Hereditary structural completeness of weakly transitive modal logics

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

In a deductive system, a rule is said to be *admissible* if the tautologies of the system are closed under its applications, and *derivable* if the rule itself holds in the system. Although every derivable rule is admissible, the converse is not true in general. A classical problem in the area is to determine which deductive systems have the property of all admissible rules being derivable, i.e. are *structurally complete*. Early results on this problem suggest that even though a full characterisation of the structurally complete modal and superintuitionistic logics is out of reach, it might be possible to characterise the *hereditarily structurally complete* systems, those which are not only structurally complete themselves but whose finitary extensions are too.

Hereditarily structurally complete intermediate logics were characterised by Citkin (1978). This result was generalised to logics extending the transitive modal logic $\mathsf{K4}$ by Rybakov (1995). Carr (2022) revisited Rybakov's result and corrected some of its errors. This thesis gives a full characterisation of the hereditarily structurally complete extensions of the modal logic $\mathsf{wK4}$ of weakly transitive frames. The logic $\mathsf{wK4}$ is a "close neighbour" of $\mathsf{K4}$. It inherits many of its properties, yet there are essential differences. We also give a description of the n-universal models of $\mathsf{wK4}$ and compare it to the n-universal models of $\mathsf{K4}$.

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Contents

Chapter 1

Introduction

In this thesis, we study hereditary structural completeness in the setting of modal logic. Given a deductive system $\vdash$ over a set of formula Fm, a *rule* is an expression of the form $\Gamma \rhd \phi$, where $\Gamma \cup \{\phi\}$ is a finite set of formulas. A rule is said to be *admissible* if the set of tautologies of the system is closed under its application. A rule $\Gamma \rhd \phi$ is derivable if the rule itself holds in the system, i.e. if $\Gamma \vdash \phi$. Every derivable rule is admissible, but the converse fails in general. The converse holds in the classical propositional calculus (CPC), but fails in the intuitionistic propositional calculus (IPC) and in many deductive systems in between. This has motivated a study of criteria for admissibility in superintuitionistic and modal logics. This work was undertaken by Rybakov in the 1980's, see [41] for an overview of these results. The problem of finding *bases* for admissible rules was solved for IPC by Iemhoff in [30], building on the work of Ghilardi [27]. This was obtained independently by Rozière [40]. Similar results were obtained for modal and Łukasiewicz logics (see [31]).

A deductive system which has the property of all admissible rules being derivable is *structurally complete.* These deductive systems are in a sense optimal. One cannot improve a proof search of a structurally complete system by adding new admissible rules, they are all derivable already. A classical problem is to characterise the structurally complete systems. Prucnal proved in [38] that finitary extensions of the $\wedge, \rightarrow$-fragments of IPC are structurally complete.

While structurally complete logics are difficult to characterise, one can study a stronger property of *hereditary structural completeness.* These are the systems which are not only structurally complete, but whose finitary extensions are all structurally complete as well. The first result in this direction was obtained by Citkin in [18], who characterised the hereditarily structurally complete superintuitionistic logics. Citkin proved that a logic is hereditarily structurally complete if and only if the variety of Heyting algebras associated with it omits five finite algebras. This was later generalised by Rybakov in [42, 41], who characterised the hereditarily structurally complete extensions of $\mathsf{K4}$. Rybakov proved that a transitive logic is hereditarily structurally complete if and only if the variety of modal algebras associated with it omits twenty algebras.. Both proofs are quite involved. Rybakov's proof is not self-contained as it relies on Fine's completeness theorem for extensions of $\mathsf{K4}$ of finite width [23]. Recently, in [11], Bezhanishvili and Moraschini gave a new proof of Citkin's result, based on Esakia duality. Esakia duality acts as a bridge between an algebraic perspective on the problem and an order-topological one, thus allowing one to investigate the problem through both lenses. In [16], Carr used the proof technique presented by Bezhanishvili and Moraschini to provide a new proof of Rybakov's result. This allowed him to correct a mistake in Rybakov's results and make the proof more transparent. In this thesis, we use a similar technique to characterise the hereditary structurally complete extension of $\mathsf{wK4}$ (weak $\mathsf{K4}$), thus generalising the results in [16].

The logic $\mathsf{wK4}$ is defined as $\mathsf{K} + p \wedge \Box p \rightarrow \Box\Box p$. It is a logic of 1-*transitive* frames, where a relation is 1-*transitive* if

$$\forall x\ \forall y\ \forall z\ (x\ R\ y \wedge y\ R\ z \to x\ R\ z \vee x = z).$$

The logic wK4 is a weakening of K4, thus its extensions form a larger class than the extensions of K4. This logic plays an important role in topological semantics of modal logic via the derived set operator. More precisely, it is the logic of all topological spaces when modal $\Diamond$ is interpreted as the derived set operator [20, 7, 21]. Recently, topological completeness of μwK4 (wK4 with the modal fixpoint operator) has been established in [1]. The logic wK4 is also the first of a descending sequence of the so-called n-transitive logics [35, Section 3.4], which converges to K. The logic wK4 is a "close neighbour" of K4 and inherits many properties of K4. However, there are important differences. Filtration does not go trough in the 1-transitive case, although wK4 still has the finite model property. However, it is not known whether n-transitive logics, for $n > 1$, have the finite model property [17, Problem 11.2] In K4, by transitivity, every non degenerate cluster contains reflexive points only, while this is not the case for wK4. On the other hand, Bezhanishvili, Ghilardi and Jibladze showed that the method of selective filtration and of subframe logics can be extended from extensions of K4 to the extensions of wK4 [8].

Similarly to Rybakov and Carr, we characterise hereditarily structurally complete extension of wK4 through forbidden configurations. Our collection of forbidden configurations strictly contains the collection of forbidden configurations of Carr, which itself is larger than the collection obtained by Citkin for IPC. One direction of the proof consists in showing that if a logic contains one of the forbidden configuration, then it cannot be hereditarily structurally complete. The other direction is a bit more involved. One first has to use the forbidden configurations to give a precise description of the structure of frames of the logic, then use that description to prove that every extension of the logic has the finite model property, before finishing the work and prove hereditary structural completeness.

We also give a description of n-universal models for wK4. The n-universal models (for $n < \omega$) are important tools in the study of superintuitionistic and modal logics (see [17, 10]). They consist of the upper part (all the points of finite depth) of the n-canonical models, which are dual to n-generated free algebras [17]. For instance, the 1-universal model for IPC is the famous Rieger-Nishimura ladder [10, Fig. 3.1]. n-universal models contain all of the finite n-generated models as generated submodels. If the logic has the finite model property, then the n-universal models are dense in the n-canonical ones, and thus they characterise the logic. n-universal models have been used in the study of properties of logics related to admissibility and structural completeness [41, 17].

The n-universal models for transitive logics are well understood. They have a layer-wise recursive definition [17, Section 8.7]. However, the structure of n-universal models for logics below K4 is less clear. In Chapter 3, we describe the structure of the n-universal models of the logic wK4. The key method is to identify minimal p-morphisms. These are α-, β- and γ-reduction. α- and β-reduction were already identified by de Jongh and Troelstra in [32]. Those two types of reduction are also used in [10] to give a description of the n-universal models for IPC. With K4 and wK4, a third type of reduction arises, due to the possibility of quotienting points from the same cluster. The difference between the n-universal models for K4 and wK4 essentially boils down to the allowed shapes of clusters. Indeed, in wK4, proper clusters can have irreflexive elements, which is not the case in K4. However, the 0-universal model for wK4 is the same as for K4.

This thesis is organised as follows. In Chapter 2, we formally discuss hereditary structural completeness and 1-transitive logics, and introduce some of the algebraic and topological tools used to study them. In Chapter 3, we investigate and precisely describe n-universal models for wK4. In Chapter 4, we give and prove our characterisation of the hereditarily structurally complete extensions of wK4. We finish with a short chapter on future work. One obvious generalisation of our results is to study n-transitive logics. However, the notion of cluster is not well defined for such logics, and the number of prohibited frames can greatly expand when going even from 1-transitive to 2-transitive. Another problem which is interesting to investigate is to characterise hereditarily structurally complete logics in $\wedge, \to, \Box$-fragments of modal logic.

Chapter 2

Preliminaries

In this chapter, we introduce all the notions that are required to read this thesis and formally discuss the main topics. The first section covers the basics of universal algebra. Then, in the second section, we give a formal definition of hereditary structural completeness of a logic. In the third section, we use the tools provided by algebraic logic to translate the logical properties of interest into algebraic ones. In the fourth section, we finally discuss the basics of modal logic, give a brief overview of weakly transitive logics, and introduce modal algebras, the algebraic equivalent of modal and weakly transitive logics. As it turns out, these algebraic structures can be represented by topological means, which is presented in the fifth and sixth sections. In the last section, we combine all those tools together to obtain various theorems and characterisations that we will use in this thesis.

2.1 Universal algebra

We begin by recalling elementary notions of universal algebra. This section is by no means exhaustive and the reader may consult [6, 15] to familiarise themselves with the basics of universal algebra.

Definition 2.1. A *signature* is a set $\mathcal{L}$ of functional symbols together with an *arity* function $n\colon \mathcal{L} \to \mathbb{N}$.

Given a signature $\mathcal{L}$, an $\mathcal{L}$-*algebra* is a nonempty set A together with a function $f^A\colon A^n \to A$ for each symbol $f \in \mathcal{L}$ with arity n (if $n = 0$, then f is just a constant). We usually use A to denote the algebra and its underlying set interchangeably.

A *homomorphism* between $\mathcal{L}$-algebras is a map $h\colon A \to B$ that preserves the operations of $\mathcal{L}$, i.e. for each symbol $f \in \mathcal{L}$ with arity n, and for all $a_1, \dots, a_n \in A$, we have $h(f^A(a_1, \dots, a_n)) = f^B(h(a_1), \dots, h(a_n))$.

Given a set of variables Prop, we let $Fm_{\mathcal{L}}(\text{Prop})$ (or Fm when $\mathcal{L}$ and Prop are clear from context) denote the $\mathcal{L}$-algebra of formulas (or *terms*) in Prop. Its underlying set is the set of formulas in Prop, i.e. the least set such that $\text{Prop} \subseteq Fm_{\mathcal{L}}(\text{Prop})$ and if $\phi_1, \dots, \phi_n \in Fm_{\mathcal{L}}(\text{Prop})$ and f is a symbol of arity n, then $f(\phi_1, \dots, \phi_n) \in Fm_{\mathcal{L}}(\text{Prop})$. This set can be can be endowed with the structure of a $\mathcal{L}$-algebra by stipulating that $f^{Fm_{\mathcal{L}}(\text{Prop})}(\phi_1, \dots, \phi_n) = f(\phi_1, \dots, \phi_n)$, for every n-ary operation f of $\mathcal{L}$ and every formulas $\phi_1, \dots, \phi_n$.

From now on we will work with a fixed signature $\mathcal{L}$. All algebras, classes of algebras and formulas will be expected to have the same signature.

We denote by $\mathbb{I}$, $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{P}_U$ the class operators of closure under isomorphism, homomorphic images, subalgebras, direct products, and ultraproducts, respectively.

There are interesting connections between the class operators $\mathbb{I}$, $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$, and $\mathbb{P}_U$ and the axiomatisation of classes.

Definition 2.2. An *equation* (or *identity*) is an expression of the form $\varepsilon \approx \delta$, where $\varepsilon, \delta \in Fm$.

We say that an algebra A satisfies an equation $\varepsilon \approx \delta$, denoted $A \models \varepsilon \approx \delta$, if for all morphism $h\colon Fm \to A$, we have $h(\varepsilon) = h(\delta)$.

Given a class of algebras K, we define the equational consequence relation of K, $\models_K$, as follows. Let Θ be a set of equations and $\varepsilon \approx \delta$ an equation. We have $\Theta \models_K \varepsilon \approx \delta$ if for each algebra $A \in K$ and morphism $h\colon Fm \to A$, if $h(\phi) = h(\psi)$ for all $\phi \approx \psi \in \Theta$, then $h(\varepsilon) = h(\delta)$.

Theorem 2.3 (Birkhoff)**.** *A class of algebras K is closed under $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$ iff it can be axiomatised by equations.*

Proof. See [15, Thm II.11.9]. □

Definition 2.4. A class of algebras K satisfying the equivalent conditions from the previous theorem is called a *variety*. Given a class of algebras K, we let $\mathbb{V}(K)$ denote the smallest variety containing K.

Theorem 2.5 (Tarski)**.** *Given a class of algebras K, we have $\mathbb{V}(K) = \mathbb{HSP}(K)$.*

Proof. See [15, Thm II.9.5] □

This situation can be generalised a bit further.

Definition 2.6. A *quasi-equation* (or *quasi-identity*) is an expression of the form $\bigwedge_{i=1}^{k} \phi_i \approx \psi_i \to \varepsilon \approx \delta$.

We say that an algebra satisfies a quasi-equation $\Phi = \bigwedge_{i=1}^{k} \phi_i \approx \psi_i \to \varepsilon \approx \delta$, denoted $A \models \Phi$, if for each morphism $h\colon Fm \to A$, if $h(\psi_i) = h(\psi_i)$ for all $i \in \{1, \ldots, k\}$, then $h(\varepsilon) = h(\delta)$.

Theorem 2.7 (Maltsev)**.** *A class of algebras K is closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{P}_U$ iff it can be axiomatised by quasi-equations.*

Proof. See [15, Thm V.2.25]. □

Definition 2.8. A class of algebras K satisfying the equivalent conditions from the previous theorem is called a *quasi-variety*. Given a class of algebras K, we let $\mathbb{Q}(K)$ denote the smallest variety containing K.

Theorem 2.9 (Maltsev)**.** *Given a class of algebras K, we have $\mathbb{Q}(K) = \mathbb{ISPP}_U(K)$.*

Proof. See [15, Thm V.2.23]. □

Definition 2.10. Let K and M be quasi-varieties. We say that M is a *subquasi-variety* of K if $M \subseteq K$.

If in addition M can be axiomatised relative to K by a set of equations, M is a *relative subvariety* of K.

If K and M are varieties and $M \subseteq K$, we say that M is a *subvariety* of K.

Remark 2.11. If K is a variety, its relatives subvarieties are simply its subvarieties.

2.2 Hereditary structural completeness

Before introducing the concept of hereditary structurally complete logic, we first need to make clear what we mean by a logic. To this end, we introduce the notions of consequence relation and deductive system.

Definition 2.12. Given a set F, a *(finitary) consequence relation* on F is a relation $\vdash\ \subseteq \mathcal{P}(F) \times F$ such that

(i) if $x \in X$, then $X \vdash x$, for all $X \subseteq F$ and $x \in F$,

(ii) if $X \vdash y$ for all $y \in Y$ and $Y \vdash z$, then $X \vdash z$, for all $X, Y \subseteq F$ and $z \in F$,

(iii) if $X \vdash y$, then there is a finite set $X' \subseteq X$ such that $X' \vdash y$.

Let us now fix a signature $\mathcal{L}$ and a countably infinite set of propositional variables Prop, we consider consequence relations over the set of formulas Fm.

Definition 2.13. A consequence relation $\vdash$ on Fm is a *deductive system* if for every substitution σ (i.e. morphism $\sigma\colon Fm \to Fm$), if $\Gamma \vdash \phi$, then $\sigma[\Gamma] \vdash \sigma(\phi)$.

A *(finite) rule* is an expression of the form $\Gamma \rhd \phi$, where $\Gamma \subseteq Fm$ is finite and $\phi \in Fm$.

A rule $\Gamma \rhd \phi$ is said to be *admissible* in a deductive system $\vdash$ if for all substitutions σ, if $\emptyset \vdash \sigma(\gamma)$ for all $\gamma \in \Gamma$, then $\emptyset \vdash \sigma(\phi)$.

A rule $\Gamma \rhd \phi$ is said to be *derivable* in a deductive system $\vdash$ if $\Gamma \vdash \phi$.

Any derivable rule is admissible, but the converse is not true in general. For example, it fails in IPC. The deductive systems in which the converse is true ought to be singled out.

Definition 2.14. A deductive system $\vdash$ is *structurally complete* if all of its admissible rules are derivable.

We can now work towards a definition of hereditary structural completeness.

Definition 2.15. Given a deductive system $\vdash$, a deductive system $\vdash'$ is said to be an *extension* of $\vdash$ if $\Gamma \vdash \phi$ implies $\Gamma \vdash' \phi$, or in other words $\vdash\ \subseteq\ \vdash'$.

Given a deductive system $\vdash$, a deductive system $\vdash'$ is said to be an *axiomatic* extension of $\vdash$ if there is a set of formulas Δ which is closed under substitutions such that $\Gamma \vdash' \phi$ iff $\Gamma \cup \Delta \vdash \phi$.

Theorem 2.16. *Let $\vdash$ be a deductive system. The following are equivalent.*

(i) Every extension of $\vdash$ is structurally complete.

(ii) Every axiomatic extension of $\vdash$ is structurally complete.

(iii) Every extension of $\vdash$ is an axiomatic extension of $\vdash$.

Proof. This is proved in [37, Thm 2.6]. □

Definition 2.17. A deductive system $\vdash$ which satisfies any of the equivalent conditions of the previous theorem is said to be *hereditarily structurally complete*.

2.3 Algebraisable logics

In this section, we introduce the notions of algebraisable logics. The reader may consult [14] for reference. The theory of algebraisable logics will act as a bridge between logic and universal algebra, as we will see in the second half of this section. The first investigation of structural completeness from an algebraic point of view can be found in [5].

For every set of formulas $\Delta(x, y)$, equation $\varepsilon \approx \delta$ and set of equations Θ, we define

$$\Delta(\varepsilon, \delta) = \{\phi(\varepsilon, \delta) : \phi(x, y) \in \Delta(x, y)\}$$

and

$$\Delta[\Theta] = \bigcup_{\phi \approx \psi \in \Theta} \Delta(\phi, \psi).$$

For every set of equations $\tau(x)$, formula ϕ and set of formulas Γ, we define

$$\tau(\phi) = \{\varepsilon(\phi) \approx \delta(\phi) : \varepsilon(x) \approx \delta(x) \in \tau(x)\}$$

and

$$\tau[\Gamma] = \bigcup_{\psi \in \Gamma} \tau(\psi).$$

Definition 2.18. A finitary deductive system $\vdash$ is *algebraisable* if there exist a quasi-variety K, a set of equations $\tau(x)$ and a set of formulas $\Delta(x, y)$ such that for all set of equation Θ, all equation $\varepsilon \approx \delta$, all set of formulas Γ and all formula ϕ, we have

(i) $\Gamma \vdash \phi$ iff $\tau[\Gamma] \models_K \tau(\phi)$,

(ii) $\Theta \models_K \varepsilon \approx \delta$ iff $\Delta[\Theta] \vdash \Delta(\varepsilon, \delta)$,

(iii) $\phi \vdash \Delta[\tau(\phi)]$ and $\Delta[\tau(\phi)] \vdash \phi$,

(iv) $\varepsilon \approx \delta \models_K \tau[\Delta(\varepsilon, \delta)]$ and $\tau[\Delta(\varepsilon, \delta)] \models_K \varepsilon \approx \delta$.

An equivalent[1] requirement is that

(i) $\Gamma \vdash \phi$ iff $\tau[\Gamma] \models_K \tau(\phi)$,

(ii) $x \approx y \models_K \tau[\Delta(x, y)]$,

(iii) $\tau[\Delta(x, y)] \models_K x \approx y$.

We call the quasi-variety K the *equivalent algebraic semantics* of $\vdash$. When it exists, the equivalent algebraic semantics of a deductive system is unique, as proved in [14, Thm 2.15].

Theorem 2.19. *Every algebraisable deductive system has a unique equivalent algebraic semantics.*

Proof. This is proved in [14, Thm 2.15]. □

Theorem 2.20 (Blok & Pigozzi)**.** *Let $\vdash$ be an algebraisable deductive system and K its equivalent axiomatic semantics. The lattice of finitary extensions of $\vdash$ is dually isomorphic to the lattice of subquasi-varieties of K, under the map that sends a finitary extension to its equivalent algebraic semantics. This dual isomorphism restricts to one between the lattice of axiomatic extensions of $\vdash$ and the lattice of relative subvarieties of K.*

[1] See [14, Corollary 2.9]

Proof. The reader may consult [25, Thm 3.33] for a detailed proof. □

Combining Theorem 2.16, Remark 2.11 and Theorem 2.20, we obtain the following theorems.

Theorem 2.21. *Let $\vdash$ be an algebraisable deductive system with variety K as its equivalent axiomatic semantics. Then the lattice of axiomatic extensions of $\vdash$ is dually isomorphic to the lattice of subvarieties of K, under the map that sends an axiomatic extension to its equivalent algebraic semantics.*

Definition 2.22. A variety is *primitive* if all of its subquasi-varieties are varieties.

Theorem 2.23. *Let $\vdash$ be an algebraisable deductive system with variety K as its equivalent algebraic semantics. Then $\vdash$ is hereditarily structurally complete iff K is primitive.*

2.4 Weakly transitive modal logics

From now on we work in the language of modal logic, i.e. the language of propositional logic with a unary modality $\Box$. For an introduction to modal logic, the reader may consult [12].

Definition 2.24. A *normal modal logic* is a set of formulas Λ such that

(i) the set Λ contains all the classical tautologies,

(ii) the axiom $\Box(p \to q) \to \Box p \to \Box q$ is in Λ,

(iii) the set Λ is closed under Modus Ponens, Necessitation and Substitution.

Normal modal logics are defined as sets of formulas, but we are interested in deductive systems. We turn normal modal logics into deductive systems as follows.

Definition 2.25. Given a normal modal logic Λ, we define the finitary deductive system $\vdash_\Lambda$ by $\Gamma \vdash_\Lambda \phi$ iff ϕ is derivable from Γ using the theorems of Λ and the inference rules Modus Ponens and Necessitation.

A normal modal logic is said to be *transitive* if it contains the axiom $\Box p \to \Box\Box p$. In that case, its frames are transitive (see [12, Thm 4.27]). The least transitive logic is $\mathsf{K4} = \mathsf{K} + \Box p \to \Box\Box p$, and it is sound and complete with respect to the class of transitive frames. The notion of transitivity can be weakened. For a deeper overview of weakly transitive logics, the reader may consult [35, Section 3.4 & 4.3].

We first define compound modalities $\Box^k p$ for $k < \omega$ as follows: $\Box^0 p = p$, and $\Box^{k+1} p = \Box(\Box^k p)$. We also define $\Box^{\leq n} p$ as $\bigwedge_{k \leq n} \Box^k p$.

Definition 2.26. A *compound modality* $\boxplus p$ is a finite conjunction of modalities of the form $\Box^k p$ for $k < \omega$.

A compound modality $\Box^+ p$ is a *master modality* if it implies any other compound modality, i.e. for any compound modality $\boxplus p$, the formula $\Box^+ p \to \boxplus p$ is a tautology. All master modalities are equivalent, so we will sometimes refer to 'the' master modality.

Definition 2.27. A normal modal logic is said to be *n-transitive* (for $n < \omega$) if it contains the axiom

$$\mathsf{4}_n = \Box^{\leq n} p \to \Box^{n+1} p.$$

In that case, the master modality $\Box^+$ is definable as $\Box^{\leq n}$. In fact, a logic is n-transitive iff $\Box^{\leq n}$ is a master modality.

A normal modal logic is *weakly transitive* if it is n-transitive for some n.

A relation R on a set X is *n-transitive* if for any two points x and y, if there is a path from x to y then there is a (possibly empty) path of length at most n from x to y. When R is an n-transitive relation, the reflexive and transitive closure of the relation, denoted R^+, is definable as

$$\Delta_X \cup R \cup R^2 \cup \cdots \cup R^n,$$

where $\Delta_X = \{(x,x) : x \in X\}$ is the diagonal.

A Kripke frame (X, R) is *n-transitive* if R is n-transitive.

Remark 2.28. Being 1-transitive is weaker than being transitive. This is better reflected at the frame level. In a 1-transitive frame, we can have a configuration where $x\ R\ y\ R$, but x is irreflexive, while transitivity would require x to be reflexive.

We denote by $\mathsf{K4}_n$ the least n-transitive logic, $\mathsf{K} + 4_n$. By the Sahlqvist Completeness Theorem [12, Thm 5.91], $\mathsf{K4}_n$ is sound and complete with respect to n-transitive frames.

The logic $\mathsf{K4}_1$ will play a very important role in this thesis.

Definition 2.29. We refer to the logic $\mathsf{K4}_1$ as $\mathsf{wK4}$.

Proposition 2.30. *The weakly transitive modal logics form a non principal filter in the lattice of normal modal logics. Furthermore, we have*

$$\bigcap_{n<\omega} \mathsf{K4}_n = \mathsf{K}.$$

Proof. See [35, Prop. 3.4.2]. □

Definition 2.31. A deductive system $\vdash$ admits a *deduction detachment theorem* if there exists a finite set of formulas $I(x,y)$ such that for every set of formula Γ and formulas ϕ, ψ,

$$\Gamma, \phi \vdash \psi \qquad \text{iff} \qquad \Gamma \vdash I(\phi, \psi).$$

In this case, we say that $I(x,y)$ *witnesses* the deduction detachment theorem for $\vdash$.

Theorem 2.32 (Blok & Pigozzi)**.** *Let Λ be a normal modal logic. The deductive system $\vdash_\Lambda$ admits a deduction detachment theorem iff Λ is weakly transitive. In that case, the deduction detachment theorem is witnessed by $\{\Box^+ x \to y\}$.*

Proof. This is proved in [35, Thm 3.4.4]. □

It is now time to introduce the algebraic counterpart of modal logics.

Definition 2.33. A *modal algebra* (or *K-algebra*) is an algebra $(A, \wedge, \vee, \neg, \bot, \top, \Box)$ such that

(i) $(A, \wedge, \vee, \neg, \bot, \top)$ is a Boolean algebra,

(ii) $\Box\top = \top$,

(iii) $\Box(a \wedge b) = \Box a \wedge \Box b$ for all $a, b \in A$.

We let MA denote the category of modal algebras with their homomorphisms.

Given a normal modal logic Λ, the deductive system $\vdash_\Lambda$ is algebraisable, with $\tau(\phi)$ as $\{\phi \approx 1\}$, $\Delta(\varepsilon, \delta)$ as $\{\varepsilon \to \delta, \delta \to \varepsilon\}$ and K as the variety of modal algebras axiomatised by $\tau[\Lambda]$. Conversely, every variety of modal algebras K defines a normal modal logic.

Definition 2.34. A modal algebra A is *n-transitive* if $\Box^{\leq n} a \leq \Box^{n+1} a$ for all $a \in A$.

Definition 2.35. Given a normal modal logic Λ, a modal algebra is a Λ-*algebra* if it validates $\phi \approx 1$ for each $\phi \in \Lambda$.

The following claims are easy consequences of the previous definitions.

Theorem 2.36. *Given a normal modal logic Λ, the deductive system $\vdash$ is algebraised by the variety of Λ-algebras.*

Theorem 2.37. *A normal modal logic Λ is n-transitive iff the equivalent algebraic semantics of $\vdash_\Lambda$ is a variety of n-transitive algebras.*

2.5 Modal spaces

As we will see in the next section, categories of modal algebras have topological duals. We now introduce the objects and morphisms of the dual category. Throughout this section, we assume familiarity with topological dualities. The reader may consult [19] for a gentle introduction. One may also consult [35, Chapter 4].

Definition 2.38. A *modal space* (or *descriptive frame*) is a triple (X, τ, R) such that

(i) (X, τ) is a Stone space,

(ii) (X, R) is a Kripke frame,

(iii) $R[x]$ is closed for all $x \in X$,

(iv) $\Box_R[U] = R^{-1}[U^c]^c = \{x \in X : R[x] \subseteq U\}$ is clopen for all clopen $U \subseteq X$.

Points in $R[x]$ will be refered to as *successors* of x. Sometimes, $\Box_R[U]$ will be referred to as just $\Box U$ when R is clear from the context.

The following separation property of modal spaces will be useful later.

Proposition 2.39. *Let X be a modal space and let $x, y \in X$ such that $x \not R\, y$. Then there are clopen sets U, V such that $x \in U$, $y \in V$ and $R[U] \cap V = \emptyset$. In particular, R is closed in $X \times X$.*

Proof. The set $R[x]$ is closed and does not contain y, hence there is a clopen set W such that $R[x] \subseteq W$ and $y \notin W$. Then $\Box_R W$ is a clopen set containing x, W^c is a clopen set containing y and $R[\Box_R W] \cap W^c = \emptyset$.

We now show that R is closed. Take $(x, y) \notin R$. Then there are clopen sets U, V such that $x \in U$, $y \in V$ and $R[U] \cap V = 0$, i.e. $(x, y) \in U \times V$ and $U \times V \cap R = \emptyset$. $\Box$

Definition 2.40. A map $f\colon X \to Y$ between frames is a *p-morphism* if

(i) $x \; R_X \; y$ implies $f(x) \; R_Y \; f(y)$ for all $x, y \in X$,

(ii) $f(x) \; R_Y \; y$ implies that there exists $z \in X$ such that $x \; R_X \; z$ and $f(z) = y$, for all $x \in X, y \in Y$.

