quad x \to B}{x \to A \wedge B} \qquad x \text{ non-empty.}$$

Otherwise everything is defined as in the case of LC.

Theorem 2.1.9. *For each sequent φ of the language of* LCC, *and for each set Γ of sequents of* LCC,

$$\Gamma \vdash_{\mathsf{LCC}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathsf{R}} \varphi.$$

Proof: The proof of Theorem 2.1.9 is based on Theorem 2.2.4, exactly the same way as the proof of Theorem 2.1.5 was based on Theorem 2.2.3. ∎

Now we turn to investigating adding *static disjunction* $\vee$ to LCC. In relational semantics, we interpret $\vee$ as follows. Let $\langle W, C, v\rangle$ be any relational model for LC, and let $x \in W$. Then

$$x \Vdash A \vee B \quad \text{iff} \quad (x \Vdash A \text{ or } x \Vdash B).$$

In view of the above, the natural thing would be if adding $\vee$ to LCC, we would get completeness by adding the following two axioms and rule:

$$(\vee r) \qquad A \to A \vee B \qquad B \to A \vee B$$

$$(\vee l) \qquad \frac{A \to C \quad B \to C}{A \vee B \to C}.$$

This is not the case as, e.g., Theorem 2.1.10 below shows, where we prove that no finitely many axioms or rules can ensure strong completeness if we add $\vee$ to the set of operations of LCC.

Theorem 2.1.10. *Let* Q *denote any extension (in the expanded language) of* LCC *with a finite set of axioms or sequent rules for $\vee$. Then* Q *cannot be sound and strongly complete w.r.t. relational semantics.*

Proof: Let Q be any extension of LCC with a finite set of axioms and sequent rules for $\vee$. We denote derivability in Q by $\vdash_{\mathsf{Q}}$, and $\models_{\mathsf{R}}$ denotes consequence in relational semantics as before. Assume that Q is sound, i.e., $\vdash_{\mathsf{Q}} \varphi$ implies $\models_{\mathsf{R}} \varphi$. It can be shown that $\vdash_{\mathsf{Q}}$ is not strongly complete, i.e., there are Γ and φ such that

$$\Gamma \models_{\mathsf{R}} \varphi \quad \text{but} \quad \Gamma \not\vdash_{\mathsf{Q}} \varphi,$$

in the following way.

In the next section, we will define a class RRD of (ordered) algebras that reflects relational semantics, cf. Lemma 2.3.4(i):

$$\Gamma \models_{\mathsf{R}} \varphi \quad \text{iff} \quad \Gamma \models_{\mathsf{RRD}} \varphi.$$

Given Q, we can define a class QRS of algebras that reflects $\vdash_{\mathsf{Q}}$, i.e., for any Γ and φ,

$$\Gamma \vdash_{\mathsf{Q}} \varphi \quad \text{iff} \quad \Gamma \models_{\mathsf{QRS}} \varphi.$$

We will prove that the quasi-equational theory of RRD is not finitely axiomatizable, i.e., there is a quasi-equation q such that

$$\mathsf{RRD} \models q \quad \text{and} \quad \mathsf{QRS} \not\models q,$$

cf. Theorem 2.2.5. Then, by Lemma 2.3.4(ii), there are Γ and φ such that

$$\Gamma \models_{\mathsf{RRD}} \varphi \quad \text{but} \quad \Gamma \not\models_{\mathsf{QRS}} \varphi.$$

∎

We note that Theorem 2.1.10 remains true when we add any set containing $\vee$ of connectives definable in (the full language of) square arrow logic. We also note that Theorem 2.1.10 above remains true if we replace 'relational semantics' in it with 'square relational semantics' (which will be defined below).

2.1.3 COMPLETENESS W.R.T. SQUARE RELATIONAL SEMANTICS

In the definition of relational models $\langle W, C, v\rangle$ for LC, we required W to be transitive only. The reason for this is that transitivity ensures that composition is associative. The question of what other nice properties of W in a relational model we can require naturally arises. Below we will show that if we require W to be reflexive, or a Cartesian space (a square), then we lose completeness of LC. The other natural question that arises is what happens if we omit the conditions of x, y, z being non-empty in the definition of LC. It turns out that if we allow generalized sequents of the form $\rightarrow A$, i.e., delete the conditions on being non-empty in the definition of LC, then this version LC^0 of the Lambek calculus is still decidable (the original cut-elimination proof in [La58] works) and complete w.r.t. square relational semantics, cf. Theorem 2.1.13(ii). We will show that another strengthening, LC^+, by adding four new rules is complete w.r.t. square RelSem as well[5], cf. Theorem 2.1.13(i).

Definition 2.1.11. (Square relational semantics, RelSem^+**)** Let $\mathfrak{W} = \langle W, C, v\rangle$ be a relational model for LC. We say that $\mathfrak{W}$ is a *square* model if W is a Cartesian space, i.e., $W = U \times U$ for some set U. Let RelSem^+ denote relational semantics for LC where we allow only square relational models. Let Γ be a set of sequents, and let φ be a sequent of LC. Then $\models_{\mathsf{R}^+} \varphi$ denotes that φ is true in all square relational models of LC, and similarly, $\Gamma \models_{\mathsf{R}^+} \varphi$ denotes that φ is true in every square relational model of LC in which Γ is also true. ∎

Proposition 2.1.12. LC *is not complete w.r.t.* RelSem^+.

[5] We conjecture that the cut-elimination proof can be modified so that it works in this case as well.

Proof: Let us consider the sequent $\varphi = p \to p \bullet (p\backslash p)$, where p is a propositional variable. We will show that $\models_{\mathsf{R}^+} \varphi$ while not $\vdash_{\mathrm{LC}} \varphi$.

Indeed, let $\mathfrak{W} = \langle W, C, v\rangle$ be any square model with $W = U \times U$. Let $\mathsf{Id} = \{\langle u, u\rangle : u \in U\}$. Then $\mathsf{Id} \subseteq v(p\backslash p)$, thus $v(p) = v(p) \circ \mathsf{Id} \subseteq v(p) \circ v(p\backslash p)$, showing $\mathfrak{W} \models \varphi$. This shows $\models_{\mathsf{R}^+} \varphi$.

On the other hand, if $x \to A \bullet B$ is a theorem of LC, then x must be a compound formula by $(LC \bullet r)$. ∎

The odd behavior of the above sequent was already known in the literature, see [Do92]. By Theorem 2.1.13 below, (in this respect) these are the only "odd" sequents. (So this means that the second kind of "odd" sequents $(P\backslash P)\backslash Q \to Q$, mentioned in [Do92], can be derived from the first ones.)

Now we define two strengthenings, LC^+ and LC^0, of the Lambek calculus. Let LC^+ be LC together with the following four rules. The intuitive idea behind these rules is that if we have two binary relations A and B such that $A \subseteq B$, then $A\backslash B$ and B/A contain the identity relation.

$$\frac{A \to B}{C \to C \bullet (A\backslash B)} \qquad \frac{A \to B}{C \to (A\backslash B) \bullet C}$$

$$\frac{A \to B}{C \to C \bullet (B/A)} \qquad \frac{A \to B}{C \to (B/A) \bullet C}$$

We introduce another strengthening, LC^0, of LC. Let $A_0, A_1, \ldots, A_n \in \mathrm{Form}_{\mathrm{LC}}$. We call $A_1, \ldots, A_n \to A_0$ a *generalized sequent* (or sequent in the wider sense), if $n \geq 0$. That is, we allow $n = 0$ as well. These sequents with $n = 0$ will be denoted by $\to A_0$. Let $\mathfrak{W} = \langle W, C, v\rangle$ be a relational model for LC, and let $\langle u, u'\rangle \in W$. Then we define satisfaction of the generalized sequent $\to A$ as

$$\langle u, u'\rangle \Vdash\ \to A \quad \text{iff} \quad (u = u' \Rightarrow \langle u, u'\rangle \Vdash A).$$

The motivation coming from dynamic semantics for natural languages is the following. If starting from a state u we did not move at all ($n = 0$), then this transition (i.e., $\langle u, u\rangle$) is in A, cf. [vB89b]. Let LC^0 denote the calculus we obtain from LC by omitting all the conditions stating non-emptyness in it, and where at the same time by a sequent we mean a sequent in the wider sense.

Theorem 2.1.13. *Both* LC^+ *and* LC^0 *are strongly complete w.r.t.* RelSem^+. *That is,* (i), (ii) *below hold.*

(i) *Let φ be a (non-generalized) sequent and Γ be a set of (non-generalized) sequents of* LC. *Then*

$$\Gamma \vdash_{\mathrm{LC}^+} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathsf{R}^+} \varphi.$$

(ii) *Let φ be a generalized sequent and let Γ be a set of generalized sequents. Then*

$$\Gamma \vdash_{\mathrm{LC}^0} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathsf{R}^+} \varphi.$$

The proof of Theorem 2.1.13 is based on algebraic representation theorems, just as in earlier cases. The proof of Theorem 2.1.13(i) will be a close parallel to that of Theorem 2.1.5, while the proof of Theorem 2.1.13(ii) will be a refined version of that.

Definition 2.1.14. Let Γ be an arbitrary set of generalized sequents. We define an analogue, $\mathfrak{L}^0_\Gamma$, of the Lindenbaum–Tarski algebra, $\mathfrak{L}_\Gamma$, of LC.

Let $e, 0, 1$ be three new elements not in $\mathrm{Form_{LC}}$, and let $T = \mathrm{Form_{LC}} \cup \{e, 0, 1\}$.[6] Let

$$\mathfrak{T} = \langle T, \backslash, /, \bullet, \leq_\Gamma, e, 0\rangle$$

where the definitions of the operations and the relation $\leq_\Gamma$ go as follows. On $A, B \in \mathrm{Form_{LC}}$ these are defined as before, i.e., $A \leq_\Gamma B$ iff $\Gamma \vdash_{\mathrm{LC^0}} A \to B$. For every $x \in T$ and $A \in \mathrm{Form_{LC}}$, let $0 \leq_\Gamma x \leq_\Gamma 1$ and $e \leq_\Gamma e$, and let $e \leq_\Gamma A$ iff $\Gamma \vdash_{\mathrm{LC^0}} \ \to A$. Let $0 \bullet x = x \bullet 0 = 0$ and $e \bullet x = x \bullet e = x$, and if $x \neq 0$, then let $1 \bullet x = x \bullet 1 = 1$. Let $0\backslash x = 1$ and $e\backslash x = x$, and if $x \neq 0$, then let $x\backslash 0 = 0$. Further, if $x \notin \{e, 0\}$, then let $x\backslash e = 0$ and $x\backslash 1 = 1$. Finally, if $x \neq 1$, then let $1\backslash x = 0$. The other slash, $/$, can be defined in a similar way. Let

$$x \equiv_\Gamma y \quad \text{iff} \quad (x \leq_\Gamma y \text{ and } y \leq_\Gamma x),$$

and let $\mathfrak{L}^0_\Gamma = (\mathfrak{T}/{\equiv_\Gamma})$. ∎

Proof of Theorem 2.1.13: The proof of (i) proceeds exactly as the proof of Theorem 2.1.5, but now we use Theorem 2.2.7 instead of Theorem 2.2.3.

(ii): Soundness of $\mathrm{LC^0}$ w.r.t. RelSem^+ is easy to check. To prove strong completeness, let Γ be a set of generalized sequents, φ be a generalized sequent, and assume that $\Gamma \models_{\mathsf{R}^+} \varphi$. We want to show that $\Gamma \vdash_{\mathrm{LC^0}} \varphi$.

Let us consider the Lindenbaum–Tarski algebra $\mathfrak{L}^0_\Gamma$. We will turn this algebra into a square relational model $\mathfrak{W}$ such that $\mathrm{LC^0}$-provability from Γ will coincide with validity in this model. By Lemma 2.3.3 and Theorem 2.2.9, $\mathfrak{L}^0_\Gamma$ is isomorphic to an RRS^0, where RRS^0 denotes the class of algebras (of binary relations) corresponding to relational semantics of $\mathrm{LC^0}$, cf. next section. Let $h : \mathfrak{L}^0_\Gamma \longrightarrow \mathfrak{M}$ be such an isomorphism, where $\mathfrak{M} = \langle M, \circ, \backslash, /, \mathsf{Id}, \emptyset, \subseteq\rangle \in \mathsf{RRS}^0$. To turn $\mathfrak{M}$ into a relational model, let $W = U \times U$ where $U = \{u : \langle u, u\rangle \in R \text{ for some } R \in M\}$, and let $v(p) = h(p/{\equiv_\Gamma})$ for all $p \in P$. Then $\mathfrak{W} = \langle W, C, v\rangle$ is a square relational model of LC. Then, by Lemma 2.3.5, for every generalized sequent $A \to B$,

$$h(A/{\equiv_\Gamma}) \subseteq h(B/{\equiv_\Gamma}) \quad \text{iff} \quad \mathfrak{W} \models (A \to B).$$

By h being an isomorphism, we have

$$(+) \qquad \Gamma \vdash_{\mathrm{LC^0}} \varphi \quad \text{iff} \quad \mathfrak{W} \models \varphi$$

for any generalized sequent φ.

Recall that $\Gamma \models_{\mathsf{R}^+} \varphi$. By (+) we have $\mathfrak{W} \models \Gamma$, hence $\mathfrak{W} \models \varphi$ by $\Gamma \models_{\mathsf{R}^+} \varphi$. Then applying (+) once more we get $\Gamma \vdash_{\mathrm{LC^0}} \varphi$, and we are done. ∎

Remark 2.1.15. We note that if we add the axioms $(\wedge l)$ and the rule $(\wedge r)$ to LC^+, then we get completeness theorem for this expanded language, just as in the case of Theorem 2.1.9, cf. Remark 2.2.8. ∎

[6] The element 1 has a role only in the definitions of the operations on the algebra.

2.1.4 LANGUAGE MODELS

Now we prove that LC is not weakly complete w.r.t. language models (LM) and that there is no extension of LC which is sound w.r.t. $U \times U$ type relational semantics and is strongly complete w.r.t. LM. First, we recall the definition of language models from [vB91] p.189.

Definition 2.1.16. (Language model) A *language* is a set of finite, possibly empty, sequences. A *family of languages* is a set $\{L_i : i \in I\}$, where L_i is a set of finite sequences (words) over a finite alphabet.

A *language model* is a family of languages enriched with the following operations.

$$\begin{aligned} L_a \bullet L_b &\stackrel{\text{def}}{=} \{xy : x \in L_a,\ y \in L_b\} \\ L_a \backslash L_b &\stackrel{\text{def}}{=} \{x : (\forall y \in L_a) yx \in L_b\} \\ L_b / L_a &\stackrel{\text{def}}{=} \{x : (\forall y \in L_a) xy \in L_b\} \end{aligned}$$

A sequent $A_1, \ldots, A_n \to A_0$ is *true* in a language model if

$$v(A_1) \bullet \ldots \bullet v(A_n) \subseteq v(A_0)$$

where v is the valuation function defined in the obvious way. The consequence relation $\models_{\mathsf{LM}}$ is the usual as well. ∎

Proposition 2.1.17. LC *is not weakly complete w.r.t. language models.*

Proof: By the definition of $\backslash$, the empty sequence is in $L\backslash L$ for every language L. Thus $x \to x \bullet (x\backslash x)$ is valid in every language model. On the other hand, it is not a theorem of LC, cf. the proof of Proposition 2.1.12. ∎

Proposition 2.1.18. *There is no calculus containing* LC *which is strongly complete w.r.t. language models and sound w.r.t.* RelSem$^+$.

Proof: We will show that there are a set Γ of sequents and a sequent φ such that $\Gamma \not\models_{\mathsf{R}^+} \varphi$ but $\Gamma \models_{\mathsf{LM}} \varphi$.

It is easy to check that

$$\{x \to x \bullet x,\ y \to x\} \not\models_{\mathsf{R}^+} y \to x \bullet y$$

(let $v(x) = \{\langle 1,0\rangle, \langle 0,0\rangle\}$ and $v(y) = \{\langle 1,0\rangle\}$).

On the other hand,

$$\{x \to x \bullet x,\ y \to x\} \models_{\mathsf{LM}} y \to x \bullet y$$

because of the following. Let L_x, L_y be two languages and assume that $L_x \subseteq L_x \bullet L_x$, $L_y \subseteq L_x$. We want to show $L_y \subseteq L_x \bullet L_y$. If $L_x = \emptyset$, then $L_y = \emptyset$, and we are done. So we can assume that $L_x \neq \emptyset$. Let $w \in L_x$. Then, since $L_x \subseteq L_x \bullet L_x$, there are $u_1, v_1 \in L_x$ such that $w = u_1 v_1$. By the same argument, for each number i, there are $u_{i+1}, v_{i+1} \in L_x$ with $u_i = u_{i+1} v_{i+1}$. Sooner or later, since w is a finite string, either u_i or v_i is the empty sequence. That is, L_x contains the empty sequence. Hence $L_y \subseteq L_x \bullet L_y$. ∎

Corollary 2.1.19. LC^0, *that version of the Lambek Calculus where we admit sequents with empty antecedent, is not strongly complete w.r.t.* LM.

Proof: LC^0 contains LC. ∎

We note that S-semantics of [Bu86] differs from our LM-semantics in that in S-semantics the empty word is not allowed in any language. A version of Proposition 2.1.18 is proved in [Bu86], Lemma 11, for S-semantics instead of LM-semantics. We note that the sequent $P \to P \bullet (P\backslash P)$ is LM-valid but not S-valid. (This shows also, by the results in [Bu86], that the calculus LSC_1 of [Bu86] is not a conservative extension of LC, an interesting fact.)

2.2 REPRESENTATION OF ORDERED RESIDUATED SEMIGROUPS

In this section, we turn to investigating the finite axiomatizability of the classes of algebras corresponding to the logics of the previous section.[7]

The class RRA of representable relation algebras (cf. Definition 3.2.1) is not finitely axiomatizable, cf. [Mo69], reflecting the fact that square arrow logic does not have a strongly sound and complete Hilbert-style calculus. However, there are relativized versions of RRA (cf. Chapter 3) that are finitely axiomatizable. In this section, we choose another way to find finitely axiomatizable versions of classes of algebras of binary relations. We will consider *reducts* of RRA, and prove finite axiomatizability theorems. The advantage of this approach is that the operations keep their original meanings. Thus, composition remains an associative operation, while in relativized RRA's usually only a weakened version of associativity holds.

Below, we will investigate ordered algebras (RRS's) of binary relations the operations of which are relational compositions, and its two residuals (and sometimes intersection). The main results of this section are Theorem 2.2.3 and Theorem 2.2.7 stating the finite axiomatizability of the classes of transitive and square versions of RRS's.

Definition 2.2.1. (Representable ordered residuated semigroup, RRS) Let W be a transitive binary relation and let $R, S \subseteq W$ be subrelations of W. The left and right *residuals* relative to (or relativized to) W are defined as follows:

$$R \backslash_W S \overset{\text{def}}{=} \{\langle x,y\rangle \in W : \forall z(\langle z,x\rangle \in R \Rightarrow \langle z,y\rangle \in S)\}$$
$$R/_W S \overset{\text{def}}{=} \{\langle x,y\rangle \in W : \forall z(\langle y,z\rangle \in S \Rightarrow \langle x,z\rangle \in R)\}$$

and $\circ$ denotes *relation composition*, i.e.,

$$R \circ S \overset{\text{def}}{=} \{\langle x,y\rangle : \exists z(\langle x,z\rangle \in R \;\&\; \langle z,y\rangle \in S)\}.$$

[7] In this section, we use the same symbols for the operations of the algebras as for the connectives of the logics of the previous section. This way we would like to emphasize the connection between the algebras and the logics of this chapter.

We will deal with ordered algebras whose elements are binary relations, whose operations are $\circ, \backslash_W, /_W$, and whose ordering is the set-theoretical inclusion relation $\subseteq$. We will call such structures *representable*. In more detail, we call $\mathfrak{A} = \langle A, \bullet, \backslash, /, \leq \rangle$ a *representable ordered residuated semigroup*, an RRS, iff

1. A is a set of binary relations,
2. $\bullet, \backslash, /$ are binary operations on A, $\leq$ is a binary relation on A,
3. $\bullet, \backslash, /, \leq$ coincide on A with $\circ, \backslash_W, /_W, \subseteq$, respectively, where

$$W = \bigcup A = \{\langle x, y \rangle : (\exists R \in A)\langle x, y \rangle \in R\}.$$

We note that W is not necessarily reflexive or symmetric. If $\mathfrak{A} = \langle A, \circ, \backslash_V, /_V, \subseteq \rangle$ is an algebra for an arbitrary transitive V such that $A \subseteq \mathcal{P}(V)$, then V can be taken to be $W = \bigcup A$, i.e., for all $R, S \in A$ we have $R\backslash_V S = R\backslash_W S$, $R/_V S = R/_W S$.

We will often omit the index W from $\backslash_W, /_W$. ∎

We note that the operations $\backslash$ and $/$ are highly dependent on W, i.e., $R\backslash_W S$ changes if we change W but leave R, S fixed. This relative behavior is inherent in $\backslash, /$ just as in Boolean complementation. However, later we will speak of unrelativized $\backslash$ and $/$. By this we will understand that we choose W in a natural way (to be a Cartesian space). On the other hand, transitivity of W ensures that $\circ$ does not change if consider larger relations, e.g., $U \times U \supseteq W$.

Definition 2.2.2. (Ordered residuated semigroup, ORS) (i) We call an algebra with three binary operations and a binary relation on it an RS. We usually denote the operations of an RS by $\bullet, \backslash, /$ and its relation by $\leq$. Thus $\mathfrak{A} = \langle A, \bullet, \backslash, /, \leq \rangle \in$ RS iff A is an arbitrary non-empty set, $\bullet, \backslash, /$ are arbitrary binary operations on A, and $\leq$ is an arbitrary binary relation on A.

(ii) Σ denotes the following set (A1)–(A7) of formulas (in the first-order language with equality of RS), where x, y, z, u are variables.

(I) $\leq$ is an ordering, i.e.,

(A1) $x \leq x$

(A2) $x \leq y \;\&\; y \leq z \Rightarrow x \leq z$

(A3) $x \leq y \;\&\; y \leq x \Rightarrow x = y$.

(II) $\bullet$ is a semigroup operation which is monotonic in both arguments w.r.t. $\leq$, i.e.,

(A4) $(x \bullet y) \bullet z = x \bullet (y \bullet z)$

(A5) $x \leq y \;\&\; z \leq u \Rightarrow x \bullet z \leq y \bullet u$.

(III) $\backslash$ and $/$ are the so-called left and right residuals of $\bullet$, i.e.,

(A6) $x \bullet y \leq z \iff y \leq x\backslash z$

(A7) $x \bullet y \leq z \iff x \leq z/y$.

If $\mathfrak{A} \in$ RS, then $\mathfrak{A} \models \Sigma$ denotes that the set Σ of (open) formulas is valid in $\mathfrak{A}$, i.e., that the universal closures of elements of Σ are valid in $\mathfrak{A}$. For instance, $\mathfrak{A} \models x \leq x$ iff $\mathfrak{A} \models \forall x(x \leq x)$. If $\mathfrak{A} \models \Sigma$, then we call $\mathfrak{A}$ an *ordered residuated semigroup* (see, e.g., [Bu86]), and ORS denotes the class of all ordered residuated semigroups. ∎

The operations $\backslash, /$ of taking residuals in semigroups have long been investigated in semigroup theory. In algebraic logic, they correspond to some kinds of implications, see, e.g., [Pr90]. Recently, they came into focus in several works, see, e.g., [Pr92], [JT92], [JJR92], [Ji92].

2.2.1 REPRESENTATION WITH TRANSITIVE RELATIONS

We are ready to formulate the following representation theorem, which states that the two classes RRS and ORS coincide (up to isomorphism).

Theorem 2.2.3. ORS = IRRS, *i.e., for every* $\mathfrak{A} \in$ RS,

$$\mathfrak{A} \models \Sigma \quad \textit{iff} \quad \mathfrak{A} \in \mathsf{IRRS}.$$

Proof: It is easy to check that Σ is valid in every RRS.

For the other direction, let us assume that $\mathfrak{A} \in$ RS and $\mathfrak{A} \models \Sigma$. Step by step we will build a directed graph $G = \langle U, E, \ell \rangle$ the edges (E) of which will be labeled (ℓ) by the elements of our structure $\mathfrak{A}$. We will use this graph to define a representation function rep, which will be an isomorphism from $\mathfrak{A}$ to a structure of binary relations on U.

In each step α, we will define a directed graph $G_\alpha = \langle U_\alpha, E_\alpha, \ell_\alpha \rangle$, where U_α is the set of nodes, $E_\alpha \subseteq U_\alpha \times U_\alpha$ is the set of edges, $\ell_\alpha : E_\alpha \longrightarrow A$ is the labeling function (A is the universe of $\mathfrak{A}$) such that

(I) E_α is irreflexive and transitive
(II) $\langle x, y \rangle, \langle y, z \rangle \in E_\alpha$ implies $\ell_\alpha\langle x, z \rangle \leq \ell_\alpha\langle x, y \rangle \bullet \ell_\alpha\langle y, z \rangle$.

The final graph, G, will have the following additional properties ensuring that the labeling respects composition and the residuals:

(III) $(\forall a \in A)(\forall x \in U)(\exists u \in U)\ell\langle u, x \rangle = a$
(IV) $(\forall a \in A)(\forall y \in U)(\exists v \in U)\ell\langle y, v \rangle = a$
(V) $(\forall a, b, c \in A)(\forall x, y \in U)[\langle x, y \rangle \in E \;\&\; \ell\langle x, y \rangle = c \;\&\; c \leq a \bullet b \Rightarrow$ $(\exists z \in U)(\ell\langle x, z \rangle = a \;\&\; \ell\langle z, y \rangle = b)]$.

When building G step by step, we will "maintain" properties (I), (II) and will "bring-about" (III)–(V) by putting in appropriate points.

We will need a "scheduling" function σ which will help us in the construction of G. Choose an infinite cardinal κ such that $|A| \leq \kappa$. Let V be a set of cardinality κ, and let $\sigma : \kappa \longrightarrow {}^3A \times {}^2V \times 3$ be such that

$$(\forall \langle a, b, c, x, y, i \rangle \in {}^3A \times {}^2V \times 3)(\forall \lambda < \kappa)(\exists \nu < \kappa)[\lambda \leq \nu \;\&\; \sigma(\nu + 1) = \langle a, b, c, x, y, i \rangle].$$

To see that there is such a function σ, let $f : \kappa \longrightarrow {}^3A \times {}^2V \times 3 \times \kappa$ be a bijection. If we fix a, b, c, x, y, i, then, for κ many ordinals γ, $f(\gamma) = \langle a, b, c, x, y, i, \delta \rangle$ for some $\delta < \kappa$. So, for each $\lambda < \kappa$, there is $\nu \geq \lambda$ such that $f(\nu) = \langle a, b, c, x, y, i, \delta' \rangle$ for some $\delta' < \kappa$. Let $g : {}^3A \times {}^2V \times 3 \times \kappa \longrightarrow {}^3A \times {}^2V \times 3$ with $g\langle a, b, c, x, y, i, \lambda \rangle = \langle a, b, c, x, y, i \rangle$ for each $\lambda < \kappa$. If we define $\sigma(\nu + 1) = g(f(\nu))$ and $\sigma(\nu)$ arbitrary for limit $\nu < \kappa$, then σ meets the requirements.

We will build G in κ steps, V will be a universe from which we will choose our new elements to put into U, and σ will be the "scheduling" for building the graph: $\sigma(\lambda)$ is the "task" to take care of in the λth step. The condition on σ is that each task recurs after each step (this will be needed because we will care for a task only if its conditions are already met). If $\sigma(\lambda) = \langle a, b, c, x, y, i\rangle$, then in the λth step we will examine the edge $\langle x, y\rangle$ from the point of view of labels a, b, c, and i indicates the "type of the activity" to be carried through.

0TH STEP. For each element c of A, we choose two different elements from V, say u_c and v_c such that u_c, v_c are all different for different c's. Let $U_0 = \{u_c, v_c : c \in A\}$. We can assume that $|V \setminus U_0| = \kappa$. Let $E_0 = \{\langle u_c, v_c\rangle : c \in A\}$ and $\ell_0\langle u_c, v_c\rangle = c$. Clearly, (I) and (II) hold.

$\alpha + 1$ST STEP. Let $\sigma(\alpha+1) = \langle c, a, b, x, y, i\rangle$. If $\langle x, y\rangle \notin E_\alpha$ or $\ell_\alpha\langle x, y\rangle \neq c$, then let $G_{\alpha+1} = G_\alpha$. Otherwise we have three subcases according to the value of i.

$i = 0$. See Figure 2.2. Choose an element from $V \setminus U_\alpha$, say, u. Let

$$\begin{aligned} U_{\alpha+1} &= U_\alpha \cup \{u\} \\ E_{\alpha+1} &= E_\alpha \cup \{\langle u, p\rangle : \langle x, p\rangle \in E_\alpha\} \cup \{\langle u, x\rangle\} \\ \ell_{\alpha+1} &= \ell_\alpha \cup \{\langle\langle u, p\rangle, a \bullet \ell_\alpha\langle x, p\rangle\rangle : \langle x, p\rangle \in E_\alpha\} \cup \{\langle\langle u, x\rangle, a\rangle\}. \end{aligned}$$

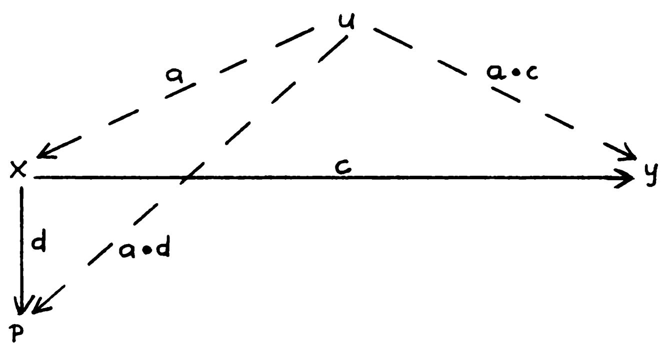

Figure 2.2: $i = 0$

$i = 1$. See Figure 2.3. Choose an element from $V \setminus U_\alpha$, say, v. Let

$$\begin{aligned} U_{\alpha+1} &= U_\alpha \cup \{v\} \\ E_{\alpha+1} &= E_\alpha \cup \{\langle q, v\rangle : \langle q, y\rangle \in E_\alpha\} \cup \{\langle y, v\rangle\} \\ \ell_{\alpha+1} &= \ell_\alpha \cup \{\langle\langle q, v\rangle, \ell_\alpha\langle q, y\rangle \bullet a\rangle : \langle q, y\rangle \in E_\alpha\} \cup \{\langle\langle y, v\rangle, a\rangle\}. \end{aligned}$$

$i = 2$. See Figure 2.4. If $c \not\leq a \bullet b$, then let $G_{\alpha+1} = G_\alpha$. Otherwise let $z \in V \setminus U_\alpha$, and let

$$\begin{aligned} U_{\alpha+1} &= U_\alpha \cup \{z\} \\ E_{\alpha+1} &= E_\alpha \cup \{\langle r, z\rangle : \langle r, x\rangle \in E_\alpha\} \cup \{\langle z, s\rangle : \langle y, s\rangle \in E_\alpha\} \cup \{\langle x, z\rangle, \langle z, y\rangle\} \\ \ell_{\alpha+1} &= \ell_\alpha \cup \{\langle\langle x, z\rangle, a\rangle, \langle\langle z, y\rangle, b\rangle\} \cup \\ &\quad \{\langle\langle r, z\rangle, \ell_\alpha\langle r, x\rangle \bullet a\rangle : \langle r, x\rangle \in E_\alpha\} \cup \{\langle\langle z, s\rangle, b \bullet \ell_\alpha\langle y, s\rangle\rangle : \langle y, s\rangle \in E_\alpha\}. \end{aligned}$$

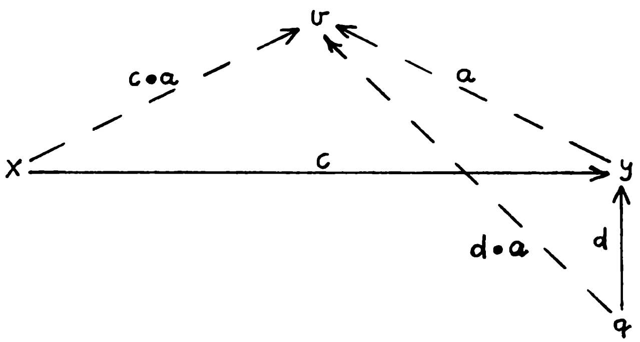

Figure 2.3: $i = 1$

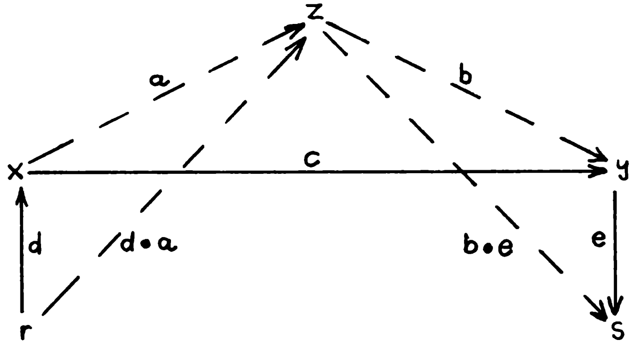

Figure 2.4: $i = 2$

It is easy to check that property (I) is preserved in the $\alpha + 1$st step.

We also have to prove that the new transitive triangles constructed in the $\alpha + 1$st step have property (II). We have to check only the new triangles, i.e., triangles in which new edges occur. We have three cases according to the value of i above.

$i = 0$. The new edges are $\{\langle u, p\rangle : p = x \text{ or } \langle x, p\rangle \in E_\alpha\}$. The typical situation is represented in Figure 2.5, where the two kinds of new triangles are the ones determined by uxp_1, and by up_1p_2. We have to show that $a_4 \leq a \bullet a_1$ and $a_5 \leq a_4 \bullet a_3$. By the construction of the graph we have $a_4 = a \bullet a_1$, $a_5 = a \bullet a_2$; and by our induction hypothesis that (II) holds for E_α we have that $a_2 \leq a_1 \bullet a_3$. Thus $a_4 \leq a \bullet a_1$ by (A1) (reflexivity of $\leq$), and $a_5 = a \bullet a_2 \leq a \bullet (a_1 \bullet a_3) = (a \bullet a_1) \bullet a_3 = a_4 \bullet a_3$ by (A5), (A1), (A4) (i.e., monotonicity and associativity of $\bullet$).

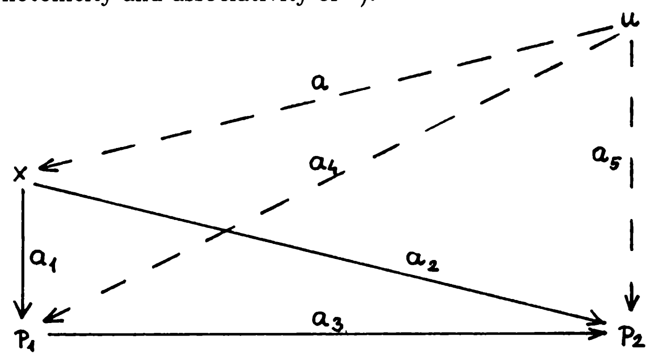

Figure 2.5: triangles ($i = 0$)

$i = 1$. This case is completely analogous to the case $i = 0$.

$i = 2$. The new edges are $\{\langle r, z\rangle : r = x \text{ or } \langle r, x\rangle \in E_\alpha\} \cup \{\langle z, s\rangle : s = y \text{ or } \langle y, s\rangle \in E_\alpha\}$, and the typical situation is represented on Figure 2.6. The new triangles to be

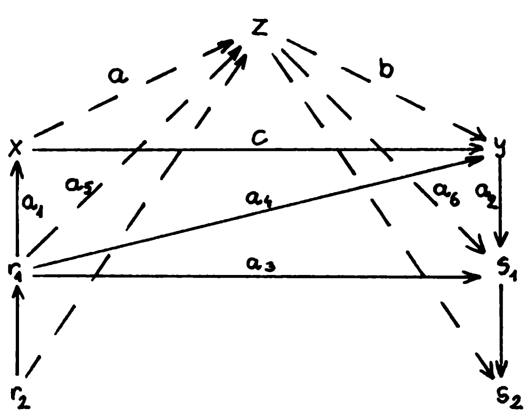

Figure 2.6: triangles ($i = 2$)

checked are the ones determined by the following triples of points: r_1xz, r_2r_1z, xzy, xzs_1, r_1zy, r_1zs_1, zys_1, zs_1s_2. Checking these is very similar to the previous cases. As an example, we check the triangle r_1zs_1. We show that $a_3 \leq a_5 \bullet a_6$. By the construction of the graph we have $a_5 = a_1 \bullet a$, $a_6 = b \bullet a_2$, $c \leq a \bullet b$ and by our induction hypothesis on E_α we have $a_3 \leq a_4 \bullet a_2$ and $a_4 \leq a_1 \bullet c$. So $a_3 \leq a_4 \bullet a_2 \leq (a_1 \bullet c) \bullet a_2 \leq (a_1 \bullet (a \bullet b)) \bullet a_2 = (a_1 \bullet a) \bullet (b \bullet a_2) = a_5 \bullet a_6$ by monotonicity and associativity of $\bullet$.

