S4.3 AND HEREDITARILY EXTREMALLY DISCONNECTED SPACES

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On the occasion of the one hundredth anniversary of the birth of George Chogoshvili.[1]

ABSTRACT. The modal logic **S4.3** defines the class of hereditarily extremally disconnected spaces (HED-spaces). We construct a countable HED-subspace X of the Gleason cover of the real closed unit interval $[0, 1]$ such that **S4.3** is the logic of X.

1. INTRODUCTION

It is well known that if we interpret modal diamond as topological closure (and hence modal box as topological interior), then the modal logic **S4** defines the class of all topological spaces. The celebrated McKinsey-Tarski theorem [14] states that **S4** is the logic of any dense-in-itself (separable) metrizable space. In particular, **S4** is the logic of the real closed unit interval $\mathbb{I} = [0, 1]$.

We recall that a topological space is extremally disconnected (ED-space) provided the closure of each open set is open. Compact Hausdorff ED-spaces are of major importance in the category of compact Hausdorff spaces as they are the projective objects in the category. In fact, each compact Hausdorff space has the projective cover, called the Gleason cover (see [12, 13, 16]). The class of ED-spaces is definable by the modal logic $\mathbf{S4.2} = \mathbf{S4} + \Diamond\Box p \to \Box\Diamond p$ (see, e.g., [3, p. 253]). As was shown in [1], ED-spaces play an important role in modeling full belief. It is a consequence of [5, Prop. 4.3] that **S4.2** is the logic of the Gleason cover of $\mathbb{I}$.

Our main interest in this paper is the modal logic $\mathbf{S4.3} = \mathbf{S4} + \Box(\Box p \to q) \vee \Box(\Box q \to p)$. This system plays an important role in tense logic. It was studied in detail by Bull [7], Fine [11], and others. In particular, it is known that Kripke frames of **S4.3** are those **S4.2**-frames whose subframes are also **S4.2**-frames. Similarly, we will see that topological spaces satisfying **S4.3** are those ED-spaces whose subspaces are also ED-spaces. Because of this, **S4.3** was recently proposed as the logic of updatable full belief [2].

ED-spaces whose subspaces are also ED-spaces are called hereditarily extremally disconnected spaces (HED-spaces). Unlike compact Hausdorff ED-spaces, which are in abundance, the only compact Hausdorff HED-spaces are finite (see, e.g., [6, p. 82]). On the other hand, there are plenty of non-compact HED-spaces. In fact, as follows from [6, Prop. 2.3], every dense-in-itself topology is contained in a dense-in-itself HED-topology. As we already pointed out above, **S4.2** is the logic of the Gleason cover of $\mathbb{I}$. Our main result yields a countable Hausdorff HED-subspace of the Gleason cover of $\mathbb{I}$ whose logic is **S4.3**.

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[1]George Chogoshvili (1914–1998) was the founder of the Georgian topological school, and one of the most influential Georgian mathematicians of the twentieth century. He has been a source of inspiration for many generations. In particular, Chogoshvili's ideas influenced Leo Esakia (1934–2010), who was one of the pioneers in developing *topological modal logic*, an area representing a fruitful cross-fertilization of tools and techniques of topology and logic. Our paper continues this tradition. We are honored to dedicate it to the memory of Professor Chogoshvili.

In proving our main result we utilize two tools, one logical and one topological. On the one hand, we use the fact that **S4.3** is characterized by finite rooted **S4.3**-frames (see, e.g., [8, Ch. 5]). On the other hand, we use Efimov's theorem [9] (see also [15, Thm. 1.4.7]) that each compact Hausdorff ED-space of weight no greater than continuum can be embedded in the Čech-Stone compactification of the natural numbers.

2. Preliminaries

In this section we recall some basic definitions and facts about modal logic and topology. As basic references we use [8] for modal logic and [10] for topology.

