Polyhedral semantics of modal logic

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

Polyhedral semantics is a way of interpreting modal formulas on polyhedra. This semantics has recently been introduced by N. Bezhanishvili, D. Gabelaia, S. Adam-Day, V. Marra and others. We introduce a new variation on this semantics, and derive metamathematical properties of both semantics.

We show that the polyhedral logic of a piecewise linear manifold-with-boundary is determined by its dimension, and that there are $2^{\aleph_0}$ polyhedrally-complete logics. We prove that p-morphisms between simplicial complexes factor through subdivisions. Using this, we show that the problem of comparing the logics of two polyhedra reduces to the problem of checking the validity of a formula on a polyhedron. We establish decidability of these problems for polyhedra embeddable in $\mathbb{R}^3$. Moreover, we demonstrate the difficulty of checking the validity of a formula on a polyhedron in $\mathbb{R}^4$, and leave the decidability of this as an open problem.

Contents

Chapter 1

Introduction

An often observed property of modal logic is that it has many semantics: there are many different ways of interpreting modal formulas. Perhaps the most famous one is the Kripke semantics using possible worlds ([CZ97, part II]). Some other main trends have used algebraic models, temporal models or spatial models. In fact, the aptitude of topological models for interpreting modal formulas was already discovered around 1940 (see [Sto38], [Tsa38], [Tar39], [McK41] and the influential [MT44]). Topology can be considered a form of geometry, with the following properties:

- There are many different topological spaces. There are topological spaces over universes of any cardinality, and there are $2^{\aleph_0}$ pairwise non-homeomorphic topological spaces with countable universes.

- There are many homeomorphisms between topological spaces. For example, every open interval in $\mathbb{R}$ is homeomorphic to $\mathbb{R}$.

Hence, topology is a rather *abstract* type of geometry. More recently, logicians became interested in exploring the connections between modal logic and more *concrete* types of geometry. For example, [BG02] studied a spatial semantics for modal logics using affine and projective geometry. Also, [BBG03] specifically considered finite unions of open, half-open and closed hyperrectangles for interpreting modal formulas. In this semantics, the formula **grz** is valid. For more elaborate surveys of different logical studies of space, we refer the reader to the *Handbook of spatial logics*, contributions [BB07] and [Bal+07].

In this thesis, we study a spatial semantics for basic modal logic, called *polyhedral semantics*, that can be thought of as a marriage of Kripke semantics and piecewise linear geometry (PL-geometry). Some important publications in this area so far were [Bez+18], [Gab+19] and [Ada19]. The concepts that were developed in the latter are the main source of inspiration for this thesis. Triangles are typical figures of PL-geometry in the plane. Similarly, d-simplices are typical figures of the PL-geometry of $\mathbb{R}^d$. *Simplicial complexes* are particularly nice sets of simplices. From a simplicial complex one can distil a finite poset. A poset can be considered a Kripke frame, and this builds the bridge from PL-geometry to modal logic. Crucially, we shall work with infinite sets of simplicial complexes. More precisely, a simplicial complex triangulates a figure in space called a *polyhedron*. For a given polyhedron, one can consider the set of all simplicial complexes that triangulate it. This set is typically infinite, and hence produces an infinite set of posets, which determines the modal logic of the polyhedron in question. This infinite flavour creates challenges in polyhedral semantics.

Similarly to polyhedral semantics, [LQ21] and [Bez+22] also worked with the poset structure of simplicial complexes, but did not take the perspective of polyhedra and infinite sets of simplicial complexes. In their case, challenges arose from the inclusion of more modalities into the language. The same holds for [Sin22], where simplicial complexes were used as models with distinguished sets of vertices (agent nodes). For us, however, there is more than enough work to do for the basic modal language only.

An advantage of concrete geometrical models is that they can be input into a computer, especially if they live in three-or-lower-dimensional space. For simplicial complexes, this was done by [Bez+22], and they programmed an efficient model checker for modal formulas with the standard unary modality and the binary reachability modality. Since many real life objects can be approximated by a concrete simplicial complex, this allows a computer model to check for certain properties of objects. In the context of polyhedral semantics, it is generally more challenging to find such algorithms. For, one does not consider models based on some fixed simplicial complex, but models based on *any* triangulation of a given polyhedron. Even if we fix a polyhedron $\mathbf{P}$, it is an interesting question whether there exists *any* algorithm that decides whether or not a given formula is valid on $\mathbf{P}$. This seems to be the case for many natural classes of polyhedra (cf. Theorem 3.20 and Corollary 7.17), but there are some polyhedra for which the answer is not apparent. (See Notation 7.5 and Open problems 7.19 and 7.20 below.)

The formula **grz** is valid in polyhedral semantics, so the Blok-Esakia theorem implies that polyhedral semantics can also be used for intuitionistic propositional formulas (see section 2.4 below). This perspective was taken in [Bez+18], facilitating algebraic motivation for polyhedral semantics. More precisely, if $\mathbf{P}$ is a polyhedron, the set of all open subpolyhedra of $\mathbf{P}$ is a Heyting algebra. The logic of $\mathbf{P}$ is the logic of this Heyting algebra. This duality also descends to the level of complexes. If Σ is a simplicial complex, then the set of all Σ-definable open polyhedra is also a Heyting algebra, whose Esakia dual is isomorphic to Σ as a poset.

The basic question driving this research is to understand which modal logics can be obtained from polyhedra and which logics can not. Furthermore, we would like to axiomatize the logics of some particularly important classes of polyhedra. In [Ada19] it was shown that some of these questions can be answered by appealing to poset combinatorics. However, most of the tools used in this thesis are of geometrical nature. As mentioned above, we shall also touch upon various interesting computational questions. It is likely that future research in polyhedral semantics will also make use of tools from complexity theory.

We shall not only study polyhedral semantics in its known form, but also a novel variation on it, that automatically arises once the empty set is accepted as a simplex. A logic arising from this new semantics is called *quasi-polyhedrally-complete*. We list the main new results of this thesis.

A. All d-dimensional polyhedra that are manifolds-with-boundary yield the same polyhedrally-complete logic (Theorem 3.20). This strengthens a theorem by [Ada19].

B. Every polyhedrally-complete logic is quasi-polyhedrally-complete (Corollary 5.34). This is important because it implies that traditional polyhedral semantics falls under the scope of some new techniques developed in this thesis.

C. Every polyhedrally-complete logic is modelled by pyramids (Theorem 5.38). A *pyramid* with base $\mathbf{P}$, where $\mathbf{P}$ is a polyhedron, is obtained by adding an independent point $\mathbf{x}$ and drawing the lines between $\mathbf{x}$ and all points in $\mathbf{P}$.

D. There are $2^{\aleph_0}$ polyhedrally-complete logics (Theorem 6.32). Moreover, this many examples of polyhedrally-complete logics can be obtained from two-dimensional polyhedra living in the three-dimensional ambient space. Before, only countably many polyhedrally-complete logics were known to exist (cf. [Ada19], [Ada+21]).

E. P-morphisms between simplicial complexes factor through subdivisions (Theorem 4.1).

F. In the presence of an oracle for checking the validity of any formula on any polyhedron, we can decide whether the logics of two given polyhedra are equal (Proposition 7.7-1). This will follow from E.

G. Validity checking of a formula on a polyhedron living in the three-dimensional ambient space, is decidable (Corollary 7.17). This means that real-life applications of polyhedral semantics will not have to worry about any issues regarding decidability.

H. Validity checking of a formula on a three-dimensional polyhedron is nontrivial in general (Theorem 7.23). To make precise what we mean by this, given a polyhedron $\mathbf{P}$ and a triangulation Σ of $\mathbf{P}$, we shall introduce *barycentric subdivisions* as a mechanical process for calculating a sequence $\Sigma, \Sigma^+, \Sigma^{++}, \ldots$ of finer and finer triangulations of $\mathbf{P}$. We show that there is a formula φ such that for some infinite list of three-dimensional polyhedra, increasingly many barycentric subdivisions are needed for establishing the invalidity of φ. This answers a question raised by [Ada19, p. 82].

Structure of the thesis

In chapter 2 we develop the geometry that is necessary for defining polyhedral semantics and for understanding the other chapters. Chapter 3 calculates the logic of a manifold-with-boundary. Chapter 4 presents a useful technical result that emphasizes the importance of p-morphisms in the context of polyhedral semantics. Chapter 5 proves general results about the connection between the two forms of polyhedral semantics. Chapter 6 zooms in on one-dimensional[1] polyhedra in order to prove results about the number of polyhedrally-complete logics. Finally, chapter 7 considers computability and zooms in on two-dimensional[2] polyhedra.

[1] Traditional polyhedral semantics of two-dimensional polyhedra corresponds to new polyhedral semantics of one-dimensional polyhedra.

[2] viz. three-dimensional from the viewpoint of traditional polyhedral semantics

Chapter 2

Preliminaries

In this chapter we review the basic mathematical tools on which the thesis builds. The central goal of this chapter is to define polyhedral semantics. In fact, we shall define two different forms of polyhedral semantics. The relationship between those two semantics and some of their properties are then probed in later chapters.

One of the two polyhedral semantics was already defined and studied in [Bez+18], [Ada19] and [Ada+21]. The other semantics is a slight variation on this line of work. Except for the new polyhedral semantics, all material in this chapter consists of known ideas and simple observations.

2.1 Notation

We start by fixing some notation.

Notation **2.1**. If A and B are disjoint sets, we write $A \cup B = A \sqcup B$. Similarly, when $\mathscr{F}$ is a family of pairwise disjoint sets, we write $\bigsqcup \mathscr{F} = \bigcup \mathscr{F}$, and if $\{A_i : i \in I\}$ is an indexed family of sets such that $A_i \cap A_j = \varnothing$ for all distinct $i, j \in I$, we write

$$\bigcup_{i \in I} A_i = \bigsqcup_{i \in I} A_i.$$

Notation **2.2**. If f is a map, then $\operatorname{dom} f$ is the domain of f and $\operatorname{Im} f$ is the image of f. For $\mathscr{A} \subseteq \mathscr{P} \operatorname{dom} f$, we write

$$f[\![\mathscr{A}]\!] = \{f[A] : A \in \mathscr{A}\}.$$

Notation **2.3**. If A is a set, then $\#A$ is the cardinality of A.

Notation **2.4**. If A is a set, $\odot \in \{=, <, \leq, >, \geq\}$ and κ is a cardinal number, we set

$$[A]^{\odot \kappa} = \{B \subseteq A : \#B \odot \kappa\}.$$

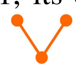
2.2 Posets

Posets (see Definition 2.5 below) can be viewed as special types of Kripke frames, and thus lend themselves for interpreting formulas of the basic modal language. Polyhedral semantics (to be introduced in Definition 2.69 below) can be considered as a special case of this semantics. Therefore, it is important to equip ourselves with some terminology for posets.

Definition **2.5** (**poset**). A *poset* is a pair $(P, \leq)$ where P is a set and $\leq$ is a partial order on P. We often write P instead of $(P, \leq)$ when the order $\leq$ is clear from the context. An *upset* in P is a subset $U \subseteq P$ such that $u \in U$ and $u \leq p \in P$ implies $p \in U$. The notion of a *downset* is defined similarly. If $p \leq q$, we say that q is a *successor* of p and p is a *predecessor* of q. If $p < q$, we say that q is a *proper successor* of p and p is a *proper predecessor* of q. In this case, q is an *immediate successor* of p and p an *immediate predecessor* of q if there exists no $r \in P$ such that $p < r < q$. If p has no proper successors, it is called a *maximal* point, and if it has no proper predecessors it is called a *minimal* point.

Notation **2.6**. If P is a poset and $X \subseteq P$, we denote the smallest upset $U \subseteq P$ for which $X \subseteq U$ by $\uparrow^P(X)$. We often omit the superscript when it is clear from the context. If X is a singleton $\{x\}$, we usually write $\uparrow(X) = \uparrow(x)$ (by abuse of notation). In this case, we furthermore set $\Uparrow(x) = \uparrow(x) \setminus \{x\}$. The meaning of the symbols $\downarrow$ and $\Downarrow$ is defined similarly.

Definition **2.7** (**height and depth**). If P is a finite poset and $p \in P$, the *height* $\mathrm{hgt}^P(p)$ of p in P is the largest cardinality of a chain in $\Downarrow(p)$. Dually, the *depth* $\mathrm{dpt}^P(p)$ of p in P is the largest cardinality of a chain in $\Uparrow(p)$. Again superscripts may be dropped if they are clear from the context. The *height* or *depth* of P, $\mathrm{hgt}(P) = \mathrm{dpt}(P)$, is the largest cardinality of a chain in P.

Hence, if P is a finite poset of height $d+1$, its elements have heights (and depths) ranging from 0 to d inclusive. For example, a two-fork  has a root of height 0 and depth 1, and two leaves, each of height 1 and depth 0. It holds

$$\mathrm{hgt}\Big(\quad \Big) = 2.$$


Notation **2.8**. If the posets P and Q are isomorphic we write $P \cong Q$.

Definition **2.9** (**connected poset**). A *path* through a poset is a finite sequence $(x_0, \dots, x_k)$ of elements of the poset such that for each $i < k$ we have $x_i \leq x_{i+1}$ or $x_i \geq x_{i+1}$. A poset P is *connected* if for all $p, q \in P$ there exists a path through P from p to q. If P is finite, the largest connected subposets of P are called the *components* of P.

2.3 Polyhedral geometry

This section supplies basic geometric tools. Some of these tools are required to set up the basic definitions around the polyhedral semantics; others are needed only in later chapters to prove results about the polyhedral semantics.

2.3.1 Affine geometry

In order to explain polyhedral geometry, we first have to cover the basics of affine geometry.

Notation **2.10**. We denote points in a Euclidean space by lowercase boldface letters, and label the coordinates starting from zero:

$$\mathbf{x} = (x_0, \dots, x_{d-1}) \in \mathbb{R}^d.$$

We write the zero vector as $\mathbf{0}^d = (0, \dots, 0) \in \mathbb{R}^d$, or simply $\mathbf{0}$ if the dimension is irrelevant or clear from the context.

Notation **2.11**. The standard basis for $\mathbb{R}^d$ is written $\{\mathbf{e}^{0,d},\dots,\mathbf{e}^{d-1,d}\}$. More precisely, the point $\mathbf{e}^{i,d} \in \mathbb{R}^d$ is defined by

$$e^{i,d}_j = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j). \end{cases}$$

Definition **2.12** (**combinations**). Let $X \subseteq \mathbb{R}^d$. An *affine combination* of X is a point of the form

$$\mathbf{y} = \sum_{\mathbf{x}\in X} \alpha(\mathbf{x}) \cdot \mathbf{x},$$

where $\alpha : X \to \mathbb{R}$ is a map with finite support that satisfies

$$\sum_{\mathbf{x}\in X} \alpha(\mathbf{x}) = 1.$$

If α moreover satisfies $\operatorname{Im}\alpha \subseteq [0,1]$, then the point $\mathbf{y}$ is called a *convex combination* of X. In this case, we call the numbers $\alpha(\mathbf{x})$ the *barycentric coordinates* of the convex combination. The set X is called affinely *dependent* if there exists a point $\mathbf{x} \in X$ that is an affine combination of $X \setminus \{\mathbf{x}\}$. Further, X is called affinely *independent* if it is not affinely dependent. The affine (resp. convex) *hull* of X is the set of all affine (resp. convex) combinations of X. The convex hull of X is denoted $\operatorname{Conv}(X)$. Intuitively, the operator Conv "fills up" the space between the points in X. See Figure 2.1.

Definition **2.13** (**convex set**). A set $X \subseteq \mathbb{R}^d$ is *convex* if $\operatorname{Conv}(X) = X$. Equivalently, X is convex iff there exists $Y \subseteq \mathbb{R}^d$ such that $\operatorname{Conv}(Y) = X$.

The following is well known:

Lemma **2.14**. An arbitrary intersection of convex sets is convex.

As most forms of geometry, affine geometry also has its own type of transformations:

Definition **2.15** (**affine map**). A map $\phi : \mathbb{R}^d \supseteq X \to \mathbb{R}^{d'}$ is an *affine map* if there exist a matrix $M \in \mathbb{R}^{d' \times d}$ and a point $\mathbf{y} \in \mathbb{R}^{d'}$ such that

$$\forall \mathbf{x} \in X : f(\mathbf{x}) = M\mathbf{x} + \mathbf{y}$$

(where we identify points in Euclidean space with column vectors).

Example **2.16**. If $i < d$, let

$$\pi^d_i : \mathbb{R}^d \to \mathbb{R} : \\ \mathbf{x} \mapsto x_i$$

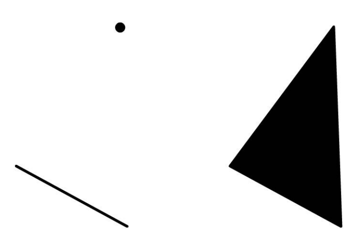


Figure 2.1: a subset of $\mathbb{R}^2$ (left) and its convex hull (right)

and

$$\begin{aligned}\iota_i^d : \mathbb{R}^i &\to \mathbb{R}^d : \\ \mathbf{y} &\mapsto (\mathbf{y}, \mathbf{0}^{d-i}).\end{aligned}$$

These are affine maps.

The following lemma is elementary to check. It is the basic principle underlying various logic-preserving maps that we shall construct later on.

Lemma **2.17**. If $X \subseteq \mathbb{R}^d$ and $\phi : X \to \mathbb{R}^{d'}$ is an injective affine map, then X is affinely indepenent iff $\phi[X]$ is affinely independent.

2.3.2 Simplices

We are now ready to introduce the basic geometric building blocks for polyhedral semantics.

Definition **2.18** (**simplex**). A *simplex* is a set σ such that there exists an affinely independent set X with $\sigma = \mathrm{Conv}(X)$. We denote simplices by lowercase Greek letters.

In the definition of a simplex, it is useful to know that the set X is unique. This is proved in the next lemma.

Lemma **2.19**. If $X, Y \subseteq \mathbb{R}^d$ are affinely independent sets and $\mathrm{Conv}(X) = \mathrm{Conv}(Y)$, then $X = Y$.

Proof. Suppose that $\mathbf{x} \in X \setminus Y$. Then $\mathbf{x}$ is a convex combination of Y, as witnessed by some map $\alpha : Y \to [0,1]$. The support $S = \alpha^{-1}\big[(0,1]\big]$ of α must have cardinality at least 2. Hence pick distinct $\mathbf{s}, \mathbf{t} \in S$. For each $\mathbf{y} \in Y$ there is a map $\alpha_{\mathbf{y}} : X \to [0,1]$ expressing $\mathbf{y}$ as a convex combination of X. Then

$$\begin{aligned}\mathbf{x} &= \sum_{\mathbf{y}\in Y} \alpha(\mathbf{y}) \cdot \mathbf{y} \\ &= \sum_{\mathbf{y}\in Y} \sum_{\mathbf{z}\in X} \alpha(\mathbf{y})\alpha_{\mathbf{y}}(\mathbf{z}) \cdot \mathbf{z} \\ &= \sum_{\mathbf{z}\in X} \sum_{\mathbf{y}\in Y} \alpha(\mathbf{y})\alpha_{\mathbf{y}}(\mathbf{z}) \cdot \mathbf{z}.\end{aligned}$$

Since $\mathbf{x}$ is not an affine combination of $X \setminus \{\mathbf{x}\}$, it follows that

$$\alpha(\mathbf{y})\alpha_{\mathbf{y}}(\mathbf{z}) = 0$$

for all $\mathbf{z} \in X \setminus \{\mathbf{x}\}$ and $\mathbf{y} \in Y$. In particular,

$$\alpha_{\mathbf{s}}(\mathbf{z}) = 0 = \alpha_{\mathbf{t}}(\mathbf{z})$$

for all $\mathbf{z} \in X \setminus \{\mathbf{x}\}$. But this implies $\mathbf{s} = \mathbf{x} = \mathbf{t}$, a contradiction. □

Definition **2.20** (**properties of simplices**). Let $\sigma \subseteq \mathbb{R}^d$ be a simplex. Then the set $\mathrm{vtc}(\sigma)$ of *vertices* of σ is the unique (in view of Lemma 2.19) affinely independent set $X \subseteq \mathbb{R}^d$ for which $\sigma = \mathrm{Conv}(X)$. The *dimension* of σ is

$$\dim \sigma = \#\big(\mathrm{vtc}(\sigma)\big) - 1.$$

(In particular, $\dim\varnothing = -1$.) Figure 2.2 shows simplices of dimensions zero (a point), one (a line segment), two (a triangle) and three (a tetrahedron). The *barycentre* of $\sigma \neq \varnothing$ is

$$\mathbf{b}(\sigma) = \sum_{\mathbf{x}\in\mathrm{vtc}(\sigma)} \frac{1}{\#\,\mathrm{vtc}(\sigma)} \cdot \mathbf{x}.$$

The empty simplex has no barycentre. A simplex θ is said to be a *face* of σ if $\mathrm{vtc}(\theta) \subseteq \mathrm{vtc}(\sigma)$. The set of all faces of σ is $\mathrm{fac}(\sigma)$. If in addition $\theta \neq \sigma$, then θ is a *proper* face of σ. The set of all proper faces of σ if $\mathrm{propfac}(\sigma)$. The union of all proper faces of σ is denoted by $\partial\sigma$ and is called the *boundary* of σ. The *relative interior* of σ is $\mathrm{relInt}\,\sigma = \sigma \setminus \partial\sigma$. See Figure 2.3. A 0-dimensional simplex σ has only one point $\mathbf{x}$, and in this case we often write $\mathbf{x}$ instead of σ.

Remark **2.21**. The definition of dimension of a simplex given in Definition 2.20 agrees with more general notions of dimension. For details on dimension theory we refer the reader to [HW41].

Remark **2.22**. If $\sigma \subseteq \mathbb{R}^d$ is a d-dimensional simplex, its relative interior is equal to its interior in the topological space $\mathbb{R}^d$. However, if the dimension of the simplex $\sigma \subseteq \mathbb{R}^d$ is less than d, its interior in this latter sense is empty.

To be able to work with properties of simplices, it is an instructive exercise to establish the following equation:

Lemma **2.23**. Let σ be a nonempty simplex. Then

$$\mathrm{relInt}\,\sigma = \Big\{ \sum_{\mathbf{x}\in\mathrm{vtc}(\sigma)} \alpha(\mathbf{x})\cdot\mathbf{x} \,\Big|\, \alpha : \mathrm{vtc}(\sigma) \to (0,1]\ \&\ \sum_{\mathbf{x}\in\mathrm{vtc}(\sigma)} \alpha(\mathbf{x}) = 1 \Big\}.$$

Proposition **2.24**. Let σ be a nonempty simplex. Then

$$\mathbf{b}(\sigma) \in \mathrm{relInt}\,\sigma.$$

2.3.3 Simplicial complexes

Next, we use various simplices together to compose *simplicial complexes*: structures that we can later use to interpret logical formulas.

Definition **2.25** (**simplicial complex**). A *simplicial complex* is a nonempty finite set $\Sigma \subseteq \mathscr{P}\mathbb{R}^d$ of simplices (called *cells* of Σ) such that the following two conditions hold:

- Σ is closed under taking faces: for each $\sigma \in \Sigma$ and for each face δ of σ we have $\delta \in \Sigma$;
- intersections in Σ behave well: for all $\sigma, \delta \in \Sigma$ we have that $\sigma \cap \delta$ is a face of σ.[1]

[1] Equivalently: $\sigma \cap \delta$ is a face of both σ and δ.

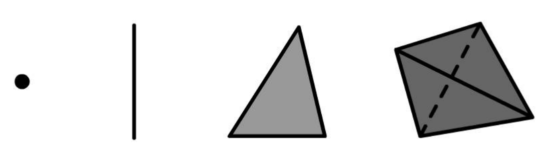


Figure 2.2: simplices of dimensions 0, 1, 2 and 3

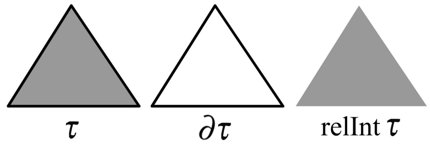


Figure 2.3: a triangle, its boundary and its relative interior

We denote simplicial complexes by uppercase Greek letters. The set of all simplicial complexes is **cmplx**. (This is a set because

$$\bigcup_{d=0}^{\infty} \mathscr{P}\mathscr{P}\mathbb{R}^d$$

is a set.)

Example **2.26**. If σ is any simplex, then the set $\operatorname{fac}(\sigma)$ of all faces of σ is the smallest simplicial complex of which σ is a cell.

Definition **2.27** (**properties of simplicial complexes**). We also view a simplicial complex as a poset whose order is the inclusion of simplices. Note that this poset is rooted since we consider $\varnothing$ as a simplex. If Σ is a simplicial complex, we write

$$\operatorname{vtc}(\Sigma) = \bigcup_{\sigma\in\Sigma} \operatorname{vtc}(\sigma)$$

for the set of *vertices* of Σ. The union $\bigcup\Sigma$ is commonly called the *carrier* of Σ and is denoted by $|\Sigma|$. A complex is said to *triangulate* its carrier. The *dimension* $\dim\Sigma$ of Σ is the largest dimension of a cell in Σ. Hence $\dim\Sigma + 2 = \operatorname{hgt}\Sigma$. Let

$$\mathbf{cmplx}_d = \{\Sigma \in \mathbf{cmplx} : \dim\Sigma \leq d\},$$

$$\mathbf{cmplx}^{d'} = \{\Sigma \in \mathbf{cmplx} : \Sigma \subseteq \mathscr{P}\mathbb{R}^{d'}\}$$

and

$$\mathbf{cmplx}_d^{d'} = \mathbf{cmplx}_d \cap \mathbf{cmplx}^{d'}.$$

Notice that, in $\mathbb{R}^d$, there are simplices and hence complexes of any dimension up to and including d. Hence

$$\mathbf{cmplx}_d \neq \mathbf{cmplx}_d^d = \mathbf{cmplx}^d.$$

We assume that $\mathbb{R}^{-1} = \varnothing$, so that this is also true for $d = -1$.

If complexes Σ and Δ satisfy $\Sigma \subseteq \Delta$, then Σ is said to be a *subcomplex* of Δ. If complexes Σ and Δ satisfy $|\Sigma| = |\Delta|$ and

$$\forall\sigma \in \Sigma : \exists\delta \in \Delta : \sigma \subseteq \delta,$$

then Σ is said to be a *subdivision* of Δ. Intuitively, this means that Σ is obtained from Δ by splitting every cell of Δ into a number of smaller cells in such a way that Σ is again a simplicial complex.

If Σ is a subdivision of Δ and Σ is also a subcomplex of Δ (or vice versa), then $\Sigma = \Delta$ (this follows from Corollary 2.33 below). An illustrative example of a simplicial complex as a poset can be found in Figure 2.4. An example of subcomplex and subdivision can be found in Figure 2.5. Here, notice that (d) is not a subdivision of (a) since (d) is not a simplicial complex because in (d)

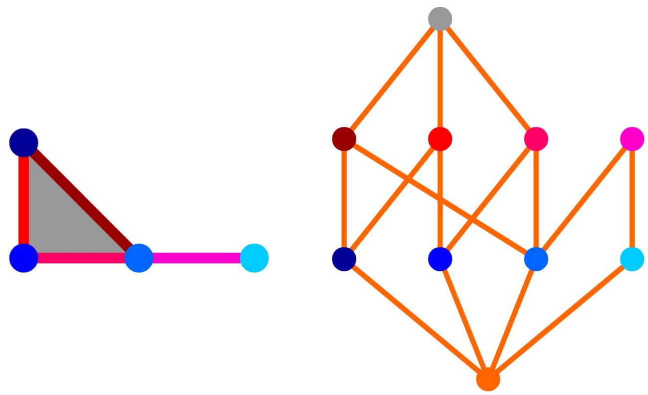


Figure 2.4: a simplicial complex pictured geometrically (left) and as a poset (right)

the intersection of the triangle on the left with one of the other triangles is not a face of the triangle on the left. It is not even clear which set of simplices exactly is depicted by (d).

As another example, the complexes Σ_0 and Σ_1 in Figure 2.7 have the same carrier while neither is a subdivision of the other.

The next lemmas will be applied frequently in later proofs. To start, from the definitions we immediately have:

Lemma **2.28**. Let Σ be a simplicial complex. A subset of Σ is a subcomplex iff it is a downset of Σ.

If we have a subdivision of some complex Σ, we automatically receive a subdivision of any subcomplex of Σ, since subcomplexes of Σ have only fewer cells. In formal terms this gives:

Lemma **2.29**. Let Σ, Σ', Δ be simplicial complexes such that Σ' is a subdivision of Σ and Δ is a subcomplex of Σ. Then Σ' has a subcomplex which is a subdivision of Δ.

Proof. See [Spa66, Corollary 3.3.5]. □

Example **2.30**. Let Σ be the complex (a) from Figure 2.5, let Σ' be the complex (c) and let Δ be the complex (b). Then the subcomplex given by lemma 2.29 is as in Figure 2.6.

Suppose that Σ_0 and Σ_1 are complexes with the same carrier. Then a less trivial classical result is that there exists a complex that is a subdivision of Σ_0 and simultaneously a subdivision of Σ_1. See Figure 2.7 for an example where Σ is the complex of all faces of some triangle. More generally, we have the following:

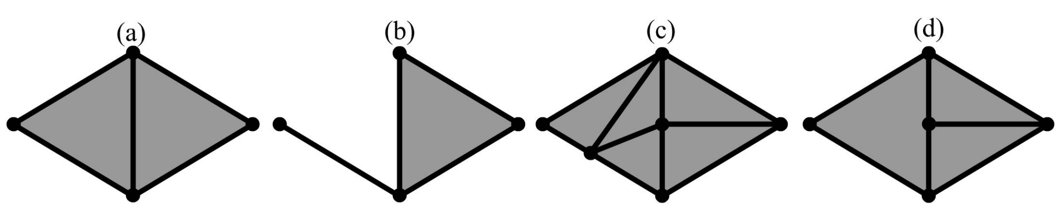


Figure 2.5: (a): simplicial complex; (b): a subcomplex of (a); (c): a subdivision of (a); (d): not a complex

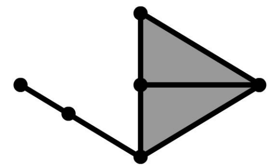


Figure 2.6: a subdivision of the complex (b) in Figure 2.5

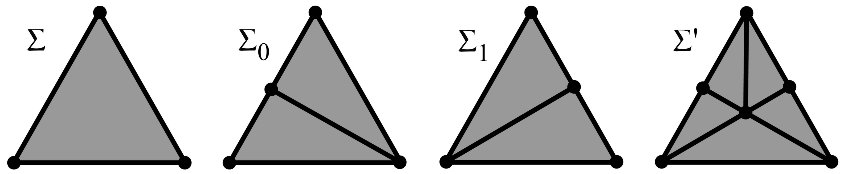


Figure 2.7: a complex Σ, two subdivisions Σ_0, Σ_1 of Σ and a common subdivision Σ' of Σ_0 and Σ_1

Lemma **2.31**. Let $\mathscr{S} \subseteq \mathbf{cmplx}^d$ be a finite set of simplicial complexes. Then there exists a simplicial complex Δ such that for each complex $\Sigma \in \mathscr{S}$ there exists a subcomplex of Δ that is a subdivision of Σ.

Proof. See [RS72, Theorem 2.11 and Addendum 2.12]. □

From the pictures we have provided so far it is tempting to say that a simplicial complex is a type of partitioning of some area in space. This is true in the following sence:

Lemma **2.32**. Let Σ be a simplicial complex. Then

$$|\Sigma| = \bigsqcup_{\sigma \in \Sigma} \operatorname{relInt} \sigma.$$

Proof. See [Mau70, Proposition 2.3.6]. □

Corollary **2.33**. If Σ is a subcomplex of Δ with $|\Sigma| = |\Delta|$, then $\Sigma = \Delta$.

The property of simplicial complexes provided by Lemma 2.32 is essential enough that is has a type of converse:

Lemma **2.34**. Let $\Sigma \subseteq \mathscr{P}\mathbb{R}^d$ be a set of simplices closed under taking faces such that $(\operatorname{relInt} \sigma) \cap (\operatorname{relInt} \delta) = \varnothing$ for all distinct $\sigma, \delta \in \Sigma$. Then Σ is a simplicial complex.

Proof. To show that intersections behave well in Σ, let $\sigma, \delta \in \Sigma$ such that $\sigma \cap \delta \neq \varnothing$. We have to show that $\sigma \cap \delta$ is a face of σ. Let σ_1 be the smallest face of σ that contains $\sigma \cap \delta$. Then $\delta \cap \operatorname{relInt} \sigma_1 \neq \varnothing$, because $\sigma \cap \delta$ is convex by Lemma 2.14. Hence, by Example 2.26 and Lemma 2.32, δ has a face δ_1 such that $(\operatorname{relInt} \sigma_1) \cap (\operatorname{relInt} \delta_1) \neq \varnothing$. Hence $\sigma_1 = \delta_1$ by the assumption. This implies $\sigma_1 \subseteq \sigma \cap \delta$, so $\sigma \cap \delta = \sigma_1$ is a face of σ. □

In order to further sophisticate our understanding of subdivisions, we introduce a canonical map on any subdivision of a given complex:

Notation **2.35**. Let Σ be a simplicial complex and Δ a subdivision of Σ. Then let $\operatorname{sub}(\Delta, \Sigma) : \Delta \to \Sigma$ be the map which sends a cell $\delta \in \Delta$ to the smallest cell of Σ that contains δ.

To understand what this map does, it may be helpful to imagine every cell of Σ in a different colour. Then the value of a cell $\delta \in \Delta$ under the map $\mathrm{sub}(\Delta,\Sigma)$ is the cell in Σ that has the colour that one would like to use for δ as well. See Figure 2.8. If Δ is a subdivision of Σ, the partition $\{\mathrm{relInt}\,\delta : \delta \in \Delta\}$ is finer than the partition $\{\mathrm{relInt}\,\sigma : \sigma \in \Sigma\}$. This refinement is made more precise by the following lemma:

Lemma **2.36**. Let Σ be a simplicial complex and Δ a subdivision of Σ. Then for all $\delta \in \Delta$,

$$\mathrm{relInt}\,\delta \subseteq \mathrm{relInt}\left(\mathrm{sub}(\Delta,\Sigma)(\delta)\right).$$

An easy consequence of this is the following:

Lemma **2.37**. Let $\Sigma,\Sigma',\Sigma'' \in \mathbf{cmplx}$ such that Σ' is a subdivision of Σ and Σ'' is a subdivision of Σ'. Then

$$\mathrm{sub}(\Sigma',\Sigma)\circ\mathrm{sub}(\Sigma'',\Sigma') = \mathrm{sub}(\Sigma'',\Sigma).$$

We next set up to construct a classical specific type of subdivision: the barycentric subdivision. This notion is important in the context of logic, since it is a mechanical construction that generates all possible subdivisions in the sense of Lemma 2.56 below.

Notation **2.38**. If P is a finite poset such that $\varnothing = \min P$, let

$$\mathscr{C}(P) = \left\{c \subseteq P \setminus \{\varnothing\} : c \text{ totally ordered}\right\}.$$

The operator $\mathscr{C}$ is analogous to the nerve operator on posets ([Ale98]), but adapted for a setting with rooted posets. Note that $\mathscr{C}(P)$ also has root $\varnothing$.

Lemma **2.39**. If Σ is a simplicial complex and $c \in \mathscr{C}(\Sigma)$, then the set $\mathbf{b}[c]$ of barycentra of elements of c is affinely independent.

Proof. An elementary exercise in affine geometry. See also Figure 2.9 in case $\Sigma = \mathrm{fac}(\tau)$ for some triangle τ. □

Before we build the barycentric subdivision of Σ from $\mathscr{C}(\Sigma)$, we introduce some more helpful notation.

Notation **2.40**. Let σ and δ be simplices such that $\mathrm{vtc}(\sigma)\cup\mathrm{vtc}(\delta)$ is affinely independent. Then define the simplex

$$\sigma \vee \delta = \mathrm{Conv}(\sigma\cup\delta) = \mathrm{Conv}\left(\mathrm{vtc}(\sigma)\cup\mathrm{vtc}(\delta)\right).$$

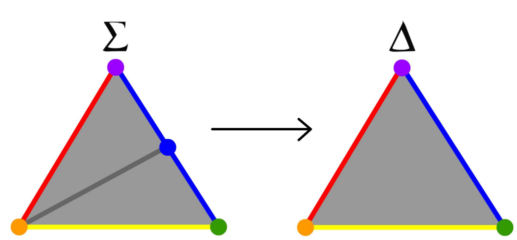


Figure 2.8: an example of using colours to visualize the map $\mathrm{sub}(\Delta,\Sigma)$

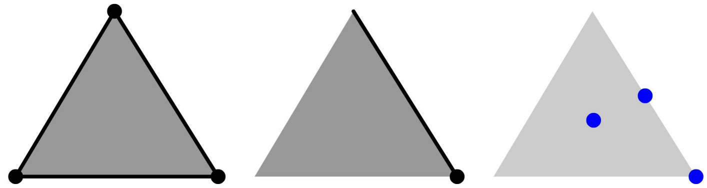


Figure 2.9: Σ (left), an element c of $\mathscr{C}(\Sigma)$ (centre) and $\mathbf{b}[c]$ (blue points on the right)

If moreover $\mathrm{vtc}(\sigma)$ and $\mathrm{vtc}(\delta)$ are disjoint, we write $\sigma \underline{\vee} \delta = \sigma \vee \delta$. We extend the notations $\vee$ and $\underline{\vee}$ to suitable families of simplices, in a fashion similar to Notation 2.1. If Σ, Δ are simplicial complexes such that $\sigma \underline{\vee} \delta$ is defined for all $\sigma \in \Sigma$ and $\delta \in \Delta$, then let

$$\Sigma * \Delta = \{\sigma \underline{\vee} \delta : \sigma \in \Sigma \;\&\; \delta \in \Delta\}.$$

(This need not be a simplicial complex, but it is in certain circumstances – for example in Lemma 2.41 below.)

Lemma **2.41**. Let σ be a simplex, Σ a triangulation of $\partial\sigma$ and $\mathbf{x} \in \mathrm{relInt}\,\sigma$. Then $\Sigma * \{\varnothing, \mathbf{x}\}$ is a triangulation Δ of σ satisfying $\Sigma \subseteq \Delta$ and $\mathrm{vtc}(\Delta) = \mathrm{vtc}(\Sigma) \sqcup \{\mathbf{x}\}$.

Proof. See [Spa66, Lemma 3.3.8]. Also see Figure 2.10. □

The following lemma is an important general technique for "expanding" a subdivision of a subcomplex to the entire complex; i.e. a type of converse to Lemma 2.29.

Lemma **2.42**. Let Δ be a simplicial complex and Σ a subcomplex of Δ such that $\delta \cap |\Sigma|$ is a face of δ for each $\delta \in \Delta$. Then, for every subdivision Σ' of Σ there exists a unique subdivision Δ' of Δ such that $\Sigma' \subseteq \Delta'$ and $\mathrm{vtc}(\Delta') = \mathrm{vtc}(\Sigma') \cup \mathrm{vtc}(\Delta)$.

Proof. For an example, see Figure 2.11. Let

$$\Delta' = \left\{\sigma' \underline{\vee} \delta^- : \delta, \delta^- \in \Delta; \sigma \in \Sigma; \sigma \cup \delta^- \subseteq \delta; \delta^- \cap |\Sigma| = \varnothing; \sigma' \in \mathrm{sub}(\Sigma', \Sigma)^{-1}\left[\{\sigma\}\right]\right\}.$$

That is, we take faces δ^- and σ of δ, such that σ lies inside $|\Sigma|$ while δ^- is disjoint from $|\Sigma|$. Then we take some $\sigma' \in \Sigma'$ with $\mathrm{relInt}\,\sigma' \subseteq \mathrm{relInt}\,\sigma$. In particular $\sigma' \subseteq \sigma$. Since $\mathrm{vtc}(\sigma) \sqcup \mathrm{vtc}(\delta^-)$ is affinely independent, also $\mathrm{vtc}(\sigma') \sqcup \mathrm{vtc}(\delta^-)$ is affinely independent. Hence $\sigma' \underline{\vee} \delta^- \in \Delta'$ is well-defined.

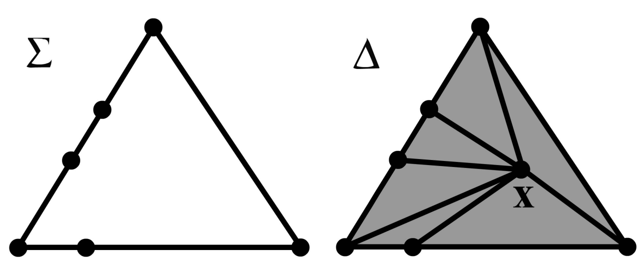


Figure 2.10: example to Lemma 2.41

Clearly Δ' is closed under taking faces. For $i = 0, 1$, let $\delta_i, \delta_i^-, \sigma_i, \sigma_i'$ be as in the definition of Δ'. Then we show that the intersection γ of the simplices

$$\sigma_0' \veebar \delta_0^- \quad \text{and} \quad \sigma_1' \veebar \delta_1^-$$

is a common face of them. Clearly we have $\gamma \supseteq (\sigma_0' \cap \sigma_1') \veebar (\delta_0^- \cap \delta_1^-)$. The other inclusion also holds, for suppose that $\mathbf{x} \in \gamma$. Let δ be the largest face of $\delta_0 \cap \delta_1$ that is disjoint from $|\Sigma|$. Since $\gamma \subseteq \delta_0 \cap \delta_1$, there is a unique $\mathbf{y} \in |\Sigma| \cap (\delta_0 \cap \delta_1)$ and a unique $\mathbf{z} \in \delta$ such that $\mathbf{x}$ is a convex combination of $\mathbf{y}$ and $\mathbf{z}$. Therefore we must have $\mathbf{y} \in \sigma_i'$ and $\mathbf{z} \in \delta_i'$ for each $i = 0, 1$. It follows that $\mathbf{z} \in (\sigma_0' \cap \sigma_1') \veebar (\delta_0^- \cap \delta_1^-)$. This proves that Δ' is as claimed.

For uniqueness, consider a subdivision Δ'' of Δ with $\Sigma' \subseteq \Delta''$ and $\operatorname{vtc}(\Delta'') = \operatorname{vtc}(\Sigma') \cup \operatorname{vtc}(\Delta)$. Then any $\delta^- \in \Delta$ that is disjoint from $|\Sigma|$ must be a cell of Δ''. From this we can conclude that Δ'' is a subcomplex of Δ', and so $\Delta' = \Delta''$ by Corollary 2.33. □

Remark **2.43**. It follows from the proof of Lemma 2.42 that any cell δ of Δ that is disjoint from $|\Sigma|$, will also be a cell of Δ'.

Definition **2.44** (**barycentric subdivision**). Let $\Sigma \in \mathbf{cmplx}^d$. In view of Lemma 2.39, define the map

$$\begin{aligned} \operatorname{cb}_\Sigma : \mathscr{C}(\Sigma) &\to \mathscr{P}\mathbb{R}^d : \\ c &\mapsto \veebar \mathbf{b}[c]. \end{aligned}$$

The *barycentric subdivision* of Σ is $\Sigma^+ = \operatorname{Im} \operatorname{cb}_\Sigma$. The nth barycentric subdivision Σ^{+n} of Σ is defined recursively by $\Sigma^{+0} = \Sigma$ and $\Sigma^{+(n+1)} = (\Sigma^{+n})^+$. See Figure 2.12.

Lemma **2.45**. Let $\Sigma \in \mathbf{cmplx}$.

1. The set Σ^+ is a subdivision of Σ.

2. cb_Σ is a poset isomorphism $\mathscr{C}(\Sigma) \cong \Sigma^+$.

Proof. See [Spa66, Theorem 3.3.9]. □

The construction of the barycentric subdivision can also be done step by step, or rather vertex by vertex. To express this, we introduce the notion of an *elementary subdivision*.

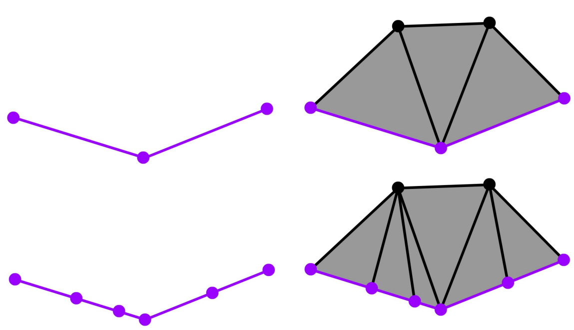


Figure 2.11: example to Lemma 2.42

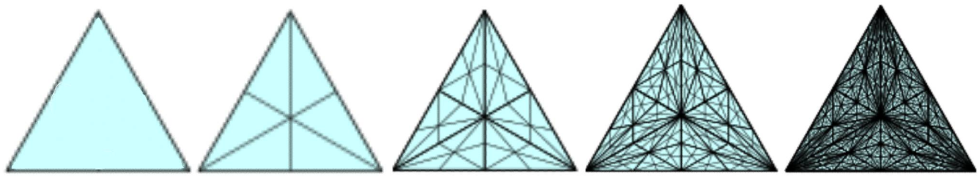


Figure 2.12: $\mathrm{fac}(\tau)^{+n}$ for a triangle τ and $n \leq 4$

Lemma **2.46**. If Δ is a subdivision of Σ then $\mathrm{vtc}(\Sigma) \subseteq \mathrm{vtc}(\Delta)$.

Proof. This follows from the remark preceding Lemma 2.36. □

Definition **2.47** (**elementary subdivision**). A subdivision Δ of Σ is called an *elementary subdivision* if

$$\#\big(\mathrm{vtc}(\Delta) \setminus \mathrm{vtc}(\Sigma)\big) = 1.$$

Lemma **2.48**. Let $\Sigma \in \mathbf{cmplx}$.

1. For each $\mathbf{x} \in |\Sigma| \setminus \mathrm{vtc}(\Sigma)$ there exists a unique elementary subdivision $\Sigma[\mathbf{x}]$ of Σ such that $\mathrm{vtc}\left(\Sigma[\mathbf{x}]\right) = \mathrm{vtc}(\Sigma) \sqcup \{\mathbf{x}\}$.

2. For each $\sigma, \delta \in \Sigma$ and $\mathbf{x} \in \mathrm{relInt}\,\delta$, the complex $\Sigma[\mathbf{x}]$ has a subcomplex with carrier σ that equals
$$\big\{\gamma \in \mathrm{fac}(\sigma) : \delta \not\subseteq \gamma\big\} * \{\varnothing, \mathbf{x}\}$$
if $\delta \subseteq \sigma$ and $\mathrm{fac}(\sigma)$ otherwise.

Proof. It is easy to derive this from Lemmas 2.41 and 2.42. In particular, uniqueness in part 1 is true because we also have uniqueness in Lemma 2.41 provided $\Sigma = \mathrm{propfac}(\sigma)$. □

Corollary **2.49**. Let Σ be a simplicial complex and let $X \subseteq |\Sigma|$ be a finite set of points. Then there is a subdivision Δ of Σ such that

$$\mathrm{vtc}(\Delta) = \mathrm{vtc}(\Sigma) \cup X.$$

Remark **2.50**. Uniqueness of Δ may fail in Corollary 2.49. See Figure 2.13 for an example when Σ is the complex of all faces of some triangle.

We are now in a position to state two technical properties of the barycentric subdivision that will be used in proofs in Chapters 4 and 7.

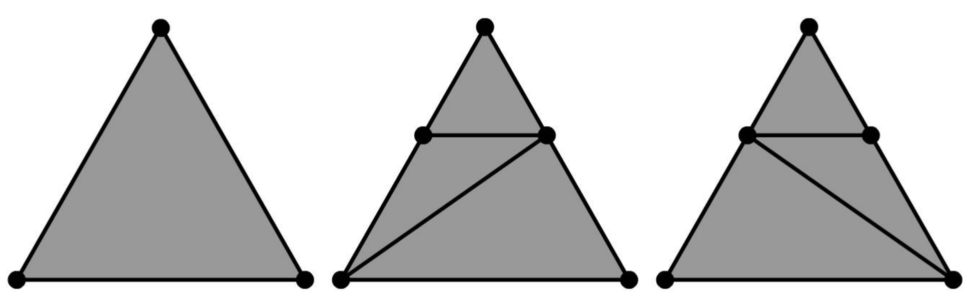


Figure 2.13: different subdivisions with the same vertices

Lemma **2.51**. Let $\Sigma \in$ **cmplx**.

1. There exists a sequence $\Sigma = \Sigma_0, \Sigma_1, \dots, \Sigma_k = \Sigma^+$ such that Σ_{i+1} is an elementary subdivision of Σ_i for each $i < \#\Sigma - 1$.

2. If Δ is a subcomplex of Σ, then Δ^+ is a subcomplex of Σ^+.

Proof. Figure 2.14 presents an example to part 1. We omit a formal proof.
Part 2 follows from the definitions. □

As logicians we are interested in the *structure* of simplical complexes, and not in the precise sizes or shapes of their cells. Therefore we need a notion of equivalence between complexes, namely *simplicial isomorphism*. The following two definitions are taken from [Spa66, ch. 3 sect. 1].

Definition **2.52** (**simplicial map**). Let Σ, Δ be simplicial complexes. A *simplicial map* ϕ from Σ to Δ is a map $\phi : \mathrm{vtc}(\Sigma) \to \mathrm{vtc}(\Delta)$ such that for any $\sigma \in \Sigma$,

$$\mathrm{Conv}\left(\phi\left[\mathrm{vtc}(\sigma)\right]\right) \in \Delta.$$

Definition **2.53** (**simplicial isomorphism**). The set of simplicial complexes with simplicial maps forms a category. In particular we have a notion of *simplicial isomorphism*: a simplicial map f from some complex Σ to some complex Δ for which there exists an inverse simplicial map f^{-1} from Δ to Σ satisfying $f \circ f^{-1} = \mathrm{id}_{\mathrm{vtc}(\Delta)}$ and $f^{-1} \circ f = \mathrm{id}_{\mathrm{vtc}(\Sigma)}$.

It is easy to check that a map $f : \mathrm{vtc}(\Sigma) \to \Delta$ is a simplicial isomorphism from Σ to Δ iff it can be extended to a poset isomorphism $\Sigma \cong \Delta$. Hence two complexes are simplicially isomorphic iff they are isomorphic as posets. The following lemma is elementary to verify.

Lemma **2.54**. Let Σ be a simplicial complex and let ϕ be an injection from $|\Sigma|$ to some Euclidean space such that $\Delta = \phi[\![\Sigma]\!]$ is a simplicial complex. Then:

1. $\phi|\,\mathrm{vtc}(\Sigma)$ is a simplicial isomorphism from Σ to Δ.

2. If Γ is a subcomplex of Σ, then $\phi[\![\Gamma]\!]$ is a subcomplex of Δ.

Lemma 2.54 is commonly applied when ϕ is a "piecewise affine" map:

Lemma **2.55**. Let Σ be a simplicial complex and let ϕ be an injection from $|\Sigma|$ to some Euclidean space such that the restriction $\phi|\sigma$ is affine for each $\sigma \in \Sigma$. Then

1. $\phi[\![\Sigma]\!]$ is a simplicial complex.

