ON THE PROOF THEORY OF INQUISITIVE LOGIC

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis is a proof-theoretical study of various systems of inquisitive logic. In the first part, we consider the basic system of propositional inquisitive logic, denoted by InqB (see, e.g., Ciardelli 2022). We construct a natural deduction system for InqB, prove a normalization theorem and establish a restricted subformula property. Our system is based on an extended natural deduction formalism in which not only *formulas*, but also *rules* can act as assumptions that may be discharged in the course of a derivation. We then present a G3-style labelled sequent calculus with internalized support semantics for InqB. Our system is shown to satisfy a number of convenient structural properties such as cut-admissibility, height-preserving admissibility of weakening and contraction, and height-preserving invertibility of all rules. Afterwards, we modularly adapt our sequent calculus to other systems of inquisitive logic, including an intuitionistic variant of InqB described by Ciardelli et al. (2020) and extensions of InqB with Kripke modalities. We provide a general method that allows to construct cut-free labelled sequent calculi for all inquisitive Kripke logics characterized by a certain type of first-order formulas, known as *geometric implications*. This generalizes a famous result for ordinary modal logic established by Negri (2005).

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Contents

Introduction

This thesis is devoted to the proof theory of inquisitive logic. Inquisitive logic provides a semantic framework for propositional and first-order logic that allows to formalize not only the *informative content* of sentences in natural language, but also the *issues* raised by such a sentence. This makes it possible to account for the meaning of both *statements* and *questions* in a uniform way. The basic system of propositional inquisitive logic was first described by Ciardelli (2009), Groenendijk and Roelofsen (2009) as well as Ciardelli and Roelofsen (2011). Roughly speaking, this system is obtained by enriching classical propositional logic with a new connective $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, referred to as the *inquisitive disjunction operator.* This connective is used in order to form alternative questions such as 'Have you lost your wallet or your keys?' within the language of the system. Thus, intuitively, $\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi$ stands for the question *whether* φ *or* ψ. The familiar truth-conditional semantics of classical logic is extended to a more general state-based semantics centered around the notion of *support.* That is, rather than specifying what it means for a formula to be true or false with respect to an atomic valuation (or with respect to a *possible world*), inquisitive semantics specifies what it means for a formula to be supported by an *information state*, representing a certain body of information. In inquisitive logic, information states are modelled as sets of possible worlds—namely, those worlds that are compatible with the information conveyed by the state. Intuitively, a formula φ is said to be *supported* by an information state s, just in case s implies the information conveyed by φ and s resolves the issue raised by φ. From a technical point of view, standard inquisitive logic can be conceived from two different perspectives: on the one hand, it can be seen as a conservative extension of classical logic; on the other hand, it can be seen as a logic that is intermediate between intuitionistic and classical logic. However, inquisitive logic is not an *intermediate logic* in the usual sense, since it is not closed under uniform substitution. This makes it particularly difficult to define simple, analytic proof systems for inquisitive logic.

Recent years have seen a growing interest in the semantic features of inquisitive logic, its linguistic applications and its relationship to other frameworks (see, e.g., Ciardelli et al. 2019; Ciardelli 2022). However, the *proof-theoretical* properties of the system have received little attention in the literature so far. In fact, apart from a few exceptions (namely, Frittella et al. 2016; Chen and Ma 2017), there exists no work on the proof theory of inquisitive logic. The present thesis aims to fill this gap, by providing an extensive proof-theoretical investigation of basic inquisitive logic and related systems. The achievements of this thesis are manifold: on the one hand, we will construct various analytic proof systems for basic inquisitive logic and some of its variants, including a system of intuitionistic inquisitive logic described by Ciardelli et al. (2020) and a wide range of inquisitive modal logics; on the other hand, we will carefully investigate the properties of our proof systems and provide a solid ground for further research in this direction.

Roughly speaking, a proof system is said to be *analytic*, if every valid formula φ of the underlying logic has a proof containing only formulas that are, in some sense, 'relevant' to φ. In most cases, this simply means that the proof system satisfies a suitable version of the *subformula property*: every valid formula φ should have a proof containing only subformulas of φ. Since this is a very demanding requirement, weaker forms of the subformula property are usually also accepted as being sufficient for an analytic proof system. In the case of natural deduction systems, analyticity is usually established by means of a *normalization theorem*, stating that every deduction in the system can be transformed into a deduction without 'detours' (cf. Troelstra and Schwichtenberg 1996, pp. 178–189). In the case of sequent calculi, there are two different methods. If the cut rule is assumed to be included in the calculus, then analyticity is typically established by a suitable *cut-elimination theorem*, i.e., by providing a constructive procedure that allows to transform any derivation in the system into a cut-free derivation. If the cut rule is not included in the calculus, then the system is already analytic *by design*. However, in order to establish the completeness of such a system proof-theoretically, one has to show that the cut rule is *admissible* in the system, i.e., whenever the premises of the rule are derivable, then so is the conclusion of the rule (cf. Negri and Von Plato 2001; Troelstra and Schwichtenberg 1996).[1] In this thesis, we will always opt for the second strategy, so all our sequent calculi are cut-free by definition.

While analyticity is taken to be the most important criterion for the quality of a proof system in this thesis, we will occasionally also evaluate our systems against other criteria. For example, in a well-designed system of natural deduction, we expect each connective to have exactly one introduction rule and exactly one elimination rule, and these rules should exhibit some kind of 'harmony' or 'symmetry' (a similar requirement should be adopted for the left and the right rules of a sequent calculus). Another important property is *modularity*: if one logic is an extension or a slight modification of another logic, then there should be an easy way to turn a proof system for the latter into a proof system for the former (ideally just by adding a few rules).

The main contributions of this thesis can be summarized as follows. In the first part, we consider the basic system of propositional inquisitive logic, denoted by InqB. We provide an elegant natural deduction system for InqB, establish a normalization theorem and derive a weak form of the subformula property for our system. This is achieved by adopting an extended natural deduction formalism inspired by the so-called *calculus of higher-level rules* developed by Schroeder-Heister (1981; 1984). In the extended setting, not only *formulas*, but also *rules* can serve as assumptions that may be discharged in the course of a derivation. Afterwards, we construct a cut-free labelled sequent calculus for InqB. Labelled sequent calculi extend the traditional sequent-style formalism introduced by Gentzen (1935a; 1935b) with labels, allowing to incorporate the semantics of a logic directly into the syntax of the proof system. Our sequent calculus is shown to have a number of convenient structural properties. In particular, we will see that the structural rules of weakening, contraction and cut are *admissible* in our system, i.e., whenever the premises of these rules are derivable, then also the conclusion is derivable. In the case of weakening and contraction, admissibility also preserves the height of derivations. We also show that each rule of our system is *height-preserving invertible*, i.e., if the conclusion of one of these rules is derivable, then so is each premise of the rule, with at most the same derivation height.

In the second part, we consider various extensions and modifications of basic inquisitive logic. First, we provide a labelled sequent calculus for an intuitionistic variant of InqB described by Ciardelli et al. (2020). Our sequent calculus for this variant is obtained from the sequent calculus for InqB in a modular way, and is shown to have the same structural properties. Finally, we consider various systems of inquisitive Kripke logic, i.e., extensions of InqB with modal operators interpreted over ordinary Kripke models (see Ciardelli 2016b, Chapter 6). We provide a generic method that allows to construct cut-free labelled sequent calculi for all inquisitive Kripke logics characterized by a certain type of first-order frame conditions, known as *geometric implications*. This generalizes a well-known result of Negri (2005) from the proof theory of ordinary (i.e., non-

[1] Alternatively, one can use a semantic argument in order to give a direct completeness proof for the cut-free calculus.

inquisitive) modal logic. Each of our sequent calculi is shown to satisfy cut-admissibility, height-preserving admissibility of weakening and contraction, and height-preserving invertibility of all rules. Below, we provide a more detailed, chapter-by-chapter overview of the thesis.

Structure of the Thesis

Chapter 1. Preliminaries. In this chapter, we give a short introduction to inquisitive semantics and provide a detailed exposition of the basic system of propositional inquisitive logic (InqB). We also present some standard axiomatizations of InqB, including a natural deduction system introduced by Ciardelli (2016b) and a Hilbert-style system described by Ciardelli and Roelofsen (2011). However, from a proof-theoretical point of view, none of these axiomatizations is well-behaved.

Chapter 2. Natural Deduction for InqB. We define a new natural deduction system for InqB and establish a normalization theorem for our system. More precisely, we provide a constructive procedure that allows to transform every deduction in our system into a deduction without 'detours'. Our system is based on an extended natural deduction formalism in which not only *formulas*, but also *rules* can act as assumptions. The basic idea is inspired by the work of Schroeder-Heister (1981; 1984; 2014). We also prove a restricted subformula property for our system, which turns out to be rather weak though. However, for a certain special case, a full subformula property will be obtained. Moreover, we will see that the subformula property is still strong enough to establish some interesting properties of inquisitive logic in a purely proof-theoretical way.

Chapter 3. Labelled Sequents for InqB. In this chapter, we provide a cut-free labelled sequent calculus for InqB. Our system, denoted by GLinqB, can be regarded as a G3-style sequent calculus in the sense of Ketonen (1944) and Kleene (1952), so weakening and contraction are fully 'absorbed' into the axioms and the remaining rules of the system. Labelled formulas will be used in order to incorporate the support semantics of InqB directly into the proof rules of GLinqB. We carefully investigate the structural properties of our system. In particular, we show that GLinqB enjoys cut-admissibility, height-preserving admissibility of weakening and contraction, and height-preserving invertibility of all rules. The completeness of GLinqB is established proof-theoretically, by exploiting the admissibility of the cut rule in our system. We also discuss a possible proof search strategy for GLinqB and prove a normal form result for the labels used in our system.

Chapter 4. Intuitionistic Inquisitive Logic. We define a labelled sequent calculus for a variant of InqB in which the background logic for statements is no longer classical logic, but intuitionistic logic. This variant is denoted by InqI and was first described by Ciardelli et al. (2020). Our sequent calculus for InqI is obtained from the system GLinqB in a very elegant way and is shown to have the same structural properties. The completeness is again established proof-theoretically.

Chapter 5. Inquisitive Kripke Logic. In this chapter, we consider various inquisitive logics obtained by enriching InqB with a modal operator $\Box$, interpreted over ordinary Kripke models. For every normal modal logic $\mathfrak{L}$, we will define a corresponding inquisitive system $\mathsf{Inq}\mathfrak{L}$. The weakest logic obtained in this way is denoted by InqK and can be seen as an inquisitive extension of the basic modal logic K. The most important contribution of this chapter is a general strategy that allows to construct a cut-free labelled sequent calculus $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ for every inquisitive Kripke logic $\mathsf{Inq}\mathfrak{L}$ determined by a finite set A of *geometric implications*, i.e., first-order frame conditions of the form $\forall\vec{w}(\varphi \to \psi)$, where φ and ψ do not contain implications or universal quantifiers. The construction is based on a method described by Negri (2003; 2005), which allows to generate sequent rules from geometric implications in a schematic way. Each of the systems $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ is shown to enjoy cut-admissibility, height-preserving admissibility of weakening and contraction, and height-preserving invertibility of all rules. Our completeness proof is based on the construction of an infinite proof search tree and the extraction of a countermodel from an open branch of this tree.

Chapter 1

Preliminaries

In this chapter, we will give a short introduction to inquisitive logic, explain the semantics of the system and describe some of its standard axiomatizations. The basic system of propositional inquisitive logic is denoted by InqB and goes back to the work of Ciardelli (2009), Groenendijk and Roelofsen (2009) as well as Ciardelli and Roelofsen (2011).[1] From a technical point of view, InqB is obtained by enriching classical propositional logic with an *inquisitive disjunction operator* $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, allowing to form alternative question within the language of the system. Thus, intuitively, a formula $\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi$ represents the question *whether* φ *or* ψ. The ordinary truth-conditional semantics of classical logic is replaced by a more general *support semantics*, specifying the conditions under which a formula is supported by some body of information. This allows to define a generalized notion of entailment, capable of dealing not only with statements, but also with questions.

The chapter is organized as follows. In Section 1.1, we will sketch the basic ideas underlying inquisitive logic and motivate the semantic setup from an informal point of view. We will see that, in inquisitive logic, the main semantic difference between questions and assertions lies in the concept of truth-conditionality: a proposition is non-inquisitive, just in case it is truth-conditional. In Section 1.2, we will then provide a formal exposition of the system InqB. The key concept is the so-called support relation, determining the support conditions for all formulas of InqB. Afterwards, in Section 1.3, we will define the notion of a *Harrop formula* and show that any such formula is guaranteed to be truth-conditional in InqB. In Section 1.4, we will describe some further properties of the system. It will turn out that InqB can be considered from two different angles: on the one hand, it may be seen as a conservative extension of classical logic; on the other hand, it can be seen as a non-standard intermediate logic. In Section 1.5, finally, we will present some well-known axiomatizations of InqB, including a non-normalizing natural deduction system and a Hilbert-style axiomatization based on the Kreisel-Putnam axiom. We also provide a new Hilbert-style system for InqB and establish its completeness in a proof-theoretical way.

1.1 Information States and Inquisitive Propositions

We start by giving an informal explanation of the fundamental semantic concepts and the basic ideas underlying inquisitive logic. A more comprehensive exposition of the material is provided by Ciardelli (2016b, Chapter 1) as well as Ciardelli et al. (2019, Chapter 2).

Traditionally, logic is considered to be the study of valid inference patters between a specific type of linguistic entities, namely *declarative sentences*, or *statements*, or *assertions*.[2] In classical

[1] Important predecessors of modern inquisitive logic include the partition theory of questions (Groenendijk and Stokhof 1984), Groenendijk's *logic of interrogation* (Groenendijk 1999; ten Cate and Shan 2007) and inquisitive pair semantics (Groenendijk 2009; Mascarenhas 2009). For further details, we refer to Ciardelli (2022, pp. 41–49).

[2] Throughout this thesis, the terms 'declarative sentence', 'statement' and 'assertion' will be used synonymously.

logic, the semantic content of such a sentence is usually assumed to be given by its *truth conditions*, i.e., by the conditions that must be satisfied by a state of affairs in order to make the sentence true. In this section, any formal specification of a complete state of affairs will be referred to as a *possible world*. In particular, we do not care about the concrete way in which possible worlds are represented; instead, we simply assume an intuitive understanding of this concept.

It is well known that there is also an alternative way of formalizing the meaning of a declarative sentence. Under this approach, which is very common in the literature on *intensional logic*, the semantic content of an assertion is identified with its *truth-set*, i.e., the set of all possible worlds making the assertion true. The truth-set of an assertion α is also referred to as the *proposition expressed by* α (cf. Stalnaker 1976, p. 80). Thus, in this setting, a proposition simply amounts to a subset $P \subseteq W$, where W stands for the entire logical space (i.e., the set of all possible worlds).

This standard way of formalizing the semantic content of sentences works well, if only *assertions* are involved. However, it falls short as soon as one wishes to formalize also the meaning of *questions*. For one thing, it is not clear what it could mean for an interrogative sentence to be true or false at a possible world; for another, an appropriate formalization of questions should take into account not only the *informative content* of a sentence, but also the *issues* it raises.

Inquisitive logic overcomes this limitation by employing a more general semantics centered around the notion of *support*. That is, rather than specifying what it means for a sentence to be true or false at a possible world, inquisitive semantics specifies what it means for a sentence to be supported by an *information state*. Formally, an information state is modelled as a set of possible worlds—namely, those worlds that are compatible with the information conveyed by the state.[3] Writing again W for the entire logical space, we thus adopt the following definition.

Definition 1.1.1 (Information State)**.** An *information state* is a set of possible worlds $s \subseteq W$.

Intuitively, every information state represents a certain *body of information*, since it locates the actual world within a particular sphere of the logical space. More formally, a state $s \subseteq W$ conveys the information that the actual world is one of the worlds in s and that all worlds in $W \setminus s$ are *ruled out*. Note that, if we have $s \subseteq t$ for some states $s, t \subseteq W$, then every world ruled out by t is also ruled out by s, so s contains at least as much information as t. For this reason, any subset $s \subseteq t$ of an information state t is also referred to as an *enhancement* of t.

Observe that the empty set, $\emptyset$, and the set of all possible worlds, W, are also information states. Intuitively, W is the least informative state, because it does not rule out any candidate for the actual world. In this sense, one might say that W is the *trivial* information state. The empty state $\emptyset$, on the other hand, rules out every candidate for the actual world, so it represents an inconsistent body of information. For this reason, we refer to $\emptyset$ as the *inconsistent state*.

Using the notion of an information state, it is now possible to represent the semantic content of both *questions* and *assertions* in a uniform way. In inquisitive semantics, this is achieved by enriching the structure of propositions, enabling them to encode not only the information conveyed by a sentence, but also the issues it raises. But how could such an issue be modelled formally? The basic idea is to identify an inquisitive proposition with a *set of information states*: those states that contain enough information to resolve the issue raised by the proposition.

There are now two things to consider. First, it seems natural to assume that every issue can be resolved by at least one information state, so propositions should always be *non-empty*. Secondly, if the issue raised by a proposition is resolved by some state s, then it should also be resolved by every enhancement of s. In other words, we should require propositions to be *downward closed*.

Definition 1.1.2 (Proposition)**.** A *proposition* is a set of information states $P \subseteq \mathcal{P}(W)$ such that:

(i) P is *non-empty*, i.e., $P \neq \emptyset$,

(ii) P is *downward closed*, i.e., for all $s, t \subseteq W$, if $s \in P$ and $t \subseteq s$, then also $t \in P$.

[3] This way of modelling information is also used outside inquisitive logic (see, e.g., Hintikka 1962; Stalnaker 1978).

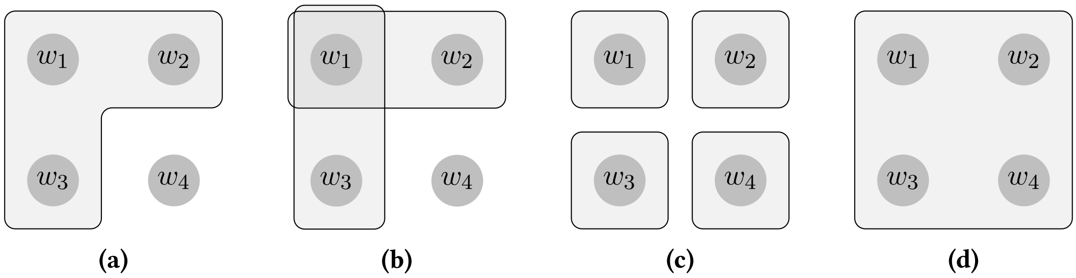

Figure 1.1: Some propositions.

The *informative content* of a proposition P is denoted by $\mathsf{info}(P)$ and defined to be the union of all states in P. That is, $\mathsf{info}(P)$ is the information state given by $\mathsf{info}(P) := \bigcup P$. Clearly, every proposition P must contain the inconsistent state: because P is non-empty, there must be at least one state $s \in P$. But then, since P is downward closed and $\emptyset \subseteq s$, it also holds $\emptyset \in P$. In inquisitive logic, the most fundamental semantic concept is taken to be *support*, rather than *truth*. Intuitively, a state s supports a proposition P, just in case it implies the information conveyed by P, i.e., $s \subseteq \mathsf{info}(P)$, and it resolves the issue raised by P, i.e., $s \in P$. Observe that the first of these two conditions is implied by the second. Thus, support can simply be defined as follows.

Definition 1.1.3 (Support)**.** An information state s *supports* a proposition P, if $s \in P$.

Using the notion of support, one can now also define a suitable notion of *truth*. Intuitively, P is true at a world w, if w is compatible with the information conveyed by P, i.e., $w \in \mathsf{info}(P)$. But this just means that P is supported by the singleton state $\{w\}$, so we define truth as follows.

Definition 1.1.4 (Truth)**.** A proposition P is *true* at a world $w \in W$, if P is supported by $\{w\}$.

In order to make sense of the notions just introduced, let us consider some examples. Figure 1.1 depicts a number of propositions over the set of worlds $W = \{w_1, w_2, w_3, w_4\}$. For simplicity, only the *maximal* elements of the propositions are displayed. The reader should bear in mind, however, that all enhancements of these maximal elements are also assumed to be included in the propositions.[4] The maximal elements of a proposition P are also referred to as the *alternatives* of P. The proposition in Figure 1.1 (a) has only one alternative, so it does not raise any issue, but simply conveys the information that the actual world is a member of $\{w_1, w_2, w_3\}$. The very same information is conveyed by the proposition in Figure 1.1 (b), but this proposition also raises the issue as to whether the actual world is contained in $\{w_1, w_2\}$ or in $\{w_1, w_3\}$. In order to resolve this issue consistently, an information state has to establish either that the actual world is in $\{w_1, w_2\}$ or that the actual world is in $\{w_1, w_3\}$. The proposition in Figure 1.1 (c) does not convey any (non-trivial) information, but raises the issue as to *which* of the four worlds is the actual one. Thus, an information state supports this issue, just in case it is either inconsistent or it contains exactly one candidate for the actual world. The proposition in Figure 1.1 (d), finally, corresponds to the power set $\mathcal{P}(W)$, so it is trivially supported by every state $s \subseteq W$.

In inquisitive logic, the semantic difference between questions and assertions is captured by the concept of *inquisitiveness*. Intuitively, a proposition P is *inquisitive*, if the information conveyed by P does not suffice to resolve the issue raised by P. This leads to the following definition.

Definition 1.1.5. A proposition P is *inquisitive*, if $\mathsf{info}(P) \notin P$.

Intuitively, an inquisitive proposition represents the semantic content of a *question*, whereas a non-inquisitive proposition represents the semantic content of a *statement*. One readily sees that,

[4] So, for example, the proposition in Figure 1.1 (a) actually corresponds to the set of states $\mathcal{P}(\{w_1, w_2, w_3\})$.

for propositions containing only *finitely many* states, the concept of inquisitiveness is closely related to the number of alternatives of a proposition: a finite proposition P is inquisitive if and only if P has at least two alternatives. For *infinite* propositions, however, this connection breaks down, since there may also be inquisitive propositions that do not have any alternatives.[5] To see some examples, consider again the propositions depicted in Figure 1.1. An easy inspection shows that the propositions in (b) and (c) are inquisitive, but the ones in (a) and (d) are not.

So far, we have characterized non-inquisitive propositions solely in terms of their informative content. However, it is also possible to characterize non-inquisitive propositions differently, by using the concept of *truth-conditionality*. This concept is defined in the following way.

Definition 1.1.6 (Truth-Conditionality)**.** A proposition P over W is *truth-conditional*, if for every state $s \subseteq W$, we have: s supports P if and only if, for all worlds $w \in s$, P is true at w.

In other words, a truth-conditional proposition is a proposition for which support at a state s simply comes down to truth at every world in s. It is now easy to show that P is truth-conditional if and only if $\mathsf{info}(P)$ is the unique alternative in P, i.e., if P is non-inquisitive. Hence, truth-conditionality represents the fundamental semantic difference between questions and assertions: every non-inquisitive proposition is truth-conditional, and every inquisitive proposition is not.

Fact 1.1.7. *A proposition P is non-inquisitive if and only if P is truth-conditional.*

Finally, one can now also define a notion of semantic *entailment* between propositions. Intuitively, a proposition P entails another proposition Q, if the information conveyed by P implies the information conveyed by Q, i.e., $\mathsf{info}(P) \subseteq \mathsf{info}(Q)$, and every state that resolves the issue raised by P also resolves the issue raised by Q, i.e., $P \subseteq Q$. But note that the first of these two conditions is implied by the second, so entailment can simply be defined as follows.

Definition 1.1.8 (Entailment)**.** A proposition P *entails* another proposition Q, if $P \subseteq Q$.

Note that, in inquisitive semantics, entailment is not restricted to *statements*, but may also involve *questions*. For example, a statement α entails a question μ, if the information conveyed by α resolves the issue raised by μ. And a question μ entails a statement α, if μ presupposes the information conveyed by α. For further information, we refer to Ciardelli (2022, p. 16).

1.2 Propositional Inquisitive Logic

Let us now give a more formal exposition of the system. The basic framework of propositional inquisitive logic is denoted by InqB and was first described by Ciardelli (2009), Groenendijk and Roelofsen (2009) as well as Ciardelli and Roelofsen (2011). It may be conceived as the result of enriching classical propositional logic with a question-forming operator $\vee\!\!\!\vee$, referred to as the *inquisitive disjunction operator*. Intuitively, $\vee\!\!\!\vee$ is used in order to form *alternative questions* within the language of InqB, so a formula of the form $\varphi \vee\!\!\!\vee \psi$ is intended to denote the question *whether φ or ψ*. To make things precise, we henceforth assume a countably infinite set P of atomic propositions, denoted by the meta-variables p, q, r, etc. The formulas of InqB are now built up from the atoms in P and the falsum constant $\bot$ by means of the binary connectives $\wedge$, $\to$ and $\vee\!\!\!\vee$.

Definition 1.2.1 (Language of InqB)**.** The language of InqB is denoted by $\mathcal{L}^{\mathsf{B}}$ and consists of all formulas generated by the following grammar, where p ranges over atomic propositions from P:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \vee\!\!\!\vee \varphi.$$

[5] To see an example, consider the infinite proposition P defined by $P := \{[n] \mid n \in \mathbb{N}\}$, where $[n] := \{0, \ldots, n\}$ for every $n \in \mathbb{N}$. Clearly, we have $\mathsf{info}(P) = \mathbb{N}$ and $\mathbb{N} \notin P$, so P is inquisitive. But since every element of P is included in some larger element, P does not have any alternatives (cf. Ciardelli et al. 2019, p. 20).

We will adopt the usual abbreviations familiar from classical logic. That is, the *verum constant* $\top$, the *negation operator* $\neg$ and the *classical disjunction operator* $\vee$ are taken to be defined by $\top := \neg\bot$, $\neg\varphi := \varphi \to \bot$ and $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, respectively. As we will see below, the semantics for $\vee$ is quite different from the one for $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, so these two operators should not be confused. In addition, we also introduce the *question mark operator* $?$, which is defined as follows.

Definition 1.2.2 (Question Mark Operator)**.** For every $\varphi \in \mathcal{L}^{\mathsf{B}}$, we put $?\varphi := \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg\varphi$.

Intuitively, $?\varphi$ represents the polar question *whether* φ. By what was said above, the connective $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ plays a special role in InqB, because it allows us to form interrogative sentences within the language of the system. Conversely, we thus expect any formula not containing $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ to be purely declarative. In the system InqB, the underlying background logic for declarative sentences is assumed to be classical logic, so $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-free formulas will also be referred to as *classical formulas*.

Definition 1.2.3 (Classical Formula)**.** A formula $\varphi \in \mathcal{L}^{\mathsf{B}}$ is said to be *classical*, if φ does not contain any occurrences of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$. The set of all classical formulas in $\mathcal{L}^{\mathsf{B}}$ is denoted by $\mathcal{L}^{\mathsf{B}}_c$.

In what follows, we will always use α, β, γ, etc., as meta-variables for classical formulas, while φ, ψ, χ, etc., will be used for arbitrary formulas of InqB. Let us now turn to the semantics of the system. The formulas of InqB are evaluated with respect to so-called *propositional information models*. Any such model consists of a non-empty set of possible worlds W and a valuation function V, assigning a truth value to each atomic proposition $p \in \mathsf{P}$ at each possible world $w \in W$.

Definition 1.2.4 (Information Model)**.** An *information model* is a pair $M = \langle W, V\rangle$, where W is a non-empty set of *possible worlds*, and $V : W \times \mathsf{P} \to \{0, 1\}$ is a *valuation function*.

Recall that, in inquisitive logic, the most fundamental semantic concept is taken to be *support*, not *truth*. Thus, instead of defining what it means for a formula to be true at a possible world, we have to define what it means for a formula to be supported by an information state. Formally, information states are modelled as sets of possible worlds, so we adopt the following definition.

Definition 1.2.5. Let $M = \langle W, V\rangle$ be a model. An *information state* over M is a subset $s \subseteq W$.

Following the terminology introduced in the previous section, we will refer to $\emptyset$ as the *inconsistent state* and to every non-empty state $s \neq \emptyset$ as a *consistent state*. Moreover, a subset $s \subseteq t$ of a state t is also said to be an *enhancement* of t. We are now ready to give an inductive definition of the support conditions for all formulas in the language of InqB (cf. Ciardelli 2016b, pp. 47–50).

Definition 1.2.6 (Support Semantics for InqB)**.** Let $M = \langle W, V\rangle$ be a model. The *support relation* $\models$ between states $s \subseteq W$ and formulas $\varphi \in \mathcal{L}^{\mathsf{B}}$ is inductively defined in the following way:

(i) $M, s \models p \;:\Leftrightarrow\; V(w, p) = 1$ for all $w \in s$,

(ii) $M, s \models \bot \;:\Leftrightarrow\; s = \emptyset$,

(iii) $M, s \models \varphi \wedge \psi \;:\Leftrightarrow\; M, s \models \varphi$ and $M, s \models \psi$,

(iv) $M, s \models \varphi \to \psi \;:\Leftrightarrow\;$ for all $t \subseteq s$, if $M, t \models \varphi$, then $M, t \models \psi$,

(v) $M, s \models \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi \;:\Leftrightarrow\; M, s \models \varphi$ or $M, s \models \psi$.

If $M, s \models \varphi$ holds, then we say that φ is *supported* by s in M. The support clauses can be read as follows. An atomic formula p is supported by a state s, just in case p is true at every world in s. The falsum constant $\bot$ is supported only by the inconsistent state $\emptyset$. A conjunction is supported by a state s, if each of the two conjuncts is supported by s. An implication is supported by s, if every enhancement of s which supports the antecedent also supports the consequent. And an alternative question is supported by s, if s supports at least one of the two alternatives.

Using Definition 1.2.6, one can now also derive support conditions for the defined connectives $\neg$ and $\vee$. For the sake of simplicity, let us say that a state s over some model M is *incompatible* with a formula φ, notation $s \ddagger \varphi$, if there exists no consistent enhancement $t \subseteq s$ such that $M, t \models \varphi$. It is now possible to prove the following proposition (cf. Ciardelli 2022, p. 58).

Proposition 1.2.7. *Let M be a model, let s be a state over M and let $\varphi, \psi \in \mathcal{L}^{\mathsf{B}}$ be formulas.*

(i) $M, s \models \neg\varphi \quad \Leftrightarrow \quad s \ddagger \varphi$,

(ii) $M, s \models \varphi \vee \psi \quad \Leftrightarrow$ *there is no consistent $t \subseteq s$ such that both $t \ddagger \varphi$ and $t \ddagger \psi$.*

As outlined in Section 1.1, the notion of *truth* can be recovered from the notion of support in a natural way: a formula is true at a world w, just in case it is supported by the singleton state $\{w\}$.

Definition 1.2.8 (Truth)**.** Let $M = \langle W, V \rangle$ be a model, let $w \in W$ be a world and let $\varphi \in \mathcal{L}^{\mathsf{B}}$ be a formula. We write $M, w \models \varphi$ and say that φ is *true* at w, if we have $M, \{w\} \models \varphi$.

By an easy inspection of the support conditions given above, one can now spell out the *truth conditions* for all formulas of InqB. We summarize the resulting clauses in the next proposition.

Proposition 1.2.9 (Truth Conditions)**.** *Let $M = \langle W, V \rangle$ be a model and let $w \in W$ be a world.*

(i) $M, w \models p \quad \Leftrightarrow \quad V(w, p) = 1$,

(ii) $M, w \not\models \bot$,

(iii) $M, w \models \varphi \wedge \psi \quad \Leftrightarrow \quad M, w \models \varphi$ *and* $M, w \models \psi$,

(iv) $M, w \models \varphi \to \psi \quad \Leftrightarrow \quad M, w \not\models \varphi$ *or* $M, w \models \psi$,

(v) $M, w \models \varphi \vee\!\!\!\vee \psi \quad \Leftrightarrow \quad M, w \models \varphi$ *or* $M, w \models \psi$,

(vi) $M, w \models \neg\varphi \quad \Leftrightarrow \quad M, w \not\models \varphi$,

(vii) $M, w \models \varphi \vee \psi \quad \Leftrightarrow \quad M, w \models \varphi$ *or* $M, w \models \psi$.

As can be seen, the classical connectives simply have their usual truth conditions familiar from classical logic. Moreover, the truth conditions for the inquisitive disjunction $\vee\!\!\!\vee$ are exactly the same as those for the classical disjunction $\vee$. That is, if we restrict ourselves to *singleton states*, then $\vee\!\!\!\vee$ and $\vee$ become indistinguishable. This, however, is no longer the case as soon as we consider states of *arbitrary size*. For example, if s is a state containing a world where p is true and another world where p is false, then $p \vee \neg p$ is supported by s, but $?p = p \vee\!\!\!\vee \neg p$ is not.

The truth-set of a formula φ in a model M is now defined to be the set of all worlds in M where φ is true. And the support-set of φ in M is the set of all states that support φ in M.

Definition 1.2.10. Let $M = \langle W, V \rangle$ be a model and let $\varphi \in \mathcal{L}^{\mathsf{B}}$ be a formula.

(i) The *truth-set* of φ in M is the information state given by $|\varphi|_M := \{w \in W \mid M, w \models \varphi\}$.

(ii) The *support-set* of φ in M is the set of states given by $\langle\varphi\rangle_M := \{s \subseteq W \mid M, s \models \varphi\}$.

An important feature of support in InqB is *persistency*: if a formula is supported by an information state s, then it is also supported by every enhancement $t \subseteq s$. In addition, it is possible to show that each formula of InqB is supported by the inconsistent state $\emptyset$. In a sense, this may be seen as a semantic version of the well-known *principle of explosion* (*ex falso quodlibet*).

Proposition 1.2.11. *Let M be a model, let s and t be states over M and let $\varphi \in \mathcal{L}^{\mathsf{B}}$ be a formula.*

(i) Persistency: if $M, s \models \varphi$ and $t \subseteq s$, then $M, t \models \varphi$.

(ii) Empty state property: $M, \emptyset \models \varphi$.

Both statements are proved by induction on the structure of φ. As a consequence of this result, one readily sees that, for any formula $\varphi \in \mathcal{L}^{\mathsf{B}}$ and any model M, the support-set $\langle\varphi\rangle_M$ is in fact a *proposition* in the sense of Definition 1.1.2: it is *downward closed* by the persistency of support

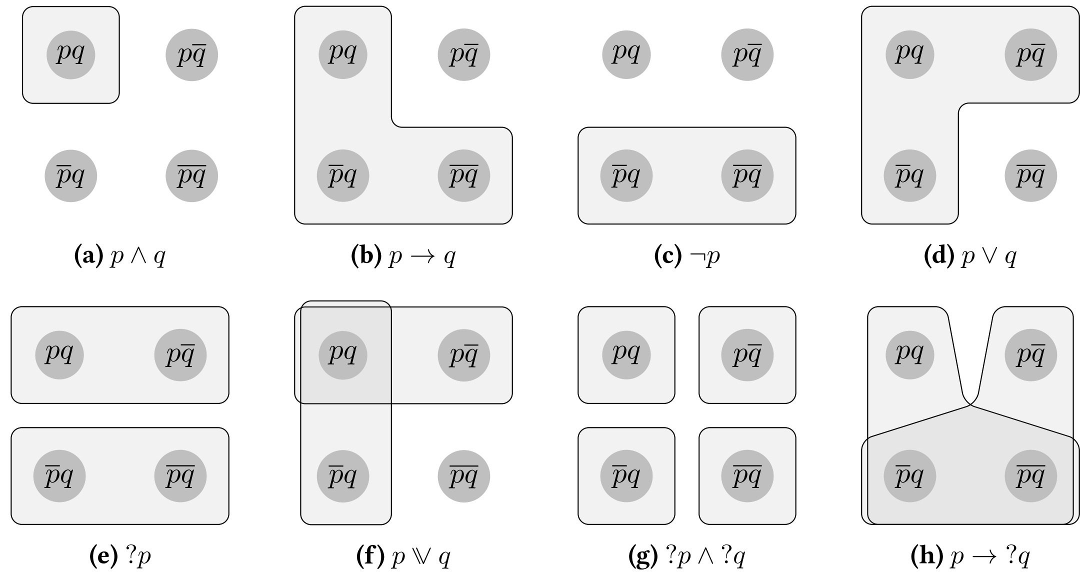

Figure 1.2: The propositions expressed by some formulas of InqB. In the figures, pq stands for a world where p and q are both true, $p\overline{q}$ for a world where p is true and q is false, etc.

in InqB; and it is *non-empty*, since φ is always supported by $\emptyset$. Clearly, the truth-set $|\varphi|_M$ simply amounts to the *informative content* of this proposition, so we have $|\varphi|_M = \bigcup\langle\varphi\rangle_M$.

Let us now consider some examples. Figure 1.2 depicts a number of propositions expressible by formulas in the language of InqB. The underlying model is assumed to contain four possible worlds: pq represents a world where both p and q are true, $p\overline{q}$ represents a world where p is true and q is false, and so on. As usual, only maximal elements are displayed, so all enhancements of the depicted states are also assumed to be included in the propositions. First, consider the formulas in Figure 1.2 (a–d). Each of these formulas is *classical* and the associated propositions have exactly one alternative, so they are all *non-inquisitive* in the sense of Definition 1.1.5. In fact, as we shall see below, this can be generalized: classical formulas are always truth-conditional in InqB. Note that the maximal elements are simply the usual truth-sets familiar from classical logic. So, for example, the unique alternative for $p \wedge q$ is just the set of worlds where p and q are both true, and the unique alternative for $p \to q$ is the set of worlds where p is false or q is true.

Consider now the formulas in Figure 1.2 (e–h). Each of these formulas is *non-classical* and the associated propositions are *inquisitive*. Figure 1.2 (e) depicts a polar question $?p = p \mathbin{\vee\!\!\!\vee} \neg p$ such as 'Have you lost your wallet?' An information state resolving this question should either establish that p or that $\neg p$. Thus, in particular, it does *not* suffice to establish that at least one of p and $\neg p$ must be true: an information state really has to *choose* one of the two alternatives. A similar observation holds for the formula $p \mathbin{\vee\!\!\!\vee} q$, depicted in Figure 1.2 (f). Intuitively, this formula represents an alternative question such as 'Have you lost your wallet or your keys?' The formula in Figure 1.2 (g), on the other hand, is a conjoined question, which corresponds to a sentence of the form 'Have you lost your wallet, and are you upset?' In order to resolve the issue expressed by this question, an information state has to resolve each of the two conjuncts. The formula in Figure 1.2 (h), finally, is a conditional question of the form $p \to ?q$. In order to resolve the issue expressed by this question, one has to resolve $?q$ under the assumption that p. That is, a state s supports $p \to ?q$, just in case q is either true at all p-worlds in s, or it is false at all p-worlds in s. This corresponds to a sentence of the form 'If you have lost your wallet, will you be upset?'

Using the support relation for InqB, one can now also define a suitable notion of *entailment*. In fact, entailment can simply be characterized as *preservation of support*: a formula φ entails another formula ψ, just in case every state that supports φ in a model M also supports ψ in M.

Definition 1.2.12 (Entailment)**.** Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ be a set of formulas. We write $\Gamma \models \varphi$ and say that Γ *entails* φ, if for every model $M = \langle W, V \rangle$ and for every state $s \subseteq W$, we have: $M, s \models \Gamma$ implies $M, s \models \varphi$. Here, $M, s \models \Gamma$ is used as an abbreviation for '$M, s \models \psi$ for all $\psi \in \Gamma$'.

Observe that we have $\varphi \models \psi$ if and only if $\langle \varphi \rangle_M \subseteq \langle \psi \rangle_M$ for every model M. Hence, with respect to the semantic content of formulas, entailment in InqB simply amounts to set-inclusion, as anticipated in Definition 1.1.8. The notions of *logical equivalence* and *validity* are now defined in the usual way. That is, two formulas are called equivalent, if they mutually entail each other. And a formula is said to be valid, if it is supported by every information state over ever model.

Definition 1.2.13 (Equivalence and Validity)**.** Let $\varphi, \psi \in \mathcal{L}^{\mathsf{B}}$ be formulas.

(i) We say that φ and ψ are *equivalent*, notation $\varphi \equiv \psi$, if we have both $\varphi \models \psi$ and $\psi \models \varphi$.

(ii) We say that φ is *valid*, notation $\models \varphi$, if it holds $M, s \models \varphi$ for all models M and all states s.

Finally, it is worth noting that InqB also validates the well-known *deduction theorem*, i.e., for every subset $\Gamma \subseteq \mathcal{L}^{\mathsf{B}}$ and for all formulas $\varphi, \psi \in \mathcal{L}^{\mathsf{B}}$, we have: $\Gamma, \varphi \models \psi$ if and only if $\Gamma \models \varphi \to \psi$.

1.3 Truth-Conditionality

In Section 1.1, we have already seen that truth-conditionality represents the fundamental semantic difference between questions and assertions: a proposition is non-inquisitive if and only if it is truth-conditional. For formulas of InqB, the concept of truth-conditionality is defined as follows.

Definition 1.3.1 (Truth-Conditionality)**.** A formula $\varphi \in \mathcal{L}^{\mathsf{B}}$ is *truth-conditional*, if for all models M and for all states s over M, we have: $M, s \models \varphi$ if and only if $M, w \models \varphi$ for all $w \in s$.

Observe that, by persistency, the left-to-right direction of the equivalence is satisfied by *every* formula. Hence, the important part of the definition is the converse implication: the support conditions of a truth-conditional formula are always completely determined by its truth conditions. We might now say that a formula $\varphi \in \mathcal{L}^{\mathsf{B}}$ is an *assertion*, if φ is truth-conditional, and we might say that φ is a *question* otherwise. As pointed out in the discussion above, formulas not involving $\mathbin{\vee\!\!\!\!\vee}$ behave in essentially the same way as in classical logic. Thus, it should not come as a surprise that *classical formulas*—in the sense of Definition 1.2.3—are always truth-conditional in inquisitive logic. In fact, it is even possible to identify a richer syntactic fragment of InqB that is guaranteed to have this property. The formulas in this fragment are known as *Harrop formulas.*[6]

Definition 1.3.2 (Harrop Formulas)**.** The set of *Harrop formulas* is denoted by $\mathcal{L}^{\mathsf{B}}_H$ and consists of all formulas generated by the following grammar, where $\varphi \in \mathcal{L}^{\mathsf{B}}$ ranges over arbitrary formulas:

$$\alpha ::= p \mid \bot \mid \alpha \wedge \alpha \mid \varphi \to \alpha.$$

In other words, by a Harrop formula, we simply mean any formula of InqB in which all occurrences of $\mathbin{\vee\!\!\!\!\vee}$ are contained in the antecedent of an implication. Observe that, in particular, every classical formula is also a Harrop formula. However, the converse of this statement is *not* true. For example, $(p \mathbin{\vee\!\!\!\!\vee} q) \to p$ is a Harrop formula, but it is not a classical formula, because it contains an occurrence of $\mathbin{\vee\!\!\!\!\vee}$. By overloading notation, we will henceforth use the meta-variables α, β, γ, etc., for both *classical formulas* and *Harrop formulas.* No confusion will arise, since it will always be clear from the context whether a classical formula or a Harrop formula is meant. We are now ready to prove the desired statement: in InqB, Harrop formulas are always truth-conditional.

[6] Harrop formulas were introduced by Ronald Harrop (1956; 1960) in order to strengthen the well-known disjunction and existence property for intuitionistic logic (cf. Troelstra and Schwichtenberg 1996, p. 107). We will come back to this concept in Chapter 2 in order to give a purely syntactical proof of the disjunction property for InqB.

Proposition 1.3.3. *Every Harrop formula $\alpha \in \mathcal{L}^{\mathsf{B}}_{H}$ is truth-conditional.*

Proof. By induction on the structure of α. The base case and the inductive step for $\wedge$ are straightforward. Thus, we only need to consider the case in which α is of the form $\alpha = \varphi \to \beta$ for some arbitrary formula $\varphi \in \mathcal{L}^{\mathsf{B}}$ and some Harrop formula $\beta \in \mathcal{L}^{\mathsf{B}}_{H}$. Let M be an arbitrary model and let s be an arbitrary state. By what was said above, it suffices to show that, if $M, w \models \varphi \to \beta$ for all $w \in s$, then $M, s \models \varphi \to \beta$. We prove the contrapositive: if $M, s \not\models \varphi \to \beta$, then there exists a world $w \in s$ such that $M, w \not\models \varphi \to \beta$. Suppose that we have $M, s \not\models \varphi \to \beta$, i.e., there exists some $t \subseteq s$ such that $M, t \models \varphi$ and $M, t \not\models \beta$. By induction hypothesis, we know that β is truth-conditional. Hence, because $M, t \not\models \beta$, there must be some world $w \in t$ such that $M, w \not\models \beta$. Since $M, t \models \varphi$ and $w \in t$, this world also satisfies $M, w \models \varphi$ by persistency (see Proposition 1.2.11). But then, by Proposition 1.2.9, we may conclude $M, w \not\models \varphi \to \beta$, as desired. □

As an immediate corollary, it follows that every classical formula of InqB is truth-conditional and therefore purely declarative. Note that, since classical formulas have their usual truth conditions in InqB, we thus arrive at the following general conclusion: a classical formula $\alpha \in \mathcal{L}^{\mathsf{B}}_{c}$ is supported by an information state if and only if α is classically true at every world in the state.

Corollary 1.3.4. *Every classical formula $\alpha \in \mathcal{L}^{\mathsf{B}}_{c}$ is truth-conditional.*

In addition, recall that $\neg\varphi$ was defined to be an abbreviation for $\varphi \to \bot$, which is a Harrop formula. Hence, not only classical formulas, but also every negated formula is truth-conditional in InqB. In order to conclude this section, let us now mention some alternative ways of characterizing truth-conditionality in inquisitive logic. First, we introduce the following terminology: the *classical variant* of a formula φ is denoted by φ^{cl} and defined to be the classical formula obtained from φ by replacing every occurrence of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ by an occurrence of $\vee$. It is now possible to show that a formula φ is truth-conditional if and only if φ is equivalent to φ^{cl}. Consequently, every truth-conditional formula can equivalently expressed as a classical formula. Using this fact, one easily checks that there is a tight connection between truth-conditionality and the *double negation law*: a formula φ is truth-conditional if and only if φ is equivalent to $\neg\neg\varphi$ (cf. Ciardelli 2022, pp. 65–66).

Proposition 1.3.5. *Let $\varphi \in \mathcal{L}^{\mathsf{B}}$ be a formula. The following three conditions are equivalent:*

(i) *φ is truth-conditional,*
(ii) *φ satisfies $\varphi \equiv \varphi^{cl}$,*
(iii) *φ satisfies $\varphi \equiv \neg\neg\varphi$.*

1.4 Properties of InqB

We now want to give a brief outline of some further properties of inquisitive logic. First of all, a characteristic feature of InqB is the fact that truth-conditional assumptions always distribute over inquisitive disjunctions: an alternative question $\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi$ is entailed by some set of truth-conditional formulas Γ if and only if at least one of the two alternatives is entailed by Γ. This is usually referred to as the *split property* and may be seen as a variant of the famous *disjunction property under hypotheses* known from intuitionistic logic (cf. Troelstra and Schwichtenberg 1996, p. 106).

Proposition 1.4.1 (Split Property)**.** *Let $\Gamma \subseteq \mathcal{L}^{\mathsf{B}}$ be a set of truth-conditional formulas and let $\varphi, \psi \in \mathcal{L}^{\mathsf{B}}$ be arbitrary formulas. It holds $\Gamma \models \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi$ if and only if $\Gamma \models \varphi$ or $\Gamma \models \psi$.*

A proof is provided by Ciardelli (2022, p. 80). Closely related to this is another interesting property: if α is truth-conditional, then any formula of the form $\alpha \to (\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi)$ is equivalent to the formula $(\alpha \to \varphi) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} (\alpha \to \psi)$. This is known as the *split equivalence* (cf. Ciardelli 2022, p. 82).

Proposition 1.4.2 (Split Equivalence). *Let $\alpha \in \mathcal{L}^{\mathsf{B}}$ be a truth-conditional formula and let $\varphi, \psi \in \mathcal{L}^{\mathsf{B}}$ be arbitrary formulas. It holds $\alpha \to (\varphi \vee\!\!\!\vee \psi) \equiv (\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi)$.*

Proof. Let $\alpha, \varphi, \psi \in \mathcal{L}^{\mathsf{B}}$ be arbitrary such that α is truth-conditional. Moreover, let M be an arbitrary model and let s be an arbitrary state. First of all, one easily checks that, for any $\chi \in \mathcal{L}^{\mathsf{B}}$, we have $M, s \models \alpha \to \chi$ iff $M, s \cap |\alpha|_M \models \chi$ by the truth-conditionality of α. Hence, it follows:

$$\begin{aligned} M, s \models \alpha \to (\varphi \vee\!\!\!\vee \psi) \quad &\Leftrightarrow \quad M, s \cap |\alpha|_M \models \varphi \vee\!\!\!\vee \psi \\ &\Leftrightarrow \quad M, s \cap |\alpha|_M \models \varphi \text{ or } M, s \cap |\alpha|_M \models \psi \\ &\Leftrightarrow \quad M, s \models \alpha \to \varphi \text{ or } M, s \models \alpha \to \psi \\ &\Leftrightarrow \quad M, s \models (\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi). \end{aligned}$$ □

Observe that, in inquisitive logic, the split equivalence has a very natural interpretation: a declarative sentence α resolves an alternative question, just in case the information conveyed by α establishes at least one of the two alternatives. Another important property is a well-known *normal form result*, established by Ciardelli (2016b, pp. 56–57). It says that every formula of InqB is equivalent to an inquisitive disjunction of classical formulas. As we shall see below, this normal form of a formula plays a crucial role in the completeness proof for the standard axiomatization of InqB (see Section 1.5). In order to prove this result, one first defines, for every formula $\varphi \in \mathcal{L}^{\mathsf{B}}$, a finite set of classical formulas $\mathcal{R}(\varphi)$, representing the different 'ways' in which φ might be settled by an information state. The elements of $\mathcal{R}(\varphi)$ are also referred to as the *resolutions* of φ.

Definition 1.4.3 (Resolutions). Let $\varphi \in \mathcal{L}^{\mathsf{B}}$ be a formula. The set of *resolutions* of φ is denoted by $\mathcal{R}(\varphi)$ and inductively defined in the following way:
(i) $\mathcal{R}(p) := \{p\}$ for all $p \in \mathsf{P}$,
(ii) $\mathcal{R}(\bot) := \{\bot\}$,
(iii) $\mathcal{R}(\varphi \wedge \psi) := \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi) \text{ and } \beta \in \mathcal{R}(\psi)\}$,
(iv) $\mathcal{R}(\varphi \to \psi) := \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)}(\alpha \to f(\alpha)) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$,
(v) $\mathcal{R}(\varphi \vee\!\!\!\vee \psi) := \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$.

By induction on φ, one easily verifies that $\mathcal{R}(\varphi)$ is in fact finite and contains only classical formulas. Resolutions can also be defined for *sets of formulas*. Given any subset $\Gamma \subseteq \mathcal{L}^{\mathsf{B}}$, we say that $f : \Gamma \to \mathcal{L}^{\mathsf{B}}_c$ is a *resolution function* of Γ, just in case f satisfies $f(\varphi) \in \mathcal{R}(\varphi)$, for all $\varphi \in \Gamma$. A *resolution* of Γ is now defined to be a set of classical formulas $\Delta \subseteq \mathcal{L}^{\mathsf{B}}_c$ such that, for some resolution function f of Γ, we have $\Delta = \{f(\varphi) \mid \varphi \in \Gamma\}$. In other words, a resolution of Γ is just a set of classical formulas containing one resolution for each element of Γ. The set of all resolutions of Γ is denoted by $\mathcal{R}(\Gamma)$. It is now possible to prove the desired normal form result: every formula is equivalent to the inquisitive disjunction of its resolutions (cf. Ciardelli 2016b, p. 57).

Proposition 1.4.4. *For any $\varphi \in \mathcal{L}^{\mathsf{B}}$, it holds $\varphi \equiv \alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \alpha_n$, where $\mathcal{R}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$.*

Finally, let us say a bit more about the relationship between inquisitive logic and classical propositional logic, henceforth denoted by CPL. We already observed that, in InqB, classical formulas are always truth-conditional, with the same truth conditions as in classical logic. Consequently, if we treat $\vee\!\!\!\vee$ as a new connective that is added to the language of classical logic, then InqB can be seen as a *conservative extension* of CPL: if we restrict ourselves to classical formulas, then entailment in InqB simply amounts to entailment in classical logic (cf. Ciardelli 2022, p. 78).

Proposition 1.4.5 (Conservativity over Classical Logic). *Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}^{\mathsf{B}}_c$ be a set of classical formulas. We have $\Gamma \models \alpha$ if and only if α is entailed by Γ in classical propositional logic.*

However, one can also take a different perspective on inquisitive logic. This perspective was examined in detail by Ciardelli (2009) as well as Ciardelli and Roelofsen (2011) and treats InqB as logic which is *intermediate* between CPL and intuitionistic propositional logic (IPL). Let us elaborate a bit more on this. First of all, if we identify $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ with the ordinary disjunction operator of classical logic, then every formula falsified by some possible world in CPL is clearly also falsified by the corresponding singleton state in InqB. As a consequence, CPL can be conceived as an extension of InqB, so we have InqB $\subseteq$ CPL.[7] Observe that this inclusion must be strict: for example, the formula $?p = p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg p$ is only valid in CPL, but not in InqB. On the other hand, it is also possible to show that, if $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ is identified with the ordinary disjunction of intuitionistic logic, then InqB can be seen as an extension of IPL, i.e., we have IPL $\subseteq$ InqB. Again, note that this inclusion is strict: for instance, the formula $\neg\neg p \to p$ is only valid in InqB, but not in IPL. Putting things together, we thus obtain the following proposition (cf. Ciardelli and Roelofsen 2011, p. 71).

Proposition 1.4.6. *Suppose that the inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ is identified with the disjunction operator of intuitionistic and classical logic, respectively. Then we have* IPL $\subsetneq$ InqB $\subsetneq$ CPL.

In this sense, inquisitive logic is in fact 'intermediate' between IPL and CPL. This, however, does *not* mean that InqB is also an *intermediate logic* in the usual sense.[8] The reason is that InqB is not closed under uniform substitution: for example, the classical formula $\neg\neg p \to p$ is valid in InqB, but the substitution instance $\neg\neg ?p \to ?p$ is not (cf. Ciardelli and Roelofsen 2011, p. 67).

1.5 Standard Axiomatizations of InqB

In preparation of the main part of this thesis, we now want to recall some standard axiomatizations of inquisitive logic. Throughout this thesis, we will often make use of these axiomatizations in order to establish the completeness of our proof systems in a proof-theoretical manner. Arguably the most widespread axiomatization of InqB today is a system of natural deduction, which was first described by Ciardelli (2016b). This system, henceforth denoted by NinqB, is obtained by extending a standard natural deduction system for intuitionistic propositional logic (see Figure 1.3) with two additional rules, denoted by (split) and (dne), respectively (see Figure 1.4).

Let us briefly comment on the rules of this system. As can be seen, the intuitionistic base calculus presented in Figure 1.3 simply comprises the usual introduction and elimination rules for the connectives, together with the intuitionistic absurdity rule (efq), accounting for the validity of the principle of explosion (*ex falso quodlibet*). Observe that, in the rules $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} I$ and $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} E$, the inquisitive disjunction operator $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ now takes the role of the 'ordinary' disjunction of intuitionistic logic. Consider now the special rules depicted in Figure 1.4. Importantly, these rules come with a side condition, saying that α must be a *classical formula* in the sense of Definition 1.2.3. Without this restriction, neither of the two rules would be sound for InqB.[9] The *split rule*, denoted by (split), allows to distribute a classical antecedent over an inquisitive disjunction, which accounts for the left-to-right direction of the split equivalence established in Proposition 1.4.2 (the other direction of the equivalence is already derivable by means of the intuitionistic rules). The *double negation rule*, on the other hand, is denoted by (dne) and allows to infer a classical formula α from its dou-

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[7] Here and in the following, we adopt the convention of identifying a logic with the set of its validities. Thus, InqB refers to the set of all formulas valid in inquisitive logic and CPL refers to the set of all formulas valid in classical logic.

[8] Recall that an *intermediate logic* is a consistent extension of IPL that is closed under modus ponens and uniform substitution (see, e.g., Chagrov and Zakharyaschev 1997, p. 109). For further information about the connections between intermediate logics and InqB, we refer to Ciardelli (2009) as well as Ciardelli and Roelofsen (2011).

[9] For example, if we would allow α to range over arbitrary formulas in the rule (split), then we would be able to derive the invalid formula $(?p \to p) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} (?p \to \neg p)$ from the intuitionistically valid formula $?p \to ?p$. And if we would allow α to be non-classical in the rule (dne), then NinqB would simply be a proof system for classical logic.

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge I \qquad \frac{\varphi_1 \wedge \varphi_2}{\varphi_i} \wedge E \quad (i = 1, 2) \qquad \frac{\varphi_i}{\varphi_1 \vee \varphi_2} \vee I \quad (i = 1, 2)$$

$$\frac{\varphi \vee \psi \quad \begin{matrix}[\varphi] \\ \vdots \\ \chi\end{matrix} \quad \begin{matrix}[\psi] \\ \vdots \\ \chi\end{matrix}}{\chi} \vee E \qquad \frac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi} \to I \qquad \frac{\varphi \to \psi \quad \varphi}{\psi} \to E \qquad \frac{\bot}{\varphi} \text{(efq)}$$

Figure 1.3: A natural deduction system for intuitionistic propositional logic (IPL). The inquisitive disjunction operator $\vee$ is identified with the ordinary intuitionistic disjunction.

$$\frac{\alpha \to (\varphi \vee \psi)}{(\alpha \to \varphi) \vee (\alpha \to \psi)} \text{(split)} \qquad \frac{\neg\neg\alpha}{\alpha} \text{(dne)}$$

Figure 1.4: Special rules of the natural deduction system NinqB described by Ciardelli (2016b). In either case, we require α to be a classical formula, so we must have $\alpha \in \mathcal{L}_c^{\mathsf{B}}$.

ble negation $\neg\neg\alpha$. This accounts for the fact that, in virtue of Proposition 1.3.5, every classical formula does indeed behave 'classically' in InqB, in the sense that we have $\alpha \equiv \neg\neg\alpha$ for all $\alpha \in \mathcal{L}_c^{\mathsf{B}}$.

Definition 1.5.1 (The System NinqB)**.** We define NinqB to be the natural deduction system consisting of the 'intuitionistic' rules given in Figure 1.3 and the special rules given in Figure 1.4.

The *provability relation* of NinqB is denoted by $\vdash_{\mathsf{N}}$ and defined in the usual way. That is, given any set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$, we write $\Gamma \vdash_{\mathsf{N}} \varphi$ and say that φ is *provable* from Γ in NinqB, if there exists a deduction $\mathcal{D}$ in NinqB such that φ is the conclusion of $\mathcal{D}$ and all open hypotheses of $\mathcal{D}$ are contained in Γ. In this case, $\mathcal{D}$ is also said to be a *deduction for* $\Gamma \vdash_{\mathsf{N}} \varphi$.

The completeness of NinqB is established by a canonical model construction. To make things precise, let $\Gamma \subseteq \mathcal{L}_c^{\mathsf{B}}$ be a set of classical formulas.[10] We say that Γ is *consistent*, if $\Gamma \nvdash_{\mathsf{N}} \bot$. And we say that Γ is *maximally consistent*, if Γ is consistent and there is no proper extension $\Delta \supsetneq \Gamma$ with $\Delta \subseteq \mathcal{L}_c^{\mathsf{B}}$ such that Δ is also consistent. The *canonical model* for InqB is defined to be the pair $M_c := \langle W_c, V_c \rangle$, where W_c is the set of all maximally consistent sets of classical formulas and V_c is given by $V_c(\Gamma, p) = 1 :\Leftrightarrow p \in \Gamma$, for all $\Gamma \in W_c$ and $p \in \mathsf{P}$. As shown by Ciardelli (2016b, pp. 90–91), one can now prove a support-based generalization of the well-known *truth lemma*, familiar from completeness proofs for classical logic. This is known as the *support lemma* and says that, for every state $S \subseteq W_c$ and for every formula $\varphi \in \mathcal{L}^{\mathsf{B}}$, we have $M_c, S \models \varphi$ if and only if $M_c, \bigcap S \vdash_{\mathsf{N}} \varphi$. Using this fact, it is now possible to establish the desired completeness result.

Theorem 1.5.2 (Soundness and Completeness)**.** *The system* NinqB *is sound and complete with respect to* InqB*. That is, for every* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$*, we have:* $\Gamma \vdash_{\mathsf{N}} \varphi$ *if and only if* $\Gamma \models \varphi$*.*

A proof is provided by Ciardelli (2016b, pp. 85–92). The soundness is shown by a straightforward induction on the structure of a deduction for $\Gamma \vdash_{\mathsf{N}} \varphi$ in NinqB. For the completeness part,

[10] The restriction to classical formulas is not essential here. It is only used in order to simplify the discussion a bit.

Axioms:

(A1) $\varphi \to (\psi \to \varphi)$,
(A2) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$,
(A3) $(\varphi \wedge \psi) \to \varphi$ and $(\varphi \wedge \psi) \to \psi$,
(A4) $\varphi \to (\psi \to (\varphi \wedge \psi))$,
(A5) $\varphi \to (\varphi \vee\!\!\!\vee \psi)$ and $\psi \to (\varphi \vee\!\!\!\vee \psi)$,
(A6) $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee\!\!\!\vee \psi) \to \chi))$,
(A7) $\bot \to \varphi$.

The only rule of inference is *modus ponens*: from $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, infer $\Gamma \vdash \psi$.

Figure 1.5: A Hilbert-style system for intuitionistic propositional logic (IPL). The inquisitive disjunction operator $\vee\!\!\!\vee$ is again identified with the ordinary intuitionistic disjunction.

Axioms:

(IPL) All axioms of the 'intuitionistic' system given in Figure 1.5,
(Split) $(\alpha \to (\varphi \vee\!\!\!\vee \psi)) \to ((\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi))$, where $\alpha \in \mathcal{L}_c^{\mathsf{B}}$ is classical,
(DN) $\neg\neg\alpha \to \alpha$, where $\alpha \in \mathcal{L}_c^{\mathsf{B}}$ is classical.

The only rule of inference is *modus ponens*: from $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, infer $\Gamma \vdash \psi$.

Figure 1.6: The Hilbert-style system HinqB.

assume that $\Gamma \nvdash_{\mathsf{N}} \varphi$. One can then show that there exists some resolution $\Delta \in \mathcal{R}(\Gamma)$ such that, for all $\alpha \in \mathcal{R}(\varphi)$, it holds $\Delta \nvdash_{\mathsf{N}} \alpha$. Suppose now that $\mathcal{R}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$ and let i with $1 \leq i \leq n$ be arbitrary. Since $\Delta \nvdash_{\mathsf{N}} \alpha_i$, it is clear that the set $\Delta \cup \{\neg\alpha_i\}$ must be consistent. Hence, by a suitable version of Lindenbaum's lemma, it can be extended to a maximally consistent set $\Theta_i \subseteq \mathcal{L}_c^{\mathsf{B}}$. Let now $S \subseteq W_c$ be the state given by $S := \{\Theta_1, \ldots, \Theta_n\}$. Using the support lemma, one can then show that we have $M_c, S \models \Gamma$ and $M_c, S \not\models \varphi$, so it follows $\Gamma \not\models \varphi$.

For our purposes, it will often be more convenient to base our considerations on a *Hilbert-style system* for InqB, rather than on a natural deduction system. Such a Hilbert-style system can easily be obtained by converting the natural deduction rules of NinqB into corresponding axiom schemes. The resulting proof system, henceforth denoted by HinqB, is presented in Figure 1.6. As can be seen, HinqB is obtained by extending a standard Hilbert-style system for intuitionistic logic (see Figure 1.5) with two additional axiom schemes, denoted by (Split) and (DN), respectively. These axiom schemes are used in order to simulate the effect of the split rule and the double negation rule included in the natural deduction system NinqB. Note that, as before, we require α to be a *classical formula*. Furthermore, in HinqB, the only rule of inference is modus ponens.

Definition 1.5.3 (The System HinqB)**.** We define HinqB to be the Hilbert-style system depicted in Figure 1.6. The provability relation of HinqB is denoted by $\vdash_{\mathsf{H}}$ and inductively defined in the usual way. That is, for every set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$, we write $\Gamma \vdash_{\mathsf{H}} \varphi$ and say that φ is *provable* from Γ in HinqB, if at least one of the following three conditions is satisfied:

(i) φ is an element of Γ,
(ii) φ is an instance of one of the axiom schemes of HinqB,

(iii) There exists some $\psi \in \mathcal{L}^{\mathsf{B}}$ such that $\Gamma \vdash_{\mathsf{H}} \psi$ and $\Gamma \vdash_{\mathsf{H}} \psi \to \varphi$.

If the last condition is satisfied, we also say that $\Gamma \vdash_{\mathsf{H}} \varphi$ is obtained from $\Gamma \vdash_{\mathsf{H}} \psi$ and $\Gamma \vdash_{\mathsf{H}} \psi \to \varphi$ by an application of modus ponens. The easiest way to prove the soundness and completeness of this system is to show that HinqB is *equivalent* to the natural deduction system NinqB, in the sense that everything provable in HinqB is also provable in NinqB and vice versa. First of all, using induction on the definition of $\Gamma \vdash_{\mathsf{H}} \varphi$, it is easy to show that the provability relation $\vdash_{\mathsf{H}}$ is *monotonic*, i.e., for all $\Gamma, \Delta \subseteq \mathcal{L}^{\mathsf{B}}$ and $\varphi \in \mathcal{L}^{\mathsf{B}}$, if we have $\Gamma \vdash_{\mathsf{H}} \varphi$ and $\Gamma \subseteq \Delta$, then also $\Delta \vdash_{\mathsf{H}} \varphi$. Using this fact, it is now possible to prove the *deduction theorem* for HinqB.

Theorem 1.5.4 (Deduction Theorem)**.** *In* HinqB*, we have* $\Gamma, \varphi \vdash_{\mathsf{H}} \psi$ *if and only if* $\Gamma \vdash_{\mathsf{H}} \varphi \to \psi$*.*

Proof. We first prove the right-to-left direction. Suppose that $\Gamma \vdash_{\mathsf{H}} \varphi \to \psi$. By monotonicity, this yields $\Gamma, \varphi \vdash_{\mathsf{H}} \varphi \to \psi$. Furthermore, by definition of $\vdash_{\mathsf{H}}$, we also have $\Gamma, \varphi \vdash_{\mathsf{H}} \varphi$. Now, from $\Gamma, \varphi \vdash_{\mathsf{H}} \varphi$ and $\Gamma, \varphi \vdash_{\mathsf{H}} \varphi \to \psi$, it follows $\Gamma, \varphi \vdash_{\mathsf{H}} \psi$ by an application of modus ponens.

In order to prove the left-to-right direction, suppose $\Gamma, \varphi \vdash_{\mathsf{H}} \psi$. We show $\Gamma \vdash_{\mathsf{H}} \varphi \to \psi$ by induction on the definition of $\Gamma, \varphi \vdash_{\mathsf{H}} \psi$. By Definition 1.5.3, there are the following possibilities.

Case 1: Suppose that $\psi \in \Gamma \cup \{\varphi\}$, i.e., we have $\psi \in \Gamma$ or $\psi = \varphi$. If it holds $\psi \in \Gamma$, then we also have $\Gamma \vdash_{\mathsf{H}} \psi$ by definition of $\vdash_{\mathsf{H}}$. Using axiom (A1) and an application of modus ponens, this yields $\Gamma \vdash_{\mathsf{H}} \varphi \to \psi$. On the other hand, if $\psi = \varphi$, then $\Gamma \vdash_{\mathsf{H}} \varphi \to \psi$ follows immediately from the fact that $\varphi \to \varphi$ is already provable in the intuitionistic base calculus given in Figure 1.5.

Case 2: Suppose that ψ is an axiom of HinqB. Then, by definition of $\vdash_{\mathsf{H}}$, we also have $\Gamma \vdash_{\mathsf{H}} \psi$. Now, using axiom (A1) and an application of modus ponens, it follows $\Gamma \vdash_{\mathsf{H}} \varphi \to \psi$.

Case 3: Suppose that $\Gamma, \varphi \vdash_{\mathsf{H}} \psi$ is obtained from $\Gamma, \varphi \vdash_{\mathsf{H}} \chi$ and $\Gamma, \varphi \vdash_{\mathsf{H}} \chi \to \psi$ by an application of modus ponens. Then, by induction hypothesis, we must also have $\Gamma \vdash_{\mathsf{H}} \varphi \to \chi$ and $\Gamma \vdash_{\mathsf{H}} \varphi \to (\chi \to \psi)$. Because $(\varphi \to (\chi \to \psi)) \to ((\varphi \to \chi) \to (\varphi \to \psi))$ is an instance of axiom scheme (A2), we now obtain $\Gamma \vdash_{\mathsf{H}} \varphi \to \psi$ by two applications of modus ponens. □

We are now ready to prove that HinqB is equivalent to the natural deduction system NinqB.

Theorem 1.5.5. *Let* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ *be a set of formulas. We have* $\Gamma \vdash_{\mathsf{H}} \varphi$ *in the Hilbert-style system* HinqB *if and only if* $\Gamma \vdash_{\mathsf{N}} \varphi$ *holds in the natural deduction system* NinqB*.*

Proof. For the left-to-right direction, one proceeds by induction on the definition of $\Gamma \vdash_{\mathsf{H}} \varphi$. This is very easy, since all axioms of HinqB are clearly derivable in NinqB and modus ponens corresponds to $\to E$. For the right-to-left direction, one can use induction on the structure of a natural deduction proof $\mathcal{D}$ for $\Gamma \vdash_{\mathsf{N}} \varphi$. This is also straightforward, since most of the rules of NinqB correspond directly to some axiom of HinqB and the discharging of hypotheses can be 'simulated' using the deduction theorem for HinqB. For example, suppose that $\mathcal{D}$ ends with an application of $\to I$. In this case, φ is of the form $\varphi = \psi \to \chi$ and $\mathcal{D}$ contains an immediate subdeduction for $\Gamma, \psi \vdash_{\mathsf{N}} \chi$. By induction hypothesis, we now have $\Gamma, \psi \vdash_{\mathsf{H}} \chi$ in our Hilbert-style system HinqB. But then, by the deduction theorem, it follows $\Gamma \vdash_{\mathsf{H}} \psi \to \chi$ and therefore $\Gamma \vdash_{\mathsf{H}} \varphi$. □

Corollary 1.5.6 (Soundness and Completeness)**.** *The system* HinqB *is sound and complete with respect to* InqB*. That is, for every* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$*, we have:* $\Gamma \vdash_{\mathsf{H}} \varphi$ *if and only if* $\Gamma \vDash \varphi$*.*

Proof. The statement follows immediately from Theorems 1.5.2 and 1.5.5. □

It is worth noting that there is also an alternative Hilbert-style axiomatization for InqB, which was shown to be sound and complete by Ciardelli (2009) as well as Ciardelli and Roelofsen (2011). This system, presented in Figure 1.7, is defined in the same way as our system HinqB, except that the double negation axiom is now restricted to atomic formulas and the split axiom is replaced

Axioms:

(IPL)	All axioms of the 'intuitionistic' system given in Figure 1.5,
(KP)	$(\neg\varphi \to (\psi \vee\!\!\!\vee \chi)) \to ((\neg\varphi \to \psi) \vee\!\!\!\vee (\neg\varphi \to \chi))$,
(ADN)	$\neg\neg p \to p$, where $p \in \mathsf{P}$ is atomic.

The only rule of inference is *modus ponens*: from $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, infer $\Gamma \vdash \psi$.

Figure 1.7: The Hilbert-style system $\mathsf{HinqB}_{\mathsf{KP}}$.

by the so-called *Kreisel-Putnam axiom* (KP).[11] Observe that (KP) has almost the same form as our split axiom: the only difference is that the classical formula involved in (Split) is now replaced by a negated formula $\neg\varphi$. In virtue of Proposition 1.3.5, the two axiom schemes are easily seen to be equivalent in InqB. For one thing, every negated formula $\neg\varphi$ is equivalent to its classical variant $\neg\varphi^{cl}$, so (Split) entails (KP). For another, every classical formula α is equivalent to its double negation $\neg\neg\alpha$, so (KP) entails (Split). However, later on in this thesis, we will also consider an *intuitionistic* variant of InqB, in which $\vee\!\!\!\vee$-free formulas do not in general validate the double negation law. In order to allow for a smooth transition to this modified setting, we decided to adopt the Hilbert-style system HinqB rather than the alternative system presented in Figure 1.7.

[11] The Kreisel-Putnam axiom has some historical significance: Łukasiewicz (1952) famously conjectured that IPL is the strongest intermediate logic having the disjunction property. This was disproved by Kreisel and Putnam (1957), who used the axiom scheme (KP) in order to construct a proper extension of IPL which still satisfies this property.

Chapter 2

Natural Deduction for InqB

Readers familiar with proof theory may have noticed that there is a serious lack of 'harmony' in the standard natural deduction system for inquisitive logic: the split rule (split), depicted in Figure 1.4, appears to be rather artificial and does not fit into the scheme of introduction and elimination rules typical of natural deduction systems in the style of Gentzen (1935a; 1935b) and Prawitz (1965). As a result, it is difficult to come up with a reasonable notion of a 'detour' in a deduction and to establish a suitable normal form result for NinqB. In order to overcome these limitations, we will now present a more well-behaved natural deduction system for InqB. Our system, henceforth referred to as NinqB^+, is based on an extended natural deduction formalism in which not only *formulas*, but also *rules* can act as assumptions that may be discharged in the course of a derivation. The basic idea is inspired by the so-called *calculus of higher-level rules* developed by Schroeder-Heister (1981; 1984; 2014) as a 'natural extension of natural deduction'. We will see that, in the extended setting, the split rule can be fully 'absorbed' by a suitably reformulated version of the elimination rule for $\mathbb{V}$. This makes it possible to prove a *normalization theorem* for our system: every deduction in NinqB^+ can be reduced to a deduction without detours.

The chapter is structured as follows. In Section 2.1, we will define some basic notions and give a detailed exposition of the natural deduction system NinqB^+. In Section 2.2, we will then show that NinqB^+ is equivalent to the standard natural deduction system for inquisitive logic: everything which is provable in NinqB^+ is also provable in NinqB and vice versa. The soundness and completeness of NinqB^+ then follows as an immediate corollary. In Section 2.3, we will provide a precise definition of the notion of a detour and give a brief outline of the normalization procedure for our system. Afterwards, in Section 2.4, we will turn to the proof of our normalization theorem. More precisely, we will describe an effective procedure that allows to transform any deduction in our system into a deduction containing no detours. This requires an extension of the technique used in classical and intuitionistic logic and will make up the main part of this chapter. In Section 2.5, finally, we will derive several corollaries from our normalization theorem. It will turn out that our system only satisfies a weak form of the subformula property, so it is not an analytic proof system in a strict sense. However, for a certain special case, a full subformula property will be obtained. Moreover, the subformula property for NinqB^+ is still strong enough to establish various properties of inquisitive logic in a purely proof-theoretical way.

2.1 The Natural Deduction System NinqB^+

We start by providing a formal exposition of our proof system. As explained above, we will use an extended natural deduction formalism which is strongly inspired by the so-called *calculus of higher-level rules* introduced by Schroeder-Heister (1981; 1984; 2014). The basic idea can be explained as follows: in ordinary natural deduction systems, formulas can be used as *assumptions* or *hypotheses*, and by applying certain rules of inference, these assumptions may become *discharged* in the course of a derivation. The extended setting considered here generalizes this idea: instead of allowing only *formulas* to act as assumptions, we now also allow *rules* to be assumed and discharged in a natural deduction proof. More precisely, every proof tree in our system will be constructed from two types of rules: on the one hand, our system comprises a finite number of *primitive rules*—namely, an introduction rule and an elimination rule for each connective and certain additional rules for the falsum constant $\perp$. The primitive rules are considered to be a 'static' component of our proof system, so applications of these rules never get discharged by other rules. On the other hand, we also allow proof trees to contain one or more applications of so-called *non-primitive rules*. These rules are treated as *assumptions* (made purely for the sake of argument) and have to be discharged by other rule applications in the proof tree.

Let us now turn to the technical details. In many respects, our system is much simpler than the framework described by Schroeder-Heister (1981; 1984). Most importantly, Schroeder-Heister also considers non-primitive rules that are able to discharge other non-primitive rules, which in turn may discharge further non-primitive rules, and so on. In other words, Schroeder-Heister's system is capable of dealing with rules of *arbitrary level*, where the level of a rule is taken to represent the complexity of the rule: formulas are identified with rules of level 0 and a rule of level $n \geq 1$ is built up from rules of level at most $n-1$ (cf. Schroeder-Heister 1981, p. 47).[1] In our system, the discharging of non-primitive rules will be accomplished only by one rule—namely, a suitably reformulated version of the elimination rule for the connective $\mathbin{\vee\mkern-13mu\vee}$. Thus, in particular, non-primitive rules are never allowed to discharge any other non-primitive rules or open hypotheses in a proof tree. Instead, every non-primitive rule of our system will simply be of the form

$$\frac{\alpha_1 \qquad \alpha_2 \qquad \cdots \qquad \alpha_n}{\varphi}$$

where $\alpha_1, \ldots, \alpha_n \in \mathcal{L}_c^{\mathsf{B}}$ are classical formulas (acting as the *premises* of the rule) and $\varphi \in \mathcal{L}^{\mathsf{B}}$ is an arbitrary formula (acting as the *conclusion* of the rule).[2] Given any finite set of classical formulas $\Theta = \{\alpha_1, \ldots, \alpha_n\}$ and an arbitrary formula φ, we will also use '$\Theta \Rightarrow \varphi$' as a name for the non-primitive rule with premises $\alpha_1, \ldots, \alpha_n$ and conclusion φ.[3] There are now two important things to note here. First of all, we always require the premises of a non-primitive rule to be *classical formulas*, so these premises are not allowed to contain occurrences of the inquisitive disjunction operator $\mathbin{\vee\mkern-13mu\vee}$ (see Definition 1.2.3). The importance of this restriction will become apparent later on, when we will establish the soundness of our proof system. Secondly, every non-primitive rule is assumed to be *non-schematic*, in the sense that both the premises $\alpha_1, \ldots, \alpha_n$ and the conclusion φ are considered to be *fixed* formulas, rather than syntactic meta-variables that may be substituted by arbitrary formulas from the language. In other words, the non-primitive rules of our system are always *specific*: they allow us to infer one specific formula φ from a fixed set of n specific formulas $\alpha_1, \ldots, \alpha_n$. As soon as one of the formulas $\varphi, \alpha_1, \ldots, \alpha_n$ is replaced by some other formula, we obtain a *distinct* non-primitive rule that will *not* be identified with the original one.

Definition 2.1.1 (Non-Primitive Rule)**.** By a *non-primitive rule*, we mean any (non-schematic) rule that allows to infer a specific formula $\varphi \in \mathcal{L}^{\mathsf{B}}$ from a specific set of n classical formulas $\alpha_1, \ldots, \alpha_n \in \mathcal{L}_c^{\mathsf{B}}$. Given any finite set of classical formulas $\Theta \subseteq \mathcal{L}_c^{\mathsf{B}}$ and an arbitrary $\varphi \in \mathcal{L}^{\mathsf{B}}$, we will also write $\Theta \Rightarrow \varphi$ for the non-primitive rule with conclusion φ and set of premises Θ.

As we will see below, every application of a non-primitive rule in a proof tree will be either *open* or it will be *discharged*. The notational conventions regarding the discharging of non-primitive

[1] In his 1984 paper, Schroeder-Heister uses a slightly different level assignment, but the basic idea is the same.

[2] So, using the terminology of Schroeder-Heister (1981), one could say that all our non-primitive rules are of level ≤ 1.