2.1. **Logical background.** The modal logic **S4** is the least set of formulas containing the classical tautologies, the formulas

$$\Box(p \to q) \to (\Box p \to \Box q),$$
$$\Box p \to p,$$
$$\Box p \to \Box\Box p,$$

and closed under Modus Ponens (MP) $\frac{\varphi,\ \varphi\to\psi}{\psi}$, substitution (S) $\frac{\varphi(p_1,\dots,p_n)}{\varphi(\psi_1,\dots,\psi_n)}$, and necessitation (N) $\frac{\varphi}{\Box\varphi}$. As it is customary, we use $\Diamond\varphi$ to abbreviate $\neg\Box\neg\varphi$. Let

$$\begin{aligned}\mathbf{S4.2} &= \mathbf{S4} + \Diamond\Box p \to \Box\Diamond p,\\ \mathbf{S4.3} &= \mathbf{S4} + \Box(\Box p \to q) \vee \Box(\Box q \to p).\end{aligned}$$

A *Kripke frame* is a pair $\mathfrak{F} = (W, R)$, where W is a nonempty set and R is a binary relation on W. A *valuation* in $\mathfrak{F}$ is a function ν assigning subsets of $\mathfrak{F}$ to propositional letters. This assignment extends recursively to all formulas, where Boolean connectives $\wedge, \neg$ are interpreted as set-theoretic intersection and complement, and we set

$$\begin{aligned}w \vDash \Box\varphi \quad &\text{iff} \quad (\forall v)(wRv \to v \vDash \varphi),\\ w \vDash \Diamond\varphi \quad &\text{iff} \quad (\exists v)(wRv \wedge v \vDash \varphi).\end{aligned}$$

A *model* on $\mathfrak{F}$ is a pair $\mathfrak{M} = (\mathfrak{F}, \nu)$, where ν is a valuation in $\mathfrak{F}$. A formula φ is *true* in a model $\mathfrak{M} = (\mathfrak{F}, \nu)$ provided $w \vDash \varphi$ for each $w \in W$; and φ is *valid* in a frame $\mathfrak{F}$ provided φ is true in every model on $\mathfrak{F}$. If φ is valid in $\mathfrak{F}$, we write $\mathfrak{F} \vDash \varphi$. If φ is not valid in $\mathfrak{F}$, then we say that $\mathfrak{F}$ *refutes* φ and write $\mathfrak{F} \nvDash \varphi$.

Let $\mathfrak{F} = (W, R)$ be a Kripke frame. We call $\mathfrak{F}$ a *quasi-order* provided R is reflexive and transitive. It is well-known (see, e.g., [8, Ch. 3]) that $\mathfrak{F} \vDash \mathbf{S4}$ iff $\mathfrak{F}$ is a quasi-order, that $\mathfrak{F} \vDash \mathbf{S4.2}$ iff $\mathfrak{F}$ is a quasi-order satisfying

$$(\forall u, v, v')(uRv \wedge uRv') \to (\exists w)(vRw \wedge v'Rw), \tag{1}$$

and that $\mathfrak{F} \vDash \mathbf{S4.3}$ iff $\mathfrak{F}$ is quasi-order satisfying

$$(\forall u, v, w)(uRv \wedge uRw) \to (vRw \vee wRv). \tag{2}$$

It is easy to see that a quasi-order $\mathfrak{F}$ satisfies (2) iff each subframe of $\mathfrak{F}$ satisfies (1).

For a quasi-order $\mathfrak{F} = (W, R)$, define an equivalence relation $\sim$ on W by setting $w \sim v$ iff wRv and vRw. The equivalence classes of $\sim$ are called *clusters* of $\mathfrak{F}$. One can partially order the clusters by setting $C \le C'$ iff there exist $w \in C$ and $w' \in C'$ such that wRw'. The resulting partial order is known as the *skeleton* of $\mathfrak{F}$. We say that a cluster C of $\mathfrak{F}$ is *maximal* provided C is a maximal element of the skeleton, and we call $\mathfrak{F}$ a *quasi-chain* provided the skeleton of $\mathfrak{F}$ is a chain.

