The Topological Theory of Belief

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Abstract. Stalnaker introduced a combined epistemic-doxastic logic that can formally express a strong concept of belief, a concept which captures the 'epistemic possibility of knowledge'. In this paper we first provide the most general extensional semantics for this concept of 'strong belief', which validates the principles of Stalnaker's epistemic-doxastic logic. We show that this general extensional semantics is a topological semantics, based on so-called extremally disconnected topological spaces. It extends the standard topological interpretation of knowledge (as the interior operator) with a new topological semantics for belief. Formally, our belief modality is interpreted as the 'closure of the interior'. We further prove that in this semantics the logic **KD45** is sound and complete with respect to the class of extremally disconnected spaces and we compare our approach to a different topological setting in which belief is interpreted in terms of the derived set operator. In the second part of the paper we study (static) belief revision as well as belief dynamics by providing a topological semantics for conditional belief and belief update modalities, respectively. Our investigation of dynamic belief change, is based on hereditarily extremally disconnected spaces. The logic of belief **KD45** is sound and complete with respect to the class of hereditarily extremally disconnected spaces (under our proposed semantics), while the logic of knowledge is required to be **S4.3**. Finally, we provide a complete axiomatization of the logic of conditional belief and knowledge, as well as a complete axiomatization of the corresponding dynamic logic.

Keywords: epistemic logic, doxastic logic, topological semantics, (hereditarily) extremally disconnected spaces, conditional beliefs, updates, completeness, axiomatization.

1. Introduction

Edmund Gettier's famous counterexamples against the *justified true belief* (JTB) account of knowledge [29] invited an interesting and extensive discussion among formal epistemologists and philosophers concerned with understanding the correct relation between knowledge and belief, and, in particular, with identifying the exact properties and conditions that distinguishes a piece of belief from a piece of knowledge and vice versa. Various proposals in the literature analysing the knowledge-belief relation can be classified in two categories: (1) the ones that start with the weakest notion of true justified (or justifiable) belief and add conditions in order to argue that they establish a "good" (e.g. factive, correctly-justified, unrevisable, coherent, stable, truth-sensitive) notion of knowledge by enhancing the conditions in the JTB analysis of knowledge; and (2) the ones that take knowledge as the primitive concept and start from a chosen notion of knowledge and weaken it to obtain a "good" (e.g. consistent, introspective, possibly false) notion of belief. Most research in formal epistemology follows the first approach. In particular, the standard *topological semantics for knowledge* (in terms of the interior operator) can be included within this first approach, as based on a notion of knowledge as "correctly justified belief": according to the interior semantics, a proposition (set of possible worlds) P is known at the real world x if there exists some "true evidence" (i.e. an open set U containing the real world x) that entails P (i.e. $U \subseteq P$). Other responses to the Gettier challenge falling under this category include, among others, the *defeasibility analysis of* *knowledge* [34, 32], the *sensitivity account* [37], the *contextualist account* [22] and the *safety account* [42][1].

While most research in formal epistemology follows the first approach, the second approach has to date received much less attention from formal logicians. This is rather surprising, since such a "knowledge-first" approach, which challenges "conceptual priorty of belief over knowledge", has been persuasively defended by one of the most influential contemporary epistemologists (Williamson [50]). The only formal account following this second approach that we are aware of (prior to our own work) is the one given by Stalnaker [44], using a relational semantics for knowledge, based on Kripke models in which the accessibility relation is a directed preorder. In this setting, Stalnaker argues that the "true" logic of knowledge is the modal logic **S4.2** and that belief can be defined as the *epistemic possibility of knowledge*[2]. In other words, *believing p is equivalent to "not knowing that you don't know" p*:

$$Bp = \neg K \neg Kp.$$

Stalnaker justifies this identity from first principles based on a particular notion of belief, namely *belief as "subjective certainty"*. Stalnaker refers to this concept as "strong belief", but we prefer to call it *full belief*[3]. What is important about this type of belief is that it is *subjectively indistinguishable from knowledge*: an agent "fully believes" p iff in fact she "believes that she knows" p.

Indeed, Stalnaker proceeds to formalize AGM belief revision [1], based on a special case of the above semantics, in which the accessibility relation is assumed to be a weakly connected preorder, and (conditional) beliefs are defined by minimization. This validates the AGM principles for belief revision.

In this paper we generalize Stalnaker's formalization, making it independent from the concept of plausibility order and from relational semantics, to a topological setting. In fact, we are looking for *the most general extensional* (*i.e., canonical*) *semantics for "full belief"* (in the above-mentioned sense), validating Stalnaker's principles for epistemic-doxastic logic. By an "extensional" semantics we mean here any semantics that assigns the same meaning to sentences having the same extension. Essentially, an extensional semantics takes the meaning of a sentence to be given by a "U.C.L.A. proposition" in the sense of Anderson-Belnap-Dunn[4]: a set of possible worlds (intuitively thought of as the set of worlds at which the proposition is true). We prove that the most general extensional semantics is a *topological* one, that extends the standard topological interpretation of knowledge as *interior* operator with a new topological semantics for belief, given by the *closure of the interior* operator with respect to an extremally disconnected topology. We compare our new semantics with the older topological interpretation of belief in terms of Cantor derivative, giving several arguments in favour of our semantics. We prove that the logic of knowledge and belief with respect to our semantics is completely axiomatized by Stalnaker's epistemic-doxastic principles. Furthermore, we show that the complete logic of knowledge in this setting is the system **S4.2**, while the complete logic of belief on extremally disconnected spaces is the standard system **KD45**.

[1] For an overview of responses to the Gettier challenge and a detailed discussion, we refer the reader to [30, 40].

[2] Note that Stalnaker considers in his work [44] also several other variations such as **S4.4, S4F, S5**.

[3] We adopt this terminology both because we want to avoid the clash with the very different notion of strong belief (due to Battigalli and Siniscalchi [8]) that is standard in epistemic game theory, and because we think that the intuitions behind Stalnaker's notion are very similar to the ones behind Van Fraassen's probabilistic concept of full belief [28].

[4] Dunn [24] explains this name as follows: 'The name honours the university that has had both R. Carnap and R. Montague in its faculty, since in modern times they (together with others, e.g. S. Kripke and R. Stalnaker) have been proponents of this construction. But the idea actually originates with Boole, who suggested thinking of propositions as "sets of cases" (...).'

We moreover focus on a topological semantics for belief revision, assuming the distinction between *static* and *dynamic conditioning* made in, e.g., [9, 5, 4]. We examine the corresponding "static" conditioning, by giving a topological semantics for conditional belief $B^{\varphi}\psi$. We formalize a notion of conditional belief $B^{\varphi}\psi$ by *relativizing* the semantic clause for a simple belief modality to the extension of the learnt formula φ and first give a complete axiomatization of the logic of knowledge and conditional beliefs based on extremally disconnected spaces. This topological interpretation of conditional belief also allows us to model static belief revision of a more general type than axiomatized by the AGM theory: the topological model validates the (appropriate versions of) AGM axioms 1-7, but not necessarily the axiom 8, though it does validate a weaker version of this axiom[5].

The above setting, however, comes with a problem when extended to a dynamic one by adding update modalities in order to capture the action of learning (conditioning with) new "hard" (true) information P. In general, conditioning with new "hard" (true) information P is modeled by simply deleting the "non-P" worlds from the initial model. Its natural topological analogue, as recognized in [6, 7, 51] among others, is a topological update operator, using the restriction of the original topology to the subspace induced by the set P. This interpretation, however, cannot be implemented smoothly on extremally disconnected spaces due to their non-hereditary nature: we cannot guarantee that the subspace induced by any arbitrary true proposition P is extremally disconnected since extremally disconnectedness is not a hereditary property and thus the structural properties, in particular extremally disconnectedness, of our topological models might not be preserved. We proposed a different solution for this problem in [2] via arbitrary topological spaces. In particular, [2] introduces a different topological semantics for belief based on *all* topological spaces in terms of the *interior of the closure of the interior operator* and models updates on arbitrary topological spaces. In this paper, however, we propose another solution for this problem via hereditarily extremally disconnected spaces. Hereditarily extremally disconnected spaces are those whose subspaces are still extremally disconnected. By restricting our attention to this class of spaces, we guarantee that any model restriction preserves the important structural properties that make the axioms of the corresponding system sound, in this case, extremally disconnectedness of the initial model. We then interpret updates $\langle !\varphi\rangle\psi$ again as a topological update operator using the *restriction* of the initial topology to its subspace induced by the new information φ and show that we no longer encounter the problem with updates that rises in the case of extremally disconnected spaces: *hereditarily extremally disconnected spaces admit updates*. Further, we show that while the complete logic of knowledge on hereditarily extremally disconnected spaces is actually **S4.3**, the complete logic of belief is still **KD45**. We moreover give a complete axiomatization of the logic of knowledge and conditional beliefs with respect to the class of hereditarily extremally disconnected spaces, as well as a complete axiomatization of the corresponding dynamic logic. In addition, we show that hereditarily disconnected spaces validate the AGM axiom 8, and that therefore our proposed semantics for knowledge and conditional beliefs captures the AGM theory as a theory of static belief revision.

This work can be seen as an extension of [3]: while the results in [3] and Section 3 of the current paper coincide, the proofs of our results can only be found in the latter. The soundness and completeness results presented in [3] are merely based on extremally

[5] AGM theory is considered to be static in the sense that it captures "the agent's changing beliefs about an unchanging world" [5, p. 14]. This static interpretation of AGM theory is mimicked by conditional beliefs in a modal framework, in the style of dynamic epistemic logic, and embedded in the complete system *Conditional Doxastic Logic* (**CDL**) introduced by Baltag and Smets in [4, 5]. The reader who is not familiar with the logic **CDL** can find its syntax and proof system introduced in [5] in Appendix A.

disconnected spaces. In this paper, however, we go further. We model knowledge and belief on hereditarily extremally disconnected spaces and propose a topological semantics for conditional beliefs and updates based on these spaces. We also provide the corresponding soundness and completeness results together with the proof details.

The paper is organized as follows. In Section 2 we provide the topological preliminaries used throughout this paper and the interior-based topological semantics for knowledge as well as the topological soundness and completeness results for the systems **S4** and **S4.2**. Section 3 introduces Stalnakers combined logic and briefly outlines his analysis regarding the relation between knowledge and belief. We then propose a topological semantics for the system, in particular a topological semantics for full belief. We continue with investigating the unimodal fragments **S4.2** for knowledge and **KD45** for belief of Stalnakers system, and give topological completeness results for these logics, again with respect to the class of extremally disconnected spaces. We also compare our topological belief semantics with Steinsvold's co-derived set semantics [45] in this section. Section 4 focuses on a topological semantics for belief revision, assuming the distinction between static and dynamic belief revision and presents the semantics for conditional beliefs and updates, respectively. Finally we conclude with Section 5 by giving a brief summary of this work and pointing out a number of directions for future research.

The proofs of our results are presented in the Appendices. More precisely, Appendix A includes some introductory material referred to in Section 1. Appendix B includes a brief overview of the standard Kripke semantics and the proofs of the results stated in Section 2. Finally, the proofs of the results of Sections 3 and 4 are presented in Appendices C and D, respectively.

2. Background

2.1. Topological Preliminaries

We start by introducing the basic topological concepts that will be used throughout this paper. For a more detailed discussion of general topology we refer the reader to [23, 25].

A ***topological space*** is a pair (X, τ), where X is a non-empty set and τ is a family of subsets of X containing X and $\emptyset$ and is closed under finite intersections and arbitrary unions. The set X is called a ***space***. The subsets of X belonging to τ are called ***open sets*** (or ***opens***) in the space; the family τ of open subsets of X is also called a ***topology*** on X. Complements of opens are called ***closed sets***. An open set containing $x \in X$ is called an ***open neighbourhood*** of x. The ***interior*** $\mathrm{Int}(A)$ of a set $A \subseteq X$ is the largest open set contained in A whereas the ***closure*** $\mathrm{Cl}(A)$ of A is the least closed set containing A. In other words,

- $\mathrm{Int}(A) = \bigcup\{U \in \tau : U \subseteq A\}$
- $\mathrm{Cl}(A) = \bigcap\{F : X \setminus F \in \tau, A \subseteq F\}$

It is easy to see that Cl is the De Morgan dual of Int (and vice versa) and can be written as $\mathrm{Cl}(A) = X \setminus \mathrm{Int}(X \setminus A)$.

Example. For any non-empty set X, $(X, \mathcal{P}(X))$ is a topological space and every set $A \subseteq X$ is both closed and open (i.e., *clopen*). Another standard example of a topological space is the real line $\mathbb{R}$ with the family τ of open intervals and their countable unions. If $A = [1, 2)$, then $\mathrm{Int}([1, 2)) = (1, 2)$ (the largest open interval included in [1, 2)) and $\mathrm{Cl}([1, 2)) = [1, 2]$ (the least closed interval containing [1, 2)).

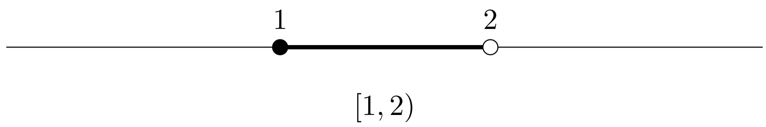
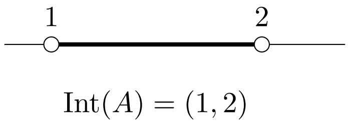
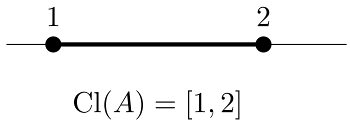


Figure 1.: Real line and $A = [1, 2)$

(a) 

(b) 

2.2. The Interior Semantics for Modal Logic

In this section, we introduce the formal background for the standard topological semantics of basic modal (epistemic) logic, originating in the work of McKinsey and Tarski [35]. In this semantics, the knowledge modality (the $\Box$-type modality) is interpreted as the interior operator on topological spaces. Referring to this fact and in order to make the distinction between different topological semantics for the basic modal language clearer, we call this semantics the ***interior semantics***[6]. While presenting some important completeness results (concerning logics of knowledge) of previous works, we also explain the connection between the interior semantics and standard Kripke semantics and focus on the topological (evidence-based) interpretation of knowledge.

Syntax and Semantics. We consider the standard unimodal language $\mathcal{L}_K$ with a countable set of propositional letters Prop, Boolean operators $\neg, \wedge$ and a modal operator K. Formulas of $\mathcal{L}_K$ are defined as usual by the following grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi$$

where $p \in \mathrm{Prop}$. Abbreviations for the connectives $\vee$, $\rightarrow$ and $\leftrightarrow$ are standard. Moreover, the *epistemic possibility* operator $\langle K \rangle$ is defined as $\neg K \neg$ and $\bot := p \wedge \neg p$.

Given a topological space (X, τ), we define a ***topological model*** or simply a ***topo-model*** (based on (X, τ)) as $\mathcal{M} = (X, \tau, \nu)$ where X and τ as before and $\nu : \mathsf{Prop} \rightarrow \mathcal{P}(X)$ is a valuation function.

DEFINITION 1. *Given a topo-model $\mathcal{M} = (X, \tau, \nu)$ and a state $x \in X$, we define the **interior semantics** for the language $\mathcal{L}_K$ recursively as:*

$$\begin{array}{lll}
\mathcal{M}, x \models p & \textit{iff} & x \in \nu(p) \\
\mathcal{M}, x \models \neg\varphi & \textit{iff} & \textit{not } \mathcal{M}, x \models \varphi \\
\mathcal{M}, x \models \varphi \wedge \psi & \textit{iff} & \mathcal{M}, x \models \varphi \textit{ and } \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models K\varphi & \textit{iff} & (\exists U \in \tau)(x \in U \wedge \forall y \in U,\ \mathcal{M}, y \models \varphi)
\end{array}$$

where $p \in \mathsf{Prop}$[7].

We let $[\![\varphi]\!]^{\mathcal{M}} = \{x \in X \mid \mathcal{M}, x \models \varphi\}$ denote the ***extension*** of a modal formula φ in a topo-model $\mathcal{M}$, i.e., the *extension* of a formula φ in a topo-model $\mathcal{M}$ is defined as the set of points in $\mathcal{M}$ satisfying φ. We skip the index when it is clear in which model we are working. It is now easy to see that $[\![K\varphi]\!] = \mathrm{Int}([\![\varphi]\!])$ and $[\![\langle K \rangle\varphi]\!] = \mathrm{Cl}([\![\varphi]\!])$. We use this extensional notation throughout the paper as it makes clear the fact that the modalities,

[6] We will also discuss a topological semantics based on the *derived set operator* in future sections.

[7] Originally, McKinsey and Tarski [35] introduce the interior semantics for the basic modal language. Since we talk about this semantics in the context of *knowledge*, we use the basic *epistemic* language.

K and $\langle K \rangle$, are interpreted in terms of specific and *natural* topological operators. In particular, as stated above, in the case of the interior semantics we interpret K as the interior operator and, dually, $\langle K \rangle$ as the closure operator.

We say that φ is *true* in $\mathcal{M}$ if it is true in all the states of $\mathcal{M}$. We say that φ is *valid* in a topological space (X, τ) if it is true in every model based on (X, τ). Finally, we say that φ is valid in a class of topological spaces if it is valid in every member of the class [10]. Equivalently,

- φ is true in $\mathcal{M} = (X, \tau, \nu)$ if $[\![\varphi]\!]^{\mathcal{M}} = X$,
- φ is valid in (X, τ) if $[\![\varphi]\!]^{\mathcal{M}} = X$ for all topo-models $\mathcal{M}$ based on (X, τ), and
- φ is valid in a class of topological spaces if φ is valid in every member of the class.

Soundness and completeness with respect to the interior semantics are defined as usual.

Topo-completeness of S4 and S4.2. Epistemic logics **S4** and **S4.2** are of particular interest in this paper and our work is built on previously given topological semantics - the interior semantics - for knowledge and topological completeness results of the aforementioned logics under this semantics[8]. We now briefly state these results and prepare the ground for ours.

It is well known that the interior (Int) and the closure (Cl) operators of a topological space (X, τ) satisfy the following properties (the so-called Kuratowski axioms) for any $A, B \subseteq X$ (see, e.g., [25, pp. 14-15]):

(I1) $\mathrm{Int}(X) = X$	(C1) $\mathrm{Cl}(\emptyset) = \emptyset$
(I2) $\mathrm{Int}(A) \subseteq A$	(C2) $A \subseteq \mathrm{Cl}(A)$
(I3) $\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$	(C3) $\mathrm{Cl}(A \cup B) = \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$
(I4) $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$	(C4) $\mathrm{Cl}(\mathrm{Cl}(A)) = \mathrm{Cl}(A)$

Given the interior semantics, it is not hard to see that the above properties (Kuratowski axioms) of the interior operator are the axioms of the system **S4** written in topological terms. This implies the soundness of **S4** with respect to the class of all topological spaces under the interior semantics (see, e.g., [10, 39, 16]). For completeness, we further need to investigate the connection between Kripke frames and topological spaces.

Connection between Kripke frames and topological spaces. The interior semantics is closely related to the standard Kripke semantics of **S4** (and of its normal extensions): every reflexive and transitive Kripke frame corresponds to a special kind of (namely, Alexandroff) topological spaces.