[3] Thus, in particular, we do not assign any order to the premises $\alpha_1, \ldots, \alpha_n$ of a non-primitive rule.

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}\wedge I \qquad \frac{\varphi_1 \wedge \varphi_2}{\varphi_i}\wedge E \quad (i=1,2) \qquad \frac{\varphi_i}{\varphi_1 \Vee \varphi_2}\Vee I \quad (i=1,2) \qquad \frac{\bot}{\varphi}\,(\mathsf{efq})$$

$$\frac{\begin{array}{c}[\varphi]\\ \mathcal{D}\\ \psi\end{array}}{\varphi \to \psi}\to I \qquad \frac{\varphi \to \psi \qquad \varphi}{\psi}\to E \qquad \frac{\begin{array}{c}[\Theta]\\ \mathcal{D}_1\\ \varphi \Vee \psi\end{array} \qquad \begin{array}{c}[\Theta \Rightarrow \varphi]\\ \mathcal{D}_2\\ \chi\end{array} \qquad \begin{array}{c}[\Theta \Rightarrow \psi]\\ \mathcal{D}_3\\ \chi\end{array}}{\chi}\Vee E_+ \qquad \frac{\begin{array}{c}[\neg\alpha]\\ \mathcal{D}\\ \bot\end{array}}{\alpha}\,(\mathsf{raa})$$

Figure 2.1: Primitive rules of the natural deduction system NinqB^+. In the rule $\Vee E_+$, we require Θ to be a finite set of classical formulas. And in the rule (raa), α must be classical.

rules will be essentially the same as those used for ordinary hypotheses in natural deduction. So, given a non-primitive rule $\Theta \Rightarrow \varphi$ and a proof tree $\mathcal{D}$ with conclusion ψ, we will use the notation

$$\begin{array}{c}\Theta \Rightarrow \varphi\\ \mathcal{D}\\ \psi\end{array}$$

in order to emphasize the fact that $\mathcal{D}$ contains some (possibly zero) undischarged applications of the rule $\Theta \Rightarrow \varphi$. By enclosing a non-primitive rule in square brackets, as in $[\Theta \Rightarrow \varphi]$, we indicate that the corresponding applications of the rule have been *discharged*. Note that, as an important special case, we also allow the set of premises Θ of a non-primitive rule to be *empty*. Clearly, using a non-primitive rule of the form $\emptyset \Rightarrow \varphi$ in a proof tree has exactly the same effect as assuming φ as a *hypothesis* in the tree, so $\emptyset \Rightarrow \varphi$ may simply be identified with φ.

We are now ready to define our natural deduction system NinqB^+. The *primitive rules* of our system are presented in Figure 2.1. As can be seen, NinqB^+ is obtained from the natural deduction system given in Figure 1.3 by omitting the ordinary elimination rule for $\Vee$, and by adding two special rules, denoted by $\Vee E_+$ and (raa), respectively.[4] Let us give a brief explanation of these special rules. The rule (raa) is a restricted version of *reductio ad absurdum* and formalizes the principle of proof by contradiction: if we can derive $\bot$ from a hypothesis $\neg\alpha$, then we are entitled to infer α and to discharge all open occurrences of the hypothesis $\neg\alpha$. Note that, in order to make sure that NinqB^+ is sound with respect to InqB, this rule is restricted to *classical formulas* (without this restriction, we would simply obtain a proof system for classical logic).

The rule $\Vee E_+$, on the other hand, is a generalization of the ordinary elimination rule for $\Vee$ and should be read as follows: let $\mathcal{D}_1$ be a proof tree ending with some formula $\varphi \Vee \psi$ and suppose that there exists a finite set of classical formulas $\Theta \subseteq \mathcal{L}_c^{\mathsf{B}}$ such that each element of Θ occurs as an open hypothesis in $\mathcal{D}_1$. Moreover, let $\mathcal{D}_2$ and $\mathcal{D}_3$ be two proof trees ending with χ such that $\mathcal{D}_2$ contains some undischarged applications of the non-primitive rule $\Theta \Rightarrow \varphi$, and $\mathcal{D}_3$ contains some undischarged applications of the non-primitive rule $\Theta \Rightarrow \psi$. Then, using the rule $\Vee E_+$, we are entitled to infer χ and to discharge all open occurrences of the hypotheses from Θ in $\mathcal{D}_1$, all undischarged applications of the non-primitive rule $\Theta \Rightarrow \varphi$ in $\mathcal{D}_2$, and all undischarged applications of the non-primitive rule $\Theta \Rightarrow \psi$ in $\mathcal{D}_3$. Note that, by choosing $\Theta = \emptyset$ in an application of $\Vee E_+$, one can also obtain the ordinary elimination rule for $\Vee$ as a special case:

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \varphi \Vee \psi\end{array} \qquad \begin{array}{c}\overline{\varphi}\,u\\ \mathcal{D}_2\\ \chi\end{array} \qquad \begin{array}{c}\overline{\psi}\,u\\ \mathcal{D}_3\\ \chi\end{array}}{\chi}\Vee E_+^u$$

[4] Of course, by the 'ordinary elimination rule for $\Vee$', we always mean the rule which is just like the standard elimination rule for disjunction in intuitionistic logic, except that $\Vee$ is identified with the intuitionistic disjunction operator.

where the label u is used in order to indicate that the corresponding applications of the rules $\emptyset \Rightarrow \varphi$ and $\emptyset \Rightarrow \psi$ are discharged by the application of $\mathbin{\vee\!\!\vee} E_+$ at the bottom of the proof tree.

Definition 2.1.2 (Primitive Rules of the System NinqB^+)**.** We define NinqB^+ to be the 'higher-level' natural deduction system whose primitive rules are the ones given in Figure 2.1.

Any proof tree built up using the primitive rules of our system and arbitrary (open or discharged) applications of non-primitive rules will be referred to as a *quasi-deduction*. Throughout this chapter, quasi-deductions will be denoted by the meta-variables $\mathcal{D}$, $\mathcal{D}_1$, $\mathcal{D}_2$, etc. Given a formula $\varphi \in \mathcal{L}^{\mathsf{B}}$ and an arbitrary set of formulas and non-primitive rules Γ, we will write $\Gamma \vdash^+_{\mathsf{N}} \varphi$ and say that φ *is derivable from* Γ, if there exists a quasi-deduction $\mathcal{D}$ such that φ is the conclusion of $\mathcal{D}$ and all open hypotheses and open applications of non-primitive rules in $\mathcal{D}$ are contained in Γ. In this case, $\mathcal{D}$ is also said to be a *quasi-deduction for* $\Gamma \vdash^+_{\mathsf{N}} \varphi$. By a *deduction* in the system NinqB^+, we mean a quasi-deduction in which all applications of non-primitive rules are discharged by some application of $\mathbin{\vee\!\!\vee} E_+$. So, in particular, every deduction is also a quasi-deduction, but not the other way around. Given any set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$, we will say that φ *is provable from* Γ in our system, if there exists a deduction $\mathcal{D}$ for $\Gamma \vdash^+_{\mathsf{N}} \varphi$ in NinqB^+. In order to see a concrete example, let $\alpha, \beta \in \mathcal{L}^{\mathsf{B}}_c$ be classical formulas and let $\varphi, \psi, \chi \in \mathcal{L}^{\mathsf{B}}$ be arbitrary formulas. We may then construct the following quasi-deduction in our system:

$$
\dfrac{
\dfrac{\dfrac{\alpha \to (\beta \to (\varphi \mathbin{\vee\!\!\vee} \psi)) \quad [\alpha]^u}{\beta \to (\varphi \mathbin{\vee\!\!\vee} \psi)}{\to}E \quad [\beta]^u}{\varphi \mathbin{\vee\!\!\vee} \psi}{\to}E
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\chi \to \alpha \quad [\chi]^v}{\alpha}{\to}E \quad \beta}{\varphi}\,u}{\chi \to \varphi}{\to}I^v}{(\chi \to \varphi) \vee (\neg\chi \to \psi)}\vee I
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\alpha \wedge \chi}{\alpha}\wedge E \quad \dfrac{\dfrac{[\neg\chi]^w \quad \chi}{\bot}{\to}E}{\beta}\text{(efq)}}{\psi}\,u}{\neg\chi \to \psi}{\to}I^w}{(\chi \to \varphi) \vee (\neg\chi \to \psi)}\vee I
}{(\chi \to \varphi) \vee (\neg\chi \to \psi)}\mathbin{\vee\!\!\vee} E^u_+
$$

As indicated by the label u, all applications of the non-primitive rules $\alpha, \beta \Rightarrow \varphi$ and $\alpha, \beta \Rightarrow \psi$ in this quasi-deduction are discharged by the application of $\mathbin{\vee\!\!\vee} E_+$ at the bottom. Hence, this quasi-deduction contains no undischarged applications non-primitive rules, so it is also a *deduction*.

In order to conclude this section, let us now introduce some useful terminology. The set of classical formulas Θ involved in an application of $\mathbin{\vee\!\!\vee} E_+$ will be referred to as the set of *auxiliary formulas* of this application. By the *auxiliary rules* of an application of $\mathbin{\vee\!\!\vee} E_+$, we will henceforth mean the non-primitive rules $\Theta \Rightarrow \varphi$ and $\Theta \Rightarrow \psi$ which are discharged in the course of this application. And the *side formulas* of an application of $\mathbin{\vee\!\!\vee} E_+$ are the two formulas φ and ψ serving as the conclusions of the auxiliary rules $\Theta \Rightarrow \varphi$ and $\Theta \Rightarrow \psi$. It is easy to see that applications of the rule $\mathbin{\vee\!\!\vee} E_+$ may also involve some *redundancy*. For example, if one of the auxiliary rules $\Theta \Rightarrow \varphi$ and $\Theta \Rightarrow \psi$ of such an application does not occur in the corresponding quasi-deduction ending with χ, then the application of $\mathbin{\vee\!\!\vee} E_+$ can always be eliminated. The process is illustrated below, where we assume that $\Theta \Rightarrow \varphi$ does not occur as an undischarged rule in the subtree $\mathcal{D}_2$:

$$
\dfrac{\begin{matrix}[\Theta]^u \\ \mathcal{D}_1 \\ \varphi \mathbin{\vee\!\!\vee} \psi\end{matrix} \qquad \begin{matrix} \\ \mathcal{D}_2 \\ \chi\end{matrix} \qquad \begin{matrix}[\Theta \Rightarrow \psi]^u \\ \mathcal{D}_3 \\ \chi\end{matrix}}{\chi}\mathbin{\vee\!\!\vee} E^u_+
\qquad \text{converts to} \qquad
\begin{matrix}\mathcal{D}_2 \\ \chi\end{matrix}
$$

Furthermore, if an auxiliary formula α does not occur as an open hypothesis in the quasi-deduction ending with the premise $\varphi \mathbin{\vee\!\!\vee} \psi$, then one can always simplify the application of $\mathbin{\vee\!\!\vee} E_+$ by removing α from the set of auxiliary formulas. This is illustrated by the following conversion:

$$
\dfrac{\begin{matrix}[\Theta]^u \\ \mathcal{D}_1 \\ \varphi \mathbin{\vee\!\!\vee} \psi\end{matrix} \qquad \begin{matrix}[\Theta, \alpha \Rightarrow \varphi]^u \\ \mathcal{D}_2 \\ \chi\end{matrix} \qquad \begin{matrix}[\Theta, \alpha \Rightarrow \psi]^u \\ \mathcal{D}_3 \\ \chi\end{matrix}}{\chi}\mathbin{\vee\!\!\vee} E^u_+
\qquad \text{converts to} \qquad
\dfrac{\begin{matrix}[\Theta]^u \\ \mathcal{D}_1 \\ \varphi \mathbin{\vee\!\!\vee} \psi\end{matrix} \qquad \begin{matrix}[\Theta \Rightarrow \varphi]^u \\ \mathcal{D}'_2 \\ \chi\end{matrix} \qquad \begin{matrix}[\Theta \Rightarrow \psi]^u \\ \mathcal{D}'_3 \\ \chi\end{matrix}}{\chi}\mathbin{\vee\!\!\vee} E^u_+
$$

where α is assumed not to occur as an open hypothesis in $\mathcal{D}_1$, and the quasi-deductions $\mathcal{D}_2'$ and $\mathcal{D}_3'$ are obtained from $\mathcal{D}_2$ and $\mathcal{D}_3$ by replacing all open occurrences of the rules $\Theta, \alpha \Rightarrow \varphi$ and $\Theta, \alpha \Rightarrow \psi$ by the simplified rules $\Theta \Rightarrow \varphi$ and $\Theta \Rightarrow \psi$, respectively.[5] For technical reasons, we will henceforth assume that applications of $\mathbb{V}E_+$ never involve any redundancy of this kind. In particular, every quasi-deduction under consideration is assumed to satisfy the conditions mentioned in the following lemma, which is a direct consequence of the observations made above.

Lemma 2.1.3. *Let $\varphi \in \mathcal{L}^{\mathrm{B}}$ be a formula and let Γ be a set of formulas and non-primitive rules. If φ is derivable from Γ in our system, then there exists a quasi-deduction $\mathcal{D}$ for $\Gamma \vdash_{\mathsf{N}}^{+} \varphi$ such that every auxiliary rule and every auxiliary formula of an application of $\mathbb{V}E_+$ in $\mathcal{D}$ does in fact have at least one undischarged occurrence in the corresponding subtree belonging to this application.*

For later purposes, we also need to define a suitable notion of *substitution*. To this end, let $\mathcal{D}$ be a quasi-deduction containing some undischarged applications of a non-primitive rule $\Theta \Rightarrow \varphi$ and let $\mathcal{D}'$ be a quasi-deduction for $\Gamma, \Theta \vdash_{\mathsf{N}}^{+} \varphi$, where Γ is an arbitrary set of formulas and non-primitive rules. In this case, we will write $\mathcal{D}\{\mathcal{D}' : \Theta \Rightarrow \varphi\}$ for the result of replacing every undischarged application of the rule $\Theta \Rightarrow \varphi$ in $\mathcal{D}$ by an occurrence of the tree $\mathcal{D}'$. For example, let Θ be the set of classical formulas $\Theta = \{\alpha, \beta\}$ and let $\mathcal{D}$ be the following quasi-deduction:

$$
\begin{prooftree}
\AxiomC{α}
\AxiomC{α}\AxiomC{$[\beta]^u$}\RightLabel{$\#$}\BinaryInfC{φ}\AxiomC{$[\beta]^u$}\RightLabel{$\wedge I$}\BinaryInfC{$\varphi\wedge\beta$}\RightLabel{$\to I$}\UnaryInfC{$\varphi\to(\varphi\wedge\beta)$}
\AxiomC{α}\AxiomC{$[\neg\beta]^v$}\AxiomC{$[\beta]^u$}\RightLabel{$\to E$}\BinaryInfC{$\bot$}\RightLabel{(raa)^v}\UnaryInfC{β}\RightLabel{$\#$}\BinaryInfC{φ}
\RightLabel{$\to E$}\BinaryInfC{$\varphi\wedge\beta$}\RightLabel{$\wedge E$}\UnaryInfC{β}
\RightLabel{$\#$}\BinaryInfC{φ}\RightLabel{$\to I^u$}\UnaryInfC{$\beta\to\varphi$}
\end{prooftree}
$$

In this quasi-deduction, there are three undischarged applications of the non-primitive rule $\Theta \Rightarrow \varphi$, marked with the symbol $\#$. Let now Γ be a set of formulas and non-primitive rules and let $\mathcal{D}'$ be a quasi-deduction for $\Gamma, \Theta \vdash_{\mathsf{N}}^{+} \varphi$. The quasi-deduction $\mathcal{D}\{\mathcal{D}' : \Theta \Rightarrow \varphi\}$ is then of the form

$$
\begin{prooftree}
\AxiomC{α}
\AxiomC{α}\AxiomC{$[\beta]^u$}\noLine\BinaryInfC{$\mathcal{D}'$}\noLine\UnaryInfC{φ}\AxiomC{$[\beta]^u$}\RightLabel{$\wedge I$}\BinaryInfC{$\varphi\wedge\beta$}\RightLabel{$\to I$}\UnaryInfC{$\varphi\to(\varphi\wedge\beta)$}
\AxiomC{α}\AxiomC{$[\neg\beta]^v$}\AxiomC{$[\beta]^u$}\RightLabel{$\to E$}\BinaryInfC{$\bot$}\RightLabel{(raa)^v}\UnaryInfC{β}\noLine\BinaryInfC{$\mathcal{D}'$}\noLine\UnaryInfC{φ}
\RightLabel{$\to E$}\BinaryInfC{$\varphi\wedge\beta$}\RightLabel{$\wedge E$}\UnaryInfC{β}
\noLine\BinaryInfC{$\mathcal{D}'$}\noLine\UnaryInfC{φ}\RightLabel{$\to I^u$}\UnaryInfC{$\beta\to\varphi$}
\end{prooftree}
$$

In other words, $\mathcal{D}\{\mathcal{D}' : \Theta \Rightarrow \varphi\}$ is obtained from $\mathcal{D}$ by simultaneously substituting the quasi-deduction $\mathcal{D}'$ for every undischarged occurrence of the rule $\Theta \Rightarrow \varphi$. Note that, if some formula from Θ does not actually occur as an open hypothesis in $\mathcal{D}'$, then every subtree in $\mathcal{D}$ ending with the corresponding premise of $\Theta \Rightarrow \varphi$ will simply be omitted in the course of the substitution (in this case, some open hypotheses or undischarged rules in $\mathcal{D}$ may get 'lost', but this is unproblematic). Clearly, by performing such a substitution, we do not change the conclusion of the quasi-deduction $\mathcal{D}$, and $\mathcal{D}\{\mathcal{D}' : \Theta \Rightarrow \varphi\}$ cannot contain any new undischarged rules or open hypotheses that were not already present in $\mathcal{D}$ or $\mathcal{D}'$. In fact, it is easy to verify the following claim.

[5] Note that, by performing this conversion, all subtrees ending with the premise α of the corresponding applications of the rules $\Theta, \alpha \Rightarrow \varphi$ and $\Theta, \alpha \Rightarrow \psi$ are simply omitted from $\mathcal{D}_2$ and $\mathcal{D}_3$. The details are left to the reader.

Fact 2.1.4. *Let Γ and Δ be arbitrary sets of formulas and non-primitive rules, let $\Theta \subseteq \mathcal{L}_c^{\mathsf{B}}$ be a finite set of classical formulas and let $\varphi, \psi \in \mathcal{L}^{\mathsf{B}}$ be arbitrary formulas. If $\mathcal{D}$ is a quasi-deduction for $\Delta, \Theta \Rightarrow \varphi \vdash_{\mathsf{N}}^{+} \psi$ (so $\mathcal{D}$ possibly contains some undischarged applications of the rule $\Theta \Rightarrow \varphi$) and if $\mathcal{D}'$ is a quasi-deduction for $\Gamma, \Theta \vdash_{\mathsf{N}}^{+} \varphi$, then $\mathcal{D}\{\mathcal{D}' : \Theta \Rightarrow \varphi\}$ is a quasi-deduction for $\Gamma, \Delta \vdash_{\mathsf{N}}^{+} \psi$.*

2.2 Soundness and Completeness

Before turning to the main part of this chapter, let us first establish the soundness and completeness of our proof system. We have to show that, for any set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$, there exists a deduction for $\Gamma \vdash_{\mathsf{N}}^{+} \varphi$ in NinqB^{+} if and only if φ is entailed by Γ in InqB. To this end, we will prove that NinqB^{+} is *equivalent* to the standard natural deduction system NinqB introduced in Definition 1.5.1, i.e., everything that is provable in NinqB^{+} is also provable in NinqB and vice versa. First, we show that NinqB^{+} is *sound* with respect to NinqB: if a formula φ is provable from a set of hypotheses in NinqB^{+}, then it is also provable from these hypotheses in NinqB.

Lemma 2.2.1. *Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ be a set of formulas. If it holds $\Gamma \vdash_{\mathsf{N}}^{+} \varphi$ in our natural deduction system* NinqB^{+}*, then we also have $\Gamma \vdash_{\mathsf{N}} \varphi$ in the standard natural deduction system* NinqB.[6]

Proof. We will show that every deduction in NinqB^{+} can be transformed into a deduction in NinqB. For this purpose, let NinqB^{a} be the *auxiliary system* which is just like NinqB^{+}, except that NinqB^{a} also includes the ordinary elimination rule $\mathbb{V}E$ from Figure 1.3 and the special rules (split) and (dne) from Figure 1.4. As before, by a *quasi-deduction* in NinqB^{a}, we mean any proof tree built up using the primitive rules of NinqB^{a} and arbitrary (open or discharged) applications of non-primitive rules. And a *deduction* in NinqB^{a} is a quasi-deduction in which all applications of non-primitive rules are discharged by applications of $\mathbb{V}E_{+}$. The derivability relation of NinqB^{a} is denoted by $\vdash_{\mathsf{N}}^{a}$, so we write $\Gamma \vdash_{\mathsf{N}}^{a} \varphi$, if there exists a quasi-deduction $\mathcal{D}$ in NinqB^{a} such that φ is the conclusion of $\mathcal{D}$ and all undischarged rules and open hypotheses of $\mathcal{D}$ are in Γ.

We will prove the following more general claim: for any set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$, if there exists a deduction $\mathcal{D}$ for $\Gamma \vdash_{\mathsf{N}}^{a} \varphi$ in the auxiliary system NinqB^{a}, then there also exists a deduction $\mathcal{D}'$ for $\Gamma \vdash_{\mathsf{N}} \varphi$ in the standard system NinqB (since every deduction in NinqB^{+} is also a deduction in NinqB^{a}, this is sufficient to establish the claim). Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ be an arbitrary set of formulas and let $\mathcal{D}$ be an arbitrary deduction for $\Gamma \vdash_{\mathsf{N}}^{a} \varphi$ in the auxiliary system NinqB^{a} (so, in particular, $\mathcal{D}$ cannot contain any undischarged applications of non-primitive rules). Moreover, let n be the number of applications of the modified elimination rule $\mathbb{V}E_{+}$ in $\mathcal{D}$. Using induction on n, we show that there is also a deduction $\mathcal{D}'$ for $\Gamma \vdash_{\mathsf{N}} \varphi$ in the standard system NinqB.

For the base case, assume that $\mathcal{D}$ contains $n = 0$ applications of $\mathbb{V}E_{+}$. In this case, $\mathcal{D}$ cannot contain any applications of non-primitive rules since every such application would have to be discharged by some application of $\mathbb{V}E_{+}$. Hence, we only need to get rid of all applications of (raa) in $\mathcal{D}$. Let $\mathcal{D}'$ be the proof tree obtained from $\mathcal{D}$ by rewriting every application of (raa) as follows:

$$
\dfrac{\begin{matrix}[\neg\alpha]^u \\ \mathcal{D}'' \\ \bot\end{matrix}}{\alpha}\,(\mathsf{raa})^u
\qquad\text{converts to}\qquad
\dfrac{\dfrac{\begin{matrix}[\neg\alpha]^u \\ \mathcal{D}'' \\ \bot\end{matrix}}{\neg\neg\alpha}\,{\to}I^u}{\alpha}\,(\mathsf{dne})
$$

Then, clearly, $\mathcal{D}'$ is a deduction for $\Gamma \vdash_{\mathsf{N}}^{a} \varphi$ which contains no applications of non-primitive rules and no applications of the rules $\mathbb{V}E_{+}$ and (raa). Consequently, $\mathcal{D}'$ is also a deduction in the standard natural deduction system NinqB, so we may conclude $\Gamma \vdash_{\mathsf{N}} \varphi$, as desired.

[6] Recall that we write $\vdash_{\mathsf{N}}^{+}$ for the provability relation of NinqB^{+} and $\vdash_{\mathsf{N}}$ for the provability relation of NinqB.

For the inductive step, assume that $\mathcal{D}$ contains $n \geq 1$ applications of the rule $\mathbb{V}E_+$. In this case, we select a topmost application of $\mathbb{V}E_+$ in $\mathcal{D}$. In other words, we write $\mathcal{D}$ in the form

$$\begin{array}{c}\dfrac{\begin{array}{c}[\Theta]^u\\ \mathcal{D}_3\\ \psi_1 \mathbin{\mathbb{V}} \psi_2\end{array} \qquad \begin{array}{c}[\Theta \Rightarrow \psi_1]^u\\ \mathcal{D}_1\\ \chi\end{array} \qquad \begin{array}{c}[\Theta \Rightarrow \psi_2]^u\\ \mathcal{D}_2\\ \chi\end{array}}{\chi}\,\mathbb{V}E^u_+\\ \mathcal{D}_4\end{array}$$

where none of the quasi-deductions $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ contains an application of $\mathbb{V}E_+$. By assumption, Θ is a finite set of classical formulas, so we must have $\Theta = \{\alpha_1, \dots, \alpha_k\}$ for some $\alpha_1, \dots, \alpha_k \in \mathcal{L}^{\mathsf{B}}_c$. Let now $i = 1, 2$ be arbitrary and consider the subtree $\mathcal{D}_i$ ending with the premise χ of the selected application of $\mathbb{V}E_+$. Let Δ_i be the set of all open hypotheses and undischarged rules other than $\Theta \Rightarrow \psi_i$ in this subtree. That is, Δ_i is the smallest set such that $\mathcal{D}_i$ is a quasi-deduction for $\Delta_i, \Theta \Rightarrow \psi_i \vdash^a_{\mathsf{N}} \chi$. Furthermore, let $\mathcal{D}^*_i$ be the quasi-deduction obtained from $\mathcal{D}_i$ by rewriting every undischarged application of the rule $\Theta \Rightarrow \psi_i$ in the following way:

$$\begin{array}{c}\dfrac{\begin{array}{c}\mathcal{D}'_1\\ \alpha_1\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}'_k\\ \alpha_k\end{array}}{\psi_i}\,\Theta \Rightarrow \psi_i\\ \mathcal{D}''\end{array} \qquad \text{converts to} \qquad \begin{array}{c}\dfrac{\bigwedge\Theta \to \psi_i \qquad \begin{array}{c}\mathcal{D}'_1 \qquad\quad \mathcal{D}'_k\\ \dfrac{\alpha_1 \quad \cdots \quad \alpha_k}{\bigwedge\Theta}\,\wedge I\end{array}}{\psi_i}\,{\to}E\\ \mathcal{D}''\end{array}$$

where $\bigwedge\Theta$ stands for the conjunction of the elements of Θ and the dashed line indicates $k-1$ applications of the rule $\wedge I$. Then, clearly, $\mathcal{D}^*_i$ contains no undischarged applications of the rule $\Theta \Rightarrow \psi_i$ anymore, but it contains an additional open hypothesis $\bigwedge\Theta \to \psi_i$ instead. In other words, $\mathcal{D}^*_i$ must be a quasi-deduction for $\Delta_i, \bigwedge\Theta \to \psi_i \vdash^a_{\mathsf{N}} \chi$. Using the quasi-deductions $\mathcal{D}^*_1$ and $\mathcal{D}^*_2$ thus obtained, we may now transform the whole deduction $\mathcal{D}$ into the proof tree $\mathcal{D}^*$ of the form

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{[\bigwedge\Theta]^u}{\Theta}\,\wedge E\\ \mathcal{D}_3\\ \psi_1 \mathbin{\mathbb{V}} \psi_2\end{array}}{\bigwedge\Theta \to (\psi_1 \mathbin{\mathbb{V}} \psi_2)}\,{\to}I^u}{(\bigwedge\Theta \to \psi_1) \mathbin{\mathbb{V}} (\bigwedge\Theta \to \psi_2)}\,(\mathsf{split}) \qquad \begin{array}{c}[\bigwedge\Theta \to \psi_1]^v\\ \mathcal{D}^*_1\\ \chi\end{array} \qquad \begin{array}{c}[\bigwedge\Theta \to \psi_2]^v\\ \mathcal{D}^*_2\\ \chi\end{array}}{\chi}\,\mathbb{V}E^v\\ \mathcal{D}_4\end{array}$$

where $\mathbb{V}E$ refers to the ordinary elimination rule for $\mathbb{V}$. Note that, since every element of Θ is assumed to be a classical formula, the conjunction $\bigwedge\Theta$ of these elements must also be a classical formula, so the indicated application of (split) is in fact correct. It is now easy to see that $\mathcal{D}^*$ does not contain any new undischarged rules or open hypotheses that were not already present in $\mathcal{D}$. Hence, $\mathcal{D}^*$ must still be a deduction for $\Gamma \vdash^a_{\mathsf{N}} \varphi$. Furthermore, by construction, $\mathcal{D}^*$ contains only $n-1$ applications of the modified elimination rule $\mathbb{V}E_+$. Therefore, using the induction hypothesis, we now obtain the desired deduction $\mathcal{D}'$ for $\Gamma \vdash_{\mathsf{N}} \varphi$ in the standard system NinqB. ☐

Next, we will show that our system is also *complete* with respect to NinqB: if φ is provable from a set of hypotheses in NinqB, then it is also provable from these hypotheses in NinqB$^+$.

Lemma 2.2.2. *Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ be a set of formulas. If $\Gamma \vdash_{\mathsf{N}} \varphi$ holds in the standard natural deduction system* NinqB, *then we also have $\Gamma \vdash^+_{\mathsf{N}} \varphi$ in our new natural deduction system* NinqB$^+$.

Proof. By what was said in the previous section, every instance of the ordinary elimination rule $\mathbb{V}E$ can be identified with an instance of the generalized rule $\mathbb{V}E_+$ in which the set of auxiliary formulas Θ is taken to be the empty set. Therefore, it suffices to show that the special rules (split) and (dne), depicted in Figure 1.4, are derivable in our system NinqB^+.[7] In order to show the derivability of (split), let $\alpha \in \mathcal{L}_c^{\mathsf{B}}$ be a classical formula and let $\varphi, \psi \in \mathcal{L}^{\mathsf{B}}$ be arbitrary formulas. Using the generalized rule $\mathbb{V}E_+$, we may then construct the following deduction in NinqB^+:

$$\dfrac{\dfrac{\alpha \to (\varphi \mathbin{\mathbb{V}} \psi) \qquad [\alpha]^w}{\varphi \mathbin{\mathbb{V}} \psi}{\to}E \qquad \dfrac{\dfrac{\dfrac{[\alpha]^u}{\varphi}\,w}{\alpha \to \varphi}{\to}I^u}{(\alpha \to \varphi) \mathbin{\mathbb{V}} (\alpha \to \psi)}\mathbb{V}I \qquad \dfrac{\dfrac{\dfrac{[\alpha]^v}{\psi}\,w}{\alpha \to \psi}{\to}I^v}{(\alpha \to \varphi) \mathbin{\mathbb{V}} (\alpha \to \psi)}\mathbb{V}I}{(\alpha \to \varphi) \mathbin{\mathbb{V}} (\alpha \to \psi)}\mathbb{V}E_+^w$$

Note that, as indicated by the label w, all applications of the non-primitive rules $\alpha \Rightarrow \varphi$ and $\alpha \Rightarrow \psi$ in this deduction are discharged by the application of $\mathbb{V}E_+$ at the bottom. Hence, the rule (split) is in fact derivable in our system. The derivability of the rule (dne), on the other hand, can be established by the following deduction in NinqB^+, where $\alpha \in \mathcal{L}_c^{\mathsf{B}}$ is again a classical formula:

$$\dfrac{\dfrac{\neg\neg\alpha \qquad [\neg\alpha]^u}{\bot}{\to}E}{\alpha}\,(\mathsf{raa})^u$$

As we have seen, each of the rules $\mathbb{V}E$, (split) and (dne) is derivable in NinqB^+. Consequently, given any subset $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ and a deduction $\mathcal{D}$ for $\Gamma \vdash_{\mathsf{N}} \varphi$ in the standard system NinqB, one can always transform $\mathcal{D}$ into a corresponding deduction $\mathcal{D}'$ for $\Gamma \vdash_{\mathsf{N}}^+ \varphi$ in NinqB^+. □

By combining the previous two lemmas, we may now conclude that our system is in fact equivalent to the standard natural deduction system for inquisitive logic: everything that is provable in NinqB^+ is also provable in the standard system NinqB and vice versa. Since NinqB is sound and complete with respect to InqB, this yields the desired completeness result for our system.

Theorem 2.2.3 (Soundness and Completeness)**.** *The system* NinqB^+ *is sound and complete with respect to* InqB*. That is, for any set of formulas* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$*, we have:* $\Gamma \vdash_{\mathsf{N}}^+ \varphi$ *if and only if* $\Gamma \models \varphi$*.*

Proof. The statement follows directly from Theorem 1.5.2, Lemma 2.2.1 and Lemma 2.2.2. □

2.3 Cut Segments and Conversions

In this and the next section, we will establish a so-called *normalization theorem* for our system, i.e., we will show that every deduction in NinqB^+ can be transformed into a detour-free deduction. The underlying technique goes back to Gentzen, who started thinking about normalization in the early 1930s as part of his work on the consistency of Peano arithmetic (cf. Von Plato 2008; 2012). Unfortunately, Gentzen did not succeed in proving normalization for classical natural deduction, which led him to develop the sequent calculus and to prove a cut-elimination theorem instead. (For intuitionistic logic, however, Gentzen had already found a direct normalization proof, but never published it. This became known in 2005, when a handwritten manuscript of his doctoral thesis was found. See Von Plato 2008). The first published proof of normalization for both classical and intuitionistic logic was provided by Prawitz (1965). Let us start by giving a brief explanation of the basic idea. Roughly speaking, by a *cut formula* in a deduction, we mean an occurrence of a formula that is introduced by an application of an introduction rule or an application of a rule for the falsum constant, and which is eliminated right away by an application

[7] Roughly speaking, a rule is said to be *derivable* in our system, if it satisfies the condition that, whenever there is a deduction for each of the premises of the rule, then there is also a deduction for the conclusion of the rule.

of an elimination rule (so, in a sense, a cut formula may be regarded as a 'detour' in a deduction, since a formula which has just been introduced gets instantly eliminated again). For example, consider the following deduction in our system, where α is assumed to be a classical formula and the two applications of $\alpha \Rightarrow \varphi$ and $\alpha \Rightarrow \psi$ are discharged by the application of $\mathbb{V}E_+$ at the end:

$$\dfrac{\dfrac{(1)\ \dfrac{\dfrac{\alpha \to \varphi \qquad [\alpha]^v}{\varphi}\to E}{\varphi \mathbin{\mathbb{V}} \psi}\mathbb{V}I \qquad (2)\ \dfrac{\dfrac{\dfrac{\neg\varphi \qquad \dfrac{[\alpha]^w}{\varphi}v}{\bot}\to E}{\varphi \wedge \psi}\text{(efq)}}{\psi}\wedge E \qquad (3)\ \dfrac{\dfrac{\dfrac{[\alpha]^u}{\psi}v}{\alpha \to \psi}\to I^u \qquad [\alpha]^w}{\psi}\to E}{\psi}\mathbb{V}E_+^v}{\alpha \to \psi}\to I^w$$

This deduction contains three cut formulas, marked with (1), (2) and (3), respectively. The first one is the formula marked with (1), since this formula is obtained by the introduction rule for $\mathbb{V}$ and eliminated right away by the corresponding elimination rule. The second cut formula is (2), because this formula is introduced by an application of a falsum rule and eliminated by the elimination rule for conjunction. And the last cut formula is (3), since this formula is obtained by the introduction rule for $\to$ and instantly eliminated again by the respective elimination rule.

In order to make things precise, let us introduce some further terminology. Given any application of a (primitive or non-primitive) rule of inference, we will call the formula occurrences directly above the line the *premises*, and the formula occurrence directly below the line the *conclusion* of this application. For each of the binary connectives $\wedge$, $\to$ and $\mathbb{V}$, our system comprises exactly one *introduction rule* and exactly one *elimination rule*.[8] In addition, NinqB$^+$ also includes the rules (efq) and (raa), which we will refer to as *falsum rules*. In elimination rules, we distinguish between two types of premises, referred to as *major premises* and *minor premises*, respectively.

Definition 2.3.1 (Major Premise, Minor Premise)**.** In an application of an elimination rule, the premise which contains the corresponding occurrence of the logical connective is said to be the *major premise* of this application. All other premises (if any) will be referred to as *minor premises*.

So, for example, in an application of $\to E$, the major premise is the formula of the form $\varphi \to \psi$ and the minor premise is the formula φ. And in an application of $\mathbb{V}E_+$, the major premise is the formula $\varphi \mathbin{\mathbb{V}} \psi$ and the minor premises are the two occurrences of χ standing immediately above the line. Without loss of generality, we will assume that the major premise of an elimination rule is always the *leftmost* premise of this rule. It might now seem that a cut could simply be defined as an occurrence of a formula which is both the conclusion of an introduction rule or the conclusion of a falsum rule, and the major premise of an elimination rule. However, things are actually a bit more complicated, since applications of the rule $\mathbb{V}E_+$ allow us to produce *repetitions* of formulas in deductions. For example, one could also construct a deduction of the following form:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \theta_1 \mathbin{\mathbb{V}} \theta_2\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ \chi_1 \mathbin{\mathbb{V}} \chi_2\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \varphi\end{matrix} \qquad \begin{matrix}\mathcal{D}'\\ \psi\end{matrix}}{\varphi \wedge \psi}\wedge I \qquad \begin{matrix}\vdots\\ \varphi \wedge \psi\end{matrix}}{\varphi \wedge \psi}\mathbb{V}E_+ \qquad \begin{matrix}\vdots\\ \varphi \wedge \psi\end{matrix}}{\varphi \wedge \psi}\mathbb{V}E_+}{\varphi}\wedge E$$

This deduction contains a sequence of three consecutive occurrences of the formula $\varphi\wedge\psi$, starting with the conclusion of an introduction rule and ending with the major premise of an elimination rule. Clearly, our definition of a cut should also account for detours of this kind. Thus, rather than

[8] Note that, for the inquisitive disjunction operator $\mathbb{V}$, the elimination rule is now the 'higher-level' rule $\mathbb{V}E_+$, rather than the ordinary elimination rule for disjunction used in classical and intuitionistic natural deduction.

considering single occurrences of formulas, we must actually consider sequences of consecutive occurrences of the same formula. A sequence of this kind will be referred to as a *segment*.

Definition 2.3.2 (Segment)**.** Let $\mathcal{D}$ be a quasi-deduction in our system. By a *segment* of length n in $\mathcal{D}$, we mean a sequence $\xi_1, \ldots, \xi_n$ of n consecutive occurrences of a formula φ in $\mathcal{D}$ such that:
(i) ξ_1 is not the conclusion of an application of $\mathbb{V}E_+$,
(ii) each ξ_i with $1 \leq i < n$ is a minor premise of an application of $\mathbb{V}E_+$,
(iii) ξ_n is not a minor premise of an application of $\mathbb{V}E_+$.

A *cut segment* is now defined to be a segment that begins with the conclusion of an introduction rule or the conclusion of a falsum rule, and ends with the major premise of an elimination rule.

Definition 2.3.3 (Cut Segment)**.** Let $\mathcal{D}$ be a quasi-deduction in NinqB$^+$. By a *cut segment* of length n in $\mathcal{D}$, we mean a segment $\xi_1, \ldots, \xi_n$ in $\mathcal{D}$ such that ξ_1 is the conclusion of an introduction rule or the conclusion of a falsum rule, and ξ_n is the major premise of an elimination rule. If π is a cut segment, then the unique formula occurring in π will also be called the *cut formula* of π.

The main goal of this chapter is to show that any deduction can be transformed into a deduction containing no cut segments. A cut-free deduction will also be referred to as a *normal deduction*.

Definition 2.3.4 (Normal Deduction)**.** A deduction $\mathcal{D}$ is said to be a *normal deduction*, if $\mathcal{D}$ contains no cut segments and all conclusions of falsum rule applications in $\mathcal{D}$ are atomic formulas.

The overall structure of our normalization proof will be as follows: first, we will show that the falsum rules can be restricted to instances in which the conclusion is an atomic formula. Afterwards, we will describe a procedure that allows to get rid of all cut segments starting with the conclusion of an introduction rule. The argument is based on the observation that any cut segment of length $n > 1$ can be transformed into a cut segment of length one by permuting the major premise at the end of the segment over the minor premises of $\mathbb{V}E_+$ in the middle of the segment. This is achieved by performing the following *permutation conversion*, where (E) stands for an arbitrary instance of an elimination rule with major premise χ and conclusion μ:

$$\dfrac{\dfrac{\begin{matrix}[\Theta]^u\\ \mathcal{D}_3\\ \varphi\mathbb{V}\psi\end{matrix}\quad \begin{matrix}[\Theta\Rightarrow\varphi]^u\\ \mathcal{D}_1\\ \chi\end{matrix}\quad \begin{matrix}[\Theta\Rightarrow\psi]^u\\ \mathcal{D}_2\\ \chi\end{matrix}}{\chi}\,\mathbb{V}E_+^u\quad \mathcal{D}_4}{\begin{matrix}\mu\\ \mathcal{D}_5\end{matrix}}\,\text{(E)} \qquad \text{converts to} \qquad \dfrac{\begin{matrix}[\Theta]^u\\ \mathcal{D}_3\\ \varphi\mathbb{V}\psi\end{matrix}\quad \begin{matrix}[\Theta\Rightarrow\varphi]^u\\ \mathcal{D}_1\\ \dfrac{\chi\quad \mathcal{D}_4}{\mu}\,\text{(E)}\end{matrix}\quad \begin{matrix}[\Theta\Rightarrow\psi]^u\\ \mathcal{D}_2\\ \dfrac{\chi\quad \mathcal{D}_4}{\mu}\,\text{(E)}\end{matrix}}{\begin{matrix}\mu\\ \mathcal{D}_5\end{matrix}}\,\mathbb{V}E_+^u$$

Unfortunately, naive applications of the permutation conversion do not necessarily yield the desired result. The reason is that (E) itself might be an instance of $\mathbb{V}E_+$ which discharges some hypotheses occurring in the subtree $\mathcal{D}_3$. In this case, permuting the major premise of (E) upwards would cause the respective hypotheses in $\mathcal{D}_3$ to become *open* hypotheses. For example, consider the following deduction in our system, which contains two consecutive applications of $\mathbb{V}E_+$:

$$\dfrac{\dfrac{\begin{matrix}[\Theta]^u\ [\Omega]^v\\ \mathcal{D}_3\\ \varphi_1\mathbb{V}\varphi_2\end{matrix}\quad \begin{matrix}[\Theta\Rightarrow\varphi_1]^u\ [\Omega]^v\\ \mathcal{D}_1\\ \psi_1\mathbb{V}\psi_2\end{matrix}\quad \begin{matrix}[\Theta\Rightarrow\varphi_2]^u\ [\Omega]^v\\ \mathcal{D}_2\\ \psi_1\mathbb{V}\psi_2\end{matrix}}{\psi_1\mathbb{V}\psi_2}\,\mathbb{V}E_+^u\quad \begin{matrix}[\Omega\Rightarrow\psi_1]^v\\ \mathcal{D}_4\\ \chi\end{matrix}\quad \begin{matrix}[\Omega\Rightarrow\psi_2]^v\\ \mathcal{D}_5\\ \chi\end{matrix}}{\begin{matrix}\chi\\ \mathcal{D}_6\end{matrix}}\,\mathbb{V}E_+^v$$

In this deduction, the upper application of $\mathbb{V}E_+$ (i.e., the one with label u) is associated with the set of auxiliary formulas Θ and the lower application of $\mathbb{V}E_+$ (i.e., the one with label v) is associated with the set of auxiliary formulas Ω. Since some of the auxiliary formulas from Ω also occur as hypotheses in the subtree $\mathcal{D}_3$, we cannot simply apply the aforementioned permutation conversion, since this would cause these hypotheses to become open. However, this problem can easily be resolved, since it is always possible to 'shift' all occurrences of the hypotheses from Ω in $\mathcal{D}_3$ to the subtrees $\mathcal{D}_1$ and $\mathcal{D}_2$. To make this more precise, let us assume that it holds $\Theta = \{\alpha_1, \ldots, \alpha_n\}$ and $\Omega = \{\beta_1, \ldots, \beta_m\}$, where each α_i and each β_i is a classical formula. Moreover, let $k = 1, 2$ be arbitrary and let $\mathcal{D}_k^*$ be the quasi-deduction obtained from $\mathcal{D}_k$ by rewriting every open application of the non-primitive rule $\Theta \Rightarrow \varphi_k$ in the following way:

$$\dfrac{\begin{array}{c}\mathcal{D}_1' \\ \alpha_1\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_n' \\ \alpha_n\end{array}}{\begin{array}{c}\varphi_k \\ \mathcal{D}''\end{array}} \,\Theta \Rightarrow \varphi_k \qquad \text{converts to} \qquad \dfrac{\begin{array}{c}\mathcal{D}_1' \\ \alpha_1\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_n' \\ \alpha_n\end{array} \quad \beta_1 \quad \cdots \quad \beta_m}{\begin{array}{c}\varphi_k \\ \mathcal{D}''\end{array}} \,\Theta, \Omega \Rightarrow \varphi_k$$

In other words, we simply replace all undischarged applications of the rule $\Theta \Rightarrow \varphi_k$ in $\mathcal{D}_k$ by undischarged applications of $\Theta, \Omega \Rightarrow \varphi_k$. Note that, in the course of this conversion, we add a number of new occurrences of the open hypotheses $\beta_1, \ldots, \beta_m$ to the proof tree. Using the quasi-deductions $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$ thus obtained, we can now transform the whole deduction into

$$\dfrac{\dfrac{\begin{array}{c}[\Theta, \Omega]^u \\ \mathcal{D}_3 \\ \varphi_1 \mathbin{\mathbb{V}} \varphi_2\end{array} \quad \begin{array}{c}[\Theta, \Omega \Rightarrow \varphi_1]^u \ [\Omega]^v \\ \mathcal{D}_1^* \\ \psi_1 \mathbin{\mathbb{V}} \psi_2\end{array} \quad \begin{array}{c}[\Theta, \Omega \Rightarrow \varphi_2]^u \ [\Omega]^v \\ \mathcal{D}_2^* \\ \psi_1 \mathbin{\mathbb{V}} \psi_2\end{array}}{\psi_1 \mathbin{\mathbb{V}} \psi_2} \,\mathbb{V}E_+^u \quad \begin{array}{c}[\Omega \Rightarrow \psi_1]^v \\ \mathcal{D}_4 \\ \chi\end{array} \quad \begin{array}{c}[\Omega \Rightarrow \psi_2]^v \\ \mathcal{D}_5 \\ \chi\end{array}}{\begin{array}{c}\chi \\ \mathcal{D}_6\end{array}} \,\mathbb{V}E_+^v$$

In this deduction, the lower application of $\mathbb{V}E_+$ does not discharge any hypotheses in $\mathcal{D}_3$ anymore, since all open hypotheses from Ω in $\mathcal{D}_3$ are now discharged by the upper application. Therefore, we can now safely apply our permutation conversion in order to permute the major premise of the lower application of $\mathbb{V}E_+$ over the minor premises of the upper application.

Using successive applications of the permutation conversion, it is possible to transform any cut segment into a cut segment containing only a single occurrence of the cut formula. The cut itself can then be eliminated using the following *logical conversions* for cut segments of length one:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1 \\ \varphi_1\end{array} \quad \begin{array}{c}\mathcal{D}_2 \\ \varphi_2\end{array}}{\varphi_1 \wedge \varphi_2} \,\wedge I}{\begin{array}{c}\varphi_i \\ \mathcal{D}_3\end{array}} \,\wedge E \qquad \text{converts to} \qquad \begin{array}{c}\mathcal{D}_i \\ \varphi_i \\ \mathcal{D}_3\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}[\varphi]^u \\ \mathcal{D}_1 \\ \psi\end{array}}{\varphi \to \psi} \,\to I^u \quad \begin{array}{c}\mathcal{D}_2 \\ \varphi\end{array}}{\begin{array}{c}\psi \\ \mathcal{D}_3\end{array}} \,\to E \qquad \text{converts to} \qquad \begin{array}{c}\mathcal{D}_2 \\ \varphi \\ \mathcal{D}_1 \\ \psi \\ \mathcal{D}_3\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}[\Theta]^u \\ \mathcal{D}_3 \\ \varphi_i\end{array}}{\varphi_1 \mathbin{\mathbb{V}} \varphi_2} \,\mathbb{V}I \quad \begin{array}{c}[\Theta \Rightarrow \varphi_1]^u \\ \mathcal{D}_1 \\ \chi\end{array} \quad \begin{array}{c}[\Theta \Rightarrow \varphi_2]^u \\ \mathcal{D}_2 \\ \chi\end{array}}{\begin{array}{c}\chi \\ \mathcal{D}_4\end{array}} \,\mathbb{V}E_+^u \qquad \text{converts to} \qquad \begin{array}{c}\mathcal{D}_i\{\mathcal{D}_3 : \Theta \Rightarrow \varphi_i\} \\ \chi \\ \mathcal{D}_4\end{array}$$

Observe that, in the last conversion, we use the substitution operator defined in Section 2.1, so $\mathcal{D}_i\{\mathcal{D}_3 : \Theta \Rightarrow \varphi_i\}$ stands for the result of replacing every undischarged application of the non-primitive rule $\Theta \Rightarrow \varphi_i$ in $\mathcal{D}_i$ by an occurrence of the quasi-deduction $\mathcal{D}_3$ (this is in fact legitimate, since $\mathcal{D}_3$ is assumed to end with the conclusion φ_i). Using Fact 2.1.4, it is easy to verify that this conversion does not affect the conclusion or the set of open hypotheses of the whole deduction, i.e., for any set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$, if the full proof tree was a deduction for $\Gamma \vdash^{+}_{\mathsf{N}} \varphi$ before the conversion, then it will also be such a deduction after the conversion.

The main difficulty in our normalization proof arises from the fact that each of the logical conversions may produce new cut segments and the last two conversions may even *increase* the total number of cut segments in a deduction. Therefore, it is not obvious from the outset that repeated applications of these conversions will eventually yield a normal deduction. For example, in the conversion rule for cut formulas of the form $\varphi \to \psi$, each of the two formulas φ and ψ may become a new cut formula and all cut segments occurring in the subtree $\mathcal{D}_2$ might be *duplicated*, depending on how often φ occurs as an open hypothesis in $\mathcal{D}_1$. Furthermore, in the conversion rule for cut formulas of the form $\varphi_1 \vee \varphi_2$, not only φ_i but also every formula in the set Θ may become a new cut formula, and these formulas can be arbitrarily 'complex'. In addition, all cut segments in $\mathcal{D}_3$ might be duplicated, depending on how often $\Theta \Rightarrow \varphi_i$ occurs as an undischarged rule in $\mathcal{D}_i$.

2.4 Normalization

Let us now turn to the proof of normalization for our system. We first show that the falsum rules, (efq) and (raa), can be restricted to instances in which the conclusion is an atomic formula.[9] Note that, since major premises of elimination rules are never atomic, this also shows that cut segments starting with the conclusion of a falsum rule can always be avoided in a deduction. In order to prove the desired claim, we will argue that one can successively reduce the complexity of the conclusion of a falsum rule application until it finally becomes an atomic formula. Here, the complexity of a formula will be measured in terms of its *degree*, which is defined in the following way.

Definition 2.4.1. The *degree* of a formula φ, notation $deg(\varphi)$, is inductively defined as follows:
(i) $deg(p) := 0$ for all atoms $p \in \mathsf{P}$, and $deg(\bot) := 1$,
(ii) $deg(\psi \otimes \chi) := deg(\psi) + deg(\chi) + 1$ for $\otimes \in \{\wedge, \to, \vee\}$.

In other words, the degree of a formula simply amounts to the number of occurrences of the logical symbols $\bot$, $\wedge$, $\to$, $\vee$ in this formula. The basic idea of our argument is to decrease the degree of the conclusion of a falsum rule application step by step, by applying suitable conversions to the deduction. To make this more precise, let us first introduce some useful terminology. The conclusion of a falsum rule application will also be referred to as a *falsum rule conclusion*. A falsum rule conclusion φ in a deduction $\mathcal{D}$ is said to be *maximal*, if there is no falsum rule conclusion of higher degree in the deduction, i.e., if for every falsum rule conclusion ψ in $\mathcal{D}$, we have $deg(\psi) \leq deg(\varphi)$. And the *falsum rank* of a deduction $\mathcal{D}$ is the pair of natural numbers $fr(\mathcal{D}) = (m, n)$ defined in the following way: if all falsum rule conclusions in $\mathcal{D}$ are atomic, then we put $fr(\mathcal{D}) := (0, 0)$. Otherwise, we put $fr(\mathcal{D}) := (m, n)$, where m is the degree of a maximal falsum rule conclusion in $\mathcal{D}$ and n is the number of maximal falsum rule conclusions in $\mathcal{D}$.

We will show that any deduction of falsum rank greater than $(0, 0)$ can be transformed into a deduction of lower falsum rank (by iterating this argument, we will then finally arrive at a deduction containing only atomic falsum rule conclusions). In order to make this notion of 'greater than' and 'lower than' precise, we will assume that falsum ranks are ordered *lexicographically*. That is, given a deduction $\mathcal{D}$ with $fr(\mathcal{D}) = (m, n)$ and another deduction $\mathcal{D}'$ with $fr(\mathcal{D}') = (m', n')$, we will write $fr(\mathcal{D}) < fr(\mathcal{D}')$ and say that the falsum rank of $\mathcal{D}$ is *smaller* than the falsum rank of $\mathcal{D}'$, if we either have $m < m'$, or we have both $m = m'$ and $n < n'$. We will also write $>$ for the *inverse relation* of the lexicographic ordering $<$. Using these notions, we

[9] To be clear: by an *atomic formula*, we will always mean a propositional letter from the set P introduced at the beginning of Section 1.2. Thus, in particular, the falsum constant $\bot$ will *not* be treated as an atomic formula.

can now prove the desired result: if φ is provable from a set of hypotheses Γ in our system, then one can always find a deduction of φ from Γ in which all falsum rule conclusions are atomic.

Lemma 2.4.2. *Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ be a set of formulas. If $\Gamma \vdash^{+}_{\mathsf{N}} \varphi$ holds in our system, then there exists a deduction for $\Gamma \vdash^{+}_{\mathsf{N}} \varphi$ in which the conclusion of every falsum rule application is atomic.*

Proof. Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ be arbitrary and suppose that there exists a deduction $\mathcal{D}$ for $\Gamma \vdash^{+}_{\mathsf{N}} \varphi$ in our system. If all falsum rule conclusions in $\mathcal{D}$ are atomic, then we are finished. Otherwise, $\mathcal{D}$ contains at least one non-atomic falsum rule conclusion and we must have $fr(\mathcal{D}) = (m, n) > (0, 0)$ for some integers $m, n \geq 0$. We first show that $\mathcal{D}$ can be transformed into a deduction $\mathcal{D}'$ with smaller falsum rank. To this end, let θ be a maximal falsum rule conclusion in $\mathcal{D}$ such that all falsum rule conclusions above θ in $\mathcal{D}$ are of smaller degree than θ. That is, θ satisfies $deg(\theta) = m$, and it holds $deg(\mu) < m$ for all falsum rule conclusions μ in the subtree ending with θ.

First, suppose that θ is the conclusion of an application of (raa). In this case, θ must be a *classical formula.* Depending on the form of θ, we now perform one of the following conversions:

$$\begin{array}{c}[\neg(\alpha\wedge\beta)]^u\\ \mathcal{D}_1\\ \dfrac{\bot}{\alpha\wedge\beta}\,(\mathsf{raa})^u\\ \mathcal{D}_2\end{array} \qquad \text{converts to} \qquad \begin{array}{c}\dfrac{\begin{array}{c}\dfrac{\dfrac{[\neg\alpha]^v \quad \dfrac{[\alpha\wedge\beta]^u}{\alpha}\,{\wedge E}}{\bot}\,{\to E}}{\neg(\alpha\wedge\beta)}\,{\to I^u}\\ \mathcal{D}_1\\ \dfrac{\bot}{\alpha}\,(\mathsf{raa})^v\end{array} \qquad \begin{array}{c}\dfrac{\dfrac{[\neg\beta]^x \quad \dfrac{[\alpha\wedge\beta]^w}{\beta}\,{\wedge E}}{\bot}\,{\to E}}{\neg(\alpha\wedge\beta)}\,{\to I^w}\\ \mathcal{D}_1\\ \dfrac{\bot}{\beta}\,(\mathsf{raa})^x\end{array}}{\alpha\wedge\beta}\,{\wedge I}\\ \mathcal{D}_2\end{array}$$

$$\begin{array}{c}[\neg(\alpha\to\beta)]^u\\ \mathcal{D}_1\\ \dfrac{\bot}{\alpha\to\beta}\,(\mathsf{raa})^u\\ \mathcal{D}_2\end{array} \qquad \text{converts to} \qquad \begin{array}{c}\dfrac{\dfrac{[\neg\beta]^v \quad \dfrac{[\alpha\to\beta]^u \quad [\alpha]^w}{\beta}\,{\to E}}{\bot}\,{\to E}}{\neg(\alpha\to\beta)}\,{\to I^u}\\ \mathcal{D}_1\\ \dfrac{\dfrac{\bot}{\beta}\,(\mathsf{raa})^v}{\alpha\to\beta}\,{\to I^w}\\ \mathcal{D}_2\end{array}$$

$$\begin{array}{c}[\neg\bot]^u\\ \mathcal{D}_1\\ \dfrac{\bot}{\bot}\,(\mathsf{raa})^u\\ \mathcal{D}_2\end{array} \qquad \text{converts to} \qquad \begin{array}{c}\dfrac{[\bot]^u}{\neg\bot}\,{\to I^u}\\ \mathcal{D}_1\\ \bot\\ \mathcal{D}_2\end{array}$$

Let $\mathcal{D}'$ be the deduction arising from the conversion and let $fr(\mathcal{D}') = (m', n')$ be the falsum rank of $\mathcal{D}'$. In the first two conversions, the old application of (raa) is replaced by one or more new applications of (raa) and the conclusions of these new applications are of lower degree than the conclusion θ of the old application. Moreover, although the quasi-deduction $\mathcal{D}_1$ is used twice in the first conversion, this cannot increase the the total number of maximal falsum rule conclusions: by assumption, all falsum rule conclusions occurring in $\mathcal{D}_1$ are of lower degree than θ. On the other hand, in the last conversion, the application of (raa) is completely eliminated. Hence, if θ was the only maximal falsum rule conclusion in $\mathcal{D}$, then the degree of a maximal falsum rule conclusion in $\mathcal{D}'$ must be smaller than m, so it holds $m' < m$. And if θ was not the only maximal falsum rule conclusion in $\mathcal{D}$, then the number of maximal falsum rule conclusions in $\mathcal{D}'$ must be smaller than n, so we have both $m' = m$ and $n' < n$. In either case, it follows $fr(\mathcal{D}') < fr(\mathcal{D})$, as desired.

Assume now that θ is the conclusion of an application of (efq). In this case, the conversions for the classical connectives are similar to the ones presented above, so we only need to consider the case in which θ is of the shape $\theta = \psi \vee\!\!\!\vee \chi$. The corresponding conversion has the following form:

$$\begin{array}{c} \mathcal{D}_1 \\ \dfrac{\bot}{\psi \vee\!\!\!\vee \chi}\,(\text{efq}) \\ \mathcal{D}_2 \end{array} \qquad \text{converts to} \qquad \begin{array}{c} \mathcal{D}_1 \\ \dfrac{\dfrac{\bot}{\psi}\,(\text{efq})}{\psi \vee\!\!\!\vee \chi}\,\vee\!\!\!\vee I \\ \mathcal{D}_2 \end{array}$$

In this conversion, the new falsum rule conclusion is again of lower degree than the old one. Therefore, if $\mathcal{D}'$ is the deduction arising from the conversion, then we must have $fr(\mathcal{D}') < fr(\mathcal{D})$.

Now, as we have seen, each of the conversions reduces the falsum rank of $\mathcal{D}$. Thus, by repeating the procedure, we can construct a sequence of deductions with strictly decreasing falsum ranks. Since the lexicographic ordering on falsum ranks is well-founded, this finally yields a deduction $\mathcal{D}''$ for $\Gamma \vdash^{+}_{\mathsf{N}} \varphi$ such that $fr(\mathcal{D}'') = (0, 0)$, so all falsum rule conclusions in $\mathcal{D}''$ are atomic. □

It is easy to check that this proof would fail, if the conclusion of (raa) would be allowed to contain the inquisitive disjunction operator $\vee\!\!\!\vee$. This is an interesting observation, because the restriction posed on (raa), tracing back to Ciardelli's (2016) double negation rule, was initially motivated on purely *semantic* grounds. It thus seems that there might be a deeper connection between non-inquisitiveness in natural language and normalization in natural deduction systems.

Next, we need to show that one can also eliminate all cut segments starting with the conclusion of an introduction rule. The overall strategy will be similar to the technique employed in the previous lemma: we always select a cut segment with maximum complexity, transform the deduction in such a way that the complexity is decreased and repeat the procedure until each cut segment has been eliminated. However, as we have seen in Section 2.3, the 'higher-level' elimination rule $\vee\!\!\!\vee E_{+}$ now causes some additional complications. In particular, by performing the logical conversion for cut segments ending with the major premise of an application of $\vee\!\!\!\vee E_{+}$, each of the auxiliary formulas of this application (i.e., each of the formulas from the set Θ) may become a new cut formula, and these formulas can have an arbitrarily large degree. In order to solve this problem, we need a more sophisticated measure of the 'complexity' of a formula.

Definition 2.4.3 (Inquisitive Degree, Classical Degree, Rank)**.** Let $\varphi \in \mathcal{L}^{\mathsf{B}}$ be a formula.

(i) The *inquisitive degree* of φ is denoted by $deg_i(\varphi)$ and defined to be the number of occurrences of the inquisitive disjunction $\vee\!\!\!\vee$ in φ. And the *classical degree* of φ is denoted by $deg_c(\varphi)$ and defined to be the number of occurrences of the connectives $\wedge$ and $\rightarrow$ in φ.

(ii) The *rank* of φ is defined to be the ordered pair given by $rank(\varphi) := (deg_i(\varphi), deg_c(\varphi))$.

In what follows, we will assume that ranks of formulas are ordered lexicographically. That is, we will write $rank(\varphi) < rank(\psi)$ and say that the rank of φ is *smaller* than the rank of ψ, if we either have $deg_i(\varphi) < deg_i(\psi)$, or we have both $deg_i(\varphi) = deg_i(\psi)$ and $deg_c(\varphi) < deg_c(\psi)$. The following lemma summarizes some immediate consequences of the preceding definitions.

Lemma 2.4.4. *Let $\varphi_1, \varphi_2 \in \mathcal{L}^{\mathsf{B}}$ be arbitrary formulas. It holds:*

(i) $rank(\varphi_i) < rank(\varphi_1 \otimes \varphi_2)$ for $i = 1, 2$ and $\otimes \in \{\wedge, \rightarrow, \vee\!\!\!\vee\}$,

(ii) $rank(\alpha) < rank(\varphi_1 \vee\!\!\!\vee \varphi_2)$ for every classical formula $\alpha \in \mathcal{L}^{\mathsf{B}}_c$.

Proof. We only prove statement (ii), the other part is trivial. Let $\varphi_1, \varphi_2 \in \mathcal{L}^{\mathsf{B}}$ be arbitrary formulas and let $\alpha \in \mathcal{L}^{\mathsf{B}}_c$ be a classical formula. Since $\varphi_1 \vee\!\!\!\vee \varphi_2$ contains an occurrence of $\vee\!\!\!\vee$, its inquisitive degree must satisfy $deg_i(\varphi_1 \vee\!\!\!\vee \varphi_2) \geq 1$. On the other hand, because α is a classical formula, we also know that it holds $deg_i(\alpha) = 0$. Therefore, it follows $deg_i(\alpha) < deg_i(\varphi_1 \vee\!\!\!\vee \varphi_2)$, so we may conclude $rank(\alpha) < rank(\varphi_1 \vee\!\!\!\vee \varphi_2)$ by definition of the lexicographic ordering. □

Most importantly, this lemma tells us that the logical conversions described in Section 2.3 do indeed decrease the complexity of cut segments in a deduction: if a new cut formula arises from one of these conversions, then it must always be of smaller rank than the original one. Using the rank of a formula, we may now define some useful concepts. First of all, by the *rank of a cut segment*, we will henceforth mean the rank of the cut formula occurring in this segment. For convenience, the rank of a cut segment π will also be denoted by $rank(\pi)$. The notion of a *maximal cut segment* and the *cut rank* of a deduction are now defined in the following way.

Definition 2.4.5 (Maximal Cut Segment, Cut Rank)**.** Let $\mathcal{D}$ be a deduction in our system.

(i) By a *maximal cut segment* in $\mathcal{D}$, we mean a cut segment π in $\mathcal{D}$ such that π has maximal rank, i.e., for every cut segment σ in $\mathcal{D}$ holds $rank(\sigma) < rank(\pi)$ or $rank(\sigma) = rank(\pi)$.

(ii) The *cut rank* of $\mathcal{D}$ is the triple $cr(\mathcal{D}) = (m, n, p)$ defined as follows: if $\mathcal{D}$ contains no cut segments, then we put $cr(\mathcal{D}) := (0, 0, 0)$. Otherwise, we put $cr(\mathcal{D}) := (m, n, p)$, where (m, n) is the rank of a maximal cut segment in $\mathcal{D}$ and p is the number of maximal cut segments in $\mathcal{D}$.

As with the ranks of formulas, we assume that cut ranks of deductions are ordered *lexicographically*. In other words, given a deduction $\mathcal{D}$ with $cr(\mathcal{D}) = (m, n, p)$ and another deduction $\mathcal{D}'$ with $cr(\mathcal{D}') = (m', n', p')$, we will write $cr(\mathcal{D}) < cr(\mathcal{D}')$ and say that the cut rank of $\mathcal{D}$ is *smaller* than the cut rank of $\mathcal{D}'$, if one of the following three conditions is satisfied: (1) $m < m'$, or (2) $m = m'$ and $n < n'$, or (3) $m = m'$, $n = n'$ and $p < p'$. It is now possible to show that cut ranks of non-normal deductions can always be decreased: every deduction containing at least one cut segment can be transformed into a deduction of smaller cut rank.

Lemma 2.4.6. *Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ be a set of formulas and let $\mathcal{D}$ be a deduction for $\Gamma \vdash_{\mathsf{N}}^{+} \varphi$ such that all falsum rule conclusions in $\mathcal{D}$ are atomic. If $cr(\mathcal{D}) > (0, 0, 0)$, then there exists a deduction $\mathcal{D}'$ for $\Gamma \vdash_{\mathsf{N}}^{+} \varphi$ such that $cr(\mathcal{D}') < cr(\mathcal{D})$ and all falsum rule conclusions in $\mathcal{D}'$ are still atomic.*

Proof. Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ be an arbitrary set of formulas, let $\mathcal{D}$ be a deduction for $\Gamma \vdash_{\mathsf{N}}^{+} \varphi$ and suppose that all falsum rule conclusions in $\mathcal{D}$ are atomic. So, in particular, $\mathcal{D}$ cannot contain any cut segments starting with the conclusion of a falsum rule application.[10] Furthermore, let $cr(\mathcal{D}) = (m, n, p)$ be the cut rank of $\mathcal{D}$ and assume that it holds $cr(\mathcal{D}) > (0, 0, 0)$. Then, clearly, $\mathcal{D}$ must contain at least one cut segment (which, by assumption, must start with the conclusion of an introduction rule). Let now π be a maximal cut segment in $\mathcal{D}$ such that all cut segments occurring above π in $\mathcal{D}$ are of lower rank than π. In other words, π is a cut segment with $rank(\pi) = (m, n)$, and it holds $rank(\sigma) < rank(\pi)$ for every cut segment σ occurring in the subtree ending with the conclusion of the elimination rule at the bottom of π. Let ξ be the cut formula occurring in π and let k be the length of the segment π. If $k > 1$, then we first perform the following *permutation conversion*, possibly in combination with a prior 'shifting' of hypotheses, in order to avoid hypotheses in $\mathcal{D}_3$ to become open in the course of the permutation:[11]

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}[\Theta]^u\\ \mathcal{D}_3\\ \varphi_1 \mathbin{\vee\!\!\!\vee} \varphi_2\end{matrix} \quad \begin{matrix}[\Theta \Rightarrow \varphi_1]^u\\ \mathcal{D}_1\\ \xi\end{matrix} \quad \begin{matrix}[\Theta \Rightarrow \varphi_2]^u\\ \mathcal{D}_2\\ \xi\end{matrix}}{\xi}\,{\vee\!\!\!\vee}E_+^u \quad \mathcal{D}_4}{\mu}\,(\mathsf{E})\\ \mathcal{D}_5\end{matrix} \quad \text{converts to} \quad \begin{matrix}\dfrac{\begin{matrix}[\Theta]^u\\ \mathcal{D}_3\\ \varphi_1 \mathbin{\vee\!\!\!\vee} \varphi_2\end{matrix} \quad \dfrac{\begin{matrix}[\Theta \Rightarrow \varphi_1]^u\\ \mathcal{D}_1\\ \xi\end{matrix} \quad \mathcal{D}_4}{\mu}\,(\mathsf{E}) \quad \dfrac{\begin{matrix}[\Theta \Rightarrow \varphi_2]^u\\ \mathcal{D}_2\\ \xi\end{matrix} \quad \mathcal{D}_4}{\mu}\,(\mathsf{E})}{\mu}\,{\vee\!\!\!\vee}E_+^u\\ \mathcal{D}_5\end{matrix}$$

Note that, although this conversion produces two identical copies of $\mathcal{D}_4$, this cannot increase the cut rank of the deduction: by assumption, π is an uppermost maximal cut segment, so all cut segments occurring in $\mathcal{D}_4$ are of lower rank than π. Each application of the permutation conversion

[10] Again, this follows immediately from the fact that major premises of elimination rules are always non-atomic.

[11] That is, if (E) is an instance of ${\vee\!\!\!\vee}E_+$ that discharges some open hypotheses in the subtree $\mathcal{D}_3$, then we first apply the procedure described in Section 2.3 in order to 'shift' these hypotheses from $\mathcal{D}_3$ to the subtrees $\mathcal{D}_1$ and $\mathcal{D}_2$.

reduces the length of the cut segment by one, so after $k-1$ steps, we must arrive at a deduction $\mathcal{D}^*$ for $\Gamma \vdash^+_{\mathsf{N}} \varphi$ in which π contains only a single occurrence of the cut formula ξ. Since neither the 'shifting' of hypotheses nor the permutation itself can produce any new maximal cut segments, this deduction $\mathcal{D}^*$ must have the same cut rank as $\mathcal{D}$, so it holds $cr(\mathcal{D}^*) = (m, n, p)$. Depending on the form of the cut formula ξ, we now perform the corresponding *logical conversion*:

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \varphi_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \varphi_2\end{array}}{\varphi_1 \wedge \varphi_2}\,\wedge I}{\varphi_i}\,\wedge E \\ \mathcal{D}_3 \end{array} \qquad \text{converts to} \qquad \begin{array}{c}\mathcal{D}_i\\ \varphi_i\\ \mathcal{D}_3\end{array}$$

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c}[\psi]^u\\ \mathcal{D}_1\\ \chi\end{array}}{\psi \to \chi}\,{\to} I^u \quad \begin{array}{c}\mathcal{D}_2\\ \psi\end{array}}{\chi}\,{\to} E \\ \mathcal{D}_3 \end{array} \qquad \text{converts to} \qquad \begin{array}{c}\mathcal{D}_2\\ \psi\\ \mathcal{D}_1\\ \chi\\ \mathcal{D}_3\end{array}$$

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c}[\Theta]^u\\ \mathcal{D}_3\\ \varphi_i\end{array}}{\varphi_1 \vee \varphi_2}\,\vee I \quad \begin{array}{c}[\Theta \Rightarrow \varphi_1]^u\\ \mathcal{D}_1\\ \chi\end{array} \quad \begin{array}{c}[\Theta \Rightarrow \varphi_2]^u\\ \mathcal{D}_2\\ \chi\end{array}}{\chi}\,\vee E^u_+ \\ \mathcal{D}_4 \end{array} \qquad \text{converts to} \qquad \begin{array}{c}\mathcal{D}_i\{\mathcal{D}_3 : \Theta \Rightarrow \varphi_i\}\\ \chi\\ \mathcal{D}_4\end{array}$$

We consider in detail only the case in which ξ is of the form $\xi = \varphi_1 \vee \varphi_2$, so we assume that the last of these three conversions has been performed. Let $\mathcal{D}'$ be the deduction arising from the conversion and let $cr(\mathcal{D}') = (m', n', p')$ be the cut rank of $\mathcal{D}'$. First of all, it is easy to see that $\mathcal{D}'$ is still a deduction for $\Gamma \vdash^+_{\mathsf{N}} \varphi$ and all falsum rule conclusions in $\mathcal{D}'$ are still atomic. Moreover, recall that $\mathcal{D}_i\{\mathcal{D}_3 : \Theta \Rightarrow \varphi_i\}$ stands for the result of replacing every undischarged application of the rule $\Theta \Rightarrow \varphi_i$ in $\mathcal{D}_i$ by an occurrence of $\mathcal{D}_3$. Hence, the only new cut formulas possibly arising from the conversion are the formula φ_i and the classical formulas contained in the set Θ. By Lemma 2.4.4, we know that it holds $rank(\varphi_i) < rank(\varphi_1 \vee \varphi_2)$ and $rank(\alpha) < rank(\varphi_1 \vee \varphi_2)$ for every classical formula $\alpha \in \mathcal{L}^{\mathsf{B}}_c$. Therefore, all new cut formulas must be of lower rank than the original cut formula $\xi = \varphi_1 \vee \varphi_2$. Furthermore, by assumption, all cut segments occurring in $\mathcal{D}_3$ are of rank smaller than (m, n), so we are not in danger of duplicating maximal cut segments. Thus, if ξ was the only maximal cut segment in $\mathcal{D}^*$, then we have $(m', n') < (m, n)$. Otherwise, we have $(m', n') = (m, n)$ and $p' < p$. In either case, it follows $cr(\mathcal{D}') < cr(\mathcal{D})$, as desired. $\square$

Observe that this proof would fail, if the set of auxiliary formulas Θ of an application of $\vee E_+$ would be allowed to contain non-classical formulas. Again, this is an interesting observation, since the restriction to classical formulas, inherited from the split rule depicted in Figure 1.4, was initially motivated purely semantically. We are now ready to prove the main result of this chapter: every deduction in our natural deduction system reduces to a normal deduction.

Theorem 2.4.7 (Normalization Theorem)**.** *Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ be a set of formulas. Every deduction $\mathcal{D}$ for $\Gamma \vdash^+_{\mathsf{N}} \varphi$ in the system* NinqB^+ *can be transformed into a normal deduction $\mathcal{D}'$ for $\Gamma \vdash^+_{\mathsf{N}} \varphi$.*

Proof. Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ be arbitrary and let $\mathcal{D}$ be an arbitrary deduction for $\Gamma \vdash^+_{\mathsf{N}} \varphi$. By Lemma 2.4.2, we may assume that all falsum rule conclusions in $\mathcal{D}$ are atomic. If $\mathcal{D}$ is already normal, then we are finished. Otherwise, we can repeatedly apply Lemma 2.4.6 in order to construct a sequence $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$, etc., of deductions for $\Gamma \vdash^+_{\mathsf{N}} \varphi$ such that the corresponding cut ranks form a descending chain $cr(\mathcal{D}) > cr(\mathcal{D}_1) > cr(\mathcal{D}_2) > \ldots$ After a finite number of steps, this must result in a deduction $\mathcal{D}'$ with $cr(\mathcal{D}') = (0, 0, 0)$. But then, clearly, $\mathcal{D}'$ is a normal deduction. $\square$

2.5 Properties of Normal Deductions

In order to conclude our treatment of natural deduction, let us now investigate some consequences of our normalization theorem. To this end, we first need to introduce some terminology. Recall that, by an *auxiliary rule* of an application of $\mathbb{V}E_+$ with major premise $\varphi \mathbb{V} \psi$ and set of auxiliary formulas Θ, we mean any occurrence of the non-primitive rules $\Theta \Rightarrow \varphi$ and $\Theta \Rightarrow \psi$ which is discharged in the course of this application. And the *side formulas* of such an application of $\mathbb{V}E_+$ are the occurrences of the formulas φ and ψ serving as the conclusions of the auxiliary rules $\Theta \Rightarrow \varphi$ and $\Theta \Rightarrow \psi$. The notion of a *track* in a deduction is now defined in the following way.

Definition 2.5.1 (Track)**.** Let $\mathcal{D}$ be a deduction in NinqB^+. By a *track* in $\mathcal{D}$, we mean a sequence of formula occurrences $\varphi_1, \ldots, \varphi_n$ in $\mathcal{D}$ such that each of the following conditions is satisfied:

(i) φ_1 is an open or discharged hypothesis in $\mathcal{D}$.

(ii) Each φ_i with $1 \leq i < n$ is not a premise of a non-primitive rule and not the minor premise of an application of $\to E$, and it holds:

  (a) If φ_i is not the major premise of an application of $\mathbb{V}E_+$, then its successor φ_{i+1} is simply the formula occurrence standing immediately below φ_i.

  (b) If φ_i is the major premise of an application of $\mathbb{V}E_+$, then its successor φ_{i+1} is one of the side formulas of this application.

(iii) φ_n is either a premise of a non-primitive rule, or it is the minor premise of an application of $\to E$, or it is the conclusion of the deduction $\mathcal{D}$.

In other words, a track in a deduction $\mathcal{D}$ is a sequence of consecutive formula occurrences in $\mathcal{D}$, beginning with a hypothesis and possibly ending with the conclusion of $\mathcal{D}$, except that (1) a track passing through the major premise of an application of $\mathbb{V}E_+$ is always continued at one of the side formulas of this application, and (2) a track always stops as soon as a premise of a non-primitive rule or the minor premise of an application of $\to E$ is encountered. So, as a consequence, none of the formula occurrences in a track, except possibly the last one, can be a premise of a non-primitive rule or the minor premise of an application of $\to E$. In what follows, a track τ in a deduction $\mathcal{D}$ will be referred to as a *main track*, if the last formula occurrence in τ is the conclusion of the deduction $\mathcal{D}$. To see an example, let us consider the following deduction:

$$
\dfrac{
  \dfrac{
    \dfrac{
      (1)\; (\alpha \mathbb{V} \beta) \to (\varphi \mathbb{V} \psi)
      \qquad
      \dfrac{(6)\; [\alpha]^u}{(7)\; \alpha \mathbb{V} \beta}\,\mathbb{V}I
    }{(2)\; \varphi \mathbb{V} \psi}\,{\to}E
    \qquad
    \dfrac{\dfrac{(8)\; [\alpha\wedge\beta]^v}{(9)\; \alpha}\,\wedge E}{(3)\; \varphi}\,u
    \qquad
    \dfrac{\dfrac{(12)\; \neg\psi \qquad \dfrac{\dfrac{(10)\; [\alpha\wedge\beta]^v}{(11)\; \alpha}\,\wedge E}{(15)\; \psi}\,u}{(13)\; \bot}\,{\to}E}{(14)\; \varphi}\,\text{(efq)}
  }{(4)\; \varphi}\,\mathbb{V}E_+^u
}{(5)\; (\alpha\wedge\beta)\to\varphi}\,{\to}I^v
$$

A straightforward inspection shows that this deduction contains exactly six tracks. The main tracks of the deduction are the two sequences $\tau_1 = (1, 2, 3, 4, 5)$ and $\tau_2 = (12, 13, 14, 4, 5)$. And the other four tracks are the sequences $\tau_3 = (1, 2, 15)$, $\tau_4 = (6, 7)$, $\tau_5 = (8, 9)$ and $\tau_6 = (10, 11)$. It is easy to see that every deduction in our system must have at least one main track. In fact, such a main track can always be found by going upwards from the conclusion of the deduction in such a way that, if a side formula of an application of $\mathbb{V}E_+$ or the conclusion of an application of $\to E$ is encountered, then the search is always continued at the major premise of this application. In addition, one can show that every formula occurrence in a deduction belongs to at least one track.