Thus $G_{\alpha+1}$ satisfies (II) as well.

LIMIT STEP. If α is a limit ordinal, then let $U_\alpha = \bigcup_{\beta<\alpha} U_\beta$, $E_\alpha = \bigcup_{\beta<\alpha} E_\beta$ and $\ell_\alpha = \bigcup_{\beta<\alpha} \ell_\beta$.

Let $G = G_\kappa$, i.e.,

$$U = \bigcup_{\alpha<\kappa} U_\alpha, \qquad E = \bigcup_{\alpha<\kappa} E_\alpha \qquad \text{and} \qquad \ell = \bigcup_{\alpha<\kappa} \ell_\alpha.$$

Clearly, G satisfies (I) and (II), and (III)–(V) hold by the construction.

Now, we are ready to define the representation function rep. For every $c \in A$, let

$$\mathsf{rep}(c) = \{\langle u, v\rangle : \ell\langle u, v\rangle \leq c\}.$$

We have to show that rep is an isomorphism from $\mathfrak{A}$ to a structure whose elements are binary relations on the set of nodes of our graph. Clearly, $\mathsf{rep}(c)$ is a binary relation on U for any $c \in A$.

We prove that rep is an isomorphism w.r.t. $\leq$, i.e.,

$$a \leq b \quad \text{iff} \quad \mathsf{rep}(a) \subseteq \mathsf{rep}(b).$$

Indeed, if $\ell\langle u, v\rangle \leq a$, then, by transitivity of $\leq$, $\ell\langle u, v\rangle \leq b$, so $\langle u, v\rangle \in \mathsf{rep}(a)$ implies $\langle u, v\rangle \in \mathsf{rep}(b)$. If $\mathsf{rep}(a) \subseteq \mathsf{rep}(b)$, then for every $\langle u, v\rangle \in E$, if $\ell\langle u, v\rangle \leq a$, then $\ell\langle u, v\rangle \leq b$. Since $\ell\langle u_a, v_a\rangle = a$ (see the 0th step), we have $a \leq b$.

Now we show that rep is one-one, i.e.,

$$a \neq b \quad \text{implies} \quad \mathsf{rep}(a) \neq \mathsf{rep}(b).$$

Assume $\mathsf{rep}(a) = \mathsf{rep}(b)$. Then $\mathsf{rep}(a) \subseteq \mathsf{rep}(b)$ and $\mathsf{rep}(b) \subseteq \mathsf{rep}(a)$, so $a \leq b$ and $b \leq a$ by the previous paragraph, thus $a = b$ by (A3) (antisymmetry of $\leq$).

We check that rep preserves the operations too.

Checking the operation $\bullet$:

$$\begin{array}{rcl} \mathsf{rep}(a \bullet b) & = & \{\langle u, v\rangle : \ell\langle u, v\rangle \leq a \bullet b\} = \\ \text{[by } (i) \text{ below]} & = & \{\langle u, v\rangle : \exists z(\ell\langle u, z\rangle \leq a \ \&\ \ell\langle z, v\rangle \leq b)\} = \\ & = & \{\langle u, z\rangle : \ell\langle u, z\rangle \leq a\} \circ \{\langle z, v\rangle : \ell\langle z, v\rangle \leq b\} = \\ & = & \mathsf{rep}(a) \circ \mathsf{rep}(b). \end{array}$$

(i): $\subseteq$: By property (V). In more detail: let $c = \ell\langle u, v\rangle$. Then, for some $\alpha + 1$, $\sigma(\alpha + 1) = \langle c, a, b, u, v, 2\rangle$. So in the $\alpha + 1$st step we put a z into the graph such that

$\ell\langle u,z\rangle = a$ and $\ell\langle z,v\rangle = b$. $\supseteq$: By properties (I) and (II), by the transitivity of $\leq$, and by (A5).

Checking the operation $\backslash$:

$$\begin{array}{rcl} \mathsf{rep}(a\backslash b) & = & \{\langle u,v\rangle : \ell\langle u,v\rangle \leq a\backslash b\} = \\ \text{[by } (ii)] & = & \{\langle u,v\rangle : a \bullet \ell\langle u,v\rangle \leq b\} = \\ \text{[by } (iii)] & = & \{\langle u,v\rangle : \forall z(\ell\langle z,u\rangle \leq a \Rightarrow \ell\langle z,u\rangle \bullet \ell\langle u,v\rangle \leq b)\} = \\ \text{[by } (iv)] & = & \{\langle u,v\rangle : \forall z(\ell\langle z,u\rangle \leq a \Rightarrow \ell\langle z,v\rangle \leq b)\} = \\ & = & \mathsf{rep}(a)\backslash\mathsf{rep}(b). \end{array}$$

(ii): $c \leq a\backslash b$ iff $a \bullet c \leq b$ by (A6).

(iii): $\subseteq$: By monotonicity of $\bullet$. $\supseteq$: By property (III), the triangle in Figure 2.7 is in the graph.

(iv): $\subseteq$: By properties (I) and (II). $\supseteq$: The triangle in Figure 2.7 is in G, so if $\ell\langle z,u\rangle \leq a$, then $\ell\langle z,u\rangle \bullet \ell\langle u,v\rangle \leq a \bullet \ell\langle u,v\rangle = \ell\langle z',u\rangle \bullet \ell\langle u,v\rangle = \ell\langle z',v\rangle \leq b$.

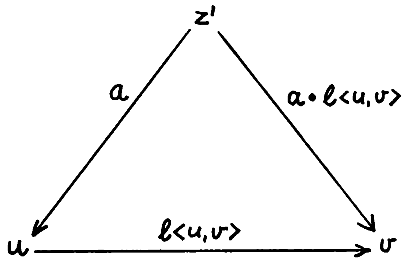

Figure 2.7: $(iii), (iv)$

Checking the operation $/$:

$$\begin{array}{rcl} \mathsf{rep}(b/a) & = & \{\langle u,v\rangle : \ell\langle u,v\rangle \leq b/a\} = \\ \text{[by } (v)] & = & \{\langle u,v\rangle : \ell\langle u,v\rangle \bullet a \leq b\} = \\ \text{[by } (vi)] & = & \{\langle u,v\rangle : \forall z(\ell\langle v,z\rangle \leq a \Rightarrow \ell\langle u,v\rangle \bullet \ell\langle v,z\rangle \leq b)\} = \\ \text{[by } (vii)] & = & \{\langle u,v\rangle : \forall z(\ell\langle v,z\rangle \leq a \Rightarrow \ell\langle u,z\rangle \leq b)\} = \\ & = & \mathsf{rep}(b)/\mathsf{rep}(a). \end{array}$$

(v): By (A7).

(vi): $\subseteq$: By monotonicity of $\bullet$. $\supseteq$: By property (IV), Figure 2.8 is in G.

(vii): $\subseteq$: By properties (I) and (II). $\supseteq$: By Figure 2.8, $\ell\langle u,v\rangle \bullet a \leq b$.

Thus rep is the desired isomorphism, since the rep-image of $\mathfrak{A}$,

$$\langle\{\mathsf{rep}(a) : a \in A\}, \backslash, /, \circ, \subseteq\rangle,$$

is in RRS. Thus Theorem 2.2.3 is proved. ∎

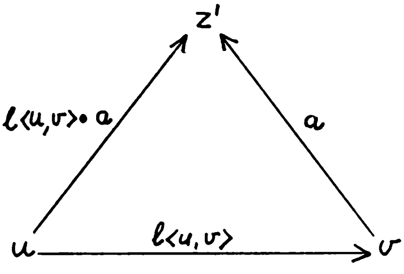

Figure 2.8: $(vi), (vii)$

2.2.2 EXTENDING THE SIMILARITY TYPE

Let RRC denote the class of all RRS's endowed with the operation of taking intersection. That is, an ordered algebra $\langle A, \circ, \backslash, /, \cap, \subseteq \rangle$ is in RRC iff $\langle A, \circ, \backslash, /, \subseteq \rangle \in$ RRS and A is closed under taking intersection, i.e., for all $R, S \in A$, we have $R \cap S \in A$.

Let Θ be Σ together with the following axiom:

(A8) $(z \leq x \ \& \ z \leq y) \Longleftrightarrow z \leq x \wedge y$.

Thus $\mathfrak{A} \models \Theta$ means that $\mathfrak{A}$ is a *semilattice-ordered residuated semigroup.*

Theorem 2.2.4. *Every semilattice-ordered residuated semigroup is isomorphic to a representable one, i.e., for any algebra $\mathfrak{A}$ (of the right similarity type),*

$$\mathfrak{A} \models \Theta \quad \textit{iff} \quad \mathfrak{A} \in \mathsf{IRRC}.$$

Proof: We use the construction in the proof of Theorem 2.2.3. We have to show that the function rep defined there is an isomorphism w.r.t. the operation $\wedge$ as well. Indeed, by (A8),

$$\begin{aligned} \mathsf{rep}(a \wedge b) &= \{\langle u, v \rangle : \ell\langle u, v \rangle \leq a \wedge b\} = \\ &= \{\langle u, v \rangle : \ell\langle u, v \rangle \leq a\} \cap \{\langle u, v \rangle : \ell\langle u, v \rangle \leq b\} = \\ &= \mathsf{rep}(a) \cap \mathsf{rep}(b). \end{aligned}$$

∎

Note that the above rep does not work for disjunction $\vee$. The reason for this is that $a \leq b \vee c$ does not imply that $a \leq b$ or $a \leq c$. Thus, it may happen that $\langle u, v \rangle \in \mathsf{rep}(b \vee c)$ while $\langle u, v \rangle \notin \mathsf{rep}(b) \cup \mathsf{rep}(c)$.

Now we give the necessary definitions and the non-finite axiomatizability theorem that we used in the proof of Theorem 2.1.10.

Let RRD denote the class of all RRC's endowed with the operation of taking union. That is, an ordered algebra $\langle A, \circ, \backslash, /, \cap, \cup, \subseteq \rangle$ is in RRD iff $\langle A, \circ, \backslash, /, \cap, \subseteq \rangle \in$ RRC and A is closed under taking union.

The axioms and sequent rules of Q (cf. Theorem 2.1.10) translate into a finite set Δ of equational implications (or, in other words, quasi-equations) in the language of RRD, by using the standard techniques in algebraic logic (see Section 1.3). For example, the sequent rule $(\vee l)$ would translate into the quasi-equation

$$x \leq z \,\&\, y \leq z \Rightarrow (x \vee y) \leq z.$$

Then Δ defines a class QRS of algebras which is analogous to ORS in that it reflects $\vdash_{\mathsf{Q}}$, i.e., for any Γ and φ,

$$\Gamma \vdash_{\mathsf{Q}} \varphi \quad \text{iff} \quad \Gamma \models_{\mathsf{QRS}} \varphi.$$

Soundness of Q implies that $\mathsf{RRD} \models \Delta$, i.e., that $\mathsf{RRD} \subseteq \mathsf{QRS}$.

Next we show that results of [An91] imply that the quasi-equational theory of RRD is not finitely axiomatizable, i.e., there is a quasi-equation q such that

$$\mathsf{RRD} \models q \quad \text{and} \quad \mathsf{QRS} \not\models q.$$

Theorem 2.2.5. *The quasi-equational theory of* RRD *is not finitely axiomatizable.*

Proof: Let us assume that there is a finite set Δ of quasi-equations axiomatizing the quasi-equational theory of RRD.

In the proof of Theorem 4 in [An91], a sequence of algebras $\mathfrak{A}_n$, and quasi-equations q_n are defined for which the following hold:

- the operations of $\mathfrak{A}_n$ are $\vee, \wedge, \bullet, -, \smile, 0, 1'$ the first three being binary, the next two unary, and the last two are constants;
- q_n contains only the operation symbols $\vee, \wedge, \bullet$, and $\mathsf{RRD} \models q_n$ while $\mathfrak{A}_n \not\models q_n$;
- any non-principal ultraproduct of the $\mathfrak{A}_n$'s is isomorphic to a *relation set algebra* on some set U, i.e., to an algebra $\mathfrak{B} = \langle B, \cup, \cap, \circ, \sim, ^{-1}, \emptyset, \mathsf{Id}\rangle$ where B is a set of binary relations on U, $\sim$ denotes the operation of taking complement w.r.t. $U \times U$, $^{-1}$ denotes the operation of taking converse of a relation (i.e., $R^{-1} = \{\langle a, b\rangle : \langle b, a\rangle \in R\}$), and Id is the identity relation on U (i.e., $\mathsf{Id} = \{\langle u, u\rangle : u \in U\}$).

For any n, we define the ordered algebra $\mathfrak{A}'_n$ as follows:

$$\mathfrak{A}'_n = \langle A_n, \bullet, \backslash, /, \wedge, \vee, \leq\rangle$$

where A_n is the universe of $\mathfrak{A}_n$, $\vee, \wedge, \bullet$ are the original operations of $\mathfrak{A}_n$, while $\backslash, /, \leq$ are defined from the original operations of $\mathfrak{A}_n$ as follows:

$$a \backslash b = -(a^{\smile} \bullet (-b))$$
$$a / b = -((-a) \bullet b^{\smile})$$
$$a \leq b \quad \text{iff} \quad a \wedge b = a.$$

Let $\mathfrak{B}'$ denote the algebra we obtain from $\mathfrak{B}$ likewise. Then $\mathfrak{B}' \in \mathsf{RRD}$, $\mathfrak{B}$ is isomorphic to a non-principal ultraproduct of the $\mathfrak{A}'_n$'s and $\mathfrak{A}'_n \not\models q_n$. By $\mathfrak{B}' \in \mathsf{RRD}$ we have $\mathfrak{B}' \models \Delta$, and then by Δ being finite we have $\mathfrak{A}'_n \models \Delta$ for some n, i.e., $\mathfrak{A}'_n \in \mathsf{QRS}$. Let $q = q_n$. Then $\mathsf{QRS} \not\models q$ by $\mathfrak{A}'_n \not\models q_n$, but $\mathsf{RRD} \models q$, contradiction. ∎

Remark 2.2.6. We note that the above theorem remains true if we add any operation expressible in relation set algebras, i.e., expressible from $\cup, \cap, \circ, ^{-1}$, and Id, to the set of operations of RRD. ∎

2.2.3 Representation with squares

Let RRS^+ be the *square* version of RRS: $\mathfrak{A} = \langle A, \bullet, \backslash, /, \leq \rangle \in \mathsf{RRS}^+$ iff there is a set U such that

1. A is a set of binary relations on U,
2. $\bullet, \backslash, /$ are binary operations on A coinciding with $\circ, \backslash_{U\times U}, /_{U\times U}$, respectively,
3. $\leq$ is a binary relation on A coinciding with $\subseteq$.

We note that in general $W = \bigcup A \neq U \times U$, all we can know is that W is a reflexive, transitive relation on U (i.e., $\{\langle u, u\rangle : u \in U\} \subseteq W$ and $W \circ W \subseteq W$). Yet, $R\backslash_W S = R\backslash_{U\times U} S$ for all $R, S \in A$, by $R\backslash_{U\times U} S \subseteq W$. Thus, $\mathsf{RRS}^+ \subseteq \mathsf{RRS}$.

Let Σ^+ be Σ together with the following four formulas.

$$x \leq y \Rightarrow z \leq z \bullet (x\backslash y) \qquad x \leq y \Rightarrow z \leq (x\backslash y) \bullet z$$
$$x \leq y \Rightarrow z \leq z \bullet (y/x) \qquad x \leq y \Rightarrow z \leq (y/x) \bullet z$$

The following theorem says that RRS^+ is axiomatized by the above four axioms together with the axioms for RRS.

Theorem 2.2.7. *For every* $\mathfrak{A} \in \mathsf{RS}$,

$$\mathfrak{A} \models \Sigma^+ \quad \textit{iff} \quad \mathfrak{A} \in \mathbf{I}\mathsf{RRS}^+.$$

Proof: The 'if' part is easy and omitted.

Assume that $\mathfrak{A} \models \Sigma^+$. We will construct, as in the case of Theorem 2.2.3, a directed and labeled graph, and we will define the representation function using this graph.

Let $G = \langle V, E, \ell\rangle$, where V is the set of nodes, $E = V \times V$ is the set of edges and $\ell : E \longrightarrow \mathcal{P}(A)$ is the labeling function. So one difference from the proof of Theorem 2.2.3 is that G is a full graph, and another difference is that we label with sets of elements of A, and not only with elements of A.

G will have the following five properties. (I) and (II) ensure that the labeling respects composition. (III) and (IV) take care of the residuals, and (V) corresponds to the new axioms: $a\backslash b$ and b/a contain the identity relation whenever $a \leq b$.

$$\begin{array}{rl}
\text{(I)} & (\forall u, v, w \in V)(\forall a, b)(a \in \ell\langle u, w\rangle \;\&\; b \in \ell\langle w, v\rangle \Rightarrow \\
& \exists c(c \leq a \bullet b \;\&\; c \in \ell\langle u, v\rangle)) \\
\text{(II)} & (\forall u, v \in V)(\forall a, b, c \in A)(a \leq b \bullet c \;\&\; a \in \ell\langle u, v\rangle \Rightarrow \\
& (\exists w \in V) b \in \ell\langle u, w\rangle \;\&\; c \in \ell\langle w, v\rangle) \\
\text{(III)} & (\forall u \in V)(\forall a \in A)\exists w(a \in \ell\langle w, u\rangle \;\&\; \\
& (\forall v \in V) u \neq v \Rightarrow \ell\langle w, v\rangle = \{a \bullet h : h \in \ell\langle u, v\rangle\}) \\
\text{(IV)} & (\forall v \in V)(\forall a \in A)\exists w(a \in \ell\langle v, w\rangle \;\&\; \\
& (\forall u \in V) u \neq v \Rightarrow \ell\langle u, w\rangle = \{h \bullet a : h \in \ell\langle u, v\rangle\}) \\
\text{(V)} & (\forall u \in V)\ell\langle u, u\rangle \supseteq I, \text{ where } I = \{a\backslash b : a \leq b\} \cup \{b/a : a \leq b\}
\end{array}$$

We will define G by recursion. Let κ and σ be as in the proof of Theorem 2.2.3. We will use the following notation. If $X, Y \subseteq A$, then we let $X \bullet Y = \{x \bullet y : x \in X,\ y \in Y\}$.

0TH STEP. Let $V_0 = \{u_a, v_a : a \in A\}$, $E_0 = V_0 \times V_0$ and $W = \{\langle u_a, v_a\rangle, \langle u_a, u_a\rangle, \langle v_a, v_a\rangle : a \in A\}$ where u_a, v_a $(a \in A)$ are all different. Moreover, let $\ell_0\langle u_a, v_a\rangle = \{a\}$ and $\ell_0\langle u_a, u_a\rangle = \ell_0\langle v_a, v_a\rangle = I$, and let $\ell_0\langle u, v\rangle = \emptyset$ if $\langle u, v\rangle \in (V_0 \times V_0) \smallsetminus W$.

(I) holds because of the new formulas in Σ^+, and (V) is satisfied as well.

$\alpha + 1$ST STEP. Let $\sigma(\alpha+1) = \langle a, b, c, x, y, i\rangle$. We have three subcases according to the value of i.

$i = 0$. See Figure 2.9. Let z be a new point $(z \notin V_\alpha)$, and let

$$\begin{aligned}
V_{\alpha+1} &= V_\alpha \cup \{z\} \\
E_{\alpha+1} &= V_{\alpha+1} \times V_{\alpha+1} \\
\ell_{\alpha+1} &= \ell_\alpha \cup \{\langle\langle z, z\rangle, I\rangle, \langle\langle z, x\rangle, \{a\} \bullet \ell_\alpha\langle x, x\rangle \cup \{a\}\rangle\} \cup \\
&\quad \{\langle\langle z, p\rangle, \{a\} \bullet \ell_\alpha\langle x, p\rangle\rangle : p \in V_\alpha \ \&\ p \neq x\} \cup \{\langle\langle p, z\rangle, \emptyset\rangle : p \in V_\alpha\}.
\end{aligned}$$

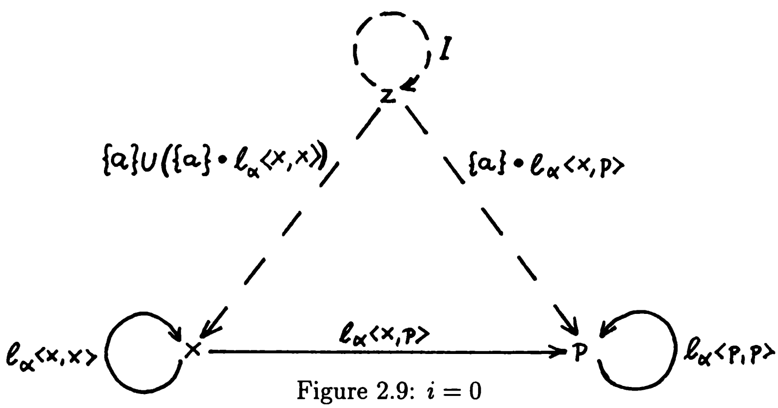

Figure 2.9: $i = 0$

$i = 1$. See Figure 2.10. Let z be a new point, and let

$$\begin{aligned}
V_{\alpha+1} &= V_\alpha \cup \{z\} \\
E_{\alpha+1} &= V_{\alpha+1} \times V_{\alpha+1} \\
\ell_{\alpha+1} &= \ell_\alpha \cup \{\langle\langle z, z\rangle, I\rangle,\ \langle\langle y, z\rangle, \ell_\alpha\langle y, y\rangle \bullet \{a\} \cup \{a\}\rangle\} \cup \\
&\quad \{\langle\langle q, z\rangle, \ell_\alpha\langle q, y\rangle \bullet \{a\}\rangle : q \in V_\alpha \ \&\ q \neq y\} \cup \{\langle\langle z, q\rangle, \emptyset\rangle : q \in V_\alpha\}.
\end{aligned}$$

$i = 2$. See Figure 2.11. If $a \not\leq b \bullet c$, or $a \notin \ell_\alpha\langle x, y\rangle$, then let $G_{\alpha+1} = G_\alpha$. Otherwise let z be a new point, and let

$$\begin{aligned}
V_{\alpha+1} &= V_\alpha \cup \{z\} \\
E_{\alpha+1} &= V_{\alpha+1} \times V_{\alpha+1} \\
\ell_{\alpha+1} &= \ell_\alpha \cup \{\langle\langle z, z\rangle, \{c\} \bullet \ell_\alpha\langle y, x\rangle \bullet \{b\} \cup I\rangle\} \cup \\
&\quad \{\langle\langle r, z\rangle, \ell_\alpha\langle r, x\rangle \bullet \{b\}\rangle : r \in V_\alpha \ \&\ r \neq x\} \cup \\
&\quad \{\langle\langle z, s\rangle, \{c\} \bullet \ell_\alpha\langle y, s\rangle\rangle : s \in V_\alpha \ \&\ s \neq y\} \cup \\
&\quad \{\langle\langle x, z\rangle, \ell_\alpha\langle x, x\rangle \bullet \{b\} \cup \{b\}\rangle\} \cup \{\langle\langle z, y\rangle, \{c\} \bullet \ell_\alpha\langle y, y\rangle \cup \{c\}\rangle\}.
\end{aligned}$$

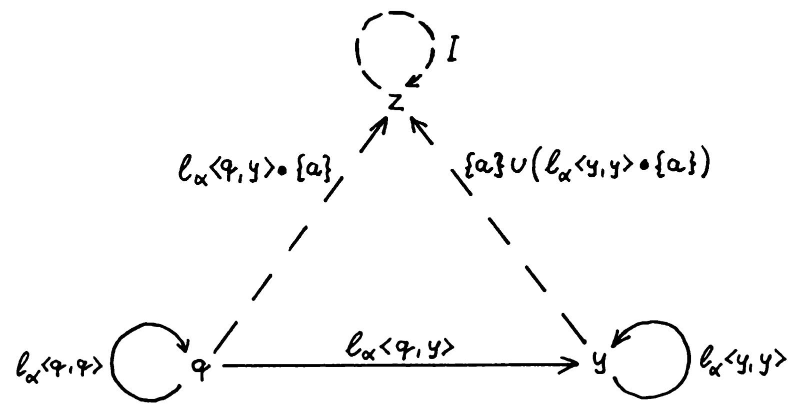

Figure 2.10: $i = 1$

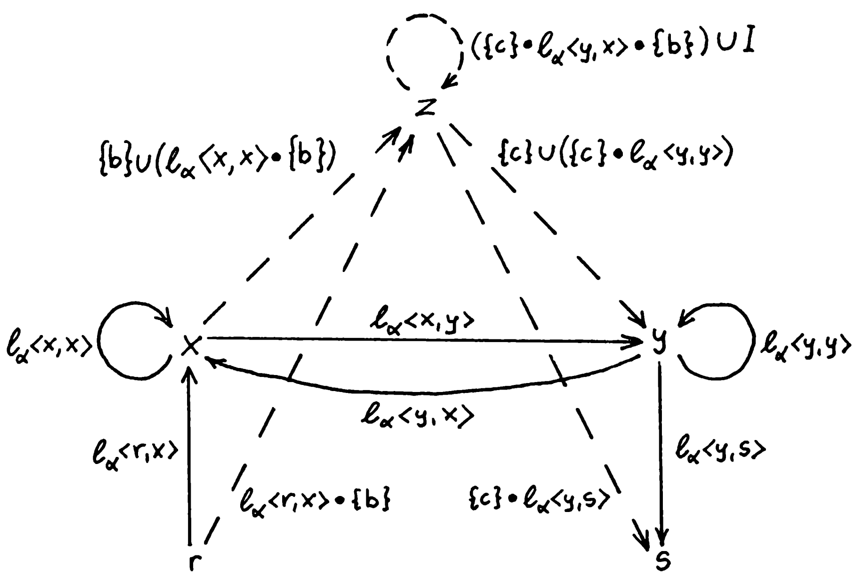

Figure 2.11: $i = 2$

LIMIT STEP. If α is a limit ordinal, then let

$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta, \qquad E_\alpha = \bigcup_{\beta<\alpha} E_\beta, \qquad \ell_\alpha = \bigcup_{\beta<\alpha} \ell_\beta.$$

We note that, if in the case $i = 2$ we have $x = y$, then $\ell_{\alpha+1}\langle x, z\rangle \neq \emptyset$, $\ell_{\alpha+1}\langle z, x\rangle \neq \emptyset$, hence we may not assume that G is directed in the sense that $(\forall u, v \in V, u \neq v)[\ell\langle u, v\rangle = \emptyset$ or $\ell\langle v, u\rangle = \emptyset]$. Because of this, in the case $i = 2$, we also may have $\ell_{\alpha+1}\langle z, z\rangle \supset I$.

Let $G = G_\kappa$. Clearly, G satisfies (V). G also satisfies (I), since in each step this property was preserved; checking this is a mechanical and tiresome calculation. As an example, we check one case. Assume we are in case $i = 2$, with the above notation, we want to check the triangle xxz. Assume $d \in \ell_\alpha\langle y, x\rangle$, we want to show that $e \leq b \bullet (c \bullet d)$ for some $e \in \ell_\alpha\langle x, x\rangle$. (See Figure 2.12.) Indeed, $a \in \ell_\alpha\langle x, y\rangle$, thus $e \leq a \bullet d$ for some $e \in \ell_\alpha\langle x, x\rangle$ by our induction hypothesis, thus $e \leq a \bullet d \leq (b \bullet c) \bullet d = b \bullet (c \bullet d)$ by $a \leq b \bullet c$ and Σ^+. Further, (II), (III) and (IV) hold for G by the construction.

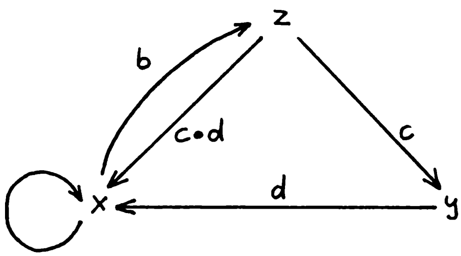

Figure 2.12: checking (I)

Let, for every $a \in A$,

$$\mathsf{rep}(a) = \{\langle u, v\rangle : (\exists h \in \ell\langle u, v\rangle) h \leq a\}.$$

Then rep clearly preserves $\leq$, and is one-one because of the 0th step in the construction.

Now we show that rep is a homomorphism. First we show that

$$\mathsf{rep}(a) \circ \mathsf{rep}(b) = \mathsf{rep}(a \bullet b).$$

Indeed, if $\langle u, v\rangle \in \mathsf{rep}(a) \circ \mathsf{rep}(b)$, then

$$\exists w((\exists h_a \in \ell\langle u, w\rangle) h_a \leq a \;\&\; (\exists h_b \in \ell\langle w, v\rangle) h_b \leq b)$$

and, by (I),

$$\exists w(\exists h \in \ell\langle u, v\rangle)(\exists h_a \in \ell\langle u, w\rangle)(\exists h_b \in \ell\langle w, v\rangle) h \leq h_a \bullet h_b \leq a \bullet b,$$

i.e., $\langle u, v\rangle \in \mathsf{rep}(a \bullet b)$. The other direction is a straightforward consequence of (II).

We also have

$$\mathsf{rep}(a\backslash b) \subseteq \mathsf{rep}(a)\backslash\mathsf{rep}(b),$$

since if $\langle u, v\rangle \in \mathsf{rep}(a\backslash b)$, then $(\exists h \in \ell\langle u, v\rangle)h \leq a\backslash b$, so, by (I),

$$\forall w((\exists h_a \in \ell\langle w, u\rangle)h_a \leq a \Rightarrow (\exists h' \in \ell\langle w, v\rangle)h' \leq a \bullet (a\backslash b) \leq b),$$

i.e., $\forall w\langle w, u\rangle \in \mathsf{rep}(a) \Rightarrow \langle w, v\rangle \in \mathsf{rep}(b)$, whence $\langle u, v\rangle \in \mathsf{rep}(a)\backslash\mathsf{rep}(b)$.

To show that

$$\mathsf{rep}(a)\backslash\mathsf{rep}(b) \subseteq \mathsf{rep}(a\backslash b)$$

we have to distinguish two cases. In the first case, we assume that $u \neq v$ and $\langle u, v\rangle \in \mathsf{rep}(a)\backslash\mathsf{rep}(b)$. Then

$$\forall w(\langle w, u\rangle \in \mathsf{rep}(a) \Rightarrow \langle w, v\rangle \in \mathsf{rep}(b)),$$

i.e.,

$$\forall w((\exists h_a \in \ell\langle w, u\rangle)h_a \leq a \Rightarrow (\exists h_b \in \ell\langle w, v\rangle)h_b \leq b),$$

so, by (III),

$$\exists w((\exists h \in \ell\langle u, v\rangle)(\exists h_b \in \ell\langle w, v\rangle)a \bullet h = h_b \leq b).$$

Thus $(\exists h \in \ell\langle u, v\rangle)a \bullet h \leq b$, so $(\exists h \in \ell\langle u, v\rangle)h \leq a\backslash b$, i.e., $\langle u, v\rangle \in \mathsf{rep}(a\backslash b)$.

Now we assume that $u = v$, i.e., $\langle u, u\rangle \in \mathsf{rep}(a)\backslash\mathsf{rep}(b)$. By the construction $\exists w(\ell\langle w, u\rangle = \{a\} \bullet \ell\langle u, u\rangle \cup \{a\})$, so we conclude that

$$\exists w(a \leq b \text{ or } (\exists h \in \ell\langle u, u\rangle)(\exists h_b \in \ell\langle w, u\rangle)a \bullet h = h_b \leq b).$$

Then, by (V), and because $\ell\langle u, u\rangle \subseteq I$, $(\exists h \in \ell\langle u, u\rangle)h \leq a\backslash b$, i.e., $\langle u, u\rangle \in \mathsf{rep}(a\backslash b)$.

Similar argument, using (IV), shows that

$$\mathsf{rep}(a/b) = \mathsf{rep}(a)/\mathsf{rep}(b).$$

Let $B = \{\mathsf{rep}(a) : a \in A\}$. Then B is a set of binary relations on V, by the definition of rep. Also, B is closed under the operations $\circ, \backslash_{V\times V}, /_{V\times V}$ because $\mathfrak{A}$ is closed under $\bullet, \backslash, /$ and rep is a homomorphism w.r.t. these operations. (That is, we checked that $\mathsf{rep}(a\backslash b) = \mathsf{rep}(a)\backslash_{V\times V}\mathsf{rep}(b)$ etc. for all $a, b \in A$). Thus $\mathfrak{B} = \langle B, \circ, \backslash, /, \subseteq\rangle \in \mathsf{RRS}^+$, and rep is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. Therefore, Theorem 2.2.7 has been proved. ∎

Remark 2.2.8. We note that, just in the case of ORS, we can add $\wedge$ to the set of operations without losing finite axiomatizability. Actually, Σ^+ plus (A8) works as a set of axioms.

On the other hand, we cannot add $\vee$ without losing finite axiomatizability, since the algebras defined in the proof of Theorem 2.2.5 have $U \times U$ top elements. ∎

In the representation theorem we used to prove Theorem 2.1.13(ii), we have two additional constants $e, 0$ denoting the identity relation and the empty relation, respectively.

RRS^0 denotes the class of all RRS's expanded with $\mathsf{Id}, \emptyset$ as constants, i.e.,

$$\mathfrak{A} = \langle A, \bullet, \backslash, /, e, 0, \leq\rangle \in \mathsf{RRS}^0$$

iff $\langle A, \bullet, \backslash, /, \leq\rangle \in \mathsf{RRS}^{+}$ and $e = \mathsf{Id} = \{\langle u,u\rangle : (\exists R \in A)\langle u,u\rangle \in R\}$ and 0 is the empty set $\emptyset$.

RS^0 denotes the class of all ordered algebras expanded with two constants, i.e., $\mathfrak{A} = \langle A, \bullet, \backslash, /, e, 0, \leq\rangle \in \mathsf{RS}^0$ iff $\langle A, \bullet, \backslash, /, \leq\rangle \in \mathsf{RS}$ and $e, 0 \in A$.

Let Σ^0 be Σ together with

$$e \bullet x = x \bullet e = x \qquad 0 \bullet x = x \bullet 0 = 0 \qquad 0 \leq x.$$

Let Δ be the set of the following formulas

$$x \bullet y = 0 \Longleftrightarrow (x = 0 \text{ or } y = 0)$$
$$x \bullet y \leq e \Longleftrightarrow (x = 0 \text{ or } y = 0 \text{ or } x = y = e).$$

Note that Δ is not valid in RRS^0 (while Σ^0 is). That is why, in the following theorem, only one direction is stated.

Theorem 2.2.9. *For every* $\mathfrak{A} \in \mathsf{RS}^0$,

$$\mathfrak{A} \models \Sigma^0 \cup \Delta \quad \textit{implies} \quad \mathfrak{A} \in \mathsf{IRRS}^0.$$

Proof: We make essentially the same construction as in the proof of Theorem 2.2.7 with some modifications.

We will construct a directed and labeled graph, $G = \langle V, E, \ell\rangle$, satisfying the following six properties. Properties (I), (II) and (V) will be the same as in the proof of Theorem 2.2.7. We require properties (III) and (IV) only for $a \in A \setminus \{e, 0\}$. The graph will have this feature too:

$$\text{(VI)} \quad (\forall\langle u,v\rangle \in E)0 \notin \ell\langle u,v\rangle \ \&\ (e \in \ell\langle u,v\rangle \Rightarrow u = v).$$

Let σ and κ be as before. We define the graph by recursion using the original construction in the proof of Theorem 2.2.7.

0TH STEP. This is the same as before, we just choose v_a and u_a for $a \in A \setminus \{e, 0\}$ only.

$\alpha + 1$ST STEP. Let $\sigma(\alpha+1) = \langle a, b, c, x, y, i\rangle$. We have three subcases according to the value of i again.

$i = 0$ or $i = 1$. Do the original construction, provided $a \notin \{0, e\}$. Otherwise, let $G_{\alpha+1} = G_\alpha$.

$i = 2$. If $a \not\leq b \bullet c$, or $a \notin \ell_\alpha\langle x,y\rangle$, then let $G_{\alpha+1} = G_\alpha$. Otherwise, by property (VI), we have that $0 \notin \{b, c\}$. If $b = e$ or $c = e$, then let $G_{\alpha+1} = G_\alpha$. Otherwise, by Δ, $b \not\leq e$ and $c \not\leq e$. In this case, do the original construction.

LIMIT STEP. Take the union as before.

Let $G = G_\kappa$. Then properties (I)–(V) are achieved. (This is an easy consequence of the original construction.) Further, (VI) is clearly preserved in each step.