Let $\mathfrak{F} = (W, R)$ be a quasi-order. Then $w \in W$ is a *root* of $\mathfrak{F}$ provided wRv for each $v \in W$, and $\mathfrak{F}$ is *rooted* provided $\mathfrak{F}$ has a root. It is easy to see that a finite rooted quasi-order satisfies (1) iff it has a unique maximal cluster, and it satisfies (2) iff it is a quasi-chain. It is well known (see, e.g., [8, Ch. 5]) that **S4** is characterized by finite rooted quasi-orders, that **S4.2** is characterized by finite rooted quasi-orders having a unique maximal cluster, and that **S4.3** is characterized by finite quasi-chains.

2.2. **Topological background.** Topological semantics generalizes Kripke semantics for **S4**. Indeed, we can view quasi-orders as special topological spaces, in which each point has a least neighborhood, namely $R[w] := \{v \mid wRv\}$. Such spaces are often referred to as *Alexandroff spaces* and can equivalently be characterized as those topological spaces in which the intersection of an arbitrary family of opens is open. The quasi-order associated with an Alexandroff space X is the *specialization order* of a topological space defined by xRy iff x belongs to the closure of $\{y\}$.

Given a topological space X, we interpret formulas as subsets of X, Boolean connectives as the corresponding set-theoretic operations, $\Box$ as interior, and $\Diamond$ as closure. Consequently, for $x \in X$, we have

$$x \vDash \Box\varphi \quad \text{iff} \quad \text{there is an open neighborhood } U \text{ of } x \text{ such that } y \vDash \varphi \text{ for all } y \in U,$$
$$x \vDash \Diamond\varphi \quad \text{iff} \quad \text{for each open neighborhood } U \text{ of } x \text{ there is } y \in U \text{ such that } y \vDash \varphi.$$

Since **S4**-axioms correspond to Kuratowski's axioms, we see that **S4** defines the class of all topological spaces. Moreover, since for an Alexandroff space, the topological semantics coincides with the Kripke semantics of the associated quasi-order, and since **S4** is Kripke complete, we see that **S4** is the logic of all topological spaces. In fact, by the McKinsey-Tarski theorem [14], **S4** is the logic of an arbitrary dense-in-itself separable metric space. Rasiowa and Sikorski proved in [17] that separability can be dropped from the assumptions, and hence **S4** is the logic of an arbitrary dense-in-itself metric space.

A topological space X is *extremally disconnected* (ED-space) if the closure of each open subset of X is open, and it is *hereditarily extremally disconnected* (HED-space) if every subspace of X is an ED-space. Let $\mathbf{i}$ and $\mathbf{c}$ denote the interior and closure. Since X is an ED-space iff $\mathbf{ci}(A) \subseteq \mathbf{ic}(A)$ for each $A \subseteq X$, we see that **S4.2** defines the class of all ED-spaces. In addition, since **S4.2** is Kripke complete, we see that **S4.2** is the logic of all ED-spaces. It is a corollary of the McKinsey-Tarski theorem that **S4** is the logic of the real closed unit interval $\mathbb{I} = [0, 1]$. It follows from [5] that **S4.2** is the logic of the Gleason cover of $\mathbb{I}$.

The *Gleason cover* of a compact Hausdorff space X is a pair (Y, π), where Y is a compact Hausdorff ED-space and $\pi : Y \to X$ is an *irreducible map* (an onto continuous map such that the image of a proper closed subset of the domain is proper). The Gleason cover of X is unique up to homeomorphism, and can be constructed as follows. A subset U of X is *regular open* if $U = \mathbf{ic}(U)$. Let $\mathcal{RO}(X)$ be the collection of regular open subsets of X. Ordered by inclusion, $\mathcal{RO}(X)$ is a complete Boolean algebra, where $\bigvee_I U_i = \mathbf{ic}\left(\bigcup_I U_i\right)$ and $\neg U = \mathbf{i}(X \setminus U)$. Let Y be the Stone space of $\mathcal{RO}(X)$ (the space of ultrafilters of $\mathcal{RO}(X)$). By Stone duality, since $\mathcal{RO}(X)$ is complete, Y is a compact Hausdorff ED-space. Define $\pi : Y \to X$ by setting $\pi(\nabla) = \bigcap\{\mathbf{c}_X(U) \mid U \in \nabla\}$. Then (Y, π) is the Gleason cover of X [12].

