

**SOME EXTENSIONAL  
TERM MODELS FOR  
COMBINATORY LOGICS  
AND  
 $\lambda$  - CALCULI**

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FOR COMBINATORY LOGICS  
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## Introduction

This thesis consists of two parts which are, physically, separated into the present 'proefschrift' and a supplementary part II published in Barendregt [1971].

The subjects studied are formal properties of extensions of combinatory logic and of the  $\lambda$ -calculus, and (formal) relations between them. But neither the choice of problems nor even of the techniques introduced in the proofs can be properly understood without a description of the notions which we intend to study. Such a description will be given in this introduction. The summaries of the text which follow the introduction not only quote the main formal results but interpret them in terms of the aims described in the introduction. Strictly speaking, we do not only summarize results actually stated in the text, but include background material and, sometimes, alternative proofs.

Rules: the intended interpretations. Whatever other interesting models the formal theories considered here may have, in particular the projective limits recently introduced by Scott [1970], the original intention of the founders of our subject was to study rules; or, in other words, to study the old-fashioned notion of 'function' in the sense of definition. In contrast to Dirichlet's notion (of graph, that is the set of pairs of argument and associated value) the older notion referred also to the process of stepping from argument to value, a process coded by a definition. Generally we think of such definitions as given by words in ordinary English, applied to arguments also expressed by words (in English); or, more specifically, we may think of the

definitions as programmes for machines, applied to, that is operating on, such programmes. In both cases we have to do with a type free structure, permitting self application, a feature which is often held to be 'responsible' for the contradictions in the first formulations of set theory (by Frege). Similarly the first formulation of the  $\lambda$ -calculus turned out to be inconsistent (paradox of Kleene-Rosser [1935]). In contrast the first formulation of combinatory logic was consistent (but not some of its later extensions as shown by the paradox of Curry [1942]).

Evidently, the type free character as such is not problematic. We have not only the examples informally described in the preceding paragraph, but also many familiar and natural structures in elementary algebra where an object acts both as argument and as function. Specifically, in the theory of semigroups an element  $a$  determines the function with the action

$$x \rightarrow ax$$

(and in group theory an 'element' determines its co'set').

It seems plausible that the contradictions are connected with the aim of providing foundations for the whole of mathematics.

Before we distinguish between different meanings of such general concepts as 'set' or 'rule' employed in different areas of mathematics, it is tempting to put down axioms and rules, some of which are valid for one meaning, some for another. In this way contradictions are liable to arise.

Be that as it may, here we do not propose to use combinatory logic or the  $\lambda$ -calculus as a foundation for the whole of mathematics; but rather in the study of those parts of the

subject which actually present themselves as being about rules.

A paradigm of such parts is numerical arithmetic which is

'about' rules in the literal sense that

$$5 + 7 = 2 + 10$$

asserts:

The LHS and RHS reduce to the same numeral, that is to the same normal (or canonical) form when the computation rules for addition are applied.

From this point of view, perhaps the single most important feature of a language (for a theory of rules) is that each term should code its own reduction procedure; not necessarily a deterministic one, but a class of 'equivalent' ones. Thus implicit in an interpretation or 'model' of the language is an immediate reduction relation or 'multiplication table'. In fact, in CL or the  $\lambda$ -calculus the intended reduction procedure is only implicit, that is we may assign a specific procedure to each term metamathematically, but the intended immediate reduction relation cannot be expressed in the purely equational languages of CL or the  $\lambda$ -calculus. Moreover, throughout this thesis (usually in connection with certain 'conservative extension results') we introduce additional symbols for reduction relations and axioms which make more explicit the intended meaning of the formalism. By the way our use of such additional 'structure' is parallel to the use in set theory of ordinals and their order, and of the relation between sets  $x$  and ordinals  $\alpha$  ( $\alpha$  is the rank of  $x$ ); the only difference is that in the case of set theory, at least in the presence of the axiom of foundation, the extensions considered are not merely conservative but definitional (familiar from

Tarski, Mostowski, Robinson [1953]).

Term Models. In the light of the preceding paragraph, one fundamental role of terms is clear: they are the objects on which we, literally, operate and they are the objects which code the operations. Another role, to be distinguished from that of terms-as-elements-of-the-intended-reduction-relation, is the use of terms in a formal theory of this relation, as is clear from the assertion about numerical arithmetic mentioned earlier on. The difference is particularly striking for terms of the formal theory which contain variables, when we think of the reduction relation as being a relation between closed terms (also called: interior of the term models considered below).

The reason for speaking of term models, which suggests model theory, is this. Whatever our specific intended interpretations may be, our formal theory can also be interpreted in traditional model theoretic style. In fact, some of the more general results are consequences of such superficial syntactic features as these: When regarded as theories on 'standard' formalization, CL is a purely equational system and, a fortiori, axiomatized by universal formulae, and so is the  $\lambda$ -calculus (if for each term  $\lambda x t$  we associate a function symbol  $f_t$  with  $n$  arguments, if  $\lambda x t$  contains  $n$  free variables and the axiom  $f_t a = [x/a]t^*$ , where  $t^*$  results from  $t$  by replacing the  $\lambda$ -expressions in  $t$  by the associated function symbols).

By what has just been said we do not think of these 'applications' of model theory as profound but rather as separating general (some would say 'trivial') properties from the specific properties of our intended interpretation. Adapting a phrase from Bourbaki, model theory provides here the 'hygiene' of proof theory (used for establishing the more specific properties).

A typical example of such uses of elementary model theory occurs in the analysis of extensionality. This property (of a term model) cannot be expressed equationally, but needs the logically complicated form,

$$(\forall x E_1) \rightarrow E_2$$

for suitable equations  $E_1$  and  $E_2$ . Hence it cannot be expressed directly in CL. However, as far as equational consequences of (this axiom of) extensionality are concerned, they can also be generated by the purely equational rule of extensionality. Thus, in contrast to the defects of equational theories, mentioned above, for expressing the intended reduction relation, these theories are adequate for formulating and solving problems of extensionality, in consequence of the following quite general 'model theoretic' result. (For arbitrary quantifier-free  $A_1$  and  $A_2$ , in place of equations  $E_1$  and  $E_2$ , Shepherdson [1965] finds a more delicate rule to replace axioms  $\forall x A_1 \rightarrow A_2$ .)

Let  $S$  be any set of atomic formulae of a predicate logic, closed under logical deduction (that is substitution of a term of the language for variables), and under the rule: derive  $P_0^{(i)}$  for  $P_1^{(i)}, \dots, P_{k_i}^{(i)}$  ( $i \in I$ ) where the  $P$ 's are all atomic.

Then the term model of the theory (in which  $P$  is interpreted as  $S \vdash P$ ), satisfies the axioms corresponding to the rules:

$$\forall \vec{x}(P_1 \wedge \dots \wedge P_k) \rightarrow P_0^{(i)}.$$

Proof. Suppose that  $\forall \vec{x}(P_1 \wedge \dots \wedge P_k)$  is satisfied in the term model, then  $P_1(\vec{t}), \dots, P_k(\vec{t})$  hold for all terms  $\vec{t}$ , hence in particular for  $\vec{t} = \vec{x}$ . Thus  $P_1, \dots, P_k$  all hold in that model, hence  $P_1, \dots, P_k$  are provable in S. Hence by the rule  $P_0$  is provable in S and therefore satisfied in the model.

Thus, in particular, if  $k = 1$ ,  $P_1$  is the equation  $Mx = M'x$  and  $P_0$  is  $M = M'$ , the result mentioned above follows.

Such hygienic uses of model theory are to be distinguished from model theoretic constructions formulated by use of sophisticated notions studied in mathematical practice and (hence) well-understood; for example the use of ultraproducts. Here we may, so to speak, look at the constructions and read off properties of these models which are not at all evident from the axioms. Particularly when consistency, that is the existence of suitable models, is involved, it is not at all necessary that the constructions provide all models of the theory considered: it is more important that we have 'enough' models (for some specific purpose) and that they be manageable. Similarly Scott's lattice theoretic models are useful despite the fact established in §3.2 that not all extensional models of CL are among them.

But it is of interest to observe that the particular model left out from Scott's collection of models, is very natural from a computational view. In this model the so called fixed-points which all act in the same computational way are identified.

To conclude this discussion of the role of model theory in our study of CL and the  $\lambda$ -calculus, we may perhaps compare it with the use of non-standard models of arithmetic (as developed at the present time). We establish consistency results by analyzing computation procedures and then restate them as properties of suitable term models. Similarly, one establishes consistency results for formal arithmetic by proof theory or recursion theory (Gödel's incompleteness results) and may then apply the completeness theorem for predicate logic to infer the existence of some non-standard model of arithmetic with required properties. Probably it is fair to say that the known non-standard models are not of intrinsic interest. In contrast, our formulations for term models are directly relevant to what we are talking about since, as we have said already, we study operations on terms and operations coded by terms.

Finally a comment on versions of the  $\lambda$ -calculus or CL with types (cf. Sanchis [1967]). Here the situation is much simpler. In terms of models, the theory with types can be immediately interpreted in familiar mathematical terms. One model consists of the collections of all set-theoretic functions of finite type (over an infinite domain). For the more interesting models HRO and in extensional version HEO see Troelstra [1971].

In terms of computation, it is shown in Sanchis [1967] that in the theories with types all terms have a normal form, contrary to the situation in the type free theory.

This normalizability of all terms of the typed  $\lambda$ -calculus has also an interesting 'negative' consequence for definability problems, specifically of recursion operators (for any given types). In the type free calculus we have an R with the property that, for (variable) M,N and each numerical  $\underline{n}$

$$(*) \quad \begin{aligned} \text{RMN}\underline{0} &= N \\ \text{RMN}\underline{n+1} &= M(\text{RMN}\underline{n})\underline{n} \end{aligned}$$

In contrast, there is a specific type  $\tau$  such that no term of the typed  $\lambda$ -calculus satisfies (\*) for M,N of relevant types. (A refinement is possible showing that (\*) do not hold for certain closed M,N.)

The proof uses the following facts.

1. The normalizability of the terms in the typed  $\lambda$ -calculus can be established by the principles of first order arithmetic. Consequently there is a provably total valuation function  $v$ , assigning to each (Gödel number of a) term, the (Gödel number of) its normal form.

2. By 1. all number theoretic functions which are (locally) definable are primitive recursive in  $v$  because if the term  $F$  defines  $f$ , then the Gödel number of the term  $F\underline{n}$  is  $\phi(n)$  for a primitive recursive function  $\phi$ , and from  $v(\phi(n))$  it is possible to recover  $f(n)$  primitive recursively.

Corollary. Not all functions provably recursive in first order arithmetic are represented in the typed  $\lambda$ -calculus.

3. Let  $f_0$  be a provably total function, with Gödel number  $e_0$ , which is not primitive recursive in  $v$ . Such an  $f_0$  can be defined by the use of recursion operators. For,  $f_0$  being

provably total we have

$\vdash \forall x \exists y T(\underline{e}_0, x, y)$  in intuitionistic arithmetic.

Hence by the dialectica interpretation of Gödel [1958] there exists a term  $t$  in the typed  $\lambda$ -calculus with  $R$  such that  $T(\underline{e}_0, \underline{n}, \underline{t}(\underline{n}))$  is valid for all  $n \in \omega$ .

Hence  $f$  is represented by  $\lambda n. u(t(n))$  where  $u$  represents the  $U$  of Kleene's normal form.

Now we discuss in more detail the subjects treated in the text.

Consistency. The first formulation of the  $\lambda$ -calculus being inconsistent, a revised formulation (the  $\lambda I$ -calculus) was proved consistent by Church and Rosser [1936]. More informatively their theorem establishes the uniqueness of the normal forms and the fact that the normal form of a normalizable term  $M$  can be found simply by reducing  $M$ . This reduction can be made deterministic, by the standardization theorem, cf. Curry, Feys [1958] Ch 4E. However, the fine-structure (i.e. whether it is from the human point of view the shortest possible reduction) of the standard reduction is not discussed.

Finally, cf. remark 1.2.18, there is now a simple proof of the Church-Rosser theorem by Martin-Löf [1971]. This will be given in appendix II.

The relation between the  $\lambda$ -calculus and CL. Several possibilities of mapping  $\lambda$ -terms into CL (see 1.4.6) have been discussed (cf. Curry, Feys [1958] Ch 6A). But, without extensionality, they do not preserve the set of provable equations. They do, if

extensionality is included but not the provable reductions nor the normal forms. On the other hand the translations do preserve application and therefore by 3.2.20 $\frac{1}{2}$  the solvability of closed terms. The importance of this is seen below. In particular, in the extensional case consistency results can be transferred. In this way the  $\omega$ -consistency of the  $\lambda$ -calculus follows from the corresponding result for CL.

$\lambda$ -definability. We begin with the  $\lambda$ -calculus since, traditionally,  $\lambda$ -definability and not combinatory definability is treated.

A number theoretic partial function  $f$  is said to be  $\lambda$ -definable if there is a term  $F$  such that  $\lambda \vdash F\bar{n} = \bar{m} \iff f(n) = m$  and  $F\bar{n}$  has no normal form if  $f(n)$  is undefined. (Here  $\bar{n}$  is the  $n$ th numeral). We changed this definition, requiring not only  $\lambda \vdash F\bar{n} = m \iff f(n) = m$ , but also that  $F\bar{n}$  be unsolvable if  $f(n)$  is undefined. In this case we say that  $F$  strongly defines  $f$ .

The concept of strong  $\lambda$ -definability has several advantages.

(i) If  $f_1, f_2$  are defined by terms  $F_1, F_2$ , then it is not true that  $f_1 \circ f_2$  is defined by  $\lambda x. F_1(F_2x)$ . For example let  $f_1$  be the constant zero function and let  $f_2$  be everywhere undefined. Then  $f_1 \circ f_2$  is totally undefined but  $\lambda x F_1(F_2x) = \lambda x \underline{0}$  represents the constant zero function.

Hence it is not immediate that the  $\lambda$ -definable functions are closed under composition.

Use of strong definability is made by representing the composition  $f_1 \circ f_2$  by  $F = \lambda x. (F_2x \text{I } F_1(F_2x))$ . If  $f_2(n)$  is undefined  $F_2\bar{n}$  is unsolvable and hence  $F\bar{n}$  is unsolvable, and if  $f_2(n)$  is defined  $F_2\bar{n} \text{I}$  is essentially the same as  $I$ .

ii) Traditionally the  $\lambda$ -definability of the partial recursive functions was proved by use of Kleene's normal form theorem. The representation thus obtained is not intensional with respect to definitional equality. Using strong  $\lambda$ -definability we give a representation of the partial recursive functions preserving their definition trees. This is not to be regarded as a mere technical improvement but simply central to the objects which are here intended. In this context, see Kearns [1969] who uses an extension of CL to give a faithful representation of the computations of a Turing machine.

iii) Application to undecidability results, see below.

In contrast to the representation of say the primitive recursive functions in the predicate calculus, their representation in the  $\lambda$ -calculus is not global, that is, their defining recursion equations are not derivable for the representing terms with a free variable, but only for each numerical instance.

For example in the extensional case (this is not essential) the term  $F$  with  $Fxy = xy$  represents exponentiation since  $\lambda + \text{ext} \vdash \underline{nm} = \underline{m}^n$ . Hence the function  $\underline{1}^x$  is represented by  $G$  with  $Gx = x\underline{1}$ . Now we have for all numerals  $\lambda + \text{ext} \vdash G\underline{n} = \underline{1}$ , but not  $\lambda + \text{ext} \vdash Gx = \underline{1}$  since  $\lambda \vdash G(\underline{K0}) = \underline{K0\underline{1}} = \underline{0}$ .

(It should be admitted that  $F$  is not the standard representation of exponentiation as this will be done in §1.3, but similar examples can be given there.) It is unlikely that there exists a global representation of the primitive recursive functions at all, since in contrast to the representation in predicate calculus, all models of the  $\lambda$ -calculus must contain 'non-

standard' elements (such as the element denoted by K).

Undecidability results. As was mentioned above, the translation going from the  $\lambda$ -calculus to CL preserves solvability of closed terms. Thus undecidability results about the  $\lambda$ -calculus as in §1.3 transfer to CL, without any need for a parallel development of CL-definability.

We note that the role of unsolvable closed terms in the  $\lambda$ K-calculus is similar to the role of those closed terms in the  $\lambda$ I-calculus which have no normal form. In particular

i) In the  $\lambda$ K-calculus it is consistent to equate all unsolvable terms (see §3.2) and in the  $\lambda$ I-calculus it is consistent to equate all terms without a normal form (see part II).

ii) For some purposes unsolvable terms can replace familiar arguments involving negation. See for example 3.2.19 where it is proved that Con, the set of equations consistent with the  $\lambda$ -calculus is complete  $\Pi_1^0$ .

An example of an undecidability result which does not need strong representability is 1.3.17 for which we give here an alternative proof (cf. Smullyan [1961]).

Let A,B be disjoint r.e. sets, recursively inseparable.

Define  $f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \\ \uparrow & \text{else} \end{cases}$

then  $f$  is partial recursive. Let  $f$  be defined by the term  $F$ .

If  $T$  were a recursive consistent extension of the  $\lambda$ -calculus, then  $C = \{n \mid T \vdash F_n = \underline{0}\}$  would be a recursive separation of A and B.

A minimal extensional term model. In Chapter II we define a model of CL which is minimal in two respects. Firstly its domain consists of the closed terms only, which are, as was stated in the introduction, principal object of our study. Secondly the model is minimal with respect to the equality relation, that is to the set of terms that are equated. It is clear, that the model is generated by the  $\omega$ -rule described in §2.1. Formally the theory enriched with the  $\omega$ -rule is a strengthening of the rule of extensionality: to infer  $M = M'$  we do not require a derivation of  $Mx = M'x$  with variable  $x$ , but only derivations of  $MZ = M'Z$  for all closed terms  $Z$  (with no uniformity on these derivations). Two questions, so to speak, at opposite poles are

a) Is the theory consistent?

b) Is the theory conservative over the  $\lambda$ -calculus + ext?

Ad a) In §2.2, 2.3 it is shown by transfinite induction, that the extension of CL (or the  $\lambda$ -calculus) by the  $\omega$ -rule is consistent.

Ad b) In §2.5 it is established for equations  $M = M'$  where  $M$  and  $M'$  are not universal generators that the  $\omega$ -rule is conservative. We prove this by showing the existence of variable like closed terms  $E$  such that  $ME = M'E \Rightarrow M = M'$  .

This result includes the known special case (a consequence of the theorem of Böhm [1968], cf. §2.1) of equations between normal terms since they are not universal generators.

In this connection it is to be understood that our use of closed terms which are not in normal form is not haphazard: as we see it, different meanings, that is programmes, are to be given to

certain (different) terms which have no normal form; for example if  $M \equiv \lambda x.x\underline{0}\Omega$  and  $N \equiv \lambda x.x\underline{1}\Omega$ , where  $\Omega$  does not have a normal form, then  $M, N$  have no normal form but  $MK = \underline{0}$  and  $NK = \underline{1}$ .

Unsolvable terms. These terms were already mentioned in 'λ-definability' above, in connection with the definability of functions, where they were needed to provide particularly hereditarily undefined values. From the computational point of view this means that such terms do not do much. In accordance with this we now consider the possibility of putting them equal and establish (3.2.16) that this can be done consistently, even in the presence of extensionality (added in print: or the ω-rule). It is the term model of this theory that is not among Scott's collection of models mentioned above. We note that in general the addition of extensionality not only adds new theorems (cf. the remark following 1.1.16) but can be problematic. In particular in 3.2.24 we provide a consistent extension of CL which becomes inconsistent when extensionality is added.

Recursion theoretic structures for the λK- and the λI-calculus.

By a recursion theoretic structure we mean here a combinatory structure where the domain consists of  $\omega \cup \{*\}$  and the application operator is interpreted as Kleene brackets, i.e.

$n \cdot m = \{n\}(m) = U(\mu z T(n,m,z))$ . \* has to be treated as the undefined element, it serves to make the application operator total, with the additional property that  $\{n\}(*) = (*)$  (cf. the theory of Wagner and Strong (Strong [1968])).

This kind of construction can of course be made more general. Instead of Kleene's brackets which come from his particular

equation calculus for computing recursive functions from recursion equations, an other equational calculus (and other numberings of equations) may be considered. In this way one discovers exactly which properties of numberings and equation calculi are relevant to our subject.

Perhaps the most essential difference between the  $\lambda I$ -calculus and the  $\lambda K$ -calculus is that the former is 'more' systematic or deterministic, not allowing short-cuts. All programmes that are 'represented' in a term of the  $\lambda I$ -calculus have to be performed, while in the  $\lambda K$ -calculus some subprogrammes can be skipped. For example we reduce  $KMN$  to  $M$  without looking at the value of  $N$ . (This may be compared to a 'stupid' evaluation of  $0.(2^{15} + 3^6)$  in numerical arithmetic where we evaluate  $2^{15} + 3^6$  and a short cut using the reduction  $0.x \rightarrow 0$ ; actually the  $\lambda I$ -calculus itself does 'stupid' numerical computations.)

Because of this systematic feature of the  $\lambda I$ -calculus we can find a recursion theoretic model for it or better for its combinatory equivalent, described in Rosser [1936], here called  $CL_I$ . It is like  $CL$  but with primitive constants  $I, J$  and the axioms  $IM = M$  and  $JMNLP = MN(MPL)$ . For suitable numbers  $i, j$  the structure  $\mathcal{R} = \langle \omega \cup \{*\}, i, j, \cdot \rangle$  is a model for  $CL_I$ . In part II of our thesis thesis we will prove that

(1)  $M$  has no  $CL_I$  normal form  $\iff \mathfrak{N}_R(M) = *$ .

The proof uses a formalization of the description of the recursive functions as is in Kleene [1959].

This implies that it is consistent with  $CL_I$  to equate all terms without a normal form.

The recursion theoretic structure obtained by use of Kleene's particular equation calculus is not a model for CL. Though there are numbers  $i, k, s \in \omega$  which satisfy  $i \cdot x = x$ ,  $k \cdot x \cdot y = x$  and  $s \cdot x \cdot y \cdot z = x \cdot z \cdot (y \cdot z)$  for all  $x, y, z \in \omega$  no  $k$  satisfies  $k \cdot x \cdot y = x$  for  $y = *$ .

However the structure  $\mathcal{Q}' = \langle \omega \cup \{*\}, i, k, s, \cdot \rangle$  realizes the language of CL, but is not a model of it, and satisfies an analogue to (1) namely

$$(2) M \text{ has no CL-normal form} \iff \square_R(M) = *.$$

Considering finally the canonical mapping:  $M \rightarrow M^{WS}$  of the language CL into the language of Wagner and Strong, we find a term  $M$  in Part II such that

$M$  has no CL (and hence no  $CL_I$ ) normal form but  $M^{WS} = *$  is not a theorem of the theory of Wagner-Strong.

Since however properties (1) and (2) hold for our intended objects of study, we should wish to extend the theory of Wagner-Strong; specifically to extend their language. Since the proofs of (1) and (2) for  $\mathcal{Q}$  and  $\mathcal{Q}'$  required an analysis of computations, the extended language should refer to the latter, in particular to length of computations (as pointed out to us by C.Gordon).



## CHAPTER I

### Preliminaries

#### §1.1. The $\lambda$ -calculus.

The  $\lambda$ -calculus is a theory studied thoroughly in the thirties (by Church, Kleene, Rosser and others). It has been designed to describe a class of functions  $V$ , where the domain of all functions is  $V$  itself. Therefore the objects we consider are at the same time function and at the same time argument. (A similar situation we have in most set theories: there the objects are at the same time set and at the same time element.) Hence we have the feature that a function can be applied to itself. In the usual conception of a function in mathematics, for example in Zermelo Fraenkel set theory, this is impossible (because of the axiom of foundation).

The  $\lambda$ -calculus defines (or better represents) a class of (partial) functions ( $\lambda$ -definable functions) which turns out to be the class of (partial) recursive functions. In fact the equivalence between the Turing computable functions and the  $\mu$ -recursive functions was proved via the  $\lambda$ -definable functions:

Kleene [1936] proved: the  $\mu$ -recursive functions are exactly the  $\lambda$ -definable functions.

Turing [1937] proved: the computable functions are exactly the  $\lambda$ -definable functions.

In this sense the  $\lambda$ -calculus played a central role in the early investigations of the theory of the recursive functions.

The consistency of the  $\lambda$ -calculus was proved by Church and

Rosser [1936]. Because some theories related to the  $\lambda$ -calculus turned out to be inconsistent (paradox of Kleene-Rosser [1935], paradox of Curry [1942]), we see that this consistency proof is not a luxury. In contrast to most mathematical theories, and like set theory, the  $\lambda$ -calculus was initiated before any models were known. In group theory for example many concrete groups were known long before formal group theory was developed. Not until the end of 1969 however, were the first models in ordinary mathematical terms for the  $\lambda$ -calculus constructed (Scott [1970]).

In the  $\lambda$ -calculus we have the fundamental operation of application. The application of a function  $f$  to  $a$  will be written as  $fa$ .

Apart from this application we have an abstraction operator  $\lambda$ . The intuitive meaning of  $\lambda x \dots$  is the function which assigns  $\dots$  to  $x$ . Its use is illustrated by the following formula (not a formula of the  $\lambda$ -calculus by the way)

$$(\lambda x. x^2 + 2x + 1)3 = 16.$$

Now we give a formal description of the  $\lambda$ -calculus.

### 1.1.1 Definition

We define the following language  $L_K$ :

Alphabet:  $a, b, c, \dots$  variables  
 $\lambda, (, )$  improper symbols  
 $=$  equality

Terms: Terms are inductively defined by

- 1) a variable is a term
- 2) if  $M, N$  are terms, then  $(MN)$  is a term

3) if  $M$  is a term, then  $(\lambda xM)$  is a term ( $x$  is an arbitrary variable).

Formulas: if  $M, N$  are terms then  $M = N$  is a formula.

### 1.1.2 Definition

An occurrence of a variable  $x$  in a term is called bound if this  $x$  is "in the scope of  $x$ ". Otherwise we call this occurrence of  $x$  free.

$BV(M)$  ( $FV(M)$ ) is the set of all variables in  $M$  that occur in  $M$  as a bound (free) variable.

$BV$  and  $FV$  can be defined inductively as follows:

$$BV(x) = \emptyset$$

$$BV(MN) = BV(M) \cup BV(N)$$

$$BV(\lambda xM) = BV(M) \cup \{x\}$$

$$FV(x) = \{x\}$$

$$FV(MN) = FV(M) \cup FV(N)$$

$$FV(\lambda xM) = FV(M) - \{x\}.$$

A variable can occur free and bound in the same term (e.g.  $((\lambda x(xy))x)$ ).

Note that, as is common, we use for equality in the object-language and in the metalanguage the same symbol " $=$ ". When we need to distinguish between them we write (as Curry does) " $\equiv$ " for equality in the metalanguage. This is mainly the case when we mean syntactic identity.

1.1.3 Notation

i)  $M_1 M_2 \dots M_n$  stands for  $((\dots(M_1 M_2)\dots)M_n)$  (association to the left).

ii)  $\lambda x_1 x_2 \dots x_n \cdot M$  stands for  $(\lambda x_1 (\lambda x_2 (\dots (\lambda x_n M)\dots)))$

iii)  $[x/N]M$  stands for the result of substituting  $N$  in those places of  $x$  in  $M$  which are free:

$$[x/N]x = N$$

$$[x/N]y = y$$

$$[x/N](M_1 M_2) = ([x/N]M_1)([x/N]M_2)$$

$$[x/N](\lambda x M) = \lambda x M$$

$$[x/N](\lambda y M) = \lambda y([x/N]M).$$

In the above  $x \neq y$ .

1.1.4 Definition

The  $\lambda$ -calculus is the theory in  $L_K$  defined by the following axioms and rules:

- I
1.  $\lambda x M = \lambda y [x/y] M$  if  $y \notin FV(M)$
  2.  $(\lambda x M)N = [x/N]M$  if  $BV(M) \cap FV(N) = \emptyset$ .

- II
1.  $M = M$
  2.  $\frac{M = N}{N = M}$
  3.  $\frac{M = N, N = L}{M = L}$
  4.  $\frac{M = M'}{ZM = ZM'} \quad \frac{M = M'}{MZ = M'Z} \quad \frac{M = M'}{\lambda x M = \lambda x M'}$

In the above  $M, M', N, L, Z$  denote arbitrary terms and  $x, y$  denote arbitrary variables.

We say that ...2 is a direct consequence of ...1 if  $\frac{\dots 1}{\dots 2}$ .

### 1.1.5 Remarks

1. Axiom I 1. allows us the change of bound variables (like  $\int_0^1 x^2 dx = \int_0^1 y^2 dy$  ).

If  $M = N$  is provable using only axiom I 1. (and the rules of II) then we say that  $M$  is an  $\alpha$ -variant of  $N$  (notation  $M \equiv_{\alpha} N$ ).

If  $M \equiv_{\alpha} N$  then only the bound variables are renamed. Note that  $M \equiv_{\alpha} N$  is a statement of the metalanguage.

Axiom I 2. expresses the essential feature of the  $\lambda$  operator. The Axiom and rules of II express that  $=$  is an equality.

2. If we would drop the restriction in I 1. we would get  $(\lambda a. ab) = (\lambda b. bb)$  which is in conflict with our intuitive interpretation of  $\lambda$ .

If we would drop the restriction in I 2. we would get

$$a) (\lambda a(\lambda b.ba))b = \lambda b.bb$$

$$b) (\lambda a(\lambda c.ca))b = \lambda c.cb$$

which is also undesirable. In (a) there is a difference between the "local" variable  $b$  and the "global" variable  $b$ . The restriction in I 2. is just to prevent confusion of variables.

3. If we would restrict the definition of terms as follows:

3') If  $M$  is a term and if  $x \in FV(M)$ , then  $\lambda xM$  is a term, then we get a restricted form of the  $\lambda$ -calculus, the so called  $\lambda I$ -calculus. Our form of the  $\lambda$ -calculus is called the  $\lambda K$ -calculus because we can define a term  $K = \lambda ab.a$ , with the property  $KMN = M$ , which is impossible in the  $\lambda I$ -calculus. In the absence of  $K$  many theorems are a bit harder to prove; see Church [1941] who treats the  $\lambda I$ -calculus.

4. Axioms I.1. and I 2. are called  $\alpha$ -resp.  $\beta$ -reduction.
5.  $\lambda \vdash M = N$  means that  $M = N$  is a provable formula of the  $\lambda$ -calculus.

#### 1.1.6 Examples

Define  $I = \lambda a.a$

$K = \lambda ab.a$

$S = \lambda abc.ac(bc)$

Then  $\lambda \vdash IM = M$

$\lambda \vdash KMN = M$

$\lambda \vdash SMNL = ML (NL)$

A nice exercise is the following.

#### 1.1.7 Theorem (fixed point theorem)

For every term  $M$  there exists a term  $\Omega$  such that  $\lambda \vdash M\Omega = \Omega$ .

Proof.

Define  $\omega = (\lambda x M(xx))$  with  $x \notin FV(M)$

$\Omega = \omega\omega$

Then  $\lambda \vdash \Omega = (\lambda x M(xx))\omega = M(\omega\omega) = M\Omega$ . ☒

This fixed point theorem, so simple to prove, is strongly related to the fixed point theorem in recursion theory (recursion theorem). In 1.1.7 we have proved more than we formulated. The fixed point can be found in a uniform way.

#### 1.1.8 Corollary

There exists a term  $FP$  such that for every term  $M$  we have

$\lambda \vdash M(FPM) = FPM$

Proof.

Define  $FP = \lambda a.((\lambda b.a(bb))(\lambda a.b(aa)))$  ☒

The Russellparadox and Gödel's self referring sentence can be considered as applications of FP. Therefore in Curry Feys [1958] such a term FP is called a paradoxical combinator.

### 1.1.9 Corollary

There exists a term M such that  $\lambda \vdash Mx = M$ . Such a term is called a fixed point.

Proof.

Define  $M = FP K$  (where K is defined as in 1.1.6).

Then  $\lambda \vdash Mx = KMx = M$ . ☒

### 1.1.10 Definition (See also 1.2.6.)

1) A term M is in normal form if it has no part of the form  $(\lambda xP)Q$ .

2) A term M has a normal form if there exists a term M' which is in normal form such that  $\lambda \vdash M = M'$ . In this case we say that M' is a normal form of M.

In the next § we state a theorem (1.2.9) which has as consequence:

### 1.1.11 Theorem

If M has a normal form, then this normal form is unique up to  $\alpha$ -reduction (i.e. change of bound variables).

Examples.

1.  $(\lambda a.a)(\lambda b.bb)$  is not in normal form but has the normal form  $\lambda b.bb$ .

2.  $c(\lambda a.a)(\lambda b.bb)$  is in normal form.

In § 1.2 we will give examples of terms that do not have normal forms.