Equivalently, a map $f\colon X \to Y$ is a p-morphism if $f[R_X[x]] = R_Y[f[x]]$ for all $x \in X$.

We let MS be the category of modal spaces and continuous p-morphisms.

We now investigate a few of the elementary constructions on modal spaces. As we will see in the next section, those are dual to the algebraic construction of subalgebras and quotient algebras.

Definition 2.41. Given an equivalence relation E on a set X, a subset $U \subseteq X$ is called E-*saturated* (or *saturated* for short) if it is an union of equivalence classes of E, or equivalently, if $E[U] = U$.

An equivalence relation E on a modal space is a *bisimulation equivalence* if

(i) $x\ E\ y$ and $x\ R\ z$ implies that there exists $t \in X$ such that $y\ R\ t$ and $z\ E\ t$,

(ii) $x \not\!E\ y$ implies that there exists a saturated clopen set U such that $x \in U$ and $y \notin U$.

Remark 2.42. Given a bisimulation equivalence on a modal space $X = (X, \tau, R)$, let $\pi_E : X \to X/E$ be the quotient map. We turn X/E into a modal space $(X/E, \tau_E, R_E)$ as follows. A set $U \subseteq X/E$ is open in τ_E iff $\pi_E^{-1}(U)$ is open in τ, i.e. τ_E is the finest topology on X/E making π_E continuous. We define $x/E\ R_E\ y/E$ iff there is $x'\ R\ x$ and $y'\ R\ y$ such that $x'\ R\ y'$.

Conversely, any surjective continuous p-morphism $\pi : X \to Y$ between modal spaces induces a bisimulation equivalence $\ker \pi$ on X, and these two constructions are (up to isomorphism) each other's inverse, i.e. $\pi_{\ker \pi} \cong \pi$ and $\ker \pi_E = E$. See e.g. [10, Thm 2.3.9] for more details.

Most of the quotient that we will consider in this thesis have a very specific form, given by the following definition.

Definition 2.43. Let $U_1, \dots, U_k$ be pairwise disjoint clopen subsets of a modal space X. We say that this collection is a *M-partition* of X if, for all $i, j \in \{1, \dots, k\}$, we have

(i) for all $x, y \in U_i$ and $z \in U_j$ such that $x\ R\ z$, there is $t \in U_j$ such that $y\ R\ t$,

(ii) for all $x, y \in U_i$ and $z \in X \setminus (U_1 \cup \dots \cup U_k)$ such that $x\ R\ z$, we have $y\ R\ z$.

It is easy to check that in this case, the relation

$$E = \Delta_X \cup (U_1 \times U_1) \cup \dots \cup (U_k \times U_k)$$

is a bisimulation equivalence.

Remark 2.44. If a subset U of a modal space X is a clopen upset (closed under R), then it trivially satisfies the condition to be an M-partition provided that every point in U has a successor, or equivalently that every maximal point in U is reflexive, and it becomes a maximal point in the quotient space.

The dual notion of a quotient algebra is that of generated subspace.

Definition 2.45. A subspace Y of a modal space X is a *generated subspace* if

(i) Y is closed in X, (equivalently, Y is compact),

(ii) Y is an upset, i.e. the inclusion map from Y to X is a p-morphism.

Finally, we introduce the topological equivalent of weak transitivity.

Definition 2.46. A modal space (X, τ, R) is *n-transitive* if so is (X, R).

Remark 2.47. Notice that if (X, τ, R) is an n-transitive space, then (X, τ, R^+) is a modal space, because R^+ is definable as a finite combination of R^k. The space (X, τ, R^+) is reflexive and transitive.

Proposition 2.48. *Let X be an n-transitive space and let $x, y \in X$ be such that $x \not\!R^+ y$. Then there is a clopen upset U such that $x \in U$ and $y \notin U$.*

Proof. By Proposition 2.39, there are clopen sets U, V such that $x \in U$, $y \in V$ and $R^+[U] \cap V = \emptyset$. Then $(R^+)^{-1}[V]$ is a clopen downset containing y but not x, hence $(R^+)^{-1}[V]^c$ is the desired upset. $\square$

2.6 Jónsson-Tarski duality

In this section, we introduce a topological duality between categories of modal algebras and modal spaces. This duality will act as a bridge between the algebraic and order-topological approaches to modal logic. The results in this section can be found in [34], [12, Sections 5.3 & 5.4] or [35, Chapter 4]

Theorem 2.49 (Jónsson-Tarski Duality)**.** *The category* MA *is dually equivalent to the category* MS*. This duality restricts to a dual equivalence between the categories of n-transitive modal algebras and n-transitive modal spaces.*

Proof sketch. The functor $-_*\colon \mathsf{MA} \to \mathsf{MS}$ is defined as follows. Given a modal algebra A, we let A_* be the set of ultrafilters of A, topologised with the sets $\phi(a) = \{F \in A_* : a \in F\}$ for $a \in A$ as a basis. The relation R on A_* is defined by $F\ R\ G$ iff $\Box a \in F$ implies $a \in G$ for all $a \in A$. A morphism $f\colon A \to B$ is sent to the morphism $f_*\colon B_* \to A_*$ defined by $f_*(F) = f^{-1}(F)$ for any ultrafilter $F \in B_*$.

The functor $-^*\colon \mathsf{MS} \to \mathsf{MA}$ is defined as follows. Given a modal space X, we let X^* be the set of clopen subsets of X. Meet, join and negation are given by intersection, union and complement, respectively. The $\Box$ operator is given by $\Box_R U = R^{-1}[U^c]^c = \{x \in X : R[x] \subseteq U\}$. A morphism $f\colon X \to Y$ is sent to the morphism $f^*\colon Y^* \to X^*$ defined by $f^*(C) = f^{-1}(C)$ for any clopen set $C \in Y^*$. $\Box$

Theorem 2.50. *Under this dual equivalence, the following correspond*

(i) subalgebras and quotient spaces,

(ii) quotient algebras and generated subspaces,

(iii) direct products indexed by a finite set and disjoint union of spaces,

(iv) quotient algebras and modal filters (filters F such that $a \in F$ implies $\Box a \in F$,

(v) generated subspaces and closed upsets.

Proof. The first three correspondences follow from the duality. However, it will be useful later to have a more practical description of some correspondences. Given an algebra A and a subalgebra B, we define a map $\pi\colon A_* \to B_* \quad F \mapsto F \cap B$. This map is a quotient map, therefore B_* is a quotient of A_*. Conversely, given a space X and a quotient Y of X (i.e. a bisimulation equivalence E), then Y^* is isomorphic to the subalgebra of X^* consisting of the E-saturated clopen sets of X.

Given an algebra A and a quotient map $\pi\colon A \to B$, we define a map $\iota\colon B_* \to A_* \quad F \mapsto \pi^{-1}(F)$. This map is an embedding, therefore B_* is a generated subspace of A_*. Conversely, given a space X and a generated subspace Y of X, we can define a map $\pi\colon X^* \to Y^* \quad C \mapsto C \cap Y$, which is a quotient map.

That generated subspaces and closed upsets correspond to each other is trivial, by definition of generated subspace. Given a congruence on a modal algebra A, the equivalence class of 1 is a modal filter. Conversely, given a modal filter, the relation E defined by $a\ E\ b$ iff $a \to b \wedge b \to a \in F$ is a congruence. $\Box$

It will also be useful for us to characterise the reflexive and transitive closure of the relation R on A_*. We give a series of lemmas.

Lemma 2.51. *Let A be a modal algebra and A_* its dual space. Then for ultrafilters F, G and $k < \omega$, the following are equivalent.*

(i) $F\ R^k\ G$,

(ii) $\Box^k a \in F$ *implies* $a \in G$ *for all* $a \in A$,

(iii) $a \in G$ *implies* $\Diamond^k a \in F$ *for all* $a \in A$.

Proof. Clearly the $\Box$ condition is equivalent to the $\Diamond$ condition. Furthermore, $F\ R^k\ G$ clearly implies both of these conditions. For the converse, we proceed by induction on n. If $k = 0$, the $\Box$ condition implies $F \subseteq G$, thus $F = G$ since F and G are ultrafilters.

Now assume that the equivalence is true for k and assume that $\Box^{k+1} a \in F$ implies $a \in G$. Consider the filter H generated by the set $\{a \in A : \Box a \in F\} \cup \{\Diamond^k b : b \in G\}$. It is proper. Indeed, if not there would be $a \in A$ such that $\Box a \in F$ and $b \in G$ such that $a \wedge \Diamond^k b = \bot$. But then, $\Box a \wedge \Diamond^{k+1} b = \bot$, and by assumption $\Box a$ and $\Diamond^{k+1} b$ belong to F, contradicting the fact that F is a proper filter. Therefore H is a proper filter and it can be extended to an ultrafilter H'. One easily checks that $F\ R\ H'$ and $H'\ R^k\ G$. $\square$

We give a similar characterisation for the reflexive and transitive closure of R.

Proposition 2.52. *Let A be a modal algebra, A_* its dual space, and R^+ the reflexive and transitive closure of the relation R on A_*. Then for ultrafilters F and G, we have $F\ R^+\ G$ iff there is a compound modality $\boxplus$ such that $\boxplus a \in F$ implies $a \in G$ for all $a \in A$.*

Proof. First assume that $F\ R^+\ G$. Then there is some $k < \omega$ such that $F\ R^k\ G$, and by the previous lemma the compound modality $\Box^k$ does the job.

For the converse, assume that the right hand side of the statement is true for a compound modality $\boxplus p$ which is the conjunction of $\Box^{k_1} p, \ldots, \Box^{k_n} p$. We show that the claim also holds for one of the compound modalities $\Box^{k_1} p, \ldots, \Box^{k_n} p$. Assume not. Then for each $i \in \{1, \ldots, n\}$, there is some $a_i \in A$ such that $\Box^{k_i} a_i \in F$ but $a_i \notin G$. Consider $a = a_1 \vee \cdots \vee a_n$. Then for each $i \in \{1, \ldots, n\}$, we have $\Box^{k_i} a \in F$ since $\Box^{k_i} a \geq \Box^{k_i} a_i$. Therefore $\boxplus a \in F$. However, $a \notin G$ as $a_i \notin G$ for $i \in \{1, \ldots, n\}$ and G is an ultrafilter.

Therefore, the claim holds for some compound modality $\Box^{k_i}$, and by the previous lemma, we have $F\ R^{k_i}\ G$, thus $F\ R^+\ G$. $\square$

Combining this result with the remark that in an n-transitive logic, $\Box^{\leq n}$ implies every compound modality, we get the following.

Corollary 2.53. *Let A be an n-transitive algebra, A_* its dual space, and R^+ the reflexive and transitive closure of the relation R on A_*. Then for ultrafilters F and G, we have $F\ R^+\ G$ iff $\Box^{\leq n} a \in F$ implies $a \in G$ for all $a \in A$.*

Through the Jónsson-Tarksi duality, we can translate the algebraic semantics for modal logic into a topological semantics using modal spaces. Indeed, given a modal space (X, τ, R), we can consider the set of clopen sets, X^*, which induces a general frame[2] (X, R, X^*).

Definition 2.54. Given a modal space (X, τ, R), we define a consequence relation as follows. We have $\Gamma \models_X \phi$ iff for all valuation $V \colon \mathrm{Prop} \to X^*$, if for all $x \in X$, $(X, R, V), x \models \Gamma$, then for all $x \in X$, $(X, R, V), x \models \phi$.

Given a normal modal logic Λ and a modal space X, we say that X is a Λ-*space* if $\Gamma \vdash_\Lambda \phi$ implies $\Gamma \models_X \phi$, or equivalently, if $\models_X \Lambda$.

Given a class $\mathcal{C}$ of space, we define a consequence relation $\models_\mathcal{C}$ by $\Gamma \models_\mathcal{C} \phi$ iff $\Gamma \models_X \phi$ for all $X \in \mathcal{C}$.

[2]General frames are covered in [12, Section 1.4].

Theorem 2.55. *Let Λ be a normal modal logic with K as its equivalent algebraic semantics and let X be a modal space. Then $X^* \in K$ iff X is a Λ-space.*

Corollary 2.56. *Let Λ be a normal modal logic and let S be the class of Λ-spaces. Then Λ is sound and complete with respect to S, in the sense that $\Gamma \vdash_\Lambda \phi$ iff $\Gamma \models_S \phi$.*

2.7 Primitive varieties

By Theorems 2.21 and 2.23, in order to characterise the hereditary structurally complete axiomatic extensions of a modal logic, it is sufficient to characterise the primitive subvarieties of its equivalent axiomatic semantics. We now look at how we can characterise primitive varieties of algebras, and how the duality can help us in this task. Let us first introduce some special elements of a variety.

Definition 2.57. An algebra A is said to be *subdirectly irreducible* (resp. *finitely subdirectly irreducible*) if the identity relation is completely $\wedge$-irreducible (resp. $\wedge$-irreducible) in the congruence lattice of A.

Remark 2.58. Notice that if A is a finite algebra, then A is subdirectly irreducible iff it is finitely subdirectly irreducible, as the congruence lattice of A is finite.

Definition 2.59. An algebra A in a variety K is *weakly projective* in K if for all $B \in K$, if $A \in \mathbb{H}(B)$ then $A \in \mathbb{IS}(B)$.

We can already formulate a straightforward necessary condition for a variety to be primitive.

Lemma 2.60. *Let K be a primitive variety of finite signature. Then the finite subdirectly irreducible members of K are weakly projective in K.*

Proof. See [11, Lemma 2.1]. □

To be able to give a sufficient condition, we need to introduce a special class of varieties.

Definition 2.61. Given an algebra A, we denote by $\mathrm{Cg}^A((c_1, d_1), \dots, (c_n, d_n)$ the smallest congruence containing the pairs $(c_1, d_1), \dots, (c_n, d_n)$.

A variety K is said to have *equationally definable principal congruences* if there is a finite set of equations $\Phi(x, y, z, t)$ such that for any $A \in K$ and any $a, b, c, d \in A$, we have

$$(a, b) \in \mathrm{Cg}^A(c, d) \qquad \text{iff} \qquad A \models \Phi(c, d, a, b).$$

When a variety K has equationally definable principal congruences witnessed by $\Phi(x, y, z, t)$, it is possible to define a finite set of equations $\Phi_n(x_1, y_1, \dots, x_n, y_n, z, t)$ such that for any $A \in K$ and $a, b, c_1, d_1, \dots, c_n, d_n \in A$,

$$(a, b) \in \mathrm{Cg}^A((c_1, d_1), \dots, (c_n, d_n)) \qquad \text{iff} \qquad A \models \Phi_n(c_1, d_1, \dots, c_n, d_n, a, b).$$

Varieties with equationally definable principal congruence have the following properties.

Theorem 2.62. *Let K be a variety with equationally definable principal congruences. Then for any equations $\phi_1 \approx \psi_1, \dots, \phi_n \approx \psi_n$ and $\varepsilon \approx \delta$ and set of equations Θ, we have $\Theta, \phi_1 \approx \psi_1, \dots, \phi_n \approx \psi_n \models_K \varepsilon \approx \delta$ iff $\Theta \models_K \Phi_n(\phi_1, \psi_1, \dots, \phi_n, \psi_n, \varepsilon, \delta)$.*

Proof. See [13, Thm 5.4]. □

Theorem 2.63 (Blok & Pigozzi)**.** *Let $\vdash$ be a deductive system with the variety K as its equivalent algebraic semantics. Then $\vdash$ has a deduction theorem iff K has equationally definable principal congruences.*

Proof. See [13, Thm 5.5]. $\square$

Together with Theorem 2.32, this tells us that the varieties of weakly transitive algebras all have equationally definable principal congruences.

Theorem 2.64. *Let $\vdash$ be a deductive system with K as its equivalent algebraic semantics. Then $\vdash$ is structurally complete iff $K = \mathbb{Q}(F_K(\omega))$, where $F_K(\omega)$ is the free countably generated algebra of K.*

Proof. This is proved in [39, Thm 64]. $\square$

This results can also be expressed in purely algebraic terms.

Proposition 2.65. *A variety K is primitive iff for all subvariety L of K, we have $L = \mathbb{Q}(F_L(\omega))$.*

Proof. First assume that K is primitive, and let L be a subvariety of K. Since $F_L(\omega) \in L$, it is clear that $\mathbb{Q}(F_L(\omega)) \subseteq L$. Since K is primitive, $\mathbb{Q}(F_L(\omega))$ is a variety. But by definition of $F_L(\omega)$, we have $L \subseteq \mathbb{V}(F_L(\omega))$, thus $L \subseteq \mathbb{V}\mathbb{Q}(F_L(\omega)) = \mathbb{Q}(F_L(\omega))$.

For the converse, let M be a subquasi-variety of K with free algebra $F_M(\omega)$, and let $L = \mathbb{V}(M)$ be the variety generated by M. We want to prove that $F_M(\omega) = F_L(\omega)$. The construction of the free algebra $F_M(\omega)$ (resp. $F_L(\omega)$) is carried out as follows. We take the term algebra F and identify terms that are equivalent when interpreted in M (resp. L). Therefore, to prove $F_M(\omega) = F_L(\omega)$, it suffices to show that an equation holds in M iff it holds in L. As L contains M, it suffices to show that if an equation holds in M, then it holds in L. But this follows from the fact that L is the variety generated by M. By our assumption, we get $L = \mathbb{Q}(F_L(\omega)) = \mathbb{Q}(F_M(\omega))$. But $\mathbb{Q}(F_M(\omega)) \subseteq M$, as $F_M(\omega) \in M$. This implies that $M = L$, thus M is a variety. $\square$

Another property of varieties that is important for our sufficient condition is that of finite model property.

Definition 2.66. A variety K has the *finite model property* iff for any equation $\varepsilon \approx \delta$ such that $A \not\models \varepsilon \approx \delta$ for some $A \in K$, there is a finite algebra $A \in K$ such that $A \not\models \varepsilon \approx \delta$. In other words, K has the finite model property if $K = \mathbb{V}(K_{\mathrm{Fin}})$, where K_{Fin} is the class of finite members of K.

Lemma 2.67. *Let K be a variety with the finite model property and equationally definable principal congruences, and let K_{FinSI} be the class of finite subdirectly irreducible members of K. Then $K = \mathbb{Q}(K_{\mathrm{FinSI}})$.*

Proof. Clearly $\mathbb{Q}(K_{\mathrm{FinSI}}) \subseteq K$. To show the reverse inclusion, since quasi-varieties are axiomatised by quasi-equations, it is sufficient to show that any quasi-equation that fails in K also fails in K_{FinSI}.

Take a quasi-equation $\bigwedge_{i=1}^{n} \phi_i \approx \psi_i \to \varepsilon \approx \delta$ that fails in K. Then $\phi_1 \approx \psi_1, \dots, \phi_n \approx \psi_n \not\models_K \varepsilon \approx \delta$. By Theorem 2.62, $\not\models_K \Phi_n(\phi_1, \psi_1, \dots, \phi_n, \psi_n, \varepsilon, \delta)$. Since K has the finite model property, there is some finite $A \in K$ such that $A \not\models \Phi_n(\phi_1, \psi_1, \dots, \phi_n, \psi_n, \varepsilon, \delta)$, therefore there is an assignment $h\colon Fm \to A$ such $\Phi_n(h(\phi_1), h(\psi_1), \dots, h(\phi_n), h(\psi_n), h(\varepsilon), h(\delta))$ does not holds in A. By definition of Φ_n, this implies that $(h(\varepsilon), h(\delta)) \notin \mathrm{Cg}^{A}((h(\phi_1), h(\psi_1)), \dots, (h(\phi_n), h(\psi_n))) =$

E. Therefore, under the assignment $\pi \circ h$, where $\pi \colon A \to A/E$ is the quotient map associated to E, the quasi-equation $\wedge_{i=1}^{n} \phi_i \approx \psi_i \to \varepsilon \approx \delta$ fails in the quotient algebra A/E.

The algebra A/E can be written as a subdirect product of subdirectly irreducible elements[3] $B_1, \dots, B_m$, and the quasi-equation has to fails in one of them, say B_j. Since B_j is a quotient of B, which itself is a quotient of A, and A is finite, so is B_j. Furthermore, $B_j \in K$ since K is closed under $\mathbb{H}$. Therefore, $B_j \in K_{\mathrm{FinSI}}$ and $\bigwedge_{i=1}^{n} p\phi_i \approx \psi_i \to \varepsilon \approx \delta$ fails in K_{FinSI}. $\square$

Theorem 2.68. *Let K be a variety with equationally definable principal congruences, and assume that all its subvarieties have the finite model property. If the finite subdirectly irreducible members of K are weakly projective in K, then K is primitive.*

Proof. By Theorems 2.21, 2.23 and 2.65, it is sufficient to prove that for any subvariety L of K, we have $L = \mathbb{Q}(F_L(\omega))$. Let L be a subvariety of K. The right to left inclusion is trivial. For the converse, notice that L has the finite model property by assumption and equationally definable principal congruences because this property is hereditary, hence $L = \mathbb{Q}(L_{\mathrm{FinSI}})$. We only need to prove $\mathbb{Q}(L_{\mathrm{FinSI}}) \subseteq \mathbb{Q}(F_L(\omega))$, which is equivalent to $L_{\mathrm{FinSI}} \subseteq \mathbb{Q}(F_L(\omega))$ since $\mathbb{Q}$ is a closure operator.

So let $A \in L_{\mathrm{FinSI}}$. Since A is finite, it is countably generated hence $A \in \mathbb{H}(F_L(\omega))$. As A is also subdirectly irreducible, by assumption it is weakly projective and $A \in \mathbb{IS}(F_L(\omega))$. Therefore $A \in \mathbb{Q}(F_L(\omega))$. $\square$

Recalling that varieties of weakly transitive algebras always have equationally definable principal congruences, we get the following.

Corollary 2.69. *Let K be a variety of n-transitive modal algebras such that all its subvarieties have the finite model property. If the finite subdirectly irreducible members of K are weakly projective in K, then K is primitive.*

Finally, while characterising the dual of a weakly projective algebra is rather easy, since weak projectivity is a category theoretical property, characterising the dual of subdirectly irreducible and finitely subdirectly irreducible algebras requires a bit more work.

Lemma 2.70. *(i) An n-transitive algebra A is subdirectly irreducible iff the set of roots is a nonempty open set iff its interior is nonempty.*

(ii) An n-transitive algebra A is finitely subdirectly irreducible iff A_* is rooted.

Proof. (i) This is established in [44].

(ii) First assume that A_* is rooted. By Corollary 2.50, we only need to show that A_* is $\vee$-irreducible in the lattice of closed upset of A_*. Let $C_1, \dots, C_k$ be closed upsets such that $A_* = C_1 \cup \dots \cup C_k$. Let r be a root of A_*. There must be some $i \in \{1, \dots, k\}$ such that $r \in C_i$. Since r is a root and C_i is an upset, we must have $C_i = A_*$, thus showing that A_* is $\vee$-irreducible.

For the converse, assume that A is finitely subdirectly irreducible. By Corollary 2.50, this means that the modal filter $\{\top\}$ is $\wedge$-irreducible. Consider the ideal I generated by

$$\{\Box^{\leq n} a_1 \vee \dots \vee \Box^{\leq n} a_k : a_1, \dots, a_k \in A \setminus \{\top\}\}.$$

We prove that this is a proper ideal. Assume not, then there are $a_1, \dots, a_k \in A \setminus \{\top\}$ such that $\Box^{\leq n} a_1 \vee \dots \vee \Box^{\leq n} a_k = \top$. Consider the modal filters generated by $a_1, \dots, a_k$,

[3] See [15, Section II.8] for more details on subdirect products and subdirect decompositions.

respectively. That is, we consider the modal filters $\uparrow\Box^{\leq n}a_1, \dots, \uparrow\Box^{\leq n}a_k$. They are all proper, since they contain $a_i \neq \top$, but their intersection is $\{\top\}$. Thus the modal filter $\{\top\}$ is not $\wedge$-irreducible, and by Corollary 2.50, the identity relation on A is not $\wedge$-irreducible. This contradicts the assumption that A is finitely subdirectly irreducible.

Therefore I is proper, and we can find an ultrafilter F such that $F \cap I = \emptyset$. We have $\Box^{\leq n}a \notin F$ for any $a \in A \setminus \{\top\}$, hence by Corollary 2.53, F is a root.

□

This concludes the preliminaries. We have introduced three different perspective on our main problem: the more standard logical perspective, the algebraic perspective and the more practical topological perspective. This is possible thanks to two connections between these perspectives: the theory of algebraisable logics provides a bridge between logical properties of logics and algebraic perspectives of their equivalent algebraic semantics, and the theory of topological dualities provides a categorical duality between varieties of algebras and classes of topological spaces with a relation. Using different formulations of our initial problem, we have obtained several useful tools for the rest of this thesis, including Theorems 2.21 and 2.23, as well as Lemma 2.60. We also introduced weakly transitive logics, together with their algebraic and topological equivalent. By specifying some results to weakly transitive logics, we obtained some extra tools, such as Corollary 2.69 and Lemma 2.70.

Chapter 3

Universal models of wK4

In this chapter, we take a closed look at the wK4-spaces. We extensively study the ones dual to finitely generated algebras. Finally, we define the universal models for wK4 and prove some of their properties. One of the first investigation of n-generated models is [33]. n-universal models were defined and investigated in [43, 2, 4, 28]. Most of this section is an adaptation of Sections 3.1 and 3.2 of [10]. Some results can also be found (in a slightly different format) in [17], where universal models for transitive logics are treated.

3.1 Finitely generated modal spaces

It is quite common to have a notion of 'finitely generated' for algebraic structures. Using duality, this notion can be translated to modal spaces.

Definition 3.1. Let A be a modal algebra and let $G \subseteq A$. We say that G *generates* A if the smallest subalgebra of A containing G is A itself. The elements of G are called the *generators* of A. We say that A is *finitely generated* if it has a finite set of generators. We say that A is *n-generated* if it has a set of generators of size at most n.

Definition 3.2. Let A be a modal algebra and X its dual modal space. The space X is said to be *finitely generated* (resp. *n-generated*) if A is.

Our aim for the rest of this section is to characterise the finitely generated modal spaces. To this end, we introduce the notion of a *colour*.

For each $n < \omega$, let Prop_n denote the set $\{p_1, \dots, p_n\}$ of propositional variables, and Fm_n the set of modal formulas over Prop_n. Let A be a modal algebra and X be its dual space. Fix $g_1, \dots, g_n \in A$. We can think of A together with these fixed elements as a modal algebra with a valuation $v \colon \mathrm{Prop}_n \to A$ defined by $v(p_i) = g_i$ for $i \in \{1, \dots, n\}$. This valuation $v \colon \mathrm{Prop}_n \to A$ corresponds to a valuation $V \colon \mathrm{Prop}_n \to X^*$. Let $\mathbb{M} = (X, V)$ be the model corresponding to (A, v).

Definition 3.3. For every point w of $\mathbb{M}$, we define a sequence $c_1 \dots c_n$ by

$$c_i = \begin{cases} 1 & \text{if } w \models p_i, \\ 0 & \text{if } w \not\models p_i \end{cases}$$

for $i \in \{1, \dots, n\}$. We call this sequence the *colour* of w and denote it by $\mathrm{col}(w)$.

We now have the necessary tools to characterise the finitely generated spaces. The following proof is an adaptation of [10, Thm 3.1.5]. It was first proved by Esakia and Grigolia in [22].

Theorem 3.4 (Colouring Theorem)**.** *Let A be a modal algebra, $g_1, \ldots, g_n$ be fixed elements of A, and (X, V) be the corresponding model. Then the following are equivalent:*

(i) The algebra A is generated by $g_1, \ldots, g_n$.

(ii) Every proper continuous onto p-morphism with domain X identifies points of different colours.

(iii) Every proper bisimulation equivalence on X identifies points of different colours.

Proof. (ii) $\Leftrightarrow$ (iii) follows from Remark 2.42. We show (i) $\Leftrightarrow$ (iii). Assume that A is generated by $g_1, \ldots, g_n$ and that E is a proper bisimulation equivalence. Let X_E be the quotient space of X by E, and let A_E be the corresponding modal algebra, i.e. the algebra of all saturated clopens. Because E is proper, A_E is a proper subalgebra of E. As A is generated by $g_1, \ldots, g_n$, there is some i such that $g_i \notin A_E$, i.e. $V(p_i) \notin A_E$. In other words, $V(p_i)$ is not saturated, hence there are points u, v such that $u\ E\ v$, $u \in V(p_i)$ and $v \notin V(p_i)$, and thus $\mathrm{col}(u) \neq \mathrm{col}(v)$.