Let

$$\mathsf{rep}(a) = \{\langle u,v\rangle : (\exists h \in \ell\langle u,v\rangle)h \leq a\}.$$

It is easy to prove, using property (VI), that $\mathsf{rep}(e) = \{\langle u,u\rangle : u \in V\}$, $\mathsf{rep}(0) = \emptyset$ and $\mathsf{rep}(0\backslash y) = \mathsf{rep}(x/0) = V \times V$. The other cases, in checking that rep is an isomorphism, are the same as in the proof of Theorem 2.2.7. ∎

2.3 THE LAMBEK BRIDGE

In this section, we state and prove several lemmas explaining the connections between the logics and the algebras of this chapter.

First we show that the Lindenbaum–Tarski algebra of LC is in the class ORS, cf. Definition 2.1.7 and Definition 2.2.2.

Lemma 2.3.1. *$\mathfrak{L}_\Gamma$ is an ordered residuated semigroup, i.e., $\mathfrak{L}_\Gamma \models \Sigma$, for any set Γ of sequents.*

Proof: $\leq_\Gamma$ is an ordering on L because it is reflexive and transitive by $(LC0), (LC1)$, and it is asymmetric because we factorized by $\equiv_\Gamma$.

Later in this proof we will use the following $(*)$ several times: for any $A, B, C \in$ $\mathrm{Form}_{\mathrm{LC}}$,

$$(*) \qquad \Gamma \vdash_{\mathrm{LC}} A, B \to C \quad \text{iff} \quad \Gamma \vdash_{\mathrm{LC}} A \bullet B \to C.$$

Indeed, the 'only if' direction follows immediately by using $(LC \bullet l)$, while the other direction follows from $(LC0), (LC \bullet l), (LC1)$.

To show associativity of $\bullet$, we get $\Gamma \vdash_{\mathrm{LC}} A, B, C \to (A \bullet B) \bullet C$ by using $(LC0), (LC \bullet r)$, then we get $\Gamma \vdash_{\mathrm{LC}} A \bullet (B \bullet C) \to (A \bullet B) \bullet C$ by using $(LC \bullet l)$ twice. The proof of $\Gamma \vdash_{\mathrm{LC}} (A \bullet B) \bullet C \to A \bullet (B \bullet C)$ is similar.

Monotonicity of $\bullet$: From $\Gamma \vdash_{\mathrm{LC}} A \to B$ by $(LC0)$ and $(LC \bullet r)$ we get $\Gamma \vdash_{\mathrm{LC}} A, C \to B \bullet C$, from which $\Gamma \vdash_{\mathrm{LC}} A \bullet C \to B \bullet C$ by $(*)$. To show monotonicity in the other argument is similar.

Residual property: Assume $\Gamma \vdash_{\mathrm{LC}} A \bullet B \to C$. Then $\Gamma \vdash_{\mathrm{LC}} A, B \to C$ by $(*)$, hence $\Gamma \vdash_{\mathrm{LC}} B \to A \backslash C$ by $(LC\backslash r)$. Assume now $\Gamma \vdash_{\mathrm{LC}} B \to A \backslash C$. Then $\Gamma \vdash_{\mathrm{LC}} A \bullet B \to A \bullet (A \backslash C)$ by monotonicity of $\bullet$. By $(LC0), (LC\backslash l)$ it is easy to show $\Gamma \vdash_{\mathrm{LC}} A \bullet (A \backslash C) \to C$, so one application of $(LC1)$ gives $\Gamma \vdash_{\mathrm{LC}} A \bullet B \to C$. The proof for / is completely analogous. ∎

For any class $\mathsf{K} \subseteq \mathsf{RS}$ we define a semantics $\models_{\mathsf{K}}$ for LC, as follows. A K-model for LC is a pair $\langle \mathfrak{G}, v \rangle$ where $\mathfrak{G} \in \mathsf{K}$ and $v : \mathfrak{F} \longrightarrow \mathfrak{G}$ is a homomorphism (we recall that $\mathfrak{F}$ is the formula algebra of LC), i.e., if $\mathfrak{G} = \langle G, \bullet, \backslash, /, \leq \rangle$, then $v : \mathrm{Form}_{\mathrm{LC}} \longrightarrow G$ such that for any $A, B \in \mathrm{Form}_{\mathrm{LC}}$, $v(A \bullet B) = v(A) \bullet v(B)$, $v(A \backslash B) = v(A) \backslash v(B)$, $v(A/B) = v(A)/v(B)$. $M(\mathsf{K})$ denotes the class of all K-models of LC. Let φ be a sequent, say, φ is $A_1, \ldots, A_n \to A_0$, and let $\mathfrak{M} = \langle \mathfrak{G}, v \rangle \in M(\mathsf{K})$. Then we define $\mathfrak{M} \models \varphi$ iff $v((A_1 \bullet A_2) \ldots \bullet A_n) \leq v(A_0)$ in $\mathfrak{G}$. Let Γ be a set of sequents, φ a sequent of LC. Then $\mathfrak{M} \models \Gamma$ iff $\mathfrak{M} \models \psi$ for all $\psi \in \Gamma$, and $\Gamma \models_{\mathsf{K}} \varphi$ iff $\mathfrak{M} \models \varphi$ for all $\mathfrak{M} \in M(\mathsf{K})$ such that $\mathfrak{M} \models \Gamma$. We note that K and **I**K define the same semantical consequence relation.

Lemma 2.3.2. *Let Γ be a set of sequents and let φ be a sequent of* LC. *Then* (i), (ii) *below hold.*

(i) $\Gamma \models_{\mathsf{RRS}} \varphi$ *iff* $\Gamma \models_{\mathsf{R}} \varphi$.
(ii) $\Gamma \models_{\mathsf{ORS}} \varphi$ *iff* $\Gamma \vdash_{\mathrm{LC}} \varphi$.

Proof: (i): Assume $\Gamma \models_{\mathsf{R}} \varphi$ and let $\mathfrak{M} \in M(\mathsf{RRS})$ be such that $\mathfrak{M} \models \Gamma$. We want to show $\mathfrak{M} \models \varphi$. Let $\mathfrak{M} = \langle \mathfrak{G}, v\rangle$ with $\mathfrak{G} = \langle G, \circ, \backslash_W, /_W, \subseteq\rangle \in \mathsf{RRS}$, $W = \bigcup G$. Then W is a transitive relation, hence $\mathfrak{W} = \langle W, C, v'\rangle$ is a relational model for LC (here, v' denotes the restriction of v to P: $v\lceil P$). For any $A \in \mathrm{Form}_{\mathrm{LC}}$ define

$$w(A) = \{\langle a, b\rangle \in W : \langle a, b\rangle \Vdash A\}.$$

Then by the definition of $\Vdash$ we immediately have that $w(A \bullet B) = w(A) \circ w(B)$, $w(A\backslash B) = w(A)\backslash_W w(B)$, $w(A/B) = w(A)/_W w(B)$. Thus $w = v$ because $w\lceil P = v\lceil P$ and v is a homomorphism. Hence $\mathfrak{M} \models \psi$ iff $\mathfrak{W} \models \psi$, for any sequent ψ of LC. Thus $\mathfrak{W} \models \Gamma$ by $\mathfrak{M} \models \Gamma$, hence $\mathfrak{W} \models \varphi$ by $\Gamma \models_{\mathsf{R}} \varphi$, hence $\mathfrak{M} \models \varphi$ by $\mathfrak{W} \models \varphi$, and we are done. The proof of the other direction is very similar, we omit it.

(ii): Assume $\Gamma \models_{\mathsf{ORS}} \varphi$, we want to show $\Gamma \vdash_{\mathrm{LC}} \varphi$. Let $v(B) = B/\equiv_\Gamma$ for any $B \in \mathrm{Form}_{\mathrm{LC}}$. Then $\mathfrak{M} = \langle \mathfrak{L}_\Gamma, v\rangle \in M(\mathsf{ORS})$ by Lemma 2.3.1. Let ψ be any sequent of LC of the form $A_1, \ldots, A_n \to A_0$, and let $A = ((A_1 \bullet A_2) \ldots \bullet A_n)$. By definition, $\mathfrak{M} \models \psi$ iff $(A/\equiv_\Gamma \leq A_0/\equiv_\Gamma)$ in $\mathfrak{L}_\Gamma$. By a generalization of $(*)$ in the proof of Lemma 2.3.1, we also have that $\Gamma \vdash_{\mathrm{LC}} A \to A_0$ iff $\Gamma \vdash_{\mathrm{LC}} \psi$. Thus for any sequent ψ,

$$(\star) \qquad \mathfrak{M} \models \psi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{LC}} \psi.$$

Now $\mathfrak{M} \models \Gamma$ by $(\star)$, since $\Gamma \vdash_{\mathrm{LC}} \psi$ for any $\psi \in \Gamma$, thus $\mathfrak{M} \models \varphi$ by $\Gamma \models_{\mathsf{ORS}} \varphi$ and $\mathfrak{M} \in M(\mathsf{ORS})$, and then $\Gamma \vdash_{\mathrm{LC}} \varphi$ by $(\star)$ again. The proof of the other direction goes by an easy induction along the steps of the $\vdash_{\mathrm{LC}}$-derivation. We omit that part. ∎

Next, we show that the Lindenbaum–Tarski algebra $\mathfrak{L}^0_\Gamma$ satisfies the axioms $\Sigma^0 \cup \Delta$, thus it is representable as an RRS^0, cf. Definition 2.1.14 and Theorem 2.2.9.

Lemma 2.3.3.

$$\mathfrak{L}^0_\Gamma \models \Sigma^0 \cup \Delta.$$

Proof: It is easy to check, using the definition, that $\leq_\Gamma$ is a partial ordering, and that $\bullet$ is an associative operation which is monotonic w.r.t. $\leq_\Gamma$. It is not difficult to show, by case distinction, that if $a \leq_\Gamma b$ and $c \leq_\Gamma d$, then $a \bullet c \leq_\Gamma b \bullet d$. The rest of Σ^0 is easy, by the definitions of the slashes, and Δ holds, by the definition of $\bullet$, as well. ∎

Now we show how QRS and RRD reflects $\vdash_{\mathsf{Q}}$ and $\models_{\mathsf{R}}$, respectively, cf. Theorem 2.1.10 and Theorem 2.2.5.

Lemma 2.3.4. (i) *For any set* $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash_{\mathsf{Q}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathsf{QRS}} \varphi$$
$$\Gamma \models_{\mathsf{R}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathsf{RRD}} \varphi.$$

(ii) *For any quasi-equation* q, *there is a set* $\Gamma \cup \{\varphi\}$ *of sequents such that*

$$\mathsf{QRS} \models q \quad \textit{iff} \quad \Gamma \models_{\mathsf{QRS}} \varphi$$
$$\mathsf{RRD} \models q \quad \textit{iff} \quad \Gamma \models_{\mathsf{RRD}} \varphi.$$

Proof: (i): It is easy to see, just as before, that for any Γ and φ,

$$\Gamma \models_{\mathsf{R}} \varphi \quad \text{iff} \quad \Gamma \models_{\mathsf{RRD}} \varphi$$

and

$$\Gamma \vdash_{\mathsf{Q}} \varphi \quad \text{iff} \quad \Gamma \models_{\mathsf{QRS}} \varphi.$$

(ii): Let q be a quasi-equation. By $[x = y \text{ iff } (x \leq y \ \& \ y \leq x)]$, we may assume that q is of the form $(\tau_1 \leq \sigma_1 \ \& \ \ldots \ \& \ \tau_n \leq \sigma_n) \Rightarrow \tau_0 \leq \sigma_0$. Now, by the standard translation techniques, τ_i, σ_i translate into formulas A_i, B_i (by just replacing the variables with primitive symbols in P). Let $\Gamma = \{A_1 \to B_1, \ldots, A_n \to B_n\}$ and φ be $A_0 \to B_0$. Then $\mathsf{RRD} \models q$ means that $\Gamma \models_{\mathsf{RRD}} \varphi$, while $\mathsf{QRS} \models q$ means that $\Gamma \models_{\mathsf{QRS}} \varphi$. ∎

Lemma 2.3.5. *Let $h : \mathfrak{L}^0_\Gamma \longrightarrow \mathfrak{M} = \langle M, \circ, \backslash, /, \mathsf{Id}, \emptyset, \subseteq \rangle \in \mathsf{RRS}^0$ be an isomorphism. Then there is a square relational model $\mathfrak{W} = \langle W, C, v \rangle$ such that, for every generalized sequent $A \to B$,*

$$h(A/{\equiv_\Gamma}) \subseteq h(B/{\equiv_\Gamma}) \quad \textit{iff} \quad \mathfrak{W} \models A \to B.$$

Proof: Let $W = U \times U$ where $U = \{u : \langle u, u \rangle \in R \text{ for some } R \in M\}$, and let $v(p) = h(p/{\equiv_\Gamma})$ for all $p \in P$. Then $\mathfrak{W}$ is a square relational model. It is easy to check by induction that, for all $x \in W$ and $A \in \mathrm{Form}_{\mathrm{LC}}$,

$$(**) \qquad x \Vdash A \quad \text{iff} \quad x \in h(A/{\equiv_\Gamma}).$$

Let $A, B \in \mathrm{Form}_{\mathrm{LC}}$. By the definition of $\mathfrak{L}^0_\Gamma$, we have that $\Gamma \vdash_{\mathrm{LC}^0} (A \to B)$ iff $(A/{\equiv_\Gamma}) \leq (B/{\equiv_\Gamma})$ in $\mathfrak{L}^0_\Gamma$, and by h being an isomorphism the second statement holds iff $h(A/{\equiv_\Gamma}) \subseteq h(B/{\equiv_\Gamma})$. By $(**)$ we have that $h(A/{\equiv_\Gamma}) \subseteq h(B/{\equiv_\Gamma})$ iff $\mathfrak{W} \models A \to B$.

Now look at the sequent $\to B$. The above argument remains valid if we replace A with the empty sequence, and $(A/{\equiv_\Gamma})$ with e (of $\mathfrak{L}^0_\Gamma$) in it. Thus we get

$$\Gamma \vdash_{\mathrm{LC}^0} (\to B) \quad \text{iff} \quad \mathfrak{W} \models (\to B).$$

Let finally ψ be any sequent of form $A_1, \ldots, A_n \to B$ with $n > 1$, and let $A = ((A_1 \bullet A_2) \bullet \ldots \bullet A_n)$. Then, it is easy to see that (cf. $(*)$ in the proof of Lemma 2.3.1)

$$\Gamma \vdash_{\mathrm{LC}^0} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{LC}^0} (A \to B),$$

and clearly $\mathfrak{W} \models \psi$ iff $\mathfrak{W} \models (A \to B)$, thus we obtained

$$\Gamma \vdash_{\mathrm{LC}^0} \psi \quad \text{iff} \quad \mathfrak{W} \models \psi,$$

for any generalized sequent ψ. ∎

3

Counting arrows

In this chapter[1], we will introduce our second (and more sophisticated) taming strategy. We will apply this strategy to find well-behaved versions of pair arrow logic.

The situation concerning pair arrow logic can be described as follows. Let us consider its square version, PAL_{sq}, cf. Definition 1.1.3. PAL_{sq} has undesirable metalogical properties: its validities form an undecidable set (cf., e.g., [AKNSS]), there is no Hilbert-style calculus which is strongly complete and strongly sound for PAL_{sq} (cf. [Mo64] for the equivalent algebraic result), and the Craig interpolation and Beth definability properties fail (cf., e.g., [Ma95]). Our aim is to find such a version of pair arrow logic that (a) its power is close to that of PAL_{sq}, and (b) it has nicer metalogical properties. The taming strategy will consist of two major steps.

(i) First, we will define weakened versions of pair arrow logic by widening the class of models of PAL_{sq}, and recall results from the literature stating that these weakened versions have nice properties: decidability, Hilbert-style completeness, Beth definability and Craig interpolation properties.

(ii) Next, we will strengthen these logics by (re-)introducing connectives (the difference operator and graded modalities) that are not definable after weakening. Of course, our goal is to make these strengthenings in such a way that (some of) the nice properties are preserved. We will concentrate on decidability and completeness, and prove that these features indeed remain true.

The (variety generated by the) algebraic counterpart of PAL_{sq} is the class of representable relation algebras, RRA, cf. Definition 3.2.1. Although RRA is a variety, it is not finitely axiomatizable, and its equational theory is undecidable, reflecting the fact that PAL_{sq} is incomplete and undecidable. We will define relativized versions of RRA (cf. Definition 3.2.1) that form finitely axiomatizable and decidable varieties. Then we will expand the similarity type of these algebras, and show that the above properties hold for this enriched language as well.

3.1 Arrow logic with graded modalities

Arrow logic as defined in [vB94] is intended to be the authentic basic "computational core" for logical systems used for reasoning about dynamic aspects of the subject matter of our thinking, e.g., about processes, actions, and programs. See also [Ve92] for linguistic motivations for arrow logic.

[1]This chapter is (partially) based on the following papers: [MMN94], [AMN94], and [MMNSi].

Since arrow logic is intended to "speak" about transitions, it is natural to give a semantics where the worlds are ordered pairs, reflecting the dynamic character of the logic. Further, it is a natural idea to define the frames for this pair version of arrow logic as Cartesian spaces, i.e., the universes have the form $U \times U$. The bad news is that the square version PAL_{sq} of pair arrow logic defined in the above way, cf. Definition 1.1.3, does not behave in a nice way. As we mentioned above, PAL_{sq} is undecidable, Hilbert-style incomplete, etc.

Thus, it is a natural question whether pair arrow logic has versions such that (a) they have nicer properties and (b) their power is close to PAL_{sq}. The results of [AKNSS] and Theorem 2.1.10 suggest that associativity of composition makes pair arrow logic undecidable and incomplete. If we insist on the transitivity of the frames for pair arrow logic (as in the case of PAL_{sq}), then composition is associative. Thus, to find nicer versions of pair arrow logic, one should apply to "non-square" approach.

FIRST TAMING STEP. First we widen the class of models by allowing frames whose universes are not necessarily Cartesian spaces. In the logics PAL_H ($H \subseteq \{r, s\}$), cf. Definition 1.1.3, associativity of composition does not hold, since the universes are arbitrary or reflexive and/or symmetric relations, and so are not necessarily transitive. Thus, there is a chance that these *relativized* versions have nicer properties. Indeed, for $H \subseteq \{r, s\}$, PAL_H has desirable metalogical properties as the following theorem states. For proofs and precise references we refer to [Ma95].

Theorem 3.1.1. *Let $H \subseteq \{r, s, t\}$ be arbitrary. Then*

1. PAL_H *has a strongly sound and strongly complete Hilbert-style calculus iff $t \notin H$;*
2. PAL_H *is decidable iff $t \notin H$;*
3. PAL_H *has the Craig interpolation property iff $t \notin H$;*
4. PAL_H *has the Beth definability property iff $t \notin H$.*

We mention that the above negative results hold for PAL_{sq} as well, for the logics PAL_{sq} and $\mathsf{PAL}_{\{r,s,t\}}$ are equivalent.

This way we achieved our first goal, i.e., we found well-behaved versions of pair arrow logic. The problem is that the above relativized versions PAL_H are remarkably weaker than PAL_{sq}. An example is the universal modality $\Diamond\varphi$ that is definable in PAL_{sq} as $\top \bullet \varphi \bullet \top$, but cannot be expressed on non-square frames. $\Diamond$ is important from theoretical point of view, since the deduction term is definable by means of $\Diamond$, cf. [Si92] for more details. This example motivates the following.

SECOND TAMING STEP. In this phase of taming, we try to strengthen the weakened logics by introducing new connectives. We will add the *difference operator* D to PAL_H, and show that Hilbert-style completeness and decidability are preserved, cf. Theorem 3.1.2 and Theorem 3.1.4. We will also expand the similarity type by the *graded modalities*. Decidability holds in this case too, cf. Theorem 3.1.4. Hilbert-style completeness is an open problem for these graded logics, but we will show that non-Hilbert-style weak completeness can be achieved, cf. Theorem 3.1.3.

Let us recall that the logics *graded pair arrow logic* ${}^{\kappa}\mathsf{PAL}_H^{grad}$ and *counting pair arrow logic* ${}^{\kappa}\mathsf{PAL}_H^{count}$ are defined using PAL_H as L in Definition 1.1.5. $\mathsf{PAL}_H^{\mathsf{D}}$ is defined in a similar way, using Definition 1.1.7. By Theorem 1.1.6, graded and counting PAL_H's are equivalent, and the same holds for $\mathsf{PAL}_H^{\mathsf{D}}$ and ${}^{3}\mathsf{PAL}_H^{grad}$.

Now, let us formulate the main results of this section.

Theorem 3.1.2. *There exist strongly sound and strongly complete Hilbert-style calculi for* ${}^{3}\mathsf{PAL}_{\{r,s\}}^{grad}$, ${}^{3}\mathsf{PAL}_{\{r,s\}}^{count}$ *and* $\mathsf{PAL}_{\{r,s\}}^{\mathsf{D}}$.

Theorem 3.1.3. *Let* $0 < \kappa \leq \omega$. ${}^{\kappa}\mathsf{PAL}_{\{r,s\}}^{grad}$ *and* ${}^{\kappa}\mathsf{PAL}_{\{r,s\}}^{count}$ *have weakly sound a weakly complete calculi.*

Note that, in the case of $\kappa = \omega$, the languages of ${}^{\kappa}\mathsf{PAL}_H^{grad}$ and ${}^{\kappa}\mathsf{PAL}_H^{count}$ contain infinitely many connectives. Thus, it is impossible to give a finite calculus (in the sense of Definition 1.1.10) for these logics. But we can slightly generalize the definition of a calculus by defining instances of formula schemata by allowing substitutions not just formulas (for formula variables) but connectives as well. For instance, the instances of the formula schema $\langle N\rangle A \to \langle N+1\rangle A$ are the formulas $\langle n\rangle\varphi \to \langle n+1\rangle\varphi$ for every formula φ and natural number n. Completeness for ${}^{\omega}\mathsf{PAL}_H^{grad}$ and ${}^{\omega}\mathsf{PAL}_H^{count}$ is meant in this sense in the theorem above. For $\kappa < \omega$ the original definition works.

Theorem 3.1.4. *Let* $0 < \kappa \leq \omega$. *Then the logics* ${}^{\kappa}\mathsf{PAL}_H^{grad}$, ${}^{\kappa}\mathsf{PAL}_H^{count}$, *and* $\mathsf{PAL}_H^{\mathsf{D}}$ *are decidable iff* $t \notin H$.

The proofs of the above theorems are based on the corresponding algebraic results of the following section, using the bridge developed in Section 1.3.

The following proposition tells us what the algebraic counterparts of different versions of PAL_H are, cf. Definition 3.2.1 for the definitions of the classes of algebras.

Proposition 3.1.5. *1.* $\mathsf{Alg}(\mathsf{PAL}_H^{\mathsf{D}}) = \mathbf{S}(\mathbf{Rl}_H\mathsf{RRA})^{\mathsf{D}}$.
2. $\mathsf{Alg}({}^{\kappa}\mathsf{PAL}_H^{grad}) = \mathbf{S}(\mathbf{Rl}_H\mathsf{RRA})^{<\kappa}$.

Proof: Straightforward. ∎

Now we are in the position to prove the main theorems of this section. The idea of the completeness proof is to use the completeness theorem for PAL_H, and then duplicating the worlds in the (canonical) model so that the accessibility relation of D becomes inequality.

Proof of Theorem 3.1.2: By Theorem 3.2.3 and Proposition 3.1.5 we get that $\mathbf{SP}\mathsf{Alg}(\mathsf{PAL}_{\{r,s\}}^{\mathsf{D}}) = \mathbf{SP}\mathsf{WAD}$ (cf. Definition 3.2.1) is a finitely axiomatizable variety. Then, by the equivalence Theorem 1.3.7, the logic $\mathsf{PAL}_{\{r,s\}}^{\mathsf{D}}$ has a strongly sound and strongly complete Hilbert-style calculus. Then so do the equivalent logics ${}^{3}\mathsf{PAL}_{\{r,s\}}^{grad}$ and ${}^{3}\mathsf{PAL}_{\{r,s\}}^{count}$. ∎

The following proof depends on the fact that every model for $\mathsf{PAL}_{\{r,s\}}^{\mathsf{D}}$ can be considered as a model for ${}^{\kappa}\mathsf{PAL}_{\{r,s\}}^{grad}$ by interpreting the connectives $\langle n\rangle$ in the standard way.

Proof of Theorem 3.1.3: This is an easy consequence of Theorem 3.2.17 and the bridge Theorem 1.3.11. ∎

The idea of the decidability proof is to construct a model from finitely many "small" partial models (called mosaics). Note that this method does not prove finite model property, since one mosaic may be used infinitely many times during the construction of the model. Actually, the finite model property is open even for PAL_H (H any subset of $\{r, s\}$).

Proof of Theorem 3.1.4: By Theorem 3.2.24 and Proposition 3.1.5, the equational theory of $\mathsf{Alg}({}^{\kappa}\mathsf{PAL}_H^{grad}) = \mathbf{S}(\mathbf{Rl}_H\mathsf{RRA})^{<\kappa}$ is decidable iff $t \notin H$. Then, by the equivalence Theorem 1.3.12, ${}^{\kappa}\mathsf{PAL}_H^{grad}$ is decidable iff $t \notin H$. Since ${}^{\kappa}\mathsf{PAL}_H^{grad}$ is equivalent to ${}^{\kappa}\mathsf{PAL}_H^{count}$, the same holds for the latter logic. We already saw, cf. Proposition 1.1.8, that $\mathsf{PAL}_H^{\mathsf{D}}$ is equivalent to ${}^{3}\mathsf{PAL}_H^{grad}$, thus the theorem holds for $\mathsf{PAL}_H^{\mathsf{D}}$ as well. ∎

3.2 Relativized relation algebras with counting operations

The algebraic counterpart of PAL_{sq} generates the variety RRA of *representable relation algebras*, cf. Definition 3.2.1. The unit of an RRA is an equivalence relation, which guarantees the associativity of composition. Associativity seems to cause both non-finite axiomatizability, cf., e.g., Theorem 2.1.10, and undecidability, cf. [AKNSS]. Indeed, if we allow not necessarily transitive units, then we get a finitely axiomatizable and decidable class, cf. [Kr91] and [Ma95] for proofs and a survey of related results. This class is called *relativized representable relation algebras* **Rl**RRA. Moreover, if we require that the unit is a symmetric and/or reflexive relation, finite axiomatizability and decidability remain true. These classes are denoted by $\mathbf{Rl}_H$RRA where $H \subseteq \{r, s\}$, and r and s abbreviate 'reflexive' and 'symmetric', respectively. If $H = \{r, s\}$, then we get the class of *weakly associative relation algebras* WA, since instead of the associativity axiom $x \circ (y \circ z) = (x \circ y) \circ z$ only its weakened version holds where one of the arguments is less than identity.[2] The reason for this is that in a **Rl**RRA the composition of two elements a and b is computed *relativized* to the unit W of the algebra: $a \circ b = \{\langle u, v\rangle \in W : \exists w(\langle u, w\rangle \in a \;\&\; \langle w, v\rangle \in b)\}$.

Thus, completeness and decidability can be regained. But what was the price that we had to pay for this? Well, we had to abandon associativity, a necessary sacrifice. But we also lost the discriminator term. Indeed, in RRA, $1 \circ x \circ 1$ defines a complemented closure operator, whence RRA is a discriminator variety.[3] On the other hand, in WA, $1 \circ (x \circ 1) = (1 \circ x) \circ 1$ does not hold and the discriminator term is not definable. But a discriminator term is really useful, e.g., using it we can code quasi-equations as

[2]Using the notation above $\mathsf{WA} = \mathbf{Rl}_{\{r,s\}}\mathsf{RRA}$.

[3]In **Sir**RRA the discriminator term τ and the complemented closure operator $\Diamond$ are mutually term-definable: $\tau(x, y, u, v) = (\Diamond(x \oplus y) \cdot v) + (u \cdot -\Diamond(x \oplus y))$ and $\Diamond x = \tau(x, 0, 0, 1)$.

equations.[4] Moreover, the complemented closure operator corresponds to the universal modality of the logic, and we already mentioned that expressibility of the universal modality implies the deduction theorem. Thus, the question of whether the loss of the discriminator term is a "must" naturally arises.

In this section, we will add the unary operations *at least n-times*, $\langle n\rangle$, $(n \in \omega \smallsetminus 1)$ and the *difference operator* D to $\mathbf{Rl}_H\mathsf{RRA}$, yielding the classes $(\mathbf{Rl}_H\mathsf{RRA})^{<\omega}$ and $(\mathbf{Rl}_H\mathsf{RRA})^{\mathsf{D}}$, respectively. We will also investigate reducts $(\mathbf{Rl}_H\mathsf{RRA})^{<n}$ of $(\mathbf{Rl}_H\mathsf{RRA})^{<\omega}$ where the operations $\langle i\rangle$ are included only to a certain bound $n \in \omega \smallsetminus 1$. If we consider WA instead of $\mathbf{Rl}_H\mathsf{RRA}$, then we get the classes $\mathsf{WA}^{<\omega}$, WAD, and $\mathsf{WA}^{<n}$, respectively. See Definition 3.2.1 below.

We will prove that the equational theories of $\mathsf{WA}^{<\omega}$ and $\mathsf{WA}^{<n}$ are decidable, which will yield the same result for WAD. An easy modification of this decidability proof yields the decidability of the classes obtained by replacing WA by $\mathbf{Rl}_H\mathsf{RRA}$, where H is any subset of $\{r, s\}$. On the other hand, if $t \in H$ (i.e., we relativize with transitive units), then decidability does not hold. Further, we will show that WAD (and the term-equivalent $\mathsf{WA}^{<3}$) generates a finitely axiomatizable discriminator variety, while the same question for $\mathsf{WA}^{<n}$ $(n > 3)$ and $(\mathbf{Rl}_H\mathsf{RRA})^{<n}$ $(2 < n$ and $H \subseteq \{r, s\})$ is still open. For ω only finite schema axiomatizability may hold, an interesting open problem. However, we will prove that the $\mathbf{SP}$-closure of $\mathsf{WA}^{<\alpha}$ $(\alpha \leq \omega)$ coincides with the $\mathbf{SP}$-closure of the subclass of singleton-dense (i.e., where there is a singleton atom below every non-zero element) members of a finitely schema axiomatizable variety.

Using D we can express the complemented closure operator $\Diamond x$, also denoted as $\langle 1\rangle x$, (as $x + \mathsf{D}x$), *once* $\langle 1!\rangle x$ (as $\Diamond(x \cdot -\mathsf{D}x)$) and *at least twice* $\langle 2\rangle x$ (as $\mathsf{D}(x \cdot \mathsf{D}x)$) that cannot be defined in WA. The expressive power of $\mathsf{WA}^{<n}$ is even stronger. With $\langle 1\rangle$ and $\langle 2\rangle$ we can define D: $\mathsf{D}x = (-x \cdot \langle 1\rangle x) + \langle 2\rangle x$; and we can express *precisely k-times* as well: $\langle k!\rangle x = \langle k\rangle x \cdot -\langle k+1\rangle x$ (for $k + 1 < n$).

Below we will define RRA (cf. [HMT85]), its relativized versions, and expansions with the difference operator and graded modalities.

Definition 3.2.1. (Set algebras of relations)

1. By a *relation set algebra*, an Rs, we mean an algebra

$$\mathfrak{A} \subseteq \langle \mathcal{P}(W), \cap, \sim, \circ, {}^{-1}, \mathsf{Id}\rangle$$

   where $W = U \times U$ is a non-empty set, $\cap$ is intersection, $\sim$ is complement w.r.t. $U \times U$, $\circ$ is relational composition, ${}^{-1}$ is relational converse, and Id is the identity relation on U. More formally, $a \circ b = \{\langle u, v\rangle \in W : \exists w(\langle u, w\rangle \in a \;\&\; \langle w, v\rangle \in b)\}$, $a^{-1} = \{\langle u, v\rangle \in W : \langle v, u\rangle \in a\}$ and $\mathsf{Id} = \{\langle u, v\rangle \in W : u = v\}$. We denote the class of all relation set algebras by Rs.
2. Let the class RRA of *representable relation algebras* be defined as

$$\mathsf{RRA} \stackrel{\text{def}}{=} \mathbf{SP}\mathsf{Rs}.$$

[4] $(\tau_1 = \sigma_1 \;\&\; \ldots \;\&\; \tau_n = \sigma_n) \Rightarrow \tau_0 = \sigma_0$ is equivalent, on subdirect irreducible algebras, to $\tau_0 \oplus \sigma_0 \leq \Diamond(\tau_1 \oplus \sigma_1) + \ldots + \Diamond(\tau_n \oplus \sigma_n)$.

3. Let r, s, t abbreviate 'reflexive', 'symmetric', and 'transitive', respectively, and let $H \subseteq \{r, s, t\}$. $\mathfrak{A}$ is a *full* $\mathbf{Rl}_H$RRA, a *full relativized representable relation algebra*, if
$$\mathfrak{A} = \langle \mathcal{P}(W), \cap, \sim, \circ, {}^{-1}, \mathsf{Id} \rangle,$$
where W is a non-empty binary relation satisfying the condition H. If $H = \{r, s\}$, we get the class of *(full) weakly associative relation algebras* WA.
4. The class $(\mathbf{Rl}_H\mathsf{RRA})^{\mathsf{D}}$ of algebras is defined as $\{\langle \mathfrak{A}, \mathsf{D}\rangle : \mathfrak{A}$ a full set $\mathbf{Rl}_H\mathsf{RRA}\}$, where the difference operator D on $\mathfrak{A}$ is defined as
$$\mathsf{D}a \overset{\text{def}}{=} \begin{cases} \emptyset & \text{if } a = \emptyset \\ \sim a & \text{if } |a| = 1 \\ W & \text{otherwise} \end{cases}$$
and W denotes the unit of $\mathfrak{A}$.
5. The class $(\mathbf{Rl}_H\mathsf{RRA})^{<\omega}$ is defined as $\{\langle \mathfrak{A}, \langle n\rangle\rangle_{n\in\omega\smallsetminus 1} : \mathfrak{A}$ a full set $\mathbf{Rl}_H\mathsf{RRA}\}$, where
$$\langle n\rangle a \overset{\text{def}}{=} \begin{cases} \emptyset & \text{if } |a| < n \\ W & \text{otherwise} \end{cases}$$
with unit W of $\mathfrak{A}$.
6. The class $(\mathbf{Rl}_H\mathsf{RRA})^{<n}$ $(n \in \omega \smallsetminus 1)$ is defined as the appropriate reduct of the class $(\mathbf{Rl}_H\mathsf{RRA})^{<\omega}$:
$$(\mathbf{Rl}_H\mathsf{RRA})^{<n} \overset{\text{def}}{=} \{\langle \mathfrak{A}, \langle k\rangle\rangle_{k\in n\smallsetminus 1} : \mathfrak{A} \text{ full set } \mathbf{Rl}_H\mathsf{RRA}\}.$$

Substituting WA for $\mathbf{Rl}_H$RRA, we get the classes WAD, $\mathsf{WA}^{<\omega}$, and $\mathsf{WA}^{<n}$, respectively. ∎

3.2.1 Axiomatization of WAD

Below we give a finite set Ax of equations axiomatizing the variety **SPWAD** generated by WAD. Let $\cdot, +, \oplus$, and $-$ denote Boolean meet, join, symmetric difference, and complement, respectively. We will denote abstract composition, converse, and identity by $;$, $\smile$, and id, respectively. Throughout, $\Diamond x \overset{\text{def}}{=} x + \mathsf{D}x$, $\mathsf{dom}x \overset{\text{def}}{=} (x;1)\cdot \mathsf{id}$ and $\mathsf{ran}x \overset{\text{def}}{=} (1;x)\cdot\mathsf{id}$. Instead of the equation $x \cdot y = x$, we will use $x \le y$. We note that axioms (1) and (D1)–(D4) axiomatize the class BAD of Boolean set algebras with the difference operator (defined analogously to WAD).

Definition 3.2.2. **(WAD-axioms)** Let Ax consist of the following equations:

1. axioms of WA:
   (1) Boolean axioms
   (2) $((x\cdot \mathsf{id});y);z = (x\cdot\mathsf{id});(y;z)$
   (3) $(x+y);z = (x;z)+(y;z)$
   (4) $x;\mathsf{id} = x$
   (5) $x^{\smile\smile} = x$
   (6) $(x+y)^{\smile} = x^{\smile} + y^{\smile}$
   (7) $(x;y)^{\smile} = y^{\smile};x^{\smile}$
   (8) $x^{\smile};(-(x;y)) \le -y$

2. axioms for the difference operator ensuring that D is a modality and that $\Diamond$ is a complemented closure operator:
   (D1) $\mathsf{D}(x+y) = \mathsf{D}x + \mathsf{D}y$
   (D2) $\mathsf{DD}x \leq \Diamond x$
   (D3) $\Diamond x \leq -\mathsf{D} - \Diamond x$
   (D4) $\mathsf{D}x \cdot \Diamond y = \mathsf{D}(x \cdot \Diamond y)$
3. axioms ensuring that the relativization with $\Diamond a$ is a homomorphism (cf. Proposition 3.2.6), and describing the connection between D, and ran and dom (for more intuition on (D*c*) and (D*d*) see the beginning of the proof of Theorem 3.2.3):
   (D*a*) $(x;y) \cdot \Diamond z = (x \cdot \Diamond z);(y \cdot \Diamond z)$
   (D*b*) $x^{\smile} \leq \Diamond x$
   (D*c*) $\Diamond(\mathsf{Ddom}x \oplus -\mathsf{dom}x) + \Diamond(\mathsf{Dran}x \oplus -\mathsf{ran}x) = \Diamond(\mathsf{D}x \oplus -x)$
   (D*d*) $((\mathsf{dom}(x;\mathsf{dom}y);1) \cdot (1;\mathsf{ran}y)) \oplus (x;y) \leq \Diamond(\mathsf{Ddom}y \oplus -\mathsf{dom}y)$.

