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MINIMAL DEONTIC LOGICS
(abstract)

§1. Variations on the minimal modal logic

In propositional modal logic (cf.[5]) the language of propositional logic (on a set PL of proposition letters) is enriched by adding modal operators $\Box$ ("necessarily") and $\Diamond$ ("possibly"). This language is interpreted in Kripke models $M=\langle W,R,V\rangle$, where $W\neq\emptyset$ is the set of "worlds", $R\subseteq W\times W$ is the "alternative relation" and $V:PL\to\{X\mid X\subseteq W\}$ is a "valuation". The key clauses in the truth definition are as follows:

$M\models\Box\varphi[w]$ iff $M\models\varphi[v]$ for all $v\in W$ such that Rwv

$M\models\Diamond\varphi[w]$ iff $M\models\varphi[v]$ for some $v\in W$ such that Rwv.

These clauses allow for a transcription of modal formulas into first-order ones; e.g., $\Diamond p\to\Box q$ would become $\exists y(Rxy\wedge Py)\to\forall z(Rxz\to Qz)$. In view of the known recursive axiomatizability of universal validity in first-order logic, that of modal propositional logic follows at once. A natural purely modal complete system is the minimal modal logic K consisting of a complete propositional calculus (with the rule of Modus Ponens) with the modal superstructure (1) $\Diamond\varphi=\neg\Box\neg\varphi$ (definition), (2) $\Box(\varphi\to\psi)\to(\Box\varphi\to\Box\psi)$ (axiom) and (3) "from φ to $\Box\varphi$" (rule of inference).

In propositional tense logic (cf.[7]) tense operators G ("always from now on") and H ("always up to now") are added to the propositional language, as well as defined operators F ($=\neg G\neg$) and P ($=\neg H\neg$). The interpretation is similar, W now standing for a set of moments and R for the relation of temporal precedence. The important clauses are

$M \models G\varphi[w]$ iff $M \models \varphi[v]$ for all $v \in W$ such that Rwv

$M \models H\varphi[w]$ iff $M \models \varphi[v]$ for all $v \in W$ such that Rvw.

Again, an obvious first-order transcription exists. The <u>minimal tense logic</u> K_t consists of K-for-G and K-for-H, together with the "mixing principles" $\varphi \to HF\varphi$ and $\varphi \to GP\varphi$. (When G and H are interpreted in models $\langle W, R_G, R_H, V\rangle$, both according to the clause for $\Box$, then validity of the mixing principles ensures that R_H is the converse relation of R_G.)

In <u>propositional dynamic logic</u> (cf.[6]), with an added operator $\Box^*$, the above transcription does not work any more, witness the following clause:

$M \models \Box^*\varphi[w]$ iff $M \models \varphi[v]$ for all $v \in W$ such that $\langle w, v\rangle$ belongs to the transitive closure of R. (W now formalizes a set of computer states, R the set of possible transitions under some non-deterministic program.) Still, there is a <u>minimal dynamic logic</u> K_a consisting of K and K-for-$\Box^*$, together with the mixing principles $\Box^*\varphi \to \Box\varphi$, $\Box^*\varphi \to \Box^*\Box^*\varphi$ and $(\Box\varphi \wedge \Box^*(\varphi \to \Box\varphi)) \to \Box^*\varphi$. (When $\Box$ and $\Box^*$ are interpreted in models $\langle W, R, R^*, V\rangle$, both according to the clause for $\Box$, then validity of these principles ensures that R^* be the transitive closure of R.)

Finally, in <u>propositional deontic logic</u> (cf.[4] and [2]), one adds operators O("obligatory") and P("permitted") to the propositional base. W now stands for possible states of the actual world, and R relates such states w to "deontically perfect" alternatives (from w's point of view). The key clause for obligation symbolizes that this notion amounts to truth in all perfect alternatives:

$M \models O\varphi\,[w]$ iff $M \models \varphi[v]$ for all $v \in W$ such that Rwv.