In the  $\lambda$ -calculus we can represent the natural numbers:

1.1.12 Definition

$$\begin{aligned}\underline{0} &= \lambda ab \cdot b \\ \underline{1} &= \lambda ab \cdot ab \\ \underline{2} &= \lambda ab \cdot a(ab) \\ \underline{3} &= \lambda ab \cdot a(a(ab)) \\ &\text{etc.}\end{aligned}$$

Hence  $\underline{n} = \lambda ab \cdot \underbrace{a(\dots(ab)\dots)}_{n \text{ times}} = \lambda ab \cdot a^n b$

Note that  $\underline{n}$  is in normal form and has the property:

1.1.13

$$\lambda \vdash \underline{n}fx = f^n x.$$

1.1.14 Definition

Let  $f: \omega^n \rightarrow \omega$  be a partial function.

$f$  is called  $\lambda$ -definable iff there exists a term  $F$  such that

$$\begin{aligned}\lambda \vdash F\underline{k}_1 \dots \underline{k}_n &= \underline{m} && \text{if } f(k_1, \dots, k_n) = m \\ F\underline{k}_1 \dots \underline{k}_n &\text{ has no normal form} && \text{if } f(k_1, \dots, k_n) \text{ is undefined.}\end{aligned}$$

In this case we say that  $F$  defines  $f$ .

If  $F$  defines  $f$ , but we no longer know  $f$ , can we recover  $f$  from  $F$ ? The following gives an affirmative answer.

1.1.15 Theorem

If  $F$  defines  $f$  then

$$\lambda \vdash F\underline{k}_1 \dots \underline{k}_n = \underline{m} \iff f(k_1, \dots, k_n) = m .$$

Proof.

$\Leftarrow$  By definition.

$\Rightarrow$  Suppose  $\lambda \vdash F_{\underline{k}_1} \dots \underline{k}_n = \underline{m}$

Then  $F_{\underline{k}_1} \dots \underline{k}_n$  has a normal form

so  $f(k_1, \dots, k_n)$  is defined, say

$$f(k_1, \dots, k_n) = m' .$$

Hence  $\lambda \vdash F_{\underline{k}_1} \dots \underline{k}_n = \underline{m}'$ .

By 1.1.11 it follows that  $\underline{m} \equiv_\alpha \underline{m}'$ .

Hence  $m = m'$  and therefore  $f(k_1, \dots, k_n) = m$ . □

Now we consider the following rule of extensionality:

1.1.16

$$\text{ext} \quad \frac{Mx = Nx \quad x \notin FV(MN)}{M = N}$$

ext is not provable in the  $\lambda$ -calculus:

Let  $M = \lambda a \cdot ba$

$N = b$ .

Then  $\lambda \vdash Ma = Na$ , but not  $\lambda \vdash M = N$  because  $M$  and  $N$  are distinct normal forms. So we may take ext as an extra axiom for the  $\lambda$ -calculus.

$\lambda + \text{ext} \vdash M = N$  means that  $M = N$  is a provable formula of the  $\lambda$ -calculus + extensionality.

1.1.17 Theorem

$\lambda + \text{ext} \vdash (\lambda x(Mx)) = M \quad \text{if } x \notin FV(M)$

(In the literature, this is called  $\eta$ -reduction.)

Proof.

$(\lambda x(Mx))x = Mx, \quad x \notin FV(M)$ .

Hence by extensionality

$(\lambda x(Mx)) = M$ . □

Note that extensionality follows in turn from 1.1.17:

### 1.1.18 Theorem

If we assume 1.1.17 as an extra axiom for the  $\lambda$ -calculus, then we can prove extensionality.

Proof.

Let  $Mx = Nx$   $x \notin FV(MN)$ .

Then  $\lambda x(Mx) = \lambda x(Nx)$  by rule II.4.

Hence  $M = N$  by 1.1.17. □

### 1.1.19 Remark

Often in the literature the  $\lambda$ -calculus we described is called the  $\lambda$ - $\beta$ -calculus. (Because its essential axiom is I.2.:  $\beta$ -reduction.) The  $\lambda$ -calculus with 1.1.17 as an extra axiom is called the  $\lambda$ - $\beta\eta$ -calculus. When it is needed to stress that we are working with the  $\lambda K$ -calculus, we speak about the  $\lambda K$ - $\beta$ -calculus and the  $\lambda K$ - $\beta\eta$ -calculus.

Of course we have also the  $\lambda I$ - $\beta$ -calculus and the  $\lambda I$ - $\beta\eta$ -calculus.

## §1.2. The Church-Rosser theorem. The consistency of the $\lambda$ -calculus.

Since we do not have a negation in the  $\lambda$ -calculus we cannot define the concept of a contradiction. Therefore we define the consistency as follows.

### 1.2.1 Definition

The  $\lambda$ -calculus is consistent if we cannot prove  $a = b$ .

(If  $a = b$  were provable every formula would be provable.)

As was remarked in the introduction some theories related to the  $\lambda$ -calculus turned out to be inconsistent. Fortunately the  $\lambda$ -

calculus itself is consistent as was proved by Church and Rosser [1936].

We will now give an idea of how this was done. Let us go back to the informal discussion in the beginning of §1.

We had the expression  $(\lambda x \cdot x^2 + 2x + 1)3$ . After computing we replace this expression by 16. No one would replace 16 by the more complicated  $(\lambda x \cdot x^2 + 2x + 1)3$ . Hence we can assign a certain asymmetry to the relation  $=$ .

This will be expressed by

$$(\lambda x \cdot x^2 + 2x + 1)3 \geq 16.$$

$\geq$  is called reduction.

Note that reduction is a stronger relation than equality. I.e. if M reduces to N then M is equal to N. Now we will describe an extension of the  $\lambda$ -calculus in which we formalize this reduction relation.

### 1.2.2 Definition

We define the following language  $L'_K$ .

The alphabet of  $L'_K$  consists of that of  $L_K$  together with the symbol " $\geq$ ".

The terms of  $L'_K$  are the same as those of  $L_K$ .

The formulas of  $L'_K$  are defined as follows:

If M, N are terms of  $L'_K$  then  $M = N$  and  $M \geq N$  are formulas.

### 1.2.3 Definition

In the language  $L'_K$  we define an extension of the  $\lambda$ -calculus by the following axioms and rules (see appendix):

I 1.  $\lambda xM \geq \lambda y[x/y]M$  if  $y \notin FV(M)$  ( $\alpha$ -reduction)

2.  $(\lambda xM)N \geq [x/N]M$  if  $BV(M) \cap FV(N) = \emptyset$  ( $\beta$ -reduction)

II Same as in 1.1.4. (These state that "=" is an equality.)

III 1.  $M \geq M$

2.  $\frac{M \geq N, N \geq L}{M \geq L}$

3.  $\frac{M \geq M'}{ZM \geq ZM'}$ ,  $\frac{M \geq M'}{MZ \geq M'Z}$ ,  $\frac{M \geq M'}{\lambda xM \geq \lambda xM'}$

4.  $\frac{M \geq N}{M = N}$

Again in the above  $M, M', N, L, Z$  denote arbitrary terms and  $x, y$  arbitrary variables.

If we want to include extensionality we add the axiom

1.2.4

I 3.  $\lambda x(Mx) \geq M$  if  $x \notin FV(M)$  ( $\eta$ -reduction)

It is easy to see that this extended  $\lambda$ -calculus (with or without extensionality) is in fact a conservative extension of the  $\lambda$ -calculus (with or without extensionality) considered in §1.1.

For this reason we also write for the extended  $\lambda$ -calculus

$\lambda \vdash M = N$  and  $\lambda + \text{ext} \vdash M = N$ .

Henceforth, if we refer to the  $\lambda$ -calculus, we mean the extended  $\lambda$ -calculus.

Note that if we have  $\lambda + \text{ext} \vdash M \geq N$ ,  $\eta$ -reduction may be used.

With the help of  $\geq$  we can express an important property of normal forms:

### 1.2.5 Lemma

If  $M$  is in normal form and  $\lambda \vdash M \geq N$  then  $M \equiv_{\alpha} N$ .

Proof.

By induction to the length of proof of  $\lambda \vdash M \geq N$ . □

Remark. It is not true that if  $\forall N[\lambda \vdash M \geq N \Rightarrow M \equiv_{\alpha} N]$ , then  $M$  is in normal form.

Consider for example  $(\lambda a \cdot aa)(\lambda a \cdot aa)$ .

### 1.2.6 Definition

If we consider the  $\lambda$ - $\beta\eta$ -calculus we can define

A term is in  $\eta$ -normal form if it has no part of the form  $\lambda x(Px)$ .

The concept of normal form as defined in 1.1.8 is then called

$\beta$ -normal form.

A term is in  $\beta\eta$ -normal form if it is both in  $\beta$ - and in  $\eta$ -normal form.

A term  $M$  has a  $\beta\eta$ -normal form if there exists a term  $M'$  which is in  $\beta\eta$ -normal form such that  $\lambda + \text{ext} \vdash M = M'$ .

If we just speak of normal form, we mean  $\beta$ -normal form.

Analogous to 1.2.5 we have

### 1.2.7 Lemma

If  $M$  is in  $\beta\eta$ -normal form and if  $\lambda + \text{ext} \vdash M \geq N$  then  $M \equiv_{\alpha} N$ .

### 1.2.8 Lemma

If  $M$  has a  $\beta$ -normal form, then  $M$  has a  $\beta\eta$ -normal form.

Proof.

Let  $M'$  be a  $\beta$ -normal form of  $M$ . Then  $\lambda \vdash M = M'$ , hence a fortiori  $\lambda + \text{ext} \vdash M = M'$ .

By applying I.3 ( $\eta$ -reduction) a finite number of times to  $M'$ , we obtain a  $M''$  which is in  $\eta$ -normal form and  $\lambda + \text{ext} \vdash M = M''$ .

Because  $\eta$ -reductions do not introduce subterms of the form

$(\lambda xP)Q$ ,  $M''$  will be in  $\beta\eta$ -normal form, hence it is a  $\beta\eta$ -normal form of  $M$ . □

In Curry Hindley Seldin [1917] Ch.11 E, lemma 13.1 it is proved that the converse of 1.2.8 also holds.

Now we state without proof:

1.2.9 Theorem (Church-Rosser theorem [1936])

If  $\lambda \vdash M = N$  then there exists a term  $Z$  such that  $\lambda \vdash M \geq Z$  and  $\lambda \vdash N \geq Z$ .

See Mitschke [1970] for an elegant proof.

Mitschke's proof is still rather long, but much simpler than the original one. In §1.5 we will use Mitschke's ideas to prove a Church-Rosser theorem for combinatory logic.

1.2.10 Corollary

If  $M, N$  are terms both in normal form and  $M \not\equiv_{\alpha} N$ , then  $\lambda \not\vdash M = N$ .

Proof.

Suppose  $\lambda \vdash M = N$ , then by 1.2.8 there exists a  $Z$  such that

$$\lambda \vdash M \geq Z, \lambda \vdash N \geq Z.$$

$M, N$  are in normal form, hence by 1.2.5  $M \equiv_{\alpha} Z$  and  $N \equiv_{\alpha} Z$ .

But then  $M \equiv_{\alpha} N$  contradiction. □

From this 1.1.11 readily follows:

1.1.11 Theorem

If  $M$  has a normal form, then this normal form is unique up to  $\alpha$ -reduction.

Proof.

If  $\lambda \vdash M = N_1$ ,  $\lambda \vdash M = N_2$  with  $N_1, N_2$  in normal form, then  $\lambda \vdash N_1 = N_2$ . Hence by 1.2.10,  $N_1 \equiv_\alpha N_2$ .  $\square$

### 1.2.11 Corollary

The  $\lambda$ -calculus is consistent.

Proof.

Because the terms  $a, b$  are in normal form and  $a \not\equiv_\alpha b$  we have  $\lambda \not\vdash a = b$  by 1.2.10.  $\square$

Also the question of the consistency of the  $\lambda$ -calculus with extensionality arises. In Curry Feys [1958] Ch.4D an extension of the Church-Rosser theorem is proved for the  $\lambda$ -calculus with  $\eta$ -reduction. We state this here without proof.

### 1.2.12 Theorem

If  $\lambda + \text{ext} \vdash M = N$ , then there exists a term  $Z$  such that  $\lambda + \text{ext} \vdash M \geq Z$  and  $\lambda + \text{ext} \vdash N \geq Z$ .

Analogous to corollaries 1.2.10, 1.1.11 and 1.2.11 we have:

### 1.2.13 Corollary

If  $M, N$  are terms, both in  $\beta\eta$ -normal form and  $M \not\equiv_\alpha N$ , then  $\lambda + \text{ext} \not\vdash M = N$ .

### 1.2.14 Corollary

If  $M$  has a  $\beta\eta$ -normal form, then this  $\beta\eta$ -normal form is unique up to  $\alpha$ -reduction.

### 1.2.15 Corollary

The  $\lambda$ -calculus with extensionality is consistent.

In chapter II we will prove a theorem stronger than 1.2.15.

Now we state a corollary of the Church-Rosser theorem with the help of which we can show that some terms have no normal form:

### 1.2.16 Corollary

If  $M$  has the normal form  $N$ , then  $\lambda \vdash M \geq N$ .

Proof.

If  $\lambda \vdash M = N$  and  $N$  is in normal form, then by 1.2.8 there exists a  $Z$  such that  $\lambda \vdash M \geq Z$  and  $\lambda \vdash N \geq Z$ , hence by 1.2.5.

$$N \equiv_{\alpha} Z.$$

So we have  $\lambda \vdash M \geq N$ . □

### 1.2.17 Examples of terms without normal form.

From 1.2.16 follows that the term  $\omega_2\omega_2$  with  $\omega_2 = \lambda a \cdot aa$  has no normal form. Because  $\omega_2\omega_2$  reduces only to itself and it is not in normal form. When we reduce  $\omega_3\omega_3$  with  $\omega_3 = \lambda a \cdot aaa$ , then the result will grow larger and larger. Along this line we will prove in chapter II, the following extreme result:

There exists a term  $M$  such that  $\forall N \exists M' \lambda \vdash M \geq M'$  and  $N$  is sub-term of  $M'$ . Such a term is called a universal generator.

### 1.2.18 Remarks <sup>1)</sup>

1. Apart from the proofs already mentioned  
two abstract forms of the Church-Rosser

1) (Added in proof.) Recently a very simple proof of the Church-Rosser theorem is given by Martin-Löf [1971]. See appendix II.

theorem were proved in Newman [1942] and Curry [1952] which were supposed to imply the original theorem as a corollary. However, as was pointed out by Schroer (see Rosser [1956]) and Newman [1952], it turned out that these general theorems did not have as corollary the original theorem.

Two adequate abstract forms of the Church-Rosser theorem were proved by Schroer [1965] and Hindley [1969].

Mitschke [1970] gave a new proof of the Church-Rosser theorem, which is conceptually simpler than the original one. In §1.5 we will use Mitschke's ideas to give a proof of the Church-Rosser theorem for combinatory logic.

Finally there is another proof of the Church-Rosser theorem by Rosen [1971]. He proves also an abstract form of the Church-Rosser property, which is applicable to the  $\lambda$ -calculus. But this application needs a construction similar to that of Mitschke [1970] and this makes the proof rather long. On the other hand, Rosens general theorem is also applicable to other fields (like the McCarthy calculus of recursive definitions). See also Curry, Feys [1958] Ch 4 S and Kleene [1962] for historical remarks.

2. Before the Church-Rosser theorem was proved the consistency of combinatory logic was shown already by Curry [1930]. It was shown in Rosser [1935] that this implies the consistency of the  $\lambda$ -calculus.

In part II of our thesis a very simple consistency proof for combinatory logic, due to Scott, is given.

§1.3. The  $\lambda$ -definability of the partial recursive functions.

Undecidability results.

The (partial) functions  $f$  considered in this § are all number theoretic (i.e.  $f: \omega^n \rightarrow \omega$ ).

1.3.1 Definition

We define some standard functions:

- 1)  $U_i^n(x_1, \dots, x_n) = x_i$  are the projection functions.
- 2)  $S^+(x) = x+1$  is the successor function.
- 3)  $Z(x) = 0$  is the zero function.

1.3.2 Definition

The partial recursive functions can be defined as the smallest class  $\mathcal{R}$  of partial functions such that

- 1)  $U_i^n \in \mathcal{R}$
- 2)  $S^+ \in \mathcal{R}$
- 3)  $Z \in \mathcal{R}$
- 4) If  $g, h_1, \dots, h_m \in \mathcal{R}$  and  $f$  is defined by
 
$$f(x_1, \dots, x_n) = g(h_1(x_1, \dots, x_n), \dots, h_m(x_1, \dots, x_n))$$
 then  $f \in \mathcal{R}$  ( $\mathcal{R}$  is closed under substitution).
- 5) If  $g, h \in \mathcal{R}$ ,  $g$  and  $h$  are total and  $f$  is defined by
 
$$f(0, \vec{x}) = g(\vec{x}) \quad (\text{where } \vec{x} = x_1, \dots, x_n)$$

$$f(k+1, \vec{x}) = h(f(k, \vec{x}), k, \vec{x})$$
 then  $f \in \mathcal{R}$  ( $\mathcal{R}$  is closed under recursion).
- 6) If  $g \in \mathcal{R}$ ,  $g$  is total and  $f$  is defined by
 
$$f(\vec{x}) = \mu y [g(\vec{x}, y) = 0]$$
 (i.e. the least  $y$  such that  $g(\vec{x}, y) = 0$ ; if there does not,

exist such a  $y$ , then  $\mu y[g(\vec{x}, y) = 0]$  is undefined)  
 then  $f \in \mathcal{R}$ . ( $\mathcal{R}$  is closed under minimalisation).

The primitive recursive functions are defined by 1) - 5) only.  
 We write  $f(\vec{x})\downarrow$  if  $f(\vec{x})$  is defined and  $f(\vec{x})\uparrow$  if  $f(\vec{x})$  is undefined.

In order to show that the partial recursive functions are  $\lambda$ -definable, we will have to show a slightly stronger fact.

### 1.3.3 Definition

Let  $f$  be a partial function which is  $\lambda$ -definable by a term  $F$ .

We say that  $F$  strongly defines  $f$  iff

$f(k_1, \dots, k_n)\uparrow \Rightarrow \forall Z_1 \dots Z_s \quad Fk_1 \dots k_n Z_1 \dots Z_s$  has no normal form.

In this case  $f$  is called strongly  $\lambda$ -definable.

Note 1. Our concept of strongly  $\lambda$ -definable should not be confused with that of Curry Hindley and Seldin [1971] Ch 13 A. Our notion of  $\lambda$ -definability corresponds to their strong definability.

2. If a total function is  $\lambda$ -definable it is automatically strongly  $\lambda$ -definable.

Now we will prove that the partial recursive functions are all strongly  $\lambda$ -definable. In order to do this we have to show that:

- 1)  $U_i^n, Z, S^+$  are strongly  $\lambda$ -definable.
- 2) The strongly  $\lambda$ -definable functions are closed under substitution, recursion and minimalisation.

### 1.3.4 Lemma

The functions  $U_i^n, S^+, Z$  are  $\lambda$ -definable. (Hence they are strongly  $\lambda$ -definable.)

Proof.

1)  $U_i^n$  is defined by  $\underline{U}_i^n = \lambda a_1, \dots, a_n \cdot a_i$  :

$$\begin{aligned} \lambda \vdash \underline{U}_i^n \underline{k}_1 \dots \underline{k}_n &= (\lambda a_1 \dots a_n \cdot a_i) \underline{k}_1 \dots \underline{k}_n = \\ &= (\lambda a_2 \dots a_n \cdot a_i) \underline{k}_2 \dots \underline{k}_n = \\ &= \dots \\ &= (\lambda a_{i+1} \dots a_n \cdot a_i) \underline{k}_{i+1} \dots \underline{k}_n = \\ &= \dots = \underline{k}_i \end{aligned}$$

2)  $Z$  is defined by  $\underline{Z} = \lambda a \underline{0}$ :

$$\lambda \vdash \underline{Z} \underline{k} = \underline{0}.$$

3)  $S^+$  is defined by  $\underline{S}^+ = abc \cdot b(abc)$ :

$$\begin{aligned} \lambda \vdash \underline{S}^+ \underline{n} &= \lambda bc \cdot b(\underline{n}bc) = \lambda bc \cdot b(b^n c) \\ &= \lambda bc \cdot b^{n+1} c = \underline{n+1}. \end{aligned}$$

□

### 1.3.5 Lemma

The strongly  $\lambda$ -definable functions are closed under substitution

(i.e. if  $g, h_1, \dots, h_n$  are  $\lambda$ -definable and  $f(\vec{x}) = g(h_1(\vec{x}), \dots, h_m(\vec{x}))$  (where  $\vec{x} = x_1, \dots, x_n$ ), then  $f$  is  $\lambda$ -definable).

Proof.

Let  $g, h_1, \dots, h_m$  be strongly  $\lambda$ -defined by  $G, H_1, \dots, H_m$  respectively. Then  $f$  is defined by

$$\begin{aligned} F &= \lambda a_1 \dots a_n \cdot (H_1 a_1 \dots a_n I) \dots (H_m a_1 \dots a_n I) G(H_1 a_1 \dots a_n) \dots \\ &\dots (H_m a_1 \dots a_n) \end{aligned}$$

For suppose that  $f(k_1, \dots, k_n) = k$ .

Then  $\forall i \exists n_i \quad h_i(k_1, \dots, k_n) = n_i$ , therefore  $\lambda \vdash H_i \underline{k}_1, \dots, \underline{k}_n = \underline{n}_i$ .

Furthermore  $g(n_1, \dots, n_m) = k$ . Thus

$$\begin{aligned} \lambda \vdash F \underline{k}_1 \dots \underline{k}_n &= (H_1 \underline{k}_1 \dots \underline{k}_n I) \dots (H_m \underline{k}_1 \dots \underline{k}_n I) G(H_1 \underline{k}_1 \dots \underline{k}_n) \dots \\ &\dots (H_m \underline{k}_1 \dots \underline{k}_n) \\ &= (\underline{n}_1 I) \dots (\underline{n}_m I) G \underline{n}_1 \dots \underline{n}_m \end{aligned}$$

$$\begin{aligned}
&= I^{n_1} \dots I^{n_m} \underline{k} \\
&= \underline{k}
\end{aligned}$$

Suppose  $f(k_1, \dots, k_n) \uparrow$ . Then there are two cases:

1.  $\forall i \exists n_i \ h_i(k_1, \dots, k_n) = n_i$ .

Then  $g(n_1, \dots, n_m) \uparrow$  and

$$\lambda \vdash F_{\underline{k}_1} \dots F_{\underline{k}_n} = G_{\underline{n}_1} \dots G_{\underline{n}_m}$$

Hence  $\forall Z_1 \dots Z_s \ F_{\underline{k}_1} \dots F_{\underline{k}_n} Z_1 \dots Z_s$  has no normal form, because  $G$  strongly defines  $g$ .

2.  $h_i(k_1, \dots, k_n) \uparrow$  for some  $i$ .

Then  $\forall Z_1 \dots Z_s \ (H_i k_1 \dots k_n) Z_1 \dots Z_s$  has no normal form.

Hence  $\forall Z_1 \dots Z_s \ F_{\underline{k}_1} \dots F_{\underline{k}_n} Z_1 \dots Z_s$  has no normal form.

(For this case it was needed to introduce the notion of strong definability.)

In the  $\lambda$ -calculus it is possible to define ordered pairs.

### 1.3.6 Definitions

$[M, N] = \lambda z \cdot zMN$ , where  $z \notin FV(MN)$

$K = \lambda ab \cdot a$ ,  $\tau_1 = \lambda a \cdot aK$

$K' = \lambda ab \cdot b$ ,  $\tau_2 = \lambda a \cdot aK'$

Then we have

$\lambda \vdash KMN = M$  hence  $\lambda \vdash [M, N]K = M$ , thus  $\lambda \vdash \tau_1[M, N] = M$  and

$\lambda \vdash K'MN = N$  hence  $\lambda \vdash [M, N]K' = N$ , thus  $\lambda \vdash \tau_2[M, N] = N$ .

Hence  $[M, N]$  is an ordered pair function with projections  $\tau_1, \tau_2$ .

Note that  $M = [\tau_1 M, \tau_2 M]$  is not provable.

### 1.3.7 Theorem (Bernays)

The strongly  $\lambda$ -definable functions are closed under recursion.

Proof.

Because in 1.3.2. 5) only total functions are considered we do not need to bother about strong definability.

Let  $f$  be defined by

$$f(0) = n_0$$

$$f(n+1) = g(f(n), n)$$

and suppose  $g$  is  $\lambda$ -defined by  $G$ .

(For the sake of simplicity we treat the case that  $f$  has no additional parameters. The proof in the general case is analogous).

We will prove that  $f$  is  $\lambda$ -definable.

Consider  $M = \lambda a[\underline{S}^+(\tau_1 a), G(\tau_2 a)(\tau_1 a)]$ . Then  $M$  has the property  $\lambda \vdash M[\underline{n}, \underline{f(n)}] = [\underline{S}^+ \underline{n}, G \underline{f(n)} \underline{n}] = [\underline{n+1}, \underline{f(n+1)}]$ .

Now we have  $\lambda \vdash [\underline{0}, \underline{f(0)}] = [\underline{0}, \underline{n_0}]$

$$\lambda \vdash [\underline{1}, \underline{f(1)}] = M[\underline{0}, \underline{n_0}]$$

$$\lambda \vdash [\underline{2}, \underline{f(2)}] = M[\underline{1}, \underline{f(1)}] = M^2[\underline{0}, \underline{n_0}]$$

.....

$$\lambda \vdash [\underline{n}, \underline{f(n)}] = M^n[\underline{0}, \underline{n_0}] = \underline{n} M[\underline{0}, \underline{n_0}]$$

Hence

$$\lambda \vdash \underline{f(n)} = \tau_2[\underline{n}, \underline{f(n)}] = \tau_2(\underline{n} M[\underline{0}, \underline{n_0}]).$$

Therefore  $f$  can be  $\lambda$ -defined by

$$F = \lambda a \cdot \tau_2(a M[\underline{0}, \underline{n_0}]). \quad \square$$

By 1.3.4, 1.3.5 and 1.3.7 we can already state:

1.3.8 All primitive recursive functions are  $\lambda$ -definable.

### 1.3.9 Theorem

The strongly  $\lambda$ -definable functions are closed under minimalisation.

(This theorem was first proved for the  $\lambda I$ -calculus in Kleene [1934]. We give here a simplification for the  $\lambda K$ -calculus due to Turing [1937a]).

Proof.

Let  $f(x) = \mu y [g(x,y) = 0]$  where  $g$  is a total  $\lambda$ -definable function. (Again for simplicity we suppose that  $f$  has only one argument.) We will prove that  $f$  is strongly  $\lambda$ -definable.

Let  $g'(x,y) = sg(g(x,y))$  where  $sg(0) = 0$  and  $sg(n+1) = 1$ .

From 1.3.8 and 1.3.5 it follows that  $g'$  is  $\lambda$ -definable, say by  $G'$ . Hence

$$G' \underline{k} \underline{n} \equiv \begin{cases} \underline{0} & \text{if } g(k,n) = 0 \\ \underline{1} & \text{if } g(k,n) \neq 0. \end{cases}$$

By the fixed point theorem there exists a term  $M$  such that

$$\lambda \vdash M = \lambda a b \cdot G' a b (\lambda c \cdot M a (\underline{S}^+ c)) b$$

(Define  $N = \lambda m a b \cdot G' a b (\lambda c \cdot m a (\underline{S}^+ c)) b$  and take  $M = FPN$ .)

Define  $F = \lambda a \cdot M a \underline{0}$ .

Then  $F$  strongly defines  $f$ :

$$\begin{aligned} \lambda \vdash F \underline{k} &= M \underline{k} \underline{0} = G' \underline{k} \underline{0} (\lambda c \cdot M \underline{k} (\underline{S}^+ c)) \underline{0} = \underline{0} && \text{if } g(k,0) = 0 \\ &= (\lambda c \cdot M \underline{k} (\underline{S}^+ c)) \underline{0} && \text{if } g(k,0) \neq 0 \\ &= M \underline{k} \underline{1} = G' \underline{k} \underline{1} (\lambda c \cdot M \underline{k} (\underline{S}^+ c)) \underline{1} = \underline{1} && \text{if } g(k,1) = 0 \\ &= (\lambda c \cdot M \underline{k} (\underline{S}^+ c)) \underline{1} && \text{if } g(k,1) \neq 0 \\ &= M \underline{k} \underline{2} = \\ &\quad \text{etc.} \end{aligned}$$

If  $f(k) \uparrow$  then  $\forall y \ g'(k,y) = 1$ , hence  $\forall Z_1 \dots Z_S \ F \underline{k} \ Z_1 \dots Z_S$  has no normal form, as is readily seen.  $\square$

From 1.3.4, 1.3.5, 1.3.7 and 1.3.9 we get:

### 1.3.10 Theorem

All partial recursive functions are  $\lambda$ -definable.

Remark. The converse of 1.3.10: All  $\lambda$ -definable functions are partial recursive, is intuitively clear from Church thesis. For a proof see Kleene [1936].

We now state some corollaries to 1.3.10 which concern undecidability.

Let  $\ulcorner \urcorner$  denote the Gödel number of a term or formula in some Gödelisation. For the definition of some notions in recursion theory the reader is referred to Rogers [1967] Ch 7.

With the following we answer a question of Mostowski.

### 1.3.11 Theorem

The set  $\{\ulcorner M = N \urcorner \mid \lambda \vdash M = N\}$  is 1-complete and hence creative.

Proof.

Let  $X$  be an arbitrary r.e. set.

Define  $f(x) = \begin{cases} 0 & \text{if } x \in X \\ \uparrow & \text{else} \end{cases}$

Then  $f$  is partial recursive. Let  $f$  be defined by  $F$ .

Then

$$k \in X \iff f(k) = 0 \iff \lambda \vdash F\underline{k} = \underline{0}$$

Hence  $X \leq_1 \{\ulcorner M = N \urcorner \mid \lambda \vdash M = N\}$  via the function  $h(k) = \ulcorner F\underline{k} = \underline{0} \urcorner$ .

### 1.3.12 Corollary

The  $\lambda$ -calculus is undecidable.

Grzegorzczak [1970] has proved even that the  $\lambda$ -calculus is essentially undecidable (see 1.3.17).

1.3.13 Lemma (Kleene [1936])

There exists a term  $E$  such that

$$\forall M [ FV(M) = \emptyset \Rightarrow \exists n \lambda \vdash E_n = M ]$$

See for a proof Kleene[1936] theorem(24), pg.351 or Church[1941] theorem 14 III, pp.47-48.

1.3.14 Lemma

For any terms  $M_0, M_1$  there exists a term  $M$  such that

$$\lambda \vdash M \underline{0} = M_0$$

$$\lambda \vdash M \underline{1} = M_1 .$$

Proof.

Define  $M = \lambda a \cdot a(KM_1)M_0$  (where  $K$  is defined as in 1.3.6).

$$\text{Then } \lambda \vdash M \underline{0} = \underline{0} (KM_1)M_0 = M_0$$

$$\lambda \vdash M \underline{1} = \underline{1} (KM_1)M_0 = KM_1M_0 = M_1 .$$

□

1.3.15 Theorem (Scott [1963])

Let  $A$  be a set of terms such that

- 1)  $A$  is not trivial (i.e.  $M_0 \in A$  and  $M_1 \notin A$  for some  $M_0, M_1$ ).
- 2) If  $M \in A$  and  $\lambda \vdash M = M'$ , then  $M' \in A$ .

Then  $\ulcorner A \urcorner = \{ \ulcorner M \urcorner \mid M \in A \}$  is not recursive.

(Compare this theorem to the theorem of Rice [1953]).

Proof.

Suppose  $\ulcorner A \urcorner$  is recursive and let  $M_0 \in A$  and  $M_1 \notin A$ . By 1.3.14 there exists a term  $M$  such that  $\lambda \vdash M \underline{0} = M_0$  and  $\lambda \vdash M \underline{1} = M_1$ .

Define  $B = \{ n \mid M(E_{\underline{n}}) \in A \}$ , with  $E$  as in 1.3.13.

Then  $B$  is recursive, hence there exists a term  $\underline{c}_B$  such that

$$\lambda \vdash \underline{c}_B \underline{n} = \underline{0} \quad \text{if } n \in B \quad \text{and}$$

$$\lambda \vdash \underline{c}_B \underline{n} = \underline{1} \quad \text{if } n \notin B .$$

We can assume that  $FV(\underline{c}_B) = \emptyset$ .

Let  $n_0$  be such that  $\lambda \vdash E_{\underline{n}_0} = \underline{c}_B$ .