Conversely, assume that A is not generated by $g_1, \ldots, g_n$. Let B be the proper subalgebra of A generated by $g_1, \ldots, g_n$, and let E_B be the corresponding proper bisimulation equivalence on X. Since $g_i \in B$ for all $i \in \{1, \ldots, n\}$, every $V(p_i)$ is E_B-saturated, hence equivalence classes only contain points of the same colour. □

3.2 Clusters, skeleton and depth

From now on we restrict to 1-transitive spaces. We first define some notions of clusters for 1-transitive spaces, then define the frame of clusters (i.e. the skeleton) and introduce the notion of depth. All these concepts will be useful later.

Clusters are defined as equivalence classes of a certain equivalence relation.

Definition 3.5. Given a 1-transitive space X, consider the relation $\equiv$ on X defined by $x\ E\ y$ iff $x = y$ or $x\ R\ y$ and $y\ R\ x$. This is an equivalence relation. An equivalence class of $\equiv$ is called a *cluster*. A cluster is called *proper* if it contains more than one element, and *improper* otherwise.

Let C and D be clusters. We say that C *sees* D, denoted $C\ \tilde{R}\ D$, if there are $x \in C$ and $y \in D$ such that $x\ R\ y$. When C and D are distinct, C sees D iff for all $x \in C$ and $y \in D$, we have $x\ R\ y$. As $\tilde{R}$ is transitive and antisymmetric, the reflexive and transitive closure $\tilde{R}^+$ is a partial order on the set of clusters.

Proposition 3.6. *The relation $\equiv$ is a bisimulation equivalence.*

Proof. Clearly $\equiv$ is an equivalence relation.

First assume that $x\ E\ y$ and $x\ R\ z$. If $x = y$, take $t = z$, then $y\ R\ t$ and $z\ E\ t$. If $x \neq y$, then $x\ R\ y$ and $y\ R\ x$. If $z = y$, take $t = x$, then $y\ R\ t$ and $z\ E\ t$. If $z \neq y$, then we have $y\ R\ z$ (by weak transitivity if $x \neq z$, and trivially if $x = z$), thus we can take $t = z$.

Now assume that $x \not\!E\ y$. Without loss of generality we may assume $x \neq y$ and $x \not\!R\ y$. By Proposition 2.39, there are clopen sets U, V with $x \in U$, $y \in V$ and $R[U] \cap V = \emptyset$, or equivalently, $U \cap R^{-1}[V] = \emptyset$. Because $x \neq y$, we can also find disjoint clopen sets U', V' such that $x \in U'$, $y \in V'$. Setting $U'' = U \cap U'$ and $V'' = V \cap V'$, we have $x \in U''$ and $y \in V''$ and $U'' \cap (V'' \cup R^{-1}[V'']) = \emptyset$. It follows that $W = V'' \cup R^{-1}[V'']$ is a clopen set separating x and y. By weak transitivity, W is a downset hence it is saturated, since $E \subseteq R^{-1} \cup \Delta_X$. □

Since $\equiv$ is a bisimulation equivalence, we can look at the space of clusters $X/\equiv$.

Definition 3.7. The quotient $X/\equiv$ is called the *skeleton* (or *cluster space*) of X and is denoted $\mathrm{sk}(X)$. It is based on the Kripke frame whose states are clusters and whose relation is the $\tilde{R}$ relation.

The quotient map $\kappa\colon X \to \mathrm{sk}(X)$ is a continuous onto p-morphism.

We will now prove that the skeleton of a space always has maximal and minimal elements. We first need to prove a lemma.

Lemma 3.8. *Let X be a 1-transitive space. Clusters are closed sets. If C is a closed set, then $R[C]$ and $R^{-1}[C]$ are closed. Therefore, $R^+[C]$ and $(R^+)^{-1}[C]$ are closed for any cluster C.*

Proof. Clusters are the inverse image through κ of singletons, which are closed.

If C is a cluster, then $R[C] = \pi_2[(C \times X) \cap R]$ and $R^{-1}[C] = \pi_1[(X \times C) \cap R]$. We know that $C \times X$, $X \times C$ and R are closed in $X \times X$, and π_1, π_2 are continuous maps between compact Hausdorff spaces, hence they are closed maps.

If C is a cluster, we have $R^+[C] = C \cup R[C]$ and $(R^+)^{-1}[C] = R^{-1}[C] \cup C$. Thus those two sets are closed as they can be written as a finite union of closed sets. □

We can prove that there are maximal clusters, in a similar fashion to [10, Thm 2.3.24]

Theorem 3.9. *Let X be a 1-transitive space. Then for every cluster C in X, there is a maximal cluster D such that $C\ \tilde{R}^+\ D$. Similarly, for every cluster C in X, there is a minimal cluster D such that $D\ \tilde{R}^+\ C$.*

Proof. We only prove the first half of the claim, the proof of the second half is analogous. Let A be an arbitrary $\tilde{R}^+$ chain of $\mathrm{sk}(X)$ and consider the set $\mathcal{F} = \{\tilde{R}^+[D] : D \in A\}$. Because A is a chain, $\mathcal{F}$ has the finite intersection property and by the previous proposition, its elements are closed. By compactness, $\bigcap \mathcal{F}$ is nonempty, and any cluster $E \in \bigcap \mathcal{F}$ is an upper bound for A. Therefore, by Zorn's lemma, every cluster is below a maximal cluster. □

Remark 3.10. The previous result can be formulated in a slightly more general setting. Indeed, let C be a closed subset of a skeleton X. Then a similar reasoning shows the existence of maximal points in C.

Theorem 3.9 motivates a notion of *depth* of a cluster.

Definition 3.11. Let X be a 1-transitive space. A cluster C has *depth* $m < \omega$, denoted $\mathrm{d}(C) = m$, if there is a chain in $\mathrm{sk}(X)$ of length m starting at C, and no chain of greater length starting at C. A cluster that does not have depth m for any $m < \omega$ has *infinite depth*, denote $\mathrm{d}(C) = \omega$.

The *depth* of a point $x \in X$ is defined as the depth of its cluster, and is denoted $\mathrm{d}(x)$. We let $D_m = \{x \in X : \mathrm{d}(x) = m\}$ and $D_{\leq m} = \{x \in X : \mathrm{d}(x) \leq m\}$.

In certain cases, identifying points of the same depth is a bisimulation equivalence.

Lemma 3.12. *Let X be a 1-transitive space such that for all $m < \omega$, D_m is clopen and consists either only of reflexive clusters or only of irreflexive clusters. Then the equivalence relation defined by $x\ E\ y$ iff $\mathrm{d}(x) = \mathrm{d}(y)$ is a bisimulation equivalence.*

Proof. The requirement that the sets D_m are clopen takes care of the second condition for a bisimulation equivalence. For the first one, observe that for any $x \in X$ such that $\mathrm{d}(x) > 0$ and for each $m < \mathrm{d}(x)$, there is $y \in D_m$ such that $x\ R\ y$, otherwise x would have depth less than or equal to m. To prove the first condition for a bisimulation equivalence, consider x, y, z such that $x\ E\ y$ and $x\ R\ z$.

If $\mathrm{d}(z) < \mathrm{d}(x)$, we can use the observation we just made to find $t \in X$ such that $\mathrm{d}(z) = \mathrm{d}(t)$ and $y \mathrel{R} t$.

If $\mathrm{d}(z) = \mathrm{d}(x) < \omega$, then $D_{\mathrm{d}(x)}$ contains a reflexive cluster, hence every cluster in $D_{\mathrm{d}(x)}$ is reflexive and there is $t \in D_{\mathrm{d}(x)}$ such that $y \mathrel{R} t$.

If $\mathrm{d}(z) = \mathrm{d}(x) = \omega$, we need to find $t \in D_\omega$ such that $y \mathrel{R} t$. Assume that $R[y] \cap D_\omega = \emptyset$. The set $R[y]$ is closed, hence compact, and the sets $R[y] \cap D_m$, for $m < \omega$, give an open cover of $R[y]$. By compactness, we have $R[y] \subseteq \bigcup_{m \leq k} D_m$ for some $k < \omega$, contradicting that fact that $y \in D_\omega$. □

3.3 Reductions

To use the full potential of the Colouring Theorem, we should be able to easily determine whether or not there is a proper continuous onto p-morphism identifying points of the same colour only. That is the aim of this section and the next.

In this section, we characterise the continuous onto p-morphisms. We first identify the basic equivalences on finite spaces, the so called *reductions*, then investigate how they behave with other equivalences. This section is a generalisation and simplification of Section 2.3.1 of [16]. Reductions were first defined for IPC by de Jongh and Troelstra in [32].

Lemma 3.13. *Let X be a 1-transitive space.*

(i) Let $C = \{c_1, \dots, c_n\}$ and $D = \{d_1, \dots, d_m\}$ be distinct clusters of X with $m \leq n$ such that

(a) D sees C,

(b) c_i is reflexive, for all $i \in \{1, \dots, m\}$,

(c) for all $x \notin C \cup D$, we have $x \in R[C]$ iff $x \in R[D]$.

*We define the relation $E = \Delta_X \cup \{(c_i, d_i), (d_i, c_i) : 1 \leq i \leq m\}$. The relation E is a bisimulation equivalence and the quotient map $\pi_E \colon X \to X/E$ is an α-*reduction*.*

(ii) Let $C = \{c_1, \dots, c_n\}$ and $D = \{d_1, \dots, d_n\}$ be distinct clusters of X such that

(a) C and D do not see each other,

(b) c_i is reflexive iff d_i is reflexive, for all $i \in \{1, \dots, n\}$,

(c) for all $x \notin C \cup D$, we have $x \in R[C]$ iff $x \in R[D]$.

*We define the relation $E = \Delta_X \cup \{(c_i, d_i), (d_i, c_i) : 1 \leq i \leq n\}$. It is a bisimulation equivalence and the quotient map $\pi_E \colon X \to X/E$ is a β-*reduction*.*

*(iii) Let $x, y \in X$ be distinct elements of the same cluster. We define the relation $E = \Delta_X \cup \{(x, y), (y, x)\}$. It is a bisimulation equivalence and the quotient map $\pi_E \colon X \to X/E$ is a γ-*reduction*.*

Proof. In all three cases, the first condition of a bisimulation equivalence is guaranteed by the conditions imposed. The second condition is trivial because X is finite, hence discrete. □

Remark 3.14. Notice that in the category of finite $\mathsf{wK4}$-spaces, reductions are regular epimorphisms.

Reductions can also be characterised as those proper epimorphisms f such that whenever $f = gh$ where g and h are epimorphisms, either g or h is an isomorphism. Given the next lemma, which asserts that every epimorphism factors as a sequence of reductions, we can view reductions as the prime elements of epimorphisms.

The continuous onto p-morphisms can be characterised in term of reductions.

Lemma 3.15. *Let X and Y be finite 1-transitive spaces. Any onto p-morphism $f\colon X \to Y$ factors as a sequence of α-, β- and γ-reductions.*

Proof. We proceed by induction on the number of points that are identified by f, which we call the index of f. If the index is 0, then f is an isomorphism and there is nothing to do. Otherwise, f identifies at least two points. We prove that there is a reduction $\pi_E\colon X \to X/E$ with $E \subseteq \ker f$. The map f then factors through π_E as $f = f_E \circ \pi_E$. The map $f_E\colon X/E \to Y$ is onto, and its index is less than the index of f, hence we can apply the induction hypothesis.

So let us show that given a proper onto morphism $f\colon X \to Y$, we can find a reduction $\pi_E\colon X \to X/E$ such that $E \subseteq \ker f$. We work through a series of cases. If f identifies two points u, v from the the same cluster, we can consider the γ-reduction $\pi_E\colon X \to X/E$ defined from u and v.

Otherwise, let m be a maximal[1] point which in mapped onto by distinct points and let $B = \{b_1, \ldots, b_n\}$ be its cluster. We consider the maximal clusters[2] of X intersecting $f^{-1}(B)$. Let C be such a cluster.

The function f restricts to a bijection from C to B. Because f does not identify points from the same cluster, it is injective on C. We show that it is also surjective on B. Let $x \in C$ such that $f(x) \in B$. For any $y \in B$ such that $y \neq f(x)$, we have $f(x)\ R\ y$. Since f is a p-morphism, there is $x \in X$ such that $x_i\ R\ x$ and $f(x) = y$. By maximality of C, we have $x \in C$. We can thus enumerate C as $\{c_1, \ldots, c_n\}$ such that $f(c_i) = b_i$ for all $i \in \{1, \ldots, n\}$.

For any $i \in \{1, \ldots, n\}$, c_i is reflexive iff b_i is (and similarly for d_i). Left to right is obvious. For right to left, assume that b_i is reflexive. Then there is $x \in X$ such that $c_i\ R\ x$ and $f(x) = b_i$. By maximality of C, we have $x \in C$ and $x = c_i$, hence c_i is reflexive.

Assume that there are at least two maximal clusters $C = \{c_1, \ldots, c_n\}$ and $D = \{d_1, \ldots, d_n\}$. We consider the β-reduction defined from C and D. Obviously neither C nor D sees the other. The second requirement is guaranteed since c_i is reflexive iff b_i is reflexive iff d_i is reflexive.

For any $x \notin C \cup D$, we show that $x \in R[C]$ implies $x \in R[D]$, the converse is done similarly. Since $x \notin C$, we have $c_1\ R\ x$, hence $b_1\ R\ f(x)$. Notice that $f(x) \notin B$ since $x \notin C$ and C is maximal. Because f is a p-morphism, there is $z \in X$ such that $d_1\ R\ z$ and $f(z) = f(x)$. By maximality of m, we must have $x = z$, hence $x \in R[D]$.

Finally, assume that there is a single maximal cluster $C = \{c_1, \ldots, c_n\}$. By definition of B, there are at least two clusters intersecting $f^{-1}(B)$. We can thus pick a cluster D which is maximal among the clusters intersecting $f^{-1}(B)$ and distinct from C. We consider the α-reduction defined from C and D, and show that the requirements are met.

We know that f is injective, hence $|D| \leq |C|$. Up to a renumbering of B and C, we may assume $D = \{d_1, \ldots, d_m\}$ and $f(d_i) = b_i$ for $i \in \{1, \ldots, m\}$. Obviously D sees C, since C is the unique maximal cluster intersecting $f^{-1}(B)$. For any $i \in \{1, \ldots, m\}$, we have $d_i\ R\ c_i$, hence b_i is reflexive, which implies c_i reflexive.

For any $x \notin C \cup D$, we show that $x \in R[D]$ iff $x \in R[C]$. Right to left is obvious since D sees C. For the converse, assume $x \in R[D]$. Since $x \notin D$, we have $d_1\ R\ x$ hence $b_1\ R\ f(x)$. Because f is a p-morphism, there is $z \in X$ such that $c_1\ R\ z$ and $f(z) = f(x)$. If we show that $z = x$, we are done. So assume that $z \neq x$. Then $f(x) \in B$ by maximality of m. By maximality of C and D, we then have $z \in C$ and $x \in C \cup D$, which is a contradiction. ☐

[1] In the sense that for any point $y \in R[x]$ which is mapped onto by distinct points, we have $y\ R\ m$.

[2] There is at least one.

3.4 Cluster patterns

The previous section allows us to simplify the Colouring Theorem. Indeed, the existence of a proper continuous onto p-morphism identifying points of the same colour only is equivalent to the existence of a reduction identifying points of the same colour only. If a finite space is n-generated, then there are no reductions identifying points of the same colour only, which implies that clusters can only have finitely many patterns.

This motivates us to formally define what a *pattern* is. Afterwards, we will define formulas associated with patterns, which will be useful in proving our next theorem.

Definition 3.16. A *cluster pattern* is a pair (P, r) where P is a (nonempty) set of colours and r is an assignment from P to $\mathbf{2} = \{0, 1\}$. We let $\mathcal{P}_n$ denote the set of all cluster patterns.

Given a cluster C in an n-generated frame, we define its *pattern* $\mathrm{pat}(C) = (P, r)$ where $P = \mathrm{col}[C]$ is the set of colours occurring in C, and $r\colon P \to \mathbf{2}$ is defined by

$$r(\mathrm{col}(x)) = \begin{cases} 1 & \text{if } x \text{ is reflexive,} \\ 0 & \text{if } x \text{ is irreflexive.} \end{cases}$$

Because C belongs to an n-generated frame, it does not contain two states with the same colour, thus r is well defined.

We define a (transitive and antisymmetric) relation $\sqsubseteq$ on patterns by $(P, r) \sqsubseteq (Q, s)$ iff $P \subseteq s^{-1}(1)$.

There are exactly 2^n colours. Given $k \in \{1, \dots, 2^n\}$, there are $\binom{2^n}{k} 2^k$ patterns[3] of size k. Hence the total number of patterns is

$$\sum_{k=1}^{2^n} \binom{2^n}{k} 2^k = 3^{2^n} - 1.$$

Remark 3.17. We can pair this notion with Lemma 3.13. Let X be a finite n-generated space. Then if C and D are clusters such that C is the immediate successor of D (in other words, D sees C and C sees everything else that D sees), we have $(P_D, r_D) \not\sqsubseteq (P_C, r_c)$, otherwise we would have an α-reduction identifying points of the same colour only. Similarly, if C and D are clusters such that neither sees the other but they see the same things, then $(P_C, r_c) \neq (P_D, r_D)$, otherwise we would have a β-reduction identifying points of the same colour only.

Clearly, every colour induces a propositional formula. We show that every pattern induces a modal formula. These formulas will be useful in the next section, when we prove that $\max(X)$ is closed for any finitely generated space X.

Definition 3.18. Let c be a colour. We define the *colour formula* $\widehat{c} = \bigwedge_{c_i=1} p_i \wedge \bigwedge_{c_i=0} \neg p_i$. We also define the *extended colour formulas* $c^0 = \widehat{c} \wedge \neg\Diamond\widehat{c}$ and $c^1 = \widehat{c} \wedge \Diamond\widehat{c}$.

Given a cluster pattern (P, r), and a colour $c \in P$, we define the *pattern formula*

$$(P, r)^c = c^{r(c)} \wedge \bigwedge_{d \in P \setminus \{c\}} \Diamond d^{r(d)} \wedge \Box \bigvee_{d \in P} \left(d^{r(d)} \wedge \bigwedge_{e \in P \setminus \{d\}} \Diamond e^{r(e)} \right).$$

Remark 3.19. These formulas are similar to the so-called Fine normal normal forms for modal logic [24]. They can be seen as modal variants of de Jongh formulas for intuitionistic logic [33] (see also [10, Section 3.3]). For various applications these formulas in modal logic, see [36, 26].

[3] We first have to select k colours among the 2^n that are available, then for each of them, choose whether or not it is reflexive.

It is not hard to see that any point x of an n-generated frame satisfies exactly one colour formula (the one associated with $\mathrm{col}(x) = c$) and one extended colour formula (c^0 if x if irreflexive, c^1 if x is reflexive). We will show that x also satisfies at most one pattern formula.

Lemma 3.20. *The pattern formulas are mutually exclusive. More precisely, if X is an n-generated frame, $x \in X$ and $x \models (P,r)^c \wedge (Q,s)^d$, then $c = d$, $P = Q$ and $r = s$.*

Proof. Clearly we have $c = d = \mathrm{col}(x)$. We show that $P \subseteq Q$ and $s{\restriction}P = r$. The converse is done similarly. Clearly $c \in Q$ and $r(c) = s(d)$, for otherwise $c^{r(c)}$ and $d^{s(d)}$ would be mutually exclusive. For $e \in P \setminus \{c\}$, we have $x \models \Diamond e^{r(e)}$, hence there is a successor y of x such that $y \models e^{r(e)}$. We also have

$$y \models \bigvee_{f \in Q} f^{s(f)}.$$

Because the extended colour formulas are mutually exclusive, we have $e \in Q$ and $s(e) = r(e)$. □

3.5 Structure of n-generated spaces

We will now investigate the structure of finitely generated spaces. We show that any finitely generated space X can be decomposed into ω-many *layers*, together with a *lower part* of ω-deep points. We also prove some topological properties of this structure.

We let $\mathrm{Up}(X) = \{x \in X : \mathrm{d}(x) < \omega\}$ and $\mathrm{Low}(X) = \{x \in W : \mathrm{d}(x) = \omega\}$. Clearly those two sets partition X. By Theorem 3.9, $\mathrm{Up}(X) \neq \emptyset$. We will later show that if X is infinite, then $\mathrm{Low}(X) \neq \emptyset$. Clearly the sets D_m partition $\mathrm{Up}(X)$.

We will show that the sets D_m are finite clopen sets. We first prove the following lemma, whose proof draws inspiration from [10, Thm 3.1.8]

Lemma 3.21. *Let X be an n-generated space. Then $\max(X)$ is a finite clopen set containing at most $3^{2^n} - 1$ clusters.*

Proof. Let A be the dual of X, $g_1, \ldots, g_n \in A$ be its generators and $v\colon \mathrm{Prop}_n \to A$ a valuation such that $v(p_i) = g_i$ for $i \in \{1, \ldots, n\}$. This defines a colouring of X.

A maximal cluster C in X has size at most 2^n, otherwise it contains two points of the same colour and we can consider the γ-reduction identifying those two points. Then by the Colouring Theorem, X is not n-generated, which is a contradiction. Moreover, different maximal clusters in X have different patterns, otherwise we can consider the β-reduction identifying them, once again contradicting the fact that X is n-generated. Thus, there are at most $3^{2^n} - 1$ clusters in $\max(X)$, and at most $3^{2^n} - 1$ points in $\max(\mathrm{sk}(X))$.

Now consider the formula

$$\mu = \bigvee_{(P,r) \in \mathcal{P}_n, c \in P} (P,r)^c.$$

We prove that $V(\mu) = \max(X)$. One easily checks that if $x \in \max(X)$ and C is the cluster containing x, then $x \models (P,r)^c$, where $c = \mathrm{col}(x)$ and $(P,r) = \mathrm{pat}(C)$.

Conversely, assume that there is a point $x \in X$ such that $x \models \mu$. Then there is a pattern formula $(P,r)^c$ that x satisfies. Consider the formulas $\varepsilon_d = (P,r)^c$ for $d \in P$, and the sets $V(\varepsilon_d) = \{y \in X : y \models \varepsilon_d\}$. Clearly $x \in V(\varepsilon_c)$. Consider the equivalence relation

$$E = \Delta_X \cup \bigcup_{d \in P} V(\varepsilon_d) \times V(\varepsilon_d).$$

Clearly it only identifies points of the same colour.

Claim 3.22. *The equivalence relation E is a bisimulation equivalence.*

Proof. The second condition is relatively easy. Assume that $y \not E z$. If $y \in V(\varepsilon_d)$ for some $d \in P$, then $z \notin V(\varepsilon_d)$, hence $V(\varepsilon_d)$ is a saturated clopen set separating y and z. If $z \in V(\varepsilon_d)$ for some $d \in P$, we proceed similarly. Otherwise, we have $y, z \notin V(\mu)$ and $y \neq z$. We can find a clopen set U separating y and z, and the set $U \cup V(\mu)$ is a saturated clopen set separating y and z.

For the first condition, assume that $y \, R \, z$ and $y \, R \, u$. We want to show that there is $v \in X$ such that $u \, E \, v$ and $z \, R \, v$.

First assume that $u \models (P, r)^c$. Then $r(c) = 1$ hence $z \models c^1$. Thus there exists $v \in X$ such that $z \, R \, v$ and $v \models \widehat{c}$. Since

$$z \models \Box \bigvee_{d \in P} \left(d^{r(d)} \wedge \bigwedge_{e \in P \setminus \{d\}} \Diamond e^{r(e)} \right),$$

we have

$$v \models \Box^+ \bigvee_{d \in P} \left(d^{r(d)} \wedge \bigwedge_{e \in P \setminus \{d\}} \Diamond e^{r(e)} \right)$$

by weak transitivity. Because $v \models \widehat{c}$, we must have

$$v \models c^{r(c)} \wedge \bigwedge_{d \in P \setminus \{c\}} \Diamond d^{r(d)}.$$

Hence $v \models (P, r)^c$, i.e. $v \in V(\varepsilon_c)$ and $u \, E \, v$.

Otherwise, by weak transitivity

$$u \models \bigvee_{d \in P} (P, r)^d,$$

hence there is $d \in P \setminus \{c\}$ such that $u \models (P, r)^d$. We have $z \models \Diamond d^{r(d)}$ hence there is $v \in X$ such that $z \, E \, v$ and $v \models d^{r(d)}$. By weak transitivity,

$$v \models \Box^+ \bigvee_{d \in P} \left(d^{r(d)} \wedge \bigwedge_{e \in P \setminus \{d\}} \Diamond e^{r(e)} \right).$$

Because $v \models d^{r(d)}$, we have $v \models (P, r)^d$, i.e. $v \in V(\varepsilon_d)$ and $u \, E \, v$. This finishes the proof of the claim. $\square$

Assume that x is not maximal. Then there is $y \in X$ such that $x \, R \, y$, $y \not R \, x$ and $y \models (P, r)^d$ for some $d \in P$. If $d = c$, set $z = y$, otherwise, there is $z \in X$ such that $y \, R \, z$ and $z \models (P, r)^c$. In both cases, we have $x \, R \, z$ and $z \not R \, x$ hence $x \neq z$. We also have $x \, R \, z$, hence E identifies distinct point of the same colour, contradicting the fact that X is n-generated. $\square$

We can now move to the main theorem of this section, which is an adaptation of [10, Thm 2.1.10]

Theorem 3.23. *Let X be an infinite n-generated space. Then*

(i) For every $m < \omega$, D_m is a finite clopens set.

(ii) The set $\mathrm{Low}(X)$ *is a nonempty closed set.*

(iii) For every $x \in \mathrm{Low}(X)$ and $m < \omega$, there is a point $y \in D_m$ such that $x \, R \, y$.

Proof. Let A be the dual of X, let $g_1, \ldots, g_n \in A$ be its generators and $v\colon \mathrm{Prop}_n \to A$ a valuation such that $v(p_i) = g_i$ for $i \in \{1, \ldots, n\}$. This defines a colouring of X. We first prove (i) by induction on m. The base case is established by the previous lemma. Now assume that (i) holds for m, we show that it holds for $m + 1$.

Consider the subframe $X_m = X \setminus D_{\leq m}$ together with its dual A_m. Clearly these are a wK4-space and wK4-algebra. Because $D_{\leq m}$ is clopen there is a formula δ_m that defines it. Furthermore, since $D_{\leq m}$ is finite, each of its subset is clopen. Let $\phi_1, \ldots, \phi_k$ be the formulas defining those subsets.

Claim 3.24. *The space X_m is finitely generated.*

Proof. Consider the elements

$$g'_i = \delta_m \vee p_i$$

for $i \in \{1, \ldots, n\}$ and

$$g'_{n+i} = \delta_m \vee \Box(\delta_m \to \phi_i)$$

for $i \in \{1, \ldots, k\}$. Those elements provide a new colouring of X, and also of X_m. Let $g''_1, \ldots, g''_{n+k}$ be the elements of A_m corresponding to this new colouring. For $x \in X$, we let $\mathrm{col}(x)$ denote the colour according to the old colouring and $\mathrm{col}_N(x)$ the colour according to the new one. We show that A_m is generated by $g''_1, \ldots, g''_{n+k}$.

Assume not. By the Colouring Theorem, there is a proper bisimulation equivalence E on X_m such that $x\ E\ y$ implies $\mathrm{col}_N(x) = \mathrm{col}_N(y)$, for all $x, y \in X_m$. We extend E to an equivalence relation on X by setting

$$E' = E \cup \Delta_X.$$

We show that E' is a bisimulation equivalence.

For the first condition, assume that $x\ E'\ y$ and $y\ R\ z$. If $x = y$, then $x\ R\ z$. Otherwise, we have $x, y \in X_m$ and $x\ E\ y$. If $z \in X_m$, as E is a bisimulation equivalence on X_m, there is $t \in X_m$ such that $x\ R\ t$ and $t\ E\ z$. If $z \in D_{\leq m}$, we show that $x\ R\ z$. Assume not, and let ϕ_i be the formula defining $R[x] \cap D_m$. Then $z \nvDash \phi_i$, and $t \models \phi_i$ for all $t \in R[x] \cap D_m$. It follows that $x \models \Box(\delta_m \to \phi_i)$ and $y \nvDash \Box(\delta_m \to \phi_i)$. Hence x and y have different colours, which is a contradiction.

For the second condition, assume that $x \not\!E'\ y$. If $x, y \in D_{\leq m}$, then $x \neq y$ and there is a clopen set U separating x and y. Then $U \cup X_m$ is a saturated clopen set separating x and y. If $x \in D_m$, $y \in X_m$ or $x \in X_m$, $y \in D_m$, then D_m is a saturated clopen set separating x and y. If $x, y \in X_m$, then since E is a bisimulation equivalence there is a saturated clopen set U of X_m separating x and y. The set U is also a saturated clopen set in X, hence we are done proving that E' is a bisimulation equivalence.