Let us now fix some notation and terminology. We denote a *Kripke frame* by $\mathcal{F} = (X, R)$, a *Kripke model* by $M = (X, R, \nu)$ and we let $\|\varphi\|^M$ denote the *extension* of a formula φ in a Kripke model $M = (X, R, \nu)$[9]. A topological space (X, τ) is called ***Alexandroff*** if τ is closed under arbitrary intersections, i.e., $\bigcap \mathcal{A} \in \tau$ for any $\mathcal{A} \subseteq \tau$. Equivalently, a topological space (X, τ) is Alexandroff iff every point in X has a least open neighborhood. As mentioned, there is a one-to-one correspondence between reflexive and transitive Kripke frames and Alexandroff spaces. More precisely, given a reflexive and transitive Kripke frame $\mathcal{F} = (X, R)$, we can construct a topological space, indeed an Alexandroff space, $\mathcal{X} = (X, \tau_R)$ by defining τ_R to be the set of all upsets[10] of $\mathcal{F}$. The

[8] Axioms and inference rules of the system **S4** can be found in Appendix A and the system **S4.2** is defined below.

[9] The reader who is not familiar with standard Kripke semantics is referred to Appendix B.1 for a brief introduction of the aforementioned notions.

[10] A set $A \subseteq X$ is called an *upset* of (X, R) if for each $x, y \in X$, xRy and $x \in A$ imply $y \in A$.

set $R(x) = \{y \in X \mid xRy\}$ forms the least open neighborhood containing the point x. Conversely, for *every* topological space (X, τ), the relation R_τ defined by

$$xR_\tau y \text{ iff } x \in \mathrm{Cl}(\{y\})$$

is reflexive and transitive. The pair (X, R_τ) thus constitutes a reflexive and transitive Kripke frame.

Moreover, the evaluation of modal formulas in a reflexive and transitive Kripke model coincides with their evaluation in the corresponding (Alexandroff) topological space:

PROPOSITION 1. *For all reflexive and transitive Kripke models $M = (X, R, \nu)$ and all $\varphi \in \mathcal{L}_K$,*

$$\|\varphi\|^M = [\![\varphi]\!]^{\mathcal{M}_{\tau_R}}$$

where $\mathcal{M}_{\tau_R} = (X, \tau_R, \nu)$.

Proof. See [39, p. 306].

THEOREM 1 (McKinsey and Tarski, 1944). **S4** *is sound and complete with respect to the class of all topological spaces under the interior semantics.*

Proof. See Appendix B.2

In fact, aforementioned one-to-one correspondence between Alexandroff spaces and reflexive and transitive Kripke frames implies the following stronger result (see, e.g., [10, p. 238]):

PROPOSITION 2. *Every normal extension of* **S4** *that is complete with respect to the standard Kripke semantics is also complete with respect to the interior semantics.*

Since the normal extension **S4.2** of **S4** is of particular interest in our work, we also elaborate on the topological soundness and completeness of **S4.2**.

S4.2 is a *strengthening* of **S4** defined as

$$\mathbf{S4.2} = \mathbf{S4} + (\langle K\rangle K\varphi \to K\langle K\rangle\varphi)$$

where $\mathbf{L} + \varphi$ is the smallest normal modal logic containing $\mathbf{L}$ and φ. It is well known, see e.g., [17] or [20] that **S4.2** is sound and complete with respect to reflexive, transitive and directed Kripke frames. Recall that a Kripke frame (X, R) is called *directed*[11] (see Figure 2) if

$$(\forall x, y, z)(xRy \wedge xRz) \to (\exists u)(yRu \wedge zRu).$$

$$\begin{tikzcd}
& u & \\
y \arrow[ur, dashed] & & z \arrow[ul, dashed] \\
& x \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Figure 2.: Directedness

The directedness condition on Kripke frames is needed to ensure the validity of (.2)-axiom $\langle K\rangle K\varphi \to K\langle K\rangle\varphi$ (see, e.g., [17, 21]), however, in the interior semantics it is a special case of a more general condition called *extremally disconnectedness*:

[11] Directedness is also called *confluence* or the *Church-Rosser property*.

DEFINITION 2. *A topological space (X,τ) is called* ***extremally disconnected*** *if the closure of each open subset of X is open.*

We give a few examples of extremally disconnected spaces. Alexandroff spaces corresponding to reflexive, transitive and directed Kripke frames are extremally disconnected. More precisely, for a given reflexive, transitive and directed Kripke frame (X,R), the space (X,τ_R) is extremally disconnected (see, [38, Proposition 3] for its proof)[12]. Another interesting example of an extremally disconnected space is the topological space $(\mathbb{N},\tau)$ where $\mathbb{N}$ is the set of natural numbers and $\tau = \{\emptyset, \text{ all cofinite subsets of } \mathbb{N}\}$. In this space, the set of all finite subsets of $\mathbb{N}$ together with $\emptyset$ and X completely describes the set of closed subsets with respect to $(\mathbb{N},\tau)$. It is not hard to see that for any $U \in \tau$, $\mathrm{Cl}(U) = \mathbb{N}$ and $\mathrm{Int}(F) = \emptyset$ for any closed F with $F \neq X$. Also it is well known that topological spaces that are Stone-dual to complete Boolean algebras and the Stone-Čech compactification $\beta(\mathbb{N})$ of the set of natural numbers with a discrete topology are extremally disconnected [41]. Similarly to the case of the standard Kripke semantics, in the interior semantics extremally disconnectedness is needed in order to ensure the validity of the (.2)-axiom. More accurately, (.2)-axiom characterizes extremally disconnected spaces under the interior semantics:

PROPOSITION 3. *For any topological space (X,τ),*

$$\langle K\rangle K\varphi \to K\langle K\rangle\varphi \text{ is valid in } (X,\tau) \text{ iff } (X,\tau) \text{ is extremally disconnected.}$$

Proof. See Appendix B.3.

Proposition 3 and topological soundness of **S4** imply that **S4.2** is sound with respect to the class of extremally disconnected spaces. As reflexive, transitive and directed Kripke frames correspond to extremally disconnected Alexandroff spaces, the following topological completeness result follows from the completeness of **S4.2** with respect to the standard Kripke semantics and Proposition 2:

THEOREM 2 (Folklore). **S4.2** *is sound and complete with respect to the class of extremally disconnected spaces under the interior semantics.*

Epistemic Interpretation: open sets as pieces of evidence. The original reason for interpreting interior as knowledge was that the Kuratowski axioms for interior match exactly the **S4** axioms for knowledge, and in particular the principles

$$(T) \quad Kp \to p$$

of Truthfulness of Knowledge ("factivity") and

$$(KK) \quad Kp \to KKp$$

of Positive Introspection of Knowledge (known as axiom-(4) in modal logic).

Philosophically, one of the best arguments in favor of the topological semantics is negative: namely, the fact that it does *not* validate the principle

$$\neg Kp \to K\neg Kp.$$

This principle, known as axiom-(5) or Negative Introspection, is rejected by essentially all philosophers. One of its undesirable consequences is that it makes it impossible for a rational agent to have wrong beliefs about her knowledge: *she always knows whatever*

[12] This correspondence between extremally disconnected spaces and reflexive, transitive and directed Kripke frames will be used in our completeness proof for **KD45** in Section 3.2.

she believes that she knows. This is known in the literature as Voorbraak's paradox [49]: it contradicts the day-to-day experience of encountering agents who believe they know things that they do not actually know[13].

But, even beyond the issue of negative introspection, the topological semantics can arguably give us a deeper insight into the nature of knowledge and its evidential basis than the usual Kripke semantics. From an extensional point of view, the properties U that are directly observable by an agent naturally form an *open basis* for a topology: closure under finite intersections captures an agent's ability to combine finitely many pieces of evidence into a single piece[14]. A proposition P is true at world w if $w \in P$. If an open U is included in a set P, then we can say that *proposition P is entailed (supported, justified) by evidence U*. Open neighbourhoods U of the actual world w play the role of *sound (correct, truthful) evidence*. The actual world w is in the interior of P iff there exists such a sound piece of evidence U that supports P. So *the agent "knows" P if she has a correct justification for P* (based on a sound piece of evidence supporting P). Moreover, open sets will then correspond to properties that are in principle *verifiable* by the agent: whenever they are true they can be known. Dually, closed sets will correspond to *falsifiable* properties. See Vickers [48] and Kelly [31] for more on this interpretation and its connections to Epistemology, Logic and Learning Theory.

So the knowledge-as-interior conception can be seen as an implementation of one of the most widespread intuitive responses to Gettier's challenge: knowledge is "correctly justified belief" (rather than being simply true justified belief). To qualify as knowledge, not only the content of one's belief has to be truthful, but its evidential justification has to be sound.

3. The Topology of Full Belief and Knowledge

3.1. Stalnaker's Combined Logic of Knowledge and Belief

In his paper [44], Stalnaker focuses on the properties of (justified or justifiable) belief and knowledge and proposes an interesting analysis regarding the relation between the two. As also pointed out in the introduction, most research in the formal epistemology literature concerning the relation between knowledge and belief, in particular, dealing with the attempt to provide a definition of the one in terms of the other, takes belief as a primitive notion and tries to determine additional properties which render a piece of belief knowledge (see, e.g., [32, 34, 37, 22, 40]). In contrast, Stalnaker chooses to start with a notion of knowledge and weakens it to have a "good" notion of belief. He *initially* considers knowledge to be an **S4**-type modality and analyzes belief based on the conception of "subjective certainty": from the point of the agent in question, her belief is subjectively indistinguishable from her knowledge.

The *bimodal language $\mathcal{L}_{KB}$* of *knowledge and (full) belief* is given by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid B\varphi$$

where $p \in \mathsf{Prop}$. Abbreviations for the connectives $\vee$, $\to$ and $\leftrightarrow$ are standard. The existential modalities $\langle K \rangle$ and $\langle B \rangle$ are defined as $\neg K \neg$ and $\neg B \neg$ respectively. We will also consider two unimodal fragments $\mathcal{L}_K$ (having K as its only modality) and $\mathcal{L}_B$ (having only B) of the language $\mathcal{L}_{KB}$ in later sections.

[13] This common experience can be considered the starting point of all epistemological reflection, and historically played such a role, see e.g. in Platonic dialogues.

[14] See van Benthem and Pacuit [12] for a more general logical account of evidence-management which relaxes this assumption: by using instead a neighbourhood semantics, this account can deal with agents who have not yet managed to combine all their pieces of evidence.

We call Stalnaker's epistemic-doxastic system, given in the following table, **KB**:

	Stalnaker's Axioms	
(K)	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	Knowledge is additive
(T)	$K\varphi \to \varphi$	Knowledge implies truth
(KK)	$K\varphi \to KK\varphi$	Positive introspection for K
(CB)	$B\varphi \to \neg B\neg\varphi$	Consistency of belief
(PI)	$B\varphi \to KB\varphi$	(Strong) positive introspection of B
(NI)	$\neg B\varphi \to K\neg B\varphi$	(Strong) negative introspection of B
(KB)	$K\varphi \to B\varphi$	Knowledge implies Belief
(FB)	$B\varphi \to BK\varphi$	Full Belief

	Inference Rules	
(MP)	From φ and $\varphi \to \psi$ infer ψ.	Modus Ponens
(K-Nec)	From φ infer $K\varphi$.	Necessitation

Table I.: Stalnaker's System **KB**

The axioms seem very natural and uncontroversial: the first three are the **S4** axioms for knowledge; (CB) captures the consistency of beliefs, and in the context of the other axioms will be equivalent to the modal axiom (D) for beliefs: $\neg B\bot$; (PI) and (NI) capture strong versions of introspection of beliefs: the agent knows what she believes and what not; (KB) means that agents believe what they know; and finally, (FB) captures the essence of "full belief" as subjective certainty (the agent believes that she knows all the things that she believes). Finally, the rules of Modus Ponens and Necessitation seem uncontroversial (for implicit knowledge, if not for explicit knowledge) and are accepted by a majority of authors (and in particular, they are implicitly used by Stalnaker). The above axioms yield the belief logic **KD45**:

PROPOSITION 4 (Stalnaker, 2006). *All axioms of the standard belief system* **KD45** *are provable in the system* **KB**. *More precisely, the axioms*

(K) $B(\varphi \to \psi) \to (B\varphi \to B\psi)$

(D) $B\varphi \to \neg B\neg\varphi$

(4) $B\varphi \to BB\varphi$

(5) $\neg B\varphi \to B\neg B\varphi$

are provable in **KB**.

Moreover, *belief can be defined in terms of knowledge*:

PROPOSITION 5. *The following equivalence is provable in the system* **KB***:*

$$B\varphi \leftrightarrow \langle K\rangle K\varphi$$

Proof. See Appendix C.1.

Proposition 5 constitutes one of the most important features of Stalnaker's combined system **KB**. This equivalence allows us to have a combined logic of knowledge and belief in which the only modality is K and the belief modality B is defined in terms of the former. We therefore obtain "...a more economical formulation of the combined belief-knowledge logic..." [44, p. 179]. Moreover, substituting $\langle K\rangle K$ for B in the axiom (CB) results in the modal axiom

$$\langle K\rangle K\varphi \to K\langle K\rangle\varphi$$

also known as the (.2)-axiom in the modal logic literature [17]. Recall that we obtain the logic of knowledge **S4.2** by adding the (.2)-axiom to the system **S4**. If we substitute $\langle K\rangle K$ for B in all the other axioms of **KB**, they turn out to be theorems of **S4.2** [44]. Therefore, given the equivalence $B\varphi \leftrightarrow \langle K\rangle K\varphi$, we can obtain the unimodal logic of knowledge **S4.2** by substituting $\langle K\rangle K$ for B in all the axioms of **KB** implying that the logic **S4.2** by itself forms a *unimodal* combined logic of knowledge and belief. Stalnaker then argues that his analysis of the relation between knowledge and belief suggests that the "true" logic of knowledge should be **S4.2** and that belief can be defined as the *epistemic possibility of knowledge*:

$$B\varphi := \langle K\rangle K\varphi.$$

This equation leads to our proposal for a topological semantics for (full) belief.

3.2. Our Topological Semantics for Full Belief

In this section, we introduce a new topological semantics for the language $\mathcal{L}_{KB}$, which is an extension of the interior semantics for knowledge with a new topological semantics for belief given by the *closure of the interior operator.*

DEFINITION 3 (Topological Semantics for Full Belief and Knowledge). *Given a topo-model $\mathcal{M} = (X, \tau, \nu)$, the semantics for the formulas in $\mathcal{L}_{KB}$ is defined for Boolean cases and $K\varphi$ the same way as in the interior semantics. The semantics for $B\varphi$ is defined as*

$$[\![B\varphi]\!]^{\mathcal{M}} = \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})).$$

Truth and validity of a formula is defined the same way as in the interior semantics.

PROPOSITION 6. *A topological space validates all the axioms and rules of Stalnaker's system* **KB** (*under the semantics given above*) *iff it is extremally disconnected.*

Proof. See Appendix C.2.

We now generalize the above semantics given on topological spaces to an extensional framework independent from topologies and show that the most general extensional (and compositional) semantics validating the axioms of the system **KB** is again topological and based on extremally disconnected spaces.

DEFINITION 4 (Extensional Semantics for $\mathcal{L}_{KB}$). *An* ***extensional (and compositional) semantics*** *for the language $\mathcal{L}_{KB}$ of knowledge and full belief is a triple $(X, \mathrm{K}, \mathrm{B})$, where X is a set of possible worlds and $\mathrm{K} : \mathcal{P}(X) \to \mathcal{P}(X)$ and $\mathrm{B} : \mathcal{P}(X) \to \mathcal{P}(X)$ are unary operations on (sub)sets of worlds.*

Any extensional semantics $(X, \mathrm{K}, \mathrm{B})$, together with a valuation $\nu : \mathsf{Prop} \to \mathcal{P}(X)$, gives us an ***extensional model*** *$\mathcal{M} = (X, \mathrm{K}, \mathrm{B}, \nu)$, in which we can interpret the formulas φ of $\mathcal{L}_{KB}$ in the obvious way: the clauses for Boolean formulas are the same as in the interior semantics, and the remaining cases are given by*

$$\begin{array}{rcl} [\![K\varphi]\!]^{\mathcal{M}} & = & \mathrm{K}[\![\varphi]\!]^{\mathcal{M}} \\ [\![B\varphi]\!]^{\mathcal{M}} & = & \mathrm{B}[\![\varphi]\!]^{\mathcal{M}}. \end{array}$$

As usual, a formula $\varphi \in \mathcal{L}_{KB}$ is valid in an extensional semantics $(X, \mathrm{K}, \mathrm{B})$ if $[\![\varphi]\!]^{\mathcal{M}} = X$ for all extensional models $\mathcal{M}$ based on $(X, \mathrm{K}, \mathrm{B})$.

A special case of extensional semantics for the language $\mathcal{L}_{KB}$ is *our proposed topological semantics*:

DEFINITION 5 (Topological Extensional Semantics). *A **topological extensional semantics** for the language $\mathcal{L}_{KB}$ is an extensional semantics $(X, \mathrm{K}_\tau, \mathrm{B}_\tau)$, where (X, τ) is a topological space, $\mathrm{K}_\tau = \mathrm{Int}$ is the interior operator and $\mathrm{B}_\tau = \mathrm{Cl}(\mathrm{Int})$ is the closure of the interior operator with respect to the topology τ.*

We can now state one of the main results of this section; a Topological Representation Theorem for extensional models of **KB**:

THEOREM 3 (Topological Representation Theorem). *An extensional semantics $(X, \mathrm{K}, \mathrm{B})$ validates all the axioms and rules of Stalnaker's system* **KB** *iff it is a topological extensional semantics given by an extremally disconnected topology τ on X, such that $\mathrm{K} = \mathrm{K}_\tau = \mathrm{Int}$ and $\mathrm{B} = \mathrm{B}_\tau = \mathrm{Cl}(\mathrm{Int})$.*

Proof. See Appendix C.3.

Theorem 3 shows that Stalnaker's axioms form an alternative axiomatization of extremally disconnected spaces, in which both the interior and the closure of the interior are taken to be primitive operators (corresponding to the primitive modalities K and B in $\mathcal{L}_{KB}$, respectively). The conclusion is that *our topological semantics is indeed the most general extensional (and compositional) semantics validating Stalnaker's axioms.*

THEOREM 4. *The sound and complete logic of knowledge and belief on extremally disconnected spaces is given by Stalnaker's system* **KB**.

Proof. See Appendix C.4.

Unimodal Case: The belief logic KD45. As emphasized in the beginning of Section 3, Stalnaker's logic **KB** yields the system **S4.2** as the logic of knowledge and **KD45** as the logic of belief (Proposition 4). It has already been proven that **S4.2** is sound and complete with respect to the class of extremally disconnected spaces under the interior semantics (Theorem 2). In this section, we investigate the case for **KD45** under our proposed semantics for belief. More precisely, we focus on the unimodal case for belief and consider the topological semantics for the unimodal language $\mathcal{L}_B$ in which we interpret belief as the *closure of interior* operator. We name our proposed semantics in this section ***topological belief semantics***. We then prove topological soundness and completeness results for **KD45** under the aforementioned semantics. Let us first recall the basic doxastic language $\mathcal{L}_B$, the system **KD45** and the topological belief semantics for the language $\mathcal{L}_B$.

The language $\mathcal{L}_B$ is given by

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid B\varphi$$

and we again denote $\neg B \neg$ with $\langle B \rangle$. Recall that

$$\mathbf{KD45} = \mathbf{K} + (B\varphi \to \langle B\rangle\varphi) + (B\varphi \to BB\varphi) + (\langle B\rangle\varphi \to B\langle B\rangle\varphi)$$

and given a topo-model $\mathcal{M} = (X, \tau, \nu)$, the semantic clauses for the propositional variables and the Boolean connectives are the same as in the interior semantics. For the modal operator B, we put

$$[\![B\varphi]\!]^{\mathcal{M}} \quad = \quad \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}}))$$

and the semantic clause for $\langle B\rangle$ is easily obtained as

$$[\![\langle B\rangle\varphi]\!]^{\mathcal{M}} \;=\; \mathrm{Int}(\mathrm{Cl}([\![\varphi]\!]^{\mathcal{M}})).$$

We are now ready to state the main results of this section:

THEOREM 5. *The belief logic* **KD45** *is sound with respect to the class of extremally disconnected spaces under the topological belief semantics. In fact, a topological space (X,τ) validates all the axioms and rules of the system* **KD45** *(under the topological belief semantics) iff (X,τ) is extremally disconnected.*

Proof. See Appendix C.5.