Then

$$\begin{aligned} n_0 \in B &\Rightarrow \lambda \vdash E_{\underline{n}_0 \underline{n}_0} = \underline{0} \Rightarrow \lambda \vdash M(E_{\underline{n}_0 \underline{n}_0}) = M_0 \Rightarrow \\ &\Rightarrow M(E_{\underline{n}_0 \underline{n}_0}) \in A \Rightarrow n_0 \notin B \\ n_0 \notin B &\Rightarrow \lambda \vdash E_{\underline{n}_0 \underline{n}_0} = \underline{1} \Rightarrow \lambda \vdash M(E_{\underline{n}_0 \underline{n}_0}) = M_1 \Rightarrow \\ &\Rightarrow M(E_{\underline{n}_0 \underline{n}_0}) \notin A \Rightarrow n_0 \in B. \end{aligned}$$

This is a contradiction. ☒

### 1.3.16 Corollary (Church [1936])

The set  $\{\ulcorner M \urcorner \mid M \text{ has a normal form}\}$  is not recursive.

Proof.

Take  $A = \{M \mid M \text{ has a normal form}\}$  in 1.3.15. ☒

### 1.3.17 Corollary (Grzegorzczuk [1970])

The  $\lambda$ -calculus is essentially undecidable (i.e. has no decidable consistent extension).

Proof.

Suppose  $T$  is a decidable consistent extension of the  $\lambda$ -calculus (i.e. the set of theorems of  $T$  is recursive).

Define  $A = \{M \mid T \vdash M = \underline{0}\}$ . Then  $\ulcorner A \urcorner$  is recursive because  $T$  is.

But  $A$  satisfies 1) and 2) of 1.3.15. Hence  $\ulcorner A \urcorner$  is not recursive.

Contradiction. ☒

Remark. 1.3.12, 1.3.15 and 1.3.17 were proved independently.

## §1.4. Combinatory logic.

Combinatory logic is a theory closely related to the  $\lambda$ -calculus. Considerable parts of it were developed by Curry. See Curry Feys [1958] and Curry Hindley Seldin [1971] for an extensive treatment of the subject.

Combinatory logic is intended to be a foundation for mathematical logic. Therefore it includes "illative" notions corresponding to concepts like equality, quantification etcetera.

However, we will be concerned only with the combinatorial part of combinatory logic or as Curry calls it: "pure combinatory logic". For illative combinatory logic the reader is referred to the above standard texts.

#### 1.4.1 Definition

We define the following language  $L_H$ :

Alphabet: a,b,c,... variables  
           I,K,S constants  
           (,) improper symbols  
           = equality  
            $\geq$  reduction

Terms: Terms are inductively defined by

- 1) any variable or constant is a term
- 2) if M,N are terms, then (MN) is a term.

Formulas: if M,N are terms, then  $M = N$  and  $M \geq N$  are formulas.

Again  $M_1 M_2 \dots M_n$  stands for  $((\dots(M_1 M_2)\dots)M_n)$ .

To be explicit terms of CL are called sometimes CL-terms, terms of the  $\lambda$ -calculus are called then  $\lambda$ -terms.

#### 1.4.2 Definition

Combinatory logic (CL) is the theory defined in  $L_H$  by the following axioms and rules (see appendix).

- I
1.  $IM \supseteq M$
  2.  $KMN \supseteq M$
  3.  $SMNL \supseteq ML(NL)$
- II Same as in 1.2.3.
- III
1. Same as in 1.2.3
  2. Same as in 1.2.3.
  3.  $\frac{MM \supseteq M'}{ZM \supseteq ZM'} , \frac{M \supseteq M'}{MZ \supseteq M'Z}$
  4. Same as in 1.2.3.

In the above  $M, M', N, L, Z$  denote arbitrary terms.

Notation. We write  $x \in M$  if  $x$  occurs in the term  $M$ .

As in §1.1. we can adjoin to CL extensionality.

1.4.3

$$\text{ext } \frac{Mx=Nx, x \in MN}{M=N}$$

$CL (+ \text{ ext}) \vdash \dots$  means that  $\dots$  is provable in  $CL (+ \text{ ext})$ .

For example

1.4.4

$$CL + \text{ ext} \vdash S(S(KS)K)(KI) = I$$

Curry proved in his thesis that from a finite number of such theorems as 1.4.4 extensionality is provable. These axioms are called the combinatory axioms. See Curry Feys [1958] Ch 6 C.

We will show that the  $\lambda$ -calculus and combinatory logic are interpretable in each other.

1.4.5 Theorem

There exists a mapping  $\phi: CL \rightarrow \lambda$  (i.e. from the set of terms of  $L_H$  into the set of terms of  $L_K$ ) such that

1.  $CL (+ \text{ ext}) \vdash M = N \Rightarrow \lambda (+ \text{ ext}) \vdash \phi(M) = \phi(N)$
2.  $CL \vdash M \geq N \Rightarrow \lambda \vdash \phi(M) \geq \phi(N)$

Proof.

Define  $\phi(x) = x$  for any variable  $x$ .

$$\phi(I) = I \quad (\equiv \lambda a.a)$$

$$\phi(K) = K \quad (\equiv \lambda ab.a)$$

$$\phi(S) = S \quad (\equiv \lambda abc.ac(bc))$$

$$\phi(MN) = \phi(M)\phi(N)$$

Then it is clear that  $\phi$  has the required properties.

Remark. The converse of 1. and 2. do not hold. For example:

$$\lambda \vdash \phi(SKK) = \phi(I) \quad \text{but} \quad CL \not\vdash SKK = I$$

as follows from the Church-Rosser theorem for CL (see §1.5.).

In order to obtain a reverse interpretation, we need first a simulation of the  $\lambda$ -operator in CL.

#### 1.4.6 Definition

For any term  $M$  of  $L_H$  and any variable  $x$  we define inductively a term  $\lambda^*xM$ .

$$\lambda^*x x = I$$

$$\lambda^*x M = KM \quad \text{if} \quad x \notin M$$

$$\lambda^*x (MN) = S(\lambda^*x M)(\lambda^*x N)$$

Remarks.

1. This method of defining  $\lambda$ -terms with  $I$ ,  $K$  and  $S$  comes from Schönfinkel [1924] although it is not in general formulated there.

2. It is possible to formulate CL with K and S alone. In that case we define

$$\lambda^*x X = SKK$$

because  $CL \vdash SKK x = x$

#### 1.4.7 Lemma

$\lambda^*$  has the following properties:

- 1)  $x$  does not occur in  $\lambda^*x M$
- 2)  $\lambda^*x M = \lambda^*y[x/y]M$  if  $y \notin M$
- 3)  $CL \vdash (\lambda^*x M)N = [x/N]M$
- 4)  $\lambda^*x[y/N]M = [y/N]\lambda^*x M$  if  $x \neq y$  and  $x \notin N$ .

Proof.

The proof uses induction on the complexity of  $M$  and is in all the four cases similar.

As an example we prove 4).

$M = x$  Then  $\lambda^*x[y/N]M = \lambda^*x x = I$

and  $[y/N]\lambda^*x M = [y/N]I = I$ .

$x \notin M$  Then  $\lambda^*x[y/N]M = K[y/N]M = [y/N]KM$

and  $[y/N]\lambda^*x M = [y/N]KM$ .

$M = M_1M_2$  and  $x \in M$ .

Then  $\lambda^*x[y/N]M = \lambda^*x[y/N]M_1M_2$

$= \lambda^*x([y/N]M_1[y/N]M_2)$

$= S(\lambda^*x[y/N]M_1)(\lambda^*x[y/N]M_2)$

$= S([y/N]\lambda^*x M_1)([y/N]\lambda^*x M_2)$  by the induction hypothesis

$= [y/N] S(\lambda^*x M_1)(\lambda^*x M_2)$

$= [y/N]\lambda^*x M$

□

1.4.8 Lemma

$$\frac{M = N}{\lambda^*x M = \lambda^*x N} \quad \text{is a derived rule for CL + ext}$$

Proof.

Let  $CL + \text{ext} \vdash M = N$ Then  $CL + \text{ext} \vdash (\lambda^*x M)x = M = N = (\lambda^*x N)x$ 

Hence by ext

 $CL + \text{ext} \vdash \lambda^*x M = \lambda^*x N$ 

□

1.4.9 TheoremThere exists a mapping  $\psi: \lambda \rightarrow CL$  such that $\lambda + \text{ext} \vdash M = N \Rightarrow CL + \text{ext} \vdash \psi(M) = \psi(N)$ 

Proof.

Define  $\psi$  inductively: $\psi(x) = x$  for any variable  $x$  $\psi(MN) = \psi(M)\psi(N)$  $\psi(\lambda x M) = \lambda^*x \psi(M)$ .Sublemma $\psi([x/N]M) \equiv [x/\psi(N)] \psi(M)$  if  $BV(M) \cap FV(N) = \emptyset$ .

Proof.

Induction on the structure of  $M$ . (Use:  $x \in FV(M) \iff x \in \psi(M)$ and 1.4.7.4.) Now suppose  $\lambda + \text{ext} \vdash M = N$ . We will show by induction on the length  $l$  of the proof of  $M = N$  that(1)  $CL + \text{ext} \vdash \psi(M) = \psi(N)$ .1 = 0  $M = N$  is axiom of the  $\lambda$ -calculus.case 1.  $M \equiv N$ , then  $\psi(M) \equiv \psi(N)$ ,hence  $CL \vdash \psi(M) = \psi(N)$ .case 2.  $M \equiv \lambda x P$  and  $N \equiv \lambda y[x/y]P$ , with  $y \notin FV(P)$ .

Then  $\psi(M) \equiv \lambda^*x \psi(P)$  and

$\psi(N) \equiv \lambda^*y[x/y]\psi(P)$  by the sublemma.

Hence by 1.4.7.2)  $CL \vdash \psi(M) = \psi(N)$ .

case 3.

$M \equiv (\lambda x P)Q$  and  $N \equiv [x/Q]P$ , with

$BV(P) \cap FV(Q) = \emptyset$ .

Then  $CL \vdash \psi(M) = (\lambda^*x \psi(P))\psi(Q) = [x/\psi(Q)]\psi(P)$

by 1.4.7. 3)

and  $CL \vdash \psi(N) = \psi([x/Q]P) = [x/\psi(Q)]\psi(P)$  by the sublemma, hence  $CL \vdash \psi(M) = \psi(N)$ .

l = k and the theorem holds for  $l' < k$ .

$M = N$  is the consequence of a rule of inference. Because of 1.4.8 the rules of inference for the  $\lambda$ -calculus + ext and  $CL + ext$  are the same, hence (1) follows immediately by the induction hypothesis.

### Remarks

1. It is not true that

$\lambda \vdash M = N \Rightarrow CL \vdash \psi(M) = \psi(N)$ .

For example  $\lambda \vdash \lambda a((\lambda b \cdot b)a) = \lambda a \cdot a$ .

But  $CL \not\vdash S(KI)I = I$  as follows from §1.5.

2. Also we do not have  $\lambda \vdash M \geq N \Rightarrow CL \vdash \psi(M) \geq \psi(N)$ .

Because of this peculiarity a special theory of strong reducibility (notation:  $\succ$ ) for  $CL$  is developed.

Then we have  $\lambda \vdash M \geq N \Rightarrow \psi(M) \succ \psi(N)$  in  $CL$ .

(See Curry Feys [1958] Ch 6 F.)

In order to distinguish it from  $\succ$ ,  $\geq$  is called weak reduction.

1.4.10 Theorem

- 1)  $\lambda \vdash \phi(\psi(M)) \geq M$
- 2)  $CL + ext \vdash \psi(\phi(M)) = M$

Proof.

- 1) Induction on the structure of  $M$ .

Prove first  $\lambda \vdash \phi(\lambda * x M) \geq \lambda x \phi(M)$  for  $M$  a term of  $CL$ .

- 2) Induction on the structure of  $M$ .

The essential step is to show that

$CL + ext \vdash S = \lambda * xyz \cdot xz(yz)$  and

$CL + ext \vdash K = \lambda * xy \cdot x$  .

☒

Remark. It is not true that  $CL \vdash \psi(\phi(M)) = M$ .

Take for example  $M = K$ .

Then  $CL \not\vdash S(KK)I = K$  as follows from §1.5.

1.4.11 Corollary

The  $\lambda$ -calculus + ext and  $CL + ext$  are equivalent:

- 1)  $CL + ext \vdash \psi(\phi(M)) = M$
- 2)  $\lambda + ext \vdash \phi(\psi(M)) = M$
- 3)  $CL + ext \vdash M = N \iff \lambda + ext \vdash \phi(M) = \phi(N)$
- 4)  $\lambda + ext \vdash M = N \iff CL + ext \vdash \psi(M) = \psi(N)$

Proof.

- 1) and 2) follow directly from 1.4.10.

- 3)  $CL + ext \vdash M = N \Rightarrow \lambda + ext \vdash \phi(M) = \phi(N)$  by 1.4.5  
 $\Rightarrow CL + ext \vdash \psi(\phi(M)) = \psi(\phi(N))$  by 1.4.9  
 $\Rightarrow CL + ext \vdash M = N$  by 1).

- 4) Similar proof as 3).

☒

Because of this equivalence the names for the  $\lambda$ -calculus and combinatory logic are sometimes interchanged.

Scott [1970] in fact constructs a model for combinatory logic + extensionality.

#### 1.4.12 Definition

- 1) A term  $M$  of CL is in normal form if it has no part of the form  $IM$ ,  $KMN$  or  $SMNL$ .
- 2) A term  $M$  has a normal form if there exists a term  $M'$  in normal form such that  $CL \vdash M = M'$ .

Remark. The terms in normal form can be defined inductively as follows:

- 1)  $I$ ,  $K$  and  $S$  are in normal form
- 2) If  $M$  is in normal form, then  $KM$  and  $SM$  are in normal form
- 3) If  $M$  and  $N$  are in normal form then  $SMN$  is in normal form.

Similarly to 1.2.5 we have

#### 1.4.13 Theorem

If  $M$  is in normal form and if  $CL \vdash M \geq N$ , then  $M \equiv N$ .

As in §1.1. we can represent the natural numbers in CL. Then we can define similar to definition 1.1.14 the concept of CL-definability. Analogous to §1.3. it is possible to prove that the partial CL-definable functions are exactly the partial recursive functions. See Curry Hindley Seldin [1971] Ch 13 for details.

## §1.5. The Church-Rosser property for combinatory logic à la

### Mitschke.

In this § we will use the ideas of Mitschke [1970] to give a proof of the Church-Rosser property for combinatory logic (with weak reduction).

In Hindley [197?] this will be proved as an application of an abstract Church-Rosser theorem proved in Hindley [1969].

(Added in print. We have realized too late that for the combinatory equivalent of the  $\lambda I$ -calculus the Church-Rosser property was already proved in Rosser [1935] T 12, p.144. This proof carries over immediately to CL (cf. Curry, Hindley, Seldin [1971]). Compare Rosser's proof with that of Martin-Löf in appendix II.)

The theorem we are about to prove is:

#### 1.5.1 Theorem (Church-Rosser property for CL)

If  $CL \vdash M = N$ , then there exists a term  $Z$  such that

$CL \vdash M \geq Z$  and  $CL \vdash N \geq Z$ .

In order to show this we have to define several auxiliary languages and theories.

#### 1.5.2 Definition

$CL'$  is an extension of  $CL$  where the alphabet of  $CL'$  has the additional symbol " $\geq_1$ " (one step reduction); the language of  $CL'$  has the same terms as  $CL$ ;  $CL'$  has as extra formulas  $M \geq_1 N$  (for arbitrary terms  $M$  and  $N$ ).

$CL'$  has the following axioms and rules (see appendix):

I 1.  $IM \geq_1 M$

2.  $KMN \geq_1 M$

3.  $SMNL \geq_1 ML(NL)$

II Same as in 1.4.2.

III Same as in 1.4.2.

IV 1.  $M \geq_1 M$

$$2. \frac{M \geq_1 M'}{ZM \geq_1 ZM'} , \quad \frac{M \geq_1 M'}{MZ \geq_1 M'Z}$$

$$3. \frac{M \geq_1 M'}{M \geq M'}$$

Again  $M, M', N, L, Z$  denote arbitrary terms.

### 1.5.3 Lemma

$CL'$  is a conservative extension of  $CL$ . Hence in order to prove

1.5.1 it is sufficient to prove the Church-Rosser property for  $CL'$ .

Proof.

First show that  $CL' \vdash M \geq_1 N \Rightarrow CL \vdash M \geq N$ .

Then the rest follows easily. □

Now we will introduce a theory  $CL^*$  which plays the same role as  $\lambda^*$  in Mitschke [1970].

### 1.5.4 Definition

We define the following language  $L_H^*$ .

Alphabet: The alphabet of  $CL'$  together with the extra symbol  
 ", " .

Terms: Terms are inductively defined by

- 1) Any variable or constant is a term.
- 2) If  $M$  and  $N$  are terms, then  $(MN)$  is a term.
- 3) If  $M, N$  and  $L$  are terms, then  $S(M,N,L)$  is a term.

Formulas: If  $M$  and  $N$  are terms, then

$M \geq_1 N, M \geq N$  and  $M = N$  are formulas.

1.5.5 Definition

CL\* is the theory with the language  $L_H^*$  defined by the following axioms and rules (see appendix)

I 1.  $IM \geq_1 M$

2.  $KMN \geq_1 M$

3.  $SMNL \geq_1 S(M,N,L)$

II Same as in 1.4.2 II

III Same as in 1.4.2 III

IV 1. }  
 2. } Same as in 1.5.2 IV  
 3. }

4. 
$$\frac{M \geq_1 M'}{S(M,N,L) \geq_1 S(M',N,L)} \quad , \quad \frac{N \geq_1 N'}{S(M,N,L) \geq_1 S(M,N',L)}$$

$$\frac{L \geq_1 L'}{S(M,N,L) \geq_1 S(M,N,L')}$$

M, M', N, N', L, L' and Z denoting arbitrary terms.

The essential axiom of this theory is I 3. It freezes the action of S.

Our method of proving 1.5.1 is the following:

First, we (almost) prove the Church-Rosser property for CL\*; then with the help of an homomorphism argument, we obtain the corresponding result for CL', and hence, by 1.5.3 for CL.

1.5.6 Lemma

$CL^* \vdash M \geq_1 N \leftrightarrow$  there exists a term L with exactly one occurrence of a variable x and terms N, N' such that  $M \equiv [x/N]L$ ,  $M' \equiv [x/N']L$  and  $N \geq_1 N'$  is an axiom of CL\*.

Proof.

$\Rightarrow$  Induction on the length of proof of  $M \geq_1 M'$ .

$\Leftarrow$  Induction on the structure of  $L$ . □

### 1.5.7 Lemma

If  $CL^* \vdash M_1 \geq_1 M_2$  and  $CL^* \vdash M_1 \geq_1 M_3$ , then there exists a term  $M_4$ , such that

$CL^* \vdash M_2 \geq_1 M_4$  and  $CL^* \vdash M_3 \geq_1 M_4$  (see fig.1)

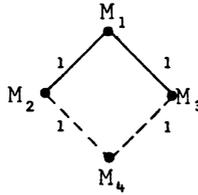


Figure 1

Proof.

First we consider the possibility that  $M_1 \geq_1 M_2$  is an axiom.

(By repeated use of 1.5.6 we obtain the subcases.)

case 1.  $M_1 \equiv IM$ ,  $M_2 \equiv M$

subcase 1.1.  $M_3 \equiv M_1$  or  $M_3 \equiv M_2$ . Take  $M_4 \equiv M_2$ .

subcase 1.2.  $M_3 \equiv IM'$  and  $CL^* \vdash M \geq_1 M'$ . Take  $M_4 \equiv M'$ .

case 2.  $M_1 \equiv KMN$ ,  $M_2 \equiv M$

subcase 2.1.  $M_3 \equiv M_1$  or  $M_3 \equiv M_2$ . Take  $M_4 \equiv M_2$ .

subcase 2.2.  $M_3 \equiv KM'N$  and  $CL^* \vdash M \geq_1 M'$ . Take  $M_4 \equiv M'$ .

subcase 2.3.  $M_3 \equiv KMN'$  and  $CL^* \vdash N \geq_1 N'$ . Take  $M_4 \equiv M$ .

case 3.  $M_1 \equiv SMNL$ ,  $M_2 \equiv S(M,N,L)$

subcase 3.1.  $M_3 \equiv M_1$  or  $M_3 \equiv M_2$ . Take  $M_4 \equiv M_2$ .

subcase 3.2.  $M_3 \equiv SM'NL$  and  $CL^* \vdash M \geq_1 M'$ .

Take  $M_4 \equiv S(M',N,L)$

subcases 3.3, 3.4  $M_3 \equiv SMN'L$  or  $M_3 \equiv SMNL'$

Similar to subcase 3.2.

case 4.  $M_1 \equiv M_2$ . Take  $M_4 \equiv M_3$ .

Now we have proved that the lemma for  $M_1 \geq_1 M_2$  is an axiom. Since  $CL^* \vdash M_1 \geq_1 M_2$  it follows from 1.5.6 that  $M_1 \equiv \dots M \dots$ ,  $M_2 \equiv \dots \dots M' \dots$  and  $M \geq_1 M'$  is an axiom of  $CL^*$ .

case 1.  $M_3 \equiv M_1$  or  $M_3 \equiv M_2$ . Take  $M_4 \equiv M_2$ .

case 2.  $M_1 \equiv \dots N \dots$ ,  $M_3 \equiv \dots N' \dots$  and  $N \geq_1 N'$  is an axiom of  $CL^*$ .

subcase 2.1.  $M$  and  $N$  are disjoint subterms of  $M_1$ .

Then  $M_1 \equiv \dots M \dots N \dots$ ,

$M_2 \equiv \dots M' \dots N \dots$  and

$M_3 \equiv \dots M \dots N' \dots$ .

Take  $M_4 \equiv \dots M' \dots N' \dots$ .

subcase 2.2.  $N$  is a subterm of  $M$ .

Then  $M \geq_1 M'$  is an axiom, and

$CL^* \vdash M \geq_1 M''$

by the reduction  $CL^* \vdash N \geq_1 N'$ .

Hence there exists a term  $M'''$  such that

$CL^* \vdash M' \geq_1 M'''$  and

$CL^* \vdash M'' \geq_1 M'''$ .

Take  $M_4 \equiv \dots M''' \dots$ .

subcase 2.3.  $M$  is a subterm of  $N$ .

Analogous to subcase 2.2. □

### 1.5.8 Lemma

If  $CL^* \vdash M_1 \geq M_2$  and  $CL^* \vdash M_1 \geq M_3$ , then there exists a term  $M_4$  such that

(\*)  $CL^* \vdash M_2 \geq M_4$  and  $CL^* \vdash M_3 \geq M_4$ .

Proof.

Note that  $CL^* \vdash M_1 \geq M_2 \iff \exists N_1 \dots N_k \quad CL^* \vdash M_1 \equiv N_1 \geq_1 N_2 \geq_1 \dots \dots \geq_1 N_k \equiv M_2$ .

The same holds for  $CL^* \vdash M_1 \geq M_3$ .

Then repeated use of 1.5.7 yields (\*). (See fig. 2.)

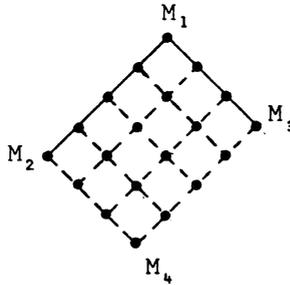


Figure 2.

⊠

Remark. 1.5.7 does not hold for  $CL'$ , but only

$CL' \vdash M_1 \geq_1 M_2$  and  $CL' \vdash M_1 \geq_1 M_3 \Rightarrow$

$CL' \vdash M_2 \geq M_4$  and  $CL' \vdash M_3 \geq M_4$  for some term  $M_4$ .

Therefore an analogue of 1.5.8 for  $CL'$  is much harder to prove.

From 1.5.8 we can easily derive the Church-Rosser property for

$CL^*$ . However we do not need to do so.

### 1.5.9 Definition

We define inductively a mapping  $\theta: CL^* \rightarrow CL'$  (in fact from the set of terms of  $CL^*$  into the set of terms of  $CL'$ ):

$\theta(c) = c$  if  $c$  is a variable or constant

$\theta(MN) = \theta(M)\theta(N)$

$\theta(S, M, N, L) = \theta(M)\theta(L)(\theta(N)\theta(L))$

It is clear that if  $M$  is a term of  $CL'$ , then  $\theta(M) = M$ .

### 1.5.10 Lemma

1)  $CL^* \vdash M \geq_1 N \Rightarrow CL' \vdash \theta(M) \geq \theta(N)$

2)  $CL^* \vdash M \geq N \Rightarrow CL' \vdash \theta(M) \geq \theta(N)$

3)  $CL^* \vdash M = N \Rightarrow CL' \vdash \theta(M) = \theta(N)$

Proof.

In all cases the result follows by induction on the length of

proof in  $CL^*$ . ☒

### 1.5.11 Lemma

If  $CL' \vdash M_1 \succcurlyeq_1 M_2$  then there exists a term  $M_2^*$  of  $CL^*$  such that  $CL^* \vdash M_1 \succcurlyeq_1 M_2^*$  and  $\Theta(M_2^*) = M_2$  (see fig. 3).

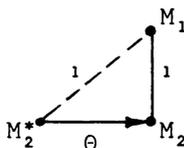


Figure 3.

Proof.

Induction on the length of proof of  $M_1 \succcurlyeq_1 M_2$ . Suppose first that  $M_1 \succcurlyeq_1 M_2$  is an axiom of  $CL'$ .

case 1,2.  $M_1 \equiv IM$  or  $M_1 \equiv KMN$ . Take  $M_2^* \equiv M$ .

case 3.  $M_1 \equiv SMNL$ ,  $M_2 \equiv ML(NL)$ . Take  $M_2^* \equiv S(M,N,L)$ .

case 4.  $M_1 \equiv M_2$ . Take  $M_2^* \equiv M_2$ .

Suppose now that  $M_1 \succcurlyeq_1 M_2$  is  $ZM_1' \succcurlyeq_1 ZM_2'$  and is a direct consequence of  $M_1' \succcurlyeq_1 M_2'$ .

By the induction hypothesis there exists a term  $M_2'^*$  such that  $CL^* \vdash M_1' \succcurlyeq_1 M_2'^*$  and  $\Theta(M_2'^*) = M_2'$ . Then take  $M_2^* \equiv ZM_2'^*$ .

The case that  $M_1 \succcurlyeq_1 M_2$  is  $M_1'Z \succcurlyeq_1 M_2'Z$  is treated analogously. ☒

### 1.5.12 Definition

We write  $M \succcurlyeq^* N$  for

$\exists M^*, N^* \quad CL^* \vdash M^* \succcurlyeq_1 N^* \quad \text{and} \quad \Theta(M^*) = M, \Theta(N^*) = N.$

### 1.5.13 Main Lemma

Suppose  $M \succcurlyeq^* N$ .

Then there exists a term  $N_1^*$  such that

$CL^* \vdash M \succcurlyeq_1 N_1^* \quad \text{and} \quad \Theta(N_1^*) = N \quad (\text{see fig. 4.})$

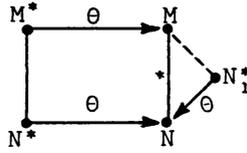


Figure 4

We will postpone the proof of the main lemma until 1.5.18.

#### 1.5.14 Lemma

If  $CL' \vdash M_1 \geq_1 M_2$  and  $CL' \vdash M_1 \geq M_3$ , then there exists a term  $M_4$  such that

$CL' \vdash M_2 \geq M_4$  and  $M_3 \geq^* M_4$ , and hence by 1.5.10 2)

$CL' \vdash M_3 \geq M_4$ .

Proof.

$CL' \vdash M_1 \geq M_3 \iff \exists N_1 \dots N_k \quad CL' \vdash M_1 \equiv N_1 \geq_1 N_2 \geq_1 \dots \geq_1 N_k \equiv M_3$ .

We prove the lemma by induction on  $k$ .

If  $k = 1$  then  $M_1 \equiv M_3$  and we can take  $M_4 \equiv M_2$ , then

$M_3 \geq^* M_4$  follows from 1.5.11.

Suppose now that  $\exists N_1 \dots N_{k+1} \quad CL' \vdash M_1 \equiv N_1 \geq_1 \dots \geq_1 N_{k+1} \equiv M_3$ .

By the induction hypothesis, there exists a term  $N'_k$  such that

$CL' \vdash M_2 \geq N'_k$  and  $N_k \geq^* N'_k$ .

By 1.5.13 there exists a  $N'^*_k$  such that

$CL^* \vdash N_k \geq N'^*_k$  and  $\theta(N'^*_k) = N'_k$  (see fig. 5).

By 1.5.11 there exists a  $N^*_{k+1}$  such that

$CL^* \vdash N_k \geq N^*_{k+1}$  and  $\theta(N^*_{k+1}) = N_{k+1}$  (see fig. 5).

Now it follows from 1.5.8 that there exists a  $N'^*_{k+1}$  such that

$CL^* \vdash N'^*_k \geq N'^*_{k+1}$  and  $CL^* \vdash N^*_{k+1} \geq N'^*_{k+1}$ .



case 1.  $M \equiv N$ . Take  $Z \equiv M$ .

case 2.  $M = N$  is a direct consequence of  $N = M$ . By the induction hypothesis there exists a  $Z$  such that  $CL' \vdash N \geq Z$  and  $CL' \vdash M \geq Z$ .

case 3.  $M = N$  is a direct consequence of  $M = L$  and  $L = N$ . By the induction hypothesis there exists terms  $Z_1, Z_2$  such that  $CL' \vdash M \geq Z_1$ ,  $CL' \vdash L \geq Z$ ,  $CL' \vdash L \geq Z_2$  and  $CL' \vdash N \geq Z_2$  (see fig. 6).

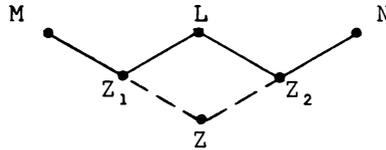


Figure 6.

By 1.5.15 there exists a  $Z$  such that  $CL' \vdash Z_1 \geq Z$  and  $CL' \vdash Z_2 \geq Z$ , hence  $CL' \vdash M \geq Z$  and  $CL' \vdash N \geq Z$ .

case 4.  $M = N$  is  $Z_1 M' = Z_1 N'$  (or  $M' Z_1 = N' Z_1$ ) and is a direct consequence of  $M' = N'$ .

By the induction hypothesis there exists a term  $Z'$  such that  $CL' \vdash M' \geq Z'$  and  $CL' \vdash N' \geq Z'$ .

Take  $Z \equiv Z_1 Z'$  (resp.  $\equiv Z' Z_1$ )

case 5.  $M = N$  is a direct consequence of  $M \geq N$ . Take  $Z \equiv N$ .  $\square$

### 1.5.17 Corollary

The Church-Rosser property for CL (see 1.5.1) holds.

Proof.

This follows from 1.5.16 by 1.5.3.  $\square$

In the remainder of this section we will prove the main lemma.

In order to do this, we introduce a new theory CL\*.

1.5.18 Definition

We define the following language  $L_H^*$ .

Alphabet: The alphabet for  $L_H^*$ , with as extra symbol "\_".

Terms are inductively defined by

1), 2), 3) same clauses as in definition 1.5.4.

4) If  $S(M,N,L)$  is a term, then  $\underline{S(M,N,L)}$  is a term.

Formulas are defined in the same way as in 1.5.4.

1.5.19 Definition (see appendix)

$CL^*$  is a theory in  $L_H^*$  defined by the same axioms and rules as  $CL^*$  except that

I 3 is replaced by

I 3'.  $SMNL \geq_1 \underline{S(M,N,L)}$

and there are the additional rules:

$$IV\ 5. \quad \frac{M \geq_1 M' \quad N \geq_1 N'}{\underline{S(M,N,L)} \geq_1 \underline{S(M',N',L)}} \quad , \quad \frac{N \geq_1 N'}{\underline{S(M,N,L)} \geq_1 \underline{S(M,N',L)}}$$

$$\frac{L \geq_1 L'}{\underline{S(M,N,L)} \geq_1 \underline{S(M,N,L')}}$$

1.5.20 Definition

We define inductively two mappings  $\underline{\phantom{x}}$  and  $|\dots| : CL^* \rightarrow CL^*$  as follows:

$$\underline{\phantom{x}}(c) = c \quad \text{if } c \text{ is a variable or constant}$$

$$\underline{\phantom{x}}(MN) = \underline{\phantom{x}}(M)\underline{\phantom{x}}(N)$$

$$\underline{\phantom{x}}(S(M,N,L)) = \underline{\phantom{x}}(M)\underline{\phantom{x}}(L)(\underline{\phantom{x}}(N)\underline{\phantom{x}}(L))$$

$$\underline{\phantom{x}}(\underline{S(M,N,L)}) = S(\underline{\phantom{x}}(M), \underline{\phantom{x}}(N), \underline{\phantom{x}}(L))$$

$$|c| = c \quad \text{if } c \text{ is a variable or constant}$$

$$|MN| = |M| |N|$$

$$|S(M,N,L)| = S(|M|, |N|, |L|)$$

$$|\underline{S(M,N,L)}| = |S(M,N,L)|$$

$|M|$  is apart from the underlining the same as  $M$ .