Proposition 2.5.2. *Let $\mathcal{D}$ be a deduction in our system. Then $\mathcal{D}$ has at least one main track. Furthermore, every formula occurrence in $\mathcal{D}$ belongs to some track in $\mathcal{D}$.*

Proof. The basic idea is as follows: first, we define a *pre-track* in a quasi-deduction $\mathcal{D}$ in exactly the same way as a track, except that a pre-track is also allowed to start with the conclusion of an undischarged application of a non-primitive rule. A pre-track is said to be *maximal*, if it ends with the conclusion of the deduction. Using induction on the structure of a quasi-deduction, it is then possible to show that each quasi-deduction must have at least one maximal pre-track and that every formula occurrence in a quasi-deduction belongs to at least one pre-track. The desired statement now follows from the observation that, if $\mathcal{D}$ is a *deduction* (i.e., if all applications of non-primitive rules in $\mathcal{D}$ are discharged), then every pre-track in $\mathcal{D}$ is also a track. □

Obviously, every track can also be divided into a sequence of *segments* rather than into a sequence of *formula occurrences*. The first segment in a track will be referred to as the *top segment* of the track and the last segment will be referred to as its *end segment*. For simplicity, we also say that a segment is the *conclusion* of a rule of inference, if the first formula occurrence in the segment is such a conclusion. And a segment is said to be a (major or minor) *premise* of a rule application, if the last formula occurrence in the segment is such a premise. Recall that, in a normal deduction, no segment can be both the conclusion of an introduction rule or the conclusion of a falsum rule, and the major premise of an elimination rule. As a consequence, we now obtain the following fact.

Proposition 2.5.3 (Structure of Tracks)**.** *Let $\mathcal{D}$ be a normal deduction and let τ be a track in $\mathcal{D}$. All major premises of elimination rules in τ precede all premises of falsum rules and all premises of introduction rules in τ. Moreover, there can be at most one premise of a falsum rule in τ and this premise succeeds all major premises of elimination rules and precedes all premises of introduction rules.*

Proof. Let $\mathcal{D}$ be a normal deduction, let τ be a track in $\mathcal{D}$ and let $\pi_1, \dots, \pi_n$ be the sequence of segments in τ. We start by proving the first claim. Towards a contradiction, suppose that there exists a premise of an introduction rule or a premise of a falsum rule in τ that precedes a major premise of an elimination rule. Then, clearly, there must be a segment π_i which is both the conclusion of an introduction rule or falsum rule, and the major premise of an elimination rule. But this is a contradiction to the assumption that $\mathcal{D}$ is normal. Hence, all major premises of elimination rules in τ precede all premises of falsum rules and all premises of introduction rules in τ.

For the second claim, suppose for a contradiction that there is a premise of an introduction rule which precedes a premise of a falsum rule in τ. Consider the last such premise π_i. By what was said above, the successor π_{i+1} cannot be the major premise of an elimination rule, so it must be the premise of a falsum rule. But in this case, the formula occurring in π_{i+1} is $\bot$, which is impossible since $\bot$ cannot be the conclusion of an introduction rule. Thus, all premises of falsum rules succeed all major premises of elimination rules and precede all premises of introduction rules in τ. This also shows that there can be at most one premise of a falsum rule in τ: otherwise, there would be a premise of a falsum rule that is immediately followed by another premise of a falsum rule, which is a contradiction to the assumption that all falsum rule conclusions in $\mathcal{D}$ are atomic. □

Proposition 2.5.3 tells us that every track in a normal deduction can be divided into two (possibly empty) parts: an *elimination part* containing only major premises of elimination rules, and an *introduction part* containing only premises of introduction rules and the end formula of the track. In order to strengthen this result, we first define the concept of a *strictly positive subformula*.

Definition 2.5.4. The *strictly positive subformulas* of $\varphi \in \mathcal{L}^{\mathsf{B}}$ are inductively defined as follows:
(i) φ is a strictly positive subformula of itself.
(ii) If $\psi \otimes \chi$ with $\otimes \in \{\wedge, \mathbin{\vee\!\!\!\vee}\}$ is a strictly positive subformula of φ, then so are both ψ and χ.
(iii) If $\psi \to \chi$ is a strictly positive subformula of φ, then so is χ.

In other words, an occurrence of a subformula is strictly positive, if it is not contained in the antecedent of an implication. Note that, in particular, a *Harrop formula* might now simply be described as a formula that does not contain any strictly positive subformulas of the form $\varphi \mathbin{\vee\!\!\!\vee} \psi$

(see Definition 1.3.2). By inspection of the primitive rules of our system, it is now easy to check that, if a track contains a premise of an introduction rule, then this premise must always be a strictly positive subformula of its immediate successor in the track. And if a track contains a major premise of an elimination rule, then this premise always contains its immediate successor as a strictly positive subformula.[12] As a consequence, we now obtain the following proposition.

Proposition 2.5.5. *Let $\mathcal{D}$ be a normal deduction, let τ be a track in $\mathcal{D}$ and let $\pi_1, \dots, \pi_n$ be the sequence of segments in τ. There is a segment π_k in τ, called the* minimum segment *of τ, such that:*

(i) Each π_i with $1 \leq i < k$ is a major premise of an elimination rule and the formula occurring in π_i is a strictly positive subformula of the formula occurring in the top segment π_1.

(ii) π_k is a premise of an introduction rule or the premise of a falsum rule or it is the conclusion of $\mathcal{D}$.

(iii) Each π_i with $k < i < n$ is a premise of an introduction rule and the formula occurring in π_i is a strictly positive subformula of the formula occurring in the end segment π_n.

Proof. Let $\mathcal{D}$ be a normal deduction, let τ be a track in $\mathcal{D}$ and let $\pi_1, \dots, \pi_n$ be the sequence of segments in τ. If τ contains a premise of an introduction rule or a premise of a falsum rule, then let π_k be the segment that ends with the first such formula in τ. Otherwise, let π_k be the end segment of the track τ. Then, by Proposition 2.5.3, all segments preceding π_k in τ must be major premises of elimination rules and all segments succeeding π_k in τ are premises of introduction rules. The second part of statements (i) and (iii) can now be established by a straightforward induction on the length of the elimination part and the introduction part, respectively. □

Note that, by definition, every track in a deduction either ends with a premise of a non-primitive rule, or with the minor premise of an application of $\to E$, or it ends with the conclusion of the deduction. Thus, one can assign an *order* to all tracks in a deduction in the following way: a main track is assigned the order 0, and a track ending with the minor premise of an instance of $\to E$ is assigned the order $m+1$, if the major premise of this instance belongs to a track of order m. Finally, if τ is a track ending with a premise α of a non-primitive rule $\Theta \Rightarrow \varphi$, discharged by some application of $\mathbb{V}E_+$, then we assign τ the order $m+1$, where m is the maximum order of all tracks starting with the auxiliary formula α of this application of $\mathbb{V}E_+$. So, for example, let $\mathcal{D}$ be an arbitrary deduction and suppose that $\mathcal{D}$ contains an application of $\mathbb{V}E_+$ of the following form:

$$\dfrac{\begin{matrix}[\Theta]^u \\ \mathcal{D}_1 \\ \varphi \mathbin{\mathbb{V}} \psi\end{matrix} \qquad \begin{matrix}[\Theta \Rightarrow \varphi]^u \\ \mathcal{D}_2 \\ \chi\end{matrix} \qquad \begin{matrix}[\Theta \Rightarrow \psi]^u \\ \mathcal{D}_3 \\ \chi\end{matrix}}{\chi}\ \mathbb{V}E_+^u$$

Consider an arbitrary auxiliary formula $\alpha \in \Theta$ and let $\tau_1, \dots, \tau_n$ be the collection of all tracks in $\mathcal{D}_1$ that start with an occurrence of the discharged hypothesis α. In this case, every track in $\mathcal{D}_2$ or $\mathcal{D}_3$ that ends with the premise α of one of the auxiliary rules $\Theta \Rightarrow \varphi$ and $\Theta \Rightarrow \psi$ will be assigned the order $m+1$, where m is the maximum order of the tracks $\tau_1, \dots, \tau_n$. In what follows, the order of a track τ will also be denoted by $ord(\tau)$. We can now prove the following statement.

Proposition 2.5.6. *Let $\mathcal{D}$ be a normal deduction and let τ be a track in $\mathcal{D}$. If τ contains the major premise of an application of $\mathbb{V}E_+$, then every track starting with an auxiliary formula of this application is of order strictly greater than $ord(\tau)$. As a consequence, a main track of a normal deduction can never begin with an auxiliary formula of an application of $\mathbb{V}E_+$.*

Proof. Let $\mathcal{D}$ be a normal deduction and suppose that $\mathcal{D}$ contains a major premise $\varphi \mathbin{\mathbb{V}} \psi$ of some application of $\mathbb{V}E_+$ with set of auxiliary formulas Θ. Let $\tau_1, \dots, \tau_n$ be the collection of all tracks containing this major premise $\varphi \mathbin{\mathbb{V}} \psi$ and let $\mathcal{D}'$ be the subtree ending with $\varphi \mathbin{\mathbb{V}} \psi$. Then, clearly,

[12] Note that, for this observation to hold, it is in fact essential that tracks passing through the major premise of an application of $\mathbb{V}E_+$ are always continued at one of the side formulas (rather than at the conclusion) of this application.

the initial parts of $\tau_1, \dots, \tau_n$ ending with $\varphi \vee \psi$ are the only main tracks of $\mathcal{D}'$ and the auxiliary formulas from Θ can occur as hypotheses only in $\mathcal{D}'$. Therefore, either there exists an $\alpha \in \Theta$ which is the top formula of some τ_i, or every track starting with an auxiliary formula from Θ is of order greater than $ord(\tau_i)$ for all $1 \leq i \leq n$. Towards a contradiction, assume that there exists an $\alpha \in \Theta$ which is the top formula of some τ_i. By Proposition 2.5.5, $\varphi \vee \psi$ occurs in the elimination part of τ_i, so it must be a subformula of α. But this is impossible, since α is a classical formula and $\varphi \vee \psi$ is not. Thus, every track starting with an auxiliary formula from Θ must be of order greater than $ord(\tau_i)$, for all $1 \leq i \leq n$. This establishes the first claim. The second claim follows as an immediate consequence: for, if a main track of a normal deduction would start with an auxiliary formula of an application of $\vee E_+$, then also the major premise of this application would have to belong to this track. However, as we have just seen, this cannot be the case. □

It is well known that, for the standard system of intuitionistic natural deduction, one can prove an unrestricted version of the so-called *subformula property*: every formula occurring in a normal intuitionistic deduction is either a subformula of the conclusion of the deduction, or it is a subformula of some open hypothesis of the deduction (cf. Prawitz 1965, p. 53). Unfortunately, for the natural deduction system NinqB^+, this is not in general the case. As in classical logic, one problem arises from the presence of the *reductio ad absurdum rule*, since this rule allows to construct normal deductions like the one below, where $p \in \mathsf{P}$ is an arbitrary atomic formula:

$$\dfrac{\dfrac{[\neg p]^u \qquad p}{\bot}\rightarrow \mathrm{E}}{p}\,(\mathsf{raa})^u$$

As can be seen, this normal deduction contains an occurrence of the negated formula $\neg p$ and a subsequent occurrence of $\bot$, but neither of these two formulas is a subformula of the conclusion or a subformula of an open hypothesis of the deduction. Consequently, in our system, a normal deduction for $\Gamma \vdash^+_{\mathsf{N}} \varphi$ might in general also contain the constant $\bot$ or the negation of an atomic formula occurring in φ or in Γ.[13] For convenience, we will henceforth say that a formula φ is a *weak subformula* of some formula ψ, just in case φ is either a subformula of ψ, or it is the negation of an atomic subformula of ψ, or it is of the form $\varphi = \bot$. So, in our system, a normal deduction may contain not only subformulas, but also weak subformulas of the conclusion or the open hypotheses of the deduction. However, there is also a more severe problem arising from the intricate structure of the 'higher-level' elimination rule $\vee E_+$. For example, in the system NinqB^+, one can also construct a normal deduction of the following form, where $\alpha, \beta \in \mathcal{L}_c^{\mathsf{B}}$ are classical formulas:

$$\dfrac{\dfrac{\alpha \to (\varphi \vee \varphi) \qquad \dfrac{[\alpha \wedge \beta]^u}{\alpha}\wedge E}{\varphi \vee \varphi}\to E \qquad \dfrac{\dfrac{\alpha \qquad \beta}{\alpha \wedge \beta}\wedge I}{\varphi}\,u \qquad \dfrac{\dfrac{\alpha \qquad \beta}{\alpha \wedge \beta}\wedge I}{\varphi}\,u}{\varphi}\vee E_+^u$$

In this deduction, there are three occurrences of the auxiliary formula $\alpha \wedge \beta$, belonging to the application of $\vee E_+$ at the bottom of the deduction. However, this formula is neither a subformula of the conclusion nor a subformula of an open hypothesis of the proof tree. Note that, in a sense, $\alpha \wedge \beta$ behaves just like a *cut formula* here: it is the conclusion of an introduction rule in the two subtrees ending with the minor premise φ; and it is, at the same time, the major premise of an elimination rule in the subtree on the left-hand side. Given these observations, we cannot hope to obtain a strong subformula property for NinqB^+. In fact, in our system, the subformula property only holds up to the first track starting with an auxiliary formula of an application of $\vee E_+$.

[13] Note that, since all falsum rule applications in normal deductions are restricted to instances in which the conclusion is an atomic formula, this is in fact sufficient in order to account for the effect of the rule (raa).

Theorem 2.5.7 (Restricted Subformula Property). *Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ be a set of formulas and let $\mathcal{D}$ be a normal deduction for $\Gamma \vdash^{+}_{\mathsf{N}} \varphi$. Moreover, let m be the largest number such that no track of order $\leq m$ begins with an auxiliary formula of an application of $\mathbb{V}E_{+}$.*[14] *Then every formula occurring in a track τ of $\mathcal{D}$ with $ord(\tau) \leq m$ is a weak subformula of some formula in $\Gamma \cup \{\varphi\}$.*

Proof. Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ be arbitrary, let $\mathcal{D}$ be a normal deduction for $\Gamma \vdash^{+}_{\mathsf{N}} \varphi$ and let m be the largest number such that no track of order $\leq m$ in $\mathcal{D}$ begins with an auxiliary formula of some application of $\mathbb{V}E_{+}$. Consider an arbitrary track τ in $\mathcal{D}$ such that $ord(\tau) \leq m$. Then, clearly, τ cannot start with an auxiliary formula of an application of $\mathbb{V}E_{+}$ and it cannot end with a premise of a non-primitive rule. We now proceed by induction on the order of τ. Let $ord(\tau) = k$ and suppose that the statement holds for all tracks of order less than k. Moreover, let $\psi_1, \ldots, \psi_n$ be the sequence of formulas in τ and let ψ_i be arbitrary. First, assume that ψ_i occurs below the minimum segment in τ. Then, by Proposition 2.5.5, ψ_i is a subformula of the end formula ψ_n, which must be either the conclusion of $\mathcal{D}$ or a minor premise of $\to E$. If ψ_n is the conclusion of $\mathcal{D}$, then we are finished. And if ψ_n is a minor premise of $\to E$, then the associated major premise must belong to a track of order less than k, so the claim follows from the induction hypothesis. Assume now that ψ_i occurs in the minimum segment or above the minimum segment. Then, by Proposition 2.5.5, ψ_i is a subformula of the hypothesis ψ_1 at the top of τ. If this hypothesis is not discharged, then we are done. Otherwise, it must be discharged by an application of (raa) or $\to I$. If ψ_1 is discharged by (raa), then ψ_i must either be of the form $\bot$ or it must be the negation of an atom occurring in the introduction part of τ or in the introduction part of some track of order less than k. Hence, using the induction hypothesis, we may conclude that ψ_i is a weak subformula of some formula in $\Gamma \cup \{\varphi\}$. And if ψ_1 is discharged by an application of $\to I$, then the conclusion of this application must belong to the introduction part of τ or to the introduction part of some track of order less than k, so the statement follows in the same way as above. $\square$

This result is rather weak and shows that applications of $\mathbb{V}E_{+}$ may have a very disturbing impact on the structure of normal deductions. However, there is also an important special case in which the subformula property for our system can be strengthened: if $\Gamma \vdash^{+}_{\mathsf{N}} \varphi$ can be established by a deduction in which all auxiliary formulas of applications of $\mathbb{V}E_{+}$ are *implication-free*, then one can always find a normal deduction for $\Gamma \vdash^{+}_{\mathsf{N}} \varphi$ which satisfies a full subformula property.[15]

Proposition 2.5.8. *Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ and let $\mathcal{D}$ be a deduction for $\Gamma \vdash^{+}_{\mathsf{N}} \varphi$ such that all auxiliary formulas of applications of $\mathbb{V}E_{+}$ in $\mathcal{D}$ are implication-free. Then there is a normal deduction $\mathcal{D}'$ for $\Gamma \vdash^{+}_{\mathsf{N}} \varphi$ such that every formula occurring in $\mathcal{D}'$ is a weak subformula of some formula in $\Gamma \cup \{\varphi\}$.*

Proof. We only sketch the basic idea. Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ be arbitrary and let $\mathcal{D}$ be a deduction for $\Gamma \vdash^{+}_{\mathsf{N}} \varphi$ such that all auxiliary formulas of applications of $\mathbb{V}E_{+}$ in $\mathcal{D}$ are implication-free. Moreover, let us say that a formula ψ is a *prime formula*, if it is either atomic of it is of the form $\psi = \bot$. We first show that all applications of $\mathbb{V}E_{+}$ in $\mathcal{D}$ can be restricted to instances in which the auxiliary formulas are prime. To this end, we select a topmost application of $\mathbb{V}E_{+}$ in $\mathcal{D}$ which has at least one non-prime auxiliary formula γ. By assumption, γ is implication-free, so we must have $\gamma = \alpha \wedge \beta$ for some classical formulas $\alpha, \beta \in \mathcal{L}^{\mathsf{B}}_{c}$. Let $\Theta = \{\gamma_1, \ldots, \gamma_n\}$ be the set of all other auxiliary formulas of this application of $\mathbb{V}E_{+}$. We can then write $\mathcal{D}$ in the form

$$
\cfrac{\begin{array}{c}[\Theta]^u \;\; [\alpha \wedge \beta]^u \\ \mathcal{D}_3 \\ \psi_1 \mathbin{\mathbb{V}} \psi_2\end{array} \qquad \begin{array}{c}[\Theta, \alpha \wedge \beta \Rightarrow \psi_1]^u \\ \mathcal{D}_1 \\ \chi\end{array} \qquad \begin{array}{c}[\Theta, \alpha \wedge \beta \Rightarrow \psi_2]^u \\ \mathcal{D}_2 \\ \chi\end{array}}{\begin{array}{c}\chi \\ \mathcal{D}_4\end{array}} \;\mathbb{V}E^{u}_{+}
$$

[14] Recall that, by what was said in Proposition 2.5.6, no main track of $\mathcal{D}$ can begin with an auxiliary formula of an application of $\mathbb{V}E_{+}$. Thus, because main tracks are the only tracks of order 0, we must always have $m \geq 1$.

[15] Of course, by an *implication-free* formula, we mean a formula not containing any occurrences of the connective $\to$.

Let now $i = 1, 2$ be arbitrary and let $\mathcal{D}_i^*$ be the quasi-deduction obtained from $\mathcal{D}_i$ by rewriting every undischarged application of the non-primitive rule $\Theta, \alpha \wedge \beta \Rightarrow \psi_i$ in the following way:

$$\begin{array}{c}\dfrac{\begin{array}{cccc}\mathcal{D}'_1 & & \mathcal{D}'_n & \mathcal{D}'\\ \gamma_1 & \cdots & \gamma_n & \alpha\wedge\beta\end{array}}{\psi_i}\,\Theta,\alpha\wedge\beta\Rightarrow\psi_i\\ \mathcal{D}''\end{array} \quad \text{converts to} \quad \begin{array}{c}\dfrac{\begin{array}{cccc}\mathcal{D}'_1 & & \mathcal{D}'_n & \begin{array}{c}\mathcal{D}'\\ \dfrac{\alpha\wedge\beta}{\alpha}\wedge E\end{array} \quad \begin{array}{c}\mathcal{D}'\\ \dfrac{\alpha\wedge\beta}{\beta}\wedge E\end{array}\\ \gamma_1 & \cdots & \gamma_n & \end{array}}{\psi_i}\,\Theta,\alpha,\beta\Rightarrow\psi_i\\ \mathcal{D}''\end{array}$$

Thus, in the resulting quasi-deduction $\mathcal{D}_i^*$, every undischarged application of the rule $\Theta, \alpha \wedge \beta \Rightarrow \psi_i$ is replaced by an undischarged application of the rule $\Theta, \alpha, \beta \Rightarrow \psi_i$. Using the quasi-deductions $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$ thus obtained, we can now transform the whole deduction $\mathcal{D}$ into

$$\begin{array}{c}\dfrac{\begin{array}{c}[\Theta]^u \; \dfrac{[\alpha]^u \quad [\beta]^u}{\alpha\wedge\beta}\wedge I\\ \mathcal{D}_3\\ \psi_1 \vee\!\!\vee \psi_2\end{array} \qquad \begin{array}{c}[\Theta,\alpha,\beta\Rightarrow\psi_1]^u\\ \mathcal{D}_1^*\\ \chi\end{array} \qquad \begin{array}{c}[\Theta,\alpha,\beta\Rightarrow\psi_2]^u\\ \mathcal{D}_2^*\\ \chi\end{array}}{\chi}\,\vee\!\!\vee E_+^u\\ \mathcal{D}_4\end{array}$$

Observe that, in the resulting deduction, the old auxiliary formula $\alpha \wedge \beta$ is now replaced by two new auxiliary formulas α and β, which are of lower degree than the old one. Therefore, by repeating the procedure, we must finally arrive at a deduction $\mathcal{D}^*$ for $\Gamma \vdash_{\mathsf{N}}^{+} \varphi$ in which all auxiliary formulas of applications of $\vee\!\!\vee E_+$ are prime. We now apply the procedure described in Section 2.4, in order to transform $\mathcal{D}^*$ into a normal deduction $\mathcal{D}'$ for $\Gamma \vdash_{\mathsf{N}}^{+} \varphi$. Clearly, this does not affect the auxiliary formulas in the deduction, so these formulas must still be prime in $\mathcal{D}'$.

We will now prove that every formula in $\mathcal{D}'$ is a weak subformula of some formula in $\Gamma \cup \{\varphi\}$. To this end, let ξ be an arbitrary formula occurrence in $\mathcal{D}'$. Then, by Proposition 2.5.2, ξ must belong to some track $\tau = \psi_1, \dots, \psi_n$ in $\mathcal{D}$. We now proceed by induction on the order of τ. Most cases are treated in the same way as in the proof of Theorem 2.5.7. We only need to consider the case in which τ starts with an auxiliary formula discharged by an application of $\vee\!\!\vee E_+$, and the case in which τ ends with a premise of a non-primitive rule. First, assume that τ starts with an auxiliary formula. By assumption, this formula is prime, so the elimination part of τ must be empty and ξ must occur in the introduction part of τ. But then, by Proposition 2.5.5, ξ is a subformula of the formula ψ_n occurring at the bottom of τ, which must be either the conclusion of $\mathcal{D}'$, or a minor premise of $\rightarrow E$, or a premise of a non-primitive rule. If ψ_n is the conclusion of $\mathcal{D}'$, then we are finished. And if ψ_n is a minor premise of $\rightarrow E$ or a premise of a non-primitive rule, then ψ_n must also occur as a subformula on some track of order less than $ord(\tau)$, so the statement follows from the induction hypothesis.[16] Assume now that τ ends with a premise of a non-primitive rule, belonging to some application of $\vee\!\!\vee E_+$. By assumption, this premise is prime, so the introduction part of τ must be empty and ξ occurs in the elimination part of τ. Hence, by Proposition 2.5.5, ξ is a subformula of the hypothesis ψ_1 at the top of τ. If this hypothesis is open, then we are done. Otherwise, ψ_1 must be discharged by some application of $\vee\!\!\vee E_+$ or it must be discharged by an application of (raa) or $\rightarrow I$. If ψ_1 is discharged by $\vee\!\!\vee E_+$, then we use the same argument as above. And if ψ_1 is discharged by (raa) or $\rightarrow I$, then ξ must occur as a weak subformula in some track of order less than $ord(\tau)$, so the statement follows from the induction hypothesis. $\square$

Although one cannot in general hope to obtain a strong subformula property for our system, it is nevertheless possible to establish some interesting properties of inquisitive logic in a purely

[16] Note that, if ψ_n is a premise of a non-primitive rule, then this rule must be discharged by some application of $\vee\!\!\vee E_+$. Consequently, ψ_n also occurs as a (discharged) hypothesis in the subtree ending with the major premise of this application of $\vee\!\!\vee E_+$. And by definition of $ord(\tau)$, every track starting with such a hypothesis is of order less than $ord(\tau)$.

proof-theoretical way. For example, using Theorem 2.5.7, we can now prove that our system is *conservative* over classical logic: if a classical formula is provable from a set of classical hypotheses, then it is also provable from these hypotheses in classical logic (see also Proposition 1.4.5).

Proposition 2.5.9 (Conservativity over Classical Logic)**.** *Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_c^{\mathsf{B}}$ be a set of classical formulas. If it holds $\Gamma \vdash_{\mathsf{N}}^{+} \alpha$ in our system, then α is provable from Γ in classical natural deduction.*

Proof. Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_c^{\mathsf{B}}$ be a set of classical formulas and suppose that $\Gamma \vdash_{\mathsf{N}}^{+} \alpha$. Then, by Theorem 2.4.7, there exists a normal deduction $\mathcal{D}$ for $\Gamma \vdash_{\mathsf{N}}^{+} \alpha$. Let m be the largest number such that no track of order $\leq m$ in $\mathcal{D}$ begins with an auxiliary formula of an application of $\mathbb{V}E_{+}$. Towards a contradiction, suppose that there is a formula with main connective $\mathbb{V}$ in $\mathcal{D}$. Let τ be a track containing such a formula $\varphi \mathbb{V} \psi$ and suppose that no track of order less than $ord(\tau)$ also contains such a formula. Then, in particular, all major premises of applications of $\mathbb{V}E_{+}$ must occur on tracks of order $\geq ord(\tau)$. Hence, by Proposition 2.5.6, the corresponding auxiliary formulas can occur only on tracks of order $> ord(\tau)$, so it follows $ord(\tau) \leq m$. Thus, by Theorem 2.5.7, $\varphi \mathbb{V} \psi$ must be a weak subformula of some formula in $\Gamma \cup \{\alpha\}$. But this is impossible, since all formulas in $\Gamma \cup \{\alpha\}$ are classical. Therefore, $\mathcal{D}$ does not contain a formula with main connective $\mathbb{V}$. Consequently, $\mathcal{D}$ contains no applications of $\mathbb{V}I$ or $\mathbb{V}E_{+}$, so it is also a deduction in classical logic. □

Furthermore, by exploiting the structure of normal deductions, one can also give a purely syntactical proof of the *disjunction property* for inquisitive logic: if an inquisitive disjunction is provable from a set of Harrop formulas, then at least one of the two disjuncts must be provable from this set. This can be seen as a special case of the split property described in Proposition 1.4.1.

Proposition 2.5.10 (Disjunction Property)**.** *Let $\Gamma \subseteq \mathcal{L}_H^{\mathsf{B}}$ be a set of Harrop formulas and let $\varphi, \psi \in \mathcal{L}^{\mathsf{B}}$ be arbitrary. If it holds $\Gamma \vdash_{\mathsf{N}}^{+} \varphi \mathbb{V} \psi$ in our system, then also $\Gamma \vdash_{\mathsf{N}}^{+} \varphi$ or $\Gamma \vdash_{\mathsf{N}}^{+} \psi$.*

Proof. Let $\Gamma \subseteq \mathcal{L}_H^{\mathsf{B}}$ be a set of Harrop formulas, so no formula in Γ has a strictly positive subformula with main connective $\mathbb{V}$. Moreover, let $\varphi, \psi \in \mathcal{L}^{\mathsf{B}}$ be arbitrary and suppose that it holds $\Gamma \vdash_{\mathsf{N}}^{+} \varphi \mathbb{V} \psi$. Then, by Theorem 2.4.7, there exists a normal deduction $\mathcal{D}$ for $\Gamma \vdash_{\mathsf{N}}^{+} \varphi \mathbb{V} \psi$. Consider the conclusion $\varphi \mathbb{V} \psi$ of this deduction. We show that $\varphi \mathbb{V} \psi$ must have been obtained by an application of $\mathbb{V}I$. First, suppose for a contradiction that $\varphi \mathbb{V} \psi$ is the conclusion of an application of $\mathbb{V}E_{+}$. Consider the major premise $\chi_1 \mathbb{V} \chi_2$ of this application and let $\tau = \xi_1, \ldots, \xi_n$ be a track to which $\chi_1 \mathbb{V} \chi_2$ belongs. By Proposition 2.5.5, $\chi_1 \mathbb{V} \chi_2$ occurs in the elimination part of τ, so it must be a strictly positive subformula of the hypothesis ξ_1 at the top of τ. As a consequence, this hypothesis cannot be discharged by $\mathbb{V}E_{+}$, since this would require ξ_1 to be a classical formula. In addition, ξ_1 can also not be discharged by an application of (raa) or $\rightarrow I$, since no such application occurs below $\chi_1 \mathbb{V} \chi_2$ in $\mathcal{D}$. Thus, ξ_1 must be an open hypothesis. But this implies $\xi_1 \in \Gamma$, which is a contradiction to the assumption that Γ contains only Harrop formulas. Hence, $\varphi \mathbb{V} \psi$ is not the conclusion of an application of $\mathbb{V}E_{+}$. However, it can also not be the conclusion of another elimination rule: for, if it were, then $\varphi \mathbb{V} \psi$ would be the minimum segment of a main track τ in $\mathcal{D}$ and therefore a strictly positive subformula of the hypothesis at the top of τ. Using the same argument as above, one could then show that this hypothesis must be open, in contradiction to the assumption that Γ is a set of Harrop formulas. Furthermore, since $\varphi \mathbb{V} \psi$ is non-atomic, it can also not be the conclusion of a falsum rule. Hence, $\varphi \mathbb{V} \psi$ must be the conclusion of an application of $\mathbb{V}I$. But then, clearly, $\mathcal{D}$ contains an immediate subdeduction for $\Gamma \vdash_{\mathsf{N}}^{+} \varphi$ or for $\Gamma \vdash_{\mathsf{N}}^{+} \psi$. □

Chapter 3

Labelled Sequents for InqB

In this chapter, we will provide a cut-free labelled sequent calculus for InqB. Labelled sequent calculi play an important role in the proof theory of non-classical logics and can be seen as an extension of the traditional sequent-style formalism developed by Gentzen (1935a; 1935b). The systematic study of these proof systems was primarily promoted by Negri, who developed labelled sequent calculi for a wide range of modal and intermediate logics (see Negri 2005; Dyckhoff and Negri 2012).[1] The basic idea is to enrich the language of ordinary sequent calculi in order to incorporate the semantics of a logic directly into the syntax of the proof system. In the extended setting, sequents are no longer built up from ordinary formulas, but from expressions of the form $\pi : \varphi$, where π is a label and φ is a formula. In our sequent calculus, labels will be used to represent information states and $\pi : \varphi$ will be interpreted as 'φ is supported by π'.

The chapter is structured as follows. In Section 3.1, we will provide a detailed description of our labelled sequent calculus for InqB. Our system will be denoted by GLinqB and can be regarded as a G3-style sequent calculus in the sense of Ketonen (1944) and Kleene (1952), so weakening and contraction are fully 'absorbed' into the axioms and the remaining rules of the system. In Section 3.2, we will investigate some important properties of our sequent calculus. Above all, we will prove that the structural rules of weakening, contraction and cut are *admissible* in GLinqB, i.e., whenever the premises of these rules are derivable in our system, then also the conclusion is derivable. In the case of weakening and contraction, admissibility also preserves the height of derivations. In addition, we will show that each rule of GLinqB is *height-preserving invertible*, i.e., whenever the conclusion of one of these rules is derivable, then also each premise of the rule is derivable, with at most the same derivation height. In Section 3.3, we will then prove the soundness and completeness of our sequent calculus. The completeness will be established proof-theoretically, i.e., instead of giving a semantic argument, we will exploit the admissibility of the cut rule in our system in order to show that GLinqB is complete with respect to the Hilbert-style system HinqB introduced in Chapter 1. In Section 3.4, finally, we will outline a possible proof search strategy for our sequent calculus and discuss some issues that have to be resolved in order to make sure that the procedure is terminating. We also establish a normal form result for the labels used in our system, which might play an important role in an effective proof search procedure for GLinqB. However, the full specification of the desired algorithm will be left for future work.

3.1 The Sequent Calculus GLinqB

Let us start by giving a formal exposition of our labelled sequent calculus for InqB. Our system, henceforth denoted by GLinqB, can be seen as a non-trivial extension of a labelled sequent cal-

[1] Important predecessors of modern-day labelled sequent calculi include the modal sequent calculus with 'spotted formulas' provided by Kanger (1957), the 'prefixed tableaux' of Fitting (1983), the 'indexed' sequent calculi described by Mints (1997) and the labelled natural deduction systems of Viganò (2000). See also Goré and Ramanayake (2012).

culus for intuitionistic logic previously described by Dyckhoff and Negri (2012). An alternative labelled sequent calculus for InqB has been provided by Chen and Ma (2017), but their system is not as well-behaved as the system GLinqB presented here. For one thing, Chen and Ma (2017) do not provide a modular construction, capable of being easily adapted to variants of InqB such as the system of intuitionistic inquisitive logic discussed in the next chapter. For another, our system is also much simpler and allows for a more elegant cut-admissibility proof.

From a technical point of view, GLinqB may be regarded as a G3-style sequent calculus in the sense of Ketonen (1944) and Kleene (1952). Sequent calculi of this type usually enjoy very strong structural properties, arising from the fact that the structural rules of weakening and contraction are fully 'absorbed' into the axioms and the remaining rules of these systems. For further details about the family of G3-style calculi, we refer to Troelstra and Schwichtenberg (1996, pp. 77–85).

Labelled sequent calculi were systematically developed by Negri (2005) and may be seen as an extension of the traditional sequent-style formalism originating with Gentzen (1935a; 1935b). The basic idea is to enrich the underlying language of a sequent calculus so as to incorporate some kind of semantic information directly into the syntax of the proof system. In the extended setting, sequents are no longer built up from ordinary formulas, but from expressions of the form $\pi : \varphi$, where π is some label and φ is a formula. The interpretation of the extended language now varies, depending on the semantics of the underlying logic. So, for example, in a labelled sequent calculus for *modal logic*, π might be a representation of a possible world in a Kripke model and an expression of the form $\pi : \varphi$ could stand for the statement 'φ is true at π' (cf. Negri 2005).

In our labelled sequent calculus for InqB, we have to incorporate the more complicated *support semantics* described in Chapter 1. For this reason, labels will not be used in order to represent *possible worlds*, but in order to represent *information states* (i.e., sets of worlds). An expression of the form $\pi : \varphi$ is then interpreted as 'φ is supported by π'. In order to make this precise, we henceforth assume two countably infinite sets of *state variables*, defined by $\mathfrak{S} := \{w_i \mid i \in \mathbb{N}\}$ and $\mathfrak{V} := \{x_i \mid i \in \mathbb{N}\}$, respectively. In what follows, the variables in $\mathfrak{S}$ are used for *singleton states* and the variables in $\mathfrak{V}$ are used for states of *arbitrary size*. In order to avoid confusion, we will always use the meta-variables u, v, w, etc., for elements of $\mathfrak{S}$ and the meta-variables x, y, z, etc., for elements of $\mathfrak{V}$. The set of labels is now built up from the variables in $\mathfrak{S}$ and $\mathfrak{V}$ by means of additional symbols representing the intersection and the union of two information states.

Definition 3.1.1 (Labels)**.** The set of *labels* is denoted by $\Lambda(\mathfrak{S}, \mathfrak{V})$ and consists of all expressions generated by the following grammar, where $w \in \mathfrak{S}$ and $x \in \mathfrak{V}$ are arbitrary variables:

$$\pi ::= w \mid x \mid \emptyset \mid \pi \cdot \pi \mid \pi + \pi.$$

Throughout this thesis, we will use the meta-variables π, σ, τ, etc., for arbitrarily complex labels. Intuitively, a label of the form $\pi \cdot \sigma$ represents the *intersection* of the states described by π and σ, while $\pi + \sigma$ stands for the *union* of π and σ. The constant $\emptyset$, on the other hand, represents the *inconsistent state*. For simplicity, we will also use $\pi\sigma$ as an abbreviation for $\pi \cdot \sigma$. We can now define two types of expressions, referred to as *labelled formulas* and *relational atoms*, respectively.

Definition 3.1.2 (Labelled Formula, Relational Atom)**.** By a *labelled formula*, we mean an expression of the form $\pi : \varphi$, where $\pi \in \Lambda(\mathfrak{S}, \mathfrak{V})$ is a label and $\varphi \in \mathcal{L}^{\mathsf{B}}$ is a formula. A *relational atom*, on the other hand, is an expression of the form $\pi \leqslant \sigma$, where $\pi, \sigma \in \Lambda(\mathfrak{S}, \mathfrak{V})$ are both labels.

As outlined above, labelled formulas will be used in order to incorporate the semantic support relation of InqB directly into the syntax of our proof system. Thus, intuitively, $\pi : \varphi$ should be read as 'φ is supported by π'. Relational atoms, on the other hand, are used in order to represent the *subset ordering* on information states, so $\pi \leqslant \sigma$ stands for the statement 'π is a subset of σ'.

The elements of our proof system will be referred to as *sequents*. Formally, a sequent is defined to be an expression of the form $\Gamma \Rightarrow \Delta$, where Γ is a finite multiset containing labelled formulas and relational atoms, and Δ is a finite multiset containing *only* labelled formulas (but no relational atoms). Hence, in particular, we allow relational atoms to occur only on the left-hand side of a sequent, but not on the right-hand side. This will simplify our treatment and is more in line with the intuitionistic system described by Dyckhoff and Negri (2012).[2] Given any sequent $\Gamma \Rightarrow \Delta$, we also call Γ the *antecedent* and Δ the *succedent* of the sequent. As in ordinary sequent calculi, sequents are assumed to have a 'conjunctive' reading on the left and a 'disjunctive' reading on the right. Moreover, the sequent arrow $\Rightarrow$ will be interpreted as an implication in the meta-language. Thus, intuitively, a sequent $\Gamma \Rightarrow \Delta$ is considered to be 'valid', if it satisfies the condition that, whenever all expressions in Γ are 'true', then at least one expression in Δ is 'true'.[3] A precise definition of this notion of 'validity' and 'truth' will be provided in Section 3.3. For the moment, we content ourselves with an intuitive interpretation of sequents.

We are now ready to define our labelled sequent calculus for InqB. Our system will be denoted by GLinqB and can be found in Figure 3.1. Throughout this chapter, any finite tree of sequents built up from the axioms and the rules of our system will be referred to as a *derivation* or a *proof tree* in GLinqB. Moreover, given any finite set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$, we will say that φ *is provable from* Γ, if there exists a derivation for the sequent $x : \Gamma \Rightarrow x : \varphi$ in our system, where x is an arbitrary state variable from $\mathfrak{V}$ and $x : \Gamma$ is the set of labelled formulas defined by $(x : \Gamma) := \{x : \psi \mid \psi \in \Gamma\}$. Intuitively, this accounts for the fact that, in InqB, entailment is defined as *preservation of support*: φ is entailed by Γ, if every state which supports all formulas in Γ also supports φ (see Definition 1.2.12). As we shall see below, the concrete choice of the variable x is actually irrelevant when it comes to provability in our system: if a sequent $x : \Gamma \Rightarrow x : \varphi$ is derivable in GLinqB, then also $y : \Gamma \Rightarrow y : \varphi$ will be derivable, for every state variable $y \in \mathfrak{V}$.

Definition 3.1.3 (The System GLinqB)**.** We define GLinqB to be the labelled sequent calculus depicted in Figure 3.1. A sequent $\Gamma \Rightarrow \Delta$ is called *derivable* in our system, if there exists a proof tree in GLinqB ending with this sequent. For any finite subset $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$, we also say that φ *is provable from* Γ in GLinqB, if there is some $x \in \mathfrak{V}$ such that $x : \Gamma \Rightarrow x : \varphi$ is derivable in GLinqB.[4]

A proof tree ending with a sequent $\Gamma \Rightarrow \Delta$ is also referred to as a *derivation for* $\Gamma \Rightarrow \Delta$. In what follows, derivations will be denoted by the meta-variables $\mathcal{D}$, $\mathcal{D}_1$, $\mathcal{D}_2$, etc. By a *branch* in a derivation $\mathcal{D}$, we mean any sequence β of consecutive sequents in $\mathcal{D}$ such that the first sequent in β is the conclusion (i.e., the root node) of $\mathcal{D}$ and the last sequent in β is one of the axioms (i.e., a leaf node) in $\mathcal{D}$. The *length of a branch* β is taken to be the number of sequents occurring in β. And the *height of a derivation* $\mathcal{D}$ is defined to be the length of a longest branch in $\mathcal{D}$.

As can be seen from Figure 3.1, the components of our system may roughly be divided into three groups. The first group consists of *axioms* or, as they are sometimes called, *initial sequents*. Each axiom reflects the idea that certain sequents are trivially 'valid' given the intuitive interpretation of labelled formulas and relational atoms outlined above. For example, $Ax^{\bot}$ accounts for the fact that $\bot$ is only supported by the inconsistent state $\emptyset$. Therefore, if w is a variable representing a *singleton*, then $w : \bot$ must always be 'false', so any sequent of the form $w : \bot, \Gamma \Rightarrow \Delta$ will be 'valid'. Observe that, in the first axiom Ax, we require p to be an *atomic formula*.

Let us now turn to the second group, consisting of the *logical rules* of our system. As in traditional sequent calculi, each logical connective is assigned a pair of rules, consisting of a *left rule* and a *right rule*. In addition to that, we now also have a left and a right rule for atomic formulas and for the falsum constant $\bot$. Intuitively, the logical rules are used in order to reflect the *support conditions* for the formulas of InqB presented in Definition 1.2.6. So, for example, the

[2] Nevertheless, it also *possible* to define an extension of our system in which relational atoms may occur on both sides of a sequent. For a large class of modal logics, systems of this kind have been described in detail by Negri (2005).

[3] Note that, in particular, every sequent in our system should be interpreted as a statement in the *meta-language*, rather than as a formula in the *object language*. This is an important difference to ordinary (i.e., label-free) sequent calculi, where the meaning of a sequent $\Gamma \Rightarrow \Delta$ can usually also be described by the formula $\bigwedge \Gamma \to \bigvee \Delta$.

[4] Note that, since every set is also a multiset, the notion of derivability is actually defined for $x : \Gamma \Rightarrow x : \varphi$.

Axioms:

$$\frac{}{w : p, \Gamma \Rightarrow \Delta, w : p}\,Ax \qquad \frac{}{w : \bot, \Gamma \Rightarrow \Delta}\,Ax^{\bot} \qquad \frac{}{w \leqslant \emptyset, \Gamma \Rightarrow \Delta}\,Ax^{\emptyset}$$

Logical Rules:

$$\frac{w : p, w \leqslant \pi, \pi : p, \Gamma \Rightarrow \Delta}{w \leqslant \pi, \pi : p, \Gamma \Rightarrow \Delta}\,Lp \qquad \frac{w \leqslant \pi, \Gamma \Rightarrow \Delta, w : p}{\Gamma \Rightarrow \Delta, \pi : p}\,Rp$$

$$\frac{w : \bot, w \leqslant \pi, \pi : \bot, \Gamma \Rightarrow \Delta}{w \leqslant \pi, \pi : \bot, \Gamma \Rightarrow \Delta}\,L\bot \qquad \frac{w \leqslant \pi, \Gamma \Rightarrow \Delta, w : \bot}{\Gamma \Rightarrow \Delta, \pi : \bot}\,R\bot$$

$$\frac{\pi : \varphi, \pi : \psi, \Gamma \Rightarrow \Delta}{\pi : \varphi \wedge \psi, \Gamma \Rightarrow \Delta}\,L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, \pi : \varphi \qquad \Gamma \Rightarrow \Delta, \pi : \psi}{\Gamma \Rightarrow \Delta, \pi : \varphi \wedge \psi}\,R\wedge$$

$$\frac{\pi : \varphi, \Gamma \Rightarrow \Delta \qquad \pi : \psi, \Gamma \Rightarrow \Delta}{\pi : \varphi \vee \psi, \Gamma \Rightarrow \Delta}\,L\vee \qquad \frac{\Gamma \Rightarrow \Delta, \pi : \varphi, \pi : \psi}{\Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi}\,R\vee$$

$$\frac{\pi \leqslant \sigma, \sigma : \varphi \to \psi, \Gamma \Rightarrow \Delta, \pi : \varphi \qquad \pi \leqslant \sigma, \sigma : \varphi \to \psi, \pi : \psi, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \sigma : \varphi \to \psi, \Gamma \Rightarrow \Delta}\,L\!\to$$

$$\frac{x \leqslant \pi, x : \varphi, \Gamma \Rightarrow \Delta, x : \psi}{\Gamma \Rightarrow \Delta, \pi : \varphi \to \psi}\,R\!\to$$

Order Rules:

$$\frac{\pi \leqslant \tau, \pi \leqslant \sigma, \sigma \leqslant \tau, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \sigma \leqslant \tau, \Gamma \Rightarrow \Delta}\,\textsf{(tr)} \qquad \frac{\pi \leqslant \pi\sigma + \pi\tau, \pi \leqslant \sigma + \tau, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma + \tau, \Gamma \Rightarrow \Delta}\,\textsf{(dis)}$$

$$\frac{\pi \leqslant \sigma\tau, \pi \leqslant \sigma, \pi \leqslant \tau, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \pi \leqslant \tau, \Gamma \Rightarrow \Delta}\,\textsf{(in)} \qquad \frac{\pi + \tau \leqslant \sigma, \pi \leqslant \sigma, \tau \leqslant \sigma, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \tau \leqslant \sigma, \Gamma \Rightarrow \Delta}\,\textsf{(un)}$$

$$\frac{\pi\sigma \leqslant \pi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textsf{(il)} \qquad \frac{\sigma\pi \leqslant \pi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textsf{(ir)} \qquad \frac{\pi \leqslant \pi + \sigma, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textsf{(ul)}$$

$$\frac{\pi \leqslant \sigma + \pi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textsf{(ur)} \qquad \frac{\pi \leqslant \emptyset, \pi \leqslant w, \Gamma \Rightarrow \Delta \qquad w \leqslant \pi, \pi \leqslant w, \Gamma \Rightarrow \Delta}{\pi \leqslant w, \Gamma \Rightarrow \Delta}\,\textsf{(sg)}$$

$$\frac{\pi \leqslant \pi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textsf{(rf)} \qquad \frac{w \leqslant \pi, w \leqslant \pi + \sigma, \Gamma \Rightarrow \Delta \qquad w \leqslant \sigma, w \leqslant \pi + \sigma, \Gamma \Rightarrow \Delta}{w \leqslant \pi + \sigma, \Gamma \Rightarrow \Delta}\,\textsf{(cd)}$$

Figure 3.1: The system GLinqB. In each case, w ranges over variables from $\mathfrak{S}$, x ranges over variables from $\mathfrak{V}$, and π, σ, τ, etc., stand for arbitrary labels. In applications of Rp and $R\bot$, w must be a fresh variable not occurring in the conclusion of the rule and π is required to be a non-singleton label, so we must have $\pi \notin \mathfrak{S}$. Similarly, x must be a fresh variable in $R\!\to$.

rules Lp and Rp account for the fact that an atom $p \in \mathsf{P}$ is supported by a state π if and only if this atom is true at every world in π, i.e., if p is supported by every singleton state $w \subseteq \pi$. The rules $L\!\to$ and $R\!\to$, on the other hand, reflect the support clause for implication: a formula $\varphi \to \psi$ is supported by a state π, if every enhancement of π that supports φ also supports ψ.

Importantly, each of the rules Rp, $R\bot$ and $R\!\to$ is subject to a *side condition*, imposing certain restrictions on the variables involved in applications of these rules. In particular, in an application of Rp or $R\bot$, we require $w \in \mathfrak{S}$ to be a *fresh variable*, i.e., a variable not occurring in the conclusion of the rule. And in an application of $R\!\to$, we require $x \in \mathfrak{V}$ to be fresh. Throughout this thesis, the fresh variables involved in applications of Rp, $R\bot$ and $R\!\to$ will also be referred to as the *eigenvariables* of these rules. Moreover, a rule with eigenvariables is said to be a *dynamic* *rule*, and all other rules are called *static rules*. Note that, for the rules Rp and $R\bot$, there is also another restriction, saying that π must be a non-singleton label satisfying $\pi \notin \mathfrak{S}$. As we shall see below, this restriction is necessary in order to make sure that the rules Rp and $R\bot$ are height-preserving invertible, in the sense that, whenever the conclusion of one of these rules is derivable, then also the corresponding premise is derivable, with at most the same derivation height.[5]

Finally, let us consider the third group, consisting of the so-called *order rules*. The purpose of these rules is to formalize the set-theoretic properties of information states. For example, the rules (rf) and (tr) account for the *reflexivity* and the *transitivity* of the subset ordering $\subseteq$. And the rule (in) says that, if a state π is a subset of two other states σ and τ, then π is also a subset of their intersection $\sigma \cap \tau$. A similar property of the union operator is formalized by the rule (un): if two states π and τ are both subsets of some other state σ, then also their union $\pi \cup \tau$ must be a subset of σ. The rules (sg) and (cd) are somewhat special, since they allow to make a *case distinction* in a derivation. Intuitively, (sg) says that, if a state π is a subset of a singleton w, then π must either be empty (in which case π is a subset of $\emptyset$) or π must be equal to w (in which case w is also a subset of π). The rule (cd), on the other hand, accounts for the fact that, if a singleton w is a subset of the union of two states π and σ, then w must also be a subset of at least one of π and σ.

Note that, by instantiating the meta-variables π, σ, τ, etc., with labels, one might sometimes also produce a *duplication* of relational atoms (i.e., two copies of the same relational atom) in the conclusion of an order rule. In order to avoid certain technicalities arising from such a duplication, we henceforth need to adopt the so-called *closure condition* going back to Negri (2003).[6]

Convention 3.1.4 (Closure Condition)**.** If an instance of an order rule produces a duplication of an atom $\pi \leqslant \sigma$ in the conclusion of the rule, then also the contracted instance of the rule (in which the two copies of $\pi \leqslant \sigma$ are replaced by a single $\pi \leqslant \sigma$) is assumed to be part of our system.

So, for example, the instance of the rule (un) depicted on the left-hand side below is also allowed to be replaced by the corresponding contracted instance shown on the right-hand side:

$$\dfrac{\pi + \pi \leqslant \sigma, \pi \leqslant \sigma, \pi \leqslant \sigma, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \pi \leqslant \sigma, \Gamma \Rightarrow \Delta}\,(\mathsf{un}) \quad \text{can be replaced by} \quad \dfrac{\pi + \pi \leqslant \sigma, \pi \leqslant \sigma, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \Gamma \Rightarrow \Delta}\,(\mathsf{un})$$

Clearly, for every order rule, there is only a finite number of contracted rules to be added to our system, so this convention is unproblematic (see also Negri and Von Plato 1998; Negri 2005).

As we have seen in Chapter 2, our natural deduction system for InqB only satisfies a weak form of the subformula property, so it is not an *analytic* proof system in a strict sense. For the system GLinqB, this problem no longer occurs. In fact, using induction on the structure of a derivation, one can easily prove that GLinqB satisfies an unrestricted version of the subformula property, provided that we ignore all labels (and therefore all relational atoms) occurring in a derivation.

Fact 3.1.5 (Subformula Property)**.** *If all labels are ignored, then every formula occurring in a derivation $\mathcal{D}$ for $\Gamma \Rightarrow \Delta$ is a subformula of some formula in Γ or a subformula of some formula in Δ.*

In this sense, our proof system is indeed fully 'analytic'. However, for labelled sequent calculi, it is also reasonable to consider another kind of analyticity, which can be expressed by the so-called *subterm property*. Roughly speaking, a labelled sequent calculus is said to satisfy the subterm property, if in this proof system, every label occurring in a derivation for $\Gamma \Rightarrow \Delta$ is either an eigenvariable or a label occurring in $\Gamma \Rightarrow \Delta$ (cf. Negri and Von Plato 2011; Dyckhoff and Negri 2012). Clearly, this constraint is *not* satisfied by the proof system GLinqB. For example, by

[5] Strictly speaking, the constraint $\pi \notin \mathfrak{S}$ is only needed for the rule Rp, but not for the rule $R\bot$. However, adopting the constraint for both Rp and $R\bot$ has some technical advantages that will become apparent later on.

[6] The importance of the closure condition will become clearer in Section 3.2.2. For the moment, it suffices to understand that, without the closure condition, our system would lose some important structural properties.

performing a root-first application of the order rule (un) in a derivation $\mathcal{D}$, one can create a new label of the form $\pi + \sigma$, which is not guaranteed to occur in the conclusion of $\mathcal{D}$.

We conclude this section by introducing some useful terminology. In each of the rules and axioms presented in Figure 3.1, the multiset Γ is called the *left context* and the multiset Δ is called the *right context*. In the conclusion of each rule, and also in the axioms, the labelled formulas and relational atoms not belonging to the context are said to be *principal*. The corresponding expressions occurring in the premises of a rule are called *active*. So, for example, in an application of $R\!\rightarrow$ with premise $x \leqslant \pi, x : \varphi, \Gamma \Rightarrow \Delta, x : \psi$ and conclusion $\Gamma \Rightarrow \Delta, \pi : \varphi \rightarrow \psi$, the labelled formula $\pi : \varphi \rightarrow \psi$ is principal, and each of the expressions $x \leqslant \pi$, $x : \varphi$ and $x : \psi$ is active. On the other hand, in an application of (ul) with premise $\pi \leqslant \pi + \sigma, \Gamma \Rightarrow \Delta$ and conclusion $\Gamma \Rightarrow \Delta$, there is no principal expression and the only active expression is $\pi \leqslant \pi + \sigma$. Observe that, in the order rules and in each of the logical rules Lp, $L\bot$ and $L\!\rightarrow$, the principal expressions occurring in the conclusion are always *repeated* in each of the premises of the rule (a rule of this type is also said to be 'cumulative'). This repetition of expressions is necessary in order to make sure that these rules are *invertible*, in the sense that, whenever their conclusion is derivable, then also each of their premises is derivable. We will return to the concept of invertibility in the next section.

3.2 Properties of GLinqB

In this section, we will investigate some important properties of our proof system. Most importantly, we will show that the cut rule and the rules of weakening and contraction are *admissible* in GLinqB. That is, whenever the premises of one of these rules are derivable in our system, then also the conclusion of the rule is derivable. In the case of weakening and contraction, admissibility also preserves the height of derivations. In addition to that, we will prove that all rules of GLinqB are *height-preserving invertible*, i.e., whenever the conclusion of one of these rules is derivable, then also each premise of the rule is derivable, with at most the same derivation height.

3.2.1 Generalized Initial Sequents

We first show that one can derive generalized versions of the initial sequents of our system. For the initial sequent Ax, this is accomplished by the following lemma. Note that, intuitively, the first sequent in the lemma also reflects the *persistency* of the support relation: if φ is supported by a state σ and if π is an enhancement of σ, then φ is also supported by π (see Proposition 1.2.11).

Lemma 3.2.1. *All sequents of the following form are derivable in* GLinqB*:*

(i) $\pi \leqslant \sigma, \sigma : \varphi, \Gamma \Rightarrow \Delta, \pi : \varphi$,

(ii) $\pi : \varphi, \Gamma \Rightarrow \Delta, \pi : \varphi$.

Proof. The derivability of (i) is established by induction on the structure of φ. For the base case, assume that $\varphi = p$ is atomic. If π satisfies $\pi \notin \mathfrak{S}$, then we construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{w : p, w \leqslant \sigma, w \leqslant \pi, \pi \leqslant \sigma, \sigma : p, \Gamma \Rightarrow \Delta, w : p}\,Ax}{w \leqslant \sigma, w \leqslant \pi, \pi \leqslant \sigma, \sigma : p, \Gamma \Rightarrow \Delta, w : p}\,Lp}{w \leqslant \pi, \pi \leqslant \sigma, \sigma : p, \Gamma \Rightarrow \Delta, w : p}\,(\mathsf{tr})}{\pi \leqslant \sigma, \sigma : p, \Gamma \Rightarrow \Delta, \pi : p}\,Rp$$

If we have $\pi \in \mathfrak{S}$, then the derivation is the same, except that we omit the applications of (tr) and Rp at the bottom of the derivation. The case $\varphi = \bot$ is treated in essentially the same way. In the inductive step for conjunction and implication, we construct the following two derivations:[7]

[7] In the following, we will use dashed lines in derivations in order to indicate the use of an assumption or the application of a previously established result (such as, e.g., a lemma saying that a certain rule is admissible). If necessary, we will also write '. . .' to indicate that some expressions in a sequent are not shown explicitly due to a lack of space.

$$\dfrac{\dfrac{\dfrac{\text{By ind. hyp.}}{\pi \leqslant \sigma, \sigma : \psi, \sigma : \chi, \Gamma \Rightarrow \Delta, \pi : \psi}}{\pi \leqslant \sigma, \sigma : \psi \wedge \chi, \Gamma \Rightarrow \Delta, \pi : \psi}\, L\wedge \qquad \dfrac{\dfrac{\text{By ind. hyp.}}{\pi \leqslant \sigma, \sigma : \psi, \sigma : \chi, \Gamma \Rightarrow \Delta, \pi : \chi}}{\pi \leqslant \sigma, \sigma : \psi \wedge \chi, \Gamma \Rightarrow \Delta, \pi : \chi}\, L\wedge}{\pi \leqslant \sigma, \sigma : \psi \wedge \chi, \Gamma \Rightarrow \Delta, \pi : \psi \wedge \chi}\, R\wedge$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{By ind. hyp.}}{x \leqslant x, \dots, x : \psi, \Gamma \Rightarrow \Delta, x : \chi, x : \psi}}{\dots, x : \psi, \Gamma \Rightarrow \Delta, x : \chi, x : \psi}\,(\mathsf{rf}) \qquad \dfrac{\dfrac{\text{By ind. hyp.}}{x \leqslant x, \dots, x : \chi, x : \psi, \Gamma \Rightarrow \Delta, x : \chi}}{\dots, x : \chi, x : \psi, \Gamma \Rightarrow \Delta, x : \chi}\,(\mathsf{rf})}{x \leqslant \sigma, x \leqslant \pi, \pi \leqslant \sigma, \sigma : \psi \to \chi, x : \psi, \Gamma \Rightarrow \Delta, x : \chi}\, L{\to}}{x \leqslant \pi, \pi \leqslant \sigma, \sigma : \psi \to \chi, x : \psi, \Gamma \Rightarrow \Delta, x : \chi}\,(\mathsf{tr})}{\pi \leqslant \sigma, \sigma : \psi \to \chi, \Gamma \Rightarrow \Delta, \pi : \psi \to \chi}\, R{\to}$$

The inductive step for inquisitive disjunction is similar to the inductive step for conjunction. The sequent in (ii) can now be obtained from the sequent in (i) by an application of the rule (rf). □

The following lemma shows that there are also derivable generalizations of $Ax^{\emptyset}$ and $Ax^{\perp}$. Intuitively, the first two sequents account for the fact that, in InqB, every formula is supported by the inconsistent state $\emptyset$. This corresponds to the *empty state property* expressed by Proposition 1.2.11. The last two sequents reflect the support conditions for the falsum constant: since $\perp$ is supported only by $\emptyset$, any enhancement of a state supporting $\perp$ also supports every other formula.

Lemma 3.2.2. *All sequents of the following form are derivable in* GLinqB*:*

(i) $\pi \leqslant \emptyset, \Gamma \Rightarrow \Delta, \pi : \varphi$*,*
(ii) $\Gamma \Rightarrow \Delta, \emptyset : \varphi$*,*
(iii) $\pi \leqslant \sigma, \sigma : \perp, \Gamma \Rightarrow \Delta, \pi : \varphi$*,*
(iv) $\pi : \perp, \Gamma \Rightarrow \Delta, \pi : \varphi$*.*

Proof. The derivability of (i) and (iii) is established by induction on φ. We only prove the derivability of (iii). For the base case, let $\varphi = p$ be atomic. If $\pi \notin \mathfrak{S}$, then we construct the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{w : \perp, w \leqslant \sigma, w \leqslant \pi, \pi \leqslant \sigma, \sigma : \perp, \Gamma \Rightarrow \Delta, w : p}\, Ax^{\perp}}{w \leqslant \sigma, w \leqslant \pi, \pi \leqslant \sigma, \sigma : \perp, \Gamma \Rightarrow \Delta, w : p}\, L\perp}{w \leqslant \pi, \pi \leqslant \sigma, \sigma : \perp, \Gamma \Rightarrow \Delta, w : p}\,(\mathsf{tr})}{\pi \leqslant \sigma, \sigma : \perp, \Gamma \Rightarrow \Delta, \pi : p}\, Rp$$

If $\pi \in \mathfrak{S}$, then the derivation is the same, except that we leave out the applications of (tr) and Rp. The case $\varphi = \perp$ is similar. In the inductive step for $\wedge$ and $\vee\!\!\!\vee$, the statement follows directly from the induction hypothesis. And in the inductive step for $\to$, we construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\text{By ind. hyp.}}{x \leqslant \sigma, x \leqslant \pi, \pi \leqslant \sigma, \sigma : \perp, x : \psi, \Gamma \Rightarrow \Delta, x : \chi}}{x \leqslant \pi, \pi \leqslant \sigma, \sigma : \perp, x : \psi, \Gamma \Rightarrow \Delta, x : \chi}\,(\mathsf{tr})}{\pi \leqslant \sigma, \sigma : \perp, \Gamma \Rightarrow \Delta, \pi : \psi \to \chi}\, R{\to}$$

This concludes the induction. For the sequent in (i), the induction works in essentially the same way. The sequents in (ii) and (iv) can be derived from (i) and (iii) by an application of (rf). □

3.2.2 Basic Admissibility and Invertibility Results

Next, we want to establish some important structural properties of our proof system. In particular, we will see that the structural rules of weakening and contraction are height-preserving admissible in our system and that each rule of GLinqB is height-preserving invertible. For the sake of clarity, let us first define the relevant notions. By the rules of *weakening* and *contraction*, we will henceforth mean the six rules depicted in Figure 3.2. As can be seen, the weakening rules

$$\dfrac{\Gamma \Rightarrow \Delta}{\pi : \varphi, \Gamma \Rightarrow \Delta}\ LW \qquad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \pi : \varphi}\ RW \qquad \dfrac{\Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \Gamma \Rightarrow \Delta}\ W^{\leqslant}$$

$$\dfrac{\pi : \varphi, \pi : \varphi, \Gamma \Rightarrow \Delta}{\pi : \varphi, \Gamma \Rightarrow \Delta}\ LC \qquad \dfrac{\Gamma \Rightarrow \Delta, \pi : \varphi, \pi : \varphi}{\Gamma \Rightarrow \Delta, \pi : \varphi}\ RC \qquad \dfrac{\pi \leqslant \sigma, \pi \leqslant \sigma, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \Gamma \Rightarrow \Delta}\ C^{\leqslant}$$

Figure 3.2: The structural rules of weakening and contraction.

(displayed in the first row) allow us to introduce a new expression in the antecedent or the succedent of a sequent. And the contraction rules (displayed in the second row) are used in order to remove multiple occurrences of the same expression in a sequent. Note that, in particular, weakening and contraction are applicable not only to *labelled formulas*, but also to *relational atoms*.

A rule of inference is said to be *admissible* in GLinqB, if it satisfies the condition that, whenever each premise of the rule is derivable, then also the conclusion of the rule is derivable. Thus, intuitively, an admissible rule may be regarded as a rule that is *redundant*: if something is derivable by using an admissible rule, then it is also derivable without using the admissible rule. If the admissibility of a rule preserves the height of derivations, then the rule is also said to be *height-preserving admissible* (cf. Troelstra and Schwichtenberg 1996; Negri and Von Plato 2001).[8]

Definition 3.2.3 (Admissibility)**.** Let $\mathcal{R}$ be a rule with premises $P_1, \ldots, P_m$ and conclusion C.

(i) We say that $\mathcal{R}$ is *admissible* in our system if, whenever an instance of $P_1, \ldots, P_m$ is derivable in GLinqB, then also the corresponding instance of C is derivable in GLinqB.

(ii) We say that $\mathcal{R}$ is *height-preserving admissible* (or *hp-admissible*) in GLinqB if, whenever an instance of $P_1, \ldots, P_m$ is derivable by a proof tree of height at most n in GLinqB, then also the corresponding instance of C is derivable by a proof tree of height at most n in GLinqB.

On the other hand, we will say that a rule is *invertible* if, whenever the conclusion of the rule is derivable, then also each of the premises of the rule is derivable. And a rule is said to be *height-preserving invertible*, if the invertibility of the rule preserves the height of derivations.

Definition 3.2.4 (Invertibility)**.** Let $\mathcal{R}$ be a rule with premises $P_1, \ldots, P_m$ and conclusion C.

(i) We say that $\mathcal{R}$ is *invertible* in GLinqB if, whenever an instance of C is derivable in GLinqB, then also the corresponding instance of P_i is derivable in GLinqB, for every $1 \leq i \leq m$.

(ii) We say that $\mathcal{R}$ is *height-preserving invertible* (or *hp-invertible*) in GLinqB if, whenever an instance of C is derivable by a proof tree of height at most n in our system, then also the corresponding instance of P_i is derivable by a tree of height at most n, for every $1 \leq i \leq m$.

For technical reasons, we also need to show that our system allows to perform *height-preserving substitutions* on labels. To this end, we first define a substitution operator in the following way.

Definition 3.2.5 (Substitution)**.** Let $s \in \mathfrak{S} \cup \mathfrak{V}$ be a variable and let $\pi \in \Lambda(\mathfrak{S}, \mathfrak{V})$ be an arbitrary label. The result of substituting π for s in a label is inductively defined by the following clauses:

(i) for any variable $t \in \mathfrak{S} \cup \mathfrak{V}$, we put $t(\pi/s) := \pi$, if $s = t$, and we put $t(\pi/s) := t$, if $s \neq t$,

(ii) $\emptyset(\pi/s) := \emptyset$,

(iii) $(\sigma \circledast \tau)(\pi/s) := \sigma(\pi/s) \circledast \tau(\pi/s)$ for $\circledast \in \{\cdot, +\}$.

If Γ is a multiset of labelled formulas and relational atoms, we also write $\Gamma(\pi/s)$ for the result of substituting π for s in every label occurring in Γ. The *substitution rules* are defined to be the rules

$$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma(u/w) \Rightarrow \Delta(u/w)}\ (u/w) \qquad \text{and} \qquad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma(\pi/x) \Rightarrow \Delta(\pi/x)}\ (\pi/x)$$

[8] Recall that, by the *height* of a derivation $\mathcal{D}$, we mean the length of a longest branch in $\mathcal{D}$.

where u and w are variables from $\mathfrak{S}$, x is a variable from $\mathfrak{V}$, and π is an arbitrary label. Thus, in particular, we allow singleton variables to be replaced only by other singleton variables, but not by any other label. Variables from $\mathfrak{V}$, on the other hand, can be replaced by arbitrary labels.

Proposition 3.2.6. *The substitution rules are hp-admissible in* GLinqB*.*

Proof. For the sake of brevity, we only prove the hp-admissibility of the first substitution rule (for the second rule, the proof is similar). Let $\mathcal{D}$ be an arbitrary derivation for $\Gamma \Rightarrow \Delta$ and let $u, w \in \mathfrak{S}$ be arbitrary singleton variables. Using induction on the height of $\mathcal{D}$, we show that $\Gamma(u/w) \Rightarrow \Delta(u/w)$ is also derivable, with at most the same derivation height. For the base case, assume that $\mathcal{D}$ has height $n = 1$. In this case, $\Gamma \Rightarrow \Delta$ must be an instance of an axiom. But then, clearly, the sequent $\Gamma(u/w) \Rightarrow \Delta(u/w)$ is also an instance of this axiom, so the claim follows.[9]

For the inductive step, suppose that $\mathcal{D}$ has height $n > 1$. We consider the last rule applied in $\mathcal{D}$. If this rule does not have an eigenvariable from the set $\mathfrak{S}$, then we simply apply the induction hypothesis to the premises of the rule, and then the rule itself. On the other hand, if the last rule in $\mathcal{D}$ has an eigenvariable from $\mathfrak{S}$, then we first use the induction hypothesis in order to rename the eigenvariable, before performing the desired substitution (u/w). To see an example, let us suppose that the last step in $\mathcal{D}$ is an application of Rp. In this case, $\mathcal{D}$ must be of the form

$$\dfrac{\begin{matrix}\mathcal{D}' \\ v \leqslant \pi, \Gamma \Rightarrow \Theta, v : p\end{matrix}}{\Gamma \Rightarrow \Theta, \pi : p}\; Rp$$

where $v \in \mathfrak{S}$ is a fresh variable, π is a non-singleton label and $\mathcal{D}'$ is a derivation of height $n-1$. By applying the induction hypothesis to the subderivation $\mathcal{D}'$, we first replace the eigenvariable v by a fresh variable $v' \in \mathfrak{S}$ satisfying the condition $v' \neq w$. This yields a derivation $\mathcal{D}''$ of height at most $n - 1$ for $v' \leqslant \pi, \Gamma \Rightarrow \Theta, v' : p$. We can now apply the induction hypothesis again in order to perform the substitution (u/w) in the derivation $\mathcal{D}''$. Using a subsequent application of Rp, we thus obtain the desired derivation of height at most n for $\Gamma(u/w) \Rightarrow \Delta(u/w)$. $\square$

Recall that, according to Definition 3.1.3, a formula φ is said to be *provable* from Γ in GLinqB, if for some variable $x \in \mathfrak{V}$, the sequent $x : \Gamma \Rightarrow x : \varphi$ is derivable. The previous proposition tells us that, in fact, the concrete choice of this variable does not matter: if the sequent $x : \Gamma \Rightarrow x : \varphi$ is derivable in GLinqB, then also $y : \Gamma \Rightarrow y : \varphi$ is derivable, for any $y \in \mathfrak{V}$. We are now ready to prove the desired admissibility and invertibility results: weakening and contraction are height-preserving admissible in our system and each rule of GLinqB is height-preserving invertible.

Proposition 3.2.7. *The weakening rules are hp-admissible in* GLinqB*.*

Proof. We only prove the hp-admissibility of LW. For the other two weakening rules, the proof is similar. Let $\mathcal{D}$ be an arbitrary derivation for $\Gamma \Rightarrow \Delta$ and let $\pi : \varphi$ be an arbitrary labelled formula. Using induction on the height of $\mathcal{D}$, we show that $\pi : \varphi, \Gamma \Rightarrow \Delta$ is also derivable, with at most the height of $\mathcal{D}$. For the base case, suppose that $\mathcal{D}$ has height $n = 1$. This means that $\Gamma \Rightarrow \Delta$ is an instance of an axiom. But then, clearly, $\pi : \varphi, \Gamma \Rightarrow \Delta$ is also an axiom.

For the inductive step, assume that $\mathcal{D}$ has height $n > 1$. We consider the last rule applied in $\mathcal{D}$. If this rule does not have an eigenvariable, then we apply the induction hypothesis to the premises of the rule, and the the rule itself. Otherwise, we first use Proposition 3.2.6 in order to introduce a fresh eigenvariable not occurring in the weakening formula $\pi : \varphi$. For example, suppose that the last step in $\mathcal{D}$ is an application of $R{\to}$ with eigenvariable x, so $\mathcal{D}$ is of the form

$$\dfrac{\begin{matrix}\mathcal{D}' \\ x \leqslant \sigma, x : \psi, \Gamma \Rightarrow \Theta, x : \chi\end{matrix}}{\Gamma \Rightarrow \Theta, \sigma : \psi \to \chi}\; R{\to}$$

[9] Note that, for this step, it is in fact essential that u is also a variable from $\mathfrak{S}$. Otherwise, it might be the case that $\Gamma \Rightarrow \Delta$ is an instance of an axiom, but the sequent resulting from the substitution is not an axiom anymore.

where $\mathcal{D}'$ is a derivation of height $n-1$. By performing a height-preserving substitution in the subderivation $\mathcal{D}'$, we obtain a derivation of height at most $n-1$ for $y \leqslant \sigma, y : \psi, \Gamma \Rightarrow \Theta, y : \chi$, where $y \in \mathfrak{V}$ is a fresh variable not occurring in π. By induction hypothesis, this yields a derivation of height at most $n-1$ for $\pi : \varphi, y \leqslant \sigma, y : \psi, \Gamma \Rightarrow \Theta, y : \chi$. Hence, by an application of $R\rightarrow$, we obtain the desired derivation of height at most n for $\pi : \varphi, \Gamma \Rightarrow \Theta, \sigma : \psi \rightarrow \chi$. □

Proposition 3.2.8. *All rules of* GLinqB *are hp-invertible.*

Proof. For the 'cumulative' rules of our system (that is, for the rules Lp, $L\bot$, $L\rightarrow$ and for the order rules), hp-invertibility follows immediately from the hp-admissibility of weakening. For all other rules of GLinqB, the proof proceeds by induction on the height of a derivation for the conclusion of the rule. We illustrate the basic idea for the rule Rp. Let $\pi \notin \mathfrak{S}$ be a non-singleton label, let $\mathcal{D}$ be an arbitrary derivation for the sequent $\Gamma \Rightarrow \Delta, \pi : p$ and let n be the height of $\mathcal{D}$. Moreover, let $w \in \mathfrak{S}$ be an arbitrary variable not occurring in $\Gamma \Rightarrow \Delta, \pi : p$. Using induction on n, we show that $w \leqslant \pi, \Gamma \Rightarrow \Delta, w : p$ is also derivable by a proof tree of height at most n.

For the base case, assume that $\mathcal{D}$ has height $n = 1$. Then, clearly, $\Gamma \Rightarrow \Delta, \pi : p$ must be an instance of an axiom. Since we have $\pi \notin \mathfrak{S}$ by assumption, one readily sees that the labelled formula $\pi : p$ cannot be principal in this axiom (recall that, in an instance of Ax, the principal formula must always be labelled with a variable from $\mathfrak{S}$). Hence, the sequent $w \leqslant \pi, \Gamma \Rightarrow \Delta, w : p$ is also an instance of an axiom and therefore derivable by a proof tree of height $n = 1$.[10]

For the inductive step, suppose that $\mathcal{D}$ has height $n > 1$. If the last step in $\mathcal{D}$ is a rule for which $\pi : p$ is not principal, then we apply the induction hypothesis to the premises of the rule (possibly together with a height-preserving substitution), and we then use the same rule again. On the other hand, if $\mathcal{D}$ ends with an application of Rp for which $\pi : p$ is principal, then $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\mathcal{D}'\\ u \leqslant \pi, \Gamma \Rightarrow \Delta, u : p\end{array}}{\Gamma \Rightarrow \Delta, \pi : p}\ Rp$$

where $u \in \mathfrak{S}$ is a fresh variable and $\mathcal{D}'$ is of height $n-1$. By substituting w for u in the subderivation $\mathcal{D}'$, we now obtain the desired derivation of height at most n for $w \leqslant \pi, \Gamma \Rightarrow \Delta, w : p$. □

Proposition 3.2.9. *The contraction rules are hp-admissible in* GLinqB.

Proof. The hp-admissibility of the three contraction rules is established *simultaneously*, by induction on the height of a derivation for the premise of the respective rule. More generally, let $\mathcal{D}$ be an arbitrary derivation for some sequent $\Gamma \Rightarrow \Delta$, let n be the height of $\mathcal{D}$ and suppose that either the antecedent or the succedent of $\Gamma \Rightarrow \Delta$ contains a duplication of some relational atom or a duplication of some labelled formula. Using induction on n, we show that also the contracted version of the sequent $\Gamma \Rightarrow \Delta$ is derivable by a proof tree of height at most n.

For the base case, assume that $\mathcal{D}$ has height $n = 1$, so $\Gamma \Rightarrow \Delta$ is an instance of an axiom. But then, clearly, also the contracted version of $\Gamma \Rightarrow \Delta$ must be an instance of this axiom.

For the inductive step, suppose that $\mathcal{D}$ has height $n > 1$. There are the following possibilities.

Case 1: Suppose that the duplicated expression in $\Gamma \Rightarrow \Delta$ is a relational atom $\pi \leqslant \sigma$. In this case, $\Gamma \Rightarrow \Delta$ must be of the form $\pi \leqslant \sigma, \pi \leqslant \sigma, \Theta \Rightarrow \Delta$. We now consider the last rule applied in $\mathcal{D}$. If at most one of the two occurrences of $\pi \leqslant \sigma$ is principal in this rule, then we simply apply the induction hypothesis to the premises of the rule, and then the same rule again. And if both occurrences of $\pi \leqslant \sigma$ are principal in the last rule applied in $\mathcal{D}$, then we use the *closure*

[10] Note that, without the restriction $\pi \notin \mathfrak{S}$, this argument would no longer work. In fact, in this case, it might be the case that $\Gamma \Rightarrow \Delta, \pi : p$ is an axiom, but $w \leqslant \pi, \Gamma \Rightarrow \Delta, w : p$ is not (namely, if $\pi : p$ is the principal formula of the axiom). Thus, the restriction $\pi \notin \mathfrak{S}$ makes sure that the invertibility of Rp is in fact *height-preserving*.

condition (Convention 3.1.4). For example, suppose that the last step in $\mathcal{D}$ is an application of (un) for which both occurrences of $\pi \leqslant \sigma$ are principal. In this case, $\mathcal{D}$ must be of the form

$$\dfrac{\begin{array}{c}\mathcal{D}' \\ \pi + \pi \leqslant \sigma, \pi \leqslant \sigma, \pi \leqslant \sigma, \Theta \Rightarrow \Delta\end{array}}{\pi \leqslant \sigma, \pi \leqslant \sigma, \Theta \Rightarrow \Delta}\ (\mathsf{un})$$

where $\mathcal{D}'$ is of height $n-1$. By applying the induction hypothesis to $\mathcal{D}'$, we get a derivation of height at most $n-1$ for $\pi+\pi \leqslant \sigma, \pi \leqslant \sigma, \Theta \Rightarrow \Delta$. Now, by the closure condition, we can use the *contracted version* of (un) in order to obtain a derivation of height at most n for $\pi \leqslant \sigma, \Theta \Rightarrow \Delta$.

Case 2: Suppose that the duplicated expression in $\Gamma \Rightarrow \Delta$ is a labelled formula $\pi : \varphi$. Without loss of generality, assume that the duplication occurs in the *succedent*, so $\Gamma \Rightarrow \Delta$ is of the form $\Gamma \Rightarrow \Theta, \pi : \varphi, \pi : \varphi$ (if the duplication is in the antecedent, the proof is similar). We now consider the last rule applied in $\mathcal{D}$. If $\pi : \varphi$ is not principal in this rule, then both occurrences of $\pi : \varphi$ also appear in each of the premises of the rule. Thus, by applying the induction hypothesis to the premises and then the same rule again, we obtain the desired derivation for $\Gamma \Rightarrow \Theta, \pi : \varphi$. On the other hand, if $\pi : \varphi$ is principal in the last rule applied in $\mathcal{D}$, we consider the following cases.