But what about permission? The usual procedure is to define $P\varphi$ as $\neg O \neg \varphi$, thus creating a "weak permission", satisfying laws like $P(q \vee r) \leftrightarrow (Pq \vee Pr)$. According to another intuition, however, permission should rather satisfy the law $P(q \vee r) \leftrightarrow Pq \wedge Pr$. ("If I may skate or ski, I may skate <u>and</u> I may ski!") These two concepts of permission cannot be reconciled; for, then, anything becomes permitted: for any q, $P(q \vee \neg q)$ (which is plausible for weak permission) implies Pq (a plausible transition for strong permission). This is the so-called "Ross Paradox". Henceforward, our concern will be with strong permission. A formal analogy with the tautology $((q \vee r) \rightarrow s) \leftrightarrow ((q \rightarrow s) \wedge (r \rightarrow s))$ inspired the following semantic clause:

$M \models P\varphi\,[w]$ iff Rwv for all $v \in W$ such that $M \models \varphi\,[v]$.

(φ is permitted whenever it is "totally safe".) In this case too, there is an obvious first-order transcription; but we will look for a pure <u>minimal deontic logic</u>.

§2. <u>Permissions only</u>

Using the Henkin method of [5] as a heuristic device, one obtains the following completeness result for the propositional language plus strong permission.

THEOREM 1. Universal validity is axiomatized by K_d.

Here K_d consists of a complete propositional base like the one for K, together with the axioms

$P\bot$ ($\bot$ is the "Falsum")

$(P\varphi \wedge P\psi) \rightarrow P(\varphi \vee \psi)$

and the rule of inference

from $\varphi \rightarrow \psi$ to $P\psi \rightarrow P\varphi$.

Compare the analogous formulation of K in terms of $\Box\top$ ($\top$ is the "Verum"), $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$ and "from $\varphi \rightarrow \psi$ to $\Box\varphi \rightarrow \Box\psi$".

§3. Permissions and obligations

In the full language with both P and O- or, equivalently, $\Box$-, the situation becomes more complicated. There is a marked increase in expressive power (as compared to plain modal logic; cf. [1]), witness the definability of $\forall x \forall y Rxy$ (by $\Diamond p \vee Pp$). This subject remains to be studied. The Henkin proof has no immediate application here, since one is confronted with the necessity of omitting certain types. Instead, Beth's method of semantic tableaus may be used to find the following result.

THEOREM 2. Universal validity is axiomatized by K_D.

Here K_D consists of K, K_d and the mixing principles

$$\neg((\neg P, \Diamond)_1 (Pq \wedge \Box r) \wedge (\neg P, \Diamond)_2 (q \wedge \neg r)),$$

where $(\neg P, \Diamond)_{1,2}$ stand for arbitrary finite sequences of $\neg P$ and $\Diamond$. For example, $\neg(\Diamond(Pq \wedge \Box\neg q) \wedge (q \wedge \neg\neg q))$, i.e., $\Diamond(Pq \wedge \Box\neg q) \rightarrow \neg q$, is a valid mixing principle.

§4. Permissions, obligations and modalities

There are ancient connections between the concepts of logical and ethical necessity. Cf. Leibniz' "mondes possibles" and- among these- his "le meilleur des mondes possibles"; which is reflected in the current distinction between "ordinary" and "perfect" worlds. Modal reductions have been proposed for obligation and permission by several deontic logicians; e.g., $O\varphi = \Box(I \to \varphi)$ and $P\varphi = \Box(\varphi \to I)$, where I is the "Good". This may be mirrored in models $\langle W, R, I, V \rangle$, where I is a subset of W (the "good" worlds). We prefer a slightly different approach, however, using a relation R_1 of modal alternativity and a relation R_2 of deontic alternativity. All deontic ("allowed") alternatives are modally possible, though the converse need not hold. Formally, then, models will be 4-tuples $M = \langle W, R_1, R_2, V \rangle$ such that $R_2 \subseteq R_1$. The key clauses become

$M \models \Box\varphi[w]$ iff $M \models \varphi[v]$ for all $v \in W$ such that R_1wv

$M \models O\varphi[w]$ iff $M \models \varphi[v]$ for all $v \in W$ such that R_2wv

(!) $M \models P\varphi[w]$ iff R_2wv for all $v \in W$ such that R_1wv and $M \models \varphi[v]$.