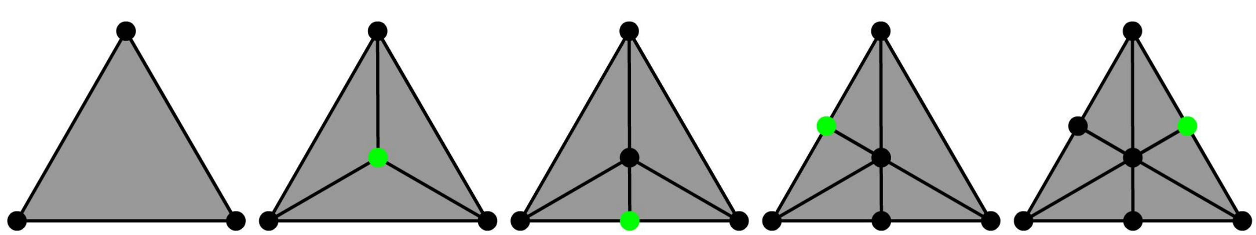


Figure 2.14: building $\mathrm{fac}(\tau)^+$ (where τ is a triangle) vertex by vertex

2. $\phi[\![\Sigma]\!]^+ = \phi[\![\Sigma^+]\!]$.

3. For all $c \in \mathscr{C}(\Sigma)$,

$$\mathrm{cb}_{\phi[\![\Sigma]\!]}\left(\phi[\![c]\!]\right) = \phi\left[\mathrm{cb}_\Sigma(c)\right].$$

Proof. Part 1 follows from Lemma 2.17.

For parts 2 and 3, note that $\phi\left(\mathbf{b}(\sigma)\right) = \mathbf{b}\left(\phi[\sigma]\right)$ for $\sigma \in \Sigma$. □

Using the notion of isomorphism, the following lemma expresses the fact that iterated barycentric subdivisions are arbitrarily fine.

Lemma **2.56**. If Σ is a simplicial complex and Δ is a subdivision of Σ, then there exists n such that Σ^{+n} is isomorphic to a subdivision of Δ.

Proof. See [Ada19, section 2.1]. □

Example **2.57**. Let τ be a triangle, $\Sigma = \mathrm{fac}(\tau)$ and let Δ be a subdivision of Σ that splits τ into four triangles sharing a vertex. See Figure 2.15. Then Σ^{+2} is isomorphic to a subdivision Δ' of Δ.

Vertices of simplicial complexes deserve some special attention since we shall often use them as points of evaluation for logical formulas. The next lemma says that it suffices to take only one barycentric subdivision when it comes to vertices.

Lemma **2.58**. Let Σ be a simplicial complex, Δ a subdivision of Σ and $\mathbf{x} \in \mathrm{vtc}(\Delta)$. Then there is $\mathbf{y} \in \mathrm{vtc}(\Sigma^+)$ and a subdivision Γ of Σ such that there is a simplicial isomorphism f from a subdivision of Δ to Γ satisfying $f(\mathbf{x}) = \mathbf{y}$.

Proof. By Lemma 2.56, we may assume w.l.o.g. that $\Delta = \Sigma^{+n}$ for some n.

By Lemma 2.32, find $\sigma \in \Sigma$ such that $\mathbf{x} \in \mathrm{relInt}\,\sigma$. Let $\mathbf{y} = \mathbf{b}(\sigma)$. We have to find a subdivision Γ of Σ such that there is a simplicial isomorphism f from Δ to Γ satisfying $f(\mathbf{x}) = \mathbf{y}$. In fact, we shall only use that $\mathbf{y} \in \mathrm{relInt}\,\sigma$.

By Lemma 2.51-1 there exists a sequence $\Sigma = \Sigma_0, \ldots, \Sigma_k = \Sigma^{+n}$ such that Σ_{i+1} is an elementary subdivision of Σ_i for each $i < k$. For each $i \leq k$, by Lemma 2.32 find $\sigma_i \in \Sigma_i$ such that $\mathbf{x} \in \mathrm{relInt}\,\sigma_i$. We shall inductively construct a sequence $\Sigma = \Gamma_0, \ldots, \Gamma_k = \Gamma$ such that for each $i \leq k$ there is an isomorphism $f_i : \Sigma_i \to \Gamma_i$ with $\mathbf{y} \in \mathrm{relInt}\, f_i(\sigma_i)$. Obviously, f_0 we just take to be the identity on Σ. Suppose that Γ_i and f_i have been constructed. Let $\Sigma_{i+1} = \Sigma_i[\mathbf{z}]$. Find $\theta \in \Sigma_i$ such that $\mathbf{z} \in \mathrm{relInt}\,\theta$. Distinguish two cases:

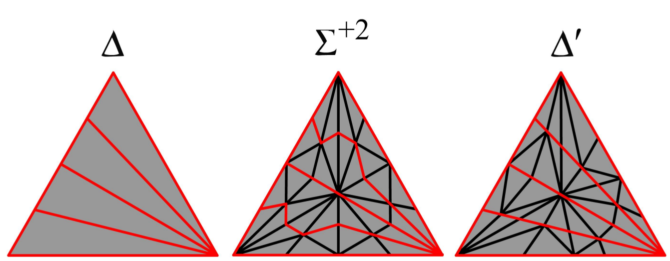


Figure 2.15: using barycentric subdivisions to obtain an arbitrary subdivision

- $\theta = \sigma_i$. Then $\sigma_{i+1} \subsetneq \sigma_i$. Since $\mathbf{x} \in \operatorname{relInt} \sigma_i$ and $\mathbf{x} \in \operatorname{relInt} \sigma_{i+1}$, Lemma 2.48-2 implies that $\mathbf{z} \in \operatorname{vtc}(\sigma_{i+1})$. Thus

  $$\mathbf{x} \in \operatorname{relInt} \underline{\bigvee} \Big(\{\mathbf{z}\} \sqcup \big(\operatorname{vtc}(\sigma_{i+1}) \cap \operatorname{vtc}(\sigma_i) \big) \Big),$$

  and therefore we can write $\mathbf{z}$ as an affine combination of

  $$\{\mathbf{x}\} \sqcup \operatorname{vtc}(\sigma_{i+1}) \cap \operatorname{vtc}(\sigma_i).$$

  Using the same coefficients (which are all nonzero), we find a corresponding affine combination $\mathbf{u}$ of

  $$\{\mathbf{y}\} \sqcup f_i \big[\operatorname{vtc}(\sigma_{i+1}) \cap \operatorname{vtc}(\sigma_i) \big],$$

  so that

  $$\mathbf{y} \in \operatorname{relInt} \underline{\bigvee} \Big(\{\mathbf{u}\} \sqcup f_i \big[\operatorname{vtc}(\sigma_{i+1}) \cap \operatorname{vtc}(\sigma_i) \big] \Big)$$

  and $\mathbf{u} \in \operatorname{relInt} f_i(\sigma_i)$. Let $\Gamma_{i+1} = \Gamma_i[\mathbf{u}]$. By Lemma 2.48, the isomorphism $f_{i+1} : \Sigma_{i+1} \to \Gamma_{i+1}$ with $\mathbf{y} \in \operatorname{relInt} f_{i+1}(\sigma_{i+1})$ exists.

- $\theta \neq \sigma_i$. Then $\sigma_i = \sigma_{i+1}$. Here we pick an arbitrary $\mathbf{u} \in \operatorname{relInt} g(\theta)$ and let $\Gamma_{i+1} = \Gamma_i[\mathbf{u}]$. Again the desired map f_{i+1} exists by Lemma 2.48.

This completes the construction of the complexes Γ_i and the isomorphisms f_i. Now let $f = f_k$. Since $\sigma_k = \mathbf{x}$, we have $\mathbf{y} = f(\mathbf{x})$. □

It might even be possible to strengthen the statement of Lemma 2.58 to saying that f is a simplicial isomorphism from Δ itself to Γ, but we shall not need this. Figure 2.16 provides an example in case Σ is the complex of all faces of some triangle.

As a final piece of background on simplicial complexes, we consider the following question. Given a complex Σ, what is the smallest $d \geq -1$ such that $\mathbf{cmplx}^d$ contains a complex isomorphic to Σ? The following theorem was stated and proved in purely topological terms by Menger in 1928 (see [Men13, chapter IX section 3]). The version stated here has an easier folklore proof.

Theorem **2.59** (**Menger–Nöbeling–Pontryagin theorem for simplicial complexes**). For every d-dimensional simplicial complex, there exists an isomorphic simplicial complex in $\mathbf{cmplx}^{2d+1}$.

Proof. Consider the curve

$$\begin{aligned} \gamma : \mathbb{R} &\to \mathbb{R}^{2d+1} \\ t &\mapsto (t, t^2, \ldots, t^{2d+1}). \end{aligned}$$

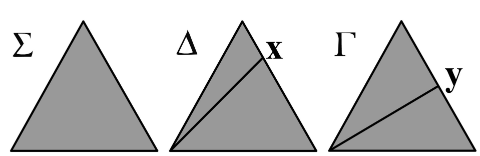


Figure 2.16: example to Lemma 2.58

(*) Any subset of the image $\operatorname{Im}\gamma$ of size at most $2d+2$ is affinely independent. (This follows from the determinant formula of Vandermonde matrices. See [Poo15, Exploration "Geometric Applications of Determinants", Exercise 19, p. 291].)

Let Σ be a d-dimensional simplicial complex. Choose any injection $\psi : \mathrm{vtc}(\Sigma) \to \operatorname{Im}\gamma$. For $\sigma \in \Sigma$, let

$$\chi(\sigma) = \mathrm{Conv}\Big(\psi\big[\mathrm{vtc}(\sigma)\big]\Big).$$

Define $\Delta = \chi[\Sigma]$. Then by (*), Δ is a set of simplices closed under taking faces. To show that Δ is a simplicial complex, suppose that $\delta, \delta' \in \Delta$. Find $\sigma, \sigma' \in \Sigma$ such that $\delta = \chi(\sigma)$ and $\delta' = \chi(\sigma')$. Then

$$\#\big(\mathrm{vtc}(\sigma) \cup \mathrm{vtc}(\sigma')\big) \leq 2d+2,$$

so $\psi\big[\mathrm{vtc}(\sigma) \cup \mathrm{vtc}(\sigma')\big]$ is affinely independent by (*). It follows that

$$\begin{aligned}\delta \cap \delta' &= \chi(\sigma) \cap \chi(\sigma')\\ &= \chi(\sigma \cap \sigma')\end{aligned}$$

which is a common face of δ and δ'.

By definition, ψ is a simplicial map from Σ to Δ, and ψ is clearly bijective. To show that it is a simplicial isomorphism, we have to show that ψ^{-1} is a simplicial map. Suppose that $X \subseteq \mathrm{vtc}(\Delta)$ such that

$$\delta := \mathrm{Conv}(X) \in \Delta.$$

Then (*) implies $\#X \leq d+1$: if $\#X > d+1$, pick $Y \in [X]^{=d+2}$ and note that Y is affinely independent by (*), so that

$$\dim\delta \geq \dim\mathrm{Conv}(X) = d+1 > d,$$

which is impossible by definition of Δ. Now again, (*) implies that X is affinely independent. There exists $\sigma \in \Sigma$ such that $\delta = \chi(\sigma)$. Then $\psi\big[\mathrm{vtc}(\sigma)\big]$ is affinely independent and has the same convex hull as does X (namely δ). Hence $\psi\big[\mathrm{vtc}(\sigma)\big] = X$, whence $\mathrm{vtc}(\sigma) = \psi^{-1}[X]$, so $\mathrm{Conv}\left(\psi^{-1}[X]\right) \in \Sigma$ as desired. □

It is known that the bound $2d+1$ is tight in general ([Flo33]). Also substantial work has been done on lowering this dimension in particular circumstances (see e.g. [Kra91], [Hor71]).

2.3.4 Polyhedra

From a geometric point of view, it is natural to pay special attention to the carriers of complexes, since these are actual figures in space.

Definition **2.60** (**polyhedron**). If Σ is a simplicial complex, then the carrier $\mathbf{P} = |\Sigma|$ is called a *polyhedron*. We let **plhdr** be the set of all polyhedra. We define the *dimension* $\dim\mathbf{P}$ of $\mathbf{P}$ to be the dimension of Σ. By standard dimension theory this is well-defined and compatible with the definition of the dimension of a simplex. For $d, d' \geq -1$ let

$$\mathbf{plhdr}_d = \{\mathbf{P} \in \mathbf{plhdr} : \dim\mathbf{P} \leq d\},$$

$$\mathbf{plhdr}^{d'} = \{\mathbf{P} \in \mathbf{plhdr} : \mathbf{P} \subseteq \mathbb{R}^{d'}\}$$

and

$$\mathbf{plhdr}_d^{d'} = \mathbf{plhdr}_d \cap \mathbf{plhdr}^{d'}.$$

As for complexes, we have

$$\mathbf{plhdr}_d \neq \mathbf{plhdr}_d^d = \mathbf{plhdr}^d.$$

Fortunately, in polyhedral semantics we can often choose to work with simplicial complexes or polyhedra as we please. For polyhedra, we also have a notion of equivalence:

Definition **2.61** (**PL-homeomorphism**). If $\mathbf{P},\mathbf{Q}$ are polyhedra, a map $\phi : \mathbf{P} \to \mathbf{Q}$ is called *piecewise linear*[2] if there exists a triangulation Σ of $\mathbf{P}$ such that the restriction $\phi|\sigma$ is affine for each $\sigma \in \Sigma$. A *PL-homeomorphism* is a piecewise linear homeomorphism. If there is a PL-homeomorphism from $\mathbf{P}$ to $\mathbf{Q}$, we call these polyhedra PL-homeomorphic and write $\mathbf{P} \cong \mathbf{Q}$.

The following lemmas describe the connection between simplicial isomorphisms and PL-homeomorphisms.

Lemma **2.62**. If f is a simplicial isomorphism from Σ to Δ, then f can be extended to a PL-homeomorphism $|\Sigma| \to |\Delta|$.

Proof. Using Lemmas 2.32 and 2.23, we can define an extension $\overline{f}$ of f on $|\Sigma|$ by

$$\overline{f}\Big(\sum_{\mathbf{x}\in \mathrm{vtc}(\sigma)} \alpha(\mathbf{x})\cdot \mathbf{x}\Big) = \sum_{\mathbf{x}\in \mathrm{vtc}(\sigma)} \alpha(\mathbf{x})\cdot f(\mathbf{x})$$

for $\sigma \in \Sigma$ and $\alpha : \mathrm{vtc}(\sigma) \to (0,1]$ summing to 1. It is a basic exercise in affine geometry to check that $\overline{f}$ is affine on each cell of Σ, and that $\overline{f}$ is a homeomorphism. □

Lemma **2.63**. Let Σ,Δ be simplicial complexes and let f be a poset isomorphism $\Sigma \cong \Delta$. Then, if Σ' is a subdivision of Σ, there exists a subdivision Δ' of Δ such that there is a poset isomorphism $g : \Sigma' \cong \Delta'$ satisfying

$$f \circ \mathrm{sub}(\Sigma',\Sigma) = \mathrm{sub}(\Delta',\Delta) \circ g. \tag{2.1}$$

Proof. Let $f_0 = f|\mathrm{vtc}(\Sigma)$. Let $\overline{f_0} : |\Sigma| \to |\Delta|$ be the map from the proof of Lemma 2.62. By Lemmas 2.55-1 and 2.54-1, $\overline{f_0}$ restricts to a simplicial isomorphism g_0 from Σ' to some complex Δ' that is clearly a subdivision of Δ. Next g_0 extends to a poset isomorphism $g : \Sigma' \cong \Delta'$ that satisfies (2.1). □

Lemma **2.64**.

1. Polyhedra $\mathbf{P},\mathbf{Q}$ are PL-homeomorphic iff there exist a triangulation Σ of $\mathbf{P}$ and a triangulation Δ of $\mathbf{Q}$ such that $\Sigma \cong \Delta$.

2. $\cong$ is an equivalence relation on **plhdr**.

Proof. Part 1: $\Rightarrow$ follows from Lemma 2.55-1, and $\Leftarrow$ follows from Lemma 2.62.

In view of the characterization in part 1, reflexivity and symmetry in part 2 are trivial, and transitivity follows from Lemmas 2.31 and 2.63. □

[2] We remark that "piecewise affine" might be a more logical name, but we stick to the traditional name.

As we noted earlier, many operations can be done with polyhedra as well as with simplicial complexes. In particular, this is the case for the Menger-Nöbeling-Pontryagin theorem:

Corollary **2.65**. Let $d \geq 0$. For $\mathbf{P} \in \mathbf{plhdr}_d$ there exists $\mathbf{Q} \in \mathbf{plhdr}_d^{2d+1}$ such that $\mathbf{P} \cong \mathbf{Q}$.

Proof. Theorem 2.59 and Lemma 2.64-1. □

Also we have a notion that is parallel to that of a subcomplex:

Definition **2.66** (**subpolyhedron**). A polyhedron $\mathbf{Q}$ is a *subpolyhedron* of the polyhedron $\mathbf{P}$ if $\mathbf{Q} \subseteq \mathbf{P}$.

Remark **2.67**. If $\mathbf{Q}$ is a subpolyhedron of $\mathbf{P}$, then by Lemma 2.31 there exist triangulations Δ of $\mathbf{Q}$ and Σ of $\mathbf{P}$ such that Δ is a subcomplex of Σ.

2.4 Modal logic

In this section, we introduce some modal logics for which we can define polyhedral semantics. We assume that the reader is familiar with the most basic notions of modal logic, including Kripke semantics, frame validity and p-morphisms (the latter of which will be recalled in Definition 2.71). If (X,R) is a Kripke frame and $V : \mathrm{Prop} \to \mathscr{P}X$ is a valuation, we write $V^\top$ for the transpose of V, i.e. the marking $V^\top : X \to \mathscr{P}\mathrm{Prop}$ defined by $p \in V^\top(x)$ iff $x \in V(p)$.

Discussion **2.68**. Our development of polyhedral semantics builds on the observation that a poset is nothing but a Kripke frame with a reflexive, antisymmetric, transitive accessibility relation. For any set $\mathscr{Q}$ of posets, we can consider the normal modal logic $\mathrm{Log}(\mathscr{Q})$ consisting of all formulas in the basic modal language that are valid on each poset in $\mathscr{Q}$. Then it is well known that

$$\mathbf{S4} \subseteq \mathrm{Log}(\mathscr{Q})$$

(see [CZ97, p. 92]). If each poset in $\mathscr{Q}$ is finite, it is easy to show that

$$\mathbf{grz} = \Box\big(\Box(p \to \Box p) \to p\big) \to p$$

is valid on each poset in $\mathscr{Q}$, so that

$$\mathbf{S4.Grz} = \mathbf{S4} + \mathbf{grz} \subseteq \mathrm{Log}(\mathscr{Q}),$$

i.e. $\mathrm{Log}(\mathscr{Q})$ is a normal extension of **S4.Grz**. In fact, the celebrated Blok-Esakia theorem ([Blo76], [Esa76]) states that the collection of normal extensions of **S4.Grz** is in a natural 1-1 correspondence with the collection of superintuitionistic logics, so polyhedral semantics can be viewed as a semantics for propositional formulas as well as a semantics for basic modal formulas.

We are now ready to expose the fundamentals of polyhedral semantics.

2.4.1 Logics for polyhedra

Definition **2.69** (**polyhedrally-complete logic**). If $\mathfrak{p} \subseteq \mathbf{plhdr}$, let $\mathrm{Trian}(\mathfrak{p})$ be the set of all triangulations of members of $\mathfrak{p}$. Next define

$$\mathrm{Log}(\mathfrak{p}) = \mathrm{Log}\left\{\Sigma \setminus \{\varnothing\} : \Sigma \in \mathrm{Trian}(\mathfrak{p})\right\}$$

and

$$\mathrm{Log}_{\varnothing}(\mathfrak{p}) = \mathrm{Log}\left(\mathrm{Trian}(\mathfrak{p})\right).$$

We omit curly brackets if $\mathfrak{p}$ is a singleton. A logic of the form $\mathrm{Log}(\mathfrak{p})$ is said to be *polyhedrally-complete* if $\mathfrak{p} \neq \varnothing$. A logic of the form $\mathrm{Log}_{\varnothing}(\mathfrak{p})$ is said to be *quasi-polyhedrally-complete* if $\mathfrak{p} \neq \varnothing$.

With this definition of polyhedrally-complete logics, we follow the tradition in the literature. The notion of a quasi-polyhedrally-complete logic appears to be novel. In chapters 5, 6 and 7 the importance of quasi-polyhedrally-complete logics will become apparent.

Remark **2.70**. If $\mathfrak{P} \subseteq \mathscr{P}\mathbf{plhdr}$, then

$$\mathrm{Trian}\left(\bigcup \mathfrak{P}\right) = \bigcup_{\mathfrak{p} \in \mathfrak{P}} \mathrm{Trian}(\mathfrak{p}),$$

$$\mathrm{Log}\left(\bigcup \mathfrak{P}\right) = \bigcap_{\mathfrak{p} \in \mathfrak{P}} \mathrm{Log}(\mathfrak{p})$$

and

$$\mathrm{Log}_{\varnothing}\left(\bigcup \mathfrak{P}\right) = \bigcap_{\mathfrak{p} \in \mathfrak{P}} \mathrm{Log}_{\varnothing}(\mathfrak{p}).$$

Notice that every (quasi-)polyhedrally-complete logic is a normal extension of **S4.Grz**, by Discussion 2.68.

2.4.2 P-morphisms

P-morphisms between Kripke frames (or in our case, posets) are important for logic because they preserve the truth of modal formulas and can also be expressed by logic to a certain extent (see Definition 2.73 below).

Definition **2.71** (**p-morphism**). Let P, Q be posets, $f : P \to Q$ and $X \subseteq P$. The map f is said to satisfy the *back-property* on X if for each $p \in X$ and $f(p) < q \in Q$ there exists $p_1 \in P$ such that $p < p_1$ and $f(p_1) = q$. If f is monotone and f satisfies the back-property (on the entire universe P), f is called a p-morphism.

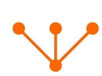
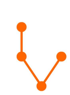
Consider the following variation of a p-morphism:

Definition **2.72**. Let F, G be frames. An *up-p-morphism* from F to G is a p-morphism $U \to G$, where U is some generated subframe of F. An *up-reduction* from F to G is a surjective up-p-morphism from F to G.

Definition **2.73** (**Jankov-Fine formula – [Fin74]**). If G is a finite rooted frame, there exists a formula $\chi(G)$, the *Jankov-Fine formula* of G, such that for any frame F it holds $F \vDash \chi(G)$ iff there is *no* up-reduction from F to G.

Using Jankov-Fine formulas, we can axiomatize some logics which are known to by polyhedrally-complete.

Notation **2.74**. Let $\mathbf{BD}_{d+1}$ be the logic of all posets of height at most $d+1$. Let

$\mathbf{PL}_d = \mathbf{BD}_{d+1} + \chi\Big($  $\Big) + \chi\Big($  $\Big).$

It is well-known that $\mathbf{BD}_d$ is finitely axiomatizable ([CZ97, Proposition 2.38]), so $\mathbf{PL}_d$ is finitely axiomatizable.

Proposition **2.75**. Let $d \geq 0$. Then $\mathbf{PL}_d$ is the largest polyhedrally-complete logic of height $d+1$.

Proof. [Ada19, Proposition 4.8]. □

Remark **2.76**. Suppose that $\mathbf{P} \in \mathbf{plhdr}$ has dimension d. Note that the posets in $\mathrm{Trian}(\mathbf{P})$ have height $d+2$ whereas $\Sigma \setminus \{\varnothing\}$ for $\Sigma \in \mathrm{Trian}(\mathbf{P})$ has height $d+1$. Thus Proposition 2.75 implies $\mathrm{Log}(\mathbf{P}) \subseteq \mathbf{PL}_d$ and $\mathrm{Log}_\varnothing(\mathbf{P}) \subseteq \mathbf{PL}_{d+1}$.

We can extend Proposition 2.75 as follows:

Proposition **2.77**. Let $d \geq 0$. Then $\mathbf{PL}_d$ is the largest quasi-polyhedrally-complete logic of height $d+1$.

Proof sketch. We omit a formal proof, as this proposition is inessential to this thesis. The reader may want to skip this proof before reading chapter 5. The idea is to apply the fact that $\mathrm{Log}_\varnothing(\sigma) = \mathbf{PL}_d$ for a $(d-1)$-simplex σ (Lemma 5.44 below). Next, the key observation is that, given $\varphi \notin \mathrm{Log}_\varnothing(\sigma)$, we can find a triangulation Σ of σ and a valuation $V : \mathrm{Prop} \to \mathscr{P}\Sigma$ such that $\Sigma, V, \varnothing \not\models \varphi$ and there exists a colour $\mathbb{P} \subseteq \mathrm{Prop}$ with

$$V^\top(\delta) = \mathbb{P}$$

for all $\delta \in \Sigma$ which intersect $\partial\sigma$. If we think of σ as lying in some arbitrary polyhedron $\mathbf{P}$, this means that it is easy to derive from V a suitable valuation on some triangulation of $\mathbf{P}$ using the constant value $\mathbb{P}$. Hence $\varphi \notin \mathrm{Log}_\varnothing(\mathbf{P})$.

The reason why the claim holds is that we can first guarantee that all cells that intersect $\partial\sigma \setminus \mathrm{relInt}\,\upsilon$ have the same colour, where υ is some $(d-2)$-dimensional face of σ. Then we take a mirrored copy of σ, and glue it to σ along the face υ, and the result is again PL-homeomorphic to a $(d-1)$-simplex. □

Apart from the logical importance of p-morphisms, these maps also arise naturally from polyhedral geometry in the following way:

Lemma **2.78**. Let Σ be a simplicial complex and Δ a subdivision of Σ.

1. The map $\mathrm{sub}(\Delta, \Sigma) : \Delta \to \Sigma$ is a surjective p-morphism.

2. Consequently, this map restricts to a surjective p-morphism $\Delta \setminus \{\varnothing\} \to \Sigma \setminus \{\varnothing\}$.

We postpone the proof of Lemma 2.78 to the next chapter. To demonstrate the importance of this lemma, suppose that $\mathbf{P} \in \mathbf{plhdr}$. Then we can define a partial order on $\mathrm{Trian}(\mathbf{P})$ by $\Sigma \leq \Sigma'$ iff Σ' is a subdivision of Σ. By Lemma 2.31, this is a directed poset. By Lemma 2.78, we have $\mathrm{Log}\{\Sigma \setminus \{\varnothing\} : \Sigma \in \mathscr{C}\} = \mathrm{Log}(\mathbf{P})$ and $\mathrm{Log}(\mathscr{C}) = \mathrm{Log}_\varnothing(\mathbf{P})$ for any $\mathscr{C} \subseteq \mathrm{Trian}(\mathbf{P})$ that is cofinal in $\big(\mathrm{Trian}(\mathbf{P}), \leq\big)$. Moreover, if we fix a $\Sigma \in \mathrm{Trian}(\mathbf{P})$, the set $\{\Sigma^{+0}, \Sigma^{+1}, \Sigma^{+2}, \dots\}$ of barycentric subdivisions is an example of such a cofinal set $\mathscr{C}$, by Lemma 2.56.

A first application of p-morphisms is that the notion of equivalence that we introduced on **plhdr** harmonizes with the logics:

Lemma **2.79**. If $\mathbf{P} \cong \mathbf{Q}$, then $\mathrm{Log}(\mathbf{P}) = \mathrm{Log}(\mathbf{Q})$ and $\mathrm{Log}_\varnothing(\mathbf{P}) = \mathrm{Log}_\varnothing(\mathbf{Q})$.

Proof. Lemmas 2.64-1, 2.31 and 2.78. □

Another preliminary application of p-morphisms is the following lemma, which says that we can always take vertices (as opposed to any simplices) as points of evaluation for modal formulas.

Lemma **2.80**. We have $\varphi \notin \mathrm{Log}(\mathbf{P})$ iff there exists a triangulation Σ of $\mathbf{P}$ and a vertex $\mathbf{x}$ of Σ such that φ can be falsified in Σ at $\mathbf{x}$.

Proof. Lemmas 2.48-1 and 2.78. □

This concludes the general set-up. In the next chapters we focus on various specific aspects of the polyhedral semantics.

Chapter 3

Manifold-with-boundary

In chapter 2 we have introduced the logics $\mathbf{PL}_d$ as examples of polyhedrally-complete logics. In this chapter we shall "rediscover" these logics by investigating a large class of polyhedra, namely the manifolds-with-boundary (to be defined below). Note that every polyhedron in $\mathbf{plhdr}^d$ inherits a subspace topology from the standard topology on $\mathbb{R}^d$ and thus can be considered as a topological space. From there, we can ask the question whether or not a given polyhedron is a manifold-with-boundary. Manifolds form a standard subject of interest in geometric topology ([Bre93], [Fen83]). A polyhedron can be a manifold. A polyhedron in $\mathbf{plhdr}^d$ can be a d-dimensional manifold-with-boundary, but it cannot be a d-dimensional manifold. Furthermore, the basic building blocks of polyhedral semantics, viz. simplices, are examples of manifolds-with-boundary (Proposition 3.18). Viewed in this way, polyhedral semantics for manifolds-with-boundary can be thought of as a simpler kind of polyhedral semantics. That is why we study this class before moving to a more general picture in the next chapters.

The main result of this chapter is that the logic $\mathrm{Log}(\mathbf{P})$ of any d-dimensional manifold-with-boundary $\mathbf{P}$ (if $\mathbf{P}$ is also a polyhedron!) is exactly $\mathbf{PL}_d$. Not every compact manifold is homeomorphic to a polyhedron ([KS69], [KS77]), but every compact two-dimensional manifold is ([Rad25]).

To prove our result, we rely on the maximality property of $\mathbf{PL}_d$, i.e. Proposition 2.75, and hence on [Ada19]. Furthermore, some topological arguments are needed. We need to prove that the topological definition of manifold-with-boundary is strong enough that it enforces certain structural properties of the poset structure of triangulations (Lemma 3.16 below). Some technical results from this chapter will also be applied in later chapters. More generally, the geometry used in this chapter is valuable for understanding polyhedral semantics.

3.1 Topological preparation

The following concept is crucial to the definition of a manifold-with-boundary:

Definition **3.1**. For $d \geq 1$, the *d-dimensional closed halfspace* is

$$\mathbb{H}_d = \left\{(x_0, \dots, x_{d-1}) \in \mathbb{R}^d : x_{d-1} \geq 0\right\}.$$

Remark **3.2**.

$$\mathbb{R}^d \times \mathbb{H}_{d'} = \mathbb{H}_{d+d'}.$$

Remark **3.3**. An open ball $B(\mathbf{x},\varepsilon) \subseteq \mathbb{R}^d$ is homeomorphic to $\mathbb{R}^d$, and $B(\mathbf{x},\varepsilon) \cap \mathbb{H}_d$ is homeomorphic to $\mathbb{R}^d$ or $\mathbb{H}_d$.

The concept of a manifold-with-boundary (Definition 3.14) is a topological one. Obviously $\mathbb{R}^d \neq \mathbb{H}_d$, but in the context of topology it is important to observe that these spaces are also not homeomorphic. Formulated in an even stronger way, we have the following:

Lemma **3.4**. No open subset of $\mathbb{H}_d$ that intersects $\mathbb{R}^{d-1} \times \{0\}$ is homeomorphic to an open subset of $\mathbb{R}^d$.

Proof. Let $U \subseteq \mathbb{H}_d$ be an open set and $\mathbf{x} \in U$ with $x_{d-1} = 0$. Let V be the connected component of U that contains $\mathbf{x}$. For a contradiction, suppose that U is homeomorphic to an open subset of $\mathbb{R}^d$. Then V is homeomorphic to an open subset of $\mathbb{R}^d$. Hence V has an open subset $W \ni \mathbf{x}$ that is homeomorphic to $\mathbb{R}^d$. Find $\varepsilon > 0$ such that the open ball $B(\mathbf{x},\varepsilon) \cap \mathbb{H}_d \subseteq W$. Let S be the boundary of $B(\mathbf{x},\varepsilon/2)$. Then S is a small $(d-1)$-sphere that intersects W. Notice that $S \cap \mathbb{H}_d$ is a proper closed subset of S that separates W. By the Jordan separation theorem ([Dug66, chapter XVII Theorem 2.4]), it follows that W is not homeomorphic to $\mathbb{R}^d$. □

We aim to generalize the spaces $\mathbb{R}^d$ and $\mathbb{H}_d$, making a sequence $\mathbb{H}_d, \mathbb{R}^d, \ldots$ of increasingly large spaces. For $d = 2$, the next element in this list will look like the space in Figure 3.1.

Definition **3.5**. For $n \geq 1$, let the *n-star* be the set

$$\text{star}_n = \{re^{\rho \cdot 2\pi i/n} : \rho \in \mathbb{Z} \;\&\; r \in \mathbb{R}_{\geq 0}\} \subseteq \mathbb{C}.$$

See Figure 3.2. The *rays* of the n-star are the sets of the form

$$\{re^{\rho \cdot 2\pi i/n} : \mathbb{R}_{>0}\},$$

for $\rho \in \mathbb{Z}$. Let the 0-*star* be the set $\{0_{\mathbb{C}}\}$.

Example **3.6**. The 1-star is homeomorphic to $\mathbb{H}_1 = [0,\infty)$ and the 2-star is homeomorphic to $\mathbb{R}$ (in fact $\text{star}_1 = \iota_1^2\big[[0,\infty)\big]$ and $\text{star}_2 = \iota_1^2[\mathbb{R}]$ using the notation from Example 2.16). Hence Remark 3.2 implies that $\mathbb{H}_d$ is homeomorphic to $\mathbb{R}^{d-1} \times \text{star}_1$ and $\mathbb{R}^d$ is homeomorphic to $\mathbb{R}^{d-1} \times \text{star}_2$. Thus $\mathbb{R}^{d-1} \times \text{star}_n$ generalizes $\mathbb{R}^d$ and $\mathbb{H}_d$.

Since $(0,\infty)$ is homeomorphic to $\mathbb{R}$, the following is easy to see:

Remark **3.7**. The rays of a star are each homeomorphic to $\mathbb{R}$.

Removing four rays from the 8-star may produce the space on the left of Figure 3.3. Clearly this is homeomorphic to the 4-star, pictured on the right of the same figure. More generally, we have the following lemma:

Lemma **3.8**. If $Y_0, \ldots, Y_{k-1}$ are distinct rays of star_n, then $\text{star}_n \setminus (Y_0 \cup \cdots \cup Y_{k-1})$ is homeomorphic to star_{n-k}.

The following lemma gives an essential property of these stars, that we will later use to prove a property of the triangulations of manifolds-with-boundary (namely Lemma 3.16-3). Roughly speaking, it says that the centre of a star is topologically different from Euclidean space, even after we take a product with $\mathbb{R}^{d-1}$.

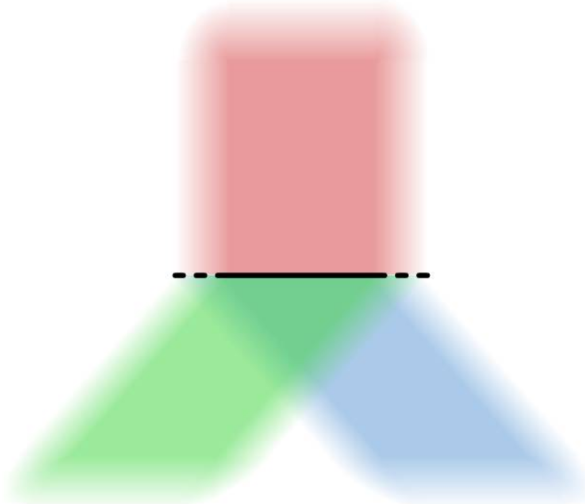


Figure 3.1: $\mathbb{R} \times \text{star}_3$

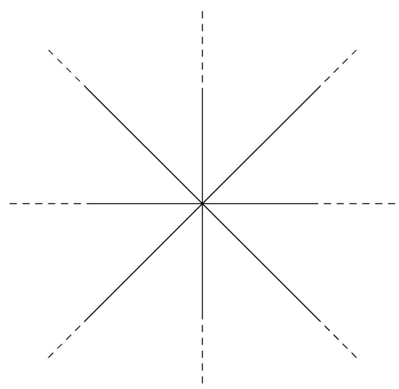


Figure 3.2: star_8

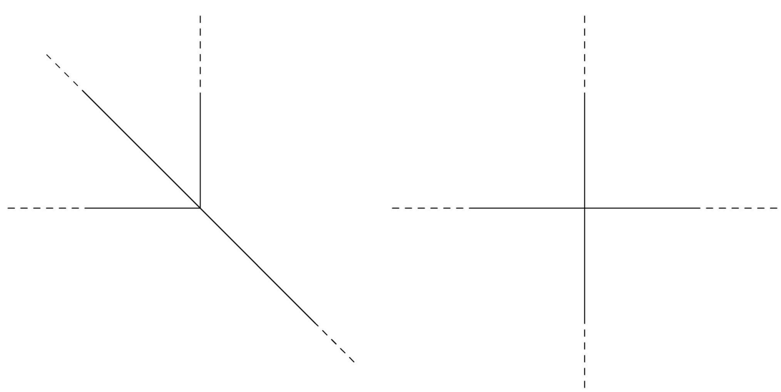


Figure 3.3: star_8 minus four rays (left); star_4 (right)

Lemma **3.9**. Let $n \geq 3$ and $d \geq 1$, and let U be an open subset of the product $\mathbb{R}^{d-1} \times \mathrm{star}_n$ such that U intersects $\mathbb{R}^{d-1} \times \{0_{\mathbb{C}}\}$. Then U is not homeomorphic to a subset of $\mathbb{R}^d$.

We prove Lemma 3.9 using the following classical result of Brouwer [Bro12].

Theorem **3.10** (**invariance of domain**). Let $U \subseteq \mathbb{R}^d$ be open, and $f : U \to \mathbb{R}^d$ an injective continuous map. Then $V := \operatorname{Im} f$ is open in $\mathbb{R}^d$ and f is a homeomorphism between U and V.

Proof of Lemma 3.9. U has a connected component that intersects $\mathbb{R}^{d-1} \times \{0_{\mathbb{C}}\}$. Hence it suffices to prove that U is not homeomorphic to a *connected* subset of $\mathbb{R}^d$.

The assumption that U intersects $\mathbb{R}^{d-1} \times \{0_{\mathbb{C}}\}$ implies that U intersects $\mathbb{R}^{d-1} \times Y$ for every ray Y of star_n, since U is open. Fix two distinct $\rho_0, \rho_1 \in \{1, \dots, n\}$. Then observe that

$$L = \{re^{\rho \cdot 2\pi i/n} : \rho \in \{\rho_0, \rho_1\} \ \& \ r \in \mathbb{R}_{\geq 0}\}$$

and

$$B = \{re^{\rho \cdot 2\pi i/n} : \rho \in \{1, \dots, n\} \setminus \{\rho_0, \rho_1\} \ \& \ r \in \mathbb{R}_{>0}\}$$

satisfy that $L \sqcup B = \mathrm{star}_n$, that L is homeomorphic to $\mathbb{R}$ (by Lemma 3.8 and Example 3.6) and that B is open in star_n. We can lift these properties to the product: we have that

$$(\mathbb{R}^{d-1} \times L) \sqcup (\mathbb{R}^{d-1} \times B) = \mathbb{R}^{d-1} \times \mathrm{star}_n,$$

that $\mathbb{R}^{d-1} \times L \subseteq \mathbb{R}^{d-1} \times \mathrm{star}_n$ is homeomorphic to $\mathbb{R}^d$ and that $\mathbb{R}^{d-1} \times B \subseteq \mathbb{R}^{d-1} \times \mathrm{star}_n$ is open in $\mathbb{R}^{d-1} \times \mathrm{star}_n$. Next we can also pass to the subset U: we have that

$$\big((\mathbb{R}^{d-1} \times L) \cap U\big) \sqcup \big((\mathbb{R}^{d-1} \times B) \cap U\big) = U,$$

that $(\mathbb{R}^{d-1} \times L) \cap U$ is homeomorphic to an open subset of $\mathbb{R}^d$ and that $(\mathbb{R}^{d-1} \times B) \cap U$ is open in U.

Suppose that $f : U \to X \subseteq \mathbb{R}^d$ is a homeomorphism. Then, in particular, the restriction of f to $(\mathbb{R}^{d-1} \times L) \cap U$ is a continuous injection into $\mathbb{R}^d$. By Theorem 3.10, the set $f\big[(\mathbb{R}^{d-1} \times L) \cap U\big]$ is open in $\mathbb{R}^d$, and hence open in X. But $f\big[(\mathbb{R}^{d-1} \times B) \cap U\big]$ is also open in X, and

$$f\big[(\mathbb{R}^{d-1} \times L) \cap U\big] \sqcup f\big[(\mathbb{R}^{d-1} \times B) \cap U\big] = X,$$

so X is disconnected. □

Now we can generalize the fact that $\mathbb{R}^d$ is not homeomorphic to $\mathbb{H}_d$.

Corollary **3.11**. If $n, m, d \geq 0$ and $\mathbb{R}^d \times \mathrm{star}_n$ is homeomorphic to $\mathbb{R}^d \times \mathrm{star}_m$, then $n = m$.

Proof. Suppose that $n \geq m$. First, if $m = 0$ then $n = m$ follows by a simple dimension argument. If $m \in \{1, 2\}$ then $\mathbb{R}^d \times \mathrm{star}_m$ is isomorphic to $\mathbb{R}^{d+1}$ or $\mathbb{H}_{d+1}$, by Example 3.6, which implies $n \in \{1, 2\}$ by Lemma 3.9. Then $n = m$ by Lemma 3.4. Hence assume that $m \geq 3$. Let $\phi : \mathbb{R}^d \times \mathrm{star}_m \to \mathbb{R}^d \times \mathrm{star}_n$ be a homeomorphism. Let $Y_0, \dots, Y_{m-1}$ be the rays of star_m. Then $\mathrm{star}_m \setminus (Y_0 \cup \cdots \cup Y_{m-1}) = \mathrm{star}_0$, so

$$(\mathbb{R}^d \times \mathrm{star}_m) \setminus \big((\mathbb{R}^d \times Y_0) \cup \cdots \cup (\mathbb{R}^d \times Y_{m-1})\big) = \mathbb{R}^d \times \big(\mathrm{star}_m \setminus (Y_0 \cup \cdots \cup Y_{m-1})\big)$$

is homeomorphic to $\mathbb{R}^d$. It follows that

$$(\mathbb{R}^d \times \mathrm{star}_n) \setminus \big(\phi[\mathbb{R}^d \times Y_0] \cup \cdots \cup \phi[\mathbb{R}^d \times Y_{m-1}]\big) \tag{3.1}$$

is homeomorphic to $\mathbb{R}^d$. Each $\phi[\mathbb{R}^d \times Y_i]$ is a connected open subset of $\mathbb{R}^d \times \mathrm{star}_n$ that is homeomorphic to $\mathbb{R}^{d+1}$ (Remark 3.7), so Lemma 3.9 implies that for each $i < m$ there is a ray Z_i of star_n such that

$$\phi[\mathbb{R}^d \times Y_i] \subseteq \mathbb{R}^d \times Z_i.$$

Hence, if $n > m$ then the set in (3.1) has a subset of the form $\mathbb{R}^d \times Z$, where Z is a ray of star_n. But this set $\mathbb{R}^d \times Z$ is homeomorphic to $\mathbb{R}^{d+1}$. □

The following lemma demonstrates how spaces of the form $\mathbb{R}^d \times \mathrm{star}_n$ arise from simplicial complexes.

Lemma **3.12**. Let Σ be a d-dimensional simplicial complex and $\sigma \in \Sigma$ a $(d-1)$-dimensional simplex. If σ has exactly $n \geq 0$ proper successors, then $o(\sigma) = \bigcup\big(\uparrow(\sigma)\big)$ is homeomorphic to $\mathbb{R}^d \times \mathrm{star}_n$ via a homeomorphism χ that maps $\mathrm{relInt}\,\sigma$ onto $\mathbb{R}^d \times \{0_{\mathbb{C}}\}$.

Proof. Let $\sigma_0, \ldots, \sigma_{n-1}$ be the proper successors of σ. Then

$$o(\sigma) = (\mathrm{relInt}\,\sigma) \sqcup (\mathrm{relInt}\,\sigma_0) \sqcup \cdots \sqcup (\mathrm{relInt}\,\sigma_{n-1}).$$

Find $\mathbf{x}^0, \ldots, \mathbf{x}^{n-1}$ such that $\sigma_\rho = \sigma \veebar \mathbf{x}^\rho$ for each $\rho < n$. Let $Y_0, \ldots, Y_{n-1}$ be the rays of star_n. Let $\phi : \mathrm{relInt}\,\sigma \to \mathbb{R}^{d-1}$ be a homeomorphism. Furthermore let $\psi_0 : [0,1) \to [0,\infty)$ be an order-isomorphism. Then define

$$\psi : o(\sigma) \to \mathbb{R}^{d-1} \times \mathrm{star}_n :$$
$$(1-\alpha)\cdot\mathbf{x} + \alpha\cdot\mathbf{v}^\rho \mapsto \big(\phi(\mathbf{x}), \psi_0(\alpha)\cdot e^{\rho\cdot 2\pi i/n}\big).$$

It is easy to prove that ψ is a homeomorphism. □

The last lemma of this section provides a useful translation between topology and posets.

Lemma **3.13**. Let Σ be a simplicial complex and $S \subseteq \Sigma$. Then S is a connected poset iff

$$\bigsqcup_{\sigma \in S} \mathrm{relInt}\,\sigma$$

is connected.

Proof. An easy exercise. For the direction $\Leftarrow$, use normality of Euclidean space and the fact that any two cells in Σ are either disjoint closed sets or they share a nonempty face. □

3.2 Manifold-with-boundary

We next meet the main actors of this chapter.

Definition **3.14** (**manifold-with-boundary**). Let $d \geq 1$. A (topological) *d-dimensional manifold-with-boundary* is a topological space X such that every point $x \in X$ has a neighbourhood that is homeomorphic to $\mathbb{H}_d$ or $\mathbb{R}^d$. The *interior* of X is the set of points in X that have a neighbourhood that is homeomorphic to $\mathbb{R}^d$. The *boundary* of X is the complement of the interior of X. If moreover the boundary of X is empty, then X is a d-dimensional *manifold*.

We also have a direct definition of the boundary:

Remark **3.15**. If X is a d-dimensional manifold-with-boundary, then Lemma 3.4 implies that the boundary of X consists of all points $x \in X$ that have a neighbourhood that is homeomorphic to $\mathbb{H}_d$ via a homeomorphism ϕ satisfying $\phi(x) \in \mathbb{R}^{d-1} \times \{0\}$.

A basic example of a two-dimensional manifold-with-boundary in $\mathbf{plhdr}^2$ is depicted in Figure 3.4. A famous example of a two-dimensional manifold-with-boundary is a Möbius strip. See Figure 3.5.

Lemma **3.16**. Let $\mathbf{P}$ be a d-dimensional polyhedron that is also a manifold-with-boundary, Σ a triangulation of $\mathbf{P}$ and $\sigma \in \Sigma$. Then:

1. there is $\delta \in \Sigma$ of dimension d with $\sigma \subseteq \delta$ (i.e. the poset Σ has uniform height $d+1$);
2. if $\dim \sigma < d-1$, then $\Uparrow(\sigma)$ is a connected poset;
3. if $\dim \sigma = d-1$, then σ has at most two proper successors. The relative interior of σ is either contained in the boundary of $\mathbf{P}$ (in which case σ has exactly one proper successor), or in the interior of $\mathbf{P}$ (in which case σ has exactly two proper successors).

Proof. Choose $\mathbf{x} \in \text{relInt}(\sigma)$. Then $\mathbf{x}$ (as a point in the topological space $\mathbf{P}$) has a neighbourhood V which is disjoint from every simplex in Σ except the ones that contain σ. Since $\mathbf{P}$ is a manifold-with-boundary, $\mathbf{x}$ also has a neighbourhood $U \subseteq V$ that is homeomorphic to $\mathbb{R}^d$ or $\mathbb{H}_d$.

To prove 1, note that $U \subseteq \mathbf{P} = |\Sigma|$. Therefore there exists $\delta \in \Sigma$ of dimension d such that $\delta \cap U \neq \varnothing$. By choice of V, it follows that $\sigma \subseteq \delta$.

2: by Lemma 3.13, it suffices to show that

$$C := \bigcup_{\delta \in \Uparrow(\sigma)} \text{relInt}\, \delta$$

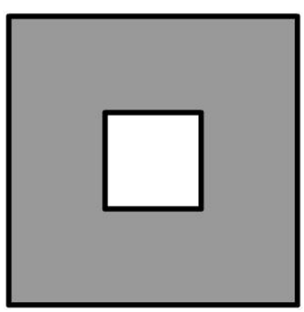


Figure 3.4: a two-dimensional manifold-with-boundary

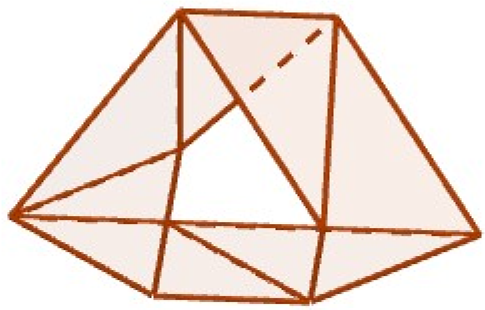


Figure 3.5: a triangulation of a Möbius strip in **cmplx**^{3}

is topologically connected. By [HW41, Theorem IV 4][1], the set

$$U \setminus \operatorname{relInt} \sigma$$

is connected since $\dim \sigma \leq d-2$. But this set intersects $\operatorname{relInt} \delta$ for each $\delta \in \Uparrow(\sigma)$, and is contained in C by choice of V. Hence C is connected.

The first statement in 3 follows from Lemmas 3.9 and 3.12.

The second statement in 3 follows from Lemma 3.12 and Remark 3.15. □

Example **3.17**. For $i, j \in \{-1, 1\}$ let

$$\mathbf{P}_{i,j} = \operatorname{Conv}\left\{(0,0,0),(i,-1,0),(i,1,0),(0,-1,j),(0,1,j)\right\} \subseteq \mathbb{R}^3.$$

Let

$$\mathbf{P} = \mathbf{P}_{1,1} \cup \mathbf{P}_{-1,1} \cup \mathbf{P}_{1,-1} \cup \mathbf{P}_{-1,-1}.$$

$\mathbf{P}$ is the closure of the complement in $[-1,1]^3$ of a squarebased hourglass. See Figure 3.6. Then $\mathbf{P}$ is not a manifold-with-boundary, because any neighbourhood V of the point $\mathbf{0} \in \mathbf{P}$ has the property that $V \setminus \{\mathbf{0}\}$ is not simply connected. However, if Σ is any triangulation of $\mathbf{P}$ and $\sigma \in \Sigma$, then one can verify that the properties 1, 2, 3 in Lemma 3.16 all hold.