∎

Since in the above definition of WAD we did not close it under subalgebras and products, WAD itself cannot be a variety. That is why we have to take its **SP**-closure.

Theorem 3.2.3. **SP**WAD *is a finitely axiomatizable discriminator variety:*

$$\mathbf{SP}\mathsf{WAD} = \mathrm{Mod}(Ax).$$

That is, an algebra $\langle A, \cdot, -, ;, ^{\smile}, \mathsf{id}, \mathsf{D}\rangle$ *satisfies the above axioms iff it is (isomorphic to) a subalgebra of a direct product of* WAD*'s.*

Corollary 3.2.4. **SP**$\mathsf{WA}^{<3}$ *is a finitely axiomatizable variety.*

Proof of Corollary 3.2.4: This is an immediate consequence of Theorem 3.2.3, since the classes WAD and $\mathsf{WA}^{<3}$ are term-definitionally equivalent (as we pointed out above). ∎

In the following proof we will use the following axiomatization theorem for WA, cf. [Ma82].

Theorem 3.2.5.

$$\mathbf{I}\mathsf{WA} = \mathrm{Mod}((1) - (8)).$$

First we will represent an abstract WAD as a set WA (with an abstract D on it), and then modify the representation so that it behaves correctly w.r.t. D as well.

Proof of Theorem 3.2.3: It may be useful (to gain some intuition) to reformulate the axioms (D*c*) and (D*d*). The following formulas are equivalent to the corresponding equations on subdirectly irreducible algebras.[5] Below we say that an element x is a *singleton* iff $\mathsf{D}x = -x$. (D*c*) says that an element is a singleton iff both its domain and its range are singletons:

$$\mathsf{D}x = -x \iff \mathsf{Ddom}x = -\mathsf{dom}x \ \&\ \mathsf{Dran}x = -\mathsf{ran}x.$$

[5]Here we use the standard technique of discriminator varieties explained above.

(Dd) ensures that we can form the composition of two arrows (i.e., ordered pairs) if the range of the first one (x) and the domain of the second one (y) intersect the same singleton element ($\mathsf{dom}y$):

$$\mathsf{Ddom}y = -\mathsf{dom}y \Rightarrow (\mathsf{dom}(x;\mathsf{dom}y);1)\cdot(1;\mathsf{ran}y) = x;y.$$

We will need this axiom when we want to collapse different points in the representation of a singleton element.

It is easy to check that Ax is valid in WAD, and so in **SPWAD**. It may help to check the equations only in the subdirectly irreducible algebras, and use their reformulation above.

To prove the other direction we show that every subdirectly irreducible algebra $\mathfrak{A}$ satisfying Ax can be represented as a subalgebra of a WAD. This is enough, since every element of $\mathrm{Mod}(Ax)$ is a subdirect product of subdirectly irreducible members of $\mathrm{Mod}(Ax)$. Since the class $\mathbf{Sir}\mathrm{Mod}(Ax)$ of the subdirectly irreducible members of $\mathrm{Mod}(Ax)$ is a discriminator class (cf. Proposition 3.2.6 below), $\mathrm{Mod}(Ax)$ is a discriminator variety.

First we note some straightforward consequences of the axioms.

1. $\Diamond$ is a complemented closure operator, i.e., $x \le \Diamond x \le \Diamond(x+y)$, $\Diamond\Diamond x \le \Diamond x$, and $\Diamond - \Diamond x \le -\Diamond x$.
2. $\mathsf{D}0 = 0$.

Proposition 3.2.6. *Let $\mathfrak{A} \in \mathbf{Sir}\mathrm{Mod}(Ax)$ and $a \in A$. Then $0 < a$ implies $\Diamond a = 1$. Thus, on every subdirectly irreducible member of $\mathrm{Mod}(Ax)$, there is a discriminator term.*

Proof: It is easy to show that relativizing $\mathfrak{A}$ with $\Diamond a$ (and with $-\Diamond a$) is a homomorphism. Thus $\mathfrak{A}$ can be embedded into the product of the relativizations of $\mathfrak{A}$ with $\Diamond a$ and with $-\Diamond a$. Further, this is a subdirect embedding. Thus one of the algebras must be isomorphic to $\mathfrak{A}$. This yields $\Diamond a = 1$. Then the discriminator term τ can be easily defined: $\tau(x,y,u,v) \stackrel{\mathrm{def}}{=} (\Diamond(x \oplus y)\cdot v) + (u \cdot -\Diamond(x\oplus y))$. ∎

Claim 3.2.7. *Let $\mathfrak{A} \in \mathbf{Sir}\mathrm{Mod}(Ax)$ and $a \in A$. Then the following hold.*

(i) $\mathsf{D}a \in \{0, -a, 1\}$; *if* $\mathsf{D}a = -a$, *then a is an atom; if* $0 < \mathsf{D}x$, *then* $0 < x$.
(ii) *If* $|A| > 2$, *then* $\mathsf{D}1 = 1$, *and* $0 < x$ *implies* $0 < \mathsf{D}x$.

Proof: This is an easy calculation using the fact that, in subdirectly irreducible algebras, $\mathsf{D}x + x = \Diamond x = 1$ whenever $0 < x$. ∎

Clearly, any two-element algebra $\langle\{0,1\},\ldots\rangle \in \mathrm{Mod}(Ax)$ can be represented (choose $W = \{\langle 0,0\rangle\}$ or $W = \{\langle 0,0\rangle,\langle 1,1\rangle\}$ according to whether $\mathsf{D}1 = 0$ or $\mathsf{D}1 = 1$). Thus, from now on, we may assume that $|A| > 2$.

Let $\mathfrak{A} \in \mathrm{Mod}(Ax)$ be a subdirectly irreducible algebra and $\mathfrak{A}_0$ be the D-free (i.e., WA-) reduct of $\mathfrak{A}$. By the WA-representation Theorem 3.2.5, there are a reflexive and symmetric relation $W \subseteq U \times U$ for some set U, and an algebra $\mathfrak{A}_0'$ such that $\mathfrak{A}_0 \cong \mathfrak{A}_0' \subseteq \langle \mathcal{P}(W), \cap, \sim, \circ, {}^{-1}, \mathsf{Id}\rangle$. For $a \in A$ we denote its representation by a'.

First we show that there is a reflexive and symmetric relation $W^- \subseteq U^- \times U^-$ for some set U^- such that there is an embedding $g : \mathfrak{A}_0 \longrightarrow \langle \mathcal{P}(W^-), \cap, \sim, \circ, ^{-1}, \mathsf{Id}\rangle$, and that, for every $x \in A$, if $\mathsf{D}(x) = -x$, then $g(x) = \{w\}$ for some $w \in W^-$. This is of course not the end of the story, since there still may be elements such that $\mathsf{D}y = 1$ and $|g(y)| = 1$, but we will deal with that problem later.

FIRTS STEP. We define the relation $\equiv$ on U to collapse those distinct pairs which are in the representation of a singleton element. It is enough to consider those singleton elements which are below the identity, because (Dc) ensures that singletons have singleton ranges and domains.

Definition 3.2.8. Define the relation $\equiv$ on U as follows: for $u, v \in U$, $u \equiv v$ iff (i) $u = v$ or (ii) there is a $y \in A$ such that $\mathsf{D}(y) = -y$ and $\langle u, u\rangle, \langle v, v\rangle \in y'$. Note that y must be an atom, and thus it is below the identity, and $\equiv$ is an equivalence relation.

Let $U^- \overset{\text{def}}{=} U/\!\equiv$ and $W^- \overset{\text{def}}{=} \{\langle u/\!\equiv, v/\!\equiv\rangle : \langle u, v\rangle \in W\}$.

Let g be the function mapping A to $\mathcal{P}(W^-)$ defined by the formula

$$g(a) = \{\langle u/\!\equiv, v/\!\equiv\rangle : (\exists u' \in u/\!\equiv)(\exists v' \in v/\!\equiv)\langle u', v'\rangle \in a'\}.$$

∎

It is easy to see that W^- is indeed a reflexive and symmetric relation.

Claim 3.2.9. *If $\langle u, v\rangle, \langle u', v'\rangle \in W$ and $u \equiv u', v \equiv v'$, then $\langle u, v\rangle \in a'$ implies $\langle u', v'\rangle \in a'$, for every $a \in A$.*

Proof: We have four cases according to whether $u = u'$ and $v = v'$. To avoid triviality, let us assume $u \neq u'$ or $v \neq v'$.

First let $u \neq u'$ and $v \neq v'$. By the definition of $\equiv$, there are identity atoms y, z such that $\mathsf{D}y = -y$, $\mathsf{D}z = -z$, $\langle u, u\rangle, \langle u', u'\rangle \in y'$ and $\langle v, v\rangle, \langle v', v'\rangle \in z'$. Let $b = (y; 1) \cdot (1; z)$. Note that then $\langle u, v\rangle, \langle u', v'\rangle \in b'$. Since y and z are atoms, $\mathsf{dom}b = y$ and $\mathsf{ran}b = z$, so we have $\mathsf{D}(b) = -b$ because of axiom (Dc). Then b is an atom in $\mathfrak{A}$, and, since $b \cdot a \neq 0$, $\langle u', v'\rangle \in a'$.

Now let $u = u'$ and $v \neq v'$. Then there is an identity atom z such that $\mathsf{D}z = -z$ and $\langle v, v\rangle, \langle v', v'\rangle \in z'$. Assume $\langle u, v'\rangle \notin a'$. Let $x = a \cdot (1; z)$ and $y = (-a)^\smile \cdot (z; 1)$. Then $\mathsf{dom}y = z$, whence $\mathsf{D}\mathsf{dom}y = -\mathsf{dom}y$ and $\langle v, v\rangle \in \mathsf{dom}y'$. Thus $\langle u, u\rangle \in \mathsf{dom}(x' \circ \mathsf{dom}y') \circ W$. On the other hand, $\langle u, u\rangle \in W \circ \mathsf{ran}y'$, since $\langle v', u\rangle \in y'$. Then $\langle u, u\rangle \in x' \circ y'$ by (Dd). Since $x; y \leq a; (-a)^\smile$ and $(a; (-a)^\smile) \cdot \mathsf{id} = 0$, we derived a contradiction.

If $u \neq u'$ and $v = v'$, the above argument shows that $\langle v, u'\rangle \in (a')^{-1}$, i.e., $\langle u', v\rangle \in a'$.
∎

Proposition 3.2.10. *g is an embedding of $\mathfrak{A}_0$ into $\langle \mathcal{P}(W^-), \cap, \sim, \circ, ^{-1}, \mathsf{Id}\rangle$.*

Proof: The only problematic case is composition. For the other connectives use the previous claim.

$g(a; b) \subseteq g(a) \circ g(b)$ is easy. For the other direction let us assume $\langle u/\!\equiv, v/\!\equiv\rangle \in g(a) \circ g(b)$. Then $\exists w/\!\equiv(\langle u/\!\equiv, w/\!\equiv\rangle \in g(a)$ & $\langle w/\!\equiv, v/\!\equiv\rangle \in g(b)$ & $\langle u/\!\equiv, v/\!\equiv\rangle \in W^-)$, i.e., $\exists w/{\equiv}(\exists u, u', v, v')(\exists w', w'' \in w/{\equiv}) u \equiv u'$ & $v \equiv v'$ & $\langle u, v\rangle \in W$ & $\langle u', w'\rangle \in a'$ & $\langle w'', v'\rangle \in b'$.

CASE 1: $w' \neq w''$. Then $\exists l(\mathsf{D}l = -l$ & $\langle w', w'\rangle, \langle w'', w''\rangle \in l')$. First we show that there are t and t' such that $\langle u, t\rangle \in a'$, $\langle t', v\rangle \in b'$ and $t \equiv t'$.

If $u = u'$ and $v = v'$, then $t = w$ and $t' = w'$ will do. So assume that $u \neq u'$. Then $\exists k(\mathsf{D}k = -k$ & $\langle u, u\rangle, \langle u', u'\rangle \in k')$. Let $x = a \cdot ((k; 1) \cdot (1; l))$. Then $\langle u', w'\rangle \in x'$, whence $\langle u', u'\rangle \in \mathsf{dom} x'$. Hence $\langle u, u\rangle \in \mathsf{dom} x'$, since $k' \cap \mathsf{dom} x' \neq \emptyset$. Thus there exists t such that $\langle u, t\rangle \in x' \subseteq a'$, whence $\langle t, t\rangle \in l'$, i.e., $t \equiv w'$. By the same argument we can find y and t' such that $t' \equiv w''$ and $\langle t', v\rangle \in y' \subseteq b'$.

By $\langle t, t\rangle \in l' = \mathsf{dom} y'$, we have $\langle u, t\rangle \in x' \circ \mathsf{dom} y'$, whence $\langle u, u\rangle \in \mathsf{dom}(x' \circ \mathsf{dom} y')$. Thus we have $\langle u, v\rangle \in (\mathsf{dom}(x' \circ \mathsf{dom} y') \circ W) \cap (W \circ \mathsf{ran} y')$. Then, by ($\mathsf{D}d$), we get $\langle u, v\rangle \in x' \circ y' = (x; y)' \subseteq (a; b)'$, i.e., $\langle u/{\equiv}, v/{\equiv}\rangle \in g(a; b)$.

CASE 2: $w' = w''$. If $u = u'$ & $v = v'$, then $\langle u, v\rangle \in (a; b)'$, whence $\langle u/{\equiv}, v/{\equiv}\rangle \in g(a; b)$.

Now assume that $u \neq u'$, i.e., $\exists k\ \mathsf{D}k = -k$ and $\langle u, u\rangle, \langle u', u'\rangle \in k'$. Let $x = (k; 1) \cdot a$. Then $\mathsf{dom} x = k$. By case 1, $\langle w', v'\rangle \in (x^{\smile})' \circ W$, whence $\exists u''\ \langle w', u''\rangle \in (x^{\smile})'$ and $\langle u'', w'\rangle \in W$. Then $\langle u'', w'\rangle \in x'$, and $u'' \equiv u$ by $\langle u'', u''\rangle \in \mathsf{dom} x' = k'$. Thus $\langle u'', v'\rangle \in x' \circ b' \subseteq a' \circ b'$ by $\langle w', v'\rangle \in b'$. Since $u'' \equiv u$ and $v' \equiv v$, we have $\langle u/{\equiv}, v/{\equiv}\rangle \in g(a; b)$.

Thus we proved that $\langle u/{\equiv}, v/{\equiv}\rangle \in g(a) \circ g(b)$ implies $\langle u/{\equiv}, v/{\equiv}\rangle \in g(a; b)$, finishing the proof. ∎

Proposition 3.2.11. *If $a \in A$ and $\mathsf{D}(a) = -a$, then $|g(a)| = 1$.*

Proof: Since $a \neq 0$, there is a pair $\langle u, v\rangle \in a'$. We claim that $g(a) = \{\langle u/{\equiv}, v/{\equiv}\rangle\}$. $\supseteq$ is true by the definition of g. Now suppose that $\langle u'/{\equiv}, v'/{\equiv}\rangle \in g(a)$. Then for some $u'' \in u'/{\equiv}$ and $v'' \in v'/{\equiv}$, $\langle u'', v''\rangle \in a'$ by the definition of g. Now $\mathsf{D}(a) = -a$ gives $\mathsf{D}(\mathsf{dom} a) = -\mathsf{dom} a$ and $\mathsf{D}(\mathsf{ran} a) = -\mathsf{ran} a$ by axiom ($\mathsf{D}c$). Since $\langle u, u\rangle, \langle u'', u''\rangle \in \mathsf{dom} a'$ and $\langle v, v\rangle, \langle v'', v''\rangle \in \mathsf{ran} a'$, we have $u \equiv u''$ and $v \equiv v''$, by the definition of $\equiv$. So $\langle u'/{\equiv}, v'/{\equiv}\rangle = \langle u/{\equiv}, v/{\equiv}\rangle$, finishing the proof. ∎

SECOND STEP. Now we have to achieve our second goal, to find a symmetric and reflexive relation W^+ such that there is an embedding h of $\mathfrak{A}_0$ into $\langle \mathcal{P}(W^+), \cap, \sim, \circ, ^{-1}, \mathsf{Id}\rangle$ with the following properties: (i) if $\mathsf{D}a = -a$, then $|h(a)| = 1$ and (ii) if $\mathsf{D}a = 1$, then $|h(a)| > 1$. We need this, because there may be elements $a \in A$ such that $\mathsf{D}a = 1$ while $|g(a)| = 1$. Below we will duplicate those points u for which $\{\langle u, u\rangle\} = g(a)$ for some non-singleton a.

Definition 3.2.12. Let

$$X \stackrel{\text{def}}{=} \{u \in U^- : \exists a(\{\langle u, u\rangle\} = g(a) \;\&\; \mathsf{D}a = 1)\}$$

and let $\bar{}$ be a 1–1 function with range disjoint from U^-. Let $\bar{X} \stackrel{\text{def}}{=} \{\bar{x} : x \in X\}$, $U^+ \stackrel{\text{def}}{=} U^- \cup \bar{X}$ and

$$W^+ \stackrel{\text{def}}{=} W^- \cup \{\langle \bar{x}, \bar{x}\rangle : x \in X\} \cup \{\langle u, \bar{x}\rangle, \langle \bar{x}, u\rangle : \langle u, x\rangle \in W^- \;\&\; u \neq x \in X\}.$$

∎

Clearly W^+ is a reflexive and symmetric relation. For every $a \in A$, let

$$f(g(a)) \stackrel{\text{def}}{=} g(a) \cup f_0(g(a)) \cup f_1(g(a)) \cup f_2(g(a))$$

where

$$\begin{aligned} f_0(g(a)) &\stackrel{\text{def}}{=} \{\langle \bar{x}, \bar{x}\rangle : \langle x, x\rangle \in g(a) \ \&\ x \in X\} \\ f_1(g(a)) &\stackrel{\text{def}}{=} \{\langle \bar{x}, u\rangle : \langle x, u\rangle \in g(a) \ \&\ u \neq x \in X\} \\ f_2(g(a)) &\stackrel{\text{def}}{=} \{\langle u, \bar{x}\rangle : \langle u, x\rangle \in g(a) \ \&\ u \neq x \in X\}. \end{aligned}$$

Proposition 3.2.13. (i) *If* $\mathsf{D}a = 1$, *then* $|f(g(a))| > 1$.
(ii) *If* $\mathsf{D}a = -a$, *then* $|f(g(a))| = 1$.
(iii) f *is an embedding.*

Proof: (i): By (Dc), $\mathsf{Ddom}a = 1$ or $\mathsf{Dran}a = 1$. W.l.o.g. we can assume $\mathsf{Ddom}a = 1$. If $|g(a)| > 1$, then so is $|f(g(a))|$. Now assume $g(a) = \{\langle x, y\rangle\}$. Then $g(\mathsf{dom}a) = \{\langle x, x\rangle\}$. By the definition of X, $x \in X$, and then not just $\langle x, y\rangle \in f(g(a))$ but $\langle \bar{x}, y\rangle \in f_1(g(a)) \subseteq f(g(a))$ or $\langle \bar{x}, \bar{x}\rangle \in f_0(g(a)) \subseteq f(g(a))$ according to whether $x = y$, or x and y are different.

(ii): By (Dc), $\mathsf{Ddom}a = -\mathsf{dom}a$ and $\mathsf{Dran}a = -\mathsf{ran}a$. Thus if $g(a) = \{\langle x, y\rangle\}$, then $\{\langle x, x\rangle\} = g(\mathsf{dom}a)$ and $\{\langle y, y\rangle\} = g(\mathsf{ran}a)$, whence neither x nor y is in X. That is, $f(g(a)) = g(a)$.

(iii): This is a somewhat long but easy calculation. We just give hints. Complement is easy. Preservation of $\cap$ follows from $f_i(g(a) \cap g(b)) = f_i(g(a)) \cap f_i(g(b))$ for $i \in 3$. The case of $\circ$ amounts to prove:

$$\begin{aligned} g(a) \circ g(b) &= g(a) \circ g(b) \cup f_2(g(a)) \circ f_1(g(b)) \\ f_0(g(a) \circ g(b)) &= f_0(g(a)) \circ f_0(g(b)) \cup f_1(g(a)) \circ f_2(g(b)) \\ f_1(g(a) \circ g(b)) &= f_0(g(a)) \circ f_1(g(b)) \cup f_1(g(a)) \circ g(b) \\ f_2(g(a) \circ g(b)) &= f_2(g(a)) \circ f_0(g(b)) \cup g(a) \circ f_2(g(b)). \end{aligned}$$

For $^{-1}$ it suffices to show:

$$f_0((g(a))^{-1}) = (f_0(g(a)))^{-1} \text{ and } f_i((g(a))^{-1}) = (f_j(g(a)))^{-1} \quad (\{i, j\} = \{1, 2\}).$$

Finally,

$$f(g(\mathsf{id})) = g(\mathsf{id}) \cup \{\langle \bar{x}, \bar{x}\rangle : \langle x, x\rangle \in g(\mathsf{id})\} = \mathsf{Id}^{U^+ \times U^+}.$$

where $\mathsf{Id}^{U^+ \times U^+}$ denotes the identity relation on U^+. ∎

Now let h be the composition of f and g, i.e., $h(a) \stackrel{\text{def}}{=} f(g(a))$. Then h is the desired embedding of $\mathfrak{A}_0$ into $\langle \mathcal{P}(W^+), \cap, \sim, \circ, ^{-1}, \mathsf{Id}\rangle \in \mathsf{WA}$.

Let $\mathfrak{B}_0$ be the h-image of $\mathfrak{A}_0$ and $\mathfrak{B} \stackrel{\text{def}}{=} \langle \mathfrak{B}_0, \mathsf{D}\rangle$ where D is the restriction of the difference operator defined on $\mathcal{P}(W^+)$ to B. Then $\mathfrak{A} \cong \mathfrak{B}$, i.e., we embedded $\mathfrak{A}$ into $\langle \mathcal{P}(W^+), \cap, \sim, \circ, ^{-1}, \mathsf{Id}, \mathsf{D}\rangle \in \mathsf{WAD}$. This finishes the proof of Theorem 3.2.3. ∎

Remark 3.2.14. We conjecture that the above theorem can be extended to any $(\mathbf{Rl}_H\mathsf{RRA})^{<3}$ with $H \subseteq \{r, s\}$. For axiomatization of $(\mathbf{Rl}_H\mathsf{RRA})^{<2}$ with $H \subseteq \{r, s\}$ see [Ma95]. ∎

3.2.2 AXIOMATIZATION OF $\mathsf{WA}^{<\alpha}$

As we will see $\mathbf{SPWA}^{<\alpha}$ is a variety for any $\alpha \leq \omega$, cf. Theorem 3.2.16. It is an open problem whether these varieties can be axiomatized by finitely many (universal) equations, provided $3 < \alpha$. If $\alpha = \omega$, then finite axiomatization is impossible, since the language is infinite, there are infinitely many $\langle n\rangle$'s. Then a natural question is whether one can find finitely many *schemata*, formulas containing variables for which terms can be substituted, axiomatizing the class.

From now on, let $3 \leq \alpha$. We will show that $\mathbf{SPWA}^{<\alpha}$ is finitely axiomatizable as an *existential variety*. By this we mean the following. We will define a variety K axiomatizable by finitely many schemas. Then we will add one more existential equation, i.e., an equation of the form $\forall x \exists y(\tau = \sigma)$ for some terms σ and τ. This defines a subclass DK of K, and we will show that $\mathbf{SPWA}^{<\alpha} = \mathbf{SPDK}$. We note that if α is finite, then the set of schemata yields a *finite* set of axioms.

The meaning of the above-mentioned existential equation is that an algebra satisfying it has a *density* property: below every non-zero element there is a special element. Representation of this kind of algebras has been investigated extensively, cf. [HMT85] and Chapter 4, since these representation theorems yield completeness results for logics.

In the proof of Theorem 3.2.3, we had a relatively easy task: after representing the WA-reduct we modified the representation so that it worked for D as well. Since by D we can express only $\langle 1\rangle$ and $\langle 2\rangle$, in the second step it did not matter how many copies we made. On the other hand, if we have all the $\langle n\rangle$'s in the similarity type, than we have to be more careful with the copying. It is not clear how to make, say, precisely three copies of a pair (preserving isomorphism). In spite of this, we can apply Theorem 3.2.3 to achieve a representation theorem for (abstract) algebras with all the $\langle n\rangle$'s. Namely, we will represent those algebras in which, below every element, there is a singleton element — the representations of the singletons will induce the representations for the elements above the singletons.

Definition 3.2.15. Let Σ_α be the following set of equations:

1. finitely many equations axiomatizing $\mathbf{SPWA}^{<3}$,
2. equations axiomatizing the counting operations:
   $(C1)$ $\langle 1\rangle\langle n\rangle x \leq \langle n\rangle x \qquad (n < \alpha)$
   $(C2)$ $\langle n+1\rangle x \leq \langle n\rangle x \qquad (n+1 < \alpha)$
   $(C3)$ $\langle n\rangle x \leq \langle n\rangle(x+y) \qquad (n < \alpha)$
   $(C4)$ $-\langle 1\rangle(x \cdot y) \cdot \langle m!\rangle x \cdot \langle n!\rangle y \leq \langle (m+n)!\rangle(x+y) \qquad (m+n+1 < \alpha)$
   $(C5)$ $-\langle 1\rangle(x \cdot y) \cdot \langle m\rangle x \cdot \langle n\rangle y \leq \langle l\rangle(x+y) \qquad (l,m,n < \alpha \;\&\; l \leq m+n)$

where $\langle k!\rangle x$ abbreviates $\langle k\rangle x \cdot -\langle k+1\rangle x$.

Let $\mathsf{K}_\alpha = \mathrm{Mod}(\Sigma_\alpha)$ and $\mathsf{DK}_\alpha = \mathrm{Mod}(\Sigma_\alpha + (d))$, where

$$(d) \qquad (\forall 0 < a \in A)(\exists 0 < b \in A) b \leq a \;\&\; \langle 2\rangle b = 0.$$

∎

The intuitive meaning of the above (d) is that below every non-zero element a, there is a singleton element b, i.e., (d) expresses a $d$ensity property.

We note that, for $\alpha = \omega$, $(C5)$ is superfluous both in the proof of Lemma 3.2.21 and of Theorem 3.2.17 below.

First let us prove that $\mathbf{SPWA}^{<\alpha}$ is a variety. The main result is Theorem 3.2.17 below.

Theorem 3.2.16. *$\mathbf{SPWA}^{<\alpha}$ is a variety.*

Proof: We will show that $\mathbf{IWA}^{<\alpha}$ is a pseudoaxiomatizable class, i.e., it is a reduct of first-order axiomatizable class. Hence, it is closed under ultraproducts: $\mathbf{UpWA}^{<\alpha} = \mathbf{IWA}^{<\alpha}$. Then $\mathbf{SPWA}^{<\alpha} = \mathbf{SPUpWA}^{<\alpha}$. Since $\mathbf{IWA}^{<\alpha}$ is a discriminator class, the quasi-variety $\mathbf{SPUpWA}^{<\alpha} = \mathbf{SPWA}^{<\alpha}$ generated by it is a variety.

First let us prove that $\mathsf{WA}^{<\alpha}$ is a discriminator class. Indeed, the term $\tau(x, y, u, v) \stackrel{\text{def}}{=} (\langle 1\rangle(x \oplus y) \cdot v) + (u \cdot -\langle 1\rangle(x \oplus y))$ is a discriminator term, since $\langle 1\rangle z = 1$ for any element $z > 0$.

To show that $\mathbf{IWA}^{<\alpha}$ is closed under $\mathbf{Up}$, we define two-sorted structures. Let $\mathfrak{A} \in \mathsf{WA}^{<\alpha}$ and U be the base of $\mathfrak{A}$: $A \subseteq \mathcal{P}(U \times U)$. Then we will describe the class of structures of the form

$$\langle A, U, \cap, \sim, \circ, {}^{-1}, \mathsf{Id}, \langle n\rangle, \in\rangle_{n\in\alpha\setminus 1}$$

where the $\{U, \in\}$-free reduct is a $\mathsf{WA}^{<\alpha}$ and $A \subseteq \mathcal{P}(U \times U)$ and $\in\, \subseteq (U \times U) \times A$ is the (set-theoretic) element relation. Indeed, the following set of first-order formulas does the job:

1. a formula expressing the extensionality of $\in$: for every two distinct elements a, b of A, there is an element w of $U \times U$ such that exactly one of the following holds: $w \in a$, or $w \in b$;
2. formulas ensuring that A is a WA-universe and that the operations work properly on A (below, 1 denotes the top element of $\mathfrak{A}$):

$$\begin{aligned}
&\forall u, v(\langle u, v\rangle \in 1 \Rightarrow \langle u, u\rangle, \langle v, v\rangle, \langle v, u\rangle \in 1)\\
&(\forall\langle u, v\rangle \in 1)(\langle u, v\rangle \in x \cap y \iff \langle u, v\rangle \in x \;\&\; \langle u, v\rangle \in y)\\
&(\forall\langle u, v\rangle \in 1)(\langle u, v\rangle \in \sim x \iff \langle u, v\rangle \notin x)\\
&(\forall\langle u, v\rangle \in 1)(\langle u, v\rangle \in x \circ y \iff \exists w(\langle u, w\rangle \in x \;\&\; \langle w, v\rangle \in y))\\
&(\forall\langle u, v\rangle \in 1)(\langle u, v\rangle \in x^{-1} \iff \langle v, u\rangle \in x)\\
&(\forall\langle u, v\rangle \in 1)(\langle u, v\rangle \in \mathsf{Id} \iff u = v)\\
&(\forall\langle u, v\rangle \in 1)(\langle u, v\rangle \in \langle n\rangle x \iff \exists u_0, v_0, \dots, u_{n-1}, v_{n-1}\\
&\qquad |\{\langle u_0, v_0\rangle, \dots, \langle u_{n-1}, v_{n-1}\rangle\}| = n \;\&\; \langle u_0, v_0\rangle \in x \;\&\; \dots \;\&\; \langle u_{n-1}, v_{n-1}\rangle \in x)
\end{aligned}$$

   for every $n \in \alpha \setminus 1$.

Then a two-sorted structure satisfies the above formulas iff its appropriate ($\{U, \in\}$-free) reduct is in $\mathsf{WA}^{<\alpha}$. ∎

Although, we do not know whether $\mathbf{SPWA}^{<\alpha}$ is a finitely (schema) axiomatizable variety, we can prove the following.

Theorem 3.2.17. $\mathbf{SPWA}^{<\alpha} = \mathbf{SPDK}_\alpha$.

Proof: $\subseteq$: Let $\mathfrak{A} \in \mathsf{WA}^{<\alpha}$. An easy verification shows that $\mathfrak{A} \in \mathrm{Mod}(\Sigma_\alpha)$. Further, let $a \in A$ and $|a| = 1$ (note that $\mathfrak{A}$ is a full algebra). Then $\langle 2\rangle a = 0$, whence $\mathfrak{A} \in \mathsf{DK}_\alpha$. Hence $\mathbf{SP}\mathsf{WA}^{<\alpha} \subseteq \mathbf{SP}\mathsf{DK}_\alpha$.

$\supseteq$: Now, let $\mathfrak{A} \in \mathsf{DK}_\alpha$. We will show $\mathfrak{A} \in \mathbf{SP}\mathsf{WA}^{<\alpha}$. Assume that $\mathfrak{A} \notin \mathbf{SP}\mathsf{WA}^{<\alpha}$. By Theorem 3.2.16 above, $\mathbf{SP}\mathsf{WA}^{<\alpha}$ is a variety. Thus, there is an equation e valid in $\mathbf{SP}\mathsf{WA}^{<\alpha}$ that is not valid in $\mathfrak{A}$, i.e., for some assignment k of variables of e, $\mathfrak{A} \not\models e[k]$. By the downward Löwenheim–Skolem theorem, there is a countable subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that (i) $\mathfrak{B} \not\models e[k]$, and (ii) $\mathfrak{B} \in \mathsf{DK}_\alpha$, since (d) is a first-order property. By Theorem 3.2.20 below, every countable element of DK_α is representable as a $\mathbf{SP}\mathsf{WA}^{<\alpha}$. Thus $\mathfrak{B} \in \mathbf{SP}\mathsf{WA}^{<\alpha}$, whence $\mathfrak{B} \models e$, a contradiction. ∎

It remains to prove that every countable DK_α is representable as a $\mathbf{SP}\mathsf{WA}^{<\alpha}$. First we prove that representability holds for simple algebras.

Theorem 3.2.18. *Let $\mathfrak{A} \in \mathsf{DK}_\alpha$ be a simple algebra. Then $\mathfrak{A} \in \mathbf{IS}\mathsf{WA}^{<\alpha}$.*

Proof: Let $\mathfrak{A} \in \mathsf{DK}_\alpha$ be a simple algebra. Then $\mathfrak{A}$ satisfies the $\mathsf{WA}^{<3}$-axioms. These axioms together with $(C2)$ ensures that DK_α is in a discriminator variety. This yields that $(\forall 0 < x \in A)\langle 1\rangle x = 1$. Then (d) guarantees that $(\forall 0 < a \in A)(\exists 0 < b \in A)(b \le a \;\&\; \mathsf{D}b = -b)$ (recall that $\mathsf{D}x \stackrel{\text{def}}{=} (-x\cdot\langle 1\rangle x) + \langle 2\rangle x)$. Since $\mathfrak{A}$ satisfies the $\mathsf{WA}^{<3}$-axioms, $\mathsf{D}b = -b \Rightarrow b \in At(\mathfrak{A})$. Thus $\mathfrak{A}$ is atomic and $(\forall a \in At(\mathfrak{A}))(\mathsf{D}a = -a)$.

Let $\mathfrak{A}'$ be the $\{\langle n\rangle : 2 < n < \alpha\}$-free reduct of $\mathfrak{A}$. Since $\mathfrak{A}'$ satisfies the $\mathbf{SP}\mathsf{WA}^{<3}$-axioms, it can be represented as a subalgebra of a $\mathsf{WA}^{<3}$, cf. Corollary 3.2.4:

$$\mathfrak{A}' \subseteq \langle \mathcal{P}(W), \cap, \sim, \circ, {}^{-1}, \mathsf{Id}, \langle 1\rangle, \langle 2\rangle\rangle.$$

Let us denote this embedding by rep. We will show that rep works for the other $\langle n\rangle$'s as well.

First we need a claim. We note that, in simple algebras, $\langle n\rangle x \in \{0,1\}$ by $(C1)$ and $(C2)$.

Claim 3.2.19. *$\langle n!\rangle x = 1$ iff there are precisely n atoms below x.*

Proof: Let us assume that $\langle n!\rangle x = 1$. If $n = 1$, then x is an atom. Now, assume $n \ge 2$, and that there are less than n, say k, atoms below x: $x_1, \dots, x_k$. Then, applying $(C4)$ $k-1$ times, we get $\langle k!\rangle x = 1$, i.e., $\langle k+1\rangle x = 0$, a contradiction. If there were at least $n+1$ atoms below x, then, applying $(C4)$ n times, we get $1 = \langle n+1!\rangle(x_1 + \dots + x_{n+1}) \le \langle n+1\rangle x$ (use $(C3)$ for $\le$), a contradiction.

If there are exactly n atoms below x, then, applying $(C4)$ $n-1$ times, we get $\langle n!\rangle x = 1$. ∎

Let x be an arbitrary element of A. We have two cases.

Assume there exists n such that $\langle n!\rangle x = 1$. Then, by Claim 3.2.19, there are exactly n atoms below x, and these atoms are singletons: $|\mathsf{rep}(x_i)| = 1$ for all $i \in n$. Since $\mathsf{rep}(x) = \bigcup\{\mathsf{rep}(x_i) : i \in n\}$, we have $|\mathsf{rep}(x)| = n$. Then, for every $k \le n$, $\mathsf{rep}(\langle k\rangle x) = \mathsf{rep}(1) = W = \langle k\rangle \mathsf{rep}(x)$ by $(C2)$. Since $\mathsf{rep}(\langle k\rangle x) = \mathsf{rep}(0) = \emptyset = \langle k\rangle\mathsf{rep}(x)$ $(n < k)$ (again, use $(C2)$), rep works for every $\langle k\rangle$.