Since E is proper, so is E'. Since $\mathrm{col}_N(x) = \mathrm{col}_N(y)$ implies $\mathrm{col}(x) = \mathrm{col}(y)$. Hence E' is a proper bisimulation equivalence on X identifying points of the same (old) colour, contradicting the fact that X is n-generated and finishing the proof of the claim. □

The proof of (i) follows by the previous lemma, since $D_{m+1} = \max(X_m)$.

The set $\mathrm{Low}(X)$ is nonempty, otherwise the sets D_m give an open cover of X with no finite subcover, since X is infinite while the sets D_m are finite.

The last statement follows from the claim and Theorem 3.9. □

3.6 n-universal models

We have all the tools required to define the n-universal models for wK4. We proceed as in Section 3.2.1 of [10].

We say that a set of clusters A *totally covers* a cluster C, denoted $C \prec A$, if $\tilde{R}^+[A] = \tilde{R}^+[C] \setminus \{C\}$, i.e. A is the set of immediate successors of C. We use the shorthand $C \prec D$ for $C \prec \{D\}$. In a 1-transitive frame (X, R) where every point only has finitely many successors, the relation $\prec$ on clusters characterises $\tilde{R}$. Indeed, $\tilde{R}^+$ is the reflexive and transitive closure of the immediate successor relation, and it is thus characterised by $\prec$. Only the clusters containing a single irreflexive point are irreflexive, hence $\prec$ characterises $\tilde{R}$.

Definition 3.25. We define the *n-universal model* $\mathcal{U}(n)$, by induction on layers, as follows

(a) $\max(\mathcal{U}(n))$ consists of $3^{2^n} - 1$ clusters of distinct patterns,

(b) for every cluster C in $\mathcal{U}(n)$ and every pattern $(P, r) \not\sqsubseteq \mathrm{pat}(C)$, there is a unique cluster D such that $D \prec C$ and $\mathrm{pat}(D) = (P, r)$,

(c) for every finite antichain A in $\mathcal{U}(n)$ of clusters and every pattern (P, r), there is a unique cluster D such that $D \prec A$ and $\mathrm{pat}(D) = (P, r)$.

The n-universal model is the least model with that satisfy those requirements.

Remark 3.26. The 0-universal model for wK4 coincides with the 0-universal model for K4. This is not the case for $n \geq 1$. In fact, even the first layer is different. Moreover, the second layer of the 1-universal model for wK4 already has more than 2000 points.

We give a partial representation of the 1-universal model for wK4 in Figure 3.1 below. Reflexive points are represented as $\circ$ and irreflexive points as $\bullet$. Green and red represent the colours 1 and 0, respectively.

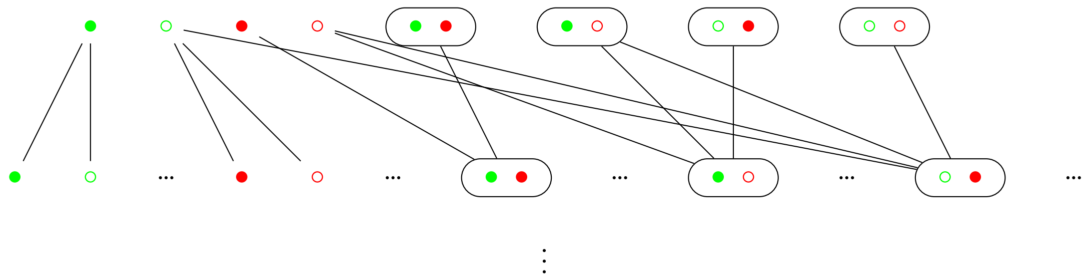


Figure 3.1: A partial representation of the 1-universal model for wK4

Lemma 3.27. *For every $m < \omega$, the generated submodel $\mathcal{U}_m(n)$ consisting of the first m layers of $U(n)$ is n-generated.*

Proof. From the construction of $\mathcal{U}(n)$, any reduction identifies points of different colours. By Lemma 3.15, every proper onto p-morphism identifies points of different colours. It follows by the Colouring Theorem that $U_m(n)$ is n-generated. $\square$

3.7 Free algebras and n-canonical models

In this section, we show that the universal models are the upper part of the canonical models, i.e. the dual of the free generated algebras. We also show that the universal models characterise wK4.

Definition 3.28. Let $F(n)$ be the free n-generated $\mathsf{wK4}$-algebra, i.e. the Lindenbaum-Tarski algebra of $\mathsf{wK4}$ over Prop_n. Let $H(n)$ denote the dual modal space of $F(n)$. The space $H(n)$ is called the *n-canonical frame* (or *n-canonical frame*) of $\mathsf{wK4}$.

The generators of $F(n)$ induce a colouring of $H(n)$. We call the n-canonical frame with this colouring the *n-canonical model* (or *n-canonical model*), and denote it $\mathcal{H}(n)$.

Proposition 3.29. *Let X be an n-generated space. Then X is (up to isomorphism) a generated subspace of $H(n)$.*

Proof. Any n-generated algebra is a quotient of $F(n)$. By duality, any n-generated space is a generated subspace of $H(n)$. □

We can now prove that the n-universal model is isomorphic to the upper part of the n-canonical model.

Theorem 3.30. *The generated submodel of $\mathcal{H}(n)$ consisting of the points of finite depth is isomorphic to the n-universal model $U(n)$. That is,* $\mathrm{Up}(\mathcal{H}(n))$ *is isomorphic to* $\mathcal{U}(n)$.

Proof. We proceed by induction on layers. By Lemma 3.27 and the previous proposition, the generated submodel $\max(\mathcal{U}(n))$ of $\mathcal{U}(n)$ is isomorphic to a generated submodel of $\mathcal{H}(n)$. By definition, $|\max(\mathcal{U}(n))| = 3^{2^n} - 1$ and by Lemma 3.21, $|\max(\mathcal{H}(n))| \leq 3^{2^n} - 1$, thus $\max(\mathcal{H}(n))$ and $\max(\mathcal{U}(n))$ are isomorphic.

Now assume that the first m layers are isomorphic. We show that the first $m+1$ layers are isomorphic as well. Once again by Lemma 3.27 and the previous proposition, we know that the first $m+1$ layers of $\mathcal{U}(n)$ form an n-generated submodel $\mathbb{M}_{m+1}$ of $\mathcal{U}(n)$ which is isomorphic to a generated submodel of $\mathcal{H}(n)$. Let us identify $\mathbb{M}_{m+1}$ with its isomorphic image in $\mathcal{H}(n)$. Assume that there is x of depth $m+1$ in $\mathcal{H}(n)$ which does not belong to $\mathbb{M}_{m+1}$. Let $\{y_1, \ldots, y_k\}$ be the (nonempty) set of immediate successors of x. By induction hypothesis, $\{y_1, \ldots, y_k\} \subseteq \mathbb{M}_{k+1}$.

If $k = 1$ and $\mathrm{col}(x) \sqsubseteq \mathrm{col}(y_1)$, we consider the α-reduction identifying x and y_1.
If $k = 1$ and $\mathrm{col}(x) \not\sqsubseteq \mathrm{col}(y_1)$, by definition of $\mathcal{U}(n)$ there is $z \in \mathbb{M}_{m+1}$ such that y_1 is the only immediate successor of z and $\mathrm{col}(x) = \mathrm{col}(z)$. We can thus consider the β-reduction identifying x and z.
If $k > 1$, $\{y_1, \ldots, y_k\}$ is an antichain and by definition of $\mathcal{U}(n)$, there is $z \in \mathbb{M}_{m+1}$ which is totally covered by $\{y_1, \ldots, y_k\}$ and $\mathrm{col}(x) = \mathrm{col}(z)$. We can then consider the β-reduction identifying x and z.
In either case, by the Colouring Theorem, we get a contradiction since $F(n)$ is n-generated. Therefore, the first $m+1$ layers of $\mathcal{U}(n)$ and $\mathcal{H}(n)$ are isomorphic, which finishes the induction. □

It is well known that a logic is characterised by either of its Lindenbaum-Tarski algebra or its canonical model.

Theorem 3.31. *For every modal formula ϕ in Fm_n, we have* $\mathsf{wK4} \vdash \phi$ *iff* $F(n) \models \phi \approx 1$ *iff* $\mathcal{H}(n) \models \phi$.

Proof. See [17, Thm 5.5] □

Finally, let us show that the n-universal models for $\mathsf{wK4}$ also characterise the logic.

Theorem 3.32. *For every formula ϕ in Fm_n, we have* $\mathsf{wK4} \vdash \phi$ *iff* $\mathcal{U}(n) \models \phi$.

Proof. Clearly if $\mathsf{wK4} \vdash \phi$ then $\mathcal{H}(n) \models \phi$, hence $\mathcal{U}(n) \models \phi$. For the converse, assume that $\mathsf{wK4} \nvdash \phi$. Then there is a finite model $\mathbb{M} = (X, V)$ such that $\mathbb{M} \nvDash \phi$. Let A be the dual algebra of X, and $v\colon \mathrm{Prop}_n \to A$ the valuation corresponding to V. Let A' be the subalgebra of A generated by $v(p_i)$ for $i \in \{1, \dots, n\}$. The algebra A' is a finite n-generated space such that $\phi \not\approx 1$, hence it is a quotient of $F(n)$. Letting $\mathbb{M}' = (X', V')$ be the model defined from (A', v'), we have $\mathbb{M}' \nvDash \phi$. Then $\mathbb{M}'$ is a finite generated submodel of $\mathcal{H}(n)$, hence also of $\mathcal{U}(n)$. It follows that $\mathcal{U}(n) \nvDash \phi$. □

Remark 3.33. The n-canonical models can be defined for any modal logic L, and are denoted $\mathcal{H}_L(n)$. We can also always define $\mathcal{U}_L(n)$ as the set of points x in $\mathcal{H}_L(n)$ such that the submodel generated by x is finite. The following are equivalent.

(i) The logic L has the finite model property.

(ii) The sets $\mathcal{U}_L(n)$ are dense in $\mathcal{H}_L(n)$.

(iii) The logic L is sound and complete with respect to the models $\mathcal{U}_L(n)$.

Since $\mathsf{wK4}$ has the finite model property, $\mathcal{U}(n)$ is dense in $\mathcal{H}(n)$. In fact, $\mathcal{U}(n)$ consist exactly of the isolated points of $\mathcal{H}(n)$. Indeed, any point in $\mathcal{U}(n)$ is isolated by Theorem 3.23, and a point in $\mathcal{H}(n) \setminus \mathcal{U}(n)$ cannot be isolated, since $\mathcal{U}(n)$ is dense in $\mathcal{H}(n)$.

3.8 Singleton formulas

From Theorem 3.23, we know that for any $x \in \mathrm{Up}(\mathcal{H}(n))$, the singleton $\{x\}$ is clopen, hence there is a formula defining x. Combining this with Theorem 3.30, we obtain that any singleton in $\mathcal{U}(n)$ is definable by a modal formula in n variables. In this section, we give an explicit description of those formulas. These formulas have already been defined for IPC by de Jongh [33] (see [10, Section 3.3] and also [4]). Similar formulas also appear in Bellissima [3] in the context of n-universal models for $\mathsf{K4}$. In [17], the authors show that such formulas exist for $\mathsf{K4}$ as well, without writing them out explicitly.

The construction will be recursive on the depth of the point. We begin by defining a few useful shorthand notations.

Definition 3.34. Let Y be a finite upset of $\mathcal{U}(n)$ and assume that for each $y \in Y$, there is a formula ψ_y in n variables such that $V(\psi_y) = \{y\}$[4]. We set

$$\psi_Y = \bigvee_{y \in Y} \psi_y.$$

We have $V(\psi_Y) = Y$.

Definition 3.35. Let C be a cluster in $\mathcal{U}(n)$ and Y the set of its strict successors, i.e. $Y = R[C] \setminus C$. Assume that for each $y \in Y$, there is a formula ψ_y in n variables such that $V(\psi_y) = \{y\}$. For any $x \in C$, we define $\widehat{x} = \bigwedge_{x \in V(p_i)} p_i \wedge \bigwedge_{x \notin V(p_i)} \neg p_i$. We also define

$$\psi'_x = \begin{cases} \widehat{x} \wedge \neg\psi_Y \wedge \Diamond(\widehat{x} \wedge \neg\psi_Y) & \text{if } x \text{ is reflexive,} \\ \widehat{x} \wedge \neg\psi_Y \wedge \neg\Diamond(\widehat{x} \wedge \neg\psi_Y) & \text{if } x \text{ is irreflexive.} \end{cases}$$

and

$$\psi''_x = \psi'_x \wedge \bigwedge_{y \in C \setminus \{x\}} \Diamond\psi'_y \wedge \bigwedge_{y \in Y} \Diamond\psi_y.$$

[4] $V(\phi)$ is the truth set of ϕ.

Finally, we define

$$\psi_x = \psi''_x \wedge \Box \left(\psi_Y \vee \bigvee_{y \in C} \psi''_y \right).$$

Using these definitions, we can recursively define a formula ψ_x in n variables for each $x \in \mathcal{U}(n)$ (if x is maximal, then $Y = \emptyset$, $\psi_Y = \bot$, and ψ_x corresponds to a formulas defined in the proof of Lemma 3.21). The only thing left is to show that these are the right formulas, that is, $V(\psi_x) = \{x\}$ for each $x \in \mathcal{U}(n)$.

Proposition 3.36. *For each $x \in \mathcal{U}(n)$, we have $V(\psi_x) = \{x\}$.*

Proof. We proceed by induction on the depth. Assume that the statement is true for states of depth strictly less than m and let C be a cluster of depth m. Let $Y = R[C] \setminus C$. By induction hypothesis, we have that $V(\psi_Y) = Y$. It is not hard to check that for each $x \in C$, $x \in V(\psi_x)$. So we only have to show that $V(\psi_x)$ is a singleton, for each $x \in C$.

Consider the equivalence relation

$$E = \Delta_{\mathcal{U}(n)} \cup \bigcup_{x \in C} V(\psi_x) \times V(\psi_x).$$

We claim that it is a bisimulation equivalence. By the definition of the ψ_x formulas, it can only identify points of the same colour. Because the finite generated submodels of $\mathcal{U}(n)$ are n-generated, E can only be the identity, otherwise this would contradict the Colouring Theorem. But E being the identity implies that the sets $V(\psi_x)$ are singletons.

So let us show that E is a bisimulation equivalence. The second condition of a bisimulation is easy to check since the sets $V(\psi_x)$ are clopen and saturated. Let us focus on the first condition. Assume that $u \mathrel{E} v$ and $u \mathrel{R} w$. If $u = v$, then $v \mathrel{R} w$ and we are done. Otherwise, there is some $x \in C$ such that $u, v \in V(\psi_x)$. Since $u \mathrel{R} w$ and

$$\mathcal{U}(n), u \models \Box \left(\psi_Y \vee \bigvee_{y \in C} \psi''_y \right),$$

we have either $w \in V(\psi_y)$ for some $y \in Y$ or $w \in V(\psi''_z)$ for some $z \in C$. In the first case, since $\mathcal{U}(n), v \models \Diamond \psi_y$, there is some $t \in \mathcal{U}(n)$ such that $v \mathrel{R} t$ and $t \in V(\psi_y)$. By induction hypothesis, $V(\psi_Y)$ is a singleton hence $w = t$ and $w \mathrel{R} t$.

In the second case, if $z \neq x$, then $\mathcal{U}(n), v \models \Diamond \psi'_z$, hence there is some $t \in \mathcal{U}(n)$ such that $v \mathrel{R} t$ and $t \in V(\psi'_z)$. If $z = x$, then x must be reflexive. Indeed, we have $\mathcal{U}(n), w \models \widehat{x} \wedge \neg \psi_Y$, thus $\mathcal{U}(n), u \models \Diamond(\widehat{x} \wedge \neg \psi_Y)$. Since ψ'_x contains either $\Diamond(\widehat{x} \wedge \neg \psi_Y)$ or its negation, and $\mathcal{U}(n), u \models \psi'_x$, it must be the former, which means that x is reflexive. But then, as $\mathcal{U}(n), v \models \Diamond(\widehat{x} \wedge \neg \psi_Y)$, there must be some $t \in \mathcal{U}(n)$ such that $v \mathrel{R} t$ and $t \in V(\widehat{x} \wedge \neg \psi_Y)$. In both cases, we have

$$\mathcal{U}(n), t \models \psi_Y \vee \bigvee_{y \in C} \psi''_y,$$

which implies that $t \in V(\psi''_z)$, because all the other disjuncts are mutually exclusive with ψ'_z (if $z \neq x$) or $\widehat{x} \wedge \neg \psi_Y$ (if $z = x$). Also recall that $w \in V(\psi''_z)$. By weak transitivity, and $u, v \in V(\psi_x)$, we have

$$\mathcal{U}(n), w \models \Box \left(\psi_Y \vee \bigvee_{y \in C} \psi''_y \right) \qquad \text{and} \qquad \mathcal{U}(n), t \models \Box \left(\psi_Y \vee \bigvee_{y \in C} \psi''_y \right),$$

hence $w, t \in V(\psi_z)$. This means that $w \mathrel{E} t$, thus finishing the proof. $\square$

Remark 3.37. The previous proof tells us that any singleton in the n-universal model is definable by a modal formula in n-variables. Since the n-universal model contains all finite models as generated submodels, it also allows us to generate formulas defining singletons in finite n-generated models.

This finishes our investigation of n-generated 1-transitive spaces, along with our description of the n-universal models. Having introduced n-generated spaces and their characterisation, we described some of their properties. We have provided several useful tools, including a characterisation of onto maps between finite spaces (Lemmas 3.13 and 3.15), a collection of formulas defining maximal clusters and other singletons (Definitions 3.18 and 3.35), a structural result for n-generated spaces (Theorem 3.23) as well as an explicit description of the n-universal models of $\mathsf{wK4}$ together with their properties (Definition 3.25 and Section 3.7).

Chapter 4

Hereditary structurally complete 1-transitive logics

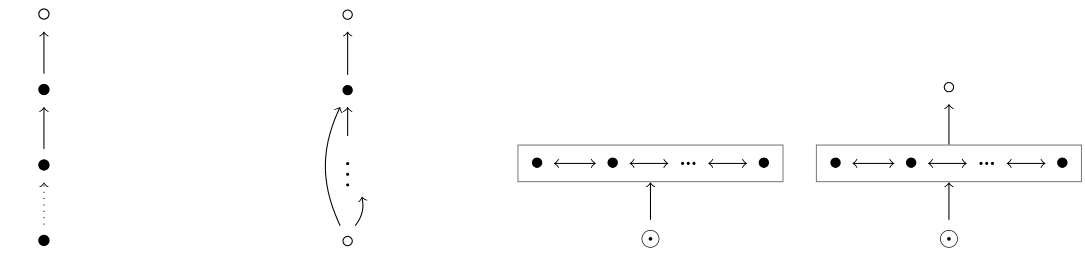
In this chapter, we characterise the hereditary structurally complete extension of $\mathsf{wK4}$. We know from Section 2.3 that the problem is equivalent to characterising primitive varieties of $\mathsf{wK4}$-algebras. The proof follows the strategy of [16], which studies hereditary structurally complete extension of $\mathsf{K4}$. The primitive varieties are characterised through forbidden configurations. We made corrections to some results, improved on numerous results and proofs, and obtained generalisations to adjust to the $\mathsf{wK4}$ case.

It is also worth comparing our proof strategy to the one of Rybakov. Some parts of the proof are very similar, especially the structural work done in Sections 4.4 and 4.5, which is similar to [42, Section 3]. Rybakov also has to take irreflexive points into account (Lemmas 4.12, 4.15 and 4.16 correspond to Lemmas 3.1 and 3.2 for Rybakov), put a bound on width (Theorem 4.21 corresponds to Lemma 3.3 for Rybakov) and provide extra results about the structure of spaces (Theorems 4.23 and 4.25 correspond to Lemma 3.4). We then use these results to give a precise description of the structure of spaces in the dual of primitive varieties (Theorem 4.29 corresponds to Lemma 3.6 and Corollary 3.8. The technique used to prove that logics omitting our frames have the finite model property is the same here as in Rybakov's paper. However, there is a significant difference in the way those results are used to complete the proof. While our approach focuses on weak projectivity, Rybakov uses the n-universal models of $\mathsf{K4}$ to complete the proof. For the sufficient condition, we rely on Theorem 4.33 and weak projectivity, while Rybakov reduces to problem to showing that every finitely subdirectly irreducible algebra embeds into a freely finitely generated algebra. For the necessary condition, we rely on the Lemma in Section 4.2, while Rybakov also relies on universal models. This different approach allowed Carr [16] to identify some frames that were missing in Rybakov's characterisation. As we will see, the list of prohibited frames becomes larger when working with $\mathsf{wK4}$ instead of $\mathsf{K4}$.

In the first section, we introduce the characterisation and prove some preliminary results. In the second section, we prove the easy direction of the characterisation. In the third section, we explain the proof strategy for the other direction. In the fourth and fifth sections, we prove some structural properties of the spaces we are working with, leading to a full description of their structure in the sixth section. We then finish the work in the last section.

4.1 Characterisation of hereditary structurally complete logics over wK4

In this section, we introduce the characterisation of hereditary structurally complete extensions of wK4. The frames and spaces depicted in Figures 4.1 and 4.2 will play a special role in our characterisation. Recall that $\circ$ represents a reflexive point, $\bullet$ an irreflexive one and $\odot$ a point which is either (frames including $\odot$ thus represent a several frames rather than just one).



(a) G_n (b) G_ω (c) I_n (d) J_n

Figure 4.1: Countably-many frames

Remark 4.1. The frame G_n, with $n < \omega$, is a chain of n irreflexive points followed by a reflexive maximum. The space G_ω is the transitive space $(\mathbb{N} \cup \{\omega\}, \tau, R)$ where $R[\omega] = \mathbb{N} \cup \{\omega\}$, $R[0] = \{0\}$ and $R[n] = \{m \in \mathbb{N} : m < n\}$ for $0 < n < \omega$. The topology τ is the one-point compactification of $\mathbb{N}$ equipped with the discrete topology, the limit being ω. Consequently, a set is clopen if it is finite and omits ω or if it is cofinite and includes ω.

The frame I_n, with $n > 1$, is a root followed by a cluster of n irreflexive point. The frame J_n is the same as I_n with a top, reflexive element.

We have the following characterisation of primitive varieties of wK4-algebras.

Theorem 4.2. *Let K be a variety of* wK4*-algebras. Then K is primitive iff it omits F^* for each F depicted in Figure 4.2, omits G_n^* for some $n < \omega$ and omits I_n and J_n for all $n > 1$.*

Remark 4.3. It is worth noting that Rybakov's characterisation of hereditarily structurally complete transitive logics [42] only contained the spaces F_i for $i \in \{1, \ldots, 17\}$. This is a mistake, which was corrected in Carr's thesis [16], by adding the spaces G_n.

The frames I_n and J_n do not appear in the characterisation for K4, because they are not K4-spaces. However, when working with wK4, new configurations can arise, which forces us to prohibit those spaces.

Together with the perspective introduced in the preliminaries, this establishes our characterisation of hereditary structurally complete extensions of wK4.

Corollary 4.4. *Let Λ be a normal modal logic extending* wK4 *and let K be the equivalent algebraic semantics of its deductive system $\vdash_\Lambda$. The following are equivalent.*

(i) The system $\vdash_\Lambda$ is hereditary structurally complete.

(ii) The variety K is primitive.

(iii) Each frame depicted in Figure 4.2 is not a Λ-space, for some $n < \omega$, G_n is not a Λ-space and for all $n > 1$, I_n and J_n are not Λ-spaces.

Before starting the proof of Theorem 4.2, let us reformulate it slightly.

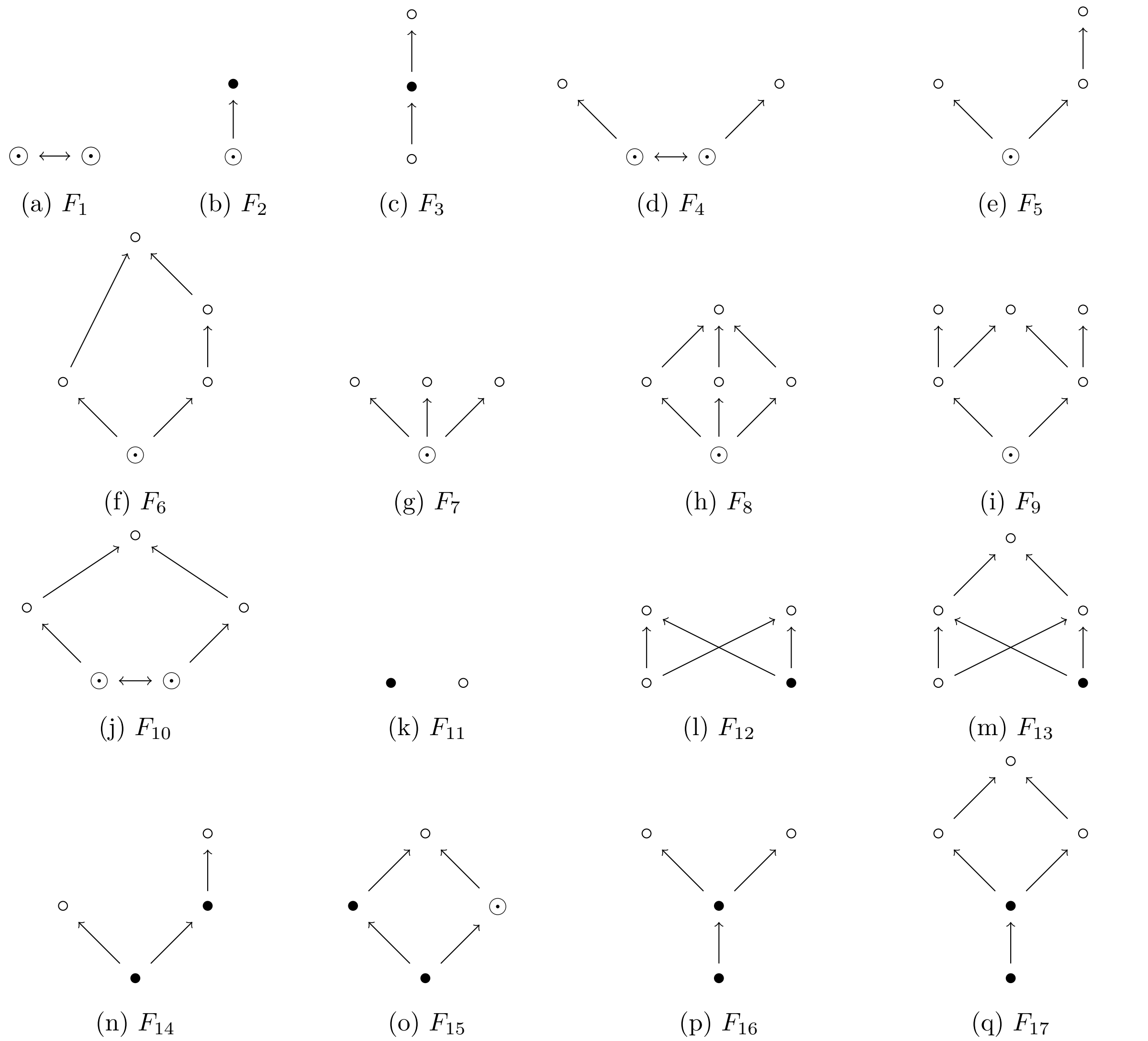


Figure 4.2: Seventeen frames

Lemma 4.5. *A variety K omits G_ω iff it omits G_n for some n.*

Proof. For any $n < \omega$, G_n is a generated subspace of G_ω, hence $G_n^* \in \mathbb{H}(G_\omega^*)$. The corresponding onto map $h_n : G_\omega^* \to G_n^*$ is given by $U \mapsto U \cap G_n$. The right to left direction follows.