Proposition **3.18**. A d-simplex σ is a d-dimensional manifold-with-boundary with boundary $\partial\sigma$ and interior $\operatorname{relInt}\sigma$.

Proof. See [Mau70, Examples 5.4.2]. □

We use this example of a manifold-with-boundary to prove Lemma 2.78.

Proof of Lemma 2.78. It is clear that $\operatorname{sub}(\Delta,\Sigma)$ is monotone. To show that it has the back-property, let $\delta \in \Delta$, $\sigma = \operatorname{sub}(\Delta,\Sigma)(\delta)$ and suppose that σ_1 is a successor of σ in Σ. Let $d = \dim \sigma_1$. Since $\operatorname{fac}(\sigma_1)$ is a subcomplex of Σ, Lemma 2.29 implies that Δ has a subcomplex Θ which is a subdivision of $\operatorname{fac}(\sigma_1)$. By Proposition 3.18, σ_1 is a d-dimensional manifold-with-boundary. By Lemma 2.32 we must have $\delta \in \Theta$. Thus by Lemma 3.16-1, δ has a d-dimensional successor $\delta_1 \in \Theta$. Then $\operatorname{sub}(\Delta,\Sigma)(\delta_1) = \sigma_1$. □

[1]The theorem there is only stated for $\mathbb{R}^d$, but the same proof goes through for $\mathbb{H}_d$.

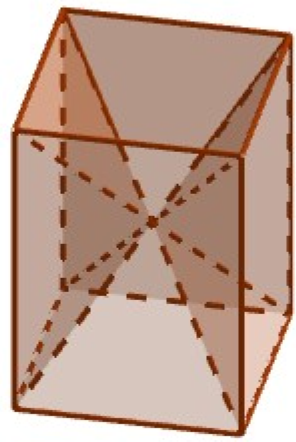


Figure 3.6: A nonexample of a manifold-with-boundary

3.3 The logic of a manifold-with-boundary

We first introduce more examples of manifolds-with-boundary by means of the following proposition:

Proposition **3.19**. Every convex polyhedron is a manifold-with-boundary.

Proof. By [Ada19, Corollary 4.4], any convex polyhedron **C** is homeomorphic to a simplex. Thus Proposition 3.18 implies that **C** is a manifold-with-boundary, since being a manifold-with-boundary is a topological property. □

Hence, quite a lot of polyhedra are manifolds-with-boundary.

In [Ada19, Corollary 4.5 and Theorem 4.10], it was shown that $\mathrm{Log}(\mathbf{C}) = \mathbf{PL}_d$ for every convex d-dimensional polyhedron **C**. We can extend this result to manifolds-with-boundary.

Theorem **3.20**. Consider $d \geq 0$. Let **P** be a d-dimensional polyhedron that is also a manifold-with-boundary. Then

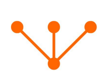
$$\mathrm{Log}(\mathbf{P}) = \mathbf{PL}_d.$$

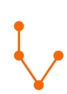
Proof. The inclusion $\mathrm{Log}(\mathbf{P}) \subseteq \mathbf{PL}_d$ follows from Proposition 2.75. For the other inclusion, it suffices to show that

$$\Sigma \setminus \{\varnothing\} \vDash \mathbf{PL}_d$$

(where Σ is a triangulation of **P**), or

$$\Sigma \setminus \{\varnothing\} \vDash \chi\Big(\quad \Big) \wedge \chi\Big(\quad \Big).$$





By [Ada19, Proposition 3.29] it suffices to show that Σ is "1^3-connected" and "$1 \cdot 2$-connected", which together means that for any $\sigma \in \Sigma$ the strict upset $\Uparrow(\sigma)$ is either connected or has exactly two components of height 1 each (in which case it is just an antichain with two elements). Thus let $\sigma \in \Sigma$. If $\dim \sigma < d-1$, then Lemma 3.16-2 implies that $\Uparrow(\sigma)$ is connected. If $\dim \sigma = d-1$, then the first statement in Lemma 3.16-3 implies that $\Uparrow(\sigma)$ is either an antichain with two elements or a singleton. If $\dim \sigma = d$, then $\Uparrow(\sigma) = \varnothing$ is connected. □

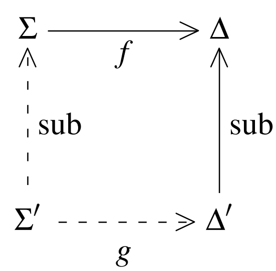
This emphasizes the importance of the logic $\mathbf{PL}_d$. By Proposition 2.75, $\mathbf{PL}_d$ can be thought of as the maximum of the set of all polyhedrally-complete logics of a given height $d+1$. Theorem 3.20 says that this maximum is attained not just for one particular set of polyhedra: the maximum is attained for many different sets of polyhedra, namely at least for every set of polyhedra that are d-dimensional manifolds-with-boundary. In fact, there are even many more sets of polyhedra that have this logic. For example, if $\mathbf{P}$ is a disjoint union of two polyhedra that are respectively a one-dimensional manifold-with-boundary and a two-dimensional manifold-with-boundary, then $\mathbf{P}$ is not a manifold-with-boundary but still $\mathrm{Log}(\mathbf{P}) = \mathbf{PL}_2$.

Chapter 4

Subdividing a p-morphism

The goal of this chapter is to prove Theorem 4.1, which gives us an important connection between subdivisions of complexes and p-morphisms between complexes. We saw in chapter 2 that both subdivisions and p-morphisms are closely related to polyhedral semantics. Hence Theorem 4.1 is of interest in this thesis.

Theorem **4.1**. Let Σ, Δ be simplicial complexes and consider an up-p-morphism $f : \Sigma \rightharpoonup \Delta$. Let Δ' be a subdivision of Δ. Then there exists a subdivision Σ' of Σ such that there is an up-p-morphism $g : \Sigma' \rightharpoonup \Delta'$ with $f \circ \mathrm{sub}(\Sigma', \Sigma) = \mathrm{sub}(\Delta', \Delta) \circ g$ and $\mathrm{dom}\, g = \mathrm{sub}(\Sigma', \Sigma)^{-1}[\mathrm{dom}\, f]$.



Corollary **4.2**. Let $\mathbf{P}, \mathbf{Q} \in \mathbf{plhdr}$ such that some triangulation of $\mathbf{Q}$ is a p-morphic image of some triangulation of $\mathbf{P}$.

1. Every triangulation of $\mathbf{Q}$ is a p-morphic image of some triangulation of $\mathbf{P}$.
2. $\mathrm{Log}(\mathbf{P}) \subseteq \mathrm{Log}(\mathbf{Q})$.
3. $\mathrm{Log}_{\varnothing}(\mathbf{P}) \subseteq \mathrm{Log}_{\varnothing}(\mathbf{Q})$.

Proof. Let Σ be a triangulation of $\mathbf{P}$ and Δ a triangulation of $\mathbf{Q}$ such that there is a surjective p-morphism $f : \Sigma \to \Delta$.

1: let Γ be a triangulation of $\mathbf{Q}$. By Lemma 2.31, there is a common subdivision Δ' of Δ and Γ. By Theorem 4.1, there is a subdivision Σ' of Σ and a p-morphism $g : \Sigma' \to \Delta'$ with

$$\begin{aligned}\big(\mathrm{sub}(\Delta', \Delta) \circ g\big)(\varnothing) &= \big(f \circ \mathrm{sub}(\Sigma', \Sigma)\big)(\varnothing)\\ &= f(\varnothing)\\ &= \varnothing,\end{aligned}$$

which implies $g(\varnothing) = \varnothing$ and hence g is surjective. By Lemma 2.78, $\mathrm{sub}(\Delta', \Gamma) \circ g$ is a surjective p-morphism $\Sigma' \to \Gamma$.

Items 2 and 3 follow. For example, if $\varphi \notin \mathrm{Log}(\mathbf{Q})$, then φ can be falsified in $\Delta \setminus \{\varnothing\}$ for some triangulation Δ of $\mathbf{Q}$. Then by 1, there exists a triangulation Σ of $\mathbf{P}$ such that Δ is a p-morphic image of Σ. It follows that $\Delta \setminus \{\varnothing\}$ is a p-morphic image of $\Sigma \setminus \{\varnothing\}$, and therefore φ can be falsified in $\Sigma \setminus \{\varnothing\}$ whence $\varphi \notin \mathrm{Log}(\mathbf{P})$. □

Discussion **4.3**. We shall prove Theorem 4.1 by taking "a lot" of barycentric subdivisions. By no means do we claim that our construction is efficient. To make more concrete what we mean by this, consider an up-p-morphism $f : \Sigma \rightharpoonup \Delta$ and assume that Theorem 4.1 holds. Consider $\Delta' = \Delta^+$. Then by Lemmas 2.56 and 2.78, we can find Σ', g as in Theorem 4.1 with the additional property that $\Sigma' = \Sigma^{+n}$ for some n. But what is the smallest possible n? This quantitative question is particularly relevant because Theorem 4.1 and Corollary 4.2 are applied in computational observations (chapter 7): we speculate that it will also pop up in future applications and implementations of polyhedral semantics. We leave it to future research to investigate this quantitative issue (in generality as well as for restricted classes of complexes). For now, we only mention without proof that $n = 2$ suffices if $\Sigma, \Delta \in \mathbf{cmplx}_1$.

We start with two basic properties of the double barycentric subdivision.

Claim **4.4**. Let Σ be a simplicial complex and $\sigma \in \Sigma$. Let

$$f : \mathscr{C}^2(\Sigma) \to \Sigma^{++} : \quad c \mapsto \mathrm{cb}_{\Sigma^+}\left(\mathrm{cb}_\Sigma[c]\right)$$

be the canonical isomorphism and $c \in \mathscr{C}^2(\Sigma)$. Then $\sigma \cap f(c) \neq \varnothing$ iff c has an element that is a subset of $\mathrm{fac}(\sigma)$.

Proof. From the definitions,

$$f(c) = \underline{\bigvee} \mathbf{b}\left[\mathrm{cb}_\Sigma[c]\right].$$

It holds that $\sigma \cap f(c) \neq \varnothing$, iff: $f(c)$ has a vertex that lies in σ, iff: there exists $\sigma' \in \mathrm{cb}_\Sigma[c]$ such that $\mathbf{b}(\sigma') \in \sigma$, iff: there exists $c' \in c$ such that $\underline{\bigvee} \mathbf{b}[c'] \in \sigma$. This relation is equivalent to $\mathbf{b}[c'] \subseteq \sigma$, and hence to $c' \subseteq \mathrm{fac}(\sigma)$ (by Lemma 2.32). □

Claim **4.5**. Let Σ be a simplicial complex and $\sigma \in \Sigma$. Then there exists $\sigma^{++} \in \Sigma^{++}$ such that $\sigma^{++} \subseteq \mathrm{relInt}\,\sigma$ and $\dim \sigma^{++} = \dim \sigma$.

Proof. W.l.o.g., we assume that $\Sigma = \mathrm{fac}(\sigma)$. Let $\{\sigma_0 < \cdots < \sigma_d = \sigma\} \in \mathscr{C}(\Sigma)$ be a maximal chain, so $d = \dim \sigma$. Let

$$c = \left\{\{\sigma_i < \cdots < \sigma_d\} : 0 \leq i \leq d\right\} \in \mathscr{C}^2(\Sigma).$$

Let $\sigma^{++} \in \Sigma^{++}$ be the cell that corresponds to c. Then $\dim \sigma^{++} = d$. Claim 4.4 implies that σ^{++} is disjoint from every proper face of σ. It follows that $\sigma^{++} \subseteq \mathrm{relInt}\,\sigma$. □

We aim to prove Theorem 4.1 by building Σ' "bottom-up": first subdividing all the line segments in Σ, then subdividing all the triangles in Σ, etc.

Notation **4.6**. Let $\Sigma \in \mathbf{cmplx}$. Let

$$\mathscr{U}(\Sigma) = \left\{c \in \mathscr{C}(\Sigma) \setminus \{\varnothing\} : \text{for all } c' \in \uparrow^{\mathscr{C}(\Sigma)}(c), \text{ if } \min c' = \min c \text{ then } c = c'\right\}.$$

Equivalently, $c \in \mathscr{U}(\Sigma)$ means that for all $\sigma \in c$ either σ is maximal in Σ or c contains an immediate successor of $\sigma \in \Sigma$, see Figure 4.1.

If $c \in \mathscr{U}(\Sigma)$, then $\dim(\min c) + \#c = \dim(\max c) + 1$. This equals $\dim(\Sigma) + 1$ if Σ has uniform height. So in this case we have $\#\text{vtc}(\min c) + \#\mathbf{b}[c] = \dim \Sigma + 2$. However, $\text{vtc}(\min c) \cup \mathbf{b}[c]$ is usually affinely dependent, since $\mathbf{b}(\min c)$ is an affine combination of $\text{vtc}(\min c)$. Lemma 4.9 will combine a *translated* copy of $\min c$ with $\mathbf{b}[c]$, to build a simplex of dimension $\dim \Sigma + 1$, for every $c \in \mathscr{U}(\Sigma)$, and these together will form a new complex. This is the basic idea of each inductive step in the proof of Theorem 4.1.

Lemma **4.7**. Let σ be a d-dimensional simplex with $d \geq 0$, $\mathbf{x} \in \sigma$ and $\alpha \in [0,1]$. Then there is a unique $\mathbf{y} \in \sigma$ such that for some $c = \{\sigma_0 < \cdots < \sigma_k = \sigma\} \in \mathscr{U}\big(\text{fac}(\sigma)\big)$ and some sequence $\eta_0, \ldots, \eta_k$ of nonnegative real numbers we have $\mathbf{y} \in \sigma_0$ and

$$\alpha + \sum_{i=0}^{k} \eta_i = 1 \quad \& \quad \mathbf{x} = \alpha \mathbf{y} + \sum_{i=0}^{k} \eta_i \mathbf{b}(\sigma_i). \tag{4.1}$$

Proof. The proof goes by induction on d. Write $\text{vtc}(\sigma) = \{\mathbf{x}^0, \ldots, \mathbf{x}^d\}$. Let $\beta_0, \ldots, \beta_d \in [0,1]$ be the barycentric coordinates of $\mathbf{x}$:

$$\sum_{i=0}^{d} \beta_i = 1 \quad \& \quad \mathbf{x} = \sum_{i=0}^{d} \beta_i \mathbf{x}^i.$$

Let j be an index i for which β_i is smallest, and let $\rho = (d+1)\beta_j$. Distinguish two cases:

- $\rho \geq 1 - \alpha$. Take $c = \{\sigma\}$,

  $$\mathbf{y} = \sum_{i=0}^{d} \left(\beta_i - \frac{1-\alpha}{d+1}\right) \mathbf{x}^i$$

  and $\eta_0 = 1 - \alpha$. It is easy to see that the choice of $\mathbf{y}$ is forced if we take $c = \{\sigma\}$. If instead we took c to be a larger chain $\{\sigma_0 < \cdots < \sigma_k = \sigma\}$ with $k > 0$, then we could find $i \leq d$ with

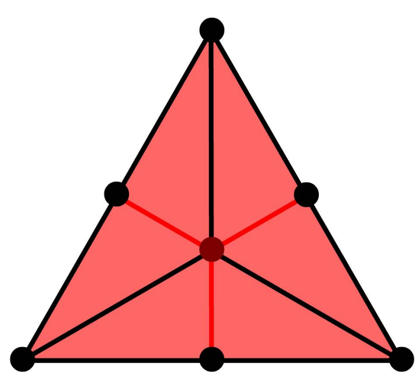


Figure 4.1: Σ^+ with $\text{cb}_\Sigma\left[\mathscr{U}(\Sigma)\right]$ coloured red, where $\Sigma = \text{fac}(\tau)$ for a triangle τ

$\mathbf{x}^i \notin \sigma_{k-1}$, whence $\mathbf{x}^i \notin \sigma_0, \ldots, \sigma_{k-1}$, and (4.1) (written out in the barycentric coordinates $\beta_0, \ldots, \beta_d$) would imply that β_i is at most

$$\frac{\eta_k}{d+1} \leq \frac{1-\alpha}{d+1} \leq \beta_i,$$

so $\eta_k = 1 - \alpha$ hence we could have chosen the chain $\{\sigma\}$ after all with the same $\mathbf{y}$.

- $\rho < 1 - \alpha$. Consider

$$\mathbf{x}' = \frac{1}{1-\rho} \sum_{i=0}^{d} (\beta_i - \beta_j)\mathbf{x}^i.$$

Here $\mathbf{x}'$ is written in barycentric coordinates, since

$$\frac{1}{1-\rho} \sum_{i=0}^{d} (\beta_i - \beta_j) = \frac{1}{1-\rho}(1-\rho) = 1.$$

Hence let σ' be a proper face of σ such that $\mathbf{x}' \in \operatorname{relInt} \sigma'$ (Lemma 2.32). Let

$$\alpha' = \frac{\alpha}{1-\rho}.$$

By inductive hypothesis, there is a $\mathbf{y}' \in \sigma'$ such that for some $c' = \{\sigma'_0 < \cdots < \sigma'_{k'} = \sigma'\} \in \mathscr{U}\big(\operatorname{fac}(\sigma')\big)$ and some sequence $\eta'_0, \ldots, \eta'_{k'}$ of nonnegative real numbers we have $\mathbf{y}' \in \sigma_0$ and

$$\alpha' + \sum_{i=0}^{k'} \eta'_i = 1 \quad \& \quad \mathbf{x}' = \alpha'\mathbf{y}' + \sum_{i=0}^{k'} \eta'_i \mathbf{b}'(\sigma_i).$$

Let $\mathbf{y} = \mathbf{y}'$, $k = k' + (d - \dim \sigma')$, let $c = \{\sigma_0 < \cdots < \sigma_k = \sigma\}$ be some extension of c' that lies in $\mathscr{U}\big(\operatorname{fac}(\sigma)\big)$ (so $\sigma_i = \sigma'_i$ for $i \leq k'$) and

$$\eta_i = \begin{cases} \eta'_i(1-\rho) & (i \leq k') \\ 0 & (k' < i < k) \\ \rho & (i = k). \end{cases}$$

Then

$$\sum_{i=0}^{k} \eta_i = (1-\rho)\Big(\sum_{i=0}^{k'} \eta'_i\Big) + \rho = (1-\rho) + \rho = 1$$

and

$$\begin{aligned} \mathbf{x} &= \Big(\sum_{i=0}^{d} (\beta_i - \beta_j)\mathbf{x}^i\Big) + \Big(\sum_{i=0}^{d} \beta_j \mathbf{x}^i\Big) \\ &= (1-\rho)\mathbf{x}' + \rho\mathbf{b}(\sigma) \\ &= (1-\rho)\alpha'\mathbf{y}' + (1-\rho)\Big(\sum_{i=0}^{k'} \eta'_i \mathbf{b}(\sigma'_i)\Big) + \eta_k \mathbf{b}(\sigma) \\ &= \alpha\mathbf{y} + \Big(\sum_{i=0}^{k'} \eta_i \mathbf{b}(\sigma_i)\Big) + \eta_k \mathbf{b}(\sigma) \\ &= \alpha\mathbf{y} + \sum_{i=0}^{k} \eta_i \mathbf{b}(\sigma_i). \end{aligned}$$

To prove uniqueness of $\mathbf{y}$, suppose that $e = \{\gamma_0 < \cdots < \gamma_\ell = \sigma\} \in \mathscr{U}\big(\operatorname{fac}(\sigma)\big)$ and $v_0, \ldots, v_\ell$ are nonnegative numbers such that for some $\mathbf{z} \in \gamma_0$ we have

$$\alpha + \sum_{i=0}^{\ell} v_i = 1 \quad \& \quad \mathbf{x} = \alpha\mathbf{z} + \sum_{i=0}^{\ell} v_i \mathbf{b}(\gamma_i).$$

We show that $\mathbf{y} = \mathbf{z}$. First note that $e \neq \{\sigma\}$, or else we would have

$$\beta_j \geq \frac{v_\ell}{d+1} = \frac{1-\alpha}{d+1} > \frac{\rho}{d+1} = \beta_j.$$

At least one barycentric coordinate of

$$\alpha\mathbf{z} + \sum_{i=0}^{\ell-1} v_i \mathbf{b}(\gamma_i)$$

(as a convex combination of $\operatorname{vtc}(\sigma)$) is zero. It follows that $v_\ell = \rho = \eta_k$ and hence that

$$\alpha\mathbf{y} + \sum_{i=0}^{k-1} \eta_i \mathbf{b}(\sigma_i) = \alpha\mathbf{z} + \sum_{i=0}^{\ell-1} v_i \mathbf{b}(\gamma_i).$$

Thus

$$\mathbf{x}' = \frac{1}{1-\rho}\Big(\alpha\mathbf{z} + \sum_{i=0}^{\ell-1} v_i \mathbf{b}(\gamma_i)\Big) \in \gamma_{\ell-1}.$$

Therefore $\sigma' \subseteq \gamma_{\ell-1}$, by choice of σ' and Lemma 2.36. Find a chain $e' = \{\sigma_0' < \cdots < \sigma_{k-1}' = \gamma_{\ell-1}\} \in \mathscr{U}\big(\operatorname{fac}(\gamma_{\ell-1})\big)$ with $c' \subseteq e'$. Further, letting $v_i' = v_i/(1-\rho)$ for $i < \ell$ and $\eta_i' = 0$ for $k' < i < k$ we have

$$\alpha'\mathbf{z} + \sum_{i=0}^{\ell-1} v_i' \mathbf{b}(\gamma_i) = \mathbf{x}',$$

$$\begin{aligned}\alpha' + \sum_{i=0}^{\ell-1} v_i' &= \frac{1}{1-\rho}\Big(\alpha + \sum_{i=0}^{\ell-1} v_i\Big) \\ &= \frac{1}{1-\rho}\big(\alpha + (1-\alpha-v_\ell)\big) \\ &= 1,\end{aligned}$$

$$\alpha'\mathbf{y} + \sum_{i=0}^{k-1} \eta_i' \mathbf{b}(\sigma_i') = \mathbf{x}'$$

and

$$\alpha' + \sum_{i=0}^{k-1} \eta_i' = 1.$$

Thus, by the inductive hypothesis applied to the simplex $\gamma_{\ell-1}$, it follows that $\mathbf{y} = \mathbf{z}$.

$\square$

Notation **4.8**. In this chapter, write

$$\phi_\rho : \mathbb{R}^d \to \mathbb{R}^d : \\ \mathbf{x} \mapsto \rho \cdot \mathbf{x}$$

for $\rho \in \mathbb{R}$.

This is an affine map, and it is bijective unless $\rho = 0$. We shall now see how to use the operation $\mathscr{U}$ to build complexes.

Lemma **4.9**. Let $\theta \subseteq \mathbb{R}^d$ be a d-simplex with $\mathbf{b}(\theta) = \mathbf{0}$. Let Σ be a simplicial complex such that $|\Sigma| = \partial\theta$. For any $\mu \in (0,1)$, there is a unique simplicial complex Δ such that $\Sigma \sqcup \phi_\mu [\![\Sigma]\!]^+ \subseteq \Delta$,

$$|\Delta| = \bigcup_{\rho \in [\mu,1]} \phi_\rho[\partial\theta] \tag{4.2}$$

and the set of d-dimensional cells of Δ equals

$$\Big\{ \min(c) \veebar \phi_\mu \left[\mathrm{cb}_\Sigma(c) \right] : c \in \mathscr{U}(\Sigma) \Big\}. \tag{4.3}$$

See Figure 4.2.

Proof. First we sketch the basic idea underlying this proof. Consider a triangle $\tau_0 \subseteq \mathbb{R}^3$. (Think of τ_0 as an element of Σ in case $d = 3$.) See Figure 4.3(b). Let τ_1 be a translation of τ_0 in a direction orthogonal to the plane through τ_0. See Figure 4.3(a). Recall that

$$U = \mathrm{cb}_{\mathrm{fac}(\tau_1)} \left[\mathscr{U}\left(\mathrm{fac}(\tau_1)\right) \right] \subseteq \mathrm{fac}(\tau_1)^+$$

consists of the red elements in Figure 4.1. Then we can fill up the space between τ_0 and τ_1, i.e. the prism $\mathrm{Conv}(\tau_0 \sqcup \tau_1)$, using U: first we construct a tetrahedron that has $\mathbf{b}(\tau_1) \in U$ as a vertex and τ_0 as a face, as in Figure 4.3(c). Next we add three tetrahedra that have the three respective line segments in U as faces, as in Figure 4.3(d). The one-dimensional faces of τ_0 are also faces of these three tetrahedra. Finally we add six tetrahedra that have the six respective triangles in U as faces, as in Figure 4.3(e). Each vertex of τ_0 is a vertex of exactly two of these six tetrahedra. This construction generalizes to arbitrary dimensions.

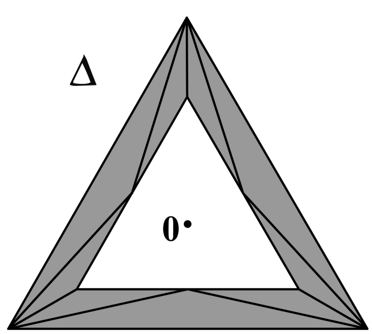


Figure 4.2: illustration of Lemma 4.9 in case $d = 2$ and $\Sigma = \mathrm{propfac}(\theta)$ (θ is the largest triangle and $\phi_\mu[\theta]$ is the inner triangle)

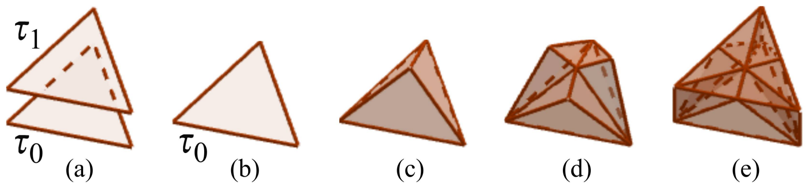


Figure 4.3: Filling up the space between two triangles

Formally, let

$$\begin{aligned} f : \mathscr{C}(\Sigma) &\to \phi_\mu [\![\Sigma]\!]^+ \\ c &\mapsto \phi_\mu \left[\mathrm{cb}_\Sigma(c) \right] = \mathrm{cb}_{\phi_\mu [\![\Sigma]\!]} \left(\phi_\mu [\![c]\!] \right) \end{aligned}$$

by Lemma 2.55-3. For $c \in \mathscr{U}(\Sigma)$ write $\delta(c) = \min(c) \veebar f(c)$. This is well-defined because $\mathrm{vtc}\left(\min(c)\right) \sqcup \mathbf{b}\left[\phi_\mu [\![c]\!]\right]$ is affinely independent since $\mathrm{vtc}\left(\min(c)\right) \sqcup \mathbf{b}\left[c \setminus \{\min c\}\right]$ is affinely independent. Further, we count that $\dim \delta(c) = d$. The set in (4.3) is the set of all the various $\delta(c)$. Let Δ be the set of all the various $\delta(c)$ and their faces. To check that Δ is a simplicial complex, suppose that $c, c' \in \mathscr{U}(\Sigma)$. We show that

$$\delta(c) \cap \delta(c') = \left((\min c) \cap (\min c')\right) \veebar \left(f(c) \cap f(c')\right).$$

The inclusion $\supseteq$ is obvious. Suppose that $\mathbf{x} \in \delta(c) \cap \delta(c')$. Then there are $\mathbf{y} \in \min(c), \mathbf{z} \in f(c), \mathbf{y}' \in \min(c'), \mathbf{z}' \in f(c'), \nu, \nu' \in [0,1]$ satisfying

$$\nu \mathbf{y} + (1-\nu)\mathbf{z} = \mathbf{x} = \nu' \mathbf{y}' + (1-\nu')\mathbf{z}'.$$

Note that

$$\mathbf{y} \in \max c \quad \& \quad \mathbf{z} \in f(c) \subseteq \phi_\mu [\max c]$$

since $\mathrm{cb}_\Sigma(c) \subseteq \max c$. It follows that

$$\mathbf{x} \in \phi_{\nu + (1-\nu)\mu} [\max c] \subseteq \phi_{\nu + (1-\nu)\mu} [\partial \theta]. \tag{4.4}$$

Similarly,

$$\mathbf{x} \in \phi_{\nu' + (1-\nu')\mu} [\partial \theta].$$

Hence $\nu = \nu'$. Write $c = \{\sigma_0 < \cdots < \sigma_k\}$ and $c' = \{\sigma_0' < \cdots < \sigma_{k'}'\}$. There are nonnegative numbers ν_i, ν_i' such that both sequences sum to 1 and

$$\mathbf{z} = \sum_{i=0}^{k} \nu_i \mu \cdot \mathbf{b}(\sigma_i) \quad \& \quad \mathbf{z}' = \sum_{i=0}^{k'} \nu_i' \mu \cdot \mathbf{b}(\sigma_i').$$

With

$$\zeta := \frac{1}{\nu + (1-\nu)\mu},$$

we have

$$\begin{aligned} \zeta \mathbf{x} &= \zeta \nu \mathbf{y} + \zeta (1-\nu) \left(\sum_{i=0}^{k} \nu_i \mu \cdot \mathbf{b}(\sigma_i) \right) \\ &= \zeta \nu \mathbf{y} + \left(\sum_{i=0}^{k} \zeta (1-\nu) \nu_i \mu \cdot \mathbf{b}(\sigma_i) \right) \end{aligned}$$

and

$$\zeta\nu + \sum_{i=0}^{k} \zeta(1-\nu)\nu_i\mu = \zeta\nu + \zeta(1-\nu)\mu = 1,$$

so $\zeta\mathbf{x} \in \max c$. Similarly, $\zeta\mathbf{x} \in \max c'$. So $\zeta\mathbf{x} \in \sigma := (\max c) \cap (\max c')$. By Lemma 4.7, there exist $\mathbf{w} \in \sigma$, $e = \{\gamma_0 < \cdots < \gamma_r = \sigma\} \in \mathscr{U}\big(\operatorname{fac}(\sigma)\big)$ and $\mu_0, \ldots, \mu_r \geq 0$ such that

$$\zeta\nu + \sum_{i=0}^{r} \mu_i = 1 \quad \& \quad \zeta\mathbf{x} = \zeta\nu\mathbf{w} + \sum_{i=0}^{r} \mu_i \mathbf{b}(\gamma_i).$$

Let $\{\gamma_0 < \cdots < \gamma_\ell = \max c\}$ be a superset of e that belongs to $\mathscr{U}\big(\operatorname{fac}(\max c)\big)$. Putting $\mu_i = 0$ for $r < i \leq \ell$, we get

$$\zeta\mathbf{x} = \zeta\nu\mathbf{w} + \sum_{i=0}^{\ell} \mu_i \mathbf{b}(\gamma_i).$$

Now uniqueness of $\mathbf{y}$ in Lemma 4.7 implies that $\mathbf{y} = \mathbf{w}$. Similarly $\mathbf{y}' = \mathbf{w}$. Thus $\mathbf{y} = \mathbf{y}'$, and hence also $\mathbf{z} = \mathbf{z}'$. It follows $\mathbf{y} \in (\min c) \cap (\min c')$ and $\mathbf{z} \in f(c) \cap f(c')$, and so

$$\mathbf{x} \in \big((\min c) \cap (\min c')\big) \vee f(c \cap c').$$

This confirms that Δ is a simplicial complex.

Clearly $\Sigma \sqcup \phi_\mu[\![\Sigma]\!]^+ \subseteq \Delta$.

We proceed to prove (4.2). We have already seen the inclusion $\subseteq$ in (4.4). For the inclusion $\supseteq$, let $\rho \in [\mu, 1]$ and $\mathbf{x} \in \phi_\rho[\partial\theta]$. Let $\zeta = 1/\rho$ and

$$\nu = \frac{\rho - \mu}{1 - \mu} \in [0, 1].$$

In view of Proposition 3.18 and Lemma 3.16-1, find a $(d-1)$-dimensional $\sigma \in \Sigma$ with $\zeta\mathbf{x} \in \sigma$. By Lemma 4.7, there exist $\mathbf{y} \in \sigma$, $c = \{\sigma_0 < \cdots < \sigma_k = \sigma\} \in \mathscr{U}\big(\operatorname{fac}(\sigma)\big)$ and $\eta_0, \ldots, \eta_k \geq 0$ such that $\mathbf{y} \in \sigma_0$ and

$$\zeta\nu + \sum_{i=0}^{k} \eta_i = 1 \quad \& \quad \zeta\mathbf{x} = \zeta\nu\mathbf{y} + \sum_{i=0}^{k} \eta_i \mathbf{b}(\sigma_i).$$

Therefore

$$\mathbf{x} = \nu\mathbf{y} + (1-\nu)\mu \cdot \sum_{i=0}^{k} \frac{\eta_i}{\zeta\mu(1-\nu)} \mathbf{b}(\sigma_i)$$

while

$$\sum_{i=0}^{k} \frac{\eta_i}{\zeta\mu(1-\nu)} = \frac{1-\zeta\nu}{\zeta\mu(1-\nu)} = \frac{\frac{\zeta\mu-\mu}{1-\mu}}{\zeta\mu \cdot \frac{1-1/\zeta}{1-\mu}} = \frac{\zeta\mu - \mu}{\zeta\mu(1-1/\zeta)} = \frac{\zeta - 1}{\zeta(1-1/\zeta)} = 1,$$

so $\mathbf{x} \in \delta(c)$. □

Remark **4.10**. In Lemma 4.9 and equation (4.2), note that $|\Delta|$ is a d-dimensional manifold-with-boundary. (The proof of this is similar to the proof that a simplex is a manifold-with-boundary.)

The next lemma is an important tool for further subdividing cells in a simplicial complex. The idea is that every existing cell σ of non-maximal dimension receives a new successor that is not a successor of other existing cells (except the faces of σ). In proving Theorem 4.1 (more specifically the key Lemma 4.14 below), this will be useful in making sure that the map under construction enjoys the back-property.

Lemma **4.11**. Let δ be a d-simplex and Σ a simplicial complex with $|\Sigma| = \partial\delta$. Then there exist a triangulation Δ of δ such that $\Sigma \subseteq \Delta$ and a map $h : \Sigma \to \Delta$ such that

$$\forall\sigma \in \Sigma : \sigma \subseteq h(\sigma) \ \&\ \dim h(\sigma) = d$$

and

$$\forall\sigma, \sigma' \in \Sigma : \sigma \neq \sigma' \implies h(\sigma) \cap h(\sigma') = \sigma \cap \sigma'.$$

In particular,

$$\forall\sigma \in \Sigma : h(\sigma) \cap (\partial\delta) = \sigma.$$

See Figure 4.4.

Proof. W.l.o.g., assume that $\mathbf{b}(\delta) = \mathbf{0}$ (Lemmas 2.55-1 and 2.63). Choose $\mu \in (0,1)$. By Lemma 4.9, there exists a simplicial complex Γ such that $\Sigma \sqcup \phi_\mu[\![\Sigma]\!]^+ \subseteq \Gamma$,

$$|\Gamma| = \bigcup_{\rho\in[\mu,1]} \phi_\rho[\partial\delta]$$

and the set of d-dimensional cells in Γ equals (4.3). Then for each $\gamma \in \Gamma$, the set $\gamma \cap \phi_\mu[\partial\delta]$ is a face of γ. Also, $\phi_\mu[\![\Sigma]\!]^+$ is a subcomplex of Γ with carrier $\phi_\mu[\partial\delta]$. By Lemma 2.42, there exists a subdivision Γ' of Γ such that $\phi_\mu[\![\Sigma]\!]^{+++} \subseteq \Gamma'$ and

$$\mathrm{vtc}(\Gamma') = \mathrm{vtc}(\Gamma) \cup \mathrm{vtc}\left(\phi_\mu[\![\Sigma]\!]^{+++}\right).$$

By Remark 2.43 we have $\Sigma \subseteq \Gamma'$. Choose $g : \Sigma \setminus \{\varnothing\} \to \mathscr{U}(\Sigma)$ such that $\sigma = \min g(\sigma)$ for each $\sigma \in \Sigma \setminus \{\varnothing\}$. Let f be as in the proof of Lemma 4.9. By Claim 4.5, there exists $g_0 : \Sigma \setminus \{\varnothing\} \to \phi_\mu[\![\Sigma]\!]^{+++}$ such that $g_0(\sigma) \subseteq \mathrm{relInt}\, f\big(g(\sigma)\big)$ and $\dim g_0(\sigma) = \dim f\big(g(\sigma)\big)$ for each $\sigma \in \Sigma \setminus \{\varnothing\}$. Now, by the definition of Γ' (taken from the proof of Lemma 2.42) we can let

$$h(\sigma) = \sigma \veebar g_0(\sigma) \in \Gamma'.$$

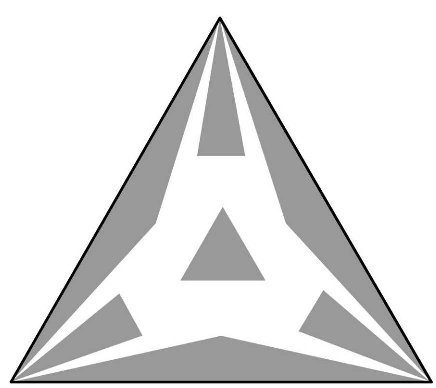


Figure 4.4: $\Sigma \sqcup \mathrm{Im}\, h$ in Lemma 4.11 in case $d = 2$ and $\Sigma = \mathrm{propfac}(\delta)$

Then $\dim h(\sigma) = d$. Choose $\nu \in (0,1)$. By Lemma 4.9, there exists a simplicial complex Θ_0 with carrier

$$\bigcup_{\rho\in[\nu,1]} \phi_\rho\left[\partial\phi_\mu[\delta]\right]$$

such that $\phi_\mu[\![\Sigma]\!]^{+++} \subseteq \Theta_0$ and Θ_0 has a subcomplex Π with carrier $\phi_{\mu\nu}[\partial\delta]$. Then $\Theta = \Gamma' \cup \Theta_0$ is a simplicial complex with carrier

$$\bigcup_{\rho\in[\mu\nu,1]} \phi_\rho[\partial\delta].$$

By Lemma 2.41, there exists a triangulation Θ_1 of $\phi_{\mu\nu}[\delta]$ such that $\Pi \subseteq \Theta_1$. Hence $\Delta = \Theta \cup \Theta_1$ is a triangulation of δ. Let $h(\varnothing)$ be some d-dimensional cell of Θ_1.

Suppose that $\sigma, \sigma' \in \Sigma$ are distinct. If $\sigma = \varnothing$, then $h(\sigma) \subseteq \phi_{\mu\nu}[\delta]$ while

$$h(\sigma') \subseteq \bigcup_{\rho\in[\mu,1]} \phi_\rho[\partial\delta],$$

so $h(\sigma) \cap h(\sigma') = \varnothing$. Therefore, assume that $\sigma, \sigma' \in \Sigma \setminus \{\varnothing\}$. Then $g(\sigma) \neq g(\sigma')$, so $f\big(g(\sigma)\big) \neq f\big(g(\sigma')\big)$, so $g_0(\sigma) \cap g_0(\sigma') = \varnothing$ by Lemma 2.32. Since $h(\sigma) \cap h(\sigma')$ is a face of $h(\sigma)$ and of $h(\sigma')$, this must be a face of $\sigma \cap \sigma'$; and therefore $h(\sigma) \cap h(\sigma') = \sigma \cap \sigma'$. □

We are now sufficiently prepared to state and prove the key Lemma 4.14. In order to simplify the presentation of this lemma a little, we introduce a poset called *topless Boolean algebra*:

Notation **4.12**. For $d \geq 0$,

$$\mathrm{BA}_d^- = \big(\mathscr{P}\{a_0, \ldots, a_{d-1}\}\big) \setminus \{a_0, \ldots, a_{d-1}\}$$

where $a_0, a_1, \ldots$ are some arbitrary distinct objects. BA_d^- is partially ordered by inclusion. See Figure 4.5.

Remark **4.13**. If σ is a d-simplex, then the complex $\mathrm{propfac}(\sigma)$ is isomorphic to BA_{d+1}^-, since $\mathrm{fac}(\sigma) \cong \mathscr{P}\{a_0, \ldots, a_{d-1}\}$.

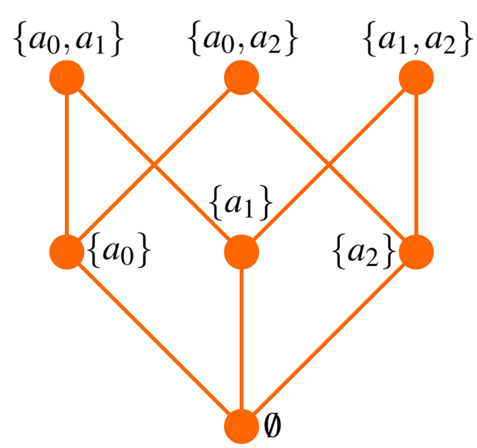


Figure 4.5: BA_3^-

The following lemma is the key to each inductive step in the proof of Theorem 4.1.

Lemma **4.14**. Let θ be a d-dimensional simplex with $d > 0$ and Σ a simplicial complex satisfying $|\Sigma| = \partial\theta$. Let $S \subseteq \Sigma \setminus \{\varnothing\}$ be a subset and $f : S \to P := \mathrm{BA}^{-}_{d'+1}$ be a monotone map. Then there exists a triangulation Δ of θ such that $\Sigma \subseteq \Delta$ and f has a monotone extension $\overline{f} : S \sqcup (\Delta \setminus \Sigma) \to P$ that is surjective if $d \geq d'$, that satisfies the back-property on $\Delta \setminus \Sigma$ and if

$$\sigma \in S \;\&\; \mathrm{dpt}^{\Sigma}(\sigma) \geq \max\left\{ \mathrm{dpt}^{P}\left(f(\sigma)\right), 1 \right\} \tag{4.5}$$

then σ has a (proper) successor $\delta \in \Delta \setminus \Sigma$ with $\overline{f}(\delta) = \overline{f}(\sigma)$.

Example **4.15**. Let θ be a triangle and $\Sigma = \mathrm{propfac}(\theta)$. Then there exist only finitely many partial maps $\Sigma \setminus \{\varnothing\} \rightharpoonup \mathrm{BA}^{-}_{2}$. In particular, there are only finitely many monotone maps $\Sigma \setminus \{\varnothing\} \to \mathrm{BA}^{-}_{2}$. If we colour the three points of BA^{-}_{2} as in Figure 4.7, we can depict a map $\Sigma \setminus \{\varnothing\} \to \mathrm{BA}^{-}_{2}$ by colouring $\Sigma \setminus \{\varnothing\}$ using the same three colours. It turns out that, up to a suitable notion of equivalence, there are exactly eleven monotone maps $f : \Sigma \setminus \{\varnothing\} \to \mathrm{BA}^{-}_{2}$. For every such map f, we can apply Lemma 4.14 to find a triangulation of θ and an extension of f. Again such an extension can be displayed using the colour code in Figure 4.7. See Figure 4.6.

Proof of Lemma 4.14. W.l.o.g. we may assume that $\theta \subseteq \mathbb{R}^d$ and $\mathbf{b}(\theta) = \mathbf{0}$ (Lemmas 2.55-1 and 2.63).

We start by choosing an arbitrary $\mu \in (0,1)$ and letting Δ_0 be the complex from Lemma 4.9. Write $\Sigma_0 = \phi_\mu[\![\Sigma]\!]^{+}$. Let

$$\begin{aligned} g : \mathscr{C}(\Sigma) &\to \Sigma_0 : \\ c &\mapsto \phi_\mu\left[\mathrm{cb}_\Sigma(c)\right] \end{aligned}$$

be the canonical isomorphism. Let $\Sigma_1 = \Sigma_0^{++}$. As in the proof of Lemma 4.11, by Lemma 2.42 there exists a unique subdivision Δ_1 of Δ_0 such that $\Sigma_1 \subseteq \Delta_1$ and $\mathrm{vtc}(\Delta_1) = \mathrm{vtc}(\Delta_0) \cup \mathrm{vtc}(\Sigma_1)$, and we have $\Sigma \subseteq \Delta_1$. Next we prove that we can find a monotone extension $f_1 : S \sqcup (\Delta_1 \setminus \Sigma) \to P$ of f such that

$$\mathrm{dpt}^{\Delta_1}(\delta) \geq \mathrm{dpt}^{P}\left(f_1(\delta)\right) \tag{4.6}$$

for all $\delta \in \Delta_1 \setminus \Sigma$. For each $\sigma \in \Sigma$, fix some maximal element $p_\sigma \in P$ in such a way that $f(\sigma) \leq p_\sigma$ for all $\sigma \in S$. The value of $f_1(\delta)$, where $\delta \in \Delta_1$ is d-dimensional, is set as follows: find a d-dimensional $\delta_0 \in \Delta_0$ such that $\delta \subseteq \delta_0$; then (by definition of Δ_0) find $c \in \mathscr{U}(\Sigma)$ such that $\delta_0 = g(c) \veebar (\min c)$; then by the definition of Δ_1 (from the proof of Lemma 2.42) there exists a cell $\sigma_1 \in \Sigma_1$ such that $\sigma_1 \subseteq g(c)$ and $\delta = \sigma_1 \veebar (\min c)$. Then $\dim \sigma_1 = \dim g(c)$. Then let $f_1(\delta)$ be p_σ, where $\sigma = \max g^{-1}(\sigma_0)$, where σ_0 is the smallest face of $g(c)$ that intersects σ_1. To see that this is well-defined, note that the smallest face of $g(c)$ that intersects σ_1 exists by Claim 4.4. It is a nonempty face of $g(c)$ and therefore $g^{-1}(\sigma_0) \neq \varnothing$. We extend f_1 on simplices in S by copying the values of f, and on $(< d)$-dimensional simplices in $\Delta_1 \setminus \Sigma$ by taking meets:

$$f_1(\gamma) = \bigcap_{\gamma \leq \delta \in \Delta_1, \dim \delta = d} f_1(\delta)$$

for $\gamma \in \Delta_1 \setminus \Sigma$. This is well-defined by Remark 4.10 and Lemma 3.16-1. (We shall silently apply Lemma 3.16-1 in the sequel.) Then f_1 is an extension of f. To see that f_1 is monotone, suppose

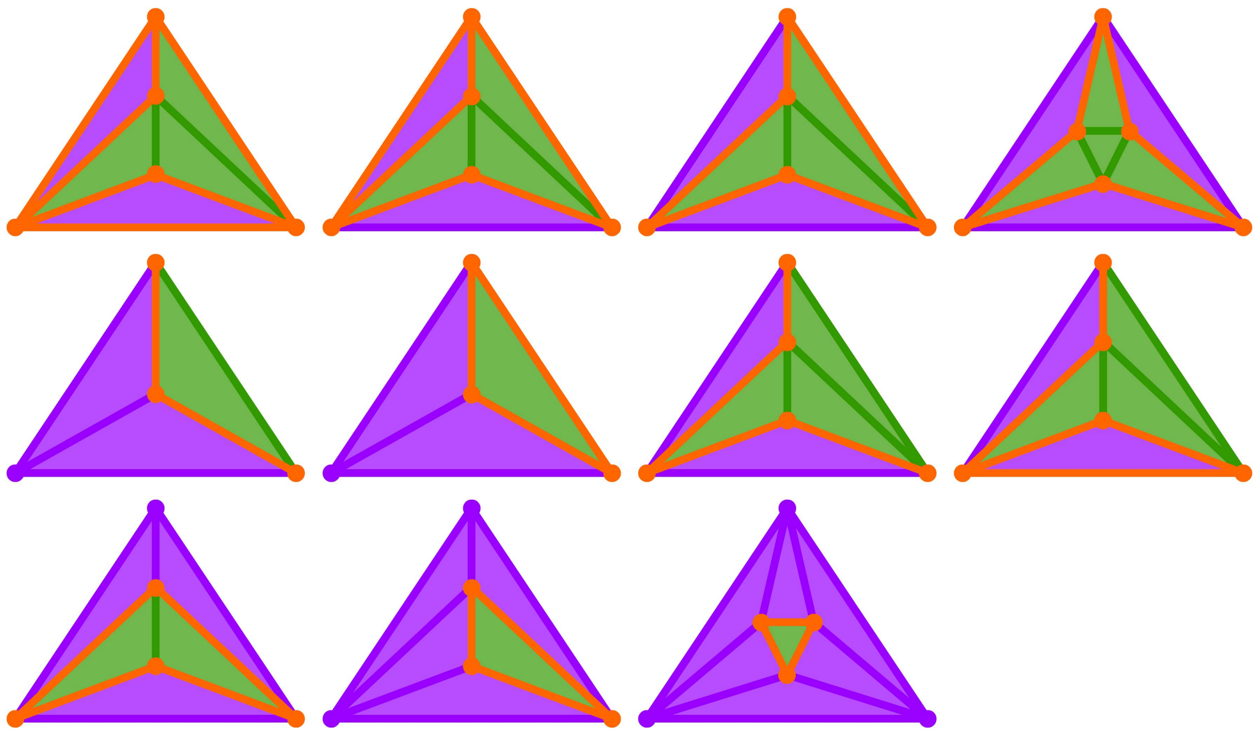


Figure 4.6: instances of Lemma 4.14 for $d = 2$, $d' = 1$ and $S = \Sigma = \text{propfac}(\theta)$

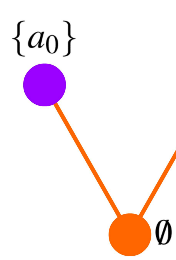


Figure 4.7: BA_2^- coloured

that $\underline{\sigma} \in S$ and $\delta \in \Delta_1 \setminus \Sigma$ such that $\underline{\sigma} \subseteq \delta$. W.l.o.g., $\dim \delta = d$. As before, find $c \in \mathscr{U}(\Sigma)$ and $\sigma_1 \in \Sigma_1$ such that $\sigma_1 \subseteq g(c)$ and $\delta = \sigma_1 \veebar (\min c)$. Then $\underline{\sigma} \subseteq \min c$. Let σ_0 be the smallest face of $g(c)$ that intersects σ_1. We have $\sigma := \max g^{-1}(\sigma_0) \in c$ (since $g^{-1}(\sigma_0) \subseteq c$), so

$$f_1(\delta) = p_\sigma \geq f(\sigma) \geq f(\min c) \geq f(\underline{\sigma}) = f_1(\underline{\sigma}).$$

Hence f_1 is monotone.