3. Main results

Our goal is to obtain results about **S4.3** and HED-spaces that are similar to the ones about **S4.2** and ED-spaces. We start by showing that **S4.3** defines the class of all HED-spaces (see also [2]). We recall that $A, B \subseteq X$ are *separated* provided $\mathbf{c}(A) \cap B = \varnothing = A \cap \mathbf{c}(B)$. By [6, Prop. 2.1], X is HED iff any two separated subsets of X have disjoint closures.

Proposition 3.1. *For a topological space X, the following are equivalent:*

1. *X is an HED-space.*
2. *$X \vDash \Box(\Box p \to q) \vee \Box(\Box q \to p)$.*
3. *$\mathbf{c}(A \setminus \mathbf{c}B) \cap \mathbf{c}(B \setminus \mathbf{c}A) = \varnothing$ for any $A, B \subseteq X$.*

Proof. It is straightforward to verify that when interpreting $\Box$ as $\mathbf{i}$, then $X \vDash \Box(\Box p \to q) \vee \Box(\Box q \to p)$ iff $\mathbf{c}(A \setminus \mathbf{c}B) \cap \mathbf{c}(B \setminus \mathbf{c}A) = \varnothing$ for any $A, B \subseteq X$. Thus, (2) is equivalent to (3). To see that (1) implies (3), suppose that X is an HED-space. Since $A \setminus \mathbf{c}B$ and $B \setminus \mathbf{c}A$ are separated, [6, Prop. 2.1] yields $\mathbf{c}(A \setminus \mathbf{c}B) \cap \mathbf{c}(B \setminus \mathbf{c}A) = \varnothing$. Finally, to see that (3) implies (1), suppose that $A, B \subseteq X$ are separated. Then $A \setminus \mathbf{c}(B) = A$ and $B \setminus \mathbf{c}A = B$. Therefore, $\mathbf{c}(A) \cap \mathbf{c}(B) = \mathbf{c}(A \setminus \mathbf{c}B) \cap \mathbf{c}(B \setminus \mathbf{c}A) = \varnothing$. Thus, X is an HED-space by [6, Prop. 2.1]. $\square$

As a corollary, we obtain that **S4.3** defines the class of all HED-spaces. Since **S4.3** is Kripke complete, it follows that **S4.3** is the logic of all HED-spaces. As we pointed out in the introduction, **S4.2** is the logic of the Gleason cover of $\mathbb{I}$. We will construct a countable HED-subspace X of the Gleason cover of $\mathbb{I}$ whose logic is **S4.3**.

To see that the logic of an HED-space X is **S4.3**, in view of Proposition 3.1, it is sufficient to show that each non-theorem of **S4.3** is refuted on X. But since **S4.3** is the logic of finite quasi-chains, each non-theorem of **S4.3** is refuted on a finite quasi-chain. We call a topological space Y an *interior image* of X provided there is a continuous open surjection $f : X \to Y$. Note that f is continuous and open iff $\mathbf{c}_X f^{-1}(B) = f^{-1}\mathbf{c}_Y(B)$ for each $B \subseteq Y$ (see, e.g., [17, pp. 99–100]); and B can be replaced by singletons provided Y is finite. It is well known (see, e.g., [4, Prop. 2.9]) that interior images reflect refutation. Therefore, to conclude that **S4.3** is the logic of X, it is sufficient to show that each finite quasi-chain, viewed as a topological space, is an interior image of X.