THEOREM 6. *In the topological belief semantics,* **KD45** *is the complete logic of belief with respect to the class of extremally disconnected spaces.*

Theorem 6, though unsurprising, states technically the hardest and intriguing result in this paper. The interested reader can find all the details of both soundness and completeness proofs in Appendix C.5 and C.6, respectively. The proof details however can be skipped without loss of continuity.

3.3. Comparison with Related Work

Although (the interior-based) topological interpretation of knowledge has been studied extensively together with its extensions to multi-agent cases [13, 11], to common knowledge [11], to logics of learning known as topo-logic [39, 36], topological semantics for belief has not been as deeply investigated and is a rather non-standard and new approach. We compare now our topological interpretation of belief with a different (and older) topological semantics that has been proposed for doxastic logic, using Cantor's derivative operator [45].

Cantor's Derivative and its Dual. Let (X,τ) be a topological space. We recall that a point x is called a ***limit point*** (limit points are also called ***accumulation points***) of a set $A \subseteq X$ if for each open neighbourhood U of x we have $(U \setminus \{x\}) \cap A \neq \emptyset$. Let $d(A)$ denote the set of all limit points of A. This set is called the ***derived set*** and d is called the ***derived set operator***. For each $A \subseteq X$ we let $t(A) = X \setminus d(X \setminus A)$. We call t the ***co-derived set operator***. Also recall that there is a close connection between the derived and co-derived set operators and the closure and interior operators. In particular, for each $A \subseteq X$ we have $\mathrm{Cl}(A) = A \cup d(A)$ and $\mathrm{Int}(A) = A \cap t(A)$. Unlike the closure operator there may exist elements of A that are not its limit points. In other words, in general $A \not\subseteq d(A)$. Also note that for each $x \in X$ we have $x \notin d(x)$, where $d(x)$ is a shorthand for $d(\{x\})$.

DEFINITION 6. *Given a topo-model $\mathcal{M} = (X,\tau,\nu)$ and a state $x \in X$, the* ***co-derived set semantics for $\mathcal{L}_{KB}$*** *is obtained by extending the interior semantics for $\mathcal{L}_K$ with the following clause:*

$$\mathcal{M}, x \models B\varphi \quad \textit{iff} \quad (\exists U \in \tau)(x \in U \wedge \forall y \in U \setminus \{x\},\ \mathcal{M}, y \models \varphi)$$

This immediately gives us that $[\![B\varphi]\!]^{\mathcal{M}} = t([\![\varphi]\!]^{\mathcal{M}})$ and that $[\![\langle B\rangle\varphi]\!]^{\mathcal{M}} = d([\![\varphi]\!]^{\mathcal{M}})$. We again skip the index $\mathcal{M}$ if it is clear from the context[15].

[15] This semantics was also first suggested by McKinsey and Tarski in [35], and later developed by Esakia and his colleagues (see, e.g., [26, 15, 27]) among others.

See [10, 39, 16] for an overview of the results on the co-derived set semantics. Here we only mention the completeness results for the unimodal language $\mathcal{L}_B$ with the co-derived set semantics: the complete logic of belief over all topological spaces is $\mathbf{wK4} = \mathbf{K} + ((\varphi \wedge B\varphi) \to BB\varphi)$ [26], while the doxastic logic **KD45** is complete with respect to so-called DSO-spaces. Here, a ***DSO*-space** is a topological space (X, τ) satisfying the following conditions: the T_D-separation axiom[16]; for every $A \subseteq X$ the set $d(A)$ is open; and (X, τ) is dense-in-itself, i.e., $d(X) = X$. See [45, 47, 46] for more details.

Criticism and comparison with our conception. Under the co-derived set semantics, as it is easy to notice, (justifiable) belief is modeled *"just like knowledge except that it may be false (in the actual world)."* This interpretation yields one of the most desirable properties of belief; namely the property of its being non-factive. All authors agree that a right notion of belief should hold the *possibility of error*: it must be possible for an agent to have false beliefs. In other words, any good semantics for belief should allow for models and worlds at which some beliefs are false. However, we claim that, according to the co-derived semantics *the existence of false beliefs is a necessary fact* (holding for all possible agents at all possible worlds in all possible models!). To explain: as we pointed out above, for any topological space (X, τ), any subset $A \subseteq X$ and any $x \in X$, we have $x \notin d(x)$. Thus, for any singleton proposition $\{x\}$[17], $x \notin \langle B \rangle(\{x\})$. Hence, $x \in B(X \setminus \{x\})$ meaning that *the agent believes the proposition* $X \setminus \{x\}$ *at the world* x. However, $X \setminus \{x\}$ is in fact false at x since $x \notin X \setminus \{x\}$. This argument holds for any topological space (X, τ) and any $x \in X$ implying that the co-derived set semantics entails not only the possibility of error but also the *necessity of error*: *"the actual world is always dis-believed"* [3].

We think this consequence is an intuitively undesirable property. It generally prevents any act of learning (updating with) the actual world. Indeed, the main problem of Formal Learning Theory (learning the true world, or the correct possibility, from a given set of possibilities) becomes automatically unattainable. Similarly, the physicist's dream of finding a true "theory of everything" is declared impossible by fiat, *as a matter of logic*. More importantly, even if necessity of error might seem realistic within a Lewisian "large-world interpretation" of possible-world semantics (in which each world must really come with a full description of all the myriad of ontic facts of the world), this property seems completely unrealistic when we adopt the more down-to-earth "small-world" models that are common in Computer Science, Game theory and other applications. In these fields, the "worlds" in any usable model come only with the description of the facts that are *relevant* for the problem at hand: e.g. in a scenario involving the throwing of a fair coin, the relevant fact is the upper face of the coin. A model for this scenario will involve typically only two possible worlds: Head and Tail. Requiring that the agent must always have a false belief means in this context that the agent can never find out which of the coin's faces is the upper one: an obviously absurd conclusion!

There is another objection, maybe even more decisive, against the co-derived set semantics, namely that it can be easily "Gettierized". For any topological space (X, τ) and any $A \subseteq X$, we have

$$\mathrm{Int}(A) = t(A) \cap A.$$

Assuming that the interior operator corresponds to the knowledge modality, the above topological identity of Int leads to

$$KP := BP \wedge P$$

[16] Recall that the T_D separation axiom states that every point is the intersection of a closed and open set. This condition is equivalent to $d(d(A)) \subseteq d(A)$, see e.g., [25].

[17] We can consider the singleton proposition $\{x\}$ as the complete description of the world x.

for any proposition P. Therefore, the co-derived set interpretation of belief together with the interior-based interpretation of knowledge yields that *knowledge is true belief.* Even if true belief comes with a canonical justification, it can easily be 'Gettierized'.

The last argument concerning the advantages of our proposal over the co-derived set semantics is of a more technical nature. While the belief logic **KD45** is sound and complete with respect to the class of extremally disconnected spaces under the topological belief semantics, it is sound and complete with respect to only the class of DSO-spaces under the co-derived semantics. Therefore, as the following proposition shows, *our topological interpretation "works" on a larger class of models than the co-derived set semantics*:

PROPOSITION 7. *Every DSO-space is extremally disconnected. However, not every extremally disconnected space is a DSO-space.*

Proof. See Appendix C.7.

4. Topological Models for Belief Revision: Static and Dynamic Conditioning

Conditioning (with respect to some qualitative plausibility order or to a probability measure) is the most widespread way to model the learning of "hard" information[18]. The prior plausibility/probability assignment (encoding the agent's original beliefs before the learning) is changed to a new such assignment, obtained from the first one by conditioning with the new information P. In the qualitative case, this means just restricting the original order to P-worlds; while in the probabilistic case, restriction has to be followed by re-normalization (to ensure that the probabilities newly assigned to the remaining worlds add up to 1). In Dynamic Epistemic Logic (DEL), one makes also a distinction between *simple* ("*static*") *conditioning* and *dynamic conditioning* (also known as "update"). The first essentially corresponds to conditional beliefs: the change is made only locally, affecting only one occurrence of the belief operator $B\varphi$ (which is thus locally replaced by conditional belief $B^P\varphi$) or of the probability measure (which is locally replaced by conditional probability). In contrast, an update is a global change, at the level of the whole model (thus recursively affecting the meaning of all occurrences of the belief/probability operators). In this section, we investigate the natural topological analogues of static and dynamic conditioning.

4.1. Static Conditioning: Conditional Beliefs

Conditional Beliefs. In DEL, static belief revision captures the agent's revised beliefs about how the world was before learning new information and is implemented by conditional belief operators $B^\varphi\psi$. Using van Benthem's terminology, "[c]onditional beliefs pre-encode beliefs that we would have if we learnt certain things." [9, p. 139]. The statement $B^\varphi\psi$ says that *if the agent would learn φ, then she would come to believe that ψ was the case before the learning* [5, p. 12]. That means conditional beliefs are hypothetical by nature, hinting at possible future belief changes of the agent. In the DEL literature, the semantics for conditional beliefs is generally given in terms of plausibility models (or equivalently, in terms of sphere models), see, e.g., [9, 5, 12].

In this section, we explore the topological analogue of static conditioning by providing a topological semantics for conditional belief modalities. As conditional beliefs capture

[18] This term is used to denote information that comes with an inherent warranty of veracity, e.g. because of originating from an infallibly truthful source.

hypothetical belief changes of an agent in case she would learn a piece of new information φ, we can obtain the semantics for a conditional belief modality $B^{\varphi}\psi$ in a natural and standard way by relativizing the semantics for the simple belief modality to the extension of the learnt formula φ. By *relativization* we mean a local change in the sense that it only affects one occurrence of the belief modality $B\varphi$. It does not cause a change in the model, i.e. it does not lead to a global change, due to its static nature.

Semantics of conditional beliefs. We start by recalling some properties of extremally disconnected spaces and the topological belief semantics. As we know a topological space (X, τ) is extremally disconnected if the closure of every open set in it is open. Therefore, a topological space (X, τ) is extremally disconnected if and only if for any $A \subseteq X$ we have

$$\mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))).$$

Hence, given a topological extensional frame $(X, \mathrm{K}_\tau, \mathrm{B}_\tau)$ based on an extremally disconnected topology τ, we obtain

$$\mathrm{B}_\tau(A) \overset{(1)}{=} \mathrm{Cl}(\mathrm{Int}(A)) \overset{(2)}{=} \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$$

for any $A \subseteq X$. Therefore, a topological extensional frame based on an extremally disconnected space provides two (*extensionally*) equivalent meaning for the belief modality B. However, when we generalize the belief operator B by relativizing the closure and the interior operators to the extension of a learnt formula φ in order to obtain a semantics for conditional belief modalities, the resulting clauses no longer remain equivalent.

We now briefly look at the relativization of Cl(Int) and explain why we do not think it provides a sufficiently "good" semantics for conditional beliefs. We then continue with our main proposal for *topological conditional belief semantics*: *the relativization of* Int(Cl(Int)).

The basic topological semantics for conditional beliefs. As pointed out above, for every subset P of a topological space (X, τ), we can generalize the belief operator B on the topological extensional frames in a natural way by relativizing the closure and the interior operators to the set P. More precisely, we define the *conditional belief operator* $\mathrm{B}^P : \mathcal{P}(X) \to \mathcal{P}(X)$ as

$$\mathrm{B}^{\mathrm{P}}(A) = \mathrm{Cl}(P \cap \mathrm{Int}(P \to A))$$

for any $A \subseteq X$ where $P \to A := (X \setminus P) \cup A$ is the set-theoretic version of material implication. This immediately gives us a topological semantics for the language $\mathcal{L}_{KCB}$ of knowledge and conditional beliefs obtained by adding the conditional belief modalities $B^{\varphi}\psi$ to $\mathcal{L}_{KB}$. Given a topological model $\mathcal{M} = (X, \tau, \nu)$, the additional semantic clause reads

$$[\![B^{\varphi}\psi]\!]^{\mathcal{M}} = \mathrm{B}^{[\![\varphi]\!]^{\mathcal{M}}}[\![\psi]\!]^{\mathcal{M}} = \mathrm{Cl}([\![\varphi]\!]^{\mathcal{M}} \cap \mathrm{Int}([\![\varphi]\!]^{\mathcal{M}} \to [\![\psi]\!]^{\mathcal{M}})).$$

As hinted, we do not find this semantics for conditional beliefs sufficiently "good" for several reasons we are about to explain. First of all, it validates the equivalence

$$K\varphi \leftrightarrow \neg B^{\neg\varphi}\top \leftrightarrow \neg B^{\neg\varphi}\neg\varphi$$

which gives a rather unusual definition of knowledge in terms of conditional beliefs: this identity corresponds neither to the definition of knowledge in [4, 5] in terms of conditional beliefs nor to the definition of "necessity" in [43] in terms of doxastic conditionals (see also, e.g., [19]). Moreover, the first of these equivalences shows that the *conditional belief operator is not a normal modality*: it does not obey the Necessitation Rule, and in particular the formula $B^{\varphi}\top$ is not in general a validity. The second equivalence above shows that in our theory the AGM Success Postulate $B^{\varphi}\varphi$ written in terms of conditional beliefs is not always valid. Ideally, we would like to have all the AGM postulates in the appropriate form stated in terms of conditional beliefs to be valid with respect to our semantics. However, one can show that while the AGM Postulates 2-6, written in terms of conditional beliefs, are valid with respect to the above semantics, the postulates 1, 7 and 8 are not[19]. The basic topological semantics for conditional beliefs is thus not optimal in capturing all of the AGM postulates for static belief revision. This motivates the search for an alternative semantics for conditional beliefs which captures more of the AGM postulates and is compatible with the notion of belief in Stalnaker's system. Fortunately, as mentioned, the definition of extremally disconnected spaces suggests an alternative semantics for conditional beliefs: *the relativization of* Int(Cl(Int)).

A 'refined' topological semantics for conditional beliefs. For every $P \subseteq X$, we can define the *new conditional belief operator* $\mathrm{B}^P : \mathcal{P}(X) \to \mathcal{P}(X)$ as

$$\mathrm{B}^P(A) = \mathrm{Int}(P \to \mathrm{Cl}(P \cap \mathrm{Int}(P \to A)))$$

for any $A \subseteq X$. This again immediately gives us a topological semantics for the language $\mathcal{L}_{KCB}$. Given a topological model $\mathcal{M} = (X, \tau, \nu)$, the additional semantic clause reads

$$[\![B^{\varphi}\psi]\!]^{\mathcal{M}} = \mathrm{Int}([\![\varphi]\!]^{\mathcal{M}} \to \mathrm{Cl}([\![\varphi]\!]^{\mathcal{M}} \cap \mathrm{Int}([\![\varphi]\!]^{\mathcal{M}} \to [\![\psi]\!]^{\mathcal{M}}))).$$

We consider this semantics an improvement of the basic topological semantics of conditional beliefs and knowledge, since, as we will see in Theorem 8, it is more successful in capturing the rationality postulates of AGM theory. We refer to this semantics as *the refined topological semantics for conditional beliefs and knowledge*. Another, and simpler, possible justification for the above semantics of conditional belief is that it validates an equivalence that generalizes the one for belief in a natural way:

PROPOSITION 8. *The following equivalence is valid in all topological spaces with respect to the refined topological semantics for conditional beliefs and knowledge*

$$B^{\varphi}\psi \;\leftrightarrow\; K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \psi))).$$

Proof. Follows immediately from the semantic clauses of conditional beliefs and knowledge.

This shows that, just like simple beliefs, conditional beliefs can be defined in terms of knowledge and this identity corresponds to the definition of the "conditional connective ⇒" in [19]. Moreover, as a corollary of Proposition 8, we obtain that the equivalences

$$B^{\top}\psi \;\overset{(1)}{\leftrightarrow}\; K(\top \to \langle K\rangle(\top \wedge K(\top \to \psi))) \;\overset{(2)}{\leftrightarrow}\; K\langle K\rangle K\psi \overset{(3)}{\leftrightarrow} \langle K\rangle K\psi$$

are valid in the class extremally disconnected spaces[20]. Interestingly, unlike the case of simple belief, *knowledge can be defined in terms of conditional belief:*

PROPOSITION 9. *The following equivalences are valid in all topological spaces with respect to the refined topological semantics for conditional beliefs and knowledge*

$$K\varphi \;\leftrightarrow\; B^{\neg\varphi}\bot \;\leftrightarrow B^{\neg\varphi}\varphi.$$

[19] The interested reader can find a more detailed discussion about this semantics in [38].

[20] In fact, equivalences (1) and (2) are valid in the class of all topological spaces, however, equivalence (3) is valid only in the class of extremally disconnected spaces.

Proof. Follows immediately from the semantic clauses of conditional beliefs and knowledge.

Proposition 9 constitutes another argument in favor of the refined semantics for conditional beliefs over the basic one: as also stated in [5], this identity coincides with the definition of "necessity" in [43] in terms of doxastic conditionals (see also, e.g., [4], [19]).

Therefore, the logic **KCB** of knowledge and conditional beliefs, **KB**, and even the unimodal fragment of **KB** having K as the only modality (which is in fact the system **S4.2** in this setting) and the unimodal fragment **CB** having only *conditional belief modalities*, have the same expressive power, since we can define simple beliefs and conditional beliefs in terms of knowledge (Proposition 5 and Proposition 8, respectively), and we can define simple beliefs and knowledge in terms of conditional beliefs (Proposition 8 and Proposition 9, respectively). As neither knowledge nor conditional beliefs can be defined in terms of simple beliefs, the unimodal fragment of **KB** having B as the only modality (which is in fact the system **KD45** in this setting) is less expressive than the aforementioned systems.

$$
\begin{tikzcd}
\mathbf{KCB} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[dd, leftrightarrow] & & \mathbf{KB} \arrow[dl, leftrightarrow] \\
& \mathbf{CB} & \\
\mathbf{S4.2} \arrow[ur, leftrightarrow] & & \mathbf{KD45} \arrow[ll] \arrow[ul] \arrow[uu]
\end{tikzcd}
$$

Figure 3.: Expressivity diagram

As for completeness, this can be obtained trivially:

THEOREM 7. *The logic* **KCB** *of knowledge and conditional beliefs is axiomatized completely by the system* **S4.2** *for the knowledge modality* K *together with the following equivalences:*

1. $B^{\varphi}\psi \leftrightarrow K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \psi)))$
2. $B\varphi \leftrightarrow B^{\top}\varphi$

Proof. See Appendix D.1.

Finally, we evaluate the success of the refined semantics in capturing the rationality postulates of AGM theory.

THEOREM 8. *The following formulas are valid in all topological spaces with respect to the refined topological semantics for conditional beliefs and knowledge*

Normality:	$B^{\theta}(\varphi \to \psi) \to (B^{\theta}\varphi \to B^{\theta}\psi)$
Truthfulness of Knowledge:	$K\varphi \to \varphi$
Persistence of Knowledge:	$K\varphi \to B^{\theta}\varphi$
Strong Positive Introspection:	$B^{\theta}\varphi \to KB^{\theta}\varphi$
Success of Belief Revision:	$B^{\varphi}\varphi$
Consistency of Revision:	$\neg K\neg\varphi \to \neg B^{\varphi}\bot$
Inclusion:	$B^{\varphi\wedge\psi}\theta \to B^{\varphi}(\psi \to \theta)$
Cautious Monotonicity:	$B^{\varphi}\psi \wedge B^{\varphi}\theta \to B^{\varphi\wedge\psi}\theta$

Moreover, the Necessitation rule for conditional beliefs:

$$\textit{From } \vdash \varphi \textit{ infer } \vdash B^{\psi}\varphi$$

preserves validity.