Let us check (4.6) for some $\delta \in \Delta_1 \setminus \Sigma$. Again find $c \in \mathscr{U}(\Sigma)$ and a cell $\sigma_1 \in \Sigma_1$ such that $\sigma_1 \subseteq g(c)$ and δ is a face of $\sigma_1 \veebar (\min c)$. Since $\delta \notin \Sigma$, we have $\sigma_1' := \delta \cap \sigma_1 \neq \varnothing$. Furthermore σ_1' only depends on δ:

$$\sigma_1' = \delta \cap \phi_\mu[\partial \theta].$$

How many different values does f_1 assume on d-dimensional successors of δ? Recall that σ_1' corresponds to some chain $c_1' \in \mathscr{C}^2(\Sigma_0)$, more precisely

$$c_1' \in \mathscr{C}^2\Big({\downarrow}^{\Sigma_0}\big(g(c)\big)\Big)$$

(since $\sigma_1' \in {\downarrow}^{\Sigma_0}\big(g(c)\big)^{++}$ by Lemma 2.51-2), and that $\dim \sigma_1' = \#c_1' - 1$. Since $\dim g(c) = \#c - 1$, the poset $\operatorname{fac}\big(g(c)\big)$ has height $\#c + 1$ (recall that the empty simplex is also in there). Hence the elements of c_1' have size at most $\#c$, because $c_1' \subseteq \mathscr{C}\Big(\operatorname{fac}\big(g(c)\big)\Big)$. Thus

$$\#\min c_1' \leq \#c - (\#c_1' - 1) = \#c - \dim \sigma_1'.$$

Also σ_1 corresponds to a chain

$$c_1 \in \mathscr{C}^2\Big({\downarrow}^{\Sigma_0}\big(g(c)\big)\Big),$$

and $c_1' \subseteq c_1$. Let σ_0 be the smallest face of $g(c)$ that intersects σ_1, and $\sigma = \max g^{-1}(\sigma_0)$. Then σ is determined by σ_0. That σ_0 intersects σ_1 is to say that c_1 has an element that is a subset of ${\downarrow}(\sigma_0)$ (Claim 4.4), or in other words, $\max\big(\min(c_1)\big) \leq \sigma_0$. By minimality of σ_0, this is an equality. In particular, $\sigma_0 \in \min c_1$. But $\min(c_1) \subseteq \min(c_1')$, so also $\sigma_0 \in \min(c_1')$. And $f_1\big(\sigma_1 \veebar (\min c)\big) = p_\sigma$, while σ is determined by $\sigma_0 \in \min(c_1')$ and $\min(c_1')$ depends only on δ (because σ_1' is determined by δ). We conclude that f_1 takes at most $\#\min c_1' \leq \#c - \dim \sigma_1'$ values on d-dimensional cells containing δ. Each of these values is a subset of $\{a_0, \dots, a_{d'}\}$ with a complement of size 1, so their intersection is a subset of $\{a_0, \dots, a_{d'}\}$ with a complement of size at most $\#c - \dim \sigma_1'$. Thus

$$\begin{aligned}
\operatorname{dpt}^P\big(f_1(\delta)\big) &\leq \#c - \dim \sigma_1' - 1 \\
&\overset{*}{=} (d - \dim \min c) - \dim \sigma_1' - 1 \\
&= d - (\dim \min c + \dim \sigma_1' + 1) \\
&\leq d - \dim \delta \\
&= \operatorname{dpt}^{\Delta_1}(\delta).
\end{aligned}$$

Equality * used the facts that $c \in \mathscr{U}(\Sigma)$ and $\dim \Sigma = d - 1$. Here, moreover note that, if $\delta \in \Sigma_1$, then $\delta = \sigma_1'$ whence $d - (\dim \min c + \dim \sigma_1' + 1) \leq d - \dim \delta - 1$ and so the above display could be improved to

$$\operatorname{dpt}^P\big(f_1(\delta)\big) \leq \operatorname{dpt}^{\Delta_1}(\delta) - 1. \tag{4.7}$$

Choose some $\nu \in (0,1)$. Let $\Sigma_2 = \phi_\nu[\![\Sigma_1]\!]^+$. By Lemma 4.9, there exists a triangulation Δ_2 of

$$\bigcup_{\rho\in[\nu,1]} \phi_\rho\left[\partial\phi_\mu[\theta]\right]$$

such that $\Sigma_1 \cup \Sigma_2 \subseteq \Delta_2$. By Lemma 2.41, there is a triangulation Δ_3 of $\phi_{\mu\nu}[\theta]$ such that $\Sigma_2 \subseteq \Delta_3$ and $\mathrm{vtc}(\Delta_3) = \mathrm{vtc}(\Sigma_2) \sqcup \{\mathbf{0}\}$. See Figure 4.8. Let $\Delta_4 = \Delta_1 \cup \Delta_2 \cup \Delta_3$. This is a triangulation of θ. Let $\mathbf{y}^0, \dots, \mathbf{y}^d$ be the vertices of $\phi_{\mu\nu}[\theta]$. For $i \le d$, define the d-simplex

$$\overline{\upsilon}_i = \mathbf{0} \veebar \mathbf{y}^0 \veebar \dots \veebar \mathbf{y}^{i-1} \veebar \mathbf{y}^{i+1} \veebar \dots \veebar \mathbf{y}^d.$$

Since $\Delta_3 = \Sigma_2 * \{\varnothing, \mathbf{0}\}$, every cell of Δ_3 lies in some $\overline{\upsilon}_i$. Let p be a constant maximal value in P, say $p = \{a_1, \dots, a_{d'}\}$. We define a new map $f_4 : S \sqcup (\Delta_4 \setminus \Sigma) \to P$. This map will be mostly an extension of f_1, but it might disagree with f_1 on Σ_1. The idea is to use the value p as default, and, if $d \ge d'$, ensure that $f_4(\mathbf{0}) = \varnothing$ by using all maximal elements of P as values of d-dimensional cells around $\mathbf{0}$. Formally: on d-dimensional cells $\delta \in \Delta_4 \setminus \Delta_1$, let

$$f_4(\delta) = \begin{cases} \{a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_{d'}\} & \big(\text{if } d \ge d' \ \&\ \delta \subseteq \overline{\upsilon}_i \text{ for some } i \le d'\big), \\ p & \text{otherwise.} \end{cases}$$

Extend f_4 by copying the values of f_1 on

$$S \sqcup \big(\Delta_1 \setminus (\Sigma \sqcup \Sigma_1)\big)$$

and by taking meets:

$$f_4(\gamma) = \bigcap_{\gamma \le \delta \in \Delta_4, \dim\delta = d} f_4(\delta)$$

for $\gamma \in \Delta_2 \cup \Delta_3$. Then $f_4 : \Delta_4 \to P$ is monotone because f_1 is. We again claim that

$$\mathrm{dpt}^P\left(f_4(\delta)\right) \le \mathrm{dpt}^{\Delta_4}(\delta) \tag{4.8}$$

for all $\delta \in \Delta_4 \setminus \Sigma$. To check this, distinguish cases:

- $\delta \in \Delta_1 \setminus (\Sigma \sqcup \Sigma_1)$. Then (4.8) follows from (4.6).

- $\delta \in \Sigma_1$. A d-dimensional successor of δ is a cell of Δ_1 or of Δ_2. But f_4 maps d-dimensional cells of Δ_2 to p, so (4.8) follows from (4.7).

- $\delta \in \Delta_2 \setminus \Sigma_1$. If δ' is a d-dimensional successor of δ, then either $f_4(\delta') = p$ or $f_4(\delta') = \{a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_{d'}\}$ where $\delta \subseteq \overline{\upsilon}_i$. We have $\delta \in \Delta_2$ and

  $$\overline{\upsilon}_i \cap |\Delta_2| = \mathbf{y}^0 \veebar \dots \veebar \mathbf{y}^{i-1} \veebar \mathbf{y}^{i+1} \veebar \dots \veebar \mathbf{y}^d,$$

  so there are at most $d - \dim\delta = \mathrm{dpt}^{\Delta_4}(\delta)$ values of i for which $\delta \subseteq \overline{\upsilon}_i$. Therefore f_4 assumes at most $\mathrm{dpt}^{\Delta_4}(\delta) + 1$ values on d-dimensional cells containing δ. (4.8) follows.

- $\delta \in \Delta_3 \setminus \Sigma_2$. Then all d-dimensional successors δ' of δ lie in Δ_3. Then $f_4(\delta') = \{a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_{d'}\}$ where $\delta \subseteq \overline{\upsilon}_i$. There are at most $d + 1 - \dim\delta = \mathrm{dpt}^{\Delta_4}(\delta) + 1$ values of i for which $\delta \subseteq \overline{\upsilon}_i$. Therefore f_4 assumes at most $\mathrm{dpt}^{\Delta_4}(\delta) + 1$ values on d-dimensional cells containing δ. (4.8) follows.

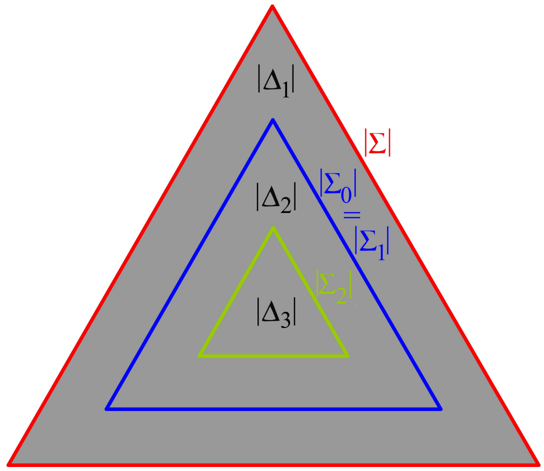


Figure 4.8: overview of the proof of Lemma 4.14

Thus the map f_4 is monotone and well-behaved, but it need not satisfy the back-property on $\Delta_4 \setminus \Sigma$. To remedy this, we shall first construct a subdivision Δ_5 such that many cells δ have a unique associated successor $\delta_\sim$. Then, for every δ, the final map $\overline{f}$ will assume different values on a subdivision of $\delta_\sim$, in such a way that $\overline{f}$ does satisfy the back-property at δ. (This will be similar to the definition of f_4 on Δ_3.)

For each d-dimensional $\delta \in \Delta_4$, apply Lemma 4.11 to find a triangulation $\Delta_5(\delta)$ of δ that contains $\Downarrow^{\Delta_4}(\delta) = \operatorname{propfac}(\delta)$, and a map $h_\delta : \operatorname{propfac}(\delta) \to \Delta_5(\delta)$. Then

$$\Delta_5 = \bigcup_{\delta \in \Delta_4, \dim \delta = d} \Delta_5(\delta)$$

is a subdivision of Δ_4. Let $\underline{\Delta}_4$ be the set of all $\delta \in \Delta_4$ of dimension at most $d-2$ with $\operatorname{dpt}^{\Delta_4}(\delta) \geq \operatorname{dpt}^P\big(f_4(\delta)\big)$. For each $\delta \in \underline{\Delta}_4$, choose a d-dimensional successor $\delta^* \in \Delta_4$ of δ and let $\delta_\sim = h_{\delta^*}(\delta)$. Observe that

$$\delta_\sim \cap \delta'_\sim = \delta \cap \delta' \tag{4.9}$$

for all distinct $\delta, \delta' \in \underline{\Delta}_4$: if $\delta^* = (\delta')^*$ this follows from the choice of h, and if $\delta^* \neq (\delta')^*$ we have $\delta^* \cap (\delta')^* \subseteq \partial\delta^*, \partial(\delta')^*$ whence

$$\begin{aligned} \delta_\sim \cap \delta'_\sim &= \big(\delta_\sim \cap \delta^* \cap (\delta')^*\big) \cap \big(\delta'_\sim \cap \delta^* \cap (\delta')^*\big) \\ &\subseteq (\delta_\sim \cap \partial\delta^*) \cap \big(\delta'_\sim \cap \partial(\delta')^*\big) \\ &= \delta \cap \delta'. \end{aligned}$$

Let δ_* be the opposite face of $\delta_\sim$ w.r.t. δ, i.e. $\delta_\sim = \delta \veebar \delta_*$. Then $\delta_* \subseteq \operatorname{relInt} \delta^*$ by nature of the map h_{δ^*}, so $\mathbf{b}(\delta_*) \in \operatorname{relInt} \delta^*$. Moreover, we have

$$\delta_\sim \cap \delta'_* = \varnothing \tag{4.10}$$

for all distinct $\delta, \delta' \in \underline{\Delta}_4$, by (4.9). Let $\mathbf{x}^{\delta,0}, \ldots, \mathbf{x}^{\delta, d-\dim\delta-1}$ be a faithful listing of the vertices of δ_*. By Corollary 2.49, let Δ be a subdivision of Δ_5 that satisfies

$$\operatorname{vtc}(\Delta) = \operatorname{vtc}(\Delta_5) \sqcup \big\{\mathbf{b}(\delta_*) : \delta \in \underline{\Delta}_4\big\}.$$

By Lemma 2.29, for each $\delta \in \underline{\Delta_4}$, Δ has a subcomplex $\Delta(\delta)$ with carrier $\delta_\sim$. By (4.10),

$$\operatorname{vtc}\big(\Delta(\delta)\big) = \operatorname{vtc}(\delta_\sim) \sqcup \big\{\mathbf{b}(\delta_*)\big\}.$$

By Lemma 2.48-2 we conclude that

$$\Delta(\delta) = \big\{\gamma \in \operatorname{fac}(\delta_\sim) : \delta_* \nsubseteq \gamma\big\} * \big\{\varnothing, \mathbf{b}(\delta_*)\big\}. \tag{4.11}$$

Note that every cell of Δ_4 of dimension $< d$ is also a cell of Δ, since such a cell belongs to Δ_5 and contains no point of the form $\mathbf{b}(\delta_*)$ (with $\delta \in \underline{\Delta_4}$).

For $\delta \in \underline{\Delta_4}$, let $k = k(\delta) = \operatorname{dpt}^P\big(f_4(\delta)\big) \leq d - \dim\delta$. Enumerate

$$\{a_0, \dots, a_{d'}\} \setminus f_4(\delta) = \{a_{i(\delta,0)}, \dots, a_{i(\delta,k)}\}$$

in such a way that

$$f_4(\delta^*) = \{a_0, \dots, a_{d'}\} \setminus \{a_{i(\delta,k)}\},$$

which is possible since $f_4(\delta) \leq f_4(\delta^*)$.

Define the map $\overline{f}$ as follows: consider a d-dimensional $\underline{\delta} \in \Delta$ and distinguish two cases.

- There exists $\delta \in \underline{\Delta_4}$ such that $\underline{\delta} \subseteq \delta_\sim$. Then δ is unique by (4.9). We have $\delta_* \nsubseteq \underline{\delta}$, so find j such that $\mathbf{x}^{\delta,j} \notin \underline{\delta}$. With $k = k(\delta)$, set

$$\overline{f}(\underline{\delta}) = \{a_0, \dots, a_{d'}\} \setminus \{a_{i(\delta,\min\{j,k\})}\} \geq f_4(\delta).$$

- There is no such $\delta \in \underline{\Delta_4}$. Find $\delta \in \Delta_4$ such that $\underline{\delta} \subseteq \delta$ and let $\overline{f}(\underline{\delta}) = f_4(\delta)$.

Extend $\overline{f}$ on S by copying the values of f, and on $(< d)$-dimensional elements of $\Delta \setminus \Sigma$ by taking meets like before. The maps $f, f_1, f_4, \overline{f}$ all agree on S.

For $\delta \in \underline{\Delta_4}$ and $j < d - \dim\delta$ we observe that

$$\gamma(\delta, j) = \delta \,\underline{\vee}\, \mathbf{b}(\delta_*) \,\underline{\vee}\, \mathbf{x}^{\delta,0} \,\underline{\vee} \dots \underline{\vee}\, \mathbf{x}^{\delta,j-1} \,\underline{\vee}\, \mathbf{x}^{\delta,j+1} \,\underline{\vee} \dots \underline{\vee}\, \mathbf{x}^{\delta,d-\dim\delta-1}$$

is a d-dimensional cell in $\Delta(\delta)$ by (4.11), and that $\overline{f}\big(\gamma(\delta, j)\big) = \{a_0, \dots, a_{d'}\} \setminus \{a_{i(\delta,\min\{j,k\})}\}$ (for $k = k(\delta)$). Moreover, every d-dimensional cell in $\Delta(\delta)$ is equal to $\gamma(\delta, j)$ for some j.

If $\upsilon \in \Delta(\delta)$ is $(d-1)$-dimensional and $\upsilon \subseteq \partial\delta_\sim$, then Proposition 3.18 and Lemma 3.16-3 imply that υ has a successor $\underline{\delta} \in \Delta$ that does not lie in $\delta_\sim$. Since $\dim\delta < d-1$ we have $\upsilon \nsubseteq \delta$, and therefore $\upsilon \cap \operatorname{relInt}\delta^* \neq \varnothing$ by choice of $\delta_\sim$, which implies that $\underline{\delta} \subseteq \delta^*$. The simplex $\underline{\delta}$ cannot lie in $\delta'_\sim$ for any $\delta' \in \underline{\Delta_4}$, since this would imply $\delta \neq \delta'$ whence $\upsilon \subseteq \delta \cap \delta'$ by (4.9). Thus we have $\overline{f}(\underline{\delta}) = f_4(\delta^*)$, whence $\overline{f}(\upsilon) \leq f_4(\delta^*)$, i.e.

$$a_k \notin \overline{f}(\upsilon). \tag{4.12}$$

To show that $\overline{f}$ is monotone, suppose that $\underline{\sigma} \in S$ and $\underline{\delta} \in \Delta \setminus \Sigma$ such that $\underline{\sigma} \subseteq \underline{\delta}$. W.l.o.g. we assume that $\dim\underline{\delta} = d$. First suppose that there exists $\delta \in \underline{\Delta_4}$ such that $\underline{\delta} \subseteq \delta_\sim$. Then $\underline{\sigma} \subseteq \delta_\sim \subseteq \delta^*$, but also $\underline{\sigma} \in \Delta_4$, so $\underline{\sigma} \in \operatorname{fac}(\delta^*)$. In particular $\underline{\sigma} \subseteq \partial\delta^*$, so $\underline{\sigma} \subseteq \delta$ by choice of $\delta_\sim$. It holds

$$\overline{f}(\underline{\sigma}) = f_4(\underline{\sigma}) \subseteq f_4(\delta) \subseteq \overline{f}(\underline{\delta}).$$

Next suppose that there is no such $\delta \in \underline{\Delta}_4$. Find $\delta \in \Delta_4$ such that $\underline{\delta} \subseteq \delta$. Then $\underline{\sigma} \subseteq \delta$, so

$$\overline{f}(\underline{\sigma}) = f_4(\underline{\sigma}) \subseteq f_4(\delta) = \overline{f}(\underline{\delta}).$$

Thus $\overline{f}$ is monotone.

We claim that $\overline{f}$ and f_4 agree on $(< d)$-dimensional cells of Δ_4. To check this, suppose that $\delta \in \Delta_4 \setminus \Sigma$ with $\dim \delta < d$. First assume $\dim \delta = d-1$. By Proposition 3.18 and Lemma 3.16-3, δ has exactly two proper successors γ_0, γ_1 in Δ_4 and exactly two proper successors γ_0', γ_1' in Δ, such that $\gamma_i' \subseteq \gamma_i$. Note that γ_i' cannot be contained in any cell of the form $\delta'_\sim$, since this would imply that $\delta \subseteq \delta'_\sim \subseteq (\delta')^*$ so $\delta \subseteq \delta'_\sim \cap \partial(\delta')^* = \delta'$ and therefore $\dim \delta' \geq d-1$ and $\delta' \notin \underline{\Delta}_4$. Hence $\overline{f}(\gamma_i') = f_4(\gamma_i)$ and $\overline{f}(\delta) = f_4(\delta)$. Next assume $\dim \delta < d-1$. By (4.8), it follows that $\delta \in \underline{\Delta}_4$. Let υ be a $(d-1)$-dimensional face of $\delta_\sim$ such that $\delta \subseteq \upsilon$. Then $\upsilon \in \Delta$. From (4.12) it follows that $a_{i(\delta,k)} \notin \overline{f}(\delta)$ (for $k = k(\delta)$). Furthermore we have $\overline{f}(\delta) \subseteq \overline{f}\big(\gamma(\delta,j)\big) = \{a_0,\dots,a_{d'}\} \setminus \{a_{i(\delta,j)}\}$ for $j < k$. Therefore $\overline{f}(\delta) \subseteq f_4(\delta)$. For the other inclusion, suppose that $\underline{\delta} \in \Delta$ is some d-dimensional successor of δ. If there exists $\delta' \in \underline{\Delta}_4$ with $\underline{\delta} \subseteq \delta'_\sim$, then $\delta \subseteq \delta'_\sim$, so $\delta \subseteq \delta'$ by (4.9) and it follows that

$$\overline{f}(\underline{\delta}) \geq f_4(\delta') \geq f_4(\delta).$$

On the other hand, if there exists no such $\delta' \in \underline{\Delta}_4$, then find $\delta' \in \Delta_4$ with $\underline{\delta} \subseteq \delta'$ and note that $\delta \subseteq \delta'$ so

$$\overline{f}(\underline{\delta}) = f_4(\delta') \geq f_4(\delta).$$

This proves that $\overline{f}(\delta) \supseteq f_4(\delta)$.

Suppose that $\underline{\delta} \in \Delta$ is d-dimensional and there exists $\delta \in \underline{\Delta}_4$ such that $\mathbf{x}^{\delta,j} \in \underline{\delta}$ for some j or $\mathbf{b}(\delta_*) \in \underline{\delta}$. Then $\underline{\delta}$ intersects relInt δ^*, so $\underline{\delta} \subseteq \delta^*$ and $\underline{\delta}$ intersects $\delta_\sim \setminus \delta$, so $\underline{\delta}$ cannot lie in $\delta'_\sim$ for any $\delta' \in \underline{\Delta}_4$ other than δ by (4.9); hence we have (*): $\underline{\delta}$ is either $\gamma(\delta,\ell)$ for some $\ell < d - \dim \delta$ or $\overline{f}(\underline{\delta}) = f_4(\delta^*)$.

To show that $\overline{f}$ satisfies the back-property on $\Delta \setminus \Sigma$, let $\delta \in \Delta \setminus \Sigma$. Distinguish some cases:

- $\dim \delta = d$. Then $\overline{f}(\delta)$ is maximal.

- $\dim \delta = d-1$. By Proposition 3.18 and Lemma 3.16-3, δ has exactly two proper successors in Δ. If $\overline{f}(\delta)$ is not maximal, it also has exactly two proper successors in P, and they must be the values of the proper successors of δ under $\overline{f}$.

- $\dim \delta < d-1$. Let q be an immediate successor of $\overline{f}(\delta)$. So there exists $i \leq d'$ with $q = \overline{f}(\delta) \sqcup \{a_i\}$. Distinguish two subcases:

  - $\delta \in \underline{\Delta}_4$. Find $j \leq k = k(\delta) \leq d - \dim \delta$ such that $i = i(\delta,j)$ (in virtue of the choice of the sequence $i(\delta,\cdot)$). Distinguish two subsubcases:

    * $j = k$. Consider

    $$\gamma = \delta \veebar \mathbf{b}(\delta_*) \veebar \mathbf{x}^{\delta,k} \veebar \dots \veebar \mathbf{x}^{\delta,d-\dim\delta-1} \in \Delta$$

    by (4.11). Then $\delta \leq \gamma$ and all successors of $\gamma \in \Delta$ lie in $\delta_\sim$ (since $\delta \veebar \mathbf{b}(\delta_*) \subseteq \gamma$), so

$$\begin{aligned}\overline{f}(\gamma) &= \bigcap_{\ell<k} \overline{f}\big(\gamma(\delta,\ell)\big) \\ &= \bigcap_{\ell<k} \{a_0,\dots,a_{d'}\} \setminus \{a_{i(\delta,\ell)}\} \\ &= f_4(\delta) \sqcup \{a_{i(\delta,k)}\} \\ &= \overline{f}(\delta) \sqcup \{a_i\} \\ &= q.\end{aligned}$$

* $j < k$. Consider
$$\gamma = \delta \,\underline{\vee}\, \mathbf{x}^{\delta,j} \in \Delta.$$
Then $\delta \leq \gamma$, and γ has a $(d-1)$-dimensional successor $\upsilon \in \Delta$ with $\upsilon \subseteq \partial\delta_\sim$ so by (4.12)
$$\begin{aligned}\overline{f}(\gamma) &\leq \overline{f}(\upsilon) \cap \bigcap_{\ell<k;\ell\neq j} \overline{f}\big(\gamma(\delta,\ell)\big) \\ &\leq \big(\{a_0,\dots,a_{d'}\} \setminus \{a_{i(\delta,k)}\}\big) \\ &\cap \big(\{a_0,\dots,a_{d'}\} \setminus \{a_{i(\delta,0)},\dots,a_{i(\delta,j-1)},a_{i(\delta,j+1)},\dots,a_{i(\delta,k-1)}\}\big) \\ &= \{a_0,\dots,a_{d'}\} \setminus \{a_{i(\delta,0)},\dots,a_{i(\delta,j-1)},a_{i(\delta,j+1)},\dots,a_{i(\delta,k)}\} \\ &= f_4(\delta) \sqcup \{a_{i(\delta,j)}\} \\ &= \overline{f}(\delta) \sqcup \{a_i\} \\ &= q.\end{aligned}$$
The other inequality, $\overline{f}(\gamma) \geq q$, also holds, because $\overline{f}(\gamma) \geq \overline{f}(\delta)$ by monotonicity and $a_i \in \overline{f}(\gamma)$ because of the following. Let $\underline{\delta}$ be a d-dimensional successor of γ. Then $\mathbf{x}^{\delta,j} \in \underline{\delta}$, so by (*), $\underline{\delta}$ is either $\gamma(\delta,\ell)$ for some $\ell < d - \dim\delta$ with $\ell \neq j$ or $\overline{f}(\underline{\delta}) = f_4(\delta^*)$, and in both cases it follows that $a_i \in \overline{f}(\underline{\delta})$.

– $\delta \notin \underline{\Delta}_4$. Since $\delta \notin \Sigma$, this implies $\delta \notin \Delta_4$ by (4.8). Then $\upsilon = \mathrm{sub}(\Delta,\Delta_4)(\delta)$ is d-dimensional, because $\varnothing \neq \mathrm{relInt}\,\delta \subseteq \mathrm{relInt}\,\upsilon$ by Lemma 2.36 and $\dim\upsilon < d$ would imply that $\upsilon \in \Delta$ while $\delta \neq \upsilon$, contradicting Lemma 2.32. Now distinguish subsubcases:

  * There is a (unique) $\delta' \in \underline{\Delta}_4$ such that $\delta \subseteq \delta'_\sim$. Since $\delta \notin \underline{\Delta}_4$ we have $\delta \not\subseteq \delta'$. Then $\upsilon = (\delta')^*$. Find j such that $i = i(\delta', j)$ and let $k = k(\delta')$. We claim that
  $$\xi = \delta' \,\underline{\vee}\, \mathbf{b}(\delta'_*) \,\underline{\vee}\, \mathbf{x}^{\delta',k} \,\underline{\vee}\, \dots \,\underline{\vee}\, \mathbf{x}^{\delta',d-\dim\delta'-1}$$
  is the smallest cell of $\Delta(\delta')$ for which $f_4\big((\delta')^*\big) \not\geq \overline{f}(\xi)$. Indeed, all d-dimensional successors of ξ lie in $\delta'_\sim$, so $a_{i(\delta',k)} \in \overline{f}(\xi)$; on the other hand if $\xi' \in \Delta(\delta')$ and $f_4\big((\delta')^*\big) \not\geq \overline{f}(\xi')$, then by (4.12) ξ' has no $(d-1)$-dimensional successor lying in $\partial\delta'_\sim$, so $\delta' \,\underline{\vee}\, \mathbf{b}(\delta'_*) \subseteq \xi'$, and we must have $\gamma(\delta',\ell) \not\supseteq \xi'$ for $k \leq \ell < d - \dim\delta'$ so $\mathbf{x}^{\delta',\ell} \in \xi'$. Distinguish subsubsubcases:

· $j < k$. Take $\gamma = \delta \vee \mathbf{x}^{\delta',j}$. We have $\gamma \in \Delta$, because $a_i \notin \overline{f}(\delta)$ implies that δ has a successor containing $\mathbf{x}^{\delta',j}$, by (*) since $\delta \not\subseteq \delta'$. Moreover $\gamma \supseteq \xi$ iff $\delta \supseteq \xi$, so $f_4\big((\delta')^*\big) \geq \overline{f}(\gamma)$ iff $f_4\big((\delta')^*\big) \geq \overline{f}(\delta)$. For all $\ell < k$ with $\ell \neq j$, we have $\delta \subseteq \gamma(\delta',\ell)$ iff $\gamma \subseteq \gamma(\delta',\ell)$. Moreover we have $\gamma \not\subseteq \gamma(\delta',j)$. Thus (*) implies that $\overline{f}(\gamma) = \overline{f}(\delta) \cup \{a_{i(\delta',j)}\} = q$.

· $j = k$. Take $\gamma = \delta \vee (\mathbf{b}(\delta'_*) \underline{\vee} \delta' \underline{\vee} \mathbf{x}^{\delta',k} \underline{\vee} \ldots \underline{\vee} \mathbf{x}^{\delta',d-\dim\delta'-1})$. To show that γ is a well-defined cell of Δ, suppose for a contradiction that $\mathbf{x}^{\delta',\ell} \in \delta$ for all $\ell < k$. Since $\delta \not\subseteq \delta'$, (*) then implies that $a_{i(\delta',\ell)} \in \overline{f}(\delta)$ for all $\ell < k$, so $q = \{a_0, \ldots, a_{d'}\} \notin P$.
Since $\gamma \supseteq \xi$, we have $a_i = a_{i(\delta',k)} \in \overline{f}(\gamma)$. For all $\ell < k$, we have $\delta \subseteq \gamma(\delta',\ell)$ iff $\gamma \subseteq \gamma(\delta',\ell)$, so (*) implies that $\overline{f}(\gamma) = \overline{f}(\delta) \cup \{a_i\} = q$.

$*$ There is no $\delta' \in \underline{\Delta}_4$ such that $\delta \subseteq \delta'_\sim$. Then for all d-dimensional successors δ' of δ it holds $\mathrm{sub}(\Delta,\Delta_4)(\delta') = \upsilon$ so $\overline{f}(\delta') = f_4(\upsilon)$. Hence δ is maximal.

We show that $\overline{f}$ is surjective if $d \geq d'$. In this case we claim that $\overline{f}(\mathbf{0}) = \varnothing$. Since $\dim \mathbf{0} = 0 < d$ and $\mathbf{0} \in \Delta_4$, we have $\overline{f}(\mathbf{0}) = f_4(\mathbf{0})$ and it suffices to show that $f_4(\mathbf{0}) = \varnothing$. To this end, let $i \leq d' \leq d$. There exists a d-dimensional cell $\delta \in \Delta_3 \subseteq \Delta_4$ such that $\delta \subseteq \overline{\upsilon}_i$ and $\mathbf{0} \in \delta$. Then $f_4(\mathbf{0}) \leq f_4(\delta) = \{a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{d'}\}$.

We shall now verify the final statement in the lemma. Suppose (4.5). Then $\sigma \in \underline{\Delta}_4$. Write $k = k(\sigma) < d - \dim\sigma$. Then (4.11) implies that $\delta = \sigma \underline{\vee} \mathbf{b}(\sigma_*) \in \Delta$. Hence

$$\begin{aligned}\overline{f}(\delta) &\leq \bigcap_{j \leq k} \overline{f}(\gamma(\sigma,j)) \\ &= \bigcap_{j \leq k} \{a_0, \ldots, a_{d'}\} \setminus \{a_{i(\sigma,j)}\} \\ &= f_4(\sigma) \\ &= f(\sigma) \\ &= \overline{f}(\sigma),\end{aligned}$$

so $\overline{f}(\delta) = \overline{f}(\sigma)$ by monotonicity. □

Proof of Theorem 4.1. By Lemmas 2.51-1, 2.56 and 2.37 and induction, we may assume that Δ' is an elementary subdivision of Δ. Let $\{\mathbf{w}\} = \mathrm{vtc}(\Delta') \setminus \mathrm{vtc}(\Delta)$. By Lemma 2.32, find $\underline{\delta} \in \Delta$ such that $\mathbf{w} \in \mathrm{relInt}\,\underline{\delta}$. Then, by Lemma 2.48-2, for each $\delta \in \Delta$, Δ' has a subcomplex $\Delta'(\delta)$ with carrier δ such that

$$\Delta'(\delta) = \begin{cases} \big\{\gamma \in \mathrm{fac}(\delta) : \underline{\delta} \not\subseteq \gamma\big\} * \{\varnothing, \mathbf{w}\} & (\underline{\delta} \subseteq \delta) \\ \mathrm{fac}(\delta) & (\text{otherwise}). \end{cases} \tag{4.13}$$

For $\delta' \in \Delta'$ it holds

$$\mathrm{sub}(\Delta',\Delta)(\delta') = \delta' \quad \Leftrightarrow \quad \delta' \in \Delta \quad \Leftrightarrow \quad \mathbf{w} \notin \delta'. \tag{4.14}$$

Let $d' = \dim \underline{\delta}$ and let

$$Y = \mathrm{vtc}(\underline{\delta}) = \{\mathbf{y}^0, \ldots, \mathbf{y}^{d'}\}.$$

Let $f : \Sigma \rightharpoonup \Delta$ be an up-p-morphism. We construct a subdivision Σ' of Σ and an up-p-morphism $g : \Sigma' \rightharpoonup \Delta'$. The idea is that the lower-dimensional cells are easier to take care of. In particular, there will be no hassle defining g on vertices. Then, we shall add the one-and-higher-dimensional cells of Σ one by one. At each nontrivial step, we apply Lemma 4.14 to triangulate the new cell $\sigma \in \Sigma$ that is being added, and to pick values for g on σ. Because of this, the final subdivision Σ' may be quite fine (in the sense that it has many cells), but in the process we never need to further subdivide parts of Σ' that we have already constructed. Formally speaking, enumerate

$$\Sigma = \{\sigma_0, \dots, \sigma_m\}$$

in such a way that

$$\sigma_i \subseteq \sigma_j \Rightarrow i \leq j.$$

Then, by induction on $0 < i \leq m+1$, we construct a subdivision Σ'_i of the subcomplex $\Sigma_i = \{\sigma_0, \dots, \sigma_{i-1}\}$ and a monotone map $g_i : \Sigma'_i \rightharpoonup \Delta'$ such that

$$\operatorname{dom} g_i = \operatorname{sub}(\Sigma'_i, \Sigma_i)^{-1}[\Sigma_i \cap \operatorname{dom} f]$$

and

$$\operatorname{sub}(\Delta', \Delta) \circ g_i = f \circ \operatorname{sub}(\Sigma'_i, \Sigma_i).$$

Furthermore Σ'_i will be a subcomplex of Σ'_{i+1} and g_{i+1} will be an extension of g_i.

Let $\Sigma'_1 = \{\varnothing\} = \Sigma_1$ and suppose that we have defined Σ_i and g_i for some $0 < i \leq m$. First assume that $\dim \sigma_i = 0$ or $\sigma_i \notin \operatorname{dom} f$ or $f(\sigma_i) \in \Delta$. Let $\Sigma'_{i+1} = \Sigma'_i \sqcup \{\sigma_i\}$ and let g_{i+1} be an extension of g_i such that, if $\sigma_i \in \operatorname{dom} f$ then

$$g_{i+1}(\sigma_i) \in \operatorname{sub}(\Delta', \Delta)^{-1}\left[\{f(\sigma_i)\}\right]$$

of the same dimension as $f(\sigma_i)$. We have to prove that Σ'_{i+1} is a simplicial complex. We can already prove this recursively, even though we haven't yet covered all cases of the recursive definition of Σ'_{i+1}. We have $\operatorname{propfac}(\sigma_i) \subseteq \Sigma_i$, and $\sigma_i \notin \operatorname{dom} f$ implies that $\operatorname{propfac}(\sigma_i) \cap \operatorname{dom} f = \varnothing$ while $f(\sigma_i) \in \Delta$ implies that $f\big[\operatorname{propfac}(\sigma_i)\big] \subseteq \Delta$. Hence our recursive definition (so far) implies that $\operatorname{propfac}(\sigma_i) \subseteq \Sigma'_i$. It follows that Σ'_{i+1} is a simplicial complex. Since $\operatorname{sub}(\Sigma'_{i+1}, \Sigma_{i+1})|\Sigma'_i = \operatorname{sub}(\Sigma'_i, \Sigma_i)$ we have

$$\begin{aligned}
\operatorname{dom} g_{i+1} &= (\operatorname{dom} g_i) \sqcup \big(\{\sigma_i\} \cap \operatorname{dom} f\big) \\
&= \operatorname{sub}(\Sigma'_i, \Sigma_i)^{-1}[\Sigma_i \cap \operatorname{dom} f] \sqcup \operatorname{sub}(\Sigma'_{i+1}, \Sigma_{i+1})^{-1}\big[\{\sigma_i\} \cap \operatorname{dom} f\big] \\
&= \operatorname{sub}(\Sigma'_{i+1}, \Sigma_{i+1})^{-1}[\Sigma_{i+1} \cap \operatorname{dom} f]
\end{aligned}$$

and

$$\operatorname{sub}(\Delta', \Delta) \circ g_{i+1} = f \circ \operatorname{sub}(\Sigma'_{i+1}, \Sigma_{i+1}).$$

Next suppose that $\dim \sigma_i > 0$ and $f(\sigma_i) \notin \Delta$. We have $\mathbf{w} \in f(\sigma_i)$ so $\underline{\delta} \subseteq f(\sigma_i)$. Let $\overline{\delta}_i$ be the face of $f(\sigma_i)$ that is opposite to $\underline{\delta}$. We already have a triangulation $\Sigma_i^- \subseteq \Sigma'_i$ of $\partial \sigma_i$ (Lemma 2.29). We put

$$f_i(\sigma) = g_i(\sigma) \cap Y$$

for $\sigma \in \Sigma_i^- \cap \operatorname{dom} g_i = S_i$. Let $a_i = \mathbf{y}^i$ for $i \le d'$. Then $f_i : S_i \to P = \mathrm{BA}_{d'+1}^-$ is monotone. By Lemma 4.14, σ_i has a triangulation Σ_i^* such that $\Sigma_i^- \subseteq \Sigma_i^*$ and there is an extension $\overline{f}_i : S_i \sqcup (\Sigma_i^* \setminus \Sigma_i^-) \to P$ of f_i that satisfies the back-property on $\Sigma_i^* \setminus \Sigma_i^-$ and moreover:

(*) whenever

$$\sigma \in S_i \ \& \ \operatorname{dpt}^{\Sigma_i^-}(\sigma) \ge \max\left\{\operatorname{dpt}^P\left(f_i(\sigma)\right), 1\right\}$$

then σ has a strict successor $\gamma \in \Sigma_i^* \setminus \Sigma_i^-$ with $\overline{f}_i(\gamma) = \overline{f}_i(\sigma)$.

Next we can choose $\Sigma'_{i+1} = \Sigma'_i \cup \Sigma_i^*$. Let g_{i+1} be the smallest extension of g_i with

$$g_{i+1}(\sigma) = \overline{\delta}_i \veebar \mathbf{w} \veebar \left(\underline{\bigvee}\left(\overline{f}_i(\sigma)\right)\right)$$

when $\sigma \in \Sigma'_{i+1} \setminus \Sigma'_i$. By (4.13), $g_{i+1} : \Sigma'_{i+1} \to \Delta'$ is well-defined. For $\sigma \in \Sigma_i^-$ we have $\operatorname{sub}(\Sigma'_i, \Sigma_i)(\sigma) \subseteq \sigma_i$ so

$$\begin{aligned}\operatorname{sub}(\Delta', \Delta)\big(g_i(\sigma)\big) &= f\big(\operatorname{sub}(\Sigma'_i, \Sigma_i)(\sigma)\big)\\ &\le f(\sigma_i)\end{aligned}$$

whence

$$g_{i+1}(\sigma) = g_i(\sigma) \subseteq \overline{\delta}_i \veebar \mathbf{w} \veebar \left(\underline{\bigvee}\left(f_i(\sigma)\right)\right) = \overline{\delta}_i \veebar \mathbf{w} \veebar \left(\underline{\bigvee}\left(\overline{f}_i(\sigma)\right)\right),$$

so g_{i+1} is monotone because $\overline{f}_i$ is monotone.

For all $\sigma \in \Sigma_{i+1} \setminus \Sigma_i$ we have $\overline{\delta}_i \veebar \mathbf{w} \subseteq g_{i+1}(\sigma)$ and therefore $\operatorname{relInt} g_{i+1}(\sigma) \subseteq \operatorname{relInt} f(\sigma_i)$ so $\operatorname{sub}(\Delta', \Delta)\big(g_{i+1}(\sigma)\big) = f(\sigma_i)$ by Lemmas 2.36 and 2.32; and on the other hand $\operatorname{sub}(\Sigma'_{i+1}, \Sigma_{i+1})(\sigma) = \sigma_i$. Also $\operatorname{sub}(\Sigma'_{i+1}, \Sigma_{i+1})|\Sigma'_i = \operatorname{sub}(\Sigma'_i, \Sigma_i)$. In total, it follows that

$$\operatorname{sub}(\Delta', \Delta) \circ g_{i+1} = f \circ \operatorname{sub}(\Sigma'_{i+1}, \Sigma_{i+1}).$$

Furthermore

$$\begin{aligned}\operatorname{dom} g_{i+1} &= (\operatorname{dom} g_i) \sqcup (\Sigma'_{i+1} \setminus \Sigma'_i)\\ &= \operatorname{sub}(\Sigma'_i, \Sigma_i)^{-1}[\Sigma_i \cap \operatorname{dom} f] \sqcup \operatorname{sub}(\Sigma'_{i+1}, \Sigma_{i+1})^{-1}\big[\{\sigma_i\}\big]\\ &= \operatorname{sub}(\Sigma'_{i+1}, \Sigma_{i+1})^{-1}[\Sigma_{i+1} \cap \operatorname{dom} f].\end{aligned}$$

This completes the construction of the sequences $\Sigma'_1, \ldots, \Sigma'_{m+1}$ and $g_1, \ldots, g_{m+1}$.

Let $\Sigma' = \Sigma'_{m+1}$ and $g = g_{m+1}$. To prove that g has the back-property, suppose that $g(\sigma') = \delta'$ and δ'_1 is an immediate successor of δ' in Δ'. We look for a successor of σ' in Σ' that is mapped to δ'_1 by g. Let $\sigma = \operatorname{sub}(\Sigma', \Sigma)(\sigma')$, $\delta = \operatorname{sub}(\Delta', \Delta)(\delta')$ and $\delta_1 = \operatorname{sub}(\Delta', \Delta)(\delta'_1)$. Then

$$\begin{aligned}f(\sigma) &= f\big(\operatorname{sub}(\Sigma', \Sigma)(\sigma')\big)\\ &= \operatorname{sub}(\Delta', \Delta)\big(g(\sigma')\big)\\ &= \operatorname{sub}(\Delta', \Delta)(\delta')\\ &= \delta.\end{aligned}$$

Therefore σ has a successor θ with $f(\theta) = \delta_1$ and

$$\dim\theta - \dim\sigma \ge \dim\delta_1 - \dim\delta$$

(θ is found by taking a chain $\delta = \delta^0, \ldots, \delta^k = \delta_1$ of immediate successors and finding a chain $\sigma = \sigma^0, \ldots, \sigma^k = \theta$ of proper successors such that $f(\sigma^\ell) = \delta^\ell$ for all $\ell \le k$). Find i, j such that $\sigma = \sigma_j$ and $\theta = \sigma_i$. Then $j \le i$. Distinguish some cases:

- $\delta_1 = \delta_1'$. Then $\delta_1' \in \Delta$, so $\Sigma_{i+1}' = \Sigma_i' \sqcup \{\theta\}$. σ is a face of θ, so also $\sigma \in \Sigma'$. It follows that $\sigma = \sigma'$. Conclude that θ is a successor of σ' in Σ'. We have $g(\theta) = f(\theta) = \delta_1 = \delta_1'$.

- $\delta_1 \neq \delta_1'$ and $\delta = \delta'$. Then $\delta_1' = \delta' \underline{\vee} \mathbf{w}$ by (4.14), and $\delta_1 = \delta \vee \underline{\delta}$. Furthermore $\sigma \in \Sigma'$, so $\sigma = \sigma'$. Distinguish two subcases:

  - δ_1 is an immediate successor of δ. Let σ_1' be a successor of σ' in Σ' such that $\sigma_1' \subseteq \theta$ and $\dim \sigma_1' = \dim \theta$. Since $\mathbf{w} \in f(\theta)$, we have $f(\theta) \notin \Delta'$ and therefore $\mathbf{w} \in g(\sigma_1')$ by construction. Since g is monotone, it follows that $g(\sigma_1') \geq \delta_1'$. However,

$$\begin{aligned}\dim g(\sigma_1') &\leq \dim \operatorname{sub}(\Delta', \Delta)\big(g(\sigma_1')\big)\\ &= \dim f\big(\operatorname{sub}(\Sigma', \Sigma)(\sigma_1')\big)\\ &= \dim f(\theta)\\ &= \dim \delta_1\\ &= \dim \delta + 1\\ &= \dim \delta' + 1\\ &= \dim \delta_1';\end{aligned}$$

    so $g(\sigma_1') = \delta_1'$.

  - $\dim \delta_1 \geq \dim \delta + 2$. Then $\dim \theta \geq \dim \sigma + 2 = \dim \sigma' + 2$. Hence

$$1 \leq \dim \partial\theta - \dim \sigma' = \operatorname{dpt}^{\Sigma_i^-}(\sigma').$$

    Furthermore $\dim g(\sigma') = \dim \delta' = \dim \delta_1' - 1$ so

$$\begin{aligned}\operatorname{dpt}^P\big(f_i(\sigma')\big) &= d' - \#\Big(\operatorname{vtc}\big(g(\sigma')\big) \cap Y\Big)\\ &= d' - \#\big(\operatorname{vtc}(\delta') \cap Y\big)\\ &= d' - \#\big(\operatorname{vtc}(\delta) \cap Y\big)\\ &= -1 + \dim \delta_1 - \dim \delta\\ &\leq -1 + \dim \theta - \dim \sigma\\ &= \dim \partial\theta - \dim \sigma'\\ &= \operatorname{dpt}^{\Sigma_i^-}(\sigma').\end{aligned}$$

    Thus by (*), σ' has a successor $\sigma_1' \subseteq \theta$ such that $\mathbf{w} \in g(\sigma_1')$ and $\overline{f}_i(\sigma_1') = f_i(\sigma')$, i.e.

$$\begin{aligned}\operatorname{vtc}\big(g(\sigma_1')\big) \cap Y &= \operatorname{vtc}\big(g(\sigma')\big) \cap Y\\ &= \operatorname{vtc}(\delta') \cap Y.\end{aligned}$$

    Since $\delta_1' = \delta' \underline{\vee} \mathbf{w}$, the fact that $\mathbf{w} \in g(\sigma_1')$ implies $g(\sigma_1') \supseteq \delta_1'$ by monotonicity, and also $g(\sigma_1') \subseteq f(\theta) = \delta_1 = \delta' \vee \underline{\delta}$ so $g(\sigma_1') \subseteq \delta' \vee \mathbf{w} = \delta_1'$ by the above display.

- $\delta_1 \neq \delta_1'$ and $\delta \neq \delta'$. Distinguish two subcases:

– $\delta = \delta_1$. Then we claim that

$$\mathrm{vtc}(\delta') \setminus Y = \mathrm{vtc}(\delta_1') \setminus Y.$$

For suppose that $\mathbf{x} \in \mathrm{vtc}(\delta_1') \setminus Y$. We have $\mathbf{w} \in \mathrm{vtc}(\delta')$ by (4.14), so suppose $\mathbf{x} \neq \mathbf{w}$. Then $\mathbf{x} \in \mathrm{vtc}(\delta_1)$ since $\delta_1' \in \Delta'(\delta_1)$, so $\mathbf{x} \in \mathrm{vtc}(\delta)$. Hence $\xi = \underline{\bigvee}\big(\mathrm{vtc}(\delta) \setminus \{\mathbf{x}\}\big)$ is a proper predecessor of δ, and if $\mathbf{x} \notin \mathrm{vtc}(\delta')$ then $\delta' \in \Delta'(\xi)$ because $\delta' \in \Delta'(\delta)$, so in particular $\delta' \subseteq \xi$ which contradicts $\delta = \mathrm{sub}(\Delta', \Delta)(\delta')$. This proves the claim. Furthermore $\delta' \notin \Delta$, so f_j is defined. Recall that $\overline{f}_j$ satisfies the back-property on $\Sigma_j^* \setminus \Sigma_j^- = \mathrm{sub}(\Sigma', \Sigma)^{-1}\big[\{\sigma_j\}\big] \ni \sigma'$. Hence σ' has a successor σ_1' with $\overline{f}_j(\sigma_1') = \mathrm{vtc}(\delta_1') \cap Y$, which implies

$$\mathrm{vtc}\big(g(\sigma_1')\big) = \mathrm{vtc}\big(g(\sigma')\big) \cup \big(\mathrm{vtc}(\delta_1') \cap Y\big) = \mathrm{vtc}(\delta_1').$$

– $\delta \neq \delta_1$. Then $\sigma \neq \theta$. Since $\overline{f}_j$ has the back-property on $\mathrm{sub}(\Sigma', \Sigma)^{-1}\big[\sigma_j\}\big] \ni \sigma'$, we have $\mathrm{dpt}^P\big(\overline{f}_j(\sigma')\big) \leq \mathrm{dpt}^{\Sigma_j^*}(\sigma')$. Notice that $\overline{f}_j \subseteq f_i$. Distinguish two subsubcases:
  * $\dim \sigma' + 2 \leq \dim \theta$. Then

    $$\mathrm{dpt}^{\Sigma_i^-}(\sigma') \geq \max\Big\{1, \mathrm{dpt}^P\big(\overline{f}_j(\sigma')\big)\Big\},$$

    so by (*), σ' has a successor σ_0' with $\mathrm{sub}(\Sigma', \Sigma)(\sigma_0') = \theta$ and $\overline{f}_i(\sigma_0') = \overline{f}_i(\sigma')$ whence

    $$\mathrm{vtc}\big(g(\sigma_0')\big) \cap Y = \mathrm{vtc}\big(g(\sigma')\big) \cap Y \subseteq \mathrm{vtc}(\delta_1') \cap Y.$$

    Next pick a successor σ_1' of σ_0' with $\mathrm{sub}(\Sigma', \Sigma)(\sigma_1') = \theta$ and $\overline{f}_i(\sigma_1') = \mathrm{vtc}(\delta_1') \cap Y$. Then

    $$\mathrm{vtc}\big(g(\sigma_1')\big) \setminus \underline{\delta} = \mathrm{vtc}(\overline{\delta}_i) = \mathrm{vtc}(\delta_1') \setminus \underline{\delta},$$

    and $\mathbf{w} \in \mathrm{vtc}\big(g(\sigma_1')\big) \cap \mathrm{vtc}(\delta_1')$; so $g(\sigma_1') = \delta_1'$ using the formula for $\Delta'(\delta_1)$ (see (4.13)).
  * $\dim \sigma' + 1 = \dim \theta$. Then $\dim \sigma' = \dim \sigma$, so $\overline{f}_j(\sigma') \in P$ is maximal. Let σ_1' be any successor of σ' such that $\mathrm{sub}(\Sigma', \Sigma)(\sigma_1') = \theta$. Then

    $$\mathrm{vtc}\big(g(\sigma_1')\big) \setminus \underline{\delta} = \mathrm{vtc}(\overline{\delta}_i) = \mathrm{vtc}(\delta_1') \setminus \underline{\delta}.$$

    On the other hand, both $g(\sigma_1')$ and δ_1' are successors of $\delta' \supseteq \overline{f}_j(\sigma')$, so

    $$\mathrm{vtc}\big(g(\sigma_1')\big) \cap Y = \mathrm{vtc}(\delta_1') \cap Y.$$

    Since $\mathbf{w} \in \mathrm{vtc}\big(g(\sigma_1')\big) \cap \mathrm{vtc}(\delta_1')$ it follows that $g(\sigma_1') = \delta_1'$ using the formula for $\Delta'(\delta_1)$.