Let (Y, π) be the Gleason cover of $\mathbb{I}$. As $\mathbb{I}$ is a dense-in-itself separable space, so is Y. Moreover, since Y is an infinite compact Hausdorff ED-space, it contains a copy of the Čech-Stone compactification of the natural numbers $\beta\omega$ (see, e.g., [10, Exercise 6.2.G(b)]).[2] Therefore, the weight of Y is at least that of continuum. But since Y is separable, its weight is at most that of continuum. Thus, the weight of Y is that of continuum. Furthermore, $\pi^{-1}(x)$ is infinite for each $x \in \mathbb{I}$. To see this, take a pairwise disjoint family $\{U_n \in \mathcal{RO}(\mathbb{I}) \mid n \in \omega\}$ such that $x \in \mathbf{c}(U_n)$ for each $n \in \omega$.[3] The filter in $\mathcal{RO}(\mathbb{I})$ generated by the regular open neighborhoods of x together with U_n is proper, hence extends to an ultrafilter in $\mathcal{RO}(\mathbb{I})$. Each such ultrafilter ∇ contains all regular open neighborhoods of x, so $\pi(\nabla) = x$. Since the U_n's are disjoint, these ultrafilters are distinct, producing infinitely many points in $\pi^{-1}(x)$. In fact, each $\pi^{-1}(x)$ has a large cardinality because as an infinite closed set of a compact Hausdorff ED-space, it contains a copy of $\beta\omega$ (see, e.g., [10, Exercise 6.2.G(b)]).

Lemma 3.2. *There is a pairwise disjoint family $\{E_n \subseteq Y \mid n \in \omega\}$ such that each E_n is countably infinite and dense in Y.*

Proof. Let D be a countably infinite dense subset of $\mathbb{I}$ (for example, take $D = \mathbb{Q} \cap \mathbb{I}$). For each $x \in D$, since $\pi^{-1}(x)$ is infinite, there is a countably infinite subset $D_x = \{x_n \mid n \in \omega\}$

[2] This can be seen by observing that a compact Hausdorff ED-space is an F-space [10, Exercise 6.2.G(f)] and then applying [18, p. 37, Prop. 1.64].

[3] To see that such a family exists, let $m \in \omega$. Put $V_m^- = (x - \frac{1}{2m+1}, x - \frac{1}{2(m+1)})$ and $V_m^+ = (x + \frac{1}{2(m+1)}, x + \frac{1}{2m+1})$. Let $\theta : \omega \to \omega$ be any sequence such that $\theta^{-1}(n)$ is infinite for all $n \in \omega$. Finally, for $n \in \omega$, set $U_n = \bigcup\{(V_m^- \cup V_m^+) \cap \mathbb{I} \mid m \in \theta^{-1}(n)\}$. Since any two distinct intervals in $\{V_m^-, V_m^+ \mid m \in \omega\}$ do not share endpoints, and hence have disjoint closures, we see that each U_n is regular open.

of $\pi^{-1}(x)$. For each $n \in \omega$, define $E_n = \{x_n \mid x \in D\}$ (see Figure 1). Clearly each E_n is

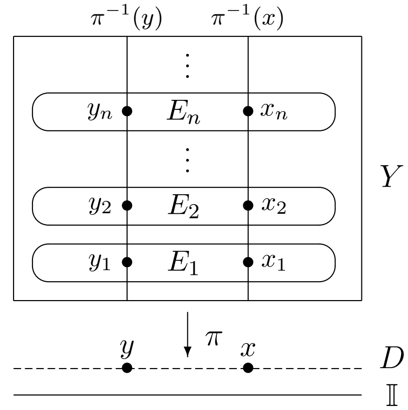

Figure 1

countably infinite and $\{E_n \mid n \in \omega\}$ is pairwise disjoint. It remains to be shown that E_n is dense in Y for each $n \in \omega$. By construction, $\pi(E_n) = \{\pi(x_n) \mid x \in D\} = D$. Since π is a closed map, $\pi(\mathbf{c}E_n)$ is a closed set in $\mathbb{I}$ containing D. As D is dense in $\mathbb{I}$, we see that $\pi(\mathbf{c}E_n) = \mathbb{I}$. Thus, since π is irreducible, $\mathbf{c}E_n = Y$. □

We are ready to construct an HED-subspace X of Y such that each finite quasi-chain is an interior image of X. The space X will be the union of the spaces $\{X_n \mid n \in \omega\}$, defined recursively. In defining X_n we will also define a decreasing sequence Y_n of subspaces of Y such that each Y_n is homeomorphic to Y. Let D be a fixed countable dense subset of $\mathbb{I}$, and fix $x \in \mathbb{I} \setminus D$.