Proof. See Appendix D.2.

The validity of the Normality principle and the Necessitation rule shows that, unlike in case of the basic topological semantics for conditional beliefs, the conditional belief modality is a normal modal operator with respect to the refined semantics. Moreover, the refined semantics also validates the Success Postulate. However, in this case, we have to restrict the principle of Consistency of Belief Revision to the formulas that are consistent with the agent's knowledge. This is in fact a desirable restriction taking into account the agent's knowledge and is perfectly compatible with the corresponding dynamic system that we will present in the next section. Intuitively, if the agent knows $\neg\varphi$ with some degree of certainty, she should not revise her beliefs with φ. As conditional beliefs *pre-encode* possible future belief changes of an agent and the future belief changes must be based on the new information consistent with the agent's knowledge, her consistent conditional beliefs must pre-encode the possibilities that are in fact *consistent with her knowledge.*

More generally, all the axioms of the system **CDL** except for Strong Negative Introspection and Rational Monotonicity are valid on all topological spaces with respect to the refined topological semantics for conditional beliefs and knowledge. In fact, the failure of Strong Negative Introspection is an expected result for the following reasons. First of all, observe that Theorem 7 and Theorem 8 imply that all the formulas stated in Theorem 8 are theorems of the system **KCB**. Recall that

$$\neg B^{\theta}\varphi \to K\neg B^{\theta}\varphi$$

is the principle of Strong Negative Introspection. If this principle were a theorem of **KCB**, then in particular $\neg B^{\neg\varphi}\varphi \to K\neg B^{\neg\varphi}\varphi$ would be a theorem of **KCB**. Then, by Proposition 9, we would obtain

$$\neg K\varphi \to K\neg K\varphi$$

as a theorem of **KCB**. However, Theorem 7 states that the knowledge modality of **KCB** is an **S4.2**-type modality implying that $\neg K\varphi \to K\neg K\varphi$ is not a theorem of the system.

Moreover, even the extremally disconnected spaces fail to validate Rational Monotonicity, which captures the AGM postulate of Superexpansion, with respect to the refined topological semantics for conditional beliefs and knowledge. However, a weaker principle, namely, the principle of Cautious Monotonicity is valid in all topological spaces. This principle says that *if the agent would come to believe ψ and would also come to believe θ if she would learn φ, her learning ψ should not defeat her belief in θ and vice versa.* In [33], the authors state that D. Gabbay also gives a convincing argument to accept Cautious Monotonicity: "if φ is an enough reason to believe ψ and also to believe θ, then φ and ψ should also be enough to make us believe θ, since φ was enough anyway and, on this basis, ψ was accepted" [33, p. 178].

The refined conditional belief semantics therefore captures the AGM postulates 1-7 together with a weaker version of 8 on all topological spaces. It is thus more successful than the basic one in modeling static belief change of a rational agent. Moreover, we will show in Section 4.2 that the refined semantics for conditional beliefs and knowledge validates Rational Monotonicity in a restricted class of topological spaces, namely in the class of hereditarily extremally disconnected spaces, and therefore it is able to capture the AGM postulates 1-8 stated in terms of conditional beliefs.

Summing up the work that has been done so far in this paper, a new topological semantics for belief on *extremally disconnected* spaces is proposed in [3, 38] and it has been proven, in this setting, that the complete logic of knowledge and belief is Stalnaker's system **KB**, the complete logic of knowledge is **S4.2** and the complete logic of belief is **KD45** in this setting. Moreover, we provided a semantics for conditional beliefs again on *extremally disconnected* spaces as well as complete axiomatizations of the corresponding *static* systems. These results on extremally disconnected spaces, however, encounter problems when extended to a dynamic setting by adding update modalities formalized as model restriction by means of subspaces.

4.2. Dynamic Conditioning: Updates

In DEL, update (dynamic conditioning) corresponds to change of beliefs through learning hard information. Unlike the case for conditional beliefs, update induces a global change in the model.

The most standard topological analogue of this corresponds to taking the restriction of a topology τ on X to a subset $P \subseteq X$. This way, we obtain a *subspace* of a given topological space.

DEFINITION 7 (Subspace). *Given a topological space (X, τ) and a non-empty set $P \subseteq X$, a space (P, τ_P) is called a **subspace** of (X, τ) where $\tau_P = \{U \cap P : U \in \tau\}$.*

We can define the closure operator Cl_{τ_P} and the interior operator Int_{τ_P} of the subspace (P, τ_P) in terms of the closure and the interior operators of the space (X, τ) as follows[21]:

$$\mathrm{Cl}_{\tau_P}(A) = \mathrm{Cl}(A) \cap P$$

$$\mathrm{Int}_{\tau_P}(A) = \mathrm{Int}(P \to A) \cap P.$$

Topological semantics for update modalities. We now consider the language $\mathcal{L}_{!KCB}$ obtained by adding to the language $\mathcal{L}_{KCB}$ (existential) dynamic update modalities $\langle !\varphi \rangle \psi$ associated with updates. $\langle !\varphi \rangle \psi$ means that φ *is true and after the agent learns the new information* φ*,* ψ *becomes true*. The dual $[!\varphi]$ is defined as $\neg \langle !\varphi \rangle \neg$ as usual and $[!\varphi]\varphi$ means that *if* φ *is true then after the agent learns the new information* φ*,* ψ *becomes true.*

Given a topo-model (X, τ, ν) and $\varphi \in \mathcal{L}_{!KCB}$, we denote by $\mathcal{M}_\varphi$ the ***restricted model*** $\mathcal{M}_\varphi = (\llbracket \varphi \rrbracket, \tau_{\llbracket \varphi \rrbracket}, \nu_{\llbracket \varphi \rrbracket})$ where $\llbracket \varphi \rrbracket = \llbracket \varphi \rrbracket^{\mathcal{M}}$, $\tau_{\llbracket \varphi \rrbracket} = \{U \cap \llbracket \varphi \rrbracket \mid U \in \tau\}$ and $\nu_{\llbracket \varphi \rrbracket}(p) = \nu(p) \cap \llbracket \varphi \rrbracket$ for any $p \in \mathrm{Prop}$. Then, the semantics for the dynamic language $\mathcal{L}_{!KCB}$ is obtained by extending the semantics for $\mathcal{L}_{KCB}$ with:

$$\llbracket \langle !\varphi \rangle \psi \rrbracket^{\mathcal{M}} = \llbracket \psi \rrbracket^{\mathcal{M}_\varphi}.$$

To explain the problem: Given that the underlying *static* logic of knowledge and (conditional) belief is the logic of extremally disconnected spaces (see e.g., Theorems 2, 4, 6 and 7) and extremally disconnectedness is not inherited by arbitrary subspaces[22], we cannot guarantee that the restricted model induced by an arbitrary formula φ remains extremally disconnected. As we work with rational, highly idealized, logically omniscient agents, we demand our agents not to lose logical omniscience and require them to hold consistent beliefs after an update with true, new information. Under our proposed topological belief semantics, we satisfy these requirements if and only if the resulting structure is extremally

[21] See [25, pp. 65-74].

[22] In other words, extremally disconnectedness is, in general, not a hereditary property where a topological property is said to be *hereditary* if for any topological space (X, τ) that has the property, every subspace of (X, τ) also has it [25, p. 68].

disconnected: under the topological belief semantics, both the (K)-axiom (also known as the axiom of *Normality*)

$$B(\varphi \wedge \psi) \leftrightarrow (B\varphi \wedge B\psi)$$

and the axiom of *Consistency of Belief*

$$B\varphi \to \neg B\neg\varphi$$

characterize extremally disconnected spaces (see, Propositions 12 and 13, respectively, in Appendix C.4). Therefore, if the restricted model is not extremally disconnected, the agent comes to have inconsistent beliefs after an update with true information. To be more precise, we illustrate this problem with the following example:

> Consider the topo-model $\mathcal{M} = (X, \tau, \nu)$ where $X = \{x_1, x_2, x_3, x_4\}$, $\tau = \{X, \emptyset, \{x_4\}, \{x_2, x_4\}, \{x_3, x_4\}, \{x_2, x_3, x_4\}\}$ and $\nu(p) = \{x_4\}$ and $\nu(q) = \{x_2, x_4\}$ for some $p, q \in$ Prop (see Figure 4). It is easy to check that (X, τ) is an extremally disconnected space and $Bq \to \neg B\neg q$ is true in $\mathcal{M}$. We stipulate that x_1 is the actual world and the agent receives the information $\neg p$ from an infallible, truthful source. The updated (i.e., restricted) model is then $\mathcal{M}_{\neg p} = (\llbracket \neg p \rrbracket^{\mathcal{M}}, \tau_{\neg p}, \nu_{\neg p})$ where $\llbracket \neg p \rrbracket^{\mathcal{M}} = \{x_1, x_2, x_3\}$, $\tau_{\neg p} = \{\llbracket \neg p \rrbracket^{\mathcal{M}}, \emptyset, \{x_2\}, \{x_3\}, \{x_2, x_3\}\}$, $\nu_{\neg p}(p) = \emptyset$ and $\nu_{\neg p}(q) = \{x_2\}$. Here, $(\llbracket \neg p \rrbracket^{\mathcal{M}}, \tau_{\neg p})$ is not an extremally disconnected space since $\{x_3\}$ is an open subset of $(\llbracket \neg p \rrbracket^{\mathcal{M}}, \tau_{\neg p})$ but $\mathrm{Cl}_{\tau_{\neg p}}(\{x_3\}) = \{x_1, x_3\}$ is not open in $(\llbracket \neg p \rrbracket^{\mathcal{M}}, \tau_{\neg p})$. Moreover, as $x_1 \in \llbracket Bq \rrbracket^{\mathcal{M}_{\neg p}} = \mathrm{Cl}_{\tau_{\neg p}}(\mathrm{Int}_{\tau_{\neg p}}(\{x_2\})) = \{x_1, x_2\}$ and $x_1 \in \llbracket B\neg q \rrbracket^{\mathcal{M}_{\neg p}} = \mathrm{Cl}_{\tau_{\neg p}}(\mathrm{Int}_{\tau_{\neg p}}(\{x_1, x_3\})) = \{x_1, x_3\}$, the agent comes to believe both q and $\neg q$.

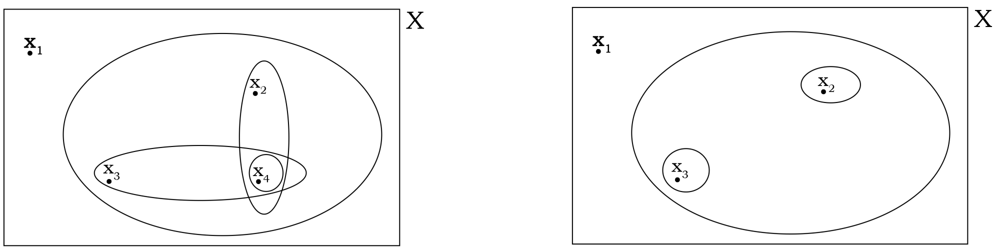


Figure 4.: (X, τ) and $(\llbracket \neg p \rrbracket^{\mathcal{M}}, \tau_{\neg p})$

One possible solution for this problem is extending the class of spaces we work with: we can focus on all topological spaces instead of working with only extremally disconnected spaces and provide semantics for belief in such a way that the aforementioned axioms which were problematic on extremally disconnected spaces would be valid on all topological spaces. This way, we do not need to worry about any additional topological property that is supposed to be inherited by subspaces. This solution, however unsurprisingly, leads to a weakening of the underlying static logic of knowledge and belief. It is very well-known that the knowledge logic of all topological spaces under the interior semantics is **S4** and we explored the (weak) belief logic of all topological spaces under the topological belief semantics in [2]. In this work, we propose another solution which approaches the issue from the opposite direction: we further restrict our attention to *hereditarily extremally disconnected spaces*, thereby, we guarantee that no model restriction leads to inconsistent beliefs. As the logic of hereditarily extremally disconnected spaces under the interior semantics is **S4.3**, the underlying static logic, in this case, would consist in **S4.3** as the logic of knowledge but again **KD45** as the logic of belief (see Theorems 9 and 10 below).

DEFINITION 8. *A topological space (X, τ) is called* ***hereditarily extremally disconnected (h.e.d.)*** *if every subspace of (X, τ) is extremally disconnected.*

For hereditarily extremally disconnected spaces, we can think of Alexandroff spaces corresponding to total preorders, in particular, corresponding to reflexive, transitive and linear Kripke frames. Recall that a Kripke frame (X, R) is called *linear* if

$$(\forall x, y, z)((xRy \wedge xRz) \rightarrow (yRz \vee zRy \vee y = z))^{23}.$$

Another interesting and non-Alexandroff example of a hereditarily extremally disconnected space is the topological space $(\mathbb{N}, \tau)$ where $\mathbb{N}$ is the set of natural numbers and $\tau = \{\emptyset, \text{ all cofinite subsets of } \mathbb{N}\}$. We elaborated on this topological space in Section 2.2. Furthermore, every countable Hausdorff extremally disconnected space is hereditarily extremely disconnected [18]. For more examples of hereditarily extremally disconnected spaces, we refer to [18].

Recall that **S4.3** as well is a *strengthening* of **S4** (and also of **S4.2**) defined as

$$\mathbf{S4.3} := \mathbf{S4} + K(K\varphi \rightarrow \psi) \vee K(K\psi \rightarrow \varphi).$$

THEOREM 9 ([14]). **S4.3** *is sound and complete with respect to the class of hereditarily extremally disconnected spaces under the interior semantics.*

Proof. See Appendix D.3

As Stalnaker also observed in [44], the derived logic of belief with belief modality defined as *epistemic possibility of knowledge*, i.e., as $\langle K \rangle K$, is **KD45** in case K is an **S4.3** modality:

THEOREM 10. *In the topological belief semantics,* **KD45** *is the complete logic of belief with respect to the class of hereditarily extremally disconnected spaces.*

Proof. See Appendix D.4

Then, we again obtain a complete logic $\mathbf{KCB}'$ of knowledge and conditional beliefs trivially, yet, with respect to the class of hereditarily extremally disconnected spaces:

THEOREM 11. *The logic* $\mathbf{KCB}'$ *of knowledge and conditional beliefs is axiomatized completely by the system* **S4.3** *for the knowledge modality K together with the following equivalences:*

1. $B^{\varphi}\psi \leftrightarrow K(\varphi \rightarrow \langle K \rangle(\varphi \wedge K(\varphi \rightarrow \psi)))$
2. $B\varphi \leftrightarrow B^{\top}\varphi$

PROPOSITION 10. *The following formula*

$$B^{\varphi}(\psi \rightarrow \theta) \wedge \neg B^{\varphi}\neg\psi \rightarrow B^{\varphi \wedge \psi}\theta,$$

called the axiom of Rational Monotonicity for conditional beliefs, is valid on hereditarily extremally disconnected spaces.

[23] This property is also called *no branching to the right* (see, e.g., [17, p. 195]) and it boils down to $(\forall x, y, z)((xRy \wedge xRz) \rightarrow (yRz \vee zRy))$, if R is reflexive.

Proof. See Appendix D.5

We can then conclude, by Theorem 8 and Proposition 10, that the refined semantics for conditional beliefs on hereditarily extremally disconnected spaces captures the AGM postulates 1-8 (written in terms of conditional belief modalities as given in the system **CDL** in [4, 5]).

We now implement updates on hereditarily extremally disconnected spaces and show that the problems occurred when we work with extremally disconnected spaces do not arise here: we in fact obtain a complete dynamic logic of knowledge and conditional beliefs with respect to the class of hereditarily extremally disconnected spaces. We again consider the language $\mathcal{L}_{!KCB}$ and semantics for update modalities $\langle !\varphi \rangle \psi$ by means of subspaces exactly the same way as formalized in the beginning of the current section, i.e., by using the restricted model $\mathcal{M}_\varphi$ with the semantic clause

$$[\![\langle !\varphi \rangle \psi]\!]^{\mathcal{M}} = [\![\psi]\!]^{\mathcal{M}_\varphi}.$$

In this setting, however, as the underlying static logic $\mathbf{KCB'}$ is the logic of hereditarily extremally disconnected spaces, we implement updates on those spaces. Since the resulting restricted model $\mathcal{M}_\varphi$ is always based on a hereditarily extremally disconnected (sub)space, we do not face the problem of loosing some validities of the corresponding static system: all the axioms of $\mathbf{KCB'}$ (and, in particular, of **S4.3** and **KD45**) will still be valid in the restricted space. Moreover, we obtain a complete axiomatization of the dynamic logic of knowledge and conditional beliefs:

THEOREM 12. *The complete and sound dynamic logic* $\mathbf{!KCB'}$ *of knowledge and conditional beliefs with respect to the class of hereditarily extremally disconnected spaces is obtained by adding the following reduction axioms to any complete axiomatization of the logic* $\mathbf{KCB'}$*:*

1. $\langle !\varphi \rangle p \leftrightarrow (\varphi \wedge p)$
2. $\langle !\varphi \rangle \neg \psi \leftrightarrow (\varphi \wedge \neg \langle !\varphi \rangle \psi)$
3. $\langle !\varphi \rangle (\psi \wedge \theta) \leftrightarrow (\langle !\varphi \rangle \psi \wedge \langle !\varphi \rangle \theta)$
4. $\langle !\varphi \rangle K\psi \leftrightarrow (\varphi \wedge K(\varphi \rightarrow \langle !\varphi \rangle \psi))$
5. $\langle !\varphi \rangle B^\theta \psi \leftrightarrow (\varphi \wedge B^{\langle !\varphi \rangle \theta} \langle !\varphi \rangle \psi)$
6. $\langle !\varphi \rangle \langle !\psi \rangle \chi \leftrightarrow \langle ! \langle !\varphi \rangle \psi \rangle \chi$

Proof. See Appendix D.6.

5. Conclusion and Future Work

Summary. In this work, we proposed a new topological semantics for belief in terms of the *closure of the interior operator*. Combining it with the interior semantics for knowledge, our topological semantics for (full) belief constitutes *the most general extensional semantics for Stalnaker's system of full belief and knowledge*. Moreover, our proposal provides an intuitive interpretation of Stalnaker's conception of (full) belief as *subjective certainty* due to the nature of topological spaces, in particular, through the definitions of interior and closure operators. Recall that for any subset P of a topological space (X, τ) and any $x \in X$,

$$x \in \mathrm{Int}(P) \text{ iff } (\exists U \in \tau)(x \in U \wedge U \subseteq P).$$

In other words, a state x is in the interior of P iff there is an open neighborhood U of x such that $U \cap (X \setminus P) = \emptyset$, i.e., x can be sharply distinguished from all non-P states by an open neighborhood U. Therefore, under this interpretation, we can say that an agent knows P at a world x iff she can sharply distinguish it from all the non-P worlds. Dually,

$$x \in \mathrm{Cl}(P) \text{ iff } (\forall U \in \tau)(x \in U \to U \cap P \neq \emptyset)$$

meaning that a state x is in the closure of P iff it is very close to P, i.e., it cannot be sharply distinguished from P states. Thus, according to our topological belief semantics, an agent (fully) believes P at a state x iff she cannot sharply distinguish x from the worlds in which she has knowledge of P, i.e., the agent cannot sharply distinguish the states in which she has belief of P from the states in which she has knowledge of P. Belief, under this semantics, therefore becomes subjectively indistinguishable from knowledge, implying that our topological semantics perfectly captures the conception of belief as "subjective certainty". The majority of approaches to knowledge and belief take belief – the weaker notion, – as basic and then strengthen it to obtain a "good" concept of knowledge. Our work provides a semantics for Stalnaker's system which approaches the issue from the other direction, i.e. taking knowledge as primitive. The formal setting developed in our studies therefore adds a precise semantic framework to a rather non-standard approach to knowledge and belief, providing a novel semantics to Stalnaker's system and imparting if not additional momentum at least an additional interpretation of it.