Hence g has the back-property, so g is a p-morphism. □

We achieved the goal of this chapter, namely proving Theorem 4.1. Informally speaking, it means that we can now take arbitrarily fine subdivisions of p-morphisms between simplicial complexes. This indicates that p-morphisms are important maps not only in the context of complexes but also in the context of polyhedra. In chapter 7, we shall use this to draw conclusions from the invalidity of Jankov-Fine formulas in a polyhedron.

Chapter 5

Pyramids

Polyhedra of low dimensions can already have rather complicated shapes. Suppose that we are interested in the logical properties of some specific set $\mathfrak{p} \subseteq \mathbf{plhdr}_4$ of four-dimensional polyhedra. One would expect difficulties examining the logic $\mathrm{Log}(\mathfrak{p})$, since four-dimensional objects are hard to imagine. In this chapter we establish some general theory that can be used for mitigating this problem. Recall from Remark 2.76 that the logic $\mathrm{Log}_{\varnothing}(\mathfrak{p})$ is of height 6 whereas the logic $\mathrm{Log}(\mathfrak{p})$ is of height only 5. We shall show that there is a set $\mathfrak{q} \subseteq \mathbf{plhdr}_3$ such that $\mathrm{Log}(\mathfrak{p}) = \mathrm{Log}_{\varnothing}(\mathfrak{q})$ (Theorem 5.33). Hence we can study three-dimensional polyhedra instead of four-dimensional polyhedra.

All of this can be generalized to any dimensions. In view of the applications mentioned in chapter 1, this implies that some problems related to three-dimensional modelling will be reducible to problems staged in the plane $\mathbb{R}^2$.

First, section 5.1 shows how to go from a three-dimensional polyhedron to a four-dimensional polyhedron. Then, section 5.2 shows how to go from a four-dimensional polyhedron to a three-dimensional polyhedron. Next, section 5.3 derives some relationships between these two geometrical operations. Finally, section 5.4 puts everything together to obtain results for polyhedral logics.

5.1 Pyramids

Notation **5.1**. We write $\iota = \iota_d^{d+1}$, $\pi = \pi_d^{d+1}$ and $\mathbf{e} = \mathbf{e}^{d,d+1}$.

A traditional pyramid has a square as its base. For our purpose, a pyramid can have any polyhedron as a base. In the context of logic, given a triangulation of a polyhedron, it is convenient to preserve the structure of the triangulation while building the pyramid. Therefore we start by introducing pyramids as simplicial complexes.

Definition **5.2** (**standard (bi)pyramid**). Let $\Sigma \subseteq \mathscr{P}\mathbb{R}^d$ be a simplicial complex. Then the *standard pyramid* with base Σ is

$$\Sigma\blacktriangleright = \iota[\![\Sigma]\!] * \{\mathbf{e}, \varnothing\}.$$

(Recall that $\iota[\![\Sigma]\!]$ is a simplicial complex by Lemma 2.55-1 and Example 2.16.) Symmetrically,

$$\blacktriangleleft\Sigma = \iota[\![\Sigma]\!] * \{-\mathbf{e}, \varnothing\}.$$

The *standard bipyramid* with base Σ is

$$\blacktriangleleft\Sigma\blacktriangleright = \iota[\![\Sigma]\!] * \{-\mathbf{e},\mathbf{e},\varnothing\} = (\blacktriangleleft\Sigma)\cup(\Sigma\blacktriangleright).$$

We write

$$^{0}\blacktriangleleft\Sigma\blacktriangleright^{0} = \Sigma \quad \& \quad ^{k+1}\blacktriangleleft\Sigma\blacktriangleright^{k+1} = \blacktriangleleft\left(^{k}\blacktriangleleft\Sigma\blacktriangleright^{k}\right)\blacktriangleright.$$

See Figure 5.1.

Lemma **5.3**. For any simplicial complex Σ, the sets $\Sigma\blacktriangleright$, $\blacktriangleleft\Sigma$ and $\blacktriangleleft\Sigma\blacktriangleright$ are simplicial complexes. Moreover, $\Sigma\blacktriangleright$ and $\blacktriangleleft\Sigma$ are subcomplexes of $\blacktriangleleft\Sigma\blacktriangleright$.

Proof. The least trivial part is to show that intersections behave well in these complexes. In particular, we prove the inclusion $\subseteq$ in the equality

$$\left(\iota[\sigma_0]\veebar\mathbf{e}\right)\cap\left(\iota[\sigma_1]\veebar\mathbf{e}\right) = \iota[\sigma_0\cap\sigma_1]\veebar\mathbf{e}$$

(for $\sigma_0,\sigma_1\in\Sigma$). Let $\mathbf{x}$ be an element of the left-hand side. We either have $\mathbf{x}=\mathbf{e}$, or there exists a unique $\mathbf{y}\in\iota[\mathbb{R}]$ such that $\mathbf{x}\in\mathbf{y}\veebar\mathbf{e}$. In the latter we have $\mathbf{y}\in\iota[\sigma_0]\cap\iota[\sigma_1]$. □

Remark **5.4**. Let $\Sigma\subseteq\mathscr{P}\mathbb{R}^d$ be a simplicial complex. Then:

1. $$\mathrm{vtc}(\Sigma\blacktriangleright) = \iota\left[\mathrm{vtc}(\Sigma)\right]\sqcup\{\mathbf{e}\} \quad \& \quad \mathrm{vtc}(\blacktriangleleft\Sigma) = \iota\left[\mathrm{vtc}(\Sigma)\right]\sqcup\{-\mathbf{e}\}$$
2. $$\mathrm{vtc}(\blacktriangleleft\Sigma\blacktriangleright) = \iota\left[\mathrm{vtc}(\Sigma)\right]\sqcup\{-\mathbf{e},\mathbf{e}\}.$$
3. For $k\geq 0$,
$$\mathrm{vtc}(^{k}\blacktriangleleft\Sigma\blacktriangleright^{k}) = \iota_d^{d+k}\left[\mathrm{vtc}(\Sigma)\right]\sqcup\{-\mathbf{e}_i^{d+k},\mathbf{e}_i^{d+k} : d\leq i<d+k\}.$$

A straightforward set-theoretic calculation allows an explicit formulation of the iterated bipyramids:

Lemma **5.5**. Let $\Sigma\subseteq\mathscr{P}\mathbb{R}^d$ be a simplicial complex and $k\geq 0$. Then

$$^{k}\blacktriangleleft\Sigma\blacktriangleright^{k} = \iota_d^{d+k}[\![\Sigma]\!] * \left\{\underline{\bigvee} X : X\subseteq Y \;\&\; X\cap(-1\cdot X)=\varnothing\right\}$$

where

$$Y = \{-\mathbf{e}_i^{d+k},\mathbf{e}_i^{d+k} : d\leq i<d+k\}.$$

Remark **5.6**. Let $\Sigma\subseteq\mathscr{P}\mathbb{R}^d$ be a simplicial complex. Then the map which fixes the points in $\iota\left[\mathrm{vtc}(\Sigma)\right]$ and swaps the points $\mathbf{e}$ and $-\mathbf{e}$, is a simplicial isomorphism from $\blacktriangleleft\Sigma\blacktriangleright$ to $\blacktriangleleft\Sigma\blacktriangleright$ that restricts to a simplicial isomorphism from $\Sigma\blacktriangleright$ to $\blacktriangleleft\Sigma$.

Pyramids nicely co-operate with basic relations between complexes, as the following lemma shows.

Lemma **5.7**. Let Σ be a simplicial complex.

1. If Σ' is a subdivision of Σ, then $\Sigma'\blacktriangleright$ is a subdivision of $\Sigma\blacktriangleright$, $\blacktriangleleft\Sigma'$ is a subdivision of $\blacktriangleleft\Sigma$ and $\blacktriangleleft\Sigma'\blacktriangleright$ is a subdivision of $\blacktriangleleft\Sigma\blacktriangleright$.

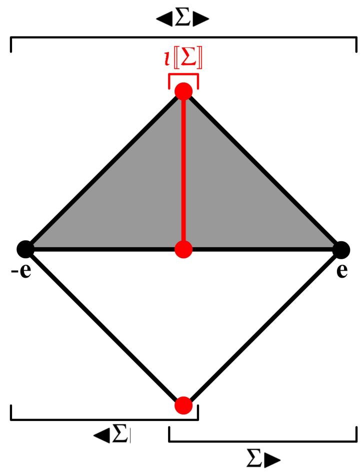


Figure 5.1: standard bipyramid with base $\Sigma = \{\varnothing, -1, 0, 1, [0,1]\}$

2. If Δ is a subcomplex of Σ, then $\Delta\blacktriangleright$ is a subcomplex of $\Sigma\blacktriangleright$, $\blacktriangleleft\Delta$ is a subcomplex of $\blacktriangleleft\Sigma$ and $\blacktriangleleft\Delta\blacktriangleright$ is a subcomplex of $\blacktriangleleft\Sigma\blacktriangleright$.

Proof. Trivial. See Figure 5.2 for an example. □

If we reformulate the endomorphism from Remark 5.6 in terms of posets, we can get slightly more information:

Lemma **5.8**. Let $\Sigma \subseteq \mathscr{P}\mathbb{R}^d$ be a simplicial complex. Then

$$\uparrow^{\Sigma\blacktriangleright}(\mathbf{e}) = \uparrow^{\blacktriangleleft\Sigma\blacktriangleright}(\mathbf{e}) \cong \Sigma \cong \uparrow^{\blacktriangleleft\Sigma\blacktriangleright}(-\mathbf{e}) = \uparrow^{\blacktriangleleft\Sigma}(-\mathbf{e}).$$

Proof. Trivial. □

Since we shall be looking at the logics of pyramids, we want to understand how p-morphisms behave on them. The following two lemmas provide a basic idea.

Lemma **5.9**. Let Σ be a simplicial complex. Then the map which fixes the points in $\mathrm{vtc}(\Sigma\blacktriangleright)$ and sends $-\mathbf{e}$ to $\mathbf{e}$, is a simplicial map from $\blacktriangleleft\Sigma\blacktriangleright$ to $\Sigma\blacktriangleright$ that can be extended to a surjective p-morphism.

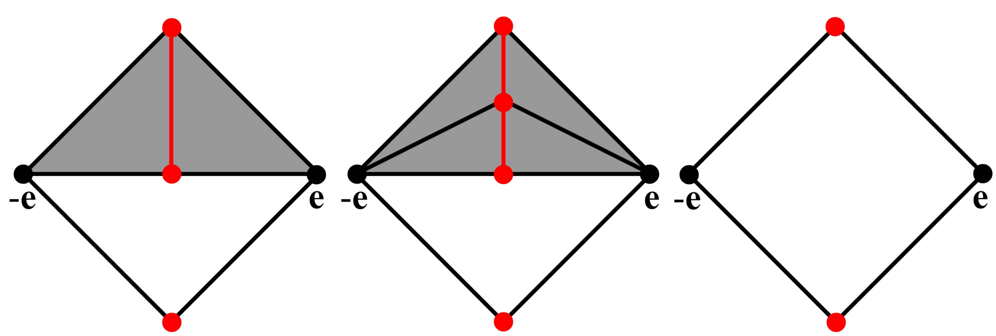


Figure 5.2: standard bipyramids with bases $\{\varnothing, -1, 0, 1, [0,1]\}$, $\{\varnothing, -1, 0, 1/2, 1, [0,1/2], [1/2,1]\}$ and $\{\varnothing, -1, 1\}$

Proof. Call this map f. It is clear that f is a simplicial map. Let

$$\begin{aligned} &◀\Sigma▶ \to \Sigma▶ : \\ &\qquad \delta \mapsto \underline{\bigvee}\Big(f\big[\operatorname{vtc}(\delta)\big]\Big). \end{aligned}$$

It is easy to verify that this is a surjective p-morphism extending f. □

Lemma **5.10**. Let $\Sigma, \Delta \in$ **cmplx**. If there is a surjective p-morphism $\Sigma \to \Delta$, then there are surjective p-morphisms $\Sigma▶ \to \Delta▶$ and $◀\Sigma▶ \to ◀\Delta▶$.

Proof. Let $f : \Sigma \to \Delta$ be a surjective p-morphism. Find d, d' such that $\Sigma \in \mathbf{cmplx}^d$ and $\Sigma' \in \mathbf{cmplx}^{d'}$. Define $g : \Sigma▶ \to \Delta▶$ by

$$g\big(\iota_d^{d+1}[\sigma]\big) = \iota_{d'}^{d'+1}\big[f(\sigma)\big] \quad \& \quad g\big(\iota_d^{d+1}[\sigma] \underline{\vee} \mathbf{e}^{d,d+1}\big) = \iota_{d'}^{d'+1}\big[\big[f(\sigma)\big] \underline{\vee} \mathbf{e}^{d',d'+1}$$

for $\sigma \in \Sigma$. Then it is easy to verify that g is a surjective p-morphism. Similarly, there is a surjective p-morphism $◀\Sigma▶ \to ◀\Delta▶$. □

As promised at the start of this chapter, we can also build pyramids using polyhedra.

Lemma **5.11**. If $|\Sigma| = |\Delta|$, then $|\Sigma▶| = |\Delta▶|$, $|◀\Sigma| = |◀\Delta|$ and $|◀\Sigma▶| = |◀\Delta▶|$.

Proof. We have

$$|\Sigma▶| = \bigcup_{\mathbf{x} \in |\Sigma|} \iota(\mathbf{x}) \underline{\vee} \mathbf{e},$$

and similar equations for $|◀\Sigma|$ and $|◀\Sigma▶|$. □

Notation **5.12**. Hence, if $\mathbf{P} = |\Sigma|$, write $\mathbf{P}▶ = |\Sigma▶|$, $◀\mathbf{P} = |◀\Sigma|$ and $◀\mathbf{P}▶ = |◀\Sigma▶|$.

Remark **5.13**. Let $\mathbf{P} \in$ **plhdr**. Then $-\mathbf{e}$ and $\mathbf{e}$ are the only points in $◀\mathbf{P}▶$ whose final coordinate lies outside of $(-1, 1)$.

Lemma **5.14**. If $\mathbf{P} \cong \mathbf{Q}$ then $\mathbf{P}▶ \cong \mathbf{Q}▶$, $◀\mathbf{P} \cong ◀\mathbf{Q}$ and $◀\mathbf{P}▶ \cong ◀\mathbf{Q}▶$.

Proof. By Lemma 2.64-1, $\mathbf{P}$ and $\mathbf{Q}$ have some isomorphic triangulations. Then by Notation 5.12, $\mathbf{P}▶$ and $\mathbf{Q}▶$ have some isomorphic triangulations (and analogous claims hold for the other cases). Hence $\mathbf{P}▶ \cong \mathbf{Q}▶$ by Lemma 2.64-1. □

Up until this point, nothing seems to be special or useful about *bi*pyramids as opposed to pyramids. The following lemma gives a nice property of bipyramids. This will be important for understanding the connection between polyhedrally-complete logics and quasi-polyhedrally-complete logics later in this chapter.

Lemma **5.15**. Let $\mathbf{P} \in$ **plhdr**. Let $\psi : [-1, 1] \to [-1, 1]$ be a piecewise linear order-isomorphism. Then there is a PL-homeomorphism $\phi : ◀\mathbf{P}▶ \to ◀\mathbf{P}▶$ such that $\pi \circ \phi = \psi \circ \pi$. See Figure 5.3.

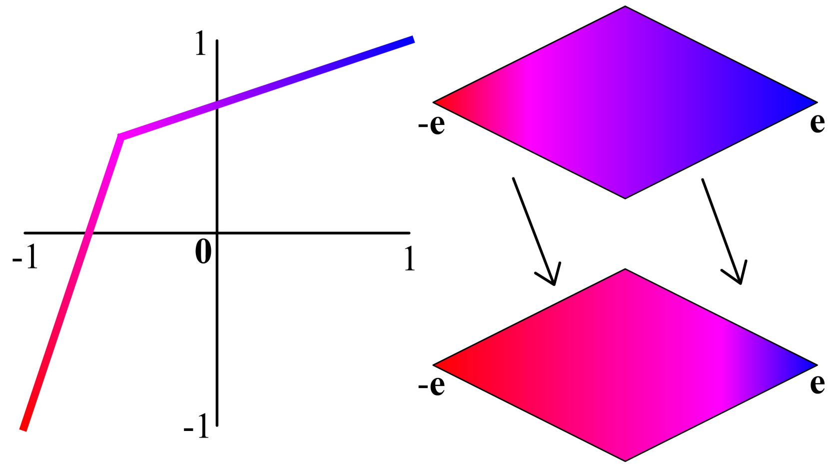


Figure 5.3: example of Lemma 5.15 (left: ψ, right: ϕ)

Proof. Let $\mathbf{P} \subseteq \mathbb{R}^d$. For $\mathbf{x} \in \mathbb{R}^d$ and $t \in [-1,1]$, let

$$\phi_t(\mathbf{x}) = \Big(\big(1-|t|\big)\cdot\mathbf{x}, t\Big) \in \mathbb{R}^{d+1}.$$

Then

$$\blacktriangleleft\mathbf{P}\blacktriangleright = \bigsqcup_{t\in[-1,1]} \phi_t[\mathbf{P}],$$

and ϕ_t is injective for $t \in (-1,1)$. Let T be a finite subset of $[-1,1]$ that includes the points where ψ changes slope and the points -1, 0, 1 and $\psi^{-1}(0)$. Let Σ be a triangulation of $\mathbf{P}$. By Lemma 2.31, there is a subdivision Δ of $\blacktriangleleft\Sigma\blacktriangleright$ such that for each $t \in T \setminus \{0,1\}$ there is a subcomplex of Δ which is a subdivision of $\phi_t[\![\Sigma]\!]$ (for the latter is a simplicial complex by Lemma 2.55-1). Define the map ϕ by

$$\phi\big(\phi_t(\mathbf{x})\big) = \phi_{\psi(t)}(\mathbf{x})$$

for all $t \in [-1,1]$ and $\mathbf{x} \in \mathbf{P}$. This is well-defined since 0 and 1 are fixpoints of ψ. Whenever $t < s$ such that $[t,s] \cap T = \{t,s\}$, the restriction $\psi|[t,s]$ is an affine map, so also $\phi\big|\pi^{-1}\big[[t,s]\big]$ is an affine map (note that the absolute value poses no problem since $0 \notin (t,s)$ and $0 \notin \big(\psi(t),\psi(s)\big)$). Hence ϕ is piecewise linear. It is easy to check that ϕ is a homeomorphism. $\square$

We calculate some simple examples of (bi)pyramids, some of which will be used in proofs later in this chapter.

Lemma **5.16**. Let σ_d be a d-simplex and σ_{d+1} a $(d+1)$-simplex.

1. $(\partial\sigma_d)\blacktriangleright$ is PL-homeomorphic to σ_d via a PL-homeomorphism that maps $\mathbf{e}$ into $\operatorname{relInt}\sigma_d$.

2. $\blacktriangleleft(\partial\sigma_d)\blacktriangleright$ is PL-homeomorphic to $\partial\sigma_{d+1}$.

3. $\sigma_d\blacktriangleright$ is PL-homeomorphic to σ_{d+1}.

4. $\blacktriangleleft\sigma_d\blacktriangleright$ is PL-homeomorphic to σ_{d+1}.

Proof. W.l.o.g., we may assume that σ_d is a face of σ_{d+1}.

By Lemma 2.48,

$$\mathrm{propfac}(\sigma_d)\blacktriangleright \cong \mathrm{fac}(\sigma_d)\big[\mathbf{b}(\sigma_d)\big]$$

via an isomorphism that maps $\mathbf{e}$ to $\mathbf{b}(\sigma_d)$. Hence item 1 follows from Lemma 2.64-1. It also follows that

$$\mathrm{propfac}(\sigma_{d+1})\big[\mathbf{b}(\sigma_d)\big] \cong \blacktriangleleft\mathrm{propfac}(\sigma_d)\blacktriangleright,$$

implying 2.

The proofs of 3 and 4 are similar. □

5.2 Local subcomplexes

It turns out that, for a polyhedron $\mathbf{P}$, the logic $\mathrm{Log}(\mathbf{P})$ describes the *local* structure of $\mathbf{P}$ while $\mathrm{Log}_{\varnothing}(\mathbf{P})$ describes the *global* structure of $\mathbf{P}$. This will be made precise in section 5.4. In the present section, we introduce the relevant geometrical notions of locality.

Notation **5.17**. Let Σ be a simplicial complex and let $\mathbf{x}$ be a vertex of Σ. Then write

$$\mathrm{star}(\Sigma,\mathbf{x}) = \downarrow\big(\uparrow(\mathbf{x})\big)$$

and

$$\mathrm{link}(\Sigma,\mathbf{x}) := \mathrm{star}(\Sigma,\mathbf{x}) \setminus \big(\uparrow(\mathbf{x})\big).$$

Furthermore set

$$\Sigma\textcircled{\mathbf{x}} = \blacktriangleleft\mathrm{link}(\Sigma,\mathbf{x})\blacktriangleright.$$

See Figure 5.4.

The reason for introducing the compound notion $\Sigma\textcircled{\mathbf{x}}$ is that this turns out to provide a suitable closure operator on sets of simplicial complexes, that we use to establish a connection between polyhedrally-complete and quasi-polyhedrally-complete logics (see Lemma 5.36 and Theorem 5.38).

Remark **5.18**. If $\Sigma \in \mathbf{cmplx}_d^{d'}$ then $\Sigma\textcircled{\mathbf{x}} \in \mathbf{cmplx}_d^{d'+1}$.

It is easy to describe a link as a poset:

Proposition **5.19**. Let Σ be a simplicial complex and let $\mathbf{x}$ be a vertex of Σ. Then

$$\mathrm{link}(\Sigma,\mathbf{x}) \cong \uparrow^{\Sigma}(\mathbf{x}).$$

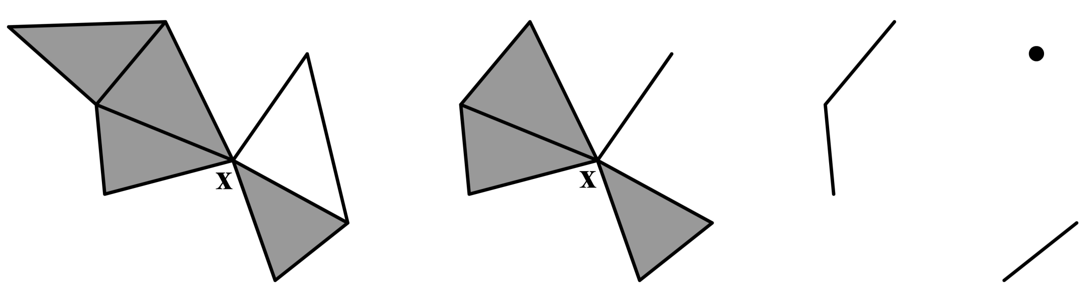


Figure 5.4: example of a complex Σ, $\mathrm{star}(\Sigma,\mathbf{x})$ and $\mathrm{link}(\Sigma,\mathbf{x})$

Proof. We have

$$\mathrm{star}(\Sigma,\mathbf{x}) = {\uparrow}^{\Sigma}(\mathbf{x}) \sqcup \Big\{ \underline{\bigvee} \big(\mathrm{vtc}(\sigma) \setminus \{\mathbf{x}\}\big) : \sigma \in {\uparrow}^{\Sigma}(\mathbf{x}) \Big\}.$$

Hence the right-hand side of this disjoint union is $\mathrm{link}(\Sigma,\mathbf{x})$. □

Hence a link can be thought of as a "generated subframe", which already hints at the connection with modal logic.

Corollary **5.20**. Let Σ be a simplicial complex and let $\mathbf{x}$ be a vertex of Σ. Then

$$\dim \mathrm{link}(\Sigma,\mathbf{x}) = \dim \mathrm{star}(\Sigma,\mathbf{x}) - 1.$$

Lemma **5.21**. Let Σ be a simplicial complex and let $\mathbf{x}$ be a vertex of Σ. Then for each

$$\mathbf{y} \in \big|\mathrm{star}(\Sigma,\mathbf{x})\big|$$

there is a unique $\alpha \in [0,1]$ such that $\mathbf{y} = \alpha\cdot\mathbf{x} + (1-\alpha)\cdot\mathbf{z}$ for some $\mathbf{z} \in \mathrm{link}(\Sigma,\mathbf{x})$. If $\alpha < 1$, then $\mathbf{z}$ is also unique. See Figure 5.5.

Proof. This can easily be derived using Lemma 2.32. □

Suppose $\Sigma \in \mathbf{cmplx}^4$ and $\mathbf{x} \in \mathrm{vtc}(\Sigma)$, and that we are interested in $\mathrm{link}(\Sigma,\mathbf{x})$. We know that $\dim \mathrm{link}(\Sigma,\mathbf{x}) < \dim \Sigma$, so $\mathrm{link}(\Sigma,\mathbf{x}) \in \mathbf{cmplx}_3$. However, it would be nice if $\mathrm{link}(\Sigma,\mathbf{x})$ is also isomorphic to a complex in $\mathbf{cmplx}^3$. We shall show in Lemma 5.23 that this is usually the case. In section 5.4, we show how to use this for studying $\mathrm{Log}(\mathfrak{p})$ given $\mathfrak{p} \subseteq \mathbf{cmplx}^4$.

Lemma **5.22**. Let σ be a $(d+1)$-simplex and let $\mathbf{P}$ be a subpolyhedron of $\partial\sigma$ that is PL-homeomorphic to a d-simplex. Then the closure of $\partial\sigma \setminus \mathbf{P}$ is PL-homeomorphic to a d-simplex.

Proof. See [Bry, Corollary 3.8]. □

Lemma **5.23**. Let $\Sigma \subseteq \mathscr{P}\mathbb{R}^d$ be a simplicial complex where $d \geq 1$, and let $\mathbf{x}$ be a vertex of Σ.

1. $\big|\mathrm{link}(\Sigma,\mathbf{x})\big|$ is PL-homeomorphic to a subpolyhedron of the boundary of a d-simplex.
2. Any proper subpolyhedron of the boundary of a d-simplex is PL-homeomorphic to a polyhedron in $\mathbb{R}^{d-1}$.
3. $\big|\mathrm{link}(\Sigma,\mathbf{x})\big|$ is PL-homeomorphic to the boundary of a d-simplex or to a polyhedron in $\mathbb{R}^{d-1}$. [The former is the case iff $\mathbf{x}$ lies in the interior of $|\Sigma| \subseteq \mathbb{R}^d$.]

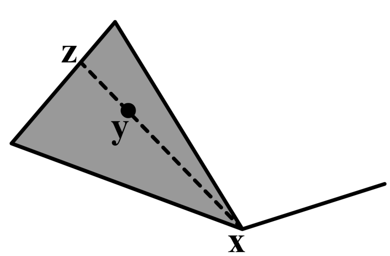


Figure 5.5: example to Lemma 5.21

Proof. 1: let $\sigma \subseteq \mathbb{R}^d$ be a d-simplex such that $\mathbf{x} = \mathbf{b}(\sigma)$ and $\left|\text{link}(\Sigma, \mathbf{x})\right| \subseteq \sigma$. Then for every $\mathbf{y} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, there exists $r > 0$ such that $\mathbf{x} + r\mathbf{y} \in \partial\sigma$. Using this "scaling" method, we obtain a map $\phi : \left|\text{link}(\Sigma, \mathbf{x})\right| \to \partial\sigma$ given by $\phi(\mathbf{x}+\mathbf{y}) = \mathbf{x} + r\mathbf{y}$. Figure 5.6(a) shows an example of this map ϕ in case Σ is the complex of Figure 5.4. Using the fact that $\mathbf{x}$ is not an affine combination of the vertices of any particular cell in $\text{link}(\Sigma, \mathbf{x})$, one can conclude that ϕ is injective. Furthermore, we claim that ϕ is piecewise linear. ϕ need not be affine on each cell of $\text{link}(\Sigma, \mathbf{x})$. However, by Lemma 2.48-1 there is a subdivision Δ of $\text{fac}(\sigma)$ with $\text{vtc}(\Delta) = \text{vtc}(\sigma) \sqcup \{\mathbf{x}\}$. By Lemma 2.31 we can find a subdivision of Δ that has a subcomplex Σ' that is a subdivision of $\text{link}(\Sigma, \mathbf{x})$. Then ϕ is affine on each cell of Σ'. See Figure 5.6(b).

2: let σ be a d-simplex and $\mathbf{P} \subsetneq \partial\sigma$ a polyhedron. Then there exists a $(d-1)$-simplex $\delta \subseteq \partial\sigma \setminus \mathbf{P}$. Then $\mathbf{P}$ is a subpolyhedron of the closure of $\sigma \setminus \delta$. By Lemma 5.22, $\sigma \setminus \delta$ is PL-homeomorphic to a $(d-1)$-simplex. By Lemmas 2.31 and 2.64-1, it follows that $\mathbf{P}$ is PL-homeomorphic to a subpolyhedron of a $(d-1)$-simplex.

3 follows from 1 and 2. □

The next lemma shows that links cooperate nicely with barycentric subdivisions. This will be relevant in chapter 7.

Lemma **5.24**. Let Σ be a simplicial complex and let $\mathbf{x}$ be a vertex of Σ. Then

$$\text{link}(\Sigma, \mathbf{x})^+ \cong \text{link}(\Sigma^+, \mathbf{x}).$$

Proof. We have

$$\begin{aligned}
\text{link}(\Sigma^+, \mathbf{x}) &= \text{cb}_\Sigma\left[\left\{c \in \mathscr{C}(\Sigma) : c \sqcup \{\mathbf{x}\} \in \mathscr{C}(\Sigma)\right\}\right] \\
&\cong \text{cb}_\Sigma\left[\left\{c \in \mathscr{C}(\Sigma) : \mathbf{x} \in c\right\}\right] \\
&= \text{cb}_\Sigma\left[\mathscr{C}\left(\text{link}(\Sigma, \mathbf{x})\right)\right] \\
&= \text{link}(\Sigma, \mathbf{x})^+.
\end{aligned}$$

□

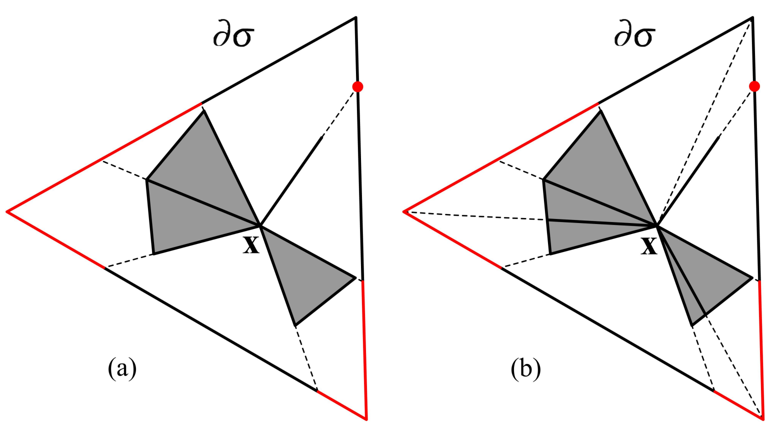


Figure 5.6: impression of the map ϕ in the proof of Lemma 5.23-1

5.3 Pyramids and local complexes

We next study some relations between the constructions presented in sections 5.1 and 5.2. The first attractive property of a pyramid with some arbitrary base Σ is that it immediately "realizes" Σ as a link:

Remark **5.25**. Let Σ be a simplicial complex. Then

$$\operatorname{link}(\Sigma\blacktriangleright,\mathbf{e}) = \operatorname{link}(\blacktriangleleft\Sigma,-\mathbf{e}) = \operatorname{link}(\blacktriangleleft\Sigma\blacktriangleright,\mathbf{e}) = \operatorname{link}(\blacktriangleleft\Sigma\blacktriangleright,-\mathbf{e}) = \iota[\![\Sigma]\!] \cong \Sigma.$$

Moreover, a star is, up to isomorphism, a pyramid with a link as base:

Lemma **5.26**. Let Σ be a simplicial complex and let $\mathbf{x}$ be a vertex of Σ. The map

$$\mathbf{y} \mapsto \begin{cases} \mathbf{e} & (\mathbf{y}=\mathbf{x}) \\ \iota\mathbf{y} & (\mathbf{y}\neq\mathbf{x}) \end{cases}$$

is a simplicial isomorphism from $\operatorname{star}(\Sigma,\mathbf{x})$ to $\operatorname{link}(\Sigma,\mathbf{x})\blacktriangleright$.

Proof. This follows easily from the definitions. See also the display in the proof of Proposition 5.19. □

A first application of pyramids is a proof of a harmony between links and subdivisions.

Lemma **5.27**. Let Σ be a simplicial complex and let $\mathbf{x}$ be a vertex of Σ.

1. If Δ is a subdivision of Σ, then $\operatorname{link}(\Delta,\mathbf{x})$ is isomorphic to a subdivision of $\operatorname{link}(\Sigma,\mathbf{x})$.

2. If Γ is a subdivision of $\operatorname{link}(\Sigma,\mathbf{x})$, there exists a subdivision Δ of Σ such that $\operatorname{link}(\Delta,\mathbf{x})$ is a subdivision of Γ.

Proof. 1: by Lemmas 5.26, 2.63 and 2.29, we may assume w.l.o.g. that $\Sigma = \Theta\blacktriangleright$ for some $\Theta \in \mathbf{cmplx}^d$ and that $\mathbf{x}=\mathbf{e}$.

For $\mathbf{z} \in \operatorname{vtc}\big(\operatorname{link}(\Delta,\mathbf{x})\big) \subseteq \mathbb{R}^{d+1}$, let $\phi(\mathbf{z})$ be the unique point in $\iota[\mathbb{R}^d]$ such that $\mathbf{e},\mathbf{z},\phi(\mathbf{z})$ are collinear. Then ϕ is defined on

$$\operatorname{vtc}\big(\operatorname{link}(\Delta,\mathbf{x})\big).$$

Moreover, ϕ maps affinely independent sets to affinely independent sets. A straightforward exercise in PL-geometry using barycentric coordinates shows that ϕ can be extended to a bijection

$$\overline{\phi} : \big|\operatorname{link}(\Delta,\mathbf{x})\big| \to \iota\big[|\Theta|\big]$$

that is affine on each cell of $\operatorname{link}(\Delta,\mathbf{x})$. Hence by Lemmas 2.55-1 and 2.54-1,

$$\Xi = \overline{\phi}\big[\![\operatorname{link}(\Delta,\mathbf{x})\big]\!]$$

is a simplicial complex isomorphic to $\operatorname{link}(\Delta,\mathbf{x})$. Moreover, if $\delta \in \operatorname{link}(\Delta,\mathbf{x})$ and $\delta \subseteq \iota[\theta]\veebar\mathbf{e}$ for some $\theta\in\Theta$, then $\overline{\phi}[\delta] \subseteq \iota[\theta]$. Hence Ξ is a subdivision of $\iota[\![\Theta]\!]$, as desired. This proves 1.

2: let Θ be the subcomplex of Σ consisting of all cells that do not contain $\mathbf{x}$. By Lemma 2.31 there exists a subdivision Θ' of Θ that has a subcomplex Γ' that is a subdivision of Γ. By Lemma 2.42 there is a unique subdivision Σ' of Σ such that $\Theta' \subseteq \Sigma'$ and $\operatorname{vtc}(\Sigma') = \operatorname{vtc}(\Theta') \sqcup \{\mathbf{x}\}$. From the definition of Σ' (taken from the proof of Lemma 2.42) we may conclude that $\operatorname{link}(\Sigma',\mathbf{x}) = \Gamma'$. □

Corollary **5.28**. Let Σ be a simplicial complex, Δ a subdivision of Σ and let $\mathbf{x}$ be a vertex of Σ. Then $\Delta\textcircled{\mathbf{x}}$ is isomorphic to a subdivision of $\Sigma\textcircled{\mathbf{x}}$.

Proof. Lemmas 5.27-1 and 5.7-1. □

Another relationship between local subcomplexes and pyramids is that the operations "commute", in the following sense:

Lemma **5.29**. Let $\Sigma \subseteq \mathscr{P}\mathbb{R}^d$ be a simplicial complex and let $\mathbf{x}$ be a vertex of Σ.

1. $$\operatorname{star}(\Sigma,\mathbf{x})\blacktriangleright = \operatorname{star}\left(\Sigma\blacktriangleright, \iota(\mathbf{x})\right)$$

2. $$\blacktriangleleft\operatorname{star}(\Sigma,\mathbf{x})\blacktriangleright = \operatorname{star}\left(\blacktriangleleft\Sigma\blacktriangleright, \iota(\mathbf{x})\right)$$

3. $$\operatorname{link}(\Sigma,\mathbf{x})\blacktriangleright = \operatorname{link}\left(\Sigma\blacktriangleright, \iota(\mathbf{x})\right)$$

4. $$\Sigma\textcircled{\mathbf{x}} = \operatorname{link}\left(\blacktriangleleft\Sigma\blacktriangleright, \iota(\mathbf{x})\right)$$

5. $$^k\blacktriangleleft\operatorname{link}(\Sigma,\mathbf{x})\blacktriangleright^k = \operatorname{link}(^k\blacktriangleleft\Sigma\blacktriangleright^k, \iota_d^{d+k}\mathbf{x})$$

Proof. All of this can easily be calculated from the definitions. See also Figure 5.7. □

So far in this chapter, we have focused on one simplicial complex or polyhedron at a time. However, most quasi-polyhedrally-complete logics are given by infinite sets of polyhedra. We now introduce notation for handling links of sets of polyhedra.

Notation **5.30**. If $\mathscr{S}$ is a set of simplicial complexes, let

$$\operatorname{links}(\mathscr{S}) = \left\{\operatorname{link}(\Sigma,\mathbf{x}) : \Sigma \in \mathscr{S} \;\&\; \mathbf{x} \in \operatorname{vtc}(\Sigma)\right\},$$

$$\mathfrak{S}(\mathscr{S}) = \left\{\Sigma\textcircled{\mathbf{x}} : \Sigma \in \mathscr{S} \;\&\; \mathbf{x} \in \operatorname{vtc}(\Sigma)\right\},$$

$\mathfrak{S}^0(\mathscr{S}) = \mathscr{S}$ and $\mathfrak{S}^{k+1}(\mathscr{S}) = \mathfrak{S}\big(\mathfrak{S}^k(\mathscr{S})\big)$. For $\mathfrak{p} \subseteq \mathbf{plhdr}$, define

$$\operatorname{links}(\mathfrak{p}) = \left\{|\Sigma| : \Sigma \in \operatorname{links}\left(\operatorname{Trian}(\mathfrak{p})\right)\right\}$$

and

$$\mathfrak{S}(\mathfrak{p}) = \left\{|\Sigma| : \Sigma \in \mathfrak{S}\left(\operatorname{Trian}(\mathfrak{p})\right)\right\}.$$

We call $\mathfrak{p}$ closed under $\mathfrak{S}$ if $\mathfrak{S}(\mathfrak{p}) \subseteq \mathfrak{p}$. In view of Lemma 2.79, it is equally good to require that every polyhedron in $\mathfrak{S}(\mathfrak{p})$ is PL-homeomorphic to a polyhedron in $\mathfrak{p}$: then we call $\mathfrak{p}$ closed under $\mathfrak{S}$ up to PL-homeomorphisms. We are interested in such sets $\mathfrak{p}$ because we shall prove that a quasi-polyhedrally-complete logic $\operatorname{Log}_{\varnothing}(\mathfrak{p})$ is in fact polyhedrally-complete provided $\mathfrak{p}$ is closed under $\mathfrak{S}$ (Theorem 5.38). Moreover, we shall prove that every polyhedrally-complete logic is of this form (Lemma 5.36 and Theorem 5.33).

For complexes, it turns out that the dimension already bounds the number of iterations one needs to perform before reaching a set that is essentially closed under $\mathfrak{S}$.

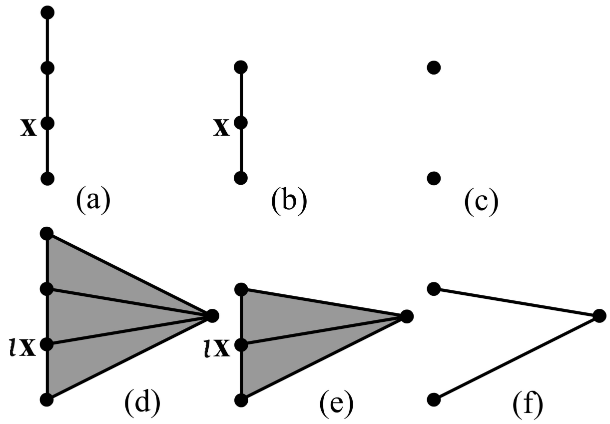


Figure 5.7: example to Lemma 5.29. (a): a complex Σ with a vertex $\mathbf{x}$; (b): $\operatorname{star}(\Sigma,\mathbf{x})$; (c): $\operatorname{link}(\Sigma,\mathbf{x})$; (d): $\Sigma\blacktriangleright$; (e): the complex in Lemma 5.29-1; (f): the complex in Lemma 5.29-3

Proposition **5.31**. Let $\mathscr{S} \subseteq \mathbf{cmplx}_d$. Then up to simplicial isomorphisms, the set

$$\bigcup_{k=0}^{d+1} \mathfrak{S}^k(\mathscr{S})$$

is closed under $\mathfrak{S}$.

Proof. We prove by induction on $k \geq 0$ that a complex in $\mathfrak{S}^k(\mathscr{S})$ which is not isomorphic to a complex in

$$\bigcup_{i=0}^{k-1} \mathfrak{S}^i(\mathscr{S}),$$

is isomorphic to a complex of the form ${}^k\blacktriangleleft\Delta\blacktriangleright^k$ for some subcomplex Δ of some complex in $\mathscr{S}$ satisfying $\dim\Delta \leq d-k$. For $k=0$, we take $\Sigma \in \mathscr{S}$ and let $\Delta = \Sigma$. Suppose that the claim holds for k. Let $\Gamma \in \mathfrak{S}^{k+1}(\mathscr{S})$, not isomorphic to a complex in

$$\bigcup_{i=0}^{k} \mathfrak{S}^i(\mathscr{S}).$$

Find $\Sigma \in \mathfrak{S}^k(\mathscr{S})$ and $\mathbf{x} \in \operatorname{vtc}(\Sigma)$ such that $\Gamma = \Sigma\textcircled{\mathbf{x}}$ and Σ is not isomorphic to a complex in

$$\bigcup_{i=0}^{k-1} \mathfrak{S}^i(\mathscr{S}).$$

By inductive hypothesis, find a subcomplex Δ of some complex in $\mathscr{S}$ satisfying $\dim\Delta \leq d-k$, such that Σ is isomorphic to ${}^k\blacktriangleleft\Delta\blacktriangleright^k$. W.l.o.g., assume that $\Sigma = {}^k\blacktriangleleft\Delta\blacktriangleright^k$. Find d' such that $\Delta \subseteq \mathscr{P}\mathbb{R}^{d'}$. Then Remark 5.4-3 implies

$$\operatorname{vtc}(\Sigma) = \iota_{d'}^{d'+k}\left[\operatorname{vtc}(\Delta)\right] \sqcup \{\mathbf{e}_i^{d'+k}, -\mathbf{e}_i^{d'+k} : d' \leq i < d'+k\}.$$

We claim that $\mathbf{x} \in \iota_{d'}^{d'+k}\big[\mathrm{vtc}(\Delta)\big]$. To check this, suppose otherwise. Then $k \geq 1$, and by Lemma 5.5 we can assume w.l.o.g. that $\mathbf{x} = \mathbf{e}_{d'+k-1}^{d'+k}$. Hence Remark 5.25 implies that

$$\begin{aligned}\Gamma &= {}^{k}\blacktriangleleft\Delta\blacktriangleright^{k}ⓧ \\ &\cong \blacktriangleleft^{k-1}\blacktriangleleft\Delta\blacktriangleright^{k-1}\blacktriangleright \\ &= \Sigma,\end{aligned}$$

a contradiction. Hence there is $\mathbf{y} \in \mathrm{vtc}(\Delta)$ such that $\mathbf{x} = \iota_{d'}^{d'+k}(\mathbf{y})$. Then Lemma 5.29-5 implies that

$$\begin{aligned}\Gamma &= {}^{k}\blacktriangleleft\Delta\blacktriangleright^{k}ⓧ \\ &= \blacktriangleleft\mathrm{link}({}^{k}\blacktriangleleft\Delta\blacktriangleright^{k}, \mathbf{x})\blacktriangleright \\ &= \blacktriangleleft^{k}\blacktriangleleft\mathrm{link}(\Delta, \mathbf{y})\blacktriangleright^{k}\blacktriangleright \\ &= {}^{k+1}\blacktriangleleft\mathrm{link}(\Delta, \mathbf{y})\blacktriangleright^{k+1}.\end{aligned}$$

And $\dim \mathrm{link}(\Delta, \mathbf{y}) \leq d - (k+1)$ by Corollary 5.20. □

For polyhedra, we shall give a stronger result (Proposition 5.37).

Even before closing off a set of polyhedra under $\mathfrak{S}$, there is no need to distinguish between PL-homeomorphic polyhedra. More precisely, we have:

Lemma **5.32**. If $\mathbf{P} \cong \mathbf{Q}$, then the elements of $\mathfrak{S}\{\mathbf{P}\}$ are PL-homeomorphic to the elements of $\mathfrak{S}\{\mathbf{Q}\}$.

Proof. By Lemma 2.64-1, there exist triangulations Σ, Δ of $\mathbf{P}, \mathbf{Q}$ resp. such that $\Sigma \cong \Delta$. Let $\mathbf{S} \in \mathfrak{S}\{\mathbf{P}\}$. Then there is a triangulation Γ of $\mathbf{P}$ and $\mathbf{x} \in \mathrm{vtc}(\Gamma)$ such that $\mathbf{S} = |\Gamma ⓧ|$. By Lemma 2.31, Σ and Γ have a common subdivision Θ. By Lemma 2.63, there is a subdivision Ξ of Δ that is isomorphic to Θ. Let f be a simplicial isomorphism from Θ to Ξ and $\mathbf{y} = f(\mathbf{x})$. By Corollary 5.28, $\Theta ⓧ \cong \Xi ⓨ$ is isomorphic to a subdivision of $\Gamma ⓧ$. Hence Lemma 2.64-1 implies that $|\Xi ⓨ| \in \mathfrak{S}\{\mathbf{Q}\}$ is PL-homeomorphic to $\mathbf{S}$. □

5.4 Modelling polyhedrally-complete logics with pyramids

We arrive at our first important result, that describes how the operator $\mathrm{links}(\cdot)$ can be used to view a polyhedrally-complete logic as a quasi-polyhedrally-complete logic.

Theorem **5.33**. For $\mathfrak{p} \subseteq \mathbf{plhdr}$,

$$\mathrm{Log}(\mathfrak{p}) = \mathrm{Log}_{\varnothing}\big(\mathrm{links}(\mathfrak{p})\big).$$

Proof. $\subseteq$. Suppose that $\varphi \notin \mathrm{Log}_{\varnothing}\big(\mathrm{links}(\mathfrak{p})\big)$. Then there is a $\mathbf{Q} \in \mathrm{links}(\mathfrak{p})$ and a triangulation Δ of $\mathbf{Q}$ such that φ can be falsified in Δ. Find $\mathbf{P} \in \mathfrak{p}$, a triangulation Σ of $\mathbf{P}$ and $\mathbf{x} \in \mathrm{vtc}(\Sigma)$ such that $\mathbf{Q} = \big|\mathrm{link}(\Sigma, \mathbf{x})\big|$. By Lemma 2.31, find a common subdivision Γ of Δ and $\mathrm{link}(\Sigma, \mathbf{x})$. By Lemma 2.78-1, φ can be falsified in Γ. By Lemma 5.27-2, there is a subdivision Σ' of Σ such that $\mathrm{link}(\Sigma', \mathbf{x})$ is isomorphic to a subdivision of Γ. By Lemma 2.78-1, φ can be falsified in $\mathrm{link}(\Sigma', \mathbf{x})$. Hence Proposition 5.19 implies that φ can be falsified in Σ' at $\mathbf{x}$, and therefore $\varphi \notin \mathrm{Log}(\mathfrak{p})$.

$\supseteq$. Suppose that $\varphi \notin \mathrm{Log}(\mathfrak{p})$. By Lemma 2.80, find $\mathbf{P} \in \mathfrak{p}$, a triangulation Σ of $\mathbf{P}$ and $\mathbf{x} \in \mathrm{vtc}(\Sigma)$ such that φ can be falsified in Σ at $\mathbf{x}$. By Proposition 5.19, φ can be falsified in $\mathrm{link}(\Sigma, \mathbf{x})$, so that $\varphi \notin \mathrm{Log}_\varnothing\big(\mathrm{links}(\mathfrak{p})\big)$. □

Corollary **5.34**. Every polyhedrally-complete logic is quasi-polyhedrally-complete.

We claim that every polyhedrally-complete logic can be obtained from pyramids. We already know that a pyramid with base Σ has a link that is isomorphic to Σ (Remark 5.25). It is now time to examine *all* links of a given pyramid $\mathbf{P}\blacktriangleright$. By Theorem 5.33, this is crucial to understanding the polyhedrally-complete logic $\mathrm{Log}(\mathbf{P}\blacktriangleright)$ of a pyramid.