#### 1.5.21 Lemma

1)  $\theta(\underline{\theta(M)}) = \theta(|M|)$  for all terms  $M$  of  $\underline{CL}^*$ .

2)  $\underline{\theta(M)} = \theta(M)$  if  $M$  is a term of  $CL^*$ .

Proof.

Induction on the structure of  $M$ . ☒

#### 1.5.22 Lemma

If  $CL^* \vdash M_1 \succcurlyeq_1 M_2$  or  $CL^* \vdash M_1 \succcurlyeq M_2$ , then there exists a term  $M_2'$  of  $\underline{CL}^*$  such that  $|M_2'| = M_2$  and  $\underline{CL}^* \vdash M_1 \succcurlyeq M_2'$ .

Proof.

Induction on the length of proof of  $M_1 \succcurlyeq_1 M_2$  or  $M_1 \succcurlyeq M_2$ . ☒

#### 1.5.23 Lemma

If  $\underline{CL}^* \vdash M_1 \succcurlyeq_1 M_2$  or  $\underline{CL}^* \vdash M_1 \succcurlyeq M_2$ , then  $CL^* \vdash \underline{\theta(M_1)} \succcurlyeq \underline{\theta(M_2)}$ .

Proof.

Induction on the length of proof of  $M_1 \succcurlyeq_1 M_2$  or  $M_1 \succcurlyeq M_2$ . ☒

Now we are able to prove the main lemma.

#### 1.5.13 Main Lemma.

If  $M \succcurlyeq^* N$ , then there exists a term  $N^*$  such that

$CL^* \vdash M \succcurlyeq N^*$  and  $\theta(N^*) = N$ .

Proof.

Since  $M \succcurlyeq^* N$ , there are terms  $M_1, N_1$  of  $CL^*$  such that

$CL^* \vdash M_1 \succcurlyeq N_1$  and  $\theta(M_1) = M, \theta(N_1) = N$ .

By 1.5.22 there exists a term  $N_1'$  such that  $|N_1'| = N_1$ , and  $\underline{CL}^* \vdash M_1 \geq N_1'$ .

By 1.5.23 it follows that  $\underline{CL}^* \vdash \underline{\theta}(M_1) \geq \underline{\theta}(N_1')$ .

Take  $N^* = \underline{\theta}(N_1')$ .

By 1.5.21 2) we have  $\underline{\theta}(M_1) = \theta(M_1) = M$ , hence

$\underline{CL}^* \vdash M \geq N^*$ .

By 1.5.21 1) we have  $\theta(N^*) = \theta(\underline{\theta}(N_1')) = \theta(|N_1'|) = \theta(N_1) = N$ .  $\square$

Remarks.

The idea of using  $\underline{CL}^*$ ,  $\theta$  and the main lemma is taken from Mitschke [1970]. (He formulated them for the  $\lambda$ -calculus.)

The proof of the main lemma is new.

More extensive use of auxiliary theories like  $\underline{CL}^*$  will be made several times in the sequel.

CHAPTER IIThe  $\omega$ -rule for combinatory logic and  
 $\lambda$ -calculus§2.1. The  $\omega$ -rule.2.1.1 Definition

A term  $M$  of the  $\lambda$ -calculus is called closed if  $FV(M) = \emptyset$ .

A term  $M$  of CL is called closed if no variable occurs in  $M$ .

2.1.2 Definition

We can extend the  $\lambda$ -calculus or CL with the following rule which we call the  $\omega$ -rule

$\omega$ -rule 
$$\frac{MZ = NZ \text{ for all closed } Z}{M = N}$$

We write  $\lambda\omega$  or  $CL\omega$  for the  $\lambda$ -calculus or CL extended with the  $\omega$ -rule.

It is clear that the  $\omega$ -rule implies extensionality. For this reason it does not matter whether we consider  $\lambda\omega$  or  $CL\omega$ , because the  $\lambda$ -calculus and CL are equivalent when we have extensionality (1.4.11). In general we will formulate and prove results about the  $\omega$ -rule in CL. However when it is easier or even necessary we do this in the  $\lambda$ -calculus.

2.1.3 Definition

1. CL is  $\omega$ -consistent if  $CL\omega$  is consistent.
2. CL is  $\omega$ -complete if the  $\omega$ -rule is derivable in  $CL + \text{ext}$  (i.e. if  $CL + \text{ext} \vdash MZ = NZ$  for all closed  $Z \Rightarrow CL + \text{ext} \vdash M = N$ ).

In a personal communication professor Curry suggested the possibility that CL is  $\omega$ -complete.

In this chapter we will prove:

1. CL is  $\omega$ -consistent (§§2.2, 2.3).
2. The  $\omega$ -rule is derivable in  $\lambda + \text{ext}$  for a large class of terms M,N (in fact for all terms which are not universal generators) (§§2.4, 2.5).

It is still an open question whether CL is  $\omega$ -complete <sup>1)</sup>.

As a corollary to the following theorem of Böhm [1968] which we state here without a proof, we can show that the  $\omega$ -rule holds in  $\lambda + \text{ext}$  for terms having a normal form. This was suggested to us by R.Hindley.

2.1.4 Theorem (Böhm [1968]. Cf. Curry,Hindley,Seldin [1971] Ch.11 F.)

Let M,N be closed  $\lambda$ -terms in  $\beta\eta$ -normal form such that  $M \neq_{\alpha} N$ .

Then there exists closed terms  $Z_1, \dots, Z_n$  ( $n \geq 1$ ) such that for variables  $x, y$

$$\lambda \vdash MZ_1 \dots Z_n xy = x \quad \text{and}$$

$$\lambda \vdash MZ_1 \dots Z_n xy = y .$$

2.1.5 Corollary

For closed  $\lambda$ -terms M,N which have a  $\beta$ -normal form the  $\omega$ -rule is provable i.e. if  $\lambda + \text{ext} \vdash MZ = NZ$  for all closed Z, then

$$\lambda + \text{ext} \vdash M = N.$$

-----  
1) However there is a rumour that CL is not  $\omega$ -complete.

Proof.

By 1.2.8 it follows that both  $M, N$  have a  $\beta\eta$ -normal form

hence  $\lambda + \text{ext} \vdash M \geq M_1$ ,  $\lambda + \text{ext} \vdash N \geq N_1$  where  $M_1, N_1$  are closed and in normal forms.

Suppose that  $M_1 \not\equiv_{\alpha} N_1$ .

Then by the theorem of Böhm it follows that there exists closed terms  $Z_1, \dots, Z_n$  such that

$$\lambda \vdash M_1 Z_1 \dots Z_n xy = x \quad \text{and}$$

$$\lambda \vdash N_1 Z_1 \dots Z_n xy = y .$$

From the assumption  $\lambda + \text{ext} \vdash MZ = NZ$  for all closed  $Z$  it follows that  $\lambda + \text{ext} \vdash M_1 Z_1 = N_1 Z_1$ . Hence  $\lambda + \text{ext} \vdash x = M_1 Z_1 \dots Z_n xy = N_1 Z_1 \dots Z_n xy = y$  which contradicts the consistency of  $\lambda + \text{ext}$ .

Hence  $M_1 \equiv_{\alpha} N_1$ , therefore  $\lambda \vdash M_1 = N_1$ , and hence

$$\lambda + \text{ext} \vdash M = N. \quad \square$$

Corollary 2.1.5 will be included in the result of §2.5.

A priori we cannot state a similar result for CL because the theorem of Böhm does not extend to CL:

$$\text{Let } M = S[K(SII)][K(SII)]$$

$$N = K$$

$M$  and  $N$  are closed terms both in normal form. But then the conclusion of 2.1.4 does not hold, because  $CL \vdash MZ_1 = (SII)(SII)$  for all  $Z_1$ , hence  $MZ_1 \dots Z_n$  does not have a normal form for all  $Z_1 \dots Z_n$ .

## §2.2. The $\omega$ -consistency of combinatory logic.

In order to prove the  $\omega$ -consistency of CL we introduce a theory  $CL\omega'$  which is a conservative extension of  $CL\omega$ . In the object language of  $CL\omega'$  itself something like "the length of a proof in  $CL\omega$ " is formulated. Because the  $\omega$ -rule is an infinitary rule, this length can be a transfinite (however countable) ordinal. Ordinals will be denoted by  $\alpha, \beta \dots$  etc.

### 2.2.1 Definition

$CL\omega'$  has the following language: Alphabet =

Alphabet $_{CL}$   $\cup \{\approx_\alpha \mid \alpha \text{ countable}\} \cup \{\sim_\alpha \mid \alpha \text{ countable}\} \cup \{=_\alpha \mid \alpha \text{ countable}\}$ .

The terms are those of CL.

Formulas: If  $M, N$  are terms, then

$M \geq N$ ,  $M = N$ ,  $M \approx_\alpha N$ ,  $M \sim_\alpha N$  and  $M =_\alpha N$  are formulas.

### 2.2.2 Definition

$CL\omega'$  has the following axioms and rules (see appendix).

I Same as in 1.4.2.

- II
1.  $M =_\alpha M$  ,  $M \approx_\alpha M$  ,  $M \sim_\alpha M$
  2.  $M =_\alpha N$  ,  $M \approx_\alpha N$  ,  $M \sim_\alpha N$   
 $\frac{N =_\alpha M}{N \approx_\alpha M}$  ,  $\frac{N \sim_\alpha M}{N \approx_\alpha M}$
  3.  $M =_\alpha N, N =_\alpha L$   
 $\frac{M =_\alpha N, N =_\alpha L}{M =_\alpha L}$
  4.  $M =_\alpha M'$  ,  $M =_\alpha M'$  ,  $M \sim_\alpha M'$  ,  $M \sim_\alpha M'$   
 $\frac{ZM =_\alpha ZM', MZ =_\alpha M'Z}{ZM \sim_\alpha ZM', MZ \sim_\alpha M'Z}$  ,  $\frac{M \sim_\alpha M'}{ZM \sim_\alpha ZM'} , \frac{M \sim_\alpha M'}{MZ \sim_\alpha M'Z}$
  5.  $M =_\alpha M'$  ,  $\alpha \leq \alpha'$  ,  $M =_\alpha M'$   
 $\frac{M =_\alpha M', \alpha \leq \alpha'}{M =_{\alpha'} M}$  ,  $\frac{M =_\alpha M'}{M =_{\alpha'} M}$

III 1.  $M \geq M$

$$2. \frac{M \geq N, N \geq L}{M \geq L}$$

$$3. \frac{M \geq M'}{ZM \geq ZM'}, \frac{M \geq M'}{MZ \geq M'Z}$$

$$4. \frac{M \geq M'}{M \approx_0 M'} ; \quad \frac{M \approx_\alpha M'}{M \sim_\alpha M'}, \frac{M \sim_\alpha M'}{M =_\alpha M'}$$

$$\text{IV } \omega\text{'-rule} \quad \frac{\forall Z \text{ closed } \exists \beta < \alpha \quad MZ =_\beta NZ}{M \approx_\alpha N}$$

In the above  $M, M', N, L, Z$  are arbitrary terms, and  $\alpha, \alpha'$  arbitrary countable ordinals.

The intuitive interpretation of

$M =_\alpha N$  is:  $M = N$  is provable using the  $\omega$ -rule at most  $\alpha$  times.

$M \sim_\alpha N$  is:  $M =_\alpha N$  is provable without use of transitivity.

$M \approx_\alpha N$  is:  $M =_\alpha N$  follows directly from the  $\omega$ -rule (or is provable in CL in case  $\alpha = 0$ ).

### 2.2.3 Lemma

$$\text{CL} \vdash M \geq N \iff \text{CL}\omega \vdash M \geq N \iff \text{CL}\omega' \vdash M \geq N.$$

Proof.

Induction on the length of proof of  $M \geq N$ . ☒

### 2.2.4 Lemma

$$\text{CL}\omega' \vdash M = N \iff \exists \alpha \text{ CL}\omega' \vdash M =_\alpha N.$$

Proof. Trivial ☒

### 2.2.5 Lemma

$$\text{CL}\omega' \vdash M = N \iff \text{CL}\omega' \vdash M = N$$

Proof.

Show by induction (on the length of proof):

1.  $CL\omega \vdash M = N \Rightarrow \exists \alpha \quad CL\omega' \vdash M =_{\alpha} N \quad \text{and}$
2.  $CL\omega' \vdash M =_{\alpha} N \Rightarrow CL\omega \vdash M = N$   
 $CL\omega' \vdash M \sim_{\alpha} N \Rightarrow CL\omega \vdash M = N$   
 $CL\omega' \vdash M \approx_{\alpha} N \Rightarrow CL\omega \vdash M = N$

Then the result follows from 2.2.4. ☒

2.2.3 and 2.2.5 state that  $CL\omega'$  is a conservative extension of  $CL\omega$ .

### 2.2.6 Lemma

$$CL \vdash M = N \iff CL\omega' \vdash M =_0 N$$

Proof.

Show by induction

1.  $CL \vdash M = N \Rightarrow CL\omega' \vdash M =_0 N \quad \text{and}$
  2.  $CL\omega' \vdash M =_0 N \Rightarrow CL \vdash M = N$   
 $CL\omega' \vdash M \sim_0 N \Rightarrow CL \vdash M = N$   
 $CL\omega' \vdash M \approx_0 N \Rightarrow CL \vdash M = N$
- ☒

### 2.2.7 Lemma

$$CL\omega' \vdash M =_{\alpha} N \iff \exists N_1, \dots, N_k \exists \beta_1, \dots, \beta_k \leq \alpha$$

$$CL\omega' \vdash M \sim_{\beta_1} N_1 \sim_{\beta_2} N_2 \sim \dots \sim_{\beta_k} N_k \equiv N$$

Proof.

← Trivial.

⇒ Induction on the length of proof of  $M =_{\alpha} N$ . ☒

### 2.2.8 Lemma

- 1) If  $CL\omega' \vdash M \sim_{\alpha} N$  and  $M, N$  are closed, then  $\exists M', N', Z$  closed,  
 $[CL\omega' \vdash ZM' =_0 M, CL\omega' \vdash ZN' =_0 N \text{ and } CL\omega' \vdash M' \approx_{\alpha} N']$ .
- 2)  $CL\omega' \vdash M \approx_{\alpha} N \quad \alpha \neq 0 \iff \forall Z \text{ closed } \exists \beta < \alpha \quad CL\omega' \vdash MZ =_{\beta} NZ$ .

Proof.

1. Induction on the length of proof of  $M \sim_{\alpha} N$  (as an illustration we give the full proof):

case 1.  $M \sim_{\alpha} N$  is an instance of II 1.

Take  $M' = N' = M (= N)$  and  $Z = I$ .

case 2.  $M \sim_{\alpha} N$  is a direct consequence of  $N \sim_{\alpha} M$ .

By the induction hypothesis there are  $N', M', Z$  closed such that

$CL\omega' \vdash ZN' =_0 N$ ,  $CL\omega' \vdash ZM' = M$  and  $CL\omega' \vdash N' \approx_{\alpha} M'$ .

This is what we had to prove.

case 3.  $M \sim_{\alpha} N$  is  $Z_1 M_1 \sim_{\alpha} Z_1 N_1$  (hence  $Z_1$  is closed) and is a direct consequence of  $M_1 \sim_{\alpha} N_1$ .

By the induction hypothesis there are closed  $M'_1, N'_1, Z_0$  such that

$CL\omega' \vdash Z_0 M'_1 =_0 M_1$ ,  $CL\omega' \vdash Z_0 N'_1 =_0 N_1$  and  $CL\omega' \vdash M'_1 \approx_{\alpha} N'_1$ .

Define  $Z = \lambda^* a. Z_1(Z_0 a)$ , then the conclusion holds for  $M'_1, N'_1, Z$  as follows from 1.4.7.

case 4.  $M \sim_{\alpha} N$  is  $M_1 Z_1 \sim_{\alpha} N_1 Z_1$ . This case is treated analogous to case 3.

case 5.  $M \sim_{\alpha} N$  is a direct consequence of  $M \approx_{\alpha} N$ . Take  $M' = M$ ,  $N' = N$  and  $Z = I$ .

2. Induction on the length of proof of  $M \approx_{\alpha} N$ . □

### 2.2.9 Main Lemma

If  $CL\omega' \vdash ZM \geq K$  and  $CL\omega' \vdash M \approx_{\alpha} N$ , where  $\alpha \neq 0$  and  $M, N$  and  $Z$  are closed, then

$\exists \beta < \alpha [CL\omega' \vdash ZNKK =_{\beta} K]$ .

We will carry out the proof of the main lemma in §2.3.

2.2.10 Notation

$MK_n$  stands for  $M \underbrace{K \dots K}_n$ .

Note that  $K_n$  is not a term, because  $MK \dots K$  stands for  $(\dots((MK)K)\dots K)\dots$ .

2.2.11 Lemma

If  $CL\omega' \vdash M =_\alpha K$  and  $M$  is closed, then

$$(*) \quad \exists n \in \omega \quad CL \vdash MK_{2n} \geq K$$

Proof.

Induction on  $\alpha$ . Because we will make use of a double induction we call the induction hypothesis with respect to this induction the  $\alpha$ -ind.hyp.

case 1.  $\alpha=0$ . Then  $CL\omega' \vdash M =_\alpha K$  implies that  $CL \vdash M = K$  by 2.2.6, hence  $CL \vdash M \geq K$  by 1.5.1. So we can take  $n = 0$ .

case 2.  $\alpha > 0$ . From 2.2.7 it follows that

$$CL\omega' \vdash M =_\alpha K \iff \exists M_1 \dots M_k \exists \beta_1, \dots, \beta_k \leq \alpha$$

$$(**) \quad CL\omega' \vdash M \sim_{\beta_1} M_1 \sim_{\beta_2} M_2 \sim \dots \sim_{\beta_k} M_k \geq K.$$

We can suppose that the  $M_i$ ,  $i=1, \dots, k$  are all closed. Because if they were not, we could substitute some constant for the free variables of the  $M_i$  and then also  $(**)$  would hold.

Now we prove with induction on  $k$  that  $(**) \Rightarrow (*)$ .

The induction hypothesis w.r.t. this induction is called the  $k$ -ind.hyp.

If  $k = 0$  then there is nothing to prove, so suppose that  $k > 0$ .

subcase 2.1.  $\beta_k < \alpha$ .

Then  $CL\omega' \vdash M_{k-1} \sim_{\beta} M_k \geq K$  with  $\beta = \beta_k$

therefore  $CL\omega' \vdash M_{k-1} =_{\beta} K$ .

Hence by the  $\alpha$ -ind.hyp.

$\exists n \in \omega \quad CL \vdash M_{k-1} K_{2n} \geq K$ , because we assumed that  $M_{k-1}$  is closed.

Thus

$\exists n \in \omega \quad CL\omega' \vdash MK_{2n} \sim_{\beta_1} M_1 K_{2n} \sim \dots \sim_{\beta_{k-1}} M_{k-1} K_{2n} \geq K$ ,

hence by the  $k$ -ind.hyp.

$\exists n, n' \in \omega \quad CL \vdash MK_{2n} K_{2n'} \geq K$ , which is

$\exists n, n' \in \omega \quad CL \vdash MK_{2(n+n')} \geq K$ .

subcase 2.2.  $\beta_k = \alpha$ .

Then  $CL\omega' \vdash M_{k-1} \sim_{\alpha} M_k \geq K$ .

By 2.8.1 it follows that there are

$M'_{k-1}, M'_k, Z$  such that

$CL\omega' \vdash ZM'_{k-1} =_0 M_{k-1}$ ,  $CL\omega' \vdash ZM'_k =_0 M_k$  and

$CL\omega' \vdash M'_{k-1} \approx_{\alpha} M'_k$ .

Hence by 2.2.6 and 1.5.1 it follows that

$CL\omega' \vdash ZM'_k \geq K$ .

By the main lemma 2.2.9 it follows that

$\exists \beta < \alpha [CL\omega' \vdash ZM'_{k-1} KK =_{\beta} K]$ , thus

$\exists \beta < \alpha [CL\omega' \vdash M_{k-1} KK =_{\beta} K]$ . Hence by the

$\alpha$ -ind.hyp.  $\exists n \in \omega \quad CL \vdash M_{k-1} KKK_{2n} \geq K$ , thus

$\exists n \in \omega \quad CL\omega' \vdash MKKK_{2n} \sim_{\beta_1} M_1 KKK_{2n} \sim \dots \sim_{\beta_{k-1}} M_{k-1} KKK_{2n} \geq K$ .

Therefore by the  $k$ -ind.hyp. we have

$\exists n, n' \in \omega \quad CL \vdash MKKK_{2n} K_{2n'} \geq K$  i.e.

$\exists n, n' \in \omega \quad CL \vdash MK_{2(n+n'+1)} \geq K$ . □

### 2.2.12 Corollary

If  $CL\omega \vdash M = K$ ,  $M$  is closed, then  $\exists n \in \omega \quad CL \vdash MK_{2n} = K$ .

Proof.

This follows immediately from 2.2.11 by 2.2.4 and 2.2.5.  $\square$

### 2.2.13 Theorem

$CL$  is  $\omega$ -consistent.

Proof.

Suppose  $CL\omega$  were inconsistent, then

$CL\omega \vdash KK = K$ . Therefore by 2.2.12

$\exists n \in \omega \quad CL \vdash KKK_{2n} = K$ . Hence  $CL \vdash KK = K$ .

This contradicts the Church-Rosser theorem for  $CL$ .  $\square$

Theorem 2.2.13 implies in particular that  $\lambda\omega$ ,  $CL + \text{ext}$  and the  $\lambda$ -calculus +  $\text{ext}$  are consistent.

## §2.3. The theory $CL$ .

The most convenient way to carry out the proof of the main lemma 2.2.9, is to develop a new theory  $CL$ .

The intuitive idea behind the proof is the following.

Definition. An occurrence  $M'$  of a subterm of  $M$  is said to be active if this occurrence is in a part  $(M'N)$  of  $M$ , otherwise it is passive.

In the theory  $CL$  we keep track of the occurrences of the residuals of  $M$  in the reduction  $ZM \geq K$  by underlining them. Then we substitute  $N$  for the underlined subterms (this is done by  $\phi_N$  of 2.3.8) and we obtain a reduction of  $ZN$ . When an occurrence of a

residual of  $M$  is active, we omit the underlining, because sub-terms like  $MN$  sometimes have to be evaluated. (This is the essence of axiom VI). This is not in conflict with the substitution of  $N$  for the underlined subterms, because by 2.2.8.2) it follows that if  $M \approx_{\alpha} N$ , active occurrences of  $M$  in the reduction  $ZM \geq K$  can be replaced by  $N$  up to equality of a lower level (i.e.  $\beta < \alpha$ ).

If in the reduction of  $ZM$  to  $K$  it happens that all the residuals of  $M$  are active sooner or later, we are done, because then  $ZN =_{\beta} K$ . In the opposite case  $K$  is a residual of  $M$ , hence  $M \geq K$  and  $ZN =_{\beta} N$ . Therefore

$$ZNKK =_{\beta} NKK =_{\beta'} MKK \geq KKK \geq K, \text{ with } \beta, \beta' < \alpha.$$

### 2.3.1 Definition

CL is a theory defined in the following language:

Alphabet<sub>CL</sub> = Alphabet<sub>CL</sub>,  $\cup \{_, \approx\}$ .

Simple terms are defined inductively by

- 1) Any variable or constant is a simple term.
- 2) If  $M, N$  are simple terms, then  $(MN)$  is a simple term.

Terms are defined inductively by

- 1) Any simple term is a term.
- 2) If  $M$  is a simple term, then  $\underline{M}$  is a term.
- 3) If  $M, N$  are terms, then  $(MN)$  is a term.

Formulas: If  $M, N$  are terms, then

$$M \geq_1 N, M \geq N, M = N \text{ and } M \approx N \text{ are formulas.}$$

Remark: The simple terms of CL are exactly the terms of CL.

2.3.2 Definition

CL has the following axioms and rules (see appendix):

I Same as in 1.5.2.

II Same as in 1.5.2.

III Same as in 1.5.2.

IV Same as in 1.5.2 plus  $\frac{M \geq_1 M'}{\underline{M} \geq_1 \underline{M}'}$

V 1.  $M \approx M$

2.  $\frac{M \approx M'}{M' \approx M}$

3.  $\frac{M \approx N, N \approx L}{M \approx L}$

4.  $\frac{M \approx M'}{ZM \approx ZM'}$  ,  $\frac{M \approx M'}{MZ \approx M'Z}$

5.  $M \approx \underline{M}$

VI  $\underline{MN} \geq_1 MN$ .

In the above the restrictions on the terms are clear.

Remarks: Axiom VI is essential for CL as will become clear later on.

If  $CL \vdash M \approx M'$ , then  $M$  and  $M'$  are, except for the underlining, equal.

2.3.3 Lemma

1)  $CL \vdash M \geq M' \iff CL \vdash M \geq M'$  if  $M, M'$  are simple terms

2)  $CL \vdash M \geq M' \iff CL \vdash \underline{M} \geq \underline{M}'$  if  $M, M'$  are simple terms

3) [ $CL \vdash M \geq_1 M'$  and  $\underline{N}' \text{ sub } M'$ ]  $\Rightarrow \exists N[\underline{N} \text{ sub } M \text{ and } CL \vdash N \geq_1 N']$ .

$N \text{ sub } M$  means that  $N$  is a subterm of  $M$ .

Proof.

- 1) Induction on the length of proof of  $M \geq M'$ .
- 2) Immediate.
- 3) Immediate using 2) and 1). ☒

It follows from 2.3.3 1) that  $\underline{CL}$  is a conservative extension of  $CL$ .

#### 2.3.4 Lemma

$$\underline{CL} \vdash M \geq M' \iff \exists N_1 \dots N_k \quad \underline{CL} \vdash M \equiv N_1 \geq_1 \dots \geq_1 N_k \equiv M' .$$

Proof.

Immediate. ☒

#### 2.3.5 Lemma

If  $\underline{CL} \vdash \underline{ZM} \geq M'$ , where  $Z$  is simple, and  $\underline{N}$  sub  $M'$ , then

$$(*) \quad CL \vdash M \geq N.$$

Proof.

$$\text{By 2.3.4 } \underline{CL} \vdash \underline{ZM} \geq M' \iff \exists N_1 \dots N_k \quad \underline{CL} \vdash \underline{ZM} \equiv N_1 \geq_1 \dots \geq_1 N_k \equiv M' .$$

From lemma 2.3.3 it follows by induction on  $k$  that  $(*)$  holds. ☒

#### 2.3.6 Lemma

Let  $M, M', N$  be terms such that

- 1)  $M$  and  $M'$  are simple,
- 2)  $\underline{CL} \vdash M \geq M'$  and
- 3)  $\underline{CL} \vdash M \approx N$ ,

then there exists a term  $N'$  such that

- 4)  $\underline{CL} \vdash N \geq N'$  and
- 5)  $\underline{CL} \vdash M' \approx N'$  (see figure 7).

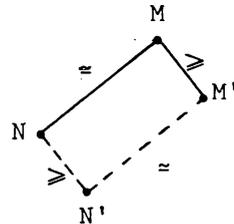


Figure 7

Proof.

Induction on the length of proof of  $M \geq M'$  with the use of axiom VI and the sublemma:

$$\begin{aligned} \underline{CL} \vdash M \approx N \iff & M \equiv N \text{ or } [M \equiv \underline{M}_1 \text{ and } \underline{CL} \vdash M_1 \approx N] \text{ or} \\ & [N \equiv \underline{N}_1 \text{ and } \underline{CL} \vdash M \approx N_1] \text{ or} \\ & [M \equiv \underline{M}_1 \underline{M}_2 \text{ and } N \equiv \underline{N}_1 \underline{N}_2 \text{ and } \underline{CL} \vdash M_1 \approx N_1 \\ & \text{and } \underline{CL} \vdash M_2 \approx N_2]. \quad \square \end{aligned}$$

### 2.3.7 Definition

Let  $A$  be a simple term of  $\underline{CL}$ .

We define a mapping  $\phi_A: \underline{CL} \rightarrow CL$  (in fact from the set of terms of  $\underline{CL}$  onto the set of terms of  $CL$ ).

$$\phi_A(c) = c \quad \text{if } c \text{ is a constant or variable.}$$

$$\phi_A(MN) = \phi_A(M)\phi_A(N)$$

$$\phi_A(\underline{M}) = A$$

### 2.3.8 Lemma

If  $\underline{CL} - \{VI\} \vdash M \geq_1 M'$ , then  $CL \vdash \phi_N(M) \geq \phi_N(M')$  for simple terms  $N$ .

Proof.

Immediate. □

### 2.3.9 Lemma

If  $\underline{CL} \vdash \underline{ZM} \geq M'$ , where  $Z$  is simple,  $Z, M, M'$  are closed and  $CL\omega' \vdash M \approx_\alpha N$ , then  $\exists \beta < \alpha \quad CL\omega' \vdash \phi_N(\underline{ZM}) =_\beta \phi_N(M')$ .

Proof.

Suppose  $\underline{CL} \vdash \underline{ZM} \geq M'$  and  $CL\omega' \vdash M \approx_\alpha N$ .

Then  $\exists N_1 \dots N_k \quad \underline{CL} \vdash \underline{ZM} \equiv N_1 \geq_1 \dots \geq_1 N_k \equiv M'$ .

We claim that

$$(*) \quad \forall i \leq k \quad \exists \beta_i < \alpha \quad CL\omega' \vdash \phi_N(N_i) =_{\beta_i} \phi_N(N_{i+1})$$

We will prove this with induction on the length of proof of

$$N_i \geq_1 N_{i+1}.$$

case 1.  $N_i \geq_1 N_{i+1}$  is an axiom.

subcase 1.1.  $N_i \geq_1 N_{i+1}$  is not an instance of axiom VI.

Then it follows from 2.3.8 and 2.2.6 that

$$CL\omega' \vdash \phi_N(N_i) =_0 \phi_N(N_{i+1}).$$

subcase 1.2.  $N_i \geq_1 N_{i+1}$  is an instance of axiom VI,

$$\text{say } \underline{M}_1 M_2 \geq_1 M_1 M_2.$$

Then we have to show that

$$(**) \quad \exists \beta < \alpha \quad CL\omega' \vdash N\phi_N(M_2) =_{\beta} M_1 \phi_N(M_2)$$

because  $M_1$  is simple and hence  $\phi_N(M_1) \equiv M_1$ .

Since  $\underline{CL} \vdash \underline{ZM} \geq_1 \underline{M}_1 M_2$  it follows from 2.3.5 that  $CL \vdash M \geq_1 M_1$ .

Hence since  $CL\omega' \vdash M \approx_{\alpha} N$  it follows from 2.2.8. 2) and 2.2.6 that

$$\exists \beta < \alpha \quad CL\omega' \vdash N\phi_N(M_2) =_{\beta} M\phi_N(M_2) =_0 M_1 \phi_N(M_2).$$

This implies (\*\*).

case 2.  $N_i \geq_1 N_{i+1}$  is  $ZM_1 \geq_1 ZM_2$  and is a direct consequence of

$M_1 \geq_1 M_2$ . By the induction hypothesis

$$\exists \beta < \alpha \quad CL\omega' \vdash \phi_N(M_1) =_{\beta} \phi_N(M_2) \quad \text{hence}$$

$$CL\omega' \vdash \phi_N(Z)\phi_N(M_1) =_{\beta} \phi_N(Z)\phi_N(M_2) \quad \text{which is}$$

$$CL\omega' \vdash \phi_N(N_i) =_{\beta} \phi_N(N_{i+1}).$$

case 3.  $N_i \geq_1 N_{i+1}$  is  $M_1 Z \geq_1 M_2 Z$ . This case is analogous to case 2.

Now we have established (\*). Let  $\beta = \text{Max}\{\beta_0, \dots, \beta_k\}$  then  $\beta < \alpha$

$$\text{and } CL\omega' \vdash \phi_N(\underline{ZM}) =_{\beta} \phi_N(N_1) =_{\beta} \dots =_{\beta} \phi_N(N_k) =_{\beta} \phi_N(M'). \quad \square$$

Now we are able to prove the main lemma.

### 2.2.9 Main Lemma

If  $CL\omega' \vdash ZM \geq K$  and  $CL\omega' \vdash M \approx_\alpha N$ , where  $\alpha \neq 0$  and  $M, N$  and  $Z$  are closed, then  $\exists \beta < \alpha [CL\omega' \vdash ZNKK =_\beta K]$ .

Proof.