Now assume that, for all n, $\langle n\rangle x = 1$. Then, by Claim 3.2.19, there are infinitely many atoms below x. This yields $|\mathsf{rep}(x)| \geq \omega$, whence $\mathsf{rep}(\langle k\rangle x) = \mathsf{rep}(1) = W = \langle k\rangle \mathsf{rep}(x)$ for every k. This finishes the proof of Theorem 3.2.18. ∎

Theorem 3.2.20. *Let $\mathfrak{A} \in \mathsf{DK}_\alpha$ be a countable algebra, $|A| \leq \omega$. Then $\mathfrak{A} \in$ $\mathbf{SP}\mathsf{WA}^{<\alpha}$.*

Proof: Let $\mathfrak{A} \in \mathsf{DK}_\alpha$ be a countable algebra. We would like to apply Theorem 1.2.6, i.e., to find simple (thus representable) algebras $\mathfrak{A}_i$ such that $\mathfrak{A} \subseteq \mathbf{SP}\{\mathfrak{A}_i : i \in I\}$.

But the $\langle n\rangle$'s are not additive. To overcome this difficulty, we will define another (equivalent) class of algebras of the right similarity type. (The situation will be similar to the case of Boolean algebras, when both $\{\cdot, -\}$ and $\{+, -\}$ can be taken as the set of primitive connectives.) The only problem is that we have to ensure that the extra-Boolean operations are additive. We will need two lemmas to achieve the above goal.

Let Σ_α^- be the following set of formulas: axioms from Definition 3.2.15 for Boolean algebras, those ensuring that $\langle 1\rangle$ is a complemented closure operator, and $(C1)$–$(C5)$. Let the class $\mathsf{BA}^{<\alpha}$ be defined as $\mathsf{WA}^{<\alpha}$ using the class of full Boolean set algebras instead of the class of full set WA's, cf. Definition 3.2.1.

Lemma 3.2.21. $\mathrm{Mod}(\Sigma_\alpha^-) = \mathbf{SP}\mathsf{BA}^{<\alpha}$.

Proof: This is a well-known result. See, e.g., [HR93], where this result (with a slightly different axiomatization) is proved in a logical form. However, we give a sketch how to prove the lemma in purely algebraic setting.

As usual, we will represent the subdirect irreducible members of $\mathrm{Mod}(\Sigma_\alpha^-)$, and then the lemma follows. So let $\mathfrak{A} \in \mathbf{Sir}\mathrm{Mod}(\Sigma_\alpha^-)$. Since, $\langle 1\rangle$ is a complemented closure operator, $0 < x \Rightarrow \langle 1\rangle x = 1$. Then, by $(C1), (C2)$ $\langle n\rangle x \in \{0, 1\}$ for every x and n.

Since $\mathfrak{A}$ satisfies the Boolean axioms, it can be represented as a field of sets, say by the Stone-representation rep. We will modify rep so that it works for the extra-Boolean operations as well. First we correct rep on the atoms. We know that, for every atom a, $|\mathsf{rep}(a)| = 1$. Let $k_a = \max\{k : \langle k\rangle a = 1\}$. Let $\mathsf{rep}'(a)$ be such that $|\mathsf{rep}'(a)| = k$ and $\mathsf{rep}'(a) \cap \mathsf{rep}'(b) = \emptyset$ for distinct atoms a and b. This can be easily done, e.g., by just putting new elements into the old representations (this technique is called *splitting* in algebraic logic, cf. [HMT85]). Now we extend rep' in the obvious way:

$$\mathsf{rep}'(x) = \bigcup\{\mathsf{rep}'(a) : a \in At(\mathfrak{A})\ \&\ a \leq x\} \cup \mathsf{rep}(x)$$

for every $x \in A$.

It is easy to see that rep' works properly on atoms. Now let x be arbitrary. If x is the sum of finitely many atoms, then a similar argument as in the proof of Claim 3.2.19 shows that the $\langle n\rangle$'s are preserved. If x is not a sum of finitely many atoms, then there are infinitely many disjoint elements below it, and, since these elements y are not zero, $|\mathsf{rep}(y)| \geq 1$. This yields that $|\mathsf{rep}'(x)| \geq \omega$. On the other hand, let $n \in \alpha$ be arbitrary. Then, applying $(C5)$ $n-1$ times for n disjoint elements below x, we have $\langle n\rangle x = 1$. This shows that rep' works in this case as well. ∎

Lemma 3.2.22. *Let* $\mathfrak{A} \in \mathsf{DK}_\alpha$ *and* $|A| \le \omega$. *Then there are* $\mathfrak{A}_i \in \mathbf{Sim}\mathsf{DK}_\alpha$ *such that* $\mathfrak{A} \in \mathbf{SP}\{\mathfrak{A}_i : i \in I\}$.

Proof: We will define classes K'_α and DK'_α such that

(i) K'_α and DK'_α satisfy the conditions of Theorem 1.2.6,
(ii) there are operations $\mathfrak{Rd}$ and $\mathfrak{Rd}'$ such that $\forall \mathfrak{A} \in \mathsf{DK}_\alpha$ and $\forall \mathfrak{A}' \in \mathsf{DK}'_\alpha$,

$$\begin{array}{ll}
(a) & \mathfrak{Rd}'(\mathfrak{A}) \in \mathsf{DK}'_\alpha \ \&\ \mathfrak{Rd}(\mathfrak{A}') \in \mathsf{DK}_\alpha \ \&\ (\mathfrak{A}' = \mathfrak{Rd}'(\mathfrak{A}) \ \&\ |A| \le \omega \Rightarrow |A'| \le \omega) \\
(b) & \mathfrak{Rd}(\mathfrak{Rd}'(\mathfrak{A})) \cong \mathfrak{A} \ \&\ \mathfrak{Rd}'(\mathfrak{Rd}(\mathfrak{A}')) \cong \mathfrak{A}' \\
(c) & \mathfrak{A}' \in \mathbf{Sim}\mathsf{DK}'_\alpha \Rightarrow \mathfrak{Rd}(\mathfrak{A}') \in \mathbf{Sim}\mathsf{DK}_\alpha \\
(d) & \mathfrak{A}' \in \mathbf{SP}\{\mathfrak{A}'_i : i \in I\} \Rightarrow \mathfrak{Rd}(\mathfrak{A}') \in \mathbf{SP}\{\mathfrak{Rd}(\mathfrak{A}'_i) : i \in I\}.
\end{array}$$

Assume that (i) and (ii) are achieved. Let $\mathfrak{A} \in \mathsf{DK}_\alpha$ and $|A| \le \omega$. Then there is an $\mathfrak{A}' \in \mathsf{DK}'_\alpha$ such that $\mathfrak{Rd}'(\mathfrak{A}) = \mathfrak{A}'$ and $|A'| \le \omega$. Then, by (i), $\mathfrak{A}' \in \mathbf{SP}\{\mathfrak{A}'_i : i \in I\}$ for some I and $\mathfrak{A}'_i \in \mathbf{Sim}\mathsf{DK}'_\alpha$. Let $\mathfrak{A}_i = \mathfrak{Rd}(\mathfrak{A}'_i)$, for every $i \in I$. Then, by (ii)(c), $\mathfrak{A}_i \in \mathbf{Sim}\mathsf{DK}_\alpha$, and, by (ii)($d$), $\mathfrak{A} \in \mathbf{SP}\{\mathfrak{A}_i : i \in I\}$.

It remains to fulfill (i) and (ii). First we define the n-ary operations $\Diamond_n$ as

$$\begin{aligned}\Diamond_n(x_0, \ldots, x_{n-1}) &\stackrel{\text{def}}{=} \langle 1\rangle x_0 \cdot \ldots \cdot \langle 1\rangle x_{n-1} \cdot \\ &\langle 2\rangle(x_0 + x_1) \cdot \ldots \cdot \langle 2\rangle(x_{n-2} + x_{n-1}) \cdot \ldots \cdot \langle n\rangle(x_0 + \ldots + x_{n-1})\end{aligned}$$

and the unary operations $\langle n\rangle' x$ as

$$\langle n\rangle' x \stackrel{\text{def}}{=} \Diamond_n(x, \ldots, x).$$

Let Σ'_α be defined by substituting $\Diamond_n(x, \ldots, x)$ for $\langle n\rangle x$ in Σ_α. Let K'_α be the class of algebras of the form

$$\langle A, \cdot, -, ;, {}^\smile, \mathsf{id}, \Diamond_n\rangle_{0<n<\alpha}$$

satisfying Σ'_α: $\mathsf{K}'_\alpha = \mathrm{Mod}(\Sigma'_\alpha)$; and let DK'_α be the class of singleton-dense elements of K'_α: $\mathsf{DK}'_\alpha = \mathrm{Mod}(\Sigma'_\alpha + (d)')$ where $(d)'$ is defined by substituting $\Diamond_2(b, b)$ for $\langle 2\rangle b$ in (d).

Let $\mathfrak{Rd}$ and $\mathfrak{Rd}'$ be defined by

$$\mathfrak{Rd}'(\mathfrak{A}) = \mathfrak{Rd}'(\langle A, \cdot, -, ;, {}^\smile, \mathsf{id}, \langle n\rangle\rangle) \stackrel{\text{def}}{=} \langle A, \cdot, -, ;, {}^\smile, \mathsf{id}, \Diamond_n\rangle$$

and

$$\mathfrak{Rd}(\mathfrak{A}') = \mathfrak{Rd}(\langle A', \cdot, -, ;, {}^\smile, \mathsf{id}, \Diamond_n\rangle) \stackrel{\text{def}}{=} \langle A', \cdot, -, ;, {}^\smile, \mathsf{id}, \langle n\rangle'\rangle.$$

It is easy to check that (ii)(b) holds and that the cardinality of the algebras are preserved under $\mathfrak{Rd}'$. By the definition of Σ'_α,

$$(\forall \mathfrak{A} \in \mathsf{DK}_\alpha)\mathfrak{Rd}'(\mathfrak{A}) \in \mathrm{Mod}(\Sigma'_\alpha + (d)')$$

and

$$(\forall \mathfrak{A}' \in \mathsf{DK}'_\alpha)\mathfrak{Rd}(\mathfrak{A}') \in \mathrm{Mod}(\Sigma_\alpha + (d)),$$

whence (ii)(a) holds.

Now we check (i). Let $\mathfrak{A}' \in \mathsf{K}'_\alpha$. We have to show that

1. every $\Diamond_n$ is additive (we know this for ; and $\smile$ by the WA-axioms);
2. $\Diamond_1$ is a complemented closure operator;
3. $f(x_0, \ldots, x_{n-1}) \leq \Diamond_1 x_0 \cdot \ldots \cdot \Diamond_1 x_{n-1}$ for every n-ary extra-Boolean operation f.

2 and 3 hold by the definition of Σ'_α and $\Diamond_1$. To prove 1, we use Lemma 3.2.21. Since $\mathfrak{Rd}(\mathfrak{A}') \in \mathsf{K}_\alpha$, $\mathfrak{Rd}(\mathfrak{A}') \in \mathrm{Mod}(\Sigma_\alpha^-)$. This implies that the $\{;, \smile, \mathsf{id}\}$-free reduct of $\mathfrak{A}'$ can be represented as a Boolean set algebra with the operations $\Diamond_n$ defined as

$$\Diamond_n(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \begin{cases} W & \text{if } \exists y_1, \ldots, y_n(|\{y_1, \ldots, y_n\}| = n \;\&\; y_1 \in x_1 \;\&\; \ldots \;\&\; y_n \in x_n) \\ \emptyset & \text{otherwise} \end{cases}$$

for arbitrary elements $x_1, \ldots x_n$ (with top element W) — an easy way to prove this is to repeat the proof of Theorem 1.1.6, i.e., to show that the definition of $\Diamond_n$ by the $\langle k \rangle$'s work on set algebras. Since the additivity axioms hold in set algebras, they are valid in K'_α as well.

(ii)(c) follows from

$$\begin{aligned} \mathfrak{A} \in \mathbf{SimDK}'_\alpha &\iff (\forall 0 < a \in A)\Diamond_1 a = 1 \iff \\ &\iff (\forall 0 < a \in A')\langle 1 \rangle a = 1 \iff \\ &\iff \mathfrak{Rd}(\mathfrak{A}') \in \mathbf{SimDK}_\alpha. \end{aligned}$$

Finally, (ii)(d) follows from the fact that $\langle n \rangle'$ was defined by an equation, and that equations are preserved under **SP**. ∎

Now we are in the position to finish the proof of Theorem 3.2.20. Let $\mathfrak{A} \in \mathsf{DK}_\alpha$ and $|A| \leq \omega$. By Lemma 3.2.22, we can embed $\mathfrak{A}$ into a product of simple elements of DK_α. Since simple algebras are representable by Theorem 3.2.18, so is $\mathfrak{A}$. Thus Theorem 3.2.20 has been proved. ∎

Remark 3.2.23. In the following chapter, we will prove a similar result for RRA: there are a finitely axiomatizable variety RA and a property called rectangularity such that the rectangularly dense elements of RA are representable. In RA, $\langle 1 \rangle$ and $\langle 2 \rangle$ are expressible. We conjecture that, by a similar argument as above, the same result holds for $\mathsf{RRA}^{<\alpha}$. ∎

3.2.3 Decidability of $(\mathbf{Rl}_H\mathsf{RRA})^{<\alpha}$

Below, we will prove that the equational theory of $\mathsf{WA}^{<\alpha}$ $(\alpha \leq \omega)$ is decidable. Easy modifications in the proof yield the same result for the other relativizations $(\mathbf{Rl}_H\mathsf{RRA})^{<\alpha}$ $(\alpha \leq \omega$ and $H \subseteq \{r, s\})$. We will use the so-called *mosaic method*, cf. [Né86], [Né92], and [Né95].

Theorem 3.2.24. *Let $H \subseteq \{r, s, t\}$, and let $n \in \omega \setminus 1$. The equational theories of $\mathbf{Rl}_H\mathsf{RRA}^{<\omega}$, $\mathbf{Rl}_H\mathsf{RRA}^{<n}$, and $\mathbf{Rl}_H\mathsf{RRA}^{\mathsf{D}}$ are decidable iff $t \notin H$.*

To prove Theorem 3.2.24 we need some definitions.

CONVENTION. By a graph we mean a directed, symmetric and reflexive graph. That is, if E is the set of edges, then $\langle u,v\rangle \in E$ implies $\langle u,u\rangle, \langle v,v\rangle, \langle v,u\rangle \in E$. A labeled graph is $G = \langle E, \ell\rangle$ where E is a graph and $\ell : E \longrightarrow \mathcal{P}(X)$ for some set X.

Definition 3.2.25. **(Consistently labeled graph,** ***CLG*****)** Let $G = \langle E, \ell\rangle$ be a labeled graph. Then G is a *consistently labeled graph*, $G \in CLG$, if the following holds.

The edges are labeled with $\mathsf{WA}^{<\alpha}$-terms, i.e., $\ell : E \longrightarrow \mathcal{P}(X)$ for some set X of $\mathsf{WA}^{<\alpha}$-terms, and for every $\langle u,v\rangle = e \in E$, $x, y \in X$ and $n \in \alpha \smallsetminus 1$,

$$\begin{array}{lrcl}
(A1) & x, y \in \ell(e) & \iff & x \cdot y \in \ell(e) \\
(A2) & x \in \ell(e) & \iff & -x \notin \ell(e) \\
(A3) & x^{\smile} \in \ell\langle u,v\rangle & \iff & x \in \ell\langle v,u\rangle \\
(A4) & \mathsf{id} \in \ell\langle u,v\rangle & \iff & u = v \\
(A5) & -(x;y) \in \ell\langle u,v\rangle & \Rightarrow & \nexists w(x \in \ell\langle u,w\rangle \ \&\ y \in \ell\langle w,v\rangle) \\
(A6) & \langle n\rangle x, -\langle n+1\rangle x \in \ell(e) & \Rightarrow & (\exists! e_0, \ldots, e_{n-1} \in E)(\forall i,j \in n) \\
 & & & (i \neq j \Rightarrow e_i \neq e_j \ \&\ x \in \ell(e_i)) \\
(A7) & -\langle n\rangle x \in \ell(e) & \Rightarrow & (\nexists e_0, \ldots, e_{n-1} \in E)(\forall i,j \in n) \\
 & & & (i \neq j \Rightarrow e_i \neq e_j \ \&\ x \in \ell(e_i)) \\
(A8) & \langle n\rangle x \in \ell(e) & \Rightarrow & (\forall e' \in E)\langle n\rangle x \in \ell(e') \\
(A9) & -\langle n\rangle x \in \ell(e) & \Rightarrow & (\forall e' \in E) - \langle n\rangle x \in \ell(e') \\
(A10) & \langle n\rangle x \in \ell(e) & \Rightarrow & (\forall e' \in E)(\forall k \in n)\langle k\rangle x \in \ell(e')
\end{array}$$

provided that the corresponding terms are in X. We recall that for $\langle n\rangle x \cdot -\langle n+1\rangle x$ we use the abbreviation $\langle n!\rangle x$. ∎

Definition 3.2.26. **(Mosaic)** Let $\mu = \langle m, \ell\rangle$ be a finite CLG. We say that μ is a *mosaic* if the following holds, cf. Figure 3.1. We can divide m into three parts: $m = m_0 \cup m_1 \cup m_2$ for some sets m_0, m_1, and m_2 satisfying the conditions below.

1. m_0 consists of "distinguished" arrows, i.e.,

$$m_0 = \{\langle u,v\rangle \in m : u, v \in p_\mu\}$$

where

$$p_\mu = \{u : \exists v \exists x (\exists n \in \alpha \smallsetminus 1)(x \cdot \langle n!\rangle x \in \ell\langle u,v\rangle \cup \ell\langle v,u\rangle\};$$

2. m_2 consists of arrows connecting m_0 and m_1, i.e.,

$$(\forall \langle u,v\rangle \in m_2)((u \in p_\mu \ \&\ \langle v,v\rangle \in m_1) \text{ or } (v \in p_\mu \ \&\ \langle u,u\rangle \in m_1)).$$

Two mosaics μ and ν are *isomorphic*, in symbols $\mu \cong \nu$, if there is a label-preserving graph-isomorphism f between them, $f : \mu \longrightarrow \nu$.

We say that μ is a *submosaic* of ν, in symbols $\mu \subseteq \nu$, if the labeling set X is the same, $m_0 = n_0$, $m_1 \subseteq n_1$, $m_2 \subseteq n_2$,

$$(\forall \langle u,v\rangle \in n)(\langle u,u\rangle \in m \ \&\ \langle v,v\rangle \in m) \Rightarrow \langle u,v\rangle \in m$$

and the labels of the common edges agree, cf. Figure 3.2. ∎

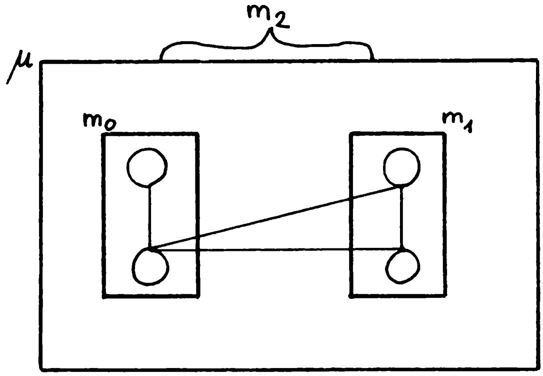

Figure 3.1: a mosaic

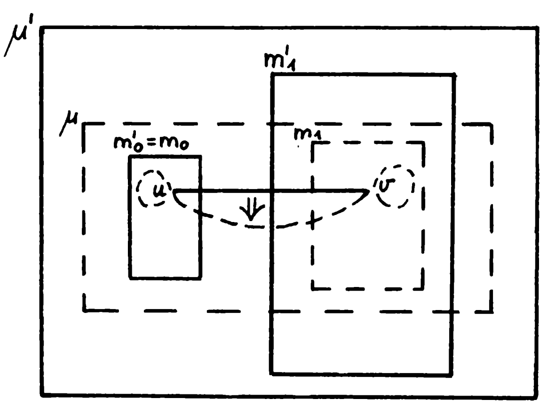

Figure 3.2: submosaic

Note that a mosaic may contain some "defects". That is, there may be an edge e such that $x;y$ or $\langle n\rangle x$ is in the label $\ell(e)$ of e, but there are no edges with the appropriate labels witnessing the label of e. That is why we define good set of mosaics (GSM) below. The idea of a GSM is that all of the defects of each element of a GSM disappear if we put the members of the GSM together in an appropriate way.

Definition 3.2.27. (Good set of mosaics, *GSM*) A finite set M of mosaics is a *good set of mosaics*, $M \in GSM$, if the following conditions hold.

1. The m_1 part of every mosaic is a triangle: for all $\mu \in M$,

$$m_1 = {}^2\{u, v, w\}$$

for not necessarily distinct points u, v, and w.

2. Condition for composition, cf. Figure 3.3:

$$\begin{aligned}&(\forall\mu \in M)(\forall e \in m)(\forall x;y \in X)x;y \in \ell(e) \Rightarrow\\&\quad(\exists\mu' \in M)(\exists\nu,\nu',\nu'' \text{ mosaics})(\exists h,h' \text{ isomorphisms})\\&\quad h : \mu \longrightarrow \nu \;\&\; h' : \mu' \longrightarrow \nu' \;\&\; \nu \subseteq \nu'' \;\&\; \nu' \subseteq \nu'' \;\&\\&\quad(\exists e_0, e_1, e_2 \in m')(\exists u, v, w)h'(e_0) = h(e) = \langle u, v\rangle \;\&\\&\quad h'(e_1) = \langle u, w\rangle \;\&\; h'(e_2) = \langle w, v\rangle \;\&\; x \in \ell(e_1) \;\&\; y \in \ell(e_2).\end{aligned}$$

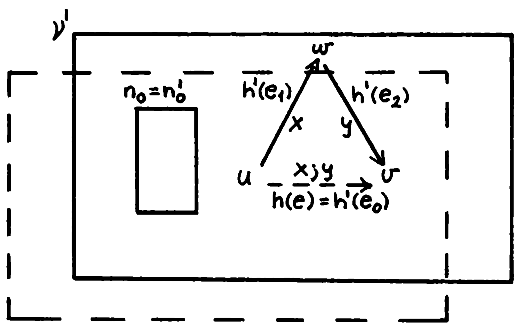

Figure 3.3: composition condition

3. Condition for $\langle n\rangle$, cf. Figure 3.4:

$$\begin{aligned}&(\forall\mu \in M)(\forall e \in m)(\forall\langle n\rangle x \in X)\langle n\rangle x \in \ell(e) \Rightarrow\\&\quad(\exists\mu^{(0)},\ldots,\mu^{(n-1)} \in M)(\exists\nu,\nu^{(0)},\ldots,\nu^{(n)} \text{ mosaics})\\&\quad(\exists h, h_0,\ldots,h_{n-1} \text{ isomorphisms})(\forall i \in n)h_i : \mu^{(i)} \longrightarrow \nu^{(i)} \;\&\\&\quad h : \mu \longrightarrow \nu \;\&\; \nu \subseteq \nu^{(n)} \;\&\; \nu^{(i)} \subseteq \nu^{(n)} \;\&\\&\quad(\exists e_0,\ldots e_{n-1})e_i \in m^{(i)} \;\&\; x \in \ell(e_i) \;\&\; (\forall j \neq i)h_j(e_j) \neq h_i(e_i).\end{aligned}$$

4. M is closed under submosaics:

$$(\forall\mu \in M)(\forall\mu' \text{ mosaic})\mu' \subseteq \mu \Rightarrow \mu' \in M.$$

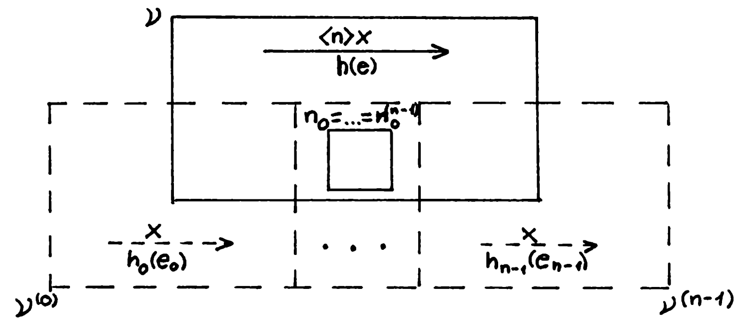

Figure 3.4: condition for $\langle n\rangle$

∎

Now we can turn to proving Theorem 3.2.24. Sometimes we will not distinguish between isomorphic copies of the same mosaic.

Proof of Theorem 3.2.24: By [AKNSS], any non-trivial extension of $\mathbf{Rl}_H\mathsf{RRA}$ has an undecidable equational theory whenever $t \in H$. Thus the equational theories of $(\mathbf{Rl}_H\mathsf{RRA})^{<\omega}$, $(\mathbf{Rl}_H\mathsf{RRA})^{<n}$, and $(\mathbf{Rl}_H\mathsf{RRA})^{\mathsf{D}}$ are undecidable provided $t \in H$.

For the other direction we give the precise proof for $H = \{r, s\}$. For the other choices of H, one should consider different kinds of graphs (not symmetric and/or reflexive ones), and modify the definitions of CLG and mosaic. Apart from this obvious modification the same proof works.

The decidability of WAD follows from that of $\mathsf{WA}^{<3}$, since the two classes are term-definitionally equivalent. To decide whether an equation of $\mathsf{WA}^{<\omega}$ is valid, it is enough to check whether it holds in every $\mathsf{WA}^{<l}$ where l is an upper bound for those k's such that $\langle k\rangle$ occurs in the equation in question. Thus to prove the theorem, it suffices to decide the equational theory of $\mathsf{WA}^{<l}$ for every $l \in \omega \smallsetminus 1$.

Let $l \in \omega \smallsetminus 1$ be fixed, and $\sigma = \tau$ be an equation in the language of $\mathsf{WA}^{<l}$. Our goal is to decide whether $\sigma = \tau$ is valid in $\mathsf{WA}^{<l}$. The above equation is valid iff so is $\sigma \oplus \tau = 0$. Thus it is enough to check whether the value of a term is empty in every $\mathsf{WA}^{<l}$.

Instead of a term ξ we can decide $\xi \cdot \mathsf{O}z$ where z is a variable not occurring in ξ and $\mathsf{O}z$ abbreviates $z \cdot \langle 1!\rangle z$, since the two terms are equivalent. So let ξ be given. Then we define an appropriate closure set X of terms:

$$\begin{aligned}
X_0 &\stackrel{\text{def}}{=} \text{subterms of } \xi \cdot \mathsf{O}z \cdot \mathsf{id}\\
X_1 &\stackrel{\text{def}}{=} \{\langle n!\rangle x : \exists x (\exists k \geq n)\langle k\rangle x \in X_0\} \cup X_0\\
X_2 &\stackrel{\text{def}}{=} \text{subterms of } X_1\\
X &\stackrel{\text{def}}{=} \text{Boolean closure of } X_2.
\end{aligned}$$

Note that X is a finite set of terms.

We say that a term is satisfiable if there is a $\mathsf{WA}^{<l}$ such that the value of this term is not empty. We will have two steps in deciding whether a term is satisfiable: (i)

Lemma 3.2.28 ensures that it is a decidable procedure to find out if there is a good set of mosaics for a given term and (ii) Lemma 3.2.29 guarantees that a term is satisfiable iff there is a good set of mosaics for this term.

Lemma 3.2.28. *Given ξ and X above, it is decidable whether there is a GSM M labeled with X for which $(\exists \mu \in M)(\exists e \in m)\xi \cdot \mathsf{O}z \in \ell(e)$.*

Proof: If there is a GSM M satisfying the conditions of Lemma 3.2.28, then the size of the mosaics of M is bounded. Indeed, let s be the size of X. Given a term $\langle n! \rangle x \cdot x \in X$ and a mosaic μ, we can (and have to) write the term on precisely n edges, cf. Definition 3.2.25. Let k be the upper bound for $\{n : \exists x(\langle n \rangle x \in X_0)\}$. Then there are at most ks edges labeled by terms of the form $\langle n! \rangle x \cdot x$. Thus $(\forall \mu \in M)|p_\mu| \leq 2ks$. Since for every $\mu \in M$, the m_1 part is a "triangle", i.e., it contains at most three points, every mosaic has size not greater than $(2ks+3)^2$. So the size of every mosaic in M is bounded. And there are only finitely many graphs of this size (up to isomorphism), and they can be labeled by X only in finitely many ways, since X is finite. That is, we have to check only finitely many finite sets of mosaics whether at least one of them satisfies the conditions of Lemma 3.2.28, and this is a decidable procedure. ∎

Thus to prove Theorem 3.2.24 it suffices to prove Lemma 3.2.29 below.

Lemma 3.2.29. *$\xi \cdot \mathsf{O}z$ is satisfiable $\Longleftrightarrow$ there is a GSM M for X such that $(\exists \mu \in M)(\exists e \in m)\xi \cdot \mathsf{O}z \in \ell(e)$.*

Proof: $\Rightarrow$: It is not hard to see that we can "cut out" the appropriate set of mosaics from a $\mathfrak{A} \in \mathsf{WA}^{<l}$ satisfying $\xi \cdot \mathsf{O}z$. Let $W \subseteq U \times U$ be the top element of $\mathfrak{A}$, and let the labeling function $\ell : W \longrightarrow \mathcal{P}(X)$ be defined as

$$x \in \ell(w) \Longleftrightarrow w \in x^{\mathfrak{A}}.$$

Then $\mathfrak{A}' = \langle W, \ell \rangle$ is a CLG without defects. For every edge $e = \langle u, v \rangle \in W$ and $w \in U$, let the mosaic $\mu_e(w) = \langle m, \ell \rangle$ be defined as follows. Let m_0 be the smallest WA-unit generated by the pairs labeled by $\langle n! \rangle x \cdot x$ terms. Let m_1 be the triangle defined by $\{u, v, w\}$, and let m_2 be the set of edges of W connecting the m_0 and m_1 parts. Since each $\mu_e(w)$ $(e \in W,\ w \in U)$ is in the same CLG $\mathfrak{A}'$ without defects, and their union is A', they satisfy the conditions of Definition 3.2.27.

$\Leftarrow$: First, using the elements of M as building blocks, we will construct a CLG $G_\omega = \langle E_\omega, \ell_\omega \rangle$ without "defects", i.e., for every $x; y$ and $\langle n \rangle x$ label there will be "witnessing" edges for these labels. More precisely, the following two conditions will hold:

$$\begin{array}{lrcl} (*) & x; y \in \ell_\omega\langle u, v \rangle & \Rightarrow & \exists w(x \in \ell_\omega\langle u, w \rangle \ \& \ y \in \ell_\omega\langle w, v \rangle) \\ (**) & \langle n \rangle x \in \ell_\omega(e) & \Rightarrow & (\exists e_0, \dots e_{n-1} \in E_\omega)(\forall i \in n)x \in \ell_\omega(e_i) \ \& \ (\forall j \neq i)e_i \neq e_j. \end{array}$$

Then we will define a $\mathsf{WA}^{<l}$ satisfying $\xi \cdot \mathsf{O}z$.

0TH STEP: Take a mosaic $\mu \in M$ such that $(\exists e \in m)\xi \cdot \mathsf{O}z \in \ell(e)$.

$2n+1$ST STEP: By induction hypothesis, the finite mosaic $G_{2n} = \langle E_{2n}, \ell_{2n} \rangle$ constructed so far consists of members of M, i.e., $(\forall e \in E_{2n})(\exists \mu \in M)e \in m$ and $\mu \subseteq G_{2n}$.

Enumerate all the edges $\langle u', v'\rangle = e' \in E_{2n}$ such that $(\exists x; y \in X)x; y \in \ell_{2n}(e')$, but there are no witnesses for this labeling, i.e., $\nexists w(x \in \ell_{2n}\langle u', w\rangle \ \&\ y \in \ell_{2n}\langle w, v'\rangle)$.

Take the first $e = \langle u', v'\rangle \in E_{2n}$ of this kind. Take $\mu \in M$ such that $e \in m$ and $\mu \subseteq G_{2n}$. Then, by the definition of a GSM, there is $\mu' \in M$ with witnesses that can be added to μ in the sense of Definition 3.2.27. We will add a submosaic of ν' to G_{2n}, cf. Definition 3.2.27 for notation.

Consider ν'. It contains points u, v, w such that $x; y \in \ell\langle u, v\rangle$, $x \in \ell\langle u, w\rangle$ and $y \in \ell\langle w, v\rangle$. We claim that $\langle w, w\rangle \notin n'_0$. Assume otherwise. Then $\langle w, w\rangle \in n_0$, since ν and ν' are submosaics of the same mosaic ν'' (that we get when we put μ and μ' together in the sense of Definition 3.2.27). Thus $\langle u, u\rangle, \langle w, w\rangle \in n$ and $\langle u, w\rangle \in n''$. Then, by $\nu \subseteq \nu''$, $\langle u, w\rangle \in n$. Similarly, $\langle w, v\rangle \in n$. That is, we have witnesses already in ν for the label $x; y \in \ell\langle u, v\rangle$. Since $\mu \cong \nu$, there are witnesses in μ as well, a contradiction.

Let $\rho = \langle r, \ell\rangle$ be the smallest submosaic of ν' containing ${}^2\{u, v, w\}$:

$$r = n'_0 \cup {}^2\{u, v, w\} \cup ((p_{\nu'} \times \{u, v, w\}) \cap n') \cup ((\{u, v, w\} \times p_{\nu'}) \cap n')$$

and the labels in ρ agree with the labels in ν'. It is easy to see that ρ is indeed a submosaic, whence $\rho \in M$.

We will add ρ to G_{2n}, yielding a new mosaic $G' = \langle E', \ell'\rangle$ satisfying the induction hypothesis, in such a way that in G' there are witnesses for $x; y \in \ell'(e)$. Since $\mu \cong \nu$, there is an isomorphism $f : \nu \longrightarrow \mu$ such that $f\langle u, v\rangle = \langle u', v'\rangle$. This f induces a map f_0 between the set of nodes of ν and that of μ: $f_0(s) = s'$ iff $f\langle s, t\rangle = \langle s', t'\rangle$ for some t and t'. Now let w' be a brand new point not occurring in the set of nodes of G_{2n} and $f_0(w) = w'$. Then f_0 induces an isomorphism from ρ: $f\langle s, t\rangle = \langle f_0(s), f_0(t)\rangle$ and $\ell(f\langle s, t\rangle) = \ell\langle s, t\rangle$ for $\langle s, t\rangle \in r$. Let the f-image of ρ be denoted by $\rho' = \langle r', \ell\rangle$. We will add the triangle $u'w'v'$ to G_{2n} with the necessary reflexive, converse and connecting edges with their labels in ρ:

$$\begin{aligned} E' &\stackrel{\text{def}}{=} E \cup r' \\ \ell' &\stackrel{\text{def}}{=} \ell \cup \{\langle f\langle s, t\rangle, \ell\langle s, t\rangle\rangle : \langle s, t\rangle \in r\}, \end{aligned}$$

cf. Figure 3.5.

We have to check that G' satisfies the induction hypothesis. Let $e \in E' \setminus E_{2n}$. Then $e \in r'$, and $\rho' \cong \rho \in M$, i.e., E' consists of members of M.

For G' being a mosaic we check that "unintended" triangles are not constructed, i.e., every triangle not in E_{2n} is in ρ'. Indeed, we added only triangles of the form $u'w'v'$, $u'w's$, $w'v's$ and $s_1w's_2$ where $s, s_1, s_2 \in p_{\rho'}$, and all of them are in ρ'. Thus conditions $(A1) - (A5)$ of Definition 3.2.25 hold.

To check that $(A6) - (A10)$ hold we use the fact that, in E_{2n}, there is an edge, say e, with $\langle 1!\rangle z \cdot z$ label. (Actually, this is the reason why we decide the term $\xi \cdot \mathsf{O}z$ instead of ξ.) Then $e \in r' \cap E_{2n}$, since $e \in m_0 = r'_0$.

Let us check $(A7)$. Let $-\langle j\rangle x \in \ell'(e')$ for some x and $e' \in E'$. Let $i = \max\{k : (\exists e \in E')\langle k\rangle x \in \ell'(e)\}$. Then $i < j$ because of the following. If there were an edge e'' such that $\langle j\rangle x \in \ell'(e'')$, then, by $(A8)$ for ρ' or G_{2n} according to where e'' is, $\langle j\rangle x \in \ell'(e)$. Then, by $(A8)$ for ρ' or G_{2n} according to where e' is, $\langle j\rangle x \in \ell(e')$, a contradiction.