Lemma **5.35**. Let $\mathfrak{p} \subseteq \mathbf{plhdr}$. Then, up to PL-homeomorphisms, the set $\mathfrak{p} \cup \mathfrak{S}(\mathfrak{p})$ is equal to

$$\mathrm{links}\{\blacktriangleleft\mathbf{P}\blacktriangleright : \mathbf{P} \in \mathfrak{p}\}.$$

Proof. To show $\supseteq$, suppose that $\mathbf{P} \in \mathfrak{p}$, that Σ is a triangulation of $\blacktriangleleft\mathbf{P}\blacktriangleright$ and $\mathbf{x} \in \mathrm{vtc}(\Sigma)$. For the moment, assume that $\mathbf{x} \in \{\mathbf{e}, -\mathbf{e}\}$. By Lemma 2.31, let Θ be a common subdivision of Σ and $\blacktriangleleft\Theta_0\blacktriangleright$, where Θ_0 is some triangulation of $\mathbf{P}$. By Lemma 5.27-1, $\mathrm{link}(\Theta, \mathbf{x})$ is isomorphic to a subdivision of $\mathrm{link}(\Sigma, \mathbf{x})$ and to a subdivision of $\mathrm{link}(\blacktriangleleft\Theta_0\blacktriangleright, \mathbf{x})$. So by Lemma 2.64-1,

$$\big|\mathrm{link}(\Sigma, \mathbf{x})\big| \cong \big|\mathrm{link}(\blacktriangleleft\Theta_0\blacktriangleright, \mathbf{x})\big|.$$

By Remark 5.25, it follows that

$$\big|\mathrm{link}(\Sigma, \mathbf{x})\big| \cong \mathbf{P}.$$

Next, we suppose that $\mathbf{x} \notin \{\mathbf{e}, -\mathbf{e}\}$ so that $x_d \notin \{-1, 1\}$ (Remark 5.13). By Lemma 5.15 there exists a PL-homeomorphism $\phi : \blacktriangleleft\mathbf{P}\blacktriangleright \to \blacktriangleleft\mathbf{P}\blacktriangleright$ such that $\mathbf{z} = \phi(\mathbf{x})$ satisfies $z_d = 0$. Find a triangulation Σ' of $\blacktriangleleft\mathbf{P}\blacktriangleright$ such that ϕ is affine on each cell of Σ'. By Lemma 2.31, we may assume that Σ' is a subdivision of Σ. By Lemma 5.27-1, $\mathrm{link}(\Sigma', \mathbf{x})$ is isomorphic to a subdivision of $\mathrm{link}(\Sigma, \mathbf{x})$. By Lemma 2.55-1, $\Delta = \phi[\![\Gamma']\!]$ is a triangulation of $\blacktriangleleft\mathbf{P}\blacktriangleright$ and $\mathbf{z} \in \mathrm{vtc}(\Delta)$. By Lemma 2.54-1, $\mathrm{link}(\Sigma', \mathbf{x}) \cong \mathrm{link}(\Delta, \mathbf{z})$. By Lemma 2.31, let Δ' be a subdivision of Δ such that Δ' has a subcomplex Ξ with carrier $\iota[\mathbf{P}]$. By Lemma 5.27-1, $\mathrm{link}(\Delta', \mathbf{z})$ is isomorphic to a subdivision of $\mathrm{link}(\Delta, \mathbf{z})$. By Lemma 2.55-1, $\Upsilon = \iota^{-1}[\![\Xi]\!]$ is a triangulation of $\mathbf{P}$. By Lemma 2.31, find a common subdivision Υ' of Δ' and $\blacktriangleleft\Upsilon\blacktriangleright$. By Lemma 5.27-1, $\mathrm{link}(\Upsilon', \mathbf{z})$ is isomorphic to a subdivision of $\mathrm{link}(\Delta', \mathbf{z})$. Hence $\mathrm{link}(\Upsilon', \mathbf{z})$ is isomorphic to a subdivision of $\mathrm{link}(\Sigma, \mathbf{x})$ and to a subdivision of $\mathrm{link}(\blacktriangleleft\Upsilon\blacktriangleright, \mathbf{z})$. Therefore

$$\big|\mathrm{link}(\Sigma, \mathbf{x})\big| \cong \big|\mathrm{link}(\blacktriangleleft\Upsilon\blacktriangleright, \mathbf{z})\big|.$$

Let $\mathbf{w} = \iota^{-1}(\mathbf{z}) \in \mathrm{vtc}(\Upsilon)$. By Lemma 5.29-4,

$$\mathrm{link}(\blacktriangleleft\Upsilon\blacktriangleright, \mathbf{z}) = \Upsilon\circledast{\mathbf{w}}.$$

Hence

$$\big|\mathrm{link}(\Sigma, \mathbf{x})\big| \cong |\Upsilon\circledast{\mathbf{w}}| \in \mathfrak{S}(\mathfrak{p}).$$

Next we prove $\subseteq$. First let $\mathbf{P} \in \mathfrak{p}$. Let Σ be a triangulation of $\mathbf{P}$. Then Remark 5.25 yiels

$$\mathrm{link}(\blacktriangleleft\Sigma\blacktriangleright, \mathbf{e}) \cong \Sigma,$$

so

$$\big|\mathrm{link}(\blacktriangleleft\Sigma\blacktriangleright, \mathbf{e})\big| \cong \mathbf{P}.$$

Next let $\mathbf{Q} \in \mathfrak{S}(\mathfrak{p})$. Then there exists $\mathbf{P} \in \mathfrak{p}$, a triangulation Σ of $\mathbf{P}$ and $\mathbf{x} \in \mathrm{vtc}(\Sigma)$ such that $\mathbf{Q} = |\Sigma\textcircled{x}|$. Let $\mathbf{y} = \iota(\mathbf{x})$. By Lemma 5.29-4,

$$\mathrm{link}(\blacktriangleleft\Sigma\blacktriangleright, \mathbf{y}) = \Sigma\textcircled{x},$$

so

$$\big|\mathrm{link}(\blacktriangleleft\Sigma\blacktriangleright, \mathbf{y})\big| = \mathbf{Q}.$$

□

The next lemma says that the links of any polyhedron are essentially closed under $\mathfrak{S}$. By Theorem 5.33, this implies that any polyhedrally-complete logic is $\mathrm{Log}_{\varnothing}(\mathfrak{q})$ for some set $\mathfrak{q}$ of polyhedra closed under $\mathfrak{S}$.

Lemma **5.36**. Let $\mathfrak{p} \subseteq \mathbf{plhdr}$. Then, up to PL-homeomorphisms, $\mathrm{links}(\mathfrak{p})$ is closed under $\mathfrak{S}$.

Proof. Let $\mathbf{Q} \in \mathrm{links}(\mathfrak{p})$. Let Σ be a triangulation of $\mathbf{Q}$ and $\mathbf{x} \in \mathrm{vtc}(\Sigma)$. We have to show that a PL-homeomorphic copy of $|\Sigma\textcircled{x}|$ lies in $\mathrm{links}(\mathfrak{p})$. Find $\Delta \in \mathbf{cmplx}$ and $\mathbf{y} \in \mathrm{vtc}(\Delta)$ such that $|\Delta| \in \mathfrak{p}$ and $\mathbf{Q} = \big|\mathrm{link}(\Delta, \mathbf{y})\big|$. By Lemma 2.31, let Σ' be a common subdivision of $\mathrm{link}(\Delta, \mathbf{y})$ and Σ. By Corollary 5.28, $\Sigma'\textcircled{x}$ is isomorphic to a subdivision of $\Sigma\textcircled{x}$. By Lemma 5.27-2, there is a subdivision Δ' of Δ such that $\Sigma_1 = \mathrm{link}(\Delta', \mathbf{y})$ is isomorphic to a subdivision of Σ'. Thus by Corollary 5.28 there is $\mathbf{z} \in \mathrm{vtc}(\Sigma_1)$ such that $\Sigma_1\textcircled{z}$ is isomorphic to a subdivision of $\Sigma'\textcircled{x}$. Thus by Lemma 2.64-1, it suffices to show that $|\Sigma_1\textcircled{z}|$ has a PL-homeomorphic copy in $\mathrm{links}(\mathfrak{p})$.

Consider the subcomplex $\Gamma = \mathrm{star}(\Delta', \mathbf{y})$. Let f be the isomorphism from Γ to $\Sigma_1\blacktriangleright$ given by Lemma 5.26. Let $\mathbf{P} = |\Sigma_1|$. Let $\psi : [-1,1] \to [-1,1]$ be a piecewise linear order-isomorphism with $\psi(0) = 1/2$. By Lemma 5.15, there is a PL-homeomorphism $\phi : \blacktriangleleft\mathbf{P}\blacktriangleright \to \blacktriangleleft\mathbf{P}\blacktriangleright$ such that $\pi \circ \phi = \psi \circ \pi$. Find a triangulation Θ of $\blacktriangleleft\mathbf{P}\blacktriangleright$ such that ϕ is affine on each cell of Θ. By Lemma 2.31, we may assume that Θ is a subdivision of $\blacktriangleleft\Sigma_1\blacktriangleright$. Let $\mathbf{w} = \iota(\mathbf{z}) \in \mathrm{vtc}(\Theta)$. By Lemma 5.27-1, $\mathrm{link}(\Theta, \mathbf{w})$ is isomorphic to a subdivision of $\mathrm{link}(\blacktriangleleft\Sigma_1\blacktriangleright, \mathbf{w})$, which equals $\Sigma_1\textcircled{z}$ by Lemma 5.29-4. By Lemma 2.55-1, $\Xi = \phi[\![\Theta]\!]$ is a triangulation of $\blacktriangleleft\mathbf{P}\blacktriangleright$. Let $\mathbf{v} = \phi(\mathbf{w}) \in \mathrm{vtc}(\Xi)$. By Lemma 2.54-1, we have $\mathrm{link}(\Theta, \mathbf{w}) \cong \mathrm{link}(\Xi, \mathbf{v})$. By Lemma 2.31, there is a subdivision Ξ' of Ξ such that Ξ' has a subcomplex Ξ_1 that is a subdivision of $\Sigma_1\blacktriangleright$. By Lemma 5.27-1, $\mathrm{link}(\Xi', \mathbf{v})$ is isomorphic to a subdivision of $\mathrm{link}(\Xi, \mathbf{v})$, hence isomorphic to a subdivision of $\Sigma_1\textcircled{z}$. By Lemma 2.63, the isomorphism f can be extended to a simplicial isomorphism g from some subdivision Γ' of Γ to Ξ_1 such that

$$\mathrm{sub}(\Xi_1, \Sigma_1\blacktriangleright) \circ g = f \circ \mathrm{sub}(\Gamma', \Gamma).$$

By Lemma 2.31, let Δ'' be a subdivision of Δ' such that Δ'' has a subcomplex Γ'' which is a subdivision of Γ'. We have $\mathbf{v} \in \mathrm{vtc}(\Xi_1)$ and – since $v_d = 1/2$ –

$$\uparrow^{\Xi'}(\mathbf{v}) = \uparrow^{\Xi_1}(\mathbf{v})$$

so, letting $\mathbf{u} = g^{-1}(\mathbf{v})$,

$$\begin{aligned}\mathrm{link}(\Xi', \mathbf{v}) &= \mathrm{link}(\Xi_1, \mathbf{v}) \\ &\cong \mathrm{link}(\Gamma', \mathbf{u}).\end{aligned}$$

By Lemma 5.27-1, it follows that also $\mathrm{link}(\Gamma'', \mathbf{u})$ is isomorphic to a subdivision of $\Sigma_1\textcircled{z}$. Also

$$\mathbf{e} \in \mathrm{sub}(\Xi_1, \Sigma_1\blacktriangleright)(\mathbf{v}) = \big(f \circ \mathrm{sub}(\Gamma', \Gamma)\big)(\mathbf{u})$$

whence

$$\begin{aligned}\mathbf{y} &= f^{-1}(\mathbf{e})\\ &\in \mathrm{sub}(\Gamma',\Gamma)(\mathbf{u})\\ &= \mathrm{sub}(\Gamma'',\Gamma)(\mathbf{u})\\ &= \mathrm{sub}(\Delta'',\Delta)(\mathbf{u})\end{aligned}$$

which implies

$$\mathrm{sub}(\Delta'',\Delta')\left[\uparrow^{\Delta''}(\mathbf{u})\right] \subseteq \uparrow^{\Delta'}(\mathbf{y})$$

so by Lemma 2.36

$$\uparrow^{\Delta''}(\mathbf{u}) \subseteq \uparrow^{\Gamma''}(\mathbf{u}).$$

Moreover, the reverse inclusion is trivial. Thus $\mathrm{link}(\Delta'',\mathbf{u}) = \mathrm{link}(\Gamma'',\mathbf{u})$ is isomorphic to a subdivision of $\Sigma_1 ⓩ$. Therefore $|\Sigma_1 ⓩ|$ is PL-homeomorphic to $\left|\mathrm{link}(\Delta'',\mathbf{u})\right| \in \mathrm{links}(\mathfrak{p})$. □

Combining what we know about links, we can obtain a pretty result that talks solely about the operator $\mathfrak{S}$ on polyhedra.

Proposition **5.37**. For $\mathfrak{p} \subseteq \mathbf{plhdr}$, the set $\mathfrak{p} \cup \mathfrak{S}(\mathfrak{p})$ is closed under $\mathfrak{S}$ up to PL-homeomorphisms.

Proof. Lemmas 5.35 and 5.36. □

We can now formally establish the result that every polyhedrally-complete logic can be obtained from pyramids.

Theorem **5.38**. Let $\mathfrak{p} \subseteq \mathbf{plhdr}$. Then

$$\mathrm{Log}_{\varnothing}\left(\mathfrak{p} \cup \mathfrak{S}(\mathfrak{p})\right) = \mathrm{Log}\{\blacktriangleleft\mathbf{P}\blacktriangleright : \mathbf{P} \in \mathfrak{p}\}, \tag{5.1}$$

and every polyhedrally-complete logic is of this form; in fact for $\mathrm{Log}(\mathfrak{q})$ we can take $\mathfrak{p} = \mathrm{links}(\mathfrak{q})$.

Proof. First, let $\mathfrak{p} \subseteq \mathbf{plhdr}$ be given. By Theorem 5.33 we have

$$\mathrm{Log}\{\blacktriangleleft\mathbf{P}\blacktriangleright : \mathbf{P} \in \mathfrak{p}\} = \mathrm{Log}_{\varnothing}\left(\mathrm{links}\{\blacktriangleleft\mathbf{P}\blacktriangleright : \mathbf{P} \in \mathfrak{p}\}\right).$$

By Lemmas 5.35 and 2.79 this equals

$$\mathrm{Log}_{\varnothing}\left(\mathfrak{p} \cup \mathfrak{S}(\mathfrak{p})\right).$$

Conversely, suppose that $\mathscr{L} = \mathrm{Log}(\mathfrak{q})$ is a polyhedrally-complete logic. Take $\mathfrak{p} = \mathrm{links}(\mathfrak{q})$. By Theorem 5.33 we have

$$\mathscr{L} = \mathrm{Log}_{\varnothing}(\mathfrak{p}).$$

By Lemma 5.36, $\mathfrak{p}$ is closed under $\mathfrak{S}$ up to PL-homeomorphisms, so Lemma 2.79 implies

$$\mathrm{Log}_{\varnothing}(\mathfrak{p}) = \mathrm{Log}_{\varnothing}\left(\mathfrak{p} \cup \mathfrak{S}(\mathfrak{p})\right).$$

□

A first application of Theorem 5.38 is that we can reduce the bound on the ambient dimension given by Corollary 2.65 by 1.

Corollary **5.39**. For $\mathfrak{p} \subseteq \mathbf{plhdr}_d$ there exists $\mathfrak{q} \subseteq \mathbf{plhdr}_d^{2d}$ such that $\mathrm{Log}(\mathfrak{p}) = \mathrm{Log}(\mathfrak{q})$.

Proof. By Corollary 5.20, we have $\mathrm{links}(\mathfrak{p}) \subseteq \mathbf{plhdr}_{d-1}$. By Corollary 2.65, there exists a set $\mathfrak{r} \subseteq \mathbf{plhdr}_{d-1}^{2d-1}$ that equals $\mathrm{links}(\mathfrak{p})$ up to PL-homeomorphisms. Theorem 5.33 yields

$$\mathrm{Log}(\mathfrak{p}) = \mathrm{Log}_{\varnothing}\big(\mathrm{links}(\mathfrak{p})\big),$$

so Lemma 2.79 implies

$$\mathrm{Log}(\mathfrak{p}) = \mathrm{Log}_{\varnothing}(\mathfrak{r}).$$

By Lemma 5.36, $\mathfrak{r}$ is closed under $\mathfrak{S}$ up to PL-homeomorphisms, so

$$\mathrm{Log}(\mathfrak{p}) = \mathrm{Log}_{\varnothing}\big(\mathfrak{r} \cup \mathfrak{S}(\mathfrak{r})\big)$$

by Lemma 2.79. Hence, by Theorem 5.38, $\mathfrak{q} = \{\blacktriangleleft\mathbf{R}\blacktriangleright : \mathbf{R} \in \mathfrak{r}\} \subseteq \mathbf{plhdr}_d^{2d}$ satisfies

$$\mathrm{Log}(\mathfrak{p}) = \mathrm{Log}(\mathfrak{q}).$$

□

We saw in section 5.1 that bipyramids often behave in a way similar to pyramids. This is also largely true for their logics:

Corollary **5.40**. For any polyhedron $\mathbf{P}$,

$$\mathrm{Log}(\mathbf{P}\blacktriangleright) = \mathrm{Log}(\blacktriangleleft\mathbf{P}\blacktriangleright) \quad \& \quad \mathrm{Log}_{\varnothing}(\mathbf{P}\blacktriangleright) \supseteq \mathrm{Log}_{\varnothing}(\blacktriangleleft\mathbf{P}\blacktriangleright).$$

Proof. The inclusion $\supseteq$ (in both cases) follows from Lemma 5.9, Corollary 4.2-1 and Lemma 2.78 (an easy direct proof without appealing to Corollary 4.2 also exists).

For the other inclusion, note that $\iota[\mathbf{P}] \in \mathrm{links}\{\mathbf{P}\blacktriangleright\}$ by Remark 5.25. By the final statement of Theorem 5.38, we have

$$\begin{aligned}\mathrm{Log}(\mathbf{P}\blacktriangleright) &= \mathrm{Log}\big\{\blacktriangleleft\mathbf{Q}\blacktriangleright : \mathbf{Q} \in \mathrm{links}\{\mathbf{P}\blacktriangleright\}\big\} \\ &\subseteq \mathrm{Log}\big(\blacktriangleleft\iota[\mathbf{P}]\blacktriangleright\big) \\ &= \mathrm{Log}(\blacktriangleleft\mathbf{P}\blacktriangleright)\end{aligned}$$

by Lemmas 5.14 and 2.79. □

We can use pyramids to make explicit that a logic $\mathrm{Log}(\mathbf{P})$ (where $\mathbf{P} \in \mathbf{plhdr}$) deals with the *local* shape of $\mathbf{P}$.

Definition **5.41**. If $X \subseteq \mathbb{R}^d$, the *diameter* of X is

$$\mathrm{diam}(X) = \sup_{\mathbf{x},\mathbf{y} \in X} d(\mathbf{x},\mathbf{y}),$$

where $d(\cdot,\cdot)$ is the Euclidean distance metric on $\mathbb{R}^d$.

Lemma **5.42**. Let $\Sigma \in \mathbf{cmplx}_d$ and $\sigma \in \Sigma^+$. Then

$$\mathrm{diam}(\sigma) \leq \frac{d}{d+1} \cdot \mathrm{diam}\big(\mathrm{sub}(\Sigma^+,\Sigma)(\sigma)\big).$$

Proof. Omitted, as this can be straightforwardly analyzed using barycentric coordinates. □

Suppose that we are interested in $\mathrm{Log}(\mathbf{P})$ for a certain $\mathbf{P} \in \mathbf{plhdr}$. We can prove that it is equivalent to look at $\mathrm{Log}(\mathbf{Q})$ for certain specific small subpolyhedra of $\mathbf{P}$; hence $\mathrm{Log}(\mathbf{P})$ is characterized locally.

Proposition **5.43**. Let $\mathbf{P} \in \mathbf{plhdr}$ and $\varepsilon > 0$. Then there exists a collection $\mathfrak{q}$ of subpolyhedra of $\mathbf{P}$ such that

$$\sup_{\mathbf{Q}\in\mathfrak{q}} \mathrm{diam}(\mathbf{Q}) \leq \varepsilon \quad \& \quad \mathrm{Log}(\mathbf{P}) = \mathrm{Log}(\mathfrak{q}).$$

Proof. Let $d = \dim \mathbf{P}$. Let $\Sigma \in \mathrm{Trian}\{\mathbf{P}\}$ and

$$n = \left\lceil \log_{d/(d+1)} \left(\frac{\varepsilon}{2 \cdot \mathrm{diam}(\mathbf{P})} \right) \right\rceil.$$

Then by Lemma 5.42, every cell of Σ^{+n} has diameter at most $\varepsilon/2$. Let $\mathscr{S}$ be the set of all subdivisions of Σ^{+n}. Let $\mathscr{D}$ be the set of all stars of complexes in $\mathscr{S}$. Let $\mathfrak{q} = \{|\Delta| : \Delta \in \mathscr{D}\}$. Then $\mathfrak{q}$ is a set of subpolyhedra of $\mathbf{P}$ of diameter at most ε. By Lemmas 2.31, 2.64-1 and 5.27, the set $\mathfrak{q}$ equals the set of all carriers of stars of triangulations of $\mathbf{P}$ up to PL-homeomorphisms. Hence by Lemma 5.26, $\mathfrak{q}$ equals the set of all standard pyramids with bases in $\mathrm{links}\{\mathbf{P}\}$ (up to PL-homeomorphisms). By Corollary 5.40, Theorem 5.38 and Lemmas 5.36 and 2.79 we have

$$\mathrm{Log}(\mathfrak{q}) = \mathrm{Log}_{\varnothing}\left(\mathrm{links}\{\mathbf{P}\}\right).$$

By Theorem 5.33, it follows that

$$\mathrm{Log}(\mathbf{P}) = \mathrm{Log}(\mathfrak{q}).$$

□

If $\mathfrak{p} \subseteq \mathbf{plhdr}^4$, then $\mathrm{links}(\mathfrak{p}) \subseteq \mathbf{plhdr}^4$. It would be even nicer if it were the case that $\mathrm{links}(\mathfrak{p}) \subseteq \mathbf{plhdr}^3$, since by Theorem 5.33 it would then be easier to study $\mathrm{Log}(\mathbf{P})$. In the remainder of this section, we proceed to present a trick with which we can slightly change $\mathrm{links}(\mathfrak{p})$ (while preserving the logic) so that it does lie in $\mathbf{plhdr}^3$. For this, we need to understand the "simplest" occurrences of quasi-polyhedrally-complete logics, namely the ones given by simplices and their boundaries.

Lemma **5.44**. Consider $d > 0$. Let σ_d be a d-simplex and σ_{d+1} a $(d+1)$-simplex. Then

$$\mathrm{Log}_{\varnothing}(\sigma_d) = \mathrm{Log}_{\varnothing}(\partial\sigma_{d+1}) = \mathbf{PL}_{d+1}.$$

Proof. Let $\mathbf{P}$ be the boundary of a $(d+2)$-simplex. Then $\mathbf{P}$ is a $(d+1)$-dimensional manifold by Proposition 3.18. It is possible to show that, up to PL-homeomorphisms, the set $\mathrm{links}\{\mathbf{P}\}$ equals $\{\partial\sigma_{d+1}\}$. So by Theorem 5.33 and Theorem 3.20,

$$\mathrm{Log}_{\varnothing}(\partial\sigma_{d+1}) = \mathrm{Log}(\mathbf{P}) = \mathbf{PL}_{d+1}.$$

Let σ_{d-1} be a $(d-1)$-simplex. By Lemma 5.16-1, $\sigma_{d-1}\blacktriangleright \cong \sigma_d$. By Lemma 5.16-2, $\blacktriangleleft\sigma_{d-1}\blacktriangleright \cong \partial\sigma_{d+1}$. By Corollary 5.40, it follows that

$$\mathrm{Log}_{\varnothing}(\sigma_d) \supseteq \mathrm{Log}_{\varnothing}(\partial\sigma_{d+1}).$$

Conversely, suppose that $\varphi \notin \mathrm{Log}_\varnothing(\partial\sigma_{d+1})$. Let Σ be a triangulation of $\partial\sigma_{d+1}$ such that φ can be falsified in Σ using some valuation $V : \mathrm{Prop} \to \mathscr{P}\Sigma$. Pick a d-dimensional cell $\sigma \in \Sigma$. Then there exists a d-dimensional cell $\delta \in \Sigma^{++}$ such that $\delta \subseteq \mathrm{relInt}\,\sigma$ (Claim 4.5). By Lemma 2.78, φ can be falsified in Σ^{++} using the valuation

$$W : \mathrm{Prop} \to \mathscr{P}\Sigma^{++} : \\ p \mapsto \mathrm{sub}(\Sigma^{++},\Sigma)^{-1}\big[V(p)\big].$$

Let $\Gamma = \Sigma^{++} \setminus \{\delta\}$, and let

$$W' : \mathrm{Prop} \to \mathscr{P}\Gamma : \\ p \mapsto W(p) \setminus \{\delta\}$$

be the induced valuation. Then the models (Σ^{++}, W) and (Γ, W') are bisimilar under the identity relation.Hence $\Gamma, W' \vDash \neg\varphi$. But by Lemma 5.22, $|\Gamma|$ is PL-homeomorphic to σ_d, so Lemma 2.79 implies $\varphi \notin \mathrm{Log}_\varnothing(\sigma_d)$. □

In view of Proposition 2.77, Lemma 5.44 says that σ_d and $\partial\sigma_d$ do not contribute to a quasi-polyhedrally-complete logic: if e.g. $\{\sigma_d\} \subsetneq \mathfrak{p} \subseteq \mathbf{plhdr}$, then $\mathrm{Log}_\varnothing(\mathfrak{p}) = \mathrm{Log}_\varnothing\big(\mathfrak{p} \setminus \{\sigma_d\}\big)$.

As promised, we can reduce the ambient dimension of links:

Proposition **5.45**. Let $\mathfrak{p} \subseteq \mathbf{plhdr}_d^{d'}$, where $d > 0$ and $d' > 1$. Then there exists $\mathfrak{q} \subseteq \mathbf{plhdr}_{d-1}^{d'-1}$ such that

$$\mathrm{Log}(\mathfrak{p}) = \mathrm{Log}_\varnothing(\mathfrak{q}).$$

Proof. Let σ be a d'-simplex. By Corollary 5.20 and Lemma 5.23-3, there exists a set $\mathfrak{r} \subseteq \mathbf{plhdr}_{d-1}^{d'-1} \sqcup \{\partial\sigma\}$ that coincides with links$(\mathfrak{p})$ up to PL-homeomorphisms. [In fact, we can put $\partial\sigma \in \mathfrak{r}$ iff $\mathfrak{p} \not\subseteq \mathbf{plhdr}_{d'-1}^{d'}$, by Lemma 2.48-1.] If $\partial\sigma \in \mathfrak{r}$, we can replace it by a $(d'-1)$-simplex lying in $\mathbb{R}^{d'-1}$, in virtue of Lemma 5.44. □

We saw how to build a $(d+1)$-dimensional polyhedron (namely a pyramid) from a d-dimensional polyhedron. Conversely, we saw how to obtain d-dimensional polyhedra (namely links) from a $(d+1)$-dimensional polyhedron. These operations allowed us to prove results about the two forms of polyhedral semantics ($\mathrm{Log}(\cdot)$ and $\mathrm{Log}_\varnothing(\cdot)$). In particular, Theorem 5.33 shows that links provide a direct translation between the two semantics. As an application, we were able to make precise what we mean by the local character of traditional polyhedral semantics (Proposition 5.43). More applications of the results in this chapter will be given in the next chapters.

Chapter 6

Cardinality of the landscape of polyhedral logics

If one wants to understand which modal logics are polyhedrally-complete or quasi-polyhedrally-complete, an ideal solution would be a classification providing axiomatizations of all such logics. In [Gab+19] this task was carried out for polyhedrally-complete logics given by polyhedra in $\mathbf{plhdr}^2$. In particular, they showed that:

Theorem **6.1**.

$$\#\{\operatorname{Log}(\mathfrak{p}) : \mathfrak{p} \subseteq \mathbf{plhdr}^2\} = \aleph_0.$$

In this chapter, we demonstrate how the picture becomes more complicated if we increase the dimension by one. We shall identify a relatively tame countable set of logics of polyhedra in $\mathbf{plhdr}^3_2$ (Corollary 6.28). We also show that there are uncountably many logics of polyhedra in $\mathbf{plhdr}^3_2$ in general (Theorem 6.32). This implies that not all polyhedrally-complete logics have a finite axiomatization.

The methodology of this chapter relies on chapter 5. In particular, if $\mathfrak{p} \subseteq \mathbf{plhdr}^3_2$, then Proposition 5.45 implies that there exists $\mathfrak{q} \subseteq \mathbf{plhdr}^2_1$ with

$$\operatorname{Log}(\mathfrak{p}) = \operatorname{Log}_{\varnothing}(\mathfrak{q}).$$

Therefore, we shall be dealing with one-dimensional complexes.

In section 6.1 we introduce some basic tools for studying the poset structure of one-dimensional complexes. In section 6.2 we explain how to view one-dimensional complexes as graphs. This makes it easier to obtain our logical results in sections 6.4 and 6.5.

6.1 One-dimensional complexes

Since one-dimensional complexes consist of vertices and line segments (and the empty simplex), it is often important to count the number of line segments that originate from a given vertex.

Lemma **6.2**. Let Λ, Π be one-dimensional complexes with the same carrier and let $\mathbf{x} \in \operatorname{vtc}(\Lambda) \cap \operatorname{vtc}(\Pi)$. Then the number of line segments in Λ originating from $\mathbf{x}$ equals the number of line segments in Π originating from $\mathbf{x}$.

Proof. This should be visually obvious. An example is provided by Figure 6.1. A formal proof can be created using Corollary 3.11. □

By lemma 6.2, one-dimensional complexes feature a notion of degree similar to the one from graph theory.

Definition **6.3** (**degree**). Let $\mathbf{L} \in \mathbf{plhdr}_1$ and let $\mathbf{x} \in \mathbf{L}$. Then the *degree* of $\mathbf{x}$ in $\mathbf{L}$ is the number of line segments originating from $\mathbf{x}$ in a triangulation of $\mathbf{L}$ that has $\mathbf{x}$ as a vertex (see Lemma 6.2). The points $\mathbf{x}, \mathbf{y} \in \mathbf{L}$ are said to be *neighbours* if there exists a path $p : [0,1] \to \mathbf{L}$ from $\mathbf{x}$ to $\mathbf{y}$ such that $p(t)$ has degree at most 2 for all $t \in (0,1)$.

Remark **6.4**. Let Λ be a one-dimensional complex and $\mathbf{x} \in |\Lambda| \setminus \mathrm{vtc}(\Lambda)$. Then $\mathbf{x}$ has degree exactly 2 in $|\Lambda|$.

Our definition of neighbour made use of the perspective on a polyhedron as a topological space. There exists an equivalent definition from the perspective of complexes as posets. This is given by the following lemma.

Lemma **6.5**. Let Λ be a one-dimensional complex and $\mathbf{x}, \mathbf{y} \in \mathrm{vtc}(\Lambda)$. Then $\mathbf{x}$ and $\mathbf{y}$ are neighbours in $|\Lambda|$ iff there is a path

$$(\mathbf{x} = \mathbf{x}^0, \lambda^0, \mathbf{x}^1, \lambda^1, \ldots, \lambda^{n-1}, \mathbf{x}^n = \mathbf{y})$$

through Λ such that λ^i is a line segment for each $i < n$ and $\mathbf{x}^i$ is a vertex of degree at most 2 in $|\Lambda|$ for each $0 < i < n$.

Proof. For the direction $\Rightarrow$, suppose that $p : [0,1] \to |\Lambda|$ is a path from $\mathbf{x}$ to $\mathbf{y}$ such that $p(t)$ has degree at most 2 for all $t \in (0,1)$. Let $L \subseteq \Lambda$ be the subset containing all line segments of Λ and all vertices of Λ that have degree at most 2. Then by Lemma 2.32, for every $t \in (0,1)$ there exists $\sigma(t) \in L$ such that $p(t) \in \mathrm{relInt}\,\sigma(t)$. Hence $\mathbf{x}$ and $\mathbf{y}$ lie in the same connected component of the set

$$\{\mathbf{x}, \mathbf{y}\} \cup \bigsqcup_{\sigma \in L} \mathrm{relInt}\,\sigma.$$

Therefore, by Lemma 3.13, $\mathbf{x}$ and $\mathbf{y}$ lie in the same connected component of the poset $\{\mathbf{x}, \mathbf{y}\} \cup L$. The claim follows.

For the direction $\Leftarrow$, use Remark 6.4. □

The notions of degree and neighbour will be important for modal logic because they are "preserved" by p-morphisms, in the sense of the following lemma.

Lemma **6.6**. Let Λ, Π be one-dimensional complexes and let $f : \Lambda \to \Pi$ be a p-morphism.

1. For $\mathbf{x} \in \mathrm{vtc}(\Lambda)$ of degree at least 1 in $|\Lambda|$, the degree of $f(\mathbf{x})$ in $|\Pi|$ is at most the degree of $\mathbf{x}$ in $|\Lambda|$.

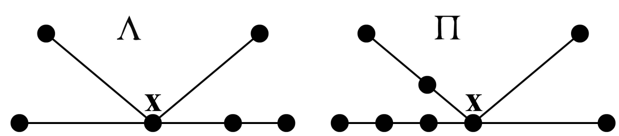


Figure 6.1: example to Lemma 6.2: $\mathbf{x}$ has four line segments in both complexes

2. If $\mathbf{x},\mathbf{y} \in \mathrm{vtc}(\Lambda)$ are neighbours in $|\Lambda|$, then $f(\mathbf{x}), f(\mathbf{y})$ are neighbours in $|\Pi|$.

Proof. 1 follows from the definitions (note that $f(\mathbf{x}) \in \mathrm{vtc}(\Pi)$ since $\mathbf{x}$ has a proper successor in Λ).

2 follows from Lemma 6.5. □

6.2 Drawings

A one-dimensional complex can be defined as a finite set of vertices in a Euclidean space, together with line segments connecting some of the points. The same thing can be said about a finite graph, except that the vertices are abstract and need not lie in a Euclidean space. We introduce graph-geometric realizations as a translation between graphs and one-dimensional complexes. From there, we can relate graphs to one-dimensional polyhedra.

Definition **6.7** (**drawing of a graph**). Let $G = (V, E)$ be a finite graph and $\Lambda \in \mathbf{cmplx}_1$. Then we say that Λ is a *graph-geometric realization* of G provided there exists a bijection $f : V \to \mathrm{vtc}(\Lambda)$ such that for all distinct $v, w \in V$ we have

$$f(v) \underline{\vee} f(w) \in \Lambda \quad \Leftrightarrow \quad \{v, w\} \in E.$$

If $\mathbf{L}$ is a polyhedron, we say that $\mathbf{L}$ is a *drawing* of G provided there exists a triangulation of $\mathbf{L}$ which is a drawing of G.

Remark **6.8**. Every graph has graph-geometric realizations and drawings. Every one-or-lower-dimensional complex is a graph-geometric realization of some graph. Hence every one- or lower-dimensional polyhedron is a drawing of some graph. In this way, graphs are a sensible way of thinking about one-dimensional polyhedra.

We consider two simple lemmas investigating how graph drawings behave with respect to notions of equivalence. This will be important as we shall look at the logics of drawings of certain graphs.

Lemma **6.9**. Two graphs are isomorphic iff their graph-geometric realizations are (simplicially) isomorphic.

Proof. This can easily be checked from the definitions. □

Lemma **6.10**. Let $\mathbf{L}_0, \mathbf{L}_1 \in \mathbf{plhdr}_1$. Then $\mathbf{L}_0 \cong \mathbf{L}_1$ iff there exists a graph G such that both $\mathbf{L}_0$ and $\mathbf{L}_1$ are a drawing of G.

Proof. Lemmas 6.9 and 2.64-1. □

As explained in the introduction of this chapter, we mostly deal with $\mathbf{plhdr}^2_1$. The graphs that have a drawing in $\mathbf{plhdr}^2_1$ are precisely the planar graphs.[1] Most readers will be familiar with planar graphs, and this can make $\mathbf{plhdr}^2_1$ easier to think about.

[1] Planar graphs are commonly defined as graphs that can be drawn in the plane using arcs, so edges need not be drawn in a straight fashion. However, Fáry's theorem ([Wag36], [Fár48]) states that planar graphs can also be drawn in the plane using straight lines.

6.3 Tree drawings

A particularly well-behaved set of one-dimensional polyhedra are the ones without "cycles", i.e. one-dimensional polyhedra that do not have a subspace that is homeomorphic to a circle. In section 6.4 shall show that such polyhedra provide only countably many logics (using both variants of polyhedral semantics).

It is easy to see that a one-dimensional polyhedron has no cycles iff it is a drawing of a graph without cycles. Such graphs can be obtained from finite trees. Let us have a look at this process. This will be important to our logical proof below, which will rely on a classical combinatorial result for graphs (Kruskal's theorem).

Definition **6.11** (**tree**). A finite *tree* is a finite rooted poset T in which every point has at most one immediate predecessor.

Notation **6.12**. If P is a finite poset, associate a graph by

$$\mathrm{graph}(P) = \Big(P, \big\{\{p,q\} \subseteq P : q \text{ is an immediate successor of } p\big\}\Big).$$

As promised, the resulting graphs have no cycles. By a *cycle* in a graph we mean a sequence $(v_0, \dots, v_k = v_0)$ without repetitions such that $k \geq 3$ and $\{v_i, v_{i+1}\}$ is an edge for each $i < k$.

Proposition **6.13**. If T is a finite tree, then $\mathrm{graph}(T)$ has no cycles.

Proof. This is well known. If $(t_0, \dots, t_k = t_0)$ is a cycle in $\mathrm{graph}(T)$, we cannot have $t_i < t_{i+1} > t_{i+2}$ for any i. Hence the cycle $(t_0, \dots, t_k)$ can be divided into a descending sequence followed by an ascending sequence. But then the sequence $(t_0, \dots, t_k)$ is either monotone or $t_1 = t_{k-1}$. □

The next step is to build the bridge towards one-dimensional polyhedra.

Definition **6.14** (**drawing of a poset**). Let P be a finite poset. A *graph-geometric realization* of P is a graph-geometric realization of $\mathrm{graph}(P)$. A *drawing* of P is a drawing of $\mathrm{graph}(P)$.

Remark **6.15**. We are now dealing with posets on two levels: a graph-geometric realization Λ of any finite poset P is again a poset. Note that $\mathrm{hgt}\,\Lambda \leq 3$, regardless of $\mathrm{hgt}\,P$.

Corollary **6.16**. The one-or-lower-dimensional complex Σ is a graph-geometric realization of a tree iff $|\Sigma| \setminus \{\mathbf{x}\}$ is disconnected for every $\mathbf{x} \in |\Sigma|$.

Proof. Lemma 3.13 and Proposition 6.13. □

Corollary **6.17**. If $\mathbf{T}$ is a drawing of a tree, then every triangulation of $\mathbf{T}$ is a graph-geometric realization of a tree.

The following proposition is well known and visually obvious.

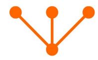
Proposition **6.18**. If T is a finite tree, then $\mathrm{graph}(T)$ is planar.

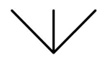
Hence every finite tree has a drawing in $\mathbf{plhdr}_1^2$.

6.4 Logics of tree drawings

We proceed to introduce some (quasi-)polyhedrally-complete logics of height 3. In this section, we do so using finite trees. The main result is that we only get countably many logics from finite trees (Theorem 6.27 and Corollary 6.28).

Definition **6.19** (**tree-(pyramid-)complete**). If $\mathfrak{t} \subseteq \mathbf{plhdr}_1$ is a collection of drawings of trees, the quasi-polyhedrally-complete logic $\mathrm{Log}_\varnothing(\mathfrak{t})$ is called *tree-complete* and the polyhedrally-complete logic $\mathrm{Log}\{\mathbf{T}\blacktriangleright : \mathbf{T} \in \mathfrak{t}\}$ is called *tree-pyramid-complete*.

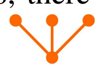
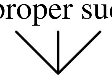
Example **6.20**. Let  be a drawing of a 3-fork  . Then

$\mathrm{Log}_\varnothing\Big($  $\Big)$ (6.1)

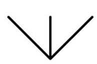
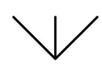
is tree-complete but not polyhedrally-complete. In contrast,

$\mathrm{Log}\Big($  $\blacktriangleright\Big)$ (6.2)

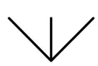
is tree-pyramid-complete but not tree-complete.

Proof. First, let $\mathscr{L}$ be the logic in (6.1). By definition, $\mathscr{L}$ is tree-complete. If it were polyhedrally-complete, then by Theorem 5.33 and Lemma 5.36, there would exist a set $\mathfrak{l} \subseteq \mathbf{plhdr}_1$ such that $\mathscr{L} = \mathrm{Log}_\varnothing(\mathfrak{l})$ and $\mathfrak{l}$ is closed under $\mathfrak{S}$. The root of  has degree 3 in the drawing  , so a triangulation Π of the drawing has a vertex with three proper successors (line segments). Trivially, we have $\chi_\Pi \notin \mathscr{L} = \mathrm{Log}_\varnothing(\mathfrak{l})$. Hence, there is a $\Pi_0 \in \mathrm{Trian}(\mathfrak{l})$ such that there exists an up-reduction $\Pi_0 \rightharpoonup \Pi$. It follows that Π_0 has a vertex $\mathbf{x}$ with at least three proper successors. Hence $\Lambda = \Pi_0 \circledx$ has *two* vertices that each have at least three proper successors. We have $|\Lambda| \in \mathfrak{l}$, so $\chi_\Lambda \notin \mathrm{Log}_\varnothing(\mathfrak{l}) = \mathscr{L}$. Hence, there is a triangulation Π_1 of  such that there exists an up-reduction $\Pi_1 \rightharpoonup \Lambda$. This implies that Π_1 has two vertices that each have at least three proper successors. But this is absurd.

Next, let $\mathscr{L}$ be the logic in (6.2). By definition, $\mathscr{L}$ is tree-pyramid-complete. If it were tree-complete, then there would be a set $\mathfrak{t}$ of drawings of trees such that $\mathscr{L} = \mathrm{Log}_\varnothing(\mathfrak{t})$. By Theorem 5.38, we have $\mathscr{L} = \mathrm{Log}_\varnothing(\mathfrak{l})$ where

$\mathfrak{l} = \Big\{$  $\Big\} \cup \mathfrak{S}\Big\{$  $\Big\}$.

As in the previous paragraph, there exists a $\Lambda \in \mathrm{Trian}(\mathfrak{l})$ that has *two* vertices that each have at least three proper successors. We have $\chi_\Lambda \notin \mathscr{L} = \mathrm{Log}_\varnothing(\mathfrak{t})$, so there are a member $\mathbf{T}$ of $\mathfrak{t}$ and a triangulation Π of $\mathbf{T}$ such that there is an up-reduction $\Pi \rightharpoonup \Lambda$. Therefore Π has two vertices $\mathbf{t}, \mathbf{s}$ that each have at least three proper successors. Then $\mathbf{T}$ is a drawing of a tree T with at least two points t, s that each have at least two proper successors. At least one of them is not the root of T. Hence Proposition 6.13 implies that the subgraph of $\mathrm{graph}(T)$ with universe $T \setminus \{t, s\}$ has at least four components. Hence (by Lemma 3.13) the set $\mathbf{T} \setminus \{\mathbf{t}, \mathbf{s}\}$ has at least four components, and therefore the poset $\Pi \setminus \{\varnothing, \mathbf{t}, \mathbf{s}\}$ has at least four components. Again, we have $\chi_\Pi \notin \mathrm{Log}_\varnothing(\mathfrak{t}) = \mathscr{L} = \mathrm{Log}_\varnothing(\mathfrak{l})$, so there exists $\Pi_1 \in \mathrm{Trian}(\mathfrak{l})$ such that there is an up-reduction $f : \Pi_1 \rightharpoonup \Pi$. Then f is a full map and

$\Pi_1 \in \mathrm{Trian}\Big(\mathfrak{S}\Big\{$  $\Big\}\Big)$.

But

links $\Big\{$  $\Big\}$

only contains (zero-dimensional) polyhedra of cardinality at most 3, so Π_1 has exactly two vertices $\mathbf{x}, \mathbf{y}$ that have three proper successors. Then $\mathbf{x}$ and $\mathbf{y}$ are mapped to $\mathbf{t}$ and $\mathbf{s}$ by f, and $f(\varnothing) = \varnothing$. Since $\Pi_1 \setminus \{\varnothing, \mathbf{x}, \mathbf{y}\}$ is a poset with only three components, we reach a contradiction. $\square$

In the proof of Example 6.20, we made heavy use of p-morphisms. In the more general proofs below, this will also be the case. In the case of finite trees, it turns out that p-morphisms are "preserved" by graph-geometric realization, in the following sense:

Lemma **6.21**. Let T_0, T_1 be finite trees such that there is a surjective p-morphism $T_0 \to T_1$. Let Π_0 be a graph-geometric realization of T_0 and let Π_1 be a graph-geometric realization of T_1. Then there is a surjective p-morphism $\Pi_0 \to \Pi_1$.

Proof. W.l.o.g. we may assume that $T_i = \mathrm{vtc}(\Pi_i)$ and for all distinct $\mathbf{x}, \mathbf{y} \in T_i$ we have $\mathbf{x} \underline{\vee} \mathbf{y} \in \Pi_i$ iff $\mathbf{y}$ is an immediate successor of $\mathbf{x}$ or $\mathbf{x}$ is an immediate successor of $\mathbf{y}$. Let $\mathbf{r}^i$ be the root of T_i. Let $f : T_0 \to T_1$ be a surjective p-morphism. Then the roots satisfy $f(\mathbf{r}^0) = \mathbf{r}^1$. For $\mathbf{x} \in T_0$, let $\mathbf{p}(\mathbf{x})$ be the smallest predecessor of $\mathbf{x}$ which has the same value under f as does $\mathbf{x}$. For each $\sigma \in \Pi_1$, fix a maximal successor $\lambda(\sigma)$ of σ in Π_1.

Let S_0 be the set of all $\mathbf{x} \in T_0$ such that $f(\mathbf{x}) > f(\mathbf{y})$ for all $\mathbf{y} > \mathbf{x}$. Let Q_0 be the set of all $\mathbf{x} \in S_0$ such that $\mathbf{p}(\mathbf{x}) = \mathbf{r}^0$ or the immediate predecessor $\mathbf{y}$ of $\mathbf{p}(\mathbf{x})$ satisfies $\mathbf{y} \in Q_0$ and $f(\mathbf{y})$ is the immediate predecessor of $f(\mathbf{x})$ (this definition makes sense by induction on the height of $\mathbf{x}$ in T_0). Observe that $\downarrow(Q_0)$ is the set of all $\mathbf{x} \in T_0$ such that $\mathbf{p}(\mathbf{x}) = \mathbf{r}^0$ or the immediate predecessor $\mathbf{y}$ of $\mathbf{p}(\mathbf{x})$ satisfies $\mathbf{y} \in Q_0$ and $f(\mathbf{y})$ is the immediate predecessor of $f(\mathbf{x})$. In particular, $\mathbf{r}^0 \in \downarrow(Q_0)$.

Define $g : \Pi_0 \to \Pi_1$ as follows. Let $g(\varnothing) = \varnothing$, and let $g(\mathbf{x}) = f(\mathbf{x})$ for $\mathbf{x} \in Q_0$. Suppose that $\mathbf{x} \in \downarrow(Q_0) \setminus Q_0$. If $\mathbf{p}(\mathbf{x}) = \mathbf{r}^0$, let $g(\mathbf{x}) = \lambda(\mathbf{r}^1)$. Otherwise let $\mathbf{y}$ be the immediate predecessor of $\mathbf{p}(\mathbf{x})$ and set $g(\mathbf{x}) = f(\mathbf{y}) \underline{\vee} f(\mathbf{x})$. This is in Π_1 since $f(\mathbf{y})$ is an immediate successor of $f(\mathbf{x})$. Notice that $g(\mathbf{x})$ is always a line segment containing $f(\mathbf{x})$, for $\mathbf{x} \in \downarrow(Q_0) \setminus Q_0$. Next suppose that $\mathbf{x} \in T_0 \setminus \downarrow(Q_0)$. Let $\mathbf{y}$ be the largest predecessor of $\mathbf{x}$ that lies in $\downarrow(Q_0)$ and set $g(\mathbf{x}) = \lambda\big(g(\mathbf{y})\big)$. For a line segment $\lambda \in \Pi_0 \setminus T_0$, we let $g(\lambda) = \mathrm{Conv}\Big(\bigcup g\big[\mathrm{vtc}(\lambda)\big]\Big)$ and claim that this is a line segment in Π_1. To check this, find $\mathbf{x}, \mathbf{u} \in T_0$ such that $\lambda = \mathbf{x} \underline{\vee} \mathbf{u}$ and $\mathbf{u}$ is an immediate successor of $\mathbf{x}$. Distinguish cases:

- $\mathbf{x} \in Q_0$. Then $g(\mathbf{x}) = f(\mathbf{x})$ and $\mathbf{p}(\mathbf{u}) = \mathbf{u}$. Distinguish subcases:
  - $\mathbf{u} \in Q_0$. Then $f(\mathbf{u})$ is an immediate successor of $f(\mathbf{x})$, and $g(\mathbf{u}) = f(\mathbf{u})$, so $g(\lambda) = f(\mathbf{x}) \underline{\vee} f(\mathbf{u}) \in \Pi_1$.
  - $\mathbf{u} \in \downarrow(Q_0) \setminus Q_0$. Then $f(\mathbf{u})$ is an immediate successor of $f(\mathbf{x})$, and $g(\mathbf{u}) = f(\mathbf{x}) \underline{\vee} f(\mathbf{u})$, so $g(\lambda) = g(\mathbf{u}) \in \Pi_1$.
  - $\mathbf{u} \in T_0 \setminus \downarrow(Q_0)$. Then $g(\mathbf{u}) = \lambda\big(g(\mathbf{x})\big)$, so $g(\lambda) = g(\mathbf{u}) \in \Pi_1$.
- $\mathbf{x} \in \downarrow(Q_0) \setminus Q_0$. Distinguish subcases:
  - $\mathbf{u} \in Q_0$. Then $\mathbf{p}(\mathbf{u}) \neq \mathbf{u}$, so $f(\mathbf{x}) = f(\mathbf{u}) = g(\mathbf{u})$. Since $f(\mathbf{x})$ is an endpoint of $g(\mathbf{x})$, it follows that $g(\lambda)$ is a line segment in Π_1.

  - $\mathbf{u} \in {\downarrow}(Q_0) \setminus Q_0$. Then $g(\mathbf{x}) = g(\mathbf{u})$ is the same line segment in Π_1.
  - $\mathbf{u} \in T_0 \setminus {\downarrow}(Q_0)$. Then $g(\mathbf{u}) = \lambda\big(g(\mathbf{x})\big)$ so $g(\lambda) = g(\mathbf{u}) \in \Pi_1$.

- $\mathbf{x} \in T_0 \setminus {\downarrow}(Q_0)$. Then $\mathbf{u} \in T_0 \setminus {\downarrow}(Q_0)$. So $g(\mathbf{x}) = g(\mathbf{u})$ is the same line segment in Π_1.

Hence $g : \Pi_0 \to \Pi_1$ is well-defined. It is immediate that g is monotone.

Since $\varnothing \in \operatorname{Im} g$, it remains to show that g has the back-property.