Furthermore, we explore topological analogues of static and dynamic conditioning by providing a topological semantics for *conditional belief* and *update modalities*. We evaluated two, *basic* and *refined*, topological semantics for conditional beliefs directly obtained from the semantics of simple belief by conditioning and argued that the latter is an improvement of the former. We demonstrated that the refined semantics for conditional beliefs quite successfully captures the rationality postulates of AGM theory: *it validates the appropriate versions of the AGM postulates 1-7 and a weaker version of postulate 8* (see Theorem 8). We moreover gave a complete axiomatization of the logic of conditional beliefs and knowledge. Although the semantics proposed for the aforementioned static notions (namely; knowledge, (full) belief and conditional beliefs) completely captures their intended meanings, modelling these notions on extremally disconnected spaces causes the problem of preserving the important structural properties of these spaces given that extremally disconnectedness is not a hereditary property as explained in Section 4.2 and also stated in [2]. In this paper, we solved this problem by restricting the class of spaces we work with to the class of hereditarily extremally disconnected spaces and we formalized knowledge, belief and conditional beliefs also on hereditarily extremally disconnected spaces together with *updates* and provide complete axiomatizations for the corresponding logics. As a result of working on hereditarily extremally disconnected spaces, the unimodal logic of knowledge becomes **S4.3** whereas the unimodal logic of belief remains to be **KD45**. We also showed that hereditarily extremally disconnected spaces validate the AGM axiom 8, stated as Rational Monotonicity in terms of conditional beliefs, and concluded that our topological semantics can capture the theory of belief revision AGM as a static one formalized in a modal setting in terms of conditional beliefs.

Future Research. In this paper, we focused on providing a topological semantics for *single agent* logics for knowledge, belief, conditional beliefs and updates. However, reasoning about knowledge, belief and especially about information change becomes particularly interesting when applied to multi-agent cases. One natural continuation of this work therefore consists in extending our framework to a multi-agent setting and providing topological semantics for operators, such as *common knowledge* and *common belief*, in line with, e.g., [13, 45].

In on-going work, we also explore topological semantics for evidence and its connection to topological (evidence-based) knowledge and belief. We therefore build *topological* evidence models generalizing those of van Benthem and Pacuit [12] and also interpret evidence dynamics on such models following the aforementioned work.

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Appendices

A. Introduction

The system S4.

Name	Axiom
K	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
T	$K\varphi \to \varphi$
4	$K\varphi \to KK\varphi$
Inference Rules	
Modus Ponens	From φ and $\varphi \to \psi$ infer ψ
Necessitation	From φ infer $K\varphi$

Table II.: The system **S4**

The logic of conditional beliefs (CDL) [4, 5][24]. The syntax of **CDL** is given by

$$\varphi := \; p \mid \neg\varphi \mid \varphi \wedge \varphi \mid B^{\varphi}\varphi$$

and the semantics is given on plausibility models as above. In this system, knowledge and belief are defined as $K\varphi := B^{\neg\varphi}\varphi$ and $B\varphi := B^{\top}\varphi$, where $\top := \neg(p \wedge \neg p)$ is some tautological sentence. A sound and complete system of **CDL** (with respect to plausibility models) is given as follows:

The inference rules and axioms of propositional logic

Necessitation Rule:	From $\vdash \varphi$ infer $\vdash B^{\psi}\varphi$
Normality:	$B^{\theta}(\varphi \to \psi) \to (B^{\theta}\varphi \to B^{\theta}\psi)$
Truthfulness of Knowledge:	$K\varphi \to \varphi$
Persistence of Knowledge:	$K\varphi \to B^{\theta}\varphi$
Strong Positive Introspection:	$B^{\theta}\varphi \to KB^{\theta}\varphi$
Strong Negative Introspection:	$\neg B^{\theta}\varphi \to K\neg B^{\theta}\varphi$
Success of Belief Revision:	$B^{\varphi}\varphi$
Consistency of Revision:	$\neg K\neg\varphi \to \neg B^{\varphi}\bot$
Inclusion:	$B^{\varphi\wedge\psi}\theta \to B^{\varphi}(\psi \to \theta)$
Rational Monotonicity:	$B^{\varphi}(\psi \to \theta) \wedge \neg B^{\varphi}\neg\psi \to B^{\varphi\wedge\psi}\theta$

[24] This system was first introduced in [4] with common knowledge and common belief operators. We work with the simplified version introduced in [5].

B. Background

B.1. The Standard Kripke Semantics

DEFINITION 9 (Kripke Frame/Model). *A* ***Kripke frame*** $\mathcal{F} = (X, R)$ *is a pair where* X *is a non-empty set and* R *is a binary relation on* X*. A* ***Kripke model*** $M = (X, R, \nu)$ *is a tuple where* (X, R) *is a Kripke frame and* ν *is a valuation, i.e. a map* $\nu : \mathrm{Prop} \to \mathcal{P}(X)$*.*

DEFINITION 10 (Standard Kripke Semantics). *Let* $M = (X, R, \nu)$ *be a Kripke model and* x *be a state in* X*. The truth of modal formulas at a world* x *in* M *is defined recursively as:*

$$\begin{array}{lll} M, x \models p & \textit{iff} & x \in \nu(p) \\ M, x \models \neg\varphi & \textit{iff} & \textit{not } M, x \models \varphi \\ M, x \models \varphi \wedge \psi & \textit{iff} & M, x \models \varphi \textit{ and } M, x \models \psi \\ M, x \models K\varphi & \textit{iff} & (\forall y \in X)(xRy \to M, y \models \varphi) \end{array}$$

It is useful to note that

$$M, x \models \langle K \rangle \varphi \quad \text{iff} \quad (\exists y \in X)(xRy \wedge M, y \models \varphi).$$

Truth and validity of a formula with respect to the standard Kripke semantics are defined as usual, i.e., the same way as in the interior semantics. We let $\|\varphi\|^M = \{x \in X : M, x \models \varphi\}$ and call $\|\varphi\|^M$ the *extension* of the modal formula φ in M.

B.2. Proof of Theorem 1

The soundness proof is a routine check and immediately follows from the Kuratowski axioms for the interior operator (see, e.g., [10, p. 237] for a detailed proof). For completeness, let $\varphi \in \mathcal{L}_K$ such that φ is not a theorem of **S4**, i.e., **S4** $\not\vdash \varphi$. Then, by the relational completeness of **S4**, there exists a reflexive and transitive Kripke model $M = (X, R, \nu)$ such that $\|\varphi\|^M \neq X$. Hence, by Proposition 1, we have that $[\![\varphi]\!]^{\mathcal{M}_{\tau_R}} \neq X$ where $\mathcal{M}_{\tau_R} = (X, \tau_R, \nu)$ is the corresponding topo-model (see also, e.g., [16]).

B.3. Proof of Proposition 3 [10, p. 253]

Let (X, τ) be a topological space and $\mathcal{M} = (X, \tau, \nu)$ be a topo-model on (X, τ). Then,

$$\begin{array}{rll} [\![\langle K \rangle K\varphi \to K\langle K \rangle \varphi]\!]^{\mathcal{M}} = X & \text{iff} & \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})) \subseteq \mathrm{Int}(\mathrm{Cl}([\![\varphi]\!]^{\mathcal{M}})) \\ & \text{iff} & \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}}))) \\ & \text{iff} & (X, \tau) \text{ is extremally disconnected.} \end{array}$$

C. The Topology of Full Belief and Knowledge

C.1. Proof of Proposition 5

($\Rightarrow$) $B\varphi \to \langle K \rangle K\varphi$

1.	$K\neg K\varphi \to B\neg K\varphi$	Ax.(KB)
2.	$B\neg K\varphi \to \neg BK\varphi$	Ax.(CB)
3.	$\neg BK\varphi \to \neg B\varphi$	Ax.(FB)
4.	$K\neg K\varphi \to \neg B\varphi$	Propositional tautology and MP, 1, 2, 3
5.	$B\varphi \to \langle K \rangle K\varphi$	Contraposition, 4

($\Leftarrow$) $\langle K\rangle K\varphi \to B\varphi$

1.	$\neg B\varphi \to K\neg B\varphi$	Ax.(NI)
2.	$\neg B\varphi \to \neg K\varphi$	Ax.(KB)
3.	$K(\neg B\varphi \to \neg K\varphi)$	K-Nec, 2
4.	$K(\neg B\varphi \to \neg K\varphi) \to (K\neg B\varphi \to K\neg K\varphi)$	Ax.(K)
5.	$K\neg B\varphi \to K\neg K\varphi$	MP, 3, 4
6.	$\neg B\varphi \to K\neg K\varphi$	Propositional tautology and MP, 1, 5
7.	$\langle K\rangle K\varphi \to B\varphi$	Contraposition, 6

C.2. Proof of Proposition 6

Observe that for any topo-model (X, τ, ν) and for any $\varphi,\ \psi \in \mathcal{L}_{KB}$,

$$[\![\varphi \to \psi]\!] = X \text{ iff } [\![\varphi]\!] \subseteq [\![\psi]\!].$$

Let (X, τ) be a topological space and ν be an arbitrary valuation on (X, τ). We know, by the soundness of **S4** under the interior semantics, that the axioms (K), (T), (KK) and the inference rules of **KB** are valid on all topological spaces. In addition, (NI), (KB) and (FB) are also valid in all topological extensional semantics. Here, we demonstrate only the proof for the validity of (NI):

(NI):

$$\begin{aligned} X = [\![\neg B\varphi \to K\neg B\varphi]\!] \quad &\text{iff} \quad [\![\neg B\varphi]\!] \subseteq [\![K\neg B\varphi]\!] \\ &\text{iff} \quad X \setminus (\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))) \subseteq \mathrm{Int}(X \setminus (\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])))) \\ &\text{iff} \quad \mathrm{Int}(\mathrm{Cl}(X \setminus [\![\varphi]\!])) \subseteq \mathrm{Int}(\mathrm{Int}(\mathrm{Cl}(X \setminus [\![\varphi]\!]))) \end{aligned}$$

Since $\mathrm{Int}(\mathrm{Cl}(X \setminus [\![\varphi]\!])) = \mathrm{Int}(\mathrm{Int}(\mathrm{Cl}(X \setminus [\![\varphi]\!])))$ is true in all topological spaces (by (I4) in Section 2.2), the result follows. The proofs for the validity of the axioms (KB) and (FB) follow similarly.

Moreover, both (CB) and (PI) are valid on (X, τ) (under our proposed semantics) iff (X, τ) is extremally disconnected:

(CB):

$$\begin{aligned} X = [\![B\varphi \to \neg B\neg\varphi]\!] \quad &\text{iff} \quad [\![B\varphi]\!] \subseteq [\![\langle B\rangle\varphi]\!] \\ &\text{iff} \quad \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])) \subseteq \mathrm{Int}(\mathrm{Cl}([\![\varphi]\!])) \\ &\text{iff} \quad \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))) \\ &\text{iff} \quad (X, \tau) \text{ is extremally disconnected.} \end{aligned}$$

(PI):

$$\begin{aligned} X = [\![B\varphi \to KB\varphi]\!] \quad &\text{iff} \quad [\![B\varphi]\!] \subseteq [\![KB\varphi]\!] \\ &\text{iff} \quad \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])) \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))) \\ &\text{iff} \quad (X, \tau) \text{ is extremally disconnected.} \end{aligned}$$

Therefore, (X, τ) validates the axioms and rules of **KB** iff it is extremally disconnected.

C.3. Proof of Theorem 3

($\Leftarrow$) This direction is proven in Proposition 6.

($\Rightarrow$) Let $(X, \mathrm{K}, \mathrm{B})$ be an extensional semantics and suppose it validates all the axioms and rules of **KB**. Then, the validity of the **S4** axioms implies that K satisfies the Kuratowski conditions for topological interior, and so it gives rise to a topology τ in which $\mathrm{K} = \mathrm{Int}$, by the Theorem 5.3 in [23, p. 74] (see also Proposition 1.2.9 in [25, p. 23]). Then, since $(X, \mathrm{K}, \mathrm{B})$ validates all the axioms of **KB**, we have $[\![B\varphi \leftrightarrow \langle K \rangle K\varphi]\!]^{\mathcal{M}} = X$ for any model $\mathcal{M} = (X, \mathrm{K}, \mathrm{B}, \nu)$ and for all $\varphi \in \mathcal{L}_{KB}$ (by Proposition 5). Hence, $[\![B\varphi]\!]^{\mathcal{M}} = \mathrm{B}[\![\varphi]\!]^{\mathcal{M}} = \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})$, i.e., $\mathrm{B} = \mathrm{Cl}(\mathrm{Int})$. Thus, $(X, \mathrm{K}, \mathrm{B})$ is a topological extensional semantics. Finally, the validity of the axiom (CB) proves that (X, τ) is extremally disconnected (see the proof of Proposition 6).

C.4. Proof of Theorem 4

Since axioms of **KB** are Sahlqvist formulas, **KB** is canonical, hence, complete with respect to its canonical model. However, the canonical model of **KB** is in fact an extensional model validating all of its axioms. Thus, Topological Representation Theorem for extensional models of **KB** (Theorem 3 in Section 3.2), we have that **KB** is sound and complete with respect to the class of extremally disconnected spaces.

C.5. Proof of Theorem 5

PROPOSITION 11. *For any topo-model $\mathcal{M} = (X, \tau, \nu)$ and any $\varphi \in \mathcal{L}_B$ we have*

1. $[\![B\varphi \to BB\varphi]\!] = X$,
2. $[\![\langle B \rangle \varphi \to B\langle B \rangle \varphi]\!] = X$.

Proof. Let $\mathcal{M} = (X, \tau, \nu)$ be a topo-model and $\varphi \in \mathcal{L}_B$. Recall that for any $\varphi, \psi \in \mathcal{L}_B$ we have

$$[\![\varphi \to \psi]\!] = X \text{ iff } [\![\varphi]\!] \subseteq [\![\psi]\!]. \tag{1}$$

1. By (1), it suffices to show that $[\![B\varphi]\!] \subseteq [\![BB\varphi]\!]$. By our semantics, we have

$$[\![B\varphi]\!] = \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])) \text{ and } [\![BB\varphi]\!] = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])))).$$

As known, the closure of an open set is a closed domain[25] [25, p. 20]. We then have

$$\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])))).$$

as $\mathrm{Int}([\![\varphi]\!])$ is open in (X, τ). Therefore, we obtain $[\![B\varphi]\!] = [\![BB\varphi]\!]$ which implies $[\![B\varphi \to BB\varphi]\!] = X$.

2. Similar to part-(a), it suffices to show that $[\![\langle B \rangle \varphi]\!] \subseteq [\![B\langle B \rangle \varphi]\!]$ and the proof follows:

$$\begin{array}{rcll} [\![\langle B \rangle \varphi]\!] & = & \mathrm{Int}(\mathrm{Cl}([\![\varphi]\!])) & \\ & \subseteq & \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}([\![\varphi]\!]))) & \text{(by (C2))} \\ & = & \mathrm{Cl}(\mathrm{Int}(\mathrm{Int}(\mathrm{Cl}([\![\varphi]\!])))) & \text{(by (I4))} \\ & = & [\![B\langle B \rangle \varphi]\!]. & \end{array}$$

Therefore, by (1), we have $[\![\langle B \rangle \varphi \to B\langle B \rangle \varphi]\!] = X$.

[25] A subset A of a topological space is called *closed domain* if $A = \mathrm{Cl}(\mathrm{Int}(A))$ [25, p. 20]. In the literature, a closed domain is also called *regular closed*.

It follows from Proposition 11 that all topological spaces validate the axioms (4) and (5) under the topological belief semantics. However, the K-axiom $B\varphi \wedge B\psi \leftrightarrow B(\varphi \wedge \psi)$ and the D-axiom $B\varphi \to \langle B \rangle \varphi$ are not valid on all topological spaces but *are* valid on all extremally disconnected spaces:

LEMMA 1. *For any topological space (X, τ), we have*

$$U \cap \mathrm{Cl}(A) \subseteq \mathrm{Cl}(U \cap A)$$

for any $U \in \tau$ and $A \subseteq X$.

Proof. Let (X, τ) be a topological space, $U \in \tau$, $A \subseteq X$ and $x \in X$. Suppose $x \in U \cap \mathrm{Cl}(A)$. Since $x \in \mathrm{Cl}(A)$, for all open neighbourhoods V of x, $V \cap A \neq \emptyset$. Let W be an open neightbourhood of x. Then, since τ is closed under finite intersection and x is an element of both W and U, the set $W \cap U$ is an open neighbouhood of x as well. Thus, by the assumption that $x \in \mathrm{Cl}(A)$, $(W \cap U) \cap A \neq \emptyset$, i.e., $W \cap (U \cap A) \neq \emptyset$. As W has been chosen arbitrarily, $x \in \mathrm{Cl}(U \cap A)$.

LEMMA 2. *The following conditions are equivalent for any topological space (X, τ):*

1. *(X, τ) is extremally disconnected.*
2. *$\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \mathrm{Cl}(U \cap V)$ for all $U, V \in \tau$.*
3. *$\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$ for all $U, V \in \tau$ with $U \cap V = \emptyset$.*

Proof. Let (X, τ) be a topological space.

$(1 \Rightarrow 2)$ Suppose (X, τ) is extremally disconnected and let $U, V \in \tau$. We always have $\mathrm{Cl}(U \cap V) \subseteq \mathrm{Cl}(U) \cap \mathrm{Cl}(V)$ by (C2). For the other direction, we have

$$\begin{array}{rcll} \mathrm{Cl}(U) \cap \mathrm{Cl}(V) & \subseteq & \mathrm{Cl}(\mathrm{Cl}(U) \cap V) & \text{(by Lemma 1 and } \mathrm{Cl}(U) \text{ being open)} \\ & \subseteq & \mathrm{Cl}(U \cap V) & \text{(by Lemma 1 and } V \text{ being open)} \end{array}$$

$(2 \Rightarrow 3)$ Suppose (2). Let $U, V \in \tau$ such that $U \cap V = \emptyset$. Then, by (C1), $\mathrm{Cl}(U \cap V) = \emptyset$. Thus, by (2), we have $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$.

$(3 \Rightarrow 1)$ Suppose (3) and let $U \in \tau$. We want to show that $\mathrm{Cl}(U)$ is open, i.e., $\mathrm{Int}(\mathrm{Cl}(U)) = \mathrm{Cl}(U)$. By (I2), we have $\mathrm{Int}(\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(U)$. For the other direction, let $x \in X$ and suppose $x \in \mathrm{Cl}(U)$ but $x \notin \mathrm{Int}(\mathrm{Cl}(U))$. $x \notin \mathrm{Int}(\mathrm{Cl}(U))$ implies that $x \in \mathrm{Cl}(\mathrm{Int}(X \setminus U))$. Hence, $\mathrm{Cl}(U) \cap \mathrm{Cl}(\mathrm{Int}(X \setminus U)) \neq \emptyset$. However, as $\mathrm{Int}(\mathrm{X} \setminus \mathrm{U})$ is open and $U \cap \mathrm{Int}(\mathrm{X} \setminus \mathrm{U}) = \emptyset$, we have $\mathrm{Cl}(U) \cap \mathrm{Cl}(\mathrm{Int}(X \setminus U)) = \emptyset$ (by (3)). Contradiction!

