General Topological Frames for Polymodal Provability Logic

MSc Thesis *(Afstudeerscriptie)*

written by

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under the supervision of **Prof. Dr. Lev Beklemishev** and **Dr. Nick Bezhanishvili**, and submitted to the Examinations Board in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam.*

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Abstract

The polymodal provability logic GLP is a system of propositional modal logic with infinitely many modalities having provability semantics. It was initially introduced by Japaridze in his PhD thesis [19]. GLP has significant applications in proof theory and arithmetic, however, it is well-known that GLP is Kripke incomplete. GLP is complete with respect to topological semantics [3], yet the relevant class of spaces is rather involved. Topological completeness of GLP under the natural class of ordinal spaces requires certain set-theoretic assumptions (the existence of large cardinals), however, it is still open whether it holds under these assumptions (see [4]). Therefore, it becomes crucial to search for some simpler models for GLP.

In this thesis, we define the concept of a general topological frame, that is, a topological space equipped with a distinguished set of admissible sets, akin to the notion of a general Kripke frame. Then, we describe a natural class of general topological frames on ordinals, that we call *periodic frames*. These frames are based on well-orderings equipped with some natural topologies introduced by Icard [18]. While GLP is known to be incomplete with respect to Icard's spaces, we show that the bimodal fragment of GLP is sound and complete with respect to the periodic frames. We hope that the results in this thesis will pave the way to further generalizations of this completeness to the whole system GLP.

Contents

Chapter 1

Introduction and Preliminary

The polymodal provability logic GLP is a system of propositional modal logic with infinitely many modalities having provability semantics. It was initially introduced by Japaridze in his PhD thesis [19]. GLP has found significant applications in proof theory and arithmetic [5, 6, 7].

However, in any Kripke frame for GLP, all relations but one have to be discrete, which means that GLP is Kripke incomplete. Therefore, it becomes crucial to explore alternative models for GLP.

Before the advent of Kripke semantics, topological semantics for modal logic was independently developed by Tang [26], McKinsey and Tarski [27, 21, 22] in 1930's and 1940's. Later, Simmons [24] and Esakia [15] independently studied the topological interpretation of provability logic GL, revealing its correspondence to a natural class of topological spaces known as scattered spaces, employing the topological d-semantics. Later, we will show that ordinals form natural examples of scattered spaces and a topological completeness result for GL was obtained independently by Abashidze [1] and Blass [11].

Topological models for GLP have also been explored by Beklemishev, Bezhanishvili, Icard and Gabelaia [18, 2, 3]. Notably, in [3], it is shown that GLP is complete with respect to a class of topological spaces known as GLP-spaces. However, the class of spaces for which the completeness is established is complicated. Topological completeness of GLP under the natural class of ordinal spaces turns out to be an even harder question. The existence of a non-discrete ordinal GLP-space is independent of **ZFC**, therefore the completeness of GLP with respect to ordinal GLP-spaces requires certain set-theoretic assumptions (the existence of large cardinals). It is still open whether the completeness holds under these assumptions (see [4]). Therefore, it becomes crucial to search for some simpler complete class of models for GLP.

In this thesis, we define the concept of a general topological frame, that is, a topological space equipped with a distinguished set of admissible sets, akin to the notion of a general Kripke frame. Then, we describe a natural class of general topological frames on ordinals that we call *periodic frames*. These frames are based on well-orderings equipped with some natural topologies introduced by Icard [18]. While GLP is known to be incomplete with respect to Icard's spaces, we show that the bimodal fragment of GLP is sound and complete with respect to the periodic frames. In the future, we hope to generalize the result to the whole system GLP.

This thesis is structured as follows:

In the rest of this chapter, we provide an introduction to the provability logics GL and GLP and their natural semantics. In Chapter 2, we recall the topological d-semantics for modal logic and introduce the results for GL in topological interpretation. Moreover, we define the notion of general topological frames, which is the new topological model in this thesis. In Chapter 3, we define four kinds of periodic sets on ordinals. Later, 0-periodic sets and 1-periodic sets will be used as the admissible sets in our general topological frames for GL and GLB. In Chapter 4, we build general topological frames for GL and GLB, in which hereditarily periodic sets are used as the admissible sets. Such kind of general topological frames will be called periodic frames and we prove that GLB is sound in all the periodic frames. In Chapter 5, we prove that GLB is complete with respect to the class of periodic frames. In Chapter 6, we give a conclusion and discuss the future work.

1.1 GL and GLP

1.1.1 Classical Provability Logic GL

The idea of provability logic originates from a short paper by Gödel [16], where he attempted to formalize the BHK-interpretation and introduced a modal calculus with informal provability semantics (equivalent to the Lewis modal system $\mathsf{S4}$).

Formal provability semantics is based on Gödel's proof predicate $\mathsf{Proof}(x, y)$, which denotes "y is the code of a proof of the formula having a code x" for a classical first order theory containing Peano Arithmetic.

The provability predicate can then be expressed as $\mathsf{Prov}(x) = \exists y \mathsf{Proof}(x, y)$. Then, if we disregard the distinction between a formula and its Gödel's number, $\mathsf{Prov}(F)$ can be viewed as a modal formula. However, Gödel showed in [16] that $\mathsf{S4}$ was not the modal logic of the formal provability predicate $\mathsf{Prov}(F)$.

Based on the previous work by Hilbert and Bernays [17], Löb [20] discovered the final principle and demonstrated that along with other natural conditions on the provability predicate, i.e. the axioms and rules in the modal logic $\mathsf{K4}$, it is sufficient for the proof of Gödel's second incompleteness theorem. Nowadays, a formalization of Löb's Theorem is known as Löb's Axiom:

$$\mathsf{L} : \Box(\Box\varphi \to \varphi) \to \Box\varphi$$

The extension of $\mathsf{K4}$ by L is usually denoted GL after Gödel and Löb. An *arithmetical interpretation* f is a mapping that assigns propositional variables to arbitrary arithmetical sentences, which commutes with boolean operators and $f(\Box\psi) := \mathsf{Prov}(f(\psi))$. Löb demonstrated that if φ is a theorem of GL, then for any arithmetical interpretation f, $f(\varphi)$ is a theorem of PA.

Then, the following question arose: whether GL contains all the provable schemata of the provability predicate Prov? This question was solved by Solovay [25].

Theorem 1.1.1 (Solovay, [25])**.** *$\mathsf{GL} \vdash \varphi$ iff $\mathsf{PA} \vdash f(\varphi)$, for all arithmetical interpretations f.*

1.1.2 Polymodal Provability Logic GLP

Some other important proof-theoretic notions lead to different types of modalities and logics. For instance, in 1980, Boolos considered the concept of ω-provability, which is dual to Gödel's notion of ω-consistency [12].

Recall that an arithmetical theory T is *ω-inconsistent* if there exists an arithmetical formula $P(x)$ such that T proves that $\varphi(n)$ holds for every standard natural number n, however, T also proves that there is some natural number n such that $\varphi(n)$ fails, that is, $\mathsf{T} \vdash \exists x \neg\varphi(x)$. In this case, T can still be consistent, because this may not generate a contradiction within T, and such n is necessarily a non-standard integer in any model of T.

T is *ω-consistent* if it is not ω-inconsistent. This concept is stronger than consistency: on the one hand, any inconsistent theory is also ω-inconsistent; on the other hand, there exist consistent theories which are not ω-consistent.

A formula φ is called ω-provable in a theory T, if the theory $\mathsf{T} + \neg\varphi$ is ω-inconsistent, i.e. if there exists a formula $\psi(x)$ s.t. $\mathsf{T} + \neg\varphi \vdash \psi(n)$ for all standard natural numbers n but $\mathsf{T} + \neg\varphi \vdash \exists x \neg\psi(x)$. On the other hand, ω-provability can also be described as the provability in arithmetic by one application of the ω-rule, i.e. provability in the theory

$$\mathsf{T}' := \mathsf{T} + \{\forall x \varphi(x) : \mathsf{T} \vdash \varphi(n), \text{ for all } n\}.$$

That provability by one application of the ω-rule implies ω-provability is obvious. For the other direction, suppose that φ is ω-provable, then $\mathsf{T} + \neg\varphi$ is ω-inconsistent, i.e., there exists a formula $\psi(x)$ such that $\mathsf{T} \vdash \neg\varphi \to \psi(n)$ for all n but $\mathsf{T} \vdash \neg\varphi \to \exists x \neg\psi(x)$. Hence, T' contains $\forall x(\neg\varphi \to \psi(x))$, i.e. $\mathsf{T}' \vdash \neg\varphi \to \forall x \psi(x)$. With $\mathsf{T} \vdash \neg\varphi \to \exists x \neg\psi(x)$, it follows that $\mathsf{T}' \vdash \neg\varphi \to \bot$, hence φ is provable by one application of the ω-rule.

In [12], Boolos proved that the logic of ω-provability coincides with GL using a Solovay-style arithmetical completeness proof. Let us write $[0]$ for normal provability and $[1]$ for ω-provability. The next natural question was to find the bimodal logic of $[0]$ and $[1]$, which was answered by Japaridze [19]. In fact, Japaridze formulated the polymodal logic GLP with infinitely many modalities $[0], [1], [2]$, etc, where $[n]$ is interpreted as the provability by n nested applications of the ω-rule.

Definition 1.1.2 (Polymodal provability logic)**.** The language of polymodal provability logic $\mathcal{L}_P$ is defined as follows:

$$\varphi ::= p \mid \bot \mid \varphi \to \varphi \mid [n]\varphi$$

where $p \in \mathsf{Prop}$ and $n \in \mathbb{N}$. Other connectives $\leftrightarrow, \vee, \wedge, \neg$ are defined as usual. The bimodal fragment of $\mathcal{L}_P$ with only $[0]$ and $[1]$ is denoted as $\mathcal{L}_B$.

An arithmetical interpretation f for $\mathcal{L}_P$ is a mapping from modal formulas to arithmetical formulas such that f commutes with boolean operators and $[n]\varphi$ is mapped to the formalization of "φ is provable by n nested applications of the ω-rule." Similarly, we can define arithmetical interpretations for $\mathcal{L}_B$.

Based on our earlier observation, we know that

$$[0]\varphi \to [1]\varphi.$$

It is also not difficult to prove that the following axiom is valid:

$$\neg[0]\varphi \to [1]\neg[0]\varphi.$$

If φ is not provable then no standard natural number n is the code of a proof of φ, i.e. $\neg\mathsf{Proof}(\varphi, n)$ is true for each n, hence, by the ω-rule, it follows that $\forall x \neg\mathsf{Proof}(\varphi, x)$, which means that $\neg[0]\varphi$ is ω-provable. Therefore, $\neg[0]\varphi \to [1]\neg[0]\varphi$ is valid.

Moreover, the relationship between $[n]$ and $[n+1]$ is the same as that between [0] and [1]. Thus, Japaridze introduced the system GLP as follows [19]:

Definition 1.1.3 (System GLP)**.** The Polymodal Provability Logic GLP is a system composed of the following axioms and rules:

Axioms: (i) Axioms of GL for each modality $[n]$;

(ii) $[m]\varphi \to [n]\varphi$, for $m \leqslant n$;

(iii) $\langle m\rangle\varphi \to [n]\langle m\rangle\varphi$, for $m < n$.

Rules: Modus Ponens,

$[n]$-necessitation: $\varphi \vdash [n]\varphi$.

Definition 1.1.4 (System GLB)**.** If we restrict GLP to the bimodal fragment $\mathcal{L}_B$, then we get the following system GLB:

Axioms: (i) Axioms of GL for [0] and [1];

(ii) $[0]\varphi \to [1]\varphi$;

(iii) $\langle 0\rangle\varphi \to [1]\langle 0\rangle\varphi$.

Rules: Modus Ponens,

[0]-necessitation and [1]-necessitation: $\varphi \vdash [0]\varphi$ and $\varphi \vdash [1]\varphi$.

With a non-trivial variation of Solovay-style completeness proof, the following theorem is proved by Japaridze.

Theorem 1.1.5 (Japaridze, [19])**.** *GLP $\vdash \varphi$ iff PA $\vdash f(\varphi)$, for all the arithmetical in terpretations f of GLP.*

1.1.3 Kripke Incompleteness

It is well-known that the logic GL is sound and complete with respect to the class of converse well-founded strict partial ordered Kripke frames [23]. Moreover, GL has the finite model property, hence GL is also complete with respect to the class of finite, transitive, irreflexive trees.

However, even GLB is Kripke incomplete. More specifically, there is no non-trivial Kripke frame for GLB [13].

Proposition 1.1.6. Consider a Kripke frame with two relations $\mathbb{F} = \langle W, R_0, R_1 \rangle$, if R_1 is non-empty then it is impossible that all the axioms of GLB are valid in F.

Proof. Suppose that R_1 is non-empty, then there are a, b such that aR_1b.

First, $[0]\varphi \to [1]\varphi$ is valid if and only if for any x, y, xR_1y implies xR_0y. Hence we have aR_0b.

Second, $\langle 0 \rangle \varphi \to [1]\langle 0 \rangle \varphi$ is valid if and only if for any x, y, z, xR_0y and xR_1z implies zR_0y. Hence, with aR_1b and aR_0b, it is followed that bR_1b.

However, if GLB is valid in $\mathbb{F}$, R_1 should be a converse well-founded relation, which contradicts with bR_1b. Hence, if R_1 is non-empty then $\mathbb{F}$ can't be a Kripke frame of GLB. ■

Corollary 1.1.7. GLB is Kripke incomplete. Moreover, GLP is also Kripke incomplete.