Case 2.1: Let $\varphi = p$ be atomic. In this case, $\mathcal{D}$ ends with an application of Rp, so it is of the form

$$\dfrac{\begin{array}{c}\mathcal{D}' \\ w \leqslant \pi, \Gamma \Rightarrow \Theta, w : p, \pi : p\end{array}}{\Gamma \Rightarrow \Theta, \pi : p, \pi : p}\ Rp$$

where $w \in \mathfrak{S}$ is a fresh variable, π satisfies $\pi \notin \mathfrak{S}$ and $\mathcal{D}'$ is of height $n-1$. By applying the hp-invertibility of Rp and a subsequent height-preserving substitution to the subderivation $\mathcal{D}'$, we get a derivation of height at most $n-1$ for the sequent $w \leqslant \pi, w \leqslant \pi, \Gamma \Rightarrow \Theta, w : p, w : p$. Now, using the induction hypothesis and a subsequent application of Rp, this yields the desired derivation of height at most n for $\Gamma \Rightarrow \Theta, \pi : p$. The case $\varphi = \bot$ is treated similarly.

Case 2.2: Let $\varphi = \psi \wedge \chi$. In this case, the last step is an application of $R\wedge$, so $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma \Rightarrow \Theta, \pi : \psi, \pi : \psi \wedge \chi\end{array} \qquad \begin{array}{c}\mathcal{D}_2 \\ \Gamma \Rightarrow \Theta, \pi : \chi, \pi : \psi \wedge \chi\end{array}}{\Gamma \Rightarrow \Theta, \pi : \psi \wedge \chi, \pi : \psi \wedge \chi}\ R\wedge$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are of height at most $n-1$. By applying the hp-invertibility of $R\wedge$ to each of the two derivations $\mathcal{D}_1$ and $\mathcal{D}_2$, we obtain derivations of height at most $n-1$ for $\Gamma \Rightarrow \Theta, \pi : \psi, \pi : \psi$ and $\Gamma \Rightarrow \Theta, \pi : \chi, \pi : \chi$. By induction hypothesis and a subsequent application of $R\wedge$, this yields the desired derivation of height at most n for $\Gamma \Rightarrow \Theta, \pi : \psi \wedge \chi$. The case $\varphi = \psi \mathbin{\mathbb{V}} \chi$ is similar.

Case 2.3: Let $\varphi = \psi \to \chi$. In this case, $\mathcal{D}$ ends with an application of $R{\to}$, so it is of the form

$$\dfrac{\begin{array}{c}\mathcal{D}' \\ x \leqslant \pi, x : \psi, \Gamma \Rightarrow \Theta, x : \chi, \pi : \psi \to \chi\end{array}}{\Gamma \Rightarrow \Theta, \pi : \psi \to \chi, \pi : \psi \to \chi}\ R{\to}$$

where $x \in \mathfrak{V}$ is fresh and $\mathcal{D}'$ is of height $n-1$. We now apply the hp-invertibility of $R{\to}$ and a subsequent height-preserving substitution to $\mathcal{D}'$ in order to obtain a derivation of height at most $n-1$ for $x \leqslant \pi, x \leqslant \pi, x : \psi, x : \psi, \Gamma \Rightarrow \Theta, x : \chi, x : \chi$. By induction hypothesis and an application of $R{\to}$, this yields the desired derivation of height at most n for $\Gamma \Rightarrow \Theta, \pi : \psi \to \chi$. $\square$

3.2.3 Admissibility of the Cut Rule

In this section, we will show that the structural rule of cut is admissible in GLinqB. This can be seen as the most important structural property of our proof system and will help us to establish the completeness of GLinqB in a purely proof-theoretical manner (see Section 3.3). First, let us introduce some terminology. By the *cut rule*, we will henceforth mean the sequent rule given by

$$\frac{\Gamma \Rightarrow \Delta, \pi : \varphi \qquad \pi : \varphi, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\ (\text{cut})$$

where $\pi : \varphi$ is an arbitrary labelled formula, referred to as the *cut formula* of the corresponding application of the cut rule. Note that, if we take $\Delta = \emptyset$ and read the sequent arrow $\Rightarrow$ as an implication in the meta-language, then the cut rule has a very natural interpretation: if a statement $\pi : \varphi$ follows from some set of assumptions Γ, and if Θ follows from $\pi : \varphi$ and Σ, then we can always infer Θ directly from Γ and Σ, without using the intermediate step represented by $\pi : \varphi$ (so, in a sense, the occurrence of $\pi : \varphi$ is 'cut out' by the rule). The formulation of the cut rule given above is also said to be *context-mixing* (or *multiplicative*), since we allow the contexts of the left and the right premise to be distinct. There is also a *context-sharing* (or *additive*) version of the cut rule, in which these contexts are required to be the same. However, due to the admissibility of weakening and contraction, the two formulations are easily seen to be equivalent: adding one version of the cut rule to GLinqB has exactly the same effect as adding the other version to GLinqB.[11]

Importantly, our admissibility proof for the cut rule will be fully *constructive*, in the sense that it can easily be turned into an effective procedure that allows to transform any given derivation containing applications of the cut rule into a corresponding cut-free derivation. The general idea goes back to Gentzen (1935a; 1935b), who first proved a cut-elimination theorem for classical and intuitionistic first-order logic (this is known today as *Gentzen's Hauptsatz*). Since then, many extensions and variations of Gentzen's method have been developed. Our argument will be somewhat similar to the standard cut-admissibility proof for G3-style systems outlined by Negri and Von Plato (2001, pp. 35–40).[12] The basic idea is to consider an arbitrary application of the cut rule and to prove the admissibility of the rule by induction on the complexity of this application.

However, when measuring the complexity of a cut formula $\pi : \varphi$, we must now also take into account the complexity of the label π, rather than considering only the complexity of the formula φ, as it is done in standard cut-admissibility proofs for labelled and unlabelled sequent calculi (see, e.g., Negri 2005). To this end, we first define the degree of a label in the following way.

Definition 3.2.10. The *degree of a label* π is denoted by $deg(\pi)$ and defined as follows: if $\pi \in \mathfrak{S}$ is a singleton variable, then we put $deg(\pi) := 0$, and if $\pi \notin \mathfrak{S}$, then we put $deg(\pi) := 1$.

In other words, we simply assign the degree 0 to every singleton variable $w \in \mathfrak{S}$ and the degree 1 to every non-singleton label $\pi \notin \mathfrak{S}$. The degree of a formula φ is now defined to be the number of occurrences of the logical symbols in φ, so we adopt the following definition.

Definition 3.2.11. The *degree of a formula* φ, notation $deg(\varphi)$, is inductively defined as follows:
- (i) $deg(p) := 0$ for all atoms $p \in \mathsf{P}$, and $deg(\bot) := 1$,
- (ii) $deg(\psi \otimes \chi) := deg(\psi) + deg(\chi) + 1$ for $\otimes \in \{\wedge, \rightarrow, \vee\!\!\!\!\vee\}$.

Finally, using the degree of a label and the degree of a formula, we define the *rank of a labelled formula* $\pi : \varphi$ to be pair of natural numbers $rank(\pi : \varphi) := (deg(\varphi), deg(\pi))$, where $deg(\varphi)$ is the degree of the formula φ and $deg(\pi)$ is the degree of the label π. In order to compare the ranks of labelled formulas, we will employ a *lexicographic ordering*. That is, we will write $rank(\pi : \varphi) < rank(\sigma : \psi)$ and say that the rank of $\pi : \varphi$ is *smaller* than the rank of $\sigma : \psi$, if we either have $deg(\varphi) < deg(\psi)$, or we have both $deg(\varphi) = deg(\psi)$ and $deg(\pi) < deg(\sigma)$. The following lemma summarizes some immediate consequences of the preceding definitions.

[11] Moreover, the cut-admissibility proof given below can easily be adapted to the context-sharing version of the rule.
[12] A corresponding cut-elimination strategy is described by Troelstra and Schwichtenberg (1996, pp. 94–101).

Lemma 3.2.12. *Let π and σ be arbitrary labels and let $w \in \mathfrak{S}$ be a singleton variable. It holds:*

(i) If $\pi \notin \mathfrak{S}$, then $rank(w : \varphi) < rank(\pi : \varphi)$,

(ii) $rank(\pi : \varphi_i) < rank(\sigma : \varphi_1 \otimes \varphi_2)$ for $i = 1, 2$ and $\otimes \in \{\wedge, \rightarrow, \mathbb{V}\}$.

Proof. For the first part, suppose that $w \in \mathfrak{S}$ and $\pi \notin \mathfrak{S}$. Then, by Definition 3.2.10, we have $deg(w) = 0$ and $deg(\pi) = 1$, so it follows $deg(w) < deg(\pi)$. Using the definition of the lexicographic ordering, this yields $rank(w : \varphi) < rank(\pi : \varphi)$, as desired. For the second part, it suffices to observe that we have $deg(\varphi_i) < deg(\varphi_1 \otimes \varphi_2)$, so the rank of $\pi : \varphi_i$ must be lexicographically smaller than the rank of $\sigma : \varphi_1 \otimes \varphi_2$, regardless of the degrees of π and σ. □

We are now ready to define a suitable measure of the complexity of a cut rule application. To this end, let us consider the following cut, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are two derivations in our system:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \pi : \varphi\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \pi : \varphi, \Sigma \Rightarrow \Theta\end{array}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\ (\mathsf{cut})$$

The *rank* of the indicated cut rule application is defined to be the rank of the associated cut formula $\pi : \varphi$. Thus, in particular, we assume that ranks of cut rule applications are ordered lexicographically. The *height* of the cut rule application is defined to be the sum $h(\mathcal{D}_1) + h(\mathcal{D}_2)$, where $h(\mathcal{D}_1)$ is the height of the derivation $\mathcal{D}_1$ and $h(\mathcal{D}_2)$ is the height of the derivation $\mathcal{D}_2$.[13] In other words, the height of a cut rule application is the sum of the heights of the two derivations for the premises of this application. In order to prove our cut-admissibility theorem for GLinqB, we will now use a main induction on the rank of a cut rule application, together with a subinduction on the height of this application. That is, given an arbitrary application of the cut rule, we will show that the corresponding derivation can be transformed in such a way that all new applications of the cut rule are either of lower rank than the original one, or they are of the same rank but have lower height. Using this strategy, we are now able to prove the desired admissibility result.

Theorem 3.2.13 (Cut-Admissibility)**.** *The cut rule is admissible in* GLinqB.

Proof. Throughout this proof, we consider an arbitrary application of the cut rule given by

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow \Delta, \pi : \varphi\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \pi : \varphi, \Sigma \Rightarrow \Theta\end{array}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\ (\mathsf{cut})$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are two derivations in GLinqB. Let $d_1 := deg(\varphi)$ be the degree of the formula φ, let $d_2 := deg(\pi)$ be the degree of the label π, and let $h := h(\mathcal{D}_1) + h(\mathcal{D}_2)$ be the height of the selected cut rule application. In order to prove the admissibility of the cut rule, we proceed by a main induction on the rank (d_1, d_2) of the cut rule application, with a subinduction on the height h of the cut.[14] In other words, we have to show that the proof tree can be transformed in such a way that all new cut rule applications are either of rank smaller than (d_1, d_2), or they are also of rank (d_1, d_2), but their height is less than h. Note that, in particular, there are now *two* induction hypotheses: we have a *main induction hypothesis*, saying that all cuts of rank smaller than (d_1, d_2) are admissible; and we have a *subinduction hypothesis*, saying that all cuts of rank (d_1, d_2) and of height smaller than h are admissible. We need to consider the following three main cases:

(1) At least one of $\mathcal{D}_1$ and $\mathcal{D}_2$ is an instance of an axiom.

(2) $\mathcal{D}_1$ and $\mathcal{D}_2$ are not axioms, and the cut formula is not principal in at least one of $\mathcal{D}_1$ and $\mathcal{D}_2$.

(3) $\mathcal{D}_1$ and $\mathcal{D}_2$ are not axioms, and the cut formula is principal on both sides.

[13] Again, by the *height* of a derivation, we mean the length of a longest branch in this derivation.

[14] Or, to put it differently, we perform a well-founded *lexicographic induction* on the triple (d_1, d_2, h).

Case 1: Suppose that at least one of $\mathcal{D}_1$ and $\mathcal{D}_2$ is an axiom. There are the following possibilities.

Case 1.1: At least one of $\mathcal{D}_1$ and $\mathcal{D}_2$ is an instance of an axiom and the cut formula $\pi : \varphi$ is not principal in this instance. In this case, the conclusion of the cut rule application must also be an axiom, so we can eliminate the cut completely. For example, suppose that the left premise of the cut rule application is an instance of Ax for which $\pi : \varphi$ is not principal, so the cut is of the form

$$\dfrac{\dfrac{}{w : p, \Gamma \Rightarrow \Delta, w : p, \pi : \varphi}\,Ax \qquad \begin{array}{c}\mathcal{D}_2\\ \pi : \varphi, \Sigma \Rightarrow \Theta\end{array}}{w : p, \Gamma, \Sigma \Rightarrow \Delta, \Theta, w : p}\,(\mathsf{cut})$$

As can be seen, the conclusion $w : p, \Gamma, \Sigma \Rightarrow \Delta, \Theta, w : p$ is also an instance of Ax, so it can be derived without any applications of the cut rule. For the other axioms, the argument is similar.

Case 1.2: $\mathcal{D}_1$ is an instance of Ax for which $\pi : \varphi$ is principal, and $\mathcal{D}_2$ ends with a rule $\mathcal{R}$ for which $\pi : \varphi$ is not principal. In this case, we permute the cut upwards over the application of $\mathcal{R}$ on the right, possibly in combination with a height-preserving substitution in order to rename the eigenvariable of $\mathcal{R}$. To see a representative case, let us assume that $\mathcal{D}_2$ ends with an application of the rule Rp for which the cut formula is not principal, so the cut is of the form

$$\dfrac{\dfrac{}{w : p, \Gamma \Rightarrow \Delta, w : p}\,Ax \qquad \dfrac{\begin{array}{c}\mathcal{D}_2'\\ u \leqslant \sigma, w : p, \Sigma \Rightarrow \Theta, u : q\end{array}}{w : p, \Sigma \Rightarrow \Theta, \sigma : q}\,Rp}{w : p, \Gamma, \Sigma \Rightarrow \Delta, \Theta, \sigma : q}\,(\mathsf{cut})$$

where σ is a non-singleton label and $u \in \mathfrak{S}$ is a fresh variable not occurring in the sequent $w : p, \Sigma \Rightarrow \Theta, \sigma : q$. Using the hp-admissibility of substitution, this proof tree is transformed into

$$\dfrac{\dfrac{\dfrac{}{w : p, \Gamma \Rightarrow \Delta, w : p}\,Ax \qquad \dfrac{\begin{array}{c}\mathcal{D}_2'\\ u \leqslant \sigma, w : p, \Sigma \Rightarrow \Theta, u : q\end{array}}{v \leqslant \sigma, w : p, \Sigma \Rightarrow \Theta, v : q}\,(v/u)}{w : p, \Gamma, v \leqslant \sigma, \Sigma \Rightarrow \Delta, \Theta, v : q}\,(\mathsf{cut})}{w : p, \Gamma, \Sigma \Rightarrow \Delta, \Theta, \sigma : q}\,Rp$$

where $v \in \mathfrak{S}$ is a fresh variable not occurring in the sequent $w : p, \Gamma, \Sigma \Rightarrow \Delta, \Theta, \sigma : q$. In this proof tree, the new application of the cut rule is of lower height than the original one, so it can be eliminated by the subinduction hypothesis in order to obtain the desired cut-free derivation.

Case 1.3: $\mathcal{D}_1$ is an instance of Ax for which $\pi : \varphi$ is principal, and $\mathcal{D}_2$ ends with an application of Lp for which $\pi : \varphi$ is also principal. In this case, the cut rule application must be of the form

$$\dfrac{\dfrac{}{w : p, \Gamma \Rightarrow \Delta, w : p}\,Ax \qquad \dfrac{\begin{array}{c}\mathcal{D}_2'\\ u : p, u \leqslant w, w : p, \Sigma \Rightarrow \Theta\end{array}}{u \leqslant w, w : p, \Sigma \Rightarrow \Theta}\,Lp}{u \leqslant w, w : p, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\,(\mathsf{cut})$$

Using the admissibility of weakening, we can now eliminate the cut by constructing the following proof tree, where 'W' refers to a sequence of multiple applications of the weakening rules:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_2'\\ u : p, u \leqslant w, w : p, \Sigma \Rightarrow \Theta\end{array}}{u \leqslant w, w : p, \Sigma \Rightarrow \Theta}\,Lp}{u \leqslant w, w : p, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\,W$$

Case 1.4: Both of the derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ are instances of Ax and the cut formula $\pi : \varphi$ is principal on both sides. In this case, the application of the cut rule must be of the following form:

$$\dfrac{\dfrac{}{w : p, \Gamma \Rightarrow \Delta, w : p}\,Ax \qquad \dfrac{}{w : p, \Sigma \Rightarrow \Theta, w : p}\,Ax}{w : p, \Gamma, \Sigma \Rightarrow \Delta, \Theta, w : p}\,(\mathsf{cut})$$

As can be seen, the conclusion is also an instance of Ax, so the cut can be eliminated completely.

Case 1.5: $\mathcal{D}_2$ is an instance of $Ax^{\perp}$ for which $\pi : \varphi$ is principal, and $\mathcal{D}_1$ ends with an application of some rule $\mathcal{R}$. Note that, since $\pi : \varphi$ is the principal formula of the instance of $Ax^{\perp}$ on the right, it must be of the form $w : \perp$ for some singleton variable $w \in \mathfrak{S}$. Consequently, the cut formula cannot be principal in the application of $\mathcal{R}$ on the left: this would only be possible, if $\mathcal{R}$ would be an instance of $R\perp$; but in this case, π would have to be a non-singleton label, i.e., $\pi \notin \mathfrak{S}$. Therefore, we may assume that $w : \perp$ is not principal in $\mathcal{D}_1$. Using this fact, we can now permute the cut upwards over the application of $\mathcal{R}$ on the left, possibly in combination with a height-preserving substitution in order to rename the eigenvariable of $\mathcal{R}$. For example, if $\mathcal{D}_1$ ends with an application of $R{\to}$ for which the cut formula is not principal, then the cut is of the form

$$\dfrac{\dfrac{\overset{\mathcal{D}_1'}{x \leqslant \sigma, x : \psi, \Gamma \Rightarrow \Delta, x : \chi, w : \perp}}{\Gamma \Rightarrow \Delta, \sigma : \psi \to \chi, w : \perp}\,R{\to} \qquad \dfrac{}{w : \perp, \Sigma \Rightarrow \Theta}\,Ax^{\perp}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta, \sigma : \psi \to \chi}\,(\text{cut})$$

where $x \in \mathfrak{V}$ is a fresh variable not occurring in the sequent $\Gamma \Rightarrow \Delta, \sigma : \psi \to \chi, w : \perp$. Using the hp-admissibility of substitution, we can now eliminate the cut by constructing the proof tree

$$\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}_1'}{x \leqslant \sigma, x : \psi, \Gamma \Rightarrow \Delta, x : \chi, w : \perp}}{y \leqslant \sigma, y : \psi, \Gamma \Rightarrow \Delta, y : \chi, w : \perp}\,(y/x) \qquad \dfrac{}{w : \perp, \Sigma \Rightarrow \Theta}\,Ax^{\perp}}{y \leqslant \sigma, y : \psi, \Gamma, \Sigma \Rightarrow \Delta, \Theta, y : \chi}\,(\text{cut})}{\Gamma, \Sigma \Rightarrow \Delta, \Theta, \sigma : \psi \to \chi}\,R{\to}$$

where $y \in \mathfrak{V}$ is some fresh variable not occurring in the conclusion. In this proof tree, the new cut is of lower height than the old one, so it can be eliminated by the subinduction hypothesis.

Case 1.6: $\mathcal{D}_2$ is an instance of Ax for which $\pi : \varphi$ is principal, and $\mathcal{D}_1$ ends with an application of some rule $\mathcal{R}$. In this case, we proceed in the same way as in the previous case.

Case 2: Suppose that neither $\mathcal{D}_1$ nor $\mathcal{D}_2$ is an axiom and the cut formula $\pi : \varphi$ is not principal in at least one of $\mathcal{D}_1$ and $\mathcal{D}_2$. Without loss of generality, assume that $\pi : \varphi$ is not principal in the left derivation $\mathcal{D}_1$, and let $\mathcal{R}$ be the last rule applied in $\mathcal{D}_1$ (if $\pi : \varphi$ is not principal on the right, then the argument is similar). Since $\pi : \varphi$ is not principal in $\mathcal{D}_1$, we can now permute the cut upwards over the application of $\mathcal{R}$ on the left, possibly in combination with a height-preserving substitution in order to rename the eigenvariable of $\mathcal{R}$. To see a concrete example, let us assume that $\mathcal{D}_1$ ends with an application of the two-premise rule (sg). In this case, the cut must be of the form

$$\dfrac{\dfrac{\overset{\mathcal{D}_1'}{\sigma \leqslant \emptyset, \sigma \leqslant w, \Gamma \Rightarrow \Delta, \pi : \varphi} \qquad \overset{\mathcal{D}_1''}{w \leqslant \sigma, \sigma \leqslant w, \Gamma \Rightarrow \Delta, \pi : \varphi}}{\sigma \leqslant w, \Gamma \Rightarrow \Delta, \pi : \varphi}\,(\text{sg}) \qquad \overset{\mathcal{D}_2}{\pi : \varphi, \Sigma \Rightarrow \Theta}}{\sigma \leqslant w, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\,(\text{cut})$$

We now permute the cut upwards over the application of (sg) by constructing the proof tree

$$\dfrac{\dfrac{\overset{\mathcal{D}_1'}{\sigma \leqslant \emptyset, \sigma \leqslant w, \Gamma \Rightarrow \Delta, \pi : \varphi} \quad \overset{\mathcal{D}_2}{\pi : \varphi, \Sigma \Rightarrow \Theta}}{\sigma \leqslant \emptyset, \sigma \leqslant w, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\,(\text{cut}) \quad \dfrac{\overset{\mathcal{D}_1''}{w \leqslant \sigma, \sigma \leqslant w, \Gamma \Rightarrow \Delta, \pi : \varphi} \quad \overset{\mathcal{D}_2}{\pi : \varphi, \Sigma \Rightarrow \Theta}}{w \leqslant \sigma, \sigma \leqslant w, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\,(\text{cut})}{\sigma \leqslant w, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\,(\text{sg})$$

Note that, in this proof tree, both of the two new applications of the cut rule are of lower height than the original one, so they can be eliminated according to the subinduction hypothesis.

Case 3: Suppose that neither $\mathcal{D}_1$ nor $\mathcal{D}_2$ is an instance of an axiom, and the cut formula $\pi : \varphi$ is principal in both premises of the cut rule application. There are the following possibilities.

Case 3.1: Suppose that $\varphi = p$ is an atomic formula. In this case, the cut must be of the form

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ w \leqslant \pi, \Gamma \Rightarrow \Delta, w : p\end{matrix}}{\Gamma \Rightarrow \Delta, \pi : p}\,Rp \qquad \dfrac{\begin{matrix}\mathcal{D}_2'\\ u : p, u \leqslant \pi, \pi : p, \Sigma \Rightarrow \Theta\end{matrix}}{u \leqslant \pi, \pi : p, \Sigma \Rightarrow \Theta}\,Lp}{u \leqslant \pi, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\,(\text{cut})$$

where $\pi \notin \mathfrak{S}$ is a non-singleton label and $w \in \mathfrak{S}$ is a fresh variable not occurring in $\Gamma \Rightarrow \Delta, \pi : p$. Using the admissibility of substitution and contraction, this proof tree is now transformed in the following way, where 'C' stands for a sequence of multiple applications of the contraction rules:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ w \leqslant \pi, \Gamma \Rightarrow \Delta, w : p\end{matrix}}{u \leqslant \pi, \Gamma \Rightarrow \Delta, u : p}\,(u/w) \qquad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ w \leqslant \pi, \Gamma \Rightarrow \Delta, w : p\end{matrix}}{\Gamma \Rightarrow \Delta, \pi : p}\,Rp \qquad \begin{matrix}\mathcal{D}_2'\\ u : p, u \leqslant \pi, \pi : p, \Sigma \Rightarrow \Theta\end{matrix}}{u : p, u \leqslant \pi, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\,(\text{cut})}{u \leqslant \pi, u \leqslant \pi, \Gamma, \Gamma, \Sigma \Rightarrow \Delta, \Delta, \Theta}\,(\text{cut})}{u \leqslant \pi, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\,C$$

In this proof tree, the uppermost of the two new cuts (i.e., the one with cut formula $\pi : p$) has lower height than the original one, so it can be eliminated according to the subinduction hypothesis. On the other hand, since $\pi \notin \mathfrak{S}$ and $u \in \mathfrak{S}$, we also have $rank(u : p) < rank(\pi : p)$ by Lemma 3.2.12 (i). Therefore, the lowermost cut (i.e., the one with cut formula $u : p$) has lower rank than the original one, so it can be eliminated by the main induction hypothesis.

Case 3.2: Let $\varphi = \bot$. This case is treated in essentially the same way as the previous case.

Case 3.3: Let $\varphi = \psi \wedge \chi$. In this case, the cut rule application must have the following form:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Gamma \Rightarrow \Delta, \pi : \psi\end{matrix} \qquad \begin{matrix}\mathcal{D}_1''\\ \Gamma \Rightarrow \Delta, \pi : \chi\end{matrix}}{\Gamma \Rightarrow \Delta, \pi : \psi \wedge \chi}\,R\wedge \qquad \dfrac{\begin{matrix}\mathcal{D}_2'\\ \pi : \psi, \pi : \chi, \Sigma \Rightarrow \Theta\end{matrix}}{\pi : \psi \wedge \chi, \Sigma \Rightarrow \Theta}\,L\wedge}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\,(\text{cut})$$

Using the admissibility of contraction, we now transform the proof tree into

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1''\\ \Gamma \Rightarrow \Delta, \pi : \chi\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_1'\\ \Gamma \Rightarrow \Delta, \pi : \psi\end{matrix} \qquad \begin{matrix}\mathcal{D}_2'\\ \pi : \psi, \pi : \chi, \Sigma \Rightarrow \Theta\end{matrix}}{\pi : \chi, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\,(\text{cut})}{\Gamma, \Gamma, \Sigma \Rightarrow \Delta, \Delta, \Theta}\,(\text{cut})}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\,C$$

In the resulting tree, all new cuts are of lower rank than the old one, so they can be eliminated by the main induction hypothesis (i.e., we first use the induction hypothesis in order to remove the topmost cut, and we then use the induction hypothesis again in order to remove the second cut).

Case 3.4: Let $\varphi = \psi \vee \chi$. This case is similar to the previous case.

Case 3.5: Let $\varphi = \psi \to \chi$. In this case, the cut rule application must have the following form:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ x \leqslant \pi, x : \psi, \Gamma \Rightarrow \Delta, x : \chi\end{matrix}}{\Gamma \Rightarrow \Delta, \pi : \psi \to \chi}\,R{\to} \qquad \dfrac{\begin{matrix}\mathcal{D}_2'\\ \sigma \leqslant \pi, \pi : \psi \to \chi, \Sigma \Rightarrow \Theta, \sigma : \psi\end{matrix} \qquad \begin{matrix}\mathcal{D}_2''\\ \sigma \leqslant \pi, \pi : \psi \to \chi, \sigma : \chi, \Sigma \Rightarrow \Theta\end{matrix}}{\sigma \leqslant \pi, \pi : \psi \to \chi, \Sigma \Rightarrow \Theta}\,L{\to}}{\sigma \leqslant \pi, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\,(\text{cut})$$

where $x \in \mathfrak{V}$ is a fresh variable. This derivation is now transformed into the proof tree

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}'\\ (\sigma \leqslant \pi)^2, \Gamma^2, \Sigma \Rightarrow \Delta^2, \Theta, \sigma : \chi\end{matrix} \qquad \begin{matrix}\mathcal{D}''\\ \sigma \leqslant \pi, \sigma : \chi, \Gamma, \Sigma \Rightarrow \Delta, \Theta\end{matrix}}{(\sigma \leqslant \pi)^3, \Gamma^3, \Sigma^2 \Rightarrow \Delta^3, \Theta^2}\,(\text{cut})}{\sigma \leqslant \pi, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\,C$$

$$\frac{\sigma : \varphi, \sigma \leqslant \pi, \pi : \varphi, \Gamma \Rightarrow \Delta}{\sigma \leqslant \pi, \pi : \varphi, \Gamma \Rightarrow \Delta}\ (\mathsf{glp}) \qquad \frac{w \leqslant \pi, \Gamma \Rightarrow \Delta, w : \alpha}{\Gamma \Rightarrow \Delta, \pi : \alpha}\ (\mathsf{grp})$$

Figure 3.3: Admissible generalizations of Lp and Rp. In the rule (grp), w must be a fresh variable not occurring in the conclusion of the rule and $\alpha \in \mathcal{L}^{\mathsf{B}}_{H}$ must be a Harrop formula.

where $\mathcal{D}'$ and $\mathcal{D}''$ are the following two derivations:

$$\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}'_1}{x \leqslant \pi, x : \psi, \Gamma \Rightarrow \Delta, x : \chi}}{\Gamma \Rightarrow \Delta, \pi : \psi \to \chi}\ R{\to} \qquad \overset{\mathcal{D}'_2}{\sigma \leqslant \pi, \pi : \psi \to \chi, \Sigma \Rightarrow \Theta, \sigma : \psi}}{\sigma \leqslant \pi, \Gamma, \Sigma \Rightarrow \Delta, \Theta, \sigma : \psi}\ (\mathsf{cut}) \qquad \dfrac{\overset{\mathcal{D}'_1}{x \leqslant \pi, x : \psi, \Gamma \Rightarrow \Delta, x : \chi}}{\sigma \leqslant \pi, \sigma : \psi, \Gamma \Rightarrow \Delta, \sigma : \chi}\ (\sigma/x)}{(\sigma \leqslant \pi)^2, \Gamma^2, \Sigma \Rightarrow \Delta^2, \Theta, \sigma : \chi}\ (\mathsf{cut})$$

$$\dfrac{\dfrac{\overset{\mathcal{D}'_1}{x \leqslant \pi, x : \psi, \Gamma \Rightarrow \Delta, x : \chi}}{\Gamma \Rightarrow \Delta, \pi : \psi \to \chi}\ R{\to} \qquad \overset{\mathcal{D}''_2}{\sigma \leqslant \pi, \pi : \psi \to \chi, \sigma : \chi, \Sigma \Rightarrow \Theta}}{\sigma \leqslant \pi, \sigma : \chi, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\ (\mathsf{cut})$$

As can be seen, the original cut is now replaced by four new cuts. An easy inspection shows that the two uppermost of these cuts (i.e., those with cut formula $\pi : \psi \to \chi$) are of lower height than the original one, and the two other cuts are of lower rank. Thus, using the subinduction hypothesis and then the main induction hypothesis, we can successively remove each of the four cuts. □

3.2.4 Further Admissibility Results

Using the results obtained in the previous sections, we now want to present a number of additional rules that can be shown to be admissible in GLinqB. First, we will prove the admissibility of the rules (glp) and (grp), depicted in Figure 3.3. In a sense, the rule (glp) can be seen as a generalization of the rule Lp, reflecting the *persistency* of the support relation in inquisitive logic: if a formula φ is supported by some state π and if σ is an enhancement of π, then φ must also be supported by σ (see Proposition 1.2.11). The rule (grp), on the other hand, generalizes the rule Rp and accounts for the *truth-conditionality* of Harrop formulas in InqB: if a Harrop formula α is true at every world in a state π, then α is supported by π (see Proposition 1.3.3).[15] Note that the truth-conditionality of α is in fact essential here (without this restriction, the rule would not be sound).

Proposition 3.2.14. *The rules* (glp) *and* (grp) *are admissible in* GLinqB.

Proof. We first show that (glp) is admissible. To this end, assume that $\sigma : \varphi, \sigma \leqslant \pi, \pi : \varphi, \Gamma \Rightarrow \Delta$ is derivable by a proof tree $\mathcal{D}$. Using this proof tree, we may then construct the following derivation, where 'C' stands for a sequence of multiple applications of the contraction rules:

$$\dfrac{\dfrac{\dfrac{\text{By Lemma 3.2.1 (i)}}{\sigma \leqslant \pi, \pi : \varphi \Rightarrow \sigma : \varphi} \qquad \overset{\mathcal{D}}{\sigma : \varphi, \sigma \leqslant \pi, \pi : \varphi, \Gamma \Rightarrow \Delta}}{\sigma \leqslant \pi, \sigma \leqslant \pi, \pi : \varphi, \pi : \varphi, \Gamma \Rightarrow \Delta}\ (\mathsf{cut})}{\sigma \leqslant \pi, \pi : \varphi, \Gamma \Rightarrow \Delta}\ C$$

Hence, by the admissibility of the cut rule, it follows that (glp) is admissible. In order to prove the admissibility of the rule (grp), we proceed by induction on the structure of the Harrop formula α.

[15] Recall that, by a *Harrop formula*, we mean any formula φ in which all occurrences of $\mathbin{\mathbb{V}}$ are contained in the antecedent of an implication. For an inductive definition of the set of all Harrop formulas, we refer to Definition 1.3.2.

For the base case, let $\alpha = p$ be an atomic formula and suppose that $w \leqslant \pi, \Gamma \Rightarrow \Delta, w : p$ is derivable, where $w \in \mathfrak{S}$ is a fresh variable not occurring in the sequent $\Gamma \Rightarrow \Delta, \pi : p$. If we have $\pi \notin \mathfrak{S}$, then the derivability of $\Gamma \Rightarrow \Delta, \pi : p$ follows immediately from the derivability of $w \leqslant \pi, \Gamma \Rightarrow \Delta, w : p$ by an application of the rule Rp. Thus, let us now assume that we have $\pi \in \mathfrak{S}$. In this case, we can apply the substitution (π/w) to the sequent $w \leqslant \pi, \Gamma \Rightarrow \Delta, w : p$ in order to obtain a derivation for $\pi \leqslant \pi, \Gamma \Rightarrow \Delta, \pi : p$. Now, using an application of the rule (rf), this yields the desired derivation for $\Gamma \Rightarrow \Delta, \pi : p$. The case $\alpha = \bot$ is treated similarly.

In the inductive step, we only have to consider the following two cases.

Case 1: Let α be of the form $\alpha = \beta \wedge \gamma$ for some Harrop formulas $\beta, \gamma \in \mathcal{L}_H^{\mathsf{B}}$. Suppose that the sequent $w \leqslant \pi, \Gamma \Rightarrow \Delta, w : \beta \wedge \gamma$ is derivable, where $w \in \mathfrak{S}$ is again a fresh variable. By the invertibility of the rule $R\wedge$, this implies that there is also a derivation $\mathcal{D}_1$ for the sequent $w \leqslant \pi, \Gamma \Rightarrow \Delta, w : \beta$ and a derivation $\mathcal{D}_2$ for the sequent $w \leqslant \pi, \Gamma \Rightarrow \Delta, w : \gamma$. Using these derivations and the induction hypothesis, we now obtain the desired derivation in the following way:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{w \leqslant \pi, \Gamma \Rightarrow \Delta, w : \beta}}{\Gamma \Rightarrow \Delta, \pi : \beta}\,\text{IH} \qquad \dfrac{\overset{\mathcal{D}_2}{w \leqslant \pi, \Gamma \Rightarrow \Delta, w : \gamma}}{\Gamma \Rightarrow \Delta, \pi : \gamma}\,\text{IH}}{\Gamma \Rightarrow \Delta, \pi : \beta \wedge \gamma}\,R\wedge$$

Case 2: Let α be of the form $\alpha = \varphi \to \beta$ for some arbitrary formula $\varphi \in \mathcal{L}^{\mathsf{B}}$ and some Harrop formula $\beta \in \mathcal{L}_H^{\mathsf{B}}$. Suppose that the sequent $w \leqslant \pi, \Gamma \Rightarrow \Delta, w : \varphi \to \beta$ is derivable, where $w \in \mathfrak{S}$ is fresh. By the invertibility of the rule $R{\to}$, this implies that, for some fresh variable $x \in \mathfrak{V}$, there is also a derivation $\mathcal{D}$ for the sequent $x \leqslant w, w \leqslant \pi, x : \varphi, \Gamma \Rightarrow \Delta, x : \beta$. Using this derivation and the induction hypothesis, we may now construct the desired derivation in the following way, where 'W' refers to a sequence of multiple applications of the weakening rules:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}}{x \leqslant w, w \leqslant \pi, x : \varphi, \Gamma \Rightarrow \Delta, x : \beta}}{w \leqslant w, w \leqslant \pi, w : \varphi, \Gamma \Rightarrow \Delta, w : \beta}\,(w/x)}{w \leqslant w, w : \varphi, w \leqslant \pi, w \leqslant x, x \leqslant \pi, x : \varphi, \Gamma \Rightarrow \Delta, w : \beta}\,W}{w : \varphi, w \leqslant \pi, w \leqslant x, x \leqslant \pi, x : \varphi, \Gamma \Rightarrow \Delta, w : \beta}\,\text{(rf)}}{w \leqslant \pi, w \leqslant x, x \leqslant \pi, x : \varphi, \Gamma \Rightarrow \Delta, w : \beta}\,\text{(glp)}}{w \leqslant x, x \leqslant \pi, x : \varphi, \Gamma \Rightarrow \Delta, w : \beta}\,\text{(tr)}}{x \leqslant \pi, x : \varphi, \Gamma \Rightarrow \Delta, x : \beta}\,\text{IH}}{\Gamma \Rightarrow \Delta, \pi : \varphi \to \beta}\,R{\to}$$

This concludes the induction. Therefore, the rule (grp) is admissible in GLinqB. □

As a corollary of this proposition, we can now prove that, if α is a Harrop formula, then every sequent of the form $\pi : \alpha, \sigma : \alpha, \Gamma \Rightarrow \Delta, \pi + \sigma : \alpha$ is derivable in GLinqB. Intuitively, this accounts again for the truth-conditionality of Harrop formulas in InqB: if a Harrop formula α is supported by two states π and σ, then it must also be supported by the union of π and σ.

Corollary 3.2.15. *If $\alpha \in \mathcal{L}_H^{\mathsf{B}}$ is a Harrop formula, then every sequent of the form $\pi : \alpha, \sigma : \alpha, \Gamma \Rightarrow \Delta, \pi + \sigma : \alpha$ is derivable in* GLinqB.

Proof. For an arbitrary Harrop formula $\alpha \in \mathcal{L}_H^{\mathsf{B}}$, we may construct the following proof tree:

$$\dfrac{\dfrac{\dfrac{\text{By Lemma 3.2.1 (i)}}{w \leqslant \pi, w \leqslant \pi + \sigma, \pi : \alpha, \sigma : \alpha, \Gamma \Rightarrow \Delta, w : \alpha} \qquad \dfrac{\text{By Lemma 3.2.1 (i)}}{w \leqslant \sigma, w \leqslant \pi + \sigma, \pi : \alpha, \sigma : \alpha, \Gamma \Rightarrow \Delta, w : \alpha}}{w \leqslant \pi + \sigma, \pi : \alpha, \sigma : \alpha, \Gamma \Rightarrow \Delta, w : \alpha}\,\text{(cd)}}{\pi : \alpha, \sigma : \alpha, \Gamma \Rightarrow \Delta, \pi + \sigma : \alpha}\,\text{(grp)}$$

Thus, by the admissibility of (grp), the desired sequent is derivable in GLinqB. □

$$\frac{\pi \circledast \sigma \approx \sigma \circledast \pi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(\text{com}) \qquad \frac{\pi \circledast (\sigma \circledast \tau) \approx (\pi \circledast \sigma) \circledast \tau, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(\text{ass})$$

$$\frac{\pi \circledast \pi \approx \pi}{\Gamma \Rightarrow \Delta}\,(\text{id}) \qquad \frac{\pi(\sigma + \tau) \approx \pi\sigma + \pi\tau}{\Gamma \Rightarrow \Delta}\,(\text{i-dis}) \qquad \frac{\pi + \sigma\tau \approx (\pi + \sigma)(\pi + \tau)}{\Gamma \Rightarrow \Delta}\,(\text{u-dis})$$

$$\frac{\pi(\pi + \sigma) \approx \pi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(\text{i-abs}) \qquad \frac{\pi + \pi\sigma \approx \pi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(\text{u-abs})$$

Figure 3.4: Some further admissible rules. In each case, we assume that $\circledast \in \{\cdot, +\}$. The notation '$\pi \approx \sigma$' is used as an abbreviation for the pair of relational atoms '$\pi \leqslant \sigma, \sigma \leqslant \pi$'.

Finally, it is worth noting that our system also allows to derive the usual algebraic properties of the intersection and the union operator. In Figure 3.4, we summarize a number of additional rules that can be shown to be admissible in our sequent calculus. Note that, in the figure, the symbol $\circledast$ acts as a placeholder, representing either the intersection or the union operator, and '$\pi \approx \sigma$' is used as a shorthand for the pair of relational atoms '$\pi \leqslant \sigma, \sigma \leqslant \pi$'. Intuitively, the rules (com), (ass) and (id) account for the commutativity, associativity and idempotence of union and intersection. The other four rules account for the usual distributivity and absorption laws.

Proposition 3.2.16. *Each of the rules depicted in Figure 3.4 is admissible in* GLinqB.

Proof. For the sake of brevity, we only show the admissibility of the two absorption rules (i-abs) and (u-abs). In order to prove the admissibility of (i-abs), let $\mathcal{D}$ be a derivation for the sequent $\pi(\pi + \sigma) \approx \pi, \Gamma \Rightarrow \Delta$. Using this derivation, we may then construct the following proof tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \pi(\pi+\sigma) \leqslant \pi, \pi \leqslant \pi(\pi+\sigma), \Gamma \Rightarrow \Delta\end{matrix}}{\pi(\pi+\sigma) \leqslant \pi, \pi \leqslant \pi(\pi+\sigma), \pi \leqslant \pi, \pi \leqslant \pi+\sigma, \Gamma \Rightarrow \Delta}\,W}{\pi \leqslant \pi(\pi+\sigma), \pi \leqslant \pi, \pi \leqslant \pi+\sigma, \Gamma \Rightarrow \Delta}\,(\text{il})}{\pi \leqslant \pi, \pi \leqslant \pi+\sigma, \Gamma \Rightarrow \Delta}\,(\text{in})}{\pi \leqslant \pi+\sigma, \Gamma \Rightarrow \Delta}\,(\text{rf})}{\Gamma \Rightarrow \Delta}\,(\text{ul})$$

Thus, (i-abs) is admissible. In order to prove the admissibility of (u-abs), let us now suppose that $\pi + \pi\sigma \approx \pi, \Gamma \Rightarrow \Delta$ is derivable by a proof tree $\mathcal{D}$. We may then construct the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \pi+\pi\sigma \leqslant \pi, \pi \leqslant \pi+\pi\sigma, \Gamma \Rightarrow \Delta\end{matrix}}{\pi+\pi\sigma \leqslant \pi, \pi \leqslant \pi+\pi\sigma, \pi \leqslant \pi, \pi\sigma \leqslant \pi, \Gamma \Rightarrow \Delta}\,W}{\pi+\pi\sigma \leqslant \pi, \pi \leqslant \pi, \pi\sigma \leqslant \pi, \Gamma \Rightarrow \Delta}\,(\text{ul})}{\pi \leqslant \pi, \pi\sigma \leqslant \pi, \Gamma \Rightarrow \Delta}\,(\text{un})}{\pi\sigma \leqslant \pi, \Gamma \Rightarrow \Delta}\,(\text{rf})}{\Gamma \Rightarrow \Delta}\,(\text{il})$$

This show that (u-abs) is admissible. For the other rules from Figure 3.4, the proof is similar. □

3.3 Soundness and Completeness

In this section, we will prove the soundness and completeness of our sequent calculus, i.e., we will show that, for every finite set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ and for every variable $x \in \mathfrak{V}$, there exists a derivation for $x : \Gamma \Rightarrow x : \varphi$ in GLinqB if and only if φ is entailed by Γ in InqB.

To begin with, let us establish the soundness of our system. For this purpose, we have to prove that every sequent derivable in GLinqB is also 'valid' with respect to the support semantics of InqB. In order to give a precise definition of what it means for a sequent to be 'valid', we first need to specify how the labels of our proof system are interpreted in an information model.

Definition 3.3.1 (Interpretation)**.** Let $M = \langle W, V \rangle$ be an information model.[16] An *interpretation* over M is a function $I : \mathfrak{S} \cup \mathfrak{V} \to \mathcal{P}(W)$ such that, for all singleton variables $w \in \mathfrak{S}$, the state $I(w) \subseteq W$ is a singleton. Given any interpretation I over some model M, it is inductively extended to a function from the set $\Lambda(\mathfrak{S}, \mathfrak{V})$ of all labels to the set $\mathcal{P}(W)$ in the following way:

(i) $I(\emptyset) := \emptyset$,
(ii) $I(\pi \cdot \sigma) := I(\pi) \cap I(\sigma)$,
(iii) $I(\pi + \sigma) := I(\pi) \cup I(\sigma)$.

In other words, an interpretation is a function that assigns a singleton state to each variable from $\mathfrak{S}$ and an arbitrary state to each variable from $\mathfrak{V}$. Such a function is then extended to arbitrary labels in the obvious way. So, in particular, the constant $\emptyset$ is interpreted as the *inconsistent state*, $\pi \cdot \sigma$ is interpreted as the *intersection* of the states $I(\pi)$ and $I(\sigma)$, and $\pi + \sigma$ is interpreted as the *union* of $I(\pi)$ and $I(\sigma)$. Given any interpretation I over some model M, we will now say that a labelled formula $\pi : \varphi$ is *satisfied* by I, just in case φ is supported by the state $I(\pi)$ in M. And a relational atom $\pi \leqslant \sigma$ is said to be satisfied by I, if the state $I(\pi)$ is an enhancement of $I(\sigma)$.

Definition 3.3.2 (Satisfaction)**.** Let I be an interpretation over some model M. The notion of *satisfaction* is defined in the following way: we say that a labelled formula $\pi : \varphi$ is satisfied by I, if we have $M, I(\pi) \models \varphi$. And a relational atom $\pi \leqslant \sigma$ is satisfied by I, if it holds $I(\pi) \subseteq I(\sigma)$.

We are now able to define a suitable notion of validity for sequents. As explained in Section 3.1, every sequent is assumed to have a 'conjunctive' reading on the left and a 'disjunctive' reading on the right. In line with this, we will call a sequent $\Gamma \Rightarrow \Delta$ *valid* in a model M, if every interpretation over M that satisfies all expressions in Γ also satisfies at least one expression in Δ.

Definition 3.3.3 (Validity)**.** Let M be a model. We say that a sequent $\Gamma \Rightarrow \Delta$ is *valid in* M, if for every interpretation I over M, the following holds: if I satisfies all labelled formulas and relational atoms in Γ, then there exists a labelled formula $\pi : \varphi$ in Δ such that I satisfies $\pi : \varphi$.

Using this notion of validity, we can now establish the soundness of our labelled sequent calculus: if a formula φ is provable from Γ in GLinqB, then φ is entailed by Γ in the basic system InqB.

Proposition 3.3.4 (Soundness)**.** *For every finite set of formulas $\Gamma \subseteq \mathcal{L}^{\mathsf{B}}$ and for every formula $\varphi \in \mathcal{L}^{\mathsf{B}}$, if the sequent $x : \Gamma \Rightarrow x : \varphi$ is derivable in* GLinqB *for some $x \in \mathfrak{V}$, then $\Gamma \models \varphi$.*[17]

Proof. We first prove that, if a sequent $\Gamma \Rightarrow \Delta$ is derivable in GLinqB, then $\Gamma \Rightarrow \Delta$ is valid in every information model M. For this purpose, let $\Gamma \Rightarrow \Delta$ be an arbitrary sequent and suppose that there exists a derivation $\mathcal{D}$ for $\Gamma \Rightarrow \Delta$ in GLinqB. Moreover, let $M = \langle W, V \rangle$ be an arbitrary model. Using induction on the structure of $\mathcal{D}$, we show that $\Gamma \Rightarrow \Delta$ is valid in M.

In the base case, we have to show that all axioms of GLinqB are valid in M. This is straightforward. For example, let us assume that $\mathcal{D}$ is an instance of $Ax^{\perp}$. In this case, $\Gamma \Rightarrow \Delta$ must be of the form $w : \perp, \Theta \Rightarrow \Delta$ for some singleton variable $w \in \mathfrak{S}$. By Definition 3.3.1, every interpretation over M must assign a singleton state to w. Hence, because $\perp$ is supported only by the inconsistent state $\emptyset$, there can be no interpretation over M that satisfies $w : \perp$, so the sequent $w : \perp, \Theta \Rightarrow \Delta$ is trivially valid in M. Similar arguments can be found for the other axioms.

[16] Recall that W stands for a non-empty set of possible worlds and V stands for a valuation function, assigning a truth value to each atomic formula $p \in \mathsf{P}$ at each possible world $w \in W$. For further details, we refer to Definition 1.2.4.

[17] As before, we write $x : \Gamma$ for the set given by $(x : \Gamma) := \{x : \psi \mid \psi \in \Gamma\}$. Moreover, recall that $\Gamma \models \varphi$ was defined to hold, if for every model M and for every state s, it holds: $M, s \models \Gamma$ implies $M, s \models \varphi$ (see Definition 1.2.12).

For the inductive step, let us assume that $\mathcal{D}$ ends with an application of one of the logical rules or one of the order rules of GLinqB. For simplicity, we only consider a few representative cases.

Case 1: Suppose that the last step in $\mathcal{D}$ is an application of the rule Rp, so $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\mathcal{D}'\\ w \leqslant \pi, \Gamma \Rightarrow \Theta, w : p\end{array}}{\Gamma \Rightarrow \Theta, \pi : p}\ Rp$$

where $\pi \notin \mathfrak{S}$ is a non-singleton label and $w \in \mathfrak{S}$ is a fresh variable not occurring in the conclusion of $\mathcal{D}$. By induction hypothesis, we know that the sequent $w \leqslant \pi, \Gamma \Rightarrow \Theta, w : p$ is valid in M. We have to show that this also holds for $\Gamma \Rightarrow \Theta, \pi : p$. Towards a contradiction, assume that $\Gamma \Rightarrow \Theta, \pi : p$ is *not* valid in M, i.e., there exists an interpretation I over M such that I satisfies all expressions in Γ, but I satisfies neither $\pi : p$ nor any expression in Θ. By the support conditions for atomic formulas (see Definition 1.2.6), this implies that there exists some world $u \in I(\pi)$ such that $M, u \not\models p$. Let now I^* be the interpretation which is just like I, except that the variable w is mapped to the singleton state $\{u\}$, so we put $I^*(w) := \{u\}$. Then, by assumption, I^* satisfies $w \leqslant \pi$ and each expression in Γ. Hence, by induction hypothesis, I^* must also satisfy $w : p$ or some expression in Θ. If I^* satisfies $w : p$, then we must have $M, u \models p$, which is a contradiction to the fact that $M, u \not\models p$. And if I^* satisfies some element of Θ, then also the original interpretation I must satisfy this element, which is a contradiction to our assumption about I.

Case 2: Suppose that $\mathcal{D}$ ends with an application of $L\bot$. In this case, $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\mathcal{D}'\\ w : \bot, w \leqslant \pi, \pi : \bot, \Theta \Rightarrow \Delta\end{array}}{w \leqslant \pi, \pi : \bot, \Theta \Rightarrow \Delta}\ L\bot$$

By induction hypothesis, $w : \bot, w \leqslant \pi, \pi : \bot, \Theta \Rightarrow \Delta$ is valid in M. Suppose for a contradiction that $w \leqslant \pi, \pi : \bot, \Theta \Rightarrow \Delta$ is *not* valid. Then, in particular, there must be an interpretation I over M such that I satisfies both $w \leqslant \pi$ and $\pi : \bot$, so we have $I(w) \subseteq I(\pi)$ and $M, I(\pi) \models \bot$. Since $\bot$ is only supported by the inconsistent state $\emptyset$, this implies $I(\pi) = \emptyset$. But then, because $I(w)$ is non-empty by Definition 3.3.1, it follows $I(w) \nsubseteq I(\pi)$, which is a contradiction to $I(w) \subseteq I(\pi)$.

Case 3: Suppose that $\mathcal{D}$ ends with an application of $L{\to}$, so $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \pi \leqslant \sigma, \sigma : \varphi \to \psi, \Theta \Rightarrow \Delta, \pi : \varphi\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \pi \leqslant \sigma, \sigma : \varphi \to \psi, \pi : \psi, \Theta \Rightarrow \Delta\end{array}}{\pi \leqslant \sigma, \sigma : \varphi \to \psi, \Theta \Rightarrow \Delta}\ L{\to}$$

By induction hypothesis, $\pi \leqslant \sigma, \sigma : \varphi \to \psi, \Theta \Rightarrow \Delta, \pi : \varphi$ and $\pi \leqslant \sigma, \sigma : \varphi \to \psi, \pi : \psi, \Theta \Rightarrow \Delta$ are both valid. In order to show that this also holds for the conclusion of $\mathcal{D}$, let I be an arbitrary interpretation over M and suppose that I satisfies $\pi \leqslant \sigma$, $\sigma : \varphi \to \psi$ and each expression in Θ. Then, by induction hypothesis, I must also satisfy $\pi : \varphi$ or some element of Δ. If the latter holds, then we are finished. Thus, let us assume that I satisfies $\pi : \varphi$. By assumption, we also have $I(\pi) \subseteq I(\sigma)$ and $M, I(\sigma) \models \varphi \to \psi$. Using the support conditions for implication (see Definition 1.2.6), one readily sees that this yields $M, I(\pi) \not\models \varphi$ or $M, I(\pi) \models \psi$. Therefore, since I satisfies $\pi : \varphi$, it follows $M, I(\pi) \models \psi$. As we have seen, I satisfies each of the expressions $\pi \leqslant \sigma, \sigma : \varphi \to \psi, \pi : \psi$ and every element of Θ. But then, by induction hypothesis, I must also satisfy some element of Δ. Because I was arbitrary, this shows that the conclusion of $\mathcal{D}$ is in fact valid in M.

Case 4: Suppose that the last step in $\mathcal{D}$ is an application of $R{\to}$, so $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\mathcal{D}'\\ x \leqslant \pi, x : \varphi, \Gamma \Rightarrow \Theta, x : \psi\end{array}}{\Gamma \Rightarrow \Theta, \pi : \varphi \to \psi}\ R{\to}$$

where $x \in \mathfrak{V}$ is a fresh variable not occurring in the conclusion. By induction hypothesis, we know that $x \leqslant \pi, x : \varphi, \Gamma \Rightarrow \Theta, x : \psi$ is valid in M. In order to show that this also holds for the conclusion of $\mathcal{D}$, suppose for a contradiction that $\Gamma \Rightarrow \Theta, \pi : \varphi \to \psi$ is *not* valid in M, i.e., there exists an interpretation I over M such that I satisfies each element of Γ, but I neither satisfies $\pi : \varphi \to \psi$ nor any element of Θ. Then, since $M, I(\pi) \not\models \varphi \to \psi$, there must be some state $s \subseteq I(\pi)$ such that $M, s \models \varphi$ and $M, s \not\models \psi$. Let now I^* be the interpretation which is just like I, except that the variable x is mapped to s, so we put $I^*(x) := s$. Then, clearly, I^* satisfies $x \leqslant \pi, x : \varphi$ and each element of Γ. Therefore, by induction hypothesis, it must also satisfy $x : \psi$ or some element of Θ. If I^* satisfies $x : \psi$, then we have $M, s \models \psi$ by definition of I^*, which is a contradiction to $M, s \not\models \psi$. And if I^* satisfies some element of Θ, then also the original interpretation I must satisfy this element, which is a contradiction to our assumption about I.

Case 5: Suppose that $\mathcal{D}$ ends with an application of the rule (un). In this case, $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\mathcal{D}'\\ \pi + \tau \leqslant \sigma, \pi \leqslant \sigma, \tau \leqslant \sigma, \Theta \Rightarrow \Delta\end{array}}{\pi \leqslant \sigma, \tau \leqslant \sigma, \Theta \Rightarrow \Delta}\ (\mathsf{un})$$

By induction hypothesis, $\pi + \tau \leqslant \sigma, \pi \leqslant \sigma, \tau \leqslant \sigma, \Theta \Rightarrow \Delta$ is valid in M. We have to show that this also holds for the conclusion of $\mathcal{D}$. To this end, I be an arbitrary interpretation over M and suppose that I satisfies $\pi \leqslant \sigma$, $\tau \leqslant \sigma$ and each expression in Θ. Then, in particular, we have $I(\pi) \subseteq I(\sigma)$ and $I(\tau) \subseteq I(\sigma)$, so it follows $I(\pi) \cup I(\tau) \subseteq I(\sigma)$. Hence, I also satisfies the relational atom $\pi + \tau \leqslant \sigma$. Therefore, by induction hypothesis, some element of Δ must be satisfied by I. Since I was arbitrary, this shows that $\pi \leqslant \sigma, \tau \leqslant \sigma, \Theta \Rightarrow \Delta$ is valid in M.

Case 6: Suppose that the last step in $\mathcal{D}$ is an application of (sg), so $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \pi \leqslant \emptyset, \pi \leqslant w, \Theta \Rightarrow \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ w \leqslant \pi, \pi \leqslant w, \Theta \Rightarrow \Delta\end{array}}{\pi \leqslant w, \Theta \Rightarrow \Delta}\ (\mathsf{sg})$$

By induction hypothesis, $\pi \leqslant \emptyset, \pi \leqslant w, \Theta \Rightarrow \Delta$ and $w \leqslant \pi, \pi \leqslant w, \Theta \Rightarrow \Delta$ are both valid in M. In order to show that this also holds for the conclusion of $\mathcal{D}$, let I be an arbitrary interpretation over M and suppose that I satisfies $\pi \leqslant w$ and each element of Θ. Since $I(w)$ is a singleton and $I(\pi) \subseteq I(w)$, we must either have $I(\pi) = \emptyset$ or $I(\pi) = I(w)$. In the first case, I satisfies $\pi \leqslant \emptyset$, and in the second case, I satisfies $w \leqslant \pi$. Hence, by induction hypothesis, I satisfies some element of Δ, as desired. Because I was arbitrary, this shows that $\pi \leqslant w, \Theta \Rightarrow \Delta$ is valid in M.

The other rules are treated similarly. This concludes the induction. Let now $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ be an arbitrary finite set and suppose that $x : \Gamma \Rightarrow x : \varphi$ is derivable in GLinqB for some $x \in \mathfrak{V}$. As we have just seen, this implies that, for every model M and for every interpretation I over M, if $M, I(x) \models \psi$ for all $\psi \in \Gamma$, then $M, I(x) \models \varphi$. But then, clearly, we also have $\Gamma \models \varphi$. ☐

Next, we will establish the completeness of our sequent calculus. This will be achieved in a purely proof-theoretical manner, i.e., instead of giving a semantic argument, we will use our cut-admissibility theorem in order to show that GLinqB is complete with respect to the Hilbert-style system HinqB depicted in Figure 3.5. As demonstrated in Section 1.5, this Hilbert-style system is sound and complete with respect to InqB. Therefore, in order to prove the completeness of our sequent calculus, it suffices to show that every formula provable in HinqB is also provable in GLinqB. To this end, we first need to show that every axiom of HinqB is derivable in GLinqB.

Lemma 3.3.5. *Let φ be an instance of one of the axiom schemes of* HinqB*. Then, φ is provable in* GLinqB*, i.e., for any variable $x \in \mathfrak{V}$, there exists a derivation for the sequent $\Rightarrow x : \varphi$ in* GLinqB.

Proof. For the 'intuitionistic' axioms given in Figure 1.5, the proof is a matter of routine (in fact, the only non-trivial case is the axiom $\bot \to \varphi$, but this axiom can be easily derived from part (iv) of Lemma 3.2.2). Therefore, we only need to show the derivability of the split axiom (Split) and

Axioms:

(IPL)	All axioms of the 'intuitionistic' system given in Figure 1.5,
(Split)	$(\alpha \to (\varphi \vvee \psi)) \to ((\alpha \to \varphi) \vvee (\alpha \to \psi))$, where $\alpha \in \mathcal{L}^{\mathsf{B}}_c$ is classical,
(DN)	$\neg\neg\alpha \to \alpha$, where $\alpha \in \mathcal{L}^{\mathsf{B}}_c$ is classical.

The only rule of inference is *modus ponens*: from $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, infer $\Gamma \vdash \psi$.

Figure 3.5: The Hilbert-style system HinqB.

the double negation axiom (DN). In order to derive the split axiom, let $\alpha \in \mathcal{L}^{\mathsf{B}}_c$ be a classical formula and let $\varphi, \psi \in \mathcal{L}^{\mathsf{B}}$ be arbitrary formulas. We may then construct the following derivation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{By Corollary 3.2.15}}{\ldots, z_1 : \alpha, z_2 : \alpha \Rightarrow z_1 : \varphi, z_2 : \psi, z_1 + z_2 : \alpha} \qquad \dfrac{\mathcal{D}}{\ldots, z_1 + z_2 : \varphi \vvee \psi \Rightarrow z_1 : \varphi, z_2 : \psi}}{z_1 + z_2 \leqslant y, z_2 \leqslant y, z_1 \leqslant y, y \leqslant x, y : \alpha \to (\varphi \vvee \psi), z_1 : \alpha, z_2 : \alpha \Rightarrow z_1 : \varphi, z_2 : \psi}\,L{\to}}{z_2 \leqslant y, z_1 \leqslant y, y \leqslant x, y : \alpha \to (\varphi \vvee \psi), z_1 : \alpha, z_2 : \alpha \Rightarrow z_1 : \varphi, z_2 : \psi}\,(\mathsf{un})}{z_1 \leqslant y, y \leqslant x, y : \alpha \to (\varphi \vvee \psi), z_1 : \alpha \Rightarrow z_1 : \varphi, y : \alpha \to \psi}\,R{\to}}{y \leqslant x, y : \alpha \to (\varphi \vvee \psi) \Rightarrow y : \alpha \to \varphi, y : \alpha \to \psi}\,R{\to}}{y \leqslant x, y : \alpha \to (\varphi \vvee \psi) \Rightarrow y : (\alpha \to \varphi) \vvee (\alpha \to \psi)}\,R{\vvee}}{\Rightarrow x : (\alpha \to (\varphi \vvee \psi)) \to ((\alpha \to \varphi) \vvee (\alpha \to \psi))}\,R{\to}
$$

where the subderivation $\mathcal{D}$ is of the form

$$
\dfrac{\dfrac{\dfrac{\text{By Lemma 3.2.1 (i)}}{z_1 \leqslant z_1 + z_2, \ldots, z_1 + z_2 : \varphi \Rightarrow z_1 : \varphi, z_2 : \psi}}{\ldots, z_1 + z_2 : \varphi \Rightarrow z_1 : \varphi, z_2 : \psi}\,(\mathsf{ul}) \qquad \dfrac{\dfrac{\text{By Lemma 3.2.1 (i)}}{z_2 \leqslant z_1 + z_2, \ldots, z_1 + z_2 : \psi \Rightarrow z_1 : \varphi, z_2 : \psi}}{\ldots, z_1 + z_2 : \psi \Rightarrow z_1 : \varphi, z_2 : \psi}\,(\mathsf{ur})}{\ldots, z_1 + z_2 : \varphi \vvee \psi \Rightarrow z_1 : \varphi, z_2 : \psi}\,L{\vvee}
$$

Note that, since α is a classical formula, it is also a Harrop formula, so Corollary 3.2.15 is in fact applicable here. In order to derive the double negation axiom (DN), let $\alpha \in \mathcal{L}^{\mathsf{B}}_c$ be again a classical formula. Then, using the admissible rule (grp) from Figure 3.3, we may construct the derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{By Lemma 3.2.2 (i)}}{z \leqslant \emptyset, \ldots \Rightarrow w : \alpha, z : \bot} \qquad \dfrac{\text{By Lemma 3.2.1 (i)}}{w \leqslant z, \ldots, z : \alpha \Rightarrow w : \alpha, z : \bot}}{z \leqslant w, w \leqslant y, y \leqslant x, y : \neg\neg\alpha, z : \alpha \Rightarrow w : \alpha, z : \bot}\,(\mathsf{sg})}{w \leqslant y, y \leqslant x, y : \neg\neg\alpha \Rightarrow w : \alpha, w : \neg\alpha}\,R{\to} \qquad \dfrac{}{\ldots, w : \bot \Rightarrow w : \alpha}\,Ax^{\bot}}{w \leqslant y, y \leqslant x, y : \neg\neg\alpha \Rightarrow w : \alpha}\,L{\to}}{y \leqslant x, y : \neg\neg\alpha \Rightarrow y : \alpha}\,(\mathsf{grp})}{\Rightarrow x : \neg\neg\alpha \to \alpha}\,R{\to}
$$

Again, observe that α is also a Harrop formula, so the application of (grp) is in fact correct. □

In a similar way, one can also derive the axioms of the alternative Hilbert-style system $\mathsf{HinqB}_{\mathsf{KP}}$ presented in Figure 1.7. The derivations are almost the same. In particular, in the derivation for the Kreisel-Putnam axiom (KP), one can again use Corollary 3.2.15, since negated formulas, too, are always Harrop formulas. In the derivation for the atomic double negation axiom (ADN), one can also use Rp instead of (grp). For the next step, let now (mp) be the *modus ponens rule* given by

$$
\dfrac{\Rightarrow x : \varphi \qquad \Rightarrow x : \varphi \to \psi}{\Rightarrow x : \psi}\,(\mathsf{mp})
$$

Using our cut-admissibility theorem, it is easy to verify that (mp) is admissible in GLinqB.

Lemma 3.3.6. *The modus ponens rule* (mp) *is admissible in* GLinqB.

Proof. Suppose that there is a derivation $\mathcal{D}_1$ for $\Rightarrow x : \varphi$ and a derivation $\mathcal{D}_2$ for $\Rightarrow x : \varphi \to \psi$. We have to show that there is also a derivation for $\Rightarrow x : \psi$. By applying the invertibility of the rule $R{\to}$ to the derivation $\mathcal{D}_2$, we first obtain a derivation $\mathcal{D}_2'$ for $y \leqslant x, y : \varphi \Rightarrow y : \psi$, where $y \in \mathfrak{V}$ is a fresh variable. Using $\mathcal{D}_1$ and $\mathcal{D}_2'$, we may now construct the following derivation:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Rightarrow x : \varphi\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2'\\ y \leqslant x, y : \varphi \Rightarrow y : \psi\end{array}}{x \leqslant x, x : \varphi \Rightarrow x : \psi}\,(x/y)}{x \leqslant x \Rightarrow x : \psi}\,(\mathsf{cut})}{\Rightarrow x : \psi}\,(\mathsf{rf})$$

Therefore, by the admissibility of the cut rule, it follows that (mp) is admissible in GLinqB. □

By combining the previous two lemmas, we can now prove that our labelled sequent calculus is weakly complete with respect to HinqB: if a formula φ is provable in the Hilbert-style system HinqB, then it is also provable in the sequent calculus GLinqB. The strong completeness of GLinqB is then obtained as an immediate corollary, by using the deduction theorem for HinqB.

Theorem 3.3.7. *For every formula $\varphi \in \mathcal{L}^{\mathsf{B}}$, if we have $\vdash_{\mathsf{H}} \varphi$ in the Hilbert-style system* HinqB, *then the sequent $\Rightarrow x : \varphi$ is derivable in* GLinqB, *for any variable $x \in \mathfrak{V}$.*[18]

Proof. The statement is proved by induction on the structure of a Hilbert-style proof for $\vdash_{\mathsf{H}} \varphi$ (see Definition 1.5.3). This is trivial, since we already know that all axioms of HinqB are provable in GLinqB and that modus ponens is admissible in GLinqB (see Lemma 3.3.5 and 3.3.6). □

Corollary 3.3.8 (Soundness and Completeness)**.** *The labelled sequent calculus* GLinqB *is sound and strongly complete with respect to* InqB. *That is, for every finite set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$, we have $\Gamma \models \varphi$ if and only if $x : \Gamma \Rightarrow x : \varphi$ is derivable in* GLinqB, *for any variable $x \in \mathfrak{V}$.*

Proof. The soundness of GLinqB has been established in Proposition 3.3.4. For the completeness part, let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{B}}$ be an arbitrary finite set of formulas and suppose that $\Gamma \models \varphi$. By the completeness of HinqB (see Corollary 1.5.6), this yields $\Gamma \vdash_{\mathsf{H}} \varphi$. Now, because Γ is finite, we have $\Gamma = \{\psi_1, \ldots, \psi_n\}$ for some formulas $\psi_1, \ldots, \psi_n \in \mathcal{L}^{\mathsf{B}}$. Thus, from $\Gamma \vdash_{\mathsf{H}} \varphi$, it follows $\vdash_{\mathsf{H}} \psi_1 \to (\psi_2 \to \ldots (\psi_n \to \varphi) \ldots)$ by the deduction theorem for HinqB (see Theorem 1.5.4). One readily sees that this implies $\vdash_{\mathsf{H}} \bigwedge \Gamma \to \varphi$, where $\bigwedge \Gamma$ is the conjunction of the formulas in Γ. Hence, by Theorem 3.3.7, we may conclude that $\Rightarrow x : \bigwedge \Gamma \to \varphi$ is derivable in GLinqB, for any $x \in \mathfrak{V}$. Therefore, by the invertibility of the rules $R{\to}$ and $L{\wedge}$, the sequent $y \leqslant x, y : \Gamma \Rightarrow y : \varphi$ is also derivable, where $y \in \mathfrak{V}$ is fresh. But then, by performing the substitution (x/y) and a subsequent application of (rf), we obtain the desired derivation for $x : \Gamma \Rightarrow x : \varphi$ in GLinqB. □

3.4 Towards a Proof Search Procedure for Inquisitive Logic

In the previous section, we have seen that GLinqB is sound and complete with respect to inquisitive logic, so a formula φ is provable in our system if and only if φ is valid in InqB. It would now be desirable to have an effective *proof search procedure* for our sequent calculus, i.e., an algorithm that, given any formula φ as input, either outputs a derivation for a sequent of the form $\Rightarrow x : \varphi$, or a finite countermodel for the formula φ. In this section, we want to take a first step towards such an algorithm. In particular, we will outline the overall structure of a possible proof search

[18] Recall that we write $\vdash_{\mathsf{H}}$ for the provability relation of the Hilbert-style system HinqB (see Definition 1.5.3).

strategy for GLinqB and discuss some problems that have to be resolved in order to make sure that the procedure is terminating. The full specification of the desired algorithm and the corresponding termination proof will be left for future work. We start by defining some basic notions.

Definition 3.4.1 (Proof Search Tree, Branch)**.** Let $\varphi \in \mathcal{L}^{\mathrm{B}}$ be a formula. A *proof search tree* for φ is a finite tree of sequents $\mathfrak{T}$, built up from a root node of the form $\Rightarrow x : \varphi$ with $x \in \mathfrak{V}$, by root-first applications of the rules of our system.[19] A *branch* in a proof search tree $\mathfrak{T}$ is a sequence β of consecutive sequents in $\mathfrak{T}$, starting with the root node and ending with one of the leaf nodes of $\mathfrak{T}$.

A branch β in a proof search tree is said to be *closed*, if the topmost sequent in β is an instance of one of the axioms of our system. Otherwise, the brach is said to be *open*. And a proof search tree $\mathfrak{T}$ is called closed, if every branch in $\mathfrak{T}$ is closed, and it is called open otherwise. Recall that, by a *dynamic rule*, we mean a sequent rule that allows to introduce a fresh variable (i.e., an *eigenvariable*) in the course of a derivation. In the sequent calculus GLinqB, this includes the logical rules Rp, $R\bot$ and $R{\to}$. All other rules of our system are said to be *static rules* (see Section 3.1).

As explained above, we are interested in an algorithm that allows to find sequent proofs for formulas in a systematic and mechanical way. The basic idea is to use a 'bottom-up' search strategy in the style of Schütte (1956) and Takeuti (1987). That is, given any formula $\varphi \in \mathcal{L}^{\mathrm{B}}$ as input, our algorithm should start to construct a proof search tree $\mathfrak{T}$ for φ by successively applying all rules of our system root-first to the topmost sequents in the tree. If the input formula φ is valid, then the search tree $\mathfrak{T}$ should become closed after a finite number of steps. In this case, our algorithm should output $\mathfrak{T}$, which is now a derivation for the root node $\Rightarrow x : \varphi$. Otherwise, the procedure should stop as soon as some open branch in $\mathfrak{T}$ satisfies a suitable *saturation condition*. Intuitively, a branch β is said to be *saturated*, if the topmost sequent in β is not an instance of an axiom and β is closed under non-redundant applications of all rules of our sequent calculus. If a branch β in the search tree $\mathfrak{T}$ becomes saturated at some stage of the construction, then our algorithm should be able to use this branch in order to construct a *finite countermodel* M_β and an interpretation function I_β such that $M_\beta, I_\beta(x) \not\models \varphi$, where φ stands for the input formula.

In order for this strategy to work, one has to design the search procedure and the saturation condition for branches in such a way that the algorithm is guaranteed to *terminate* on each input, so every branch in the proof search tree should in fact become either closed or saturated after a finite number of steps. In addition to that, a saturated branch should always allow us to 'read off' the desired countermodel for the input formula. Unfortunately, finding a suitable saturation condition for our sequent calculus turns out to be quite difficult. In order to get an idea of the difficulties, let us discuss some technical problems that may arise during the search process.

One problem is related to the complex syntax of the labels used in our proof system. In particular, by performing a root-first application of an order rule, we may introduce a new label in a branch which can then be used in order to introduce further labels and so on. This might cause our algorithm to get stuck in an infinite loop of order rule applications, producing increasingly more complex labels. For instance, using only the rule (un), we may create an infinite loop of the form

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{((x+y)+x)+y \leqslant z, (x+y)+x \leqslant z, x+y \leqslant z, x \leqslant z, y \leqslant z, \Gamma \Rightarrow \Delta}\ (\mathsf{un})}{(x+y)+x \leqslant z, x+y \leqslant z, x \leqslant z, y \leqslant z, \Gamma \Rightarrow \Delta}\ (\mathsf{un})}{x+y \leqslant z, x \leqslant z, y \leqslant z, \Gamma \Rightarrow \Delta}\ (\mathsf{un})}{x \leqslant z, y \leqslant z, \Gamma \Rightarrow \Delta}$$

From a semantic point of view, the repeated applications of (un) in this loop do not yield any new information, since the relational atoms $x+y \leqslant z$, $(x+y)+x \leqslant z$ and $((x+y)+x)+y \leqslant z$ all

[19] By a *root-first application* of a rule, we mean that the rule is applied 'bottom-up', i.e., if one of the leaf nodes of $\mathfrak{T}$ is an instance of the conclusion of the rule, then we write the corresponding premises of the rule above this leaf node.

have exactly the same 'meaning'.[20] In order to avoid loops of this kind, one has to find a suitable mechanism that prevents our algorithm from producing 'equivalent' relational atoms in a branch.

Luckily, this problem has an easy solution, since it is possible to show that every label of our system can be reduced to a suitable *normal form*. To make this more precise, let us introduce some terminology. First of all, we say that two labels π and σ are *equivalent*, notation $\pi \equiv \sigma$, if for every model M and for every interpretation I over M, it is the case that $I(\pi) = I(\sigma)$. In other words, π and σ are equivalent, if they denote exactly the same information state under every possible interpretation. By an *intersection of variables*, we mean any label of the form $s_1 s_2 \cdots s_n$, where each s_i is a variable, i.e., $s_i \in \mathfrak{S} \cup \mathfrak{V}$ for all $1 \leq i \leq n$. Given an intersection of variables $\pi = s_1 \cdots s_n$, we will also write $S(\pi)$ for the set of variables given by $S(\pi) := \{s_1, \ldots, s_n\}$. It is now possible to show that each label of our system can be transformed into an equivalent label of the form $\pi_1 + \ldots + \pi_n$, where each π_i is an intersection of variables. In order to make sure that this normal form is unique, we have to require that the intersections $\pi_1, \ldots, \pi_n$ and the variables occurring in these intersections are ordered in a fixed way. To this end, we henceforth assume a *strict total order* $<$ on the set of all labels $\Lambda(\mathfrak{S}, \mathfrak{V})$, i.e., an irreflexive, asymmetric, transitive and connected relation between labels.[21] The notion of a normal form can now be defined as follows.

Definition 3.4.2 (Normal Form)**.** A label π is said to be a *normal form*, if π is either of the form $\emptyset$, or it is of the form $\pi = \pi_1 + \ldots + \pi_n$ and each of the following three conditions is satisfied:

(i) Each π_i is an intersection of variables $\pi_i = s_1 \cdots s_k$ such that $s_i < s_j$ for all $1 \leq i < j \leq k$.

(ii) For all intersections π_i and π_j with $1 \leq i < j \leq n$, we have $\pi_i < \pi_j$.

(iii) There are no intersections π_i and π_j with $i \neq j$ such that $S(\pi_i) \subseteq S(\pi_j)$.

Observe that, by the first two conditions, no variable can occur more than once in any of the intersections π_i and these intersections must always be pairwise distinct. The last condition ensures that normal forms do not contain any 'superfluous' intersections: if we would have $S(\pi_i) \subseteq S(\pi_j)$ for two distinct intersections π_i and π_j, then it would follow $I(\pi_j) \subseteq I(\pi_i)$ for every interpretation I, so the label π_j could also be removed from the union $\pi_1 + \ldots + \pi_n$ without changing the 'meaning' of the normal form. Using the concept of a normal form, it is now possible to avoid redundant applications of order rules during the search process. Let us make this idea more precise. First of all, given any branch β in a proof search tree, we write $\mathfrak{S}_\beta$ for the set of all variables from $\mathfrak{S}$ that occur in β, and we write $\mathfrak{V}_\beta$ for the set of all variables from $\mathfrak{V}$ that occur in β. Moreover, let $\Lambda^{\mathsf{nf}}_\beta$ be the set of all normal forms built up from the variables in $\mathfrak{S}_\beta \cup \mathfrak{V}_\beta$. Note that, since every branch in a search tree is finite, both of the sets $\mathfrak{S}_\beta$ and $\mathfrak{V}_\beta$ must be finite as well. Using this fact, it is easy to see that $\Lambda^{\mathsf{nf}}_\beta$ contains only finitely many normal forms.

Proposition 3.4.3. *For every branch β in a proof search tree, $\Lambda^{\mathsf{nf}}_\beta$ is finite.*

Proof. Let β be an arbitrary branch in a proof search tree, let $V_\beta := \mathfrak{S}_\beta \cup \mathfrak{V}_\beta$ be the set of all variables occurring in β and let $n := |V_\beta|$ be the cardinality of V_β. Clearly, every normal form $\pi_1 + \ldots + \pi_k$ from $\Lambda^{\mathsf{nf}}_\beta$ can be uniquely encoded by the set $\{S(\pi_1), \ldots, S(\pi_k)\}$, which is a subset of $\mathcal{P}(V_\beta)$. Since there are exactly 2^{2^n} such subsets, $\Lambda^{\mathsf{nf}}_\beta$ can contain at most 2^{2^n} elements. □

In addition, we can now prove the desired normal form result for the labels of our system.

Proposition 3.4.4 (Existence of Normal Forms)**.** *For every label π, there exists a unique normal form σ with $\pi \equiv \sigma$. Moreover, there is an algorithm that computes this normal form, for any label π.*

Proof. Let π be an arbitrary label. In order to compute the normal form of π, we may perform the following steps: first, we use the left-to-right direction of the equivalences $\sigma(\tau_1 + \tau_2) \equiv \sigma\tau_1 + \sigma\tau_2$ and $(\tau_1 + \tau_2)\sigma \equiv \tau_1\sigma + \tau_2\sigma$ in order to distribute all intersections in π over the unions in π.

[20] This follows immediately from the way in which labels are interpreted in a model (see Definition 3.3.1).

[21] We do not care about the exact definition of this relation—all that matters is that such a relation *can* be defined.