PROPOSITION 12. *A topological space (X, τ) validates the* K*-axiom iff (X, τ) is extremally disconnected.*

Proof. Let (X, τ) be a topological space and $\mathcal{M} = (X, \tau, \nu)$ be a topo-model on (X, τ). Then,

$$\begin{array}{ll} & X = [\![B\varphi \wedge B\psi \leftrightarrow B(\varphi \wedge \psi)]\!] \\ \text{iff} & [\![B\varphi \wedge B\psi]\!] = [\![B(\varphi \wedge \psi)]\!] \\ \text{iff} & \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])) \cap \mathrm{Cl}(\mathrm{Int}([\![\psi]\!])) = \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!] \cap [\![\psi]\!])) \\ \text{iff} & \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])) \cap \mathrm{Cl}(\mathrm{Int}([\![\psi]\!])) = \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]) \cap \mathrm{Int}([\![\psi]\!])) \quad \text{(by (I3))} \\ \text{iff} & (X, \tau) \text{ is extremally disconnected} \qquad\qquad\qquad\qquad \text{(by Lemma 2)} \end{array}$$

PROPOSITION 13. *A topological space* (X, τ) *validates the* D*-axiom iff* (X, τ) *is extremally disconnected.*

Proof. Let (X, τ) be a topological space and $\mathcal{M} = (X, \tau, \nu)$ be a topo-model on (X, τ). Then,

$$\begin{aligned} X = [\![B\varphi \to \langle B\rangle\varphi]\!] \quad &\text{iff} \quad [\![B\varphi]\!] \subseteq [\![\langle B\rangle\varphi]\!] \\ &\text{iff} \quad \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])) \subseteq \mathrm{Int}(\mathrm{Cl}([\![\varphi]\!])) \\ &\text{iff} \quad \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))) \\ &\text{iff} \quad (X, \tau) \text{ is extremally disconnected.} \end{aligned}$$

It follows from Proposition 12 and Proposition 13 that the K-axiom and the D-axiom are not only valid on extremally disconnected spaces, they also characterize extremally disconnected spaces (under the topological belief semantics). Hence, the class of extremally disconnected spaces is the largest class of topological spaces which validates the K-axiom and the D-axiom. The fact that both K-axiom and the D-axiom characterizing extremally disconnectedness might seem surprising at first sight. However, given that we interpret the knowledge modality K as the interior and the belief modality B as the closure of the interior operators on topological spaces, we obtain that the formula $(B\varphi \to \langle B\rangle\varphi)$ is equivalent to $(\langle K\rangle K\varphi \to K\langle K\rangle\varphi)$, which is the (.2)-axiom, and the formula $(B\varphi \wedge B\psi \leftrightarrow B(\varphi \wedge \psi))$ is equivalent to $(\langle K\rangle K\varphi \wedge \langle K\rangle K\psi \leftrightarrow \langle K\rangle K(\varphi \wedge \psi))$. Thus, the above propositions only state that a topological space validates $(\langle K\rangle K\varphi \to K\langle K\rangle\varphi)$ iff it validates $(\langle K\rangle K\varphi \wedge \langle K\rangle K\psi \leftrightarrow \langle K\rangle K(\varphi \wedge \psi))$ iff it is extremely disconnected. In fact, these results together with Proposition 11 yield the soundness of **KD45**:

THEOREM 5. *The belief logic* **KD45** *is sound with respect to the class of extremally disconnected spaces in topological belief semantics. In fact, a topological space* (X, τ) *validates all the axioms and rules of the system* **KD45** *in the topological belief semantics iff* (X, τ) *is extremally disconnected.*

Proof. Follows from Propositions 11, 12 and 13.

C.6. Proof of Theorem 6

Throughout this proof, we use the notation $[\varphi]^{\mathcal{M}}$ for the extension of a formula $\varphi \in \mathcal{L}_K$ with respect to the *interior semantics* in order to make clear in which semantics we work. We reserve the notation $[\![\varphi]\!]^{\mathcal{M}}$ for the extension of a formula $\varphi \in \mathcal{L}_B$ with respect to the *topological belief semantics*. We skip the index when confusion is unlikely to occur.

DEFINITION 11 (Translation $(.)^* : \mathcal{L}_B \to \mathcal{L}_K$). *For any* $\varphi \in \mathcal{L}_B$*, the translation* $(\varphi)^*$ *of* φ *into* $\mathcal{L}_K$ *is defined recursively as follows:*

1. $(p)^* = p$, *where* $p \in \mathrm{Prop}$
2. $(\neg\varphi)^* = \neg\varphi^*$
3. $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$
4. $(B\varphi)^* = \langle K\rangle K\varphi^*$

It is useful to note that $(\langle B\rangle\varphi)^* = K\langle K\rangle\varphi^*$.

PROPOSITION 14. *For any topo-model $\mathcal{M} = (X, \tau, \nu)$ and for any formula $\varphi \in \mathcal{L}_B$, we have*

$$[\![\varphi]\!]^{\mathcal{M}} = [\varphi^*]^{\mathcal{M}}.$$

Proof. We prove the lemma by induction on the complexity of φ. The cases for $\varphi = p$, $\varphi = \neg\psi$ and $\varphi = \psi \wedge \chi$ are straightforward. Now let $\varphi = B\psi$, then

$$\begin{array}{rcll} [\![\varphi]\!]^{\mathcal{M}} & = & [\![B\psi]\!]^{\mathcal{M}} & \\ & = & \mathrm{Cl}(\mathrm{Int}([\![\psi]\!]^{\mathcal{M}})) & \text{(by the topological belief semantics for } \mathcal{L}_B) \\ & = & \mathrm{Cl}(\mathrm{Int}([\psi^*]^{\mathcal{M}})) & \text{(by I.H.)} \\ & = & [\langle K\rangle K\psi^*]^{\mathcal{M}} & \text{(by the interior semantics for } \mathcal{L}_K.) \\ & = & [(B\psi)^*]^{\mathcal{M}} & \text{(by the translation.)} \\ & = & [\varphi^*]^{\mathcal{M}}. & \end{array}$$

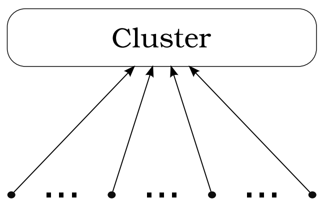
We prove the topological completeness of **KD45** by using the translation $(.)^*$ of the language $\mathcal{L}_B$ into the language $\mathcal{L}_K$ given in Definition 11 and the completeness of **S4.2** with respect to the class of extremally disconnected spaces in the interior semantics.

For the topological completeness proof of **KD45** we also make use of the completeness of **KD45** and **S4.2** in the standard Kripke semantics. We first recall some frame conditions concerning the relational completeness of the corresponding systems. Let (X, R) be a *transitive* Kripke frame. A non-empty subset $C \subseteq X$ is a ***cluster*** if

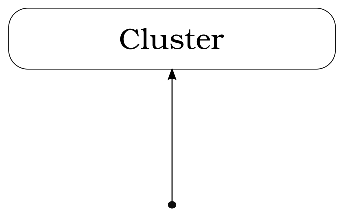
(1) for each $x, y \in C$ we have xRy, and

(2) there is no $D \subseteq X$ such that $C \subset D$ and D satisfies (1).

A point $x \in X$ is called a ***maximal point*** if there is no $y \in X$ such that xRy and $\neg(yRx)$. We call a cluster a ***final cluster*** if all its points are maximal. It is not hard to see that for any final cluster C of (X, R) and any $x \in C$, we have $R(x) = C$. A transitive Kripke frame (X, R) is called ***cofinal*** if it has a unique final cluster C such that for each $x \in X$ and $y \in C$ we have xRy. We call a cofinal frame a ***brush*** if $X \setminus C$ is an irreflexive antichain, i.e., for each $x, y \in X \setminus C$ we have $\neg(xRy)$ where C is the final cluster. A brush with a singleton $X \setminus C$ is called a ***pin***. By definition, every brush and every pin is transitive. Finally, a transitive frame (X, R) is called ***rooted***, if there is an $x \in X$, called a *root*, such that for each $y \in X$ with $x \neq y$ we have xRy. Hence, every rooted brush is in fact a pin. Figure 5 illustrates brushes and pins, respectively.



a. brush



b. pin

Figure 5.: Brush and Pin

LEMMA 3.

1. *Each reflexive and transitive cofinal frame is an* **S4.2**-*frame. Moreover,* **S4.2** *is sound and complete with respect to the class of finite rooted reflexive and transitive cofinal frames.*

2. *Each brush is a* **KD45***-frame. Moreover,* **KD45** *is sound and complete with respect to the class of finite brushes, indeed, with respect to the class of finite pins.*

Proof. See, e.g., [20, Chapter 5].

For any reflexive and transitive cofinal frame (X, R) we define R_B on X by

$$xR_By \text{ if } y \in C$$

for each $x, y \in X$ where C is the final cluster of (X, R). It is easy to see that for each $x \in X$, we have $R_B(x) = C$. Recall that we denote the extension of a modal formula φ (either in $\mathcal{L}_B$ or in $\mathcal{L}_K$) in a Kripke model $M = (X, R, \nu)$ by $\|\varphi\|^M$.

LEMMA 4. *For any reflexive and transitive cofinal frame (X, R),*

1. *(X, R_B) is a brush.*
2. *For any valuation ν on X and for each formula $\varphi \in \mathcal{L}_B$ we have*

$$\|\varphi^*\|^M = \|\varphi\|^{M_B}$$

where $M = (X, R, \nu)$ and $M_B = (X, R_B, \nu)$.

Proof. Let (X, R) be a reflexive and transitive cofinal frame.

1. By definition, R_B is transitive. We can also show that the final cluster C of (X, R) is also a cluster (X, R_B). For each $x, y \in C$, xR_By by definition of R_B. Moreover, suppose for a contradiction that there is a $D \subseteq X$ such that $C \subset D$ and for each $x, y \in D$ we have xR_By. As $C \subset D$, there is an $x_0 \in D$ such that $x_0 \notin C$ contradicting that xR_Bx_0 for all $x \in D$. Hence, C is a cluster of (X, R_B) too. By definition of R_B, we also have that for any $x \in X$, $R_B(x) = C$, i.e., for any $x \in X$ and $y \in C$ we have xR_By. Hence, (X, R_B) is a cofinal frame with the final cluster C. Now consider $X \setminus C$. Suppose there is an $x \in X \setminus C$ such that xR_Bx. This implies, by definition of R_B, that $x \in C$ contradicting our assumption. Hence, each point $x \in X \setminus C$ is irreflexive. Suppose also that $X \setminus C$ is not an antichain, i.e., there exist $x, y \in X \setminus C$ such that either xR_By or yR_Bx. W.l.o.g, assume xR_By. This also implies, by definition of R_B, that $y \in C$ contradicting $y \in X \setminus C$. Hence, (X, R_B) is a brush.

2. We prove this item by induction on the complexity of φ. Let $M = (X, R, \nu)$ be a model on (X, R). The cases for $\varphi = p$, $\varphi = \neg\psi$, $\varphi = \psi \wedge \chi$ are straightforward. Let $\varphi = B\psi$.

   ($\subseteq$) Let $x \in \|(B\psi)^*\|^M = \|\langle K\rangle K\psi^*\|^M$. Then, by the standard Kripke semantics, there is a $y \in X$ with xRy such that $R(y) \subseteq \|\psi^*\|^M$. Since (X, R) is a cofinal frame, we have $C \subseteq R(y)$, hence, $C \subseteq \|\psi^*\|^M$. Then, by induction hypothesis, $C \subseteq \|\psi\|^{M_B}$. Since $R_B(x) = C$ in the brush (X, R_B), we have $R_B(x) \subseteq \|\psi\|^{M_B}$ implying that $x \in \|B\psi\|^{M_B}$.

   ($\supseteq$) Let $x \in \|B\psi\|^{M_B}$. Then, by the standard Kripke semantics, for all $y \in X$ with xR_By we have $y \in \|\psi\|^{M_B}$, i.e., $R_B(x) \subseteq \|\psi\|^{M_B}$. Then, $C \subseteq \|\psi\|^{M_B}$, since $R_B(x) = C$. Hence, by induction hypothesis, $C \subseteq \|\psi^*\|^M$. By definition of a final cluster, we have $R(y) = C$ for any $y \in C$. Hence, $y \in \|K\psi^*\|^M$ for any $y \in C$. Since (X, R) is a cofinal frame, xRy for any $y \in C$. Thus, $x \in \|\langle K\rangle K\psi^*\|^M$, i.e., $x \in \|(B\psi)^*\|^M$.

For each Kripke frame (X, R) we let R^+ to be the reflexive closure of R defined as $R^+ = R \cup \{(x, x) \mid x \in X\}$.

LEMMA 5. *For any brush* (X, R),

1. (X, R^+) *is a reflexive and transitive cofinal frame.*
2. *For any valuation* ν *on* X *and for each formula* $\varphi \in \mathcal{L}_B$ *we have*
$$\|\varphi\|^M = \|\varphi^*\|^{M^+}$$
where $M = (X, R, \nu)$ *and* $M^+ = (X, R^+, \nu)$.

Proof. Let (X, R) be a brush.

1. Since a brush is also a transitive cofinal frame, (X, R^+) is also transitive and cofinal. Moreover, R^+ is reflexive by definition. Therefore, (X, R^+) is a reflexive and transitive cofinal frame.

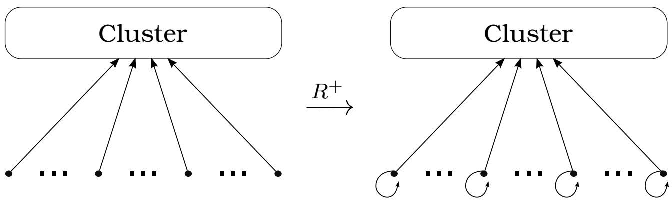


Figure 6.: From (X, R) to (X, R^+)

2. We prove (2) by induction on the complexity of φ. Let $M = (X, R, \nu)$ be a model on (X, R). The cases for $\varphi = p$, $\varphi = \neg\psi$, $\varphi = \psi \wedge \chi$ are straightforward. Let $\varphi = B\psi$.

   ($\subseteq$) Let $x \in \|B\psi\|^M$. Then, by the standard Kripke semantics, for all $y \in X$ with xRy we have $y \in \|\psi\|^M$, i.e., $R(x) \subseteq \|\psi\|^M$. This implies, since M is a model based on a brush, $C \subseteq \|\psi\|^M$. By I.H., $C \subseteq \|\psi^*\|^{M^+}$. Since (X, R^+) is in fact just a reflexive brush, $C \subseteq R^+(x)$. Hence there is a $z \in C$ such that xRz and, since $R^+(z) = C$ and $C \subseteq \|\psi^*\|^{M^+}$, $z \in \|K\psi^*\|^{M^+}$. Therefore, $x \in \|\langle K\rangle K\psi^*\|^{M^+} = \|(B\psi)^*\|^{M^+}$.

   ($\supseteq$) Let $x \in \|(B\psi)^*\|^{M^+} = \|\langle K\rangle K\psi^*\|^{M^+}$. Then, by the standard Kripke semantics, there is a $y \in X$ with xR^+y such that $R^+(y) \subseteq \|\psi^*\|^{M^+}$. Observe that either $y = x$ or xRy (equivalently, $y \in C$).

   If $x = y$, $R^+(y) \subseteq \|\psi^*\|^{M^+}$ meaning that $R^+(x) \subseteq \|\psi^*\|^{M^+}$. Then, since $R(x) \subseteq R^+(x)$, we have $R(x) \subseteq \|\psi\|^M$ by induction hypothesis. Therefore, $x \in \|B\psi\|^M$.

   If xRy, i.e., $y \in C$, we have $R(x) = R^+(y)$. Hence, by induction hypothesis, $R(x) \subseteq \|\psi\|^M$. Therefore, $x \in \|B\psi\|^M$.

LEMMA 6. *For each formula* $\varphi \in \mathcal{L}_B$,
$$\mathbf{S4.2} \vdash \varphi^* \textit{ iff } \mathbf{KD45} \vdash \varphi.$$

Proof. Let $\varphi \in \mathcal{L}_B$.

($\Rightarrow$) Suppose $\mathbf{KD45} \nvdash \varphi$. By Lemma 3(2), there exists a Kripke model $M = (X, R, \nu)$ where (X, R) is a finite pin such that $\|\varphi\|^M \neq X$. Then, by Lemma 5, M^+ is a model based on the finite reflexive and transitive cofinal frame (X, R^+) and $\|\varphi^*\|^{M^+} \neq X$. Hence, by Lemma 3(1), we have $\mathbf{S4.2} \nvdash \varphi^*$.

($\Leftarrow$) Suppose $\mathbf{S4.2} \not\vdash \varphi^*$. By Lemma 3(1), there exists a Kripke model $M = (X, R, \nu)$ where (X, R) is a finite reflexive and transitive cofinal frame such that $\|\varphi^*\|^M \neq X$. Then, by Lemma 4, M_B is a model based on the brush (X, R_B) and $\|\varphi\|^{M_B} \neq X$. Hence, by Lemma 3(2), we have $\mathbf{KD45} \not\vdash \varphi$.

THEOREM 6. *In the topological belief semantics,* **KD45** *is the complete logic of belief with respect to the class of extremally disconnected spaces.*

Proof. Let $\varphi \in \mathcal{L}_B$ such that $\mathbf{KD45} \not\vdash \varphi$. By Lemma 6, $\mathbf{S4.2} \not\vdash \varphi^*$. Hence, by topological completeness of **S4.2** with respect to the class of extremally disconnected spaces in the interior semantics, there exists a topo-model $\mathcal{M} = (X, \tau, \nu)$ where (X, τ) is an extremally disconnected space such that $[\varphi^*]^{\mathcal{M}} \neq X$. Then, by Proposition 14, $[\![\varphi]\!]^{\mathcal{M}} \neq X$. Thus, we found an extremally disconnected space (X, τ) which refutes φ in the topological belief semantics. Hence, **KD45** is complete with respect to the class of extremally disconnected spaces in the topological belief semantics.

C.7. Proof of Proposition 7

Let (X, τ) be a DSO-space and $U \in \tau$. Recall that for any $A \subseteq X$, $\mathrm{Cl}(A) = d(A) \cup A$. So $\mathrm{Cl}(U) = d(U) \cup U$. Since (X, τ) is a DSO-space, $d(U)$ is an open subset of X. Thus, since U is open as well, $\mathrm{Cl}(U)$ is open. Therefore, (X, τ) is an extremally disconnected space.

Now consider the topological space (X, τ) where $X = \{1, 2, 3\}$ and $\tau = \{X, \emptyset, \{2\}, \{1, 2\}\}$. It is easy to check that for all $U \in \tau$, $\mathrm{Cl}(U)$ is open (in fact, for each $U \in \tau$ with $U \neq \emptyset$, $\mathrm{Cl}(U) = X$). Hence, (X, τ) is an extremally disconnected space. However, as $\mathrm{Cl}(X \setminus \{2\}) = \{1, 3\}$, we have $2 \notin d(X)$. Thus, (X, τ) is not dense in itself and thus not a DSO-space.

D. Topological Models for Belief Revision: Static and Dynamic Conditioning

D.1. Proof of Theorem 7

The validity of (1) and (2) is given by Proposition 8. While (2) reduces belief to conditional belief, (1) reduces conditional beliefs to knowledge. Hence, the proof follows from the topological completeness of **S4.2**.

D.2. Proof of Theorem 8

By Proposition 8, we know that each of the axioms can be rewritten by using only the knowledge modality K. We also know that the logic of knowledge **S4** is complete with respect to to the class of reflexive and transitive Kripke frames. In this proof, we will first show that each of the axioms is a theorem of **S4** by using Kripke frames and the relational completeness of **S4**. Then, we can conclude that these axioms are also valid on all topological spaces, since **S4** is sound with respect to the class of all topological spaces under the interior semantics. Recall that the semantic clauses for knowledge in the interior semantics and in the refined topological semantics for conditional beliefs and knowledge coincide.