Chapter 2

Topological Models for GL and GLP

2.1 Topological d-semantics

Since GLP is Kripke incomplete, we need to find some other semantical tool for its investigation, such as topological semantics.

Usually, when interpreting modal logic in topological spaces, the diamond operator $\Diamond$ is translated as topological closure. However, this translation only works when the logic contains the reflexivity axiom T, because every set is a subset of its closure. For logics such as GL and GLP, instead of the closure operator, $\Diamond$ can be translated as the topological derivative operator.

Definition 2.1.1 (Derived Set)**.** Let $\langle X, \tau \rangle$ be a topological space, A a subset of X. Topological *derivative* $d_\tau(A)$ of A is the set of all the limit points of A:

$$x \in d_\tau(A) \iff \forall U \in \tau(x \in U \Rightarrow \exists y \neq x \; y \in U \cap A).$$

$i_\tau(A) := A \backslash d_\tau(A)$ is the set of isolated points of A and $c_\tau(A) := A \cup d_\tau(A)$ is the closure of A.

Definition 2.1.2 (Topological d-semantics)**.** A topological model $\mathcal{M} = (X, \tau, \nu)$ is a tuple where (X, τ) is a topological space and $\nu : \mathsf{Prop} \to \mathcal{P}(X)$ is a valuation. Then, the satisfaction relation between a point w of a topological model $\mathcal{M}$ and a formula φ is defined inductively as follows:

$\mathcal{M}, w \not\models \bot$		
$\mathcal{M}, w \models p$	$\iff$	$w \in \nu(p)$
$\mathcal{M}, w \models \neg\varphi$	$\iff$	$\mathcal{M}, w \not\models \varphi$
$\mathcal{M}, w \models \varphi \vee \psi$	$\iff$	$\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
$\mathcal{M}, w \models \Diamond\varphi$	$\iff$	$\forall U \in \tau(x \in U \Rightarrow \exists y \in U\backslash\{x\} : \mathcal{M}, y \models \varphi)$
$\mathcal{M}, w \models \Box\varphi$	$\iff$	$\exists U \in \tau(x \in U \Rightarrow \forall y \in U\backslash\{x\} : \mathcal{M}, y \models \varphi)$

In other terms, $[\![\Diamond\varphi]\!] = d_\tau[\![\varphi]\!]$. A formula φ is *valid* in a model $\mathcal{M}$ if it is true in all the points $w \in \mathcal{M}$ and φ is valid in a topological space (X, τ) if it is valid in all the model $\mathcal{M} = (X, \tau, \nu)$ based on (X, τ).

Topological d-semantics for modal logic was independently suggested by Simmons [24] and Esakia [15]. They proved that under this interpretation, GL corresponds to a natural class of topological spaces.

Definition 2.1.3. A topological space (X, τ) is *scattered* if every nonempty subspace $A \subseteq X$ has an isolated point, i.e. $i_\tau(A) \neq \varnothing$.

Theorem 2.1.4 (Esakia,[15])**.** GL $\vdash \varphi$ *if and only if* φ *is valid in all the scattered spaces.*

2.2 Ordinal Topological Spaces

Natural examples of scattered topological spaces come from orderings, especially ordinals. First, we introduce an equivalent characterization of scattered spaces in terms of the following transfinite *Cantor-Bendixson sequence* of subsets of X:

- $d_\tau^0(X) = X$;
- $d_\tau^{\alpha+1}(X) = d_\tau(d_\tau^\alpha(X))$;
- $d_\tau^\alpha(X) = \bigcap_{\beta<\alpha} d_\tau^\beta(X)$ if α is a limit ordinal.

For a scattered space, $d_\tau^{\alpha+1}(X) \subseteq d_\tau^\alpha(X)$ always holds, and it is a strict inclusion unless $d_\tau^\alpha(X) = \varnothing$.

Theorem 2.2.1 (Cantor)**.** (X, τ) *is a scattered space iff* $d_\tau^\alpha(X) = \varnothing$ *for some ordinal* α.

Definition 2.2.2 (Cantor-Bendixson rank)**.** For a scattered space X, the *Cantor-Bendixson rank* of X is the least α such that $d_\tau^\alpha(X) = \varnothing$, denoted by $\rho_\tau(X)$. The *Cantor-Bendixson rank function* $\rho_\tau : X \to \mathrm{On}$ is defined by

$$\rho_\tau(x) := \min\{\alpha : x \notin d_\tau^{\alpha+1}(X)\}.$$

Hence, ρ_τ is a map from X onto $\rho_\tau(X) = \{\alpha : \alpha < \rho_\tau(X)\}$.

Definition 2.2.3 (Order topology)**.** For an ordinal α, the *order topology* $\tau_<$ on α is the topology generated by all intervals (β_1, β_2) such that $\beta_1, \beta_2 \in \alpha \cup \{\pm\infty\}$ and $\beta_1 < \beta_2$.

Proposition 2.2.4. On any ordinal α, the order topology $\tau_<$ is scattered.

Proof. For any subset $X \subseteq \alpha$, there exists a least element x in X. Hence, it is easy to see that $x \notin d_\tau(X)$, i.e. x is an isolated point in X. So, by Definition 2.1.3, the order topology is scattered. ∎

Hence, it is very natural to consider ordinal spaces as models for the provability logic GL.

Every ordinal α has a unique representation, called the Cantor Normal Form, as a finite sum of ordinal powers of ω, i.e. any non-zero ordinal can be written as

$$\alpha = \omega^{\beta_n} \cdot k_n + \cdots + \omega^{\beta_0} \cdot k_0,$$

where $n \geqslant 0, \alpha \geqslant \beta_n > \cdots > \beta_0$, and $k_0, \cdots, k_n$ are non-zero natural numbers. The ordinal β_n is the *degree* and β_0 is the *rank* of α.

Definition 2.2.5. Let $\ell : \Omega \to \Omega$ be defined by

$$\ell(0) = 0; \ell(\alpha) = \beta \text{ if } \alpha = \gamma + \omega^\beta, \text{ for some } \gamma, \beta.$$

That is to say, when $\alpha \neq 0$, $\ell(\alpha)$ is the rank of α defined on the Cantor Normal Form. Hence, ℓ is called the *rank function*.

Proposition 2.2.6 ([3])**.** Let Ω be an ordinal equipped with its order topology. Then, the Cantor-Bendixson rank function $\rho_{\tau_<}$ coincides with the rank function ℓ.

Definition 2.2.7 (d-map)**.** A map $f : X \to Y$ between topological spaces is called a *d-map* if f is continuous, open and *pointwise discrete*, i.e. $f^{-1}(y)$ is a discrete subspace of X for each $y \in Y$.

d-maps are well-known to satisfy the following proposition (see [9] or [3]).

Proposition 2.2.8. Suppose that $f : X \to Y$ is a d-map between topological spaces, then we have the following properties:

- $f^{-1}(d_Y(A)) = d_X(f^{-1}(A))$, for any $A \subseteq Y$;
- $f^{-1} : (\mathcal{P}(Y), d_Y) \to (\mathcal{P}(X), d_X)$ is a homomorphism of modal algebras;
- If f is onto, then $\mathrm{Log}(X) \subseteq \mathrm{Log}(Y)$.

Theorem 2.1.4 was improved independently by Abashidze [1] and Blass [11] as follows. The following proof is from [10].

Theorem 2.2.9 (Abashidze, Blass)**.** *Consider an ordinal $\Omega \geqslant \omega^\omega$ equipped with the order topology. Then $\mathsf{Log}(\Omega) = \mathsf{GL}$.*

Proof. First, any Kripke frame $\mathcal{F} = \langle W, R\rangle$ of GL can also be viewed as a topological space, i.e. we can also consider $\mathcal{F}$ as the set W equipped with the upset topology w.r.t. the relation R.

Since GL is complete with respect to finite transitive irreflexive trees, we aim to prove that if T is a finite transitive irreflexive tree of depth n, then there exists an onto d-map $f : \omega^n + 1 \twoheadrightarrow T$. Then, by Proposition 2.2.8, it implies that $\mathsf{Log}(\Omega) = \mathsf{GL}$ when $\Omega \geqslant \omega^\omega$.

We prove it by induction on the depth n:

- If $n = 0$, the tree T contains only one irreflexive point a, so the result is trivial.

- Suppose $n > 0$ and the result holds for all the $k < n$. Denote the root as a and all the immediate successors of a as $a_1, a_2, \ldots, a_l$. Let T_i be the subtree of T generated by a_i for $i \in [1, l]$. Since the depth of T is n, the depth n_i of each subtree T_i will be smaller than n. WLOG, we assume that $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_l$. By the Induction Hypothesis, for each $i \in [1, l]$, there exists an onto d-map $f_i : \omega^{n_i} + 1 \twoheadrightarrow T_i$.

  When we view the tree T as a topological space, it can be thought of as the disjoint union of $\{a\}$ and all the subtrees $T_1, \ldots, T_l$. Moreover, if $U \subseteq T$ is an open set, then either $U = T$ or $U = \bigcup_{i=1}^{l} U_i$ where U_i is an open set of the subtree T_i for each i. On the other hand, we decompose $\omega^n + 1$ into a similar structure: we write $\omega^n + 1$ as the disjoint union

$\biguplus_{j=1}^{\infty} X_j \cup \{\omega^n\}$, where X_j is isomorphic to the ordinal $\omega^{n_i}+1$ if $j \equiv i$ mod l and $n_i > 0$, and X_i is a singleton if $n_i = 0$. Since all the n_i are smaller than n and we also know that at least one of them is equal to $n-1$, so the structure of $\biguplus_{i=1}^{\infty} X_i$ is the same as ω^n.

Now we construct the d-map from $\omega^n + 1$ to T: First, set $f(\omega^n) = a$. Second, for each $i \in \omega$, suppose that $i = l \cdot k + j$ where $j < l$. Then, define $f|_{X_i} : X_i \to T_j$ as the copy of the d-map $f_j : \omega^{n_j} + 1 \twoheadrightarrow T_j$. Combining them, we have the construction of $f : \omega^n + 1 \to T$. It is not hard to check that f is an onto d-map. (For more details, the readers can check [10, Lemma 3.4])

■

2.3 General Topological Frames

In this section, we introduce the concept of a general topological frame, which is inspired by the notion of a general Kripke frame. First, recall the following definition.

Definition 2.3.1. A *polytopological space* is a tuple $\langle X, \{\tau_i : i \in I\}\rangle$ where τ_i is a topology in X for each $i \in I$. A *polytopological space* is a tuple $\langle X, \{\tau_i : i \in I\}, \nu\rangle$ where $\langle X, \{\tau_i : i \in I\}\rangle$ forms a polytopological space and $\nu : \mathsf{Prop} \to \mathcal{P}(X)$ is a valuation function. In polytopological models, the satisfaction relation for polymodal formulas is defined in the same way as the topological d-semantics in Definition 2.1.2.

Proposition 2.3.2 ([2])**.** If a polytopological space $\langle X, \{\tau_n : n \in \omega\}\rangle$ is a model of GLP, it should satisfy the following conditions:

(i) $\langle X, \tau_n\rangle$ is a scattered topological space for each $n \in \omega$;

(ii) $\tau_n \subseteq \tau_{n+1}$;

(iii) For each $U \subseteq X$, $d_{\tau_n}(U)$ is τ_{n+1}-open.

Proof. By Definition 1.1.3, we know the following properties for a model of GLP:

- We have axioms of GL for each modality $[n]$, hence, $\langle X, \tau_n\rangle$ should form a scattered topological space for each $n \in \omega$.

- We have the axioms of the form $[m]\varphi \to [n]\varphi$ for $m \leqslant n$. In topological d-semantics, it is easy to see that $[n]\varphi \to [n+1]\varphi$ means that $\tau_n \subseteq \tau_{n+1}$.
- We have the axioms of the form $\langle m\rangle\varphi \to [n]\langle m\rangle\varphi$ for $m < n$. In topological d-semantics, $\langle n\rangle\varphi \to [n+1]\langle n\rangle\varphi$ means that the truth set of $\langle n\rangle\varphi$ should be τ_{n+1}-open. Therefore, for each $U \subseteq X$, $d_{\tau_n}(U)$ is τ_{n+1}-open.

■

Polytopological spaces satisfying conditions $(i)-(iii)$ in Proposition 2.3.2 are called GLP-*spaces* [3].

In [3], it is proved that GLP is complete w.r.t. the class of all GLP-spaces. However, if we consider ordinal GLP-space, the situation becomes much more complicated, the existence of a non-discrete ordinal GLP-space is independent of **ZFC**. Roughly speaking, since τ_{n+1} will be a refined topology w.r.t. τ_n such that $d_{\tau_n}(A)$ is τ_{n+1}-open, in order to make sure that τ_{n+1} is non-discrete, we will need the existence of some large cardinals. The question if GLP is complete w.r.t. ordinal GLP-spaces under some natural set-theoretic assumptions is still open.

Now, we define the notion of general topological frame, which is inspired by the notion of general Kripke frame. First, we recall the definition of a general Kripke frame.

Definition 2.3.3 (General Kripke frame)**.** A *general Kripke frame* is a tuple $\langle F, \{R_i : i \in I\}, \mathcal{A}\rangle$ such that $\langle F, \{R_i : i \in I\}\rangle$ forms a Kripke frame and $\mathcal{A} \subseteq \mathcal{P}(F)$, which is closed under finite union, finite intersection, complement and R_i^{-1} for any $i \in I$. For a model $\langle F, \{R_i : i \in I\}, \mathcal{A}, \nu\rangle$ based on the general Kripke frame $\langle F, \{R_i : i \in I\}, \mathcal{A}\rangle$, the valuation function ν should be a mapping from Prop to $\mathcal{A}$.