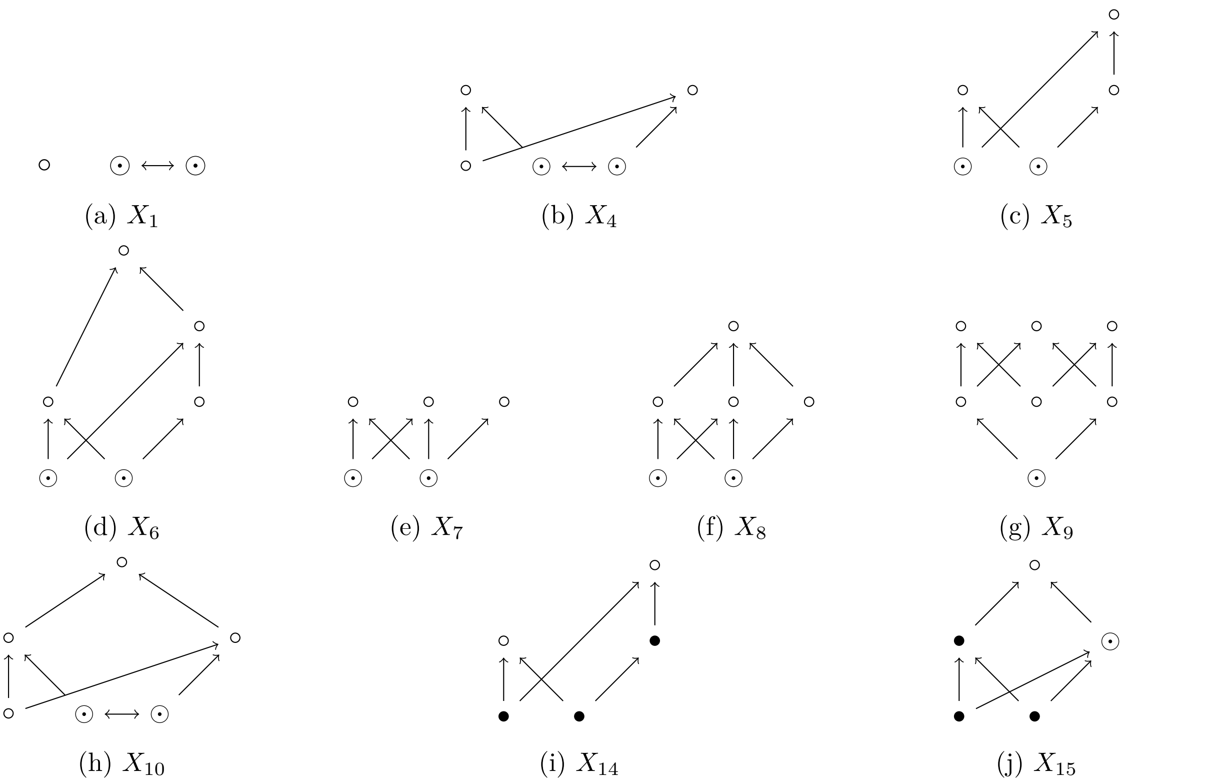
For the other direction, as K is a variety, it is definable by a set of equations Θ. If K omits G_ω^*, then there is some equation $\varepsilon \approx \delta \in \Theta$ such that $G_\omega^* \not\models \varepsilon \approx \delta$. Therefore there is an morphism $h : Fm \to G_\omega^*$ such that $h(\varepsilon) \neq h(\delta)$. We can assume without loss of generality that $h(\varepsilon) \nsubseteq h(\delta)$. We show that there is some $n < \omega$ such that $n \in h(\varepsilon)$ and $n \notin h(\delta)$. Otherwise, we must have $\omega \in h(\varepsilon)$ and $\omega \notin h(\delta)$. But that means that $h(\varepsilon)$ is cofinite while $h(\delta)$ is finite, so there must still be some $n \in h(\varepsilon)$ such that $n \notin h(\delta)$. But then $h' = h_n \circ h : Fm \to G_n$ is such that $h'(\varepsilon) \neq h'(\delta)$, thus $G_n^* \not\models \varepsilon \approx \delta$. It follows that K omits G_n^*. $\square$

4.2 First direction

The left to right implication of Theorem 4.2 is relatively easy to prove. We proceed by contraposition. Assuming that K contains F_i^* for some $i \in \{1, \ldots, 17\}$ or contains G_n^* for all $n < \omega$, we exhibit a finite, member of K which is not weakly projective in K. By Lemma 2.60, this implies that K is not primitive.

Lemma 4.6. *Let K be a variety of* wK4*-algebras. Assume that K contains F_i^* for some $i \in \{1, 4, 5, 6, 7, 8, 9, 10, 14, 15\}$, then K is not primitive.*

Proof. Assume that K contains F_i^*. Consider the frame X_i^* depicted in Figure 4.3. The frame F_i is finite and rooted, therefore F_i^* is a finite subdirectly irreducible member of K. We show that it is not weakly projective. Observe that X_i can be obtained as a quotient of two disjoint copies of F_i, therefore $X_i^* \in \mathbb{SP}(F_i^*) \subseteq K$. Furthermore, F_i is a generated subspace of X_i, thus $F_i^* \in \mathbb{H}(X_i^*)$. However, one can check that there is no sequence of reductions turning X_i into F_i, thus $F_i^* \notin \mathbb{IS}(X_i^*)$. In conclusion, F_i^* is not weakly projective, and consequently, K is not primitive.



(a) X_1 (b) X_4 (c) X_5 (d) X_6 (e) X_7 (f) X_8 (g) X_9 (h) X_{10} (i) X_{14} (j) X_{15}

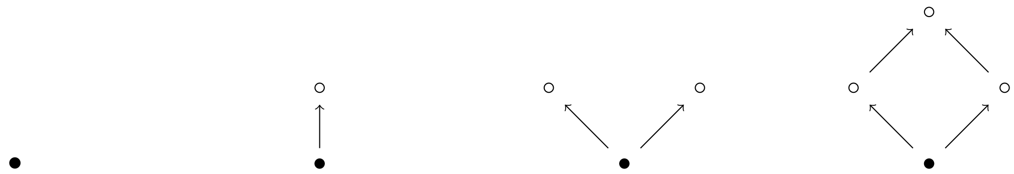
Figure 4.3: Some non-rooted frames

□

Lemma 4.7. *Let K be a variety of* wK4*-algebras. Assume that K contains F_i^* for some $i \in \{2, 3, 11, 12, 13, 16, 17\}$, then K is not primitive.*

Proof. Assume that K contains F_i^*. Consider the frame X_i^* depicted in Figure 4.4 (if F_i contains a point $\odot$, then then point $\odot$ in X_i is the same, in the sense that it is reflexive if the one in F_i is, and irreflexive otherwise). First, observe that X_i is a generated subspace of F_i, hence $X_i^* \in \mathbb{H}(F_i^*) \subseteq K$. The space X_i is finite and rooted, therefore X_i^* is a finite subdirectly irreducible member of K. To show that it is not weakly projective, it is sufficient to check that $X_i^* \notin \mathbb{IS}(F_i^*)$. One easily checks that there is no sequence of reductions turning F_i into X_i, thus proving that K is not primitive.

□



(a) $X_2 = X_{11}$ (b) X_3 (c) $X_{12} = X_{16}$ (d) $X_{13} = X_{17}$

Figure 4.4: Some rooted frames

Lemma 4.8. *Let K be a variety of* wK4*-algebras. Assume that K contains G_n^* for all $n < \omega$, then K is not primitive.*

Proof. By Lemma 4.5, K contains G_ω^*. We already know that $G_1^* \in K$, and that G_1^* is a finite subdirectly irreducible member of K. Observe that G_1 is a generated subspace of G_ω, therefore $G_1^* \in \mathbb{H}(G_\omega^*)$. However, G_1 cannot be obtained as a quotient of G_ω, since G_ω has a reflexive root and G_1 does not. Therefore, $G_1^* \notin \mathbb{IS}(G_\omega^*)$, showing that K is not primitive. ☐

Lemma 4.9. *Let K be a variety of* wK4*-algebras. Assume that K contains I_n or J_n for some $n > 1$, then K is not primitive.*

Proof. Assume that K contains I_n. Consider the subspace X of I_n generated by the cluster C consisting of n irreflexive points. Clearly $X^* \in K$, and X is finite and rooted thus X^* is finite and subdirectly irreducible. We show that $X^* \notin \mathbb{S}(I_n)$, thus proving that X^* is not weakly projective. This amounts to showing that I_n does not reduce to X. For the sake of contradiction, assume we have an onto p-morphism $f \colon I_n \to X$. Let r be the root of I_n. Then $f(n)$ has to be a root of X, so $f(r) = x$ for some $x \in C$. Take $y \in C$. Then as $r \mathrel{R} y$, we have $x \mathrel{R} f(y)$, thus $f(y) \neq x$. By the pigeonhole principle, there are $y, y' \in C$ such that $y \neq y'$ and $f(y) = f(y')$. But then $f(y)$ is reflexive, which is absurd.

We proceed similarly if K contains J_n. The only difference if when show that no onto p-morphism $f \colon J_n \to X$ exists. A point $y \in C$ could be sent to the top element. But then every point in C has to be sent to the top element, and f is not onto (it does not reach points in $C \setminus \{x\}$. ☐

4.3 Proof strategy for the other direction

The right to left direction of Theorem 4.2 is a lot trickier. Given our sufficient condition, for any variety K of wK4-algebras satisfying the condition of Theorem 4.2, we need to establish two facts. First, that all the subvarieties of K have the finite model property. Given that subvarieties of K also omit the given algebras, it is sufficient to show that any variety omitting the given algebras has the finite model property. Second, we need to prove that the finite subdirectly irreducible members of K are weakly projective in K. By Corollary 2.69, this is enough to conclude. The main part of the work lies in describing precisely the structure of the finitely generated subdirectly irreducible members of K (Theorem 4.29). This is established in the next three sections.

We first establish some results about the skeleton of the members of K, starting by dealing with irreflexive points. We then prove several central results, claiming that duals of members of K have to avoid certain configurations. The proofs of those results always follow the same pattern: we assume that the modal space does contain the specified configuration, and using generated subspaces and quotients, we exhibit one of the prohibited spaces, thus leading to a contradiction. With these results, we can give a precise description of the structure of the finitely generated subdirectly irreducible members of K.

Once those results are established, we will establish the two main facts, starting with the proof that every subvariety of K has the finite model property, then showing that every finite subdirectly irreducible member of K is weakly projective.

Let us first introduce some terminology.

Definition 4.10. A variety of $\mathsf{wK4}$-algebras is called *cobwebby* if it omits F_i^* for $i \in \{1, \dots, 17\}$ and G_n for some $n < \omega$.

A $\mathsf{wK4}$-algebra is called *cobwebby* if the variety it generates is cobwebby. A $\mathsf{wK4}$-space is *cobwebby* if its dual is cobwebby.

Notice that if X is a 1-transitive space and $\mathrm{sk}(X)$ its skeleton, we have $\mathrm{sk}(X)^* \in \mathbb{V}(X^*)$ by Proposition 3.6. Therefore, if X is cobwebby, so is its skeleton. In the next three sections, we will give some structural results about cobwebby 1-transitive spaces. For simplicity, we will refer to the skeletons of those spaces as *cobwebby skeletons*.

4.4 A few technical results

In this section, we establish a few results about the shape of clusters in a cobwebby space. We first start with some claims about maximal clusters.

Lemma 4.11. *Let X be a cobwebby space. Then the maximal clusters of X are singletons.*

Proof. If not, by taking generated subspace we obtain a space which consists of a single proper cluster. Take two distinct points x, y in that cluster. Since they are distinct, we can find a clopen set U separating them. If we collapse all points in U together and all points in U^c together, we obtain F_1. Thus, $F_1^* \in \mathbb{SH}(X^*) \subseteq \mathbb{V}(X^*)$, contradicting the assumption that X is cobwebby. □

Lemma 4.12. *Let X be a cobwebby space. Then either the maximal clusters of X are reflexive or X is an antichain of irreflexive points.*

Proof. Suppose that neither is the case. That is, $R \neq \emptyset$ (otherwise X is an antichain of irreflexive points) and there is a point $x \in X$ such that $R[x] = \emptyset$ (this is the case for any irreflexive maximal point).

First, assume that there is $y \in X$ such that for all $z \in R^+[y]$, $R[z] \neq \emptyset$. As $y \in R^+[y]$, we have $R[y] \neq \emptyset$ thus $x \neq y$. Consider the subspace X' generated by x and y, i.e. $X' = \{x\} \cup R^+[y]$. The set X' is clopen (in X'), therefore $R^{-1}[X']$ is clopen. As $R[x] = \emptyset$ and $R[z] \neq \emptyset$ for all $z \in R^+[z]$, we have $R^{-1}[X'] = R^+[y]$, thus this set is closed. Furthermore, $R^+[y]$ is an upset, and for all $z \in R^+[y]$, we have $R[z] \neq \emptyset$, thus in X' we can identify the points in $R^+[y]$. By doing that, we obtain the space F_{11}. Thus $F_{11}^* \in \mathbb{SH}(X^*) \subseteq \mathbb{V}(X^*)$, contradicting the assumption that X is cobwebby.

Figure 4.5: F_{11}

Now assume that for all $y \in X$, there is $z \in R^+[y]$ such that $R[z] = \emptyset$. We start by proving the following claim.

Claim 4.13. *For all $y \in X$, either $R[y] = \emptyset$ or there is $z \in R[y] \setminus \{y\}$ such that $R[z] \neq \emptyset$.*

Proof. Assume not. Then there is $y \in X$ such that $R[y] \neq \emptyset$ and for all $z \in R[y] \setminus \{y\}$, we have $R[z] = \emptyset$. Consider the subspace X' generated by y, i.e. $X' = R^+[y]$. The set $\emptyset$ is clopen, hence so is $\Box\emptyset$. As every point in X' except y sees nothing, $\Box\emptyset = R[y] \setminus \{y\}$, thus this set is closed. The set $R[y] \setminus \{y\}$ is also an upset, and in X' we can identify all points of this set. The resulting space is F_2, thus $F_2^* \in \mathbb{SH}(X^*) \subseteq \mathbb{V}(X^*)$, contradicting the assumption that X is cobwebby. This finishes the proof of the claim. $\square$

Figure 4.6: F_2

Returning to the proof of our lemma, we already know that for all $y \in X$, there is $z \in R^+[y]$ such that $R[z] \neq \emptyset$. We may also assume that either $R[y] = \emptyset$ or there is $z \in R[y] \setminus \{y\}$ such that $R[z] \neq \emptyset$. Those claims, combined with the fact that $\Box\emptyset$ is a clopen set, imply that the relation

$$E = \{(u, v) \in X : R[u] = \emptyset \text{ iff } R[v] = \emptyset\}$$

is a bisimulation equivalence. The quotient space is isomorphic to F_2, which again contradicts the assumption that X is cobwebby. $\square$

Corollary 4.14. *Let X be a cobwebby space. Then if X is not an antichain of irreflexive points, every point in X has a successor.*

We now prove some results about irreflexive clusters.

Lemma 4.15. *Let X be a cobwebby skeleton and let $x \in X$ be an irreflexive point. Then any chain in $R^{-1}[x]$ is finite and any $y \in R^{-1}[x]$ is irreflexive.*

Proof. Suppose, on the contrary, that there is an irreflexive point $x \in X$ such that $R^{-1}[x]$ contains an infinite chain or a reflexive point. For any $y \in R[x]$, we have $y \not R^+ x$ because we are working in a skeleton, thus we can find a clopen upset containing y but not x. Since $R[x]$ is closed, a simple compactness argument allows us to find a clopen upset U containing $R[x]$ but not x. By Corollary 4.14, this also implies that U is nonempty. Next, we define $V = U^c \cap \Box U$, which is a clopen set containing x. Notice that if $y, z \in V$, then $y \in \Box U$ while $z \notin U$ thus $y \not R z$. Therefore, V is an antichain of irreflexive points. Finally, we define $W = U \cup (R^+)^{-1}[V]^c$, which is a clopen upset by Remark 2.47.

Using the fact that W is an upset and $V \subseteq \Box U \subseteq \Box W$, we can show that V, W form an M-partition (as defined in 2.43), thus defining a bisimulation equivalence E. Indeed, the M-partition condition for W follows from Remark 2.44 as it is an upset. For V, observe that for any $x, y \in V$ and $z \in R[x]$, by $V \subseteq \Box W$ we have $z \in W$. By Corollary 4.14, we can find $t \in R[x]$, and since $V \subseteq \Box W$ we have $t \in W$, thus $z \; E \; t$ and we are done. The set W is identified with the top element $\top$, while V is identified with an irreflexive point, which we will call x (from now on, we will assume). Both of these points are isolated. Notice that $R^{-1}[x]$ still contains an infinite chain. The definition of V and W ensure that x sees $\top$ and nothing else, since $R[x] \subseteq W$. For any $y \notin V \cup W$, we have $y \notin U$ and $y \in V \cup R^{-1}[V]$ as $y \notin W$. As $y \notin V$, we have $y \in R^{-1}[V]$. Therefore, in X/E, every point besides x and $\top$ sees x. We now do a case distinction.

First assume that there is a reflexive point (different from $\top$) of finite depth in X/E. Let y be a reflexive point of minimal depth and consider the subspace generated by y. We know that y has depth at least 2, and everything besides y and $\top$ is irreflexive. We want to apply Lemma 3.12. We need to check that the sets D_m are clopen. The set $D_0 = \{\top\}$ is clopen because $\top$ is isolated. Notice that $D_{m+1} = R^{-1}[D_m] \cap \Box D_m \setminus D_m$, thus every set D_m is clopen. Applying Lemma 3.12, we obtain the following frame, which reduces to F_3, contradicting the assumption that X is cobwebby.

Now assume that all points of finite depth are irreflexive. We can apply Lemma 3.12 and obtain either the space G_n for some $n < \omega$, or the space G_ω. Since X is cobwebby, it cannot reduce to G_ω, so it must reduce to G_n for some $n \in \omega$. But G_n clearly does not contain any infinite chain, which is a contradiction. $\Box$

Lemma 4.16. *Let X be a cobwebby skeleton and let $x \in X$ be an irreflexive point. Assume that $y, z \in X$ are such that $y \mathrel{R} x$ and $y \mathrel{R} z$. Then x and z are comparable.*

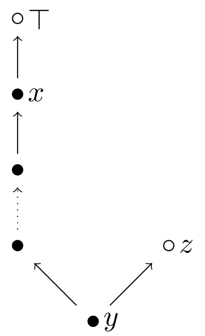
Proof. Assume not, we have $x, y, z \in X$ such that x is irreflexive and $y \mathrel{R} x$, $y \mathrel{R} z$, and y, z are incomparable. Since $R^{-1}[x]$ does not contain infinite chains, we may assume that y is maximal with the property that it has a successor which is incomparable with x. By taking the subspace generated by y, we may assume that y is the root.

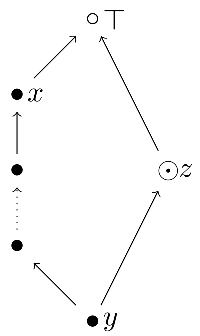
For each $t \in R[x]$, we have $t \not\mathrel{R}^+ x$ and $t \not\mathrel{R}^+ z$, thus we can find a clopen upset containing t but not x or z. Since $R[x]$ is closed, a simple compactness argument allows us to find a clopen upset U containing $R[x]$ but not x or z. Then U is an M-partition, and we may collapse it to a maximal and isolated point $\top$. Notice that $R[x] = \{\top\}$. We now do a case distinction.

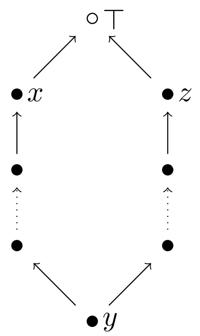
First assume that there is no $t \in R^{-1}[\top]$ such that x and t are incomparable. Then since $R[x] = \{\top\}$, any $t \in R^{-1}[\top]$ which is distinct from $\top$ or x is in $R^{-1}[x]$, that is, $R^{-1}[\top] = R^{-1}[x] \cup \{x, \top\}$. Consider the set $U = R^{-1}[\top]^c$. It is a clopen upset, containing z and not x or $\top$. We may collapse it to a single isolated point z. The resulting space has underlying set $R^{-1}[\top] \cup \{z\} = R^{-1}[x] \cup \{x, z, \top\}$. By Lemma 4.15, $R^{-1}[\top]$ has finite depth. An easy adaptation of Lemma 3.12 allows us to identify points of the same depth in $R^{-1}[\top]$, thus leaving us with the finite space depicted on the next page, on the left. That space reduces to F_{14}, contradicting the assumption that X is cobwebby.

Now assume that there is $t \in R^{-1}[\top]$ such that x and t are incomparable. By redefining z, we may assume that $z = t$. Since $\{\top\}$ is a clopen upset, so is $U = R^{-1}[\top]^c \cup \{\top\}$, not containing x, y or z. We may identify all points in U and end up with a single top element $\top$. Consider the set $V = \Box\{\top\} \setminus \{\top\}$. This is clearly a clopen set, since $\top$ is isolated, and it is an antichain of irreflexive points containing x. We do another case distinction.

First assume that $V = \{x\}$**.** Then x is isolated and for all $t \in X \setminus \{\top, x\}$, we have $R[t] \setminus \{\top\} \neq \emptyset$. Consider the set $U = R^{-1}[x]^c \setminus \{\top, x\}$. This is a clopen set, and it is nonempty because $z \in U$. We claim that U is an M-partition. Indeed, let $u, v \in U$ and let $w \in R[u]$. If $w = \top$, then $v \mathrel{R} w$. If $w \neq \top$, then $w \in U$, as $u \notin R^{-1}[x]$ so $w \notin \{x\} \cup R^{-1}[x]$. Since $v \notin \{\top, x\}$, we have $R[v] \setminus \{\top\} \neq \emptyset$, thus there is $t \in R[v]$ such that $t \neq \top$. Then by the same reasoning we did for w, we have $t \in U$. Thus U is an M-partition, and we may collapse U to a single point z. The resulting space has underlying set $R^{-1}[\top] \cup \{z\} = R^{-1}[x] \cup \{x, z, \top\}$. By Lemma 4.15, $R^{-1}[\top]$ has finite depth. An easy adaptation of Lemma 3.12 allows us to identify points of the same depth in $R^{-1}[\top]$, thus leaving us with the finite space depicted below, in the center. That space reduces to F_{15}, contradicting the assumption that X is cobwebby.







Finally, assume that $V \neq \{x\}$**.** By redefining z, we may assume that $z \in V$, that is, $R[z] = \{\top\}$. The set $(R^+)^{-1}[V]$ is a clopen upset, by Remark 2.47, therefore we can collapse it to a single maximal point $\top$. Our space now has underlying set $\{\top\} \cup V \cup R^{-1}[V]$. Since x and z are unrelated, we can find a clopen set U containing x and not z. Let $V_1 = V \cap U$ and $V_2 = V \setminus U$. Then V_1, V_2 are disjoint clopen sets, $x \in V_1$ and $z \in V_2$. For any $t \in V_1 \cup V_2 = V$, we have $R[t] = \{\top\}$. Therefore, it is easy to check that V_1 and V_2 form an M-partition. By collapsing each of them, we obtain two irreflexive and isolated points x and z. The underlying set of our space then becomes $\{\top, x, z\} \cup R^{-1}[x, z]$. By Lemma 4.15, both $R^{-1}[x]$ and $R^{-1}[z]$ have finite depth. An easy variation on Lemma 3.12 allows us to collapse the points having the same depth in $R^{-1}[x]$, and to collapse the points having the same depth in $R^{-1}[z]$. The resulting space is depicted above, on the right, and can be reduced to F_{15}, thus contradicting X being cobwebby. ☐

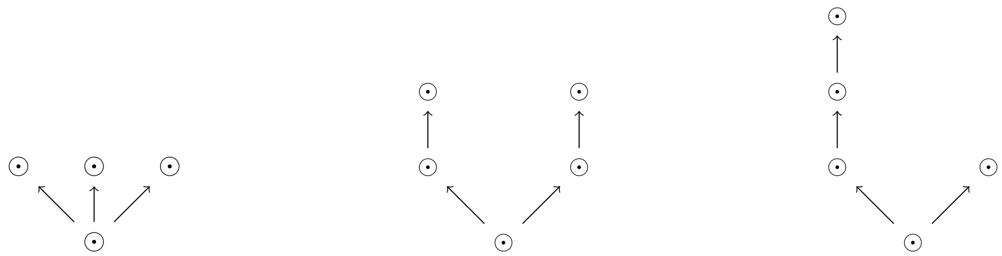
Finally, we need a result about irreflexive points in proper cluster.

Lemma 4.17. *Let X be a cobwebby finitely generated space. Then any non-minimal, proper cluster in X contains a reflexive point.*

Proof. Assume not. Then there is a cluster C in X that only contains irreflexive points. Since C is not minimal, for any $x \in C$, the set $R^{-1}[x]$ is nonempty. It is also closed, therefore we can find a maximal cluster D in this set. Then C is an immediate successor of D. Consider the subspace generated by D. The cluster D can be collapsed to a single root. Let us call Y the resulting space. If $Y = R^{-1}[C]$, then Y is equal to I_n for some $n > 1$ (since X is finitely generated, its clusters are finite). If $Y \neq R^{-1}[C]$, then $R^{-1}[C]^c$ can be collapsed to a single maximal element, and the space obtained is J_n for some $n > 1$. Both options contradict the assumption that X is cobwebby. ☐

4.5 Three central results

In this section, we wish to prove that cobwebby spaces do not contain any of the subframes depicted in Figure 4.8.



(a) X_1 (b) X_2 (c) X_3

Figure 4.8: Three frames

Let us first make clear what we mean by a subframe.

Definition 4.18. A modal space $Y = (Y, \tau_Y, R_Y)$ is a subframe of a modal space $X = (X, \tau_X, R_X)$ if (Y, τ_Y) is a topological subspace of (X, τ_X) and R_Y is the restriction to Y of R_X.

Remark 4.19. In our work, we will only be concerned about finite subframes Y of a space X. Since finite spaces are equipped with the discrete topology, the requirement that (Y, τ_Y) is a topological subspace of (X, τ_X) is always true, and we may ignore it.

Let us start by proving that a cobwebby space does not contain X_1 as a subframe. Throughout this section, we use repeatedly the fact that if X is a 1-transitive space, then the dual of its skeleton is in the variety generated by X^*, or put more simply, that the skeleton of a cobwebby space is cobwebby.

Lemma 4.20. *Let K be a cobwebby variety and let $A \in K$. Assume that A_* contains X_1 as a subframe. Then K contains an algebra B such that B_* is rooted and has three incomparable, isolated points x_1, x_2 and x_3 such that either of the following holds.*

(i) x_1, x_2 *and* x_3 *are maximal and* $B_* = R^{-1}[x_1, x_2, x_3]$,

(ii) B_* *has a maximum* $\top$ *which is isolated and* $B_* \setminus \{\top\} = R^{-1}[x_1, x_2, x_3]$,

(iii) B_* *has an isolated point* $\top$ *such that* $x_1 \; R \; \top$, $x_2 \; R \; \top$, $\top$ *and* x_3 *are maximal,* $B_* \setminus \{\top\} = R^{-1}[x_1, x_2, x_3]$ *and* $R^{-1}[\top] \cap R^{-1}[x_3] \subseteq R^{-1}[x_1, x_2]$.

Proof. We have points $\perp, x_1, x_2, x_3$ such that $\perp$ sees x_1, x_2, x_3, which are incomparable. By considering the subspace generated by $\perp$, we may assume that $\perp$ is the root of A_*. By considering the skeleton, we may also assume that A_* only contains improper clusters.

Let y be an irreflexive point. Then $\perp \; R \; y$ and $\perp \; R \; x_1$, thus by Lemma 4.16, y and x_1 are comparable. Similarly, y is comparable with x_2 and x_3. This immediately implies that $y \notin \{x_1, x_2, x_3\}$ as x_1, x_2, x_3 are incomparable. Therefore, x_1, x_2, x_3 are reflexive. But then for any irreflexive point y, we have $x_1 \not R \; y$, by Lemma 4.15. We must then have $y \; R \; x_1$. Similarly, $y \; R \; x_2$ and $y \; R \; x_3$. Therefore, irreflexive points are contained in $R^{-1}[x_1] \cap R^{-1}[x_2] \cap R^{-1}[x_3]$. We do a case distinction on the number of intersection between the sets $R[x_1], R[x_2]$ and $R[x_3]$.

First assume that these sets are pairwise disjoint. Then by modal separation and a standard compactness argument (the sets $R[x_1], R[x_2]$ and $R[x_3]$ are compact), we can find pairwise disjoint clopen upsets U_1, U_2 and U_3 such that $x_i \in U_j$ iff $i = j$ for each $i, j \in \{1, \ldots, 3\}$. Consider the sets U_1, U_2 and $R^{-1}[U_1 \cup U_2]^c$. They are all clopen upsets, therefore they form an M-partition. It is easy to check that the quotient satisfies (i).

Now assume that there is exactly one nonempty intersection. Without loss of generality, we have $R[x_1] \cap R[x_2] \neq \emptyset$, $R[x_2] \cap R[x_3] = \emptyset$ and $R[x_3] \cap R[x_1] = \emptyset$. By modal separation and a standard compactness argument, we can find clopen upset U_1, U_2 and U_3 such that U_3 is disjoint from $U_1 \cup U_2$ and $x_i \in U_j$ iff $i = j$ for each $i, j \in \{1, \ldots, 3\}$.

We define $V = U_1 \cap U_2$, $V_1 = U_1 \cap R^{-1}[V] \setminus V$, $V_2 = U_2 \cap R^{-1}[V] \setminus V$ and $W = V \cup R^{-1}[V_1 \cup V_2 \cup U_3]$. Clearly U_3 and W are upsets, and $V \subseteq W$. We prove that V_1, V_2, U_3 and W form an M-partition. We have that V_1 (resp. V_2) sees itself and W. We prove that V_1 (resp. V_2) does not see anything else. Let $x \in V_1$ and $y \in R[x]$. Then as U_1 is an upset, we have $y \in U_1$. If $y \in V$, then $y \in W$ and we are done. If $y \in R^{-1}[V]$, then $y \in V_1$ and we are done. Otherwise, we have $y \notin R^{-1}[V_1]$ and $y \notin R^{-1}[V_2]$, as the opposite would imply $y \in R^{-1}[V]$. We also have $y \notin R^{-1}[U_3]$ as U_1 is an upset disjoint from U_3. Therefore, we have $y \notin R^{-1}[V_1 \cup V_2 \cup U_3]$, thus $y \in W$ and we are done.