We claim that $f|Q_0 : Q_0 \to T_1$ is surjective. Let $\mathbf{z} \in T_1$ and proceed by induction on $\operatorname{hgt}(\mathbf{z})$. If $\mathbf{z} = \mathbf{r}^1$, find an $\mathbf{x} \in S_0$ such that $f(\mathbf{x}) = \mathbf{r}^1$ and note that $\mathbf{x} \in Q_0$. Otherwise, let $\mathbf{w}$ be the immediate predecessor of $\mathbf{z}$ and, by inductive hypothesis, find $\mathbf{y} \in Q_0$ such that $f(\mathbf{y}) = \mathbf{w}$. Since f has the back-property, there exists $\mathbf{x} > \mathbf{y}$ such that $f(\mathbf{x}) = \mathbf{z}$. We can arrange that $\mathbf{x} \in S_0$. Let $\mathbf{v}$ be the immediate successor of $\mathbf{y}$ that is a predecessor of $\mathbf{x}$. Then $\mathbf{y} \in S_0$ implies $\mathbf{w} < f(\mathbf{v}) \leq \mathbf{z}$, so $f(\mathbf{v}) = \mathbf{z}$. Thus $\mathbf{v} = \mathbf{p}(\mathbf{x})$, whence $\mathbf{x} \in Q_0$.

Thus $f|Q_0$ is surjective. It follows that g satisfies the back-property at $\varnothing$. Next let $\mathbf{x} \in T_0$; we show that g satisfies the back-property at $\mathbf{x}$. Assume that $g(\mathbf{x}) \in T_1$ and $\lambda_1 \in \Pi_1$ is a line segment containing $g(\mathbf{x})$. Then $\mathbf{x} \in Q_0$. Find $\mathbf{z} \in T_1$ such that $\lambda_1 = g(\mathbf{x}) \veebar \mathbf{z}$. Distinguish cases:

- $\mathbf{z}$ is an immediate successor of $g(\mathbf{x})$. Find a successor $\mathbf{v}$ of $\mathbf{x}$ such that $f(\mathbf{v}) = \mathbf{z}$. We can arrange that $\mathbf{v} \in S_0$. Let $\mathbf{u}$ be the immediate successor of $\mathbf{x}$ that is a predecessor of $\mathbf{v}$. Then $\mathbf{x} \in S_0$ implies $\mathbf{y} < f(\mathbf{u}) \leq \mathbf{z}$, so $f(\mathbf{u}) = \mathbf{z}$. Thus $\mathbf{u} = \mathbf{p}(\mathbf{v})$, whence $\mathbf{v} \in Q_0$. Note that $\lambda_0 = \mathbf{x} \veebar \mathbf{u}$ is a successor of $\mathbf{x}$ in Π_0. Distinguish subcases:
  - $\mathbf{u} = \mathbf{v}$. Then $\mathbf{u} \in Q_0$, so
  $$g(\lambda_0) = \operatorname{Conv}\big\{f(\mathbf{x}), f(\mathbf{u})\big\} = \lambda_1.$$
  - $\mathbf{u} \neq \mathbf{v}$. Then $\mathbf{u} \in {\downarrow}(Q_0) \setminus Q_0$, so $g(\mathbf{u}) = f(\mathbf{x}) \veebar f(\mathbf{u})$ because $\mathbf{p}(\mathbf{u}) = \mathbf{u}$. It follows that
  $$g(\lambda_0) = g(\mathbf{u}) = \lambda_1.$$

- $\mathbf{z}$ is the immediate predecessor of $g(\mathbf{x})$. Then $g(\mathbf{x}) \neq \mathbf{r}^1$, so $\mathbf{p}(\mathbf{x}) \neq \mathbf{r}^0$. Let $\mathbf{y}$ be the immediate predecessor of $\mathbf{p}(\mathbf{x})$. Then $\mathbf{y} \in Q_0$ because $\mathbf{x} \in Q_0$. Also $f(\mathbf{y})$ is the immediate predecessor of $f(\mathbf{x})$, so $f(\mathbf{y}) = \mathbf{z}$. Let $\mathbf{w}$ be the immediate predecessor of $\mathbf{x}$ and $\lambda_0 = \mathbf{x} \veebar \mathbf{w} \in \Pi_0$. Distinguish subcases:
  - $\mathbf{p}(\mathbf{x}) = \mathbf{x}$. Then $\mathbf{y} = \mathbf{w}$, so
  $$g(\lambda_0) = \operatorname{Conv}\big\{f(\mathbf{x}), f(\mathbf{w})\big\} = \operatorname{Conv}\big\{f(\mathbf{x}), f(\mathbf{y})\big\} = \lambda_1.$$
  - $\mathbf{p}(\mathbf{x}) \neq \mathbf{x}$. Then $\mathbf{w} \notin S_0$, so $\mathbf{w} \in {\downarrow}(Q_0) \setminus Q_0$, so $g(\mathbf{w}) = f(\mathbf{w}) \veebar f(\mathbf{y}) = f(\mathbf{x}) \veebar \mathbf{z} = \lambda_1$ and so
  $$g(\lambda_0) = g(\mathbf{w}) = \lambda_1. \qquad \square$$

Lemma 6.21 will be useful because it allows us to translate a combinatorial result about trees to a result about one-dimensional polyhedra. To introduce the combinatorial result about trees, we need a bit more notation.

Notation **6.22**. If T is a finite tree and $t, s \in T$, let $t \wedge s$ be the largest common predecessor of t and s in T.

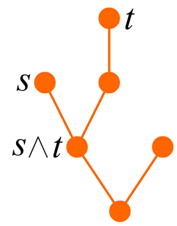


Figure 6.2: example of the operation $\wedge$

For an illustrative example, see Figure 6.2.

Definition **6.23** (**decent embedding**). Let T_0, T_1 be finite trees. A *decent embedding* from T_0 to T_1 is a monotone injection $f : T_0 \to T_1$ such that

$$f(t_0 \wedge s_0) = f(t_0) \wedge f(s_0)$$

for all $t_0, s_0 \in T_0$.

Now we can state a combinatorial result about trees.

Lemma **6.24**. Let $\mathscr{T}$ be an infinite collection of finite trees. Then there exist distinct $T_0, T_1 \in \mathscr{T}$ such that there is a decent embedding $T_0 \to T_1$.

Proof. This statement is weaker than Kruskal's theorem [Kru60]. □

Before we can translate this result to one-dimensional polyhedra, we reformulate it in terms of p-morphisms:

Lemma **6.25**. Let T_0, T_1 be finite trees such that there is a decent embedding $T_0 \to T_1$. Then there is a surjective p-morphism $T_1 \to T_0$.

Proof. Let $f : T_0 \to T_1$ be a decent embedding and let r_i be the root of T_i. Since $\uparrow(t_1)$ is a p-morphic image of T_1 for every $t_1 \in T_1$, we may assume w.l.o.g. that $f(r_0) = r_1$. Since every nonempty downset of T_1 is a p-morphic image of T_1, we may assume w.l.o.g. that $T_1 = \downarrow(\operatorname{Im} f)$. Moreover, $\operatorname{Im} f$ is closed under $\wedge$. Hence for every $t_1 \in T_1$, the point

$$s(t_1) = \min\left(\uparrow(t_1) \cap \operatorname{Im} f\right)$$

exists. Define $g : T_1 \to T_0$ by $g(t_1) = f^{-1}\big(s(t_1)\big)$. Then it is easy to see that g is monotone. We claim that g is a surjective p-morphism. We have $f(r_0) = r_1$, so $g(r_1) = r_0$. Hence it suffices to show that g has the back-property. To this end, suppose that $g(t_1) = t_0$ and $t_0 < s_0$. Then $t_1 \leq s(t_1) = f(t_0) < f(s_0)$, and $g\big(f(s_0)\big) = s_0$. □

Combining these lemmas, we obtain a combinatorial result about p-morphisms between trees:

Proposition **6.26**. Let $\mathscr{T}$ be an infinite collection of finite trees. Then there exist distinct $T_0, T_1 \in \mathscr{T}$ such that there is a surjective p-morphism $T_0 \to T_1$.

Proof. This follows from Lemmas 6.24 and 6.25. We would like to thank Ian Hodkinson for pointing out this proof idea to us. □

This allows us to calculate the cardinality of the set of tree-complete logics.

Theorem **6.27**. There exist only countably many tree-complete logics.

Proof. Fix a set $\mathscr{T}$ of finite trees that contains exactly one isomorphic copy of each finite tree. For each $T \in \mathscr{T}$, let $\Sigma(T)$ be a simplicial complex that is a graph-geometric realization of T, and $\mathbf{Q}(T)$ the carrier of $\Sigma(T)$, so that $\mathbf{Q}(T)$ is a drawing of T. If $\mathscr{L}$ is a tree-complete logic, let $\mathscr{T}(\mathscr{L})$ be the set of all $T \in \mathscr{T}$ such that

$$\mathscr{L} \nsubseteq \mathrm{Log}_{\varnothing}\big(\mathbf{Q}(T)\big).$$

Further, let $\mathscr{T}'(\mathscr{L})$ be the set of minimal trees in $\mathscr{T}(\mathscr{L})$ w.r.t. p-morphisms: $T \in \mathscr{T}'(\mathscr{L})$ iff $T \in \mathscr{T}(\mathscr{L})$ and T is the only p-morphic image of T in $\mathscr{T}(\mathscr{L})$.

For any tree-complete logic $\mathscr{L}$, the set $\mathscr{T}'(\mathscr{L})$ is finite by Proposition 6.26.

We claim that $\mathscr{T}'$ is an injection. Suppose that $\mathscr{L}_0$ and $\mathscr{L}_1$ are two tree-complete logics such that $\mathscr{T}'(\mathscr{L}_0) = \mathscr{T}'(\mathscr{L}_1)$. We show that $\mathscr{L}_0 \subseteq \mathscr{L}_1$. Find $\mathfrak{t} \subseteq \mathbf{plhdr}$ such that each polyhedron in $\mathfrak{t}$ is a drawing of a tree and $\mathscr{L}_1 = \mathrm{Log}_{\varnothing}(\mathfrak{t})$. Suppose that $\varphi \in \mathscr{L}_0 \setminus \mathscr{L}_1$. Find $\mathbf{T} \in \mathfrak{t}$ such that $\varphi \notin \mathrm{Log}_{\varnothing}(\mathbf{T})$. Find a tree T such that $\mathbf{T}$ is a drawing of T. We can pick $T \in \mathscr{T}$. We have $\mathbf{T} \cong \mathbf{Q}(T)$ by Lemma 6.10, so

$$\mathscr{L}_0 \nsubseteq \mathrm{Log}_{\varnothing}(\mathbf{P}) = \mathrm{Log}_{\varnothing}\big(\mathbf{Q}(T)\big)$$

by Lemma 2.79. It follows that $T \in \mathscr{T}(\mathscr{L}_0)$. Then there exists a p-morphic image T' of T such that $T' \in \mathscr{T}'(\mathscr{L}_0)$ (since a bijective p-morphism is an isomorphism). It follows that $T' \in \mathscr{T}'(\mathscr{L}_1) \subseteq \mathscr{T}(\mathscr{L}_1)$, so

$$\mathscr{L}_1 \nsubseteq \mathrm{Log}_{\varnothing}\big(\mathbf{Q}(T')\big).$$

By Lemma 6.21, $\Sigma(T')$ is a p-morphic image of $\Sigma(T)$. By Corollary 4.2-3 it follows that $\mathrm{Log}_{\varnothing}\big(\mathbf{Q}(T)\big) \subseteq \mathrm{Log}_{\varnothing}\big(\mathbf{Q}(T')\big)$, so

$$\mathscr{L}_1 \nsubseteq \mathrm{Log}_{\varnothing}\big(\mathbf{Q}(T)\big) = \mathrm{Log}_{\varnothing}(\mathbf{T})$$

contradicting $\mathbf{T} \in \mathfrak{t}$. Thus $\mathscr{T}'$ is injective.

The image of $\mathscr{T}'$ is a subset of $[\mathscr{T}]^{<\aleph_0}$ and $\mathscr{T}$ is countable, so the image of $\mathscr{T}'$ is countable and the claim follows. ☐

Corollary **6.28**. There exist only countably many tree-pyramid-complete logics.

Proof. Suppose that $\mathscr{L}$ is tree-pyramid-complete. Find a set $\mathfrak{t}(\mathscr{L})$ of drawings of trees such that $\mathscr{L} = \mathrm{Log}\{\mathbf{T}\blacktriangleright : \mathbf{T} \in \mathfrak{t}(\mathscr{L})\}$. By Theorem 5.38, we have $\mathscr{L} = \mathrm{Log}_{\varnothing}\Big(\mathfrak{t}(\mathscr{L}) \cup \mathfrak{S}\big(\mathfrak{t}(\mathscr{L})\big)\Big)$. Define $\mathscr{L}_* = \mathrm{Log}_{\varnothing}\big(\mathfrak{t}(\mathscr{L})\big)$ (this can depend on the choice of $\mathfrak{t}(\mathscr{L})$). Furthermore, note that $\mathfrak{S}\big(\mathfrak{t}(\mathscr{L})\big)$ is a set of standard bipyramids with zero-dimensional bases. Let $B(\mathscr{L}) \subseteq \omega$ be the set of numbers b such that $\mathfrak{S}\big(\mathfrak{t}(\mathscr{L})\big)$ contains a standard bipyramid with a collection of b points as a base. Let $k(\mathscr{L}) \leq \omega$ be the smallest strict upper bound on $B(\mathscr{L})$. We claim that

$$\big((\cdot)_*, k(\cdot)\big)$$

is an injection. By Theorem 6.27, this implies the result.

Suppose that $\mathscr{L}^0, \mathscr{L}^1$ are tree-pyramid-complete and

$$\big(\mathscr{L}^0_*, k(\mathscr{L}^0)\big) = \big(\mathscr{L}^1_*, k(\mathscr{L}^1)\big).$$

Since $\mathscr{L}_*^0 = \mathscr{L}_*^1$, it suffices to show that

$$\mathrm{Log}_{\varnothing}\Big(\mathfrak{S}\big(\mathfrak{t}(\mathscr{L}^0)\big)\Big) = \mathrm{Log}_{\varnothing}\Big(\mathfrak{S}\big(\mathfrak{t}(\mathscr{L}^1)\big)\Big).$$

We show the inclusion $\supseteq$. Suppose that $\varphi \notin \mathrm{Log}_{\varnothing}\Big(\mathfrak{S}\big(\mathfrak{t}(\mathscr{L}^0)\big)\Big)$. Then there exists $\mathbf{L}_0 \in \mathfrak{S}\big(\mathfrak{t}(\mathscr{L}^0)\big)$ and a triangulation Λ of $\mathbf{L}_0$ such that $\Lambda \not\models \varphi$. Then $\mathbf{L}_0$ is a standard bipyramid with a collection of some number b_0 of points as a base, where $b_0 < k(\mathscr{L}^0)$. Hence there exists a zero-dimensional complex Ξ_0 with b_0 vertices such that $\mathbf{L}_0 = \big|\blacktriangleleft\Xi_0\blacktriangleright\big|$. Since $b_0 < k(\mathscr{L}^1)$, there exists $\mathbf{L}_1 \in \mathfrak{S}\big(\mathfrak{t}(\mathscr{L}^0)\big)$ which is a standard bipyramid with a collection of b_1 points as a base, for some $b_1 \geq b$. Hence there exists a zero-dimensional complex Ξ_1 with b_1 vertices such that $\mathbf{L}_1 = \big|\blacktriangleleft\Xi_1\blacktriangleright\big|$. Then Ξ_0 is a p-morphic image of Ξ_1, so by Lemma 5.10, $\blacktriangleleft\Xi_0\blacktriangleright$ is a p-morphic image of $\blacktriangleleft\Xi_1\blacktriangleright$. From Corollary 4.2-2 it follows that $\varphi \notin \mathrm{Log}(\mathbf{L}_1)$ and hence $\varphi \notin \mathrm{Log}_{\varnothing}\Big(\mathfrak{S}\big(\mathfrak{t}(\mathscr{L}^1)\big)\Big)$. □

6.5 Logics of planar graphs

It turns out that many polyhedrally-complete logics of height 3 are neither tree-complete nor tree-pyramid-complete. Indeed, the goal of this section is to show that there exist continuum many such logics.

We shall rely on the following technical lemma, which is essentially a more specific version of Lemma 6.6.

Lemma **6.29**. Let Λ, Π be one-dimensional complexes, let Λ' be a subdivision of Λ and let $f : \Lambda' \to \Pi$ be a p-morphism.

1. If $\mathbf{x} \in \mathrm{vtc}(\Lambda)$ and $\mathbf{z} \in \mathrm{vtc}(\Pi)$, $f(\mathbf{x}) = \mathbf{z}$ and $\mathbf{x}$ and $\mathbf{z}$ have the same degree and all vertices in $\mathrm{star}(\Pi, \mathbf{z})$ have degree at least 3, then there exists a bijection $g : \mathrm{link}(\Lambda, \mathbf{x}) \to \mathrm{link}(\Pi, \mathbf{z})$ such that
$$f\Big[\mathrm{sub}(\Lambda', \Lambda)^{-1}\big[\{\mathbf{y} \veebar \mathbf{x}\}\big]\Big] = \big\{g(\mathbf{y}) \veebar \mathbf{z}\big\} \tag{6.3}$$
for all $\mathbf{y} \in \mathrm{link}(\Lambda, \mathbf{x})$.

2. If $\mathbf{x}, \mathbf{y} \in \mathrm{vtc}(\Lambda)$ and $\mathbf{z}, \mathbf{w} \in \mathrm{vtc}(\Pi)$ and $f(\mathbf{x}) = \mathbf{z}$ and $f(\mathbf{y}) = \mathbf{w}$ and $\mathbf{x} \veebar \mathbf{y} \in \Lambda$ and $\mathbf{z} \veebar \mathbf{w} \in \Pi$ and $\mathbf{z}$ and $\mathbf{w}$ each have degree at least 3 and $\mathbf{z}$ and $\mathbf{w}$ are not neighbours in
$$\big|\Pi \setminus \{\mathbf{z} \veebar \mathbf{w}\}\big|,$$
then
$$f\Big[\mathrm{sub}(\Lambda', \Lambda)^{-1}\big[\{\mathbf{x} \veebar \mathbf{y}\}\big]\Big] = \{\mathbf{w} \veebar \mathbf{z}\}.$$

Proof sketch. 1: vertices in $\mathrm{sub}(\Lambda', \Lambda)^{-1}\big[\{\mathbf{y} \veebar \mathbf{x}\}$ cannot be mapped to vertices of $\mathrm{star}(\Pi, \mathbf{z})$ by f, by Lemma 6.6-1. Hence a map g satisfying (6.3) exists since f is monotone. Since f satisfies the back-property at $\mathbf{x}$, Lemma 6.6-1 implies that g is injective.

2: taking the image under f of all cells of Λ' that lie in $\mathbf{x} \veebar \mathbf{y}$, we get a path from $\mathbf{z}$ to $\mathbf{w}$ through Π. By Lemma 6.6-1, this path does not pass through $\mathbf{z}$ or $\mathbf{w}$. Hence it is either the path $(\mathbf{z}, \mathbf{z} \veebar \mathbf{w}, \mathbf{w})$, or it does not pass through $\mathbf{z} \veebar \mathbf{w}$ at all. The latter case is impossible, since in view of Lemma 6.5, this would show that $\mathbf{z}$ and $\mathbf{w}$ are neighbours in $\big|\Pi \setminus \{\mathbf{z} \veebar \mathbf{w}\}\big|$. □

We next introduce some specific one-dimensional complexes that will allow us to construct a continuum of polyhedrally-complete logics.

Notation **6.30**. For $\ell \geq 1$, let $\Lambda_\ell \in \mathbf{cmplx}_1^2$ have a set of vertices faithfully labelled as

$$\mathrm{vtc}(\Lambda_\ell) = \{\mathbf{a}^i : i < \ell\} \sqcup \{\mathbf{t}^{ij} : i < \ell \;\&\; j < 10\} \sqcup \{\mathbf{c}^i : i < \ell+1\} \sqcup \{\mathbf{t}^{-11}, \mathbf{t}^{-14}, \mathbf{t}^{\ell 6}, \mathbf{t}^{\ell 9}\}$$

and whose set of line segments equals

$$\big\{\mathbf{t}^{ij} \veebar \mathbf{t}^{ik} : i < \ell \;\&\; j,k < 10 \;\&\; j+1 \cong k\,(\mathrm{mod}\ 10)\big\}$$

$$\sqcup\big\{\mathbf{a}^i \veebar \mathbf{t}^{ij} : i < \ell \;\&\; j \in \{0,2,3,5,7,8\}\big\}$$

$$\sqcup\{\mathbf{t}^{(i-1)1} \veebar \mathbf{c}^i, \mathbf{t}^{(i-1)4} \veebar \mathbf{c}^i, \mathbf{t}^{i6} \veebar \mathbf{c}^i, \mathbf{t}^{i9} \veebar \mathbf{c}^i : i \leq \ell\}.$$

Figure 6.3 should convince the reader that these complexes indeed exist in $\mathbb{R}^2$.

The crucial property of the polyhedra $|\Lambda_\ell|$ is that there are very *few* p-morphisms between them (Lemma 6.31). This is useful for our purpose, because the presence of less p-morphisms will imply that these polyhedra logically differ more from one another.

Lemma **6.31**. If there is a surjective p-morphism from some subdivision of Λ_ℓ to Λ_m, then $\ell = m$.

Proof. Let Λ' be a subdivision of Λ_ℓ and let $f : \Lambda' \to \Lambda_m$ be a surjective p-morphism. It is clear that Lemma 6.6-1 implies that $\ell \geq m$.

We claim that whenever $f(\mathbf{a}^i) = \mathbf{a}^s$, the following hold:

1. If $i > 0$, then $f(\mathbf{a}^{i-1}) \in \{\mathbf{a}^{s-1}, \mathbf{a}^{s+1}\}$.
2. If $i < \ell - 1$, then $f(\mathbf{a}^{i+1}) \in \{\mathbf{a}^{s-1}, \mathbf{a}^{s+1}\}$.
3. If $0 < i < \ell - 1$, then $f\big[\{\mathbf{a}^{i-1}, \mathbf{a}^{i+1}\}\big] = \{\mathbf{a}^{s-1}, \mathbf{a}^{s+1}\}$.

Assume this is true. Find $\mathbf{x} \in \Lambda'$ with $f(\mathbf{x}) = \mathbf{a}^0$. By Lemma 6.6-1 we have $\mathbf{x} = \mathbf{a}^i$ for some i. Then $i \in \{0, \ell-1\}$ by 3. Assume that $i = 0$ (the case $i = \ell - 1$ is symmetrical). Then 2 implies that $f(\mathbf{a}^1) = \mathbf{a}^1$. By repeated applications of 3 we obtain $f(\mathbf{a}^2) = \mathbf{a}^2$, …, $f(\mathbf{a}^{\ell-1}) = \mathbf{a}^{\ell-1}$. So $\mathbf{a}^{\ell-1} \in \Lambda_m$, i.e. $\ell \leq m$.

Next we prove the claim. By Lemma 6.29-1, there exists a permutation g of $\{0,2,3,5,7,8\}$ such that

$$f\Big[\mathrm{sub}(\Lambda', \Lambda_\ell)^{-1}\big[\{\mathbf{a}^i \veebar \mathbf{t}^{ij}\big]\Big] = \{\mathbf{a}^s \veebar \mathbf{t}^{sg(j)}\}$$

for each $j \in \{0,2,3,5,7,8\}$. In particular,

$$f\Big[\mathrm{sub}(\Lambda', \Lambda_\ell)^{-1}\big[\mathrm{star}(\Lambda_\ell, \mathbf{a}^i)\big]\Big] \subseteq \mathrm{star}(\Lambda_m, \mathbf{a}^s).$$

In fact we must have $\Big\{g\big[\{2,3\}\big], g\big[\{7,8\}\big]\Big\} = \big\{\{2,3\}, \{7,8\}\big\}$ and

$$f(\mathbf{t}^{ij}) = \mathbf{t}^{sg(j)} \tag{6.4}$$

for $j \in \{2,3,7,8\}$ by Lemma 6.6-2, since $\mathbf{t}^{i2}$ and $\mathbf{t}^{i3}$ are neighbours in $|\Lambda_\ell|$ and $\mathbf{t}^{i7}$ and $\mathbf{t}^{i8}$ are neighbours in $|\Lambda_\ell|$. Then by Lemma 6.6-2, $f(\mathbf{t}^{i1})$ is a point that is both a neighbour of $f(\mathbf{t}^{i0})$ and a

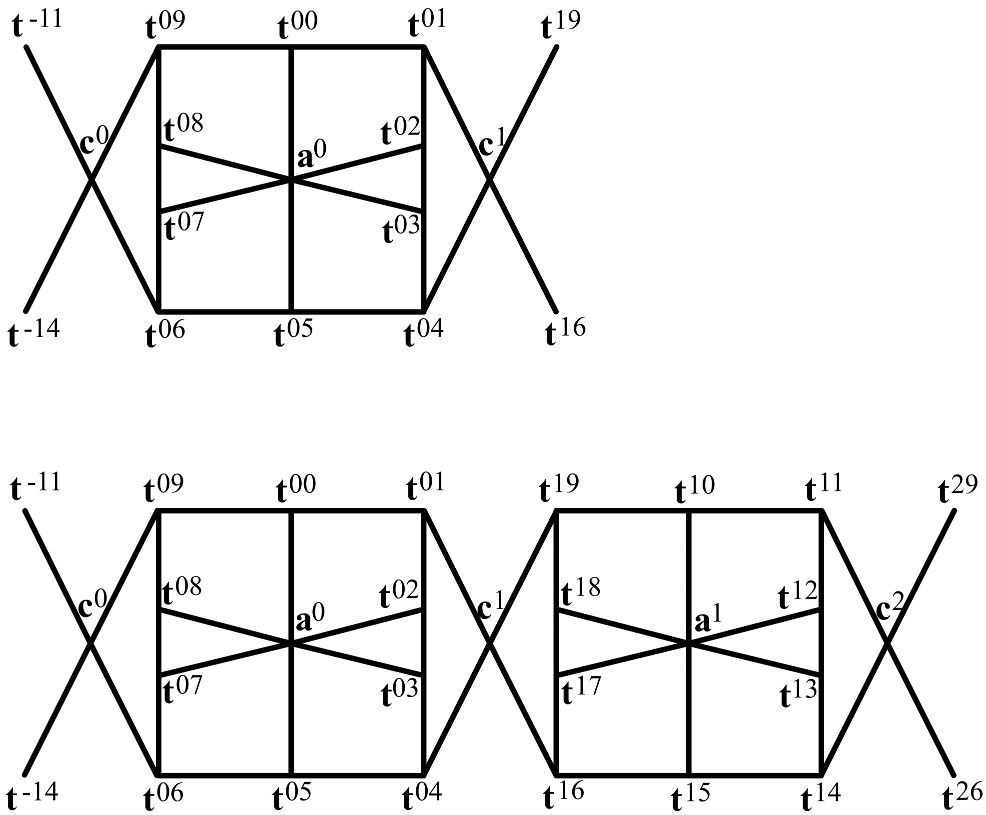


Figure 6.3: Λ_1 (top) and Λ_2 (bottom)

neighbour of $f(\mathbf{t}^{i2})$. By Lemma 6.6-1, $f(\mathbf{t}^{i1})$ has degree at most 3. Similar remarks hold for $f(\mathbf{t}^{i9})$. Thus $f(\mathbf{t}^{i0})$ has neighbours each of degree at most 3 which are neighbours of $f(\mathbf{t}^{i2})$ and $f(\mathbf{t}^{i8})$ resp. It follows that (6.4) also holds for $j = 0$ and

$$\{f(\mathbf{t}^{i1}), f(\mathbf{t}^{i9})\} = \{\mathbf{t}^{s(g(0)+1)}, \mathbf{t}^{sk}\}$$

where $g(0) - 1 \cong k \,(\mathrm{mod}\ 10)$. Similarly, (6.4) also holds for $j = 5$ and

$$\{f(\mathbf{t}^{i4}), f(\mathbf{t}^{i6})\} = \{\mathbf{t}^{s(g(5)+1)}, \mathbf{t}^{sk}\}$$

where $g(5) - 1 \cong k \,(\mathrm{mod}\ 10)$. Thus

$$f\big[\{\mathbf{t}^{i1}, \mathbf{t}^{i4}, \mathbf{t}^{i6}, \mathbf{t}^{i9}\}\big] = \{\mathbf{t}^{s1}, \mathbf{t}^{s4}, \mathbf{t}^{s6}, \mathbf{t}^{s9}\}.$$

Note that $\mathbf{c}^{i+1}$ is a neighbour of $\mathbf{t}^{i1}$ and $\mathbf{t}^{i4}$. Hence $f(\mathbf{c}^{i+1})$ is a neighbour of $f(\mathbf{t}^{i1})$ and $f(\mathbf{t}^{i4})$ by Lemma 6.6-1. Therefore $f(\mathbf{c}^{i+1}) \in \{\mathbf{c}^{s-1}, \mathbf{c}^{s+1}\}$. Suppose that $f(\mathbf{c}^{i+1}) = \mathbf{c}^{s+1}$ (the other case is similar). By Lemma 6.29-2, for each $j \in \{1,4\}$, the line segment $\lambda \in \Lambda'$ for which $\mathbf{c}^{i+1} \in \lambda \subseteq \mathbf{c}^{i+1} \veebar \mathbf{t}^{ij}$ satisfies $f(\lambda) = f(\mathbf{t}^{ij}) \veebar \mathbf{c}^{s+1}$. Hence, by Lemma 6.29-1, we can find a permutation h of $\{6,9\}$ such that

$$f\Big[\mathrm{sub}(\Lambda', \Lambda_\ell)^{-1}\big[\{\mathbf{c}^{i+1} \veebar \mathbf{t}^{(i+1)j}\big]\Big] = \{\mathbf{t}^{(s+1)h(j)} \veebar \mathbf{c}^{s+1}\}.$$

Suppose that $i < \ell - 1$. Then $\mathbf{t}^{(i+1)6}$ and $\mathbf{t}^{(i+1)9}$ have respective neighbours $\mathbf{t}^{(i+1)7}$ and $\mathbf{t}^{(i+1)8}$, each of degree 3, that are also neighbours of each other. By Lemma 6.6, it follows that $f(\mathbf{t}^{(i+1)j}) = \mathbf{t}^{(s+1)h(j)}$ for $j \in \{6,9\}$ and

$$f\big[\{\mathbf{t}^{(i+1)7}, \mathbf{t}^{(i+1)8}\}\big] = \{\mathbf{t}^{(s+1)7}, \mathbf{t}^{(s+1)8}\}.$$

By Lemma 6.29-2 applied thrice,

$$f\Big[\mathrm{sub}(\Lambda', \Lambda_\ell)^{-1}\big[\{\mathbf{t}^{(i+1)9} \veebar \mathbf{t}^{(i+1)8}, \mathbf{t}^{(i+1)8} \veebar \mathbf{t}^{(i+1)7}, \mathbf{t}^{(i+1)7} \veebar \mathbf{t}^{(i+1)6}\}\big]\Big] =$$

$$\{\mathbf{t}^{(s+1)9} \veebar \mathbf{t}^{(s+1)8}, \mathbf{t}^{(s+1)8} \veebar \mathbf{t}^{(s+1)7}, \mathbf{t}^{(s+1)7} \veebar \mathbf{t}^{(s+1)6}\}.$$

By Lemma 6.29-1 applied at $\mathbf{t}^{(i+1)7}$ and $\mathbf{t}^{(i+1)8}$, it follows that

$$f\Big[\mathrm{sub}(\Lambda', \Lambda_\ell)^{-1}\big[\{\mathbf{t}^{(i+1)j} \veebar \mathbf{a}^{i+1}\big]\Big] = \{\mathbf{t}^{(s+1)\overline{h}(j)} \veebar \mathbf{a}^{s+1}\}$$

for $j = 7,8$, where $\overline{h} : \{6,7,8,9\} \to \{6,7,8,9\}$ is the order-isomorphism extending h. Hence $f(\mathbf{a}^{i+1}) = \mathbf{a}^{s+1}$. This proves 2. The proof of 1 is similar. For 3, note that we must have

$$f\big[\{\mathbf{c}^{i-1}, \mathbf{c}^{i+1}\}\big] = \{\mathbf{c}^{s-1}, \mathbf{c}^{s+1}\}$$

in the above proof (since the points $f(\mathbf{t}^{i1})$ and $f(\mathbf{t}^{i4})$ are different from the points $f(\mathbf{t}^{i6})$ and $f(\mathbf{t}^{i9})$), and then the proof also implies $f\big[\{\mathbf{a}^{i-1}, \mathbf{a}^{i+1}\}\big] = \{\mathbf{a}^{s-1}, \mathbf{a}^{s+1}\}$. □

With Lemma 6.31, it is easy to prove that there exist $2^{\aleph_0}$ polyhedrally-complete logics.

Theorem **6.32**.

$$\#\big\{\mathrm{Log}(\mathfrak{p}) : \mathfrak{p} \subseteq \mathbf{plhdr}_2^3\big\} = 2^{\aleph_0}.$$

Proof. The inequality $\leq$ is clear. We prove the inequality $\geq$.

For $S \subseteq \{1,2,\dots\}$, define $\mathfrak{q}(S) = \{|\Lambda_\ell| : \ell \in S\}$ and

$$\mathscr{L}(S) = \mathrm{Log}_\varnothing \Big(\mathfrak{q}(S) \cup \mathfrak{S}\big(\mathfrak{q}(S)\big)\Big).$$

By Theorem 5.38, for each $S \subseteq \{1,2,\dots\}$ there exists $\mathfrak{p} \subseteq \mathbf{plhdr}_2^3$ such that $\mathscr{L}(S) = \mathrm{Log}(\mathfrak{p})$. Thus it suffices to show that $\mathscr{L}(S_0) = \mathscr{L}(S_1)$ implies $S_0 = S_1$. We show that $S_0 \subseteq S_1$. Suppose that $\ell \in S_0$. Then $\Lambda_\ell \in \mathrm{Trian}\big(\mathfrak{q}(S_0)\big)$ implies $\chi(\Lambda_\ell) \notin \mathscr{L}(S_0)$, so $\chi(\Lambda_\ell) \notin \mathscr{L}(S_1)$, so there exists $\mathbf{L} \in \mathfrak{q}(S_1) \cup \mathfrak{S}\big(\mathfrak{q}(S_1)\big)$ and a triangulation Λ of $\mathbf{L}$ such that $\Lambda \not\models \chi(\Lambda_\ell)$. This means that there is an up-reduction $\Lambda \rightharpoonup \Lambda_\ell$. Only $\varnothing$ can be mapped to $\varnothing$, so there is a surjective p-morphism $\Lambda \to \Lambda_\ell$. By Lemma 6.6-1, it follows that $\mathbf{L}$ has at least three points of degree at least 3. The members of $\mathfrak{S}\big(\mathfrak{q}(S_1)\big)$ are standard bipyramids with zero-dimensional bases, which have at most two points of degree at least 3. Hence $\mathbf{L} \in \mathfrak{q}(S_1)$. Find $m \in S_1$ such that $\mathbf{L} = |\Lambda_m|$. By Lemma 2.31, there exists a common subdivision Λ' of Λ and Λ_m. By Lemma 2.78, there is a surjective p-morphism $\Lambda' \to \Lambda_\ell$. By Lemma 6.31, we have $\ell = m \in S_1$. □

In section 6.4, we saw that finite graphs without cycles are logically tame; they yield only countably many quasi-polyhedrally-complete logics. In contrast, Theorem 6.32 shows that the number of logics explodes to $2^{\aleph_0}$ once cycles are allowed.

Future work could look for axiomatizations of some quasi-polyhedrally-complete logics of height 3. Also the logics obtained from graphs with at most some fixed number of cycles can be investigated.

Chapter 7

Computational aspects of polyhedral logics

Polyhedra can be used for computer models of certain objects or structures (cf. chapter 1). The value of such models depends on (at least) two questions:

i. Do we have algorithms for computing the operations and properties of polyhedra that we are interested in?

ii. If so, are the algorithms efficient?

For more details on the meaning and relevance of these questions, we refer to an introduction to computability theory [Rob20]. In this chapter, we focus on item i, where we are interested in the logics of polyhedra. Section 7.1 presents various decidability problems for logics of polyhedra. Moreover, section 7.1 gives some basic algorithms, which yield positive answers to some cases of those problems. Section 7.2 contains an algorithm that is tailor-made for a large subset of $\mathbf{plhdr}_2$ (containing both the one-dimensional polyhedra and the two-dimensional manifolds-with-boundary), and which answers the decidability questions positively for these polyhedra. Section 7.3 shows that the decidability problems are more complicated for the set of all polyhedra. We were not able to prove any undecidability result, but we shall show that the proof given in section 7.2 does not generalize to arbitrary two-dimensional polyhedra.

7.1 Some general algorithms

The first thing to note in the context of complexity is that it no longer makes sense to work with the logics of arbitrary infinite sets of polyhedra, since such sets cannot in general be stored in a computer. A finite set of polyhedra however can be stored in a computer "up to PL-homeomorphisms". For, if $\mathbf{P}$ is a polyhedron and Σ is a triangulation of $\mathbf{P}$, then there exists a hereditarily finite poset P that is isomorphic to Σ. By Lemma 2.64-1, the poset P then contains all information about $\mathbf{P}$ up to PL-homeomorphisms.

If polyhedra $\mathbf{P}, \mathbf{Q} \in \mathbf{plhdr}^d$ are disjoint, one can even view the finite set $\mathfrak{p} = \{\mathbf{P}, \mathbf{Q}\}$ as one polyhedron $\mathbf{P} \sqcup \mathbf{Q}$. It turns out that this does not affect $\mathrm{Log}(\mathfrak{p})$, but it may affect $\mathrm{Log}_{\varnothing}(\mathfrak{p})$. This makes sense, since we explained in chapter 5 that $\mathrm{Log}(\mathfrak{p})$ deals with the local structure of $\mathbf{P}$ and $\mathbf{Q}$ (which is the same as the local structure of $\mathbf{P} \sqcup \mathbf{Q}$), whereas $\mathrm{Log}(\mathfrak{p})$ deals with the global structure of $\mathbf{P}$ and the global structure of $\mathbf{Q}$, which are different from the global structure of $\mathbf{P} \sqcup \mathbf{Q}$. This can be formalized into the following lemma, the proof of which is an easy exercise.

Lemma **7.1**. Let $\mathfrak{p} \subseteq \mathbf{plhdr}^d$ be a finite set of pairwise disjoint polyhedra. Then

$$\mathrm{Log}\Big(\bigsqcup_{\mathbf{P}\in\mathfrak{p}} \mathbf{P}\Big) = \mathrm{Log}(\mathfrak{p}).$$

Using this disjoint union idea, we can say something about finite sets of polyhedra in general.

Lemma **7.2**. Given a finite set $\mathfrak{p} \subset \mathbf{plhdr}$, we can effectively compute a $\mathbf{Q} \in \mathbf{plhdr}$ such that

$$\mathrm{Log}(\mathfrak{p}) = \mathrm{Log}(\mathbf{Q}).$$

Proof. Find a number d such that

$$\mathfrak{p} \subseteq \bigcup_{d' \leq d} \mathbf{plhdr}^d.$$

Then we can find a system $(\varphi_{\mathbf{P}} : \mathbf{P} \in \mathfrak{p})$ of affine injective maps such that $\varphi_{\mathbf{P}} : \mathbf{P} \to \mathbb{R}^d$ and

$$\mathbf{Q} = \bigsqcup_{\mathbf{P}\in\mathfrak{p}} \varphi_{\mathbf{P}}[\mathbf{P}] \subseteq \mathbb{R}^d.$$

[If $\mathbf{P} \in \mathbf{plhdr}^{d'}$, one can construct $\varphi_{\mathbf{P}}$ by composing the map $\iota^d_{d'}$ with a suitable translation of $\mathbb{R}^d$.] Then Lemma 7.1 implies

$$\mathrm{Log}(\mathbf{Q}) = \mathrm{Log}\left\{\varphi_{\mathbf{P}}[\mathbf{P}] : \mathbf{P} \in \mathfrak{p}\right\} = \mathrm{Log}(\mathfrak{p})$$

by Lemma 2.79. □

We are now ready to prove that the operation $\mathrm{links}(\cdot)$ is essentially computable.

Lemma **7.3**. Given a finite set $\mathfrak{p} \subseteq \mathbf{plhdr}$, we can effectively compute a finite set $\mathfrak{q} \subseteq \mathbf{plhdr}$ that equals $\mathrm{links}(\mathfrak{p})$ up to PL-homeomorphisms.

Proof. Compute a triangulation $\Sigma(\mathbf{P})$ of each $\mathbf{P} \in \mathfrak{p}$. Then compute the first barycentric subdivision $\Sigma(\mathbf{P})^+$ of each $\Sigma(\mathbf{P})$. Then compute

$$\mathfrak{q} = \left\{\left|\mathrm{link}\left(\Sigma(\mathbf{P})^+, \mathbf{x}\right)\right| : \mathbf{P} \in \mathfrak{p}\ \&\ \mathbf{x} \in \mathrm{vtc}\left(\Sigma(\mathbf{P})^+\right)\right\}.$$

Clearly $\mathfrak{q} \subseteq \mathrm{links}(\mathfrak{p})$. Conversely, suppose that $\mathbf{Q} \in \mathrm{links}(\mathfrak{p})$. Find a triangulation Δ of some $\mathbf{P} \in \mathfrak{p}$ and $\mathbf{y} \in \mathrm{vtc}(\Delta)$ such that $\mathbf{Q} = \left|\mathrm{link}(\Delta, \mathbf{y})\right|$. By Lemma 2.58, there exists a triangulation Γ of $\mathbf{P}$ and $\mathbf{x} \in \mathrm{vtc}\left(\Sigma(\mathbf{P})^+\right)$ such that there is a simplicial isomorphism f from some subdivision of Δ to Γ satisfying $f(\mathbf{y}) = \mathbf{x}$. By Lemma 2.31, let Υ be a common subdivision of Γ and $\Sigma(\mathbf{P})^+$. By Lemma 2.54-2, Υ is isomorphic to a subdivision of Δ. By Lemma 5.27-1, $\mathrm{link}(\Upsilon, \mathbf{x})$ is isomorphic to a subdivision of $\mathrm{link}\left(\Sigma(\mathbf{P})^+, \mathbf{x}\right)$ and is isomorphic to a subdivision of $\mathrm{link}(\Gamma, \mathbf{x}) \cong \mathrm{link}(\Delta, \mathbf{y})$. Thus $\mathrm{link}(\Upsilon, \mathbf{x})$ is isomorphic to a triangulation of an element of $\mathfrak{q}$ and is isomorphic to a triangulation of $\mathbf{Q}$. By Lemma 2.64-1, it follows that $\mathbf{Q}$ is PL-homeomorphic with an element of $\mathfrak{q}$. □

Next, we are able to prove a computable version of Proposition 5.45.

Lemma **7.4**. Given a finite $\mathfrak{p} \subseteq \mathbf{plhdr}^{d'}_{d}$, we can effectively compute a finite $\mathfrak{q} \subseteq \mathbf{plhdr}^{d'-1}_{d-1}$ such that

$$\mathrm{Log}(\mathfrak{p}) = \mathrm{Log}_{\varnothing}(\mathfrak{q}).$$

Proof. Use Lemma 7.3 to compute a finite set $\mathfrak{r}$ that equals links$(\mathfrak{p})$ up to PL-homeomorphisms. Recall from the proof of Lemma 7.3 that for each $\mathbf{R} \in \mathfrak{r}$ we know a $\mathbf{P} \in \mathfrak{p}$, a triangulation Γ of $\mathbf{P}$ and an $\mathbf{x} \in \mathrm{vtc}(\Gamma)$ such that $\mathbf{R} = \left|\mathrm{link}(\Gamma, \mathbf{x})\right|$. Hence by Lemma 5.23-3 we can determine whether a given element of $\mathfrak{r}$ is PL-homeomorphic to the boundary of a d'-simplex. Thus we can compute a suitable $\mathfrak{q}$ using the proof of Proposition 5.45. □

We introduce a collection of problems that are closely related to (quasi-)polyhedrally-complete logics.

Notation **7.5**. If $\mathfrak{p}$ is a set of polyhedra, let *Bound*$(\mathfrak{p})$ denote the statement: "for each formula φ there exists $n(\varphi, \mathfrak{p}) < \omega$ such that for all $\mathbf{P} \in \mathfrak{p}$ and for every triangulation Σ of $\mathbf{P}$ it holds

$$\varphi \in \mathrm{Log}(\mathbf{P}) \quad \Leftrightarrow \quad \Sigma^{+n(\varphi,\mathfrak{p})} \setminus \{\varnothing\} \vDash \varphi\text{''}.$$

Let *Bound*$_\varnothing(\mathfrak{p})$ denote the statement: "for each formula φ there exists $n(\varphi, \mathfrak{p}) < \omega$ such that for all $\mathbf{P} \in \mathfrak{p}$ and for every triangulation Σ of $\mathbf{P}$ it holds

$$\varphi \in \mathrm{Log}_\varnothing(\mathbf{P}) \quad \Leftrightarrow \quad \Sigma^{+n(\varphi,\mathfrak{p})} \vDash \varphi\text{''}.$$

Furthermore let *EffBound*$_{(\varnothing)}(\mathfrak{p})$ be the statement *Bound*$_{(\varnothing)}(\mathfrak{p})$ where moreover the number $n(\varphi, \mathfrak{p})$ can be computed given φ.

Define several problems:

- VALID$_{(\varnothing)}(\mathfrak{p})$: given $\mathbf{P} \in \mathfrak{p}$ and a formula φ, does it hold $\varphi \in \mathrm{Log}_{(\varnothing)}(\mathbf{P})$?
- COMPARE$_{(\varnothing)}(\mathfrak{p})$: given $\mathbf{P}, \mathbf{Q} \in \mathfrak{p}$, does it hold $\mathrm{Log}_{(\varnothing)}(\mathbf{P}) \subseteq \mathrm{Log}_{(\varnothing)}(\mathbf{Q})$?
- EQUIV$_{(\varnothing)}(\mathfrak{p})$: given $\mathbf{P}, \mathbf{Q} \in \mathfrak{p}$, does it hold $\mathrm{Log}_{(\varnothing)}(\mathbf{P}) = \mathrm{Log}_{(\varnothing)}(\mathbf{Q})$?
- COMPARESET$_{(\varnothing)}(\mathfrak{p})$: given $\mathfrak{q}, \mathfrak{r} \in [\mathfrak{p}]^{<\aleph_0}$, does it hold $\mathrm{Log}_{(\varnothing)}(\mathfrak{q}) \subseteq \mathrm{Log}_{(\varnothing)}(\mathfrak{r})$?
- EQUIVSET$_{(\varnothing)}(\mathfrak{p})$: given $\mathfrak{q}, \mathfrak{r} \in [\mathfrak{p}]^{<\aleph_0}$, does it hold $\mathrm{Log}_{(\varnothing)}(\mathfrak{q}) = \mathrm{Log}_{(\varnothing)}(\mathfrak{r})$?

Remark **7.6**. Evidently, any problem EQUIV$_{(\varnothing)}(\mathfrak{p})$ reduces to COMPARE$_{(\varnothing)}(\mathfrak{p})$ and (by Remark 2.70) any problem EQUIVSET$_{(\varnothing)}(\mathfrak{p})$ is equivalent to COMPARESET$_{(\varnothing)}(\mathfrak{p})$. Also trivially, any problem EQUIV$_{(\varnothing)}(\mathfrak{p})$ reduces to EQUIVSET$_{(\varnothing)}(\mathfrak{p})$ and any problem COMPARE$_{(\varnothing)}(\mathfrak{p})$ reduces to COMPARESET$_{(\varnothing)}(\mathfrak{p})$. Moreover, *EffBound*$_{(\varnothing)}(\mathfrak{p})$ implies *Bound*$_{(\varnothing)}(\mathfrak{p})$ and that VALID$_{(\varnothing)}(\mathfrak{p})$ is decidable.

In general, even more reductions exist between the problems of Notation 7.5. To track them down, we employ the machinery from chapters 4 and 5.

Proposition **7.7**. Let $\mathfrak{p} \subseteq \mathbf{plhdr}$.

1. COMPARESET$_{(\varnothing)}(\mathfrak{p})$ reduces to VALID$_{(\varnothing)}(\mathfrak{p})$.
2. COMPARESET$(\mathfrak{p})$ reduces to COMPARESET$_\varnothing\big(\mathrm{links}(\mathfrak{p})\big)$.
3. EQUIVSET$(\mathfrak{p})$ reduces to EQUIVSET$_\varnothing\big(\mathrm{links}(\mathfrak{p})\big)$.
4. VALID$(\mathfrak{p})$ reduces to VALID$_\varnothing\big(\mathrm{links}(\mathfrak{p})\big)$.

5. $\mathit{Bound}_\varnothing\big(\mathrm{links}(\mathfrak{p})\big)$ implies $\mathit{Bound}(\mathfrak{p})$.

6. $\mathit{EffBound}_\varnothing\big(\mathrm{links}(\mathfrak{p})\big)$ implies $\mathit{EffBound}(\mathfrak{p})$.

Proof. We begin by proving 1 in the case with $\varnothing$ (the quasi-polyhedrally-complete case). Consider $\mathfrak{q},\mathfrak{r} \in [\mathfrak{p}]^{<\aleph_0}$. For each $\mathbf{R} \in \mathfrak{r}$, let $\Theta(\mathbf{R})$ be a triangulation of $\mathbf{R}$. We claim that

$$\mathrm{Log}_\varnothing(\mathfrak{q}) \subseteq \mathrm{Log}_\varnothing(\mathfrak{r}) \quad \Leftrightarrow \quad \Big\{\chi\big(\Theta(\mathbf{R})\big) : \mathbf{R} \in \mathfrak{r}\Big\} \cap \mathrm{Log}_\varnothing(\mathfrak{q}) = \varnothing. \tag{7.1}$$

Since Jankov-Fine formulas are computable, this will suffice. The implication $\Rightarrow$ in (7.1) is easy. For the other implication, suppose that the right-hand side of (7.1) holds and let $\varphi \notin \mathrm{Log}_\varnothing(\mathfrak{r})$. Then there exists $\mathbf{R} \in \mathfrak{r}$ such that $\varphi \notin \mathrm{Log}_\varnothing(\mathbf{R})$. Find $\mathbf{Q} \in \mathfrak{q}$ such that $\chi\big(\Theta(\mathbf{R})\big) \notin \mathrm{Log}_\varnothing(\mathbf{Q})$. By Corollary 4.2-3, it follows that $\mathrm{Log}_\varnothing(\mathbf{Q}) \subseteq \mathrm{Log}_\varnothing(\mathbf{R})$, so $\varphi \notin \mathrm{Log}_\varnothing(\mathfrak{q})$. This proves the quasi-polyhedrally-complete case of 1.