For a set W, a function $f : \mathcal{P}(W) \to \mathcal{P}(W)$ and $\mathcal{A} \subseteq \mathcal{P}(W)$, if $\mathcal{A}$ is closed under finite union, finite intersection, complement and f, $\langle \mathcal{A}, f\rangle$ is called a *modal boolean algebra*.

So, that is to say, a general Kripke frame is simply a Kripke frame with a modal boolean algebra over the frame. The subset of F containing in $\mathcal{A}$ is called *admissible set*. Only the elements in $\mathcal{A}$ are possible to be defined by a formula in a class of models based on $\langle F, R, \mathcal{A}\rangle$.

Inspired by the notion of general Kripke frame, we define general topological frames. That is to say, we aim to add a modal boolean algebra $\mathcal{A}$ to a topological (or polytopological) space.

Definition 2.3.4 (General topological frame)**.** A *general topological frame* is a tuple $\langle X, \{\tau_i : i \in I\}, \mathcal{A}\rangle$ where $\langle X, \{\tau_i : i \in I\}\rangle$ is a polytopological space and $\mathcal{A} \subseteq \mathcal{P}(X)$, which is closed under finite union, finite intersection, complement and d_{τ_i} for any $i \in I$. A *general topological model* is a tuple $\langle X, \{\tau_i : i \in I\}, \mathcal{A}, \nu\rangle$ where $\langle X, \{\tau_i : i \in I\}, \mathcal{A}\rangle$ forms a general topological frame and $\nu : \mathsf{Prop} \to \mathcal{A}$.

For a general topological frame $\langle X, \{\tau_i : i \in I\}, \mathcal{A}\rangle$ and a subset $Y \subseteq X$, we define a corresponding *subframe*: $\langle Y, \{\tau_i|_Y : i \in I\}, \mathcal{A}|_Y\rangle$, where $\tau_i|_Y$ is the subspace topology and $\mathcal{A}|_Y$ is the restriction of the elements in $\mathcal{A}$ to Y. In convenient, we will continue to denote it as $\langle Y, \{\tau_i : i \in I\}, \mathcal{A}\rangle$ where there is no ambiguity.

Definition 2.3.5. Suppose that $\langle X, \tau_x, \mathcal{A}_X\rangle$ and $\langle Y, \tau_Y, \mathcal{A}_Y\rangle$ are two general topological frames. A map $f : X \to Y$ between general topological frames is called a d-map if the following conditions hold:

- For any $V \in \mathcal{A}_Y$, $f^{-1}(V) \in \mathcal{A}_X$;
- f is continuous, open and pointwise discrete.

Proposition 2.3.6. Suppose that $f : X \to Y$ is a d-map between general topological frames, then we have the following properties:

- $f^{-1}(d_Y(U)) = d_X(f^{-1}(U))$, for any $U \subseteq Y$;
- $f^{-1} : \langle \mathcal{A}_Y, d_Y\rangle \to \langle \mathcal{A}_X, d_X\rangle$ is a homomorphism of modal algebras;
- If f is onto, then $\mathrm{Log}(X) \subseteq \mathrm{Log}(Y)$.

In the proof of Proposition 2.3.2, we show that, in the frames of GLP, the axiom $\langle n\rangle\varphi \to [n+1]\langle n\rangle\varphi$ corresponds to the frame condition that $d_{\tau_n}(U)$ is τ_{n+1}-open for each $U \subseteq X$. That is because the truth set of $\langle n\rangle\varphi$ should be τ_{n+1}-open. However, if we consider a general topological frame, the truth set of a formula can only be an element in $\mathcal{A}$. Hence, we have the following proposition, which is exactly our motivation to introduce the notion of general topological frame.

Proposition 2.3.7. If a general topological frame $\langle X, \{\tau_n : n \in \omega\}, \mathcal{A}\rangle$ satisfies the following conditions, then all the axioms of GLP are valid in $\langle X, \{\tau_n : n \in \omega\}, \mathcal{A}\rangle$:

(i) $\langle X, \tau_n \rangle$ is a scattered topological space for each $n \in \omega$;

(ii) $\tau_n \subseteq \tau_{n+1}$;

(iii) For each $U \in \mathcal{A}$, $d_{\tau_n}(U)$ is τ_{n+1}-open.

The difference between Proposition 2.3.2 and 2.3.7 will bring about a change in our study of GLP on ordinals. Now, the third condition is loosened, therefore, it is easier to find a general topological frame for GLP such that all the topologies τ_n are non-discrete.

In next chapter, we will define a suitable modal boolean algebra and begin to build general topological frames for GLB.

Chapter 3

Periodic Sets of Ordinals

In this chapter, we define the notions of periodic set, ultimately periodic set and hereditarily periodic set in ordinal spaces. Some of the notations and propositions about periodic sets is from [14], in which periodicity of transfinite words was considered.

In next chapter, hereditarily periodic sets will be used as the admissible sets in our general topological frames. In the proof of Theorem 2.2.9, we build onto d-maps from ordinals to finite transitive irreflexive trees. Moreover, in next chapter, we will prove that the preimages of any subset in these d-maps are hereditarily periodic sets. In fact, this is the reason why we are interested in periodic sets and aim to build modal algebras by hereditarily periodic sets.

3.1 Periodic Sets

First, we define periodic sets on ordinals. Suppose that A is a subset of an ordinal Ω.

It is well-known that Euclidean division for ordinals is well-defined. That is, let α, β be two ordinals, then there exists a unique pair of ordinals (τ, μ) such that $\mu < \beta$ and $\alpha = \beta \cdot \tau + \mu$.

With this, given an ordinal π, we can define an equivalence relation $\sim_\pi$ on all ordinals.

Definition 3.1.1. For an ordinal π, we define an equivalence relation $\sim_\pi$ on α: $\alpha_1 \sim_\pi \alpha_2$ if there exist three ordinals τ_1, τ_2 and $\mu < \pi$ such that $\alpha_1 = \pi \cdot \tau_1 + \mu$ and $\alpha_2 = \pi \cdot \tau_2 + \mu$.

Definition 3.1.2 (Periodic Set)**.** Let Ωbe an ordinal, a subset $A \subseteq \Omega$ is *periodic* in Ω if there exists π such that $0 < \pi < \omega^{\ell(\Omega)}$ and for any ordinals $\gamma_1, \gamma_2 < \Omega$, if $\gamma_1 \sim_\pi \gamma_2$ and $\gamma_1 \in A$ then $\gamma_2 \in A$. Such a π is called a period for the periodic set A in Ω.

So a subset A of Ω is periodic, if it is closed under the equivalence relation $\sim_\pi$ in Ω. Notice that we require that $0 < \pi < \omega^{\ell(\Omega)}$, so $\ell(\Omega)$ should be greater than 0, i.e. if A is a periodic set in Ω, Ω should be a limit ordinal.

Proposition 3.1.3. Suppose that A is a periodic set in Ω and π is a period for A, then for any natural number $n > 0$, $\pi \cdot n$ is also a period for A. Moreover, for any ordinal β, if $\pi \cdot \beta < \omega^{\ell(\Omega)}$ then $\pi \cdot \beta$ is also a period for A.

Proof. First, if $\pi < \omega^{\ell(\Omega)}$ then $\pi \cdot n < \omega^{\ell(\Omega)}$ is also true. Second, it is easy to see that for all ordinals $i, j < \Omega$, if $i \sim_{\pi \cdot n} j$ holds then $i \sim_\pi j$ also holds. So if A is closed under the equivalence relation $\sim_\pi$, it is also closed under $\sim_{\pi \cdot n}$. Hence, $\pi \cdot n$ is also a period for periodic set A in Ω.

For $\pi \cdot \beta$, it is also easy to check that A is closed under $\sim_{\pi \cdot \beta}$. Hence, if we have known that $\pi \cdot \beta < \omega^{\ell(\Omega)}$, then $\pi \cdot \beta$ is also a period for A in Ω. ∎

On the other hand, a subset A of ordinal Ω can also be viewed as a string of length Ω. That is, it is a mapping from Ω to $2 = \{0, 1\}$ such that for any ordinal $i < \Omega$, $A(i) = 1$ if and only if $i \in A$. For short, we also denote $A(i)$ as A_i.

In this way, we have the following alternative definition of periodic set.

Definition 3.1.4. A string A of length Ω is periodic if and only if there exists $0 < \pi < \omega^{\ell(\Omega)}$ such that for all ordinals $i, j < \Omega$, if $i \sim_\pi j$ then $A_i = A_j$.

It is easy to see that this definition is equivalent to Definition 3.1.2.

If A is a subset of Ω_1 and B is a subset of Ω_2, then we can define the concatenation AB of two strings A and B, which is a mapping from the ordinal $\Omega_1 + \Omega_2$ to 2 defined by

$$(AB)_i = \begin{cases} A_i & \text{if } i < \Omega_1; \\ B_j & \text{if } i = \Omega_1 + j \text{ and } j < \Omega_2. \end{cases}$$

Given a string A of length Ω, we denote by A^τ the string concatenated with itself by τ times, i.e. A^τ is a string of length $\Omega \cdot \tau$ defined by

$$A^\tau_{\Omega \cdot \gamma + i} = A_i$$

where $\gamma < \tau$ and $i < \Omega$.

Then we have the following proposition, which can be viewed as an equivalence definition of periodic set.

Proposition 3.1.5. Give a subset A of Ω, which can also be viewed as a string of length Ω. Then the following conditions are equivalent:

i) A is a periodic set in Ω.

ii) there exists a string B of length π with $0 < \pi < \omega^{\ell(\Omega)}$ such that $A = B^\tau$ for an infinite ordinal τ.

Proof. If i) holds, then there exists a period π for A and a limit ordinal τ such that $\Omega = \pi \cdot \tau$. Let B be the string of length π such that $B_i = A_i$ for ordinal $i < \pi$. By the definition of periodic set, we know that $A_{\pi\cdot\gamma+i} = A_i = B_i$. Hence, A is exactly B^τ.

Conversely, if ii) holds, then the length π for B is a period for A in Ω, because $A_{\pi\cdot\gamma+i} = B_i$ for any ordinal $\gamma < \tau$ and only they are equivalent modulo π. ■

We denote the set of all periodic sets in Ω as $\mathsf{P}(\Omega)$.

3.2 Ultimately Periodic Sets

Next, we define ultimately periodic sets on ordinals.

Definition 3.2.1 (Ultimately Periodic Set)**.** A is *ultimately periodic* in Ω if there exist a bound $\kappa < \Omega$ and a periodic set A' in Ω such that $A \cap (\kappa, \Omega) = A' \cap (\kappa, \Omega)$. In other words, A is equal to a periodic set except for an initial segment, the period π of A' is also called the period of the ultimately periodic set A.

Proposition 3.2.2. If A is ultimately periodic in Ω, then there exists a period π of the form $\omega^\beta \cdot n$ and the bound in Definition 3.2.1 can be chosen as an ordinal κ such that $\ell(\kappa) \geqslant \beta$.

Proof. Suppose that π is a period of A with bound κ, so there exists a string A' of length κ, a string B of length π and an ordinal τ such that $A = A'(B^\tau)$ and $\Omega = \kappa + \pi \cdot \tau$. We divide it to two cases:

- If $\ell^2(\Omega) > 0$, i.e. $\ell(\ell(\Omega)) > 0$, then $\ell(\Omega)$ is a limit ordinal. We also know that $\pi < \omega^{\ell(\Omega)}$.

  In this case, $\pi \cdot \omega < \omega^{\ell(\Omega)}$ is also true and there exists τ' such that $\kappa + (\pi \cdot \omega) \cdot \tau' = \Omega$. So, $\pi \cdot \omega$ can also be a period for A, because $A = A'((B^\omega)^{\tau'})$. Then, the period $\pi \cdot \omega$ will be of the form ω^β for some ordinal β. And for the bound, if $\ell(\kappa) < \beta$, we can use $\kappa + (\pi \cdot \omega)$ as the bound instead of κ, then it is obvious $\ell(\kappa + (\pi \cdot \omega)) = \beta$.

- If $\ell^2(\Omega) = 0$, then $\ell(\Omega)$ is a successor ordinal, assume that $\ell(\Omega) = \beta + 1$. Consider two subcases as follows:

  - If $\pi < \omega^\beta$, then $\pi \cdot \omega \leqslant \omega^\beta < \omega^{\ell(\Omega)}$. In this case, similarly, $\pi \cdot \omega$ can also be a period of A and A can be represented as $A'((B^\omega)^{\tau'})$ for some ordinal τ'. Hence, there exists a period of the form $\omega^{\beta'}$ and $\kappa + (\pi \cdot \omega)$ can be used as the bound instead of κ, hence it satisfies all the conditions.
  - If $\pi \geqslant \omega^\beta$, since we also know that $\pi < \omega^{\beta+1}$, we can assume that $\pi = \omega^\beta \cdot n + \gamma$ for a natural number n and an ordinal $\gamma < \omega^\beta$.