After a finite number of steps, this must result in a label of the form $\pi_1 + \ldots + \pi_n$, where each π_i is an intersection built up from variables and the constant $\emptyset$. If each π_i contains an occurrence of $\emptyset$, then the desired normal form is taken to be $\emptyset$. Otherwise, we delete all intersections π_i that contain $\emptyset$ and all intersections π_i for which there is another intersection π_j with $S(\pi_j) \subseteq S(\pi_i)$. Finally, we use the equivalences $\sigma\tau \equiv \tau\sigma$ and $\sigma + \tau \equiv \tau + \sigma$ in order to arrange the remaining intersections π_i and the variables occurring in these intersections in the right order. Clearly, each of the transformations preserves equivalence, so this finally yields a normal form σ with $\pi \equiv \sigma$.

In order to prove the *uniqueness* of normal forms, it suffices to show that distinct normal forms are always non-equivalent. To this end, let π and σ be arbitrary normal forms and suppose that $\pi \neq \sigma$. We have to show that it also holds $\pi \not\equiv \sigma$. If one of π and σ is of the form $\emptyset$, then the argument is trivial. Thus, assume that $\pi = \pi_1 + \ldots + \pi_n$ and $\sigma = \sigma_1 + \ldots + \sigma_k$, where each π_i and each σ_i is an intersection of variables. Moreover, let $\Pi := \{\pi_1, \ldots, \pi_n\}$ and $\Sigma := \{\sigma_1, \ldots, \sigma_k\}$. By a *distinguishing term* for π and σ, we mean an intersection of variables τ which is contained in only one of the two sets Π and Σ, but not in the other one. Since we have $\pi \neq \sigma$ by assumption, there clearly exists at least one distinguishing term for π and σ. Let now τ be a *smallest* distinguishing term, i.e., suppose that there is no other distinguishing term $\tau' \neq \tau$ with $S(\tau') \subseteq S(\tau)$. Without loss of generality, suppose that $\tau \in \Pi$ and $\tau \notin \Sigma$, i.e., there exists some π_i with $\tau = \pi_i$ and it holds $\tau \neq \sigma_j$ for all $1 \leq j \leq k$. Towards a contradiction, suppose that there is a σ_j with $S(\sigma_j) \subseteq S(\pi_i)$. Then, by Definition 3.4.2 (iii), this σ_j cannot be contained in Π. Hence, σ_j is a distinguishing term that is smaller than $\tau = \pi_i$, in contradiction to our assumption about τ. Thus, for every j with $1 \leq j \leq k$, we have $S(\sigma_j) \not\subseteq S(\pi_i)$. Let now M be a model with two distinct worlds w and u and let I be the interpretation that assigns the singleton $\{w\}$ to all variables in $S(\tau)$, and the singleton $\{u\}$ to every other variable. Then, clearly, we have $I(\tau) = I(\pi_i) = \{w\}$, so it follows $w \in I(\pi)$. However, since $S(\sigma_j) \not\subseteq S(\tau)$ for all $1 \leq j \leq k$, we must have $w \notin I(\sigma_j)$ for each $\sigma_j \in \Sigma$, which implies $w \notin I(\sigma)$. Now, from $w \in I(\pi)$ and $w \notin I(\sigma)$, we may conclude $\pi \not\equiv \sigma$. $\square$

In what follows, we will also write $\mathsf{nf}(\pi)$ for the unique normal form of a label π. Using our normal form result, it is now possible to circumvent the aforementioned problem in a very elegant way. The basic idea is as follows: first, we can define a variant $\mathsf{GLinqB}^{\mathsf{nf}}$ of the proof system GLinqB by reformulating all order rules in such a way that the new labels introduced by these rules are always required to be in normal form. So, for example, the old formulation of the rule (un) presented in Figure 3.1 would be replaced by the following reformulated version:

$$\frac{\mathsf{nf}(\pi + \tau) \leqslant \sigma, \pi \leqslant \sigma, \tau \leqslant \sigma, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \tau \leqslant \sigma, \Gamma \Rightarrow \Delta}\ (\mathsf{un})$$

In the modified sequent calculus $\mathsf{GLinqB}^{\mathsf{nf}}$, we can now use a suitable loop-checking mechanism in order to avoid redundant applications of order rules during the construction of a proof search tree. For instance, before performing a root-first application of the rule (un) in a branch β, we first check whether the resulting atom $\mathsf{nf}(\pi + \tau) \leqslant \sigma$ does already occur in the topmost sequent of β. If this is the case, then the application is taken to be redundant and we refrain from applying the rule. Otherwise, the application is carried out in the usual way. Note that, in the system $\mathsf{GLinqB}^{\mathsf{nf}}$, all new labels introduced by order rule applications in a branch β are taken from the finite set $\Lambda^{\mathsf{nf}}_{\beta}$, so there are only finitely many order rule applications to be performed at each step.[22] This avoids the problem of infinite loops constructed from redundant applications of order rules.

However, there is also a more severe problem that has not been fully resolved yet. This problem arises from the interaction between the dynamic rules and the rule $L{\rightarrow}$ and might cause our algorithm to generate infinitely many new variables in the course of the search process. For

[22] Strictly speaking, this also requires to impose a suitable restriction on those order rules that do not have any principal atoms in the conclusion of the rule. However, such a restriction is unproblematic. The details are left to the reader.

example, using alternating applications of the rules $R{\to}$ and $L{\to}$, our procedure might create an infinite loop of the following form (in applications of $L{\to}$, only the left premise is displayed):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ z \leqslant x, z \leqslant y, y \leqslant x, x \leqslant x, x : \neg\neg\varphi, y : \varphi, z : \varphi \Rightarrow x : \varphi, y : \bot, z : \bot, z : \neg\varphi\end{matrix}}{z \leqslant x, z \leqslant y, y \leqslant x, x \leqslant x, x : \neg\neg\varphi, y : \varphi, z : \varphi \Rightarrow x : \varphi, y : \bot, z : \bot}\,L{\to}}{z \leqslant y, y \leqslant x, x \leqslant x, x : \neg\neg\varphi, y : \varphi, z : \varphi \Rightarrow x : \varphi, y : \bot, z : \bot}\,(\mathsf{tr})}{y \leqslant x, x \leqslant x, x : \neg\neg\varphi, y : \varphi \Rightarrow x : \varphi, y : \bot, y : \neg\varphi}\,R{\to}}{y \leqslant x, x \leqslant x, x : \neg\neg\varphi, y : \varphi \Rightarrow x : \varphi, y : \bot}\,L{\to}}{x \leqslant x, x : \neg\neg\varphi \Rightarrow x : \varphi, x : \neg\varphi}\,R{\to}}{x \leqslant x, x : \neg\neg\varphi \Rightarrow x : \varphi}\,L{\to}}{x : \neg\neg\varphi \Rightarrow x : \varphi}\,(\mathsf{rf})$$

As can be seen, each application of $L{\to}$ in this loop introduces an occurrence of the formula $\neg\varphi$ in the succedent of the corresponding premise. This formula is then used in an application of $R{\to}$ in order to introduce a fresh variable, giving rise to another application of $L{\to}$, and so on. Similar loops can also be constructed by using the dynamic rules Rp or $R\bot$ instead of $R{\to}$.

In order to resolve this problem, one would have to define a suitable saturation condition that tells our algorithm to refrain from applying a dynamic rule as soon as an 'essentially identical' application of the rule already occurs in the branch. For example, in the loop given above, the variable z does not yield any new information, since everything which can be obtained from the formulas labelled with z can also be obtained from the formulas labelled with y. The desired saturation condition should be able to detect this kind of redundancy in a search tree and should block the application of dynamic rules at an appropriate stage of the construction. For an intuitionistic labelled sequent calculus, such a saturation condition has been formulated by Negri (2014, pp. 39–41). Unfortunately, this condition cannot be easily adapted to our labelled sequent calculus for InqB. The main problem lies in the complex syntax of the labels used in our system, which makes it much more difficult to detect loops of the type mentioned above and to construct the desired countermodel in an adequate way. A detailed solution to this problem is left for future work.

Chapter 4

Intuitionistic Inquisitive Logic

Recall that, in the basic inquisitive system InqB, the underlying background logic for declarative sentences was assumed to be *classical*: every truth-conditional formula of InqB behaves in essentially the same way as a formula of classical propositional logic. We will now turn to a variant of inquisitive logic in which the logic of statements is no longer classical logic, but intuitionistic logic. This system, denoted by InqI and referred to as *intuitionistic inquisitive logic*, was introduced by Ciardelli et al. (2020). Roughly speaking, InqI may be regarded as the result of adding the question-forming operator $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ to the basic system of intuitionistic propositional logic. In this chapter, we will present a cut-free labelled sequent calculus for InqI, investigate the structural properties of our proof system and establish its completeness in a proof-theoretical way.

The chapter is organized as follows. In Section 4.1, we will first provide a formal exposition of intuitionistic inquisitive logic. The language of InqI is obtained by enriching the language of basic inquisitive logic with a new primitive connective $\vee$, representing the non-inquisitive disjunction operator of the system. The formulas of InqI are evaluated with respect to ordinary intuitionistic Kripke models and the new connective $\vee$ is interpreted in the same way as the so-called *tensor disjunction* adopted in dependence logic (see, e.g., Väänänen 2007; Yang and Väänänen 2016). In Section 4.2, we will present a completeness result by Ciardelli et al. (2020) and define a sound and complete Hilbert-style system for intuitionistic inquisitive logic. Afterwards, in Section 4.3, we will turn to the description of our labelled sequent calculus for InqI. Our system will be denoted by GLinqI and can be seen as a careful modification of the sequent calculus GLinqB considered in the previous chapter. In fact, apart from a few minor changes, most of the rules of GLinqB can be easily transferred to the system GLinqI. In addition to that, GLinqI also includes a number of new rules, reflecting the semantics of the tensor disjunction $\vee$ and the properties of the accessibility relation of an intuitionistic Kripke model. In Section 4.4, we will explore the structural properties of our system. We will see that GLinqI enjoys cut-admissibility, height-preserving admissibility of weakening and contraction and height-preserving invertibility of all rules. In Section 4.5, finally, we will prove the soundness and completeness of our system. As before, the completeness will be established proof-theoretically, by exploiting the admissibility of the cut rule in GLinqI.

4.1 An Intuitionistic Variant of Inquisitive Logic

Let us first give a brief introduction to the system of intuitionistic inquisitive logic, henceforth referred to as InqI. A much more comprehensive exposition of the material is provided by Ciardelli et al. (2020). As explained above, InqI differs from the standard system of inquisitive logic in terms of the underlying background logic for declarative sentences: in InqB, the background logic was taken to be *classical logic*, whereas in InqI, it is assumed to be *intuitionistic logic*. At the syntactic level, this means that, in InqI, the non-inquisitive disjunction operator $\vee$ can no longer be viewed as a defined connective (as it was done in InqB), but has to be added to the language as a primitive operator. To make things precise, we assume again a countably infinite set P of atomic propositions, denoted by the meta-variables p, q, r, etc. The formulas of InqI are now built up from the atoms in P by means of the usual connectives of InqB and the new disjunction operator $\vee$.

Definition 4.1.1 (Language of InqI)**.** The language of InqI is denoted by $\mathcal{L}^{\mathsf{I}}$ and consists of all formulas generated by the following grammar, where p ranges over atomic propositions from P:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \mathbin{\vee\mkern-14mu\vee} \varphi \mid \varphi \vee \varphi.$$

The negation operator and the question mark operator are again treated as abbreviations, by putting $\neg\varphi := \varphi \to \bot$ and $?\varphi := \varphi \mathbin{\vee\mkern-14mu\vee} \neg\varphi$. Moreover, we will refer to $\mathbin{\vee\mkern-14mu\vee}$ as the *inquisitive disjunction* and to $\vee$ as the *standard disjunction* of InqI. As in the classical setting, the operator $\mathbin{\vee\mkern-14mu\vee}$ is used in order to form interrogative sentences within the language of our system, so $\varphi \mathbin{\vee\mkern-14mu\vee} \psi$ represents the alternative question *whether* φ *or* ψ. Conversely, any formula not containing $\mathbin{\vee\mkern-14mu\vee}$ is intended to represent a purely declarative sentence. In InqI, the underlying background logic for declarative sentences is no longer classical logic, but intuitionistic logic. For this reason, $\mathbin{\vee\mkern-14mu\vee}$-free formulas are now referred to as *standard formulas*, rather than classical formulas.

Definition 4.1.2 (Standard Formula)**.** A formula $\varphi \in \mathcal{L}^{\mathsf{I}}$ is said to be a *standard formula*, if φ contains no occurrences of $\mathbin{\vee\mkern-14mu\vee}$. The set of all standard formulas is denoted by $\mathcal{L}^{\mathsf{I}}_s$.

Throughout this chapter, standard formulas will be denoted by the meta-variables α, β, γ, etc., whereas φ, ψ, χ, etc., will be used for arbitrary formulas of InqI. The formulas of our system are evaluated with respect to ordinary intuitionistic Kripke models. As usual, any such model consists of a set of possible worlds W, a reflexive and transitive accessibility relation R between these worlds, and a valuation function V satisfying the well-known persistency requirement.

Definition 4.1.3. An *intuitionistic Kripke model* is defined to be a triple $M = \langle W, R, V\rangle$, where

(i) W is a set whose elements are referred to as *possible worlds*,

(ii) $R \subseteq W \times W$ is a *preorder*, i.e., a reflexive and transitive relation on W,[1]

(iii) $V : W \times \mathsf{P} \to \{0, 1\}$ is a valuation function satisfying the *persistency condition*: for all worlds $w, u \in W$ and for all atoms $p \in \mathsf{P}$, if $V(w, p) = 1$ and wRu, then also $V(u, p) = 1$.

Intuitively, every world in an intuitionistic Kripke model may be regarded as an incomplete state of affairs, determining certain aspects of reality and being indeterminate with respect to certain other aspects. In line with this, wRu can be interpreted as saying that u is a *refinement* of w, in the sense that everything which is determined at w is also determined at u (but not necessarily the other way around). This is also reflected by the persistency condition: if a proposition p is determinately true at some world w, then it must stay so at every refinement of w.[2]

Let us introduce some terminology. Given an intuitionistic Kripke model $M = \langle W, R, V\rangle$ and arbitrary worlds $w, u \in W$, we will say that u is a *successor* of w in M, just in case we have wRu. The *upset of a world* $w \in W$ in M is denoted by $R(w)$ and defined to be the set of all successors of w, so we put $R(w) := \{u \in W \mid wRu\}$. As usual, a set of worlds $s \subseteq W$ is also referred to as an *information state* over M. And the *upset of a state* $s \subseteq W$, notation $R(s)$, is the

[1] Strictly speaking, Ciardelli et al. (2020) also require R to be *antisymmetric* (and thus a *partial order*). However, it is easy to see that this additional requirement does not affect the set of formulas valid with respect to the support semantics given below, so the two notions of an intuitionistic Kripke model give rise to essentially the same semantics.

[2] It is well known that there is also another natural interpretation of the components of an intuitionistic Kripke model. According to this interpretation, the worlds of a model are viewed as points in time and the accessibility relation describes the temporal order of these points. If an atom p has the value 1 at some world w, then this means that p has been proved at w. And if p has value 0 at w, then this means that p has not yet been proved at w (cf. Kripke 1965).

set of all successors of the elements of s, so we define $R(s) := \bigcup_{w \in s} R(w)$. Note that, as before, every state can be thought of as representing a certain body of information, because it locates the actual world within a particular region of the logical space.[3] However, contrary to the classical setting, there are now *two* ways in which a state s may be strengthened: on the one hand, s may become more informative by eliminating some worlds from s, thus reducing the number of possible candidates for the actual world. On the other hand, s may also become more informative by replacing some of the worlds in s by other worlds that are more *refined* with respect to the preorder R. Hence, it is natural to define the notion of an enhancement in the following way.

Definition 4.1.4 (Enhancement)**.** Let $M = \langle W, R, V \rangle$ be an intuitionistic Kripke model and let $s, t \subseteq W$ be states. We say that s is an *enhancement* of t in M, just in case $s \subseteq R(t)$.

As shown by the following proposition, one can also describe the reflexivity and transitivity of an accessibility relation in terms of *states*, rather than *worlds*. This will be useful later on, when we will translate the properties of an accessibility relation into corresponding sequent rules.

Proposition 4.1.5. *Let W be a set of worlds and let $R \subseteq W \times W$ be a binary relation on W.*
(i) R is reflexive if and only if, for every state $s \subseteq W$, it holds $s \subseteq R(s)$.
(ii) R is transitive if and only if, for all states $s, t \subseteq W$, if $s \subseteq R(t)$, then $R(s) \subseteq R(t)$.

Proof. We only prove part (ii), the other part is trivial. For the left-to-right direction, suppose that R is transitive. Moreover, let $s, t \subseteq W$ be arbitrary states and assume that $s \subseteq R(t)$. We need to show that it holds $R(s) \subseteq R(t)$. To this end, let $u \in R(s)$ be an arbitrary world. Then, by definition of $R(s)$, there exists some world $v \in s$ such that vRu. Since $v \in s$ and $s \subseteq R(t)$, we also have $v \in R(t)$, so there must also be a world $w \in t$ with wRv. Now, from wRv and vRu, we may conclude wRu by the transitivity of R. But then, because $w \in t$, it follows $u \in R(t)$. Since u was an arbitrary world with $u \in R(s)$, this shows that $R(s) \subseteq R(t)$, as desired.

For the right-to-left direction, suppose that, for all states $s, t \subseteq W$, it is the case that, if $s \subseteq R(t)$, then $R(s) \subseteq R(t)$. Let $u, v, w \in W$ be arbitrary worlds and assume that it holds uRv and vRw. We have to show that uRw. For this purpose, let $s, t \subseteq W$ be the singleton states given by $s := \{v\}$ and $t := \{u\}$. Then, clearly, since uRv, we have $s \subseteq R(t)$, so it follows $R(s) \subseteq R(t)$ by assumption. On the other hand, from vRw, we may also conclude $w \in R(s)$. Together with $R(s) \subseteq R(t)$, this implies $w \in R(t)$. But then, because $t = \{u\}$, we also have uRw. □

We can now give an inductive definition of the support conditions for all formulas of InqI.

Definition 4.1.6 (Support Semantics for InqI)**.** Let $M = \langle W, R, V \rangle$ be an intuitionistic Kripke model. The support relation $\models$ between states and formulas is inductively defined as follows:
(i) $M, s \models p \quad :\Leftrightarrow \quad V(w, p) = 1$ for all $w \in s$,
(ii) $M, s \models \bot \quad :\Leftrightarrow \quad s = \emptyset$,
(iii) $M, s \models \varphi \wedge \psi \quad :\Leftrightarrow \quad M, s \models \varphi$ and $M, s \models \psi$,
(iv) $M, s \models \varphi \to \psi \quad :\Leftrightarrow$ for all $t \subseteq R(s)$, if $M, t \models \varphi$, then $M, t \models \psi$,
(v) $M, s \models \varphi \mathbin{\mathbb{V}} \psi \quad :\Leftrightarrow \quad M, s \models \varphi$ or $M, s \models \psi$,
(vi) $M, s \models \varphi \vee \psi \quad :\Leftrightarrow$ there are $t_1, t_2 \subseteq W$ such that $s = t_1 \cup t_2$, $M, t_1 \models \varphi$ and $M, t_2 \models \psi$.

If $M, s \models \varphi$ holds, then we say that φ is *supported* by the state s in M. Note that, for atomic formulas and for the logical symbols $\bot$, $\wedge$ and $\mathbin{\mathbb{V}}$, the support clauses are exactly the same as in the basic system InqB (see Definition 1.2.6). The support clause for implication, on the other hand, has now been reformulated in order to account for the different notion of an enhancement in intuitionistic inquisitive logic: whereas in InqB, we only had to consider subsets $t \subseteq s$, we

[3] More precisely, a state $s \subseteq W$ conveys the information that the actual world corresponds to one of the states of affairs in s and that all states of affairs not contained in s are ruled out as possible candidates for the actual world.

now have to consider subsets $t \subseteq R(s)$. A genuinely new component is the support clause for the standard disjunction $\vee$, which reads as follows: a formula $\varphi \vee \psi$ is supported by a state s, just in case it is possible to divide the worlds in s into two substates t_1 and t_2 such that φ is supported by t_1 and ψ is supported by t_2. This corresponds to the semantics of the so-called *tensor disjunction* adopted in dependence logic, which in turn resembles the semantics of tensor in linear logic (cf. Väänänen 2007; Yang 2014; Yang and Väänänen 2016; Ciardelli 2016a).

As in the classical setting, entailment in InqI is simply defined as preservation of support. That is, given any set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{I}}$, we will write $\Gamma \models \varphi$ and say that φ is *entailed* by Γ, if for every intuitionistic Kripke model M and for every state s over M, it is the case that $M, s \models \Gamma$ implies $M, s \models \varphi$ (where, as usual, $M, s \models \Gamma$ is used as an abbreviation for '$M, s \models \psi$ for all $\psi \in \Gamma$'). We will say that two formulas $\varphi, \psi \in \mathcal{L}^{\mathsf{I}}$ are *equivalent*, notation $\varphi \equiv \psi$, if these formulas mutually entail each other, i.e., if we have both $\varphi \models \psi$ and $\psi \models \varphi$. And a formula $\varphi \in \mathcal{L}^{\mathsf{I}}$ is said to be *valid* in InqI, if it is supported by every state over every model, i.e., if for every intuitionistic Kripke model M and for each state s, we have $M, s \models \varphi$ (cf. Ciardelli et al. 2020).

We now want to point out some interesting properties of InqI. First of all, it is easy to show that support in InqI is again *persistent*: if a formula φ is supported by a state s, then it is also supported by every enhancement of this state. Importantly, this now also holds for the refined notion of an enhancement given by Definition 4.1.4. In addition, we also have the usual *empty state property*, familiar from the basic system: in InqI, every formula is supported by the inconsistent state $\emptyset$.

Proposition 4.1.7. *Let M be an intuitionistic Kripke model, let s and t be states and let $\varphi \in \mathcal{L}^{\mathsf{I}}$.*

(i) Persistency: if $M, s \models \varphi$ and $t \subseteq R(s)$, then $M, t \models \varphi$.

(ii) Empty state property: $M, \emptyset \models \varphi$.

As before, both statements are proved by induction on φ. The notion of truth at a possible world is defined in exactly the same way as in the classical setting (see Definition 1.2.8). That is, given any formula $\varphi \in \mathcal{L}^{\mathsf{I}}$ and a world w in a model M, we will write $M, w \models \varphi$ and say that φ is *true* at w in M, just in case φ is supported by the singleton state $\{w\}$. Moreover, a formula φ is said to be *truth-conditional*, if for every model M and for every state s over M, we have $M, s \models \varphi$ if and only if $M, w \models \varphi$ for all worlds $w \in s$. In other words, a formula is truth-conditional, if its support conditions are completely determined by its truth conditions, in the sense that support at a state simply comes down to truth at every world in the state. As usual in inquisitive logic, truth-conditionality is again taken to be the fundamental semantic difference between declarative and interrogative sentences: a formula φ represents an assertion, if it is truth-conditional, and it represents a question otherwise. As anticipated above, it is now possible to show that standard formulas are always truth-conditional and therefore purely declarative in InqI.

Proposition 4.1.8. *Every standard formula $\alpha \in \mathcal{L}^{\mathsf{I}}_s$ is truth-conditional.*[4]

Proof. By induction on the structure of α. Most cases are treated in essentially the same way as in the classical setting (see the proof of Proposition 1.3.3). Thus, we only need to consider the case in which α is of the form $\alpha = \beta \vee \gamma$ for some $\beta, \gamma \in \mathcal{L}^{\mathsf{I}}_s$. Let $M = \langle W, R, V\rangle$ be an arbitrary intuitionistic Kripke model and let s be an arbitrary state. We have to show that $M, s \models \beta \vee \gamma$ if and only if $M, w \models \beta \vee \gamma$ for all $w \in s$. The left-to-right direction follows directly from the persistency of support and from the fact that, for all $w \in s$, we have $\{w\} \subseteq R(s)$ by the reflexivity of R. For the right-to-left direction, suppose that we have $M, w \models \beta \vee \gamma$ for all $w \in s$. Using the semantics of $\vee$, one readily sees that this yields $M, w \models \beta$ or $M, w \models \gamma$ for all $w \in s$. Let now $t_1, t_2 \subseteq s$ be given by $t_1 := \{w \in s \mid M, w \models \beta\}$ and $t_2 := s \setminus t_1$. Then, clearly, we must have $M, w \models \beta$ for

[4] As in our treatment of InqB, it is actually possible to extend this proposition to the class of those formulas in which $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ occurs only in the antecedent of an implication (one might refer to such a formula as an 'extended Harrop formula'). However, for our purposes, it is sufficient to prove the truth-conditionality of standard formulas only.

all $w \in t_1$, and $M, u \models \gamma$ for all $u \in t_2$. Therefore, by induction hypothesis, it follows $M, t_1 \models \beta$ and $M, t_2 \models \gamma$. Since $s = t_1 \cup t_2$, this implies $M, s \models \beta \vee \gamma$ by the semantics of $\vee$. □

Using this result, one can now also prove that, for standard formulas, the truth conditions determined by the support semantics of InqI are simply the familiar ones from intuitionistic logic.

Proposition 4.1.9 (Truth Conditions for Standard Formulas)**.** *Let M be an intuitionistic Kripke model, let w be a world in M and let $\alpha, \beta \in \mathcal{L}^{\mathsf{I}}_s$ be standard formulas. We have:*

(i) $M, w \models p \;\; \Leftrightarrow \;\; V(w, p) = 1$,
(ii) $M, w \not\models \bot$,
(iii) $M, w \models \alpha \wedge \beta \;\; \Leftrightarrow \;\; M, w \models \alpha$ *and* $M, w \models \beta$,
(iv) $M, w \models \alpha \to \beta \;\; \Leftrightarrow \;\;$ *for all* $u \in R(w)$*, if* $M, u \models \alpha$*, then* $M, u \models \beta$,
(v) $M, w \models \alpha \vee \beta \;\; \Leftrightarrow \;\; M, w \models \alpha$ *or* $M, w \models \beta$.

Proof. We only prove part (iv), the other parts are trivial. For the left-to-right direction, suppose that $M, w \models \alpha \to \beta$. Let $u \in R(w)$ be arbitrary such that $M, u \models \alpha$. Then, since $\{u\} \subseteq R(w)$ is an enhancement of $\{w\}$, it follows $M, u \models \beta$ by assumption. For the right-to-left direction, suppose that, for all worlds $u \in R(w)$, it is the case that $M, u \models \alpha$ implies $M, u \models \beta$. Let $s \subseteq R(w)$ be an arbitrary enhancement of $\{w\}$ such that $M, s \models \alpha$. Then, by Proposition 4.1.8, we must have $M, u \models \alpha$ for all $u \in s$. Using the assumption and the fact that $s \subseteq R(w)$, this yields $M, u \models \beta$ for all $u \in s$. But then, by Proposition 4.1.8, we may conclude $M, s \models \beta$. Since $s \subseteq R(w)$ was an arbitrary enhancement with $M, s \models \alpha$, this shows that $M, w \models \alpha \to \beta$, as desired. □

Thus, putting things together, it follows that standard formulas do indeed behave 'intuitionistically' in InqI: they are always truth-conditional, so their support conditions are completely determined by their truth conditions; and these truth conditions are simply the ordinary ones from intuitionistic Kripke semantics. In other words, a standard formula α is supported by an information state s if and only if α is intuitionistically true at every world in s. Using this fact, it is easy to show that InqI is a *conservative extension* of intuitionistic logic: if we restrict ourselves to standard formulas, then entailment in InqI simply amounts to intuitionistic entailment.

Proposition 4.1.10 (Conservativity over Intuitionistic Logic)**.** *Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}^{\mathsf{I}}_s$ be a set of standard formulas. We have $\Gamma \models \alpha$ if and only if α is entailed by Γ in intuitionistic propositional logic.*

Hence, with respect to standard formulas, InqI has exactly the same expressive power as ordinary intuitionistic logic. One might now ask whether this can be generalized to all truth-conditional formulas of InqI. In other words, does the presence of questions in the language allow us to express new truth-conditional meanings, going beyond the expressivity of intuitionistic logic? As shown by Ciardelli et al. (2020, p. 101), the answer is negative: for every truth-conditional formula of InqI, there exists an equivalent standard formula. Consequently, standard formulas are in fact representative of all truth-conditional meanings expressible in InqI.

Proposition 4.1.11. *A formula $\varphi \in \mathcal{L}^{\mathsf{I}}$ is truth-conditional if and only if there exists a standard formula $\alpha \in \mathcal{L}^{\mathsf{I}}_s$ such that $\varphi \equiv \alpha$.*

Finally, one can show that the inquisitive disjunction still validates the usual split property and the split equivalence familiar from our treatment of InqB (see Propositions 1.4.1 and 1.4.2).

Proposition 4.1.12 (Split Property, Split Equivalence)**.** *Let $\Gamma \subseteq \mathcal{L}^{\mathsf{I}}$ be a set of truth-conditional formulas, let $\alpha \in \mathcal{L}^{\mathsf{I}}$ be a truth-conditional formula and let $\varphi, \psi \in \mathcal{L}^{\mathsf{I}}$ be arbitrary formulas.*

(i) Split property: $\Gamma \models \varphi \mathbin{\vee\!\!\!\vee} \psi$ *if and only if* $\Gamma \models \varphi$ *or* $\Gamma \models \psi$.
(ii) Split equivalence: $\alpha \to (\varphi \mathbin{\vee\!\!\!\vee} \psi) \equiv (\alpha \to \varphi) \mathbin{\vee\!\!\!\vee} (\alpha \to \psi)$.

$$\frac{\alpha \to (\varphi \mathbin{\mathbb{V}} \psi)}{(\alpha \to \varphi) \mathbin{\mathbb{V}} (\alpha \to \psi)}\ \text{(split)} \qquad \frac{\varphi \vee \psi}{\psi \vee \varphi}\ \text{(com)} \qquad \frac{\varphi \vee (\psi \mathbin{\mathbb{V}} \chi)}{(\varphi \vee \psi) \mathbin{\mathbb{V}} (\varphi \vee \chi)}\ \text{(dis)}$$

$$\frac{\varphi_i}{\varphi_1 \vee \varphi_2}\ \vee I \quad (i = 1, 2) \qquad \frac{\varphi \vee \psi \quad \begin{array}{c}[\varphi]\\ \vdots\\ \alpha\end{array} \quad \begin{array}{c}[\psi]\\ \vdots\\ \alpha\end{array}}{\alpha}\ \vee E \qquad \frac{\varphi \vee \psi \quad \begin{array}{c}[\varphi]\\ \vdots\\ \varphi'\end{array} \quad \begin{array}{c}[\psi]\\ \vdots\\ \psi'\end{array}}{\varphi' \vee \psi'}\ \text{(ex)}$$

Figure 4.1: Special rules of the natural deduction system NinqI described by Ciardelli et al. (2020). In the rules $\vee E$ and (split), we require α to be a standard formula, i.e., $\alpha \in \mathcal{L}^!_s$.

4.2 Standard Axiomatizations of InqI

In their paper, Ciardelli et al. (2020) provide a sound and complete natural deduction system for InqI. Their proof system, henceforth denoted by NinqI, consists of the usual rules of the 'intuitionistic' base calculus given in Figure 1.3, and the special rules for the new connective $\vee$ depicted in Figure 4.1. Let us give a brief explanation of the special rules of this system. First of all, the rules $\vee I$ and $\vee E$ are very similar to the ordinary rules for disjunction in classical and intuitionistic logic. However, in the elimination rule $\vee E$, the conclusion α is now required to be a *standard formula*. This restriction is necessary in order to make sure that NinqI is sound with respect to InqI. In particular, an unrestricted elimination rule for $\vee$ would allow us to derive standard formulas that are not intuitionistically valid, in contradiction to Proposition 4.1.10.[5] Due to the restriction on the elimination rule for $\vee$, it is now necessary to include additional rules, accounting for those properties of $\vee$ that are not derivable in terms of $\vee I$ and $\vee E$. The rule (com) accounts for the commutativity of $\vee$, and (dis) accounts for the fact that standard disjunctions distribute over inquisitive disjunctions. The rule (ex), finally, allows to replace each disjunct in a standard disjunction by some other formula derivable from the disjunct. In addition to the special rules for $\vee$, the system also includes the usual split rule (split) familiar from the standard natural deduction system for InqB. The double negation rule (dne), however, is now excluded from the system: as before, this is necessary in order to make sure that NinqI is sound with respect to InqI.

Definition 4.2.1 (The System NinqI)**.** We define NinqI to be the natural deduction system comprising each of the rules presented in Figure 1.3, together with the special rules from Figure 4.1.

In what follows, we will write $\vdash_{\mathsf{N}}$ for the provability relation of NinqI. The soundness of the system is established by an easy induction on the structure of a derivation in NinqI. In order to prove the completeness of NinqI, one can first generalize the definition of resolutions and the normal form result for InqB (see Definition 1.4.3 and Proposition 1.4.4) to the extended language of InqI. The completeness of NinqI is then established in very much the same way as in the classical setting (see Theorem 1.5.2). For further details, we refer to Ciardelli et al. (2020, pp. 102–103).

Theorem 4.2.2 (Soundness and Completeness)**.** *The system* NinqI *is sound and complete with respect to* InqI. *That is, for every* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^!$*, we have:* $\Gamma \vdash_{\mathsf{N}} \varphi$ *if and only if* $\Gamma \models \varphi$*.*

For our purposes, it will be more convenient to have a Hilbert-style system for InqI, rather than a natural deduction system. As in the case of InqB, such a Hilbert-style system can be easily obtained by converting each of the natural deduction rules of NinqI into corresponding axiom

[5] A concrete example is the Kreisel-Putnam axiom $(\neg p \to (q \vee r)) \to ((\neg p \to q) \vee (\neg p \to r))$, which is known to be invalid in intuitionistic logic. By dropping the restriction on $\vee E$, this formula would become provable in NinqI.

Axioms:

(IPL)	All axioms of the 'intuitionistic' system given in Figure 1.5,
(Intro)	$\varphi \to (\varphi \vee \psi)$ and $\psi \to (\varphi \vee \psi)$,
(Elim)	$(\varphi \to \alpha) \to ((\psi \to \alpha) \to ((\varphi \vee \psi) \to \alpha))$, where α is a standard formula,
(Com)	$(\varphi \vee \psi) \to (\psi \vee \varphi)$,
(Dis)	$(\varphi \vee (\psi \mathbin{\vee\!\!\!\vee} \chi)) \to ((\varphi \vee \psi) \mathbin{\vee\!\!\!\vee} (\varphi \vee \chi))$,
(Ex)	$(\varphi \to \varphi') \to ((\psi \to \psi') \to ((\varphi \vee \psi) \to (\varphi' \vee \psi')))$,
(Split)	$(\alpha \to (\varphi \mathbin{\vee\!\!\!\vee} \psi)) \to ((\alpha \to \varphi) \mathbin{\vee\!\!\!\vee} (\alpha \to \psi))$, where α is a standard formula.

The only rule of inference is *modus ponens*: from $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, infer $\Gamma \vdash \psi$.

Figure 4.2: The Hilbert-style system HinqI.

schemes. The resulting proof system, henceforth referred to as HinqI, is presented in Figure 4.2. The provability relation of the system HinqI is denoted by $\vdash_{\mathsf{H}}$ and inductively defined in the usual way. That is, given any set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{I}}$, we will write $\Gamma \vdash_{\mathsf{H}} \varphi$ and say that φ is *provable* from Γ in HinqI, if φ is either an element of Γ, or φ is an axiom of HinqI, or there exists some formula $\psi \in \mathcal{L}^{\mathsf{I}}$ such that we have both $\Gamma \vdash_{\mathsf{H}} \psi$ and $\Gamma \vdash_{\mathsf{H}} \psi \to \varphi$ (in the last case, we also say that $\Gamma \vdash_{\mathsf{H}} \varphi$ is obtained from $\Gamma \vdash_{\mathsf{H}} \psi$ and $\Gamma \vdash_{\mathsf{H}} \psi \to \varphi$ by an application of modus ponens). Using induction on the definition of $\Gamma \vdash_{\mathsf{H}} \varphi$, it is again easy to show that the provability relation $\vdash_{\mathsf{H}}$ is *monotonic*: if we have $\Gamma \vdash_{\mathsf{H}} \varphi$ and $\Gamma \subseteq \Delta$, then also $\Delta \vdash_{\mathsf{H}} \varphi$. In order to establish the soundness and completeness of our Hilbert-style system, we will now prove that HinqI is equivalent to the natural deduction system NinqI, in the sense that everything provable in NinqI is also provable in HinqI and vice versa. To this end, one first has to establish the *deduction theorem* for HinqI.

Theorem 4.2.3 (Deduction Theorem)**.** *In* HinqI*, we have* $\Gamma, \varphi \vdash_{\mathsf{H}} \psi$ *if and only if* $\Gamma \vdash_{\mathsf{H}} \varphi \to \psi$*.*

The proof works in exactly the same way as in the classical setting (see Theorem 1.5.4). That is, the left-to-right direction is established by a straightforward induction on the definition of $\Gamma, \varphi \vdash_{\mathsf{H}} \psi$. For the right-to-left direction, one uses the fact that, from $\Gamma \vdash_{\mathsf{H}} \varphi \to \psi$, it follows $\Gamma, \varphi \vdash_{\mathsf{H}} \varphi \to \psi$ by the monotonicity of $\vdash_{\mathsf{H}}$. Together with $\Gamma, \varphi \vdash_{\mathsf{H}} \varphi$, this yields $\Gamma, \varphi \vdash_{\mathsf{H}} \psi$ by an application of modus ponens. Using the deduction theorem, it is now easy to show that our Hilbert-style system is in fact equivalent to the natural deduction system provided by Ciardelli et al. (2020). The soundness and completeness of HinqI then follows as an immediate corollary.

Theorem 4.2.4. *Let* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{I}}$ *be a set of formulas. We have* $\Gamma \vdash_{\mathsf{H}} \varphi$ *in the Hilbert-style system* HinqI *if and only if* $\Gamma \vdash_{\mathsf{N}} \varphi$ *holds in the natural deduction system* NinqI*.*

Proof. The left-to-right direction is proved by induction on the definition of $\Gamma \vdash_{\mathsf{H}} \varphi$. This is straightforward, since all axioms of HinqI are obviously derivable in NinqI and modus ponens corresponds to $\to E$. For the right-to-left direction, one proceeds by induction on a natural deduction proof for $\Gamma \vdash_{\mathsf{N}} \varphi$. This is also not difficult, since most of the natural deduction rules correspond directly to some axiom of HinqI and the discharging of hypotheses can be 'simulated' using the deduction theorem for HinqI. For further details, see the proof of Theorem 1.5.5. □

Corollary 4.2.5 (Soundness and Completeness)**.** *The system* HinqI *is sound and complete with respect to* InqI*. That is, for every* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{I}}$*, we have:* $\Gamma \vdash_{\mathsf{H}} \varphi$ *if and only if* $\Gamma \vDash \varphi$*.*

Proof. The statement follows immediately from Theorem 4.2.2 and Theorem 4.2.4. □

4.3 The Sequent Calculus GLinqI

We are now ready to introduce our labelled sequent calculus for InqI. Our proof system will be denoted by GLinqI and can be seen as a slight modification of the labelled sequent calculus GLinqB described in Section 3.1. As before, we will assume two countably infinite sets of *state variables*, denoted by $\mathfrak{S}$ and $\mathfrak{V}$, respectively. The variables in $\mathfrak{S}$ are used for *singleton states* and the variables in $\mathfrak{V}$ are used for *arbitrarily large* information states. In order to avoid confusion, we will again use the meta-variables u, v, w, etc., for variables from $\mathfrak{S}$, and the meta-variables x, y, z, etc., for variables from $\mathfrak{V}$. The *labels* of our system are now defined in the following way.

Definition 4.3.1 (Labels)**.** The set of *labels* is denoted by $\Lambda(\mathfrak{S}, \mathfrak{V})$ and consists of all expressions generated by the following grammar, where $w \in \mathfrak{S}$ and $x \in \mathfrak{V}$ are arbitrary state variables:

$$\pi ::= w \mid x \mid \emptyset \mid \pi \cdot \pi \mid \pi + \pi \mid R(\pi).$$

As in our labelled sequent calculus for InqB, labels will be denoted by the meta-variables π, σ, τ, etc. Intuitively, every label can be seen as a description of an information state. So, in particular, $\pi \cdot \sigma$ stands for the *intersection* and $\pi + \sigma$ stands for the *union* of the states represented by π and σ. A label of the form $R(\pi)$, on the other hand, is intended to denote the *upset* of the state described by π, i.e., the set of all successors of the worlds in π (see Section 4.1). Following the convention adopted in the previous chapter, we will also write $\pi\sigma$ as a shorthand for $\pi \cdot \sigma$.

Let us recall some terminology. By a *relational atom*, we will mean an expression of the form $\pi \leqslant \sigma$, where $\pi, \sigma \in \Lambda(\mathfrak{S}, \mathfrak{V})$ are arbitrary labels. A *labelled formula*, on the other hand, is defined to be an expression of the form $\pi : \varphi$, where $\pi \in \Lambda(\mathfrak{S}, \mathfrak{V})$ is a label and $\varphi \in \mathcal{L}^{\mathsf{I}}$ is a formula. Relational atoms and labelled formulas are interpreted in the usual way. That is, $\pi \leqslant \sigma$ stands for the statement 'π is a subset of σ' and $\pi : \varphi$ stands for the statement 'φ is supported by π'. By a *sequent*, we will mean any expression of the form $\Gamma \Rightarrow \Delta$, where Γ is a finite multiset containing labelled formulas and relational atoms, and Δ is a finite multiset containing only labelled formulas (but no relational atoms). The intended meaning of a sequent is the same as in our treatment of InqB. Thus, intuitively, $\Gamma \Rightarrow \Delta$ is considered to be 'valid', if it is the case that, whenever all expressions in Γ are 'satisfied', then at least one of the expressions in Δ is 'satisfied'. Given any sequent $\Gamma \Rightarrow \Delta$, we will also call Γ the *antecedent* and Δ the *succedent* of the sequent.

Our labelled sequent calculus for InqI is presented in Figure 4.3. As can be seen, most of the rules of our system are very similar to the corresponding rules of the sequent calculus GLinqB discussed in the previous chapter. However, there are also some important differences. First of all, in the rules Lp and Rp, each of the relational atoms $w \leqslant \pi$ is now replaced by a relational atom of the form $w \leqslant R(\pi)$. This change has been made in order to account for the 'internal' persistency associated with the preorder R of a Kripke model (as opposed to the 'external' persistency of the support relation of InqI). That is, an atomic formula p is supported by a state π if and only if p is true at every world in the upset $R(\pi)$ of this state. A similar modification has also been adopted for the rules $L\bot$ and $R\bot$. The rules for implication have been slightly reformulated in order to accommodate the refined notion of an *enhancement* in intuitionistic inquisitive logic: whereas in InqB, an enhancement of a state s was simply defined to be a subset $t \subseteq s$, it is now defined to be a subset $t \subseteq R(s)$ (see Definition 4.1.4). In addition to that, our sequent calculus now also includes the new rules $L\vee$ and $R\vee$, which mirror the support conditions for standard disjunctions: a formula $\varphi \vee \psi$ is supported by a state π, just in case π can be divided into two subsets such that the first subset supports φ and the second subset supports ψ. Note that, in applications of $L\vee$, we require x and y to be fresh variables not occurring in the conclusion of the rule.

The order rules of our system are now divided into two groups, referred to as *internal order rules* and *external order rules*, respectively. The internal order rules are used in order to formalize

Axioms:

$$\frac{}{w : p, \Gamma \Rightarrow \Delta, w : p}\,Ax \qquad \frac{}{w : \bot, \Gamma \Rightarrow \Delta}\,Ax^{\bot} \qquad \frac{}{w \leqslant \emptyset, \Gamma \Rightarrow \Delta}\,Ax^{\emptyset}$$

Logical Rules:

$$\frac{w : p, w \leqslant R(\pi), \pi : p, \Gamma \Rightarrow \Delta}{w \leqslant R(\pi), \pi : p, \Gamma \Rightarrow \Delta}\,Lp \qquad \frac{w \leqslant R(\pi), \Gamma \Rightarrow \Delta, w : p}{\Gamma \Rightarrow \Delta, \pi : p}\,Rp$$

$$\frac{w : \bot, w \leqslant R(\pi), \pi : \bot, \Gamma \Rightarrow \Delta}{w \leqslant R(\pi), \pi : \bot, \Gamma \Rightarrow \Delta}\,L\bot \qquad \frac{w \leqslant R(\pi), \Gamma \Rightarrow \Delta, w : \bot}{\Gamma \Rightarrow \Delta, \pi : \bot}\,R\bot$$

$$\frac{\pi : \varphi, \pi : \psi, \Gamma \Rightarrow \Delta}{\pi : \varphi \wedge \psi, \Gamma \Rightarrow \Delta}\,L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, \pi : \varphi \qquad \Gamma \Rightarrow \Delta, \pi : \psi}{\Gamma \Rightarrow \Delta, \pi : \varphi \wedge \psi}\,R\wedge$$

$$\frac{\pi : \varphi, \Gamma \Rightarrow \Delta \qquad \pi : \psi, \Gamma \Rightarrow \Delta}{\pi : \varphi \mathbin{\rotatebox[origin=c]{0}{$\mathbb{V}$}} \psi, \Gamma \Rightarrow \Delta}\,L\mathbb{V} \qquad \frac{\Gamma \Rightarrow \Delta, \pi : \varphi, \pi : \psi}{\Gamma \Rightarrow \Delta, \pi : \varphi \mathbb{V} \psi}\,R\mathbb{V}$$

$$\frac{\pi \leqslant R(\sigma), \sigma : \varphi \to \psi, \Gamma \Rightarrow \Delta, \pi : \varphi \qquad \pi \leqslant R(\sigma), \sigma : \varphi \to \psi, \pi : \psi, \Gamma \Rightarrow \Delta}{\pi \leqslant R(\sigma), \sigma : \varphi \to \psi, \Gamma \Rightarrow \Delta}\,L\to$$

$$\frac{x \leqslant R(\pi), x : \varphi, \Gamma \Rightarrow \Delta, x : \psi}{\Gamma \Rightarrow \Delta, \pi : \varphi \to \psi}\,R\to \qquad \frac{\pi \leqslant x + y, x : \varphi, y : \psi, \Gamma \Rightarrow \Delta}{\pi : \varphi \vee \psi, \Gamma \Rightarrow \Delta}\,L\vee$$

$$\frac{\pi \leqslant \sigma + \tau, \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi, \sigma : \varphi \qquad \pi \leqslant \sigma + \tau, \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi, \tau : \psi}{\pi \leqslant \sigma + \tau, \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi}\,R\vee$$

Internal Order Rules:

$$\frac{\pi \leqslant R(\pi), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textsf{(r-rf)} \qquad \frac{R(\pi) \leqslant R(\sigma), \pi \leqslant R(\sigma), \Gamma \Rightarrow \Delta}{\pi \leqslant R(\sigma), \Gamma \Rightarrow \Delta}\,\textsf{(r-tr)}$$

$$\frac{R(\pi + \sigma) \leqslant R(\pi) + R(\sigma), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textsf{(r-dis)} \qquad \frac{R(\emptyset) \leqslant \emptyset, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textsf{(r-emp)}$$

External Order Rules:

$$\frac{\pi \leqslant \tau, \pi \leqslant \sigma, \sigma \leqslant \tau, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \sigma \leqslant \tau, \Gamma \Rightarrow \Delta}\,\textsf{(tr)} \qquad \frac{\pi \leqslant \pi\sigma + \pi\tau, \pi \leqslant \sigma + \tau, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma + \tau, \Gamma \Rightarrow \Delta}\,\textsf{(dis)}$$

$$\frac{\pi \leqslant \sigma\tau, \pi \leqslant \sigma, \pi \leqslant \tau, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \pi \leqslant \tau, \Gamma \Rightarrow \Delta}\,\textsf{(in)} \qquad \frac{\pi + \tau \leqslant \sigma, \pi \leqslant \sigma, \tau \leqslant \sigma, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \tau \leqslant \sigma, \Gamma \Rightarrow \Delta}\,\textsf{(un)}$$

$$\frac{\pi\sigma \leqslant \pi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textsf{(il)} \qquad \frac{\sigma\pi \leqslant \pi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textsf{(ir)} \qquad \frac{\pi \leqslant \pi + \sigma, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textsf{(ul)}$$

$$\frac{\pi \leqslant \sigma + \pi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textsf{(ur)} \qquad \frac{\pi \leqslant \emptyset, \pi \leqslant w, \Gamma \Rightarrow \Delta \qquad w \leqslant \pi, \pi \leqslant w, \Gamma \Rightarrow \Delta}{\pi \leqslant w, \Gamma \Rightarrow \Delta}\,\textsf{(sg)}$$

$$\frac{\pi \leqslant \pi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textsf{(rf)} \qquad \frac{w \leqslant \pi, w \leqslant \pi + \sigma, \Gamma \Rightarrow \Delta \qquad w \leqslant \sigma, w \leqslant \pi + \sigma, \Gamma \Rightarrow \Delta}{w \leqslant \pi + \sigma, \Gamma \Rightarrow \Delta}\,\textsf{(cd)}$$

Figure 4.3: The system GLinqI. As usual, w ranges over variables from $\mathfrak{S}$, x and y range over variables from $\mathfrak{V}$, and π, σ, τ, etc., stand for arbitrary labels. In the rules Rp and $R\bot$, w must be a fresh variable and π must be a non-singleton label, i.e., $\pi \notin \mathfrak{S}$. Similarly, in $R\!\to$, x must be fresh. In applications of $L\vee$, we require x and y to be fresh and distinct from each other.

the model-internal properties of the preorder R of an intuitionistic Kripke model. In particular, the rules (r-rf) and (r-tr) account for the state-based characterization of *reflexivity* and *transitivity* provided in Proposition 4.1.5. And the order rules (r-emp) and (r-dis) reflect the observation that we have $R(\emptyset) = \emptyset$ and $R(\pi \cup \sigma) = R(\pi) \cup R(\sigma)$, for all states π and σ.[6] The external order rules, finally, are simply the usual order rules familiar from our labelled sequent calculus for InqB.

Definition 4.3.2 (The System GLinqI)**.** We define GLinqI to be the sequent calculus depicted in Figure 4.3. A sequent is *derivable* in GLinqI, if there exists a proof tree ending with this sequent in GLinqI. Given any finite subset $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{I}}$, we say that φ is *provable* from Γ in GLinqI, if for some (or, in fact, any) variable $x \in \mathfrak{V}$, the sequent $x : \Gamma \Rightarrow x : \varphi$ is derivable in GLinqI.[7]

In order to make sure that contraction on relational atoms is admissible in our system, we will again assume the *closure condition* discussed in the previous chapter: if an instance of an order rule produces a duplication of relational atoms in the conclusion of the rule, then also the contracted instance of the rule is added to our system (a more detailed explanation of the closure condition is provided in Section 3.1). To end this section, let us briefly recall some basic vocabulary. First of all, the fresh variables involved in applications of the rules Rp, $R\bot$, $R{\to}$ and $L\vee$ are again referred to as the *eigenvariables* of these rules.[8] Furthermore, in each of the axioms and rules depicted in Figure 4.3, the multiset Γ is referred to as the *left context* and the multiset Δ is referred to as the *right context*. In an instance of an axiom or in the conclusion of a rule of inference, all expressions not belonging to the context are said to be *principal*. The corresponding expressions in the premises of a rule are called *active*. So, for example, in an instance of $L\vee$ with premise $\pi \leqslant x+y, x : \varphi, y : \psi, \Gamma \Rightarrow \Delta$ and conclusion $\pi : \varphi \vee \psi, \Gamma \Rightarrow \Delta$, the labelled formula $\pi : \varphi\vee\psi$ is principal and each of the expressions $\pi \leqslant x+y$, $x : \varphi$ and $y : \psi$ is active. On the other hand, in an application of (r-tr) with premise $R(\pi) \leqslant R(\sigma), \pi \leqslant R(\sigma), \Gamma \Rightarrow \Delta$ and conclusion $\pi \leqslant R(\sigma), \Gamma \Rightarrow \Delta$, the relational atom $\pi \leqslant R(\sigma)$ is principal, whereas both $R(\pi) \leqslant R(\sigma)$ and $\pi \leqslant R(\sigma)$ are active.

4.4 Basic Properties of GLinqI

We will now point out some important features of our sequent calculus. As in the classical setting, we will see that GLinqI enjoys cut-admissibility, height-preserving invertibility of all rules and height-preserving admissibility of weakening and contraction. To begin with, we show the derivability of generalized initial sequents for our system. Note that, intuitively, the first sequent in the following lemma also reflects the more general notion of *persistency* in InqI: if φ is supported by a state σ and if π is a subset of $R(\sigma)$, then φ must also be supported by π (see Proposition 4.1.7).

Lemma 4.4.1. *All sequents of the following form are derivable in* GLinqI*:*

(i) $\pi \leqslant R(\sigma), \sigma : \varphi, \Gamma \Rightarrow \Delta, \pi : \varphi,$

(ii) $\pi \leqslant \sigma, \sigma : \varphi, \Gamma \Rightarrow \Delta, \pi : \varphi,$

(iii) $\pi : \varphi, \Gamma \Rightarrow \Delta, \pi : \varphi.$

Proof. The derivability of (i) is established by induction on the structure of φ. For the base case, let us suppose that $\varphi = p$ is atomic. If it holds $\pi \notin \mathfrak{S}$, then we construct the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{w : p, w \leqslant R(\sigma), w \leqslant R(\pi), R(\pi) \leqslant R(\sigma), \pi \leqslant R(\sigma), \sigma : p, \Gamma \Rightarrow \Delta, w : p}\,Ax}{w \leqslant R(\sigma), w \leqslant R(\pi), R(\pi) \leqslant R(\sigma), \pi \leqslant R(\sigma), \sigma : p, \Gamma \Rightarrow \Delta, w : p}\,Lp}{w \leqslant R(\pi), R(\pi) \leqslant R(\sigma), \pi \leqslant R(\sigma), \sigma : p, \Gamma \Rightarrow \Delta, w : p}\,(\mathsf{tr})}{w \leqslant R(\pi), \pi \leqslant R(\sigma), \sigma : p, \Gamma \Rightarrow \Delta, w : p}\,(\mathsf{r\text{-}tr})}{\pi \leqslant R(\sigma), \sigma : p, \Gamma \Rightarrow \Delta, \pi : p}\,Rp$$

[6] Both of these statements follow immediately from the definition of the upset operator given in Section 4.1.

[7] As usual, we write $x : \Gamma$ for the multiset of labelled formulas given by $(x : \Gamma) := \{x : \psi \mid \psi \in \Gamma\}$.

[8] So, in particular, in an application of the rule $L\vee$, there are now *two* eigenvariables, rather than just a single one.

If it holds $\pi \in \mathfrak{S}$, then the derivation is the same, except that we leave out the applications of (tr), (r-tr) and Rp at the bottom of the derivation. The case $\varphi = \bot$ is treated similarly. In the inductive step for $\wedge$ and $\Vee$, the statement follows immediately from the induction hypothesis. And in the inductive step for $\to$ and $\vee$, we construct the following two derivations:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{By ind. hyp.}}{x \leqslant R(x), \ldots, x:\psi, \Gamma \Rightarrow \Delta, x:\chi, x:\psi}}{\ldots, x:\psi, \Gamma \Rightarrow \Delta, x:\chi, x:\psi}\,(\text{r-rf}) \quad \dfrac{\overset{\text{By ind. hyp.}}{x \leqslant R(x), \ldots, x:\chi, x:\psi, \Gamma \Rightarrow \Delta, x:\chi}}{\ldots, x:\chi, x:\psi, \Gamma \Rightarrow \Delta, x:\chi}\,(\text{r-rf})}{x \leqslant R(\sigma), x \leqslant R(\pi), R(\pi) \leqslant R(\sigma), \pi \leqslant R(\sigma), \sigma:\psi\to\chi, x:\psi, \Gamma \Rightarrow \Delta, x:\chi}\,L{\to}}{x \leqslant R(\pi), R(\pi)\leqslant R(\sigma), \pi\leqslant R(\sigma), \sigma:\psi\to\chi, x:\psi,\Gamma\Rightarrow\Delta, x:\chi}\,(\text{tr})}{x\leqslant R(\pi), \pi\leqslant R(\sigma), \sigma:\psi\to\chi, x:\psi,\Gamma\Rightarrow\Delta,x:\chi}\,(\text{r-tr})}{\pi\leqslant R(\sigma),\sigma:\psi\to\chi,\Gamma\Rightarrow\Delta,\pi:\psi\to\chi}\,R{\to}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{By ind. hyp.}}{R(x) \leqslant R(x), \ldots, x:\psi, \Gamma \Rightarrow \Delta, \ldots, R(x):\psi}}{\ldots, x:\psi, \Gamma \Rightarrow \Delta, \ldots, R(x):\psi}\,(\text{rf}) \quad \dfrac{\overset{\text{By ind. hyp.}}{R(y) \leqslant R(y), \ldots, y:\chi, \Gamma \Rightarrow \Delta, \ldots, R(y):\chi}}{\ldots, y:\chi, \Gamma \Rightarrow \Delta, \ldots, R(y):\chi}\,(\text{rf})}{\ldots, \pi \leqslant R(x)+R(y), R(\sigma) \leqslant R(x)+R(y), \pi \leqslant R(\sigma), x:\psi, y:\chi, \Gamma \Rightarrow \Delta, \pi:\psi\vee\chi}\,R{\vee}}{\ldots, R(\sigma) \leqslant R(x)+R(y), \pi \leqslant R(\sigma), x:\psi, y:\chi, \Gamma \Rightarrow \Delta, \pi:\psi\vee\chi}\,(\text{tr})}{\ldots, R(\sigma) \leqslant R(x+y), R(x+y) \leqslant R(x)+R(y), \pi \leqslant R(\sigma), x:\psi, y:\chi, \Gamma \Rightarrow \Delta, \pi:\psi\vee\chi}\,(\text{tr})}{\ldots, R(\sigma) \leqslant R(x+y), \sigma \leqslant x+y, x+y \leqslant R(x+y), \pi \leqslant R(\sigma), x:\psi, y:\chi, \Gamma \Rightarrow \Delta, \pi:\psi\vee\chi}\,(\text{r-dis})}{\sigma \leqslant R(x+y), \sigma \leqslant x+y, x+y \leqslant R(x+y), \pi \leqslant R(\sigma), x:\psi, y:\chi, \Gamma \Rightarrow \Delta, \pi:\psi\vee\chi}\,(\text{r-tr})}{\sigma \leqslant x+y, x+y \leqslant R(x+y), \pi \leqslant R(\sigma), x:\psi, y:\chi, \Gamma \Rightarrow \Delta, \pi:\psi\vee\chi}\,(\text{tr})}{\sigma \leqslant x+y, \pi \leqslant R(\sigma), x:\psi, y:\chi, \Gamma \Rightarrow \Delta, \pi:\psi\vee\chi}\,(\text{r-rf})}{\pi \leqslant R(\sigma), \sigma:\psi\vee\chi, \Gamma \Rightarrow \Delta, \pi:\psi\vee\chi}\,L{\vee}
$$

Finally, the sequents in (ii) and (iii) can be derived from (i) by using the rules (r-rf) and (tr). $\square$

Lemma 4.4.2. *All sequents of the following form are derivable in* GLinqI*:*

(i) $\pi \leqslant \emptyset, \Gamma \Rightarrow \Delta, \pi:\varphi$,

(ii) $\Gamma \Rightarrow \Delta, \emptyset:\varphi$,

(iii) $\pi \leqslant \sigma, \sigma:\bot, \Gamma \Rightarrow \Delta, \pi:\varphi$,

(iv) $\pi:\bot, \Gamma \Rightarrow \Delta, \pi:\varphi$.

Proof. As before, the derivability of (i) and (iii) is established by induction on φ. We only show the derivability of (i). For the base case, suppose that $\varphi = p$ is atomic. If $\pi \in \mathfrak{S}$, then (i) is an instance of $Ax^{\emptyset}$, so the proof is trivial. And if $\pi \notin \mathfrak{S}$, then we construct the derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{w \leqslant \emptyset, w \leqslant R(\emptyset), w \leqslant R(\pi), R(\pi) \leqslant R(\emptyset), R(\emptyset) \leqslant \emptyset, \pi \leqslant R(\emptyset), \pi \leqslant \emptyset, \emptyset \leqslant R(\emptyset), \Gamma \Rightarrow \Delta, w:p}\,Ax^{\emptyset}}{w \leqslant R(\emptyset), w \leqslant R(\pi), R(\pi) \leqslant R(\emptyset), R(\emptyset) \leqslant \emptyset, \pi \leqslant R(\emptyset), \pi \leqslant \emptyset, \emptyset \leqslant R(\emptyset), \Gamma \Rightarrow \Delta, w:p}\,(\text{tr})}{w \leqslant R(\pi), R(\pi) \leqslant R(\emptyset), R(\emptyset) \leqslant \emptyset, \pi \leqslant R(\emptyset), \pi \leqslant \emptyset, \emptyset \leqslant R(\emptyset), \Gamma \Rightarrow \Delta, w:p}\,(\text{tr})}{w \leqslant R(\pi), R(\pi) \leqslant R(\emptyset), \pi \leqslant R(\emptyset), \pi \leqslant \emptyset, \emptyset \leqslant R(\emptyset), \Gamma \Rightarrow \Delta, w:p}\,(\text{r-emp})}{w \leqslant R(\pi), \pi \leqslant R(\emptyset), \pi \leqslant \emptyset, \emptyset \leqslant R(\emptyset), \Gamma \Rightarrow \Delta, w:p}\,(\text{r-tr})}{w \leqslant R(\pi), \pi \leqslant \emptyset, \emptyset \leqslant R(\emptyset), \Gamma \Rightarrow \Delta, w:p}\,(\text{tr})}{w \leqslant R(\pi), \pi \leqslant \emptyset, \Gamma \Rightarrow \Delta, w:p}\,(\text{r-rf})}{\pi \leqslant \emptyset, \Gamma \Rightarrow \Delta, \pi:p}\,Rp
$$

The case $\varphi = \bot$ is treated in essentially the same way. In the inductive step for $\wedge$, $\to$ and $\Vee$, the proof is easy. And in the inductive step for $\vee$, we construct the following derivation:

$$
\dfrac{\dfrac{\overset{\text{By ind. hyp.}}{\pi \leqslant \pi+\pi, \pi \leqslant \emptyset, \Gamma \Rightarrow \Delta, \pi:\psi\vee\chi, \pi:\psi} \quad \overset{\text{By ind. hyp.}}{\pi \leqslant \pi+\pi, \pi \leqslant \emptyset, \Gamma \Rightarrow \Delta, \pi:\psi\vee\chi, \pi:\chi}}{\pi \leqslant \pi+\pi, \pi \leqslant \emptyset, \Gamma \Rightarrow \Delta, \pi:\psi\vee\chi}\,R{\vee}}{\pi \leqslant \emptyset, \Gamma \Rightarrow \Delta, \pi:\psi\vee\chi}\,(\text{ul})
$$

This concludes the induction. For the sequent in (iii), the induction is similar. Finally, the sequents in (ii) and (iv) can be derived from (i) and (iii), respectively, by an application of (rf). □

Next, we will show that GLinqI also preserves the structural properties of the labelled sequent calculus considered in the previous chapter. As before, a *branch* in a derivation $\mathcal{D}$ is defined to be a sequence β of consecutive sequents in $\mathcal{D}$ such that the first sequent in β is the conclusion of $\mathcal{D}$ and the last sequent is one of the leaf nodes of $\mathcal{D}$. By the *length* of a branch β, we mean the number of sequents occurring in β. And the *height* of a derivation $\mathcal{D}$ is defined to be the length of a longest branch in $\mathcal{D}$. A rule of inference is now said to be *height-preserving admissible* (or *hp-admissible*), if it satisfies the condition that, whenever all premises of the rule are derivable by a proof tree of height at most n, then also the conclusion of the rule is derivable by a proof tree of height at most n. If the admissibility of a rule is not height-preserving, then the rule is simply called *admissible*. Furthermore, we say that a rule is *height-preserving invertible* (or *hp-invertible*), if it satisfies the condition that, whenever the conclusion of the rule is derivable by a proof tree of height at most n, then also each of the premises of the rule is derivable by such a proof tree. For a more precise definition of the relevant concepts, the reader is referred to Section 3.2.2. The substitution operator for labels is defined in exactly the same way as in the classical setting (see Definition 3.2.5), except that, for any state variable $s \in \mathfrak{S} \cup \mathfrak{V}$, we now also put $R(\sigma)(\pi/s) := R(\sigma')$, where σ' is the label given by $\sigma' = \sigma(\pi/s)$. The definition is then extended to multisets in the usual way, so we will write $\Gamma(\pi/s)$ for the result of substituting π for s in every label occurring in Γ. As before, the *substitution rules* are defined to be the rules

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma(u/w) \Rightarrow \Delta(u/w)}\ (u/w) \qquad \text{and} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma(\pi/x) \Rightarrow \Delta(\pi/x)}\ (\pi/x)$$

where u and w are variables from $\mathfrak{S}$, x is a variable from $\mathfrak{V}$, and π is an arbitrary label.

Proposition 4.4.3. *The substitution rules are hp-admissible in* GLinqI.

Proof. By induction on the height of a derivation for $\Gamma \Rightarrow \Delta$. For the base case, suppose that $\Gamma \Rightarrow \Delta$ is derivable by a proof tree of height $n = 1$. In this case, $\Gamma \Rightarrow \Delta$ must be an axiom of GLinqI. But then, clearly, $\Gamma(u/w) \Rightarrow \Delta(u/w)$ and $\Gamma(\pi/x) \Rightarrow \Delta(\pi/x)$ are axioms as well.

For the inductive step, suppose that $\Gamma \Rightarrow \Delta$ is derivable by a proof tree $\mathcal{D}$ of height $n > 1$. We consider the last rule applied in $\mathcal{D}$. If this rule does not have eigenvariables, then we simply apply the induction hypothesis to the premises of the rule, and then the same rule again. On the other hand, if the last rule in $\mathcal{D}$ has eigenvariables, then we first appeal to the induction hypothesis in order to rename the eigenvariables, before performing the desired substitution. So, for example, suppose that we want to substitute a label $\pi \in \Lambda(\mathfrak{S}, \mathfrak{V})$ for some variable $x \in \mathfrak{V}$. Moreover, assume that the last step in $\mathcal{D}$ is an application of the rule $L\vee$, so $\mathcal{D}$ is of the form

$$\frac{\begin{array}{c}\mathcal{D}' \\ \sigma \leqslant y + z, y : \varphi, z : \psi, \Theta \Rightarrow \Delta\end{array}}{\sigma : \varphi \vee \psi, \Theta \Rightarrow \Delta}\ L\vee$$

where y and z are the eigenvariables of the indicated application of $L\vee$, and $\mathcal{D}'$ is a derivation of height $n-1$. By applying the induction hypothesis to $\mathcal{D}'$, we first replace the eigenvariables y and z by fresh variables y' and z', respectively, such that x, y' and z' are pairwise distinct. This yields a derivation $\mathcal{D}''$ of height at most $n-1$ for $\sigma \leqslant y' + z', y' : \varphi, z' : \psi, \Theta \Rightarrow \Delta$. We now apply the induction hypothesis again in order to perform the substitution (π/x). By a subsequent application of $L\vee$, we then obtain the desired derivation of height at most n for $\Gamma(\pi/x) \Rightarrow \Delta(\pi/x)$. □

We are now ready to prove the desired results: the structural rules of weakening and contraction, given in Figure 4.4, are hp-admissible in GLinqI and each rule of our system is hp-invertible.

$$\frac{\Gamma \Rightarrow \Delta}{\pi : \varphi, \Gamma \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \pi : \varphi}\ RW \qquad \frac{\Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \Gamma \Rightarrow \Delta}\ W^{\leqslant}$$

$$\frac{\pi : \varphi, \pi : \varphi, \Gamma \Rightarrow \Delta}{\pi : \varphi, \Gamma \Rightarrow \Delta}\ LC \qquad \frac{\Gamma \Rightarrow \Delta, \pi : \varphi, \pi : \varphi}{\Gamma \Rightarrow \Delta, \pi : \varphi}\ RC \qquad \frac{\pi \leqslant \sigma, \pi \leqslant \sigma, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \Gamma \Rightarrow \Delta}\ C^{\leqslant}$$

Figure 4.4: The structural rules of weakening and contraction.

Proposition 4.4.4. *The weakening rules are hp-admissible in* GLinql.

Proof. For each of the three weakening rules, we proceed by induction on the height of a derivation $\mathcal{D}$ for the premise of the respective rule. In the inductive step, we distinguish cases, depending on the last rule applied in $\mathcal{D}$. If this rule does not have eigenvariables, we simply apply the induction hypothesis to the premises of the rule, and then the same rule again. Otherwise, we first use Proposition 4.4.3 in order to introduce fresh eigenvariables not clashing with the variables occurring in the weakening formula. For further details, see the proof of Proposition 3.2.7. □

Proposition 4.4.5. *All rules of* GLinql *are hp-invertible.*

Proof. The hp-invertibility of the logical rules for atomic formulas, the falsum constant and the connectives $\wedge$, $\vee\!\!\!\vee$ and $\rightarrow$ is established in the same way as in the previous chapter (see Proposition 3.2.8). Furthermore, the hp-invertibility of the 'cumulative' rules (including the new rule $R\vee$ and the internal and external order rules) follows immediately from the hp-admissibility of weakening.[9] Thus, we only need to show that $L\vee$ is hp-invertible. For this purpose, let $\mathcal{D}$ be an arbitrary derivation for $\pi : \varphi \vee \psi, \Gamma \Rightarrow \Delta$ and let n be the height of $\mathcal{D}$. Moreover, let $x, y \in \mathfrak{V}$ be arbitrary but distinct variables not occurring in $\pi : \varphi \vee \psi, \Gamma \Rightarrow \Delta$. Using induction on n, we show that there is also a derivation of height at most n for the sequent $\pi \leqslant x + y, x : \varphi, y : \psi, \Gamma \Rightarrow \Delta$.

For the base case, assume that $\mathcal{D}$ has height $n = 1$. In this case, $\pi : \varphi \vee \psi, \Gamma \Rightarrow \Delta$ must be an instance of an axiom. Since $\varphi \vee \psi$ is not atomic, the labelled formula $\pi : \varphi \vee \psi$ cannot be principal in this instance. Hence, the sequent $\pi \leqslant x + y, x : \varphi, y : \psi, \Gamma \Rightarrow \Delta$ must also be an instance of an axiom, so it is derivable by a proof tree of height $n = 1$, as desired.

For the inductive step, suppose that $\mathcal{D}$ has height $n > 1$. If the last step in $\mathcal{D}$ is a rule for which $\pi : \varphi \vee \psi$ is not principal, then we simply apply the induction hypothesis to the premises of the rule (possibly in combination with a height-preserving substitution in order to take care of eigenvariables), and we then use the very same rule again. On the other hand, if $\mathcal{D}$ ends with an application of the rule $L\vee$ for which $\pi : \varphi \vee \psi$ is principal, then $\mathcal{D}$ must be of the form

$$\frac{\begin{array}{c}\mathcal{D}' \\ \pi \leqslant z_1 + z_2, z_1 : \varphi, z_2 : \psi, \Gamma \Rightarrow \Delta\end{array}}{\pi : \varphi \vee \psi, \Gamma \Rightarrow \Delta}\ L\vee$$

where z_1 and z_2 are the eigenvariables of the indicated application of $L\vee$ and $\mathcal{D}'$ is a derivation of height $n - 1$. By substituting x for z_1 and y for z_2 in the subderivation $\mathcal{D}'$, we now obtain the desired derivation of height at most n for the sequent $\pi \leqslant x + y, x : \varphi, y : \psi, \Gamma \Rightarrow \Delta$. □

Proposition 4.4.6. *The contraction rules are hp-admissible in* GLinql.

[9] Recall that, by a 'cumulative' rule, we mean a rule in which the principal formulas and the principal atoms from the conclusion are always repeated in each of the premises of the rule. See also Section 3.1 for further details.

Proof. For each of the three contraction rules depicted in Figure 4.4, the proof is done *simultaneously*, by induction on the height of a derivation for the premise of the respective rule. For simplicity, we only sketch the inductive step for the rule LC.[10] Let $\mathcal{D}$ be a derivation for some sequent of the form $\pi : \varphi, \pi : \varphi, \Gamma \Rightarrow \Delta$ and let $n > 1$ be the height of $\mathcal{D}$. We show that there is also a derivation of height at most n for the contracted sequent $\pi : \varphi, \Gamma \Rightarrow \Delta$. To this end, consider the last rule applied in $\mathcal{D}$. If $\pi : \varphi$ is not principal in this rule, then both occurrences of $\pi : \varphi$ must also be present in each of the premises of the rule. Hence, by applying the induction hypothesis to the premises and then the same rule again, we obtain the desired derivation of height at most n for $\pi : \varphi, \Gamma \Rightarrow \Delta$. On the other hand, if $\pi : \varphi$ is principal in the last rule applied in $\mathcal{D}$, then we distinguish cases, depending on the form of φ. If φ is not a standard disjunction, then the argument is the same as in the classical setting (see the proof of Proposition 3.2.9). Therefore, let us assume that $\varphi = \psi \vee \chi$ for some $\psi, \chi \in \mathcal{L}^{\mathsf{I}}$. In this case, $\mathcal{D}$ must be of the form

$$\dfrac{\begin{array}{c}\mathcal{D}' \\ \pi \leqslant x + y, x : \psi, y : \chi, \pi : \psi \vee \chi, \Gamma \Rightarrow \Delta\end{array}}{\pi : \psi \vee \chi, \pi : \psi \vee \chi, \Gamma \Rightarrow \Delta}\ L\vee$$

where $x, y \in \mathfrak{V}$ are fresh variables and $\mathcal{D}'$ is of height $n-1$. By applying the height-preserving invertibility of $L\vee$ and subsequent height-preserving substitutions to the conclusion of the subderivation $\mathcal{D}'$, we now obtain a derivation of height at most $n-1$ for the sequent $\pi \leqslant x+y, \pi \leqslant x+y, x : \psi, x : \psi, y : \chi, y : \chi, \Gamma \Rightarrow \Delta$. Using the induction hypothesis and a subsequent application of $L\vee$, this yields the desired derivation of height at most n for $\pi : \psi \vee \chi, \Gamma \Rightarrow \Delta$. $\square$

Next, we will prove that the cut rule is admissible in GLinqI. As before, this rule is of the form

$$\dfrac{\Gamma \Rightarrow \Delta, \pi : \varphi \qquad \pi : \varphi, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\ (\mathsf{cut})$$

where $\pi : \varphi$ is an arbitrary labelled formula, referred to as the *cut formula*. The overall structure of the cut-admissibility proof will be the same as in our treatment of InqB. We thus proceed by a main induction on the rank of an arbitrary cut rule application, with a subinduction on the height of this application. Moreover, as in the previous chapter, the rank of a cut formula $\pi : \varphi$ will be measured not only in terms of the complexity of the formula φ, but also in terms of the complexity of the associated label π. To this end, we first define the degree of a label in the familiar way.

Definition 4.4.7 (Degree of a Label)**.** The *degree of a label* π is denoted by $deg(\pi)$ and defined as follows: if $\pi \in \mathfrak{S}$, then we put $deg(\pi) := 0$, and if $\pi \notin \mathfrak{S}$, then we put $deg(\pi) := 1$.

That is, as usual, we simply assign the degree 0 to every singleton variable $w \in \mathfrak{S}$ and the degree 1 to every non-singleton label $\pi \notin \mathfrak{S}$. The *degree of a formula* $\varphi \in \mathcal{L}^{\mathsf{I}}$, on the other hand, is now defined to be the number of occurrences of the logical symbols $\bot, \wedge, \to, \mathbin{\vee\mkern-10mu\vee}, \vee$ in φ. The *rank of a labelled formula* $\pi : \varphi$ is again defined to be the pair $rank(\pi : \varphi) := (deg(\varphi), deg(\pi))$, where $deg(\varphi)$ is the degree of the formula φ and $deg(\pi)$ is the degree of the label π. As in the previous chapter, ranks of labelled formulas are compared using a *lexicographic ordering*, so we will write $rank(\pi : \varphi) < rank(\sigma : \psi)$ and say that the rank of $\pi : \varphi$ is *smaller* than the rank of $\sigma : \psi$, if we either have $deg(\varphi) < deg(\psi)$, or we have both $deg(\varphi) = deg(\psi)$ and $deg(\pi) < deg(\sigma)$.

Lemma 4.4.8. *Let π and σ be arbitrary labels and let $w \in \mathfrak{S}$ be a singleton variable. It holds:*

(i) If $\pi \notin \mathfrak{S}$, then $rank(w : \varphi) < rank(\pi : \varphi)$,

(ii) $rank(\pi : \varphi_i) < rank(\sigma : \varphi_1 \otimes \varphi_2)$ for $i = 1, 2$ and $\otimes \in \{\wedge, \to, \mathbin{\vee\mkern-10mu\vee}, \vee\}$.

[10] As before, the contraction rules RC and $C^{\leqslant}$ can be treated similarly. Note that, in order to show the hp-admissibility of $C^{\leqslant}$, one has to appeal to the *closure condition*. For further details, we refer to the proof of Proposition 3.2.9.

Both statements are proved in exactly the same way as in the classical setting (see the proof of Lemma 3.2.12). We now define the *height of a cut rule application* to be the sum of the heights of the two derivations for the premises $\Gamma \Rightarrow \Delta, \pi : \varphi$ and $\pi : \varphi, \Sigma \Rightarrow \Theta$ of this application (where the height of a derivation is again taken to be the length of a longest branch in this derivation). And the *rank of a cut rule application* is defined to be the rank of the associated cut formula $\pi : \varphi$.

Theorem 4.4.9 (Cut-Admissibility)**.** *The cut rule is admissible in* GLinql.