These modal additions lead to a streamlined complete logic:

THEOREM 3. Universal validity is axiomatized by $K_{D,m}$.

Here $K_{D,m}$ consists of K, K-for-O and K_d, together with the three mixing principles

$\Box\varphi \to O\varphi$, $\Box\neg\varphi \to P\varphi$, and $(O\varphi \wedge P\psi) \to \Box(\psi \to \varphi)$.

§5. Permissions, obligations, modalities and tenses

The addition of tenses to modal logic has been effected by A.N.Prior (cf.[7]). For the case of deontic logic, this was recommended by W.Stegmüller (cf.[8]). A combination of the three is to be found in as yet unpublished work of J.A.van Eck in Groningen. Semantically, it now becomes an issue if worlds and times should be taken separately or be connected in some fashion. An elegant modelling in the latter spirit is provided by 4-tuples $M=\langle T,<,W,Q\rangle$, where T("Time") is a set of moments ordered by the precedence relation $<$, W("worlds") is a set of valuations from PL into $\{X|X\subseteq T\}$,or, alternatively, from T into $\{X|X\subseteq PL\}$, and- finally- Q is a "deontic selection function" in a sense to be explained shortly. A time-dependent alternative relation R_t is defined by $R_t w_1 w_2$ iff $w_1\restriction\{u\in T|u\leq t\}= w_2\restriction\{u\in T|u\leq t\}$ (i.e., up to and including t, w_1 and w_2 have run the same course). Now, for any world $w\in W$ and any moment $t\in T$, $Q(w,t)$ is the set of deontically perfect continuations (in the sense of R_t) of w. The main clauses in the truth definition then become:

$M\models\Box\varphi[w,t]$ iff $M\models\varphi[v,t]$ for all $v\in W$ such that $R_t wv$

$M\models G\varphi[w,t]$ iff $M\models\varphi[w,u]$ for all $u\in T$ such that $t<u$

$M\models H\varphi[w,t]$ iff $M\models\varphi[w,u]$ for all $u\in T$ such that $u<t$

$M\models O\varphi[w,t]$ iff $M\models\varphi[v,t]$ for all $v\in Q(w,t)$

$M\models P\varphi[w,t]$ iff $v\in Q(w,t)$ for all $v\in W$ such that $M\models\varphi[v,t]$ and $R_t wv$.

QUESTION. Find a recursive axiomatization for universal validity.

According to D.Lewis and B.C. van Fraassen (cf.[3]), even more semantic machinery will be needed for an adequate treatment of "counterfactual conditionals" and (deontic) "secondary obligations"; viz. a "comparability" relation between worlds. This matter has not been investigated here.

REFERENCES

[1] Benthem, J.F.A.K. van: Modal correspondence theory, dissertation, Mathematisch instituut, Universiteit van Amsterdam, 1976.

[2] Conte, A.G., Hilpinen, R. and Wright, G.H. von (eds): Deontische Logik und Semantik, Athenaion, Wiesbaden, 1977.

[3] Fraassen, B.C. van: The logic of conditional obligation, Journal of Philosophical Logic 1 (1972), pp. 417-438.

[4] Hilpinen, R. (ed): Deontic logic: introductory and systematic readings, Reidel, Dordrecht, 1971.

[5] Lemmon, E.J. and Scott, D.: An introduction to modal logic, Blackwell, Oxford, 1977, edited by K. Segerberg.

[6] Parikh, R.: The completeness of propositional dynamic logic, in Winkowski, J. (ed): Mathematical foundations of computer science 1978, Springer, Berlin, 1978, Lecture Notes in Computer Science 64, pp. 402-414.

[7] Prior, A.N.: Past, present and future, Oxford University Press, Oxford, 1967.

[8] Stegmüller, W.: Hauptströmungen der Gegenwartsphilosophie II, Kröner, Stuttgart, 1975.

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