    In this case, first, we can assume that κ is large enough such that $\Omega = \kappa + \pi \cdot \omega$. Second, the string B can be viewed as the concatenation $B_1 B_2$ in which the length of B_1 is $\omega^\beta \cdot n$ and the length of B_2 is γ. So $A = A'((B_1 B_2)^\omega)$, however, we can also represent A as $(A' B_1)((B_2 B_1)^\omega)$, which means that we use $\kappa + \omega^\beta \cdot n$ as the bound and $\omega^\beta \cdot n$ as the period, hence it satisfies all the conditions.

In all, we have proved that if A is ultimately periodic in Ω, then there exists a period π of the form $\omega^\beta \cdot n$ with a bound κ such that $\ell(\kappa) > \beta$. ∎

Corollary 3.2.3. If A is ultimately periodic in Ω and $\ell^2(\Omega) > 0$, then there exists a prime ordinal ω^β as the period and the bound can be chosen as an ordinal κ such that $\ell(\kappa) \geqslant \beta$.

Proof. In the proof of Proposition 3.2.2, we have seen that when $\ell^2(\Omega) > 0$, the period can be of the form ω^β, which is exactly the statement of this corollary. ∎

We denote the set of all ultimately periodic sets in Ω as $\mathsf{U}(\Omega)$. And we prove that $\mathsf{U}(\Omega)$ forms a boolean algebra, i.e. $\mathsf{U}(\Omega)$ is closed under finite union, finite intersection, complement.

Proposition 3.2.4. For any ordinal Ω, $\mathsf{U}(\Omega)$ is closed under finite union, finite intersection, complement and the derived set operator.

Proof. We prove it case by case:

- Finite union: If $A_1, A_2 \in \mathsf{U}(\Omega)$, then there exist κ_1, κ_2 as their bounds and π_1, π_2 as their periods in Ω. By Proposition 3.2.2, we can assume that $\pi_1 = \omega^{\beta_1} \cdot n_1, \pi_2 = \omega^{\beta_2} \cdot n_2$. Now we aim to find a common period for A_1 and A_2:

  - If $\beta_1 = \beta_2$, then by Proposition 3.1.3, $\omega^{\beta_1} \cdot \mathrm{lcm}(n_1, n_2)$ is a common period.
  - If $\beta_1 > \beta_2$, then ω^{β_1} is also a period for A_2, which implies that $\omega^{\beta_1} \cdot n_1$ is a common period.
  - If $\beta_1 < \beta_2$, similarly, $\omega^{\beta_2} \cdot n_2$ is a common period.

  Denote the common period as π, and let κ be the maximum of κ_1 and κ_2. We will now prove that $A_1 \cup A_2$ is ultimately periodic with respect to A, with a bound κ and a period π. Suppose $x \sim_\pi y$ and $x, y > \kappa$. If $x \in A_1 \cup A_2$, then $x \in A_1$ or $x \in A_2$. Without loss of generality, assume $x \in A_1$. Since π is a period for A_1, it follows that $y \in A_1$, and therefore $y \in A_1 \cup A_2$. Hence, π is a period for $A_1 \cup A_2$ with bound κ.

- Finite intersection: The argument is similar to that of finite union.

- Complement: This is trivial.

■

Remark 3.2.5. We can see that finite sets and cofinite sets are ultimately periodic in Ω. Further, since $\mathsf{U}(\Omega)$ is closed under finite union and finite intersection, if $A \in \mathsf{U}(\Omega)$ and $B \subseteq \Omega$ is finite then $A \backslash B$ and $A \cup B$ are both ultimately periodic in Ω.

3.3 Hereditarily Periodic Sets

In this section, we define hereditarily periodic set on ordinals, which will be used as the admissible sets in our general topological frames. For $\beta < \Omega$, let $A|_\beta = A \cap \beta$. $A|_\beta$ can also be viewed as a string of length β, which is an initial segment of A.

Definition 3.3.1 (Hereditarily periodic set, or 0-periodic set)**.** A is *hereditarily periodic* in Ω, if for any limit ordinal $\beta \leqslant \Omega$, $A|_\beta$ is ultimately periodic in β.

The hereditary periodic set will also be called 0-*periodic set*. We denote the set of all 0-periodic sets in Ω as $\mathsf{H}_0(\Omega)$.

Proposition 3.3.2. $\mathsf{H}_0(\Omega)$ forms a Boolean algebra, that is, $\mathsf{H}_0(\Omega)$ is closed under finite union, finite intersection and complement.

Proof. By Definition 3.3.1, $A \in \mathsf{H}_0(\Omega)$ iff $A|_\beta \in \mathsf{U}(\Lambda)$ for any limit ordinal $\beta \leqslant \Omega$. Since all the $\mathsf{U}(\beta)$ are closed under these operators, it is evident that $\mathsf{H}_0(\Omega)$ is also closed under these operators. ■

Remark 3.3.3. It is also easy to check that finite sets and cofinite sets are hereditarily periodic periodic in Ω. Hence, similar to Remark 3.2.5, if $A \in \mathsf{U}(\Omega)$ and $B \subseteq \Omega$ is finite then $A \backslash B$ and $A \cup B$ are both hereditarily periodic in Ω.

Moreover, in the next chapter, we will prove that $\langle \Omega, \tau_0, \mathsf{H}_0(\Omega) \rangle$ forms a general topological frame for GL, which will be called 0-*periodic frame* for GL. And we will show the completeness of GL with respect to the class of all the 0-periodic frames.

However, in order to achieve not only soundness but also completeness for GLB, we need to extend $\mathsf{H}_0(\Omega)$ to the set of all the 1-periodic sets, which will be denoted as $\mathsf{H}_1(\Omega)$. In the next chapter, we will define 1-periodic frames for GLB.

Definition 3.3.4 (1-periodic set)**.** For an ordinal Ω, consider the rank function $\ell : \Omega \to \Omega$. We define $\mathsf{H}_1(\Omega)$ as the Boolean algebra generated by $\mathsf{H}_0(\Omega)$ and all the sets of the form $\ell^{-1}(A)$ for $A \in \mathsf{H}_0(\Omega)$. In other words, $\mathsf{H}_1(\Omega)$ is the least set satisfying the following conditions:

- $\mathsf{H}_0(\Omega) \subseteq \mathsf{H}_1(\Omega)$;
- For any $A \in \mathsf{H}_0(\Omega)$, $\ell^{-1}(A) \in \mathsf{H}_1(\Omega)$;
- For any $A, B \in \mathsf{H}_1(\Omega)$, $\overline{A} \in \mathsf{H}_1(\Omega)$, $A \cup B \in \mathsf{H}_1(\Omega)$ and $A \cap B \in \mathsf{H}_1(\Omega)$.

If $A \in \mathsf{H}_1(\Omega)$, then we say that A is 1-*periodic* in Ω.

Proposition 3.3.5. Any 1-periodic set $A \in \mathsf{H}_1(\Omega)$ can be represented as $\bigcup_{i=1}^{k} (\ell^{-1}(A_i) \cap B_i)$ where $A_i, B_i \in \mathsf{H}_0(\Omega)$ for $i \in [1, k]$.

Proof. Since $\mathsf{H}_1(\Omega)$ is the boolean algebra generated by $\mathsf{H}_0(\Omega)$ and all the sets of the form $\ell^{-1}(A)$ for $A \in \mathsf{H}_0(\Omega)$, we assume that A can be represented as $\bigcup_{i=1}^{k} \bigcap_{j=1}^{l} A_{ij}$ where each A_{ij} is of the form $A', \overline{A'}, \ell^{-1}(A')$ or $\overline{\ell^{-1}(A')}$ for some $A' \in \mathsf{H}_0(\Omega)$.

However, $\mathsf{H}_0(\Omega)$ is closed under complement and $\overline{\ell^{-1}(A')} = \ell^{-1}(\overline{A'})$, hence, each A_{ij} must be of the form A' or $\ell^{-1}(A')$ for some $A' \in \mathsf{H}_0(\Omega)$. Moreover, we know that $\mathsf{H}_0(\Omega)$ is closed under intersection and $\ell^{-1}(A_1) \cap \ell^{-1}(A_2) = \ell^{-1}(A_1 \cap A_2)$, therefore, the intersection $\bigcap_{j=1}^{l} A_{ij}$ can be simplified as $\ell^{-1}(A_i) \cap B_i$ for some $A_i.B_i \in \mathsf{H}_0(\Omega)$. ■

We claim following lemma, which will be useful for the completeness proof in chapter 5.

Lemma 3.3.6. *Consider two ordinals λ and λ', we have the following statements: (1) If $A \in \mathsf{H}_1(\lambda + 1)$, then $\{\lambda \cdot \mu + \beta | \mu < \lambda', \beta \in A\} \in \mathsf{H}_1(\lambda \cdot \lambda' + 1)$;*

(2) If $A \in \mathsf{H}_1(\lambda' + 1)$, then $\{\lambda \cdot \mu | \ell(\mu) > 0, \mu \in A\} \in \mathsf{H}_1(\lambda \cdot \lambda' + 1)$.

Proof. (1) Denote $\{\lambda \cdot \mu + \beta | \mu < \lambda', \beta \in A\}$ as $f(A)$. It is easy to see that $f(A \cup B) = f(A) \cup f(B)$ and $f(A \cap B) = f(A) \cap f(B)$, hence we only need to prove that $f(A) \in \mathsf{H}_1(\lambda \cdot \lambda' + 1)$ and $f(\ell^{-1}(A)) \in \mathsf{H}_1(\lambda \cdot \lambda' + 1)$ for any $A \in \mathsf{H}_0(\lambda + 1)$, because $\mathsf{H}_1(\lambda + 1)$ and $\mathsf{H}_1(\lambda \cdot \lambda' + 1)$ form boolean algebras.

Fix $A \in \mathsf{H}_0(\lambda + 1)$, first, we aim to show that $\{\lambda \cdot \mu + \beta | \mu < \lambda', \beta \in A\} \in \mathsf{H}_1(\lambda \cdot \lambda' + 1)$. In fact, it is easy to see that $\{\lambda \cdot \mu + \beta | \mu < \lambda', \beta \in A\} \in \mathsf{H}_0(\lambda \cdot \lambda' + 1)$. The reason is that if we view A and $f(A)$ as strings, then $f(A) = A^{\lambda'}$, so it is easy to verify that $A^{\lambda'}$ is hereditarily periodic.

Then, we aim to show that $f(\ell^{-1}(A)) = \{\lambda \cdot \mu + \beta | \mu < \lambda', \beta < \lambda \text{ and } \beta \in \ell^{-1}(A)\} \in \mathsf{H}_1(\lambda \cdot \lambda' + 1)$. Consider 0, $f(\ell^{-1}(0))$ contains all the ordinals $\gamma \in \lambda \cdot \lambda' + 1$ such that $\ell(\gamma) = 0$ or γ is a multiple of λ. Hence, it is easy to see that $f(\ell^{-1}(0)) \in \mathsf{H}_1(\lambda \cdot \lambda' + 1)$.

Now, we can assume that $0 \notin A$, otherwise, we divide the hereditarily periodic set A as $\{0\} \cup (A \backslash \{0\})$, then $f(\ell^{-1}(A)) = f(\ell^{-1}(0) \cup \ell^{-1}(A \backslash \{0\})) = f(\ell^{-1}(0)) \cup f(\ell^{-1}(A \backslash \{0\}))$ and we only need to prove that $f(\ell^{-1}(A \backslash \{0\})) \in \mathsf{H}_1(\lambda \cdot \lambda' + 1)$.

Since $0 \notin A$, any ordinal in $f(\ell^{-1}(A))$ must be in the form of $\lambda \cdot \mu + \beta$ with $\beta \neq 0$ and $\beta < \lambda$, then $\ell(\lambda \cdot \mu + \beta) = \ell(\beta) \in A$. On the other hand, it is easy to see that any ordinal $\gamma \leqslant \lambda \cdot \lambda' + 1$ with $\ell(\gamma) \in A$

must be in the form of $\lambda \cdot \mu + \beta$ for some $\mu < \lambda'$ and $\beta \in \ell^{-1}(A), \beta < \lambda$. So $f(\ell^{-1}(A)) \in \mathsf{H}_1(\lambda \cdot \lambda' + 1)$.

(2) First, $\{\lambda \cdot \mu | \ell(\mu) > 0, \mu \in A\} = \{\lambda \cdot \mu | \mu \in A \cap \ell^{-1}((\lambda' + 1)\backslash\{0\})\}$ and $A \cap \ell^{-1}((\lambda' + 1)\backslash\{0\}) \in \mathsf{H}_1(\lambda' + 1)$. Hence, more generally, we can try to show that if $A \in \mathsf{H}_1(\lambda' + 1)$ then $\{\lambda \cdot \mu | \mu \in A\} \in \mathsf{H}_1(\lambda \cdot \lambda' + 1)$ and this is obvious.

■

Chapter 4

General Topological Frames

In this chapter, we begin to build general topological frames for GL and GLB, which will be called periodic frames.

Recall the frame condition for GLP (Proposition 2.3.7): if $\langle \Omega, \tau_0, \tau_1, \mathcal{A} \rangle$ forms a general topological frame for GLB, we have the following conditions for τ_1:

- $\tau_0 \subseteq \tau_1$;
- For any $X \in \mathcal{A}$, $d_0(X) \in \tau_1$.

Hence, the least choice of τ_1 is the topology generated by τ_0 and $\{d_0(X) | X \in \mathcal{A}\}$. In this chapter, $\mathsf{H}_0(\Omega)$ and $\mathsf{H}_1(\Omega)$ will be used as the set of admissible sets in the general topological frame for GL and GLB respectively, which will be called *periodic frames*. We find that the topology τ_1 will coincide with the topology θ_2 introduced by Icard [18]. Last, we will show the soundness of GL and GLB in the corresponding periodic frames. In the next chapter, we will prove that GLB is also complete w.r.t. the class of periodic frames.