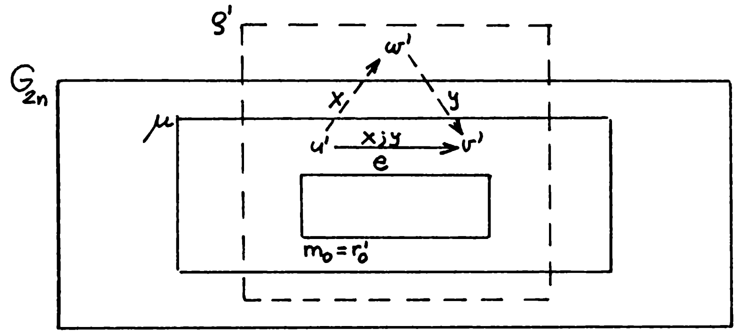

Figure 3.5: the new graph G' in the $2n+1$st step

CASE 1: $i \neq 0$. This means that there is $e'' \in E'$ such that $\langle i!\rangle x \in \ell'(e'')$. Then by $(A6)$ both in ρ' and in G_{2n} exactly i edges are labeled by x. And these edges are in the distinguished part r_0', since their labels include $\langle i!\rangle x \cdot x$. That is, these edges are in $E_{2n} \cap r'$. Now, if there were another (an $i+1$st) edge labeled with x, then either ρ' or G_{2n} would violate $(A6)$. That is, there are at most i edges in E' labeled by x.

CASE 2: $i = 0$, i.e., $(\forall e \in E')\langle 1\rangle x \notin \ell'(e)$. Then, by $(A7)$ for ρ' and G_{2n}, there is no edge labeled by x.

Conditions $(A6)$ and $(A8)-(A10)$ can be checked in a similar way using the distinguished edge e above as a bridge between the two mosaics ρ' and G_{2n}. Thus the new labeled graph E' is a mosaic.

We make the same construction with this new graph and the second enumerated edge without witnesses, etc. In finitely many steps this construction terminates, and we get a mosaic G_{2n+1} satisfying the induction hypothesis.

$2n+2$ND STEP: We enumerate the edges $e \in E_{2n+1}$ for which $(\exists\langle k\rangle x \in X)\langle k\rangle x \in \ell_{2n+1}(e)$ but without witnesses, i.e., $(\nexists e_0, \ldots, e_{k-1} \in E_{2n+1})(\forall i \in k) x \in \ell_{2n+1}(e_i)$. Take the first such e. Then by induction hypothesis $(\exists \mu \in M) e \in m$ & $\mu \subseteq G_{2n+1}$. By the definition of GSM, $(\exists \mu^{(0)}, \ldots, \mu^{(k-1)} \in M)$ satisfying the $\langle k\rangle$-condition of Definition 3.2.27. Let this new mosaic consisting of $\mu, \mu^{(0)}, \ldots, \mu^{(k-1)}$ be denoted by $\nu^{(k)}$, cf. the notation in Definition 3.2.27. Our task is to define a mosaic $G' = \langle E', \ell'\rangle$ such that in G' there are witnesses for $\langle k\rangle x \in \ell'(e)$.

Let, for every $i \in k$, $\rho^{(i)} = \langle r^{(i)}, \ell\rangle$ be the smallest submosaic of $\nu^{(i)}$ containing $h_i(e_i) = \langle u_i, v_i\rangle$:

$$r^{(i)} \stackrel{\text{def}}{=} n_0^{(i)} \cup {}^2\{u_i, v_i\} \cup ((p_{\nu^{(i)}} \times \{u_i, v_i\}) \cap n^{(i)}) \cup ((\{u_i, v_i\} \times p_{\nu^{(i)}}) \cap n^{(i)})$$

and the labels agree with those in $\nu^{(i)}$. It is easy to check that $\rho^{(i)} \subseteq \nu^{(i)}$, whence $\rho^{(i)} \in M$.

As in the previous step, there is an isomorphism $f : \nu \longrightarrow \mu$. Again, let f_0 be the map determined by f. Let U be the union of the sets of nodes of $\rho^{(i)}$ $(i \in k)$. For every $u \in U$, let $g_0(u) = f_0(u)$ if u is a node of ν and a brand new point otherwise. Let g be the isomorphism induced by g_0: for every $\langle u, v\rangle \in r^{(i)}$ $(i \in k)$, $g\langle u, v\rangle = \langle g_0(u), g_0(v)\rangle$

and $\ell(g\langle u,v\rangle) = \ell\langle u,v\rangle$. Let $G' = \langle E', \ell'\rangle$ be defined as follows:

$$\begin{array}{rcl} E' & \stackrel{\text{def}}{=} & E_{2n+1} \cup \{g\langle u,v\rangle : (\exists i \in k)\langle u,v\rangle \in r^{(i)}\} \\ \ell' & \stackrel{\text{def}}{=} & \ell_{2n+1} \cup \{\langle g\langle u,v\rangle, \ell\langle u,v\rangle\rangle : (\exists i \in k)\langle u,v\rangle \in r^{(i)}\}, \end{array}$$

cf. Figure 3.6. It is easy to see that in G' there are witnesses for the label $\langle k\rangle x \in \ell'(e)$.

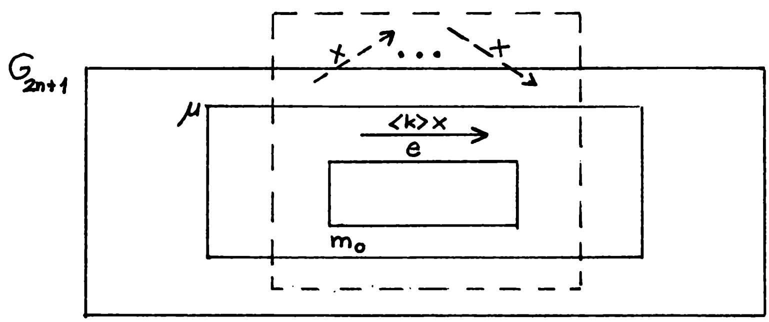

Figure 3.6: the new graph G' in the $2n+2$nd step

It remains to check that G' satisfies the induction hypothesis. Clearly every new edge is in (the g-image of) one of the $\rho^{(i)}$'s. As we mentioned, $\rho^{(i)} \in M$. It is easy to see that every new triangle ${}^2\{u,v,w\}$ (which is in E' but not in E_{2n+1}) is in (the g-image of) $\rho^{(i)}$ for some $i \in k$. That is, every triangle constructed in this step is in $\nu^{(k)}$. Since $\nu^{(k)}$ is a mosaic, it satisfies $(A5)$. Then every triangle of E' satisfies $(A5)$. $(A1)-(A4)$ hold because we constructed G' from mosaics. Finally, conditions $(A6)-(A10)$ can be checked precisely in the same way as in the $2n+1$st step.

Take the next enumerated edge, and make the same construction, etc. Again, in finitely many steps this procedure terminates yielding a mosaic G_{2n+2}.

ωTH STEP: Take the union of the already constructed mosaics. Then we get a (probably infinite) CLG $G_\omega = \langle E_\omega, \ell_\omega\rangle$ without defects, i.e., G_ω satisfies conditions $(*)$ and $(**)$. Let

$$\mathfrak{A} = \langle \mathcal{P}(E_\omega), \cap, \sim, \circ, ^{-1}, \mathsf{Id}, \langle i\rangle\rangle_{i<l} \in \mathsf{WA}^{<l}$$

and the valuation k' be defined as

$$k'(y) = \{\langle u,v\rangle \in E_\omega : y \in \ell_\omega\langle u,v\rangle\}$$

for every variable y. Let k be the obvious extension of k': for variable y and terms x, x',

$$\begin{array}{rcl} k(y) & = & k'(y) \\ k(x \cdot x') & = & k(x) \cap k(x') \\ k(-x) & = & \sim k(x) \\ k(\mathsf{id}) & = & \mathsf{Id} \\ k(x^{\smile}) & = & (k(x))^{-1} \\ k(x;x') & = & k(x) \circ k(x') \\ k(\langle i\rangle x) & = & \langle i\rangle k(x). \end{array}$$

Proposition 3.2.30. *For every* $x \in X$, $k(x) = \{\langle u, v\rangle \in E_\omega : x \in \ell_\omega\langle u, v\rangle\}$.

Proof: It is an easy induction on the complexity of x. For every connective use the corresponding condition of CLG without defects: for $\cdot$, $-$, id, $\smile$, $;$, and $\langle i\rangle$ use $(A1)$, $(A2)$, $(A4)$, $(A3)$, $(A5)$ and $(*)$, and $(A7)$ and $(**)$, respectively. ∎

By the previous proposition we have that the value of $\xi \cdot \mathsf{O}z$ in $\mathfrak{A}$ is not empty, i.e., $\xi \cdot \mathsf{O}z$ is satisfiable. ∎

As we mentioned above this finishes the proof of Theorem 3.2.24. ∎

4

Squares and rectangles

In this chapter,[1] we will introduce our third taming strategy to obtain completeness results.

The situation can be described as follows. We saw that the *square* version of pair arrow logic does not have a strongly sound and complete Hilbert-style inference system. Or, equivalently, the class RRA of representable relation algebras forms a non-finitely axiomatizable (quasi-)variety. A similar result holds for (the finite variable fragment of) classical first-order logic (if there are at least three variables). Recall that in the definition of first-order logic we require that a valuation of the variables is any member of ${}^nU$, where U is the universe of the model and n is the number of variables. That is, the set of all valuations is a Cartesian space ${}^nU$, or, in other words, a square.

In the previous section, we described how to define nice versions of pair arrow logic applying the non-square approach. There, we allowed models with non-square universes. The same strategy can be applied to first-order logic as well, cf. [Né92] and [Mi95]. The disadvantage of that approach is that we cannot preserve the full power of the logic. For a more syntactic approach to finding nice versions of first-order logic see [ABN95].

Below we will show how to obtain completeness results for the *square* versions of pair arrow logic and classical first-order logic. To achieve such results we have to allow calculi that are not Hilbert-style. Indeed, we will consider inference rules that can be applied only if some easily decidable syntactic condition is met.[2] Namely, we will require that some atomic formula does not occur as a subformula in the conclusion of the rule.

Let us have a look at the algebraic side. If we consider the square versions of our logics, then the algebraic counterparts consist of algebras of relations with Cartesian space units (of the form ${}^nU$). Both in the cases of arrow logic and of first-order logic, the (quasi-)varieties generated by these algebras form non-finitely axiomatizable classes. To overcome this difficulty we will do the following. We will define finitely axiomatizable varieties and a property called *rectangularity.* We call an algebra *rectangularly dense* if below every non-zero element, there is a non-zero rectangular element. We will show that rectangularly dense members of the above-mentioned varieties are representable as algebras of relations with square top elements.

These representability results imply completeness of the logics, cf. Theorem 1.3.11. In fact, we will define a rule corresponding to rectangular density, and the non-Hilbert-style calculi defined by this rule are weakly sound and complete w.r.t. our logics.

[1]This chapter is (partially) based on the papers [AGMNS] and [Mi93].
[2]This kind of calculi are called Gabbay-style in [Mi93], or unorthodox in [Ve91].

4.1 Weak soundness and completeness

As we mentioned above, the finite variable fragment of first-order logic and the square version of pair arrow logic do not have strongly sound and complete Hilbert-style inference systems.

In this section, we will show that weak soundness and completeness is possible. However, these calculi will not be Hilbert-style, since there is an inference rule that can be applied only if a certain, easily decidable syntactic condition is met. That is, we will consider non-Hilbert-style calculi to prove completeness theorems for classical first-order logic and pair arrow logic *without* weakening their power.

We mention that there is another, more adventurous, approach to the problem of Hilbert-style incompleteness of logics. This amounts to re-define the logic in such a way that (i) the power of the logic does not become weaker, and (ii) Hilbert-style (strong) soundness and completeness holds. This may be achieved, e.g., by choosing another similarity type such that the old connectives with their standard meanings are definable, while the logic becomes Hilbert-style complete. See [Né94] and [Si93] for this approach under the name *finitization*, and [Sa87], [Sa92], [Sa94], and [SG95] for solutions to first-order logic.

4.1.1 First-order logic

Below we will consider several versions of the finite variable fragment of classical first-order logic: ordinary and restricted versions with and without equality. Let us recall that these logics were defined in Definition 1.1.4 in Chapter 1. We will define non-Hilbert-style calculi and prove weak soundness and completeness w.r.t. three of them. As a corollary, we will get completeness results for the corresponding first-order logics with *infinitely* many variables: since every formula φ uses only finitely many variables, say n, φ is valid iff φ is valid in the n-variable fragment iff φ can be derived using the calculus for the finite variable fragment.

As usual, the results will follow from the corresponding algebraic representation theorems using the bridge Theorem 1.3.11. Thus, we have to define the appropriate algebraizations of these logics. To achieve this, we will define modal versions of first-order logics that will turn out to be equivalent to the original formalizations.

Let us formulate our main result. We note that Yde Venema proved similar results for first-order logics with equality, cf. [Ve91] and [Ve94].

Theorem 4.1.1. *Let* L *be one of the following logics:*

1. *ordinary first-order logic with* n *variables with equality,* $\mathsf{L}_n{}^{=}$,
2. *restricted first-order logic with* n *variables with equality,* ${}^{r}\mathsf{L}_n{}^{=}$,
3. *ordinary first-order logic with* n *variables without equality,* $\mathsf{L}_n{}^{\neq}$.

Let us assume that there are infinitely many relation symbols in the language of L. *Then there is a weakly sound and complete inference system for* L.

We conjecture that the above theorem can be extended to ${}^{r}\mathsf{L}_n{}^{\neq}$.

Proof of Theorem 4.1.1: This a straightforward consequence of Corollary 4.1.6 and Theorem 4.1.8 below. ∎

To define the algebraic counterpart of first-order logic, it is convenient to consider it as a multimodal logic, cf., e.g., [Ve94].

Let W be a set of n-tuples, i.e., $W \subseteq {}^nU$ for some set U. Let, for every $i < n$, the binary relation T_i on W be defined as:

$$(\forall w, w' \in W) wT_iw' \iff w\lceil n \setminus \{i\} = w'\lceil n \setminus \{i\},$$

i.e., $(\forall j \neq i) w(j) = w'(j)$. Let $\tau \in {}^nn$, i.e., τ be a map from n into n. For every $\tau \in {}^nn$, let the binary relation S_τ on W be defined as:

$$(\forall w, w' \in W) wS_\tau w' \iff w' = w \circ \tau,$$

i.e., $w' = \langle w'(0), \ldots, w'(n-1)\rangle = \langle w(\tau(0)), \ldots, w(\tau(n-1))\rangle$. For every $i, j \in n$, the unary relation D_{ij} on W is defined as:

$$(\forall w \in W) D_{ij}w \iff w(i) = w(j).$$

We are ready to define the modal versions of first-order logics. We note that our definition slightly differs from the definition in [Ve94].

Definition 4.1.2. (Modal versions of $\mathsf{L}_n{}^=$, ${}^r\mathsf{L}_n{}^=$, $\mathsf{L}_n{}^{\neq}$ and ${}^r\mathsf{L}_n{}^{\neq}$: MLIQS_n, MLIQ_n, MLQS_n and MLQ_n) The *n-dimensional modal logic of identity, quantification and substitution* MLIQS_n is defined as the ordered tuple $\langle \mathsf{F}, \mathsf{M}, \models\rangle$ for which the following hold.

F is the set of formulas built up from a set R of propositional variables using the Boolean connectives, the unary connectives $\Diamond_i$ $(i \in n)$ and σ_τ $(\tau \in {}^nn)$, and the constants δ_{ij} $(i, j \in n)$. The symbols σ_τ and δ_{ij} are called *substitution* and *identity*, respectively.

A *frame* for MLIQS_n is an ordered tuple $\langle W, T_i, S_\tau, D_{ij} : i, j \in n \;\&\; \tau \in {}^nn\rangle$ such that $W = {}^nU$ for some set U. A *model* is a frame–evaluation pair, where an evaluation $I : R \longrightarrow \mathcal{P}(W)$ is a map associating a subset of W to every propositional variable $r \in R$. M denotes the class of models.

Let $\mathfrak{M}$ be a model, and let $w \in W$ be an element of the universe of $\mathfrak{M}$. We define the *local truth* $\langle \mathfrak{M}, w\rangle \Vdash \varphi$ by induction on the complexity of φ:

- $\langle \mathfrak{M}, w\rangle \Vdash r \iff w \in I(r)$ for every $r \in R$,
- $\langle \mathfrak{M}, w\rangle \Vdash \delta_{ij} \iff D_{ij}w$,
- if $\psi_1, \psi_2 \in \mathsf{F}$, then

$$\begin{aligned}
\langle \mathfrak{M}, w\rangle \Vdash \neg\psi_1 &\iff \text{not } \langle \mathfrak{M}, w\rangle \Vdash \psi_1,\\
\langle \mathfrak{M}, w\rangle \Vdash \psi_1 \wedge \psi_2 &\iff \langle \mathfrak{M}, w\rangle \Vdash \psi_1 \;\&\; \langle \mathfrak{M}, w\rangle \Vdash \psi_2,\\
\langle \mathfrak{M}, w\rangle \Vdash \Diamond_i\psi_1 &\iff (\exists w' \in W)(wT_iw' \;\&\; \langle \mathfrak{M}, w'\rangle \Vdash \psi_1),\\
\langle \mathfrak{M}, w\rangle \Vdash \sigma_\tau\psi_1 &\iff (\exists w' \in W)(wS_\tau w' \;\&\; \langle \mathfrak{M}, w'\rangle \Vdash \psi_1).
\end{aligned}$$

Truth in a model ($\models$) and the semantical consequence relation are defined in the usual way (cf. Definition 1.1.4).

The logic MLIQ_n is defined as the substitution-free fragment of MLIQS_n. The logics MLQS_n and MLQ_n are the δ_{ij}-free fragments of MLIQS_n and MLIQ_n, respectively. ∎

In [Ve94], instead of σ_τ ($\tau \in {}^n n$), only the transpositions (i.e., permutations swapping two elements and leaving every other element in its place) are used, since the other substitutions can be defined by means of transpositions and identities. We used the above similarity type, since this formalism works for the equality-free fragment of first-order logic as well.

It is easy to see, and is well known, that every $\tau \in {}^n n$ can be expressed as a composition of transpositions $[ij]$ and replacements $[i/j]$, where

$$[ij]\langle 0, \dots, i, \dots, j, \dots, n-1\rangle = \langle 0, \dots, j, \dots, i, \dots, n-1\rangle$$

and

$$[i/j]\langle 0, \dots, i, \dots, j, \dots, n-1\rangle = \langle 0, \dots, j, \dots, j, \dots, n-1\rangle.$$

Thus, instead of all the σ_τ's ($\tau \in {}^n n$) it is enough to add the connectives $\sigma_{[ij]}$ and $\sigma_{[i/j]}$ ($i, j \in n$). We will denote these connectives by π_{ij} and σ^i_j, respectively. Their interpretations are as follows. If $i \neq j$:

$$\langle \mathfrak{M}, k\rangle \models \sigma^i_j\varphi \iff (\exists k' \in W)(\forall l \neq i)(k(l) = k'(l) \;\&\; k'(i) = k'(j) \;\&\; \langle \mathfrak{M}, k'\rangle \models \varphi),$$

and

$$\langle \mathfrak{M}, k\rangle \models \sigma^i_i\varphi \iff \langle \mathfrak{M}, k\rangle \models \varphi.$$

The interpretation of *permutation* π_{ij} is as follows.

$$\begin{aligned}\langle \mathfrak{M}, k\rangle \models \pi_{ij}\varphi \iff\; & (\exists k' \in W)(\forall l \notin \{i, j\}) \\ & (k'(l) = k(l) \;\&\; k(i) = k'(j) \;\&\; k(j) = k'(i) \;\&\; \langle \mathfrak{M}, k'\rangle \models \varphi).\end{aligned}$$

Next we define a translation between modal and first-order formalisms.

Definition 4.1.3. The translation ST from the set of modal formulas onto the set of first-order formulas is defined by recursion on the complexity of the modal formulas:

$$
\begin{aligned}
ST(r) &= r(v_0, \ldots, v_{n-1}) \\
ST(\delta_{ij}) &= (v_i = v_j) \\
ST(\neg\varphi) &= \neg ST\varphi \\
ST(\varphi \wedge \psi) &= ST(\varphi) \wedge ST(\psi) \\
ST(\Diamond_i\varphi) &= \exists v_i ST(\varphi) \\
ST(\sigma_\tau r) &= r(v_{\tau(0)}, \ldots, v_{\tau(n-1)}) \\
ST(\sigma_\tau \delta_{ij}) &= (v_{\tau(i)} = v_{\tau(j)}) \\
ST(\sigma_\tau \neg\varphi) &= \neg ST(\sigma_\tau\varphi) \\
ST(\sigma_\tau(\varphi \wedge \psi)) &= ST(\sigma_\tau\varphi) \wedge ST(\sigma_\tau\psi) \\
ST(\sigma^i_j \Diamond_i\varphi) &= ST(\Diamond_i\varphi) \\
ST(\sigma^i_j \Diamond_j\varphi) &= ST(\Diamond_j\varphi) \\
ST(\sigma^i_j \Diamond_k\varphi) &= ST(\Diamond_k\sigma^i_j\varphi) \quad (k \notin \{i,j\}) \\
ST(\pi_{ij} \Diamond_i\varphi) &= ST(\Diamond_j\pi_{ij}\varphi) \\
ST(\pi_{ij} \Diamond_j\varphi) &= ST(\Diamond_i\pi_{ij}\varphi) \\
ST(\pi_{ij} \Diamond_k\varphi) &= ST(\Diamond_k\pi_{ij}\varphi) \quad (k \notin \{i,j\}).
\end{aligned}
$$

∎

The above translation is not one-one, i.e., there are different modal formulas ($\sigma^i_j\Diamond_i\varphi$ and $\Diamond_i\varphi$) mapped to the same first-order formula ($\exists v_i ST(\varphi)$). But we would like ST to preserve truth. This is guaranteed by the following.

Proposition 4.1.4. *For every model $\mathfrak{M}$ and formulas φ, ψ,*

$$ST(\varphi) = ST(\psi) \Rightarrow (\mathfrak{M} \models \varphi \iff \mathfrak{M} \models \psi).$$

Proof: It is an easy induction following the definition of ST. We show the only non-trivial case. Assume that, for every w, $\langle\mathfrak{M}, w\rangle \Vdash \sigma^i_j\Diamond_j\varphi$. Let w' be arbitrary. We want $\langle\mathfrak{M}, w'\rangle \Vdash \Diamond_j\varphi$. Let $w'' = w' \circ [i/j]$. By assumption, $\langle\mathfrak{M}, w''\rangle \Vdash \sigma^i_j\Diamond_j\varphi$. Then $\langle\mathfrak{M}, w''\rangle \Vdash \Diamond_j\varphi$, since $w'' = w'' \circ [i/j]$. Since $w'T_jw''$, we have $\langle\mathfrak{M}, w'\rangle \Vdash \Diamond_j\varphi$ as well. The other direction can be proved similarly. ∎

We note that, in the above proposition, we cannot substitute $\Vdash$ for $\models$, as the above example shows.

The following proposition ensures that the first-order formalism and the modal formalism are equivalent.

Proposition 4.1.5. *Let φ be any formula of* MLIQS_n. *Let $U \neq \emptyset$ and $I : R \longrightarrow \mathcal{P}({}^nU)$. Then*

$$\langle {}^nU, T_i, S_\tau, D_{ij}, I\rangle \models \varphi \iff \langle U, I\rangle \models ST(\varphi).$$

The same holds for MLIQ_n, MLQS_n, *and* MLQ_n.

Proof: It is a straightforward induction on the complexity of φ. ∎

The above proposition ensures that completeness results for modal logics can be converted to completeness results for their first-order versions. Let $\vdash$ be an inference system in the language of the modal logic. Let $\vdash_{ST}$ denote the calculus defined by substituting the translations of the formulas for the formulas occurring in the axioms and inference rules of the calculus.[3]

Corollary 4.1.6. *Let $\vdash$ be a sound and complete calculus for* MLIQS_n *(*MLIQ_n*,* MLQ_n*, or* MLQS_n*). Then $\vdash_{ST}$ is a sound and complete calculus for* $\mathsf{L}_n^{=}$ *(*${}^r\mathsf{L}_n^{=}$*,* $\mathsf{L}_n^{\neq}$*, or* ${}^r\mathsf{L}_n^{\neq}$*).*

Proof: $\Gamma \vdash \varphi \iff ST(\Gamma) \vdash_{ST} ST(\varphi)$ and $\langle {}^nU, T_i, S_\tau, D_{ij}, I\rangle \models \varphi \iff \langle U, I\rangle \models ST(\varphi)$. ∎

The following proposition tells us what the algebraic counterparts of our logics are. For definitions see Definition 4.2.2.

Proposition 4.1.7. *1.* $\mathbf{SP}\mathsf{Alg}(\mathsf{MLIQS}_n) = \mathsf{RQPEA}_n$.
2. $\mathbf{SP}\mathsf{Alg}(\mathsf{MLQS}_n) = \mathsf{RQPA}_n$.
3. $\mathbf{SP}\mathsf{Alg}(\mathsf{MLIQ}_n) = \mathsf{RCA}_n$.

Proof: Straightforward. ∎

The following theorem states completeness of the modal versions of first-order logics.

Theorem 4.1.8. *Let* L *be one of the following logics:* MLIQS_n*,* MLIQ_n*, and* MLQS_n*. Let us assume that there are infinitely many relation symbols in the language of* L*. Then there is a weakly sound and complete inference system for* L*.*

Proof: The theorem follows from Theorem 1.3.11 and Theorem 4.2.4. ∎

Remark 4.1.9. In this remark we give the calculus that is weakly sound and complete w.r.t. MLIQ_n. The other calculi can be defined analogously.

Let Ax be a finite set of equations axiomatizing CA_n, cf. Definition 4.2.2. Note that by 5 below the diamonds commute, that is why we can make the following abbreviation: $\Diamond_{(\{\gamma_0 \dots \gamma_k\})}$ stands for $\Diamond_{\gamma_0} \dots \Diamond_{\gamma_k}$. Let the inference system $\vdash$ be defined as follows. Its set of axioms is the translations[4] of the elements of Ax into the language of MLIQ_n: for every $i, j, k < n$,

1. enough propositional tautologies,
2. $\neg\Diamond_i\bot$,
3. $\varphi \to \Diamond_i\varphi$,
4. $\Diamond_i(\varphi \wedge \Diamond_i\psi) \leftrightarrow \Diamond_i\varphi \wedge \Diamond_i\psi$,
5. $\Diamond_i\Diamond_j\varphi \leftrightarrow \Diamond_j\Diamond_i\varphi$,
6. δ_{ii},
7. $\delta_{ik} \leftrightarrow \Diamond_j(\delta_{ij} \wedge \delta_{jk})$ if $j \notin \{i, k\}$,
8. $\neg(\Diamond_i(\delta_{ij} \wedge \varphi) \wedge \Diamond_i(\delta_{ij} \wedge \neg\varphi))$ if $i \neq j$,

[3]Since we defined calculi for formula *schemata* and not for formulas, we have to substitute the translations of schemata for schemata. This translation between schemata is given by a straightforward modification of ST.

[4]This translation is defined by substituting formula schemata for algebraic variables, and replacing algebraic operations by the corresponding logical connectives.

its inference rules are Modus Ponens, Universal Generalization and the following rule:

$$\frac{\vdash (p \wedge \bar{\tau}(p \wedge \neg\varphi)) \to \varphi}{\vdash \varphi} \quad \text{provided } p \notin \varphi$$

where $\bar{\tau}(p \wedge \neg\varphi)$ is

$$\neg\Diamond_{(n)}\neg(\neg \bigwedge_{i\in n} \Diamond_{(n\setminus\{i\})}(p \wedge \neg\varphi) \vee (p \wedge \neg\varphi)),$$

and $p \notin \varphi$ denotes that p is an atomic formula not occurring in the formula φ. For intuition about the last rule see the discussion at Theorem 1.3.11. In the above rule, $\bar{\tau}$ is the translation of the term expressing rectangularity.

We note that Theorem 1.3.11 gives us a slightly different calculus (with other inference rules). But Modus Ponens and Universal Generalization are strong enough to "derive" the other rules. We also note that the axioms 2–4 ensures that $\Diamond$ is an $S5$ modality, while the other axioms describe the interaction between the modalities and the constants.

Note that this calculus is only weakly sound. Indeed, let φ be a non-valid formula, and let p and $\mathfrak{M}$ be such that $\mathfrak{M} \models \neg p \to \neg\varphi$. Clearly, $\neg p \models (p \wedge \bar{\tau}(p \wedge \neg\varphi)) \to \varphi$. On the other hand, $\neg p \not\models \varphi$. ∎

4.1.2 Arrow logic

Let us recall that pair arrow logic PAL and its square version PAL_{sq} were defined in Definition 1.1.3. Below, we formulate a completeness result for PAL_{sq} and its converse-free fragment. We note that Yde Venema proved a similar completeness result for square PAL in [Ve91] using a Henkin-style completeness argument. Since the completeness theorem yields a representability result about RRA's, Venema's proof shows that the bridge between logics and algebras is not one-way.

Theorem 4.1.10. *Let* L *be one of square* PAL*, or its* $\otimes$*-free fragment. Assume that there are infinitely many propositional variables in the language of* L*. Then there is a weakly sound and complete calculus for* L*.*

Proof: It is easy to see that

$$\mathbf{S}\mathbf{P}\mathsf{Alg}(\mathsf{PAL}_{sq}) = \mathsf{RRA}$$

and that the algebraic counterpart of the $\otimes$-free fragment generates the variety of RBM, cf. Definition 4.2.20. Then the theorem follows from Theorem 1.3.11 and Theorem 4.2.23. ∎

4.2 Representation of rectangularly dense algebras

Given a (quasi-)variety of (set) algebras the problem of axiomatizing it by a finite set of (quasi-)equations naturally arises. Unfortunately, in many interesting cases, such a characterization is impossible. Then one may try to find "simple", necessary and sufficient conditions for representability. Below we will show that for several classes of algebras of relations *rectangular density* is such a condition. That is, we will show that an algebra is representable iff it can be embedded into a product of rectangularly dense algebras. Luckily, this property can be expressed by an existential equation that can be translated to a rule of the corresponding logic, cf. Chapter 1. Thus, these representation theorems yield logical completeness results. For representability of dense algebras and its connection with completeness of logics see [HMT85] 3.2.

4.2.1 Cylindric and quasi-polyadic algebras

First we recall the definitions of several kinds of algebras of relations of higher arity, cf. [HMT85] and [ST91].

Definition 4.2.1. **(Cs and RCA, QPEs and RQPEA, QPs and RQPA)** Let α be an ordinal.

1. By a *cylindric set algebra of dimension* α, a Cs_α, we mean an algebra

$$\mathfrak{A} \subseteq \langle \mathcal{P}({}^\alpha U), \cap, \sim, \mathsf{C}_i, \mathsf{D}_{ij} \rangle_{i,j\in\alpha}$$

such that $U \neq \emptyset$ is any set, $\cap$ is intersection, $\sim$ is the operation taking complement w.r.t ${}^\alpha U$, for any $a \in \mathcal{P}({}^\alpha U)$ and $i \in \alpha$,

$$\mathsf{C}_i a = \{x \in {}^\alpha U : (\exists x' \in a)(\forall j \neq i) x(j) = x'(j)\},$$

and, for every $i, j \in \alpha$,

$$\mathsf{D}_{ij} = \{x \in {}^\alpha U : x(i) = x(j)\}.$$

We denote the class of cylindric set algebras of dimension α by Cs_α, and the class of all cylindric set algebras by Cs.
The class RCA_α of *representable cylindric algebras of dimension* α is defined as

$$\mathsf{RCA}_\alpha \stackrel{\text{def}}{=} \mathbf{SP}\mathsf{Cs}_\alpha.$$

RCA denotes the class of all representable cylindric algebras.

2. By a *quasi-polyadic set algebra of dimension* α, a QPs_α, we mean an algebra

$$\mathfrak{A} \subseteq \langle \mathcal{P}({}^\alpha U), \cap, \sim, \mathsf{C}_i, \mathsf{S}^i_j, \mathsf{P}_{ij} \rangle_{i,j\in\alpha}$$

such that $\langle \mathcal{P}({}^\alpha U), \cap, \sim, \mathsf{C}_i, \rangle$ is the D_{ij}-free reduct of a Cs_α, and, for any $a \in \mathcal{P}({}^\alpha U)$ and $i, j \in \alpha$,

$$\mathsf{S}^i_j a = \{x \in {}^\alpha U : (\exists x' \in a)(\forall k \neq i) x(k) = x'(k) \ \&\ x'(i) = x'(j)\},$$

and

$$\begin{aligned}\mathsf{P}_{ij} a \;=\; &\{x \in {}^\alpha U : (\exists x' \in a)(\forall k \notin \{i,j\})\\ &x(k) = x'(k) \ \&\ x(i) = x'(j) \ \&\ x(j) = x'(i)\}.\end{aligned}$$

We denote the class of quasi-polyadic set algebras of dimension α by QPs_α, and the class of all quasi-polyadic set algebras by QPs.

The class RQPA_α of *representable quasi-polyadic algebras of dimension* α is defined as
$$\mathsf{RQPA}_\alpha \stackrel{\text{def}}{=} \mathbf{SP}\mathsf{QPs}_\alpha.$$
RQPA denotes the class of all representable quasi-polyadic algebras.

3. By a *quasi-polyadic equality set algebra of dimension* α, a QPEs_α, we mean an algebra
$$\mathfrak{A} \subseteq \langle \mathcal{P}({}^\alpha U), \cap, \sim, \mathsf{C}_i, \mathsf{D}_{ij}, \mathsf{S}^i_j, \mathsf{P}_{ij}\rangle_{i,j\in\alpha}$$
such that $\langle \mathcal{P}({}^\alpha U), \cap, \sim, \mathsf{C}_i, \mathsf{D}_{ij}\rangle \in \mathsf{Cs}_\alpha$, and $\langle \mathcal{P}({}^\alpha U), \cap, \sim, \mathsf{C}_i, \mathsf{S}^i_j, \mathsf{P}_{ij}\rangle \in \mathsf{QPs}_\alpha$. We will use the notation QPs (and QPs_α) for the class of quasi-polyadic set algebras (of dimension α).
The class RQPEA_α of *representable quasi-polyadic equality algebras of dimension* α is defined as
$$\mathsf{RQPEA}_\alpha \stackrel{\text{def}}{=} \mathbf{SP}\mathsf{QPEs}_\alpha.$$
RQPEA denotes the class of all RQPEA_α (α any ordinal).

∎

For $\alpha > 2$, RCA_α, RQPEA_α and RQPA_α are non-finitely (Monk type schema) axiomatizable varieties, cf. [Mo69] Theorem 1.20 (for finite α) and Theorem 2.2 (for infinite α), [Jo69] Theorem 3.5 and [ST91] Theorem 2 for non-finite axiomatizability, and [HMT85] Theorem 3.1.103. and [Né94] for being varieties. We also note that, for $\alpha < \omega$, these classes are discriminator varieties (use $\mathsf{C}_0 \dots \mathsf{C}_{\alpha-1}x$ as $\Diamond x$, cf. Section 1.2).

In the literature, there are axiomatically given classes of algebras approximating the above classes of representable algebras. We recall the following definitions from [HMT85] and [ST91].

Definition 4.2.2. **(CA, QPA, and QPEA)** Let α be any ordinal.

1. By a *cylindric algebra of dimension* α, a CA_α, we mean an algebra
$$\mathfrak{A} = \langle A, \cdot, -, \mathsf{c}_i, \mathsf{d}_{ij}\rangle_{i,j\in\alpha}$$
such that d_{ij} is a constant (for every $i, j \in \alpha$), $-$ and c_i are unary operations (for every $i \in \alpha$), $\cdot$ is a binary operation, and such that the following postulates are satisfied, for any $x, y \in A$ and $i, j, k \in \alpha$:
$(C0)$ $\langle A, \cdot, -\rangle \in \mathsf{BA}$, i.e., it is a Boolean algebra,
$(C1)$ $\mathsf{c}_i 0 = 0$,
$(C2)$ $x \le \mathsf{c}_i x$, i.e., $x \cdot \mathsf{c}_i x = x$,
$(C3)$ $\mathsf{c}_i(x \cdot \mathsf{c}_i y) = \mathsf{c}_i x \cdot \mathsf{c}_i y$,
$(C4)$ $\mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_j\mathsf{c}_i x$,
$(C5)$ $\mathsf{d}_{ii} = 1$
$(C6)$ if $i \ne j, k$, then $\mathsf{d}_{jk} = \mathsf{c}_i(\mathsf{d}_{ij} \cdot \mathsf{d}_{ik})$,
$(C7)$ if $i \ne j$, then $\mathsf{c}_i(\mathsf{d}_{ij} \cdot x) \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot -x) = 0$.
The class of all cylindric algebras is denoted by CA, and the class of all cylindric algebras of dimension α by CA_α. The element d_{ij} is called *diagonal* element, and the operation c_i is called *cylindrification*.