If  $CL\omega' \vdash ZM \geq K$ , then by 2.2.3  $CL \vdash ZM \geq K$ , hence by 2.3.3  $\underline{CL} \vdash ZM \geq K$  and therefore by 2.3.6

$\underline{CL} \vdash \underline{ZM} \geq K'$  with  $\underline{CL} \vdash K' \approx K$  hence  $K' \equiv K$  or  $K' \equiv \underline{K}$ .

case 1.  $K' \equiv K$ .

By 2.3.9 it follows that  $\exists \beta < \alpha \quad CL\omega' \vdash ZN =_\beta K$ , hence  $\exists \beta < \alpha \quad CL\omega' \vdash ZNKK =_\beta KKK =_0 K$ .

case 2.  $K' \equiv \underline{K}$ .

Then  $\underline{CL} \vdash \underline{ZM} \geq \underline{K}$  hence by 2.3.5

(1)  $CL \vdash M \geq K$

Again by 2.3.9 we have  $\exists \beta < \alpha \quad CL\omega' \vdash ZN =_\beta N$ . Hence

(2)  $CL\omega' \vdash ZNKK =_\beta NKK$ .

Because  $CL\omega' \vdash M \approx_\alpha N$  it follows from 2.2.8.2) that

(3)  $\exists \beta' < \alpha \quad CL\omega' \vdash NKK =_{\beta'} MKK$ .

From (1), (2) and (3) it follows that

$$\exists \beta, \beta' < \alpha \quad CL\omega' \vdash ZNKK =_\beta NKK =_{\beta'} MKK \geq KKK \geq K.$$

Hence  $\exists \beta'' < \alpha \quad CL\omega' \vdash ZNKK =_{\beta''} K$ . □

### §2.4. Universal generators.

The motivation of the contents of this § will be stated in 2.4.6.

#### 2.4.1 Definition

The  $\lambda$ -family of a term  $M$  of the  $\lambda$ -calculus, notation  $\mathcal{F}_\lambda(M)$  is the following set of  $\lambda$ -terms

$\mathcal{F}_\lambda(M) = \{N \mid \exists M' \lambda \vdash M \geq M' \text{ and } N \text{ sub } M'\}$ .

Analogously we define  $\mathcal{F}_{\text{CL}}$  and  $\mathcal{F}_{\lambda+\text{ext}}$ .

#### 2.4.2 Definition

U is a universal generator (u.g.) for the  $\lambda$ -calculus if

$\mathcal{F}_\lambda(U)$  consists of all closed  $\lambda$ -terms.

Analogously we define the universal generators for CL.

Remark: If U is a universal generator for the  $\lambda$ -calculus then  $\mathcal{F}_\lambda(U)$  even consists of all  $\lambda$ -terms, since every  $\lambda$ -term M is sub-term of a closed  $\lambda$ -term (take the closure  $\lambda x_1 \dots x_n.M$ ).

#### 2.4.3 Lemma

There exists a closed term E such that

$\forall M [FV(M) = \emptyset \Rightarrow \exists n \lambda \vdash E_n \geq M]$ .

Proof.

This follows from inspection of the proof of 1.3.13.

See for details Barendregt [1970].

#### 2.4.4 Theorem

There exists a closed universal generator for the  $\lambda$ -calculus.

Proof.

We here give a modification of our original construction due to Scott.

Let E be as in lemma 2.4.3, let  $[\dots, \dots]$  be the pairing function as in 1.3.6 and let  $\underline{S}^+$  be the  $\lambda$ -defining term of the

successor function (1.3.4.3)).

Define  $A = \lambda n. [E_n, b(\underline{S}^+n)]$ .

$B = FP A$  (see 1.1.8)

Then  $\lambda \vdash B_n \geq AB_n \geq [E_n, B_{n+1}]$ . Hence

$$\begin{aligned} \lambda \vdash B_0 &\geq [E_0, B_1] \geq [E_0, [E_1, B_2]] \geq \\ &\geq [E_0, [E_1, [E_2, B_3]]] \geq \dots \end{aligned}$$

Because  $E$  enumerates all closed terms,  $B_0$  is a universal generator. Since  $E$  is closed,  $B_0$  is closed too.  $\square$

The above considerations also hold for CL. In particular Kleene's  $E$  (2.4.3) is given for CL by a term  $GD^{-1}$  in Curry, Hindley, Seldin [1971], Ch 13. The name  $GD^{-1}$  is used because it is the inverse of the Gödel numbering.

Therefore we have

#### 2.4.5 Theorem

There exists a universal generator for CL.

#### 2.4.6 Remark

The motivation for the introduction of universal generators is the following:

In the next § we will prove that, if  $M$  and  $N$  are not u.g.'s and if  $\lambda + \text{ext} \vdash MZ = NZ$  for all closed  $Z$ , then  $\lambda + \text{ext} \vdash M = N$ .

At the moment of discovery of this theorem, we were still unaware of the existence of u.g.'s. We hoped to prove that they did not exist in order to obtain, as corollary, the  $\omega$ -completeness of the  $\lambda$ -calculus.

However we subsequently found a proof of the existence of u.g.'s. The proof of the existence of u.g.'s was presented first, because

the above mentioned theorem is easier to prove with their application.

### §2.5. The provability of the $\omega$ -rule for non universal generators.

In this § we will prove a result on partial  $\omega$ -completeness. We present the proof for the  $\lambda$ -calculus because there extensionality can be axiomatized by  $\eta$ -reduction for which the Church-Rosser property holds (1.2.11). We do not know whether a similar result holds for CL, but probably we can prove it using strong reduction.

#### 2.5.1 Definition

A  $\lambda$ -term  $Z$  is said to be of order 0 if there is no term  $P$  such that  $\lambda \vdash Z \geq (\lambda x P)$ .

#### 2.5.2 Lemma

Let  $Z$  be of order 0, then:

- 1) For no term  $P$  we have  $\lambda + \text{ext} \vdash Z \geq \lambda x P$
- 2) If  $\lambda + \text{ext} \vdash Z \geq Z'$ , then  $Z'$  is of order 0
- 3) If  $\lambda + \text{ext} \vdash ZM \geq N$ , then there exist terms  $Z', M'$  such that  $N \equiv Z'M'$ ,  $\lambda + \text{ext} \vdash Z \geq Z'$  and  $\lambda + \text{ext} \vdash M \geq M'$
- 4) For all terms  $M$ ,  $ZM$  is of order 0.

Proof.

For this proof let us call a term of the first kind if it is a variable, of the second kind if it is of the form  $(MN)$  and of the third kind if it is of the form  $(\lambda x M)$ .

- 1) Suppose  $\lambda + \text{ext} \vdash Z \geq \lambda x P$  for some  $P$ . By Curry, Feys [1958] Ch 4D, theorem 2 pg 132 it follows that there exists a term  $Z'$  such that  $\lambda \vdash Z \geq Z'$  and  $\lambda + \text{ext} - I 2 \vdash Z' \geq (\lambda x P)$  (i.e.

without using  $\beta$ -reduction). Because  $Z$  is of order 0  $Z'$  is of the first or of the second kind.  $Z'$  cannot be a variable because  $\lambda + \text{ext} \vdash Z' \geq \lambda x P$ . Hence  $Z'$  is of the second kind. By induction on the length of proof in  $\lambda + \text{ext} - I_2$  of a reduction  $M \geq N$  we can show that if  $M$  is of the second kind, then  $N$  is of the second kind.

This would imply that  $(\lambda x P)$  is of the second kind; a contradiction.

2) Immediate, using 1).

3) By induction on the length of proof of  $ZM \geq N$  using 2).

4) By 3) it follows that if  $\lambda + \text{ext} \vdash ZM \geq N$ , then  $N$  is of the second kind. Hence  $ZM$  is of order 0.  $\square$

### 2.5.3 Examples

1. Any variable is of order 0.

2.  $\Omega_2 = \omega_2 \omega_2$  with  $\omega_2 = (\lambda a.aa)$  is of order 0.

Terms of order 0 behave in some sense like variables. Namely if  $\lambda + \text{ext} \vdash MZ \geq L$  where  $Z$  is of order 0, we can substitute  $x$  for the residuals of  $Z$  in this reduction and we obtain  $\lambda + \text{ext} \vdash Mx \geq L'$ .

Because  $\Omega_2$  is at the same time closed and of order 0 it will play an important role in connection with the  $\omega$ -rule. If  $\lambda + \text{ext} \vdash MZ = NZ$  for all closed  $Z$  we have in particular  $\lambda + \text{ext} \vdash M\Omega_2 = N\Omega_2$ . We hoped that this would imply  $\lambda + \text{ext} \vdash Mx = Nx$ , by substituting everywhere  $x$  for  $\Omega_2$  in the proof. The problem is that there is a difference between variables and terms of order 0. In a reduction variables can never be generated whereas closed terms can.

Therefore we have to find a term of order 0  $Z_0 \in \mathcal{F}_\lambda(M) \cup \mathcal{F}_\lambda(N)$

(see 2.4.1). This is only possible if M and N are not universal generators. Then it follows from  $\lambda + \text{ext} \vdash MZ_0 = NZ_0$  that

$\lambda + \text{ext} \vdash Mx = Nx$  and hence  $\lambda + \text{ext} \vdash M = N$ .

In order to follow the residuals of a subterm in a reduction we again make use of the underlining technic.

An outline of what happens is the following (see fig. 8).

In 2.5.4 - 2.5.14 we define and develop a theory  $\underline{\lambda}$ .

In 2.5.15 - 2.5.16 we consider a mapping  $\phi_x$  which replaces a term of order 0 by a variable x as is mentioned above.

In 2.5.21 - 2.5.24 we define the concept of closed terms which are variable like and prove their existence.

Then, to prove the main result, we assume that  $\lambda + \text{ext} \vdash M\bar{E} = N\bar{E}$ .

It follows from the Church-Rosser theorem that for some term L  $\lambda + \text{ext} \vdash M\bar{E} \geq L$  and  $\lambda + \text{ext} \vdash N\bar{E} \geq L$ . From this it follows by

the results of the theory  $\underline{\lambda}$  that  $\underline{\lambda} + \text{ext} \vdash M\underline{\bar{E}} \geq L'$  and

$\underline{\lambda} + \text{ext} \vdash N\underline{\bar{E}} \geq L''$ . The main difficulty is then to prove that

$L' \equiv L''$ . If we have  $L' \equiv L''$ , then it follows by a homomorphism

argument that  $\lambda + \text{ext} \vdash Mx = \phi_x(L') \equiv \phi_x(L'') = Nx$ .

In order to prove that  $L' \equiv L''$ , we need proposition 2.5.20, a statement about  $\mathcal{F}_{\lambda+\text{ext}}(M\underline{\bar{E}})$  and lemma 2.5.27, a statement about variable like terms.

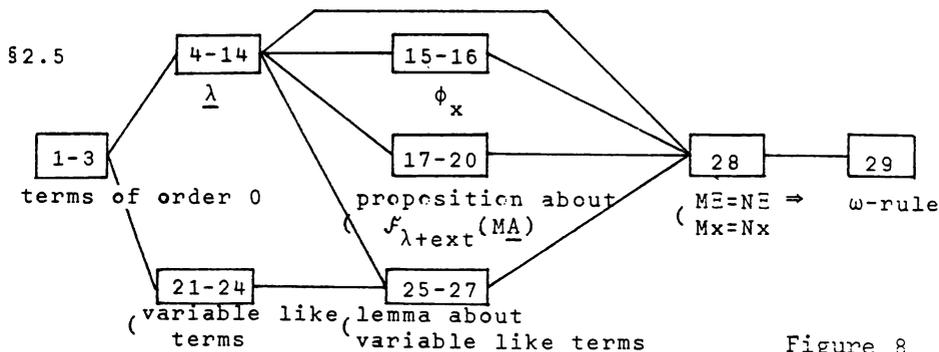


Figure 8

#### 2.5.4 Definition

We will define a theory  $\underline{\lambda} + \text{ext}$  formulated in the following language (see appendix).

Alphabet $_{\underline{\lambda} + \text{ext}}$  = Alphabet $_{\lambda}$   $\cup$   $\{\geq_1, \_, =\}$

Simple terms of the theory  $\underline{\lambda}$  are exactly the terms of the  $\lambda$ -calculus.

Terms are defined inductively by

- 1) Any simple term is a term.
- 2) If  $M$  is a simple term and  $FV(M) = \emptyset$ , then  $\underline{M}$  is a term.
- 3) If  $M, N$  are terms, then  $(MN)$  is a term.
- 4) If  $M$  is a term, then  $(\lambda x M)$  is a term ( $x$  is an arbitrary variable).

Formulas: if  $M, N$  are terms, then

$M \geq_1 N$ ,  $M \geq N$ ,  $M = N$  and  $M \approx N$  are formulas.

A term of the theory  $\underline{\lambda} + \text{ext}$  is called  $\underline{\lambda}$ -term.

The operations  $BV$ ,  $FV$  and  $[x \setminus N]$  can be extended to  $\underline{\lambda}$ -terms in the obvious way.

(Note that:  $BV(\underline{M}) = BV(M)$ ,  $FV(\underline{M}) = \emptyset$  and  $[x \setminus N]\underline{M} = \underline{M}$ .)

#### 2.5.5 Definition

The relation "...is subterm of ..." is defined in such a way that only  $\underline{M}$  is a subterm of  $\underline{M}$ . To be explicit:

Sub( $x$ ) =  $\{x\}$  for any variable  $x$ .

Sub( $MN$ ) = Sub( $M$ )  $\cup$  Sub( $N$ )  $\cup$   $\{MN\}$

Sub( $\lambda x M$ ) = Sub( $M$ )  $\cup$   $\{\lambda x M\}$

Sub( $\underline{M}$ ) =  $\{\underline{M}\}$

$N \text{ sub } M \iff N \in \text{Sub } M$

2.5.6 Definition

We define the theory  $\underline{\lambda} + \text{ext}$  by the following axioms and rules (see appendix).

- I
1.  $\lambda x M \geq_1 \lambda y[x \setminus y]M$  if  $y \notin \text{FV}(M)$
  2.  $(\lambda x M)N \geq_1 [x \setminus N]M$  if  $\text{BV}(M) \cap \text{FV}(N) = \emptyset$
  3.  $\lambda x(Mx) \geq_1 M$  if  $x \notin \text{FV}(M)$
- II Same as in 1.1.4.
- III Same as in 1.2.3.
- IV
1.  $M \geq_1 M$
  2.  $\frac{M \geq_1 M'}{ZM \geq_1 ZM'}$ ,  $\frac{M \geq_1 M'}{MZ \geq_1 M'Z}$ ,  $\frac{M \geq_1 M'}{\lambda x M \geq_1 \lambda x M'}$ ,  $\frac{M \geq_1 M'}{\underline{M} \geq_1 \underline{M}'}$
  3.  $\frac{M \geq_1 M'}{M \geq M'}$
- V
1.  $M \approx M$
  2.  $\frac{M \approx N}{N \approx M}$
  3.  $\frac{M \approx N, N \approx L}{M \approx L}$
  4.  $\frac{M \approx M'}{ZM \approx ZM'}$ ,  $\frac{M \approx M'}{MZ \approx M'Z}$ ,  $\frac{M \approx M'}{\lambda x M \approx \lambda x M'}$
  5.  $M \approx \underline{M}$

In the above the restriction on the terms is clear.

Note that we do not have a counterpart for axiom VI of 2.3.2.

$\underline{\lambda}$  is the theory which results from the above axioms and rules, omitting I 3.

2.5.7 Lemma

- 1)  $\underline{\lambda} + \text{ext} \vdash M \geq M' \iff \lambda + \text{ext} \vdash M \geq M'$  if  $M, M'$  are simple terms
- 2)  $\underline{\lambda} + \text{ext} \vdash M \geq M' \iff \underline{\lambda} + \text{ext} \vdash \underline{M} \geq \underline{M}'$  if  $M, M'$  are simple terms
- 3)  $[\underline{\lambda} + \text{ext} \vdash M \geq_1 M' \text{ and } \underline{N}' \text{ sub } M'] \Rightarrow$   
 $[\exists N \quad \underline{N} \text{ sub } M \text{ and } \lambda + \text{ext} \vdash N \geq_1 N']$ .

Proof.

- 1) Induction on the length of proof of  $M \geq M'$
- 2) Immediate
- 3) Immediate, using 1) and the following sublemma

$$\underline{N} \text{ sub}[x \setminus Q]P \iff \underline{N} \text{ sub } P \text{ or } \underline{N} \text{ sub } Q.$$

The proof of the sublemma proceeds by induction on the structure of P. ☒

#### 2.5.8 Lemma

$$\underline{\lambda} + \text{ext} \vdash M \geq M' \iff \exists N_1 \dots N_k \cdot \underline{\lambda} + \text{ext} \vdash M \equiv N_1 \triangleright_1 \dots \triangleright_k N_k \equiv M'.$$

Proof.

Immediate. ☒

#### 2.5.9 Lemma

$$[\underline{\lambda} + \text{ext} \vdash M \geq M' \text{ and } \underline{N}' \text{ sub } M'] \Rightarrow$$

$$[\exists N \quad \underline{N} \text{ sub } M \text{ and } \lambda + \text{ext} \vdash N \geq N'].$$

Proof.

By 2.5.8

$$\underline{\lambda} + \text{ext} \vdash M \geq M' \iff \exists N_1 \dots N_k \cdot \underline{\lambda} + \text{ext} \vdash M \equiv N_1 \triangleright_1 \dots \triangleright_k N_k \equiv M'.$$

From lemma 2.5.7,3) it follows by induction on k that the conclusion holds. ☒

#### 2.5.10 Lemma

Let M,N be simple terms.

- 1)  $\underline{\lambda} \vdash L \approx \underline{M} \iff [L \equiv M \text{ or } L \equiv \underline{M}]$
- 2)  $\underline{\lambda} \vdash L \approx \lambda x M \iff [\exists M' \quad L \equiv \lambda x M' \text{ and } \underline{\lambda} \vdash M \approx M'] \text{ or } L \equiv \underline{\lambda x M}$
- 3)  $\underline{\lambda} \vdash L \approx MN \iff [\exists M'N' \quad L \equiv M'N' \text{ and } \underline{\lambda} \vdash M \approx M', \underline{\lambda} \vdash N \approx N']$
- 4)  $\underline{\lambda} \vdash M \approx M' \text{ and } \underline{\lambda} \vdash N \approx N' \Rightarrow \underline{\lambda} \vdash [x \setminus N]M \approx [x \setminus N']M'.$

Proof.

1) Induction on the proof of  $L \approx \underline{M}$ , making for the induction hypothesis the statement slightly stronger.

$$[\underline{\lambda} \vdash L \approx \underline{M} \text{ or } \underline{\lambda} \vdash \underline{M} \approx L] \iff ]L \equiv M \text{ or } L \equiv \underline{M}]$$

2) 3) Similarly, making use of 1).

4) Induction on the structure of  $M$ , making use of 1), 2) and 3).  $\square$

### 2.5.11 Lemma

Let  $M, N$  be simple terms such that  $M \geq_1 N$  is an axiom of  $\underline{\lambda} + \text{ext}$  but not an instance of I 2.

Let  $\underline{\lambda} \vdash M \approx M'$ . Then  $\exists N' [\underline{\lambda} + \text{ext} \vdash M' \geq N' \text{ and } \underline{\lambda} \vdash N \approx N']$ .

(See fig. 9.)

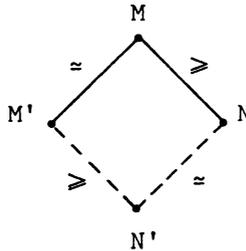


Figure 9

Proof.

By distinguishing cases and using the previous lemma.  $\square$

### 2.5.12 Lemma

Let  $M, N$  be simple terms such that  $M \geq_1 N$  is an instance of axiom I 2. Let  $\underline{\lambda} \vdash M \approx M'$ , where  $M'$  is such that if  $\underline{Z} \text{ sub } M'$ , then  $Z$  is of order 0.

Then  $\exists N' [\underline{\lambda} + \text{ext} \vdash M' \geq N' \text{ and } \underline{\lambda} \vdash N \approx N']$  (see fig. 9).

Proof.

Let  $M \geq N$  be  $(\lambda x P)Q \geq [x \setminus Q]P$ .

As  $\underline{\lambda} \vdash (\lambda x P)Q \approx M'$  we can distinguish by 2.5.10 several cases.

case 1.  $M' \equiv (\lambda x P)Q$  . Take  $N' \equiv [x \setminus Q]P$

case 2.  $M' \equiv M''Q'$  with  $\underline{\lambda} \vdash \lambda x P \approx M''$  and  $\underline{\lambda} \vdash Q \approx Q'$ .

subcase 2.1.  $M'' \equiv \lambda x P'$  with  $\underline{\lambda} \vdash P \approx P'$ .

Take  $N' \equiv [x \setminus Q']P'$ , then the result follows from 2.5.10,4).

subcase 2.2.  $M'' \equiv \lambda x P$ . This case cannot occur, because then  $\lambda x P$  sub  $M'$ , and  $\lambda x P$  is not of order 0, contrary to our assumption.  $\square$

### 2.5.13 Lemma

Let  $M, N$  be simple terms such that  $\underline{\lambda} + \text{ext} \vdash M \geq N$ .

Let  $\underline{\lambda} \vdash M \approx M'$ , where  $M'$  is such that if  $\underline{Z}$  sub  $M'$ , then  $Z$  is of order 0. Then  $\exists N'[\underline{\lambda} + \text{ext} \vdash M' \geq N', \underline{\lambda} \vdash N \approx N'$  and  $[\underline{Z}$  sub  $N' \Rightarrow Z$  is of order 0]] (see fig. 9).

Proof.

Induction on the length of proof of  $M \geq N$ .

If  $M \geq N$  is an axiom we are done by 2.5.11 or 2.5.12, since by lemma's 2.5.9 and 2.5.2.2) it follows from the assumptions that  $Z$  is of order 0 if  $\underline{Z}$  sub  $N'$ . (We need this fact for the induction step in rule III 3 (transitivity).)  $\square$

### 2.5.14 Proposition

Let  $\underline{\lambda} + \text{ext} \vdash MZ \geq L$ , where  $M, Z$  and  $L$  are simple and  $Z$  is of order 0. Then  $\exists L'[\underline{\lambda} + \text{ext} \vdash M\underline{Z} \geq L', \underline{\lambda} \vdash L \approx L'$  and  $[\underline{Z}'$  sub  $L' \Rightarrow \underline{\lambda} + \text{ext} \vdash Z \geq Z']]$  (see fig. 10).

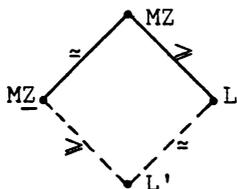


Figure 10

Proof.

This follows immediately from lemma 2.5.13 and 2.5.9.  $\square$

### 2.5.15 Definition

Let  $x$  be any variable. We define a mapping  $\phi_x: \underline{\lambda} \rightarrow \lambda$  (i.e. from the set of  $\underline{\lambda}$ -terms into the set of  $\lambda$ -terms) as follows:

$$\begin{aligned}\phi_x(y) &= y \\ \phi_x(MN) &= \phi_x(M)\phi_x(N) \\ \phi_x(\lambda y M) &= \lambda y \phi_x(M) \\ \phi_x(\underline{M}) &= x\end{aligned}$$

### 2.5.16 Lemma

If  $\underline{\lambda} + \text{ext} \vdash M \geq N$  and if  $x$  is a variable not occurring in this proof, then  $\lambda + \text{ext} \vdash \phi_x(M) \geq \phi_x(N)$ .

Proof.

Induction on the length of proof of  $M \geq N$ , using the following sublemma

$$\text{If } z \neq x, \text{ then } \phi_z([x \setminus N]M) = [x \setminus \phi_z(N)]\phi_z(M).$$

The proof of the sublemma proceeds by induction on the structure of  $M$ .  $\square$

### 2.5.17 Lemma

Let  $M, N$  be simple and  $x \notin \text{FV}(M)$ .

If  $\lambda + \text{ext} \vdash Mx \geq N$ , then  $\exists M'$  simple [ $x \notin \text{FV}(M')$ ],  $\lambda + \text{ext} \vdash M \geq M'$  and  $\underline{\lambda} + \text{ext} \vdash M'x \geq_1 N$ .

Proof.

Because  $\lambda + \text{ext} \vdash Mx \geq N$  we have by 2.5.7.1) and 2.5.8 that

$$\exists N_1 \dots N_k \quad \underline{\lambda} + \text{ext} \vdash Mx \equiv N_1 \geq_1 \dots \geq_1 N_k \equiv N.$$

If all  $N_i$ ,  $i < k$  are of the form  $Px$  with  $x \notin \text{FV}(P)$ , then we are

done.

Otherwise let  $N_{i+1}$  be the first term not of the form  $Px$  with  $x \notin FV(P)$ .

Then  $N_i$  is of the form  $(\lambda z N'_i)x$ .

By  $\alpha$ -reduction this reduces to  $(\lambda x [z \setminus x] N'_i)x$  which is  $(\lambda x N_{i+1})x$ .

Hence  $\underline{\lambda} + \text{ext} \vdash Mx \geq (\lambda z N'_i)x \geq (\lambda x N_{i+1})x \geq (\lambda x N)x \geq_1 N$ .

So we can take  $M' \equiv \lambda x N$ . □

### 2.5.18 Lemma

1) Let  $L, L'$  be  $\underline{\lambda}$ -terms such that  $\phi_x(L) = \phi_x(L')$  where  $x \notin FV(LL')$ . Let  $Z$  be a simple term such that  $Z \text{ sub } L$ . Then  $Z \text{ sub } L'$ .

2) If  $M, N$  are simple terms, then we have

$$\phi_x([x \setminus \underline{N}]M) = \phi_x(M) = M.$$

Proof.

Induction on the structure of  $L$  resp.  $M$ . □

### 2.5.19 Lemma

Let  $\underline{\lambda} + \text{ext} \vdash M\underline{A} \geq L$  and  $Z \text{ sub } L$ , where  $M, A$  and  $Z$  are simple and  $A$  is closed. Let  $L$  satisfy:  $\underline{A}' \text{ sub } L \Rightarrow \underline{A}' \equiv \underline{A}$ .

Then  $Z \in \mathcal{J}_{\lambda+\text{ext}}^f(M)$ .

Proof.

By lemma 2.5.16 we have  $\lambda + \text{ext} \vdash Mx \geq \phi_x(L)$  where  $x$  does not occur in the proof of  $M\underline{A} \geq L$ . Hence by lemma 2.5.17 there exists a simple term  $M'$  such that  $\lambda + \text{ext} \vdash M \geq M'$  and  $\underline{\lambda} + \text{ext} \vdash M'x \geq_1 \phi_x(L)$ . Hence  $\underline{\lambda} + \text{ext} \vdash M'\underline{A} \geq_1 L$ .

Suppose now  $Z \text{ sub } L$  and  $Z$  simple. By distinguishing the different possibilities for the proof of  $M'\underline{A} \geq_1 L$  we can then

show that  $Z \text{ sub } M'$  hence  $Z \in \mathcal{F}_{\lambda+\text{ext}}(M)$ . ☒

### 2.5.20 Proposition

Let  $\underline{\lambda} + \text{ext} \vdash \underline{MA} \geq L$  and  $Z \text{ sub } L$ , where  $M, A$  and  $Z$  are simple and  $A$  is closed.

Then  $Z \in \mathcal{F}_{\lambda+\text{ext}}(M)$ .

Proof.

Let  $x$  not occur in the proof of  $\underline{\lambda} + \text{ext} \vdash \underline{MA} \geq L$ .

Then by 2.5.16 we have  $\lambda + \text{ext} \vdash Mx \geq \phi_x(L)$ . Hence

$\underline{\lambda} + \text{ext} \vdash \underline{MA} \geq [x \setminus A] \phi_x(L) \equiv L'$ , say. By 2.5.18 it follows that  $Z \text{ sub } L \iff Z \text{ sub } L'$  for simple terms  $Z$ . Furthermore,  $L'$  satisfies the assumptions of 2.5.19.

Hence if  $Z \text{ sub } L$  and  $Z$  is simple, then  $Z \text{ sub } L'$  and therefore  $Z \in \mathcal{F}_{\lambda+\text{ext}}(M)$  by 2.5.19. ☒

### 2.5.21 Definition

- 1) A term  $M$  is called an  $\Omega_2$ -term if  $M$  is of the form  $\Omega_2 M'$ .
- 2) A subterm occurrence  $Z$  of  $M$  is called non- $\Omega_2$  in  $M$  if  $Z$  has no  $\Omega_2$  subterm and  $Z$  is not a subterm of an  $\Omega_2$  subterm of  $M$ .
- 3) A term  $U$  is called a heriditorily non- $\Omega_2$  universal generator if  $U$  is a closed u.g. and if  $\lambda + \text{ext} \vdash U \geq U'$ , then there is a subterm occurrence  $Z$  of  $U'$  which is a u.g. and which occurs non- $\Omega_2$  in  $U'$ .

Example: Only the second occurrence of  $Z$  in the term  $x(\Omega_2(MZ))Z$  is non- $\Omega_2$  (if  $Z$  does not have an  $\Omega_2$  subterm).

### 2.5.22 Lemma

If  $U$  is a heriditorily non- $\Omega_2$  u.g. and if  $\lambda + \text{ext} \vdash U \geq U'$ , then  $U'$  is a u.g. which is not an  $\Omega_2$ -term.

Proof.

By definition it follows that some subterm  $Z$  of  $U'$  is a u.g. Then  $U'$  itself is a u.g. That  $U'$  is not an  $\Omega_2$ -term follows from the fact that  $Z$  occurs non- $\Omega_2$  in  $U'$ .  $\square$

### 2.5.23 Proposition

There exists a heriditorily non- $\Omega_2$  universal generator.

Proof.

We introduce ordered triples as follows

$$[M, N, L] = \lambda z. zMNL.$$

Define  $A = \lambda bn[b\underline{0}, E\underline{n}, b(S^+n)]$ , where  $E$  and  $S^+$  are as in 2.4.4,

$$B = FPA \quad \text{and} \quad U = B\underline{0}.$$

We will prove that  $U$  is a heriditorily non- $\Omega_2$  u.g.

As in 2.4.4 we see that  $U$  is a closed u.g.:

$$\lambda \vdash U \equiv B\underline{0} \geq AB\underline{0} \geq [B\underline{0}, E\underline{0}, B\underline{1}] \geq [B\underline{0}, E\underline{0}, [B\underline{0}, E\underline{1}, B\underline{2}]] \geq \dots$$

Let us define  $U \geq_k U'$  to mean

$$\exists N_1 \dots N_k \quad \lambda + \text{ext} \vdash U \equiv N_1 \geq_1 \dots \geq_1 N_k \equiv U'.$$

(Here we need  $\underline{\lambda}$  only to express one step reduction  $\geq_1$ .)

Suppose now that  $\lambda + \text{ext} \vdash U \geq U'$ . Then for some  $k$  we have

$$U \geq_k U'.$$

With induction on  $k$  we can show that  $U'$  is of the form

$$1) A^P B\underline{0} \quad (\text{remember that } M^P N = \underbrace{M(\dots(M(MN))\dots)}_{p \text{ times}})$$

or 2)  $A^P(\lambda n[U'', \dots, \dots])\underline{0}$  where

$$B \geq_{k'}, \lambda n[U'', \dots, \dots] \quad \text{and} \quad U \geq_{k''} U'' \quad \text{with } k', k'' < k,$$

or 3)  $[U''', \dots, \dots]$  where  $U \geq_{k'} U'''$  with  $k' < k$ .

Now we prove with induction on  $k$  that if  $U \geq_k U'$ , then there exists a subterm occurrence  $Z_{U'}$  of  $U'$  which is a u.g. and

is non- $\Omega_2$  in  $U'$ :

If  $U'$  is of the form (1) we take  $Z_{U'} \equiv A^P B \underline{0}$ .

If  $U'$  is of the form (2) we take  $Z_{U'} \equiv Z_{U''}$ .

Finally if  $U'$  is of the form (3) we take  $Z_{U'} \equiv Z_{U'''}$ . □

#### 2.5.24 Definition

A term  $\Xi$  is called variable like if  $\Xi \equiv \Omega_2 U$ , where  $U$  is a hereditarily non- $\Omega_2$  universal generator.

#### 2.5.25 Definition

Let  $L, L'$  be  $\underline{\lambda}$ -terms such that  $L$  is simple and  $\underline{\lambda} \vdash L \approx L'$ . Then  $L$  and  $L'$  are equal except for the underlining and we can give the following informal definitions:

- 1) If  $Z'$  is a subterm occurrence of  $L'$ , then there is a unique subterm occurrence  $Z$  of  $L$  which corresponds to  $Z'$ , such that  $\underline{\lambda} \vdash Z \approx Z'$ .

Instead of giving a formal definition we illustrate this concept with an example.

Let  $L \equiv S(KS)(SKK)$  and  $L' \equiv S(\underline{KS})(\underline{SKK})$ , then  $\underline{\lambda} \vdash L \approx L'$ .