4.1 General Topological Frames for GL

In this section, we aim to prove that $\langle \Omega, \tau_0, \mathsf{H}_0(\Omega) \rangle$ forms a general topological frame, where τ_0 is the order topology and $\mathsf{H}_0(\Omega)$ is the set of all 0-periodic sets in Ω.

Proposition 4.1.1. For any ordinal Ω, $\mathsf{U}(\Omega)$ is closed under the derived set operator w.r.t. τ_0.

Proof. Suppose $\pi = \omega^\beta \cdot n$ is a period for A, and κ is a bound such that $\ell(\kappa) \geqslant \beta$. If A is bounded, then there is no element in a period, and thus $d_0(A)$ is also bounded, making it obviously ultimately periodic.

If A is unbounded, then any period is non-empty. Hence, for any ordinal $\gamma > \kappa$ with $\ell(\gamma) > \beta$, we have $\gamma \in d_0(A)$. We consider the following subcases:

- If $\ell(\Omega) > \beta + 1$, we can prove that $\omega^{\beta+1}$ is a period for $d_0(A)$ with a bound $\kappa + \omega^{\beta+1}$. For $x, y \geqslant \kappa + \omega^{\beta+1}$, if $x \sim_{\omega^{\beta+1}} y$, we have the following subcases:
  - If $\ell(x) \geqslant \beta + 1$, then y must also satisfy $\ell(y) \geqslant \beta + 1$, so both x and y belong to $d_0(A)$.
  - If $\ell(x) \leqslant \beta$, then there exist μ_x, μ_y such that $x \in (\kappa + \pi \cdot \mu_x, \kappa + \pi \cdot (\mu_x + 1)]$ and $y \in (\kappa + \pi \cdot \mu_y, \kappa + \pi \cdot (\mu_y + 1)]$. In this case, it is easy to see that $x \in d_0(A)$ if and only if $y \in d_0(A)$, as the topological structure of these two subsets is the same.
- If $\ell(\Omega) = \beta + 1$, then there exists a bound κ' such that $\Omega = \kappa' + \pi \cdot \omega$. We can prove that π is a period for $d_0(A)$ with a bound κ'. This is because if $x \sim_\pi y$ and $x, y > \kappa'$, then there exist μ_x, μ_y such that $x \in (\kappa' + \pi \cdot \mu_x, \kappa' + \pi \cdot (\mu_x + 1)]$ and $y \in (\kappa' + \pi \cdot \mu_y, \kappa' + \pi \cdot (\mu_y + 1)]$. It is easy to see that $x \in d_0(A)$ if and only if $y \in d_0(A)$.

■

Proposition 4.1.2. For any ordinal Ω, $\mathsf{H}_0(\Omega)$ is closed under finite union, finite intersection, complement and the derived set operator. Moreover, $\langle \Omega, \tau_0, \mathsf{H}_0(\Omega) \rangle$ forms a general topological frame for GL.

Proof. By Definition 3.3.1, $A \in \mathsf{H}_0(\Omega)$ iff $A|_\beta \in \mathsf{U}(\beta)$ for any limit ordinal $\beta \leqslant \Omega$. Since all the $\mathsf{U}(\beta)$ are closed under these operators, it is evident that $\mathsf{H}_0(\Omega)$ is also closed under these operators. So, by Definition 2.3.4, $\langle \Omega, \tau_0, \mathsf{H}_0(\Omega) \rangle$ forms a general topological frame.

In Proposition 2.2.4, we have shown that $\langle \Omega, \tau_0 \rangle$ is a scattered space, i.e. GL is valid in $\langle \Omega, \tau_0 \rangle$. Moreover, it is obvious that GL is also valid in the general topological frame $\langle \Omega, \tau_0, \mathsf{H}_0(\Omega) \rangle$, because any valuation based on the general topological frame $\langle \Omega, \tau_0, \mathsf{H}_0(\Omega) \rangle$ is also a suitable valuation based on $\langle \Omega, \tau_0 \rangle$. ■

Therefore, $\langle \Omega, \tau_0, \mathsf{H}_0(\Omega)\rangle$ forms a general topological frame for GL , we call it *periodic frame* for GL (or 0-periodic frame). Algebraically, this means that $\langle \mathsf{H}_0(\Omega), d_0\rangle$ forms a modal boolean algebra for GL. We call this algebra as *periodic ordinal algebra*.

Proposition 4.1.3. For any finite transitive irreflexive tree $T = \langle W, R\rangle$ of depth n, there exists an onto d-map $f : \langle \omega^n + 1, \tau_0, \mathsf{H}_0(\omega^n + 1)\rangle \to \langle W, R\rangle$.

Proof. In Theorem 2.2.9, we have constructed the onto d-map from $\langle \omega^n + 1, \tau_0\rangle$ to $\langle W, R\rangle$. In order to show that it is also a d-map between general topological frames, by Definition 2.3.5, we only need to prove that for any $A \subseteq W$, $f^{-1}(A) \in \mathsf{H}_0(\omega^n + 1)$.

When we view $\langle W, R\rangle$ as a general topological frame, it is $\langle W, \sigma, \mathcal{P}(W)\rangle$ where σ is the topology generated by all the R-upsets. Hence, we only need to show that for any $x \in W$, $f^{-1}(\{x\}) \in \mathsf{H}_0(\omega^n + 1)$. Prove it by induction on the depth n:

- If $n > 0$, the tree T contains only one irreflexive point a, so the result is trivial.

- Suppose that for any $k < n$ and a finite transitive irreflexive tree $T' = \langle W', R'\rangle$ of depth k, the result holds for the onto d-map $f' : \omega^k + 1 \twoheadrightarrow T'$. That is, for any $x \in W'$, $f_k^{-1}(\{x\}) \in \mathsf{H}_0(\omega^k + 1)$.

  Now, we prove it for the case n. In the proof of Theorem 2.2.9, $T = \bigsqcup_{i=1}^{l} T_i \cup \{a\}$ and $\omega^n + 1$ is written as the disjoint union $\biguplus_{j=1}^{\infty} X_j \cup \{\omega^n\}$ and the d-map from $\omega^n + 1$ to T is constructed as: (i) $f(\omega^n) = a$; (ii) For each $i \in \omega$, suppose that $i \equiv j \mod k$ for $j < l$. Then $f|_{X_i} : X_i \to T_j$ is specified as the map $f_j : \omega^{n_j} + 1 \twoheadrightarrow T_j$.

  In order to prove $f^{-1}(\{x\}) \in \mathsf{H}_0(\omega^n + 1)$, we divide it into two subcases:

  - If $x = a$, then it is obvious that $f^{-1}(\{x\}) = \{\omega^n\}$. Therefore, it forms a hereditarily periodic set in $\omega^n + 1$.
  - If $x \in T_j$ for $j \in [1, l]$ and the depth of T_j is n_j. By the induction hypothesis, we know that $f_j^{-1}(\{x\}) \in \mathsf{H}_0(\omega^{n_j} + 1)$. In the construction of f, we know that $f|_{X_i} : X_i \to T_j$ is specified as the d-map $f_j : \omega^{n_j} + 1 \twoheadrightarrow T_j$. Therefore, for each $i \in \omega$, $f^{-1}(x) \cap X_i$ is a hereditarily periodic set in $\omega^n + 1$. In all, $f^{-1}(x)$ is the union of them, which is also a hereditarily periodic set in $\omega^n + 1$.

In all, we prove that, the onto d-map we constructed in the proof of Theorem 2.2.9 is also an onto d-map from $\langle \omega^n + 1, \tau_0, \mathsf{H}_0(\omega^n + 1)\rangle$ to $\langle W, R\rangle$. ■

By Proposition 2.3.6, we have the following corollary.

Corollary 4.1.4. For any finite Kripke model $\langle W, R\rangle$ of GL, there exists an ordinal $\Omega < \omega^\omega$ and an embedding from $\langle \mathcal{P}(W), d_\sigma\rangle$ to $\langle \mathsf{H}_0(\Omega), d_0\rangle$. In other words, any finite algebra of GL, which is generated from a finite Kripke model of GL, can be embedded in a periodic ordinal algebra.

4.2 General Topological Frames for GLB

In this section, we aim to find a suitable τ_1 and show that $\langle \Omega, \tau_0, \tau_1, \mathsf{H}_1(\Omega)\rangle$ forms a general topological frame for GLB, which will be called periodic frame for GLB.

As we have discussed, if $\langle \Omega, \tau_0, \tau_1, \mathcal{A}\rangle$ is a general polytopological space of GLB, then the least choice of τ_1 is the topology generated by τ_0 and $\{d_0(X) | X \in \mathcal{A}\}$. Hence, let τ_1 be defined as the topology generated by τ_0 and $\{d_0(X) | X \in \mathsf{H}_1(\Omega)\}$.

Fortunately, we find that τ_1 coincides with the topology θ_2 introduced by Icard [18]. In [18], Icard introduced a topological model for the variable-free fragment of GLP (Icard's space). This model is the ordinal ϵ_0 equipped with a sequence of topologies $\theta_0, \theta_1, \ldots$ where θ_0 is the topology whose open sets are downward closed subsets of ϵ_0 and the topology θ_n is generated by θ_0 and all sets U^n_β for $\beta < \epsilon_0$, with

$$U^n_\beta := \{\gamma < \epsilon_0 : \ell^n(\gamma) > \beta\}.$$

He showed that the variable-free fragment of GLP is sound and complete w.r.t. Icard's space under the natural interpretation of modalities as the derived set operations of the corresponding topologies. In fact, Icard's space can be considered as a general topological frame for GLP where the algebra of admissible sets consists precisely of those sets definable by variable-free GLP-formulas. However, it is well-known that GLP is incomplete w.r.t. this general topological frame.

Now, we show that τ_1 coincides with topology θ_2. Later, by extending the family of admissible sets, we will define the general topological frames for which GLB will be sound and complete.

Proposition 4.2.1. In an ordinal space Ω, topology τ_1 generated by the order topology τ_0 with $\{d_0(X)|X \in \mathsf{H}_1(\Omega)\}$ is equal to topology θ_2 generated by τ_0 with all the sets of the form $U_\beta = \{\gamma : \ell(\gamma) > \beta\}$.

Proof. For the direction $\theta_2 \subseteq \tau_1$, we show that for any β, there exists $X \in \mathsf{H}_1(\Omega)$ such that $d_0(X) = U_\beta$. In fact, we can find a $X \in \mathsf{H}_0(\Omega)$ satisfying $d_0(X) = U_\beta$: let $X := \{0\} \cup \{\alpha|\ell(\alpha) \geqslant \beta\}$, then we prove the following statements:

- First, we prove that $X \in \mathsf{H}_0(\Omega)$. If so, we also have $X \in \mathsf{H}_1(\Omega)$.

  For any limit ordinal $\gamma \leqslant \Omega$, we should prove that $X|_\gamma \in \mathsf{U}(\gamma)$:

  - If $\ell(\gamma) > \beta$, then $X|_\gamma$ exactly contains all the ordinals $\alpha < \gamma$ such that $0 \equiv \alpha \mod \omega^\beta$. Hence, $X|_\gamma$ is a periodic set with period ω^β satisfying $\omega^\beta < \omega^{\ell(\gamma)}$.
  - If $\ell(\gamma) \leqslant \beta$, then we can find a bound κ such that for any ordinal $\alpha \in (\kappa, \gamma)$, $\ell(\alpha) < \beta$. Then, $X \cap (\kappa, \gamma) = \varnothing$. Therefore, $X|_\gamma \in \mathsf{U}(\gamma)$.

- Second, we prove that $d_0(X) = U_\beta$. For an ordinal α, we consider two cases:

  - $\ell(\alpha) \leqslant \beta$, then there exist μ such that $\alpha \in (\omega^\beta \cdot \mu, \omega^\beta \cdot (\mu+1)]$. Then $(\omega^\beta \cdot \mu, \omega^\beta \cdot (\mu+1)]$ is a τ_0-open set and $(\omega^\beta \cdot \mu, \omega^\beta \cdot (\mu+1)] \cap X = \{\omega^\beta \cdot (\mu+1)\}$. Hence, $d_0((\omega^\beta \cdot \mu, \omega^\beta \cdot (\mu+1)] \cap X) = \varnothing$. So α is not a limit point for X.
  - $\ell(\alpha) > \beta$, then $X|_\alpha$ is unbounded, because for any $x < \alpha$, we have $x + \omega^\beta < \alpha$. Hence, α is a limit point of the set $X|_\alpha$.

  Hence, $d_0(X) = U_\beta$.

So, we prove that $X \in \mathsf{H}_1(\Omega)$ and $d_0(X) = U_\beta$. Hence, $\theta_2 \subseteq \tau_1$.

For the other direction $\tau_1 \subseteq \theta_2$, we aim to show that for any $X \in \mathsf{H}_1(\Omega)$, $d_0(X) \in \theta_2$. By Proposition 3.3.5, we can assume that $X = \bigcup_{i=1}^k (\ell^{-1}(A_i) \cap B_i)$ where $A_i, B_i \in \mathsf{H}_0(\Omega)$ for $i \in [1, k]$, so $d_0(X) = \bigcup_{i=1}^k d_0(\ell^{-1}(A_i) \cap B_i)$. Hence, we only need to show that for any $\gamma \in d_0(\ell^{-1}(A_i) \cap B_i)$, there exists an open set $U \in \tau_1^I$ such that $\gamma \in U \subseteq d_0(\ell^{-1}(A_i) \cap B_i)$. We divide it to two cases:

- Suppose that $\ell^2(\gamma) = 0$, in [18], it is prove that such a γ is an isolated point in θ_2. Hence $\{\gamma\}$ is an open set in θ_2 and $\gamma \in \{\gamma\} \subseteq d_0(X)$.