Proof. As before, we select an arbitrary application of the cut rule and proceed by a main induction on the rank of the cut formula, with a subinduction on the height of the cut. The general structure of the argument is the same as in the proof of Theorem 3.2.13. In particular, we need to consider the same main cases, and we perform essentially the same conversions in each case. The most interesting new part of the proof is the case in which the cut formula is of the form $\pi : \varphi \vee \psi$ and principal on both sides. In this case, the cut rule application must be of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \pi \leqslant \sigma+\tau, \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \pi : \varphi \vee \psi, \Sigma \Rightarrow \Theta\end{array}}{\pi \leqslant \sigma+\tau, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\text{(cut)}$$

where the left and the right subderivation are given by

$$\mathcal{D}_1 = \left\{ \dfrac{\begin{array}{c}\mathcal{D}_1'\\ \pi \leqslant \sigma+\tau, \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi, \sigma : \varphi\end{array} \qquad \begin{array}{c}\mathcal{D}_1''\\ \pi \leqslant \sigma+\tau, \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi, \tau : \psi\end{array}}{\pi \leqslant \sigma+\tau, \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi} R\vee \right.$$

$$\mathcal{D}_2 = \left\{ \dfrac{\begin{array}{c}\mathcal{D}_2'\\ \pi \leqslant x+y, x : \varphi, y : \psi, \Sigma \Rightarrow \Theta\end{array}}{\pi : \varphi \vee \psi, \Sigma \Rightarrow \Theta} L\vee \right.$$

Note that, without loss of generality, we may assume that the variable y does not occur in the label σ (if this condition is not satisfied, then we simply perform a height-preserving substitution in the subderivation $\mathcal{D}_2'$ in order to replace y by some fresh variable). Hence, using the hp-admissibility of substitution and contraction, we may now transform the whole derivation into the proof tree

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}''\\ \pi \leqslant \sigma+\tau, \Gamma, \Sigma \Rightarrow \Delta, \Theta, \tau : \psi\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}'\\ \pi \leqslant \sigma+\tau, \Gamma, \Sigma \Rightarrow \Delta, \Theta, \sigma : \varphi\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_2'\\ \pi \leqslant x+y, x : \varphi, y : \psi, \Sigma \Rightarrow \Theta\end{array}}{\pi \leqslant \sigma+y, \sigma : \varphi, y : \psi, \Sigma \Rightarrow \Theta}(\sigma/x)}{\pi \leqslant \sigma+\tau, \sigma : \varphi, \tau : \psi, \Sigma \Rightarrow \Theta}(\tau/y)}{(\pi \leqslant \sigma+\tau)^2, \tau : \psi, \Gamma, \Sigma^2 \Rightarrow \Delta, \Theta^2}\text{(cut)}}{(\pi \leqslant \sigma+\tau)^3, \Gamma^2, \Sigma^3 \Rightarrow \Delta^2, \Theta^3}\text{(cut)}}{\pi \leqslant \sigma+\tau, \Gamma, \Sigma \Rightarrow \Delta, \Theta} C$$

where $\mathcal{D}'$ and $\mathcal{D}''$ are the following two derivations:

$$\mathcal{D}' = \left\{ \dfrac{\begin{array}{c}\mathcal{D}_1'\\ \pi \leqslant \sigma+\tau, \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi, \sigma : \varphi\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2'\\ \pi \leqslant x+y, x : \varphi, y : \psi, \Sigma \Rightarrow \Theta\end{array}}{\pi : \varphi \vee \psi, \Sigma \Rightarrow \Theta} L\vee}{\pi \leqslant \sigma+\tau, \Gamma, \Sigma \Rightarrow \Delta, \Theta, \sigma : \varphi}\text{(cut)} \right.$$

$$\mathcal{D}'' = \left\{ \dfrac{\begin{array}{c}\mathcal{D}_1''\\ \pi \leqslant \sigma+\tau, \Gamma \Rightarrow \Delta, \pi : \varphi \vee \psi, \tau : \psi\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2'\\ \pi \leqslant x+y, x : \varphi, y : \psi, \Sigma \Rightarrow \Theta\end{array}}{\pi : \varphi \vee \psi, \Sigma \Rightarrow \Theta} L\vee}{\pi \leqslant \sigma+\tau, \Gamma, \Sigma \Rightarrow \Delta, \Theta, \tau : \psi}\text{(cut)} \right.$$

$$\frac{R(\pi) \leqslant R(\sigma), \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \Gamma \Rightarrow \Delta}\text{(rel)} \qquad \frac{R(R(\pi)) \approx R(\pi), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\text{(idem)}$$

$$\frac{R(\pi + \sigma) \approx R(\pi) + R(\sigma), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\text{(g-dis)} \qquad \frac{R(\emptyset) \approx \emptyset, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\text{(g-emp)}$$

Figure 4.5: Further admissible rules. In each case, '$\pi \approx \sigma$' stands for '$\pi \leqslant \sigma, \sigma \leqslant \pi$'.

As can be seen, the original cut rule application is now replaced by four new cuts. The two uppermost of these new cuts (i.e., those with cut formula $\pi : \varphi \vee \psi$) are of lower height than the original one, and the two other cuts are of lower rank. Thus, using the subinduction hypothesis and then the main induction hypothesis, one can successively remove each of the four cuts. □

In order to conclude this section, let us point out some further properties of our sequent calculus. Figure 4.5 comprises a number of additional rules that can be shown to be admissible in GLinqI. Note that, in the figure, the notation '$\pi \approx \sigma$' is used as a shorthand for the pair of relational atoms '$\pi \leqslant \sigma, \sigma \leqslant \pi$'. Intuitively, the rule (rel) accounts for the observation that, if a state π is a subset of some state σ, then also $R(\pi)$ must be a subset of $R(\sigma)$. The rule (idem) reflects the idempotence of the upset operator: applying the R-operator multiple times to a state has exactly the same effect as applying R only once to that state. The rules (g-dis) and (g-emp), finally, can be seen as simple generalizations of the internal order rules (r-dis) and (r-emp).

Proposition 4.4.10. *Each of the rules depicted in Figure 4.5 is admissible in* GLinqI.

Proof. We only show the admissibility of the rules (rel) and (idem). For this purpose, suppose that the premise of (rel) is derivable by a proof tree $\mathcal{D}_1$ and the premise of (idem) is derivable by a proof tree $\mathcal{D}_2$. Using these proof trees, we may then construct the following two derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ R(\pi) \leqslant R(\sigma), \Gamma \Rightarrow \Delta\end{matrix}}{R(\pi) \leqslant R(\sigma), \pi \leqslant R(\sigma), \pi \leqslant \sigma, \sigma \leqslant R(\sigma), \Gamma \Rightarrow \Delta}\,W}{\pi \leqslant R(\sigma), \pi \leqslant \sigma, \sigma \leqslant R(\sigma), \Gamma \Rightarrow \Delta}\,\text{(r-tr)}}{\pi \leqslant \sigma, \sigma \leqslant R(\sigma), \Gamma \Rightarrow \Delta}\,\text{(tr)}}{\pi \leqslant \sigma, \Gamma \Rightarrow \Delta}\,\text{(r-rf)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_2\\ R(R(\pi)) \leqslant R(\pi), R(\pi) \leqslant R(R(\pi)), \Gamma \Rightarrow \Delta\end{matrix}}{R(R(\pi)) \leqslant R(\pi), R(\pi) \leqslant R(R(\pi)), R(\pi) \leqslant R(\pi), \Gamma \Rightarrow \Delta}\,W}{R(R(\pi)) \leqslant R(\pi), R(\pi) \leqslant R(\pi), \Gamma \Rightarrow \Delta}\,\text{(r-rf)}}{R(\pi) \leqslant R(\pi), \Gamma \Rightarrow \Delta}\,\text{(r-tr)}}{\Gamma \Rightarrow \Delta}\,\text{(rf)}$$

Hence, (rel) and (idem) are admissible. For the other rules from Figure 4.5, the proof is similar. □

For later purposes, we also need to show that the truth-conditionality of standard formulas can be reflected by means of formal derivations in the system GLinqI. This is accomplished by the following lemma. Intuitively, both of the sequents in the lemma express the fact that, if a standard formula α is supported by two states, then it is also supported by the union of these states. The proof proceeds by a routine induction on the structure of α and is therefore omitted.

Lemma 4.4.11. *For any standard formula $\alpha \in \mathcal{L}^{\mathsf{I}}_s$, the following sequents are derivable in* GLinqI*:*
(i) $\pi : \alpha, \sigma : \alpha, \Gamma \Rightarrow \Delta, \pi + \sigma : \alpha$*,*
(ii) $\pi \leqslant \sigma + \tau, \pi\sigma : \alpha, \pi\tau : \alpha, \Gamma \Rightarrow \Delta, \pi : \alpha$*.*

4.5 Soundness and Completeness

In order to conclude our treatment of intuitionistic inquisitive logic, we will now prove that GLinqI is sound and complete with respect to InqI. First, let us establish the soundness of our proof system. The notion of an interpretation is defined in exactly the same way as in the classical setting, except that we now also need to specify how labels of the form $R(\pi)$ are interpreted.

Definition 4.5.1 (Interpretation)**.** Let $M = \langle W, R, V \rangle$ be an intuitionistic Kripke model. An *interpretation* over M is a function $I : \mathfrak{S} \cup \mathfrak{V} \to \mathcal{P}(W)$ such that, for all singleton variables $w \in \mathfrak{S}$, the state $I(w) \subseteq W$ is a singleton. Given any interpretation I over some model M, it is inductively extended to a function from the set of all labels to the set $\mathcal{P}(W)$ in the following way:
(i) $I(\emptyset) := \emptyset$,
(ii) $I(\pi \cdot \sigma) := I(\pi) \cap I(\sigma)$,
(iii) $I(\pi + \sigma) := I(\pi) \cup I(\sigma)$,
(iv) $I(R(\pi)) := R(I(\pi))$.

Labelled formulas and relational atoms are interpreted in the usual way. That is, given any interpretation I over some intuitionistic Kripke model M, we will say that a labelled formula $\pi : \varphi$ is *satisfied* by I, just in case φ is supported by the sate $I(\pi)$, i.e., if we have $M, I(\pi) \models \varphi$. And we will say that a relational atom $\pi \leqslant \sigma$ is satisfied by I, if $I(\pi)$ is a subset of $I(\sigma)$, i.e., if it holds $I(\pi) \subseteq I(\sigma)$. A sequent $\Gamma \Rightarrow \Delta$ is said to be *valid* in a Kripke model M, if for every interpretation I over M, the following holds: if I satisfies all expressions in Γ, then there exists a labelled formula $\pi : \varphi$ in Δ such that I satisfies $\pi : \varphi$. We are now ready to prove the soundness of our sequent calculus: if a formula φ is provable from Γ in GLinqI, then φ is entailed by Γ in InqI.

Proposition 4.5.2 (Soundness of GLinqI)**.** *For every finite set of formula $\Gamma \subseteq \mathcal{L}^{\mathsf{I}}$ and for every formula $\varphi \in \mathcal{L}^{\mathsf{I}}$, if the sequent $x : \Gamma \Rightarrow x : \varphi$ is derivable in* GLinqI *for some $x \in \mathfrak{V}$, then $\Gamma \models \varphi$.*[11]

Proof. We first show that, if a sequent $\Gamma \Rightarrow \Delta$ is derivable in GLinqI, then $\Gamma \Rightarrow \Delta$ is is valid in every intuitionistic Kripke model. Let $\mathcal{D}$ be an arbitrary derivation for some sequent $\Gamma \Rightarrow \Delta$ and let $M = \langle W, R, V \rangle$ be an arbitrary intuitionistic Kripke model. Using induction on the structure of $\mathcal{D}$, we prove that $\Gamma \Rightarrow \Delta$ is valid in M. Most cases can be treated in the same way as in the classical setting (see the proof of Proposition 3.3.4). Therefore, we only consider the following cases.

Case 1: Suppose that the last step in $\mathcal{D}$ is an application of the rule $L\vee$, so $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\mathcal{D}'\\ \pi \leqslant x + y, x : \varphi, y : \psi, \Theta \Rightarrow \Delta\end{array}}{\pi : \varphi \vee \psi, \Theta \Rightarrow \Delta}\; L\vee$$

where $x, y \in \mathfrak{V}$ are fresh variables not occurring in the conclusion of $\mathcal{D}$. By induction hypothesis, we know that $\pi \leqslant x+y, x : \varphi, y : \psi, \Theta \Rightarrow \Delta$ is valid in M. In order to show that this also holds for $\pi : \varphi \vee \psi, \Theta \Rightarrow \Delta$, let I be an arbitrary interpretation over M and suppose that I satisfies $\pi : \varphi \vee \psi$ and each expression in Θ. Since I satisfies $\pi : \varphi \vee \psi$, we must have $M, I(\pi) \models \varphi \vee \psi$, so there exist two states $t_1, t_2 \subseteq W$ such that $I(\pi) = t_1 \cup t_2$, $M, t_1 \models \varphi$ and $M, t_2 \models \psi$. Let now I^* be the interpretation which is just like I, except that x is mapped to t_1 and y is mapped to t_2, so we put $I^*(x) := t_1$ and $I^*(y) := t_2$. Then, clearly, I^* satisfies $\pi \leqslant x+y, x : \varphi, y : \psi$ and each expression in Θ. Hence, by induction hypothesis, there must be some labelled formula in Δ which is also

[11] Recall that we write $x : \Gamma$ for the set of labelled formulas given by $(x : \Gamma) := \{x : \psi \mid \psi \in \Gamma\}$.

Axioms:

(IPL)	All axioms of the 'intuitionistic' system given in Figure 1.5,
(Intro)	$\varphi \to (\varphi \vee \psi)$ and $\psi \to (\varphi \vee \psi)$,
(Elim)	$(\varphi \to \alpha) \to ((\psi \to \alpha) \to ((\varphi \vee \psi) \to \alpha))$, where α is a standard formula,
(Com)	$(\varphi \vee \psi) \to (\psi \vee \varphi)$,
(Dis)	$(\varphi \vee (\psi \vee\!\!\!\vee \chi)) \to ((\varphi \vee \psi) \vee\!\!\!\vee (\varphi \vee \chi))$,
(Ex)	$(\varphi \to \varphi') \to ((\psi \to \psi') \to ((\varphi \vee \psi) \to (\varphi' \vee \psi')))$,
(Split)	$(\alpha \to (\varphi \vee\!\!\!\vee \psi)) \to ((\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi))$, where α is a standard formula.

The only rule of inference is *modus ponens*: from $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, infer $\Gamma \vdash \psi$.

Figure 4.6: The Hilbert-style system HinqI.

satisfied by I^*. Since neither x nor y occurs in Δ, this labelled formula must also be satisfied by the original interpretation I. Thus, because I was arbitrary, it follows that $\pi : \varphi \vee \psi, \Theta \Rightarrow \Delta$ is valid in M. If the last step in $\mathcal{D}$ is an application of $R\vee$, then the argument is similar.

Case 2: Suppose that $\mathcal{D}$ ends with an application of (r-tr). In this case, $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\mathcal{D}'\\ R(\pi) \leqslant R(\sigma), \pi \leqslant R(\sigma), \Theta \Rightarrow \Delta\end{array}}{\pi \leqslant R(\sigma), \Theta \Rightarrow \Delta}\ \text{(r-tr)}$$

By induction hypothesis, we know that $R(\pi) \leqslant R(\sigma), \pi \leqslant R(\sigma), \Theta \Rightarrow \Delta$ is valid in M. Let now I be an arbitrary interpretation over M and suppose that I satisfies $\pi \leqslant R(\sigma)$ and each expression in Θ. Then, in particular, we must have $I(\pi) \subseteq R(I(\sigma))$. Because R is transitive, this implies $R(I(\pi)) \subseteq R(I(\sigma))$ by Proposition 4.1.5. Hence, I also satisfies $R(\pi) \leqslant R(\sigma)$. Therefore, by induction hypothesis, there must be some labelled formula in Δ which is satisfied by I. Since I was arbitrary, this shows that $\pi \leqslant R(\sigma), \Theta \Rightarrow \Delta$ is valid in M. If $\mathcal{D}$ ends with an application of one of the other internal order rules, then the argument is similar.

This concludes the induction. Hence, if $\Gamma \Rightarrow \Delta$ is derivable in GLinqI, then $\Gamma \Rightarrow \Delta$ is valid in every intuitionistic Kripke model. Let now $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{I}}$ be an arbitrary finite set of formulas and suppose that $x : \Gamma \Rightarrow x : \varphi$ is derivable in GLinqI for some $x \in \mathfrak{V}$. As we have seen, this implies that, for every model M and for every interpretation I over M, if $M, I(x) \models \psi$ for all $\psi \in \Gamma$, then also $M, I(x) \models \varphi$. But then, by definition of entailment, we clearly have $\Gamma \models \varphi$. □

Next, we will establish the completeness of our sequent calculus. To this end, we will exploit the completeness of the Hilbert-style system HinqI introduced in Section 4.2 and displayed again in Figure 4.6. First, we need to show that each of the axioms of HinqI is provable in GLinqI.

Lemma 4.5.3. *Let φ be an instance of one of the axiom schemes of* HinqI*. Then, φ is provable in* GLinqI*, i.e., for any variable $x \in \mathfrak{V}$, there exists a derivation for the sequent $\Rightarrow x : \varphi$ in* GLinqI.

Proof. As before, showing the derivability of the axiom schemes from Figure 1.5 is straightforward. Therefore, we only need to prove that the special axioms given in Figure 4.6 are derivable. For the axiom schemes (Intro), (Com) and (Dis), we may construct the following derivations:[12]

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{By Lemma 4.4.1 (iii)}\\ \hdashline y \leqslant y+\emptyset, y \leqslant R(x), y : \varphi \Rightarrow \dots, y : \varphi\end{array} \quad \begin{array}{c}\text{By Lemma 4.4.2 (ii)}\\ \hdashline y \leqslant y+\emptyset, y \leqslant R(x), y : \varphi \Rightarrow \dots, \emptyset : \psi\end{array}}{y \leqslant y+\emptyset, y \leqslant R(x), y : \varphi \Rightarrow y : \varphi \vee \psi}\ R\vee}{y \leqslant R(x), y : \varphi \Rightarrow y : \varphi \vee \psi}\ \text{(ul)}}{\Rightarrow x : \varphi \to (\varphi \vee \psi)}\ R\to$$

[12] For simplicity, we only give a derivation for the first variant of (Intro). The other variant can be derived similarly.

$$
\begin{prooftree}
\AxiomC{By Lemma 4.4.1 (iii)}
\dashedLine
\UnaryInfC{$\ldots, z_1 : \varphi, z_2 : \psi \Rightarrow y : \psi \vee \varphi, z_2 : \psi$}
\AxiomC{By Lemma 4.4.1 (iii)}
\dashedLine
\UnaryInfC{$\ldots, z_1 : \varphi, z_2 : \psi \Rightarrow y : \psi \vee \varphi, z_1 : \varphi$}
\RightLabel{$R\vee$}
\BinaryInfC{$\ldots, y \leqslant z_2 + z_1, y \leqslant R(x), z_1 : \varphi, z_2 : \psi \Rightarrow y : \psi \vee \varphi$}
\RightLabel{(tr)}
\UnaryInfC{$\ldots, y \leqslant z_1 + z_2, z_1 + z_2 \leqslant z_2 + z_1, y \leqslant R(x), z_1 : \varphi, z_2 : \psi \Rightarrow y : \psi \vee \varphi$}
\RightLabel{(un)}
\UnaryInfC{$y \leqslant z_1 + z_2, z_1 \leqslant z_2 + z_1, z_2 \leqslant z_2 + z_1, y \leqslant R(x), z_1 : \varphi, z_2 : \psi \Rightarrow y : \psi \vee \varphi$}
\RightLabel{(ul)}
\UnaryInfC{$y \leqslant z_1 + z_2, z_1 \leqslant z_2 + z_1, y \leqslant R(x), z_1 : \varphi, z_2 : \psi \Rightarrow y : \psi \vee \varphi$}
\RightLabel{(ur)}
\UnaryInfC{$y \leqslant z_1 + z_2, y \leqslant R(x), z_1 : \varphi, z_2 : \psi \Rightarrow y : \psi \vee \varphi$}
\RightLabel{$L\vee$}
\UnaryInfC{$y \leqslant R(x), y : \varphi \vee \psi \Rightarrow y : \psi \vee \varphi$}
\RightLabel{$R\!\to$}
\UnaryInfC{$\Rightarrow x : (\varphi \vee \psi) \to (\psi \vee \varphi)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{By Lemma 4.4.1 (iii)}
\dashedLine
\UnaryInfC{$\ldots, y : \varphi \Rightarrow \ldots, y : \varphi$}
\AxiomC{By Lemma 4.4.1 (iii)}
\dashedLine
\UnaryInfC{$\ldots, z : \psi \Rightarrow \ldots, z : \psi$}
\RightLabel{$R\vee$}
\BinaryInfC{$\ldots, x' \leqslant y + z, y : \varphi, z : \psi \Rightarrow \ldots, x' : \varphi \vee \psi$}
\AxiomC{By Lemma 4.4.1 (iii)}
\dashedLine
\UnaryInfC{$\ldots, y : \varphi \Rightarrow \ldots, y : \varphi$}
\AxiomC{By Lemma 4.4.1 (iii)}
\dashedLine
\UnaryInfC{$\ldots, z : \chi \Rightarrow \ldots, z : \chi$}
\RightLabel{$R\vee$}
\BinaryInfC{$\ldots, x' \leqslant y + z, y : \varphi, z : \chi \Rightarrow \ldots, x' : \varphi \vee \chi$}
\RightLabel{$L\mathbb{V}$}
\BinaryInfC{$x' \leqslant y + z, x' \leqslant R(x), y : \varphi, z : \psi \mathbb{V} \chi \Rightarrow x' : \varphi \vee \psi, x' : \varphi \vee \chi$}
\RightLabel{$L\vee$}
\UnaryInfC{$x' \leqslant R(x), x' : \varphi \vee (\psi \mathbb{V} \chi) \Rightarrow x' : \varphi \vee \psi, x' : \varphi \vee \chi$}
\RightLabel{$R\mathbb{V}$}
\UnaryInfC{$x' \leqslant R(x), x' : \varphi \vee (\psi \mathbb{V} \chi) \Rightarrow x' : (\varphi \vee \psi) \mathbb{V} (\varphi \vee \chi)$}
\RightLabel{$R\!\to$}
\UnaryInfC{$\Rightarrow x : (\varphi \vee (\psi \mathbb{V} \chi)) \to ((\varphi \vee \psi) \mathbb{V} (\varphi \vee \chi))$}
\end{prooftree}
$$

In order to show the derivability of the axiom scheme (Elim), let now $\alpha \in \mathcal{L}^{!}_s$ be a standard formula and let $\varphi, \psi \in \mathcal{L}^{!}$ be arbitrary formulas. We may then construct the following derivation:

$$
\begin{prooftree}
\AxiomC{By Lemma 4.4.1 (ii)}
\dashedLine
\UnaryInfC{$x_0x_1 \leqslant x_1, \ldots, x_1 : \varphi \Rightarrow x_0 : \alpha, x_0x_1 : \varphi$}
\RightLabel{(ir)}
\UnaryInfC{$\ldots, x_1 : \varphi \Rightarrow x_0 : \alpha, x_0x_1 : \varphi$}
\AxiomC{$\mathcal{D}$}
\noLine
\UnaryInfC{$\ldots, x_0 \leqslant R(z), x_0x_1 : \alpha, \ldots, x_2 : \psi \Rightarrow x_0 : \alpha$}
\RightLabel{$L\!\to$}
\BinaryInfC{$\ldots, x_0x_1 \leqslant R(y), x_0 \leqslant R(z), x_0 \leqslant x_1 + x_2, y : \varphi \to \alpha, z : \psi \to \alpha, x_1 : \varphi, x_2 : \psi \Rightarrow x_0 : \alpha$}
\RightLabel{(tr)}
\UnaryInfC{$\ldots, x_0x_1 \leqslant x_0, x_0 \leqslant R(y), x_0 \leqslant R(z), x_0 \leqslant x_1 + x_2, y : \varphi \to \alpha, z : \psi \to \alpha, x_1 : \varphi, x_2 : \psi \Rightarrow x_0 : \alpha$}
\RightLabel{(il)}
\UnaryInfC{$\ldots, x_0 \leqslant R(y), x_0 \leqslant R(z), x_0 \leqslant x_1 + x_2, y : \varphi \to \alpha, z : \psi \to \alpha, x_1 : \varphi, x_2 : \psi \Rightarrow x_0 : \alpha$}
\RightLabel{(tr)}
\UnaryInfC{$\ldots, x_0 \leqslant R(z), R(z) \leqslant R(y), x_0 \leqslant x_1 + x_2, y : \varphi \to \alpha, z : \psi \to \alpha, x_1 : \varphi, x_2 : \psi \Rightarrow x_0 : \alpha$}
\RightLabel{(r-tr)}
\UnaryInfC{$x_0 \leqslant R(z), z \leqslant R(y), y \leqslant R(x), x_0 \leqslant x_1 + x_2, y : \varphi \to \alpha, z : \psi \to \alpha, x_1 : \varphi, x_2 : \psi \Rightarrow x_0 : \alpha$}
\RightLabel{$L\vee$}
\UnaryInfC{$x_0 \leqslant R(z), z \leqslant R(y), y \leqslant R(x), y : \varphi \to \alpha, z : \psi \to \alpha, x_0 : \varphi \vee \psi \Rightarrow x_0 : \alpha$}
\RightLabel{$R\!\to$}
\UnaryInfC{$z \leqslant R(y), y \leqslant R(x), y : \varphi \to \alpha, z : \psi \to \alpha \Rightarrow z : (\varphi \vee \psi) \to \alpha$}
\RightLabel{$R\!\to$}
\UnaryInfC{$y \leqslant R(x), y : \varphi \to \alpha \Rightarrow y : (\psi \to \alpha) \to ((\varphi \vee \psi) \to \alpha)$}
\RightLabel{$R\!\to$}
\UnaryInfC{$\Rightarrow x : (\varphi \to \alpha) \to ((\psi \to \alpha) \to ((\varphi \vee \psi) \to \alpha))$}
\end{prooftree}
$$

where the subderivation $\mathcal{D}$ is of the form

$$
\begin{prooftree}
\AxiomC{By Lemma 4.4.1 (ii)}
\dashedLine
\UnaryInfC{$x_0x_2 \leqslant x_2, \ldots, x_2 : \psi \Rightarrow x_0 : \alpha, x_0x_2 : \psi$}
\RightLabel{(ir)}
\UnaryInfC{$\ldots, x_2 : \psi \Rightarrow x_0 : \alpha, x_0x_2 : \psi$}
\AxiomC{By Lemma 4.4.11 (ii)}
\dashedLine
\UnaryInfC{$\ldots, x_0 \leqslant x_1 + x_2, x_0x_1 : \alpha, x_0x_2 : \alpha \Rightarrow x_0 : \alpha$}
\RightLabel{$L\!\to$}
\BinaryInfC{$\ldots, x_0x_2 \leqslant R(z), x_0 \leqslant x_1 + x_2, x_0x_1 : \alpha, z : \psi \to \alpha, x_2 : \psi \Rightarrow x_0 : \alpha$}
\RightLabel{(tr)}
\UnaryInfC{$\ldots, x_0x_2 \leqslant x_0, x_0 \leqslant R(z), x_0 \leqslant x_1 + x_2, x_0x_1 : \alpha, z : \psi \to \alpha, x_2 : \psi \Rightarrow x_0 : \alpha$}
\RightLabel{(il)}
\UnaryInfC{$\ldots, x_0 \leqslant R(z), x_0 \leqslant x_1 + x_2, x_0x_1 : \alpha, z : \psi \to \alpha, x_2 : \psi \Rightarrow x_0 : \alpha$}
\end{prooftree}
$$

The derivation for the axiom (Ex) is similar to the derivation for (Elim). And the split axiom (Split) can be derived in exactly the same way as in the classical setting, by using part (i) of Lemma 4.4.11 (see also the proof of Lemma 3.3.5). Therefore, all axioms of HinqI are provable in GLinqI. □

We are now ready to prove the desired completeness result for our sequent calculus. To this end, we first show that GLinqI is weakly complete with respect to the Hilbert-style system HinqI, i.e., if a formula φ is provable in HinqI, then it is also provable in GLinqI. The strong completeness of GLinqI then follows as an immediate corollary, by using the deduction theorem for HinqI.

Theorem 4.5.4. *Let $\varphi \in \mathcal{L}^{\mathsf{I}}$ be an arbitrary formula. If we have $\vdash_{\mathsf{H}} \varphi$ in the Hilbert-style system* Hinql*, then the sequent $\Rightarrow x : \varphi$ is derivable in* GLinql*, for any variable $x \in \mathfrak{V}$.*

Proof. By induction on the structure of a Hilbert-style proof for $\vdash_{\mathsf{H}} \varphi$. In the base case, one has to show that all axioms of Hinql are provable in GLinql. This has been done in Lemma 4.5.3. For the inductive step, it suffices to show that the rule of *modus ponens*, given by

$$\frac{\Rightarrow x : \varphi \qquad \Rightarrow x : \varphi \to \psi}{\Rightarrow x : \psi}\ (\mathsf{mp})$$

is admissible in GLinql. This follows directly from the invertibility of $R\!\to$ and the admissibility of the cut rule in GLinql. For further details, the reader is referred to the proof of Lemma 3.3.6. □

Corollary 4.5.5 (Soundness and Completeness)**.** *The labelled sequent calculus* GLinql *is sound and strongly complete with respect to* Inql*. That is, for every finite set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{I}}$, we have $\Gamma \vDash \varphi$ if and only if $x : \Gamma \Rightarrow x : \varphi$ is derivable in* GLinql*, for any variable $x \in \mathfrak{V}$.*

Proof. The soundness of GLinql has been established in Proposition 4.5.2. For the completeness part, one may use essentially the same argument as in the proof of Corollary 3.3.8. We only sketch the basic idea. Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{I}}$ be an arbitrary finite set of formulas and suppose that $\Gamma \vDash \varphi$. Then, by the completeness of Hinql (see Corollary 4.2.5), we must have $\Gamma \vdash_{\mathsf{H}} \varphi$. Now, using the deduction theorem for Hinql (see Theorem 4.2.3), one readily sees that this yields $\vdash_{\mathsf{H}} \bigwedge \Gamma \to \varphi$, where $\bigwedge \Gamma$ stands for the conjunction of the formulas in Γ. Hence, by Theorem 4.5.4, we may conclude that $\Rightarrow x : \bigwedge \Gamma \to \varphi$ is derivable in GLinql, for an arbitrary variable $x \in \mathfrak{V}$. Using the invertibility of $R\!\to$ and $L\wedge$, this implies that $y \leqslant R(x), y : \Gamma \Rightarrow y : \varphi$ is also derivable, where $y \in \mathfrak{V}$ is some fresh variable. But then, by performing the substitution (x/y) and a subsequent application of (r-rf), we obtain the desired derivation for $x : \Gamma \Rightarrow x : \varphi$ in GLinql. □

Chapter 5

Inquisitive Kripke Logic

In this chapter, we will extend our labelled sequent calculus for InqB to various systems of *modal* inquisitive logic. In the literature on this topic, two general settings have been discussed so far, both of which originate with the work of Ciardelli and Roelofsen (see Ciardelli 2014; Ciardelli and Roelofsen 2015; Ciardelli 2016b). In this chapter, we will only consider the first of these two settings, which is known as *inquisitive Kripke logic* (cf. Ciardelli 2016b, Chapter 6). The second setting extends the first one with 'properly inquisitive modalities' and will not be treated in this thesis. For further information about this second setting, we refer to Ciardelli (2016b, Chapter 7).

Roughly speaking, inquisitive Kripke logic is the result of enriching InqB with a modal operator $\Box$, interpreted over ordinary Kripke models—or, to put it differently, the result of adding the question-forming operator $\mathbb{V}$ to ordinary modal logic. The weakest logic obtained in this way will be called InqK and constitutes an inquisitive extension of the basic modal logic K. Although InqK has some advantages over ordinary modal logic when it comes to formalizing modal statements in natural language more uniformly, we will see that the presence of questions in the language does not allow us to express new *truth-conditional* meanings: given any truth-conditional formula in InqK, one can always find an equivalent formula in ordinary modal logic.

Just as in standard modal logic, it is possible to construct various extensions of InqK by restricting the semantic consequence relation to specific classes of Kripke frames. So, for example, by restricting the semantics to the class of all *reflexive* and *transitive* frames, we may construct an inquisitive extension InqKT4 of the standard modal logic KT4 (which is also known as S4). More generally, for every normal modal logic $\mathfrak{L}$, one can define a corresponding inquisitive system Inq$\mathfrak{L}$. Furthermore, as shown by Ciardelli (2016b), there exists a general strategy that allows to transform any sound and complete axiomatization of a *canonical* normal modal logic $\mathfrak{L}$ into a sound and complete axiomatization of its inquisitive counterpart, Inq$\mathfrak{L}$. Unfortunately, the proof systems obtained in this way are not *analytic* and Ciardelli's strategy only yields a *recursive* axiomatization for Inq$\mathfrak{L}$, if a decidable set of axioms for $\mathfrak{L}$ is already known in advance.

For this reason, we will henceforth focus on a specific class of inquisitive Kripke logics. This class comprises all logics Inq$\mathfrak{L}$ whose characteristic frame property can be described by a number of *geometric implications*, i.e., first-order formulas of the form $\forall\vec{w}(\varphi \to \psi)$, where φ and ψ are not allowed to contain implications or universal quantifiers. The most important contribution of this chapter is a general method that allows to construct a cut-free labelled sequent calculus $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ for every inquisitive system Inq$\mathfrak{L}$ determined by some finite set A of geometric implications. This generalizes a famous result for standard modal logic established by Negri (2005).

The chapter is structured as follows. In Section 5.1, we will define the basic system of inquisitive Kripke logic, denoted by InqK. The formulas of the language are evaluated with respect to ordinary Kripke models and the support clause for $\Box$ can be seen as a natural generalization of the ordinary truth-conditional semantics for $\Box$ familiar from standard modal logic. We will see that, although $\Box$ may now also embed questions (rather than just assertions), any formula of the form $\Box\varphi$ will be truth-conditional and therefore non-inquisitive. In Section 5.2, we will consider various extensions of the basic framework. For every normal modal logic $\mathfrak{L}$, we will define a corresponding inquisitive system $\mathsf{Inq}\mathfrak{L}$, obtained from InqK by restricting the semantics to Kripke models based on $\mathfrak{L}$-frames. After presenting a generic completeness result by Ciardelli (2016b), we then give a precise definition of the class of all *geometric extensions* of InqK. In Section 5.3, we will turn to the construction of labelled sequent calculi for the full class of geometric extensions of InqK. To this end, we will describe a general strategy that allows to transform any set of geometric implications into a corresponding set of sequent rules. For every inquisitive logic $\mathsf{Inq}\mathfrak{L}$ determined by some finite set of geometric axioms A, we thus obtain a cut-free labelled sequent calculus $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. In Section 5.4, we will investigate the properties of our sequent calculi. We will see that each of our proof systems enjoys cut-admissibility, height-preserving admissibility of weakening and contraction and height-preserving invertibility of all rules. In Section 5.5, we will prove the soundness of our calculi and show that, for some concrete choices of the underlying base logic $\mathfrak{L}$, the completeness of $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ may also be established indirectly, by using a suitable Hilbert-style system for $\mathsf{Inq}\mathfrak{L}$. In Section 5.6, finally, we will give a *general* completeness proof, covering each of the calculi $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. The argument is based on the construction of an infinite proof search tree and the extraction of a countermodel from an open branch of this tree.

5.1 Kripke Modalities in Inquisitive Logic

Let us start by introducing the basic framework of inquisitive Kripke logic. The system described in this section will be called InqK[1] and may be considered from two different angles: on the one hand, it may be seen as the result of adding the question-forming operator $\vee\!\!\!\vee$ to the standard system K of basic modal logic. On the other hand, it can be conceived as the result of adding the Kripke modality $\Box$ to the system InqB of basic inquisitive logic. As before, we will assume an infinite set P of atomic propositions, denoted by p, q, r, etc. The formulas of InqK are now built up from the atoms in P by means of the usual connectives of InqB and the modal operator $\Box$.

Definition 5.1.1 (Language of InqK)**.** The language of InqK is denoted by $\mathcal{L}^{\mathsf{K}}$ and consists of all formulas generated by the following grammar, where p ranges over atomic propositions from P:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid \Box\varphi.$$

As in the basic system InqB, the connectives $\neg$ and $\vee$ will be treated as abbreviations, by putting $\neg\varphi := \varphi \to \bot$ and $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$. In addition, the dual modality $\Diamond$ is taken to be defined by $\Diamond\varphi := \neg\Box\neg\varphi$. The symbols $\Box$ and $\Diamond$ will also be called the *box operator* and the *diamond operator*, respectively. By adopting the terminology used in the previous chapters, we will refer to $\vee\!\!\!\vee$ as the *inquisitive disjunction* and to $\vee$ as the *standard disjunction* of our system. Moreover, a formula not containing occurrences of $\vee\!\!\!\vee$ is said to be a *standard formula*. As before, standard formulas will be denoted by the meta-variables α, β, γ, etc., whereas φ, ψ, χ, etc., will be used for arbitrary formulas. Moreover, the set of all standard formulas is denoted by $\mathcal{L}^{\mathsf{K}}_s$.

For the sake of illustration, we will usually adopt an *epistemic* interpretation of the modalities. Under this interpretation, $\Box$ is used to express the *knowledge* of an agent, so $\Box\varphi$ may be read as 'The agent knows φ' and $\Diamond\varphi$ may be read as 'φ is compatible with the knowledge of the agent' or 'The knowledge of the agent does not rule out φ'. The reader should bear in mind, however, that the epistemic reading is not the only possible interpretation and different readings of $\Box$ and $\Diamond$ can be given as well. A short overview is provided by Ciardelli (2022, pp. 249–250, 252).

[1] It should be noted that Ciardelli (2016b) uses the name InqBK instead of InqK. In order to avoid an overly complicated nomenclature (especially when introducing our sequent calculi), we will henceforth deviate from this notation.

Note that, in the language $\mathcal{L}^{\mathsf{K}}$, the modality $\Box$ can be used to embed both *questions* and *assertions*. This is an important difference to standard modal logic, where $\Box$ can only be applied to statements. To appreciate this fact, let us consider the following sentences in natural language:

(1) Alice knows that Bob is lying.

(2) Alice knows whether Bob is lying.

In the first sentence, Alice's knowledge is directed towards a *purely declarative* proposition, saying that Bob is lying. Thus, if we take $\Box$ to express Alice's knowledge, then (1) can be represented by a standard formula of the shape $\Box p$. Intuitively, we expect this formula to be true at a possible world, if Alice's knowledge state at this world entails the information conveyed by p. Now, consider sentence (2). In this sentence, Alice's knowledge is directed towards an *inquisitive* proposition—namely, the proposition *whether Bob is lying*. In the language $\mathcal{L}^{\mathsf{K}}$, this corresponds to a formula of the form $\Box ?p$, where $\Box$ now embeds the alternative question $?p$.[2] Intuitively, this formula is true at a world, just in case Alice's knowledge state at this world settles the issue raised by $?p$. Note that, by the semantics of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, this is equivalent to saying that Alice's knowledge either entails p or it entails $\neg p$, so we expect $\Box ?p$ to be equivalent to the standard formula $\Box p \vee \Box \neg p$. In fact, as we will see later on, the modal operator $\Box$ always distributes over inquisitive disjunctions, turning them into standard ones, so InqK validates the equivalence $\Box(\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi) \equiv \Box\varphi \vee \Box\psi$. As a consequence, an inquisitive disjunction occurring in the scope of a box operator can always be paraphrased away by means of a standard disjunction. So, what is the point of allowing questions to be embedded under $\Box$ in InqK? The answer is that, by adding $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ to standard modal logic, InqK allows for a *uniform treatment* of sentences like (1) and (2). This is an advantage over standard modal logic, where paraphrasing is *necessary* in order to cope with sentences like (2).

We now want to give a brief outline of the semantics of InqK and sketch some important properties of the system. Our presentation mainly follows the exposition given by Ciardelli (2016b, Chapter 6), which is also an excellent source for further details. To start with, the formulas of InqK are evaluated with respect to ordinary Kripke models, which are defined in the usual way.

Definition 5.1.2 (Kripke Frame, Kripke Model)**.** A *Kripke frame* is defined to be a pair $F = \langle W, R\rangle$, where W is a set whose elements are called *worlds*, and $R \subseteq W \times W$ is a binary relation on W, referred to as the *accessibility relation* of the frame. By a *Kripke model*, we mean a triple $M = \langle W, R, V\rangle$, where $\langle W, R\rangle$ is a Kripke frame and $V : W \times \mathsf{P} \to \{0, 1\}$ is a *valuation function*.

Given a Kripke frame $F = \langle W, R\rangle$ and worlds $w, u \in W$, we will say that u is a *successor* of w in F, just in case we have wRu. Furthermore, the *neighbourhood* of a world w in F is denoted by $R(w)$ and defined to be the set of all successors of w, so we put $R(w) := \{u \in W \mid wRu\}$. As before, a set of worlds $s \subseteq W$ is also referred to as an *information state* over F.

Intuitively, every world in a Kripke model may be conceived as a possible *state of affairs*. This is similar to the interpretation of worlds in the basic inquisitive system InqB. However, in contrast to the standard setting, the state of affairs represented by a world w is now determined not only by the propositional atoms true at w, but also by the information state $R(w)$ assigned to w. Under the epistemic interpretation of $\Box$ explained above, $R(w)$ could be taken to represent the information available to an agent at world w—or, to put it differently, the set of worlds considered to be possible according to the current knowledge of the agent at that world.

Since formulas of inquisitive logic are evaluated with respect to *information states*, rather than *worlds*, we cannot simply adopt the usual truth-conditional semantics for $\Box$ known from standard modal logic. Instead, we have to find a suitable *support clause* for formulas of the shape $\Box\varphi$. But how could such a support clause look like? Recall that, under the epistemic interpretation, a

[2] As usual, $?\varphi$ is defined to be an abbreviation for $\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg\varphi$.

formula of the form $\Box\varphi$ is read as 'The agent knows φ'. Now, given a specific world w in a Kripke model, we would expect an agent to know a proposition φ at w, just in case the agent's knowledge state $R(w)$ at w entails the information conveyed by φ, and $R(w)$ settles the issue raised by φ. In other words, the agent knows φ at world w if and only if $R(w)$ *supports* φ. By generalizing this observation to information states, we thus find that a formula $\Box\varphi$ should be supported by a state s if and only if, for every world $w \in s$, the formula φ is supported by $R(w)$. This yields the following support semantics for the language of InqK (cf. Ciardelli 2016b, p. 202).

Definition 5.1.3 (Support Semantics for InqK)**.** Let $M = \langle W, R, V\rangle$ be a Kripke model. The support relation $\models$ between states $s \subseteq W$ and formulas $\varphi \in \mathcal{L}^{\mathsf{K}}$ is inductively defined as follows:

(i) $M, s \models p \quad :\Leftrightarrow \quad V(w, p) = 1$ for all $w \in s$,
(ii) $M, s \models \bot \quad :\Leftrightarrow \quad s = \emptyset$,
(iii) $M, s \models \varphi \wedge \psi \quad :\Leftrightarrow \quad M, s \models \varphi$ and $M, s \models \psi$,
(iv) $M, s \models \varphi \to \psi \quad :\Leftrightarrow \quad$ for all $t \subseteq s$, if $M, t \models \varphi$, then $M, t \models \psi$,
(v) $M, s \models \varphi \vee\!\!\!\vee\, \psi \quad :\Leftrightarrow \quad M, s \models \varphi$ or $M, s \models \psi$,
(vi) $M, s \models \Box\varphi \quad :\Leftrightarrow \quad M, R(w) \models \varphi$ for all $w \in s$.

If $M, s \models \varphi$ holds, then we say that φ is *supported* by s in M. Observe that, in InqK, the notion of support is defined in exactly the same way as in the basic system InqB, except that we now also have a support clause for the modal operator $\Box$. Using the support relation $\models$, we can now define some important semantic concepts. To begin with, given a Kripke frame $F = \langle W, R\rangle$ and a formula φ, we write $F \models \varphi$ and say that φ is *valid in the frame* F, if for every model $M = \langle F, V\rangle$, based on F, and for every state $s \subseteq W$, we have $M, s \models \varphi$. A formula φ is said to be *valid in* InqK, denoted $\models \varphi$, if φ is valid in every frame. Furthermore, given a formula φ and a set of formulas Γ, we write $\Gamma \models \varphi$ and say that φ is *entailed* by Γ, if for every Kripke model $M = \langle W, R, V\rangle$ and for every state $s \subseteq W$, it is the case that $M, s \models \Gamma$ implies $M, s \models \varphi$.[3] Finally, two formulas φ and ψ are called *equivalent*, notation $\varphi \equiv \psi$, if we have both $\varphi \models \psi$ and $\psi \models \varphi$.

We now want to highlight some basic properties of InqK. For a more detailed account, we refer to Ciardelli (2016b, Chapter 6). First of all, it is easy to check that the support relation of InqK is persistent and that every formula $\varphi \in \mathcal{L}^{\mathsf{K}}$ is supported by the empty information state.

Proposition 5.1.4. *Let M be a Kripke model, let s and t be states and let $\varphi \in \mathcal{L}^{\mathsf{K}}$ be a formula.*

(i) Persistency: if $M, s \models \varphi$ and $t \subseteq s$, then $M, t \models \varphi$.
(ii) Empty state property: $M, \emptyset \models \varphi$.

Both statements can be proved by a straightforward induction on the structure of φ. Next, we can show that InqK validates the usual principle of *necessitation*: if a formula $\varphi \in \mathcal{L}^{\mathsf{K}}$ is valid in InqK, then so is $\Box\varphi$. In addition, $\Box$ distributes over implications, so InqK validates the well-known axiom scheme K familiar from standard modal logic (cf. Blackburn et al. 2001, p. 33).

Proposition 5.1.5 (Validity of K and Necessitation)**.** *For all $\varphi, \psi \in \mathcal{L}^{\mathsf{K}}$, the following holds:*

(i) $\models \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,
(ii) if $\models \varphi$, then $\models \Box\varphi$.

Proof. The proof of the first part is easy and therefore omitted. In order to prove the second part, suppose that we have $\models \varphi$. Let M be an arbitrary Kripke model and let s be an arbitrary state over M. Because $\models \varphi$, we must have $M, R(w) \models \varphi$ for all worlds $w \in s$. Hence, by the support clause for $\Box$, it follows $M, s \models \Box\varphi$. Since M and s were arbitrary, this shows that $\models \Box\varphi$. $\Box$

We now want to work out some more exciting properties of InqK. As in our treatment of basic inquisitive logic, InqB, we will say that a formula φ is *true* at a world w of a model M,

[3] As before, $M, s \models \Gamma$ is used as a shorthand for '$M, s \models \psi$ for all $\psi \in \Gamma$'.

if φ is supported by the singleton state $\{w\}$. In this case, we also write $M, w \models \varphi$ instead of $M, \{w\} \models \varphi$. Moreover, a formula φ is said to be *truth-conditional*, if for every model M and every state s, we have: $M, s \models \varphi$ if and only if $M, w \models \varphi$ for all $w \in s$. That is, a truth-conditional formula is a formula for which support at a state simply boils down to truth at every world in the state. Now, by a straightforward inspection of the support clause for $\Box$ given in Definition 5.1.3, one readily sees that every formula of the form $\Box\varphi$ is truth-conditional in InqK.

Proposition 5.1.6. *For every $\varphi \in \mathcal{L}^{\mathsf{K}}$, the formula $\Box\varphi$ is truth-conditional.*

In our exposition of the basic system InqB, we have already seen that all standard formulas of classical logic remain truth-conditional in the inquisitive setting (see Corollary 1.3.4). Proposition 5.1.6 tells us that this can be generalized to all formulas of standard *modal* logic, i.e., to all formulas in $\mathcal{L}^{\mathsf{K}}$ not containing $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$. In fact, since $\Box$ *always* yields a truth-conditional formula—even when applied to a question—we may even identify a richer syntactic fragment of InqK that is guaranteed to have this property. The formulas in this fragment will be referred to as *declaratives*.

Definition 5.1.7. The *declarative fragment* of InqK is the set of formulas $\mathcal{L}^{\mathsf{K}}_{!}$ generated by the following grammar, where p ranges over atoms and φ ranges over arbitrary formulas in $\mathcal{L}^{\mathsf{K}}$:

$$\alpha ::= p \mid \bot \mid \Box\varphi \mid \alpha \wedge \alpha \mid \alpha \to \alpha.$$

By overloading notation, we will henceforth use the meta-variables α, β, γ, etc., for both *declarative* formulas and *standard* formulas. In any case, no confusion will arise, since it will always be clear from the context whether a declarative or a standard formula is meant. Note that every standard formula is also a declarative formula but not vice versa: for instance, $\Box(p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} q)$ is a declarative, but not a standard formula, since it contains an occurrence of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$. Now, using Proposition 5.1.6 and the fact that truth-conditionality is preserved by the connectives $\bot$, $\wedge$ and $\to$, we may conclude that every formula in the declarative fragment of InqK is truth-conditional.

Proposition 5.1.8. *Every formula in the fragment $\mathcal{L}^{\mathsf{K}}_{!}$ is truth-conditional.*

By the support clause for $\Box$, we know that $\Box\varphi$ is true at a world w of a Kripke model M, just in case φ is supported by the neighbourhood $R(w)$ of w. Consequently, if φ is a *truth-conditional* formula, then the truth conditions for $\Box\varphi$, as determined by our support semantics, are simply the familiar ones from standard modal logic, so we have $M, w \models \Box\varphi$ if and only if $M, u \models \varphi$ for all worlds u with wRu. Now, because every standard formula is truth-conditional—with the same truth conditions as in standard modal logic—it is easy to see that InqK must be a *conservative extension* of the basic modal logic K. That is, a standard formula $\alpha \in \mathcal{L}^{\mathsf{K}}_{s}$ is valid in InqK if and only if α is valid in K. As shown by Ciardelli (2016b, p. 209), there is even a more intriguing connection between InqK and standard modal logic: a formula of InqK is truth-conditional if and only if it is equivalent to some standard modal formula $\alpha \in \mathcal{L}^{\mathsf{K}}_{s}$. Therefore, with respect to *statements*, InqK has exactly the same expressive power as standard modal logic.

Proposition 5.1.9. *A formula $\varphi \in \mathcal{L}^{\mathsf{K}}$ is truth-conditional if and only if there exists a standard formula $\alpha \in \mathcal{L}^{\mathsf{K}}_{s}$ such that $\varphi \equiv \alpha$.*

The right-to-left direction of the proposition is trivial, since standard formulas are always truth-conditional by Proposition 5.1.8. For the other direction, one uses the observation that every $\varphi \in \mathcal{L}^{\mathsf{K}}$ can be translated to a standard formula $\varphi^{s} \in \mathcal{L}^{\mathsf{K}}_{s}$ in such a way that φ and φ^{s} have the same truth conditions. Thus, if φ itself is truth-conditional, then φ^{s} will be equivalent to φ (cf. Ciardelli 2016b, pp. 208–209). We already mentioned above that, intuitively, a formula of the form $\Box ?p$ should be equivalent to $\Box p \vee \Box\neg p$. The following proposition confirms this intuition: in InqK, the modality $\Box$ distributes over inquisitive disjunctions, turning them into standard ones.

Proposition 5.1.10 (Distributivity of $\Box$ over $\vee\!\!\!\vee$)**.** *For all* $\varphi, \psi \in \mathcal{L}^{\mathsf{K}}$*, it holds* $\Box(\varphi \vee\!\!\!\vee \psi) \equiv \Box\varphi \vee \Box\psi$*.*

Proof. By Proposition 5.1.8, we know that $\Box(\varphi \vee\!\!\!\vee \psi)$ and $\Box\varphi \vee \Box\psi$ are both truth-conditional. Hence, it suffices to check that they have the same truth conditions. This is achieved as follows:

$$
\begin{aligned}
M, w \models \Box(\varphi \vee\!\!\!\vee \psi) \quad &\Leftrightarrow \quad M, R(w) \models \varphi \vee\!\!\!\vee \psi \\
&\Leftrightarrow \quad M, R(w) \models \varphi \text{ or } M, R(w) \models \psi \\
&\Leftrightarrow \quad M, w \models \Box\varphi \text{ or } M, w \models \Box\psi \\
&\Leftrightarrow \quad M, w \models \Box\varphi \vee \Box\psi.
\end{aligned}
$$

Observe that, in the last step, we used the ordinary truth conditions for the defined connective $\vee$, which clearly remain valid in InqK (see also Proposition 1.2.9). $\Box$

Using the support semantics given in Definition 5.1.3, one may now also derive a support clause for the defined modality $\Diamond$ (for the other defined operators, the clauses are the same as in InqB).

Proposition 5.1.11 (Support-Conditions for $\Diamond$)**.** *For every Kripke model* M *and for every state* s*, we have* $M, s \models \Diamond\varphi$ *if and only if, for every* $w \in s$*, there exists some* $u \in R(w)$ *such that* $M, u \models \varphi$*.*

Note that, as a consequence, $\Diamond\varphi$ is true at a world w of a model M, just in case we have $R(w) \cap |\varphi|_M \neq \emptyset$, where $|\varphi|_M$ denotes the truth-set of φ with respect to M (see Definition 1.2.10). Thus, as expected, $\Diamond\varphi$ expresses that the agent's knowledge is *compatible* with the information conveyed by φ. For later purposes, we also need the following lemma, saying that failure of support can be restricted to states of *finite* size: if a formula φ is *not* supported by some state s, then one can always find a finite enhancement $t \subseteq s$ such that φ is not supported by t.

Lemma 5.1.12. *Let* M *be a Kripke model. For every formula* $\varphi \in \mathcal{L}^{\mathsf{K}}$ *and for every state* s *over* M*, if it holds* $M, s \not\models \varphi$*, then there exists a finite substate* $t \subseteq s$ *such that* $M, t \not\models \varphi$*.*

Proof. By induction on the structure of φ. The base case and the inductive step for $\wedge$ are trivial. Consider now the case in which φ is of the form $\varphi = \psi \to \chi$. Suppose that we have $M, s \not\models \psi \to \chi$ for some state s over M. By the semantics of $\to$, there must be some $r \subseteq s$ such that $M, r \models \psi$ and $M, r \not\models \chi$. Hence, by induction hypothesis, there exists a finite subset $t \subseteq r \subseteq s$ such that $M, t \not\models \chi$. Since we have $M, r \models \psi$ and $t \subseteq r$, we must also have $M, t \models \psi$ by the persistency of support in InqK. Therefore, t is a finite substate of s such that $M, t \not\models \psi \to \chi$.

Let now φ be of the form $\varphi = \psi \vee\!\!\!\vee \chi$ and suppose that we have $M, s \not\models \psi \vee\!\!\!\vee \chi$. By the semantics of $\vee\!\!\!\vee$, this yields $M, s \not\models \psi$ and $M, s \not\models \chi$. Thus, by induction hypothesis, there are finite subsets $t_1, t_2 \subseteq s$ such that $M, t_1 \not\models \psi$ and $M, t_2 \not\models \chi$. Consider the union $t := t_1 \cup t_2$. Since t_1 and t_2 are both finite, t must be finite as well. Now, suppose for a contradiction that $M, t \models \psi \vee\!\!\!\vee \chi$. This yields $M, t \models \psi$ or $M, t \models \chi$, so it follows $M, t_1 \models \psi$ or $M, t_2 \models \chi$ by the persistency of support. But this is a contradiction to our assumption about t_1 and t_2. Hence, we have $M, t \not\models \psi \vee\!\!\!\vee \chi$.

Finally, let us consider the case $\varphi = \Box\psi$. Suppose that we have $M, s \not\models \Box\psi$ for some state s over M. By the semantics of $\Box$, there must be some world $w \in s$ such that $M, R(w) \not\models \psi$. But then, clearly, for the finite state $t \subseteq s$ given by $t := \{w\}$, we also have $M, t \not\models \Box\psi$. $\Box$

5.2 Extensions of InqK

In the previous section, we constructed InqK as an inquisitive extension of the basic modal logic K. It is well known that K can be seen as the *weakest* standard modal logic, since validity in K simply boils down to validity in *all* Kripke frames. For this reason, in standard modal logic, one usually also defines various extensions of K: either by imposing further restrictions on the accessibility relation of a Kripke frame, or by requiring frames to validate certain additional axioms.

In this section, we will do something analogous for the inquisitive system InqK. More precisely, for every normal modal logic $\mathfrak{L}$, we will define a corresponding inquisitive extension Inq$\mathfrak{L}$. After introducing some standard terminology, we will see a general completeness result by Ciardelli (2016b), covering all those systems Inq$\mathfrak{L}$ for which the underlying base logic $\mathfrak{L}$ is *canonical*. At the end of this section, we will also introduce a special class of inquisitive logics, referred to as *geometric extensions* of InqK. Roughly speaking, this class comprises all systems Inq$\mathfrak{L}$ that can be determined by first-order frame conditions of the form $\forall\vec{w}(\varphi \to \psi)$, where φ and ψ are not allowed to contain implications or universal quantifiers. This will become important in Section 5.3, where we will define cut-free labelled sequent calculi for the full class of geometric extensions of InqK.

5.2.1 Some Basic Notions

To begin with, let us recall some basic terminology from standard modal logic. For a more comprehensive exposition of the material, we refer to Blackburn et al. (2001, Chapter 4). First, a *normal modal logic* is a set of standard formulas $\mathfrak{L}$ such that $\mathfrak{L}$ contains all propositional tautologies and all instances of the K-schema, and $\mathfrak{L}$ is closed under modus ponens and necessitation.

Definition 5.2.1 (Normal Modal Logic)**.** By a *normal modal logic*, we will mean any set of standard modal formulas $\mathfrak{L} \subseteq \mathcal{L}_s^{\mathsf{K}}$ such that each of the following four conditions is satisfied:

(i) $\mathfrak{L}$ contains all instances of propositional tautologies,
(ii) $\mathfrak{L}$ contains all formulas of the form $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$,
(iii) If $\alpha \in \mathfrak{L}$ and $(\alpha \to \beta) \in \mathfrak{L}$, then $\beta \in \mathfrak{L}$,
(iv) If $\alpha \in \mathfrak{L}$, then $\Box\alpha \in \mathfrak{L}$.

Observe that normal modal logics are characterized in a purely *syntactic* manner. In fact, Definition 5.2.1 can be seen as a straightforward generalization of the concept *provability* in Hilbert-style systems for modal logics. It is a generalization, since it does not talk directly about *proofs* in such a system, but focuses on what is actually important: the availability of certain axioms and the closure under modus ponens and necessitation. In line with this syntactic perspective, we will also say that a standard formula $\alpha \in \mathcal{L}_s^{\mathsf{K}}$ is a *theorem* of a normal modal logic $\mathfrak{L}$, notation $\vdash_{\mathfrak{L}} \alpha$, if α is an element of $\mathfrak{L}$. More generally, given any set of standard formulas $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_s^{\mathsf{K}}$ and a normal modal logic $\mathfrak{L}$, we write $\Gamma \vdash_{\mathfrak{L}} \alpha$ and say that α *is deducible from* Γ *in* $\mathfrak{L}$, just in case we have $\vdash_{\mathfrak{L}} \alpha$ or there are formulas $\beta_1, \dots, \beta_n \in \Gamma$ such that $\vdash_{\mathfrak{L}} (\beta_1 \wedge \dots \wedge \beta_n) \to \alpha$.

Note that, according to Definition 5.2.1, the set of *all* standard formulas is also a normal modal logic (one might call this the 'trivial' or the 'inconsistent' modal logic). Furthermore, it is easy to verify that any (finite or infinite) intersection of normal modal logics is again a normal modal logic. As a consequence, one can show that, for every set of standard formulas $\Gamma \subseteq \mathcal{L}_s^{\mathsf{K}}$, there exists a *smallest* normal modal logic $\mathfrak{L}_\Gamma$ containing Γ, i.e., $\mathfrak{L}_\Gamma$ satisfies $\Gamma \subseteq \mathfrak{L}_\Gamma$, and for every normal modal logic $\mathfrak{L}$ with $\Gamma \subseteq \mathfrak{L}$, we have $\mathfrak{L}_\Gamma \subseteq \mathfrak{L}$. In what follows, we will call $\mathfrak{L}_\Gamma$ the normal modal logic *generated* or *axiomatized* by Γ, and we say that Γ is an *axiom system* for $\mathfrak{L}_\Gamma$.

Let us consider some examples of normal modal logics. The normal modal logic generated by the empty set is denoted by K and may be conceived as the *smallest* or the *weakest* normal modal logic, since we clearly have $\mathsf{K} \subseteq \mathfrak{L}$ for every normal modal logic $\mathfrak{L}$. Now, let us turn to some extensions of K. In Table 5.1, we list a number of well-known axiom schemes from standard modal logic. For every combination of these schemes, we may obtain an extension of the basic system K by constructing the normal modal logic generated by the schemes. It is common practice to denote the systems obtained in this way simply by attaching the names of the corresponding schemes to the letter K. So, for example, KT refers to the normal modal logic generated by the scheme T, whereas KD4 stands for the modal logic generated by the schemes D and 4. However, some logics also carry traditional names that are often more common in the literature. For instance, instead of KT, KD, KT4 and KT5, one also writes T, D, S4 and S5, respectively.

Axiom Scheme		Frame Condition	First-Order Formula
T	$\Box\alpha \to \alpha$	Reflexivity	$\forall u.Ruu$
4	$\Box\alpha \to \Box\Box\alpha$	Transitivity	$\forall uvw.(Ruv \land Rvw \to Ruw)$
5	$\Diamond\alpha \to \Box\Diamond\alpha$	Euclideanness	$\forall uvw.(Ruv \land Ruw \to Rvw)$
B	$\alpha \to \Box\Diamond\alpha$	Symmetry	$\forall uv.(Ruv \to Rvu)$
D	$\Box\alpha \to \Diamond\alpha$	Seriality	$\forall u\exists v.Ruv$

Table 5.1: Some famous axiom schemes and the corresponding frame properties.

It is well known that axiom schemes of standard modal logic can also be used to characterize certain classes of Kripke frames. Roughly speaking, we say that a class C of Kripke frames is *defined* by a scheme Θ, just in case, for every frame F, we have $F \in \mathsf{C}$ if and only if F validates all instances of Θ.[4] The classes of frames defined by the schemes T, 4, 5, B and D are presented in the second column of Table 5.1. Thus, for example, one can show that a Kripke frame F is *reflexive* if and only if all instances of the T-schema are valid in F; and a frame F is *transitive* if and only if all instances of axiom scheme 4 are valid in F. As indicated in the last column of the table, many properties of frames can also be characterized by a formula of first-order logic. This will become important in Section 5.2.3, where we want to consider the class of all Kripke frames that are characterized by a certain type of such formulas, known as *geometric implications*.

So far, we have considered normal modal logics mainly from a syntactic point of view. But there is also a *semantic* perspective. To make things precise, we must introduce some further terminology. Given a normal modal logic $\mathfrak{L}$, we say that a frame F is an *$\mathfrak{L}$-frame*, if F validates every formula in $\mathfrak{L}$. Moreover, for every set of standard formulas $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_s^{\mathsf{K}}$, we write $\Gamma \models_{\mathfrak{L}} \alpha$ and say that *α is entailed by Γ in $\mathfrak{L}$*, if for every Kripke model M, based on an $\mathfrak{L}$-frame, and for every world w in M, it is the case that $M, w \models \Gamma$ implies $M, w \models \alpha$. In other words, $\models_{\mathfrak{L}}$ is the consequence relation obtained by restricting the usual notion of entailment in standard modal logic to Kripke models based on $\mathfrak{L}$-frames.[5] The link between the syntactic and the semantic perspective is provided by the concepts of *soundness* and *completeness* which are defined in the usual way.

Definition 5.2.2 (Soundness and Completeness)**.** Let $\mathfrak{L}$ be a normal modal logic. We say that $\mathfrak{L}$ is *sound*, if for every set of standard formulas $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_s^{\mathsf{K}}$, we have: $\Gamma \vdash_{\mathfrak{L}} \alpha$ implies $\Gamma \models_{\mathfrak{L}} \alpha$. And $\mathfrak{L}$ is called *complete*, if for any $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_s^{\mathsf{K}}$, it is the case that $\Gamma \models_{\mathfrak{L}} \alpha$ implies $\Gamma \vdash_{\mathfrak{L}} \alpha$.

A useful tool for showing the completeness of a normal modal logic are *canonical models*. Towards a definition of this concept, consider an arbitrary normal modal logic $\mathfrak{L}$ and an arbitrary set of standard formulas $\Gamma \subseteq \mathcal{L}_s^{\mathsf{K}}$. We say that Γ is *$\mathfrak{L}$-consistent*, if $\Gamma \not\vdash_{\mathfrak{L}} \bot$. And we say that Γ is *maximally $\mathfrak{L}$-consistent*, if Γ is $\mathfrak{L}$-consistent and no proper extension $\Delta \supsetneq \Gamma$ is also $\mathfrak{L}$-consistent.

Definition 5.2.3 (Canonical Model for $\mathfrak{L}$)**.** Let $\mathfrak{L}$ be a normal modal logic. The *canonical model* for $\mathfrak{L}$ is the Kripke model $M_{\mathfrak{L}} = \langle W_{\mathfrak{L}}, R_{\mathfrak{L}}, V_{\mathfrak{L}} \rangle$ defined as follows:

(i) $W_{\mathfrak{L}}$ is the set of all maximally $\mathfrak{L}$-consistent sets,

(ii) $(\Gamma, \Delta) \in R_{\mathfrak{L}} \quad :\Leftrightarrow \quad$ for all $\alpha \in \mathcal{L}_s^{\mathsf{K}}$, if $\Box\alpha \in \Gamma$, then $\alpha \in \Delta$,

(iii) $V_{\mathfrak{L}}(\Gamma, p) = 1 \quad :\Leftrightarrow \quad p \in \Gamma$.

The Kripke frame $F_{\mathfrak{L}} := \langle W_{\mathfrak{L}}, R_{\mathfrak{L}} \rangle$ is also referred to as the *canonical frame* for $\mathfrak{L}$. Crucially, in the canonical model for any normal modal logic $\mathfrak{L}$, truth at a world Γ simply boils down to membership in the set Γ. This is known as the *truth lemma* (cf. Blackburn et al. 2001, p. 199).

[4] Recall that, in standard modal logic, a formula α is said to be *valid* in a Kripke frame F, just in case α is true at every world of every Kripke model M based on F. Since standard modal formulas are guaranteed to be truth-conditional in InqK, this coincides with the notion of validity introduced in Section 5.1.

[5] One may also define entailment more generally, by relativizing consequence to an *arbitrary* class of frames C (cf. Blackburn et al. 2001, p. 31). Our definition can then be seen as the special case in which C is the class of all $\mathfrak{L}$-frames.

Proposition 5.2.4 (Truth Lemma)**.** *For every normal modal logic $\mathfrak{L}$, for every maximally $\mathfrak{L}$-consistent set $\Gamma \subseteq \mathcal{L}_s^{\mathsf{K}}$ and for every standard formula $\alpha \in \mathcal{L}_s^{\mathsf{K}}$, we have: $M_{\mathfrak{L}}, \Gamma \models \alpha$ if and only if $\alpha \in \Gamma$.*

The usefulness of canonical models stems from the fact that, for many modal logics, they provide us with a simple completeness proof. However, this only works if the frame $F_{\mathfrak{L}}$ actually validates the formulas in $\mathfrak{L}$. A normal modal logic satisfying this constraint is said to be *canonical*.

Definition 5.2.5 (Canonicity)**.** A normal modal logic $\mathfrak{L}$ is *canonical*, if $F_{\mathfrak{L}}$ is an $\mathfrak{L}$-frame.

It is well known that every normal modal logic generated by one or more of the axiom schemes given in Table 5.1 is canonical. However, there are also normal modal logics that are *not* canonical. A famous example of such a logic is KL, the normal modal logic generated by the *Löb axiom*: $\Box(\Box\alpha \to \alpha) \to \Box\alpha$. A proof of this fact is provided by Blackburn et al. (2001, p. 211). As outlined above, canonicity always implies completeness, so we have the following general result.

Proposition 5.2.6. *If a normal modal logic $\mathfrak{L}$ is canonical, then it is complete.*

The general idea of the proof can be described as follows: suppose that $\mathfrak{L}$ is canonical and $\Gamma \not\vdash_{\mathfrak{L}} \alpha$. Then, $\Gamma \cup \{\neg\alpha\}$ is $\mathfrak{L}$-consistent. Thus, by a suitable variant of Lindenbaum's lemma, it can be extended to a maximally $\mathfrak{L}$-consistent set Δ. We now have $M_{\mathfrak{L}}, \Delta \models \Gamma$ and $M_{\mathfrak{L}}, \Delta \not\models \alpha$ by the truth lemma. Since $\mathfrak{L}$ is canonical, $M_{\mathfrak{L}}$ is based on an $\mathfrak{L}$-frame, so this yields $\Gamma \not\models_{\mathfrak{L}} \alpha$.

5.2.2 Extensions of InqK Based on Normal Modal Logics

For every normal modal logic $\mathfrak{L}$, we may now define a corresponding inquisitive system Inq$\mathfrak{L}$ which extends the system InqK introduced above. Formally, Inq$\mathfrak{L}$ is obtained from InqK by restricting entailment to Kripke models based on $\mathfrak{L}$-frames, so we adopt the following definition.

Definition 5.2.7 (The Systems Inq$\mathfrak{L}$)**.** Let $\mathfrak{L}$ be a normal modal logic. For any set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{K}}$, we write $\Gamma \models_{\mathfrak{L}}^{\mathsf{Inq}} \varphi$ and say that φ *is entailed by* Γ *in* Inq$\mathfrak{L}$, if for every Kripke model M, based on an $\mathfrak{L}$-frame, and for every state s over M, we have: if $M, s \models \Gamma$, then $M, s \models \varphi$.

Given any normal modal logic $\mathfrak{L}$, we will also say that Inq$\mathfrak{L}$ is the inquisitive system *based on* $\mathfrak{L}$. Observe that, in particular, InqK is the system based on the smallest normal modal logic, K. Moreover, it is easy to see that, for every normal modal logic $\mathfrak{L}$, Inq$\mathfrak{L}$ is in fact an *extension* of InqK, in the sense that every formula valid in InqK is also valid in Inq$\mathfrak{L}$ but not necessarily the other way around. In other words, if InqK and Inq$\mathfrak{L}$ are identified with the sets of their validities, then InqK $\subseteq$ Inq$\mathfrak{L}$. As an immediate consequence of this, we obtain the following facts.

Proposition 5.2.8. *For every normal modal logic $\mathfrak{L}$ and for all formulas $\varphi, \psi \in \mathcal{L}^{\mathsf{K}}$, we have:*
(i) $\models_{\mathfrak{L}}^{\mathsf{Inq}} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,
(ii) $\models_{\mathfrak{L}}^{\mathsf{Inq}} \Box(\varphi \mathbin{\vee\!\!\!\vee} \psi) \to (\Box\varphi \vee \Box\psi)$,
(iii) If $\models_{\mathfrak{L}}^{\mathsf{Inq}} \varphi$, *then* $\models_{\mathfrak{L}}^{\mathsf{Inq}} \Box\varphi$.

Proof. The statements follow immediately from Propositions 5.1.5 and 5.1.10. □

In Section 5.1, we already mentioned that InqK is a *conservative extension* of the normal modal logic K. Thus, a standard formula is valid in InqK if and only if it is valid in K. The following proposition shows that this can be generalized to each of the logics Inq$\mathfrak{L}$ (cf. Ciardelli 2016b, p. 213).

Proposition 5.2.9 (Conservativity over $\mathfrak{L}$)**.** *Let $\mathfrak{L}$ be a normal modal logic. Then, for every set of standard formulas $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_s^{\mathsf{K}}$, we have: $\Gamma \models_{\mathfrak{L}}^{\mathsf{Inq}} \alpha$ if and only if $\Gamma \models_{\mathfrak{L}} \alpha$.*

$$\frac{\alpha \to (\varphi \mathbin{\vee\!\!\!\vee} \psi)}{(\alpha \to \varphi) \mathbin{\vee\!\!\!\vee} (\alpha \to \psi)}\ (\text{split}) \qquad \frac{\Box(\varphi \to \psi)}{\Box\varphi \to \Box\psi}\ (\to\text{dis})$$

$$\frac{\neg\neg\alpha}{\alpha}\ (\text{dne}) \qquad \frac{\Box(\varphi \mathbin{\vee\!\!\!\vee} \psi)}{\Box\varphi \vee \Box\psi}\ (\mathbin{\vee\!\!\!\vee}\text{dis}) \qquad \frac{\varphi}{\Box\varphi}\ (\text{nec})$$

Figure 5.1: Special rules of the natural deduction system for $\mathsf{Inq}\mathfrak{L}$ given by Ciardelli (2016b). In the rules (split) and (dne), $\alpha \in \mathcal{L}^{\mathsf{K}}_{!}$ must be a *declarative* formula. Moreover, the rule (nec) can only be applied if the subdeduction ending with φ has no open hypotheses.

However, note that this correspondence between a normal modal logic $\mathfrak{L}$ and its inquisitive counterpart, $\mathsf{Inq}\mathfrak{L}$, is restricted to *standard formulas* only. In particular, it may very well be the case that $\mathfrak{L}$ validates certain axiom schemes for *standard* formulas that do not remain valid in $\mathsf{Inq}\mathfrak{L}$ when instantiated with *arbitrary* formulas from $\mathcal{L}^{\mathsf{K}}$. For example, consider the modal logic T, axiomatized by the scheme $\Box\alpha \to \alpha$. By Proposition 5.2.9, we know that this scheme remains valid in InqT for all *standard* formulas $\alpha \in \mathcal{L}^{\mathsf{K}}_{s}$. However, for *arbitrary* $\varphi \in \mathcal{L}^{\mathsf{K}}$, the scheme $\Box\varphi \to \varphi$ turns out to be invalid in InqT. To see this, let M be a reflexive Kripke model with two worlds u and w such that $R = \{(u,u),(w,w)\}$, $V(u,p) = 0$ and $V(w,p) = 1$. Then, clearly, for the state $s = \{u,w\}$, we have $M, s \models \Box?p$ and $M, s \not\models ?p$, so it follows $\not\models^{\mathsf{Inq}}_{\mathsf{T}} \Box?p \to ?p$.

In order to conclude this section, we now want to present a generic completeness result by Ciardelli (2016b) that holds for all systems $\mathsf{Inq}\mathfrak{L}$ which are based on a *canonical* modal logic $\mathfrak{L}$. More precisely, we will describe a general strategy that allows to turn any axiom system Θ for a canonical modal logic $\mathfrak{L}$ into a sound and complete natural deduction system $\mathsf{Ninq}\mathfrak{L}_{\Theta}$ for its inquisitive counterpart, $\mathsf{Inq}\mathfrak{L}$. Recall that, by an *axiom system* for a modal logic $\mathfrak{L}$, we mean any set of standard formulas $\Theta \subseteq \mathcal{L}^{\mathsf{K}}_{s}$ such that $\mathfrak{L}$ is the smallest normal modal logic containing Θ. Thus, for example, the empty set would be an axiom system for the logic K, and the set containing all standard instances of $\Box\alpha \to \alpha$ would be an axiom system for the logic T. Note that, in principle, it would also be legitimate to take $\mathfrak{L}$ *itself* as an axiom system for $\mathfrak{L}$.[6]

Now, given any canonical modal logic $\mathfrak{L}$, Ciardelli (2016b, pp. 214–217) provides us with a general recipe that allows to transform any axiom system $\Theta \subseteq \mathcal{L}^{\mathsf{K}}_{s}$ for $\mathfrak{L}$ into a sound and complete natural deduction system for the corresponding inquisitive logic $\mathsf{Inq}\mathfrak{L}$. The system obtained by Ciardelli's method will be referred to as $\mathsf{Ninq}\mathfrak{L}_{\Theta}$ and can be defined in the following way.

Definition 5.2.10 (The Systems $\mathsf{Ninq}\mathfrak{L}_{\Theta}$)**.** Let $\mathfrak{L}$ be a canonical modal logic and let $\Theta \subseteq \mathcal{L}^{\mathsf{K}}_{s}$ be an axiom system for $\mathfrak{L}$. The natural deduction system $\mathsf{Ninq}\mathfrak{L}_{\Theta}$ comprises all rules of the system presented in Figure 1.3, together with the axioms from Θ and the special rules depicted in Figure 5.1.

In what follows, we will write $\vdash^{\mathsf{N}}_{\Theta}$ for the provability relation of the system $\mathsf{Ninq}\mathfrak{L}_{\Theta}$. Note that, for every canonical modal logic $\mathfrak{L}$, Definition 5.2.10 does not give us a *single* proof system for $\mathsf{Inq}\mathfrak{L}$, but a whole *family* of such systems—namely, one system for each possible choice of Θ.

Furthermore, by definition, all elements of the underlying axiom system Θ for $\mathfrak{L}$ are also taken to be axioms of the natural deduction system $\mathsf{Ninq}\mathfrak{L}_{\Theta}$. So, for instance, in the natural deduction system for InqT, we might adopt all instances of the schema $\Box\alpha \to \alpha$, and in the system for $\mathsf{InqK4}$, we might adopt all instances of $\Box\alpha \to \Box\Box\alpha$.[7] This is necessary in order to account

[6] As we will see later on, this has some notable consequences for the *decidability* of Ciardelli's proof systems.

[7] Note that, in either case, we require α to be a *standard formula*, i.e., $\alpha \in \mathcal{L}^{\mathsf{K}}_{s}$.

for the fact that Inq$\mathfrak{L}$ is a *conservative extension* of $\mathfrak{L}$, so Ninq$\mathfrak{L}_\Theta$ should allow us to derive the characteristic schemes of the underlying base logic $\mathfrak{L}$ and all logical consequences thereof.

Let us now say a bit more about the special rules given in Figure 5.1. First of all, in the rules (split) and (dne), the meta-variable α is now allowed to range over arbitrary *declarative* formulas and not just over *standard* formulas. Thus, α is allowed to contain occurrences of $\vee\!\!\!\!\vee$, provided that they are in the scope of a box operator. This is necessary in order to account for the truth-conditionality of declarative formulas in Inq$\mathfrak{L}$ (see Proposition 5.1.8). The rules ($\to$dis), ($\vee\!\!\!\!\vee$dis) and (nec), on the other hand, are used to account for the fact that $\Box$ distributes over both $\to$ and $\vee\!\!\!\!\vee$, and that necessitation is valid in each of the systems Inq$\mathfrak{L}$ (see Proposition 5.2.8). Note that, in the necessitation rule (nec), we require all hypotheses of the deduction to be *discharged.* Without this restriction, our systems would allow us to prove invalid formulas such as $p \to \Box p$.[8] It is now possible to show that each of the systems Ninq$\mathfrak{L}_\Theta$ is sound and complete with respect to Inq$\mathfrak{L}$.

Theorem 5.2.11 (Soundness and Completeness)**.** *Let $\mathfrak{L}$ be a canonical modal logic and let $\Theta \subseteq \mathcal{L}^{\mathsf{K}}_s$ be an axiom system for $\mathfrak{L}$. For every $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{K}}$, we have: $\Gamma \vdash^{\mathsf{N}}_\Theta \varphi$ if and only if $\Gamma \vDash^{\mathsf{Inq}}_{\mathfrak{L}} \varphi$.*

A proof is provided by Ciardelli (2016b, pp. 217–221). For the completeness part, he uses the canonical model $M_{\mathfrak{L}}$ for $\mathfrak{L}$. The basic idea is as follows: suppose that we have $\Gamma \nvdash^{\mathsf{N}}_\Theta \varphi$. Using a support-based generalization of the truth lemma, this allows us to find a state S over $M_{\mathfrak{L}}$ such that $M_{\mathfrak{L}}, S \vDash \Gamma$ and $M_{\mathfrak{L}}, S \nvDash \varphi$. But then, since $\mathfrak{L}$ is canonical, it follows $\Gamma \nvDash^{\mathsf{Inq}}_{\mathfrak{L}} \varphi$.

5.2.3 Geometric Extensions of InqK

Ciardelli's completeness result for the Kripke logics Inq$\mathfrak{L}$ obviously has some limitations. For one thing, his natural deduction systems Ninq$\mathfrak{L}_\Theta$ are clearly not *analytic* in the sense that one might prove a normalization theorem for them and derive a suitable version of the subformula property thereof. For another, Ciardelli's general strategy is only applicable *in practice*, if a suitable axiom system Θ for $\mathfrak{L}$ is already known in advance. In order to be a bit more specific about this, suppose that we want to use Ciardelli's method in order to find a sound and complete natural deduction system for some Inq$\mathfrak{L}$, where $\mathfrak{L}$ is assumed to be canonical. If an appropriate set of axioms Θ for $\mathfrak{L}$ is not given to us in advance, all we can do is to use the set $\mathfrak{L}$ *itself* as an axiom system for $\mathfrak{L}$ when constructing the natural deduction system Ninq$\mathfrak{L}_\Theta$.[9] However, in this case, we are facing the problem that $\mathfrak{L}$ might not be a *decidable* set of formulas, so the resulting axiomatization for Inq$\mathfrak{L}$ is not guaranteed to be *recursive.*[10] Furthermore, rather than considering modal logics from an axiomatic point of view, we are often more interested in modal logics given to us by a *frame condition* such as 'the logic of all reflexive frames' or 'the logic of all transitive frames'.