$S$  corresponds to  $S$ ,  $KS$  corresponds to  $\underline{KS}$  and  $(SKK)$  corresponds to  $(\underline{SKK})$ .

- 2) Let  $L''$  be another  $\underline{\lambda}$ -term with  $\underline{\lambda} \vdash L \approx L''$ . Then we say that  $L''$  has more line than  $L'$ , notation  $L' \subset L''$ , if for all subterm occurrences  $\underline{Z}'$  of  $L'$  there is a subterm occurrence  $\underline{Z}''$  of  $L''$  such that  $\underline{Z}' \text{ sub } \underline{Z}''$  where  $\underline{Z}', \underline{Z}''$  are the subterm occurrences of  $L$  corresponding to  $\underline{Z}', \underline{Z}''$  respectively.

For example, let  $L'' \equiv S(\underline{KS})(\underline{SKK})$  then  $L' \subset L''$  where  $L'$  is as in the above example.

3) Let  $Z$  be a subterm occurrence of  $L$ .

$Z$  is exactly underlined in  $L'$  if  $\underline{Z}$  is a subterm occurrence of  $L'$  and  $Z$  corresponds to  $\underline{Z}$ .

4) Let  $Z$  be a subterm occurrence of  $L$ .

$Z$  is underlined in  $L'$  if  $Z$  is a subterm of  $Z_1$  (sub  $L$ ) which is exactly underlined in  $L'$ .

For instance the first occurrence of  $K$  in  $L$  of the above example is underlined in  $L'$ .

5) Let  $Z$  be a subterm occurrence of  $L$ .

$Z$  has some line in  $L'$  if  $Z$  is underlined in  $L'$  or if there is a subterm occurrence  $Z_1$  of  $Z$  which is exactly underlined in  $L'$ .

For instance  $SKK$  sub  $L$  has some line in  $L'$  in the above example.

#### 2.5.26 Lemma

Let  $L, L', L''$  be  $\underline{\lambda}$ -terms such that  $L$  is simple and  $\underline{\lambda} \vdash L' \approx L \approx L''$

- 1) If  $L' \subset L''$  and  $L'' \subset L'$ , then  $L' \equiv L''$ .
- 2) If for all subterm occurrences  $\underline{Z}$  of  $L'$ , the corresponding subterm occurrence  $Z$  of  $L$  is underlined in  $L''$ , then  $L' \subset L''$ .
- 3) If  $Z$  is a subterm occurrence of  $L$  such that there is no corresponding subterm occurrence  $Z'$  of  $L'$  which is simple, then  $Z$  has some line in  $L'$ .

Proof.

This is clear from the definitions. ☒

2.5.27 Lemma

Let  $L, L'$  be  $\underline{\lambda}$ -terms such that  $L$  is simple and  $\underline{\lambda} \vdash L = L'$ . Let  $\Xi$  be a variable like  $\lambda$ -term.

Suppose that

- 1) If  $Z$  is a subterm occurrence of  $L$  which is exactly underlined in  $L'$ , then  $Z$  is an  $\Omega_2$ -term.
- 2) If  $Z$  is a subterm occurrence of  $L$  which is a u.g. then  $Z$  has some line in  $L'$ .

Suppose further that  $\lambda + \text{ext} \vdash \Xi \geq \Xi'$  and  $\Xi'$  is a subterm occurrence of  $L$ .

Then  $\Xi'$  is underlined in  $L'$ .

Proof.

$\Xi$  is variable like, hence  $\Xi \equiv \Omega_2 U$ , where  $U$  is a heriditarily non- $\Omega_2$  universal generator.

Since  $\Omega_2$  is of order 0 it follows from 2.5.2.3) that  $\Xi' \equiv \Omega_2 U'$ , where  $\lambda + \text{ext} \vdash U \geq U'$ .

Since  $U$  is a heriditarily non- $\Omega_2$  u.g. there is a subterm occurrence  $Z$  of  $U'$  which is a u.g. and a non- $\Omega_2$  subterm occurrence of  $U'$  (see fig.11). By our assumption 2),  $Z$  has some line in  $L'$ . The possibility that some subterm occurrence  $Z_1$  of  $Z$  is exactly underlined in  $L'$  is excluded, since by 1) then  $Z_1$  would be an  $\Omega_2$ -term whereas  $Z$  is a non- $\Omega_2$  subterm occurrence of  $L$ . Therefore  $Z$  is underlined in  $L'$ , i.e. there is a subterm occurrence  $Z_2$  of  $L$  which corresponds to  $\underline{Z_2}$  sub  $L'$  and such that  $Z \text{ sub } Z_2$ .

We claim that  $\Omega_2 U' \text{ sub } Z_2$  (see fig.11).

First note that, since  $\underline{Z_2} \text{ sub } L'$ , it follows from 1) that  $Z_2$  is an

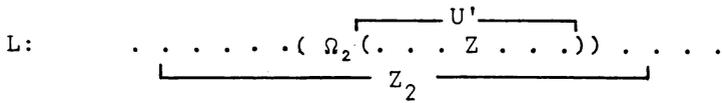


Figure 11

$\Omega_2$  term.

Hence since  $Z$  is a non- $\Omega_2$  subterm occurrence of  $U'$ ,  $Z_2$  is not a subterm of  $U'$ . Therefore  $U'$  sub  $Z_2$ , since subterms are either disjoint or comparable with respect to the relation sub.

Since by 2.5.22  $U'$  is not an  $\Omega_2$ -term  $U'$  is a proper subterm of  $Z_2$ .

Hence indeed  $\Omega_2 U'$  sub  $Z_2$ .

Therefore  $\Xi' \equiv \Omega_2 U'$  is underlined in  $L'$ . □

### 2.5.28 Theorem

Let  $M, N$  be  $\lambda$ -terms which are not universal generators and let  $E$  be a variable like  $\lambda$ -term.

If  $\lambda + \text{ext} \vdash ME = NE$ , then  $\lambda + \text{ext} \vdash Mx = Nx$  for some variable  $x \notin \text{FV}(MN)$ .

Proof.

It follows from the Church-Rosser theorem 1.2.11 and the assumption  $\lambda + \text{ext} \vdash ME = NE$ , that there exists a term  $L$  such that  $\lambda + \text{ext} \vdash ME \geq L$  and  $\lambda + \text{ext} \vdash NE \geq L$ .

Since  $\Xi \equiv \Omega_2 U$  it follows from 2.5.3 and 2.5.2.4) that  $\Xi$  is of order 0. Hence from 2.5.14 it follows that there are terms  $L', L''$  such that  $\underline{\lambda} + \text{ext} \vdash M\underline{\Xi} \geq L'$ ,  $\underline{\lambda} + \text{ext} \vdash N\underline{\Xi} \geq L''$  and  $\underline{\lambda} \vdash L' \approx L \approx L''$  (see fig. 12).

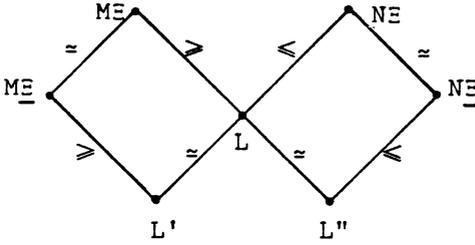


Figure 12

Now we claim that  $L' \equiv L''$ .

In order to prove this, it is sufficient to show that  $L' \subset L''$ , since by symmetry argument then also  $L'' \subset L'$  and hence by 2.5.26.1)  $L' \equiv L''$ .

We will show that for every subterm occurrence  $\underline{Z}'$  of  $L'$ ,  $Z'$  is underlined in  $L''$ , where  $Z'$  is the subterm occurrence of  $L$  corresponding to  $\underline{Z}'$ . Then it follows by 2.5.26.2) that  $L' \subset L''$ . Suppose therefore that  $\underline{Z}'$  is a subterm occurrence of  $L'$ . By 2.5.14 it follows that  $\lambda + \text{ext} \vdash \Xi \geq Z'$ .

We verify the conditions 1) and 2) of 2.5.27 for  $L, L''$ .

- 1) If  $Z$  is a subterm occurrence of  $L$  which is exactly underlined in  $L''$ , then  $\underline{Z} \text{ sub } L''$ , hence it follows by 2.5.14 that  $\lambda + \text{ext} \vdash \Xi \geq Z$ , hence  $Z$  is an  $\Omega_2$ -term.
- 2) If  $Z$  is a subterm occurrence of  $L$  which is a u.g. then  $Z \notin \mathcal{J}_{\lambda+\text{ext}}^{(N)}$  (otherwise  $N$  would be a u.g.). Hence by 2.5.20  $Z$  is not the corresponding subterm occurrence of a simple subterm of  $L''$ .

Therefore  $Z$  has some line in  $L'$ , by 2.5.26.3).

Therefore it follows from 2.5.27 that  $Z'$  is underlined in  $L''$ .

Hence we have proved that  $L' \equiv L''$ .

Let  $x$  be a variable not occurring in the reductions represented in fig. 12. Then it follows from 2.5.16 that

$$\lambda + \text{ext} \vdash Mx = \phi_x(M) \geq \phi_x(L')$$

$$\lambda + \text{ext} \vdash Nx = \phi_x(N) \geq \phi_x(L'').$$

Hence  $\lambda + \text{ext} \vdash Mx = Nx$  since  $\phi_x(L') \equiv \phi_x(L'')$ .  $\square$

Remark. We also have

Let  $M, N$  be  $\lambda$ -terms which are not u.g.'s and let  $E$  be a variable like  $\lambda$ -term.

If  $\lambda \vdash ME = NE$ , then  $\lambda \vdash Mx = Nx$  for some variable  $x \notin \text{FV}(MN)$ .

### 2.5.29 Theorem

Let  $M, N$  be  $\lambda$ -terms which are not universal generators. Then the  $\omega$ -rule for  $M$  and  $N$  is derivable in the  $\lambda$ -calculus with extensionality.

Proof.

Suppose that  $\lambda + \text{ext} \vdash MZ = NZ$  for all closed  $Z$ .

Then  $\lambda + \text{ext} \vdash ME = NE$ , for variable like terms  $E$ , since they are closed.

Hence by 2.5.28 it follows that  $\lambda + \text{ext} \vdash Mx = Nx$  for some variable  $x \notin \text{FV}(MN)$ .

Therefore, by extensionality,  $\lambda + \text{ext} \vdash M = N$ .  $\square$

### Chapter III

#### Consistency results and term models

##### §3.1. Modeltheoretic notions for combinatory logic and some of its extensions.

A combinatory structure is an algebraical structure for a reduct of the language of CL, in which we drop the relation  $\geq$ .  $=$  is always interpreted as the real equality.

A combinatory structure is called trivial if its domain consists of a single element.

A combinatory model is a non trivial combinatory structure

$\mathcal{C} = \langle C, i, k, s, \cdot \rangle$  such that  $i \cdot x = x$ ,  $k \cdot x \cdot y = x$  and  $s \cdot x \cdot y \cdot z = x \cdot z \cdot (y \cdot z)$  for all  $x, y, z \in C$  (as usual we associate to the left).

A combinatory structure  $\mathcal{C}$  assigns homomorphically to each closed CL-term  $M$  an element of  $C$  which we will denote by  $\vDash_{\mathcal{C}}(M)$ .

If  $\mathcal{C}$  is a combinatory model, its interior  $\mathcal{C}^0$  is by definition the restriction of  $\mathcal{C}$  to

$C^0 = \{x \in C \mid x = \vDash_{\mathcal{C}}(M) \text{ for some closed CL-term } M\}$ .

A combinatory model is called hard if it coincides with its own interior.

A combinatory model is an extensional model if it satisfies the axiom of extensionality i.e. if  $\forall x, y \in C [\forall z \in C (x \cdot z = y \cdot z) \Rightarrow x = y]$ .

Note that the axiom of extensionality cannot be expressed in CL, since CL has no logical connectives.

A combinatory model  $\mathcal{C}$  is an  $\omega$ -model if it satisfies the axiom corresponding to the  $\omega$ -rule, i.e. if

$$\forall x, y \in C [\forall z \in C^0 (x \cdot z = y \cdot z) \Rightarrow x = y] .$$

It is clear that an  $\omega$ -model is extensional.

From the completeness of predicate logic it follows that for every consistent extension of CL we can define a canonical model. Since the language of CL is logic free, this model is a particularly simple one, namely a term model.

### 3.1.1 Definition

Let  $\&$  be a consistent extension of CL (in the same language).

The term model of  $\&$  consists of all CL-terms (closed and open) where terms that are provably equal in  $\&$  are equated and application is defined as juxta position.

Hence the term model consists of the set of terms with the minimal equality which satisfies  $\&$ . The non-triviality of the term model follows by the consistency of  $\&$ .

An extensional model or an  $\omega$ -model can be obtained as term model of CL + ext resp. CL $\omega$ .

The restriction of an  $\omega$ -model to its interior is again an  $\omega$ -model. But the restriction of an extensional model to its interior is not necessarily extensional.

The notion of  $\omega$ -completeness should be distinguished from a stronger one. Let us call an consistent extension  $\&$  of CL strongly  $\omega$ -complete, if all extensional models of  $\&$  are in fact  $\omega$ -models. Strong  $\omega$ -completeness implies  $\omega$ -completeness, as follows by considering the term model of  $\&$  + ext.

The converse is not necessarily true.

The rumour mentioned in the footnote on page 49 was not quite

justified. Apparently Jacopini [1971] has proved that CL is not strongly  $\omega$ -complete. Hence the question of the  $\omega$ -completeness of CL still remains open.

### 3.1.2 Theorem (Grzegorzcyk [1971])

There is no recursive model for CL.

Proof.

If  $\mathcal{C}$  would be a recursive model, then  $\text{Th}(\mathcal{C})$  would form a consistent recursive extension of CL, contradicting 1.3.17.  $\square$

Since it is not clear how to interpret the  $\lambda$ -operation in a model we restricted ourselves to models of combinatory logic. It is nevertheless possible to define  $\lambda$ -abstraction in a model. This is done in the later versions of Scott [1970].

### §3.2. Term models

In this § we will answer negatively the question whether Scott's lattice theoretic method provides us with all extensional models for CL. We do this by equating all the unsolvable CL-terms and obtain an extensional term model in which there is only one fixed-point (an element  $x$  such that for all  $y$   $x \cdot y = x$  is called a fixed-point). In Scott's models there are at least two fixed-points.

Further it is shown that Con, the set of equations that can be added consistently to the  $\lambda$ -calculus, is complete  $\Pi_1^0$  (after Gödelisation). This is not an immediate consequence of the fact that  $\lambda$ -calculus is a complete  $\Sigma_1^0$  theory, since there is no negation. Unsolvable terms will play the role of negation. Finally we construct a term model for CL which cannot be embedded into nor mapped onto an extensional model.

### 3.2.1 Definition

1. A CL-term M is called CL-solvable if  $\exists N_1 \dots N_k \quad \text{CL} \vdash MN_1 \dots N_k = K$ .
2. A  $\lambda$ -term M is called  $\lambda$ -solvable if  $\exists N_1 \dots N_k \quad MN_1 \dots N_k$  has a  $\beta$ -normal form.
3. A  $\lambda$ -term M is called  $\lambda\eta$ -solvable if  $\exists N_1 \dots N_k \quad MN_1 \dots N_k$  has a  $\beta\eta$ -normal form.

In 3.2.20 $\frac{1}{2}$  we will give an alternative characterization of solvable terms.

### 3.2.2 Lemma

Let Z be a  $\lambda$ -term. Then Z is  $\lambda$ -solvable iff Z is  $\lambda\eta$ -solvable.

Proof.

By the remark following 1.2.8. □

### 3.2.3 Theorem

- 1) If  $\text{CL} \vdash ZM = K$ , then M is CL-solvable or  $\text{CL} \vdash Zx = K$  for any variable x.
- 2) If ZM has a  $\beta\eta$ -normal form N, then M is  $\lambda\eta$ -solvable or  $\lambda + \text{ext} \vdash Zx = N$  for any variable x.

The proof which, in the CL case, makes use of an auxiliary theory CL' similar to CL is postponed until §3.3.

### 3.2.4 Definition (Morris)

Let  $M, M'$  be  $\lambda$ -terms.

$M \subset M'$  if  $\forall Z [ZM \text{ has a } \beta\eta\text{-normal form} \Rightarrow \lambda + \text{ext} \vdash ZM = ZM']$ .

In his thesis Morris [1968] proved

### 3.2.5 Theorem

If  $\lambda + \text{ext} \vdash MA = A$ , then  $\text{FP } M \subset A$ .

Hence  $\text{FP } M$  is the minimal fixedpoint (in the sense of  $\subset$ ).

### 3.2.6 Theorem

If  $M$  is an unsolvable  $\lambda$ -term (i.e. if  $M$  is not  $\lambda$ -solvable) then  $M \subset M'$  for all terms  $M'$ . Hence  $M$  is minimal (in the sense of  $\subset$ ) in the set of all terms.

Proof.

This follows immediately from 3.2.3 2) and lemma 3.2.2.

As many fixedpoints are unsolvable, e.g.  $\text{FP } K$ ,  $\text{FP } S$  etc.,

3.2.6 is for those terms a sharpening of 3.2.5.

### 3.2.7 Definition

$\mathcal{K}_{\text{CL}} = \{M = M' \mid M, M' \text{ are unsolvable CL-terms}\}$

$\mathcal{K}_{\lambda} = \{M = M' \mid M, M' \text{ are unsolvable } \lambda\text{-terms}\}$ .

We will prove now that  $\text{CL} + \mathcal{K}_{\text{CL}}$  is consistent. The most convenient way to prove this is to develop a theory  $\text{CL}^+$  which is a conservative extension of  $\text{CL} + \mathcal{K}_{\text{CL}}$ .  $\text{CL}^+$  will play the same role as  $\text{CL}\omega'$  in the consistency proof of  $\text{CL} + \omega$ -rule.



3.2.10 Lemma

$$1) \text{ CL } \vdash M \geq N \iff \text{ CL } + \mathcal{H}_{\text{CL}} \vdash M \geq N \iff \text{ CL}^+ \vdash M \geq N.$$

$$2) \text{ CL } + \mathcal{H}_{\text{CL}} \vdash M = N \iff \text{ CL}^+ \vdash M = N.$$

Hence  $\text{CL}^+$  is a conservative extension of  $\text{CL} + \mathcal{H}_{\text{CL}}$ .

3.2.11 Lemma

$$\text{CL}^+ \vdash M = N \iff \exists N_1 \dots N_k \text{ CL}^+ \vdash M \sim N_1 \sim \dots \sim N_k \equiv N.$$

3.2.12 Lemma

If  $\text{CL}^+ \vdash M \sim N$ , then  $\exists M', N', Z$

[ $\text{CL}^+ \vdash ZM' \geq M$ ,  $\text{CL}^+ \vdash ZN' \geq N$  and  $\text{CL}^+ \vdash M' \approx N'$ ].

3.2.13 Lemma

If  $\text{CL}^+ \vdash M \approx N$ , then either 1)  $M, N$  are unsolvable or

2)  $\text{CL}^+ \vdash M \geq N$  or  $\text{CL}^+ \vdash N \geq M$ .

3.2.14 Theorem

If  $\text{CL}^+ \vdash M = K$ , then

$$(*) \quad \text{CL} \vdash M = K.$$

Proof.

From 3.2.11 it follows that  $\text{CL}^+ \vdash M = K$

$$(**) \quad \exists N_1 \dots N_k \text{ CL}^+ \vdash M \sim N_1 \sim \dots \sim N_k \geq K.$$

By induction on  $k$  we prove that  $(**) \Rightarrow (*)$ .

If  $k = 0$ , then there is nothing to prove. So suppose that  $k > 0$ .

Since  $\text{CL}^+ \vdash M_{k-1} \sim M_k$  it follows by 3.2.12 that  $\exists M'_{k-1}, M'_k, Z$

such that

$$\text{CL}^+ \vdash ZM'_{k-1} \geq M_{k-1}, \text{CL}^+ \vdash ZM'_k \geq M_k \text{ and } \text{CL}^+ \vdash M'_{k-1} \approx M'_k.$$

By 3.2.13 we can distinguish the following cases:

case 1.  $M'_{k-1}, M'_k$  are unsolvable.

then, since  $CL^+ \vdash ZM'_k \geq M'_k \equiv K$  it follows from 3.2.10 and 3.2.3 1) that

$CL \vdash Zx = K$  for any  $x$ .

Hence  $CL \vdash ZM'_{k-1} = K$  and therefore

$CL^+ \vdash M'_{k-1} \geq K$  as follows from

$CL^+ \vdash ZM'_{k-1} \geq M'_{k-1}$ , 3.2.10 and 1.5.1.

Thus  $CL^+ \vdash M \sim M_1 \sim \dots \sim M'_{k-1} \geq K$ , hence by the induction hypothesis

$CL \vdash M = K$ .

case 2.  $CL^+ \vdash M'_{k-1} \geq M'_k$  or  $CL^+ \vdash M'_k \geq M'_{k-1}$ .

In both cases we have

$CL \vdash M'_{k-1} = ZM'_{k-1} = ZM'_k = M'_k = K$  by 3.2.10.

Hence  $CL^+ \vdash M'_{k-1} \geq K$  by 1.5.1 and 3.2.10.

Then  $CL \vdash M = K$  follows as above from the induction hypothesis.  $\square$

### 3.2.15 Corollary

$CL + \mathcal{H}_{CL}$  is consistent.

Proof.

Suppose that  $CL + \mathcal{H}_{CL} \vdash KK = K$ , then by 3.2.10  $CL^+ \vdash KK = K$  and hence by 3.2.14  $CL \vdash KK = K$ . This contradicts the Church-Rosser theorem for  $CL$ .  $\square$

### 3.2.16 Remark

In the same way we can prove  $\text{Con}(\lambda + \mathcal{H}_\lambda)$ . With a more involved argument we can show that  $\text{Con}(CL + \mathcal{H}_{CL} + \text{ext})$  and  $\text{Con}(\lambda + \mathcal{H}_\lambda + \text{ext})$ . The idea is to use the language of  $CL\omega'$  (where  $\approx_\alpha, \sim_\alpha, =_\alpha$  are only used for finite  $\alpha$ ) and to add the rule

$$\frac{Mx =_n Nx, \quad x \in \text{FV}(MN)}{M \approx_{n+1} N}$$

The consistency of  $\text{CL} + \mathcal{K}_{\text{CL}} + \text{ext}$  is not automatically a consequence of the consistency of  $\text{CL} + \mathcal{K}_{\text{CL}}$ , as will be seen in 3.2.24 where it is shown that  $\text{Con}(\text{CL} + \mathcal{M}) \Rightarrow \text{Con}(\text{CL} + \mathcal{M} + \text{ext})$ , where  $\mathcal{M}$  is a set of equations, does not hold in general.

### 3.2.17 Remark

Let us call an element  $x$  of a combinatory model a fixed-point if  $xy = x$  holds for all  $y$  in the model.

In every combinatory model  $\mathcal{C}$ ,  $\mathfrak{K}_{\mathcal{C}}(\text{FP } K)$  is a fixed-point. Since fixed-points in a term model correspond to unsolvable terms, it is clear that in the term model of  $\text{CL} + \mathcal{K}_{\text{CL}}(+ \text{ext})$  there is only one fixed-point.

In Scott's lattice theoretic models there are always at least two fixed-points: all the elements of the initial lattice  $D_0$  become fixed-points in the limit  $D_\infty$ . See Scott [1970], p.41 theorem 2.14.

Hence Scott's method does not provide us with all models for  $\text{CL} + \text{ext}$ . This was suggested to us by professor Gross. His suggestion had inspired us to prove 3.2.15.

With the help of  $\text{Con}(\lambda + \mathcal{K}_\lambda)$  we can classify in the Kleene-Mostowski hierarchy the set of equations which can be added consistently to the  $\lambda$ -calculus.

### 3.2.18 Lemma

$K = \text{FP } K$  is not consistent with  $\text{CL}$  or the  $\lambda$ -calculus.

Proof (for CL).

Let  $Z = \text{FP } K$ , then  $\text{CL} \vdash Z = KZ$ .

Hence  $\text{CL} + K = Z \vdash K = Z = KZ = KK$ , and therefore for arbitrary

$M, N$   $\text{CL} + K = Z \vdash KIMN = KKIMN$ , thus  $\text{CL} + K = Z \vdash N = M$ .  $\square$

### 3.2.19 Theorem

$\text{Con}_\lambda = \{\ulcorner M = N \urcorner \mid \text{Con}(\lambda + M = N)\}$  is a complete  $\Pi_1^0$  set.

(As in §1.3.  $\ulcorner \dots \urcorner$  denotes the Gödel number of ... in some Gödelisation.)

Proof.

$\ulcorner M = N \urcorner \notin \text{Con}_\lambda \iff \lambda + M = N \vdash K = KK$ .

Hence the complement of  $\text{Con}_\lambda$  is r.e., therefore  $\text{Con}$  is  $\Pi_1^0$ .

To prove that  $\text{Con}_\lambda$  is complete  $\Pi_1^0$ , let  $X$  be an arbitrary  $\Pi_1^0$  set. Let  $Y = \omega - X$ , the complement of  $X$ . Then  $Y$  is r.e.

Define  $f(n) = \begin{cases} 0 & \text{if } n \in Y \\ \uparrow & \text{else} \end{cases}$

then  $f$  is partial recursive.

Hence there is a  $\lambda$ -term  $F$  which strongly defines  $f$ . Note that if

$f(n) \uparrow$ , then by 1.3.3  $F\underline{n}$  is unsolvable. Then

$n \in Y \iff f(n) = 0 \iff \lambda \vdash F\underline{n} = \underline{0} \Rightarrow \lambda \vdash F\underline{n}IK = K \Rightarrow$

$\ulcorner F\underline{n}IK = \text{FP } K \urcorner \notin \text{Con}_\lambda$  by lemma 3.2.18

$n \notin Y \iff f(n) \uparrow \iff F\underline{n}$  is unsolvable  $\Rightarrow$

$\ulcorner F\underline{n}IK = \text{FP } K \urcorner \in \text{Con}_\lambda$  by 3.2.16 since  $\text{FP } K$  is unsolvable.

Hence

$n \in X \iff n \notin Y \iff \ulcorner F\underline{n}IK = \text{FP } K \urcorner \in \text{Con}_\lambda$ .

Thus  $X \leq_1 \text{Con}_\lambda$  via the function

$h(n) = \ulcorner F\underline{n}IK = \text{FP } K \urcorner$ , i.e.  $\text{Con}_\lambda$  is complete  $\Pi_1^0$ .  $\square$

3.2.20 Remark

The same result holds for  $\text{Con}_{\lambda+\text{ext}}$  and also for  $\text{Con}_{\text{CL}}$  and  $\text{Con}_{\text{CL}+\text{ext}}$ . In the CL-case this is proved using the strong definability (in our sense) in CL of the partial recursive functions. This CL-definability is essentially proved in Curry, Hindley, Seldin [1971], Ch 13 A.

The proof of 3.2.19 suggests the following result (from which 3.2.19 follows more directly).

3.2.20 $\frac{1}{2}$  Theorem

Let  $M$  be a closed term of CL or the  $\lambda$ -calculus.

- 1)  $M$  is CL-  $(\lambda\eta-, \lambda-)$  solvable  $\iff \exists N_1 \dots N_k \quad MN_1 \dots N_k = K$  is provable in  $\text{CL}(\lambda + \text{ext}, \lambda)$ .
- 2)  $M$  is CL-  $(\lambda\eta-, \lambda-)$  solvable  $\iff M = \text{FP } K$  is inconsistent with  $\text{CL}(\lambda + \text{ext}, \lambda)$ .

Proof.

1) i) For CL this is just the definition.

ii)  $\Rightarrow$ : If  $M$  is  $\lambda\eta$ -solvable, then

$\exists N_1 \dots N_k \quad \lambda + \text{ext} \vdash MN_1 \dots N_k = N$ , where  $N$  is in  $\beta\eta$ -normal form.

By a theorem of Böhm [1968] there exists closed terms

$P_1, \dots, P_m, Z_1, \dots, Z_n$  such that for

$N' \equiv [x_1/P_1] \dots [x_m/P_m] N$

(here  $\text{FV}(N) = \{x_1, \dots, x_m\}$ )

we have

$\lambda + \text{ext} \vdash N' Z_1 \dots Z_n xy = x$ .

Define  $N'_i \equiv [x_1/P_1] \dots [x_m/P_m] N_i$ , then

$\lambda + \text{ext} \vdash MN'_1 \dots N'_k = N'$ .

Hence  $\lambda + \text{ext} \vdash MN_1' \dots N_k' Z_1 \dots Z_p KK = K$ .

⇐: By definition.

iii) ⇒: If M is  $\lambda$ -solvable, then by 3.2.2 M is  $\lambda\eta$ -solvable and therefore by ii)  $\exists N_1 \dots N_k \lambda + \text{ext} \vdash MN_1 \dots N_k = K$ . Since the  $\omega$ -rule implies the rule of extensionality, we have

$$\exists N_1 \dots N_k \lambda\omega \vdash MN_1 \dots N_k = K.$$

Hence by the analogue of 2.2.12 (which follows from the CL  $\Rightarrow$   $\lambda$ -calculus translation 1.4.11) for the  $\lambda$ -calculus

$$\exists N_1 \dots N_k \exists n \in \omega \lambda \vdash MN_1 \dots N_k K_{2n} = K$$

⇐: By definition.

2) ⇒: Let M be --- solvable, then by 1)

$$\exists N_1 \dots N_k \text{---} \vdash MN_1 \dots N_k = K.$$

Hence  $\text{---} + M = \text{FP } K \vdash K = MN_1 \dots N_k = \text{FP } K N_1 \dots N_k = \text{FP } K$   
(remember that  $\text{---} \vdash \text{FP } K x = \text{FP } K$ ).

This yields according to 3.2.18 a contradiction.

⇐: If M would be unsolvable, then  $M = \text{FP } K$  were consistent with --- by 3.2.15 and 3.2.16 (FP K is unsolvable). ☒

3.2.21 Definition

Let  $M, N$  be CL-terms.

$M$  and  $N$  are separable if  $\exists Z[CL \vdash ZM = K \text{ and } CL \vdash ZN = KK]$ .

Trivial is

## 3.2.22

$\text{Con}(CL + M = N) \Rightarrow M$  and  $N$  are not separable

but

3.2.23 Theorem

The converse of 3.2.22 is not true.

Proof.

Let  $M = \text{FP } K$  and  $N = K$ . Then, by 3.2.18, not  $\text{Con}(CL + M = N)$ .

But  $M$  and  $N$  are not separable, for suppose  $CL \vdash ZM = K$ , then by 3.2.3  $CL \vdash Zx = K$  since  $M = \text{FP } K$  is unsolvable.

Hence also  $CL \vdash ZN = K$ , which implies that  $CL \vdash ZN = KK$  is impossible. □

The rest of this § is devoted to establishing the following theorem.

3.2.24 Theorem

There is a set  $\mathcal{M}$  of equations such that  $\text{Con}(CL + \mathcal{M})$  but not  $\text{Con}(CL + \mathcal{M} + \text{ext})$ .

If  $\mathcal{M}$  is such a set of equations, then the term model of  $CL + \mathcal{M}$  can neither be embedded in nor mapped homomorphically onto an extensional model.

We will show by a Church-Rosser technique

that  $CL + K = \Omega_2(KI) + KK = \Omega_2(SK)$  is consistent (here we use par abus de langage  $\Omega_2$  as an abbreviation for  $SII(SII)$ ). If we add extensionality however, this theory becomes inconsistent, since  $CL + ext \vdash KI = SK$ .

### 3.2.25 Definition

$CL''$  is a theory with the same language as  $CL'$  (see appendix).

$CL''$  is defined by the following axioms and rules (see appendix)

I Same as in 1.5.2.

II Same as in 1.4.2.

III Same as in 1.4.2.

IV 1.  $M \supseteq_1 M$

2.  $\frac{M \supseteq_1 M', N \supseteq_1 N'}{MN \supseteq_1 M'N'} (!)$

3.  $\frac{M \supseteq_1 M'}{M \supseteq M'}$

V 1.  $\Omega_2'(KI) \supseteq_1 K$  where  $CL'' \vdash \Omega_2 \supseteq \Omega_2'$

2.  $\Omega_2'(SK) \supseteq_1 KK$  where  $CL'' \vdash \Omega_2 \supseteq \Omega_2'$

In the above  $M, M', N, L$  denote arbitrary terms and  $\Omega_2 \equiv SII(SII)$ .

### 3.2.26 Lemma

$CL'' \vdash M = N \iff CL + K = \Omega_2(KI) + KK = \Omega_2(SK) \vdash M = N$ .

Proof.