- Suppose that $\ell^2(\gamma) > 0$, then $\ell(\gamma)$ is a limit ordinal. Since $B_i \in \mathsf{H}_0(\Omega)$, so $B_i|\gamma$ is ultimately periodic in γ with a period $\pi < \omega^{\ell(\gamma)}$ and a bound κ. By Corollary 3.2.3, we can assume that $\pi = \omega^\beta$ with $\beta < \ell(\gamma)$ and $\ell(\kappa) \geqslant \beta$.

  Since $\gamma \in d_0(\ell^{-1}(A_i) \cap B_i)$, $\ell^{-1}(A_i) \cap B_i$ is unbounded in γ. Consider an ordinal $\alpha \in \ell^{-1}(A_i) \cap B_i$ such that $\alpha > \kappa, \alpha \in B_i$, $\ell(\alpha) \in A_i$ and $\ell(\alpha) < \ell(\gamma)$, then it is easy to see that $\{\delta : \delta > \kappa, \ell(\delta) = \ell(\alpha), \delta \equiv \alpha \mod \pi\} \subseteq \ell^{-1}(A_i) \cap B_i$. Finally, this means that $U_{\ell(\alpha)} \cap (\kappa, \Omega) \subseteq d_0(\ell^{-1}(A_i) \cap B_i)$, which is an open set in τ_1^I.

Hence, we prove that for any $X \in \mathsf{H}_1(\Omega)$ and any $\gamma \in d_0(X)$, there exists an θ_2-open neighbourhood of γ which contains in $d_0(X)$. Therefore, $d_0(X)$ is θ_2-open, which shows that $\tau_1 \subseteq \theta_2$.

In all, we prove that $\tau_1 = \theta_2$. ■

Since $\tau_1 = \theta_2$, we consider $\tau_0 \cup \{U_\beta : \beta < \Omega\}$ as a basis for τ_1.

The following proposition from [4, Lemma 13.1] will be very useful later.

Proposition 4.2.2. For any ordinal Ω, the following statements hold:

(1) $\ell : \langle \Omega, \tau_1 \rangle \to \langle \Omega, \tau_0 \rangle$ is a d-map;

(2) ℓ^2 is the rank function of τ_1.

Now, we aim to show that $\langle \Omega, \tau_0, \tau_1, \mathsf{H}_1(\Omega) \rangle$ forms a general topological frames, that is, we need to prove that $\mathsf{H}_1(\Omega)$ is closed under d_0 and d_1.

Proposition 4.2.3. $\mathsf{H}_1(\Omega)$ is closed under the derived set operator w.r.t. τ_0.

Proof. For $A \in \mathsf{H}_1(\Omega)$, by Definition 3.3.5, we can assume that $A = \bigcup_{i=1}^k (\ell^{-1}(A_i) \cap B_i)$ where $A_i, B_i \in \mathsf{H}_0(\Omega)$ for $i \in [1, k]$. It is easy to see that $d_0(\bigcup_{i=1}^k (\ell^{-1}(A_i) \cap B_i)) = \bigcup_{i=1}^k d_0(\ell^{-1}(A_i) \cap B_i)$. Hence, we only need to show that for any $A, B \in \mathsf{H}_0(\Omega)$, we have $d_0(\ell^{-1}(A) \cap B) \in \mathsf{H}_1(\Omega)$. We aim to show that for any limit ordinal $\gamma \leqslant \Omega$, if B is ultimately periodic with respect to γ (i.e. $B \cap \gamma$ is ultimately periodic in γ), then so is $d_0(\ell^{-1}(A) \cap B)$.

With respect to γ, suppose that there exists a bound κ and a period $\pi := \omega^\nu(n+1)$, that is to say, $B \cap \gamma$ can be viewed as a string $B_0 B_1^\alpha$ as

Definition 3.1.4 such that $\kappa = |B_0|$, $\pi = |B_1|$ and $\gamma = \kappa + \pi \cdot \alpha$. Hence, by the additivity of d_0, it is sufficient to show that $d_0(B_0 \cap \ell^{-1}(A))$ and $d_0(B_1^\alpha \cap \ell^{-1}(A))$ are both ultimately periodic in γ.

$B_0 \cap \ell^{-1}(A)$ is bounded in γ, therefore, so is $d_0(B_0 \cap \ell^{-1}(A))$. Hence, it is ultimately periodic in γ.

For $d_0(B_1^\alpha \cap \ell^{-1}(A))$, consider the set $E := \{\pi \cdot \beta : \beta < \alpha\}$. These are the coordinates of the first occurrences of each period B_1 in the string. Then, we split B_1^α into $B_1^\alpha \cap E$ and $B_1^\alpha \backslash E$. By periodicity, if $0 \in B_1$ then $B_1^\alpha \cap E = E$, otherwise $B_1^\alpha \cap E = \varnothing$. Using the additivity of d_0 again, we now deal with $d_0(B_1^\alpha \cap E \cap \ell^{-1}(A))$ and $d_0((B_1^\alpha \backslash E) \cap \ell^{-1}(A))$ separately.

For $d_0(B_1^\alpha \cap E \cap \ell^{-1}(A))$, there are two cases: if $0 \notin B_1$, then $B_1^\alpha \cap E = \varnothing$, so $d_0(B_1^\alpha \cap E \cap \ell^{-1}(A)) = \varnothing$; if $0 \in B_1$, then $d_0(B_1^\alpha \cap E \cap \ell^{-1}(A)) = d_0(E \cap \ell^{-1}(A))$, which is obviously ultimately periodic.

For $d_0((B_1^\alpha \backslash E) \cap \ell^{-1}(A))$, we observe that the ranks of points in $B_1^\alpha \backslash E$ are the same as of the corresponding points in $B_1 \backslash \{0\}$. Hence, $d_0((B_1^\alpha \backslash E) \cap \ell^{-1}(A)) = d_0(((B_1 \backslash \{0\}) \cap \ell^{-1}(A))^\alpha)$. Therefore, it is obviously ultimately periodic.

■

Proposition 4.2.4. $\mathsf{H}_1(\Omega)$ is closed under the derived set operator w.r.t. τ_1.

Proof. For $A \in \mathsf{H}_1(\Omega)$, by Definition 3.3.5, we can assume that $A = \bigcup_{i=1}^k (\ell^{-1}(A_i) \cap B_i)$ where $A_i, B_i \in \mathsf{H}_0(\Omega)$ for $i \in [1, k]$. It is easy to see that $d_1(\bigcup_{i=1}^k (\ell^{-1}(A_i) \cap B_i)) = \bigcup_{i=1}^k d_1(\ell^{-1}(A_i) \cap B_i)$. Hence, we only need to show that for any $A, B \in \mathsf{H}_0(\Omega)$, we have $d_1(\ell^{-1}(A) \cap B) \in \mathsf{H}_1(\Omega)$.

In fact, we aim to show that $d_1(\ell^{-1}(A) \cap B) = d_1(\ell^{-1}(A)) \cap d_1(B)$. It is obvious that $d_1(\ell^{-1}(A) \cap B) \subseteq d_1(\ell^{-1}(A)) \cap d_1(B)$, hence we only need to show the other direction.

Assume that $\alpha \in d_1(\ell^{-1}(A)) \cap d_1(B)$, then $\ell^2(\alpha) > 0$. Since $B \in \mathsf{H}_0(\Omega)$, we have $B|_\alpha \in \mathsf{U}(\alpha)$. Then, for $B|_\alpha$, there exists a period $\pi = \omega^\beta$ and a bound κ such that $\ell(\kappa) \geqslant \beta$. For any $\kappa < x, y < \alpha$, if $\ell(x), \ell(y) \geqslant \beta$, then $x \sim_\pi y$. Hence, either $U_\beta \cap (\kappa, \alpha) \subseteq B$ or $U_\beta \cap (\kappa, \alpha) \cap B = \varnothing$. However, if $U_\beta \cap (\kappa, \alpha) \cap B = \varnothing$ then $U_\beta \cap (\kappa, \alpha]$ is an τ_1-open neighbourhood of α such that $U_\beta \cap (\kappa, \alpha] \cap B|_\alpha = \varnothing$, which is contradict to the assumption $a \in d_1(B)$. Hence, we must have $U_\beta \cap (\kappa, \alpha) \subseteq B$.

Then we know that $U_\beta \cap (\kappa, \alpha]$ is an τ_1-open neighbourhood of α such that $U_\beta \cap (\kappa, \alpha) \subseteq B$. In this case, it is obvious that $\alpha \in d_1(\ell^{-1}(A)) \Rightarrow \alpha \in d_1(\ell^{-1}(A) \cap U_\beta \cap (\kappa, \alpha)) \Rightarrow \alpha \in d_1(\ell^{-1}(A) \cap B)$.

Now, we have $d_1(\ell^{-1}(A) \cap B) = d_1(\ell^{-1}(A)) \cap d_1(B)$. By Proposition 4.2.2, we know that ℓ is a d-map from $\langle \Omega, \tau_1 \rangle$ to $\langle \Omega, \tau_0 \rangle$. Combining with Proposition 2.2.8, we have $d_1(\ell^{-1}(A)) = \ell^{-1}(d_0(A))$. Therefore, $d_1(\ell^{-1}(A) \cap B) = d_1(\ell^{-1}(A)) \cap d_1(B) = \ell^{-1}(d_0(A)) \cap d_1(B)$. Since $A, B \in \mathsf{H}_0(\Omega)$ and $\mathsf{H}_0(\Omega)$ is closed under d_0 and d_1, we have $d_0(A) \in \mathsf{H}_0(\Omega)$ and $d_1(B) \in \mathsf{H}_0(\Omega)$. Hence, we prove that $d_1(\ell^{-1}(A) \cap B) \in \mathsf{H}_1(\Omega)$, i.e., $\mathsf{H}_1(\Omega)$ is closed under d_1. ∎

Hence, $\langle \Omega, \tau_0, \tau_1, \mathsf{H}_1(\Omega) \rangle$ forms a general topological frames. We call such frames *periodic frames* or 1-periodic frames.

Proposition 4.2.5. For any ordinal Ω, GLB is sound w.r.t. the general topological frame $\langle \Omega, \tau_0, \tau_1, \mathsf{H}_1(\Omega) \rangle$.

Proof. First, in [18], it is proved that both $\langle \Omega, \tau_0 \rangle$ and $\langle \Omega, \tau_1 \rangle$ are scattered spaces. Second, $\tau_0 \subseteq \tau_1$.

Last, for any $A \in \mathsf{H}_1(\Omega)$, we have proved that $d_0(A) \in \tau_1$. $\mathsf{H}_0(\Omega) \subseteq \mathsf{H}_1(\Omega)$, so, for $A \in \mathsf{H}_0(\Omega)$, $d_0(A) \in \tau_1$ is also true.

In all, by Proposition 2.3.7, all the axioms of GLB are valid in $\langle \Omega, \tau_0, \tau_1, \mathsf{H}_1(\Omega) \rangle$.

∎

Chapter 5

Completeness for GLB

In this chapter, our goal is to prove that GLB is complete with respect to the class of 1-periodic frames.

5.1 JB-Frame

For the topological completeness proof, we need to discuss a subsystem of GLB introduced in [8] and denoted JB. This logic is defined by weakening axiom (ii) of GLB to the following axioms (iv) and (v) both of which are theorems of GLP:

(iv) $[0]\varphi \to [1][0]\varphi$;

(v) $[0]\varphi \to [0][1]\varphi$.

The logic JB corresponds to a simple class of frames, which is established using standard methods [8, Theorem 1]. u

Lemma 5.1.1. JB *is sound and complete w.r.t the class of (finite) frames* $\langle W, R_0, R_1\rangle$ *such that, for all* $x, y, z \in W$,

1 R_0 *and* R_1 *are transitive and dually well-founded binary relations;*

2 If xR_1y, *then* xR_0z *iff* yR_0z;

3 xR_0y *and* yR_1z *imply* xR_0z.

Let R_0^* denote the transitive closure of $R_0 \cup R_1$ and let E_0, E_1 denote the reflexive, symmetric, transitive closure of R_0^* and R_1, respectively. It is clear that E_1 refines E_0. We call each E_i equivalence class a *i-sheet*. By condition 2, all points in a 1-sheet are R_0 incomparable. However, R_0 naturally defines an ordering on 1-sheets as follows: if α and β are 1-sheets, then $\alpha R_0 \beta$ iff $\exists x \in \alpha \exists y \in \beta x R_0 y$. By the standard techniques, one can improve the Lemma to show the completeness, in which the set of 1-sheets contained in each 0-sheet is a tree under R_0, and if $\alpha R_0 \beta$ then $x R_0 y$ for all $x \in \alpha, y \in \beta$. Any structure satisfying these conditions automatically becomes a JB-frame, which we refer to as a "tree-like JB-frame" (JB-tree for short).