Let (X, R) be a reflexive and transitive Kripke frame, $M = (X, R, \nu)$ a model on (X, R) and x any element of X.

1. *Normality:* $B^\theta(\varphi \to \psi) \to (B^\theta\varphi \to B^\theta\psi)$
   By Proposition 8, we can rewrite the Normality principle as

   $K(\theta \to \langle K\rangle(\theta \wedge K(\theta \to (\varphi \to \psi)))) \to$
   $(K(\theta \to \langle K\rangle(\theta \wedge K(\theta \to \varphi))) \to K(\theta \to \langle K\rangle(\theta \wedge K(\theta \to \psi)))).$

   Suppose $x \in \|K(\theta \to \langle K\rangle(\theta \wedge K(\theta \to (\varphi \to \psi))))\|$ and $x \in \|K(\theta \to \langle K\rangle(\theta \wedge K(\theta \to \varphi)))\|$. This implies,

   $$\begin{aligned} R(x) &\subseteq \|\theta \to \langle K\rangle(\theta \wedge K(\theta \to (\varphi \to \psi)))\| && (2)\\ R(x) &\subseteq \|\theta \to \langle K\rangle(\theta \wedge K(\theta \to \varphi))\| && (3)\end{aligned}$$

   We want to show that $R(x) \subseteq \|\theta \to \langle K\rangle(\theta \wedge K(\theta \to \psi))\|$

   Let $y \in X$ such that xRy, i.e. $y \in R(x)$. Suppose $y \in \|\theta\|$. Then,

   $$\begin{array}{ll} y \in \|\langle K\rangle(\theta \wedge K(\theta \to (\varphi \to \psi))\| & \text{by (2)}\\ y \in \|\langle K\rangle(\theta \wedge K(\theta \to \varphi))\| & \text{by (3)}\end{array}$$

   These imply that there exists a $y_1 \in X$ with yRy_1 such that $y_1 \in \|\theta \wedge K(\theta \to (\varphi \to \psi))\|$, and there exists a $y_2 \in X$ with yRy_2 such that $y_2 \in \|\theta \wedge K(\theta \to \varphi)\|$. Since R is transitive and $xRyRy_2$, we also have $y_2 \in \|\theta \to \langle K\rangle(\theta \wedge K(\theta \to (\varphi \to \psi))\|$, by (A.1). Similar as above, there exists $y_2' \in X$ with y_2Ry_2' such that $y_2' \in \|\theta \wedge K(\theta \to (\varphi \to \psi))\|$. Hence, we have

   $$y_2' \in \|\theta\|, \text{ and} \tag{4}$$

   $$R(y_2') \subseteq \|\theta \to (\varphi \to \psi)\|. \tag{5}$$

   As $y_2 \in \|K(\theta \to \varphi)\|$, y_2Ry_2' and R is transitive, $y_2' \in \|K(\theta \to \varphi)\|$ as well. Hence,

   $$R(y_2') \subseteq \|\theta \to \varphi\|. \tag{6}$$

   Thus, since $((\theta \to (\varphi \to \psi)) \wedge (\theta \to \varphi)) \to (\theta \to \psi)$ is a tautology, $R(y_2') \subseteq \|\theta \to \psi\|$ by (5) and (6).
   Hence, $y_2' \in \|K(\theta \to \psi)\|$. Then, by (4), $y_2' \in \|\theta \wedge K(\theta \to \psi)\|$.
   Thus, as yRy_2Ry_2' and R is transitive, we have $y \in \|\langle K\rangle(\theta \wedge K(\theta \to \psi))\|$. Therefore, $y \in \|\theta \to \langle K\rangle(\theta \wedge K(\theta \to \psi))\|$. Since we have chosen y arbitrarily from $R(x)$,

   $$R(x) \subseteq \|\theta \to \langle K\rangle(\theta \wedge K(\theta \to \psi))\|, \text{ implying that}$$

   $$x \in \|K(\theta \to \langle K\rangle(\theta \wedge K(\theta \to \psi)))\|.$$

2. *Success of Belief Revision:* $B^\varphi\varphi$
   By Proposition 8, we can rewrite this axiom as

   $$K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \varphi))).$$

   We want to show that $x \in \|K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \varphi)))\|$, i.e., that
   $R(x) \subseteq \|\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \varphi))\|$.
   Let $y \in X$ such that $y \in R(x)$ and $y \in \|\varphi\|$. As R is reflexive,

   $$y \in \|\langle K\rangle\varphi\| \tag{7}$$

Observe that $(\varphi \wedge K(\varphi \to \varphi)) \leftrightarrow \varphi$. Thus, (7) implies $y \in \|\langle K\rangle(\varphi \wedge K(\varphi \to \varphi))\|$. Therefore, $y \in \|\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \varphi))\|$. Since we have chosen y arbitrarily from $R(x)$,

$$R(x) \subseteq \|\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \varphi))\|, \text{ implying that}$$

$$x \in \|K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \varphi)))\|.$$

3. *Truthfulness of Knowledge:* $K\varphi \to \varphi$
   This is the T axiom of **S4**, hence its validity immediately follows from the soundness of **S4** with respect to the class of reflexive and transitive frames.

4. *Persistence of Knowledge:* $K\varphi \to B^{\psi}\varphi$
   By Proposition 8, we can rewrite this axiom as

$$K\varphi \to K(\psi \to \langle K\rangle(\psi \wedge K(\psi \to \varphi))).$$

   Suppose $x \in \|K\varphi\|$ and let $y \in X$ such that xRy and $y \in \|\psi\|$. By the first assumption, $y \in \|\varphi\|$ as well. As $x \in \|K\varphi\|$, $x \in \|K(\psi \to \varphi)\|$. Then, since xRy and R is transitive, $y \in \|K(\psi \to \varphi)\|$ too. Thus, $y \in \|\psi \wedge K(\psi \to \varphi)\|$ and by reflexivity of R, $y \in \|\langle K\rangle(\psi \wedge K(\psi \to \varphi))\|$. Hence, $y \in \|\psi \to \langle K\rangle(\psi \wedge K(\psi \to \varphi))\|$. As y has been chosen arbitrarily from $R(x)$, $x \in \|K(\psi \to \langle K\rangle(\psi \wedge K(\psi \to \varphi)))\|$.

5. *Strong Positive Introspection:* $B^{\psi}\varphi \to KB^{\psi}\varphi$
   By Proposition 8, we can rewrite this axiom as

$$K(\psi \to \langle K\rangle(\psi \wedge K(\psi \to \varphi))) \to KK(\psi \to \langle K\rangle(\psi \wedge K(\psi \to \varphi))).$$

   Obviously, it is an instance of the 4-axiom. Hence, it is valid.

6. *Inclusion:* $B^{\varphi \wedge \psi}\theta \to B^{\varphi}(\psi \to \theta)$
   By Proposition 8, we can rewrite this axiom as

$$K((\varphi \wedge \psi) \to \langle K\rangle(\varphi \wedge \psi \wedge K(\varphi \wedge \psi \to \theta))) \to K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to (\psi \to \theta))))$$

   Suppose $x \in \|K((\varphi \wedge \psi) \to \langle K\rangle(\varphi \wedge \psi \wedge K(\varphi \wedge \psi \to \theta)))\|$. This implies, $R(x) \subseteq \|(\varphi \wedge \psi) \to \langle K\rangle(\varphi \wedge \psi \wedge K(\varphi \wedge \psi \to \theta))\|$, i.e.,
   $R(x) \subseteq \|\varphi \to (\psi \to \langle K\rangle(\varphi \wedge \psi \wedge K(\varphi \wedge \psi \to \theta))\|$.

   We want to show that $R(x) \subseteq \|\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to (\psi \to \theta)))\|$.
   Let $y \in X$ with $y \in R(x)$ such that $y \in \|\varphi\|$. Then, by assumption,

$$y \in \|\psi \to \langle K\rangle(\varphi \wedge \psi \wedge K(\varphi \wedge \psi \to \theta))\|.$$

Case 1: $y \notin \|\psi\|$

Suppose for contradiction that $y \notin \|\langle K\rangle(\varphi \wedge K(\varphi \to (\psi \to \theta)))\|$. This implies, for every $z \in X$ with yRz, $z \notin \|\varphi \wedge K(\varphi \to (\psi \to \theta))\|$. Hence, for every $z \in X$ with yRz, $z \notin \|\varphi\|$ or $z \notin \|K(\varphi \to (\psi \to \theta))\|$. Then, since $y \in \|\varphi\|$ and R is reflexive, $y \notin \|K(\varphi \to (\psi \to \theta))\|$. Thus, there is a $z_0 \in X$ with yRz_0 such that $z_0 \notin \|\varphi \to (\psi \to \theta)\|$, i.e., $z_0 \in \|\varphi\|$, $z_0 \in \|\psi\|$ but $z_0 \notin \|\theta\|$.

On the other hand, as $xRyRz_0$ and R being transitive,
$z_0 \in \|\varphi \to (\psi \to \langle K\rangle(\varphi \wedge \psi \wedge K(\varphi \wedge \psi \to \theta)))\|$ by the first assumption.
Thus, $z_0 \in \|\langle K\rangle(\varphi \wedge \psi \wedge K(\varphi \wedge \psi \to \theta))\|$. This implies, there is a $z_1 \in X$ with

$z_0 R z_1$ such that $z_1 \in \|\varphi \wedge \psi \wedge K(\varphi \wedge \psi \to \theta)\|$. Hence, $z_1 \in \|K(\varphi \wedge \psi \to \theta)\|$.

Then, as yRz_0Rz_1, we have by the first assumption of this case that $z_1 \notin \|\varphi\|$ or $z_1 \notin \|K(\varphi \to (\psi \to \theta))\|$, which contradictions above fact. Hence,

$$y \in \|\langle K\rangle(\varphi \wedge K(\varphi \to (\psi \to \theta)))\|.$$

Case 2: $y \in \|\langle K\rangle(\varphi \wedge \psi \wedge K(\varphi \wedge \psi \to \theta))\|$
This implies that $\exists z_0 \in X$ with yRz_0 such that $z_0 \in \|\varphi \wedge \psi \wedge K(\varphi \wedge \psi \to \theta)\|$. Hence, $z_0 \in \|\varphi \wedge K(\varphi \wedge \psi \to \theta)\|$ as well. Thus,

$$y \in \|\langle K\rangle(\varphi \wedge K(\varphi \to (\psi \to \theta)))\|.$$

Therefore, $y \in \|\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to (\psi \to \theta)))\|$. Since y has been chosen arbitrarily,

$$x \in \|K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to (\psi \to \theta))))\|.$$

7. *Cautious Monotonicity:* $B^{\varphi}\psi \wedge B^{\varphi}\theta \to B^{\varphi\wedge\psi}\theta$
By Proposition 8, we can rewrite this axiom as

$K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \psi))) \wedge K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \theta))) \to$
$K((\varphi \wedge \psi) \to \langle K\rangle((\varphi \wedge \psi) \wedge K((\varphi \wedge \psi) \to \theta)))$.

Suppose $x \in \|K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \psi))) \wedge K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \theta)))\|$. Then,

$$R(x) \subseteq \|\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \psi))\|, \text{ and} \tag{8}$$
$$R(x) \subseteq \|\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \theta))\| \tag{9}$$

We want to show that $R(x) \subseteq \|(\varphi \wedge \psi) \to \langle K\rangle((\varphi \wedge \psi) \wedge K((\varphi \wedge \psi) \to \theta))\|$
Let $y \in R(x)$ such that $y \in \|\varphi\wedge\psi\|$. Then, by (8) and (9), we have $y \in \|\langle K\rangle(\varphi\wedge K(\varphi \to \psi))\|$ and $y \in \|\langle K\rangle(\varphi \wedge K(\varphi \to \theta))\|$, respectively. These imply there exists a $z_0 \in X$ with yRz_0 such that

$$z_0 \in \|\varphi \wedge K(\varphi \to \psi)\| \tag{10}$$

and there exists a $z_1 \in X$ with $z_1 R y$ such that

$$z_1 \in \|\varphi \wedge K(\varphi \to \theta)\|. \tag{11}$$

Hence, as R is reflexive, we have $z_0 \in \|\psi\|$ and thus $z_0 \in \|\varphi \wedge \psi\|$ by (10). Then, since $xRyRz_0$ and R is transitive, we have $z_0 \in \|\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \theta))\|$, by (9). Thus, as $z_0 \in \|\varphi\|$, we obtain $z_0 \in \|\langle K\rangle(\varphi \wedge K(\varphi \to \theta))\|$. This implies that these is a $z_2 \in X$ with z_0Rz_2 such that $z_2 \in \|\varphi \wedge K(\varphi \to \theta)\|$. Then, since R is transitive and $z_0 \in \|K(\varphi \to \psi)\|$, we have $z_2 \in \|K(\varphi \to \psi)\|$. Hence, since R is reflexive and $z_2 \in \|\varphi\|$, we get $z_2 \in \|\psi\|$ implying that $z_2 \in \|\varphi\wedge\psi\|$. Moreover, $z_2 \in \|K(\varphi \to \psi)\|$ and $z_2 \in \|K(\varphi \to \theta)\|$ imply that $z_2 \in \|K((\varphi\wedge\psi) \to \theta)\|$. Therefore, $z_2 \in \|(\varphi\wedge\psi)\wedge K((\varphi\wedge\psi) \to \theta)\|$. Hence, as yRz_0Rz_2 and R is transitive, $y \in \|\langle K\rangle((\varphi\wedge\psi)\wedge K((\varphi\wedge\psi) \to \theta))\|$. Hence, $y \in \|(\varphi \wedge \psi) \to \langle K\rangle((\varphi \wedge \psi) \wedge K((\varphi \wedge \psi) \to \theta))\|$. Sinca y has been chosen arbitrarily from $R(x)$, we have $R(x) \subseteq \|(\varphi \wedge \psi) \to \langle K\rangle((\varphi \wedge \psi) \wedge K((\varphi \wedge \psi) \to \theta))\|$, i.e.

$$x \in \|K((\varphi \wedge \psi) \to \langle K\rangle((\varphi \wedge \psi) \wedge K((\varphi \wedge \psi) \to \theta)))\|.$$

Therefore, each of the above axioms is valid on all reflexive and transitive Kripke frames. Thus, they are theorems of **S4**, since **S4** is complete with respect to the class of all reflexive and transitive Kripke frames (see, e.g., [17, 21]). Then, by Theorem 1, we obtain that they are valid on all topological spaces in the interior semantics. As the semantic clause of knowledge in the interior semantics and the semantic clause of knowledge in the refined topological semantics for conditional beliefs and knowledge are the same, by Proposition 8, the above axioms are also valid in all topological spaces with respect to the refined semantics .

We finally prove that the Necessitation Rule for conditional beliefs preserves validity: Let $\varphi, \psi \in \mathcal{L}_{KCB}$ such that $B^{\psi}\varphi$ is not valid. Then, there exists a topo-model $\mathcal{M} = (X, \tau, \nu)$ such that $[\![B^{\psi}\varphi]\!]^{\mathcal{M}} \neq X$, i.e., $\mathrm{Int}([\![\psi]\!] \to \mathrm{Cl}([\![\psi]\!] \wedge \mathrm{Int}([\![\psi]\!] \to [\![\varphi]\!]))) \neq X$. Now suppose $[\![\varphi]\!] = X$. Then,

$$\begin{aligned}
\mathrm{Int}([\![\psi]\!] \to \mathrm{Cl}([\![\psi]\!] \cap \mathrm{Int}([\![\psi]\!] \to [\![\varphi]\!]))) &= \mathrm{Int}((X \setminus [\![\psi]\!]) \cup \mathrm{Cl}([\![\psi]\!] \cap \mathrm{Int}((X \setminus [\![\psi]\!]) \cup X))) \\
&= \mathrm{Int}((X \setminus [\![\psi]\!]) \cup \mathrm{Cl}([\![\psi]\!])) \\
&= \mathrm{Int}(X) \\
&= X
\end{aligned}$$

contradicting $[\![B^{\psi}\varphi]\!] = X$. Hence, $[\![\varphi]\!] \neq X$

D.3. Proof of Theorem 9

The proof of this theorem can be found in [14]. We present it here in order to keep the paper self-contained. Also our proof is slightly different than that of [14].

Since **S4.3** Kripke frames correspond to hereditarily extremally disconnected spaces, we obtain the completeness by Proposition 2. For soundness, we only need to show that (.3)-axiom is sound w.r.t. the class of hereditarily extremally disconnected spaces under the interior semantics (since topological soundness of **S4** has already been proven, see Theorem 1). However, we here show a stronger result: (.3)-axiom characterizes hereditarily extremally disconnected spaces under the interior semantics.

DEFINITION 12. *Given a topological space (X, τ), any two subsets A, B of X are said to be* ***separated*** *if* $\mathrm{Cl}(A) \cap B = \emptyset = \mathrm{Cl}(B) \cap A$.

PROPOSITION 15 ([18]). *For any arbitrary topological space (X, τ), the followings are equivalent*

1. *(X, τ) is h.e.d.*
2. *any two separated subsets of X have disjoint closures.*

Therefore (by Proposition 15), a topological space (X, τ) is h.e.d. iff for any $A, B \subseteq X$ with $\mathrm{Cl}(A) \cap B = \emptyset = \mathrm{Cl}(B) \cap A$, we have $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = \emptyset$.

LEMMA 7. *For any topological space (X, τ), $Y \subseteq X$ and $U, V \in \tau$, if $(U \cap Y) \cap (V \cap Y) = \emptyset$ then $(U \cap \mathrm{Cl}(Y)) \cap (V \cap \mathrm{Cl}(Y)) = \emptyset$*

Proof. Let (X, τ) be a topological space, $Y \subseteq X$ and $U, V \in \tau$. Suppose $(U \cap Y) \cap (V \cap Y) = \emptyset$ and $(U \cap \mathrm{Cl}(Y)) \cap (V \cap \mathrm{Cl}(Y)) \neq \emptyset$. Thus, there is an $x \in X$ such that $x \in U$, $x \in V$ and $x \in \mathrm{Cl}(Y)$.

$x \in \mathrm{Cl}(Y)$ iff for all $V' \in \tau$ with $x \in V'$, $V' \cap Y \neq \emptyset$. However, we have that $x \in U \cap V$ and $U \cap V$ is an open neighbourhood of x and, by assumption, $U \cap V \cap Y = \emptyset$ leading to a contradiction. Therefore, $(U \cap \mathrm{Cl}(Y)) \cap (V \cap \mathrm{Cl}(Y)) = \emptyset$

PROPOSITION 16. *For any topological space (X, τ),*

$$K(K\varphi \to \psi) \vee K(K\psi \to \varphi) \text{ is valid in } (X,\tau) \text{ iff } (X,\tau) \text{ is h.e.d.}$$

Proof.

($\Rightarrow$) Let (X, τ) be a topological space, $\varphi, \psi \in \mathcal{L}_K$. Suppose $[\![K(K\varphi \to \psi) \vee K(K\psi \to \varphi)]\!] = X$. This means that for any $A, B \subseteq X$, we have

$$\mathrm{Int}((X \setminus \mathrm{Int}(A)) \cup B) \cup \mathrm{Int}((X \setminus \mathrm{Int}(B)) \cup A) = X \text{ and equivalently, we have}$$

$$\mathrm{Cl}(\mathrm{Int}(A) \cap (X \setminus B)) \cap \mathrm{Cl}(\mathrm{Int}(B) \cap (X \setminus A)) = \emptyset. \tag{12}$$

We want to show that (X, τ) is h.e.d., i.e., for any subspace (Y, τ_Y) of (X, τ) and for every two disjoint open subsets U_Y, V_Y of (Y, τ), we have $\mathrm{Cl}_Y(U_Y) \cap \mathrm{Cl}_Y(V_Y) = \emptyset$.