Therefore, V_1, V_2, U_3 and W form an M-partition. Let us call x_1, x_2, x_3 and $\top$ the corresponding elements in the quotient space B_*. The definition of W ensures that $\top$ and x_3 are the only maximal elements, and that $B_* \setminus \{\top\} = R^{-1}[x_1, x_2, x_3]$.

If $R^{-1}[\top] \cap R^{-1}[x_3] \subseteq R^{-1}[x_1, x_2]$, we fall under case (iii). So assume that it is not the case and define $W_1 = R^{-1}[\top] \cap R^{-1}[x_3] \setminus R^{-1}[x_1, x_2]$ and $W_2 = \{\top\} \cup R^{-1}[\top]^c$. These are both nonempty clopen sets, since x_1, x_2, x_3 and $\top$ are isolated.

We show that they form an M-partition. It is easy to check for W_2 since it is an upset. We show that W_1 does not see anything besides itself and W_2, which is enough to conclude (given $x, y \in W_1$ and $z \in R[x]$, we can always take either $\top$ or x_3 as a witness t such that $y \; R \; t$ and z, t are equivalent). Take $x \in W_1$ and $y \in R[x]$. Obviously $y \notin R^{-1}[x_1, x_2]$ since it is a downset. We have $B_* \setminus \{\top\} = R^{-1}[x_1, x_2, x_3]$ so either $y = \top$ or $y \in R^{-1}[x_3]$. In the first case, we have $y \in W_2$. In the second case, we have $y \in W_1$ if $y \in R^{-1}[\top]$ and $y \in W_2$ otherwise. One can then easily check that the definition of W_2 ensures that the quotient satisfies case (ii).

Finally, assume that there are at least two intersections. We can once again find clopen upsets U_1, U_2 and U_3 such that $x_i \in U_j$ iff $i = j$ for each $i, j \in \{1, \ldots, 3\}$. Define $V = (U_1 \cap U_2) \cup (U_2 \cap U_3) \cup (U_3 \cap U_1)$. We know that all three of U_1, U_2 and U_3 see V.

We define $V_1 = U_1 \cap R^{-1}[V] \setminus V$, $V_2 = U_2 \cap R^{-1}[V] \setminus V$, $V_3 = U_3 \cap R^{-1}[V] \setminus V$ and $W = R^{-1}[V_1 \cup V_2 \cup V_3]^c$. These are disjoint clopen sets. A similar argument to the previous case show that each of V_1, V_2 and V_3 sees exactly itself and W, and that V_1, V_2, V_3 and W form an M-partition of A_*. The quotient space then satisfies (ii). ☐

Theorem 4.21. *Let X be a cobwebby space. Then X does not contain X_1 as a subframe.*

Proof. Assume that X contains X_1 as a subframe. Then by Lemma 4.20, the variety generated by X contains an algebra B whose dual Y contains one of the subframes depicted in Figure 4.9, where the points x_1, x_2, x_3 and $\top$ are isolated.

By considering the skeleton of Y, we can still assume that every cluster in Y is improper, and since $R^{-1}[x_1] \cap R^{-1}[x_2] \cap R^{-1}[x_3]$ is closed, we may assume that $\perp$ is the only point in this set (if not, by Remark 3.10, take a maximal point y in this set and consider the subspace

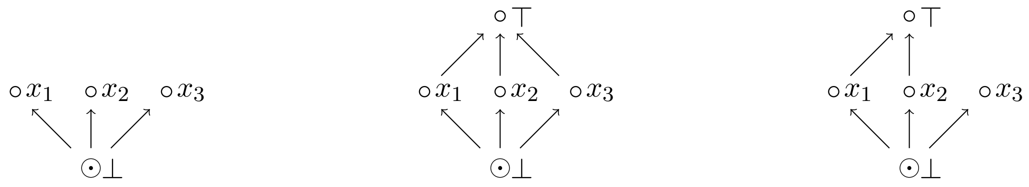


Figure 4.9: Three possible substructures

generated by y). The only irreflexive point in Y is possibly $\bot$. Consider the following sets

$$\begin{aligned} V_1 &= R^{-1}[x_1] \setminus R^{-1}[x_2, x_3], & V_2 &= R^{-1}[x_2] \setminus R^{-1}[x_3, x_1], & V_3 &= R^{-1}[x_3] \setminus R^{-1}[x_1, x_2], \\ W_1 &= R^{-1}[x_2, x_3] \setminus R^{-1}[x_1], & W_2 &= R^{-1}[x_3, x_1] \setminus R^{-1}[x_2], & W_3 &= R^{-1}[x_1, x_2] \setminus R^{-1}[x_3]. \end{aligned}$$

These sets are pairwise disjoint clopen sets, and their union, together with $\{\top, \bot\}$, covers Y. We show that they form an M-partition.

First assume $x, y \in V_1$ and $z \in R[x]$. Then either $z \in W_1$, and we may take $x_1 \in W_1$ as a witness since $y \mathrel{R} x_1$, or $z = \top$ and we may take $\top$ as a witness since $y \mathrel{R} \top$. We proceed in the same way for V_2. For V_3, if $z = \top$, then we must be in the case where $x_3 \mathrel{R} \top$ (since otherwise, $R^{-1}[\top] \cap R^{-1}[x_3] \subseteq R^{-1}[x_1, x_2]$), and we can take $\top$ as a witness. For W_1, assume $x, y \in W_1$ and $z \in R[x]$. Then $z \in V_2, V_3, W_1$ or $z = \top$, and we have $y \mathrel{R} x_2 \in V_2$, $y \mathrel{R} x_3 \in V_3$, $y \mathrel{R} y \in W_1$ and $y \mathrel{R} \top$ as witnesses. We proceed similarly for W_2 and W_3.

Let us look at the quotient induced by this M-partition. The sets V_1, V_2 and V_3 are always nonempty, while the sets W_1, W_2 and W_3 may be empty. Combining this with the three possible subframes of Y represented in Figure 4.9, we get, up to isomorphism, one of the spaces depicted in Figures 4.10 and 4.11. All of them reduce to a prohibited space through α- and β-reductions (the points that have to be identified are in red). □

We now turn to proving that a cobwebby space does not contain X_2.

Lemma 4.22. *Let K be a cobwebby variety and let $A \in K$. Assume that A_* contains X_2 as a subframe. Then K contains an algebra B such that B_* is rooted and has four isolated points x_1, x_2, y_1, y_2 such that $x_1 \mathrel{R} x_2$, $y_1 \mathrel{R} y_2$, x_i, y_j are incomparable for $i, j \in \{1, 2\}$ and either of the following holds.*

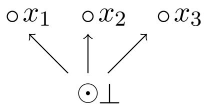
(i) *x_2 and y_2 are maximal and $B_* = R^{-1}[x_2, y_2]$,*

(ii) *B_* has a maximum $\top$ which is isolated and $B_* \setminus \{\top\} = R^{-1}[x_2, y_2]$,*

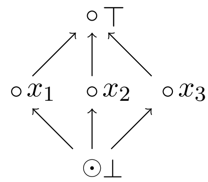
(iii) *B_* has an isolated point $\top$ such that $x_2 \mathrel{R} \top$, $y_1 \mathrel{R} \top$, $\top$ and x_2 are maximal and $B_* \setminus \{\top\} = R^{-1}[x_2, y_2]$ and $R^{-1}[\top] \cap R^{-1}[y_2] = R^{-1}[x_2, y_1]$*

Proof. By taking the subspace generated by $\bot$ and considering the skeleton, we may assume that $\bot$ is the root of A_* and that A_* only contains improper clusters. By Lemma 4.16, the irreflexive points are all in $R^{-1}[x_1] \cap R^{-1}[y_1]$. We do a case distinction on whether or not the set $R[x_1] \cap R[y_2]$ and $R[x_2] \cap R[y_1]$ are empty.

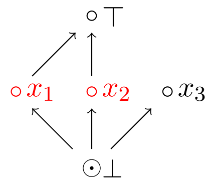
First assume that both sets are empty. Then by modal separation and a standard compactness argument, we can find disjoint clopen set U_1 and V_2 such that $x_1 \in V_1$ and $y_2 \in V_2$. We can also find disjoint clopen set U_2 and V_1 such that $x_2 \in U_2$ and $y_1 \in V_1$. By considering $U_1 \cap U_2$ instead of U_2, we may assume that $U_2 \subseteq U_1$. Similarly, we may assume $V_2 \subseteq V_1$. By using modal separation on x_1 and x_2, we find a clopen upset W such that $x_2 \in W$ and $x_1 \notin W$.



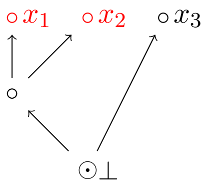
(a) Reduces to F_7



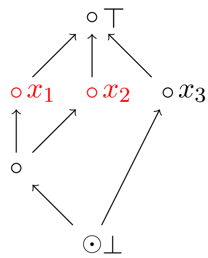
(b) Reduces to F_8



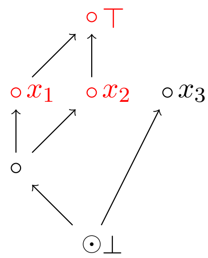
(c) Reduces to F_5



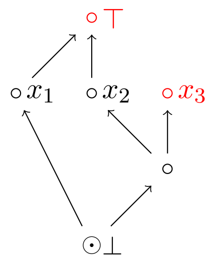
(d) Reduces to F_5



(e) Reduces to F_6

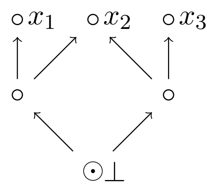


(f) Reduces to F_5

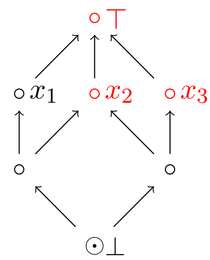


(g) Reduces to F_6

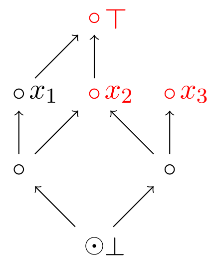
Figure 4.10: Possible configurations arising from X_1



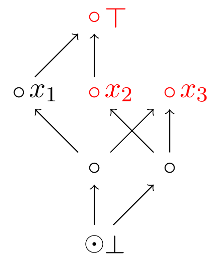
(a) Reduces to F_9



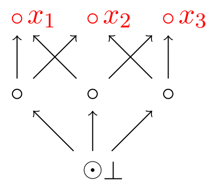
(b) Reduces to F_6



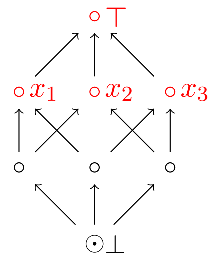
(c) Reduces to F_6



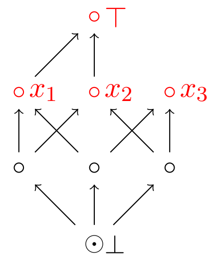
(d) Reduces to F_6



(e) Reduces to F_8



(f) Reduces to F_8



(g) Reduces to F_8

Figure 4.11: Possible configurations arising from X_1

By considering $U_2 \cap W$ instead of U_2, we may assume that U_2 omits x_1. Similarly, we may assume that V_2 omits y_1. Then the sets U_2, V_2 are disjoint clopen upsets, and therefore they form an M-partition. We let x_2 and y_2 denote the corresponding points in the quotient space B_*. We show that $B_* = R^{-1}[x_2, y_2]$. Take any point $z \in B_*$. By Theorem 4.21, B_* cannot have width 3, thus z is comparable with either x_2 or y_2. Since x_2 and y_2 are maximal, z has to be below one of them.

Now consider the sets $U_1' = U_1 \cap R^{-1}[x_2] \setminus \{x_2\}$ and $V_1' = V_1 \cap R^{-1}[y_2] \setminus \{y_2\}$. They are clopen sets, U_1' sees exactly itself and x_2 and V_1' sees exactly itself and y_2. We prove the claim for U_1'. Let $z \in U_1'$ and $t \in R[z]$. Then t is below either x_2 or y_2. If it is below y_2, then by transitivity $z\ R\ y_2$ thus $y_2 \in U_1'$, which is a contradiction. Therefore t is below x_2. If $t = x_2$, then we are done, and if $t \neq y_2$ then $t \in U_1'$. Therefore the sets U_1' and V_1' form an M-partition, and it is easy to check that the quotient satisfies (i).

Now assume that exactly one is nonempty. Without loss of generality $R[x_1] \cap R[y_2] = \emptyset$ and $R[x_2] \cap R[y_1] \neq \emptyset$. By modal separation and a standard compactness argument, we can find clopen upsets U_1, U_2, V_1, V_2 such that $U_2 \subseteq U_1$, $V_2 \subseteq V_1$, U_1 contains x_1 and x_2 but is disjoint from V_2, U_2 contains x_2 but not x_1, V_1 contains y_1 and y_2 but not x_2, V_2 contains y_2 but not y_1. We define $W = U_1 \cap V_1$. This is a nonempty clopen upset. We also define $U_2' = U_2 \cap R^{-1}[W] \setminus W$ and $W' = R^{-1}[U_2' \cup V_2]^c$. Clearly V_2 and W' are upsets and $W \subseteq W'$. We also have that U_2' sees itself and W'.

We prove that it does not see anything else. Let $z \in U_2'$ and $t \in R[z]$. Then as U_2 is an upset, we have $t \in U_2$. If $t \in W$, then $t \in W'$ and we are done. If $t \in R^{-1}[W]$, then $t \in U_2'$ and we are done. Otherwise, we have $t \notin R^{-1}[U_2']$ as $t \notin R^{-1}[W]$, and $t \notin R^{-1}[V_2]$ as U_2 is an upset disjoint from V_2. Therefore, we have $t \notin R^{-1}[U_2' \cup V_2]$, thus $t \in W'$ and we are done.

We actually just proved that U_2', V_2 and W' form an M-partition. Let us call x_2, y_2 and $\top$ the corresponding elements of the quotient space B_*. The definition of W' ensures that $\top$ and y_2 are the only maximal elements and $B_* \setminus \{\top\} = R^{-1}[x_2, y_2]$.

Now consider the clopen sets $U_1' = U_1 \cap R^{-1}[x_2] \setminus \{x_2\}$, $V_2' = R^{-1}[\top]^c$ and $V_1' = V_1 \cap R^{-1}[V_2'] \cap R^{-1}[\top]$. A proof similar to the one in the previous case shows that U_1' sees exactly $\top, x_2$ and itself. V_2' is an upset, contains y_2 but does not contain y_1 or x_2. We show that V_1' sees exactly $\top, V_2'$ and itself. Let $z \in V_1'$ and $t \in R[z]$ such that $t \neq \top$ and $t \notin V_2'$. Then t is below either x_2 or y_2, but as $t \in V_1$ and $x_2 \notin V_1$, we have $t\ R\ y_2$. Therefore, $t \in R^{-1}[V_2']$. As $t \notin V_2'$, we have $t\ R\ \top$, thus $t \in V_1'$. The sets U_1', V_2' and V_1' form an M-partition. Let us call x_1, y_2 and y_1 the corresponding points in the quotient space.

We show that it satisfies (iii). Indeed, take $z \in R^{-1}[\top] \cap R^{-1}[y_2]$. In view of Theorem 4.21, z is comparable with either x_1 or y_1. If it is below either of those, we are done. If $x_1\ R\ z$, then since $z\ R\ y_2$, we have $x_1\ R\ y_2$, which is a contradiction. If $y_1\ R\ t$, then $t = y_1$ by definition of V_1'.

Finally, assume that both sets are nonempty. By modal separation and standard compactness arguments, we can find clopen upsets U_1, U_2, V_1, V_2 such that $U_2 \subseteq U_1$, $V_2 \subseteq V_1$, U_1 contains x_1 and x_2 but not y_2, U_2 contains x_2 but not x_1, V_1 contains y_1 and y_2 but not x_2, and V_2 contains y_2 but not y_1. We define $W = U_1 \cap V_1$. This is a nonempty clopen upset. We also define $U_2' = U_2 \cap R^{-1}[W] \setminus W$, $V_2' = V_2 \cap R^{-1}[W] \setminus W$ and $W' = R^{-1}[U_2' \cup V_2']^c$. Clearly these are clopen sets and W' is an upset containing W. An argument similar to the one for the previous case shows that U_2' and V_2' see exactly themselves and W'. One can then easily check that the sets U_2', V_2' and W' form an M-partition and that the quotient space satisfies (ii). $\square$

Theorem 4.23. *Let X be a cobwebby space. Then X does not contain X_2 as a subframe.*

Proof. Assume that X contains X_2 as a subframe. Then by Lemma 4.22, the variety generated by X contains an algebra B whose dual Y contains one of the subframes represented in Figure 4.12, where the points x_1, x_2, y_1, y_2 and $\top$ are isolated.

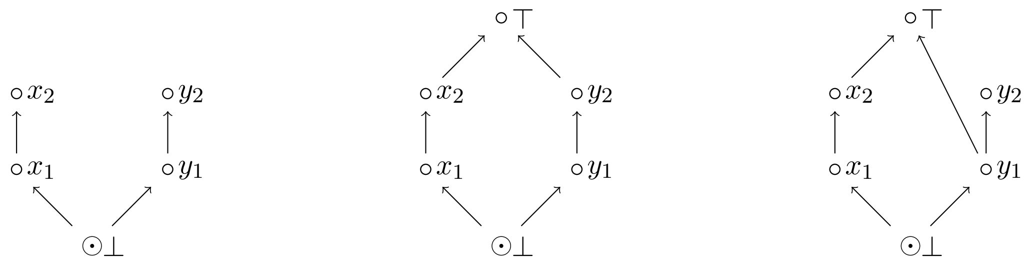


Figure 4.12: Three possible substructures

By considering the skeleton, we can still assume that every cluster in Y is improper, and since $R^{-1}[x_1] \cap R^{-1}[y_1]$ is closed, we may assume that $\bot$ is the only point in this set (if not, take a maximal point z in this set and consider the subspace generated by z). The only irreflexive point in Y is possibly $\bot$. Consider the following sets

$$
\begin{aligned}
U_2 &= R^{-1}[x_2] \setminus R^{-1}[x_1, y_2], & V_2 &= R^{-1}[y_2] \setminus R^{-1}[x_2, y_1],\\
W &= R^{-1}[x_2] \cap R^{-1}[y_2] \setminus R^{-1}[x_1, y_1], & &\\
U_1 &= R^{-1}[x_1] \setminus R^{-1}[y_2], & V_1 &= R^{-1}[y_1] \setminus R^{-1}[x_2],\\
U_0 &= R^{-1}[x_1] \cap R^{-1}[y_2] \setminus R^{-1}[y_1], & V_0 &= R^{-1}[x_2] \cap R^{-1}[y_1] \setminus R^{-1}[x_1].
\end{aligned}
$$

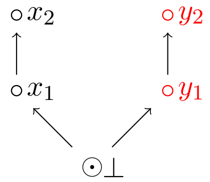
These sets are pairwise disjoint clopen sets, and their union, together with $\{\top, \bot\}$, covers Y. We show that they form an M-partition.

Assume that $x, y \in U_2$ and $z \in R[x]$. Then either $z \in U_2$, and we may take $x_2 \in U_2$ as a witness since $y \; R \; x_2$, or $z = \top$ and we may take $\top$ as a witness since $y \; R \; \top$. We proceed similarly for V_2, except that if $z = \top$ then we must have $y_2 \; R \; \top$, since otherwise $R^{-1}[y_2] \cap R^{-1}[\top] \subseteq R^{-1}[x_2, y_1]$. If $x, y \in U_1$ and $z \in R[x]$, then either $z \in U_1$ and we take x_1 as a witness, $z \in U_2$ and we take x_2 as a witness or $z = \top$ and we take $\top$ as a witness. We proceed similarly for V_1. If $x, y \in U_0$ and $z \in R[x]$, then either $z \in U_1$ and x_1 is a witness, $z \in U_2$ and x_2 is a witness, $z \in V_2$ and y_2 is a witness or $z = \top$ and $\top$ is a witness. Similarly for V_0. If $x, y \in W$ and $z \in R[x]$, then either $z \in W$ and we take y as a witness, $z \in U_2$ and we take x_2 as a witness, $z \in V_2$ and we take y_2 as a witness or $z = \top$ and we take $\top$ as a witness.

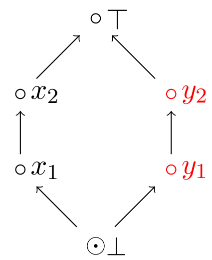
Let us look at the quotient induced by this partition. The sets U_1, U_2, V_1, V_2 are always nonempty since they contain x_1, x_2, y_1, y_2, respectively. The sets U_0, V_0 and W may be empty. Combining this with the three possible subframes of Y depicted in Figure 4.12, we get, up to isomorphism, one of the spaces depicted in Figures 4.13 and 4.14. All of them either reduce to or contain a prohibited space (the points that have to be identified or that are part of the subspace are in red). $\square$

Before proving that a cobwebby space does not contain X_3, we once again need a technical lemma, which is quite similar to the previous two.

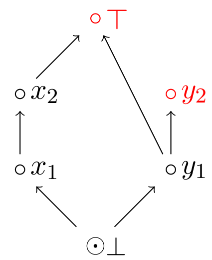
Lemma 4.24. *Let K be a cobwebby variety and let $A \in K$. Assume that A_* contains X_3 as a subframe. Then K contains an algebra B such that B_* is rooted and has four isolated points x_1, x_2, x_3, y such that $x_1 \; R \; x_2 \; R \; x_3$, x_i, y are incomparable for $i \in \{1, \ldots, 3\}$ and either of the following holds.*



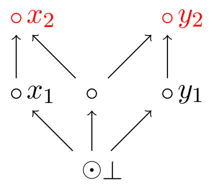
(a) Reduces to F_5



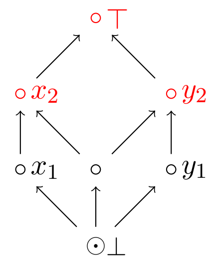
(b) Reduces to F_6



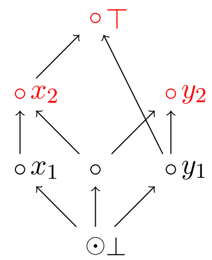
(c) Reduces to F_6



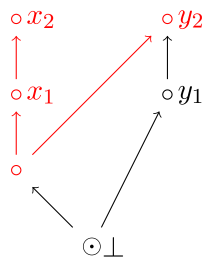
(d) Reduces to F_8



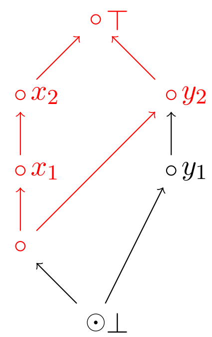
(e) Reduces to F_8



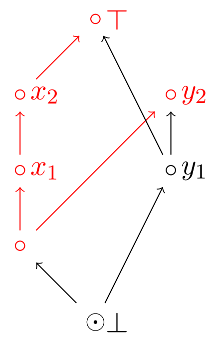
(f) Reduces to F_8



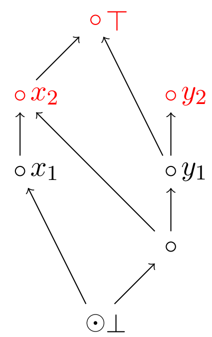
(g) Contains F_5



(h) Contains F_6

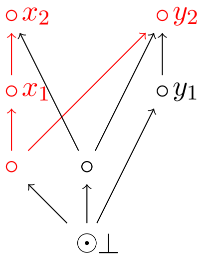


(i) Contains a space that reduces to F_5

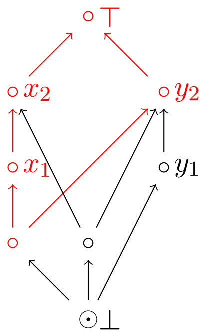


(j) Reduces to F_6

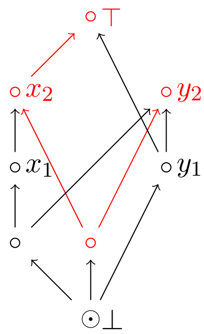
Figure 4.13: Possible configurations arising from X_2



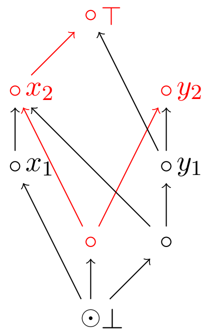
(a) Contains F_5



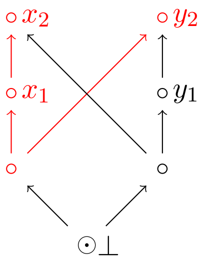
(b) Contains F_6



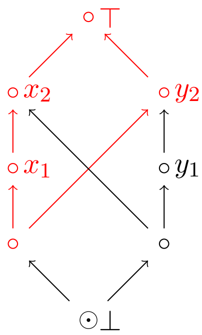
(c) Contains F_5



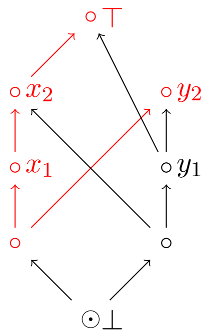
(d) Contains F_5



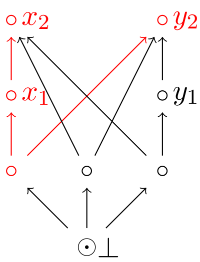
(e) Contains F_5



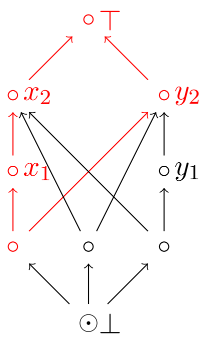
(f) Contains F_6



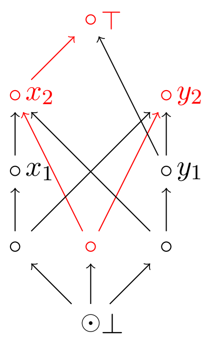
(g) Contains a space that reduces to F_5



(h) Contains F_5



(i) Contains F_6



(j) Contains F_5

Figure 4.14: Possible configurations arising from X_2

(i) x_3 *and* $\top$ *are maximal and* $B_* = R^{-1}[x_3, y]$*,*

(ii) B_* *has a maximum* $\top$ *which is isolated and* $B_* \setminus \{\top\} R^{-1}[x_3, y]$*,*

(iii) B_* *has an isolated point* $\top$ *such that* $x_2 \, R \, \top$*,* $y \, R \, \top$*,* $\top$ *and* x_3 *are maximal,* $B_* \setminus \{\top\} = R^{-1}[x_3, y]$ *and* $R^{-1}[\top] \cap R^{-1}[x_3] \subseteq R^{-1}[x_2, y]$*.*

Proof. By taking the subspace generated by $\perp$ and considering the skeleton, we may assume that $\perp$ is the root and that A_* only contains improper clusters. By Lemma 4.16, the irreflexive points are all in $R^{-1}[x_1] \cap R^{-1}[y]$. We do a case distinction on which of the sets $R[x_1], R[x_2]$ and $R[x_3]$ intersect $R[y]$.

First assume that none of them intersect $R[y]$. By modal separation and compactness, we can find clopen upsets U_1, U_2, U_3 and V such that $U_3 \subseteq U_2 \subseteq U_1$, U_1 and V are disjoint, $y \in V$, $x_1 \in U_1$, $x_2 \in U_2$ but $x_1 \notin U_2$ and $x_3 \in U_3$ but $x_2 \notin U_3$.

We define $U_2' = U_2 \cap R^{-1}[U_3] \setminus U_3$ and $U_1' = U_1 \cap R^{-1}[U_3] \setminus U_2$. These are clopen sets. The set U_2' sees exactly itself and U_3. Indeed, let $z \in U_2'$ and $t \in R[x]$. By Theorem 4.21, t has to be comparable with either x_3 or y. If it is comparable with y, we directly get that $R[y]$ intersects $R[x_1]$. Therefore t is comparable with x_3. If $x_3 \; R \; t$, then $t \in U_3$. If $t \; R \; x_3$, then $t \in R^{-1}[U_3]$ so either $t \in U_3$ or $t \in U_2'$. By a similar reasoning, we show that U_1' sees itself, U_2' and U_3. Using reflexivity of the points in U_1', U_2', U_3 and V and their definition, we easily show that these sets form an M-partition. Let x_1, x_2, x_3 and y be the corresponding points in the quotient space B_*. Because U_3 and V are upsets, x_3 and y are maximal and by Theorem 4.21, we have $B_* = R^{-1}[x_3, y]$, thus B_* satisfies (i).