For this reason, we will henceforth only consider inquisitive systems based on a certain type of canonical modal logics. The logics belonging to this type are known as *geometric extensions* of the basic system K and they are determined by a first-order frame condition of the form $\forall \vec{w}(\varphi \to \psi)$, where φ and ψ are not allowed to contain implications or universal quantifiers. A frame condition of this kind is also known as a *geometric implication* (see Negri 2003; 2005).[11] To make

[8] Strictly speaking, Ciardelli (2016b, pp. 214–217) uses a slightly more general version of the necessitation rule, which may be described in the following way: given a subdeduction of φ from Γ and additional premises $\Box\psi$ for each $\psi \in \Gamma$, we may infer $\Box\varphi$ and discharge all hypotheses in Γ. However, due to the presence of the rule ($\to$dis), the two formulations of the necessitation rule are easily seen to be equivalent.

[9] As mentioned in the previous section, this would in fact be a legitimate choice in Ciardelli's construction.

[10] Recall that an axiomatization of a logic is said to be *recursive*, if it has a decidable set of axioms (cf. Blackburn et al. 2001, p. 342). By *Craig's theorem*, this is equivalent to saying that the set of theorems of the logic is *recursively enumerable* (cf. Craig 1953; Putnam 1965). Observe that, since Ninq$\mathfrak{L}_\Theta$ comprises all formulas from Θ as axioms, it will only be a recursive axiomatization for Inq$\mathfrak{L}$, if Θ is a decidable axiom system for $\mathfrak{L}$ (see also Ciardelli 2016b, p. 216).

[11] This terminology actually comes from the field of *topos theory*, where first-order theories axiomatized by geometric implications are an important subject of study. For further details, we refer to Mac Lane and Moerdijk (1994).

things precise, let us import some further terminology from standard modal logic. In what follows, properties of frames will be described by first-order formulas taken from a so-called *frame language*. It comprises all first-order formulas built up from a binary relation symbol R, which is interpreted as the accessibility relation of a Kripke frame (cf. Blackburn et al. 2001, p. 126).

Definition 5.2.12 (Frame Language)**.** The *frame language* over a set of variables $\mathfrak{S}$ is denoted $\mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$ and consists of all first-order formulas built up from a binary relation symbol R and the variables in $\mathfrak{S}$ by means of the constants $\bot$ and $\top$, the connectives $\wedge$, $\vee$ and $\to$, and the quantifiers $\forall$ and $\exists$.

In standard modal logic, it is usually assumed that the frame language also includes the *identity predicate* $=$. For simplicity, we will henceforth neglect the issues arising from this predicate and restrict ourselves to the simplified frame language without equality. Nevertheless, this is not an essential restriction and the reader can easily adjust our results to the more general setting.

As usual in first-order logic, we will say that a formula $\varphi \in \mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$ is *closed*, if it does not contain free variables. Note that, by interpreting the binary relation symbol R as the accessibility relation of a frame, every Kripke frame may now be considered as a *model* for the first-order language $\mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$. Hence, given a frame F and a closed formula $\varphi \in \mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$, we will write $F \Vdash \varphi$ and say that F *satisfies* φ, if F satisfies φ in classical first-order logic. For any set of closed formulas $\Phi \subseteq \mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$, we will also use the notation $F \Vdash \Phi$ as an abbreviation for '$F \Vdash \varphi$ for all $\varphi \in \Phi$'.

Let now $\Phi \subseteq \mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$ be an arbitrary set of closed formulas. The *class of frames determined by* Φ is denoted by $\mathsf{Fr}(\Phi)$ and defined to be the class of all Kripke frames satisfying the formulas in Φ. In other words, we put $\mathsf{Fr}(\Phi) := \{F \mid F \text{ is a frame with } F \Vdash \Phi\}$. On the other hand, the *modal logic determined by* Φ is denoted by $\mathfrak{L}(\Phi)$ and consists of all standard formulas that are valid in all frames from $\mathsf{Fr}(\Phi)$, so we define $\mathfrak{L}(\Phi) := \{\alpha \in \mathcal{L}_s^{\mathsf{K}} \mid F \vDash \alpha \text{ for all } F \in \mathsf{Fr}(\Phi)\}$.[12] A modal logic determined by at least one set of sentences $\Phi \subseteq \mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$ is also said to be *elementarily determined*.

Let us consider some examples. As indicated in Table 5.1, the class of all *reflexive* Kripke frames is determined by the first-order sentence $\forall u.Ruu$ and the corresponding standard modal logic is known as KT or T. On the other hand, the class of all *transitive* frames is determined by the sentence $\forall uvw.(Ruv \wedge Rvw \to Ruw)$ and the very same sentence also determines the standard modal logic K4. These two first-order sentences together determine the class of all frames that are both reflexive and transitive, which corresponds to the modal logic known as KT4 or S4.

It was already mentioned that we are only interested in frame conditions of a specific type—namely, those conditions that can be described by a number of *geometric implications*. Towards a definition of this concept, let us say that a first-order formula $\varphi \in \mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$ is *geometric*, if it does not contain occurrences of $\to$ or $\forall$. We now import the following notions from Negri (2003; 2005).

Definition 5.2.13 (Geometric Implication, Geometric Axiom)**.** By a *geometric implication*, we will mean a closed formula of the shape $\forall\vec{w}(\varphi \to \psi)$, where $\varphi, \psi \in \mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$ are geometric formulas. A *geometric axiom*, on the other hand, is a closed formula of the form $\forall\vec{w}(\varphi \to \exists\vec{u}(\psi_1 \vee \ldots \vee \psi_n))$, where each $\varphi, \psi_1, \ldots, \psi_n$ is a conjunction of atomic formulas from the frame language $\mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$.[13]

In the definition of a geometric axiom, we also allow the special case in which some of the conjunctions $\varphi, \psi_1, \ldots, \psi_n$ are *empty*. In particular, if the antecedent φ of the implication is empty, we identify it with $\top$ and write the geometric axiom in the form $\forall\vec{w}\exists\vec{u}(\psi_1 \vee \ldots \vee \psi_n)$. Similarly, if each of the formulas $\psi_1, \ldots, \psi_n$ is an empty conjunction, the whole consequent $\exists\vec{u}(\psi_1 \vee \ldots \vee \psi_n)$ will be identified with $\bot$, so the geometric axiom has the form $\forall\vec{w}\neg\varphi$.

We are now ready to introduce the most important concept of this section: the class of all *geometric extensions* of the basic modal logic K. This class comprises all standard modal logics that are determined by at least one *finite* set of geometric implications from the frame language. For the sake of simplicity, such an extension of K will also be referred to as a *geometric modal logic*.

[12] Recall that $F \vDash \alpha$ was defined to mean that α is valid over F in the inquisitive system InqK. However, by the truth-conditionality of standard formulas, this is equivalent to saying that α is valid over F in the standard modal logic K.

[13] To be clear: by an *atomic formula* from $\mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$, we mean any formula of the form Ruv, where $u, v \in \mathfrak{S}$. Thus, in particular, the logical constants $\bot$ and $\top$ will *not* be considered as atomic formulas.

Definition 5.2.14 (Geometric Modal Logic)**.** A standard modal logic $\mathfrak{L} \subseteq \mathcal{L}_s^{\mathsf{K}}$ is said to be *geometric*, if there exists a finite set of geometric implications $\Phi \subseteq \mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$ such that $\mathfrak{L}$ is determined by Φ.[14]

An inquisitive system $\mathsf{Inq}\mathfrak{L}$ will be called *geometric*, if the underlying base logic $\mathfrak{L}$ is geometric. In order to see some examples, let us consider the first-order sentences depicted in Table 5.1. One readily sees that each of these sentences is a geometric implication (and, in fact, even a *geometric axiom*).[15] Consequently, every normal modal logic generated by some combination of the axiom schemes T, 4, 5, B and D is a geometric modal logic. As we have seen above, this includes a wide range of very famous modal logics such as, for example, T, B, D, S4, S5 and many others.

It is easy to see that every geometric axiom is also a geometric implication, but not the other way around. Nevertheless, it is possible to show that every geometric implication is equivalent to a conjunction of geometric axioms. As a consequence, we obtain the following proposition.

Proposition 5.2.15. *A standard modal logic $\mathfrak{L} \subseteq \mathcal{L}_s^{\mathsf{K}}$ is geometric if and only if there exists a finite set of geometric axioms $\mathsf{A} \subseteq \mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$ such that $\mathfrak{L}$ is determined by* A*.*

Proof. The right-to-left direction is trivial, since every geometric axiom is also a geometric implication. For the other direction, suppose that $\mathfrak{L} \subseteq \mathcal{L}_s^{\mathsf{K}}$ is geometric, i.e., there exists a finite set $\Phi \subseteq \mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$ of geometric implications such that $\mathfrak{L}$ is determined by Φ. Consider an arbitrary such implication $\forall\vec{w}(\varphi \to \psi)$ from Φ. As observed by Palmgren (2002, p. 298), φ and ψ are equivalent to formulas of the form $\exists\vec{v}(\varphi_1 \vee \ldots \vee \varphi_m)$ and $\exists\vec{u}(\psi_1 \vee \ldots \vee \psi_n)$, respectively, where each φ_i and ψ_i is a conjunction of atomic formulas and $\vec{u}$, $\vec{v}$ and $\vec{w}$ are assumed to be pairwise disjoint. But then, clearly, $\forall\vec{w}(\varphi \to \psi)$ is equivalent to the conjunction of the geometric axioms $\forall\vec{w}(\varphi_i \to \exists\vec{u}(\psi_1 \vee \ldots \vee \psi_n))$ where $1 \leq i \leq m$. By repeating this procedure for the other elements of Φ, we thus obtain a finite set of geometric axioms $\mathsf{A} \subseteq \mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$ that determines $\mathfrak{L}$. $\square$

Proposition 5.2.15 shows that every geometric modal logic can also be determined by a set of geometric *axioms*, rather than by a set of geometric *implications*. This will play a crucial role in the next section, where we will see a general strategy that allows to transform any set of geometric axioms into a corresponding set of sequent rules. Using this strategy, we will obtain cut-free labelled sequent calculi for all geometric extensions of the inquisitive system InqK.

Finally, let us say a bit more about the relationship between geometric modal logics and the notions introduced in Section 5.2.1. First, it is easy to verify that every geometric modal logic is also a *normal* modal logic in the sense of Definition 5.2.1. In fact, it is even possible to prove a stronger claim: every geometric modal logic is *canonical*. This follows from a more general result by Fine (1975), saying that *any* modal logic determined by first-order frame conditions is canonical.[16] As shown by Goldblatt et al. (2003), however, the converse of this statement is *not* true, i.e., there exist canonical modal logics that are not elementarily determined (and therefore not geometric). We thus arrive at the following picture: the geometric modal logics form a proper subclass of the canonical modal logics, which in turn form a proper subclass of the normal modal logics.

5.3 Sequent Calculi for Geometric Extensions of InqK

We now want to define labelled sequent calculi for the full class of geometric extensions of the basic system InqK. More specifically, for every geometric standard modal logic $\mathfrak{L}$ determined by a finite set of geometric axioms $\mathsf{A} \subseteq \mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$, we will construct a cut-free labelled sequent calculus

[14] Strictly speaking, the finiteness of Φ is not a necessary condition for the results we will obtain in this chapter. Nevertheless, we will keep this constraint, as it greatly simplifies the completeness proof given in Section 5.6.

[15] Observe that, in the first-order formulas corresponding to reflexivity and seriality, the antecedent of the implication is taken to be $\top$ and therefore not displayed in the table.

[16] This result is also known as *Fine's canonicity theorem*. See Goldblatt (2020) for further details.

$\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ that is sound and complete with respect to the corresponding inquisitive system $\mathsf{Inq}\mathfrak{L}$. The construction is based on a general strategy described by Negri (2003; 2005), which allows to generate sequent rules from geometric axioms in a schematic and uniform way.

As before, we will assume two countably infinite sets of *state variables*, denoted by $\mathfrak{S}$ and $\mathfrak{V}$, respectively. The variables in $\mathfrak{S}$ are used for *singleton states* and the variables in $\mathfrak{V}$ are used for *arbitrary* information states. For convenience, we shall assume that $\mathfrak{S}$ is also the set of variables that we used in our definition of the frame language $\mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$ (see Definition 5.2.12). Following the convention adopted in the previous chapters, we will use the meta-variables u, v, w, etc., for elements of $\mathfrak{S}$ and the meta-variables x, y, z, etc., for elements of $\mathfrak{V}$. The set of *labels* is defined in exactly the same way as in our treatment of intuitionistic inquisitive logic, except that the R-operator can no longer be used to embed arbitrary labels, but only variables from $\mathfrak{S}$.

Definition 5.3.1 (Labels)**.** The set of *labels* is denoted by $\Lambda(\mathfrak{S}, \mathfrak{V})$ and consists of all expressions generated by the following grammar, where $w \in \mathfrak{S}$ and $x \in \mathfrak{V}$ are arbitrary variables:

$$\pi ::= w \mid x \mid \emptyset \mid \pi \cdot \pi \mid \pi + \pi \mid R(w).$$

Labels are interpreted in the usual way as descriptions of information states. Thus, intuitively, $\pi \cdot \sigma$ stands for the *intersection* and $\pi + \sigma$ for the *union* of the states denoted by π and σ. As before, we will also use the notation $\pi\sigma$ as a shorthand for $\pi \cdot \sigma$. A label of the form $R(w)$ represents the *neighbourhood* of a world w, i.e., the set of all worlds accessible from w in a Kripke model.

Let us recall some basic vocabulary. A *relational atom* is defined to be an expression of the form $\pi \leqslant \sigma$, where π and σ are arbitrary labels. A *labelled formula*, on the other hand, is an expression of the form $\pi : \varphi$, where π is a label and $\varphi \in \mathcal{L}^{\mathsf{K}}$ is a formula. Relational atoms and labelled formulas have their usual (intended) meaning. That is, $\pi \leqslant \sigma$ should be read as 'π is a subset of σ' and $\pi : \varphi$ should be read as 'φ is supported by π'. By a *sequent*, we mean any expression of the form $\Gamma \Rightarrow \Delta$, where Γ is a finite multiset containing labelled formulas and relational atoms, and Δ is a finite multiset containing only labelled formulas (but no relational atoms). Given a sequent $\Gamma \Rightarrow \Delta$, we also call Γ the *antecedent* and Δ the *succedent* of the sequent.

We start by defining a labelled sequent calculus GLinqK for the basic system InqK of inquisitive Kripke logic. Afterwards, we will describe how GLinqK can be extended to a proof system for arbitrary geometric systems $\mathsf{Inq}\mathfrak{L}$.[17] Our sequent calculus for InqK is presented in Figure 5.2. As can be seen, the axioms, the rules for the propositional connectives and most of the order rules are exactly the same as in our labelled sequent calculus for InqB. The only new ingredients are the rules $L\Box$, $R\Box$ and (nb). Intuitively, (nb) says that, if a singleton state w is a subset of another singleton state u, then also the neighbourhood of w is a subset of the neighbourhood of u. The rules $L\Box$ and $R\Box$, on the other hand, mirror the support clause for $\Box$ introduced in Definition 5.1.3. Note that $R\Box$ comes with a side condition, saying that w must be a fresh variable not occurring in the conclusion of the rule. Moreover, as usual, we assume that order rules always satisfy the *closure condition*: if an instance of an order rule produces a duplication of relational atoms in the conclusion of the rule, then also the contracted instance of the rule is added to our system.[18]

Definition 5.3.2 (The System GLinqK)**.** We define GLinqK to be the sequent calculus depicted in Figure 5.2. A sequent $\Gamma \Rightarrow \Delta$ is *derivable* in our system, if there exists a proof tree in GLinqK ending with this sequent. Given any finite subset $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{K}}$, we say that φ is *provable* from Γ in GLinqK, if for some (or, in fact, any) variable $x \in \mathfrak{V}$, the sequent $x : \Gamma \Rightarrow x : \varphi$ is derivable.[19]

We adopt the usual terminology from the previous chapters. That is, in an instance of an axiom or a rule of inference, the multiset Γ is called the *left context* and the multiset Δ is called

[17] Recall that an inquisitive system $\mathsf{Inq}\mathfrak{L}$ is called *geometric*, if the underlying standard modal logic $\mathfrak{L}$ is geometric.
[18] As before, this is necessary in order to ensure that contraction on relational atoms is admissible in our system.
[19] As in the previous chapters, we write $x : \Gamma$ for the set of labelled formulas defined by $(x : \Gamma) := \{x : \psi \mid \psi \in \Gamma\}$.

Axioms:

$$\dfrac{}{w : p, \Gamma \Rightarrow \Delta, w : p}\,Ax \qquad \dfrac{}{w : \bot, \Gamma \Rightarrow \Delta}\,Ax^{\bot} \qquad \dfrac{}{w \leqslant \emptyset, \Gamma \Rightarrow \Delta}\,Ax^{\emptyset}$$

Logical Rules:

$$\dfrac{w : p, w \leqslant \pi, \pi : p, \Gamma \Rightarrow \Delta}{w \leqslant \pi, \pi : p, \Gamma \Rightarrow \Delta}\,Lp \qquad \dfrac{w \leqslant \pi, \Gamma \Rightarrow \Delta, w : p}{\Gamma \Rightarrow \Delta, \pi : p}\,Rp$$

$$\dfrac{w : \bot, w \leqslant \pi, \pi : \bot, \Gamma \Rightarrow \Delta}{w \leqslant \pi, \pi : \bot, \Gamma \Rightarrow \Delta}\,L\bot \qquad \dfrac{w \leqslant \pi, \Gamma \Rightarrow \Delta, w : \bot}{\Gamma \Rightarrow \Delta, \pi : \bot}\,R\bot$$

$$\dfrac{\pi : \varphi, \pi : \psi, \Gamma \Rightarrow \Delta}{\pi : \varphi \wedge \psi, \Gamma \Rightarrow \Delta}\,L\wedge \qquad \dfrac{\Gamma \Rightarrow \Delta, \pi : \varphi \qquad \Gamma \Rightarrow \Delta, \pi : \psi}{\Gamma \Rightarrow \Delta, \pi : \varphi \wedge \psi}\,R\wedge$$

$$\dfrac{\pi : \varphi, \Gamma \Rightarrow \Delta \qquad \pi : \psi, \Gamma \Rightarrow \Delta}{\pi : \varphi \vee\!\!\!\vee \psi, \Gamma \Rightarrow \Delta}\,L\vee\!\!\!\vee \qquad \dfrac{\Gamma \Rightarrow \Delta, \pi : \varphi, \pi : \psi}{\Gamma \Rightarrow \Delta, \pi : \varphi \vee\!\!\!\vee \psi}\,R\vee\!\!\!\vee$$

$$\dfrac{\pi \leqslant \sigma, \sigma : \varphi \to \psi, \Gamma \Rightarrow \Delta, \pi : \varphi \qquad \pi \leqslant \sigma, \sigma : \varphi \to \psi, \pi : \psi, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \sigma : \varphi \to \psi, \Gamma \Rightarrow \Delta}\,L\!\to$$

$$\dfrac{x \leqslant \pi, x : \varphi, \Gamma \Rightarrow \Delta, x : \psi}{\Gamma \Rightarrow \Delta, \pi : \varphi \to \psi}\,R\!\to$$

$$\dfrac{R(w) : \varphi, w \leqslant \pi, \pi : \Box\varphi, \Gamma \Rightarrow \Delta}{w \leqslant \pi, \pi : \Box\varphi, \Gamma \Rightarrow \Delta}\,L\Box \qquad \dfrac{w \leqslant \pi, \Gamma \Rightarrow \Delta, R(w) : \varphi}{\Gamma \Rightarrow \Delta, \pi : \Box\varphi}\,R\Box$$

Order Rules:

$$\dfrac{\pi \leqslant \tau, \pi \leqslant \sigma, \sigma \leqslant \tau, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \sigma \leqslant \tau, \Gamma \Rightarrow \Delta}\,(\mathsf{tr}) \qquad \dfrac{\pi \leqslant \pi\sigma + \pi\tau, \pi \leqslant \sigma + \tau, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma + \tau, \Gamma \Rightarrow \Delta}\,(\mathsf{dis})$$

$$\dfrac{\pi \leqslant \sigma\tau, \pi \leqslant \sigma, \pi \leqslant \tau, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \pi \leqslant \tau, \Gamma \Rightarrow \Delta}\,(\mathsf{in}) \qquad \dfrac{\pi + \tau \leqslant \sigma, \pi \leqslant \sigma, \tau \leqslant \sigma, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \tau \leqslant \sigma, \Gamma \Rightarrow \Delta}\,(\mathsf{un})$$

$$\dfrac{\pi\sigma \leqslant \pi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(\mathsf{il}) \qquad \dfrac{\sigma\pi \leqslant \pi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(\mathsf{ir}) \qquad \dfrac{\pi \leqslant \pi + \sigma, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(\mathsf{ul})$$

$$\dfrac{\pi \leqslant \sigma + \pi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(\mathsf{ur}) \qquad \dfrac{\pi \leqslant \emptyset, \pi \leqslant w, \Gamma \Rightarrow \Delta \qquad w \leqslant \pi, \pi \leqslant w, \Gamma \Rightarrow \Delta}{\pi \leqslant w, \Gamma \Rightarrow \Delta}\,(\mathsf{sg})$$

$$\dfrac{\pi \leqslant \pi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(\mathsf{rf}) \qquad \dfrac{w \leqslant \pi, w \leqslant \pi + \sigma, \Gamma \Rightarrow \Delta \qquad w \leqslant \sigma, w \leqslant \pi + \sigma, \Gamma \Rightarrow \Delta}{w \leqslant \pi + \sigma, \Gamma \Rightarrow \Delta}\,(\mathsf{cd})$$

$$\dfrac{R(w) \leqslant R(u), w \leqslant u, \Gamma \Rightarrow \Delta}{w \leqslant u, \Gamma \Rightarrow \Delta}\,(\mathsf{nb})$$

Figure 5.2: The system GLinqK. In each case, w and u range over variables from $\mathfrak{S}$, x ranges over variables from $\mathfrak{V}$, and π, σ, τ, etc., stand for arbitrary labels. In Rp and $R\bot$, w must be a fresh variable not occurring in the conclusion of the rule and π is required to be a non-singleton label, i.e., $\pi \notin \mathfrak{S}$. Similarly, w must be fresh in $R\Box$, and x must be fresh in $R\!\to$.

the *right context.* In the conclusion of each rule, and also in the axioms, the labelled formulas and relational atoms not belonging to the context are said to be *principal.* The corresponding expressions occurring in the premises of a rule are called *active.* So, for example, in an application of $R\Box$ with premise $w \leqslant \pi, \Gamma \Rightarrow \Delta, R(w) : \varphi$ and conclusion $\Gamma \Rightarrow \Delta, \pi : \Box\varphi$, the labelled formula $\pi : \Box\varphi$ is principal, while the expressions $w \leqslant \pi$ and $R(w) : \varphi$ are both active.

$$\frac{u \leqslant R(u), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathsf{T}) \qquad \frac{u \leqslant R(v), v \leqslant R(u), \Gamma \Rightarrow \Delta}{v \leqslant R(u), \Gamma \Rightarrow \Delta}\ (\mathsf{B}) \qquad \frac{v \leqslant R(u), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathsf{D})$$

$$\frac{w \leqslant R(u), w \leqslant R(v), v \leqslant R(u), \Gamma \Rightarrow \Delta}{w \leqslant R(v), v \leqslant R(u), \Gamma \Rightarrow \Delta}\ (4) \qquad \frac{w \leqslant R(v), v \leqslant R(u), w \leqslant R(u), \Gamma \Rightarrow \Delta}{v \leqslant R(u), w \leqslant R(u), \Gamma \Rightarrow \Delta}\ (5)$$

Figure 5.3: Some instances of the geometric rule scheme. The rules correspond to reflexivity, symmetry, seriality, transitivity and Euclideanness, respectively. Rule (D) has a side condition, saying that $v \in \mathfrak{S}$ must be a fresh variable not occurring in the conclusion of the rule.

We now want to extend our calculus GLinqK to a proof system for arbitrary geometric inquisitive systems Inq$\mathfrak{L}$. To this end, we will employ a general method by Negri (2003; 2005) that allows to transform any given set of geometric axioms into a corresponding set of proof rules. The basic idea is as follows: consider an arbitrary inquisitive system Inq$\mathfrak{L}$ such that $\mathfrak{L}$ is a geometric standard modal logic. Then, by Proposition 5.2.15, there exists a finite set of geometric axioms $\mathsf{A} \subseteq \mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$ such that $\mathfrak{L}$ is determined by A. We choose an arbitrary geometric axiom $\theta \in \mathsf{A}$. By Definition 5.2.13, the axiom θ must be of the form $\forall\vec{w}(\varphi \to \exists\vec{u}(\psi_1 \vee \ldots \vee \psi_n))$, where each $\varphi, \psi_1, \ldots, \psi_n$ is a conjunction of atomic formulas from the frame language. For simplicity, we restrict ourselves to the case in which $\vec{u}$ contains only one variable, and we distribute the existential quantifier over the disjunctions. More precisely, we transform θ into a formula of the shape $\forall\vec{w}(\varphi \to (\exists u_1\psi_1 \vee \ldots \vee \exists u_n\psi_n))$, where none of the variables u_i occurs free in φ. Let now Φ be the multiset of all atoms occurring in φ and let Ψ_i be the multiset of all atoms occurring in ψ_i. Furthermore, given any set of atoms from the frame language $\Sigma \subseteq \mathcal{L}_{\mathsf{F}}^{\mathfrak{S}}$, let us write Σ° for the result of replacing each atomic formula Rv_1v_2 in Σ by the corresponding relational atom $v_2 \leqslant R(v_1)$. The geometric axiom θ is then converted into the sequent rule (θ-grs) of the form

$$\frac{\Psi_1^\circ(v_1/u_1), \Phi^\circ, \Gamma \Rightarrow \Delta \qquad \cdots \qquad \Psi_n^\circ(v_n/u_n), \Phi^\circ, \Gamma \Rightarrow \Delta}{\Phi^\circ, \Gamma \Rightarrow \Delta}\ (\theta\text{-grs})$$

where each v_i is a fresh variable from $\mathfrak{S}$ and $\Psi_i^\circ(v_i/u_i)$ stands for the result of substituting v_i for u_i in each relational atom occurring in the multiset Ψ_i°. The generic rule (θ-grs) is also referred to as the *geometric rule scheme* (cf. Negri 2003; Dyckhoff and Negri 2012).[20] To see an example, consider the geometric axiom $\forall uvw.(Ruv \wedge Rvw \to Ruw)$, expressing the *transitivity* of a Kripke frame. The corresponding instance of the geometric rule scheme takes the form

$$\frac{w \leqslant R(u), w \leqslant R(v), v \leqslant R(u), \Gamma \Rightarrow \Delta}{w \leqslant R(v), v \leqslant R(u), \Gamma \Rightarrow \Delta}\ (4)$$

which can be seen as a proof-theoretical formulation of axiom scheme 4. In the same way, one can also generate instances of the geometric rule scheme for each of the other geometric axioms presented in Table 5.1. The resulting sequent rules are depicted in Figure 5.3. For every inquisitive system Inq$\mathfrak{L}$ given by some set of geometric axioms A, we may now construct a sequent calculus GLinq$\mathfrak{L}_\mathsf{A}$ by extending GLinqK with the respective instances of the geometric rule scheme.

[20] If the consequent of θ is empty (that is, if θ is of the form $\forall\vec{w}\neg\varphi$), then ($\theta$-grs) is taken to be a *zero-premise rule* (and thus an *axiom*). Moreover, one can also extend the rule scheme to the more general setting in which the frame language includes the identity predicate $=$. In this case, an atomic formula of the form $v_1 = v_2$ should be translated as the pair of relational atoms $v_1 \leqslant v_2, v_2 \leqslant v_1$. As mentioned above, however, we will not deal with this case here.

Definition 5.3.3 (The Systems $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$)**.** Let $\mathsf{Inq}\mathfrak{L}$ be a geometric inquisitive logic and let $\mathsf{A} \subseteq \mathcal{L}^{\mathfrak{G}}_{\mathsf{F}}$ be a finite set of geometric axioms such that $\mathfrak{L}$ is determined by A.[21] We define $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ to be the labelled sequent calculus obtained by adding, for each geometric axiom $\theta \in \mathsf{A}$, the corresponding instance (θ-grs) of the geometric rule scheme to the basic system GLinqK.

For example, in order to obtain a sequent calculus for the inquisitive system InqKT, we may enrich the proof system GLinqK with the sequent rule (T) presented in Figure 5.3. And in order to obtain a sequent calculus for the logic $\mathsf{InqKD4}$, we may add the rules (D) and ($\mathsf{4}$) to GLinqK.

As usual, we say that a formula φ is *provable* from a finite subset $\Gamma \subseteq \mathcal{L}^{\mathsf{K}}$ in one of the calculi $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$, if there is a variable $x \in \mathfrak{V}$ such that $x : \Gamma \Rightarrow x : \varphi$ is derivable in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. In order to make sure that contraction is admissible in each of the systems $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$, we also adopt the familiar *closure condition* for instances of the geometric rule scheme. That is, if an instance (θ-grs) of the scheme produces a duplication of relational atoms in the multiset Φ° occurring in the conclusion of the rule, then also the contracted version of (θ-grs) is taken to be part of our sequent calculi.

The rest of this chapter is organized as follows. In Section 5.4, we will highlight some remarkable structural properties of our proof systems. We will see that each of the calculi $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ enjoys cut-admissibility, height-preserving invertibility of all rules, and height-preserving admissibility of weakening and contraction. In Section 5.5, we will establish the soundness of our proof systems and show that, if $\mathfrak{L}$ is determined by some combination of the geometric axioms presented in Table 5.1, then the completeness of $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ can also be established indirectly, by using a suitable Hilbert-style system for $\mathsf{Inq}\mathfrak{L}$. In Section 5.6, finally, we will use a countermodel construction in order to provide a general completeness proof covering each of the calculi $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$.

5.4 Basic Properties

Let us start by pointing out some important features of our proof systems. Throughout this section, let $\mathsf{A} \subseteq \mathcal{L}^{\mathfrak{G}}_{\mathsf{F}}$ be an arbitrary but fixed finite set of geometric axioms and let $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ be the corresponding labelled sequent calculus introduced in Definition 5.3.3. We first show that the generalized initial sequents, familiar from the previous chapters, are derivable in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$.

Lemma 5.4.1. *All sequents of the following form are derivable in* $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$*:*
(i) $\pi \leqslant \sigma, \sigma : \varphi, \Gamma \Rightarrow \Delta, \pi : \varphi$,
(ii) $\pi : \varphi, \Gamma \Rightarrow \Delta, \pi : \varphi$.

Proof. The derivability of (i) is proved in the usual way, by induction on the structure of φ. Most cases are treated in the same way as in the classical setting (see the proof of Lemma 3.2.1). The only new case is the inductive step for $\Box$. In this case, we construct the following derivation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{By ind. hyp.}}{R(w) \leqslant R(w), R(w) : \psi, w \leqslant \sigma, w \leqslant \pi, \pi \leqslant \sigma, \sigma : \Box\psi, \Gamma \Rightarrow \Delta, R(w) : \psi}}{R(w) : \psi, w \leqslant \sigma, w \leqslant \pi, \pi \leqslant \sigma, \sigma : \Box\psi, \Gamma \Rightarrow \Delta, R(w) : \psi}\,(\mathsf{rf})}{w \leqslant \sigma, w \leqslant \pi, \pi \leqslant \sigma, \sigma : \Box\psi, \Gamma \Rightarrow \Delta, R(w) : \psi}\,L\Box}{w \leqslant \pi, \pi \leqslant \sigma, \sigma : \Box\psi, \Gamma \Rightarrow \Delta, R(w) : \psi}\,(\mathsf{tr})}{\pi \leqslant \sigma, \sigma : \Box\psi, \Gamma \Rightarrow \Delta, \pi : \Box\psi}\,R\Box
$$

The sequent in (ii), on the other hand, can be derived from (i) by an application of (rf). □

Lemma 5.4.2. *All sequents of the following form are derivable in* $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$*:*
(i) $\pi \leqslant \emptyset, \Gamma \Rightarrow \Delta, \pi : \varphi$,
(ii) $\Gamma \Rightarrow \Delta, \emptyset : \varphi$,
(iii) $\pi \leqslant \sigma, \sigma : \bot, \Gamma \Rightarrow \Delta, \pi : \varphi$,
(iv) $\pi : \bot, \Gamma \Rightarrow \Delta, \pi : \varphi$.

[21] Recall that, by Proposition 5.2.15, such a set of geometric axioms does in fact exist.

Proof. The derivability of (i) and (iii) is established by induction on φ (see Lemma 3.2.2). The only new part is the inductive step for $\Box$, which is covered by the following two derivations:

$$\dfrac{\dfrac{\dfrac{}{w \leqslant \emptyset, w \leqslant \pi, \pi \leqslant \emptyset, \Gamma \Rightarrow \Delta, R(w) : \psi}\, Ax^{\emptyset}}{w \leqslant \pi, \pi \leqslant \emptyset, \Gamma \Rightarrow \Delta, R(w) : \psi}\,(\mathsf{tr})}{\pi \leqslant \emptyset, \Gamma \Rightarrow \Delta, \pi : \Box\psi}\, R\Box$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{w : \bot, w \leqslant \sigma, w \leqslant \pi, \pi \leqslant \sigma, \sigma : \bot, \Gamma \Rightarrow \Delta, R(w) : \psi}\, Ax^{\bot}}{w \leqslant \sigma, w \leqslant \pi, \pi \leqslant \sigma, \sigma : \bot, \Gamma \Rightarrow \Delta, R(w) : \psi}\, L\bot}{w \leqslant \pi, \pi \leqslant \sigma, \sigma : \bot, \Gamma \Rightarrow \Delta, R(w) : \psi}\,(\mathsf{tr})}{\pi \leqslant \sigma, \sigma : \bot, \Gamma \Rightarrow \Delta, \pi : \Box\psi}\, R\Box$$

As before, the sequents in (ii) and (iv) can be obtained from (i) and (iii) by the rule (rf). □

Next, we want to show that $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ also preserves the structural properties of the labelled sequent calculi considered in the previous chapters. As usual, we say that a rule of inference is *height-preserving admissible* (or *hp-admissible*), if it satisfies the condition that, whenever all premises of the rule are derivable by a proof tree of height at most n, then also the conclusion of the rule is derivable by a proof tree of height at most n (where the *height* of a tree is taken to be the length of its longest branch). If the admissibility of a rule is not height-preserving, then the rule is simply called *admissible*. In addition, a rule is said to be *height-preserving invertible* (or *hp-invertible*), if it is the case that, whenever the conclusion of the rule is derivable by a proof tree of height at most n, then also each of its premises is derivable by such a proof tree.

The substitution operator for labels is defined in exactly the same way as in our sequent calculus for InqI (see Section 4.4). Thus, in particular, if $u, v, w \in \mathfrak{S}$ are singleton variables, then we put $R(w)(u/v) := R(u)$, if $v = w$, and $R(w)(u/v) := R(w)$, if $v \neq w$. The definition is extended to multisets in the usual way. The *substitution rules* are now defined to be the rules

$$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma(u/w) \Rightarrow \Delta(u/w)}\,(u/w) \qquad \text{and} \qquad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma(\pi/x) \Rightarrow \Delta(\pi/x)}\,(\pi/x)$$

where u and w are variables from $\mathfrak{S}$, x is a variable from $\mathfrak{V}$, and π is an arbitrary label.

Proposition 5.4.3. *The substitution rules are hp-admissible in* $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$.

Proof. By induction on the height of a derivation for $\Gamma \Rightarrow \Delta$. The base case is treated in the same way as in the proof of Proposition 3.2.6. For the inductive step, suppose that $\Gamma \Rightarrow \Delta$ is derivable by a proof tree $\mathcal{D}$ of height $n > 1$. Consider the last rule applied in $\mathcal{D}$. If this rule does not have eigenvariables, then we simply apply the induction hypothesis to the premises of the rule, and then the rule itself.[22] And if the last rule in $\mathcal{D}$ has eigenvariables, then we first use the induction hypothesis in order to rename the eigenvariables, before performing the desired substitution. So, for example, suppose that we want to substitute a variable $u \in \mathfrak{S}$ for some other variable $w \in \mathfrak{S}$. Moreover, assume that $\mathcal{D}$ ends with an application of the geometric rule scheme having the form

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Psi_1(v_1/u_1), \Phi, \Gamma' \Rightarrow \Delta'\end{matrix} \quad \cdots \quad \begin{matrix}\mathcal{D}_m \\ \Psi_m(v_m/u_m), \Phi, \Gamma' \Rightarrow \Delta'\end{matrix}}{\Phi, \Gamma' \Rightarrow \Delta'}\,(\theta\text{-grs})$$

where each $v_i \in \mathfrak{S}$ is an eigenvariable of the rule. By applying the induction hypothesis to the m premises, we first replace the eigenvariables v_i by fresh variables v_i' satisfying $v_i' \neq w$. For

[22] As in our labelled sequent calculus for InqB, a rule is said to have an *eigenvariable*, if a root-first application of this rule allows to introduce a fresh variable in a derivation. In the proof system $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$, this includes the rules Rp, $R\bot$, $R\!\to$, $R\Box$ and possibly some instances of the geometric rule scheme. For further details, we refer to Section 3.1.

$$\frac{\Gamma \Rightarrow \Delta}{\pi : \varphi, \Gamma \Rightarrow \Delta} LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \pi : \varphi} RW \qquad \frac{\Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \Gamma \Rightarrow \Delta} W^{\leqslant}$$

$$\frac{\pi : \varphi, \pi : \varphi, \Gamma \Rightarrow \Delta}{\pi : \varphi, \Gamma \Rightarrow \Delta} LC \qquad \frac{\Gamma \Rightarrow \Delta, \pi : \varphi, \pi : \varphi}{\Gamma \Rightarrow \Delta, \pi : \varphi} RC \qquad \frac{\pi \leqslant \sigma, \pi \leqslant \sigma, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \Gamma \Rightarrow \Delta} C^{\leqslant}$$

Figure 5.4: The structural rules of weakening and contraction.

each i with $1 \leq i \leq m$, this yields a derivation $\mathcal{D}'_i$ for $\Psi_i(v'_i/u_i), \Phi, \Gamma' \Rightarrow \Delta'$ such that $\mathcal{D}'_i$ is of height at most $n-1$. We now use the induction hypothesis again in order to perform the substitution (u/w) in each of these m derivations. Finally, by an application of the rule (θ-grs), we obtain a derivation $\mathcal{D}'$ for $\Phi(u/w), \Gamma'(u/w) \Rightarrow \Delta'(u/w)$ such that $\mathcal{D}'$ is of height at most n. This concludes the proof. For the other rules involving eigenvariables, the argument is similar. $\square$

We can now prove that the structural rules of *weakening* and *contraction*, depicted in Figure 5.4, are hp-admissible in $\mathsf{GLinq}\mathfrak{L}_\mathsf{A}$. In addition, we show that all rules of our system are hp-invertible.

Proposition 5.4.4. *The weakening rules are hp-admissible in* $\mathsf{GLinq}\mathfrak{L}_\mathsf{A}$.

Proof. For each of the three weakening rules, the proof proceeds by an easy induction on the height of a derivation $\mathcal{D}$ for the premise of the rule. In the inductive step, we consider the last rule applied in $\mathcal{D}$. If this rule does not have eigenvariables, then we apply the induction hypothesis to the premises of the rule, and then the rule itself. Otherwise, we first use Proposition 5.4.3 in order to introduce a new eigenvariable not clashing with the variables occurring in the weakening formula. For further details, the reader is referred to the proof of Proposition 3.2.7. $\square$

Proposition 5.4.5. *All rules of* $\mathsf{GLinq}\mathfrak{L}_\mathsf{A}$ *are hp-invertible.*

Proof. The hp-invertibility of the rules for the propositional connectives is established in the same way as in the proof of Proposition 3.2.8. Moreover, the hp-invertibility of $L\Box$ and the order rules (including instances of the geometric rule scheme) follows immediately from the hp-admissibility of weakening. Thus, we only need to show that $R\Box$ is hp-invertible. To this end, let $\mathcal{D}$ be an arbitrary derivation for $\Gamma \Rightarrow \Delta, \pi : \Box\varphi$ and let n be the height of $\mathcal{D}$. Using induction on n, we show that, for every singleton variable $w \in \mathfrak{S}$ not occurring in the sequent $\Gamma \Rightarrow \Delta, \pi : \Box\varphi$, there is also a derivation $\mathcal{D}'$ for $w \leqslant \pi, \Gamma \Rightarrow \Delta, R(w) : \varphi$ such that $\mathcal{D}'$ has height at most n.

For the base case, assume that $\mathcal{D}$ has height $n = 1$. In this case, $\Gamma \Rightarrow \Delta, \pi : \Box\varphi$ must be an instance of an axiom and $\pi : \Box\varphi$ cannot be principal (recall that the principal formula of an instance of Ax is always of the form $w : p$ for some atom p). Hence, $w \leqslant \pi, \Gamma \Rightarrow \Delta, R(w) : \varphi$ is also an instance of an axiom and therefore derivable by a proof tree $\mathcal{D}'$ of height $n = 1$.

For the inductive step, suppose that $\mathcal{D}$ has height $n > 1$ and let $w \in \mathfrak{S}$ be an arbitrary variable not occurring in $\Gamma \Rightarrow \Delta, \pi : \Box\varphi$. If the last step in $\mathcal{D}$ is a rule for which $\pi : \Box\varphi$ is not principal, then we apply the induction hypothesis to the premises of the rule (possibly together with a height-preserving substitution), and we then use the same rule again. On the other hand, if the last step in $\mathcal{D}$ is an application of $R\Box$ for which $\pi : \Box\varphi$ is principal, then $\mathcal{D}$ must be of the form

$$\frac{\begin{matrix}\mathcal{D}^* \\ u \leqslant \pi, \Gamma \Rightarrow \Delta, R(u) : \varphi\end{matrix}}{\Gamma \Rightarrow \Delta, \pi : \Box\varphi} R\Box$$

where $u \in \mathfrak{S}$ is a fresh variable not occurring in $\Gamma \Rightarrow \Delta, \pi : \Box\varphi$. By substituting w for u in the subderivation $\mathcal{D}^*$, we now obtain the desired derivation $\mathcal{D}'$ for $w \leqslant \pi, \Gamma \Rightarrow \Delta, R(w) : \varphi$. $\square$

Proposition 5.4.6. *The contraction rules are hp-admissible in* $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$.

Proof. For each of the three contraction rules, the proof is done simultaneously, by induction on the height of a derivation for the premise of the respective rule. The overall structure of the argument is the same as in the proof of Proposition 3.2.9. For the sake of brevity, we only consider the inductive step for the rule RC.[23] Suppose that the sequent $\Gamma \Rightarrow \Delta, \pi : \varphi, \pi : \varphi$ is derivable by a proof tree $\mathcal{D}$ of height $n > 1$. We need to show that there also exists a derivation $\mathcal{D}'$ for $\Gamma \Rightarrow \Delta, \pi : \varphi$ such that $\mathcal{D}'$ is of height at most n. For this purpose, consider the last rule applied in $\mathcal{D}$. If the labelled formula $\pi : \varphi$ is not principal in this rule, then both occurrences of $\pi : \varphi$ also appear in each of the premises of the rule. Thus, by applying the induction hypothesis to the premises and then the same rule again, we obtain the desired derivation $\mathcal{D}'$ for $\Gamma \Rightarrow \Delta, \pi : \varphi$. On the other hand, if $\pi : \varphi$ is principal in the last rule applied in $\mathcal{D}$, we distinguish cases, depending on the form of φ. If φ is atomic or if the main operator of φ is one of the propositional connectives, then the argument is the same as in the classical setting. Therefore, let us assume that φ is of the shape $\varphi = \Box\psi$ for some formula $\psi \in \mathcal{L}^{\mathsf{K}}$. In this case, the derivation $\mathcal{D}$ must be of the form

$$\dfrac{\begin{array}{c}\mathcal{D}^*\\ w \leqslant \pi, \Gamma \Rightarrow \Delta, R(w) : \psi, \pi : \Box\psi\end{array}}{\Gamma \Rightarrow \Delta, \pi : \Box\psi, \pi : \Box\psi}\ R\Box$$

where $w \in \mathfrak{S}$ is a fresh variable not occurring in the end-sequent of $\mathcal{D}$. By applying the height-preserving invertibility of $R\Box$ and a subsequent height-preserving substitution to the premise of the indicated application of $R\Box$, we now obtain a derivation of height at most $n-1$ for the sequent $w \leqslant \pi, w \leqslant \pi, \Gamma \Rightarrow \Delta, R(w) : \psi, R(w) : \psi$. Using the induction hypothesis and an application of $R\Box$, this yields the desired derivation $\mathcal{D}'$ of height at most n for $\Gamma \Rightarrow \Delta, \pi : \Box\psi$. □

Next, we will prove that the cut rule is admissible in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. A definition of this rule was given in Section 3.2.3. The overall structure of our argument is the same as in the previous chapters. We thus proceed by a main induction on the rank of the cut formula, with a subinduction on the height of the cut rule application. First, we restate our definition of the degree of a label.

Definition 5.4.7. The *degree of a label* π is denoted by $deg(\pi)$ and defined as follows: if $\pi \in \mathfrak{S}$ is a singleton variable, then we put $deg(\pi) := 0$, and if $\pi \notin \mathfrak{S}$, then we put $deg(\pi) := 1$.

The *degree of a formula* φ is denoted by $deg(\varphi)$ and defined to be the number of occurrences of the logical symbols $\bot$, $\wedge$, $\to$, $\mathbin{\vee\!\!\!\vee}$ and $\Box$ in φ. The *rank of a labelled formula* $\pi : \varphi$ is defined in the familiar way, so we put $rank(\pi : \varphi) := (deg(\varphi), deg(\pi))$, where $deg(\varphi)$ is the degree of φ and $deg(\pi)$ is the degree of π. Ranks are again compared using a *lexicographic ordering*. That is, we write $rank(\pi : \varphi) < rank(\sigma : \psi)$ and say that the rank of $\pi : \varphi$ is *smaller* than the rank of $\sigma : \psi$, if we either have $deg(\varphi) < deg(\psi)$, or we have both $deg(\varphi) = deg(\psi)$ and $deg(\pi) < deg(\sigma)$.

Lemma 5.4.8. *Let* π *and* σ *be arbitrary labels and let* $w \in \mathfrak{S}$ *be a singleton variable. It holds:*

(i) If $\pi \notin \mathfrak{S}$, *then* $rank(w : \varphi) < rank(\pi : \varphi)$,
(ii) $rank(\pi : \varphi_i) < rank(\sigma : \varphi_1 \otimes \varphi_2)$ *for* $i = 1, 2$ *and* $\otimes \in \{\wedge, \to, \mathbin{\vee\!\!\!\vee}\}$,
(iii) $rank(\pi : \varphi) < rank(\sigma : \Box\varphi)$.

Proof. The first two statements are proved in the same way as in our treatment of InqB (see Lemma 3.2.12). The last claim follows directly from the fact that we have $deg(\varphi) < deg(\Box\varphi)$. □

As in the previous chapters, the *height of a cut rule application* is taken to be the sum of the heights of the two derivations for the premises $\Gamma \Rightarrow \Delta, \pi : \varphi$ and $\pi : \varphi, \Sigma \Rightarrow \Theta$ of this application.[24] And the *rank of a cut rule application* is the rank of the associated cut formula $\pi : \varphi$.

[23] As before, the rule LC is treated symmetrically. Moreover, the hp-admissibility of the rule $C^{\leqslant}$ can be easily established by exploiting the closure condition for the order rules and for instances of the geometric rule scheme.

[24] Again, by the *height* of a proof tree, we mean the length of a longest branch in this tree.

$$\frac{\sigma : \varphi, \sigma \leqslant \pi, \pi : \varphi, \Gamma \Rightarrow \Delta}{\sigma \leqslant \pi, \pi : \varphi, \Gamma \Rightarrow \Delta}\ (\mathsf{glp}) \qquad \frac{w \leqslant \pi, \Gamma \Rightarrow \Delta, w : \alpha}{\Gamma \Rightarrow \Delta, \pi : \alpha}\ (\mathsf{grp})$$

$$\frac{u \leqslant R(w), u : \varphi, \Gamma \Rightarrow \Delta}{w : \Diamond\varphi, \Gamma \Rightarrow \Delta}\ L\Diamond \qquad \frac{u \leqslant R(w), \Gamma \Rightarrow \Delta, u : \varphi}{u \leqslant R(w), \Gamma \Rightarrow \Delta, w : \Diamond\varphi}\ R\Diamond$$

Figure 5.5: Some further admissible rules. In the rule (grp), w must be a fresh variable and α must be a declarative formula, i.e., $\alpha \in \mathcal{L}^{\mathsf{K}}_{!}$. In the rule $L\Diamond$, the variable u must be fresh.

Theorem 5.4.9 (Cut-Admissibility)**.** *The cut rule is admissible in* $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$.

Proof. We proceed by a main induction on the rank of the (labelled) cut formula with a subinduction on the height of the cut. Most cases are treated in the same way as in our sequent calculus for InqB (see Theorem 3.2.13). One of the new parts is the case in which the cut formula is of the form $\pi : \Box\varphi$ and principal on both sides. In this case, the cut rule application must be of the form

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{w \leqslant \pi, \Gamma \Rightarrow \Delta, R(w) : \varphi}}{\Gamma \Rightarrow \Delta, \pi : \Box\varphi}\ R\Box \qquad \dfrac{\dfrac{\mathcal{D}_2}{R(u) : \varphi, u \leqslant \pi, \pi : \Box\varphi, \Sigma \Rightarrow \Theta}}{u \leqslant \pi, \pi : \Box\varphi, \Sigma \Rightarrow \Theta}\ L\Box}{u \leqslant \pi, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\ (\mathsf{cut})$$

Using the hp-admissibility of substitution and contraction, this derivation is transformed into

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{w \leqslant \pi, \Gamma \Rightarrow \Delta, R(w) : \varphi}}{u \leqslant \pi, \Gamma \Rightarrow \Delta, R(u) : \varphi}\ (u/w) \qquad \dfrac{\dfrac{\dfrac{\mathcal{D}_1}{w \leqslant \pi, \Gamma \Rightarrow \Delta, R(w) : \varphi}}{\Gamma \Rightarrow \Delta, \pi : \Box\varphi}\ R\Box \qquad \dfrac{\mathcal{D}_2}{R(u) : \varphi, u \leqslant \pi, \pi : \Box\varphi, \Sigma \Rightarrow \Theta}}{R(u) : \varphi, u \leqslant \pi, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\ (\mathsf{cut})}{u \leqslant \pi, u \leqslant \pi, \Gamma, \Gamma, \Sigma \Rightarrow \Delta, \Delta, \Theta}\ (\mathsf{cut})}{u \leqslant \pi, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\ C$$

where the first application of the cut rule is of lower height and the second application is of lower rank than the original one (the latter assertion follows from part (iii) of Lemma 5.4.8). $\Box$

Finally, let us sketch some further admissibility results. Figure 5.5 comprises four additional rules that can be shown to be admissible in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. As before, the rule (glp) can be seen as a generalization of the rule Lp, reflecting the *persistency* of the support relation: if a formula φ is supported by a state π and if σ is an enhancement of π, then φ is also supported by σ. The rule (grp), on the other hand, generalizes the rule Rp and accounts for the *truth-conditionality* of declarative formulas: if a declarative α is true at every world w of a state π, then α is also supported by π. The rules $L\Diamond$ and $R\Diamond$, finally, reflect the truth conditions for $\Diamond$ described above: a formula $\Diamond\varphi$ is true at a world w, just in case there exists some world $u \in R(w)$ such that φ is true at u.

Proposition 5.4.10. *Each of the rules depicted in Figure 5.5 is admissible in* $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$.

Proof. The admissibility of (glp) is proved in exactly the same way as in the classical setting, by exploiting the admissibility of the cut rule in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ (see Proposition 3.2.14). In order to prove the admissibility of (grp), we proceed by induction on the structure of the declarative formula $\alpha \in \mathcal{L}^{\mathsf{K}}_{!}$. Again, most cases are treated in the same way as in the proof of Proposition 3.2.14. The only difference is that, in the base case, we also need to consider the case in which α is of the form $\alpha = \Box\varphi$ for some arbitrary formula $\varphi \in \mathcal{L}^{\mathsf{K}}$. For this purpose, let $w \in \mathfrak{S}$ be a fresh variable and suppose that $w \leqslant \pi, \Gamma \Rightarrow \Delta, w : \Box\varphi$ is derivable. Then, by the invertibility of the rule $R\Box$, there must also be a derivation $\mathcal{D}$ for the sequent $u \leqslant w, w \leqslant \pi, \Gamma \Rightarrow \Delta, R(u) : \varphi$, where $u \in \mathfrak{S}$ is again a fresh variable. Using this derivation, we now obtain the following proof tree:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ u \leqslant w, w \leqslant \pi, \Gamma \Rightarrow \Delta, R(u) : \varphi\end{array}}{w \leqslant w, w \leqslant \pi, \Gamma \Rightarrow \Delta, R(w) : \varphi}\;(w/u)}{w \leqslant \pi, \Gamma \Rightarrow \Delta, R(w) : \varphi}\;(\mathsf{rf})}{\Gamma \Rightarrow \Delta, \pi : \Box\varphi}\;R\Box$$

This concludes the proof. The admissibility of the rules $L\Diamond$ and $R\Diamond$ can easily be established by deriving the conclusion of the corresponding rule directly from the associated premise. $\square$

5.5 Indirect Completeness Proofs

As an interlude, we will now establish the soundness of all calculi $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ and show that, if $\mathfrak{L}$ is a standard modal logic determined by some combination A of the geometric axioms presented in Table 5.1, then the completeness of $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ can also be established indirectly, by using an appropriate Hilbert-style system for $\mathsf{Inq}\mathfrak{L}$. In Section 5.6, we will then use a countermodel construction in order to provide a general completeness proof, covering each of the systems $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$.

Let us start by proving the soundness of our calculi. First of all, if $w \in W$ is a world of a Kripke model $M = \langle W, R, V\rangle$ and if $s = \{w\}$ is the corresponding singleton state, then we will also write $R(s)$ instead of $R(w)$, where $R(w)$ refers to the neighbourhood of w in M. The labels used in our systems are now interpreted in the obvious way, so we adopt the following definition.

Definition 5.5.1 (Interpretation). Let $M = \langle W, R, V\rangle$ be a Kripke model. An *interpretation* over M is a function $I : \mathfrak{S} \cup \mathfrak{V} \to \mathcal{P}(W)$ such that, for all singleton variables $w \in \mathfrak{S}$, the state $I(w) \subseteq W$ is a singleton. Given an interpretation I over some Kripke model M, it is inductively extended to a function from the set $\Lambda(\mathfrak{S}, \mathfrak{V})$ of all labels to the set $\mathcal{P}(W)$ in the following way:

(i) $I(\emptyset) := \emptyset$,
(ii) $I(\pi \cdot \sigma) := I(\pi) \cap I(\sigma)$,
(iii) $I(\pi + \sigma) := I(\pi) \cup I(\sigma)$,
(iv) $I(R(w)) := R(I(w))$.

We will adopt the familiar terminology. In particular, given any interpretation I over some Kripke model M, we will say that a labelled formula $\pi : \varphi$ is *satisfied* by I, if φ is supported by the state $I(\pi)$, i.e., if we have $M, I(\pi) \models \varphi$. Furthermore, we say that a relational atom $\pi \leqslant \sigma$ is satisfied by I, just in case we have $I(\pi) \subseteq I(\sigma)$. A sequent $\Gamma \Rightarrow \Delta$ is called *valid in a Kripke model* M, if it satisfies the condition that, for all interpretations I over M, if I satisfies all labelled formulas and relational atoms in Γ, then there exists a labelled formula $\pi : \varphi$ in Δ such that I satisfies $\pi : \varphi$. We are now ready to prove the soundness of our calculi: if φ is provable from Γ in any of the systems $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$, then φ is entailed by Γ in the corresponding inquisitive logic $\mathsf{Inq}\mathfrak{L}$.

Proposition 5.5.2 (Soundness of $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$). *Let* $\mathsf{Inq}\mathfrak{L}$ *be a geometric system, let* A *be a finite set of geometric axioms determining* $\mathfrak{L}$*, and let* $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ *be the proof system given by Definition 5.3.3. For any finite* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{K}}$*, if* $x : \Gamma \Rightarrow x : \varphi$ *is derivable in* $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ *for some* $x \in \mathfrak{V}$*, then* $\Gamma \models^{\mathsf{Inq}}_{\mathfrak{L}} \varphi$*.*[25]

Proof. Let $\mathsf{Inq}\mathfrak{L}$ be an arbitrary geometric inquisitive system, let A be a finite set of geometric axioms such that $\mathfrak{L}$ is determined by A, and let $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ be the corresponding sequent calculus

[25] Recall that $\models^{\mathsf{Inq}}_{\mathfrak{L}}$ was defined to be the semantic consequence relation associated with the inquisitive Kripke logic $\mathsf{Inq}\mathfrak{L}$. Thus, we have $\Gamma \models^{\mathsf{Inq}}_{\mathfrak{L}} \varphi$ if and only if, for every Kripke model M based on an $\mathfrak{L}$-frame and for every state s over M, it is the case that $M, s \models \Gamma$ implies $M, s \models \varphi$. For further details, the reader is referred to Definition 5.2.7.

introduced in Definition 5.3.3. Consider an arbitrary Kripke model $M = \langle W, R, V \rangle$ such that $F = \langle W, R \rangle$ is an $\mathfrak{L}$-frame (thus, in particular, F satisfies each of the geometric axioms in A).[26]

We first prove that, if a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$, then $\Gamma \Rightarrow \Delta$ is valid in M. To this end, suppose that there exists a derivation $\mathcal{D}$ for $\Gamma \Rightarrow \Delta$ in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. Using induction on the structure of $\mathcal{D}$, we show that $\Gamma \Rightarrow \Delta$ is also valid in M. Most cases can be treated in the same way as in the classical setting. We only need to consider the case in which the last step in $\mathcal{D}$ is one of the rules for $\Box$, an instance of (nb), or an instance of the geometric rule scheme. First, assume that $\mathcal{D}$ ends with an application of the rule $R\Box$. In this case, $\mathcal{D}$ must be of the form

$$\dfrac{\begin{matrix}\mathcal{D}' \\ w \leqslant \pi, \Gamma \Rightarrow \Sigma, R(w) : \varphi\end{matrix}}{\Gamma \Rightarrow \Sigma, \pi : \Box\varphi}\, R\Box$$

where $w \in \mathfrak{S}$ is a fresh variable not occurring in $\Gamma \Rightarrow \Sigma, \pi : \Box\varphi$. By induction hypothesis, we know that the sequent $w \leqslant \pi, \Gamma \Rightarrow \Sigma, R(w) : \varphi$ is valid in M. We need to show that this also holds for the sequent $\Gamma \Rightarrow \Sigma, \pi : \Box\varphi$. Towards a contradiction, suppose that $\Gamma \Rightarrow \Sigma, \pi : \Box\varphi$ is *not* valid in M, i.e., there exists an interpretation I over M such that I satisfies all expressions in Γ, but I satisfies neither $\pi : \Box\varphi$ nor any expression in Σ. By the semantics of $\Box$, this implies that there exists some world $u \in W$ such that $u \in I(\pi)$ and $M, R(u) \not\models \varphi$. Let now I^* be the interpretation which is just like I, except that the variable w is mapped to the singleton state $\{u\}$, so we put $I^*(w) := \{u\}$. Then, clearly, I^* satisfies $w \leqslant \pi$ and each of the expressions in Γ. Therefore, by induction hypothesis, I^* must also satisfy $R(w) : \varphi$ or some labelled formula in Σ. If I^* satisfies $R(w) : \varphi$, then we must have $M, R(u) \models \varphi$ by definition of I^*, which is a contradiction to the fact that $M, R(u) \not\models \varphi$. And if I^* satisfies some element of Σ, then also the original interpretation I must satisfy this element, which is a contradiction to our assumption about I. Thus, in either case, we arrive at a contradiction, so $\Gamma \Rightarrow \Sigma, \pi : \Box\varphi$ is valid in M.

If the last step in $\mathcal{D}$ is an application of $L\Box$ or (nb), then the proof is similar. In order to show the soundness of the geometric rules, let now $\theta \in \mathsf{A}$ be an arbitrary geometric axiom. Without loss of generality, assume that θ is given to us in the form $\forall\vec{w}(\varphi \to (\exists u_1\psi_1 \vee \ldots \vee \exists u_n\psi_n))$, where each $\varphi, \psi_1, \ldots, \psi_n$ is a conjunction of atomic formulas from the frame language $\mathcal{L}^{\mathfrak{S}}_{\mathsf{F}}$. Moreover, suppose that $\mathcal{D}$ ends with an application of the corresponding geometric rule, so $\mathcal{D}$ is of the form

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Psi_1^\circ(v_1/u_1), \Phi^\circ, \Sigma \Rightarrow \Delta\end{matrix} \quad \cdots \quad \begin{matrix}\mathcal{D}_n \\ \Psi_n^\circ(v_n/u_n), \Phi^\circ, \Sigma \Rightarrow \Delta\end{matrix}}{\Phi^\circ, \Sigma \Rightarrow \Delta}\, (\theta\text{-grs})$$

where each v_i is a fresh variable not occurring in the conclusion, Φ° is the set of relational atoms corresponding to the atomic formulas in φ, and Ψ_i° is the set of relational atoms corresponding to the atomic formulas in ψ_i.[27] By induction hypothesis, we know that each of the sequents $\Psi_i^\circ(v_i/u_i), \Phi^\circ, \Sigma \Rightarrow \Delta$ is valid in the Kripke model M. We need to prove that this also holds for $\Phi^\circ, \Sigma \Rightarrow \Delta$. To this end, let I be an arbitrary interpretation over M and suppose that I satisfies all expressions in Φ° and Σ. Since M is based on a frame F satisfying the geometric axiom θ, there must be some world $w_i \in W$ such that F satisfies the first-order formula ψ_i, provided that the variable u_i is interpreted as w_i and all free variables in ψ_i are interpreted in accordance with I. Let now I^* be the interpretation which is just like I, except that the variable v_i is mapped to the singleton state $\{w_i\}$, so we put $I^*(v_i) := \{w_i\}$. Then, clearly, I^* satisfies all expressions in $\Psi_i^\circ(v_i/u_i), \Phi^\circ, \Sigma$. Hence, by induction hypothesis, there must be some element of Δ which is satisfied by I^* and therefore also by I. Since I was an arbitrary interpretation satisfying all expressions in Φ° and Σ, this shows that the sequent $\Phi^\circ, \Sigma \Rightarrow \Delta$ is valid in M, as desired.

[26] As explained above, F is an $\mathfrak{L}$-frame, if F validates every formula in $\mathfrak{L}$. Here, $\mathfrak{L}$ is assumed to be the standard modal logic determined by A, so this is equivalent to saying that F satisfies each of the geometric axioms in A when considered as a model of first-order logic. For a more detailed explanation of the relevant notions, see Section 5.2.3.

[27] By what was said in Section 5.3, given any set $\Theta \subseteq \mathcal{L}^{\mathfrak{S}}_{\mathsf{F}}$ of atomic formulas from the frame language, we write Θ° for the result of replacing every atomic formula Ruv in Θ by the corresponding relational atom $v \leqslant R(u)$.

Axioms:

(IPL)	All axioms of the 'intuitionistic' system given in Figure 1.5,
($\rightarrow$Dis)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$,
($\mathbb{V}$Dis)	$\Box(\varphi \mathbin{\mathbb{V}} \psi) \rightarrow (\Box\varphi \vee \Box\psi)$,
(Split)	$(\alpha \rightarrow (\varphi \mathbin{\mathbb{V}} \psi)) \rightarrow ((\alpha \rightarrow \varphi) \mathbin{\mathbb{V}} (\alpha \rightarrow \psi))$, where $\alpha \in \mathcal{L}^{\mathsf{K}}_{!}$ is declarative,
(DN)	$\neg\neg\alpha \rightarrow \alpha$, where $\alpha \in \mathcal{L}^{\mathsf{K}}_{!}$ is declarative.

The rules of inference of our system are *modus ponens* (from $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \rightarrow \psi$, one can infer $\Gamma \vdash \psi$) as well as *necessitation* (from $\vdash \varphi$, one can infer $\vdash \Box\varphi$).

Figure 5.6: The Hilbert-style system HinqK.

This concludes the induction. Thus, if $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$, then $\Gamma \Rightarrow \Delta$ is valid in all Kripke models M based on an $\mathfrak{L}$-frame. Let now $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{K}}$ be an arbitrary finite set and suppose that $x : \Gamma \Rightarrow x : \varphi$ is derivable in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$, where $x \in \mathfrak{V}$. As we have just seen, this implies that, for every Kripke model M, based on an $\mathfrak{L}$-frame, and for all interpretations I over M, it is the case that $M, I(x) \models \Gamma$ implies $M, I(x) \models \varphi$. But then, clearly, it follows $\Gamma \models^{\mathsf{Inq}}_{\mathfrak{L}} \varphi$. $\Box$

We now want to establish the completeness of $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ for the special case in which $\mathsf{A} \subseteq \mathcal{L}^{\mathfrak{S}}_{\mathsf{F}}$ consists of one or more of the geometric axioms presented in Table 5.1. The overall structure of our argument will be the same as in the previous chapters. Thus, given any standard modal logic $\mathfrak{L}$ determined by some combination A of the geometric axioms from Table 5.1, we will first define a Hilbert-style system $\mathsf{Hinq}\mathfrak{L}_{\Theta}$ for $\mathsf{Inq}\mathfrak{L}$, where $\Theta \subseteq \{\mathsf{T}, 4, 5, \mathsf{B}, \mathsf{D}\}$ is the set of standard axiom schemes associated with the geometric axioms in A. Afterwards, we will show that $\mathsf{Hinq}\mathfrak{L}_{\Theta}$ is equivalent to the corresponding natural deduction system $\mathsf{Ninq}\mathfrak{L}_{\Theta}$ introduced in Section 5.2.2, which allows us to conclude that $\mathsf{Hinq}\mathfrak{L}_{\Theta}$ is sound and complete with respect to $\mathsf{Inq}\mathfrak{L}$. Finally, we will use the system $\mathsf{Hinq}\mathfrak{L}_{\Theta}$ in order to establish the completeness of $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$.

A sound and complete Hilbert-style proof system for the basic logic InqK is depicted in Figure 5.6. As can be seen, our system is denoted by HinqK and comprises all axioms of the system from Figure 1.5, together with the distributivity axioms ($\rightarrow$Dis) and ($\mathbb{V}$Dis), the split axiom (Split) and the double negation axiom (DN). The only rules of inference are modus ponens and necessitation. Observe that, in the axiom schemes (Split) and (DN), α is allowed to range over arbitrary *declarative* formulas, and not just over *standard* formulas. Moreover, the necessitation rule can only be applied, if the premise of the rule is derived without using hypotheses. We will now extend HinqK to a proof system $\mathsf{Hinq}\mathfrak{L}_{\Theta}$ for the inquisitive logic $\mathsf{Inq}\mathfrak{L}$, where $\mathfrak{L}$ is an arbitrary standard modal logic generated by some combination Θ of the axiom schemes presented in Table 5.1.

Definition 5.5.3 (The Systems $\mathsf{Hinq}\mathfrak{L}_{\Theta}$)**.** Let $\mathfrak{L}$ be a standard modal logic generated by some combination $\Theta \subseteq \{\mathsf{T}, 4, 5, \mathsf{B}, \mathsf{D}\}$ of the axiom schemes from Table 5.1.[28] The Hilbert-style system $\mathsf{Hinq}\mathfrak{L}_{\Theta}$ is obtained by adding all axioms from Θ to the basic system HinqK given in Figure 5.6.

So, for example, in order to construct a Hilbert-style system for InqKT, we can add all instances of the scheme $\Box\alpha \rightarrow \alpha$ to HinqK. And in order to obtain a Hilbert-style system for InqKD4 we may add all instances of $\Box\alpha \rightarrow \Diamond\alpha$ and $\Box\alpha \rightarrow \Box\Box\alpha$ to HinqK. Note that, in each case, α must be a *standard* formula, so we require $\alpha \in \mathcal{L}^{\mathsf{K}}_{s}$. The provability relation associated with the system

[28] Recall that a standard modal logic $\mathfrak{L}$ is *generated* or *axiomatized* by Θ, if $\mathfrak{L}$ is the smallest normal modal logic containing Θ (see Section 5.2.1). Thus, strictly speaking, Θ should be a set of standard formulas, rather than a set of axiom schemes. However, for simplicity, we will also identify an axiom scheme with the collection of its instances.

$\mathsf{Hinq}\mathfrak{L}_\Theta$ is denoted by $\vdash^{\mathsf{H}}_{\Theta}$ and inductively defined in the usual way. Thus, we will write $\Gamma \vdash^{\mathsf{H}}_{\Theta} \varphi$ and say that φ *is provable from* Γ in $\mathsf{Hinq}\mathfrak{L}_\Theta$, if one of the following three conditions is satisfied:

(1) φ is an element of Γ or an axiom of $\mathsf{Hinq}\mathfrak{L}_\Theta$.

(2) There exists some formula $\psi \in \mathcal{L}^{\mathsf{K}}$ such that both $\Gamma \vdash^{\mathsf{H}}_{\Theta} \psi$ and $\Gamma \vdash^{\mathsf{H}}_{\Theta} \psi \to \varphi$.

(3) There exists some $\psi \in \mathcal{L}^{\mathsf{K}}$ such that $\varphi = \Box\psi$ and $\vdash^{\mathsf{H}}_{\Theta} \psi$.

Using induction on the definition of $\Gamma \vdash^{\mathsf{H}}_{\Theta} \varphi$, it is easy to prove that the relation $\vdash^{\mathsf{H}}_{\Theta}$ is *monotonic*, i.e., if we have $\Gamma \vdash^{\mathsf{H}}_{\Theta} \varphi$ and $\Gamma \subseteq \Delta$, then also $\Delta \vdash^{\mathsf{H}}_{\Theta} \varphi$. We now want to show that each of the calculi $\mathsf{Hinq}\mathfrak{L}_\Theta$ is equivalent to the corresponding natural deduction system $\mathsf{Ninq}\mathfrak{L}_\Theta$ given by Definition 5.2.10. To this end, one first has to prove the *deduction theorem* for $\mathsf{Hinq}\mathfrak{L}_\Theta$.

Theorem 5.5.4 (Deduction Theorem)**.** *Let Θ be some combination of the axiom schemes from Table 5.1 and let $\mathfrak{L}$ be generated by Θ. Then, in* $\mathsf{Hinq}\mathfrak{L}_\Theta$*, we have $\Gamma, \varphi \vdash^{\mathsf{H}}_{\Theta} \psi$ if and only if $\Gamma \vdash^{\mathsf{H}}_{\Theta} \varphi \to \psi$.*[29]

The argument is essentially the same as in the classical setting (see the proof of Theorem 1.5.4). Using the deduction theorem, one can now prove that each of the Hilbert-style calculi $\mathsf{Hinq}\mathfrak{L}_\Theta$ is equivalent to the corresponding natural deduction system $\mathsf{Ninq}\mathfrak{L}_\Theta$ given by Definition 5.2.10.

Theorem 5.5.5. *Let Θ be some combination of the axiom schemes from Table 5.1 and let $\mathfrak{L}$ be generated by Θ. It holds $\Gamma \vdash^{\mathsf{H}}_{\Theta} \varphi$ if and only if $\Gamma \vdash^{\mathsf{N}}_{\Theta} \varphi$, where $\vdash^{\mathsf{N}}_{\Theta}$ is the provability relation of* $\mathsf{Ninq}\mathfrak{L}_\Theta$*.*

Proof. For the left-to-right direction, one proceeds by induction on the definition of $\Gamma \vdash^{\mathsf{H}}_{\Theta} \varphi$. This is straightforward, since all axioms depicted in Figure 5.6 are clearly provable in $\mathsf{Ninq}\mathfrak{L}_\Theta$ and modus ponens and necessitation are also available in $\mathsf{Ninq}\mathfrak{L}_\Theta$. Moreover, by Definition 5.2.10, each of the special axioms from Θ is also an axiom of $\mathsf{Ninq}\mathfrak{L}_\Theta$. For the right-to-left direction, one can use induction on the structure of a natural deduction proof for $\Gamma \vdash^{\mathsf{N}}_{\Theta} \varphi$. This is also not difficult, since most of the natural deduction rules correspond directly to some axiom of $\mathsf{Hinq}\mathfrak{L}_\Theta$, and the discharging of hypotheses can be 'simulated' using the deduction theorem for $\mathsf{Hinq}\mathfrak{L}_\Theta$. $\Box$

As a corollary, it follows that $\mathsf{Hinq}\mathfrak{L}_\Theta$ is sound and complete with respect to $\mathsf{Inq}\mathfrak{L}$. We now want to use our Hilbert-style systems in order to prove the completeness of $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ for the special case in which $\mathsf{A} \subseteq \mathcal{L}^{\mathfrak{G}}_{\mathsf{F}}$ is some combination of the geometric axioms presented in Table 5.1. To this end, we first show that each of the axioms from Figure 5.6 is provable in GLinqK.

Lemma 5.5.6. *Let φ be an instance of one of the axiom schemes depicted in Figure 5.6. Then, φ is provable in* GLinqK*, i.e., for any $x \in \mathfrak{V}$, there exists a derivation for $\Rightarrow x : \varphi$ in* GLinqK*.*

Proof. For the axioms from Figure 1.5, the proof is straightforward. Thus, we only need to show the provability of the two distribution axioms, the split axiom and the double negation axiom. For the axiom schemes ($\to$Dis) and (DN), we may construct the following derivations:

By Lemma 5.4.1 (ii) $\qquad$ By Lemma 5.4.1 (ii)

$$\dfrac{\ldots, R(w) : \varphi \Rightarrow R(w) : \psi, R(w) : \varphi \qquad \ldots, R(w) : \psi, R(w) : \varphi \Rightarrow R(w) : \psi}{\ldots, R(w) \leqslant R(w), R(w) : \varphi \to \psi, R(w) : \varphi, w \leqslant y, w \leqslant z, y : \Box(\varphi \to \psi), z : \Box\varphi \Rightarrow R(w) : \psi}\ L{\to}$$

$$\dfrac{\ldots, R(w) \leqslant R(w), R(w) : \varphi \to \psi, R(w) : \varphi, w \leqslant y, w \leqslant z, y : \Box(\varphi \to \psi), z : \Box\varphi \Rightarrow R(w) : \psi}{R(w) : \varphi \to \psi, R(w) : \varphi, w \leqslant y, w \leqslant z, z \leqslant y, y \leqslant x, y : \Box(\varphi \to \psi), z : \Box\varphi \Rightarrow R(w) : \psi}\ \text{(rf)}$$

$$\dfrac{R(w) : \varphi \to \psi, R(w) : \varphi, w \leqslant y, w \leqslant z, z \leqslant y, y \leqslant x, y : \Box(\varphi \to \psi), z : \Box\varphi \Rightarrow R(w) : \psi}{R(w) : \varphi, w \leqslant y, w \leqslant z, z \leqslant y, y \leqslant x, y : \Box(\varphi \to \psi), z : \Box\varphi \Rightarrow R(w) : \psi}\ L\Box$$

$$\dfrac{R(w) : \varphi, w \leqslant y, w \leqslant z, z \leqslant y, y \leqslant x, y : \Box(\varphi \to \psi), z : \Box\varphi \Rightarrow R(w) : \psi}{w \leqslant y, w \leqslant z, z \leqslant y, y \leqslant x, y : \Box(\varphi \to \psi), z : \Box\varphi \Rightarrow R(w) : \psi}\ L\Box$$

$$\dfrac{w \leqslant y, w \leqslant z, z \leqslant y, y \leqslant x, y : \Box(\varphi \to \psi), z : \Box\varphi \Rightarrow R(w) : \psi}{w \leqslant z, z \leqslant y, y \leqslant x, y : \Box(\varphi \to \psi), z : \Box\varphi \Rightarrow R(w) : \psi}\ \text{(tr)}$$

$$\dfrac{w \leqslant z, z \leqslant y, y \leqslant x, y : \Box(\varphi \to \psi), z : \Box\varphi \Rightarrow R(w) : \psi}{z \leqslant y, y \leqslant x, y : \Box(\varphi \to \psi), z : \Box\varphi \Rightarrow z : \Box\psi}\ R\Box$$

$$\dfrac{z \leqslant y, y \leqslant x, y : \Box(\varphi \to \psi), z : \Box\varphi \Rightarrow z : \Box\psi}{y \leqslant x, y : \Box(\varphi \to \psi) \Rightarrow y : \Box\varphi \to \Box\psi}\ R{\to}$$

$$\dfrac{y \leqslant x, y : \Box(\varphi \to \psi) \Rightarrow y : \Box\varphi \to \Box\psi}{\Rightarrow x : \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)}\ R{\to}$$

[29] The reader might wonder whether this statement is in fact true, since textbook sources sometimes claim that the deduction theorem fails for modal proof systems containing the necessitation rule. However, as shown by Hakli and Negri (2012), this is actually a misconception: if necessitation is restricted to cases in which the premise does not depend on assumptions (as in our proof systems), then the usual version of the deduction theorem can be obtained.

$$
\begin{prooftree}
\AxiomC{$\Rightarrow x : \varphi$}
\AxiomC{$\Rightarrow x : \varphi \to \psi$}
\RightLabel{(mp)}
\BinaryInfC{$\Rightarrow x : \psi$}
\end{prooftree}
\qquad\qquad
\begin{prooftree}
\AxiomC{$\Rightarrow x : \varphi$}
\RightLabel{(nec)}
\UnaryInfC{$\Rightarrow x : \Box\varphi$}
\end{prooftree}
$$

Figure 5.7: Modus ponens and necessitation. In either case, we require $x \in \mathfrak{V}$.

$$
\begin{prooftree}
\AxiomC{By Lemma 5.4.2 (i)}
\dashedLine
\UnaryInfC{$z \leqslant \emptyset, \ldots \Rightarrow w : \alpha, z : \bot$}
\AxiomC{By Lemma 5.4.1 (i)}
\dashedLine
\UnaryInfC{$w \leqslant z, \ldots, z : \alpha \Rightarrow w : \alpha, z : \bot$}
\RightLabel{(sg)}
\BinaryInfC{$z \leqslant w, w \leqslant y, y \leqslant x, y : \neg\neg\alpha, z : \alpha \Rightarrow w : \alpha, z : \bot$}
\RightLabel{$R{\to}$}
\UnaryInfC{$w \leqslant y, y \leqslant x, y : \neg\neg\alpha \Rightarrow w : \alpha, w : \neg\alpha$}
\AxiomC{}
\RightLabel{$Ax^{\bot}$}
\UnaryInfC{$\ldots, w : \bot \Rightarrow w : \alpha$}
\RightLabel{$L{\to}$}
\BinaryInfC{$w \leqslant y, y \leqslant x, y : \neg\neg\alpha \Rightarrow w : \alpha$}
\dashedLine
\RightLabel{(grp)}
\UnaryInfC{$y \leqslant x, y : \neg\neg\alpha \Rightarrow y : \alpha$}
\RightLabel{$R{\to}$}
\UnaryInfC{$\Rightarrow x : \neg\neg\alpha \to \alpha$}
\end{prooftree}
$$

Note that, in the derivation for (DN), we use the admissible rule (grp) from Figure 5.5. This is legitimate, since $\alpha \in \mathcal{L}^{\mathsf{K}}_{!}$ is assumed to be declarative. The axiom ($\vvee$Dis) can be derived as follows:[30]

$$
\begin{prooftree}
\AxiomC{$\mathcal{D}$}
\noLine
\UnaryInfC{$\ldots, R(w) : \varphi \vvee \psi \Rightarrow \ldots, w : \Box\varphi, w : \Box\psi$}
\AxiomC{}
\RightLabel{$Ax^{\bot}$}
\UnaryInfC{$\ldots, w : \bot \Rightarrow \ldots, w : \Box\varphi$}
\RightLabel{$L{\to}$}
\BinaryInfC{$\ldots, w \leqslant z, R(w) : \varphi \vvee \psi, z : \neg\Box\varphi, z : \neg\Box\psi \Rightarrow w : \bot, w : \Box\varphi$}
\AxiomC{}
\RightLabel{$Ax^{\bot}$}
\UnaryInfC{$\ldots, w : \bot \Rightarrow w : \bot$}
\RightLabel{$L{\to}$}
\BinaryInfC{$w \leqslant y, w \leqslant z, z \leqslant y, y \leqslant x, R(w) : \varphi \vvee \psi, y : \Box(\varphi \vvee \psi), z : \neg\Box\varphi, z : \neg\Box\psi \Rightarrow w : \bot$}
\RightLabel{$L\wedge$}
\UnaryInfC{$w \leqslant y, w \leqslant z, z \leqslant y, y \leqslant x, R(w) : \varphi \vvee \psi, y : \Box(\varphi \vvee \psi), z : \neg\Box\varphi \wedge \neg\Box\psi \Rightarrow w : \bot$}
\RightLabel{$L\Box$}
\UnaryInfC{$w \leqslant y, w \leqslant z, z \leqslant y, y \leqslant x, y : \Box(\varphi \vvee \psi), z : \neg\Box\varphi \wedge \neg\Box\psi \Rightarrow w : \bot$}
\RightLabel{(tr)}
\UnaryInfC{$w \leqslant z, z \leqslant y, y \leqslant x, y : \Box(\varphi \vvee \psi), z : \neg\Box\varphi \wedge \neg\Box\psi \Rightarrow w : \bot$}
\RightLabel{$R\bot$}
\UnaryInfC{$z \leqslant y, y \leqslant x, y : \Box(\varphi \vvee \psi), z : \neg\Box\varphi \wedge \neg\Box\psi \Rightarrow z : \bot$}
\RightLabel{$R{\to}$}
\UnaryInfC{$y \leqslant x, y : \Box(\varphi \vvee \psi) \Rightarrow y : \neg(\neg\Box\varphi \wedge \neg\Box\psi)$}
\RightLabel{$R{\to}$}
\UnaryInfC{$\Rightarrow x : \Box(\varphi \vvee \psi) \to \neg(\neg\Box\varphi \wedge \neg\Box\psi)$}
\end{prooftree}
$$

where the subderivation $\mathcal{D}$ is of the form

$$
\begin{prooftree}
\AxiomC{By Lemma 5.4.1 (i)}
\dashedLine
\UnaryInfC{$\ldots, R(u) \leqslant R(w), R(w) : \varphi \Rightarrow \ldots, R(u) : \varphi$}
\RightLabel{(nb)}
\UnaryInfC{$\ldots, u \leqslant w, R(w) : \varphi \Rightarrow \ldots, R(u) : \varphi$}
\AxiomC{By Lemma 5.4.1 (i)}
\dashedLine
\UnaryInfC{$\ldots, R(v) \leqslant R(w), R(w) : \psi \Rightarrow \ldots, R(v) : \psi$}
\RightLabel{(nb)}
\UnaryInfC{$\ldots, v \leqslant w, R(w) : \psi \Rightarrow \ldots, R(v) : \psi$}
\RightLabel{$L\vvee$}
\BinaryInfC{$\ldots, v \leqslant w, u \leqslant w, R(w) : \varphi \vvee \psi \Rightarrow \ldots, R(u) : \varphi, R(v) : \psi$}
\RightLabel{$R\Box$}
\UnaryInfC{$\ldots, u \leqslant w, R(w) : \varphi \vvee \psi \Rightarrow \ldots, R(u) : \varphi, w : \Box\psi$}
\RightLabel{$R\Box$}
\UnaryInfC{$\ldots, R(w) : \varphi \vvee \psi \Rightarrow \ldots, w : \Box\varphi, w : \Box\psi$}
\end{prooftree}
$$

The split axiom (Split), finally, can be derived in the same way as in the proof of Lemma 3.3.5. $\Box$

Next, we need to show that modus ponens and necessitation are admissible in GLinqK.