Let (Y, τ_Y) be a subspace of (X, τ) and $U_Y, V_Y \in \tau_Y$ such that $U_Y \cap V_Y = \emptyset$. We know that $U_Y = U \cap Y$ and $V_Y = V \cap Y$ for some $U, V \in \tau$.

Then, by Lemma 7, we have $(U \cap \mathrm{Cl}(Y)) \cap (V \cap \mathrm{Cl}(Y)) = U \cap V \cap \mathrm{Cl}(Y) = \emptyset$. Thus,

$$V \cap \mathrm{Cl}(Y) \subseteq \mathrm{Cl}(Y) \setminus U \subseteq X \setminus U \tag{13}$$

and similarly,

$$U \cap \mathrm{Cl}(Y) \subseteq \mathrm{Cl}(Y) \setminus V \subseteq X \setminus V \tag{14}$$

By equation (12), we have $\mathrm{Cl}(U \cap (X \setminus V)) \cap \mathrm{Cl}(V \cap (X \setminus U)) = \emptyset$. Hence, by equations (13) and (14), we have

$$\mathrm{Cl}(U \cap (\mathrm{Cl}(Y) \setminus V)) \cap \mathrm{Cl}(V \cap (\mathrm{Cl}(Y) \setminus U)) \subseteq \mathrm{Cl}(U \cap (X \setminus V)) \cap \mathrm{Cl}(V \cap (X \setminus U)) = \emptyset. \tag{15}$$

We also have

$$\mathrm{Cl}_Y(U_Y) = \mathrm{Cl}_Y(U \cap Y) \subseteq \mathrm{Cl}(U \cap \mathrm{Cl}(Y)) \tag{16}$$

$$\mathrm{Cl}_Y(V_Y) = \mathrm{Cl}_Y(V \cap Y) \subseteq \mathrm{Cl}(V \cap \mathrm{Cl}(Y)) \tag{17}$$

Then,

$$\begin{array}{lll} & \mathrm{Cl}_Y(U_Y) \cap \mathrm{Cl}_Y(V_Y) & \\ \subseteq & \mathrm{Cl}(U \cap \mathrm{Cl}(Y)) \cap \mathrm{Cl}(V \cap \mathrm{Cl}(Y)) & \text{by eqns. (16) and (17)} \\ \subseteq & \mathrm{Cl}(U \cap (\mathrm{Cl}(Y) \setminus V)) \cap \mathrm{Cl}(V \cap (\mathrm{Cl}(Y) \setminus U)) & \text{by eqns. (13) and (14)} \\ = & \emptyset & \text{by eqn. (15)} \end{array}$$

($\Leftarrow$) Let (X, τ) be a h.e.d.-space. This boils down showing that for any $A, B \subseteq X$,

$$\mathrm{Int}((X \setminus \mathrm{Int}(A)) \cup B) \cup \mathrm{Int}((X \setminus \mathrm{Int}(B)) \cup A) = X.$$

Equivalently, we can show

$$\mathrm{Cl}(\mathrm{Int}(A) \cap (X \setminus B)) \cap \mathrm{Cl}(\mathrm{Int}(B) \cap (X \setminus A)) = \emptyset.$$

By Proposition 15, it suffices to show that

$$\mathrm{Cl}(\mathrm{Int}(A)\cap(X\setminus B))\cap(X\setminus A)\cap\mathrm{Int}(B)\overset{(1)}{=}\emptyset\overset{(2)}{=}\mathrm{Cl}(\mathrm{Int}(B)\cap(X\setminus A))\cap(X\setminus B)\cap\mathrm{Int}(A).$$

For equation (1), we have

$$\begin{aligned}
& \mathrm{Cl}(\mathrm{Int}(A)\cap(X\setminus B))\cap(X\setminus A)\cap\mathrm{Int}(B)\\
\subseteq\ & \mathrm{Cl}(\mathrm{Int}(A)\cap(X\setminus B)\cap\mathrm{Int}(B))\cap(X\setminus A) \quad \text{(by Lemma 1)}\\
=\ & \emptyset \quad (\text{since } (X\setminus B)\cap\mathrm{Int}(B)=\emptyset.)
\end{aligned}$$

Proof of equation (2) is analogous to (1). Then, since (X,τ) is h.e.d., by Proposition 15 we have

$$\mathrm{Cl}(\mathrm{Int}(A)\cap(X\setminus B))\cap\mathrm{Cl}(\mathrm{Int}(B)\cap(X\setminus A))=\emptyset,\ \text{thus}$$

$$\mathrm{Int}((X\setminus\mathrm{Int}(A))\cup B)\cup\mathrm{Int}((X\setminus\mathrm{Int}(B))\cup A)=X.$$

D.4. Proof of Theorem 10

LEMMA 8. *For each formula $\varphi\in\mathcal{L}_B$,*

$$\mathbf{S4.3}\vdash\varphi^*\ \textit{iff}\ \mathbf{KD45}\vdash\varphi,$$

where φ^* is given in Definition 11, Appendix C.6.
Proof. The proof in analogous to the proof of Lemma 6 in Appendix C.6.

Theorem 10 follows from Lemma 8 in a similar way as Theorem 6 follows from Lemma 6.

D.5. Proof of Proposition 10

We prove this proposition in a similar way as Theorem 8. By Proposition 8, we know that we can rewrite the axiom of Rational Monotonicity by using only the knowledge modality K. We also know that the logic of knowledge **S4.3** is complete with respect to the class of reflexive, transitive and linear Kripke frames (called **S4.3** frames). We first show that the axiom of Ratinal Monotonicity written by using only the K modality is a theorem of **S4.3** by using Kripke frames and the relational completeness of **S4.3**. Then, we can conclude that this axiom is also valid on all h.e.d. spaces, since **S4.3** is sound with respect to the class of h.e.d. spaces under the interior semantics (See Theorem 9). Recall that the semantic clauses for knowledge in the interior semantics and in the refined topological semantics for conditional beliefs and knowledge coincide.

Let (X,R) be a reflexive, transitive and linear Kripke frame, $M=(X,R,\nu)$ a model on (X,R) and x any element of X.

Rational Monotonicity: $B^{\varphi}(\psi\to\theta)\wedge\neg B^{\varphi}\neg\psi\to B^{\varphi\wedge\psi}\theta$.

By Proposition 8, we can rewrite the the following main subformulas of the above formula as

$$\begin{aligned}
B^{\varphi}(\psi\to\theta) &:= K(\varphi\to\langle K\rangle(\varphi\wedge K((\varphi\wedge\psi)\to\theta)))\\
\neg B^{\varphi}\neg\psi &:= \langle K\rangle(\varphi\wedge K(\varphi\to\langle K\rangle(\varphi\wedge\psi)))\\
B^{\varphi\wedge\psi}\theta &:= K((\varphi\wedge\psi)\to\langle K\rangle((\varphi\wedge\psi)\wedge K((\varphi\wedge\psi)\to\theta)))
\end{aligned}$$

Suppose $x \in \|B^{\varphi}(\psi \to \theta)\|$ and $x \in \|\neg B^{\varphi}\neg\psi\|$ and suppose toward a contradiction $x \notin \|B^{\varphi\wedge\psi}\theta\|$, i.e.,

$$x \in \|K(\varphi \to \langle K\rangle(\varphi \wedge K((\varphi \wedge \psi) \to \theta)))\| \tag{18}$$
$$x \in \|\langle K\rangle(\varphi \wedge K(\varphi \to \langle K\rangle(\varphi \wedge \psi)))\| \tag{19}$$
$$x \notin \|K((\varphi \wedge \psi) \to \langle K\rangle((\varphi \wedge \psi) \wedge K((\varphi \wedge \psi) \to \theta)))\| \tag{20}$$

(19) implies that there exist a $y_0 \in R(x)$ such that

$$y_0 \in \|\varphi \wedge K(\varphi \to \langle K\rangle(\varphi \wedge \psi))\| \tag{21}$$

and (20) implies that there exist a $x_0 \in R(x)$ such that $x_0 \in \|\varphi \wedge \psi\|$ and $x_0 \notin \|\langle K\rangle((\varphi \wedge \psi) \wedge K((\varphi \wedge \psi) \to \theta))\|$, i.e.,

$$x_0 \in \|K(\neg\varphi \vee \neg\psi \vee \langle K\rangle(\varphi \wedge \psi \wedge \neg\theta))\|. \tag{22}$$

Since R is reflexive, transitive and linear, we have either $x_0 R y_0$ or $y_0 R x_0$.

Case 1: xRx_0Ry_0

Since xRy_0 and $y_0 \in \|\varphi\|$, by (18), we obtain $y_0 \in \|\langle K\rangle(\varphi \wedge K((\varphi \wedge \psi) \to \theta))\|$. This means there exists a $z_0 \in R(y_0)$ such that $z_0 \in \|\varphi \wedge K((\varphi \wedge \psi) \to \theta)\|$. Since y_0Rz_0 and $z_0 \in \|\varphi\|$, by (21), $z_0 \in \|\langle K\rangle(\varphi \wedge \psi)\|$. Thus, there exists $t_0 \in R(z_0)$ such that $t_0 \in \|\varphi \wedge \psi\|$. Since $x_0Ry_0Rz_0Rt_0$, by transitivity, we have x_0Rt_0. Then, by (22) and $t_0 \in \|\varphi \wedge \psi\|$, we obtain $t_0 \in \|\langle K\rangle(\varphi \wedge \psi \wedge \neg\theta)\|$. This implies, there exists $w_0 \in R(t_0)$ such that $w_0 \in \|\varphi \wedge \psi \wedge \neg\theta\|$, in particular, $w_0 \in \|\varphi \wedge \psi\|$ and $w_0 \in \|\neg\theta\| = X \setminus \|\theta\|$. However, since z_0Rw_0 and $z_0 \in \|K((\varphi \wedge \psi) \to \theta)\|$, we obtain $w_0 \in \|\theta\|$, contradicting $w_0 \in \|\neg\theta\|$.

Case 2: xRy_0Rx_0

Since xRx_0 and $x_0 \in \|\varphi\|$, by (18), we obtain $x_0 \in \|\langle K\rangle(\varphi \wedge K((\varphi \wedge \psi) \to \theta))\|$. This means there exists a $z_0 \in R(x_0)$ such that $z_0 \in \|\varphi \wedge K((\varphi \wedge \psi) \to \theta)\|$. Since $y_0Rx_0Rz_0$, by transitivity, y_0Rz_0. Therefore, since $z_0 \in \|\varphi\|$, by (21), $z_0 \in \|\langle K\rangle(\varphi \wedge \psi)\|$. Thus, there exists $t_0 \in R(z_0)$ such that $t_0 \in \|\varphi \wedge \psi\|$. Since $x_0Rz_0Rt_0$, by transitivity, we have x_0Rt_0. Then, by (22) and $t_0 \in \|\varphi \wedge \psi\|$, we obtain $t_0 \in \|\langle K\rangle(\varphi \wedge \psi \wedge \neg\theta)\|$. This implies, there exists $w_0 \in R(t_0)$ such that $w_0 \in \|\varphi \wedge \psi \wedge \neg\theta\|$, in particular, $w_0 \in \|\varphi \wedge \psi\|$ and $w_0 \in \|\neg\theta\| = X \setminus \|\theta\|$. However, since z_0Rw_0 and $z_0 \in \|K((\varphi \wedge \psi) \to \theta)\|$, we obtain $w_0 \in \|\theta\|$, contradicting $w_0 \in \|\neg\theta\|$.

Therefore, $x \in \|K((\varphi \wedge \psi) \to \langle K\rangle((\varphi \wedge \psi) \wedge K((\varphi \wedge \psi) \to \theta)))\|$. We then conclude that the axiom of Rational Monotonicity written in terms of the knowledge modality is true in M. Therefore, since (X, R) and the model M on (X, R) have been chosen arbitrarily, the above formula is valid in all reflexive, transitive and linear Kripke frames. Thus, it is a theorem of **S4.3**, since **S4.3** is complete with respect to the class of all reflexive, transitive and linear Kripke frames (see, e.g, [17, 21]). Then, by Theorem 9, we obtain that it is valid in all h.e.d. spaces in the interior semantics. As the semantic clause of knowledge in the interior semantics and the semantic clause of knowledge in the refined topological semantics for conditional beliefs and knowledge are the same, by Proposition 8, the axiom of Rational Monotonicity is also valid in all h.e.d. spaces under the refined semantics.

D.6. Proof of Theorem 12

The result follows from the validity of the new axioms. The cases for (1-3) are straightforward. We only prove the validity of (4), (5) and (6). Let $\mathcal{M} = (X, \tau, \nu)$ be a topo-model and φ, ψ, θ and χ be formulas in $\mathcal{L}_{KCB}$. Then, 4.

$$\begin{aligned}
[\![\langle !\varphi\rangle K\psi]\!]^{\mathcal{M}} &= [\![K\psi]\!]^{\mathcal{M}_\varphi}\\
&= \mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\psi]\!]^{\mathcal{M}_\varphi})\\
&= \mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}})\\
&= \mathrm{Int}([\![\varphi]\!]^{\mathcal{M}} \to [\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}}) \cap [\![\varphi]\!]^{\mathcal{M}}\\
&= [\![K(\varphi \to \langle !\varphi\rangle\psi)]\!]^{\mathcal{M}} \cap [\![\varphi]\!]^{\mathcal{M}}\\
&= [\![K(\varphi \to \langle !\varphi\rangle\psi) \wedge \varphi]\!]^{\mathcal{M}}
\end{aligned}$$

5.

$$\begin{aligned}
&[\![\langle !\varphi\rangle B^\theta\psi]\!]^{\mathcal{M}}\\
=\ & [\![B^\theta\psi]\!]^{\mathcal{M}_\varphi}\\
=\ & \mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \to \mathrm{Cl}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \cap \mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \to [\![\psi]\!]^{\mathcal{M}_\varphi})))\\
=\ & \mathrm{Int}([\![\varphi]\!] \to ([\![\theta]\!]^{\mathcal{M}_\varphi} \to \mathrm{Cl}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \cap \mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \to [\![\psi]\!]^{\mathcal{M}_\varphi})))) \cap [\![\varphi]\!]\\
=\ & \mathrm{Int}(([\![\varphi]\!] \wedge [\![\theta]\!]^{\mathcal{M}_\varphi}) \to (\mathrm{Cl}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \cap \mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \to [\![\psi]\!]^{\mathcal{M}_\varphi})))) \cap [\![\varphi]\!]\\
=\ & \mathrm{Int}(([\![\varphi]\!] \wedge [\![\theta]\!]^{\mathcal{M}_\varphi}) \to (\mathrm{Cl}([\![\theta]\!]^{\mathcal{M}_\varphi} \cap \mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \to [\![\psi]\!]^{\mathcal{M}_\varphi})) \cap [\![\varphi]\!])) \cap [\![\varphi]\!]\\
=\ & \mathrm{Int}([\![\theta]\!]^{\mathcal{M}_\varphi} \to (\mathrm{Cl}([\![\theta]\!]^{\mathcal{M}_\varphi} \cap \mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \to [\![\psi]\!]^{\mathcal{M}_\varphi})) \cap [\![\varphi]\!])) \cap [\![\varphi]\!]\\
=\ & \mathrm{Int}([\![\theta]\!]^{\mathcal{M}_\varphi} \to (\mathrm{Cl}([\![\theta]\!]^{\mathcal{M}_\varphi} \cap (\mathrm{Int}([\![\varphi]\!] \cap [\![\theta]\!]^{\mathcal{M}_\varphi}) \to [\![\psi]\!]^{\mathcal{M}_\varphi}) \cap [\![\varphi]\!]))) \cap [\![\varphi]\!]\\
=\ & \mathrm{Int}([\![\langle !\varphi\rangle\theta]\!]^{\mathcal{M}} \to (\mathrm{Cl}([\![\langle !\varphi\rangle\theta]\!]^{\mathcal{M}} \cap (\mathrm{Int}([\![\varphi]\!] \cap [\![\langle !\varphi\rangle\theta]\!]^{\mathcal{M}}) \to [\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}}) \cap [\![\varphi]\!]))) \cap [\![\varphi]\!]\\
=\ & [\![B^{\langle !\varphi\rangle\theta}\langle !\varphi\rangle\psi]\!]^{\mathcal{M}} \cap [\![\varphi]\!]\\
=\ & [\![\varphi \wedge B^{\langle !\varphi\rangle\theta}\langle !\varphi\rangle\psi]\!]^{\mathcal{M}}
\end{aligned}$$

6. We will use the following lemma in the proof of this item.

LEMMA 9. *For any topo-model $\mathcal{M} = (X, \tau, \nu)$ and any $\varphi, \psi \in \mathcal{L}_{KCB}$,*

$$(\mathcal{M}_\varphi)_\psi = \mathcal{M}_{\langle !\varphi\rangle\psi}.$$

Proof. It suffices to show that the domains of $(\mathcal{M}_\varphi)_\psi$ and $\mathcal{M}_{\langle !\varphi\rangle\psi}$ are equivalent as the corresponding topologies and valuation functions are just the restriction of the initial model to the updated domains.

By definition of the restricted model,

$$(\mathcal{M}_\varphi)_\psi = ([\![\varphi]\!]^{\mathcal{M}} \cap [\![\psi]\!]^{\mathcal{M}_\varphi}, \tau_{[\![\varphi]\!]^{\mathcal{M}} \cap [\![\psi]\!]^{\mathcal{M}_\varphi}}, \nu_{[\![\varphi]\!]^{\mathcal{M}} \cap [\![\psi]\!]^{\mathcal{M}_\varphi}}).$$

Then we have,

$$[\![\varphi]\!]^{\mathcal{M}} \cap [\![\psi]\!]^{\mathcal{M}_\varphi} = [\![\varphi]\!]^{\mathcal{M}} \cap [\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}} = [\![\varphi \wedge \langle !\varphi\rangle\psi]\!]^{\mathcal{M}} = [\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}}.$$

Therefore, the domains of $(\mathcal{M}_\varphi)_\psi$ and $\mathcal{M}_{\langle !\varphi\rangle\psi}$ are equivalent, thus,

$$\begin{aligned}
(\mathcal{M}_\varphi)_\psi &= ([\![\varphi]\!]^{\mathcal{M}} \cap [\![\psi]\!]^{\mathcal{M}_\varphi}, \tau_{[\![\varphi]\!]^{\mathcal{M}} \cap [\![\psi]\!]^{\mathcal{M}_\varphi}}, \nu_{[\![\varphi]\!]^{\mathcal{M}} \cap [\![\psi]\!]^{\mathcal{M}_\varphi}})\\
&= ([\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}}, \tau_{[\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}}}, \nu_{[\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}}})\\
&= \mathcal{M}_{\langle !\varphi\rangle\psi}.
\end{aligned}$$

Then, the validity of (6) follows:

$$\begin{aligned}
[\![\langle !\varphi\rangle\langle !\psi\rangle\chi]\!]^{\mathcal{M}} &= [\![\langle !\psi\rangle\chi]\!]^{\mathcal{M}_\varphi}\\
&= [\![\chi]\!]^{(\mathcal{M}_\varphi)_\psi}\\
&= [\![\chi]\!]^{\mathcal{M}_{\langle !\varphi\rangle\psi}} \quad \text{by Lemma 9}\\
&= [\![\langle !\langle !\varphi\rangle\psi\rangle\chi]\!]^{\mathcal{M}}.
\end{aligned}$$