Now assume that $R[y]$ intersects $R[x_1]$ but not $R[x_2]$. Then $R[y] \neq \{y\}$. Consider a point $y_2 \in R[y] \setminus \{y\}$. This new point has to be incomparable with x_2 and x_3. If $y_2 \; R \; x_3$, then $y \; R \; x_3$ which is a contradiction, and by assumption $x_2 \not R \; y_2$. Then the points $\perp, x_1, x_2, y, y_2$ are as in X_2, contradicting Theorem 4.23.

Let us thus assume that $R[y]$ intersects $R[x_2]$ but not $R[x_3]$. By modal separation and compactness, we can find clopen upsets U_1, U_2, U_3 and V such that $U_3 \subseteq U_2 \subseteq U_1$, $y \in V$ but $x_3 \notin V$, $x_1 \in U_1$, $x_2 \in U_2$ but $x_1 \notin U_2$ and $x_3 \in U_3$ but $x_2 \notin U_3$.

We define $W = U_1 \cap V$, $V' = V \cap R^{-1}[W]$ and $W' = R^{-1}[U_3 \cup V']^c$. These are all clopen sets, and U_3 and W' are upsets. We claim that V' only sees itself and W'.

Take $z \in V'$ and $t \in R[z]$. If $z \in W'$, then we are done. Otherwise, we have $t \in R^{-1}[U_3 \cup V']$. Since V is an upset, we have $t \in V$, and since V is disjoint from U_3, we have $t \notin R^{-1}[U_3]$. Therefore, $t \in R^{-1}[V']$. Since $V' \subseteq R^{-1}[W]$, we have $t \in R^{-1}[W]$, thus $t \in V'$.

Therefore, the sets U_3, V', W form an M-partition. Let $x_3, y, \top$ be the corresponding points in the quotient space B_*. Since x_3 and $\top$ are maximal, and the width of B_* is bounded by 2 by Theorem 4.21, we have $B_* = R^{-1}[x_3, \top]$. Since the only immediate successor of y is $\top$, y is maximal in $B_* \setminus \{\top\}$ and we have $B_* \setminus \{\top\} = R^{-1}[x_3, y]$.

Now define the sets $U_3' = R^{-1}[\top]^c$, $U_2' = U_2 \cap R^{-1}[U_3'] \cap R^{-1}[\top]$ and $U_1' = U_1 \cap R^{-1}[U_2'] \setminus U_2'$. These are all clopen sets, and they are nonempty since $x_3 \in U_3'$, $x_2 \in U_2'$ and $x_1 \in U_1'$. Also notice that U_3' is an upset, and $U_1' \subseteq R^{-1}[U_3'] \cap R^{-1}[\top]$ by transitivity. We claim that U_2' only sees itself, U_3' and $\top$, and that U_1' only sees itself and the successors of U_2'.

Take $z \in U_2'$ and $t \in R[z]$. If $t = \top$, we are done. Otherwise, we have $t \in R^{-1}[x_3, y]$. If $t \; R \; y$, then $z \; R \; y$ so $y \in U_2$, which is a contradiction. Hence $t \in R^{-1}[x_3] \subseteq R^{-1}[U_3']$. If $t \in U_3'$, we are done, and otherwise we have $t \in R^{-1}[\top]$ thus $t \in U_2'$.

Now take $z \in U_1'$ and $t \in R[z]$. If $t = \top$, we are done. Otherwise, we have $t \in R^{-1}[x_3, y]$, but we have already seen that $t \in R^{-1}[y]$ leads to a contradiction since $y \notin U_1$. Hence $t \in R^{-1}[x_3] \subseteq U_3'$. If $t \in U_3'$, we are done. Otherwise, we have $t \in R^{-1}[\top]$. If $t \in U_2$, we are done. If $t \in R^{-1}[U_2']$, we are also done. If neither are the case, then we have in particular $x_2 \not R\, t$ and $t \not R\, x_2$, so t and x_2 are incomparable. We already know that t and y are also incomparable, so x_2, t and y are three pairwise incomparable points, contradicting Theorem 4.21.

Therefore, the sets U_3', U_2', U_1' form an M-partition. The quotient satisfies (iii). Indeed, assume that $t \in R^{-1}[\top] \cap R^{-1}[x_3]$. If $t\, R\, x_2$ or $t\, R\, y$, we are done. If $y\, R\, t$, then $y\, R\, x_3$, which is a contradiction. If $x_2\, R\, t$, then $t = x_2$ by definition of U_2'. Otherwise, the points x_2, t, y are pairwise incomparable, which is a contradiction.

Finally, assume that $R[y]$ intersects $R[x_3]$. Then we can find clopen upsets U_1, U_2, U_3 and V such that $U_3 \subseteq U_2 \subseteq U_1$, $y \in V$ but $x_3 \notin V$, $x_1 \in U_1$ but $y \notin U_1$, $x_2 \in U_2$ but $x_1 \notin U_2$ and $x_3 \in U_3$ but $x_2 \notin U_3$. We define $W = U_1 \cap V$, $U_3' = U_3 \cap R^{-1}[W] \setminus W$, $U_2' = U_2 \cap R^{-1}[U_3'] \setminus U_3'$, $U_1' = U_1 \cap R^{-1}[U_2'] \setminus U_2'$, $V' = V \cap R^{-1}[W] \setminus W$ and $W' = R^{-1}[U_3' \cup V']^c$. We claim that U_3' sees exactly itself and W', U_2' sees exactly itself and the successors of U_3', U_1' sees exactly itself and the successors of U_2', and V' sees exactly itself and W'. The proof of this claim is by an argument that should now be standard, and is omitted.

Consequently, the sets W', U_3', U_2', U_1', V' form an M-partition, and the quotient satisfies (ii). □

Theorem 4.25. *Let X be a cobwebby space. Then X does not contain X_3 as a subframe.*

Proof. Assume that X contains X_3 as a subframe. Then by Lemma 4.24, the variety generated by X contains an algebra B whose dual Y contains one of the subframes depicted in Figure 4.15, where the points x_1, x_2, x_3, y and $\top$ are isolated.

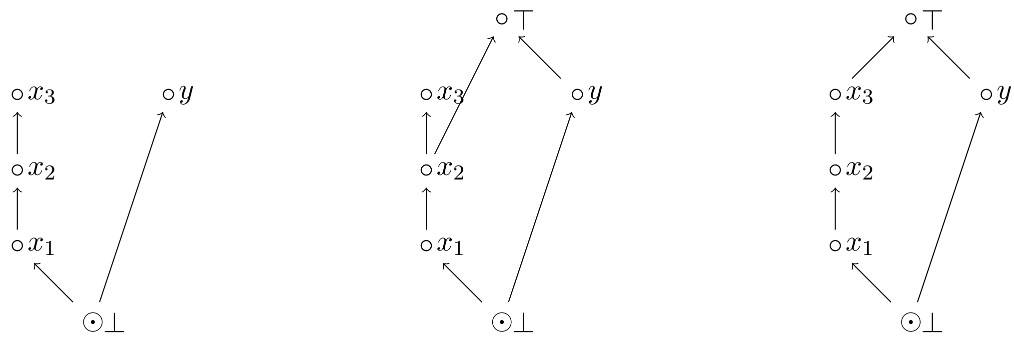


Figure 4.15: Three possible substructures

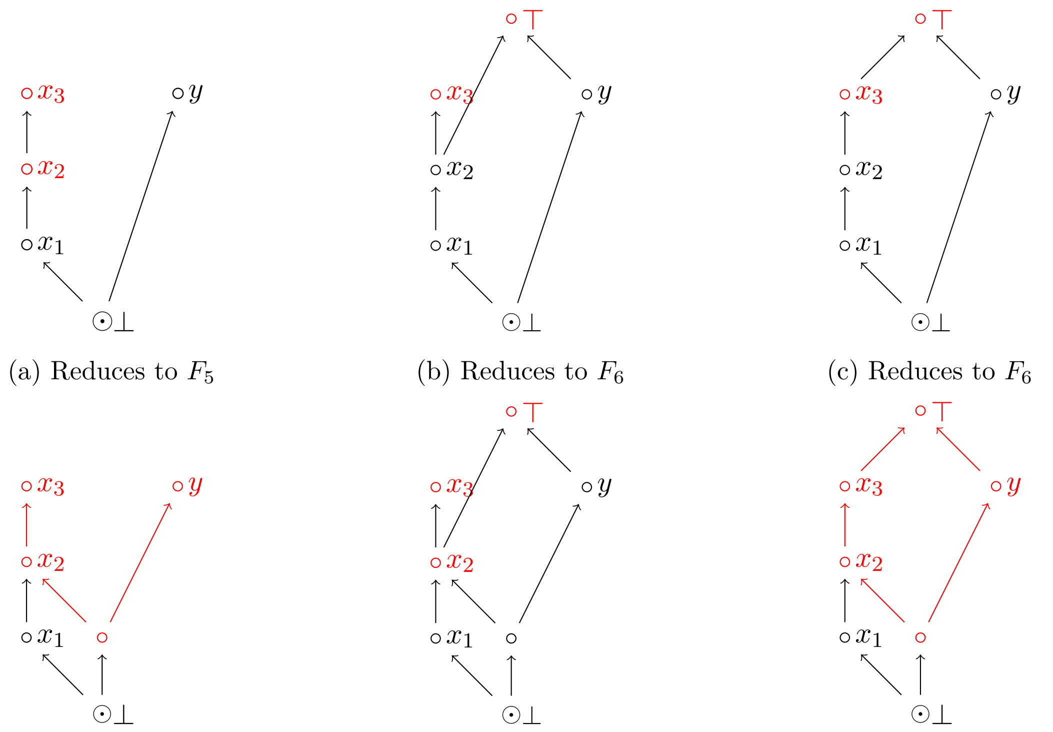
By considering the skeleton, we can assume that every cluster in Y is improper, and since $R^{-1}[x_1] \cap R^{-1}[y]$ is closed, we may assume that $\bot$ is the only point in this set (if not, take a maximal point z in this set and consider the subspace generated by z). The only irreflexive point in Y is possibly $\bot$. Consider the following sets

$$\begin{aligned} U_3 &= R^{-1}[x_3] \setminus R^{-1}[x_2, y], & W_3 &= R^{-1}[x_3] \cap R^{-1}[y] \setminus R^{-1}[x_2], \\ U_2 &= R^{-1}[x_2] \setminus R^{-1}[x_1, y], & W_2 &= R^{-1}[x_2] \cap R^{-1}[y] \setminus R^{-1}[x_1], \\ U_1 &= R^{-1}[x_1] \setminus R^{-1}[y], & V &= R^{-1}[y] \setminus R^{-1}[x_3]. \end{aligned}$$

These sets are pairwise clopen sets, and their union, together with $\{\top, \bot\}$, covers Y. We show that they form an M-partition.

Assume that $x, z \in U_3$ and $t \in R[x]$. Then either $t \in U_3$, and we may take $x_3 \in U_3$ as a witness since $y \; R \; x_3$, or $t = \top$, which implies that $x_3 \; R \; \top$ and we may take $\top$ as a witness since $z \; R \; \top$. We proceed similarly for V, without the extra consideration if $t = \top$. If $x, z \in U_2$ and $t \in R[x]$, then either $t \in U_2$ and we may take $x_2 \in U_2$ as a witness, or $t \in U_3$ and we may take $x_3 \in U_3$ as a witness, or $t = \top$ and we may take $\top$ as a witness. If $x, z \in W_3$ and $t \in R[x]$, then t is in either U_3, V, W_3 or is $\top$, and we can take x_3, y, z or $\top$, respectively, as a witness. If $x, z \in W_2$ and $t \in R[t]$, then t is in either U_3, U_2, W_3, W_2 or is $\top$, and we can take x_3, x_2, t, z or $\top$, respectively, as a witness. The only claim that requires further argumentation is that if $t \in W_3$, then $z \; R \; t$. Assume not. Then $t \not R \; z$ as $z \in R^{-1}[x_2]$ and $t \notin R^{-1}[x_2]$. Hence z and t are incomparable. Furthermore, neither z nor t see x_1 by definition of W_2 and W_3, and x_1 does not see any of them because x_1 does not see y. Hence x_1, z and t are pairwise incomparable, which contradicts Theorem 4.21.

Let us look at the quotient induced by this M-partition. The sets U_1, U_2, U_3 and V are always nonempty since they contain x_1, x_2, x_3 and y, respectively. We will prove that the set W_3 is always empty. For if not, take $t \in W_3$. Then $t \; R \; y$ and t is incomparable with x_1 and x_2, therefore Y contains X_2 as a subframe ($\perp, x_1, x_2, t$ and y), contradicting Theorem 4.23. The sets W_2 may or may not be empty. Combining this with the three possible subframes of Y represented in Figure 4.15, the quotient induced by the M-partition is one of the spaces depicted in Figure 4.16. Each of them contain or reduce to a prohibited space (the points that have to be identified or that are part of the subspace are in red). □



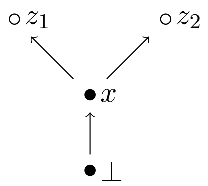
(a) Reduces to F_5 (b) Reduces to F_6 (c) Reduces to F_6

(d) Contains F_5 (e) Reduces to F_6 (f) Contains F_6

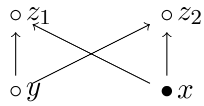
Figure 4.16: Possible configurations arising from X_3

We further show that a cobwebby space cannot contain any of the subframes in Figure 4.17, where z_1 and z_2 are immediate successors of x, (that is, $R[x] \cap R^{-1}[z_i] \subseteq R[z_i]$). The proof is split in two parts.

Let us start with X_4.



(a) X_4



(b) X_5

Figure 4.17: Two frames

Theorem 4.26. *Let X be a cobwebby space. Then X does not contain X_4 as a subframe, where z_1 and z_2 are immediate successors of x, i.e. $R[x] \cap R^{-1}[z_i] \subseteq R[z_i]$ for $i \in \{1, 2\}$.*

Proof. Suppose that X contains X_4 as a subframe. By considering its skeleton, we may assume that clusters in X is improper. Since $R^{-1}[x]$ is closed, we may assume that $\bot$ is the only point in this set. If not, take a maximal point $\bot$ in this set and take the subspace generated by $\bot$. This construction also ensures that $\bot$ is the root of X. By Lemma 4.16, and the irreflexivity of x, every point is comparable with x, and therefore $X \setminus \{\bot, x\} = R[x]$. Because we are working in a skeleton, the assumption that z_1 and z_2 are immediate successors of implies that $R[x] \cap R^{-1}[z_i] = \{z_i\}$ for $i \in \{1, 2\}$. By Theorem 4.21, every point is comparable with either z_1 or z_2, therefore $X \setminus \{\bot, x\} = R[x] = R[z_1, z_2]$.

First assume that $R[z_1]$ and $R[z_2]$ are disjoint. Then by modal separation and compactness, we can find disjoint clopen upsets U_1 and U_2 such that $z_i \in U_j$ iff $i = j$, for $i, j \in \{1, 2\}$. Then the sets U_1, U_2 form an M-partition, and the quotient is F_{16}, which contradicts X being cobwebby.

Now assume that $R[z_1]$ intersects $R[z_2]$. By modal separation and compactness, we can find clopen upsets U_1 and U_2 such that $z_i \in U_j$ iff $i = j$, for $i, j \in \{1, 2\}$. Consider the sets $W = U_1 \cap U_2$, $V_1 = U_1 \cap R^{-1}[W]$, $V_2 = U_2 \cap R^{-1}[W]$ and $W' = R^{-1}[U_1 \cup U_2]$. Using the standard argument, the sets U_1, U_2, W' form an M-partition, and the quotient is F_{17}, which contradicts X being cobwebby. $\square$

Theorem 4.27. *Let X be a cobwebby space. Then X does not contain X_5 as a subframe, where z_1 and z_2 are immediate successors of x, i.e. $R[x] \cap R^{-1}[z_i] \subseteq R[z_i]$ for $i \in \{1, 2\}$.*

Proof. Assume that X contains the subframe. As in Theorem 4.26, we may assume that every cluster in X is improper, so in particular $R[x] \cap R^{-1}[z_i] = \{z_i\}$ for $i \in \{1, 2\}$. By considering the subspace generated by x and y, we may assume that $X = R^+[x, y]$. Furthermore, since X has width at most 2 by Theorem 4.21, every point is comparable with either z_1 or z_2. Since $R[x] \cap R^{-1}[z_i] = \{z_i\}$ for $i \in \{1, 2\}$, we obtain that $R[x] = R[z_1, z_2]$. Finally, the only irreflexive point in X is x, because all other points are successors of y, z_1 or z_2, which are reflexive, so by Lemma 4.15 they have to be reflexive.

First assume that $R[z_1]$ and $R[z_2]$ are disjoint. Then by modal separation and compactness, we can find disjoint clopen upsets U_1 and U_2 such that $z_i \in U_j$ iff $i = j$, for $i, j \in \{1, 2\}$. Then the sets U_1, U_2 form an M-partition, so we may collapse them to two maximal points z_1 and z_2. By Theorem 4.21, any point is below z_1 or z_2. Consider the clopen set $V = R^{-1}[z_1] \cap R^{-1}[z_2]$. We have $x \in V \setminus R^{-1}[V]$, and any point $t \in V \setminus R^{-1}[V]$ has to be irreflexive, and therefore equal to x. Thus $V \setminus R^{-1}[V] = \{x\}$ and x is isolated. Consider the clopen sets $U_1 = R^{-1}[z_1] \setminus V$,

$U_2 = R^{-1}[z_2] \setminus V$ and $W = V \setminus \{x\}$. These are pairwise disjoint, and together with $\{x\}$ they cover X. We claim that they form an M-partition. The sets U_1 and U_2 are upsets, so they trivially satisfy the condition. If $u, v \in W$ and $w \in R[u]$, then we have either $w \in U_1$ and we can take z_1 as a witness, $w \in U_2$ and we can take z_2 as a witness of $w \in W$ and we can take v as a witness. Therefore we can collapse the sets U_1, U_2 and W, and the quotient space is F_{13}. $\square$

4.6 The main structural theorem

We can now give our main description of the structure of cobwebby spaces. We first need to define the *sequential composition* of frames.

Definition 4.28. Given a linear order I and a collection of frames $(X_i, R_i)_{i \in I}$ (for simplicity, let us assume that the sets $(X_i)_{i \in I}$ are pairwise disjoint), we define their *sequential composition* $\bigoplus_{i \in I} X_i$. The frame $\bigoplus_{i \in I} X_i$ has underlying set $\bigcup_{i \in I} X_i$, and its accessibility relation is defined by $R = \bigcup_{i \in I} R_i \cup \bigcup_{i < j \in I} X_j \times X_i$.

We also define the frame H, depicted in Figure 4.18, which will play a role in our next theorem. The symbol $*$ denotes a finite cluster.

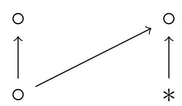


Figure 4.18: The structure of H

Our main theorem can be formulated in terms of sequential compositions.

Theorem 4.29. *Let K be a cobwebby variety, let $A \in K$ and let X be its dual. Assume that A is finitely generated and subdirectly irreducible. Then the frame underlying X is a sequential composition $\bigoplus_{\alpha \leq \beta} Q_\alpha$ of finite frames $(Q_\alpha)_{\alpha \leq \beta}$ for some ordinal $\beta = \lambda + n$, with λ a limit ordinal and $n < \omega$, such that the following hold.*

(i) Q_α is a single cluster if $\alpha = \beta$ or α is a limit ordinal.

(ii) Q_α is a single cluster, a two cluster antichain or H if $\alpha = 0$.

(iii) Q_α is a single cluster or a two cluster antichain otherwise.

(iv) Any maximal cluster is a single reflexive point, if Q_α is a two cluster antichain then $Q_{\alpha+1}$ only contains improper clusters, and any non-minimal, proper cluster contains a reflexive point.

(v) If X contains an irreflexive cluster, then $n \neq 0$ and there is some $m \in \{1, \dots, n\}$ such that for all $\alpha < \lambda + m$, Q_α does not contain any irreflexive cluster, and for all $\alpha \geq \lambda + m$, Q_α is an irreflexive cluster. Moreover, if $m < n$, then $Q_{\lambda+m-1}$ is a single cluster.

Before proceeding to the proof, we need to prove that the accessibility relation on X is conversely well-founded, thus allowing us to proceed recursively in the proof. This is a consequence of the following theorem.

Lemma 4.30. *Let X be a finitely generated cobwebby space. Then X contains no infinite ascending chain.*

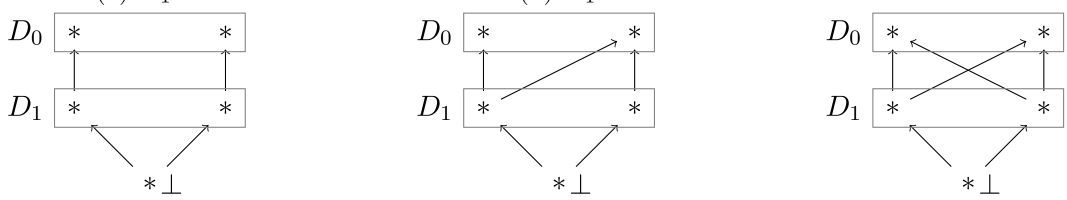
Proof. This is a particular case of in [17, Thm 10.34], as Theorem 4.21 implies that the width of cobwebby spaces is at most 2. ☐

We can now prove the main theorem.

Proof. By Lemma 4.30 and Theorem 4.21, X contains no infinite ascending chain. Therefore, any nonempty subset $Y \subseteq X$ contains maximal points. As A is subdirectly irreducible, X is rooted. Since A is finitely generated, all clusters in X are finite. We prove that X has the required structure by ordinal recursion.

Let us start with $\alpha = 0$. We look at the set D_0 of maximal clusters. Note that as X is rooted, by Theorem 4.21, D_0 contains at most two clusters. If it only contains one cluster, then we let Q_0 be that cluster. If $X = D_0$, we also set $Q_0 = D_0$ and we are done. Otherwise, let us look at D_1. The set D_1 also contains at most two clusters. Let us look at the possible subframe that can arise from this configuration. Recall that $*$ designate a finite cluster.

(a) D_1 contains one cluster (b) D_1 contains one cluster



(c) D_1 contains two clusters (d) D_1 contains two clusters (e) D_1 contains two clusters

In the second and fifth case, we let Q_0 be the two clusters in D_0, and Q_1 be the one or two clusters in D_1. We then have that $Q_0 \oplus Q_1$ is is the subframe generated by $Q_0 \cup Q_1$, and $X \setminus Q_0 \cup Q_1 = R^{-1}[Q_1]$ (or $X \setminus Q_0 \cup Q_1 = \emptyset$, in which case we are done). By Theorem 4.23, we know that the third case never arises.

We show that the first case never arises. Take a cluster C in D_2 (we know that such a cluster exists, otherwise X would not be rooted). Then either C sees both clusters in D_0 (in particular, it sees the right one), and the subspace generated by C reduces to F_5, or it only sees one of the cluster in D_0 (the left one), in which case X contains X_3 as a subframe, which is a contradiction.

In the fourth case, we let Q_0 be $D_0 \cup D_1$. We claim that Q_0 is H. Indeed, any of the maximal clusters is improper since K omits F_1, and the lower left cluster is improper since K omits F_4. We know that the maximal clusters have to be reflexive by Lemma 4.12, and the lower left cluster has to be reflexive by Lemma 4.16. Because X is rooted, we know that D_2 is nonempty, and it contains one or two clusters. This gives rise to several possible subframes.

Most cases never arise because they contradict either Theorems 4.23 or 4.25. In the two cases that can arise (the third one if D_3 contains on cluster and the fourth one if D_3 contains two clusters), we define $Q_1 = D_2$ and obtain that $Q_0 \oplus Q_1$ is is the subframe generated by $Q_0 \cup Q_1$, and $X \setminus Q_0 \cup Q_1 = R^{-1}[Q_1]$ (or $X \setminus Q_0 \cup Q_1 = \emptyset$, in which case we are done).

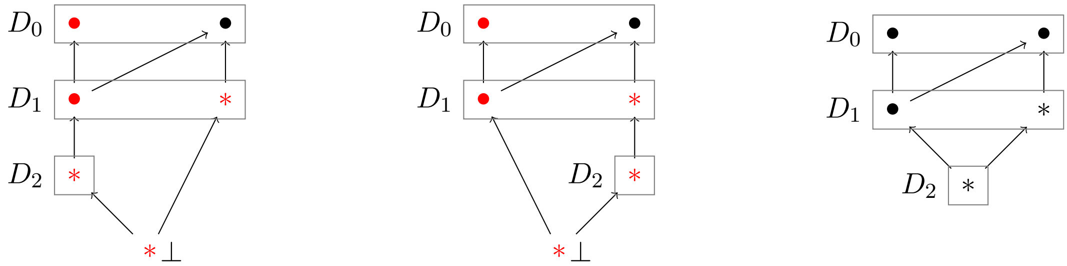


Figure 4.20: D_3 contains one cluster

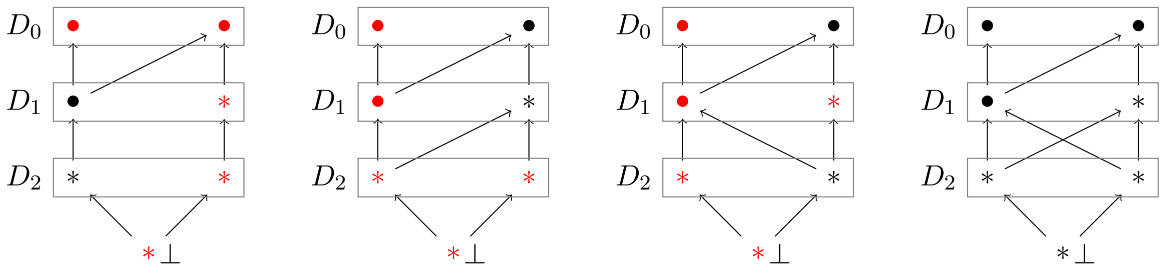


Figure 4.21: D_3 contains two clusters

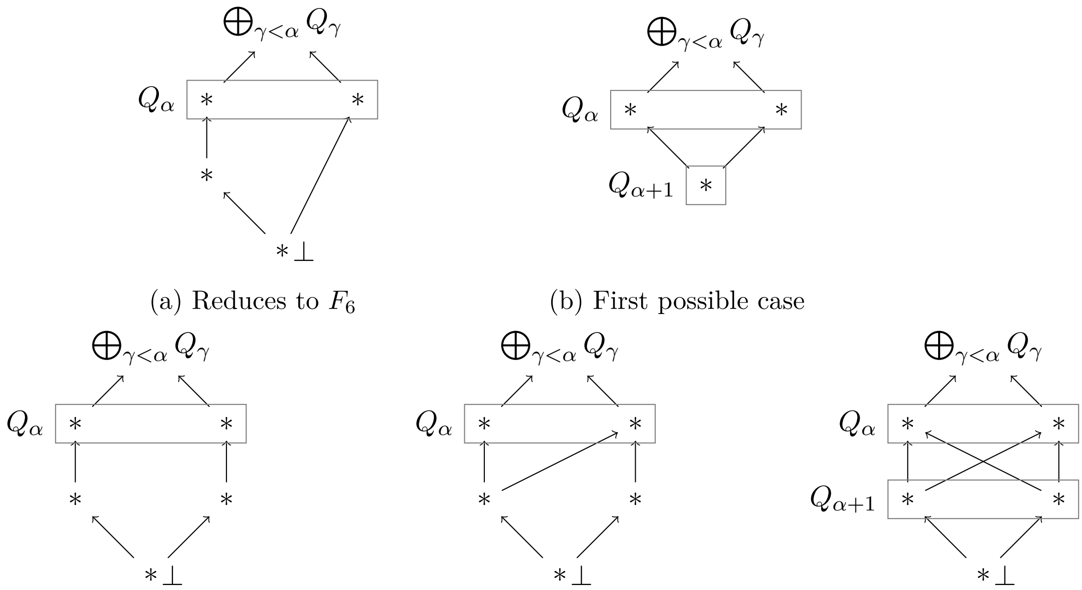
Now let us tackle the successor case. Take $\alpha \geq 1$ and assume that for $\gamma \leq \alpha$, we have a set $Q_\gamma \subseteq X$ such that $\bigoplus_{\gamma\leq\alpha} Q_\gamma$ is the subframe of X generated by $\bigcup_{\gamma\leq\alpha} Q_\gamma$ and $X \setminus \bigcup_{\gamma\leq\alpha} Q_\gamma = R^{-1}[Q_\alpha]$. By Lemma 4.30, X is conversely well-founded and we can consider the maximal clusters in $X \setminus \bigcup_{\gamma\leq\alpha} Q_\gamma$. Since X has width at most two, there are at most two such clusters. If Q_α consists of a single cluster, we let $Q_{\alpha+1}$ be the maximal cluster(s) of $X \setminus \bigcup_{\gamma\leq\alpha} Q_\gamma$. We have that $\bigoplus_{\gamma\leq\alpha+1} Q_\gamma$ is the subframe generated by $\bigcup_{\gamma\leq\alpha+1} Q_\gamma$, and either $X \setminus \bigcup_{\gamma\leq\alpha+1} Q_\gamma = R^{-1}[Q_{\alpha+1}]$, or $X \setminus \bigcup_{\gamma\leq\alpha+1} Q_\gamma = \emptyset$ and we are done.