2. By a *quasi-polyadic algebra of dimension* α, a QPA_α, we mean an algebra

$$\mathfrak{A} = \langle A, \cdot, -, \mathsf{c}_i, \mathsf{s}^i_j, \mathsf{p}_{ij} \rangle_{i,j \in \alpha}$$

such that $-$, c_i, s^i_j and p_{ij} are unary operations (for every $i, j \in \alpha$), $\cdot$ is a binary operation, and such that the following postulates are satisfied, for any $x, y \in A$ and $i, j, k \in \alpha$:
$(Q0)$ $\langle A, \cdot, - \rangle \in \mathsf{BA}$, $\mathsf{s}^i_i x = \mathsf{p}_{ii} x = x$, and $\mathsf{p}_{ij} x = \mathsf{p}_{ji} x$,
$(Q1)$ $x \leq \mathsf{c}_i x$,
$(Q2)$ $\mathsf{c}_i(x + y) = \mathsf{c}_i x + \mathsf{c}_i y$,
$(Q3)$ $\mathsf{s}^i_j \mathsf{c}_i x = \mathsf{c}_i x$,
$(Q4)$ if $i \neq j$, then $\mathsf{c}_i \mathsf{s}^i_j x = \mathsf{s}^i_j x$,
$(Q5)$ if $k \notin \{i, j\}$, then $\mathsf{s}^i_j \mathsf{c}_k x = \mathsf{c}_k \mathsf{s}^i_j x$,
$(Q6)$ s^i_j and p_{ij} are Boolean endomorphisms,
$(Q7)$ $\mathsf{p}_{ij} \mathsf{p}_{ij} x = x$,
$(Q8)$ if i, j, k are all distinct, then $\mathsf{p}_{ij} \mathsf{p}_{ik} x = \mathsf{p}_{jk} \mathsf{p}_{ij} x$,
$(Q9)$ $\mathsf{p}_{ij} \mathsf{s}^j_i x = \mathsf{s}^i_j x$.
The class of all quasi-polyadic algebras is denoted by QPA, and the class of all quasi-polyadic algebras of dimension α by QPA_α. The operations s^i_j and p_{ij} are called *substitution* and *permutation*, respectively.
3. By a *quasi-polyadic equality algebra of dimension* α, a QPEA_α, we mean an algebra $\mathfrak{A} = \langle \mathfrak{B}, \mathsf{d}_{ij} \rangle_{i,j \in \alpha}$ such that $\mathfrak{B} \in \mathsf{QPA}_\alpha$, the d_{ij}'s are new constants, and the following equations are valid, for every $x \in A$ and $i, j \in \alpha$:
$(Q10)$ $\mathsf{s}^i_j \mathsf{d}_{ij} = 1$,
$(Q11)$ $x \cdot \mathsf{d}_{ij} \leq \mathsf{s}^i_j x$.
The class of all quasi-polyadic equality algebras is denoted by QPEA, and the class of all quasi-polyadic equality algebras of dimension α by QPEA_α.

∎

In the sequel, we will use the following abbreviations: for a finite subset $\Gamma = \{\gamma_0, \ldots, \gamma_n\}$ of α, $\mathsf{c}_{(\Gamma)} x \stackrel{\text{def}}{=} \mathsf{c}_{\gamma_0 \ldots \gamma_n} x \stackrel{\text{def}}{=} \mathsf{c}_{\gamma_0} \ldots \mathsf{c}_{\gamma_n} x$. For $\Gamma = \emptyset$, we set $\mathsf{c}_{(\Gamma)} x = x$. Note that this abbreviation makes sense in CA, QPA and QPEA, since the above axioms guarantee that the cylindrifications commute.

It is well known that, for finite α, CA_α, QPA_α, and QPEA_α are discriminator varieties. As an immediate consequence of this, in simple algebras, $0 < x \Rightarrow \mathsf{c}_{(\alpha)} x = 1$.

It is easy to check that $\mathsf{RCA}_\alpha \subseteq \mathsf{CA}_\alpha$, $\mathsf{RQPA}_\alpha \subseteq \mathsf{QPA}_\alpha$, and $\mathsf{RQPEA}_\alpha \subseteq \mathsf{QPEA}_\alpha$. As we mentioned above, $\supseteq$ does not hold in either case. That is why we have to consider special subclasses of the axiomatically given classes to obtain representation results. Below for a given class V of algebras, we denote by RV the class of representable algebras of V.

Definition 4.2.3. (Rectangularity, rectangular density) Let α be any ordinal, and let V_α be one of CA_α, QPA_α, or QPEA_α. Let $\mathfrak{A} \in \mathsf{V}_\alpha$ and $a \in A$. We say that a is *rectangular* iff

$$\mathsf{c}_{(\Gamma)} a \cdot \mathsf{c}_{(\Delta)} a = \mathsf{c}_{(\Gamma \cap \Delta)} a$$

for all finite subsets Γ and Δ of α.

We say that $\mathfrak{A}$ is *rectangularly dense* iff

$$(\forall 0 \neq a \in A)(\exists 0 \neq b \in A)(b \leq a \;\&\; b \text{ is rectangular}).$$

For a given class V of algebras, we denote the class of rectangularly *d*ense elements of V by DV. ∎

The term rectangularity originates from Cs_2. Indeed, in a Cs_2, a rectangular element has the shape of a (generalized) rectangle, cf. Figure 4.1. We also note that, in a Cs_n,

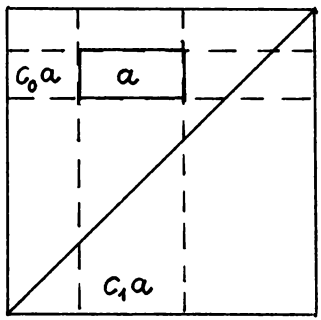

Figure 4.1: rectangularity

an element is rectangular iff it has the form $H_0 \times \ldots \times H_{n-1}$.

Now we are ready to formulate the main result of this section, a representation theorem for rectangularly dense algebras. We note that these results were known for *atomic* algebras, cf. [HMT85] 3.2.16, and 5.4.36. The novelty of the following theorem is that we could get rid of the condition of atomicity. This was important for the applications, because it is not clear how to use the result stated only for atomic algebras for obtaining completeness results for the corresponding logics. We recall the following theorem from [AGMNS].

Theorem 4.2.4. *Let α be any ordinal and $\mathsf{V}_\alpha \in \{\mathsf{CA}_\alpha, \mathsf{QPA}_\alpha, \mathsf{QPEA}_\alpha\}$. Then*

$$\mathsf{RV}_\alpha = \mathbf{SPD}\mathsf{V}_\alpha.$$

Remark 4.2.5. The above theorem can be extended to the permutation-free reducts of QPA and QPEA, cf. [AGMNS]. An interesting open problem is the case of the diagonal-free reduct of CA. ∎

The proof of the theorem for CA and QPEA consists of four steps, corresponding to the next four theorems (that we will prove a bit later). The idea of the proof is first to show that simple rectangularly dense algebras are atomic, and thus are representable. Then we show that every countable rectangularly dense algebra is embeddable into a product of simple rectangularly dense algebras, whence representability holds for these algebras as well. Then using a Löwenheim–Skolem argument representability follows for "large" algebras too. Finally, representability in infinite dimensions follows from representability in all finite dimensions.

Theorem 4.2.6. *Let $\alpha \in \omega$ and $\mathsf{V} \in \{\mathsf{CA}, \mathsf{QPEA}\}$. Let $\mathfrak{A}$ be a simple, rectangularly dense element of V_α. Then $\mathfrak{A}$ is representable.*

Theorem 4.2.7. *Let $\alpha \in \omega$ and $\mathsf{V} \in \{\mathsf{CA}, \mathsf{QPEA}\}$. Let $\mathfrak{A}$ be a rectangularly dense element of V_α, and assume that the universe of $\mathfrak{A}$ is countable, $|A| \leq \omega$. Then $\mathfrak{A}$ is embeddable into a product of simple, rectangularly dense elements of V_α.*

Theorem 4.2.8. *Let $\alpha \in \omega$, and let $\mathsf{V} \in \{\mathsf{CA}, \mathsf{QPEA}\}$. Assume that every countable element of DV_α is representable. Then every element of DV_α is representable.*

Theorem 4.2.9. *Let $\mathsf{V} \in \{\mathsf{CA}, \mathsf{QPEA}\}$. Assume that, for every finite α, $\mathsf{DV}_\alpha \subseteq \mathsf{RV}_\alpha$. Then, for every α, $\mathsf{DV}_\alpha \subseteq \mathsf{RV}_\alpha$.*

To prove the QPA case we need Theorem 4.2.10 below. This says that every rectangularly dense QPA is the identity-free reduct of a rectangularly dense QPEA. Thus, the representability of QPEA's implies the representability of QPA's.

Theorem 4.2.10. *For every rectangularly dense QPA_α $\mathfrak{A}$, there is a rectangularly dense QPEA_α $\mathfrak{B}$ such that $\mathfrak{A}$ is embeddable into the QPA-reduct of $\mathfrak{B}$.*

Now we are ready to prove Theorem 4.2.4.

Proof of Theorem 4.2.4: $\subseteq$: First we note that, in representable algebras, the singleton elements (i.e., elements a such that $|a| = 1$) are rectangular. That is, full Cs_α's, QPs_α's, and QPEs_α's are rectangularly dense. Then, by the definition of representable algebras (Definition 4.2.1), $\mathsf{RV}_\alpha \subseteq \mathbf{SP}\mathsf{DV}_\alpha$.

$\supseteq$: First, let $\mathsf{V} \in \{\mathsf{CA}, \mathsf{QPEA}\}$. If $\alpha \in \omega$, then, by Theorem 4.2.6, every simple element of DV_α is representable. By Theorem 4.2.7, every countable element of DV_α is embeddable into a direct product of simple elements of DV_α, whence countable algebras are representable. Theorem 4.2.8 ensures that representability follows for every element of DV_α. Finally, Theorem 4.2.9 yields that representability holds for $\alpha \geq \omega$ as well.

Let $\mathsf{V} = \mathsf{QPA}$. By Theorem 4.2.10, every rectangularly dense QPA is a subreduct of a rectangularly dense QPEA. Since rectangularly dense QPEA's are representable, so are the rectangularly dense QPA's.

Since RV_α is closed under $\mathbf{SP}$, we are done. ∎

Now we turn to proving Theorem 4.2.6. This time we have to give different proofs for CA_α and QPEA_α. Since in cylindric-like algebras one counts the dimensions as: "zero, one, two, many, infinite", we prove the theorem only for $\alpha = 3$. Then an easy modification of the proof works for any finite α.

Proof of Theorem 4.2.6: As we mentioned above we prove it only for $\alpha = 3$, since for other α's the proof is essentially the same.

First we mention some easy results concerning rectangularity that we will use later.

Lemma 4.2.11. *Let $\mathfrak{A}$ be an element of CA_3.*

1. *Let $a \in A$. Then a is rectangular iff there are $x, y, z \in A$ such that*

$$a = x \cdot y \cdot z \;\&\; \mathsf{c}_1 x = \mathsf{c}_2 x = x \;\&\; \mathsf{c}_0 y = \mathsf{c}_2 y = y \;\&\; \mathsf{c}_0 z = \mathsf{c}_1 z = z.$$

2. *If $\mathfrak{A}$ is rectangularly dense, then, for every $a \in A$,*

$$a = \sum\{b : b \leq a \,\&\, b \text{ is rectangular}\}.$$

Proof: 1: Let $x = \mathsf{c}_{12}a$, $y = \mathsf{c}_{02}a$ and $z = \mathsf{c}_{01}a$. The other direction is an easy calculation.

2: Clearly, a is an upper bound. Let c be an upper bound and assume that $a \not\leq c$. Then $a \cdot -c \neq 0$, so there is a rectangular $0 < d \leq a \cdot -c$. Then $d \in \{b : b \leq a \,\&\, b \text{ is rectangular}\}$, but $d \not\leq c$, a contradiction. ∎

Let $\mathsf{V}_\alpha = \mathsf{CA}_3$. We will prove that any algebra satisfying the assumptions is atomic, and then, by [HMT85] 3.2.16, is representable.

Remark 4.2.12. The representation theorem 3.2.16 in [HMT85] works for infinite α's as well, and this makes the proof rather complicated. However, we need the theorem only for finite α's, cf. Theorem 4.2.9. Then the idea of the proof is simple, cf. [HMT85] Discussion 3.2.15.

For every atom a of the atomic CA_α to be represented, let $a_i = \mathsf{c}_{(\alpha\smallsetminus\{i\})}a \cdot \prod\{\mathsf{d}_{ij} : i, j \in \alpha\}$. Let $U = \{a_i : i \in \alpha \,\&\, a \in At(\mathfrak{A})\}$. Then the map $a \mapsto \langle a_0, \ldots, a_{\alpha-1}\rangle$ induces an isomorphism into the Cs_α with top element ${}^\alpha U$.

See the proof of Theorem 4.2.24 below as well. There we will use the same strategy for representation with algebras of binary relations. ∎

Let $\mathsf{id} \stackrel{\text{def}}{=} \mathsf{d}_{12} \cdot \mathsf{d}_{02} \cdot \mathsf{d}_{01}$.

Lemma 4.2.13. *Let $\mathfrak{A} \in \mathsf{CA}_3$ and $x \in A$. Then*

$$(x \leq \mathsf{id} \,\&\, i \neq j) \Rightarrow \mathsf{c}_{ij}x \cdot \mathsf{id} = x.$$

Proof: First we note that, by [HMT85] 1.3.9, if $i \neq j$, then $x = \mathsf{d}_{ij} \cdot x = \mathsf{d}_{ij} \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot x) = \mathsf{d}_{ij} \cdot \mathsf{c}_i x$. Now, let $\{i, j, k\} = 3$. Then, since $\mathsf{c}_i\mathsf{d}_{jk} = \mathsf{d}_{jk}$, $x = \mathsf{c}_i x \cdot \mathsf{d}_{ik} = \mathsf{c}_i(\mathsf{c}_j x \cdot \mathsf{d}_{jk}) \cdot \mathsf{d}_{ik} = \mathsf{c}_{ij}x \cdot \mathsf{d}_{jk} \cdot \mathsf{d}_{ik} = \mathsf{c}_{ij}x \cdot \mathsf{id}$ by [HMT85] 1.3.7. ∎

Lemma 4.2.14. *Let $\mathfrak{A} \in \mathsf{CA}_3$ be a simple, non-atomic, rectangularly dense algebra. Then there are rectangular elements $b_n \in A$ $(n \in \omega)$ such that*

$$\mathsf{id} \geq b_0 > \ldots > b_n > \ldots.$$

Proof: Let $a \in A$ such that there is no atom below it. We will show that there is an infinite descending chain of rectangular elements below id. Below, by rectangular element we mean non-zero rectangular element.

CASE 1: $a \cdot \mathsf{id} \neq 0$. Then there is a rectangular $e_0 \leq a \cdot \mathsf{id}$. By assumption e_0 is not an atom, so there is a rectangular e_1 below it, etc. Thus, by induction, we have $\mathsf{id} \geq e_0 > e_1 > \ldots > e_n > \ldots$.

CASE 2: $a \cdot \mathsf{id} = 0$, cf. Figure 4.2. Let $b \leq a$ be a rectangular element. Now we assume that there is no descending chain of rectangular elements below id, and derive a contradiction. Since $b \neq 0$, $\mathsf{c}_0 b \neq 0$. Then, by [HMT85] 1.3.8, $\mathsf{d}_{12} \cdot \mathsf{c}_{01}b \neq 0$. Since by

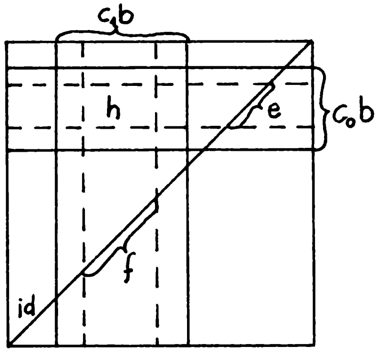

Figure 4.2: case 2 (in dimension 2)

[HMT85] 1.3.3 $\mathsf{d}_{12} = \mathsf{c}_0\mathsf{d}_{12}$, $\mathsf{c}_0(\mathsf{d}_{12} \cdot \mathsf{c}_1 b) \neq 0$. Then $\mathsf{d}_{12} \cdot \mathsf{c}_1 b \neq 0$, so, by [HMT85] 1.3.8 again, $\mathsf{d}_{01} \cdot \mathsf{c}_0(\mathsf{d}_{12} \cdot \mathsf{c}_1 b) \neq 0$; hence $\mathsf{d}_{01} \cdot \mathsf{d}_{12} \cdot \mathsf{c}_{01} b \neq 0$. By [HMT85] 1.3.7, $\mathsf{id} = \mathsf{d}_{01} \cdot \mathsf{d}_{12}$, so $0 \neq \mathsf{id} \cdot \mathsf{c}_{01} b$. By the indirect assumption, there is an atom $e \leq \mathsf{id} \cdot \mathsf{c}_{01} b$. Similarly, there are atoms f and g below $\mathsf{id} \cdot \mathsf{c}_{12} b$ and $\mathsf{id} \cdot \mathsf{c}_{02} b$, respectively.

Now we claim that $h \stackrel{\text{def}}{=} \mathsf{c}_{01} e \cdot \mathsf{c}_{12} f \cdot \mathsf{c}_{02} g$ is an atom below b, and so below a. Indeed,

$$h \leq \mathsf{c}_{01} b \cdot \mathsf{c}_{12} b \cdot \mathsf{c}_{02} b = b \leq a.$$

Now we show that $0 \neq h$. $0 = \mathsf{c}_{01} e \cdot \mathsf{c}_{12} f \cdot \mathsf{c}_{02} g$ would imply $0 = \mathsf{c}_2(\mathsf{c}_{01} e \cdot \mathsf{c}_{12} f \cdot \mathsf{c}_{02} g) = \mathsf{c}_2(\mathsf{c}_{01} e \cdot \mathsf{c}_2(\mathsf{c}_0 f \cdot \mathsf{c}_{12} g)) = \mathsf{c}_{012} e \cdot \mathsf{c}_2(\mathsf{c}_0 f \cdot \mathsf{c}_{12} g) = \mathsf{c}_{02} f \cdot \mathsf{c}_{12} g$, by simplicity. Then, by simplicity again, $0 = \mathsf{c}_1(\mathsf{c}_{02} f \cdot \mathsf{c}_{12} g) = \mathsf{c}_{012} f \cdot \mathsf{c}_{12} g = \mathsf{c}_{12} g$, a contradiction.

Now assume that h is not an atom, i.e., that there is a non-zero, rectangular $d < h$. We will show that either $\mathsf{c}_{01} d \cdot \mathsf{id} < e$, or $\mathsf{c}_{12} d \cdot \mathsf{id} < f$, or $\mathsf{c}_{02} d \cdot \mathsf{id} < g$, which yields a contradiction, since by the above argument $\mathsf{c}_{ij} d \cdot \mathsf{id} \neq 0$. First we prove that $\leq$ holds. Indeed, $d < \mathsf{c}_{01} e$ implies $\mathsf{c}_{01} d \leq \mathsf{c}_{01} e$, so, by Lemma 4.2.13, $\mathsf{c}_{01} d \cdot \mathsf{id} \leq \mathsf{c}_{01} e \cdot \mathsf{id} = e$. Similarly, $\mathsf{c}_{12} d \cdot \mathsf{id} \leq f$ and $\mathsf{c}_{02} d \cdot \mathsf{id} \leq g$. Now assume that in both cases $=$ holds, i.e., $\mathsf{c}_{01} d \cdot \mathsf{id} = e$ etc. Then, since $\mathsf{c}_1 \mathsf{id} = \mathsf{c}_1(\mathsf{d}_{12} \cdot \mathsf{d}_{01}) \cdot \mathsf{d}_{02} = \mathsf{c}_1 \mathsf{id} \cdot \mathsf{d}_{02} = \mathsf{d}_{02}$,

$$\begin{aligned} \mathsf{c}_{01} e &= \mathsf{c}_{01}(\mathsf{c}_{01} d \cdot \mathsf{id}) = \mathsf{c}_0(\mathsf{c}_{01} d \cdot \mathsf{c}_1 \mathsf{id}) = \\ [\text{by } \mathsf{c}_1 \mathsf{id} = \mathsf{d}_{02}] \quad &= \mathsf{c}_0(\mathsf{c}_{01} d \cdot \mathsf{d}_{02}) = \mathsf{c}_{01} d \cdot \mathsf{c}_0 \mathsf{d}_{02} = \\ [\text{by [HMT85] 1.3.2}] \quad &= \mathsf{c}_{01} d. \end{aligned}$$

Similarly, $\mathsf{c}_{12} f = \mathsf{c}_{12} d$ and $\mathsf{c}_{02} g = \mathsf{c}_{02} d$. Then, by the rectangularity of d,

$$d = \mathsf{c}_{01} d \cdot \mathsf{c}_{12} d \cdot \mathsf{c}_{02} d = \mathsf{c}_{01} e \cdot \mathsf{c}_{12} f \cdot \mathsf{c}_{02} g = h,$$

contradiction. So h must be an atom below a contradicting our assumption that there is no atom below a.

Thus, in both cases, there must be a descending chain of rectangular elements below id, hence Lemma 4.2.14 has been proved. ∎

Lemma 4.2.15. *Let $\mathfrak{A} \in \mathsf{CA}_3$ be simple. Then*

$$\forall a(0 < a \leq \mathsf{id} \ \&\ a \textit{ is rectangular } \Rightarrow a \in At(\mathfrak{A})).$$

Proof: This is proved in [HMT85] 1.10.13(ii). But for the reader's convenience we provide a proof here.

First we prove that if a is a rectangular element below id and $b \leq a$, then b is rectangular as well. Let a and b be as above. Then, by $b \leq \mathsf{id}$ and Lemma 4.2.13, $\mathsf{c}_{12}b \cdot \mathsf{id} = b$. By the same argument, $\mathsf{c}_{02}b \cdot \mathsf{id} = b = \mathsf{c}_{01}b \cdot \mathsf{id}$. Then

$$b = \mathsf{c}_{12}b \cdot \mathsf{c}_{02}b \cdot \mathsf{c}_{01}b \cdot \mathsf{id} = \mathsf{c}_{12}b \cdot \mathsf{c}_{02}b \cdot \mathsf{c}_{01}b,$$

since

$$\mathsf{c}_{12}b \cdot \mathsf{c}_{02}b \cdot \mathsf{c}_{01}b \leq \mathsf{c}_{12}a \cdot \mathsf{c}_{02}a \cdot \mathsf{c}_{01}a = a \leq \mathsf{id}.$$

Hence b is rectangular, by Lemma 4.2.11.1.

Now we show that if $0 \neq b < a \leq \mathsf{id}$ and a, b are rectangular, then a is the sum of pairwise disjoint, non-zero elements as below:

$$a = \sum\{\mathsf{c}_{12}x \cdot \mathsf{c}_{02}y \cdot \mathsf{c}_{01}z : x, y, z \in \{b, a \cdot -b\}\}.$$

Indeed, $=$ is easy to see, using the rectangularity of a and the additivity of the c_i's. Then every element of the sum is less than id. To see that the elements are pairwise disjoint, it is enough to show the following: for distinct i, j,

$$\mathsf{c}_{ij}b \cdot \mathsf{c}_{ij}(a \cdot -b) \cdot \mathsf{id} = 0.$$

In fact, $\mathsf{c}_{ij}b \cdot \mathsf{c}_{ij}(a \cdot -b) \cdot \mathsf{id} = b \cdot (a \cdot -b) = 0$ by Lemma 4.2.13.

Now assume that one of the elements in the sum is zero, say, that for some distinct i, j, k,

$$0 = \mathsf{c}_{ij}b \cdot \mathsf{c}_{ik}b \cdot \mathsf{c}_{jk}(a \cdot -b).$$

Then by b being rectangular,

$$0 = \mathsf{c}_{i}b \cdot \mathsf{c}_{jk}(a \cdot -b).$$

Hence

$$0 = \mathsf{c}_{ijk}(\mathsf{c}_{i}b \cdot \mathsf{c}_{jk}(a \cdot -b)) = \mathsf{c}_{ijk}b \cdot \mathsf{c}_{ijk}(a \cdot -b).$$

But we are in a simple CA_3, so $b \neq 0 \neq a \cdot -b$ implies

$$\mathsf{c}_{ijk}b = 1 = \mathsf{c}_{ijk}(a \cdot -b).$$

This implies that each member of the sum is not zero.

Now assume that a is a rectangular element below id but it is not an atom. Then a is the sum of two non-zero rectangular elements:

$$a = b + (a \cdot -b).$$

On the other hand,

$$a = \sum\{\mathsf{c}_{12}x \cdot \mathsf{c}_{02}y \cdot \mathsf{c}_{01}z : x, y, z \in \{b, a \cdot -b\}\},$$

where the members of the sum are pairwise disjoint, non-zero elements. Since b and $a \cdot -b$ are rectangular, they occur in the sum. Then

$$0 \neq \sum\{\mathsf{c}_{12}x \cdot \mathsf{c}_{02}y \cdot \mathsf{c}_{01}z : x, y, z \in \{b, a \cdot -b\}\} \cdot -(b + (a \cdot -b)) = a \cdot -a = 0,$$

contradiction. Thus a must be an atom. Thus we have proved Lemma 4.2.15. ∎

We are ready to show that every CA_3 satisfying the conditions of Theorem 4.2.6 is atomic, and so, by [HMT85] 3.2.16, is representable.

Let us assume that we have a non-atomic CA_3 satisfying the conditions. Then, by Lemma 4.2.14, there is an infinite descending chain of rectangular elements below the identity. But this is impossible, since by Lemma 4.2.15 each element of this chain must be an atom. Thus we have proved the CA case.

Now let $\mathsf{V}_\alpha = \mathsf{QPEA}_3$. Let $\mathfrak{A} \in \mathsf{QPEA}_3$ be a rectangularly dense, simple algebra. Then its cylindric reduct $\mathfrak{B} = \mathfrak{Rd}_{ca}(\mathfrak{A})$ is a rectangularly dense, simple CA_3, by Lemma 4.2.16 below. Thus, by the above argument, $\mathfrak{B}$ is atomic. Then so is $\mathfrak{A}$. Hence, by [HMT85] 5.4.36, $\mathfrak{A}$ is representable.

Lemma 4.2.16. *Let $\mathfrak{A} \in \mathsf{QPEA}_3$ be a simple algebra. Then $\mathfrak{B} = \mathfrak{Rd}_{ca}(\mathfrak{A})$ is a simple CA_3.*

Proof: By [HMT85] 5.4.3, the CA-reduct of every QPEA is a CA. We will show that any cylindric ideal on $\mathfrak{B}$ is a QPEA-ideal too. Then if $\mathfrak{B}$ is not simple, i.e., if there is a proper ideal on it, then neither is $\mathfrak{A}$ contradicting the assumption.

Let I be an ideal on $\mathfrak{B}$. By [Sa82] Proposition 7.4, it is enough to show that $x \in I$ implies

$$\mathsf{s}^i_j x = \mathsf{s}^i_j x \cdot -\mathsf{s}^i_j 0 \in I$$
$$\mathsf{p}_{ij} x = \mathsf{p}_{ij} x \cdot -\mathsf{p}_{ij} 0 \in I.$$

In fact, we will show that $\mathsf{s}^i_j x \leq \mathsf{c}_{(3)} x$, and then $\mathsf{s}^i_j x \in I$, since $x \in I$ implies $\mathsf{c}_{(3)} x \in I$, by [Sa82] again; and similarly for p_{ij} instead of s^i_j.

By $(Q6)$, $(Q3)$, we have

$$\mathsf{s}^i_j x \leq \mathsf{s}^i_j \mathsf{c}_i x = \mathsf{c}_i x \leq \mathsf{c}_{(3)} x.$$

By $(Q3)$, $(Q9)$, we get

$$\mathsf{c}_i x = \mathsf{s}^i_j \mathsf{c}_i x = \mathsf{p}_{ij} \mathsf{s}^j_i \mathsf{c}_i x.$$

Then, using $(Q6)$, $(Q3)$,

$$\mathsf{c}_{(3)} x \geq \mathsf{c}_i \mathsf{c}_j x = \mathsf{p}_{ij} \mathsf{s}^j_i \mathsf{c}_i \mathsf{c}_j x \geq \mathsf{p}_{ij} \mathsf{s}^j_i \mathsf{c}_j x = \mathsf{p}_{ij} \mathsf{c}_j x \geq \mathsf{p}_{ij} x$$

as desired. ∎

Thus we have proved the QPEA case, which finishes the proof of Theorem 4.2.6 ∎

Proof of Theorem 4.2.7: The theorem immediately follows once we showed that the conditions of Theorem 1.2.6 are satisfied. This amounts to proving that CA_α and

QPEA_α are SBAO's for finite α's, and that rectangularity is preserved under homomorphism. The latter is clear, since this property was defined by an equation. It is easy to check that CA_α and QPEA_α are normal BAO's. Finally, if we define $\Diamond$ by $\mathsf{c}_{(\alpha)}$, then $\Diamond$ is indeed a complemented closure operator which satisfies the other condition of Definition 1.2.3 as well. ∎

Proof of Theorem 4.2.8: Assume that $\mathfrak{B} \in \mathsf{DV}_\alpha$ and $\alpha < \omega$. We know that RV_α is a variety, i.e., it can be defined by an (infinite) set of equations. Now assume that $\mathfrak{B} \notin \mathsf{RV}_\alpha$. Then there is an equation e such that $\mathfrak{B} \not\models e$ while $\mathsf{RV}_\alpha \models e$. Then for some assignment k of the variables of e, $\mathfrak{B} \not\models e[k]$. Then, by the Löwenheim–Skolem theorem, cf. [Mo76], there is a countable elementary subalgebra $\mathfrak{A} \in \mathsf{V}_\alpha$ of $\mathfrak{B}$ containing every $\tau^{\mathfrak{B}}[k]$, τ a subterm of e. Since rectangular density is a first-order property, $\mathfrak{A} \in \mathsf{DV}_\alpha$. By $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A} \not\models e[k]$. On the other hand, $\mathfrak{A} \in \mathsf{RV}_\alpha$ by Theorem 4.2.6 and Theorem 4.2.7, so $\mathfrak{A} \models e$. Contradiction, thus $\mathfrak{B} \in \mathsf{RV}_\alpha$. ∎

Proof of Theorem 4.2.9: Let $\mathfrak{C} \in \mathsf{DV}_\alpha$, $\alpha \geq \omega$ and e be an equation such that $\mathsf{RV}_\alpha \models e$. Then e uses only finitely many d_{ij}'s and c_i's. Thus $|\Gamma| = n < \omega$, where $\Gamma = \{i < \alpha : \mathsf{d}_{ij}, \mathsf{d}_{ji}, \text{ or } \mathsf{c}_i \text{ occurs in } e \text{ for some } i\}$. W.l.o.g. we may assume that $\Gamma = n$. Then, it is easy to see that the n-dimensional reduct $\mathfrak{B}$ of $\mathfrak{C}$ is in V_n, cf. [HMT85] 2.6.2(i). By the definition of rectangularity, $\mathfrak{B}$ is rectangularly dense. Now assume that $\mathfrak{C} \not\models e[k]$ for some assignment k. Then $\mathfrak{B} \not\models e[k]$, but, by the assumption, $\mathfrak{B} \in \mathsf{RV}_n$, whence $\mathfrak{B} \models e[k]$. Contradiction, so $\mathfrak{C} \in \mathsf{RV}_\alpha$. ∎

Proof of Theorem 4.2.10: We will show that every (rectangularly dense) QPA is the QPA-reduct of a (rectangularly dense) QPEA. First we need a definition.

Definition 4.2.17. Let $\mathfrak{A} = \langle \mathfrak{A}_0, \mathsf{c}_i, \mathsf{s}^i_j, \mathsf{p}_{ij} \rangle_{i,j\in\alpha} \in \mathsf{QPA}_\alpha$. Then the *completion*

$$\overline{\mathfrak{A}} = \langle \overline{\mathfrak{A}}_0, \bar{\mathsf{c}}_i, \bar{\mathsf{s}}^i_j, \bar{\mathsf{p}}_{ij} \rangle_{i,j\in\alpha}$$

of $\mathfrak{A}$ is defined as follows. $\overline{\mathfrak{A}}_0$ is the completion of $\mathfrak{A}_0$ in the Boolean-algebraic sense. If o is any one of the operations c_i, s^i_j, or p_{ij} then

$$\bar{\mathsf{o}}x \stackrel{\text{def}}{=} \sum_{x \geq a \in A} \mathsf{o}a$$

for all $x \in \overline{A}$. ∎

Lemma 4.2.18. *Let $\mathfrak{A} \in \mathsf{QPA}_\alpha$. Then*

(i) $\sum^{\overline{\mathfrak{A}}} X = \sum^{\mathfrak{A}} X$ *for all* $X \subseteq A$ *such that* $\sum^{\mathfrak{A}} X$ *exists*
(ii) $x = \sum_{x \geq a \in A} a$ *for every* $x \in \overline{A}$
(iii) $\mathfrak{A}$ *is a subalgebra of* $\overline{\mathfrak{A}}$
(iv) $\bar{\mathsf{o}} \sum X = \sum_{x \in X} \mathsf{o}x$ *for all* $X \subseteq A$ *and extra-Boolean operation* o
(v) $\bar{\mathsf{s}}^i_j x = \prod_{x \leq a \in A} \mathsf{s}^i_j a$
(vi) $\bar{\mathsf{p}}_{ij} x = \prod_{x \leq a \in A} \mathsf{p}_{ij} a$.

Proof: (i) and (ii) are well-known Boolean-algebraic facts. (iii) follows easily from the definition of the extra-Boolean operators on $\overline{\mathfrak{A}}$ and the fact that all of them are monotonic. A proof for (iv) can be found in [HMT85] 2.7.21: equation (2) there is the special case of our (iv) when $\mathsf{o} = \mathsf{c}_i$. The only property of cylindrifications that is used in the proof is that it commutes with existing suprema, a fact which holds for the substitutions and the permutations, too. The proof of (vi) is exactly like the proof of (v), so we only prove the latter.

Let $x \in \overline{A}$; we have to show that

$$\sum_{x \geq a \in A} \mathsf{s}^i_j a = \prod_{x \leq a \in A} \mathsf{s}^i_j a.$$

$\prod_{x \leq a \in A} \mathsf{s}^i_j a$ is an upper bound of $\{\mathsf{s}^i_j a : x \geq a \in A\}$, so the $\leq$ direction is clear. It follows from (ii) that $\prod\{a \cdot b : a, b \in A \;\&\; x \leq a \;\&\; -x \leq b\} = 0$, whence, using (iii) and (iv) we get

$$\begin{aligned} 1 &= \bar{\mathsf{s}}^i_j 1 = \bar{\mathsf{s}}^i_j \sum\{-(a \cdot b) : a, b \in A, x \leq a, -x \leq b\} = \\ &= \sum\{\mathsf{s}^i_j - (a \cdot b) : a, b \in A, x \leq a, -x \leq b\} = \\ &= -\prod\{\mathsf{s}^i_j (a \cdot b) : a, b \in A, x \leq a, -x \leq b\}. \end{aligned}$$

Thus

$$\begin{aligned} (\textstyle\prod_{x \leq a \in A} \mathsf{s}^i_j a) \cdot -(\textstyle\sum_{x \geq a \in A} \mathsf{s}^i_j a) &= (\textstyle\prod_{x \leq a \in A} \mathsf{s}^i_j a) \cdot (\textstyle\prod_{x \geq a \in A} -\mathsf{s}^i_j a) = \\ &= \prod\{\mathsf{s}^i_j a \cdot -\mathsf{s}^i_j b : b \leq x \leq a \;\&\; a, b \in A\} = \\ &= \prod\{\mathsf{s}^i_j a \cdot \mathsf{s}^i_j b : x \leq a \;\&\; -x \leq b \;\&\; a, b \in A\} = \\ &= \prod\{\mathsf{s}^i_j (a \cdot b) : x \leq a \;\&\; -x \leq b \;\&\; a, b \in A\} = \\ &= 0, \end{aligned}$$

finishing the proof of the $\geq$ direction. ∎

Theorem 4.2.19. *The class of quasi-polyadic algebras is closed under completion. That is, if* $\mathfrak{A} \in \mathsf{QPA}_\alpha$ *then so is* $\overline{\mathfrak{A}}$.