Show by induction

1.  $CL'' \vdash M \supseteq_1 N \Rightarrow CL + \mathcal{M} \vdash M = N$

$CL'' \vdash M \supseteq N \Rightarrow CL + \mathcal{M} \vdash M = N$

$CL'' \vdash M = N \Rightarrow CL + \mathcal{M} \vdash M = N$

2.  $CL + \mathcal{M} \vdash M = N \Rightarrow CL'' \vdash M = N$

where  $\mathcal{M} = \{K = \Omega_2(KI), KK = \Omega_2(SK)\}$ . □

3.2.27 Lemma

$$CL'' \vdash M \geq N \iff \exists N_1 \dots N_k \quad CL'' \vdash M \geq_1 N_1 \geq_1 \dots \geq_1 N_k \geq N.$$

Proof.

Induction on the length of proof of  $M \geq N$ . ☒

3.2.28 Lemma

If  $CL'' \vdash MN \geq_1 L$ , then

- 1)  $L \equiv M'N'$  and  $CL'' \vdash M \geq_1 M'$ ,  $CL'' \vdash N \geq_1 N'$  or
- 2)  $M \equiv I$  and  $L \equiv N$  or
- 3)  $M \equiv KM_1$  and  $L \equiv M_1$  or
- 4)  $M \equiv SM_1M_2$  and  $L \equiv M_1N(M_2N)$  or
- 5)  $M \equiv \Omega_2'$ , where  $CL'' \vdash \Omega_2 \geq \Omega_2'$ ,  $N \equiv KI$  and  $L \equiv K$  or
- 6)  $M \equiv \Omega_2'$ , where  $CL'' \vdash \Omega_2 \geq \Omega_2'$ ,  $N \equiv SK$  and  $L \equiv KK$ .

Proof.

Induction on the length of proof of  $MN \geq_1 L$ . ☒

3.2.29 Lemma

If  $CL'' \vdash \Omega_2 \geq \Omega_2'$ , then  $CL \vdash \Omega_2' \geq \Omega_2$ , hence  $\Omega_2'$  is not of the form  $I, K, KM, S, SM_1$  or  $SM_1M_2$ .

Proof.

By induction on the length of proof of  $\Omega_2 \geq \Omega_2'$  one shows that  $\Omega_2$  is of the form  $I^n(SII)[I^m(SII)]$ . Hence  $CL \vdash \Omega_2' \geq \Omega_2$ . ☒

3.2.30 Lemma

If  $CL'' \vdash M_1 \geq_1 M_2$  and  $CL'' \vdash M_1 \geq_1 M_3$ , then there exists a term  $M_4$  such that  $CL'' \vdash M_2 \geq_1 M_4$  and  $CL'' \vdash M_3 \geq_1 M_4$ . (See fig.1, p.38.)

Proof.

Induction on the length of proof of  $M_1 \geq_1 M_2$ .

case 1.  $M_1 \supseteq_1 M_2$  is an axiom.

subcase 1.1.  $M_1 \supseteq_1 M_2$  is  $IM \supseteq_1 M$ .

By 3.2.28 and 3.2.29 it follows that either

a)  $M_3 \equiv IM'$  with  $CL'' \vdash M \supseteq_1 M'$ , then we

can take  $M_4 \equiv M'$ , or

b)  $M_3 \equiv M$ , then we can take  $M_4 \equiv M$ .

subcase 1.2,1.3.  $M_1 \supseteq_1 M_2$  is  $KMN \supseteq_1 M$  or  $SMNL \supseteq_1 ML(NL)$ .

Analogous to subcase 1.1.

subcase 1.4.  $M_1 \equiv M_2$ . Then we can take  $M_4 \equiv M_3$ .

subcase 1.5.  $M_1 \supseteq_1 M_2$  is  $\Omega_2'(KI) \supseteq_1 K$ .

By 3.2.28 and 3.2.29 it follows that either

a)  $M_3 \equiv K$ , then we can take  $M_4 \equiv K$ , or

b)  $M_3 \equiv \Omega_2''(KI)$  with  $CL'' \vdash \Omega_2' \supseteq_1 \Omega_2''$

hence we can take  $M_4 \equiv K$ .

subcase 1.6.  $M_1 \supseteq_1 M_2$  is  $\Omega_2'(SK) \supseteq_1 KK$ .

Analogous to subcase 1.5 .

case 2.  $M_1 \supseteq_1 M_2$  is  $MN \supseteq_1 M'N'$  and is a direct consequence of  $CL'' \vdash M \supseteq_1 M'$  and  $CL'' \vdash N \supseteq_1 N'$ .

If  $M_1 \supseteq_1 M_3$  is an axiom, then we are done by case 1.

Otherwise  $M_1 \supseteq_1 M_3$  is  $MN \supseteq_1 M''N''$  and is a direct consequence of  $CL'' \vdash M \supseteq_1 M''$  and  $CL'' \vdash N \supseteq_1 N''$ .

By the induction hypothesis there exist  $M''',N'''$  such that  $CL'' \vdash M' \supseteq_1 M'''$ ,  $CL'' \vdash M'' \supseteq_1 M'''$  and the same for  $N$ .

Hence we can take  $M_4 \equiv M''' N'''$ . □

### 3.2.31 Lemma

If  $CL'' \vdash M_1 \supseteq M_2$  and  $CL'' \vdash M_1 \supseteq M_3$ , then there exists a term  $M_4$  such that  $CL'' \vdash M_3 \supseteq M_4$ .

Proof.

This follows from 3.2.30 and 3.2.27 (trivially, in the same way as 1.5.8 follows from 1.5.7).  $\square$

3.2.32 Theorem (Church-Rosser theorem for  $\text{CL}''$ )

If  $\text{CL}'' \vdash M = N$ , then there exists a term  $Z$  such that  $\text{CL}'' \vdash M \geq Z$  and  $\text{CL}'' \vdash N \geq Z$ .

Proof.

Induction on the length of proof of  $M = N$  (as 1.5.16 follows from 1.5.15).  $\square$

3.2.33 Corollary

- 1)  $\text{CL}''$  is consistent.
- 2)  $\text{CL} + K = \Omega_2(\text{KI}) + \text{KK} = \Omega_2(\text{SK})$  is consistent.
- 3) Conjecture 3.2.24 is false.

Proof.

- 1) If  $\text{CL}'' \vdash K = \text{KK}$ , then there would be a  $Z$  such that  $\text{CL}'' \vdash K \geq Z$  and  $\text{CL}'' \vdash \text{KK} \geq Z$ , a contradiction.
- 2) This follows immediately from 1) and 3.2.26.
- 3) Since  $\text{CL} + \text{ext} \vdash \text{KI} = \text{SK}$ ,  
 $\text{CL} + K = \Omega_2(\text{KI}) + \text{KK} = \Omega_2(\text{SK}) + \text{ext} \vdash K = \Omega_2(\text{KI}) = \Omega_2(\text{KS}) = \text{KK}$ .  
 Hence:  $\text{CL} + \mathcal{M}$  is consistent  $\nrightarrow \text{CL} + \mathcal{M} + \text{ext}$  is consistent.  $\square$

### §3.3. The theory $\text{CL}'$ .

In this § we will prove theorem 3.2.3. We restrict ourselves to

- 1) (the  $\text{CL}$ -case), since the proof of 2) (the  $\lambda$ -case) is similar.

The most convenient way to carry out the proof is to develop an auxiliary theory  $\text{CL}'$  similar to  $\text{CL}$ .

### 3.3.1 Definition

$\underline{CL}'$  is a theory with the same language as  $\underline{CL}$ .

We define a mapping  $|\dots| : \underline{CL}' \rightarrow \underline{CL}$  as follows

$$|c| = c \quad \text{if } c \text{ is a constant or variable}$$

$$|MN| = |M||N|$$

$$|\underline{M}| = M$$

$\underline{CL}'$  is defined by the following axioms and rules (see appendix).

I, II, III, IV and V are as in 2.3.2.

$$\text{VI} \quad \underline{MN} \geq_1 \underline{M} \mid \underline{N}$$

In the above the restrictions on the terms are clear.

Axiom VI is essential for  $\underline{CL}'$ ; compare it with axiom VI for  $\underline{CL}$ .

### 3.3.2 Definition

Let  $M, M'$  be  $\underline{CL}$ -terms.

$M'$  is in the CL-solution of  $M$ , notation  $M \gg M'$  if

$$\exists N_1 \dots N_k \quad \underline{CL} \vdash \underline{MN_1 \dots N_k} \geq M' \quad (k = 0 \text{ is allowed}).$$

Note that  $\gg$  is transitive.  $M$  is  $\underline{CL}$  solvable iff  $M \gg K$ .

Now we proceed as in §2.3. When the proofs are similar to those in §2.3 we omit them.

### 3.3.3 Lemma

$$1) \quad \underline{CL}' \vdash M \geq M' \iff \underline{CL} \vdash M \geq M' \quad \text{if } M, M' \text{ are simple terms}$$

$$2) \quad \underline{CL}' \vdash M \geq M' \iff \underline{CL}' \vdash \underline{M} \geq \underline{M}' \quad \text{if } M, M' \text{ are simple terms}$$

$$3) \quad [\underline{CL}' \vdash M \geq_1 M' \quad \text{and } \underline{N}' \text{ sub } M'] \Rightarrow \exists N[\underline{N} \text{ sub } M \text{ and } N \geq N']$$

### 3.3.4 Lemma

$$\underline{CL}' \vdash M \geq M' \iff \exists N_1 \dots N_k \quad \underline{CL}' \vdash M \equiv N_1 \geq_1 \dots \geq_1 N_k \equiv M' .$$

3.3.5 Lemma

$[\underline{CL}' \vdash M \geq M' \text{ and } \underline{N}' \text{ sub } M'] \Rightarrow \exists N[\underline{N} \text{ sub } M \text{ and } N \geq N']$ .

3.3.6 Lemma

Let  $M, M', N$  be terms such that

1)  $M$  and  $M'$  are simple

2)  $\underline{CL} \vdash M \geq M'$

3)  $\underline{CL}' \vdash M \approx N$ ,

then there exists a term  $N'$  such that

4)  $\underline{CL}' \vdash N \geq N'$

5)  $\underline{CL}' \vdash M' \approx N'$  (see fig. 7, page 60).

3.3.7 Definition

Let  $M$  be a  $\underline{CL}$ -term. An  $x$  substitution of  $M$  is the result of replacing some occurrences of  $x$  in  $M$  by other terms.

3.3.8 Lemma

If  $\underline{CL} \vdash M \geq M'$  and  $N$  is an  $x$  substitution of  $M$ , then there exists a term  $N'$  which is an  $x$  substitution of  $M'$  such that  $\underline{CL} \vdash N \geq N'$ .

Proof.

Induction on the length of proof of  $M \geq M'$  . ☒

3.3.9 Lemma

If  $\underline{CL}' \vdash M \geq_1 M'$ , then  $\underline{CL} \vdash \phi_x(M) \geq M''$ , where  $M''$  is an  $x$  substitution of  $\phi_x(M')$  ( $\phi_x$  is defined in 2.3.7 with  $A \equiv x$ ).

Proof.

Induction on the length of proof of  $M \geq_1 M'$  .

case 1.  $M \geq_1 M'$  is an axiom.

subcase 1.1.  $M \geq_1 M'$  is not an instance of axiom VI 1 or 2.

Then since  $\underline{CL} - \{VI\} = \underline{CL}' - \{VI\}$  it follows from 2.3.8 that  $CL \vdash \phi_x(M) \geq \phi_x(M')$ .

Hence we can take  $M'' \equiv \phi_x(M')$ .

subcase 1.2.  $M \geq_1 M'$  is an instance of axiom VI, say

$$\underline{M}_1 M_2 \geq_1 \underline{M}_1 \mid \underline{M}_2 \mid .$$

Then  $\phi_x(M) \equiv x \phi_x(M_2)$  and  $\phi_x(M') \equiv x$ .

Hence we can take  $M'' \equiv x \phi_x(M_2)$  which is an  $x$  variant of  $\phi_x(M')$ .

case 2.  $M \geq_1 M'$  is  $ZM_1 \geq_1 ZM'_1$  and is a direct consequence of

$$M_1 \geq_1 M'_1 .$$

By the induction hypothesis we have  $CL \vdash \phi_x(M_1) \geq M''_1$ ,

where  $M''_1$  is an  $x$  substitution of  $\phi_x(M'_1)$ .

Therefore

$$CL \vdash \phi_x(M) \equiv \phi_x(Z)\phi_x(M_1) \geq \phi_x(Z)M''_1 .$$

Hence we can take  $M'' \equiv \phi_x(Z)M''_1$

since this is an  $x$  substitution of  $\phi_x(M') \equiv \phi_x(Z)\phi_x(M'_1)$ .

case 3.  $M \geq_1 M'$  is  $M_1 Z \geq_1 M'_1 Z$ . This case is analogous to

case 2. □

### 3.3.10 Lemma

Let  $M, M'$  be  $\underline{CL}'$ -terms such that  $M'$  is simple and  $\underline{CL}' \vdash M \geq M'$ .

Let  $x \notin M'$ .

Then  $CL \vdash \phi_x(M) \geq M'$ .

Proof.

Suppose  $\underline{CL}' \vdash M \geq M'$ , then  $\exists N_1 \dots N_k \underline{CL}' \vdash M \equiv N_1 \geq \dots \geq N_k \equiv M'$ .

Hence  $\exists N_1 \dots N_k \underline{CL}' \vdash M \equiv N_k \geq_1 \dots \geq_1 N_1 \equiv M'$ .

With induction on  $i \leq k$  we will prove  $CL \vdash \phi_x(N_i) \geq M'$ .

If  $i = 1$  we are done.

By lemma 3.3.9 it follows that  $CL \vdash \phi_x(N_{i+1}) \geq N_i'$  where  $N_i'$  is an  $x$  substitution of  $\phi_x(N_i)$ . Since by induction hypothesis  $CL \vdash \phi_x(N_i) \geq M'$  there exists by lemma 3.3.8 a  $M''$  which is an  $x$  variant of  $M'$  such that  $CL \vdash N_i' \geq M''$ . But  $x \notin M'$ , hence  $M'' \equiv M'$ .

Therefore  $CL \vdash \phi_x(N_{i+1}) \geq N_i' \geq M'' \equiv M'$ . □

Now we are able to prove the CL part of theorem 3:2.3.

### 3.2.3 Theorem

If  $CL \vdash ZM = K$ , then  $M$  is CL-solvable or  $CL \vdash Zx = K$  for any variable  $x$ .

Proof.

If  $CL \vdash ZM = K$ , then  $CL \vdash ZM \geq K$  by 1.5.1, hence by 3.3.6  $\underline{CL}' \vdash \underline{ZM} \geq \underline{K}'$  with  $\underline{CL}' \vdash \underline{K}' \equiv K$ , hence  $\underline{K}' \equiv K$  or  $\underline{K}' \equiv \underline{K}$ .

case 1.  $\underline{K}' \equiv K$ . Thus  $\underline{CL}' \vdash \underline{ZM} \geq K$

By 3.3.10 it follows that

$CL \vdash \phi_x(\underline{ZM}) \geq K$  i.e.  $CL \vdash Zx \geq K$ .

case 2.  $\underline{K}' \equiv \underline{K}$ . Thus  $\underline{CL}' \vdash \underline{ZM} \geq \underline{K}$ .

Hence by 3.3.5  $M \geq K$  i.e.  $M$  is CL solvable. □

Appendix I

Survey of the theories used in the text.

This appendix presents a full description of the theories considered.

In order to facilitate the locating of those descriptions we list them here with a reference to the place where they were introduced and their page in the appendix.

Theory	Introduced in	Page
$\lambda$ -calculus (with $\geq$ )	1.2.2	A1
CL	1.4.1	A2
CL'	1.5.2	A3
CL*	1.5.4	A4
<u>CL</u> *	1.5.18	A5
CLw'	2.2.2	A6
<u>CL</u>	2.3.1	A7
$\underline{\lambda}$ + ext	2.5.4	A8
CL <sup>+</sup>	3.2.8	A9
CL''	3.2.25	A10
<u>CL</u> '	3.3.1	A11

A1. The  $\lambda$ -calculusLanguage

Alphabet: a,b,c,... variables  
 $\lambda, (, )$  improper symbols  
 $=$  equality  
 $\geq$  reduction

Terms: Terms are defined inductively by

- 1) Any variable is a term.
- 2) If M,N are terms, then (MN) is a term.
- 3) If M is a term, then ( $\lambda xM$ ) is a term

Formulas: If M,N are terms, then

$M=N$  and  $M \geq N$  are formulas.

To be able to formulate the axioms we define inductively:

The set of free variables of a term:

$$\begin{aligned} FV(x) &= \{x\} \\ FV(MN) &= FV(M) \cup FV(N) \\ FV(\lambda xM) &= FV(M) - \{x\} \end{aligned}$$

The set of bound variables of a term:

$$\begin{aligned} BV(x) &= \emptyset \\ BV(MN) &= BV(M) \cup BV(N) \\ BV(\lambda xM) &= BV(M) \cup \{x\} \end{aligned}$$

Substitution of a term N in the free occurrences of the variable x in M:

$$\begin{aligned} [x/N]x &= N \\ [x/N]y &= y \\ [x/N](M_1M_2) &= [x/N]M_1 ([x/N]M_2) \\ [x/N](\lambda xM) &= \lambda xM \\ [x/N](\lambda yM) &= \lambda y([x/N]M) \end{aligned}$$

In the above x is an arbitrary variable and y is a variable different from x.

Note. In the text  $[x/N]$  sometimes was confused with  $[x \setminus N]$

The  $\lambda$ -calculus (+ extensionality, +  $\omega$ -rule)

Axioms and rules

- I
1.  $\lambda xM \geq \lambda y[x/y]M$  if  $y \notin FV(M)$
  2.  $(\lambda xM)N \geq [x/N]M$  if  $BV(M) \cap FV(N) = \emptyset$ .
- II
1.  $M = M$
  2.  $\frac{M = N}{N = M}$
  3.  $\frac{M = N, N = L}{M = L}$
  4.  $\frac{M = M'}{ZM = ZM'}, \frac{M = M'}{MZ = M'Z}, \frac{M = M'}{\lambda xM = \lambda xM'}$
- III
1.  $M \geq M$
  2.  $\frac{M \geq N, N \geq L}{M \geq L}$
  3.  $\frac{M \geq M'}{ZM \geq ZM'}, \frac{M \geq M'}{MZ \geq M'Z}, \frac{M \geq M'}{\lambda xM \geq \lambda xM'}$
  4.  $\frac{M \geq M'}{M = M'}$

In  $\lambda + \text{ext}$  we add

- I
3.  $\lambda x(Mx) \geq M$  if  $x \notin FV(M)$

In  $\lambda\omega$  we add

$\omega$ -rule

$$\frac{MZ = NZ \text{ for all } Z \text{ with } FV(Z) = \emptyset}{M = N}$$

In the above  $M, M', N, L$  and  $Z$  denote arbitrary terms and  $x$  and  $y$  arbitrary variables.

## A2. Combinatory logic (CL)

Language

Alphabet: a,b,c,... variables  
 I,K,S constants  
 ( , ) improper symbols  
 = equality  
 $\triangleright$  reduction

Terms: Terms are defined inductively by

- 1) Any variable or constant is a term.
- 2) If M,N are terms, then (MN) is a term.

Formulas: If M,N are terms, then

$M = N$  and  $M \triangleright N$  are formulas.

$M_1 M_2 \dots M_n$  stands for  $(\dots(M_1 M_2) \dots M_n)$

CL (+ extensionality, +  $\omega$ -rule)

Axioms and rules

- I
1.  $IM \geq M$
  2.  $IMN \geq M$
  3.  $SMNL \geq ML(NL)$
- II
1.  $M = M$
  2.  $\frac{M = N}{N = M}$
  3.  $\frac{M = N, N = L}{M = L}$
  4.  $\frac{M = M'}{ZM = ZM'}, \frac{M = M'}{MZ = M'Z}$
- III
1.  $M \geq M$
  2.  $\frac{M \geq N, N \geq L}{M \geq L}$
  3.  $\frac{M \geq M'}{ZM \geq ZM'}, \frac{M \geq M'}{MZ \geq M'Z}$
  4.  $\frac{M \geq M'}{M = M'}$

In CL + ext we add

$$\text{ext} \quad \frac{Mx = M'x}{M = M'} \quad \text{if } x \notin \text{FV}(MM')$$

In CL $\omega$  we add

$\omega$ -rule

$$\frac{MZ = NZ \text{ for all } Z \text{ without free variables}}{M = N}$$

In the above M, M', N, L and Z denote arbitrary terms.

A3. CL'

Language

Alphabet: a,b,c,... variables  
 I,K,S constants  
 ( , ) improper symbols  
 = equality  
 $\succ$  reduction  
 $\succ_1$  one step reduction

Terms: Terms are defined inductively by

- 1) Any variable or constant is a term.
- 2) If M,N are terms, then (MN) is a term.

Formulas: If M,N are terms, then

$M = N$ ,  $M \succ N$  and  $M \succ_1 N$  are formulas.

$M_1 M_2 \dots M_n$  stands for  $(\dots(M_1 M_2) \dots M_n)$

CL'

Axioms and rules

- I
1.  $IM \quad M$
  2.  $KMN \quad M$
  3.  $SMNL \quad ML(NL)$
- II
1.  $M = M$
  2.  $\frac{M = N}{N = M}$
  3.  $\frac{M = N, N = L}{M = L}$
  4.  $\frac{M = M'}{ZM = ZM'} , \frac{M = M'}{MZ = M'Z}$
- III
1.  $M \geq M$
  2.  $\frac{M \geq N, N \geq L}{M \geq L}$
  3.  $\frac{M \geq M'}{ZM \geq ZM'} , \frac{M \geq M'}{MZ \geq M'Z}$
  4.  $\frac{M \geq M'}{M = M'}$
- IV
1.  $M \geq_1 M$
  2.  $\frac{M \geq_1 M'}{ZM \geq_1 ZM'} , \frac{M \geq_1 M'}{MZ \geq_1 M'Z}$
  3.  $\frac{M \geq_1 M'}{M \geq_1 M'}$

In the above  $M, M', N, L$  and  $Z$  denote arbitrary terms.

A4. CL\*

Language

Alphabet: a,b,c,... variables  
 I,K,S constants  
 ( , ), , improper symbols  
 = equality  
 $\geq$  reduction  
 $\geq_1$  one step reduction

Terms: Terms are defined inductively by

- 1) Any variable or constant is a term.
- 2) If M,N are terms, then (MN) is a term.
- 3) If M,N and L are terms, then S(M,N,L) is a term.

Formulas: If M,N are terms, then

$M = N$ ,  $M \geq N$  and  $M \geq_1 N$  are formulas.

$M_1 M_2 \dots M_n$  stands for  $(\dots(M_1 M_2) \dots M_n)$

CL\*

Axioms and rules

I - 1.  $IM \supseteq_1 M$

2.  $KMN \supseteq_1 M$

3.  $SMNL \supseteq_1 S(M,N,L)$

II 1.  $M = M$

2.  $\frac{M = N}{N = M}$

3.  $\frac{M = N, N = L}{M = L}$

4.  $\frac{M = M'}{ZM = ZM'} , \frac{M = M'}{MZ = M'Z}$

III 1.  $M \supseteq M$

2.  $\frac{M \supseteq N, N \supseteq L}{M \supseteq L}$

3.  $\frac{M \supseteq M'}{ZM \supseteq ZM'} , \frac{M \supseteq M'}{MZ \supseteq M'Z}$

IV 1.  $M \supseteq_1 M$

2.  $\frac{M \supseteq_1 M'}{ZM \supseteq_1 ZM'} , \frac{M \supseteq_1 M'}{MZ \supseteq_1 M'Z}$

3.  $\frac{M \supseteq_1 M'}{M \supseteq_1 M'}$

4.  $\frac{\frac{M \supseteq_1 M'}{S(M,N,L) \supseteq_1 S(M',N,L)} , \frac{N \supseteq_1 N'}{S(M,N,L) \supseteq_1 S(M,N',L)}}{L \supseteq_1 L'}$   
 $\frac{\quad}{S(M,N,L) \supseteq_1 S(M,N,L')}$

In the above  $M, M', N, N', L, L'$  and  $Z$  denote arbitrary terms.

A5. CL\*Language

Alphabet: a,b,c,... variables  
 I,K,S constants  
 ( , ),',- improper symbols  
 = equality  
 $\geq$  reduction  
 $\geq_1$  one step reduction

Terms: Terms are defined inductively by

- 1) Any variable or constant is a term.
- 2) If M,N are terms, then (MN) is a term.
- 3) If M,N and L are terms, then  
 $S(M,N,L)$  and  $\underline{S(M,N,L)}$  are terms.

Formulas: If M,N are terms, then  
 $M = N$ ,  $M \geq N$  and  $M \geq_1 N$  are formulas.

$M_1 M_2 \dots M_n$  stands for  $(\dots(M_1 M_2) \dots M_n)$

CL\*

Axioms and rules (We give an equivalent version which is slightly different from the original one.)

- I
1.  $IM \supset_1 M$
  2.  $KMN \supset_1 M$
  3.  $SMNL \supset_1 \underline{S(M,N,L)}$
- II
1.  $M = M$
  2.  $\frac{M = N}{N = M}$
  3.  $\frac{M = N, N = L}{M = L}$
  4.  $\frac{M = M'}{ZM = ZM'} , \frac{M = M'}{MZ = M'Z}$
- III
1.  $M \supseteq M$
  2.  $\frac{M \supseteq N, N \supseteq L}{M \supseteq L}$
  3.  $\frac{M \supseteq M'}{ZM \supseteq ZM'} , \frac{M \supseteq M'}{MZ \supseteq M'Z}$
- IV
1.  $M \supseteq_1 M$
  2.  $\frac{M \supseteq_1 M'}{ZM \supseteq_1 ZM'} , \frac{M \supseteq_1 M'}{MZ \supseteq_1 M'Z}$
  3.  $\frac{M \supseteq_1 M'}{M \supseteq_1 M'}$
  4.  $\frac{M \supseteq_1 M'}{\underline{S(M,N,L)} \supseteq_1 \underline{S(M',N,L)}} , \frac{N \supseteq_1 N'}{\underline{S(M,N,L)} \supseteq_1 \underline{S(M,N',L)}}$   
 $\frac{L \supseteq_1 L'}{\underline{S(M,N,L)} \supseteq_1 \underline{S(M,N,L')}}}$
  5.  $\frac{M \supseteq_1 M'}{\underline{M} \supseteq_1 \underline{M'}}$

In the above  $M, M', N, N', L, L'$  and  $Z$  denote arbitrary terms, except in IV 5, where  $M, M'$  denote terms of the form  $S(P, Q, R)$ .

A6. CLw'

Language

Alphabet: a,b,c,... variables  
 I,K,S constants  
 ( , ) improper symbols  
 = equality  
 $\geq$  reduction  
 $\approx_\alpha, \sim_\alpha, =_\alpha$  special equalities, for every countable ordinal  $\alpha$ .

Terms: Terms are defined inductively by  
 1) Any variable or constant is a term.  
 2) If M,N are terms, then (MN) is a term.

Formulas: If M,N are terms, then  
 $M = N, M \geq N, M \approx_\alpha N, M \sim_\alpha N$   
 and  $M =_\alpha N$  are formulas.

$M_1 M_2 \dots M_n$  stands for  $(\dots(M_1 M_2) \dots M_n)$

CL $\omega'$ Axioms and rules

- I
1.  $IM \geq M$
  2.  $KMN \geq M$
  3.  $SMNL \geq ML(NL)$
- II
1.  $M =_{\alpha} M$ ,  $M \approx_{\alpha} M$ ,  $M \sim_{\alpha} M$
  2.  $M =_{\alpha} N$ ,  $M \approx_{\alpha} N$ ,  $M \sim_{\alpha} N$   
 $\frac{N =_{\alpha} M}{M =_{\alpha} M}$ ,  $\frac{N \approx_{\alpha} M}{M \approx_{\alpha} M}$ ,  $\frac{N \sim_{\alpha} M}{M \sim_{\alpha} M}$
  3.  $M =_{\alpha} N, N =_{\alpha} L$   
 $\frac{M =_{\alpha} N, N =_{\alpha} L}{M =_{\alpha} L}$
  4.  $M =_{\alpha} M'$ ,  $M =_{\alpha} M'$ ,  $M \sim_{\alpha} M'$ ,  $M \sim_{\alpha} M'$   
 $\frac{M =_{\alpha} M'}{ZM =_{\alpha} ZM'}$ ,  $\frac{M =_{\alpha} M'}{MZ =_{\alpha} M'Z}$ ,  $\frac{M \sim_{\alpha} M'}{ZM \sim_{\alpha} ZM'}$ ,  $\frac{M \sim_{\alpha} M'}{MZ \sim_{\alpha} M'Z}$
  5.  $M =_{\alpha} M', \alpha \leq \alpha'$ ,  $M =_{\alpha} M'$   
 $\frac{M =_{\alpha} M', \alpha \leq \alpha'}{M =_{\alpha'} M'}$ ,  $\frac{M =_{\alpha} M'}{M = M'}$
- III
1.  $M \geq M$
  2.  $M \geq N, N \geq L$   
 $\frac{M \geq N, N \geq L}{M \geq L}$
  3.  $M \geq M'$ ,  $M \geq M'$   
 $\frac{M \geq M'}{ZM \geq ZM'}$ ,  $\frac{M \geq M'}{MZ \geq M'Z}$
  4.  $M \geq M'$ ,  $M \approx_{\alpha} M'$ ,  $M \sim_{\alpha} M'$   
 $\frac{M \geq M'}{M \approx_0 M'}$ ,  $\frac{M \approx_{\alpha} M'}{M \sim_{\alpha} M'}$ ,  $\frac{M \sim_{\alpha} M'}{M =_{\alpha} M'}$
- IV
- $$\frac{\forall Z \text{ closed } \exists \beta < \alpha \quad MZ =_{\beta} NZ}{M \approx_{\alpha} N}$$

In the above  $M, M', N, L$  and  $Z$  denote arbitrary terms, and  $\alpha, \alpha'$  arbitrary countable ordinals.

A7. CLLanguage

Alphabet:     a,b,c,...     variables  
               I,K,S        constants  
               ( , ), \_     improper symbols  
               =            equality  
                $\geq$          reduction  
                $\geq_1$         one step reduction  
                $\approx$          intrinsic equality

Simple terms: Simple terms are defined inductively by

- 1) Any variable or constant is a simple term.
- 2) If M,N are simple terms, then (MN) is a simple term.

Terms:        Terms are defined inductively by

- 1) Any simple term is a term.
- 2) If M is a simple term, then  $\underline{M}$  is a term.
- 3) If M,N are terms, then (MN) is a term.

Formulas:     If M,N are terms, then  
 $M = N$ ,  $M \geq N$ ,  $M \geq_1 N$  and  $M \approx N$  are  
 formulas.

$M_1 M_2 \dots M_n$  stands for  $(\dots(M_1 M_2) \dots M_n)$

CLAxioms and rules

- I
1.  $IM \supseteq_1 M$
  2.  $KMN \supseteq_1 M$
  3.  $SMNL \supseteq_1 ML(NL)$
- II
1.  $M = M$
  2.  $\frac{M = N}{N = M}$
  3.  $\frac{M = N, N = L}{M = L}$
  4.  $\frac{M = M'}{ZM = ZM'}, \frac{M = M'}{MZ = M'Z}$
- III
1.  $M \supseteq M$
  2.  $\frac{M \supseteq N, N \supseteq L}{M \supseteq L}$
  3.  $\frac{M \supseteq M'}{ZM \supseteq ZM'}, \frac{M \supseteq M'}{MZ \supseteq M'Z}$
  4.  $\frac{M \supseteq M'}{M = M'}$
- IV
1.  $M \supseteq_1 M$
  2.  $\frac{M \supseteq_1 M'}{ZM \supseteq_1 ZM'}, \frac{M \supseteq_1 M'}{MZ \supseteq_1 M'Z}$
  3.  $\frac{M \supseteq_1 M'}{M \supseteq M'}$
  4.  $\frac{M \supseteq_1 M'}{\underline{M} \supseteq_1 \underline{M}'}$
- V
1.  $M \approx M$
  2.  $\frac{M \approx N}{N \approx M}$
  3.  $\frac{M \approx N, N \approx L}{M \approx L}$
  4.  $\frac{M \approx M'}{ZM \approx ZM'}, \frac{M \approx M'}{MZ \approx M'Z}$
  5.  $M \approx \underline{M}$
- VI
1.  $\underline{MN} \supseteq MN$

In the above  $M, M', N, L$  and  $Z$  denote arbitrary terms except in IV 4, V 5 and VI where  $M, M'$  denote simple terms.

A8.  $\lambda$  + extLanguage

Alphabet:  $a, b, c, \dots$  variables

$\lambda, (, ), _$  improper symbols

$=, \approx$  equality, resp. intrinsic equality

$\geq, \geq_1$  reduction, resp. one step reduction

Simple terms: Simple terms are defined inductively by

- 1) Any variable is a simple term.
- 2) If  $M, N$  are terms, then  $(MN)$  is a term.
- 3) If  $M$  is a term, then  $\lambda xM$  is a term  
( $x$  is an arbitrary variable).