If Q_α contains two clusters, then we have one of the subframes depicted in Figure 4.22.

Three of them can be eliminated because they reduce to F_6. In the remaining two cases, we let $Q_{\alpha+1}$ be the maximal cluster(s) in $X \setminus \bigcup_{\gamma\leq\alpha} Q_\gamma$. Then $\bigoplus_{\gamma\leq\alpha+1} Q_\gamma$ is the subframe generated by $\bigcup_{\gamma\leq\alpha+1}$ and $X \setminus \bigcup_{\gamma\leq\alpha+1} Q_\alpha$ is either empty or equal to $R^{-1}[Q_{\alpha+1}]$.

We now turn to the limit case. Let α be a limit ordinal and assume that for all $\gamma < \alpha$, we have a set $Q_\gamma \subseteq X$ such that $\bigoplus_{\gamma<\alpha} Q_\gamma$ is the subframe of X generated by $\bigcup_{\gamma<\alpha} Q_\gamma$, and $X \setminus \bigcup_{\gamma<\alpha} Q_\gamma \subseteq R^{-1}[Q_\delta]$ for any $\delta < \alpha$. A simple modal separation argument shows that for any $\delta < \alpha$, the set $\bigcup_{\gamma\leq\delta} Q_\gamma$ is a clopen upset. Therefore, the set $\bigcup_{\gamma<\alpha} Q_\gamma$ is not compact, since $(\bigcup_{\gamma\leq\delta} Q_\gamma)_{\delta<\alpha}$ is an open cover with no finite subcover. This implies that $\bigcup_{\gamma<\alpha} Q_\gamma \neq X$, thus we can take Q_α to be the maximal elements in $X \setminus \bigcup_{\gamma<\alpha} Q_\gamma$. We show that Q_α contains only one cluster. Suppose that Q_α contains two clusters, then we can find two unrelated points $x, y \in Q_\alpha$. The set $R[x] \cap R[y]$ is closed, thus compact. Our assumptions ensure that $R[x] \cap R[y] = \bigcup_{\gamma<\alpha} Q_\gamma$, which is not compact. This is a contradiction, therefore Q_α only contains one cluster. We can check that $\bigoplus_{\gamma\leq\alpha} Q_\gamma$ is the subframe of X generated by $\bigcup_{\gamma\leq\alpha} Q_\gamma$, and that $X \setminus \bigcup_{\gamma\leq\alpha} Q_\gamma$ is either empty or equal to $R^{-1}[Q_\alpha]$.

This establishes the first three conditions. We still need to prove the last part of the claim. Any maximal cluster in X is reflexive by Lemma 4.12. If Q_α contains two clusters, then $Q_{\alpha+1}$ can only contain improper cluster, otherwise the skeleton of the frame generated by that



(a) Reduces to F_6 (b) First possible case

(c) Reduces to F_6 (d) Reduces to F_6 (e) Second possible case

Figure 4.22: Q_α contains two clusters

cluster reduces to either F_4 or F_{10} (by collapsing $\bigcup_{\gamma<\alpha} Q_\gamma$ to a single point). Any non-minimal, proper cluster contains a reflexive point by Lemma 4.17.

First, if α is a limit ordinal, then Q_α cannot be an irreflexive cluster. Assume otherwise. Then $R[Q_\alpha] = \bigcup_{\gamma<\alpha} Q_\gamma$, thus this set is closed and compact. But we have seen in the previous paragraph that this set is not compact, thus we have a contradiction. The case of $\alpha = 0$ also showed that clusters in Q_0 cannot be irreflexive. By Lemma 4.15, we know that any cluster below an irreflexive cluster also has to be irreflexive. These fact together show that, if $\beta = \lambda + n$ with λ a limit ordinal and $n < \omega$, and X contains an irreflexive cluster, then there is $m \in \{1, \ldots, n\}$ such that Q_α does not contain irreflexive clusters for $\alpha < \lambda + m$ and Q_α only contains irreflexive clusters for $\alpha \geq \lambda + m$. By Lemma 4.16, any Q_α with $\alpha \geq \lambda + m$ contains exactly one cluster.

If $m = n$, we are done. Otherwise, we need to show that $Q_{\lambda+m-1}$ is a single cluster. Assume not. Take the subspace generated by $Q_{\lambda+m+1}$, and take the skeleton. We obtain one of the following structures.

The first one is prohibited, because it is F_{16}. The second one reduces to F_{17} after collapsing $\bigcup_{\gamma<\lambda+m-1} Q_\gamma$ to a single reflexive point. $\square$

We can specialise this result to finite spaces.

Corollary 4.31. *Let A be a cobwebby algebra and let X be its dual. Assume that A is finite and subdirectly irreducible. Then the frame underlying X is a sequential composition $Q_0 \oplus \cdots \oplus Q_n$ of finite frames $Q_0, \ldots, Q_n$ such that the following hold.*

(i) Q_k *is a single cluster if* $k = n$.

(ii) Q_k *is a single cluster, a two cluster antichain or* H *if* $k = 0$.

(iii) Q_k *is a single cluster or a two cluster antichain otherwise.*

(iv) Any maximal cluster is a single reflexive point, and if Q_k *is a two cluster antichain then* Q_{k+1} *only contains improper clusters.*

(v) If X *contains an irreflexive cluster, then* $n \neq 0$ *and there is some* $m \in \{1, \ldots, n\}$ *such that for all* $k < m$, Q_k *does not contain any irreflexive cluster, and for all* $k \geq m$, Q_k *is an irreflexive cluster. Moreover, if* $m < n$, *then* Q_{m-1} *is a single cluster.*

4.7 Primitive varieties

With our description of the finitely generated subdirectly irreducible wK4-spaces, we have all the tools we need to finish the proof. This is done in two parts. In the next theorem, we prove that any cobwebby variety K has the finite model property. Afterwards, we show that in any cobwebby variety K, the finite subdirectly irreducible elements are weakly projective in K.

Theorem 4.32. *Let K be a cobwebby variety. Then K has the finite model property.*

Proof. We use the same proof strategy as Carr [16] and Rybakov [42], which itself is a variation of Fine's drop-point technique [23].

We need to show that for any equation $\varepsilon \approx \delta$ such that $A \not\models \varepsilon \approx \delta$ for some $A \in K$, then $A \not\models \varepsilon \approx \delta$ for some finite $A \in K$. Recalling that K is the equivalent algebraic semantics of a modal logic, it is sufficient to show that for any formula $\phi \in Fm$ such that $A \not\models \phi \approx 1$ for some $A \in K$, then $A \not\models \phi \approx 1$ for some finite $A \in K$.

So take $\phi \in Fm$ and $A \in K$ such that $A \not\models \phi \approx 1$. Then there is a morphism $h\colon Fm \to A$ such that $h(\phi) \neq h(1)$. Without loss of generality, we may assume that A is generated by $h(p_1), \ldots, h(p_n)$, where $p_1, \ldots, p_n$ are the propositional letters occurring in ϕ. The algebra A can be written as a subdirect product of subdirectly irreducible quotient of itself, and the equation $\phi \approx 1$ has to fail in one of them, thus by replacing A with one of these factors we may assume that A is subdirectly irreducible. Let X be the dual of A. The valuation $h\colon Fm \to A$ induces a valuation $V\colon Fm \to \mathcal{P}(X)$ on X such that $X, V \not\models \phi$. We may apply Theorem 4.29 to get $X = \bigoplus_{\alpha \leq \beta} Q_\alpha$.

For each subformula ψ of ϕ such that $X, V \not\models \psi$, we define

$$\alpha_\psi = \min\{\alpha \leq \beta : V(\neg\psi) \cap Q_\alpha \neq \emptyset\}.$$

We also define

$$I = \{0, \beta\} \cup \{\alpha_\psi : \psi \text{ is a subformula of } \phi, \alpha_\psi \text{ is defined}\} \cup \{\alpha \leq \beta : Q_\alpha \text{ is an irreflexive cluster}\}.$$

Any $\alpha \in I$ is not a limit ordinal. Indeed, for any ψ such that α_ψ is defined, the set $(R^+)^{-1}[V(\neg\psi)]$ is clopen, and it is equal to $\bigoplus_{\alpha_\psi \leq \alpha \leq \beta} Q_\alpha$. Therefore the set $\bigoplus_{\alpha < \alpha_\psi} Q_\alpha$ is compact, and we have seen in the proof of Theorem 4.29 that this is never the case if α_ψ is a limit ordinal.

For each $\alpha \in I$ such that $\alpha+1 \leq \beta$ and $\alpha+1 \notin I$, then $Q_{\alpha+1}$ is not a single irreflexive point and $Q_{\alpha+1}$ is not minimal. Therefore, we can pick a reflexive point $x_\alpha \in Q_\alpha$. We define

$$Y = \bigcup_{\alpha \in I} Q_\alpha \cup \bigcup_{\alpha \in I, \alpha+1 \leq \beta, \alpha+1 \in I} \{x_\alpha\} \subseteq X.$$

We now have two goals. The first is to define an onto continuous p-morphism $f\colon X \to Y$, thus proving that $Y^* \in K$. The second one is to define a valuation $W\colon Fm \to \mathcal{P}(Y)$ such that $Y, W \not\models \phi$. These two results together imply that Y^* is a finite member of K such that $Y^* \not\models \phi \approx 1$, thus showing that K has the finite model property.

Let us first tackle the first task. For any $y \notin Y$, let α be such that $y \in Q_\alpha$, we define

$$\alpha_y = \max\{\gamma \in I : \gamma \leq \alpha\}.$$

We define a map $f\colon X \twoheadrightarrow Y$ by

$$f(y) = \begin{cases} y & \text{if } y \in Y, \\ x_{\alpha_y} & \text{if } y \notin Y. \end{cases}$$

We need to show that f is a continuous p-morphism. Let us start with continuity. Since singletons are a basis for the topology on Y, it is sufficient to show that $f^{-1}(y)$ is clopen in X for any $y \in Y$. First assume that $y \neq x_\alpha$ for any $\alpha \in I$. Then $f^{-1}(y) = \{y\}$. Let α be such that $y \in Q_\alpha$. We have already shown that $\bigcup_{\gamma \leq \alpha} Q_\gamma$ is clopen, and because α is a successor ordinal (or 0) so is $\bigcup_{\gamma < \alpha} Q_\gamma$. Therefore, the set Q_α is clopen in X, and being a finite set in an Hausdorff space, it has to be discrete. This implies that $\{y\}$ is clopen. Now assume that $y = x_\alpha$ for some $\alpha \in I$. Then $f^{-1}(y) = \bigcup_{\alpha < \gamma < \delta} Q_\gamma$, where $\delta = \min\{\gamma \in I : \alpha < \gamma\}$. By the same reasoning as for the other case, $\{y\}$ is clopen. We also know that $\bigcup_{\gamma \leq \alpha} Q_\gamma$ is clopen, and because δ is a successor ordinal so is $\bigcup_{\gamma < \delta} Q_\gamma$. Therefore the set $\bigcup_{\alpha < \gamma < \delta} Q_\gamma$ is clopen and so is $f^{-1}(y)$. This proves that f is continuous.

Let us now show that f is a p-morphism. We first show that $u\ R\ v$ implies $f(u)\ R\ f(v)$. If $u, v \in Y$, then $f(u) = u$, $f(v) = v$ and $u\ R\ v$ implies $f(u)\ R\ f(v)$. If $u \in Y$, $v \notin Y$, then $f(u) = u \in Q_{\alpha_u}$ and as $u\ R\ v$, $v \in Q_\alpha$ for some $\alpha \leq \alpha_u$. Since $v \notin Y$, we have $\alpha < \alpha_u$ and $f(v) = x_{\alpha_v}$, with $\alpha_v < \alpha_u$, which immediately implies that $f(u)\ R\ f(v)$. If $v \in Y$, $u \notin Y$, then $f(v) = v \in Q_{\alpha_v}$, and as $u\ R\ v$, $u \in Q_\alpha$ for some $\alpha \geq \alpha_v$. Since $u \notin Y$, we have $\alpha > \alpha_v$ and $f(v) = x_{\alpha_u}$ with $\alpha_u \geq \alpha_v$, which immediately implies that $f(u)\ R\ f(v)$. If $u, v \notin Y$, then $f(u) = x_{\alpha_u}$ and $f(v) = x_{\alpha_v}$, with $\alpha_u \geq \alpha_v$. If the inequality is strict, then clearly $f(u)\ R\ f(v)$, and otherwise $f(u) = f(v) = x_{\alpha_u}$, which is enough since x_{α_u} is reflexive. Therefore f is a p-morphism.

Let us now turn to the valuation. We define $W(p_i) = V(p_i) \cap Y$ and extend W to Fm in the natural way. We claim that for all subformula ψ of ϕ, we have that for all $y \in Y$, $X, V, y \models \psi$ iff $Y, W, y \models \psi$. By induction on the complexity of ψ. The atomic case is by definition of W, and the Boolean cases are trivial. For the $\Box$ case, let $\Box\psi$ be a subformula of ϕ and assume that for all $y \in Y$, we have $X, V, y \models \psi$ iff $Y, W, y \models \psi$.

First assume that $X, V, y \models \Box\psi$. We show that $Y, W, y \models \Box\psi$. Let $z \in Y$ be a successor of y. Then z is a successor of y in X, therefore $X, V, z \models \psi$. By induction hypothesis, we get $Y, W, z \models \psi$.

Now for the converse, assume that $X, V, y \not\models \Box\psi$. Then $V(\neg\psi)$ intersects $R[y]$ and $\alpha_\psi \leq \alpha_y$. We can take $z \in Q_{\alpha_\psi} \cap V(\neg\psi) \cap R[y]$. Then $z \in Y$, z is a successor of y and $X, V, z \not\models \psi$. By induction hypothesis, we get $Y, W, z \not\models \psi$ and therefore $Y, W, y \not\models \Box\psi$. $\Box$

Now let us turn to the second claim, namely that in a cobwebby variety K, the finite subdirectly irreducible elements are weakly projective in K.

Theorem 4.33. *Let K be a cobwebby variety. Then every finite subdirectly irreducible member of K is weakly projective in K.*

Proof. We use the same proof strategy as Carr [16]. Let $A \in K$ be a finite subdirectly irreducible algebra, and let X be its dual. Then by Corollary 4.31, we have $X = \bigoplus_{m=0}^{n} Q_m$. Let $C \in K$ be an algebra such that $A \in \mathbb{H}(C)$. Since A is finite, there is a finitely generated subalgebra B of C such that $A \in \mathbb{H}(B)$. We need to show that $A \in \mathbb{S}(C)$. It is sufficient to show that $A \in \mathbb{S}(B)$. Let Y be the dual of B. By duality, showing that $A \in \mathbb{S}(B)$ is equivalent to exhibiting an onto continuous p-morphism $f \colon Y \to X$. Our assumption that $A \in \mathbb{H}(B)$ amounts to X being a generated subspace of Y. Let k be the depth of X (recall that X is finite). The plan is to recursively collapse points in Y to layers of X. We denote by $D_0, D_1, \ldots, D_k$ the layers of depth 0 to k in X, and let $D_{\leq i} = \cup_{j \leq i} D_j$ for $i \leq k$.

We build a sequence of spaces $Y_0, Y_1, \ldots, Y_{k+1}$ such that $Y \twoheadrightarrow Y_0$ and for each $i \in \{0, \ldots, k\}$, $Y_i \twoheadrightarrow B_{i+1}$. We further require that X is a generated subspace of Y_i and $Y_i = X \cup R^{-1}[X] = X \cup R^{-1}[D_i]$, for each $i \in \{0, \ldots, k\}$. Finally, our construction is such that $Y_{k+1} = X$. We then have $Y \twoheadrightarrow Y_0 \twoheadrightarrow \cdots \twoheadrightarrow Y_{k+1} = X$, thus finishing the proof.

Let us first deal with a rather trivial case. If X is an antichain of irreflexive points, then because X is rooted it is a single irreflexive point. Because Y contains X as a generated subspace, it contains an irreflexive maximal point thus Y is also an antichain of irreflexive points by Lemma 4.12. Then Y can be collapsed to a single irreflexive point, thus $Y \twoheadrightarrow X$.

Now that we have ruled out this case, let us define Y_0. The layer D_0 consists of either one or two reflexive points, by Theorem 4.29. Let x_0 be one of them. Consider the set $R^{-1}[X]^c \cup \{x_0\}$. This is an upset, therefore we may collapse it to a single maximal point x_0. Let E_0 be the corresponding bisimulation equivalence and let $Y_0 = Y/E_0$. Then Y_0 is as required. Since the restriction of E_0 to X is the identity, X is a generated subspace of Y_0. We obviously have $Y_0 = X \cup R^{-1}[X]$ by definition of E_0, and since $X = R^{-1}[D_0]$, we have $Y_0 = X \cup R^{-1}[D_0]$.

We now do the recursive step. For simplicity, let us first assume that X does not contain irreflexive clusters. Assume that Y_i is defined for some $i \leq k$. We define Y_{i+1}. First assume that D_i is a single cluster. Then we can take a reflexive point $x_i \in D_i$, and collapse any point in $U = \{x_i\} \cup R^{-1}[D_i] \setminus (X \cup R^{-1}[D_{i+1}])$ to x_i. We check that this is a bisimulation equivalence. Let $u \in U$ and $v \in R[u]$. If $v \in U$, then $[u] = [v] = x_i$, which is reflexive, so we are done. If $v \notin U$, then $v \in D_{\leq i}$ by definition of U. Then $[u] = x_i$, $[v] = v$ and $x_i \mathrel{R} v$ by the choice of x_i. Now assume that $[u] \mathrel{R} [v]$. Since $[u] = x_i$, v can be chosen to be some point in $D_{\leq i}$. Then $x_i \mathrel{R} v$, thus $u \mathrel{R} v$ by weak transitivity (since $u \in U$, we have $u \neq v$). This proves that the map $u \mapsto [u]$ is a p-morphism. It is continuous because U is clopen. Indeed, since Y_i is finitely generated, the sets D_i and D_{i+1} are clopen by Theorem 3.23. The quotient space is Y_{i+1}. Because the restriction of the equivalence to X is the identity, X is a generated subspace of Y_{i+1}. By definition of the equivalence, we have $Y_{i+1} = X \cup R^{-1}[D_{i+1}]$.

Now assume that D_i contains two clusters, C and D. Then $i < k$, as X is rooted. Let $x_i \in C$ and $y_i \in D$ be reflexive points. Let z_i be a reflexive point in D_{i+1} which sees both x_i and y_i. We collapse $U = \{x_i\} \cup R^{-1}[C] \setminus (X \cup R^{-1}[D])$ to x_i, $V = \{y_i\} \cup R^{-1}[D] \setminus (X \cup R^{-1}[C])$ to y_i and $W = \{z_i\} \cup R^{-1}[C] \cap R^{-1}[D] \setminus X$ to z_i. We show that this is a bisimulation equivalence. For U and V, this is very similar to the previous paragraph. For W, let $u \in W$ and $v \in R[u]$. Then either $v \in D_{\leq i}$ or $v \in R^{-1}[x_i, y_i]$. In the first case, we have $v \in R[x_i, y_i]$. Then because $[u] = z_i$, $[v] = v$, $z_i \mathrel{R} x_i$ and $z_i \mathrel{R} y_i$, we get $[u] \mathrel{R} [v]$. If $v \in R^{-1}[x_i, z_i] \setminus X$, then we have either $v \in U$, $v \in V$ or $v \in W$. Thus $[u] = z_i$ and $[v] \in \{x_i, y_i, z_i\}$. Since $z_i \mathrel{R} x_i$, $z_i \mathrel{R} y_i$ and $z_i \mathrel{R} z_i$, this is enough. Now assume that $[u] \mathrel{R} [v]$. As before, v can be chosen to be some point in $R[z_i]$. If $v \neq z_i$, then $u \mathrel{R} v$ so we are done. If $v = z_i$, we need to find a point $t \in R[u]$ such that $[t] = z_i$. We prove that u is reflexive, which is sufficient. If it is not, then the points x_i, y_i, z_i and u form the subframe forbidden by Theorem 4.27. Proving continuity is as in the previous paragraph. The quotient space is Y_{i+1}. This finishes the proof if X does not contain irreflexive clusters.

If X does contain irreflexive clusters, let D_k be the lowest layer which still contains reflexive clusters. By proceeding as previously, we build a space Y_k such that $Y \twoheadrightarrow Y_k$, X is a generated subspace of Y_k and $Y_k = X \cup R^{-1}[D_k]$.

First assume that D_k contains two reflexive clusters C and D. Let $x_k \in C$ and $y_k \in D$ be reflexive points. Also let z_k be the irreflexive point in D_{k+1}. As before, we can collapse $\{x_k\} \cup R^{-1}[x_k] \setminus (X \cup R^{-1}[y_k])$ to x_k and $\{y_k\} \cup R^{-1}[y_k] \setminus (X \cup R^{-1}[x_k])$ to y_k. We also collapse $W = R^{-1}[x_k] \cap R^{-1}[y_k]$ to z_k. We need a bit more work to show that this is a bisimulation equivalence. First, observe that any cluster in W is irreflexive. Indeed, if it sees z_k, then it is irreflexive by Lemma 4.15. If it does not see z_k, then it is unrelated to z_l, it cannot be reflexive, as this would imply that Y_k contains X_5 as a subframe, thus contradicting Theorem 4.27. Furthermore, any two points in W are unrelated. Take $u \in W$ and assume that $R[u] \cap W \neq \emptyset$. Let v be a maximal point in $R[u] \cap W$. We have $u \mathrel{R} v$, and because W only contains irreflexive clusters, $v \not\mathrel{R} u$. Then the points u, v, x_k, y_k give the subframe X_4, which is prohibited by Theorem 4.26. Therefore, W is an antichain of irreflexive points, which can be reduced to a single irreflexive point. The quotient space is X, thus we have proved that $Y \twoheadrightarrow X$.

Now assume that D_k contains one reflexive cluster, and let x_k be a reflexive cluster in D_k and let z_k be the irreflexive point in D_{k+1}. We collapse point points in $U = \{x_k\} \cup R^{-1}[x_k] \setminus (X \cup R^{-1}[z_k])$ to x_k, and call the resulting space Y'. Because the bisimulation equivalence identifying point in W does not identify points in X, X is still a subspace of Y', and $Y' = X \cup R^{-1}[z_k]$. The set $R^{-1}[z_k]$ only consists of single irreflexive points by Lemma 4.15. We identify points of the same depth in $R^{-1}[z_k]$ together, thus obtaining Y''. Those sets are closed by Theorem 3.23. Let us look at the structure of X and Y. The space X is composed of an upper part $R[D_k]$, and below it a chain of p irreflexive points, with $p > 0$. The space Y is also composed of an upper part $R[D_k]$, and below it a chain of q irreflexive points. Because X is a generated subspace of Y, we have $p \leq q$. If $p = q$, then we are done, as $X = Y$. If $p < q$, then we may collapse the highest irreflexive cluster in Y to z_k, thus reducing q by one. We repeat this until $q = p$, thus showing that Y reduces to X. This finishes the proof that X is weakly projective. □

This finishes our characterisation of hereditarily structurally complete extensions of wK4.

Corollary 4.34. *Let Λ be a normal modal logic extending* wK4 *and let K be the equivalent algebraic semantics of its deductive system $\vdash_\Lambda$. The following are equivalent.*

(i) The system $\vdash_\Lambda$ is hereditary structurally complete.

(ii) The variety K is primitive.

(iii) Each frame depicted in Figure 4.2 is not a Λ-space, for some $n < \omega$, G_n is not a Λ-space and for all $n > 1$, I_n and J_n are not Λ-spaces.

With this corollary we have established our characterisation of primitive varieties of wK4 algebras, or equivalently, of hereditarily structurally complete 1-transitive logics. While the necessary condition was rather straightforward to establish, the sufficient condition required us to delve into the structure of cobwebby spaces (Sections 4.4, 4.5 and 4.6). With these results in place, we were able to establish that our varieties have the finite model property (Theorem 4.32), and that their finite subdirectly irreducible members are weakly projective in the variety (Theorem 4.33). This completed the sufficient condition and the proof of our main theorem.

Chapter 5

Conclusions and future work

By working through algebraic and topological perspective, we have been able to describe some properties of weakly transitive logics. In particular, we have been able to fully characterise the hereditarily structurally complete extensions of $\mathsf{wK4}$. We have also given a precise description of the n-universal models for $\mathsf{wK4}$. To close, let us present a few problems that could be studied further.

Going to weaker logics

The main theories that drove our investigation of hereditary structural completeness over $\mathsf{wK4}$ is the theory of algebraisable logics and Jónsson-Tarski duality. All modal logics are algebraisable, and their equivalent algebraic semantics admit a topological duality. Therefore, our investigation could be generalised to other modal logics.

While a full characterisation of the hereditarily structurally complete modal logics seems ambitious, a more modest generalisation could be a characterisation of the hereditarily structurally complete *weakly transitive* logics. As we have seen in Chapter 1, those are exactly the logics that admit equationally definable principal congruences, thus making them suitable for our proof technique. However, some issues would necessarily arise.

The first of them would be the lack of a "canonical" notion of cluster for frames that are not 1-transitive. While some candidates exist (one could consider the equivalence classes of the equivalence relation $R^+ \cap (R^+)^{-1}$, where R^+ is the reflexive and transitive closure of R), they lack a crucial property: in general, the skeleton of a space X does not belong to the variety generated by X.

Another issue is that while in $\mathsf{wK4}$, any proper cluster reduces to a two element cluster, this is not the case for any weaker logic. Therefore, the family of prohibited frames could greatly expand, possibly becoming infinite. Overall, the structural work we have done in Chapter 4 would become more difficult and lead to much longer combinatorial arguments.

Restricting to smaller signatures

Another interesting problem is to investigate the hereditarily structurally complete $\wedge, \to, \Box$-fragments of $\mathsf{K4}$, or more generally, of weakly transitive logics. First of all, this fragment can express 1 as $a \to a$ for any a, and $\vee$ by $a \vee b = (a \to b) \to b$. Adding the constant 0 to this signature gives the full signature of modal logic.

However, even though this fragment is so close to the full fragment of $\mathsf{K4}$, proving that some logics are hereditarily structurally complete is significantly easier. As an example, we show that any extension Λ of the logic $\mathsf{K4} + \Box(\Box p \to p)$ is structurally complete, using the so-called 'Prucnal's trick'. Consider an admissible rule $\Gamma \rhd \phi$. By considering conjunctions, we may assume that Γ is a single formula γ, and that the rule is just $\gamma \rhd \phi$. Consider the substitution $\sigma\colon Fm \to Fm$ defined by $\sigma(p) = \Box\gamma \to p$. We claim that for any formula ψ, $\sigma(\psi)$ is equivalent to $\Box\gamma \to \psi$. This is proved by induction on the complexity of ψ. The $\to$ and $\wedge$ cases are done in [38], and the $\Box$ case is easy, recalling that we have $\Box(\Box p \to p)$ as an axiom. Being equivalent to $\Box\gamma \to \gamma$, $\sigma(\gamma)$ is a tautology. Since $\gamma \rhd \phi$ is admissible, this implies that $\sigma(\phi)$ is a tautology as well. Therefore, $\Box\gamma \to \phi$ is a tautology, and by the deduction theorem $\gamma \vdash_\Lambda \phi$, thus $\gamma \rhd \phi$ is derivable.

In contrast, when working with the full signature, it is very hard to prove that a logic is hereditarily structurally complete, while it was relatively easy to prove that it is not. In the restricted $\wedge, \to, \Box$-signature, the roles are reversed. It is easy to prove that a logic is hereditarily structurally complete, but harder to prove that a logic is not hereditarily structurally complete. However, here are some pointers. First of all, by [25, Prop. 3.32], the $\wedge, \to, \Box$-fragment of $\mathsf{K4}$ is algebraised by the quasi-variety of subreducts of $\mathsf{K4}$-algebras, i.e. by the class of $\wedge, \to, \Box$-subalgebras (or equivalently, $\wedge, \vee, \to, \Box, 1$-subalgebra) of $\mathsf{K4}$-algebras. One would then need to find an axiomatisation for this class, and check whether it is a variety. The last step would be to exhibit a finite subdirectly irreducible algebra which is not weakly projective. As we have seen, this is more easily done by working on the dual space. Thus one would have to have a duality for the algebraising variety. This could be done by restricting the duality in [29], or directly develop a duality as in [9].

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