As shown in [8], GLB is reducible to JB in the following sense. First, assume that φ is a formula in the language $\mathcal{L}_B$ containing both [0] and [1]. Denote the set of all subformulas of φ of the form $[0]\psi$ as $\{[0]\psi_i\}_{i \in I}$. Then, let

$$M(\varphi) := \bigwedge_{i \in I} ([0]\psi_i \to [1]\psi_i).$$

When φ doesn't contain both [0] and [1], we define $M(\varphi)$ as $\top$. Also, let $M^+(\varphi) := M(\varphi) \wedge [0]M(\varphi) \wedge [1]M(\varphi)$.

Proposition 5.1.2 ([8])**.** GLB $\vdash \varphi$ iff JB $\vdash M^+(\varphi) \to \varphi$.

Consider a JB-tree $T = \langle T, R_0, R_1 \rangle$. A node $w \in T$ is called a 0-root if there is no predecessor of w with respect to R_0^*, and it is called a 1-root if there is no predecessor of w with respect to R_1.

Definition 5.1.3. We view T as a polytopological space $T = \langle T, \sigma_0, \sigma_1 \rangle$ by considering all R_i-upsets to be σ_i-open. Given a general topological space $A = \langle \Omega, \tau_0, \tau_1, \mathsf{H}_1(\Omega) \rangle$ and a map $f : \Omega \to T$ we will say that f is a JB-morphism if:

(j_1) For any subset $X \subseteq T$, $f^{-1}(X) \in \mathsf{H}_1(\Omega)$;

(j_2) $f : \langle \Omega, \tau_1 \rangle \to \langle T, \sigma_1 \rangle$ is a d-map;

(j_3) $f : \langle \Omega, \tau_0 \rangle \to \langle T, \sigma_0 \rangle$ is an open map;

(j_4) For each 1-root $w \in T$, the sets $f^{-1}(R_0^*(w))$ and $f^{-1}(R_0^*(w) \cup \{w\})$ are open in τ_0;

(j_5) For each 1-root $w \in T$, the set $f^{-1}(w)$ is a τ_0-discrete subspace of A.

A JB-morphism $f : \Omega \to T$ can be thought of as a map which is a weak kind of d-map from $\langle \Omega, \tau_i \rangle$ to $\langle T, \sigma_i \rangle$ for $i \in \{0, 1\}$.

The definition of JB-morphism is a generalization of the definition in [3], denoted as J_2-morphism there. The difference is that their J_2-morphism is a kind of morphism from a topological space to a JB-tree, but we generalize it for general topological frames. Hence, we add the condition (j_1), (j_2) – (j_5), which have existed in the definition of $\mathbf{J_2}$-morphism in [3]. Then, the following theorem is a version of Theorem 6.6 in [3].

Theorem 5.1.4. *Let $A = \langle \Omega, \tau_0, \tau_1, \mathsf{H}_1(\Omega) \rangle$ be a general topological frame for* GLB*, T a* JB*-tree, $f : \Omega \to T$ a* JB*-morphism and φ a $\mathcal{L}_B$-formula. Then $A \models \varphi$ iff $T \models M^+(\varphi) \to \varphi$.*

The proof of this theorem will be the same as the proof for Theorem 6.6 in [3].

Our aim is to prove the following lemma.

Lemma 5.1.5 (Main)**.** *For each finite* JB*-tree $\langle T, R_0, R_1 \rangle$, there exists an ordinal Ω and an onto* JB*-morphism $f : \langle [1, \Omega], \tau_0, \tau_1, \mathsf{H}_1(\Omega+1) \rangle \twoheadrightarrow \langle T, R_0, R_1 \rangle$, where $\Omega < \omega^{\omega^\omega}$.*

Using this lemma, it is easy to see that the logic GLB is complete w.r.t. the class of 1-periodic frames.

Remark 5.1.6. It is worth noting that in Lemma 5.1.5, we analyze the general topological frames based on sets of the form $[1, \Omega]$ instead of $[0, \Omega]$. Here, it should be understood as the subframe, i.e., we restrict the topologies τ_0 and τ_1, as well as the admissible sets $\mathsf{H}_1(\Omega + 1)$, to $[1, \Omega]$. For convenience, we still denote it as $\langle [1, \Omega], \tau_0, \tau_1, \mathsf{H}_1(\Omega + 1) \rangle$.

On the other hand, it is easy to see that a 1-periodic frame $\langle [0, \Omega], \tau_0, \tau_1, \mathsf{H}_1(\Omega + 1) \rangle$ is isomorphic to the subframe $\langle [1, \Omega], \tau_0, \tau_1, \mathsf{H}_1(\Omega + 1) \rangle$.

Moreover, this topological completeness theorem can also be stated in a stronger *uniform* way.

Theorem 5.1.7. *Let $\Omega = \langle \omega^{\omega^\omega}, \tau_0, \tau_1, \mathsf{H}_1(\omega^{\omega^\omega}) \rangle$. Then* $\mathsf{Log}(\Omega) = \mathsf{GLB}$*.*

5.2 Some operations on Ordinal Spaces

In this part, we introduce two operations on ordinal spaces: sum and lifting.

Definition 5.2.1 (Sum). Suppose that we have a finite number of ordinals $\lambda_1, \ldots, \lambda_k$ and d-maps $f_i : \langle [1, \lambda_i], \tau_0, \tau_1, \mathsf{H}_1(\lambda_i + 1)\rangle \to \langle T_i, R_0, R_1\rangle$ where $\langle T_i, R_0, R_1\rangle$ is a JB-tree for each $i \in [1, k]$. Define $\lambda = \lambda_1 + \ldots + \lambda_k$, and the *sum* space is $\langle [1, \lambda], \tau_0, \tau_1, \mathsf{H}_1(\lambda + 1)\rangle$. Moreover, we define the *sum* of these JB-morphisms f_i, denoting as $\bigsqcup_{i=1}^k f_i : \langle [1, \lambda], \tau_0, \tau_1, \mathsf{H}_1(\lambda + 1)\rangle \to \bigsqcup_{i=1}^k \langle T_i, R_0, R_1\rangle$, as follows:

$$\bigsqcup_{i=1}^{k} f_i(\beta) := f_i(\beta'), \text{ if } \beta = \lambda_1 + \ldots + \lambda_{i-1} + \beta', \beta' \in [1, \lambda_i].$$

The following proposition regarding the sum operation is straightforward.

Proposition 5.2.2. The sum space $\langle [1, \lambda], \tau_0, \tau_1, \mathsf{H}_1(\lambda + 1)\rangle$ is isomorphic to the topological sum $\bigsqcup_{i=1}^k [1, \lambda_i + 1]$. Moreover, the sum of JB-morphisms $\bigsqcup_{i=1}^k f_i$ is a JB-morphism.

Next, we introduce the lifting operation, similar to [3, Lemma 8.6], which allows us to construct a d-map from an ordinal equipped with the topology τ_1 to a tree-like Kripke model.

Definition 5.2.3 (Lifting Space). For an ordinal space $\langle [0, \lambda], \tau_0\rangle$, the ordinal space $\langle [1, \omega^\lambda], \tau_1\rangle$ is called the *lifting space* of $\langle [0, \lambda], \tau_0\rangle$.

Lemma 5.2.4. *The rank function* $\ell : \langle [1, \omega^\lambda], \tau_1, \mathsf{H}_1(\omega^\lambda + 1)\rangle \to \langle [0, \lambda], \tau_0, \mathsf{H}_0(\lambda + 1)\rangle$ *is a d-map.*

Proof. In Proposition 4.2.2, we have shown that ℓ is a d-map from $\langle [1, \omega^\lambda], \tau_1\rangle$ to $\langle [0, \lambda], \tau_0\rangle$. Therefore, we only need to show that for any $A \in \mathsf{H}_0(\lambda + 1)$, $l^{-1}(A) \in \mathsf{H}_1(\omega^\lambda + 1)$. This follows directly from Definition 3.3.4. ■

5.3 Proof of main lemma

For each finite JB-tree $\langle T, R_0, R_1\rangle$ with a root a, we aim to specify an ordinal $\lambda < \omega^{\omega^\omega}$ such that there exists a JB-morphism $f : \langle [1, \lambda], \tau_0, \tau_1, \mathsf{H}_1(\lambda + 1)\rangle \twoheadrightarrow \langle T, R_0, R_1\rangle$ with $f^{-1}(a) = \{\lambda\}$. We refer to such a JB-morphism as *suitable*. We divide the proof into the following lemmas.

We prove the claim by induction on the R_0-depth of T, which is denoted as $ht_0(T)$.

Lemma 5.3.1 (The Base Step)**.** *Suppose $ht_0(T) = 0$. Then there exists an ordinal $\lambda < \omega^{\omega^\omega}$ and a suitable* JB*-morphism from $[1, \lambda]$ to T.*

Proof. If $ht_0(T) = 0$, then T contains only one 1-sheet, i.e. it is a tree w.r.t. relation R_1.

We construct the model in two steps:

- Consider the tree $\langle T, R_1 \rangle$, which contains only one relation. By Theorem 2.2.9 and Proposition 4.1.3, we know that there exists an ordinal $\lambda_0 < \omega^\omega$ and a mapping $f_0 : [0, \lambda_0] \to T$ such that $f_0 : \langle [0, \lambda_0], \tau_0, \mathsf{H}_0(\lambda_0 + 1) \rangle \to \langle T, R_1 \rangle$ is a d-map.

- By Lemma 5.2.4, we have the lifting space $\langle [1, \omega^{\lambda_0}], \tau_1, \mathsf{H}_1(\omega^{\lambda_0} + 1) \rangle$ such that the rank function $\ell : \langle [1, \omega^{\lambda_0}], \tau_0, \tau_1, \mathsf{H}_1(\omega^{\lambda_0} + 1) \rangle \twoheadrightarrow \langle [0, \lambda_0], \tau_0, \mathsf{H}_0(\lambda_0 + 1) \rangle$ is a d-map.

Combining the two steps, we obtain a d-map $f_0 \circ \ell : \langle [1, \omega^{\lambda_0}], \tau_1, \mathsf{H}_1(\omega^{\lambda_0} + 1) \rangle \to \langle T, R_1 \rangle$ denoted as f. We now prove that f is also a JB-morphism from $\langle [1, \omega^{\lambda_0}], \tau_0, \tau_1, \mathsf{H}_1(\omega^{\lambda_0} + 1) \rangle$ to $\langle T, R_0, R_1 \rangle$:

(j_1) We need to prove that for any point $w \in T$, $f^{-1}(\{w\}) \in \mathsf{H}_0(\omega^{\lambda_0} + 1)$. Since $f = f_0 \circ \ell$, we have $f^{-1}(\{w\}) = \ell^{-1}(f_0^{-1}(\{w\}))$. Since f_0 is the function constructed in the proof of Theorem 2.2.9, we know that $f_0^{-1}(\{w\}) \in \mathsf{H}_0(\lambda_0 + 1)$. By Definition 3.3.4, if $A \in \mathsf{H}_0(\lambda_0 + 1)$, then $\ell^{-1}(A) \in \mathsf{H}_1(\omega^{\lambda_0} + 1)$. Therefore, $f^{-1}(\{w\})$ is a 1-periodic set in $\omega^{\lambda_0} + 1$.

(j_2) Since f is a d-map from $\langle \omega^{\lambda_0} + 1, \tau_1 \rangle$ to $\langle T, R_1 \rangle$, the condition (j_2) is satisfied.

(j_3) In this case, $R_0 = \varnothing$, so any subset of T is a σ_0-open set. Hence, it is evident that $f : \langle \omega^{\lambda_0} + 1, \tau_0 \rangle \to \langle T, R_0 \rangle$ is an open map.

(j_4) Since $R_0 = \varnothing$, so $R_0^* = R_1$. T contains only one 1-sheet, i.e. the root a of T. Hence, we only need to check that $f^{-1}(T \backslash \{a\})$ and $f^{-1}(T)$ are open in τ_0. $f^{-1}(\{a\}) = \ell^{-1}(f_0^{-1}(\{a\})) = \ell^{-1}(\{\lambda_0\}) = \{\omega^{\lambda_0}\}$, so $f^{-1}(T \backslash \{a\}) = \{\alpha | \alpha < \omega^{\lambda_0}\}$ and $f^{-1}(T) = \{\alpha | \alpha \leqslant \omega^{\lambda_0}\}$, which are both τ_0-open.

(j_5) We have checked that $f^{-1}(\{a\}) = \{\omega^{\lambda_0}\}$, so it is obviously a τ_0-discrete subspace.

Therefore, we conclude that f is a JB-morphism from $\langle[1,\omega^{\lambda_0}],\tau_0,\tau_1,\mathsf{H}_1(\omega^{\lambda_0}+1)\rangle$ to $\langle T,R_0,R_1\rangle$. Obviously, $\omega^{\lambda_0}<\omega^{\omega^\omega}$. ■

Lemma 5.3.2 (The Induction Step)**.** *Suppose that $ht_0(T)=m>0$. Then there exists an ordinal $\lambda<\omega^{\omega^\omega}$ and a suitable* JB*-morphism from $[1,\lambda]$ to T.*

Proof. If $ht_0(T)=m>0$. Let $a_1,\cdots,a_k$ be the immediate R_0-successors of a, which are 1-roots. Denote $T_i=\{a_i\}\cup R_0^*(a_i)$ for $i\in[1,k]$, and $T_0=\{a\}\cup R_1(a)$. Note that $T=\bigcup_{i=0}^k T_i$. Furthermore, for each $i\in[1,k]$ the subframe T_i of T is a JB-tree of R_0-depth less than m. By the induction hypothesis, for each $i\in[1,k]$, there exists an ordinal $\lambda_i<\omega^{\omega^\omega}$ and a JB-morphism $g_i:\langle[1,\lambda_i],\tau_0,\tau_1,\mathsf{H}_1(\lambda_i+1)\rangle\to\langle T_i,R_0,R_1\rangle$.