Base Step: Let $\{E_n \subseteq Y \mid n \in \omega\}$ be the pairwise disjoint family of countably infinite dense subsets of Y constructed in Lemma 3.2, and set $X_0 := \bigcup\{E_n \subseteq Y \mid n \in \omega\}$. Note that $X_0 \subseteq \pi^{-1}(D) \subseteq Y \setminus \pi^{-1}(x)$. Put $Y_0 = Y$ and let $h_0 : Y_0 \to Y$ be the identity homeomorphism.

Recursive Step: Suppose X_n and Y_n are already defined, $h_n : Y_n \to Y$ is a homeomorphism, $\pi_n = \pi \circ h_n$, and $X_n \subseteq Y_n \setminus (\pi_n)^{-1}(x)$. Let β_n be a closed subspace of $(\pi_n)^{-1}(x)$ homeomorphic to $\beta\omega$. Since Y is a compact Hausdorff ED-space whose weight is that of continuum, by Efimov's theorem, there is a closed subspace Y_{n+1} of β_n and a homeomorphism $h_{n+1} : Y_{n+1} \to Y$. Let X_{n+1} be the union of the pairwise disjoint family of countably infinite dense subsets of Y_{n+1} constructed in Lemma 3.2.

The recursive step of the construction is captured in Figure 2. The members of the family $\{E_i \mid i \in \omega\}$ given by Lemma 3.2 that make up X_n, which is a proper dense subset of Y_n, are labeled and depicted by collections of horizontally arranged dots. The set $(\pi_n)^{-1}(x) \subseteq Y_n$ is pictured as a 'pointed oval' appearing above the point $x \in \mathbb{I} \setminus D$. Notice that β_n, which is homeomorphic to $\beta\omega$, is contained in $(\pi_n)^{-1}(x)$, and the isolated points of β_n are depicted by the collection of vertically arranged dots 'converging' to the remainder, which is indicated by the box with thin lines. Further note that Y_{n+1}, which is homeomorphic to Y, is contained in (the remainder of) β_n, that $(\pi_{n+1})^{-1}(x) \subseteq Y_{n+1}$ is also pictured as a 'pointed oval' above the point x, and that $X_{n+1} \subseteq Y_{n+1} \setminus (\pi_{n+1})^{-1}(x)$ is also indicated by collections of horizontally arranged dots.

Observe that $\{X_n \mid n \in \omega\}$ is a pairwise disjoint family. We set $X := \bigcup\{X_n \mid n \in \omega\}$.

Lemma 3.3.

(1) *If $n \geq m$, then $X_n \subseteq Y_m$; and if $n < m$, then $X_n \cap Y_m = \varnothing$.*
(2) *X is countable.*
(3) *X is a dense subspace of Y.*