Proof: We go through the list $(Q0)$–$(Q9)$ of axioms of QPA_α and check that each of them holds in $\overline{\mathfrak{A}}$. $(Q0)$ and $(Q1)$ are obvious. To prove the non-trivial direction ($\mathsf{c}_i(x + y) \leq \mathsf{c}_i x + \mathsf{c}_i y$) of $(Q2)$, by Lemma 4.2.18(iv), it is enough to show that if $a \in A$ and $a \leq x + y$, then $\mathsf{c}_i a \leq \bar{\mathsf{c}}_i x + \bar{\mathsf{c}}_i y$. This is true, since in this case $a = a \cdot x + a \cdot y = \sum\{b \in A : b \leq a \cdot x\} + \sum\{b \in A : b \leq a \cdot y\}$ by Lemma 4.2.18(ii), so

$$\begin{aligned} \mathsf{c}_i a &= \bar{\mathsf{c}}_i a = \bar{\mathsf{c}}_i \sum\{b \in A : b \leq a \cdot x \text{ or } b \leq a \cdot y\} = \\ \text{[by Lemma 4.2.18(iv)]} \quad &= \sum\{\mathsf{c}_i b \in A : b \leq a \cdot x \text{ or } b \leq a \cdot y\} = \\ &= \sum\{\mathsf{c}_i b : b \in A, b \leq a \cdot x\} + \sum\{\mathsf{c}_i b : b \in A, b \leq a \cdot y\} = \\ \text{[by Lemma 4.2.18(iv)]} \quad &= \bar{\mathsf{c}}_i \sum\{b \in A : b \leq a \cdot x\} + \bar{\mathsf{c}}_i \sum\{b \in A : b \leq a \cdot y\} = \\ &= \bar{\mathsf{c}}_i (a \cdot x) + \bar{\mathsf{c}}_i (a \cdot y) \leq \\ &\leq \bar{\mathsf{c}}_i x + \bar{\mathsf{c}}_i y. \end{aligned}$$

The proofs of $(Q3)$–$(Q5)$ and $(Q7)$–$(Q9)$ follow the same pattern; as an example we show that $(Q3)$ holds, i.e., that $\bar{\mathsf{s}}^i_j\bar{\mathsf{c}}_i x = \bar{\mathsf{c}}_i x$. Indeed,

$$\begin{array}{rrl} \bar{\mathsf{s}}^i_j\bar{\mathsf{c}}_i x & = & \bar{\mathsf{s}}^i_j \sum\{\mathsf{c}_i a : a \in A, a \leq x\} = \\ \text{[by Lemma 4.2.18(iv)]} & = & \sum\{\mathsf{s}^i_j\mathsf{c}_i a : a \in A, a \leq x\} = \\ & = & \sum\{\mathsf{c}_i a : a \in A, a \leq x\} = \\ & = & \bar{\mathsf{c}}_i a. \end{array}$$

It remains to prove $(Q6)$, i.e., that the $\bar{\mathsf{s}}^i_j$'s and the $\bar{\mathsf{p}}_{ij}$'s are Boolean endomorphisms. The argument used to prove $(Q2)$ above can be used to show that $+$ is preserved. As for $-$, we have

$$\begin{array}{rrl} \bar{\mathsf{s}}^i_j - x & = & \sum\{\mathsf{s}^i_j a : a \in A, a \leq -x\} = \\ & = & \sum\{-\mathsf{s}^i_j a : a \in A, -a \leq -x\} = \\ \text{[by Lemma 4.2.18(v)]} & = & -\prod\{\mathsf{s}^i_j a : a \in A, x \leq a\} = \\ & = & -\bar{\mathsf{s}}^i_j x, \end{array}$$

and the same proof (using Lemma 4.2.18(vi) instead of Lemma 4.2.18(v)) shows that $\bar{\mathsf{p}}_{ij} - x = -\bar{\mathsf{p}}_{ij}x$. ∎

Let $\mathfrak{A}$ be a rectangularly dense QPA. Then $\overline{\mathfrak{A}} \in \mathsf{QPA}$ by Theorem 4.2.19, and it is also rectangularly dense by Lemma 4.2.18(ii). Since $\overline{\mathfrak{A}}$ is complete, it follows from (the proof of) [ST91] Proposition 9 (on p. 561) that $\overline{\mathfrak{A}}$ is the QPA-reduct of a QPEA $\mathfrak{B}$ (the d_{ij}'s can be defined in $\overline{\mathfrak{A}}$ as $\prod\{y \in \overline{A} : \bar{\mathsf{s}}^i_j y = 1\}$). By the definition of rectangular density $\mathfrak{B}$ is rectangularly dense. ∎

4.2.2 BOOLEAN MONOIDS AND RELATION ALGEBRAS

Let us recall that the classes Rs and RRA of relation set algebras and representable relation algebras were defined in Definition 3.2.1. We recall the definition of their converse-free reduct from [Pr90].

Definition 4.2.20. (SRBM and RBM) By a *simple representable Boolean monoid*, an SRBM, we mean an algebra $\mathfrak{A} \subseteq \langle A, \cap, \sim, \circ, \mathsf{Id}\rangle$ such that $\mathfrak{A}$ is the ${}^{-1}$-free reduct of an Rs.

The class RBM of *representable Boolean monoids* is defined as

$$\mathsf{RBM} \stackrel{\text{def}}{=} \mathbf{SP}\mathsf{SRBM}.$$

∎

RRA and RBM are non-finitely axiomatizable discriminator varieties, (see [Mo64] and [Né94] Theorem 2.1 for non-finite axiomatizability, and [Ta55] and [Né94] for being discriminator varieties). For RRA these are well-known facts. RBM is a variety, since SRBM is a pseudoelementary discriminator class, cf. [Pr90]. Actually, the same argument works as in the proof of Theorem 3.2.16 for proving that SRBM is first-order axiomatizable. If we define $\Diamond x$ as $1 \circ x \circ 1$, then $(\Diamond(x \oplus y) \cap v) \cup (u \cap \sim \Diamond(x \oplus y))$ is a discriminator term.

Definition 4.2.21. (RA, BM, and BM$^+$)

1. A *relation algebra*, an RA, is an algebra $\mathfrak{A} = \langle A, \cdot, -, ;, \smile, \mathsf{id}\rangle$ such that $\langle A, \cdot, -\rangle$ is a BA, ; (called *composition*) is a binary operation, $\smile$ (called *converse*) is a unary operation, id (called *identity*) is a constant, and, for all $x, y, z \in A$, the following equations hold:
   $(R1)$ $x;(y;z) = (x;y);z$,
   $(R2)$ $(x+y);z = (x;z) + (y;z)$,
   $(R3)$ $x;\mathsf{id} = x$,
   $(R4)$ $x^{\smile\smile} = x$,
   $(R5)$ $(x+y)^\smile = x^\smile + y^\smile$,
   $(R6)$ $(x;y)^\smile = y^\smile ; x^\smile$,
   $(R7)$ $x^\smile;(-(x;y)) \leq -y$.
   We denote the class of all relation algebras by RA.
2. By a *Boolean monoid*, a BM, we mean an algebra $\mathfrak{A} = \langle A, \cdot, -, ;, \mathsf{id}\rangle$ such that $\langle A, \cdot, -\rangle \in$ BA, and, for all $x, y, z \in A$, the following equations hold:
   $(M1)$ $x;(y;z) = (x;y);z$,
   $(M2)$ $x = x;\mathsf{id} = \mathsf{id};x$.
   We denote the class of all Boolean monoids by BM.
3. Let $\mathfrak{A}$ be a Boolean monoid. $\mathfrak{A} \in$ BM$^+$ if the following conditions hold:
   $(M3)$ $x;(y+z) = (x;y) + (x;z)$ & $(x+y);z = (x;z) + (y;z)$,
   $(M4)$ $1;-(1;x) = -(1;x)$ & $-(x;1);1 = -(x;1)$.

∎

We make the usual convention that ; binds more closely than $\cdot$ or $+$.

We note that RA and BM$^+$ are discriminator varieties (again, define $\Diamond x$ as $1;x;1$, and then the discriminator term can be defined as in the case of RBM above), while BM is not (roughly speaking, the reason for this is that $\Diamond$ is not a complemented closure operator in BM). Thus, BM is "very far" from RBM, that is why we defined its subclass BM$^+$ which satisfies more ("natural") equations valid in RBM. It is easy to see that the $\smile$-free reduct of an RA is in BM$^+$.

By an easy verification, RRA $\subseteq$ RA and RBM $\subseteq$ BM$^+$ $\subseteq$ BM. Again, $\supseteq$ does not hold, that is, there are abstract algebras that are not representable as set algebras. The class of representable elements of V will be denoted by RV.

Definition 4.2.22. (Rectangularity, rectangular density) Let $\mathfrak{A}$ be an algebra with a BM-reduct, and let $a \in A$. We say that x is *rectangular* if

$$(1;x) \cdot (x;1) = x.$$

We say that $\mathfrak{A}$ is *rectangularly dense* if

$$(\forall 0 < a \in A)(\exists 0 < b \in A) b \leq a \;\&\; b \text{ is rectangular.}$$

Given a class V of algebras we denote by DV the class of rectangularly *d*ense elements of V. ∎

Thus DRA denotes the class of rectangularly dense RA's.

We recall the following theorem from [AGMNS]. We note that the theorem for RA's is a consequence of the fact that point-dense RA's are representable (cf. [JT52] for atomic RA's, and [MT76] and [Ma91] for arbitrary RA's).

Theorem 4.2.23. *Let* $\mathsf{V} \in \{\mathsf{BM}^+, \mathsf{RA}\}$. *Then*

$$\mathsf{RV} = \mathbf{SP}\mathsf{DV}.$$

The proof of the theorem will be based on the following three theorems, cf. the case of CA.

Theorem 4.2.24. *Let* $\mathsf{V} \in \{\mathsf{RA}, \mathsf{BM}^+\}$. *Then*

$$\mathbf{Sim}\mathsf{DV} \subseteq \mathsf{RV}.$$

Theorem 4.2.25. *Let* $\mathsf{V} \in \{\mathsf{RA}, \mathsf{BM}^+\}$*, and let* $\mathfrak{A} \in \mathsf{DV}$ *be a countable algebra. Then* $\mathfrak{A}$ *can be embedded into a product of simple* DV*'s.*

Theorem 4.2.26. *Let* $\mathsf{V} \in \{\mathsf{RA}, \mathsf{BM}^+\}$*, and assume that every countable* DV *is representable. Then*

$$\mathsf{DV} \subseteq \mathsf{RV}.$$

Now we are ready to prove Theorem 4.2.23.

Proof of Theorem 4.2.23: $\subseteq$: It is easy to check that singletons, i.e., elements consisting of a single ordered pair, are rectangular in representable algebras. Then full Rs's and SRBM's are rectangularly dense. Hence RRA $=$ **SP**Rs $\subseteq$ **SP**DRA, and similarly for RBM.

$\supseteq$: By Theorem 4.2.24, every simple element of DV is representable. Then, by Theorem 4.2.25, every countable DV is representable. Finally, Theorem 4.2.26 ensures that every DV can be represented as an RV. Since RV is **SP**-closed, we are done. ∎

Proof of Theorem 4.2.24: First we prove that the cylindric-reduct of a BM^+ is a CA_2. Let the operations c_i and constants d_{ij} $(i, j \in 2)$ be defined as follows:

$$\mathsf{c}_0 x = 1; x \ \&\ \mathsf{c}_1 x = x; 1 \ \&\ \mathsf{d}_{00} = \mathsf{d}_{11} = 1 \ \&\ \mathsf{d}_{01} = \mathsf{d}_{10} = \mathsf{id}.$$

Then the cylindric-reduct $\mathfrak{Rd}_{ca}(\mathfrak{A})$ of an $\mathfrak{A} \in \mathsf{BM}^+$ is defined as

$$\mathfrak{Rd}_{ca}(\mathfrak{A}) \stackrel{\text{def}}{=} \langle A, \cdot, -, \mathsf{c}_i, \mathsf{d}_{ij} \rangle_{i,j \in 2}.$$

Lemma 4.2.27. *Let* $\mathfrak{A} \in \mathsf{BM}^+$. *Then* $\mathfrak{Rd}_{ca}(\mathfrak{A}) \in \mathsf{CA}_2$.

Proof: We check that $\mathfrak{Rd}_{ca}(\mathfrak{A})$ satisfies $(C0)$–$(C7)$. $(C1)$ (i.e., $x; 0 = 0; x = 0$) easily follows from $(M4)$ and $(M3)$:

$$0; x \leq 0; 1 \leq -(1; 1); 1 = -(1; 1) = -1 = 0$$

(and similarly $x;0=0$). For $(C3)$ (i.e., $1;(x\cdot 1;x)=(1;x)\cdot(1;x)$ etc.) it is enough to show that c_i is a complemented closure operator, cf. [Ve91] Proposition 3.5.6; and this easily follows from the axioms. For $(C7)$ it suffices to prove

$$(1;(\mathsf{id}\cdot x))\cdot(1;(\mathsf{id}\cdot -x))=0 \ \&\ ((\mathsf{id}\cdot x);1)\cdot((\mathsf{id}\cdot -x);1)=0.$$

First we prove $\mathsf{id}\cdot(1;(x\cdot\mathsf{id}))=x\cdot\mathsf{id}$. Indeed, let $z\leq\mathsf{id}$. Then

$$\begin{aligned}\mathsf{id}\cdot(1;z) &= \mathsf{id}\cdot((\mathsf{id}+-\mathsf{id});z))=\mathsf{id}\cdot(\mathsf{id};z+-\mathsf{id};z)\leq\\ &\leq \mathsf{id}\cdot(\mathsf{id};z+-\mathsf{id};\mathsf{id})=\mathsf{id}\cdot(z+-\mathsf{id})=\\ &= z,\end{aligned}$$

and $\geq$ holds by the monotonicity of $;$. Let $z=x\cdot\mathsf{id}$ and $z'=-x\cdot\mathsf{id}$. Then $z\cdot(1;z')=0$, since $z\cdot(1;z')\leq\mathsf{id}\cdot(1;z')=z'$ and $0=z\cdot z'$. Thus

$$0=1;0=1;(z\cdot 1;z')=(1;z)\cdot(1;z'),$$

i.e., $(1;(\mathsf{id}\cdot x))\cdot(1;(\mathsf{id}\cdot -x))=0$ as desired. The other part of $(C7)$ can be proved in the same way. The proofs of the other cylindric-algebraic axioms are straightforward. ∎

By $(C3)$, the following are valid in BM^+:

$$\begin{gathered}1;(x\cdot 1;y)=1;x\cdot 1;y=1;(1;x\cdot y)\\ (x\cdot y;1);1=x;1\cdot y;1=(x;1\cdot y);1.\end{gathered}$$

We will use these facts in the sequel.

Lemma 4.2.28. *Let $\mathfrak{A}\in\mathbf{Sim}\mathsf{BM}^+$ be a rectangularly dense algebra. Then $\mathfrak{A}$ is atomic.*

Proof: First we show that if $\mathfrak{A}\in\mathsf{BM}^+$ is simple, then $0<x\Rightarrow 1;x;1=1$. Assume there is an $a>0$ such that $1;a;1\neq 1$. Let $I=\{x\in A: x\leq 1;a;1\}$. Then, by the monotonicity of $;$, $x\leq 1;a;1$ implies

$$(x;y)+(y;x)\leq 1;x;1\leq 1;a;1,$$

for every $y\in A$. That is, by the normality of $;$, $x\in I$ implies $x;y=(x+y)\cdot-(x;0)\in I$ and $y;x=(y+x)\cdot-(0;x)\in I$, for every $y\in A$. Then, by [Sa82] Proposition 7.4, I is an ideal. Since $1\notin I$, this contradicts to the simplicity of $\mathfrak{A}$.

Assume that $\mathfrak{A}\in\mathbf{Sim}\mathsf{BM}^+$. Then, for every $0<a\in A$, $1;a;1=1$. Let $\mathfrak{B}$ be the cylindric-reduct of $\mathfrak{A}$. Then $\mathsf{c}_{(2)}a=1$ for every non-zero a. This implies that $\mathfrak{B}$ is simple. Furthermore, $\mathfrak{B}$ is rectangularly dense, by the definition of rectangularity in CA's and in BM's. Hence it is atomic, by Lemma 4.2.14 and Lemma 4.2.15. Then so is $\mathfrak{A}$, finishing the proof of the lemma. ∎

Now we can turn to representing simple (and thus atomic), rectangularly dense BM^+'s. For every atom $a\in At(\mathfrak{A})$, let

$$\mathsf{rep}'(a)\stackrel{\mathsf{def}}{=}\{\langle a;1\cdot\mathsf{id},1;a\cdot\mathsf{id}\rangle\},$$

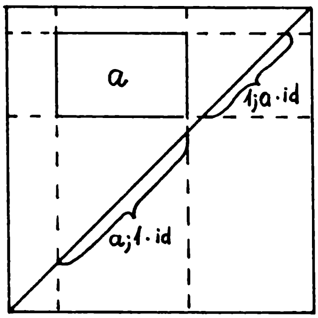

Figure 4.3: representation

cf. Figure 4.3, and, for every $x \in A$, let

$$\mathsf{rep}(x) \stackrel{\text{def}}{=} \bigcup\{\mathsf{rep}'(a) : a \in At(\mathfrak{A}) \;\&\; a \leq x\}.$$

Clearly, $\mathsf{rep}(a) = \mathsf{rep}'(a)$ for every atom a. We show that rep is an isomorphism between $\mathfrak{A}$ and a set algebra $\mathfrak{C}$ such that

$$\mathfrak{C} \subseteq \langle \mathcal{P}(U \times U), \cap, \sim, \circ, \mathsf{Id}\rangle$$

where $U = \{a \in At(\mathfrak{A}) : a \leq \mathsf{id}\}$.

Clearly, rep preserves meet and complement. For the other operations we need a claim.

Claim 4.2.29. *The following are valid in a simple* BM^+*:*

(i) $a \leq \mathsf{id} \Rightarrow a = 1; a \cdot \mathsf{id} = a; 1 \cdot \mathsf{id}$,

(ii) $a \in At(\mathfrak{A}) \Rightarrow 1; a \cdot \mathsf{id},\ a; 1 \cdot \mathsf{id} \in At(\mathfrak{A})$,

(iii) $a, b \in At(\mathfrak{A}) \Rightarrow a; 1 \cdot 1; b \in At(\mathfrak{A})$,

(iv) $a, b \in At(\mathfrak{A}) \;\&\; a + b \leq \mathsf{id} \Rightarrow a = (a; 1 \cdot 1; b); 1 \cdot \mathsf{id} \;\&\; b = 1; (a; 1 \cdot 1; b) \cdot \mathsf{id}$.

Proof: These easily follow from the fact that the CA-reduct is a CA_2.

(i): By [HMT85] 1.3.9.

(ii): If there were an atom b below $1; a \cdot \mathsf{id}$, then $1; b \cdot a$ would be an atom below a.

(iii): If there were an atom c below $a; 1 \cdot 1; b$, then there would be an atom below $a; 1 \cdot \mathsf{id}$, or below $1; b \cdot \mathsf{id}$.

(iv): By (i) and simplicity. ∎

We show that rep preserves the top element: by Claim 4.2.29(ii), (iii) and (iv),

$$\begin{aligned}
\mathsf{rep}(1) &= \bigcup\{\mathsf{rep}(a) : a \in At(\mathfrak{A})\} = \{\langle a; 1 \cdot \mathsf{id}, 1; a \cdot \mathsf{id}\rangle : a \in At(\mathfrak{A})\} = \\
&= \{\langle b, c\rangle : b, c \in At(\mathfrak{A}) \;\&\; b + c \leq \mathsf{id}\} = \\
&= U \times U.
\end{aligned}$$

rep preserves identity, since by Claim 4.2.29(i)

$$\begin{aligned}
\mathsf{rep}(\mathsf{id}) &= \{\langle a; 1 \cdot \mathsf{id}, 1; a \cdot \mathsf{id}\rangle : a \in At(\mathfrak{A}) \;\&\; a \leq \mathsf{id}\} = \\
&= \{\langle a, a\rangle : a \in At(\mathfrak{A}) \;\&\; a \leq \mathsf{id}\} = \\
&= \mathsf{Id}.
\end{aligned}$$

To check composition we need a claim.

Claim 4.2.30. *The following are valid in* BM^+*:*

(i) $(x \cdot \mathsf{id}); (y \cdot \mathsf{id}) = x \cdot y \cdot \mathsf{id}$,
(ii) $a \cdot (b; c) \leq ((a; 1) \cdot b); ((1; a) \cdot c)$.

Proof: (i): $\leq$ holds by monotonicity of ; and $(M2)$. For $\geq$ first we show that $x \cdot \mathsf{id} \leq (x \cdot \mathsf{id}); (x \cdot \mathsf{id})$. Indeed, by $(M2)$,

$$\begin{array}{rcl} x \cdot \mathsf{id} & = & (x \cdot \mathsf{id}); \mathsf{id} = (x \cdot \mathsf{id}); (x \cdot \mathsf{id} + -x \cdot \mathsf{id}) = \\ \text{[by } (M3)] & = & (x \cdot \mathsf{id}); (x \cdot \mathsf{id}) + (x \cdot \mathsf{id}); (-x \cdot \mathsf{id}), \end{array}$$

and, by $(M3)$ and $(M2)$,

$$(x \cdot \mathsf{id}); (-x \cdot \mathsf{id}) \leq (x \cdot \mathsf{id}); \mathsf{id} = x \cdot \mathsf{id} \leq x$$

and

$$(x \cdot \mathsf{id}); (-x \cdot \mathsf{id}) \leq \mathsf{id}; (-x \cdot \mathsf{id}) = -x \cdot \mathsf{id} \leq -x,$$

whence $(x \cdot \mathsf{id}); (-x \cdot \mathsf{id}) = 0$. Then

$$x \cdot y \cdot \mathsf{id} \leq (x \cdot y \cdot \mathsf{id}); (x \cdot y \cdot \mathsf{id}) \leq (x \cdot \mathsf{id}); (y \cdot \mathsf{id})$$

by monotonicity.

(ii): First we show the following:

$$x \cdot (b; (c \cdot -(1; x))) = 0 \tag{4.1}$$

Indeed, by monotonicity,

$$\begin{array}{rcl} x \cdot (b; (c \cdot -(1; x))) & \leq & x \cdot (b; -(1; x)) \leq x \cdot (1; -(1; x)) = \\ \text{[by } (M4)] & = & x \cdot -(1; x) \leq (1; x) \cdot -(1; x) = \\ & = & 0. \end{array}$$

Then we can prove the following:

$$x \cdot (b; c) \leq b; (c \cdot (1; x)). \tag{4.2}$$

Indeed,

$$\begin{array}{rcl} x \cdot (b; c) & = & x \cdot (b; (c \cdot (1; x) + c \cdot -(1; x))) = \\ & = & x \cdot (b; (c \cdot (1; x))) + x \cdot (b; (c \cdot -(1; x))) = \\ \text{[by 4.1]} & = & x \cdot (b; (c \cdot (1; x))) \leq \\ & \leq & b; (c \cdot (1; x)). \end{array}$$

Similar argument shows

$$x \cdot (b; c) \leq (b \cdot (x; 1)); c. \tag{4.3}$$

Then, by 4.2,

$$\begin{array}{rcl} a \cdot (b; c) & = & (a \cdot (b; c)) \cdot (b; (c \cdot (1; a))) \leq \\ \text{[by 4.3]} & \leq & (b \cdot ((a \cdot b; c); 1)); (c \cdot (1; a)) \leq \\ \text{[by monotonicity]} & \leq & ((a; 1) \cdot b); ((1; a) \cdot c). \end{array}$$

∎

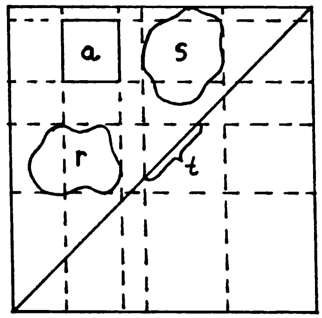

Figure 4.4: checking composition

Now we turn to checking composition.

$$\mathsf{rep}(r; s) = \{\langle a; 1 \cdot \mathsf{id}, 1; a \cdot \mathsf{id}\rangle : a \in At(\mathfrak{A}) \;\&\; a \leq r; s\}$$

and

$$\mathsf{rep}(r) \circ \mathsf{rep}(s) = \{\langle b; 1 \cdot \mathsf{id}, 1; c \cdot \mathsf{id}\rangle : b, c \in At(\mathfrak{A}) \;\&\; b \leq r \;\&\; c \leq s \;\&\; 1; b \cdot \mathsf{id} = c; 1 \cdot \mathsf{id}\}.$$

$\mathsf{rep}(r) \circ \mathsf{rep}(s) \subseteq \mathsf{rep}(r; s)$ amounts to prove that, for $b, c \in At(\mathfrak{A})$ such that $b \leq r$, $c \leq s$ and $1; b \cdot \mathsf{id} = c; 1 \cdot \mathsf{id}$, we can find $a \in At(\mathfrak{A})$ such that $a \leq r; s$, $a; 1 \cdot \mathsf{id} = b; 1 \cdot \mathsf{id}$ and $1; a \cdot \mathsf{id} = 1; c \cdot \mathsf{id}$. Let $a \stackrel{\text{def}}{=} b; 1 \cdot 1; c$. By Claim 4.2.29(iii), $b; 1 \cdot 1; c \in At(\mathfrak{A})$. We show that $b; c \neq 0$. Indeed, $0 = b; c$ would imply

$$0 = 1; (b; c); 1 \geq (1; b \cdot \mathsf{id}); (c; 1 \cdot \mathsf{id}) = 1; b \cdot \mathsf{id} \cdot c; \mathsf{id}$$

by Claim 4.2.30(i). This is a contradiction, since $1; b \cdot \mathsf{id} = c; 1 \cdot \mathsf{id} \neq 0$. Then

$$a \cdot r; s = (b; 1 \cdot 1; c) \cdot (r; s) \geq b; c \neq 0,$$

i.e., $a \leq r; s$. Finally,

$$a; 1 \cdot \mathsf{id} = (b; 1 \cdot 1; c); 1 \cdot \mathsf{id} = b; 1 \cdot \mathsf{id}$$

by simplicity; and similarly $1; a \cdot \mathsf{id} = 1; c \cdot \mathsf{id}$.

To prove $\mathsf{rep}(r; s) \subseteq \mathsf{rep}(r) \circ \mathsf{rep}(s)$, we have to find, for given $a \in At(\mathfrak{A})$ with $a \leq r; s$, two atoms $b, c \in At(\mathfrak{A})$ such that $b \leq r \;\&\; c \leq s \;\&\; b; 1 \cdot \mathsf{id} = a; 1 \cdot \mathsf{id} \;\&\; 1; c \cdot \mathsf{id} = 1; a \cdot \mathsf{id} \;\&\; 1; b \cdot \mathsf{id} = c; 1 \cdot \mathsf{id}$. Let $t \stackrel{\text{def}}{=} 1; (a; 1 \cdot r) \cdot \mathsf{id} \cdot (1; a \cdot s); 1$, and let $d \in At(\mathfrak{A})$ such that $d \leq t$, see Figure 4.4. We define $b \stackrel{\text{def}}{=} 1; d \cdot a; 1$ and $c \stackrel{\text{def}}{=} d; 1 \cdot 1; a$. First we show that $t \neq 0$. Indeed, if $t = 0$, i.e.,

$$0 = 1; (a; 1 \cdot r) \cdot \mathsf{id} \cdot (1; a \cdot s); 1,$$

then, by Claim 4.2.30(i),

$$0 = (1; (a; 1 \cdot r) \cdot \mathsf{id}); ((1; a \cdot s); 1 \cdot \mathsf{id}),$$

whence

$$0 = 1;(1;(a;1\cdot r)\cdot \mathsf{id});((1;a\cdot s);1\cdot \mathsf{id});1 \geq (a;1\cdot r);(1;a\cdot s) \geq a\cdot r;s$$

by Claim 4.2.30(ii). Since $a = a\cdot r;s$, we get $0 = r;s$, a contradiction. Then $b, c \in At(\mathfrak{A})$ by Claim 4.2.29(iii). Using the simplicity of $\mathfrak{A}$, it is easy to show that $b;1\cdot \mathsf{id} = a;1\cdot \mathsf{id}$ and $1;c\cdot \mathsf{id} = 1;a\cdot \mathsf{id}$. Further,

$$1;b\cdot \mathsf{id} = 1;(1;d\cdot a;1)\cdot \mathsf{id} = 1;d\cdot \mathsf{id} = d$$

by simplicity. Similarly we get $d = c;1\cdot \mathsf{id}$. It remains to prove that $b \leq r$ and $c \leq s$. It is enough to show that $b\cdot r \neq 0 \neq c\cdot s$, since b and c are atoms. By the definition of d, $d \leq 1;(a;1\cdot r)$, i.e., $0 \neq d\cdot 1;(a;1\cdot r)$. Then $0 \neq 1;(d\cdot 1;(a;1\cdot r))$, whence $0 \neq 1;d\cdot 1;(a;1\cdot r)$. Thus $0 \neq 1;(1;d\cdot a;1\cdot r)$, hence $0 \neq 1;d\cdot a;1\cdot r = b\cdot r$. Similar proof shows that $c \leq s$.

For the representability of simple rectangularly dense RA's it remains to prove that the above rep works for converse as well.

$$\begin{aligned}
\mathsf{rep}(s^{\smile}) &= \{\langle a;1\cdot \mathsf{id}, 1;a\cdot \mathsf{id}\rangle : a \in At(\mathfrak{A}) \ \&\ a \leq s^{\smile}\} = \\
&= \{\langle a^{\smile};1\cdot \mathsf{id}, 1;a^{\smile}\cdot \mathsf{id}\rangle : a \in At(\mathfrak{A}) \ \&\ a^{\smile} \leq s^{\smile}\} = \\
&= \{\langle 1;a\cdot \mathsf{id}, a;1\cdot \mathsf{id}\rangle : a \in At(\mathfrak{A}) \ \&\ a \leq s\} = \\
&= (\mathsf{rep}(s))^{-1},
\end{aligned}$$

since a is an atom below s iff $a^{\smile}$ is an atom below $s^{\smile}$, and

$$a^{\smile};1\cdot \mathsf{id} = (1;a)^{\smile}\cdot \mathsf{id} = (1;a\cdot \mathsf{id})^{\smile} = 1;a\cdot \mathsf{id},$$

cf. [CT51]. This finishes the proof of Theorem 4.2.24. ∎

Proof of Theorem 4.2.25: It suffices to prove that the conditions of Theorem 1.2.6 are met.

First, let $\mathsf{V} = \mathsf{BM}^+$. Clearly BM^+ is a BAO. We already showed that composition ; is a normal operator. If we define $\Diamond x$ as $1;x;1$, then $(M1)$–$(M3)$ guarantee that $\Diamond$ is a closure operator, while $(M4)$ ensures that $\Diamond$ is complemented. Finally, $(M3)$ implies that $x;y \leq \Diamond x\cdot \Diamond y$.

For $\mathsf{V} = \mathsf{RA}$ it remains to prove that $^{\smile}$ is a normal operator, and that $x^{\smile} \leq \Diamond x$. Normality follows from $(R7)$. The proof for the other equation is straightforward, since, in simple RA's, $1;x;1 = 1$ whenever $x > 0$. ∎

Proof of Theorem 4.2.26: The same argument (using the downward Löwenheim–Skolem theorem) works as in the proof of Theorem 4.2.8. ∎

EXTRO

"The torture never stops."
Frank Zappa

To conclude the dissertation we enumerate open problems, related results, and some possible further research directions connected to taming.

BRIDGE. As we mentioned in Chapter 1, the bridge between logics and algebras is worked out for more logics and metalogical properties, cf. [AKNS]. One of the most challenging tasks is to extend the bridge for even more logics. For instance, the techniques of algebraization work for the Lambek calculus, as we saw in Section 2.3. But these connections are not formulated in a general setting yet. The other direction is interesting as well: to consider even more properties. An example is to give sufficient and *necessary* conditions for not necessarily Hilbert-style completeness. This would require a classification of non-Hilbert-style calculi (e.g., by their syntactic form — what kind of condition C is).

REDUCTS. In Chapter 2, we argued that one of disjunction and composition must be left out from the set of connectives if we want to define complete and/or decidable versions of PAL with square universes. But the similarity type of the Lambek calculus is remarkably smaller than that of PAL. For instance, identity and converse are not definable. It would be interesting to see such expansions of the Lambek calculus which contains more connectives expressible in PAL_{sq} and is still complete and decidable. Another natural try would be to add the transitive closure of composition.

As we mentioned, the decidability of the Lambek calculus is proved by a syntactic argument: by cut-elimination. This does not answer the question whether the Lambek calculus has the finite model property, i.e., whether for every non-valid sequent there is a relational model with a finite universe refuting this sequent. Probably a semantical argument proving decidability would be useful to solve this open problem. One may think that the mosaic-method would work. But the problem is that the union of transitive mosaics is not necessarily transitive. Thus, in the present form, it is not clear how to use such an argument.

RELATIVIZATION. The most obvious question of Chapter 3 that we did not answer is if $(\mathbf{Rl}_H\mathsf{RRA})^{<\alpha}$ $(3 < \alpha,\ H \subseteq \{r, s\})$ generates a finitely axiomatizable (quasi-)variety. Or, equivalently, whether there is a strongly sound and complete Hilbert-style calculus for ${}^{\alpha}\mathsf{PAL}_H^{grad}$. Another interesting problem is whether the decidable logics of Chapter 3 have the finite model property, or whether $\mathsf{Alg}(\mathsf{L})$ has the finite base property. For instance, one may find a (combinatorial) argument to "shrink" the models (the bases of the algebras) we constructed using mosaics.

Again, the question of extending (even more) the similarity type arises. Transitive closure is one possibility. Another one is to define the graded modalities using accessibility relations: e.g., using T_0 and T_1 of Chapter 4. Decidability clearly holds for this version, and completeness is likely to hold as well.

One may ask if the taming strategy of Chapter 3 works for other logics as well. We have a good news: there are relativized versions of first-order logic with large (expressive) power that are Hilbert-style complete and decidable, see [Né92], [Né95],[Ma95], [MV95] and [Mi95]. In fact, it is possible to generalize the properties reflexivity and symmetry to relations of higher rank, and these relativized versions of first-order logic are decidable (even with graded modalities), and some of them are complete. Further, Beth definability and Craig interpolation properties hold for most of them.

DENSITY. The most important open problem of Chapter 4 is whether ${}^r\mathsf{L}_n{}^{\neq}$ has a weakly sound and complete calculus, or whether the diagonal-free rectangularly dense RCA_n's are representable. There are other classes of algebras for which the same question can be asked. For instance, we conjecture that the theorem holds for relativized versions of cylindric and polyadic algebras, and for Peirce algebras (cf., e.g., [dR93]).

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Samenvatting

Dit proefschrift gaat over *algebraïsche logica*, d.w.z., over *algebra's*, *logica's* en hun samenhang. In het bijzonder onderzoeken we *modale logica's* met een *dynamisch* karakter, predicaten-logica's, en de corresponderende klassen *algebra's van relaties*. We slaan een "brug" tussen logica en algebra die zowel logica's en algebra's, als metalogische en algebraïsche eigenschappen met elkaar verbindt.

Centraal staat het "temmen": het vinden van zich-goed-gedragende versies van veel onderzochte logica's. Het probleem van veel logica's is dat ze een aantal ongewenste eigenschappen hebben, zoals onvolledigheid en onbeslisbaarheid. Voorbeelden hiervan zijn de vierkante versie van *pijl-logica*, en *eerste-orde logica* (met minstens 3 variabelen). We temmen deze logica's door over de genoemde brug te gaan en de krachtige machinerie van algebraïsche logica en *universele algebra* toe te passen.

De opzet van dit proefschrift is als volgt. In hoofdstuk 1 introduceren we de logica's die we gaan onderzoeken, en werken we de brug tussen algebra en logica uit.

In hoofdstuk 2 kijken we naar *fragmenten* van pijl-logica en geven volledig- en beslisbaarheidsresultaten voor deze redukten. De meest interessante logica in dit hoofdstuk is de *Lambek calculus*. Ons belangrijkste resultaat is de volledigheid van deze calculus met betrekking tot een relationele semantiek. Het corresponderende algebraïsche resultaat geeft ons een representatie van *(semi-tralie-)geordende geresidueerde semi-groepen* als algebra's van binaire relaties.

Hoofdstuk 3 gaat over gerelativizeerde versies van pijl-logica. Eerst laten we meer modellen toe dan in de klassieke (vierkante) versie van pijl-logica, en dan voegen we connectieven toe die niet definieerbaar zijn in de zwakkere versie. We zullen bijvoorbeeld volledige en beslisbare versies van pijl-logica laten zien waarin de *difference operator* en de *graded modalities* gedefinieerd kunnen worden. Als we de brug oversteken naar algebra-land, dan vertellen deze resultaten ons dat verschillende expansies van *zwak-associatieve relatie-algebra's* en andere *gerelativizeerde versies van representeerbare relatie-algebra's* eindig axiomatiseerbaar en beslisbaar zijn.

In het laatste hoofdstuk benaderen we de volledigheids-problemen van vierkante pijl-logica en (klassieke) predicaten-logica door de regels te veranderen. In plaats van de logica te verzwakken door meer modellen toe te laten, herdefiniëren we de notie van een afleidingssysteem. Naast de standaard afleidingsregels zoals Modus Ponens staan we regels toe waarvan het gebruik beperkt is door bepaalde voorwaarden. Met behulp van deze regels zijn we in staat om simpele, eindige en volledige afleidingssystemen te geven voor bovengenoemde logica's. De algebraïsche kant van deze volledigheidsresultaten is dat die *relatie-*, *cylindrische-*, en *polyadische algebra's* die aan een bepaald *dichtheids* criterium voldoen, representeerbaar zijn als algebra's van relaties.

Tot slot noemen we een aantal open problemen in verband met dit proefschrift, verscheidene gerelateerde resultaten en mogelijk verder onderzoek.

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