Let $\lambda:=\lambda_1+\cdots+\lambda_k$ and let $g:\langle[1,\lambda],\tau_0,\tau_1,\mathsf{H}_1(\lambda+1)\rangle\to\langle\bigsqcup_{i=1}^k T_i,R_0,R_1\rangle$ be the sum of g_i, i.e. $g=\bigsqcup_{i=1}^k g_i$. We denote $\bigsqcup_{i=1}^k T_i$ as S. By Proposition 5.2.2, we know that g is a JB-morphism.

Next, consider the 1-sheet $\langle T_0,R_0,R_1\rangle$. Using the construction from the Base Step (Lemma 5.3.1), there exists an ordinal $\lambda_0<\omega^{\omega^\omega}$ and a suitable JB-morphism $g_0:\langle[1,\lambda_0],\tau_0,\tau_1,\mathsf{H}_1(\lambda_0+1)\rangle\to\langle T_0,R_0,R_1\rangle$.

Now, we aim to construct a function $f:\langle[1,\lambda\cdot\lambda_0],\tau_0,\tau_1,\mathsf{H}_1(\lambda\cdot\lambda_0+1)\rangle\to\langle T,R_0,R_1\rangle$. First, divide $[1,\lambda\cdot\lambda_0]$ into two disjoint parts: $X_0:=\{\lambda\cdot\mu+\nu|\mu<\lambda_0,\nu\in[1,\lambda]\}$ and $X_1:=\{\lambda\cdot\mu|\mu\leqslant\lambda_0\wedge\ell(\mu)>0\}$. Then, define f as follows:

$$f(\alpha):=\begin{cases}g(\nu), & \text{if } \alpha\in X_0 \text{ is of the form } \lambda\cdot\mu+\nu;\\ g_0(\mu), & \text{if } \alpha\in X_1 \text{ is of the form } \lambda\cdot\mu.\end{cases}$$

Last, we prove that f is a suitable JB-morphism from $\langle[1,\lambda\cdot\lambda_0],\tau_0,\tau_1,\mathsf{H}_1(\lambda\cdot\lambda_0+1)\rangle$ to $\langle T,R_0,R_1\rangle$:

(j_1) We need to prove that for any point $w\in T$, $f^{-1}(\{w\})\in\mathsf{H}_1(\lambda\cdot\lambda_0+1)$. We divide it to two cases:

- Suppose that $w\in T_i$ for some $i\in[1,k]$. We have known that g_i is a JB-morphism, hence $g_i^{-1}(\{w\})\in\mathsf{H}_1(\lambda_i+1)$. It follows that $g^{-1}(\{w\})\in\mathsf{H}_1(\lambda+1)$. In the product space, $f^{-1}(\{w\})$ is $\{\lambda\cdot\mu+\beta|\mu<\lambda_0,\beta\in g^{-1}(\{w\})\}$. By Lemma 3.3.6, it is a 1-periodic set in $[1,\lambda\cdot\lambda_0]$.
- Suppose that $w\in T_0$. We have known that g_0 is a JB-morphism, hence $g_0^{-1}(\{w\})\in\mathsf{H}_1(\lambda_0+1)$. In the product space, $f^{-1}(\{w\})$ is $\{\lambda\cdot\mu|\ell(\mu)>0,\mu\in g_0^{-1}(\{w\})\}$. By Lemma 3.3.6, it is a 1-periodic set in $[1,\lambda\cdot\lambda_0]$.

(j_2) We need to prove that f is a d-map from $\langle[1,\lambda\cdot\lambda_0],\tau_1\rangle$ to $\langle T,R_1\rangle$.

We divide T into two parts: $S=\bigsqcup_{i=1}^{k}T_i$ and T_0, since T_0 is a 1-sheet, $\langle T,R_1\rangle$ can be viewed as the disjoint union of $\langle S,R_1\rangle$ and $\langle T_0,R_1\rangle$. On the other hand, $\langle[1,\lambda\cdot\lambda_0],\tau_1\rangle$ can also be divided into two parts: $X_0:=\{\lambda\cdot\mu+\nu|\mu<\lambda_0,\nu\in[1,\lambda]\}$ and $X_1:=\{\lambda\cdot\mu|\mu\leqslant\lambda_0\wedge\ell(\mu)>0\}$. We know that $X_0=f^{-1}(S)$ and $X_1=f^{-1}(T_0)$.

Hence, in order to prove that f is a d-map, we only need to show that $f|_{X_0}:X_0\to S$ and $f|_{X_1}:X_1\to T_0$ are both d-maps. Since g_0 is a d-map, it is easy to see that $f|_{X_1}$ is also d-map. For $f|_{X_0}$, in fact, X_0 can be represented as the topological product $[1,\lambda]\times\{\mu|\ell(mu)=0,\mu<\lambda_0\}$. $g:\lambda+1\to S$ is a d-map, so it follows that $f|_{X_0}$ is also a d-map.

In all, we show that f is a d-map.

(j_3) In order to prove that f is an open map from $\langle[1,\lambda\cdot\lambda_0],\tau_0\rangle$ to $\langle T,R_0\rangle$, we need to show that for any open interval (β_1,β_2) in $[1,\lambda\cdot\lambda_0]$, $f((\beta_1,\beta_2))$ is R_0-upset. We divide it to two cases:

- If there exists an limit ordinal μ such that $\lambda\cdot\mu\in(\beta_1,\beta_2)$, then it is easy to see that there exists and ordinal $\mu'<\mu$ such that $[\lambda\cdot\mu',\lambda\cdot(\mu'+1)\subseteq(\beta_1,\beta_2)$ and $f([\lambda\cdot\mu',\lambda\cdot(\mu'+1))=S$. Hence, $f((\beta_1,\beta_2))$ forms a R_0-upset.
- If there is no limit ordinal μ such that $\lambda\cdot\mu\in(\beta_1,\beta_2)$, then $f((\beta_1,\beta_2))\subseteq S$. In this case, we know that the restriction of f in any interval $[\lambda\cdot\delta+1,\lambda\cdot(\delta+1)]$ is an open map. Hence, it is obvious that $f((\beta_1,\beta_2))$ is R_0-upset.

Hence, f is still an open map.

(j_4) There are two kinds of 1-root in T, it is either the root a of T or a 1-root in T_i for some $i\in[1,k]$:

- For the root a, $f^{-1}(\{a\})=\{\lambda\cdot\lambda_0\}$, so $f^{-1}(R_0^*(a))=f^{-1}(T\backslash\{a\})=[1,\lambda\cdot\lambda_0)$ and $f^{-1}(R_0^*(a)\cup\{a\})=f^{-1}(T)=[1,\lambda\cdot\lambda_0]$, which are both τ_0-open.
- For a 1-root $w\in T_i$, since g_i is JB-morphism, we have known that $g_i^{-1}(R_0^*(w))$ and $g_i^{-1}(R_0^*(w)\cup\{w\})$ are τ_0-open in $[1,\lambda_i]$, then it is easy to see that $g^{-1}(R_0^*(w))$ and $g^{-1}(R_0^*(w)\cup\{w\})$ are τ_0-open in $[1,\lambda]$. Therefore, $f^{-1}(R_0^*(w))=\{\alpha:\alpha=\lambda\cdot\mu+\nu,$ for $\nu\in$

$g^{-1}(R_0^*(w))\}$ and $f^{-1}(R_0^*(w) \cup \{w\}) = \{\alpha : \alpha = \lambda \cdot \mu + \nu,$ for $\nu \in R_0^*(w) \cup \{w\}\}$, which are both τ_0-open, because they are the union of τ_0-open sets.

(j_5) Similar to (j_4), there are two cases:

- For the root a, $f^{-1}(\{a\}) = \{\lambda \cdot \lambda_0\}$, hence, it is obviously τ_0-discrete subspace.
- For a 1-root $w \in T_i$, since g_i is JB-morphism, we have known that $g_i^{-1}(w)$ is τ_0-discrete subspace of $[1, \lambda_i]$, then it is obvious that $g^{-1}(w)$ is τ_0-discrete subspace of $[1, \lambda]$. Therefore, $f^{-1}(w) = \{\alpha : \alpha = \lambda \cdot \mu + \nu,$ for $\nu \in g^{-1}(w)\}$, which is a τ_0-discrete subspace of $[1, \lambda \cdot \lambda_0]$.

Hence, we find that f is a JB-morphism from $\langle [1, \lambda \cdot \lambda_0], \tau_0, \tau_1, \mathsf{H}_1(\lambda \cdot \lambda_0 + 1) \rangle$ to $\langle T, R_0, R_1 \rangle$. Since $\lambda_i < \omega^{\omega^\omega}$ for each $i \in [1, k]$ and $\lambda = \lambda_1 + \ldots + \lambda_k$, so $\lambda < \omega^{\omega^\omega}$. Also, $\lambda_0 < \omega^{\omega^\omega}$, hence $\lambda \cdot \lambda_0 < \omega^{\omega^\omega}$ still holds. ■

Combining Lemma 5.3.1 and 5.3.2, we finally prove the main lemma. Since all the periodic frames we used are subframes of $\Omega = \langle \omega^{\omega^\omega}, \tau_0, \tau_1, \mathsf{H}_1(\omega^{\omega^\omega}) \rangle$, thus it is sufficient to show the Theorem 5.1.7.

Chapter 6

Conclusion

In this thesis, we define the concept of a general topological frame, that is, a topological space equipped with a distinguished set of admissible sets, akin to the notion of a general Kripke frame. Then, we describe a natural class of general topological frames on ordinals, which we call *periodic frames*. These frames are based on well-orderings equipped with some natural topologies introduced by Icard [18]. While GLP is known to be incomplete w.r.t. Icard's spaces, we demonstrate that the bimodal fragment of GLP is sound and complete with respect to the periodic frames. More specifically, we present a result in the form of the Abashidze-Blass theorem: for any ordinal $\Omega \geqslant \omega^{\omega^\omega}$, the periodic frame $\langle \Omega, \tau_0, \tau_1, \mathsf{H}_1(\Omega) \rangle$ is sound and complete with respect to GLB.

In the future, our aim is to generalize this work to GLP. That is, we intend to generalize the concept of periodic frames and find a natural class of general topological frames which is sound and complete with respect to GLP. More precisely, we need to generalize the concept of 1-periodic set to n-periodic set for each $n \in \omega$, and we conjecture that with all the natural topologies introduced by Icard, we can define suitable periodic frames with an underlining set ϵ_0, which is sound and complete with respect to the whole system GLP. Additionally, we hope that our semantic tools will be useful in the application of GLP in proof theory and arithmetic.

Acknowledgements

First, I would like to express my deepest appreciation to my supervisors, Prof. Dr. Lev Beklemishev and Dr. Nick Bezhanishvili. Without their continued help and support during this period, I would not have been able to complete my thesis. This paper was completed in half a year, and I did not have much foundation or experience in this field before. Thanks for bringing me into such a wonderful academic field and patiently guiding me to learn how to work in it. Moreover, I am grateful for their patience and understanding. When they became my supervisors, I guess they must not have expected me to cause them so much trouble.

Regarding my study in Amsterdam and the writing of this thesis, I think it all began with Prof. Dr. Dick de Jongh. Due to my poor English and the epidemic, I felt that I made little progress in both study and life during the first half year in Amsterdam, but Dick gave me a lot of support during that period. Later, Dick led me into the field of provability logic and introduced me to Lev at the end of 2022. I am so lucky to have so many great mentors who have guided me along my academic path.

I am also appreciative of the other professors who helped me during my studies: Dr. Benno van den Berg and Dr. Tommaso Moraschini, whose insightful suggestions as committee members during my thesis defense greatly influenced the development of my work. Benno's Type Theory and Category Theory are my favorite lectures in ILLC, I will still try to learn and explore more in this field. I would also like to appreciate Dr. Aybüke Özgün, whose teachings on epistemic logic and topological semantics have left a lasting impact on me. I believe it will definitely be of great help to my future work.

I am grateful for my friends: Tianyi Chu, Cheng Liao, Swapnil Ghosh, Kirill Kopnev, Shin Tanaka and many others, who provided invaluable support throughout the past two years. Their camaraderie has been an anchor during both academic and personal challenges.

To my girlfriend Peiyuan Shu, in a sense, she is also my guide into logic. I fell in love with logic from the day I accompanied her to a logic class. More importantly, I cannot imagine life without her support. In the past two years in Amsterdam, there was a 6-7 hour time lag between us, but the video calls with her were a very important part of my daily life. I hope we can continue to grow together and complete our PhDs as soon as possible.

Finally, I would like to thank my family. Thanks to them for their unyielding support, which served as the foundation for my academic life. The transformations in my life over the past two years have been profound. On the one hand, I have made new growth in my studies and life. On the other hand, the conclusion of 2022 marked a period of immense loss with the departure of my beloved grandfather and grandmother, who held an unparalleled affection for me. The creation of this paper is inseparable from the void left by their absence. Almost every morning, I would awaken when I dreamed about them. When I was alone at home, every small detail seemed to trigger memories of the times we shared. If they can see me finish this paper, they will be very happy and proud. I earnestly wish for them to be at peace in heaven, free from any pain or ailment. I will carry their memory with me as I embark on the next chapter of my life.

Yunsong Wang
Beijing, 2023

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