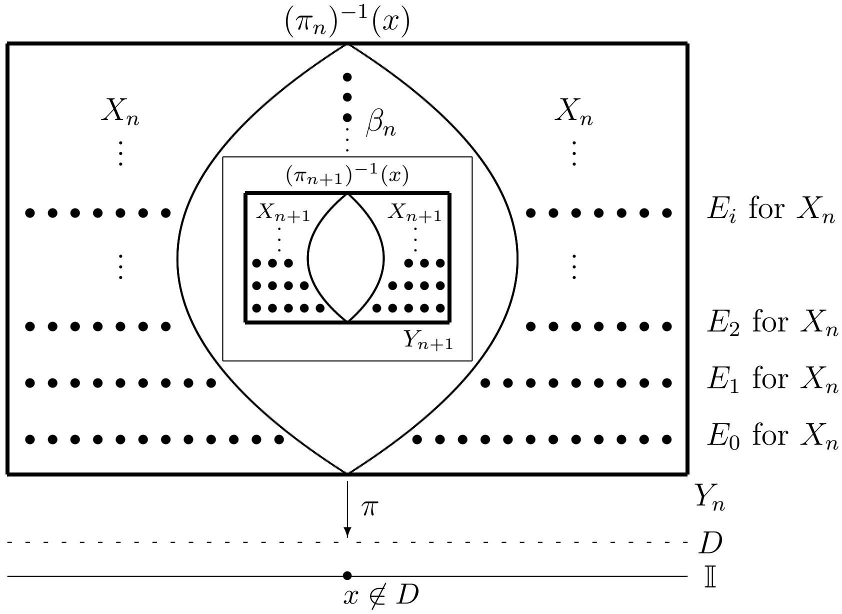

Figure 2

(4) *X is an HED-space.*

Proof. (1) By definition, $Y_{m+1} \subseteq \beta_m \subseteq (\pi_m)^{-1}(x) \subseteq Y_m$. Therefore, $n \geq m$ implies $Y_n \subseteq Y_m$. Since $X_n \subseteq Y_n$, we conclude that $n \geq m$ implies $X_n \subseteq Y_m$.

By definition, $X_n \subseteq Y_n \setminus (\pi_n)^{-1}(x)$. Since $Y_{n+1} \subseteq (\pi_n)^{-1}(x)$, we see that $X_n \cap Y_{n+1} = \varnothing$. Therefore, if $n < m$, then $n+1 \leq m$. Thus, $Y_m \subseteq Y_{n+1}$, yielding $X_n \cap Y_m \subseteq X_n \cap Y_{n+1} = \varnothing$.

(2) By definition, each X_n is a countable union of countable sets, hence is countable. Therefore, X is a countable union of countable sets, and so is countable.

(3) It is clear by the definition of X_0 and Lemma 3.2 that X_0 is dense in Y. Since $X_0 \subseteq X$, we conclude that X is dense in Y.

(4) By (3), X is a dense subspace of an ED-space, so X is an ED-space. As a subspace of a Hausdorff space, X is also clearly Hausdorff. But every countable Hausdorff ED-space is an HED-space (see, e.g., [6, p. 86]). Thus, X is an HED-space. ☐

Lemma 3.4. *Every finite quasi-chain is an interior image of X.*

Proof. Suppose $\mathfrak{F} = (W, R)$ is a finite quasi-chain, and suppose its skeleton is ordered as follows

$$C_{k-1} \leq C_{k-2} \leq \cdots \leq C_1 \leq C_0.$$

The idea is to map X_i to C_i for $i < k$, and to C_{k-1} for $i \geq k$. Recall that X_i is the countably infinite disjoint union of countable infinite dense subsets $\{E_{ij} \mid j \in \omega\}$ of Y_i.

Claim 3.5. *Let $i \in \omega$. For any nonempty $\alpha \subseteq \omega$, we have*

$$\mathbf{c}_X \left(\bigcup \{E_{ij} \mid j \in \alpha\} \right) = \bigcup \{X_n \mid n \geq i\}.$$

Proof of Claim: Let $\alpha \subseteq \omega$ be nonempty. Since each E_{ij} is dense in Y_i, the set $\bigcup\{E_{ij} \mid j \in \alpha\}$ is dense in Y_i. Therefore,

$$Y_i = \mathbf{c}_{Y_i} \left(\bigcup \{E_{ij} \mid j \in \alpha\} \right) = \mathbf{c}_Y \left(\bigcup \{E_{ij} \mid j \in \alpha\} \right) \cap Y_i \subseteq \mathbf{c}_Y \left(\bigcup \{E_{ij} \mid j \in \alpha\} \right).$$

Conversely, from $E_{ij} \subseteq Y_i$ it follows that $\bigcup\{E_{ij} \mid j \in \alpha\} \subseteq Y_i$. Therefore, since Y_i is closed in Y, we have $\mathbf{c}_Y\left(\bigcup\{E_{ij} \mid j \in \alpha\}\right) \subseteq Y_i$, hence the equality. Thus, by Lemma 3.3(1),

$$\mathbf{c}_X\left(\bigcup\{E_{ij} \mid j \in \alpha\}\right) = \mathbf{c}_Y\left(\bigcup\{E_{ij} \mid j \in \alpha\}\right) \cap X = Y_i \cap X = \bigcup\{X_n \mid n \geq i\}.$$