Lemma 5.5.7. *The rules* (mp) *and* (nec)*, depicted in Figure 5.7, are admissible in* GLinqK.

Proof. The admissibility of (mp) is proved in the usual way, by using the admissibility of the cut rule. For further details, the reader is referred to the proof of Lemma 3.3.6. In order to show the admissibility of (nec), assume that $\Rightarrow x : \varphi$ is derivable in GLinqK, where $x \in \mathfrak{V}$. For any singleton variable $w \in \mathfrak{S}$, this yields a derivation for $w \leqslant x \Rightarrow R(w) : \varphi$ by the admissibility of substitution and weakening. But then, by an application of $R\Box$, we obtain a derivation for $\Rightarrow x : \Box\varphi$. $\Box$

We now want to prove the completeness of $\mathrm{GLinq}\mathfrak{L}_{\mathsf{A}}$ for the special case in which A consists of one or more of the geometric axioms given in Table 5.1. To this end, we first show that every formula provable in the corresponding Hilbert-style system for $\mathrm{Inq}\mathfrak{L}$ is also provable in $\mathrm{GLinq}\mathfrak{L}_{\mathsf{A}}$.

[30] Recall that $\vee$ was defined by $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$.

Theorem 5.5.8. *Let $\mathfrak{L}$ be determined by some combination A of the geometric axioms from Table 5.1 and let $\Theta \subseteq \{\mathsf{T}, \mathsf{4}, \mathsf{5}, \mathsf{B}, \mathsf{D}\}$ be the corresponding set of standard axiom schemes.*[31] *If we have $\vdash^{\mathsf{H}}_{\Theta} \varphi$ in the Hilbert-style system $\mathsf{Hinq}\mathfrak{L}_{\Theta}$, then $\Rightarrow x : \varphi$ is derivable in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ for any $x \in \mathfrak{V}$.*

Proof. The statement is proved by induction on a Hilbert-style proof for $\vdash^{\mathsf{H}}_{\Theta} \varphi$. By Lemma 5.5.6 and 5.5.7, we already know that all axioms of the basic system HinqK are derivable in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ and that modus ponens and necessitation are admissible in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. Hence, it suffices to show that each of the special schemes in $\{\mathsf{T}, \mathsf{4}, \mathsf{5}, \mathsf{B}, \mathsf{D}\}$ is derivable in terms of the corresponding sequent rule depicted in Figure 5.3. For T, $\mathsf{4}$ and B, we may construct the following derivations:[32]

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{By Lemma 5.4.1 (i)}}{w \leqslant R(w), R(w) : \alpha, w \leqslant y, y \leqslant x, y : \Box\alpha \Rightarrow w : \alpha}}{R(w) : \alpha, w \leqslant y, y \leqslant x, y : \Box\alpha \Rightarrow w : \alpha}\,(\mathsf{T})}{w \leqslant y, y \leqslant x, y : \Box\alpha \Rightarrow w : \alpha}\,L\Box}{y \leqslant x, y : \Box\alpha \Rightarrow y : \alpha}\,(\mathsf{grp})}{\Rightarrow x : \Box\alpha \to \alpha}\,R{\to}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{By Lemma 5.4.1 (i)}}{v \leqslant R(w), R(w) : \alpha, v \leqslant R(u), u \leqslant R(w), w \leqslant y, y \leqslant x, y : \Box\alpha \Rightarrow v : \alpha}}{R(w) : \alpha, v \leqslant R(u), u \leqslant R(w), w \leqslant y, y \leqslant x, y : \Box\alpha \Rightarrow v : \alpha}\,(\mathsf{4})}{v \leqslant R(u), u \leqslant R(w), w \leqslant y, y \leqslant x, y : \Box\alpha \Rightarrow v : \alpha}\,L\Box}{u \leqslant R(w), w \leqslant y, y \leqslant x, y : \Box\alpha \Rightarrow R(u) : \alpha}\,(\mathsf{grp})}{w \leqslant y, y \leqslant x, y : \Box\alpha \Rightarrow R(w) : \Box\alpha}\,R\Box}{y \leqslant x, y : \Box\alpha \Rightarrow y : \Box\Box\alpha}\,R\Box}{\Rightarrow x : \Box\alpha \to \Box\Box\alpha}\,R{\to}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{By Lemma 5.4.1 (i)}}{\ldots, w \leqslant y, y \leqslant x, y : \alpha, z : \Box\neg\alpha \Rightarrow u : \bot, w : \alpha} \qquad \dfrac{}{\ldots, w : \bot \Rightarrow u : \bot}\,Ax^{\bot}}{w \leqslant R(u), u \leqslant R(w), R(u) : \neg\alpha, u \leqslant z, z \leqslant R(w), w \leqslant y, y \leqslant x, y : \alpha, z : \Box\neg\alpha \Rightarrow u : \bot}\,L{\to}}{u \leqslant R(w), R(u) : \neg\alpha, u \leqslant z, z \leqslant R(w), w \leqslant y, y \leqslant x, y : \alpha, z : \Box\neg\alpha \Rightarrow u : \bot}\,(\mathsf{B})}{R(u) : \neg\alpha, u \leqslant z, z \leqslant R(w), w \leqslant y, y \leqslant x, y : \alpha, z : \Box\neg\alpha \Rightarrow u : \bot}\,(\mathsf{tr})}{u \leqslant z, z \leqslant R(w), w \leqslant y, y \leqslant x, y : \alpha, z : \Box\neg\alpha \Rightarrow u : \bot}\,L\Box}{z \leqslant R(w), w \leqslant y, y \leqslant x, y : \alpha, z : \Box\neg\alpha \Rightarrow z : \bot}\,R\bot}{w \leqslant y, y \leqslant x, y : \alpha \Rightarrow R(w) : \neg\Box\neg\alpha}\,R{\to}}{y \leqslant x, y : \alpha \Rightarrow y : \Box\neg\Box\neg\alpha}\,R\Box}{\Rightarrow x : \alpha \to \Box\neg\Box\neg\alpha}\,R{\to}$$

Note that, in each case, the application of the admissible rule (grp) is indeed correct, since $\alpha \in \mathcal{L}^{\mathsf{K}}_{s}$ is assumed to be a standard formula (and therefore also a declarative formula). For the axiom schemes 5 and D, similar derivations can be found. The details are left to the reader. ☐

Corollary 5.5.9. *Let $\mathfrak{L}$ be determined by some combination A of the geometric axioms from Table 5.1. Then, $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ is sound and complete with respect to $\mathsf{Inq}\mathfrak{L}$, i.e., for any finite $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{K}}$, we have $\Gamma \vDash^{\mathsf{Inq}}_{\mathfrak{L}} \varphi$ if and only if $x : \Gamma \Rightarrow x : \varphi$ is derivable in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ for any $x \in \mathfrak{V}$.*

Proof. The soundness of $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ has been established in Proposition 5.5.2. For the completeness part, let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{K}}$ be an arbitrary finite set of formulas and assume that it holds $\Gamma \vDash^{\mathsf{Inq}}_{\mathfrak{L}} \varphi$. By Theorem 5.2.11, this yields $\Gamma \vdash^{\mathsf{N}}_{\Theta} \varphi$, where Θ is the set of standard axiom schemes corresponding to the geometric axioms in A, and $\vdash^{\mathsf{N}}_{\Theta}$ is the provability relation associated with the natural

[31] So, for example, if A contains the geometric axiom expressing *reflexivity*, then Θ includes all instances of $\Box\alpha \to \alpha$, and if A contains the geometric axiom corresponding to *transitivity*, then Θ includes all instances of $\Box\alpha \to \Box\Box\alpha$.

[32] In the derivation for axiom scheme B, we use the fact that $\Diamond$ was defined by $\Diamond\varphi := \neg\Box\neg\varphi$.

deduction system $\mathsf{Ninq}\mathfrak{L}_\Theta$. But then, by Theorem 5.5.5, we also have $\Gamma \vdash^{\mathsf{H}}_{\Theta} \varphi$ in the Hilbert-style system $\mathsf{Hinq}\mathfrak{L}_\Theta$. Using the deduction theorem for $\mathsf{Hinq}\mathfrak{L}_\Theta$ (see Theorem 5.5.4), one readily sees that this implies $\vdash^{\mathsf{H}}_{\Theta} \bigwedge\Gamma \to \varphi$, where $\bigwedge\Gamma$ is the conjunction of the elements of Γ. Therefore, by Theorem 5.5.8, the sequent $\Rightarrow x : \bigwedge\Gamma \to \varphi$ is derivable in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ for any $x \in \mathfrak{V}$. Now, by the invertibility of the rules $R{\to}$ and $L\wedge$, it follows that $y \leqslant x, y : \Gamma \Rightarrow y : \varphi$ is derivable in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$, where $y \in \mathfrak{V}$ is a fresh variable. But then, by performing the substitution (x/y) and a subsequent application of (rf), we obtain the desired derivation for $x : \Gamma \Rightarrow x : \varphi$ in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. □

5.6 Completeness via Countermodels

In the previous section, we have seen a restricted completeness proof, covering a limited class of the proof systems $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. We now want to give a general completeness proof, applicable to *each* of our labelled sequent calculi. More precisely, given any inquisitive logic $\mathsf{Inq}\mathfrak{L}$ such that $\mathfrak{L}$ is determined by some finite set of geometric axioms $\mathsf{A} \subseteq \mathcal{L}^{\mathfrak{G}}_{\mathsf{F}}$, we will show that $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ is sound and complete with respect to $\mathsf{Inq}\mathfrak{L}$. Consequently, the strategy described in Section 5.3 is in fact adequate in order to generate cut-free labelled sequent calculi for the full class of geometric extensions of the basic system InqK. This generalizes a famous result by Negri (2005), who provides a general method for the construction of cut-free labelled sequent calculi for all geometric extensions of the standard modal logic K (see also Negri and Von Plato 1998; Negri 2003; Dyckhoff and Negri 2012). Our argument is based on the construction of an infinite proof search tree and the extraction of a countermodel from an open branch of this tree.[33]

Throughout this section, let $\mathsf{Inq}\mathfrak{L}$ be an arbitrary geometric inquisitive system, let $\mathsf{A} \subseteq \mathcal{L}^{\mathfrak{G}}_{\mathsf{F}}$ be an arbitrary finite set of geometric axioms determining $\mathfrak{L}$, and let $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ be the sequent calculus for $\mathsf{Inq}\mathfrak{L}$ given by Definition 5.3.3. We start by introducing some basic terminology.

Definition 5.6.1 (Proof Search Tree, Branch)**.** Let $\varphi \in \mathcal{L}^{\mathsf{K}}$ be a formula. A *proof search tree* for φ is a possibly infinite tree of sequents $\mathfrak{T}$, constructed from a root node of the form $\Rightarrow x : \varphi$ with $x \in \mathfrak{V}$, by root-first applications of the proof rules of $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. By a *branch* in a proof search tree $\mathfrak{T}$, we mean any sequence β of consecutive sequents in $\mathfrak{T}$ such that the first sequent in β is the root node $\Rightarrow x : \varphi$ of $\mathfrak{T}$, and β is either infinite or it ends with one of the leaf nodes of $\mathfrak{T}$.

In what follows, proof search trees are assumed to be *upward growing*, so the root node is always written at the bottom and leaves are written at the top of the tree. Given a proof search tree $\mathfrak{T}$ and a branch β in $\mathfrak{T}$, we say that β is *closed*, if β is finite and the topmost sequent in β is an instance of one of the axioms of $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. If a branch is not closed, it is said to be *open*. The tree $\mathfrak{T}$ is called closed, if *every* branch in $\mathfrak{T}$ is closed, and it is said to be open otherwise. For any branch β of the form $\Gamma_0 \Rightarrow \Delta_0, \Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots$, we also define the sets $\Gamma^{\downarrow}_{\beta}$ and $\Delta^{\downarrow}_{\beta}$ by putting

$$\Gamma^{\downarrow}_{\beta} := \bigcup_{i \geq 0} \Gamma_i \qquad \text{and} \qquad \Delta^{\downarrow}_{\beta} := \bigcup_{i \geq 0} \Delta_i,$$

so $\Gamma^{\downarrow}_{\beta}$ is the union of all the *antecedents* of sequents in β, and $\Delta^{\downarrow}_{\beta}$ is the union of all the *succedents* of sequents in β. Note that, if β is infinite, then each of the sets $\Gamma^{\downarrow}_{\beta}$ and $\Delta^{\downarrow}_{\beta}$ can be infinite as well. Finally, for any *finite* branch β, we will also write $\Gamma^{\mathsf{top}}_{\beta} \Rightarrow \Delta^{\mathsf{top}}_{\beta}$ for the *topmost sequent* in β.

We now want to use a countermodel construction in order to establish the completeness of the calculus $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. The basic idea of our argument can be summarized as follows: first, we will describe a procedure that allows to construct proof search trees for formulas in a systematic way. More precisely, given any formula $\varphi \in \mathcal{L}^{\mathsf{K}}$ as input, our algorithm starts to construct a proof search tree $\mathfrak{T}$ for φ by successively applying the rules of $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ root-first in all possible

[33] A similar strategy was also used by Negri (2009; 2014) in order to prove the completeness of her labelled sequent calculi for geometric extensions of K and for intermediate logics characterized by geometric frame conditions.

ways. If φ is valid in $\mathsf{Inq}\mathfrak{L}$, then the search terminates and our algorithm outputs a derivation for $\Rightarrow x : \varphi$. Otherwise, the search goes on forever and $\mathfrak{T}$ becomes infinite. By Kőnig's lemma, $\mathfrak{T}$ will have at least one infinite branch, which is then used to extract a countermodel for φ.

In order for this strategy to work, we have to make sure that our proof search algorithm is *exhaustive*, i.e., if the input formula φ is provable in $\mathsf{GLinq}\mathfrak{L}_\mathsf{A}$, then the procedure should actually be able to find a derivation for $\Rightarrow x : \varphi$ after a finite number of steps. In other words, if the search tree produced by our algorithm has an infinite branch, then this branch should always be *saturated*, in the sense that every rule applicable to the branch is in fact applied at some stage.

To make sure that this condition is met, we must consider some difficulties that might arise during the search procedure. One problem is associated with the 'cumulative' rules of our system, i.e., those rules in which the principal formulas and the principal atoms are always copied in the premises of the rule. This includes all order rules of $\mathsf{GLinq}\mathfrak{L}_\mathsf{A}$, but also the logical rules Lp, $L\bot$, $L\!\rightarrow$ and $L\Box$. The problem is that these rules allow us to create repetitions of rule applications that are essentially *identical*, which might cause our algorithm to get stuck in an infinite loop. For example, naive applications of the rule $L\Box$ might produce an infinite loop of the form

$$\dfrac{\dfrac{\dfrac{\vdots}{R(w) : \varphi, R(w) : \varphi, R(w) : \varphi, w \leqslant \pi, \pi : \Box\varphi, \Gamma \Rightarrow \Delta}}{R(w) : \varphi, R(w) : \varphi, w \leqslant \pi, \pi : \Box\varphi, \Gamma \Rightarrow \Delta}\,L\Box}{\dfrac{R(w) : \varphi, w \leqslant \pi, \pi : \Box\varphi, \Gamma \Rightarrow \Delta}{w \leqslant \pi, \pi : \Box\varphi, \Gamma \Rightarrow \Delta}\,L\Box}\,L\Box$$

Clearly, such a loop is undesirable, since the repeated applications of $L\Box$ do not yield any new formulas and prevent the other rules of our system from being applied to the branch. In order to resolve this issue, we have to employ a *loop-checking mechanism* that prevents our algorithm from performing redundant applications of rules. In particular, whenever our algorithm wants to perform a root-first application of a 'cumulative' rule to a sequent in a branch β, it first checks whether the result of this application is already contained in β. If this is the case, then the application is taken to be redundant and our algorithm does not apply the rule. Otherwise, the rule application is carried out in the usual way. So, for example, before performing an application of $L\Box$ with principal expressions $w \leqslant \pi$ and $\pi : \Box\varphi$ in a branch β, we always check whether $R(w) : \varphi$ does already occur in $\Gamma_\beta^\downarrow$. If so, we refrain from applying the rule. Otherwise, the application is allowed and will be performed by our algorithm. Similarly, before performing an application of $L\!\rightarrow$ with principal expressions $\pi \leqslant \sigma$ and $\sigma : \varphi \rightarrow \psi$, we need to check whether $(\pi : \varphi) \in \Delta_\beta^\downarrow$ or $(\pi : \psi) \in \Gamma_\beta^\downarrow$. The rule is now only applied, if *neither* of these two conditions is satisfied.

Another problem arises from the fact that applications of order rules can create new labels, which can then be used for further applications of order rules, and so on. This might cause our algorithm to produce infinite sequences of rule applications, creating increasingly more complex labels. For instance, using only the rule (in), we might produce an infinite loop of the form

$$\dfrac{\dfrac{\dfrac{\vdots}{\pi \leqslant \sigma\tau\sigma\tau, \pi \leqslant \sigma\tau\sigma, \pi \leqslant \sigma\tau, \pi \leqslant \sigma, \pi \leqslant \tau, \Gamma \Rightarrow \Delta}}{\pi \leqslant \sigma\tau\sigma, \pi \leqslant \sigma\tau, \pi \leqslant \sigma, \pi \leqslant \tau, \Gamma \Rightarrow \Delta}\,(\mathsf{in})}{\dfrac{\pi \leqslant \sigma\tau, \pi \leqslant \sigma, \pi \leqslant \tau, \Gamma \Rightarrow \Delta}{\pi \leqslant \sigma, \pi \leqslant \tau, \Gamma \Rightarrow \Delta}\,(\mathsf{in})}\,(\mathsf{in})$$

In order to avoid loops of this kind, we have to make sure that, at any point in the procedure, only a finite number of order rule applications can be performed. The basic idea is to assign a *weight* to each possible instance of an order rule and to divide the construction of the proof search tree into different stages: at stage 0, only order rule applications of weight at most 0 are performed; at stage 1, only order rule applications of weight at most 1 are performed, and so forth.

Algorithm 1 The Procedure ProofSearch

Input: A formula $\varphi \in \mathcal{L}^{\mathsf{K}}$.
Output: If φ is provable, the program outputs a derivation for $\Rightarrow x : \varphi$. Otherwise, it runs forever.

$n \leftarrow 0$;
Initialize $\mathfrak{T}$ as the tree consisting only of the root node $\Rightarrow x : \varphi$, for some $x \in \mathfrak{V}$;
while *true* **do**
  $\mathfrak{T} \leftarrow$ ExtendTree$(\mathfrak{T}, n)$;
  if $\mathfrak{T}$ *is closed* **then return** $\mathfrak{T}$;
  $n \leftarrow n + 1$;
end

Let us make this idea more precise. First, recall that any label π is essentially a finite string of symbols, built up from the constant $\emptyset$ and a countable set of variables $\mathfrak{S} \cup \mathfrak{V}$ by means of a finite number of function symbols. As is well known, any such string can be encoded by a *natural number*, in such a way that the code number of each string is effectively computable and any string can be reconstructed from its code number in a unique way. A concrete way of doing this is the well-known *Gödel numbering*, i.e., the encoding used by Gödel in the proof of his famous *incompleteness theorems* (cf. Gödel 1931; Smith 2013, pp. 136–139). We thus obtain the following fact.

Fact 5.6.2. *There exists a function* $\# : \Lambda(\mathfrak{S}, \mathfrak{V}) \to \mathbb{N}$, *assigning to each label* $\pi \in \Lambda(\mathfrak{S}, \mathfrak{V})$ *some natural number* $\#(\pi) \in \mathbb{N}$, *such that each of the following three conditions is satisfied:*
(i) $\#$ *is injective, i.e., distinct labels are assigned distinct numbers under* $\#$.
(ii) $\#$ *is computable, i.e., there exists an effective procedure that computes* $\#(\pi)$, *for any label* π.
(iii) Given any number $n \in \mathbb{N}$, *one can compute the set of all labels* π *with* $\#(\pi) \leq n$.

In what follows, we will assume a fixed numbering function $\# : \Lambda(\mathfrak{S}, \mathfrak{V}) \to \mathbb{N}$ satisfying each of the conditions mentioned in Fact 5.6.2. Note that we do not care about the exact specification of this function—all that matters is that such a function *can* be defined. Given any label π, we also call $\#(\pi)$ the *weight of* π. The set of all labels of weight at most n will be denoted by Λ^n, so we put $\Lambda^n := \{\pi \mid \#(\pi) \leq n\}$. Recall that, for any natural number n, the set Λ^n is computable by the last condition mentioned in Fact 5.6.2. Now, let us say that a label π is *dominant* in an application of an order rule, if π has an occurrence in the conclusion of this application which does not belong to the context of the rule. For instance, in an application of the rule (in) with premise $\pi \leqslant \sigma\tau, \pi \leqslant \sigma, \pi \leqslant \tau, \Gamma \Rightarrow \Delta$ and conclusion $\pi \leqslant \sigma, \pi \leqslant \tau, \Gamma \Rightarrow \Delta$, each of the labels π, σ and τ would be dominant. For order rules that do not have principal atoms in the conclusion—such as, e.g., the rules (rf), (il) or (ur)—we assume that the dominant labels are simply the labels from which the active expressions of this rule are constructed. So, for example, in an application of (il) with premise $\pi\sigma \leqslant \pi, \Gamma \Rightarrow \Delta$ and conclusion $\Gamma \Rightarrow \Delta$, both of the labels π and σ are dominant (but $\pi\sigma$ is not dominant). The *weight of an order rule application* is now defined to be the maximum of the weights of the dominant labels of this application. Thus, for instance, the weight of the aforementioned application of (il) would be the number $n = \max\{\#(\pi), \#(\sigma)\}$.

We are now able to avoid the problem sketched above, by bounding the weight of legitimate order rule applications at each stage of the search procedure: at stage 0, we only allow order rule applications of weight at most 0; at stage 1, we only allow order rule applications of weight at most 1, and so on. Clearly, by the injectivity of $\#$, there are only finitely many order rule applications at each stage, but every possible application will be reached at some point.

We are now ready to describe the construction of our proof search tree. The overall structure of the procedure is presented in Algorithm 1. As can be seen, given any formula $\varphi \in \mathcal{L}^{\mathsf{K}}$ as input, our algorithm first assigns the value 0 to a variable n and initializes $\mathfrak{T}$ as the proof search tree consisting only of the root node $\Rightarrow x : \varphi$, where $x \in \mathfrak{V}$ is some state variable. Afterwards,

Algorithm 2 The Procedure ExtendTree (Excerpt)

Input: A finite proof search tree $\mathfrak{T}$ and a natural number n.
Output: An extension of $\mathfrak{T}$.

% Perform all root-first applications of Lp with loop-checking:
while *$\mathfrak{T}$ has an open branch β such that $\Gamma^{\mathrm{top}}_{\beta}$ contains $w \leqslant \pi$ and $\pi : p$ with $(w : p) \notin \Gamma^{\downarrow}_{\beta}$* **do**
  Extend $\mathfrak{T}$ by writing the sequent $w : p, \Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}$ above $\Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}$;
end
% Perform all root-first applications of Rp:
while *$\mathfrak{T}$ has an open branch β such that $\Delta^{\mathrm{top}}_{\beta}$ is of the form $\Delta, \pi : p$ for $\pi \notin \mathfrak{S}$* **do**
  Choose a fresh $w \in \mathfrak{S}$ and extend $\mathfrak{T}$ by writing $w \leqslant \pi, \Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta, w : p$ above $\Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}$;
end
% Perform all root-first applications of $L{\to}$ with loop-checking:
while *$\mathfrak{T}$ has an open β s. t. $\Gamma^{\mathrm{top}}_{\beta}$ contains $\pi \leqslant \sigma, \sigma : \varphi \to \psi$ with $(\pi : \varphi) \notin \Delta^{\downarrow}_{\beta}$, $(\pi : \psi) \notin \Gamma^{\downarrow}_{\beta}$* **do**
  Extend $\mathfrak{T}$ by writing both $\Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}, \pi : \varphi$ and $\pi : \psi, \Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}$ above $\Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}$;
end
% Perform all root-first applications of $R{\to}$:
while *$\mathfrak{T}$ has an open branch β such that $\Delta^{\mathrm{top}}_{\beta}$ is of the form $\Delta, \pi : \varphi \to \psi$* **do**
  Choose a fresh $x \in \mathfrak{V}$ and write $x \leqslant \pi, x : \varphi, \Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta, x : \psi$ above $\Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}$;
end
% Perform all root-first applications of $L\Box$ with loop-checking:
while *$\mathfrak{T}$ has an open branch β such that $\Gamma^{\mathrm{top}}_{\beta}$ contains $w \leqslant \pi, \pi : \Box\varphi$ with $(R(w) : \varphi) \notin \Gamma^{\downarrow}_{\beta}$* **do**
  Extend $\mathfrak{T}$ by writing $R(w) : \varphi, \Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}$ above $\Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}$;
end
% Perform all root-first applications of $R\Box$:
while *$\mathfrak{T}$ has an open branch β such that $\Delta^{\mathrm{top}}_{\beta}$ is of the form $\Delta, \pi : \Box\varphi$* **do**
  Choose a fresh $w \in \mathfrak{S}$ and extend $\mathfrak{T}$ by writing $w \leqslant \pi, \Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta, R(w) : \varphi$ above $\Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}$;
end
% Perform all root-first applications of (in) having weight at most n:
while *$\mathfrak{T}$ has an open β s. t. $\Gamma^{\mathrm{top}}_{\beta}$ contains $\pi \leqslant \sigma, \pi \leqslant \tau$ with $\pi, \sigma, \tau \in \Lambda^n$ and $(\pi \leqslant \sigma\tau) \notin \Gamma^{\downarrow}_{\beta}$* **do**
  Extend $\mathfrak{T}$ by writing $\pi \leqslant \sigma\tau, \Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}$ above $\Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}$;
end
% Perform all root-first applications of (cd) having weight at most n:
while *$\mathfrak{T}$ has an open β s. t. $\Gamma^{\mathrm{top}}_{\beta}$ contains $w \leqslant \pi + \sigma$ with $w, \pi + \sigma \in \Lambda^n$, $(w \leqslant \pi), (w \leqslant \sigma) \notin \Gamma^{\downarrow}_{\beta}$* **do**
  Extend $\mathfrak{T}$ by writing both $w \leqslant \pi, \Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}$ and $w \leqslant \sigma, \Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}$ above $\Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}$;
end
% Perform all root-first applications of (ul) having weight at most n:
for *each open branch β in $\mathfrak{T}$ and for all $\pi, \sigma \in \Lambda^n$ with $(\pi \leqslant \pi + \sigma) \notin \Gamma^{\downarrow}_{\beta}$* **do**
  Extend $\mathfrak{T}$ by writing $\pi \leqslant \pi + \sigma, \Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}$ above $\Gamma^{\mathrm{top}}_{\beta} \Rightarrow \Delta^{\mathrm{top}}_{\beta}$;
end
% Add similar loops for the other rules ...
return $\mathfrak{T}$

the algorithm enters a (possibly infinite) while-loop. At each iteration of the loop, the procedure calls a subroutine ExtendTree, which will be described in detail below. Roughly speaking, this subroutine updates the tree $\mathfrak{T}$, by performing all non-redundant applications of logical rules, all weight-n applications of order rules and all possible applications of the geometric rules of $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. Once the subroutine is finished, our program checks whether the updated tree $\mathfrak{T}$ is closed. If this is the case, then $\mathfrak{T}$ is a derivation for $\Rightarrow x : \varphi$, so the algorithm halts and outputs $\mathfrak{T}$. Otherwise, n is incremented and we proceed with the next iteration of the loop.

Let us now turn to the description of the subroutine ExtendTree. An excerpt from the pseudocode for this subroutine is provided in Algorithm 2. Given any finite proof search tree $\mathfrak{T}$ and any natural number n as input, the procedure ExtendTree executes the following steps:

(1) First, we perform all non-redundant applications of Lp to the open branches of $\mathfrak{T}$. That is, our algorithm selects an arbitrary open branch β in $\mathfrak{T}$ such that the topmost sequent of β is of the form $w \leqslant \pi, \pi : p, \Gamma \Rightarrow \Delta$ with $(w : p) \notin \Gamma_\beta^\downarrow$, and it then extends this branch by writing $w : p, w \leqslant \pi, \pi : p, \Gamma \Rightarrow \Delta$ above the topmost sequent of β. Afterwards, we proceed with the next branch in $\mathfrak{T}$ until no further applications of Lp are possible. This corresponds to the while-loop in lines 1–3 of Algorithm 2. Note that, due to the loop-checking mechanism, this step must terminate after a finite number of iterations of the loop.

(2) Once this is done, we perform all possible applications of Rp. Thus, while $\mathfrak{T}$ contains an open branch β such that the topmost sequent of β is of the form $\Gamma \Rightarrow \Delta, \pi : p$ with $\pi \notin \mathfrak{S}$, we select a fresh variable $w \in \mathfrak{S}$ not occurring in the tree and extend $\mathfrak{T}$ by writing $w \leqslant \pi, \Gamma \Rightarrow \Delta, w : p$ above the topmost sequent of β. This corresponds to the while-loop in lines 4–6 of Algorithm 2. Observe that, since w is a singleton label, the new formula $w : p$ cannot be used in further applications of Rp, so this step must terminate as well.[34] In the subsequent steps, we execute similar while-loops for the rules $L\bot$ and $R\bot$. The details are essentially the same as in the first two steps and therefore left to the reader.

(3) Next, we perform all possible applications of $L\wedge$ and $R\wedge$. That is, while $\mathfrak{T}$ has an open branch β with topmost sequent of the form $\pi : \varphi \wedge \psi, \Gamma \Rightarrow \Delta$, we extend the branch by writing $\pi : \varphi, \pi : \psi, \Gamma \Rightarrow \Delta$ on top of β. Similarly, while $\mathfrak{T}$ has an open branch β with topmost sequent of the form $\Gamma \Rightarrow \Delta, \pi : \varphi \wedge \psi$, we split the branch and write both $\Gamma \Rightarrow \Delta, \pi : \varphi$ and $\Gamma \Rightarrow \Delta, \pi : \psi$ on top of β. The rules $L\mathbb{V}$ and $R\mathbb{V}$ are treated analogously.

(4) Afterwards, all possible applications of the rules $L\!\rightarrow$, $R\!\rightarrow$, $L\Box$ and $R\Box$ are performed. The corresponding while-loops are displayed in lines 7–18 of Algorithm 2. Note that, for $L\!\rightarrow$ and $L\Box$, we use our loop-checking mechanism, so this step is guaranteed to terminate.

(5) In the subsequent steps, we perform all order rule applications of weight at most n. Let us consider a few representative cases. In order to perform all weight-n applications of the rule (in), we execute the loop in lines 19–21 of Algorithm 2. That is, while $\mathfrak{T}$ has an open branch β with topmost sequent of the form $\pi \leqslant \sigma, \pi \leqslant \tau, \Gamma \Rightarrow \Delta$ such that each of the labels π, σ and τ has weight at most n and $\pi \leqslant \sigma\tau$ does not already occur in $\Gamma_\beta^\downarrow$, we extend this branch by writing $\pi \leqslant \sigma\tau, \pi \leqslant \sigma, \pi \leqslant \tau, \Gamma \Rightarrow \Delta$ on top of β. The rules (tr), (dis), (un) and (nb) are treated similarly. For order rules involving a branching—that is, for (sg) and (cd)—we perform a loop-checking for each of the two premises. The corresponding while-loop for (cd) is displayed in lines 22–24. For order rules without principal atoms, we simply iterate over all weight-n instances of the rule. For example, in order to close the tree under weight-n applications of (ul), we just add all relational atoms of the form $\pi \leqslant \pi + \sigma$ with $\pi, \sigma \in \Lambda^n$ to each open branch β. This corresponds to the for-loop in lines 25–27 of our algorithm. The rules (il), (ir), (ur) and (rf) are treated in the same way. Clearly, since every order rule can only have finitely many instances of weight at most n, this step must terminate.

(6) Finally, we perform all possible applications of the geometric rules of $\mathsf{GLinq}\mathfrak{L}_\mathsf{A}$. To this end, we select an arbitrary geometric axiom $\theta \in \mathsf{A}$ and consider the corresponding rule

$$\frac{\Psi_1^\circ(v_1/u_1), \Phi^\circ, \Gamma \Rightarrow \Delta \qquad \cdots \qquad \Psi_m^\circ(v_m/u_m), \Phi^\circ, \Gamma \Rightarrow \Delta}{\Phi^\circ, \Gamma \Rightarrow \Delta}\ (\theta\text{-grs})$$

For each open branch β in $\mathfrak{T}$, our algorithm first determines all subsets $\Theta_1^\beta, \ldots, \Theta_k^\beta \subseteq \Gamma_\beta^{\mathsf{top}}$ in the topmost sequent that are instantiations of the set of principal atoms Φ°.[35] Note that we do not require these subsets to be disjoint, so some formula occurrences in $\Gamma_\beta^{\mathsf{top}}$ may also

[34] Recall that, in the rule Rp, the principal formula must be of the form $\pi : p$ for some *non-singleton* label $\pi \notin \mathfrak{S}$.

[35] By an *instantiation* of Φ°, we mean any set of relational atoms obtained from Φ° by uniform substitution of variables.

be contained in more than one of the sets Θ_i^β. For every open branch β in $\mathfrak{T}$ and for each of the subsets $\Theta_i^\beta \subseteq \Gamma_\beta^{\mathrm{top}}$, our algorithm now performs the corresponding root-first application of (θ-grs) to the respective branch β, in such a way that the k applications of (θ-grs) in β are executed *all at once* and all branches of $\mathfrak{T}$ are expanded *at the same time*. Thus, if the topmost sequent of a branch β contains k instantiations of Φ°, then this will cause the branch β to be split into m^k new branches. After that, our algorithm selects the next geometric axiom in A and proceeds in the same way. Because A is finite and since every geometric rule (θ-grs) is applied only a finite number of times, this step clearly terminates.

Once this is finished, our algorithm returns the updated tree $\mathfrak{T}$ to the program in Algorithm 1. This concludes the description of the procedure ExtendTree. Note that, by what was said above, this procedure must be *terminating*, i.e., given any tree $\mathfrak{T}$ and any natural number n as input, ExtendTree will in fact halt and return an updated version of $\mathfrak{T}$ after a finite number of steps.

Let us now consider the tree $\mathfrak{T}$ constructed by Algorithm 1 for some input formula φ. Clearly, if $\mathfrak{T}$ becomes closed at some stage of the construction, our algorithm will halt and output $\mathfrak{T}$, which is now a derivation for $\Rightarrow x : \varphi$. Otherwise, Algorithm 1 will run forever and $\mathfrak{T}$ becomes infinitely large. In this case, $\mathfrak{T}$ must contain an infinite branch β, and such a branch will always be *saturated*, in the sense that every rule applicable to the branch is in fact applied at some stage (this follows directly from the specification of ExtendTree and the iterated executions of this subroutine during the construction of $\mathfrak{T}$). In other words, an infinite branch in $\mathfrak{T}$ will always be closed under all possible applications of the rules of $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$, so we obtain the following facts.

Lemma 5.6.3 (Closure under Logical Rules). *Let β be an infinite branch in the tree constructed by Algorithm 1. Then, β is closed under root-first applications of the logical rules, so we have:*

(i) If it holds $(w \leqslant \pi) \in \Gamma_\beta^\downarrow$ and $(\pi : p) \in \Gamma_\beta^\downarrow$, then also $(w : p) \in \Gamma_\beta^\downarrow$. And if $(\pi : p) \in \Delta_\beta^\downarrow$, then there exists a variable $w \in \mathfrak{S}$ such that both $(w \leqslant \pi) \in \Gamma_\beta^\downarrow$ and $(w : p) \in \Delta_\beta^\downarrow$.

(ii) If it holds $(w \leqslant \pi) \in \Gamma_\beta^\downarrow$ and $(\pi : \bot) \in \Gamma_\beta^\downarrow$, then also $(w : \bot) \in \Gamma_\beta^\downarrow$. And if $(\pi : \bot) \in \Delta_\beta^\downarrow$, then there exists a variable $w \in \mathfrak{S}$ such that both $(w \leqslant \pi) \in \Gamma_\beta^\downarrow$ and $(w : \bot) \in \Delta_\beta^\downarrow$.

(iii) If we have $(\pi : \varphi \wedge \psi) \in \Gamma_\beta^\downarrow$, then both $(\pi : \varphi) \in \Gamma_\beta^\downarrow$ and $(\pi : \psi) \in \Gamma_\beta^\downarrow$. And if $(\pi : \varphi \wedge \psi) \in \Delta_\beta^\downarrow$, then at least one of $(\pi : \varphi) \in \Delta_\beta^\downarrow$ and $(\pi : \psi) \in \Delta_\beta^\downarrow$ is the case.

(iv) If we have $(\pi : \varphi \vee \psi) \in \Gamma_\beta^\downarrow$, then at least one of $(\pi : \varphi) \in \Gamma_\beta^\downarrow$ and $(\pi : \psi) \in \Gamma_\beta^\downarrow$ is the case. And if $(\pi : \varphi \vee \psi) \in \Delta_\beta^\downarrow$, then both $(\pi : \varphi) \in \Delta_\beta^\downarrow$ and $(\pi : \psi) \in \Delta_\beta^\downarrow$.

(v) If $(\pi \leqslant \sigma) \in \Gamma_\beta^\downarrow$ and $(\sigma : \varphi \to \psi) \in \Gamma_\beta^\downarrow$, then $(\pi : \varphi) \in \Delta_\beta^\downarrow$ or $(\pi : \psi) \in \Gamma_\beta^\downarrow$. And if $(\pi : \varphi \to \psi) \in \Delta_\beta^\downarrow$, then there is an $x \in \mathfrak{V}$ with $(x \leqslant \pi)$, $(x : \varphi) \in \Gamma_\beta^\downarrow$ and $(x : \psi) \in \Delta_\beta^\downarrow$.

(vi) If $(w \leqslant \pi) \in \Gamma_\beta^\downarrow$ and $(\pi : \Box\varphi) \in \Gamma_\beta^\downarrow$, then also $(R(w) : \varphi) \in \Gamma_\beta^\downarrow$. And if $(\pi : \Box\varphi) \in \Delta_\beta^\downarrow$, then there exists a variable $w \in \mathfrak{S}$ such that both $(w \leqslant \pi) \in \Gamma_\beta^\downarrow$ and $(R(w) : \varphi) \in \Delta_\beta^\downarrow$.

Proof. We only prove statement (vi), the other parts are similar. First, suppose that we have both $(w \leqslant \pi) \in \Gamma_\beta^\downarrow$ and $(\pi : \Box\varphi) \in \Gamma_\beta^\downarrow$. Then, at some point in the execution of Algorithm 1, some initial segment of β must end with a sequent of the form $w \leqslant \pi, \pi : \Box\varphi, \Gamma \Rightarrow \Delta$. Hence, during the next execution of the subroutine ExtendTree, our algorithm will perform the corresponding root-first application of $L\Box$ to this initial segment, so it follows $(R(w) : \varphi) \in \Gamma_\beta^\downarrow$, as desired. Now, assume that $(\pi : \Box\varphi) \in \Delta_\beta^\downarrow$. Then, at some stage, an initial segment of β must end with a sequent of the form $\Gamma \Rightarrow \Delta, \pi : \Box\varphi$. Thus, during the next execution of ExtendTree, the procedure will perform the corresponding root-first application of $R\Box$ to this initial segment, so there must be some variable $w \in \mathfrak{S}$ such that $(w \leqslant \pi) \in \Gamma_\beta^\downarrow$ and $(R(w) : \varphi) \in \Delta_\beta^\downarrow$. □

Lemma 5.6.4 (Closure under Order Rules). *Let β be an infinite branch in the tree constructed by Algorithm 1. Then, β is closed under root-first applications of the order rules, so we have:*

(i) All atoms of the form $\pi \leqslant \pi$ and $\pi_1\pi_2 \leqslant \pi_i$ and $\pi_i \leqslant \pi_1 + \pi_2$ are in $\Gamma_\beta^\downarrow$, for $i = 1, 2$.

(ii) If we have both $(\pi \leqslant \sigma) \in \Gamma_\beta^\downarrow$ and $(\sigma \leqslant \tau) \in \Gamma_\beta^\downarrow$, then $(\pi \leqslant \tau) \in \Gamma_\beta^\downarrow$.

(iii) *If* $(\pi \leqslant \sigma + \tau) \in \Gamma^{\downarrow}_{\beta}$, *then also* $(\pi \leqslant \pi\sigma + \pi\tau) \in \Gamma^{\downarrow}_{\beta}$.
(iv) *If we have both* $(\pi \leqslant \sigma) \in \Gamma^{\downarrow}_{\beta}$ *and* $(\pi \leqslant \tau) \in \Gamma^{\downarrow}_{\beta}$, *then* $(\pi \leqslant \sigma\tau) \in \Gamma^{\downarrow}_{\beta}$.
(v) *If both* $(\pi \leqslant \sigma) \in \Gamma^{\downarrow}_{\beta}$ *and* $(\tau \leqslant \sigma) \in \Gamma^{\downarrow}_{\beta}$, *then* $(\pi + \tau \leqslant \sigma) \in \Gamma^{\downarrow}_{\beta}$.
(vi) *If* $(\pi \leqslant w) \in \Gamma^{\downarrow}_{\beta}$, *then* $(\pi \leqslant \emptyset) \in \Gamma^{\downarrow}_{\beta}$ *or* $(w \leqslant \pi) \in \Gamma^{\downarrow}_{\beta}$.
(vii) *If* $(w \leqslant \pi + \sigma) \in \Gamma^{\downarrow}_{\beta}$, *then* $(w \leqslant \pi) \in \Gamma^{\downarrow}_{\beta}$ *or* $(w \leqslant \sigma) \in \Gamma^{\downarrow}_{\beta}$.
(viii) *If* $(w \leqslant u) \in \Gamma^{\downarrow}_{\beta}$, *then also* $(R(w) \leqslant R(u)) \in \Gamma^{\downarrow}_{\beta}$.

Proof. We only prove part (viii). Suppose $(w \leqslant u) \in \Gamma^{\downarrow}_{\beta}$. Then, clearly, there must be some $n \geq 0$ such that, starting from the n-th iteration of the while-loop in Algorithm 1, the topmost sequent of the current initial segment of β always contains $w \leqslant u$ in the antecedent. Let now k be given by $k := \max\{n, \#(w), \#(u)\}$. Then, during the k-th execution of ExtendTree, our algorithm performs the corresponding root-first application of (nb), so it follows $(R(w) \leqslant R(u)) \in \Gamma^{\downarrow}_{\beta}$. □

It is also possible to prove similar closure properties for the *geometric rules* of $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. The details are left to the reader. We now want to use a countermodel construction in order to prove the completeness of $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ with respect to $\mathsf{Inq}\mathfrak{L}$. In order to understand the basic idea, let $\mathfrak{T}$ be the tree constructed by Algorithm 1 for some input formula φ. We will show that, if $\mathfrak{T}$ contains an infinite branch β, then this branch can be used to construct a Kripke model M_β and an interpretation function I_β over M_β such that all expressions in $\Gamma^{\downarrow}_{\beta}$ are satisfied under I_β and all expressions in $\Delta^{\downarrow}_{\beta}$ are not satisfied under I_β. As a consequence, it then follows that the labelled formula $x : \varphi$, occurring in the root of $\mathfrak{T}$, is not satisfied under I_β, so M_β is the desired countermodel for φ.

A small technical difficulty arises from the fact that $\Gamma^{\downarrow}_{\beta}$ could also contain relational atoms of the form $w \leqslant u$, for some singleton variables $w, u \in \mathfrak{S}$. In order for these atoms to be satisfied in our model, we have to make sure that the singleton states denoted by w and u are *identified* in M_β. For this purpose, we first define an equivalence relation $\sim_\beta$ on $\mathfrak{S}$ in the following way.

Definition 5.6.5 (The Relation $\sim_\beta$)**.** Let β be a branch in some proof search tree. The binary relation $\sim_\beta$ on $\mathfrak{S}$ is defined by $w \sim_\beta u :\Leftrightarrow (w \leqslant u) \in \Gamma^{\downarrow}_{\beta}$, for all singleton variables $w, u \in \mathfrak{S}$.

Lemma 5.6.6. *If β is an infinite branch in the tree constructed by Algorithm 1, then $\sim_\beta$ is an equivalence relation on $\mathfrak{S}$, so $\sim_\beta$ is reflexive, symmetric and transitive.*

Proof. The reflexivity and transitivity of $\sim_\beta$ follows immediately from parts (i) and (ii) of Lemma 5.6.4. Thus, it suffices to show that $\sim_\beta$ is symmetric. For this purpose, let $w, u \in \mathfrak{S}$ be arbitrary and suppose that $w \sim_\beta u$, so we have $(w \leqslant u) \in \Gamma^{\downarrow}_{\beta}$. By Lemma 5.6.4 (vi), this implies $(w \leqslant \emptyset) \in \Gamma^{\downarrow}_{\beta}$ or $(u \leqslant w) \in \Gamma^{\downarrow}_{\beta}$. Because β is infinite, we cannot have $(w \leqslant \emptyset) \in \Gamma^{\downarrow}_{\beta}$, as this would mean that β contains an instance of $Ax^{\emptyset}$. Hence, it follows $(u \leqslant w) \in \Gamma^{\downarrow}_{\beta}$ and therefore $u \sim_\beta w$. □

In what follows, we will write $[w]_\beta$ for the *equivalence class* of a variable $w \in \mathfrak{S}$ with respect to $\sim_\beta$. Using the equivalence relation $\sim_\beta$, we are now able to define the desired Kripke model M_β and the interpretation I_β for each infinite branch in the tree constructed by our algorithm.

Definition 5.6.7 (The Kripke Model M_β)**.** Let β be an infinite branch in the tree constructed by Algorithm 1. We define M_β to be the Kripke model $M_\beta = \langle W_\beta, R_\beta, V_\beta \rangle$ given by:
(i) $W_\beta := \{[w]_\beta \mid w \in \mathfrak{S}\}$, so W_β is the set of all equivalence classes of the relation $\sim_\beta$,
(ii) $([w]_\beta, [u]_\beta) \in R_\beta \quad :\Leftrightarrow \quad$ there are $w' \in [w]_\beta$ and $u' \in [u]_\beta$ such that $(u' \leqslant R(w')) \in \Gamma^{\downarrow}_{\beta}$,
(iii) $V_\beta([w]_\beta, p) = 1 \quad :\Leftrightarrow \quad$ there exists some $u \in [w]_\beta$ such that $(u : p) \in \Gamma^{\downarrow}_{\beta}$.

Definition 5.6.8 (The Interpretation I_β)**.** Let β be an infinite branch in the tree constructed by Algorithm 1. For all $w \in \mathfrak{S}$ and $x \in \mathfrak{V}$, we define an interpretation I_β over M_β as follows:
(i) $I_\beta(w) := \{[w]_\beta\}$, so $I_\beta(w)$ is the singleton containing only $[w]_\beta$,
(ii) $I_\beta(x) := \{[w]_\beta \in W_\beta \mid$ there exists $u \in [w]_\beta$ such that $(u \leqslant x) \in \Gamma^{\downarrow}_{\beta}\}$.

We assume that I_β is extended to arbitrary labels in the usual way (see Definition 5.5.1). Moreover, in the following, we will also write F_β for the Kripke frame given by $F_\beta := \langle W_\beta, R_\beta \rangle$. One can now prove that, for every label π, the state $I_\beta(\pi)$ simply amounts to the set of all equivalence classes $[w]_\beta$ for which $w \leqslant \pi$ occurs in $\Gamma^{\downarrow}_\beta$. This is expressed by the following lemma.

Lemma 5.6.9. *Let β be an infinite branch in the tree constructed by Algorithm 1. For every singleton variable $w \in \mathfrak{S}$ and for every label π, we have: $[w]_\beta \in I_\beta(\pi)$ if and only if $(w \leqslant \pi) \in \Gamma^{\downarrow}_\beta$.*

Proof. Let $w \in \mathfrak{S}$ be arbitrary. We proceed by induction on the structure of the label π.

Case 1: Let $\pi = u$ for some $u \in \mathfrak{S}$. Using the definitions of I_β and $\sim_\beta$, one readily sees that we have the following equivalences: $[w]_\beta \in I_\beta(u)$ iff $[w]_\beta = [u]_\beta$ iff $w \sim_\beta u$ iff $(w \leqslant u) \in \Gamma^{\downarrow}_\beta$.

Case 2: Let $\pi = x$ for some variable $x \in \mathfrak{V}$. For the left-to-right direction, suppose that it holds $[w]_\beta \in I_\beta(x)$. Then, by definition of I_β, there exists some $u \in [w]_\beta$ such that $(u \leqslant x) \in \Gamma^{\downarrow}_\beta$. Since $u \in [w]_\beta$, we also have $(w \leqslant u) \in \Gamma^{\downarrow}_\beta$ by definition of $\sim_\beta$. Now, from $(w \leqslant u) \in \Gamma^{\downarrow}_\beta$ and $(u \leqslant x) \in \Gamma^{\downarrow}_\beta$, we may conclude $(w \leqslant x) \in \Gamma^{\downarrow}_\beta$ by Lemma 5.6.4 (ii). The converse direction is straightforward, since $(w \leqslant x) \in \Gamma^{\downarrow}_\beta$ directly implies $[w]_\beta \in I_\beta(x)$ by definition of I_β.

Case 3: Let $\pi = \emptyset$. By Definition 5.5.1, we have $I_\beta(\emptyset) = \emptyset$ and so $[w]_\beta \notin I_\beta(\emptyset)$. And since β is infinite, we also have $(w \leqslant \emptyset) \notin \Gamma^{\downarrow}_\beta$, because otherwise β would contain an instance of $Ax^{\emptyset}$.

Case 4: Let $\pi = \sigma\tau$. For the left-to-right direction, suppose that it holds $[w]_\beta \in I_\beta(\sigma\tau)$. Since we have $I_\beta(\sigma\tau) = I_\beta(\sigma) \cap I_\beta(\tau)$ by Definition 5.5.1, this yields $[w]_\beta \in I_\beta(\sigma)$ and $[w]_\beta \in I_\beta(\tau)$. Therefore, by induction hypothesis, we must have $(w \leqslant \sigma) \in \Gamma^{\downarrow}_\beta$ and $(w \leqslant \tau) \in \Gamma^{\downarrow}_\beta$, so it follows $(w \leqslant \sigma\tau) \in \Gamma^{\downarrow}_\beta$ by Lemma 5.6.4 (iv). For the converse direction, assume that it holds $(w \leqslant \sigma\tau) \in \Gamma^{\downarrow}_\beta$. By Lemma 5.6.4 (i), we also have $(\sigma\tau \leqslant \sigma) \in \Gamma^{\downarrow}_\beta$ and $(\sigma\tau \leqslant \tau) \in \Gamma^{\downarrow}_\beta$, so this implies $(w \leqslant \sigma) \in \Gamma^{\downarrow}_\beta$ and $(w \leqslant \tau) \in \Gamma^{\downarrow}_\beta$ by Lemma 5.6.4 (ii). Now, by induction hypothesis, we may conclude $[w]_\beta \in I_\beta(\sigma)$ and $[w]_\beta \in I_\beta(\tau)$. But then, because $I_\beta(\sigma\tau) = I_\beta(\sigma) \cap I_\beta(\tau)$, we also have $[w]_\beta \in I_\beta(\sigma\tau)$, as desired. If π is of the form $\pi = \sigma + \tau$, then the argument is similar.

Case 5: Let $\pi = R(u)$ for some variable $u \in \mathfrak{S}$. For the left-to-right direction, suppose $[w]_\beta \in I_\beta(R(u))$. Since we have $I_\beta(R(u)) = R_\beta([u]_\beta)$ by definition of I_β, this yields $([u]_\beta, [w]_\beta) \in R_\beta$. Hence, by definition of R_β, there are some $u' \in [u]_\beta$ and $w' \in [w]_\beta$ such that $(w' \leqslant R(u')) \in \Gamma^{\downarrow}_\beta$. Because $u' \in [u]_\beta$ and $w' \in [w]_\beta$, we must have $(u' \leqslant u) \in \Gamma^{\downarrow}_\beta$ and $(w \leqslant w') \in \Gamma^{\downarrow}_\beta$ by definition of $\sim_\beta$. From $(w \leqslant w') \in \Gamma^{\downarrow}_\beta$ and $(w' \leqslant R(u')) \in \Gamma^{\downarrow}_\beta$, it follows $(w \leqslant R(u')) \in \Gamma^{\downarrow}_\beta$ by Lemma 5.6.4 (ii). And since $(u' \leqslant u) \in \Gamma^{\downarrow}_\beta$, we also have $(R(u') \leqslant R(u)) \in \Gamma^{\downarrow}_\beta$ by Lemma 5.6.4 (viii). Now, from $(w \leqslant R(u')) \in \Gamma^{\downarrow}_\beta$ and $(R(u') \leqslant R(u)) \in \Gamma^{\downarrow}_\beta$, we may conclude $(w \leqslant R(u)) \in \Gamma^{\downarrow}_\beta$ by Lemma 5.6.4 (ii). The right-to-left direction is trivial, since $(w \leqslant R(u)) \in \Gamma^{\downarrow}_\beta$ directly implies $([u]_\beta, [w]_\beta) \in R_\beta$ and therefore $[w]_\beta \in I_\beta(R(u))$ by definition of R_β and I_β. □

Next, we can show that our interpretation does in fact have the desired properties. That is, all expressions in $\Gamma^{\downarrow}_\beta$ are satisfied under I_β and all expressions in $\Delta^{\downarrow}_\beta$ are not satisfied under I_β.

Lemma 5.6.10. *Let β be an infinite branch in the proof search tree constructed by Algorithm 1. For every relational atom $(\pi \leqslant \sigma) \in \Gamma^{\downarrow}_\beta$, it is the case that $I_\beta(\pi) \subseteq I_\beta(\sigma)$.*

Proof. Let $(\pi \leqslant \sigma) \in \Gamma^{\downarrow}_\beta$ be arbitrary. Moreover, let $w \in \mathfrak{S}$ be an arbitrary variable and suppose $[w]_\beta \in I_\beta(\pi)$. By Lemma 5.6.9, this yields $(w \leqslant \pi) \in \Gamma^{\downarrow}_\beta$. Now, because $(w \leqslant \pi) \in \Gamma^{\downarrow}_\beta$ and $(\pi \leqslant \sigma) \in \Gamma^{\downarrow}_\beta$, it follows $(w \leqslant \sigma) \in \Gamma^{\downarrow}_\beta$ by Lemma 5.6.4 (ii). But then, using Lemma 5.6.9 again, we may conclude $[w]_\beta \in I_\beta(\sigma)$. Since $[w]_\beta \in I_\beta(\pi)$ was arbitrary, this shows $I_\beta(\pi) \subseteq I_\beta(\sigma)$. □

Lemma 5.6.11. *Let β be an infinite branch in the tree constructed by Algorithm 1. For every labelled formula $\pi : \varphi$, if $(\pi : \varphi) \in \Gamma^{\downarrow}_\beta$, then $M_\beta, I_\beta(\pi) \models \varphi$, and if $(\pi : \varphi) \in \Delta^{\downarrow}_\beta$, then $M_\beta, I_\beta(\pi) \not\models \varphi$.*

Proof. Let β be an arbitrary infinite branch in the tree constructed by Algorithm 1 for some input. Moreover, let π be an arbitrary label. We proceed by induction on the structure of φ.

Case 1: Let $\varphi = p$ be atomic. For the first part, assume $(\pi : p) \in \Gamma^{\downarrow}_{\beta}$. Let $w \in \mathfrak{S}$ be arbitrary and suppose $[w]_{\beta} \in I_{\beta}(\pi)$. By Lemma 5.6.9, this implies $(w \leqslant \pi) \in \Gamma^{\downarrow}_{\beta}$. Now, as we have both $(w \leqslant \pi) \in \Gamma^{\downarrow}_{\beta}$ and $(\pi : p) \in \Gamma^{\downarrow}_{\beta}$, it follows $(w : p) \in \Gamma^{\downarrow}_{\beta}$ by Lemma 5.6.3 (i). But then, by definition of V_{β}, we also have $V_{\beta}([w]_{\beta}, p) = 1$. Thus, as $[w]_{\beta} \in I_{\beta}(\pi)$ was arbitrary, it holds $M_{\beta}, I_{\beta}(\pi) \models p$ by Definition 5.1.3. For the second part, assume that $(\pi : p) \in \Delta^{\downarrow}_{\beta}$. Then, by Lemma 5.6.3 (i), there exists some $w \in \mathfrak{S}$ such that $(w \leqslant \pi) \in \Gamma^{\downarrow}_{\beta}$ and $(w : p) \in \Delta^{\downarrow}_{\beta}$. From $(w \leqslant \pi) \in \Gamma^{\downarrow}_{\beta}$, it follows $[w]_{\beta} \in I_{\beta}(\pi)$ by Lemma 5.6.9. Towards a contradiction, suppose $V_{\beta}([w]_{\beta}, p) = 1$. Then, by definition of V_{β}, there exists some $u \in [w]_{\beta}$ such that $(u : p) \in \Gamma^{\downarrow}_{\beta}$. Thus, u satisfies $(w \leqslant u) \in \Gamma^{\downarrow}_{\beta}$ and $(u : p) \in \Gamma^{\downarrow}_{\beta}$, so it holds $(w : p) \in \Gamma^{\downarrow}_{\beta}$ by Lemma 5.6.3 (i). But now we have both $(w : p) \in \Gamma^{\downarrow}_{\beta}$ and $(w : p) \in \Delta^{\downarrow}_{\beta}$, so β must contain an instance of Ax, which is a contradiction to the assumption that β in infinite. Therefore, we have $V_{\beta}([w]_{\beta}, p) = 0$. But then, since $[w]_{\beta} \in I_{\beta}(\pi)$, it follows $M_{\beta}, I_{\beta}(\pi) \not\models p$ by Definition 5.1.3. The case $\varphi = \bot$ is easy and can be treated similarly.

Case 2: Let $\varphi = \psi \wedge \chi$. For the first part, suppose that $(\pi : \psi \wedge \chi) \in \Gamma^{\downarrow}_{\beta}$. By Lemma 5.6.3 (iii), this yields $(\pi : \psi) \in \Gamma^{\downarrow}_{\beta}$ and $(\pi : \chi) \in \Gamma^{\downarrow}_{\beta}$. Hence, by induction hypothesis, we must have $M_{\beta}, I_{\beta}(\pi) \models \psi$ and $M_{\beta}, I_{\beta}(\pi) \models \chi$, so it follows $M_{\beta}, I_{\beta}(\pi) \models \psi \wedge \chi$ by Definition 5.1.3. For the second part, assume $(\pi : \psi \wedge \chi) \in \Delta^{\downarrow}_{\beta}$. Then, by Lemma 5.6.3 (iii), we have $(\pi : \psi) \in \Delta^{\downarrow}_{\beta}$ or $(\pi : \chi) \in \Delta^{\downarrow}_{\beta}$, which implies $M_{\beta}, I_{\beta}(\pi) \not\models \psi$ or $M_{\beta}, I_{\beta}(\pi) \not\models \chi$ by induction hypothesis. Therefore, we may conclude $M_{\beta}, I_{\beta}(\pi) \not\models \psi \wedge \chi$. The case $\varphi = \psi \vee \chi$ is treated similarly.

Case 3: Let $\varphi = \psi \to \chi$. For the first part, suppose $(\pi : \psi \to \chi) \in \Gamma^{\downarrow}_{\beta}$. Towards a contradiction, assume $M_{\beta}, I_{\beta}(\pi) \not\models \psi \to \chi$, i.e., there exists a state $s \subseteq I_{\beta}(\pi)$ such that $M_{\beta}, s \models \psi$ and $M_{\beta}, s \not\models \chi$. Then, by Lemma 5.1.12, there exists a *finite* substate $t \subseteq s \subseteq I_{\beta}(\pi)$ such that $M_{\beta}, t \not\models \chi$. By persistency, t must also satisfy $M_{\beta}, t \models \psi$. Moreover, because t is finite, we can write t in the form $t = \{[w_1]_{\beta}, \ldots, [w_n]_{\beta}\}$ for some $w_1, \ldots, w_n \in \mathfrak{S}$. Hence, t can be represented by the label σ given by $\sigma := w_1 + \ldots + w_n$, so we have $I_{\beta}(\sigma) = t$. But then, from $M_{\beta}, t \models \psi$ and $M_{\beta}, t \not\models \chi$, it follows $M_{\beta}, I_{\beta}(\sigma) \models \psi$ and $M_{\beta}, I_{\beta}(\sigma) \not\models \chi$. Now, since we have $I_{\beta}(\sigma) \subseteq I_{\beta}(\pi)$ and $I_{\beta}(\sigma) = \{[w_1]_{\beta}, \ldots, [w_n]_{\beta}\}$, we must have $[w_i]_{\beta} \in I_{\beta}(\pi)$ for all $1 \leq i \leq n$. Thus, by Lemma 5.6.9, it holds $(w_i \leqslant \pi) \in \Gamma^{\downarrow}_{\beta}$ for all $1 \leq i \leq n$, so we may conclude $(\sigma \leqslant \pi) \in \Gamma^{\downarrow}_{\beta}$ by Lemma 5.6.4 (v). But then, as we have $(\sigma \leqslant \pi) \in \Gamma^{\downarrow}_{\beta}$ and $(\pi : \psi \to \chi) \in \Gamma^{\downarrow}_{\beta}$, it follows $(\sigma : \psi) \in \Delta^{\downarrow}_{\beta}$ or $(\sigma : \chi) \in \Gamma^{\downarrow}_{\beta}$ by Lemma 5.6.3 (v). By induction hypothesis, this yields $M_{\beta}, I_{\beta}(\sigma) \not\models \psi$ or $M_{\beta}, I_{\beta}(\sigma) \models \chi$, which is a contradiction to the fact that we have both $M_{\beta}, I_{\beta}(\sigma) \models \psi$ and $M_{\beta}, I_{\beta}(\sigma) \not\models \chi$. Therefore, we must have $M_{\beta}, I_{\beta}(\pi) \models \psi \to \chi$, as desired.

For the second part, assume $(\pi : \psi \to \chi) \in \Delta^{\downarrow}_{\beta}$. Then, by Lemma 5.6.4 (v), there is an $x \in \mathfrak{V}$ with $(x \leqslant \pi) \in \Gamma^{\downarrow}_{\beta}$, $(x : \psi) \in \Gamma^{\downarrow}_{\beta}$ and $(x : \chi) \in \Delta^{\downarrow}_{\beta}$. By induction hypothesis and Lemma 5.6.10, this yields $I_{\beta}(x) \subseteq I_{\beta}(\pi)$, $M_{\beta}, I_{\beta}(x) \models \psi$ and $M_{\beta}, I_{\beta}(x) \not\models \chi$. Hence, $M_{\beta}, I_{\beta}(\pi) \not\models \psi \to \chi$.

Case 4: Let $\varphi = \Box\psi$. For the first part, assume $(\pi : \Box\psi) \in \Gamma^{\downarrow}_{\beta}$. Moreover, suppose for a contradiction that $M_{\beta}, I_{\beta}(\pi) \not\models \Box\psi$, i.e., there exists some $[w]_{\beta} \in I_{\beta}(\pi)$ such that $M_{\beta}, R_{\beta}([w]_{\beta}) \not\models \psi$. Since $[w]_{\beta} \in I_{\beta}(\pi)$, we must have $(w \leqslant \pi) \in \Gamma^{\downarrow}_{\beta}$ by Lemma 5.6.9. But then, because $(w \leqslant \pi) \in \Gamma^{\downarrow}_{\beta}$ and $(\pi : \Box\psi) \in \Gamma^{\downarrow}_{\beta}$, we may conclude $(R(w) : \psi) \in \Gamma^{\downarrow}_{\beta}$ by Lemma 5.6.3 (vi). Thus, by induction hypothesis, we have $M_{\beta}, I_{\beta}(R(w)) \models \psi$ and so $M_{\beta}, R_{\beta}([w]_{\beta}) \models \psi$, which is a contradiction to the assumption that $M_{\beta}, R_{\beta}([w]_{\beta}) \not\models \psi$. Therefore, we must have $M_{\beta}, I_{\beta}(\pi) \models \Box\psi$. For the second part, assume $(\pi : \Box\psi) \in \Delta^{\downarrow}_{\beta}$. Then, by Lemma 5.6.3 (vi), there exists some $w \in \mathfrak{S}$ such that $(w \leqslant \pi) \in \Gamma^{\downarrow}_{\beta}$ and $(R(w) : \psi) \in \Delta^{\downarrow}_{\beta}$. By induction hypothesis and Lemma 5.6.9, this implies $[w]_{\beta} \in I_{\beta}(\pi)$ and $M_{\beta}, R_{\beta}([w]_{\beta}) \not\models \psi$, so it follows $M_{\beta}, I_{\beta}(\pi) \not\models \Box\psi$ by Definition 5.1.3. □

Finally, we need to prove that our countermodel M_{β} does in fact have the frame properties expressed by the geometric axioms in A. In other words, we have to show that F_{β} is an $\mathfrak{L}$-frame.[36]

[36] Recall that, by an $\mathfrak{L}$-frame, we mean any Kripke frame F that validates all formulas in the standard modal logic $\mathfrak{L}$. Here, $\mathfrak{L}$ is assumed to be determined by A, so this simply means that F satisfies each of the geometric axioms in A.

Lemma 5.6.12. *Let β be an infinite branch in the proof search tree constructed by Algorithm 1. The Kripke frame F_β, defined by $F_\beta := \langle W_\beta, R_\beta \rangle$, is an $\mathfrak{L}$-frame.*

Proof. It suffices to show that F_β satisfies each of the geometric axioms in A when considered as a model of first-order logic. To make things precise, let us recall some notation. Given any first-order formula $\theta \in \mathcal{L}^{\mathfrak{S}}_{\mathsf{F}}$ and any variable assignment $g : \mathfrak{S} \to W_\beta$, we write $F_\beta \Vdash_g \theta$, if F_β satisfies θ with respect to g in first-order logic. And if $\Sigma \subseteq \mathcal{L}^{\mathfrak{S}}_{\mathsf{F}}$ is a set of atomic formulas, we write Σ° for the result of replacing each Ruv in Σ by the relational atom $v \leqslant R(u)$.

Let now $\theta \in \mathsf{A}$ be an arbitrary geometric axiom and assume that θ is given to us in the form $\forall \vec{w}(\varphi \to (\exists u_1 \psi_1 \vee \ldots \vee \exists u_n \psi_n))$, where $\vec{w} = w_1 \cdots w_k$ is some vector of variables from $\mathfrak{S}$ and each $\varphi, \psi_1, \ldots, \psi_n$ is a conjunction of atomic formulas from $\mathcal{L}^{\mathfrak{S}}_{\mathsf{F}}$. We write Φ for the multiset of all atoms in φ and Ψ_i for the multiset of all atoms in ψ_i. Let $g : \mathfrak{S} \to W_\beta$ be an arbitrary variable assignment and suppose $F_\beta \Vdash_g \varphi$. By definition of W_β, there must be some variables $w_1^*, \ldots, w_k^* \in \mathfrak{S}$ such that $g(w_i) = [w_i^*]_\beta$ for all $1 \leq i \leq k$. Moreover, from $F_\beta \Vdash_g \varphi$, it follows $F_\beta \Vdash_g \Phi$. Using the definition of R_β and Lemma 5.6.4 (viii), one readily sees that this yields $\Phi^* \subseteq \Gamma^\downarrow_\beta$, where Φ^* is the multiset of relational atoms given by $\Phi^* := \Phi^\circ(w_1^*, \ldots, w_k^*/w_1, \ldots, w_k)$. Now, since β is infinite, it must be closed under root-first applications of the geometric rule (θ-grs). Therefore, for some multiset $\Psi_i^* := \Psi_i^\circ(w_1^*, \ldots, w_k^*/w_1, \ldots, w_k)$, we must have $\Psi_i^*(v_i/u_i) \subseteq \Gamma^\downarrow_\beta$, where $v_i \in \mathfrak{S}$ is the eigenvariable used in the corresponding application of (θ-grs). Let now g^* be the variable assignment which is just like g, except that u_i is mapped to $[v_i]_\beta$, so we have $g^*(u_i) := [v_i]_\beta$. Using the fact that $\Psi_i^*(v_i/u_i) \subseteq \Gamma^\downarrow_\beta$ and the definition of R_β, it is now straightforward to verify that we have $F_\beta \Vdash_{g^*} \Psi_i$, so it holds $F_\beta \Vdash_{g^*} \psi_i$. But then, by definition of g^*, we must also have $F_\beta \Vdash_g \exists u_i \psi_i$ and therefore $F_\beta \Vdash_g \exists u_1 \psi_1 \vee \ldots \vee \exists u_n \psi_n$. Because g was an arbitrary assignment such that $F_\beta \Vdash_g \varphi$, this shows that F_β satisfies θ, as desired. $\square$

We are now ready to give a general completeness proof, covering each of the proof systems $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. As a consequence of this, it follows that our generic method described in Section 5.3 is in fact adequate in order to generate cut-free labelled sequent calculi for all geometric extensions of the basic inquisitive system InqK. This can be seen as the main result of this chapter and generalizes the corresponding result for standard modal logic established by Negri (2005).

Theorem 5.6.13 (Soundness and Completeness)**.** *Let $\mathsf{Inq}\mathfrak{L}$ be an arbitrary geometric system, let $\mathsf{A} \subseteq \mathcal{L}^{\mathfrak{S}}_{\mathsf{F}}$ be a finite set of geometric axioms determining $\mathfrak{L}$, and let $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ be the proof system given by Definition 5.3.3. Then, $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ is sound and complete with respect to $\mathsf{Inq}\mathfrak{L}$, i.e., for any finite $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{K}}$, we have: $\Gamma \vDash^{\mathsf{Inq}}_{\mathfrak{L}} \varphi$ if and only if $x : \Gamma \Rightarrow x : \varphi$ is derivable in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$, for any $x \in \mathfrak{V}$.*

Proof. The soundness of $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$ has been established in Proposition 5.5.2. For the completeness part, let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathsf{K}}$ be an arbitrary finite set of formulas and assume that the sequent $x : \Gamma \Rightarrow x : \varphi$ is *not* derivable in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. Suppose for a contradiction that $\Rightarrow x : \bigwedge \Gamma \to \varphi$ is derivable in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$, where $\bigwedge \Gamma$ stands for the conjunction of the elements of Γ. By the invertibility of the rules $R{\to}$ and $L\wedge$, this implies that $y \leqslant x, y : \Gamma \Rightarrow y : \varphi$ is derivable, where $y \in \mathfrak{V}$ is a fresh variable. But then, by performing the substitution (x/y) and a subsequent application of (rf), we also obtain a derivation for $x : \Gamma \Rightarrow x : \varphi$, which is a contradiction to our assumption. Therefore, the sequent $\Rightarrow x : \bigwedge \Gamma \to \varphi$ is *not* derivable in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$. Let now $\mathfrak{T}$ be the proof search tree constructed by Algorithm 1 for the input $\bigwedge \Gamma \to \varphi$, so the root node of $\mathfrak{T}$ has the form $\Rightarrow x : \bigwedge \Gamma \to \varphi$ for some $x \in \mathfrak{V}$. By what was said above, this sequent is not derivable in $\mathsf{GLinq}\mathfrak{L}_{\mathsf{A}}$, so $\mathfrak{T}$ must be infinite. Hence, by Kőnig's lemma, $\mathfrak{T}$ must have an infinite branch β. Let now M_β be the Kripke model and let I_β be the interpretation defined above. Then, by Lemma 5.6.11, none of the labelled formulas in $\Delta^\downarrow_\beta$ is satisfied under I_β, so it follows $M_\beta, I_\beta(x) \not\vDash \bigwedge \Gamma \to \varphi$. Therefore, by the semantics of $\to$ and $\wedge$, there exists a state $s \subseteq I_\beta(x)$ such that $M_\beta, s \vDash \psi$ for all $\psi \in \Gamma$ and $M_\beta, s \not\vDash \varphi$. Thus, because M_β is based on an $\mathfrak{L}$-frame by Lemma 5.6.12, we may conclude $\Gamma \not\vDash^{\mathsf{Inq}}_{\mathfrak{L}} \varphi$. $\square$

Conclusion

In this thesis, we developed and investigated various proof systems for propositional inquisitive logic. After giving a short introduction to inquisitive semantics, we presented a very elegant natural deduction system for InqB, established a normalization theorem for this system and derived a restricted version of the subformula property from it. Our system was based on an extended natural deduction formalism in which not only *formulas*, but also *rules* can act as assumptions that may be discharged in the course of a derivation. Afterwards, we provided a G3-style labelled sequent calculus GLinqB for basic inquisitive logic. In this calculus, labels were used in order to incorporate the support semantics of inquisitive logic directly into the syntax of the proof system. Special attention was paid to a thorough investigation of the structural properties of our calculus. In particular, we proved that GLinqB enjoys cut-admissibility, height-preserving admissibility of weakening and contraction, and height-preserving invertibility of all rules. The completeness of our system was established proof-theoretically, by using a suitable Hilbert-style axiomatization of InqB. We also sketched a possible proof search strategy for GLinqB and established a normal form result for the labels used in our system. In the second part of the thesis, we constructed cut-free labelled sequent calculi for various extensions and modifications of basic inquisitive logic. First, we considered an intuitionistic variant of InqB introduced by Ciardelli et al. (2020). Our sequent calculus for this variant was obtained from the system GLinqB in a modular way and was shown to preserve the structural properties of the latter. Finally, we considered various systems of inquisitive Kripke logic. We provided a general method that allows to construct a cut-free labelled sequent calculus $\mathsf{GLinq}\mathfrak{L}_\mathsf{A}$ for every inquisitive Kripke logic $\mathsf{Inq}\mathfrak{L}$ determined by some finite set of geometric implications A. This generalizes a famous result for standard modal logic established by Negri (2005). Our completeness proof was based on the construction of an infinite proof search tree and the extraction of a countermodel from an open branch of this tree.

We conclude this thesis by outlining some directions for future work. As pointed out in Chapter 2, our natural deduction system NinqB^+ only satisfies a weak form of the subformula property, so it is not an analytic proof system in a strict sense. It would therefore be desirable to have a natural deduction system for InqB that allows for an unrestricted subformula property (at least to the extent to which *classical* natural deduction does). First investigations suggest that such a system might possibly be obtained by extending NinqB^+ with non-primitive rules of arbitrary level, i.e., we should maybe allow the non-primitive rules of our system to discharge other non-primitive rules in a proof tree. A solution of this kind could most likely also be adapted to other non-classical logics such as, e.g., Kreisel-Putnam logic or Gödel-Dummett logic. To the best of our knowledge, higher-level natural deduction systems have not been used so far in order to obtain normalization theorems for logics that otherwise would not have an analytic natural deduction system, so this seems to be a very promising subject for further research.

Regarding our labelled sequent calculi, it would also be of great importance to develop effective *proof search procedures*, i.e., algorithms that, given any formula φ as input, either output a derivation for a sequent of the form $\Rightarrow x : \varphi$, or a finite countermodel for φ. In Section 3.4, we have already discussed some of the problems encountered in trying to construct such an algorithm for the system GLinqB. The main difficulty arises from the complex syntax of the labels used in our system, which makes it hard to define a suitable *saturation condition* for branches in a proof search tree. Our normal form result for labels (see Proposition 3.4.4) might play an important role here, since it allows to reduce the complexity of labels in a uniform way.

Finally, it would also be interesting to extend the labelled sequent calculi presented in this thesis to other systems of inquisitive logic such as, e.g., the *classical antecedent fragment* of first-order inquisitive logic (see Grilletti 2021) or extensions of InqB with *properly inquisitive modalities* (see Ciardelli 2016b, Chapter 7). We expect this to be relatively easy. In addition, it would be desirable to construct other sequent-style proof systems for inquisitive logic such as, e.g., display calculi and nested sequent calculi (see Belnap 1982; Brünnler 2006). Concerning display calculi, a first step in this direction has already been taken by Frittella et al. (2016) and Greco et al. (2017), who provide a so-called multi-type display calculus for inquisitive logic (see also Frittella et al. 2014). As far as we know, nested sequent calculi for inquisitive logic have not been studied so far.

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