The set of free variables of a simple term is inductively defined by

$$\begin{aligned} FV(x) &= \{x\} \\ FV(MN) &= FV(M) \cup FV(N) \\ FV(\lambda xM) &= FV(M) - \{x\} \end{aligned}$$

Terms: Terms are defined inductively by

- 1) Any simple term is a term.
- 2) If  $M$  is a simple term and if  $FV(M) = \emptyset$ , then  $\bar{M}$  is a term.
- 3) If  $M, \bar{N}$  are terms, then  $(MN)$  is a term.
- 4) If  $M$  is a term, then  $\lambda xM$  is a term.

Formulas: If  $M, N$  are terms then

$M = N, M \geq N, M \geq_1 N$  and  $M \approx N$  are formulas.

$FV, BV, [x/N]$  are defined inductively by

$$\begin{aligned} FV(x) &= \{x\} \\ FV(MN) &= FV(M) \cup FV(N) \\ FV(\lambda xM) &= FV(M) - \{x\} \\ FV(\underline{M}) &= \emptyset \\ BV(x) &= \emptyset \\ BV(MN) &= BV(M) \cup BV(N) \\ BV(\lambda xM) &= BV(M) \cup \{x\} \\ BV(\underline{M}) &= BV(M) \\ [x/N]x &= N \\ [x/N]y &= y \\ [x/N](M_1M_2) &= ([x/N]M_1)([x/N]M_2) \\ [x/N](\lambda xM) &= \lambda xM \\ [x/N](\lambda yM) &= \lambda y[x/N]M \\ [x/N]\underline{M} &= \underline{M} \end{aligned}$$

In the above  $x$  is an arbitrary variable and  $y$  is a variable different from  $x$ .

$\lambda$  + ext.

Axioms and rules

- I
1.  $\lambda xM \supseteq_1 \lambda x[x/y]M$  if  $y \notin FV(M)$
  2.  $(\lambda xM)N \supseteq_1 [x/N]M$  if  $BV(M) \cap FV(N) = \emptyset$
  3.  $\lambda x(Mx) \supseteq_1 M$  if  $x \notin FV(M)$ .
- II
1.  $M = M$
  2.  $\frac{M = N}{N = M}$
  3.  $\frac{M = N, N = L}{M = L}$
  4.  $\frac{M = M'}{ZM = ZM'}$ ,  $\frac{M = M'}{MZ = M'Z}$ ,  $\frac{M = M'}{\lambda xM = \lambda xM'}$
- III
1.  $M \supseteq M$
  2.  $\frac{M \supseteq N, N \supseteq L}{M \supseteq L}$
  3.  $\frac{M \supseteq M'}{ZM \supseteq ZM'}$ ,  $\frac{M \supseteq M'}{MZ \supseteq M'Z}$ ,  $\frac{M \supseteq M'}{\lambda xM \supseteq \lambda xM'}$
  4.  $\frac{M \supseteq M'}{M = M'}$
- IV
1.  $M \supseteq_1 M$
  2.  $\frac{M \supseteq_1 M'}{ZM \supseteq_1 ZM'}$ ,  $\frac{M \supseteq_1 M'}{MZ \supseteq_1 M'Z}$ ,  $\frac{M \supseteq_1 M'}{\lambda xM \supseteq_1 \lambda xM'}$ ,  $\frac{M \supseteq_1 M'}{\underline{M} \supseteq_1 \underline{M}'}$
  3.  $\frac{M \supseteq_1 M'}{M \supseteq_1 M'}$
- V
1.  $M \approx M$
  2.  $\frac{M \approx N}{N \approx M}$
  3.  $\frac{M \approx N, N \approx L}{M \approx L}$
  4.  $\frac{M \approx M'}{ZM \approx ZM'}$ ,  $\frac{M \approx M'}{MZ \approx M'Z}$ ,  $\frac{M \approx M'}{\lambda xM \approx \lambda xM'}$
  5.  $M \approx \underline{M}$

In the above  $M, M', N, L$  and  $Z$  denote arbitrary terms except in the last item of IV 2 and in V 5 where  $M, M'$  denote simple terms.

A9. CL<sup>+</sup>Language

Alphabet: a,b,c,...: variables  
 I,K,S constants  
 ( , ) improper symbols  
 = equality  
 ≥ reduction  
 ≈, ~ special equalities

Terms: Terms are defined inductively by  
 1) Any variable or constant is a term.  
 2) If M,N are terms, then (MN) is a term.

Formulas: If M,N are terms, then  
 M = N, M ≥ N, M ≈ N and M ~ N are  
 formulas.

$M_1 M_2 \dots M_n$  stands for  $(\dots(M_1 M_2)\dots M_n)$

CL<sup>+</sup>Axioms and rules

- I
1.  $IM \geq M$
  2.  $KMN \geq M$
  3.  $SMNL \geq ML(NL)$
- II
1.  $M = M$                        $M \approx M$        $M \sim M$
  2.  $\frac{M = N}{N = M}$                        $\frac{M \approx N}{N \approx M}$        $\frac{M \sim N}{N \sim M}$
  3.  $\frac{M = N, N = L}{M = L}$
  4.  $\frac{M = M'}{ZM = ZM'}$  ,  $\frac{M = M'}{MZ = M'Z}$                        $\frac{M \sim M'}{ZM \sim ZM'}$  ,  $\frac{M \sim M'}{MZ \sim M'Z}$
- III
1.  $M \geq M$
  2.  $\frac{M \geq N, N \geq L}{M \geq L}$
  3.  $\frac{M \geq M'}{ZM \geq ZM'}$  ,  $\frac{M \geq M'}{MZ \geq M'Z}$
  4.  $\frac{M \geq M'}{M \approx M'}$                        $\frac{M \approx M'}{M \sim M'}$        $\frac{M \sim M'}{M = M'}$
- IV       $M \approx M'$       if  $M, M'$  are unsolvable terms.

In the above  $M, M', N, L$  and  $Z$  denote arbitrary terms.

A10. CL"

Language

Alphabet: a,b,c,... variables  
 I,K,S constants  
 ( , ) improper symbols  
 = equality  
 $\succ$  reduction  
 $\succ_1$  one step reduction

Terms: Terms are defined inductively by

- 1) Any variable or constant is a term.
- 2) If M,N are terms, then (MN) is a term.

Formulas: If M,N, are terms, then

$M = N$ ,  $M \succ N$  and  $M \succ_1 N$  are formulas.

$M_1 M_2 \dots M_n$  stands for  $(\dots(M_1 M_2) \dots M_n)$

$\Omega_2$  stands for SII(SII).

CL''

Axiomes and rules

- I
1.  $IM \supseteq_1 M$
  2.  $KMN \supseteq_1 M$
  3.  $SMNL \supseteq_1 ML(NL)$
- II
1.  $M = M$
  2.  $\frac{M = N}{N = M}$
  3.  $\frac{M = N, N = L}{M = L}$
  4.  $\frac{M = M'}{ZM = ZM'}, \frac{M = M'}{MZ = M'Z}$
- III
1.  $M \supseteq M$
  2.  $\frac{M \supseteq N, N \supseteq L}{M \supseteq L}$
  3.  $\frac{M \supseteq M'}{ZM \supseteq ZM'}, \frac{M \supseteq M'}{MZ \supseteq M'Z}$
  4.  $\frac{M \supseteq M'}{M = M'}$
- IV
1.  $M \supseteq_1 M'$
  2.  $\frac{M \supseteq_1 M', N \supseteq_1 N'}{MN \supseteq_1 M'N'} \quad (!)$
  3.  $\frac{M \supseteq_1 M'}{M \supseteq_1 M'}$
- V
1.  $\Omega_2'(KI) \supseteq_1 K$  where  $CL'' \vdash \Omega_2 \supseteq \Omega_2'$
  2.  $\Omega_2'(SK) \supseteq_1 KK$  where  $CL'' \vdash \Omega_2 \supseteq \Omega_2'$ .

In the above  $M, M', N, L$  and  $Z$  denote arbitrary terms, and  $\Omega_2 \equiv SII(SII)$ .

A11. CL'Language

Alphabet:    a,b,c,...    variables  
               I,K,S        constants  
               ( , ), \_    improper symbols  
               =            equality  
                $\geq$             reduction  
                $\geq_1$           one step reduction  
                $\approx$             intrinsic equality

Simple terms: Simple terms are defined inductively by

- 1) Any variable or constant is a simple term.
- 2) If M,N are simple terms, then (MN) is a simple term.

Terms:        Terms are defined inductively by

- 1) Any simple term is a term.
- 2) If M is a simple term, then  $\underline{M}$  is a term.
- 3) If M,N are terms, then (MN) is a term.

Formulas:    If M,N are terms, then  
 $M = N$ ,  $M \geq N$ ,  $M \geq_1 N$  and  $M \approx N$  are formulas.

$M_1 M_2 \dots M_n$  stands for  $(\dots(M_1 M_2) \dots M_n)$

To be able to formulate the axioms we define a mapping  $|\dots|$ : Terms  $\rightarrow$  Simple terms.

$|c| = c$         if c is a constant or variable

$|MN| = |M| |N|$

$|\underline{M}| = M$

CL'Axioms and rules

- I
1.  $IM \supseteq_1 M$
  2.  $KMN \supseteq_1 M$
  3.  $SMNL \supseteq_1 ML(NL)$
- II
1.  $M = M$
  2.  $\frac{M = N}{N = M}$
  3.  $\frac{M = N, N = L}{M = L}$
  4.  $\frac{M = M'}{ZM = ZM'}, \frac{M = M'}{MZ = M'Z}$
- III
1.  $M \supseteq M$
  2.  $\frac{M \supseteq N, N \supseteq L}{M \supseteq L}$
  3.  $\frac{M \supseteq M'}{ZM \supseteq ZM'}, \frac{M \supseteq M'}{MZ \supseteq M'Z}$
  4.  $\frac{M \supseteq M'}{M = M'}$
- IV
1.  $M \supseteq_1 M$
  2.  $\frac{M \supseteq_1 M'}{ZM \supseteq_1 ZM'}, \frac{M \supseteq_1 M'}{MZ \supseteq_1 M'Z}$
  3.  $\frac{M \supseteq_1 M'}{M \supseteq_1 M'}$
  4.  $\frac{M \supseteq_1 M'}{\underline{M} \supseteq_1 \underline{M}'}$
- V
1.  $M \approx M$
  2.  $\frac{M \approx N}{N \approx M}$
  3.  $\frac{M \approx N, N \approx L}{M \approx L}$
  4.  $\frac{M \approx M'}{ZM \approx ZM'}, \frac{M \approx M'}{MZ \approx M'Z}$
  5.  $M \approx \underline{M}$
- VI
- $$\underline{MN} \supseteq_1 \underline{M} \mid \underline{N} \mid$$

In the above  $M, M', N, L$  and  $Z$  denote arbitrary terms except in IV 4, V 5 and VI where  $M, M'$  denote simple terms.

Appendix II

The Church-Rosser theorem for the  $\lambda$ -calculus  
à la Martin-Löf

This appendix contains a proof of the Church-Rosser theorem recently discovered by Martin-Löf [1971].

This proof is strikingly simple compared to those mentioned in 1.2.18.

The idea of the proof arose from cut elimination properties of certain formal systems. In fact the Church-Rosser theorem is a kind of cut elimination theorem, the transitivity of  $=$  in the  $\lambda$ -calculus corresponding to the cut.

The trick is to define a relation  $\geq_1$  between terms in such a way that

- 1) The transitive closure of  $\geq_1$  is the (classical) reduction relation ( $\geq$ ).
- 2) If  $M_1 \geq_1 M_2$ ,  $M_1 \geq_1 M_3$ , then there exists a term  $M_4$  such that  $M_2 \geq_1 M_4$  and  $M_3 \geq_1 M_4$ .

From 1) and 2) the analogue of 2) for  $\geq$  can be derived.

From this the Church-Rosser theorem easily follows.

Definition 1.

$\lambda'$  is a theory formulated in the following language:

Alphabet: a,b,c,...	variables
$\lambda, (, )$	improper symbols
$=$	equality
$\geq$	reduction
$\geq_1$	one step reduction

Terms: The terms are defined as in the  $\lambda$ -calculus (1.1.1).

Formulas: If  $M, N$  are terms, then  $M = N$ ,  $M \geq N$  and  $M \geq_1 N$  are formulas.

As in 1.1.2 we define  $BV(M)$ ,  $FV(M)$  and  $[x/N]M$ .

Definition 2.

$\lambda'$  is defined by the following axioms and rules.

- I 1. 
$$\frac{M \geq_1 M'}{(\lambda x M) \geq_1 \lambda y [x/y] M'} \quad \text{if } y \notin FV(M')$$
2. 
$$\frac{M \geq_1 M', N \geq_1 N'}{(\lambda x M) N \geq_1 [x/N'] M'} \quad \text{if } BV(M') \cap FV(N') = \emptyset$$
- II. 1.  $M \geq_1 M$
2. 
$$\frac{M \geq_1 M', N \geq_1 N'}{MN \geq_1 M'N'}$$
3. 
$$\frac{M \geq_1 M'}{\lambda x M \geq_1 \lambda x M'}$$
4. 
$$\frac{M \geq_1 M'}{M \geq M'}$$
- III 1. 
$$\frac{M \geq N, N \geq L}{M \geq L}$$
2. 
$$\frac{M \geq M'}{ZM \geq ZM'}, \frac{M \geq M'}{MZ \geq M'Z}, \frac{M \geq M'}{\lambda x M \geq \lambda x M'}$$
3. 
$$\frac{M \geq M'}{M = M'}$$
- IV 1. 
$$\frac{M = N}{N = M}$$
2. 
$$\frac{M = N, N = L}{M = L}$$
3. 
$$\frac{M = M'}{ZM = ZM'}, \frac{M = M'}{MZ = M'Z}, \frac{M = M'}{\lambda x M = \lambda x M'}$$

In the above  $M, M', N, L$  and  $Z$  denote arbitrary terms and  $x, y$  denote arbitrary variables.

Lemma 3.

$$\lambda' \vdash M \geq N \iff \exists N_1 \dots N_k \quad \lambda' \vdash M \equiv N_1 \geq_1 \dots \geq_1 N_k \equiv N.$$

Proof.

$\Leftarrow$ : Immediate.

$\Rightarrow$ : Induction on the length of proof of  $\lambda' \vdash M \geq N$ . □

Lemma 4.

$$\lambda' \vdash M \geq_1 N \Rightarrow \lambda \vdash M \geq N$$

$$\lambda' \vdash M \geq N \iff \lambda \vdash M \geq N$$

$$\lambda' \vdash M = N \iff \lambda \vdash M = N$$

Proof.

In all cases induction on the length of proof. □

Lemma 5.

If  $\lambda' \vdash M \geq_1 M'$  and  $\lambda' \vdash N \geq_1 N'$ , then

$$\lambda' \vdash [x/N]M \geq_1 [x/N']M' .$$

Proof.

Induction on the length of proof of  $M \geq_1 M'$  using the sublemma:

$$\text{If } x \neq y \text{ then } [x/N_1]([y/N_2]M) \equiv [y/[x/N_1]N_2][x/N_1]M.$$

The proof of the sublemma proceeds by induction on the structure of  $M$ . □

Lemma 6.

1) If  $\lambda' \vdash \lambda x M \geq_1 N$ , then there exists a term  $M'$  such that

$$\lambda' \vdash M \geq_1 M' \quad \text{and} \quad N \equiv \lambda x M' \quad \text{or} \quad N \equiv \lambda y [x/y] M' \quad \text{with } y \notin \text{FV}(M').$$

2) If  $\lambda' \vdash M_1 M_2 \geq_1 N$ , then there exist  $M'_1, M'_2$  such that

$$\lambda' \vdash M_1 \geq_1 M'_1 \quad \text{and} \quad N \equiv M'_1 M'_2 \quad \text{or} \quad M_1 = (\lambda x M'_1) \quad \text{and} \quad N \equiv [x/M'_2] M''_1$$

$$\text{where } \lambda' \vdash M'_1 \geq_1 M''_1 \quad \text{and} \quad \lambda' \vdash M_2 \geq_1 M'_2 .$$

Proof.

Induction on the length of proof. □

Lemma 7.

If  $\lambda' \vdash M_1 \supseteq_1 M_2$  and  $\lambda' \vdash M_1 \supseteq_1 M_3$ , then there exists a term  $M_4$  such that  $\lambda' \vdash M_2 \supseteq_1 M_4$  and  $\lambda' \vdash M_3 \supseteq_1 M_4$ .

Proof.

Induction on the sum of lengths of proof of  $M_1 \supseteq_1 M_2$  and  $M_1 \supseteq_1 M_3$ .

case 1.  $M_1 \supseteq_1 M_2$  is an axiom. Then  $M_1 \equiv M_2$  and we can take

$$M_4 \equiv M_3.$$

case 2.  $M_1 \supseteq_1 M_2$  is  $\lambda xM \supseteq_1 \lambda y[x/y]M'$  where  $y \notin FV(M')$  and is a direct consequence of  $M \supseteq_1 M'$ .

By lemma 6 it follows that

$$M_3 \equiv \lambda y'[x/y']M'' \text{ where } \lambda \vdash M \supseteq_1 M'' \text{ and } y' \notin FV(M'') \text{ or } y' = x.$$

By the induction hypothesis there exists a  $M'''$  such that

$$\lambda' \vdash M' \supseteq_1 M''' \text{ and } \lambda' \vdash M'' \supseteq_1 M'''.$$

$$M_4 \equiv \lambda y''[x/y'']M''' \text{ with } y'' \notin FV(M''').$$

case 3.  $M_1 \supseteq_1 M_2$  is  $\lambda xM \supseteq_1 \lambda xM'$  and is a direct consequence of  $M \supseteq_1 M'$ . Analogously to case 2 we can find the required term  $M_4$ .

case 4.  $M_1 \supseteq_1 M_2$  is  $(\lambda xM)N \supseteq_1 [x/N']M'$  and is a direct consequence of  $M \supseteq_1 M'$ ,  $N \supseteq_1 N'$ .

By lemma 6 we can distinguish the following subcases.

$$\text{subcase 4.1. } M_3 \equiv (\lambda y[x/y]M'')N'', \text{ where } \lambda' \vdash M \supseteq_1 M'', \\ \lambda' \vdash N \supseteq_1 N''.$$

By the induction hypothesis there exist terms  $M'''$ ,  $N'''$  such that  $\lambda' \vdash M' \supseteq_1 M'''$ ;

$\lambda' \vdash M'' \geq_1 M'''$ ,  $\lambda' \vdash N' \geq_1 N'''$  and

$\lambda' \vdash N'' \geq_1 N'''$ .

Then by lemma 5 we can take  $M_4 \equiv [x/N''']M'''$ .

subcase 4.2.  $M_3 \equiv [x/N'']M''$  with  $\lambda' \vdash M \geq_1 M''$ ,

$\lambda' \vdash N \geq_1 N''$ .

By the induction hypothesis there exist terms  $M'''$ ,  $N'''$  such that  $\lambda' \vdash M' \geq_1 M'''$  etc.

Then by lemma 5 we can take  $M_4 \equiv [x/N''']M'''$ .

case 5.  $M_1 \geq_1 M_2$  is  $MN \geq_1 M'N'$  and is a direct consequence of  $M \geq_1 M'$ ,  $N \geq_1 N'$ .

By lemma 6 we can distinguish the following subcases.

subcase 5.1.  $M_3 \equiv M''N''$  with  $\lambda' \vdash M \geq_1 M''$ ,  $\lambda' \vdash N \geq_1 N''$ .

By the induction hypothesis there exist terms  $M'''$ ,  $N'''$  such that  $\lambda' \vdash M' \geq_1 M'''$  etc.

Then we can take  $M_4 \equiv M'''N'''$ .

subcase 5.2.  $M_1 \geq M_3$  is  $(\lambda x M_1)N \geq [x/N']M'_1$  and is a direct consequence of  $M_1 \geq_1 M'_1$ ,  $N \geq_1 N''$ .

This case is analogous to subcase 4.1.  $\square$

### Lemma 8.

If  $\lambda' \vdash M_1 \geq M_2$  and  $\lambda' \vdash M_1 \geq M_3$ , then there exists a term  $M_4$  such that  $\lambda' \vdash M_2 \geq M_4$  and  $\lambda' \vdash M_3 \geq M_4$ .

Proof.

By lemma 3  $\lambda' \vdash M_1 \geq M_2 \iff \exists N_1 \dots N_k \lambda' \vdash M_1 \equiv N_1 \geq_1 \dots \geq_1 N_k \equiv N$  and similarly for  $\lambda' \vdash M_1 \geq M_3$ .

By repeated use of lemma 7 (see figure 2, page 40) it follows that the conclusion holds.  $\square$

Lemma 9.

If  $\lambda' \vdash M = N$ , then there exists a term  $Z$  such that  
 $\lambda' \vdash M \geq Z$  and  $\lambda' \vdash N \geq Z$ .

Proof.

Induction on the length of proof of  $M = N$ , using lemma 8 in the case of transitivity of = □

Theorem 10. (Church-Rosser theorem)

If  $\lambda \vdash M = N$ , then there exists a term  $Z$  such that  
 $\lambda \vdash M \geq Z$  and  $\lambda \vdash N \geq Z$ .

Proof.

This follows immediately from lemma 9 and lemma 4. □

Remark.

In the same way we can prove the Church-Rosser theorem for  $\lambda + \text{ext}$  by adding to  $\lambda'$  the rule

$$\frac{M \geq_1 M'}{\lambda x(Mx) \geq_1 M'}$$

### References

Barendregt, Hendrik P.

[1970] A universal generator for the  $\lambda$ -calculus, underground Utrecht, published in ERCU publications nr.109.

[1971] On the interpretation of terms without a normal form. ERCU publications nr.111. Computer center, University of Utrecht.

Böhm, Corrado

[1968] Alcune proprietà della forme  $\beta$ - $\eta$ -normali del  $\lambda$ -K-calcolo. Pubblicazioni dell'istituto per le applicazioni del calcolo, no.696, Consiglio Nazionale delle Ricerche, Roma.

Church, Alonzo

[1932] A set of postulates for the foundation of logic, Ann. of Math., ser.2, vol.33, pp.346-366.

[1933] A set of postulates for the foundation of logic (second paper), ibidem, ser.2, vol.34, pp.839-864.

[1941] The calculi of lambda-conversion, Annals of mathematics studies, no.6, Princeton.

Church, Alonzo and J.Barkley Rosser

[1936] Some properties of conversion, Trans.Amer.Math.Soc., vol.39, pp.472-482.

Curry, Haskell B.

[1930] Grundlagen der kombinatorischen Logik, Amer.J.Math. vol.52, pp.509-536, 789-834.

[1942] The inconsistency of certain formal logics, J.Symbolic Logic, vol.7, pp.115-117.

[1952] A new proof of the Church-Rosser theorem, Kon.Ned. Akad.Wetensch.Proc.Ser.A, vol.55, pp.16-23 (Indag.Math., vol.14).

Curry, Haskell B. and Robert Feys

[1958] Combinatory Logic, Amsterdam.

Curry, Haskell B., J.Roger Hindley and Jonathan P.Seldin

[1971] Combinatory Logic, vol.2, Amsterdam (in process).

Grzegorzczak, Andrzej

[1970] No existence of recursive models for theory of combinators, underground Warszawa.

Hindley, J.Roger

- [1969] An abstract form of the Church-Rosser theorem I, J.Symbolic Logic, vol.34, pp.545-560.
- [197?] An abstract Church-Rosser theorem, II: Applications, in preparation.

Jacopini, Giuseppe

- [1971] Il principio di estensionalita nell'assiomatica del  $\lambda$ -calcolo, underground Roma.

Kearns, John T.

- [1969] Combinatory logic with discriminators, J.Symbolic Logic, vol.34, pp.561-575.

Kleene, Stephen C.

- [1934] A theory of positive integers in formal logic, Amer.J.Math., vol.57, pp.153-173, 219-244.
- [1936]  $\lambda$ -definability and recursiveness, Duke Math.J., vol.2, pp.340-353.
- [1959] Recursive functionals and quantifiers of finite types, I, Trans.Amer.Math.Soc., vol.91, pp.1-52.
- [1962] Lambda-definable functionals of finite types, Fund.Math.,vol.50, pp.281-303.

Kleene, Stephen C. and J.Barkley Rosser

- [1935] The inconsistency of certain formal logics, Ann.of Math., ser.2, vol.36, pp.630-636.

Martin-Löf, Per

- [1971] A theory of types, underground Stockholm.

Mitschke, Gerd

- [1970] Eine algebraische Behandlung von  $\lambda$ -K-Kalkül und kombinatorischer Logik, Ph.D.Dissertation, underground Bonn.

Newman, Maxwell H.A.

- [1942] On theories with a combinatorial definition of "equivalence", Ann. of Math., ser.2, vol.43, pp.223-243.
- [1952] Review of Curry [1952] , Math.Reviews, vol.13, p.715.

Rice, H.Gordon

[1953] Classes of recursively enumerable sets and their decision problems, Trans.Amer.Math.Soc., vol.74, pp.358-366.

Rogers, Hartley Jr.

[1967] Theory of recursive functions and effective computability, New York.

Rosen, Barry K.

[1971] Tree-manipulating systems and Church-Rosser theorems, underground Harvard University.

Rosser, J.Barkley

[1935] A mathematical logic without variables. Ann.of Math. ser.2, vol.36, pp.127-150, and Duke Math.J., vol.1, pp.328-355.

[1956] Review of Curry [1952] , J.Symbolic Logic, vol.21, p.377.

Sanchis, Luis Elpidio

[1967] Functionals defined by recursion, Notre Dame J. Formal Logic, vol.8, no 3, pp.161-174.

Schönfinkel, Moses

[1924] Über die Bausteine der mathematischen Logik, Math.Ann., vol.92, pp.305-316.

Schroer, David

[1965] The Church-Rosser theorem, Ph.D. dissertation, underground University of Illinois, Urbana Ill.

Shepherdson, John C.

[1965] Non standard models for fragments of number theory, in: The theory of models, eds J.W.Addison, L.Henkin and A.Tarski, pp. 342-358, Amsterdam.

Scott, Dana S.

[1963] A system of functional abstraction, underground Berkeley.

[1970] Lattice-theoretic models for the  $\lambda$ -calculus, underground Princeton.

Smullyan, Raymond M.

[1961] Theory of formal systems, Annals of mathematics studies, no.47, Princeton.

Tarski, Alfred, Andrzej Mostowski and Raphael M.Robinson

[1953] Undecidable theories, Amsterdam.

Troelstra, Anne S.

[1971] Notions of realizability for intuitionistic arithmetic and intuitionistic arithmetic in all finite types, Proceedings of the second Scandinavian logic symposium, ed.J.E.Fenstad, Amsterdam.

Turing, Alan M.

[1937] Computability and  $\lambda$ -definability, J.Symbolic Logic, vol.2, pp. 153-163.

[1937a] The  $p$ -function in  $\lambda$ -K-conversion, ibidem, p.164.

## Addenda

Gödel, Kurt

[1958] Über eine bisher noch nicht benutzte Erweiterung  
des finiten Standpunktes, *Dialectica*, vol.12, pp.280-287.

Morris, James H.

[1968] Lambda-calculus models of programming languages,  
Ph.D.dissertation, MIT.

Strong, H.Raymond

[1968] Algebraically generalized recursive function  
theory, IBM J.Research and Development, vol.12,  
pp.465-475.

### Samenvatting

Dit proefschrift houdt zich bezig met de combinatorische logica, niet als basis voor de rest van de wiskunde, maar als formeel systeem voor de bestudering van berekeningsprocedures.

Hoofdstuk I geeft een overzicht en uitbreiding van reeds bekend materiaal.

In Hoofdstuk II wordt de  $\omega$ -regel ingevoerd en met behulp van transfinitie inductie bewezen dat de uitbreiding van de combinatorische logica met de  $\omega$ -regel consistent is. Verder wordt de existentie van universele generatoren bewezen. Voor de termen die geen universele generatoren zijn, geldt dat de  $\omega$ -regel een afgeleide regel is.

In Hoofdstuk III worden een aantal andere consistentie resultaten bewezen, waardoor verschillende niet elementair equivalente modellen van de combinatorische logica verkregen worden.

In de bewijzen van de hierboven vermelde resultaten wordt meestal gebruik gemaakt van conservatieve uitbreidingen van de combinatorische logica. Hierbij speelt een nieuwe bewijstechniek een belangrijke rol, te weten de methode van het onderlijnen. Deze methode formaliseert het begrip residu en vermijdt aldus de anders nogal omslachtige argumenten.

Curriculum vitae

Op 18 december 1947 werd ik geboren te Amsterdam.

In 1960 kwam ik op het Montessori lyceum aldaar. Door mijn leraar F.H.Fischer werd mijn belangstelling gewekt voor de grondslagen van de wiskunde. Onder leiding van mijn leraar C.L.Scheffer studeerde ik analyse en lineaire algebra.

Na in 1965 het eindexamen gymnasium- $\beta$  gehaald te hebben, liet ik mij inschrijven als student in de wis- en natuurkunde aan de Rijksuniversiteit Utrecht.

Onder leiding van Dr.D.van Dalen legde ik mij toe op de grondslagen van de wiskunde. Daarnaast volgde ik een college van Prof.J.J.de Iongh in Nijmegen en nam ik aldaar deel aan het interuniversitaire seminarium grondslagen van de wiskunde dat samen met de Rijksuniversiteit Utrecht gegeven werd.

In 1968 kreeg ik van de Polska Akademia Nauk een studiebeurs voor een maand studie onder leiding van Prof.Mostowski in Warschau.

Van 1966 tot 1969 was ik verbonden als slagwerker aan Dansgroep Pauline de Groot, hetgeen mij financieel in staat stelde als student deel te nemen aan diverse internationale wiskunde congressen.

In december 1968 legde ik het doctoraal examen wiskunde met groot bijvak wijsbegeerte van de wiskunde af.

Sinds 1969 ben ik verbonden aan de Centrale Interfaculteit van de Rijksuniversiteit Utrecht als wetenschappelijk medewerker in de wijsbegeerte van de wiskunde.

Door de bestudering van het proefschrift van Goodman 'The interpretation of intuitionistic arithmetic in a theory of constructions' in een interuniversitaire werkgroep samen met de afdeling grondslagen van de wiskunde van de Gemeente Universiteit van Amsterdam werd mijn belangstelling gewekt voor de combinatorische logica. Deze belangstelling werd verder gestimuleerd door de hoogleraren Curry, Scott en Grzegorzcyk.

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## STELLINGEN

### I

De wijze waarop Rosenbloom het extensionaliteits principe formuleert geeft aanleiding tot verwarring, met name bij Rosenbloom zelf.

Rosenbloom: The elements of mathematical logic, blz.112.

### II

Ten onrechte schrijft Goodman aan zijn abstractie operator  $\lambda$  zekere eigenschappen met betrekking tot gedefinieerdheid toe.

Goodman: Intuitionistic arithmetic as a theory of constructions, section 8.

### III

In de intuïtionistische theorie der gelijkheid is het axioma

$$\neg \forall z (z \neq x \vee z \neq y) \rightarrow x=y$$

echt sterker dan het stabiliteits axioma

$$\neg \neg x=y \rightarrow x=y.$$

### IV

Door de axioma's van Kearns betreffende de discriminatoren in combinatorische logica iets voorzichtiger te formuleren, is het mogelijk dat de reductie- en de gelijkheidsrelatie ook rechts-monotoon zijn.

Kearns: Combinatory logic with discriminators, J.Symbolic Logic, vol.34(1969)

## V

Het begrip 'sterk definitioneel gelijk', zoals Tait dit in heeft gevoerd, is niet helemaal adequaat. De moeilijkheid is op te lossen door een variant van Curry's sterke reductie relatie in te voeren.

Tait: Intensional interpretations of functionals of finite type I, J. Symbolic Logic, vol.32, blz.204-205.

## VI

Curry's opvatting, dat de combinatorische logica een prelogica vormt die de grondslag vormt voor alle formele systemen, gaat voorbij aan de moeilijkheden in de analyse van het iteratie proces.

Curry en Feys: Combinatory logic, Introduction.

## VII

Bij de vraag of post- ook propterhypnotisch gedrag is, gaat het er niet om of de proefpersoon toneel speelt, beleefd is, bedriegt of wat dan ook. Relevant is alleen van welke stimuli alternatief gedrag afhankelijk is.

## VIII

De molens in Nederland draaien met hun wieken tegen de wijzers van de klok. Dit is een gevolg van de omstandigheid dat er in Nederland meer ruimende dan krimpwind voorkomt: wanneer de molenaar tijdens werkzaamheden tengevolge van deze ruimende wind moet kruien gaat dit lichter dan met krimpwind in verband met de gyroscopische werking van het wiekenkruis.