□

Let $\equiv_n$ be the congruence on ω modulo n. For $i < k$, let $C_i = \{w_0, \cdots, w_{n_i-1}\}$. Partition X_i into

$$\bigcup\{E_{ij} \mid j \equiv_{n_i} 0\},\ \bigcup\{E_{ij} \mid j \equiv_{n_i} 1\},\ \cdots, \bigcup\{E_{ij} \mid j \equiv_{n_i} n_i - 1\}.$$

Define $f : X \to W$ as follows. If $x \in X_i$ for $i < k$, then set $f(x) = w_n$ provided $x \in \bigcup\{E_{ij} \mid j \equiv_{n_i} n\}$. If $x \in X_i$ for $i \geq k$, then set $f(x) = v$ for some $v \in C_{k-1}$.

The map $f : X \to W$ is well defined since $\{X_i \mid i \in \omega\}$ partitions X and the sets

$$\bigcup\{E_{ij} \mid j \equiv_{n_i} 0\},\ \bigcup\{E_{ij} \mid j \equiv_{n_i} 1\},\ \cdots, \bigcup\{E_{ij} \mid j \equiv_{n_i} n_i - 1\}$$

partition X_i for $i < k$. Furthermore, for each $i < k$, we have

$$f(X_i) = f\left(\bigcup_{n<n_i}\bigcup_{j\equiv_{n_i} n} E_{ij}\right) = \bigcup_{n<n_i} f\left(\bigcup_{j\equiv_{n_i} n} E_{ij}\right) = \bigcup_{n<n_i}\{w_n\} = C_i.$$

Therefore, f is onto.

Viewing $\mathfrak{F}$ as an Alexandroff space, the closure of $w \in W$ is $R^{-1}[w] := \{v \mid vRw\}$. Therefore, to see that f is interior, since W is finite, it is sufficient to show that $\mathbf{c}_X f^{-1}(w) = f^{-1}R^{-1}[w]$ for each $w \in W$. Let $w \in W$. Then $w \in C_i$ for some $i < k$. Therefore, $w = w_m$ for some $m \leq n_i - 1$. First suppose that $x \in \mathbf{c}_X f^{-1}(w)$. Then, by Claim 3.5,

$$x \in \mathbf{c}_X f^{-1}(w_m) = \mathbf{c}_X\left(\bigcup_{j\equiv_{n_i} m} E_{ij}\right) = \bigcup_{n\geq i} X_n,$$

giving

$$f(x) \in f\left(\bigcup_{n\geq i} X_n\right) = \bigcup_{n\geq i} f(X_n) = \bigcup_{k>n\geq i} C_n = R^{-1}[C_i] = R^{-1}[w].$$

Thus, $x \in f^{-1}R^{-1}[w]$.

Conversely, suppose $x \in f^{-1}R^{-1}[w]$. Then $f(x)Rw$, giving that $f(x) \in C_j$ for $i \leq j < k$. By the definition of f, it must be the case that $x \in X_j$ when $j < k-1$ and $x \in \bigcup_{n\geq k-1} X_n$ when $j = k-1$. Therefore, by Claim 3.5,

$$x \in \bigcup_{n\geq i} X_n = \mathbf{c}_X\left(\bigcup_{j\equiv_{n_i} m} E_{ij}\right) = \mathbf{c}_X f^{-1}(w_m) = \mathbf{c}_X f^{-1}(w).$$

Thus, $\mathbf{c}_X f^{-1}(w) = f^{-1}R^{-1}[w]$, completing the proof. □

Theorem 3.6. S4.3 *is the logic of a countable HED-subspace of the Gleason cover of* $\mathbb{I}$.

Proof. Let X be the countable subspace of the Gleason cover of $\mathbb{I}$ constructed above. By Lemma 3.3(4), X is an HED-space. Therefore, by Proposition 3.1, $X \models$ **S4.3**. Suppose **S4.3** $\not\vdash \varphi$. Since **S4.3** is the logic of finite quasi-chains, there is a finite quasi-chain $\mathfrak{F}$ refuting φ. By Lemma 3.4, $\mathfrak{F}$ is an interior image of X. Since interior images reflect refutations, X refutes φ. Thus, **S4.3** is the logic of X. □

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