For the polyhedrally-complete case, suppose again that $\mathfrak{q},\mathfrak{r} \in [\mathfrak{p}]^{<\aleph_0}$. Using Lemma 7.4, compute a finite set $\mathfrak{s} \subseteq \mathbf{plhdr}$ such that $\mathrm{Log}(\mathfrak{r}) = \mathrm{Log}_\varnothing(\mathfrak{s})$. For each $\mathbf{S} \in \mathfrak{s}$, let $\Theta(\mathbf{S})$ be a triangulation of $\mathbf{S}$. We claim that

$$\mathrm{Log}(\mathfrak{q}) \subseteq \mathrm{Log}(\mathfrak{r}) \quad \Leftrightarrow \quad \Big\{\chi\big(\Theta(\mathbf{S})\big) : \mathbf{S} \in \mathfrak{s}\Big\} \cap \mathrm{Log}(\mathfrak{q}) = \varnothing. \tag{7.2}$$

Again the implication $\Rightarrow$ is easy. For the other implication, assume the right-hand side of (7.2) and let $\varphi \notin \mathrm{Log}(\mathfrak{r})$. Then $\varphi \notin \mathrm{Log}_\varnothing(\mathfrak{s})$, so there exists $\mathbf{S} \in \mathfrak{s}$ such that $\varphi \notin \mathrm{Log}_\varnothing(\mathbf{S})$. Find $\mathbf{Q} \in \mathfrak{q}$ such that $\chi\big(\Theta(\mathbf{S})\big) \notin \mathrm{Log}(\mathbf{Q})$. Find a triangulation Γ of $\mathbf{Q}$ and $\mathbf{x} \in \mathrm{vtc}(\Gamma)$ such that $\uparrow^\Gamma(\mathbf{x}) \not\models \chi\big(\Theta(\mathbf{S})\big)$. Then by Proposition 5.19 there is an up-reduction $\mathrm{link}(\Gamma,\mathbf{x}) \rightharpoonup \Theta(\mathbf{S})$. Find a triangulation Σ of $\mathbf{S}$ such that $\Sigma \not\models \varphi$. By Lemma 2.31, find a common subdivision Σ' of Σ and $\Theta(\mathbf{S})$. By Lemma 2.78 we have $\Sigma' \not\models \varphi$. By Theorem 4.1 there is an up-reduction $\Delta \rightharpoonup \Sigma'$, for some subdivision Δ of $\mathrm{link}(\Gamma,\mathbf{x})$. By Lemma 5.27-2 there is a subdivision Γ' of Γ such that $\mathrm{link}(\Gamma',\mathbf{x})$ is isomorphic to a subdivision of Δ. Then $\mathrm{link}(\Gamma',\mathbf{x}) \not\models \varphi$ by Lemma 2.78-1, so $\Gamma' \setminus \{\varnothing\} \not\models \varphi$ by Proposition 5.19, so $\varphi \notin \mathrm{Log}(\mathfrak{q})$.

2, 3 and 4 follow from Theorem 5.33 and Lemmas 7.3 and 2.79.

For 5, assume $\mathit{Bound}_\varnothing\big(\mathrm{links}(\mathfrak{p})\big)$ and let φ be a formula. Find $n < \omega$ such that for all $\mathbf{Q} \in \mathrm{links}(\mathfrak{p})$ and for every triangulation Δ of $\mathbf{Q}$ it holds

$$\varphi \in \mathrm{Log}_\varnothing(\mathbf{Q}) \quad \Leftrightarrow \quad \Delta^{+n} \models \varphi.$$

Consider $\mathbf{P} \in \mathfrak{p}$ and a triangulation Σ of $\mathbf{P}$. We claim that

$$\varphi \in \mathrm{Log}(\mathbf{P}) \quad \Leftrightarrow \quad \Sigma^{+(n+1)} \setminus \{\varnothing\} \models \varphi.$$

Suppose that $\varphi \notin \mathrm{Log}(\mathbf{P})$. Then $\varphi \notin \mathrm{Log}_\varnothing\big(\mathrm{links}\{\mathbf{P}\}\big)$ by Theorem 5.33. By Lemma 7.3 (and its proof), it follows that there exists $\mathbf{x} \in \mathrm{vtc}(\Sigma^+)$ such that $\varphi \notin \mathrm{Log}_\varnothing\big|\mathrm{link}(\Sigma^+,\mathbf{x})\big|$. Hence by assumption, $\mathrm{link}(\Sigma^+,\mathbf{x})^{+n} \not\models \varphi$. By n applications of Lemma 5.24, we conclude that $\mathrm{link}(\Sigma^{+(n+1)},\mathbf{x}) \not\models \varphi$, so $\Sigma^{+(n+1)} \setminus \{\varnothing\} \not\models \varphi$ by Proposition 5.19. This proves 5. It is clear that an analogous proof goes through for 6. □

Proposition **7.8**. If $\mathfrak{p} \subseteq \mathbf{plhdr}^d$ is closed under translations and disjoint unions, then the problems COMPARE($\mathfrak{p}$), COMPARESET($\mathfrak{p}$), EQUIV($\mathfrak{p}$) and EQUIVSET($\mathfrak{p}$) are all equivalent.

Proof. In view of Remark 7.6, it suffices to show that EQUIVSET($\mathfrak{p}$) reduces to EQUIV($\mathfrak{p}$). This follows from Lemma 7.1 and the proof of Lemma 7.2. □

7.2 Efficiently bounded triangulations in $\mathbb{R}^3$

In this section we prove $\mathit{EffBound}_{\varnothing}(\mathfrak{p})$ for a particular set $\mathfrak{p} \subseteq \mathbf{plhdr}_2$ (Theorem 7.13). To prove such a result, we need to be able to build Kripke models whose underlying frames are iterated barycentric subdivisions of triangulations of polyhedra in $\mathfrak{p}$. Since two-dimensional complexes are for an important part built from triangles, it may not be surprising that we have some technical lemmas about the behaviour of iterated barycentric subdivisions in relation to triangles.

The first lemma describes how to subdivide a triangle into areas such that some chosen vertex of the triangle is the only place where more than two areas meet. Logically, this is interesting for the local structure at the chosen vertex, without the local structure becoming too complicated elsewhere.

Lemma **7.9**. Let τ be a triangle, $\mathbf{x} \in \mathrm{vtc}(\tau)$, $n \geq 1$ and $1 \leq m \leq 2^{n-1}$. Then there exists a partition $\mathscr{T} = \{T_0, \ldots, T_{m-1}\}$ of the set of triangles in $\mathrm{fac}(\tau)^{+n}$ such that:

- $\#\mathscr{T} = m$;
- for each $T \in \mathscr{T}$ there exists $\tau \in T$ with $\mathbf{x} \in \tau$;
- whenever a triangle in T_i and a triangle in T_j intersect (other than at $\mathbf{x}$), we have $j \in \{i-1, i, i+1\}$;
- all triangles in $\mathrm{fac}(\tau)^{+n}$ that intersect $\partial\tau \setminus \{\mathbf{x}\}$ are in T_{m-1}.

We omit a proof, since everything happens within the triangle τ and is therefore easy to visualize. Some examples are depicted in Figure 7.1. The next lemma describes how to "separate" two one-dimensional polyhedra that lie within some two-dimensional polyhedron. It does so by subdividing the two-dimensional polyhedron into a list of areas such that only the first area touches the one one-dimensional polyhedron, only successive areas touch each other and only the last area touches the other one-dimensional polyhedron.

Lemma **7.10**. Let Σ be a two-dimensional complex in which each edge has at most two proper successors. Let $\Lambda_0, \Lambda_1 \subseteq \Sigma$ be two disjoint at-most-one-dimensional subcomplexes. Let $n \geq 1$ and $1 \leq m \leq 2^n$. Then there exists a partition $\mathscr{T} = \{T_0, \ldots, T_{m-1}\}$ of the set of triangles in Σ^{+n} such that:

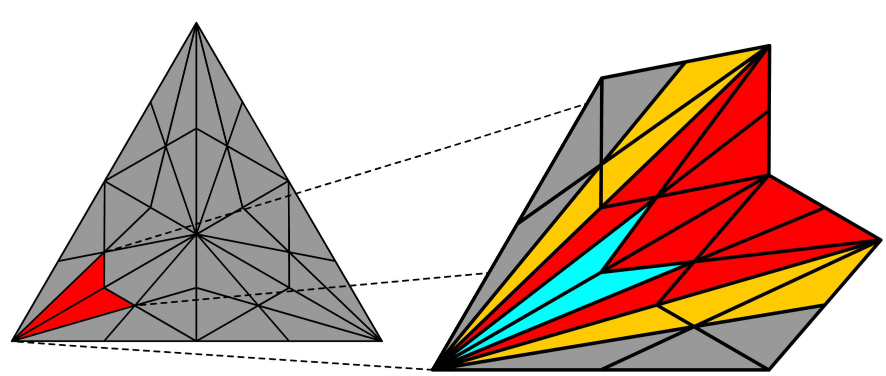


Figure 7.1: example of Lemma 7.9 when $n = 2$ (left) and $n = 3$ (right)

- $\#\mathscr{T} = m$;
- whenever a triangle in T_i and a triangle in T_j intersect, we have $j \in \{i-1, i, i+1\}$;
- all triangles in Σ^{+n} that intersect $|\Lambda_0|$ are in T_0;
- all triangles in Σ^{+n} that intersect $|\Lambda_1|$ are in T_{m-1}.

Proof sketch. For an illustrative example, see Figure 7.2. Not every two-dimensional complex is isomorphic to a complex in $\mathbf{cmplx}_2^3$, so this lemma is generally set in four-or-higher-dimensional Euclidean space. However, the proof mostly takes place in the individual triangles of the complex.

One proceeds by induction on n. For $m = 1$, the lemma is trivial. For $n = 1$ and $m = 2$, one can let T_0 be the set of triangles that intersect $|\Lambda_0|$ and T_1 the set of all other triangles in Σ^+. Since any triangle in Σ^+ contains at most one vertex of Σ and intersects at most one line segment of Σ, the partition $\mathscr{T} = \{T_0, T_1\}$ is then as desired. Suppose that the lemma holds for some $n \geq 1$. Pick $\mathscr{T} = \{T_0, \ldots, T_{2^n-1}\}$ accordingly. Let $\Lambda_{0,0}$ be the subcomplex of Σ^{+n} with carrier

$$|\Lambda_0| \cap \left(\bigcup T_0\right).$$

For $1 \leq i < 2^n$, let $\Lambda_{i,0} = \Lambda_{i-1,1}$ be the subcomplex of Σ^{+n} with carrier

$$\left(\bigcup T_i\right) \cap \left(\bigcup T_{i-1}\right).$$

Let $\Lambda_{2^n-1,1}$ be the subcomplex of Σ^{+n} with carrier

$$|\Lambda_1| \cap \left(\bigcup T_{2^n-1}\right).$$

Finally, for each $i < 2^n$, let Σ_i be the subcomplex of Σ^{+n} with carrier $\bigcup T_i$. Then $\Lambda_{i,0}, \Lambda_{i,1} \subseteq \Sigma_i$ are disjoint, because a triangle of T_{i-1} cannot touch a triangle of T_{i+1} (or a similar argument if $i = 0$ or $i = 2^n - 1$). Hence we can apply the case "n=1" of the lemma to find a partition of the set of triangles of Σ_i^+, into two parts. Combining this for all i, we find the desired subdivision of the set of triangles of Σ^{+n+1} into 2^{n+1} parts. □

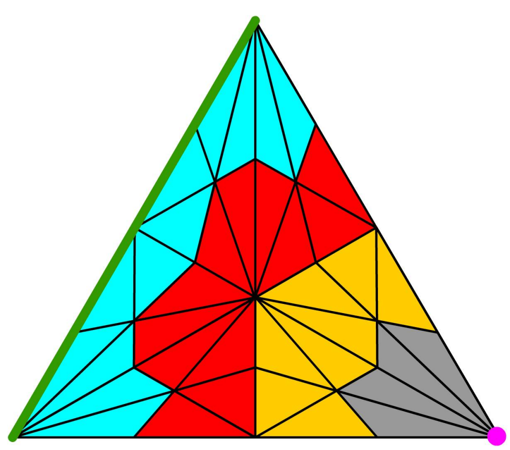


Figure 7.2: example of Lemma 7.10 when $n = 2$ with Λ_0 in green and Λ_1 in pink

It can be convenient to be able to talk about the local structure of a Kripke model without having to write down logical formulas. For this reason, we introduce types.

Notation **7.11**. If P is a finite poset of height at most 4 and $\mu : P \to \mathscr{P}\,\mathrm{Prop}$ is a marking and $p \in P$, define $\mathrm{type}_\mu(p) = \mu(p)$ if $\mathrm{hgt}^P(p) = 3$, and otherwise

$$\mathrm{type}_\mu(p) = \Big(\mu(p), \mathrm{type}_\mu\left[\{\text{immediate successors of } p\}\right]\Big).$$

Observe that, if the image of our markings is bounded[1], then only finitely many types are possible. However, with simplicial complexes, the complications arise from the "locations" of the types. For example, if τ is a triangle, $\Sigma = \mathrm{fac}(\tau)^{+10}$, $\mu : \Sigma \to \mathscr{P}\,\mathrm{Prop}$ and $\mathbf{x}, \mathbf{y}$ are distinct vertices of Σ, then there are many different paths from $\mathbf{x}$ to $\mathbf{y}$ through $\Sigma \setminus \{\varnothing\}$. Depending on the path one chooses, the cells of Σ^{+10} visited by the path may have very different values under type_μ. Had we started off with a line segment instead of the triangle τ, things would be simpler: avoiding repetitions, there would be a unique path from $\mathbf{x}$ to $\mathbf{y}$. Hence we can prove the following lemma.

Lemma **7.12**. Let $\mathbb{P} \in [\mathrm{Prop}]^{<\aleph_0}$, let $c = \#\mathscr{P}\mathbb{P}$ and let $n \geq \log_2(c^4 + c^2 + c + 1)$. Let λ be a line segment, Λ a triangulation of λ and $\mu : \Lambda \to \mathscr{P}\mathbb{P}$ a marking. Then there exists a marking $\underline{\mu} : \mathrm{fac}(\lambda)^{+n} \to \mathscr{P}\mathbb{P}$ such that $\mathrm{type}_{\underline{\mu}}$ and type_μ agree on $\varnothing$ and on the endpoints of λ.

Proof sketch. Up to simplicial isomorphisms, choosing a triangulation of λ merely amounts to choosing the number of vertices. $\mathrm{fac}(\lambda)^{+n}$ has

$$2^n + 1 \geq c^4 + c^2 + c + 2$$

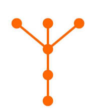
vertices. Suppose that Λ has strictly more than $c^4 + c^2 + c + 2$ vertices. Then Λ has strictly more than $c^4 + c^2 + c + 1$ line segments. Hence, by the pigeonhole principle, there exists a colour $\mathbb{C} \subseteq \mathbb{P}$ such that Λ has at least $c^3 + c + 2$ line segments which are mapped to $(\mathbb{C}, \varnothing)$ by μ. Note that

$$\#\Big(\mathrm{type}_\mu\left[\Lambda \setminus \{\varnothing\}\right]\Big) \leq c^3 + c.$$

Hence there exist two distinct line segments $\lambda_0, \lambda_1 \in \Lambda$ such that $\mu(\lambda_0) = (\mathbb{C}, \varnothing) = \mu(\lambda_1)$ and $\mathrm{type}_\mu[\Pi] = \mathrm{type}_\mu[\Lambda]$, where Π is the subcomplex of Λ consisting of all cells that do *not* lie between λ_0 and λ_1 (i.e. $|\Pi| = \lambda \setminus \mathrm{Conv}\left((\mathrm{relInt}\,\lambda_0) \cup (\mathrm{relInt}\,\lambda_1)\right)$). Then we can remove all vertices between λ_0 and λ_1 (i.e. all vertices in $\mathrm{Conv}\left((\mathrm{relInt}\,\lambda_0) \sqcup (\mathrm{relInt}\,\lambda_1)\right)$), without changing the types of $\varnothing$ and the endpoints of λ. Repeating this argument, we eventually must have that Λ has at most $c^4 + c^2 + c + 2$ vertices. This proves the lemma. $\square$

Let

$$\mathfrak{p}_2 = \left\{ \mathbf{P} \in \mathbf{plhdr}_2 : \chi\left(\quad \right) \in \mathrm{Log}_\varnothing(\mathbf{P}) \right\}.$$



If $\mathbf{P} \in \mathbf{plhdr}_2$ and Σ is a triangulation of $\mathbf{P}$, then it is easy to see that $\mathbf{P} \in \mathfrak{p}_2$ iff every line segment in Σ has at most two proper successors. If $\mathbf{P} \in \mathbf{plhdr}_1$, then every line segment in Σ has zero proper successors. If $\mathbf{P}$ is a two-dimensional manifold-with-boundary, then every line segment in Σ has one or two proper successors (Lemma 3.16-3). Hence $\mathfrak{p}_2$ can be thought of as the combination of two-dimensional manifolds-with-boundary and one-dimensional polyhedra.

[1] in the sense that $\mathrm{Im}\,\mu \subseteq \mathscr{P}\mathbb{P}$ for some fixed $\mathbb{P} \in [\mathrm{Prop}]^{<\aleph_0}$

Theorem **7.13**. *EffBound*$_\varnothing(\mathfrak{p}_2)$.

Proof sketch. Let Σ be a two-dimensional complex such that each line segment in Σ has at most two proper successors. Let Σ' be a subdivision of Σ, let $\mathbb{P} \in [\mathrm{Prop}]^{<\aleph_0}$ and let $\mu' : \Sigma' \to \mathscr{P}\mathbb{P}$ be a marking. Let $c = \#\mathscr{P}\mathbb{P}$ and $\bar{n} = c^3 + 12c + 10$. Then we claim that there exists an $n \leq \bar{n}$ and a marking $\mu : \Sigma^{+n} \to \mathscr{P}\mathbb{P}$ such that the pointed models $(\Sigma', \mu', \varnothing)$ and $(\Sigma^{+n}, \mu, \varnothing)$ are bisimilar. This is enough to prove the theorem.

Fix a total order on $\mathscr{P}\mathbb{P}$. We shall refer to this order in terms of "small" and "big".

Define an *edge-type* to be a pair $\mathbb{E} = (\mathbb{C}, \mathfrak{C})$ where $\mathbb{C} \subseteq \mathbb{P}$ and $\mathfrak{C} \in [\mathscr{P}\mathbb{P}]^{\leq 2}$. Note that $\mathrm{type}_{\mu'}(\lambda')$ is an edge-type if $\lambda' \in \Sigma'$ is a line segment. Write $\mathbb{E}_0 = \mathbb{C}$ and $\mathbb{E}_1 = \mathfrak{C}$. If $\mathfrak{E}$ is a set of edge-types, define the graph $G(\mathfrak{E})$ with universe

$$\bigcup_{\mathbb{E} \in \mathfrak{E}} \mathbb{E}_1$$

and $\mathbb{C}$ sees $\mathbb{D}$ iff $\mathbb{C} \neq \mathbb{D}$ and there exists $\mathbb{E} \in \mathfrak{E}$ such that $\mathbb{E}_1 = \{\mathbb{C}, \mathbb{D}\}$. Fix the following set of edge-types:

$$\underline{\mathfrak{E}} = \mathrm{type}_{\mu'}\left[\{\lambda' \in \Sigma' : \dim \lambda' = 1\}\right].$$

Let $\mathscr{D}$ be the finest partition of the set of triangles of Σ such that triangles that share an edge are in the same partition cell. Then, for distinct $D, D' \in \mathscr{D}$, the sets $\bigcup D$ and $\bigcup D'$ can only intersect at vertices of Σ. Furthermore let

$$\mathfrak{V}(D) = \mathrm{type}_{\mu'}\left[\left(\bigcup D\right) \cap \mathrm{vtc}(\Sigma') \setminus \mathrm{vtc}(\Sigma)\right].$$

Observe that $\#\mathfrak{V}(D) \leq c \cdot 2^{c^3}$. Furthermore let

$$\mathfrak{C}(D) = \mathrm{type}_{\mu'}\left[\left\{\tau' \in \Sigma' : \dim \tau' = 2 \ \&\ \tau' \subseteq \bigcup D\right\}\right].$$

Then $\mathfrak{C}(D)$ is a nonempty connected set of nodes in the graph $G(\underline{\mathfrak{E}})$. Pick some $\mathbb{C}(D) \in \mathfrak{C}(D)$.

Let $n(0) = \lceil \log_6(c) \rceil + c^3$. Then, for each triangle $\tau \in \Sigma$, the complex $\Sigma^{+n(0)}$ has at least $c \cdot 2^{c^3}$ triangles that are contained in τ, since every triangle splits into six parts on every barycentric subdivision. Letting $n(1) = n(0) + 1$, it follows that the complex $\Sigma^{+n(1)}$ has at least $c \cdot 2^{c^3}$ vertices in $\mathrm{relInt}\,\tau$, for every triangle $\tau \in \Sigma$. Hence for each $D \in \mathscr{D}$ there is an injection $f_D : \mathfrak{V}(D) \to \left(\bigcup D\right) \cap \mathrm{vtc}(\Sigma^{+n(1)}) \setminus \mathrm{vtc}(\Sigma)$.

For each $\mathbf{x} \in \mathrm{vtc}(\Sigma^{+n(1)})$, let $\mathscr{C}(\mathbf{x})$ be the set of connected components of the poset $\Uparrow^{\Sigma^{+n(1)}}(\mathbf{x})$. Let $n(2) = n(1) + 2$. For each $C \in \mathscr{C}(\mathbf{x})$, let $[C, \mathbf{x}]$ be the set of triangles $\tau \in \Sigma^{+n(2)}$ for which $\mathbf{x} \in \tau \subseteq \bigcup C$. Then for all $\mathbf{x}^0, \mathbf{x}^1 \in \mathrm{vtc}(\Sigma^{+n(1)})$ and $C^i \in \mathscr{C}(\mathbf{x}^i)$ we have

$$\left(\bigcup [C^0, \mathbf{x}^0]\right) \cap \left(\bigcup [C^1, \mathbf{x}^1]\right) = \begin{cases} \bigcup [C^0, \mathbf{x}^0] & (\mathbf{x}^0 = \mathbf{x}^1 \text{ and } C^0 = C^1) \\ \{\mathbf{x}^0\} & (\mathbf{x}^0 = \mathbf{x}^1 \text{ and } C^0 \neq C^1) \\ \varnothing & (\mathbf{x}^0 \neq \mathbf{x}^1). \end{cases}$$

For each $C \in \mathscr{C}(\mathbf{x})$ with $[C, \mathbf{x}] \neq \varnothing$, choose some $\tau(\mathbf{x}, C) \in [C, \mathbf{x}]$. See Figure 7.3 for the situation in three triangles of $\Sigma^{+n(1)}$. Let $n(3) = n(2) + \lceil \log_2(c^4 - c + 2) \rceil + 1$. By Lemma 7.9 (and Lemma 2.51-2) there exists a partition
$\mathscr{T}(\mathbf{x}, C) = \{T_0(\mathbf{x}, C), \ldots, T_{c^4 - c + 1}(\mathbf{x}, C)\}$ of the set of triangles in $\Sigma^{+n(3)}$ lying in $\tau(\mathbf{x}, C)$ such that

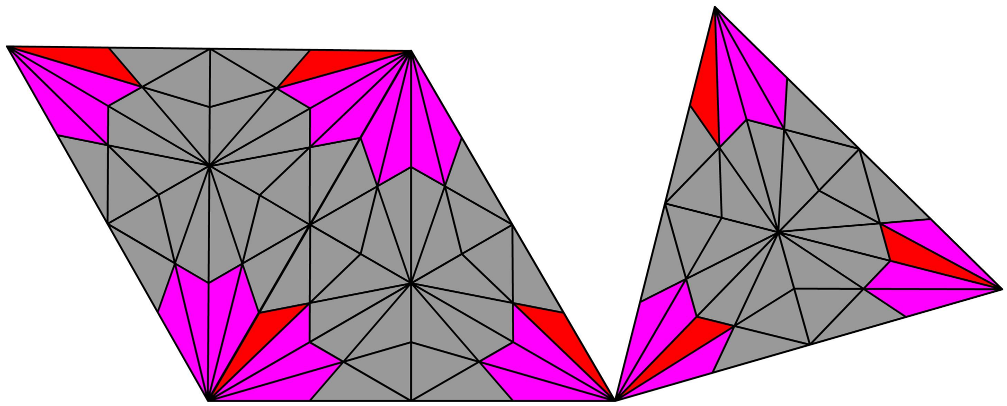


Figure 7.3: sketch of the various $[C,\mathbf{x}]$ (pink, red) and $\tau(\mathbf{x},C)$ (red)

- $\#\mathscr{T}(\mathbf{x},C) = c^4 - c + 2$;
- for each $T \in \mathscr{T}(\mathbf{x},C)$ there exists $\tau \in T$ with $\mathbf{x} \in \tau$;
- whenever a triangle in $T_i(\mathbf{x},C)$ and a triangle in $T_j(\mathbf{x},C)$ intersect (other than at $\mathbf{x}$), we have $j \in \{i-1, i, i+1\}$;
- all triangles in $\Sigma^{+n(3)}$ lying in $\tau(\mathbf{x},C)$ and intersecting $\partial\tau(\mathbf{x},C) \setminus \{\mathbf{x}\}$ are in $T_{c^4-c+1}(\mathbf{x},C)$.

Let $\Delta(\mathbf{x},C)$ be the subcomplex of $\Sigma^{+n(3)}$ with carrier

$$\left(\bigcup\left([C,\mathbf{x}] \setminus \{\tau(\mathbf{x},C)\}\right)\right) \cup \left(\bigcup\left(T_{c^4-c+1}(\mathbf{x},C)\right)\right).$$

Let

$$\Lambda_0(\mathbf{x},C) = \Delta(\mathbf{x},C) \cap {\downarrow^{\Sigma^{+n(3)}}}\left(T_{c^4-c}(\mathbf{x},C)\right).$$

Let $\Lambda_1(\mathbf{x},C)$ be the set of cells of $\Delta(\mathbf{x},C)$ that are not $\mathbf{x}$ and that have a successor (in $\Sigma^{+n(3)}$) which is not contained in $\bigcup[C,\mathbf{x}]$. Then $\Lambda_0(\mathbf{x},C)$ and $\Lambda_1(\mathbf{x},C)$ are subcomplexes of $\Delta(\mathbf{x},C)$. See Figure 7.4. It is easy to check that $\Lambda_0(\mathbf{x},C)$ and $\Lambda_1(\mathbf{x},C)$ are disjoint and each have dimension at most 1:

- Their carriers are disjoint because all successors of cells in $\left({\downarrow^{\Sigma^{+n(3)}}}\left(T_{c^4-c}(\mathbf{x},C)\right)\right) \setminus \{\mathbf{x}\}$ are contained in $\bigcup[C,\mathbf{x}]$ since $\left(\bigcup T_{c^4-c}\right) \setminus \{\mathbf{x}\} \subseteq \operatorname{relInt}\tau(\mathbf{x},C)$.
- We have $\dim\Lambda_0(\mathbf{x},C) \leq 1$ because triangles in $T_{c^4-c}(\mathbf{x},C)$ cannot be in $\Delta(\mathbf{x},C)$ because they lie in $\tau(\mathbf{x},C)$ and $T_{c^4-c}(\mathbf{x},C)$ is disjoint from $T_{c^4-c+1}(\mathbf{x},C)$.
- We have $\dim\Lambda_1(\mathbf{x},C) \leq 1$ because the carrier of $\Delta(\mathbf{x},C)$ is contained in $\bigcup[C,\mathbf{x}]$.

Let $k = \lceil \log_2(c^2) \rceil$ and $n = n(3) + k$. By Lemma 7.10 there exists a partition $\mathscr{T}'(\mathbf{x},C) = \left\{T'_0(\mathbf{x},C), \ldots, T'_{c^2-1}(\mathbf{x},C)\right\}$ of the set of triangles in $\Delta(\mathbf{x},C)^{+k}$ such that:

- $\#\mathscr{T}'(\mathbf{x},C) = c^2$;

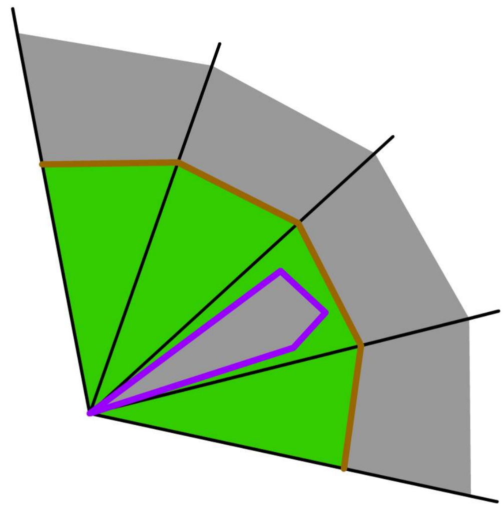


Figure 7.4: sketch of complexes $\Delta(\mathbf{x},C)$ (green), $\Lambda_0(\mathbf{x},C)$ (purple) and $\Lambda_1(\mathbf{x},C)$ (brown)

- whenever a triangle in $T_i'(\mathbf{x},C)$ and a triangle in $T_j'(\mathbf{x},C)$ intersect, we have $j \in \{i-1,i,i+1\}$;
- all triangles in $\Delta(\mathbf{x},C)^{+k}$ that intersect $\left|\Lambda_0(\mathbf{x},C)\right|$ are in $T_0'(\mathbf{x},C)$;
- all triangles in $\Delta(\mathbf{x},C)^{+k}$ that intersect $\left|\Lambda_1(\mathbf{x},C)\right|$ are in $T_{c^2-1}'(\mathbf{x},C)$.

We have $\log_2(c^4+1)+1 \leq \log_2(c^4)+2 \leq 4c+2$, so

$$n \leq c^3 + 3\cdot\big(\log_2(c^4+1)+1\big)+4 \leq c^3+12c+10 = \bar{n}.$$

Let

$$X = \mathrm{vtc}(\Sigma) \sqcup \bigsqcup_{D\in\mathscr{D}} \operatorname{Im} f_D \subseteq \mathrm{vtc}(\Sigma^{+n(1)}).$$

For $\mathbf{x} \in X$ and $C \in \mathscr{C}(\mathbf{x})$, we define a set $\mathfrak{E}(\mathbf{x},C)$ of edge-types as follows. If $\mathbf{x} \in \mathrm{vtc}(\Sigma)$, let

$$\begin{aligned}\mathfrak{E}(\mathbf{x},C) &= \left\{\mathbb{E}_0(\mathbf{x},C),\ldots,\mathbb{E}_{c^3-1}(\mathbf{x},C)\right\}\\ &= \mathrm{type}_{\mu'}\left[\left\{\lambda' \in \Sigma' : \dim\lambda' = 1 \;\&\; \mathbf{x} \in \lambda' \subseteq \bigcup C\right\}\right].\end{aligned}$$

If $\mathbf{x} = f_D(\mathbb{V})$, let $\mathfrak{E}(\mathbf{x},C) = \mathbb{V}_1$.

Observe that $G\big(\mathfrak{E}(\mathbf{x},C)\big)$ is connected in any case: if $\mathbf{x} = f_D(\mathbb{V})$, this follows from the fact that $\mathbb{V} = \mathrm{type}_{\mu'}(\mathbf{z})$ for some $\mathbf{z} \in \mathrm{vtc}(\Sigma') \setminus \mathrm{vtc}(\Sigma)$ (so that $\Uparrow^{\Sigma'}(\mathbf{z})$ is a connected poset).

For $\mathbf{x} \in X$ and $C \in \mathscr{C}(\mathbf{x})$ and $i < c^3-1$, let $\mathbb{D}_i(\mathbf{x},C,0),\ldots,\mathbb{D}_i(\mathbf{x},C,c-1)$ be a path through $G\big(\mathfrak{E}(\mathbf{x},C)\big)$ from the biggest colour in $\mathbb{E}_i(\mathbf{x},C)_1$ to the smallest colour in $\mathbb{E}_{i+1}(\mathbf{x},C)_1$. For each $i < c^3-1$ and $j < c-1$, if $\mathbb{D}_i(\mathbf{x},C,j) = \mathbb{D}_i(\mathbf{x},C,j+1)$ let $\mathbb{D}_i^1(\mathbf{x},C,j) = \mathbb{C}_i(\mathbf{x},C,j)$; otherwise find $\mathbb{E} \in \mathfrak{E}(\mathbf{x},C)$ with $\mathbb{E}_1 = \big\{\mathbb{D}_i(\mathbf{x},C,j),\mathbb{D}_i(\mathbf{x},C,j+1)\big\}$ and let $\mathbb{D}_i^1(\mathbf{x},C,j) = \mathbb{E}_0$. Let $D(C) \in \mathscr{D}$ such that $\bigcup C \subseteq \bigcup D(C)$ and note that

$$\mathbb{E}_{c^3-1}(\mathbf{x},C)_1 \subseteq \mathfrak{E}\big(D(C)\big)$$

(if $\mathbf{x} = f_D(\mathbb{V})$ for some $D \in \mathscr{D}$, this follows from the fact that $\mathbf{x} \in \bigcup D(C)$, which is true since $D = D(C)$). Hence, let $\mathbb{C}_0(\mathbf{x},C),\dots,\mathbb{C}_{c^2-1}(\mathbf{x},C)$ be a path through $G(\underline{\mathfrak{E}})$ from the biggest colour in $\mathbb{E}_{c^3-1}(\mathbf{x},C)_1$ to $\mathbb{C}\big(D(C)\big)$, that visits every colour in $\mathfrak{C}\big(D(C)\big)$. For each $i < c^2-1$, if $\mathbb{C}_i(\mathbf{x},C) = \mathbb{C}_{i+1}(\mathbf{x},C)$ let $\mathbb{C}^1_i(\mathbf{x},C) = \mathbb{C}_i(\mathbf{x},C)$; otherwise find $\mathbb{E} \in \underline{\mathfrak{E}}$ with $\mathbb{E}_1 = \big\{\mathbb{C}_i(\mathbf{x},C), \mathbb{C}_{i+1}(\mathbf{x},C)\big\}$ and let $\mathbb{C}^1_i(\mathbf{x},C) = \mathbb{E}_0$.

Let Θ be the set of cells of Σ^{+n} that have a two-dimensional successor. For each line segment $\lambda \in \Sigma \setminus \Theta$, apply Lemma 7.12 to the restriction of μ' to the subcomplex of Σ' with carrier λ, to obtain a marking μ_λ on $\mathrm{fac}(\lambda)^{+n} \subseteq \Sigma^{+n}$.

Define a marking

$$\mu \supseteq \bigcup_\lambda \mu_\lambda$$

on Σ^{+n} as follows. Let $\mu(\varnothing) = \mu'(\varnothing)$. If $\mathbf{x}$ is an isolated vertex of Σ, let $\mu(\mathbf{x}) = \mu'(\mathbf{x})$. First consider a triangle $\tau \in \Theta$. Suppose that there exists $\mathbf{x} \in X$ and $C \in \mathscr{C}(\mathbf{x})$ such that $\tau \subseteq \tau(\mathbf{x},C)$. If $\tau \subseteq \bigcup T_0(\mathbf{x},C)$ let $\mu(\tau)$ be the smallest colour in $\mathbb{E}_0(\mathbf{x},C)_1$. If $\tau \subseteq \bigcup T_{1+ic+j}(\mathbf{x},C)$ where $i < c^3-1$ and $j < c$, let $\mu(\tau) = \mathbb{D}_i(\mathbf{x},C,j)$.If $\tau \subseteq \bigcup T'_i(\mathbf{x},C)$ let $\mu(\tau) = \mathbb{C}_i(\mathbf{x},C)$. If there exist no such $\mathbf{x} \in X$ and $C \in \mathscr{C}(\mathbf{x})$, find $D \in \mathscr{D}$ with $\tau \subseteq \bigcup D$ and let $\mu(\tau) = \mathbb{C}(D)$.

Let $\theta \in \Theta \setminus \mathrm{vtc}(\Theta)$ be a vertex or edge. If there exist $\mathbf{x} \in X$ and $C \in \mathscr{C}(\mathbf{x})$ such that θ touches a triangle in $T_{ic}(\mathbf{x},C)$ and a triangle in $T_{ic+1}(\mathbf{x},C)$, let $\mu(\theta) = \mathbb{E}_i(\mathbf{x},C)_0$. If there exist $\mathbf{x} \in X$ and $C \in \mathscr{C}(\mathbf{x})$ such that θ touches a triangle in $T_{ic+j+1}(\mathbf{x},C)$ and a triangle in $T_{ic+j+2}(\mathbf{x},C)$, where $j < c-1$, let $\mu(\theta) = \mathbb{D}^1_i(\mathbf{x},C,j)$. Otherwise, if the two-dimensional successors of θ have only one value under μ, let $\mu(\theta)$ be that same value. In the remaining case, we must have that θ touches a triangle in $T'_i(\mathbf{x},C)$ and a triangle in $T'_{i+1}(\mathbf{x},C)$ for some $\mathbf{x} \in X$ and $C \in \mathscr{C}(\mathbf{x})$, and we let $\mu(\theta) = \mathbb{C}^1_i(\mathbf{x},C)$.

We extend μ to cells in $\Sigma^{+n} \setminus \Theta$ using Lemma 7.12.

Define the relation Z between Σ^{+n} and Σ' by $\sigma Z \sigma'$ iff σ satisfies the same formulas (with variables in $\mathbb{P}$) in the model (Σ^{+n},μ) as does σ' in (Σ',μ'). It follows that

$$f_D\big(\mathrm{type}_{\mu'}(\mathbf{z})\big) Z \mathbf{z}$$

whenever $D \in \mathscr{D}$ and $\mathbf{z} \in \Big(\bigcup D\Big) \cap \mathrm{vtc}(\Sigma') \setminus \mathrm{vtc}(\Sigma)$, and

$$\mathbf{x} Z \mathbf{x}$$

whenever $\mathbf{x} \in \mathrm{vtc}(\Sigma)$. It follows that Z relates every vertex of Σ' to some vertex of Σ^{+n}. Conversely, suppose that

$$\mathbf{y} \in \mathrm{vtc}(\Theta) \setminus \Big(\mathrm{vtc}(\Sigma) \cup \bigcup_{D \in \mathscr{D}} \mathrm{Im}\, f_D\Big).$$

If the triangles in Θ containing $\mathbf{y}$ have only one value $\mathbb{C}$ under μ, then also $\mu(\mathbf{y}) = \mathbb{C}$ and hence we have $\mathbf{y} Z \tau'$ for any triangle $\tau' \in \Sigma'$ with $\mu'(\tau') = \mathbb{C}$. Otherwise, there exists $\mathbb{E} \in \underline{\mathfrak{E}}$ such that $\mu(\mathbf{y}) = \mathbb{E}_0$ and $\mu\big[\uparrow^{\Sigma^{+n}}(\mathbf{y})\big] = \mathbb{E}_1 \cup \{\mathbb{E}_0\}$. □

Corollary **7.14**. *EffBound*$_\varnothing$($\mathbf{plhdr}^2$) and *EffBound*$_\varnothing$($\mathbf{plhdr}_1$).

Corollary **7.15**. VALID$_\varnothing$($\mathbf{plhdr}^2$) and VALID$_\varnothing$($\mathbf{plhdr}_1$) are decidable.

The first part of the next corollary explains the title of this section.

Corollary **7.16**. *EffBound*($\mathbf{plhdr}^3$) and *EffBound*($\mathbf{plhdr}_2$).

Corollary **7.17**. VALID($\mathbf{plhdr}^3$) and VALID($\mathbf{plhdr}_2$) are decidable.

7.3 Unbounded triangulations in $\mathbb{R}^4$

In this section, we briefly consider the question of how the notions defined in Notation 7.5 behave for larger sets $\mathfrak{p}$. We shall prove that $\mathit{Bound}_{(\varnothing)}(\mathfrak{p})$ is not always true. Hence we negatively answer the question raised by [Ada19, p. 82] of whether *Bound*(**plhdr**) holds. Since our result implies that $\mathit{EffBound}_{(\varnothing)}(\mathbf{plhdr})$ fails, it is not clear how to generalize the decidability results given in Corollaries 7.15 and 7.17. But nor is it obvious to prove undecidability of any instance of those problems. More concretely, we leave open:

Open problem **7.18**. Is EQUIV($\mathbf{plhdr}^4_3$) decidable?

Open problem **7.19**. Is VALID($\mathbf{P}$) decidable for every $\mathbf{P} \in \mathbf{plhdr}^4_3$?

Open problem **7.20**. If Problems 7.18 and 7.19 have positive answers, is even the (prima facie) harder problem $\text{VALID}_\varnothing$(**plhdr**) decidable?

In fact, we shall show that $\mathit{Bound}_\varnothing(\mathbf{plhdr}^3_2)$ fails (Theorem 7.23-2). Similarly, $\mathit{Bound}(\mathbf{plhdr}^4_3)$ fails (Theorem 7.23-1). However, this latter result is a bit stronger. To see why, suppose that $\Sigma \in \mathbf{cmplx}^4_3$, $\Delta \in \mathbf{cmplx}^3_2$ and φ is a formula that can only be satisfied at points of depth at least 3. Then to say that φ can be satisfied in Δ means that φ can be satisfied *at* $\varnothing$ in Δ, whereas to say that φ can be satisfied in $\Sigma \setminus \{\varnothing\}$ amounts to the more involved statement that there exists *some vertex* of Σ at which φ can be satisfied. For this reason, the proof of Theorem 7.23-1 will use an observation (viz. Lemma 7.22) that discards most vertices of Σ for satisfying the formula.

Lemma **7.21**. Let Σ be a complex, Σ' a subdivision of Σ and $\mathbf{x} \in \mathrm{vtc}(\Sigma') \setminus \mathrm{vtc}(\Sigma)$. Then the poset $\Uparrow^{\Sigma'}(\mathbf{x})$ has at most two components.

Proof. By Lemmas 2.56 and 2.78-1, we may assume that Σ' is an iterated barycentric subdivision of Σ. In fact it follows that $\Sigma' = \Sigma^+$ w.l.o.g. Let

$$c = \mathrm{cb}_\Sigma^{-1}(\mathbf{x}).$$

Then $c = \{\sigma\}$ for some one-or-higher-dimensional $\sigma \in \Sigma$. Now one can inspect

$$\Uparrow^{\mathscr{C}(\Sigma)}(c)$$

and conclude that this poset has exactly two components if $\dim \sigma = 1$ and σ is maximal in Σ, and exactly one component otherwise. (This is a technique from [Ada19, chapter 3].) □

Lemma **7.22**. Let Σ be a triangulation of some standard bipyramid. Let $\mathbf{x} \in \mathrm{vtc}(\Sigma) \setminus \{\mathbf{e}, -\mathbf{e}\}$. Then the poset $\Uparrow^{\Sigma}(\mathbf{x})$ has at most two components.

Proof. Find a polyhedron $\mathbf{P}$ such that $|\Sigma| = \blacktriangleleft\mathbf{P}\blacktriangleright$. By Lemma 5.15 there exists a PL-homeomorphism $\phi : \blacktriangleleft\mathbf{P}\blacktriangleright \to \blacktriangleleft\mathbf{P}\blacktriangleright$ such that $\mathbf{y} = \phi^{-1}(\mathbf{x}) \notin \mathrm{vtc}(\Sigma)$. Find a triangulation Δ of $\blacktriangleleft\mathbf{P}\blacktriangleright$ such that ϕ is affine on each cell of Δ. By Lemma 2.55-1, $\Theta = \phi[\![\Delta]\!]$ is a triangulation of $\blacktriangleleft\mathbf{P}\blacktriangleright$. By Lemma 2.31, let Θ' be a common subdivision of Θ and Σ. Then $\Delta' = \phi^{-1}[\![\Theta']\!]$ is a triangulation of $\blacktriangleleft\mathbf{P}\blacktriangleright$

by Lemma 2.55-1. Indeed Δ' is a subdivision of Δ. Let Δ'' be a common subdivision of Δ' and Σ. Then the poset $\Uparrow^{\Delta''}(\mathbf{y})$ has at most two components by Lemma 7.21. Letting $\Theta'' = \phi[\![\Delta'']\!]$, this poset is isomorphic to $\Uparrow^{\Theta''}(\mathbf{x})$. Hence by Proposition 5.19, the polyhedron $\left|\mathrm{link}(\Theta'',\mathbf{x})\right|$ has at most two connected components. By Lemma 5.27-1, it follows that also $\left|\mathrm{link}(\Sigma,\mathbf{x})\right|$ has at most two components. The desired result follows from Lemma 3.13 and Proposition 5.19. □

We are now ready for the main result of this section. We shall identify a specific formula, namely $\chi(\Delta)$, and a list $\Sigma_6, \Sigma_9, \Sigma_{12}, \ldots$ of simplicial complexes such that

$$\chi(\Delta) \notin \mathrm{Log}_{\varnothing}\left(|\Sigma_k|\right)$$

but, if $n(k)$ is the smallest natural number for which $\chi(\Delta) \notin \mathrm{Log}(\Sigma_k^{+n(k)})$ then

$$\sup\left\{n(6), n(9), n(12), \ldots\right\} = \infty.$$

This means that the property expressed by $\chi(\Delta)$ is sufficiently complex that it cannot be translated in terms of some fixed amount of iterations of the barycentric subdivision. Hence one could say that the property expressed by $\chi(\Delta)$ concerns arbitrarily fine triangulations. Theorem 7.23-1 also explains the title of this section.

Theorem **7.23**.

1. Not $\mathit{Bound}(\mathbf{plhdr}_3^4)$.
2. Not $\mathit{Bound}_{\varnothing}(\mathbf{plhdr}_2^3)$.

Proof. Let Δ be the simplicial complex depicted in Figure 7.5. Notice that this complex cannot exist in $\mathbb{R}^2$. For example, the line segment $\lambda = \mathbf{y}^3 \underline{\vee} \mathbf{y}^5 \in \Delta$ has three distinct proper successors $\lambda \underline{\vee} \mathbf{y}^6$, $\lambda \underline{\vee} \mathbf{y}^9$ and $\lambda \underline{\vee} \mathbf{y}^{14}$. If $k \geq 6$ is a multiple of 3, let Σ_k be the simplicial complex depicted in Figure 7.6. For each $i < k$ and $j \in \{3,4,5\}$ with $i \cong j \pmod 3$ we have $\mathbf{u}^j \vee \mathbf{z}^i \in \Sigma_k$. We drew these with dashed lines to indicate that they are "behind" each other; for example, $\mathbf{u}^0 \underline{\vee} \mathbf{z}^3$ does not intersect $\mathbf{z}^1 \underline{\vee} \mathbf{u}^4$.

Then one can check that Σ_k has a subdivision Γ_k such that there is a surjective p-morphism $g\Gamma_k \to \Delta$. Cf. Figure 7.7. (The vertices $\mathbf{x}^0$ and $\mathbf{x}^k$ may seem problematic, but we handle them by letting $g(\mathbf{x}^0) = \mathbf{y}^3 \underline{\vee} \mathbf{y}^5$ and $g(\mathbf{x}^k) = \mathbf{y}^4 \underline{\vee} \mathbf{y}^5$ and adjusting the values of successors of $\mathbf{x}^0$ and $\mathbf{x}^k$ under g accordingly.)

However, we claim that there is no surjective p-morphism $\Sigma_k^{+n} \to \Delta$ if $n < \log_2(k) - 1$. For, suppose that Γ is a subdivision of Σ_k such that there is a surjective p-morphism $f : \Gamma \to \Delta$. Observe that $\partial(\mathbf{y}^3 \underline{\vee} \mathbf{y}^4 \underline{\vee} \mathbf{y}^5)$ separates $|\Delta|$ into seven components. All cells of Σ_k lying in $\mathbf{x}^0 \underline{\vee} \mathbf{x}^k \underline{\vee} \mathbf{u}^6$ but not lying in $\mathbf{x}^0 \underline{\vee} \mathbf{x}^k$, must be mapped to cells in a single of these components by f. That must be the coloured component

$$\mathrm{relInt}(\mathbf{y}^3 \underline{\vee} \mathbf{y}^4 \underline{\vee} \mathbf{y}^5) \setminus \mathrm{relInt}(\mathbf{y}^{12} \underline{\vee} \mathbf{y}^{13} \underline{\vee} \mathbf{y}^{14}).$$

Hence every triangle $\tau \in \Gamma$ for which $\tau \subseteq \mathbf{x}^0 \underline{\vee} \mathbf{x}^k \underline{\vee} \mathbf{u}^6$, is mapped by f to one of the six coloured triangles in Δ. In particular, every such triangle in Γ inherits a colour (red, green or blue) in this way. We must have $f(\mathbf{u}^0) = \mathbf{y}^0$, $f(\mathbf{u}^1) = \mathbf{y}^1$ and $f(\mathbf{u}^2) = \mathbf{y}^2$. From this, it is easy to derive for $j \in \{3,4,5\}$ that $f(\mathbf{u}^j) \in \mathrm{star}(\Delta, \mathbf{y}^{j-3}) \setminus \{\mathbf{y}^{j+3}\}$. Hence, if $i < k$ and $i \cong j \pmod 3$, then also

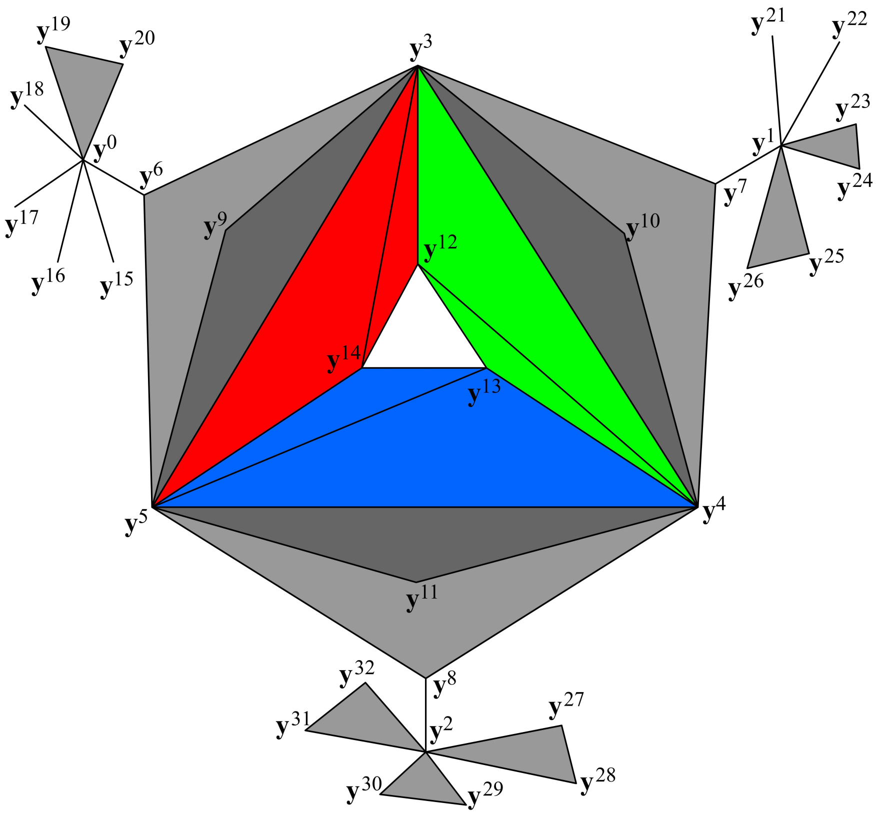


Figure 7.5: the complex Δ

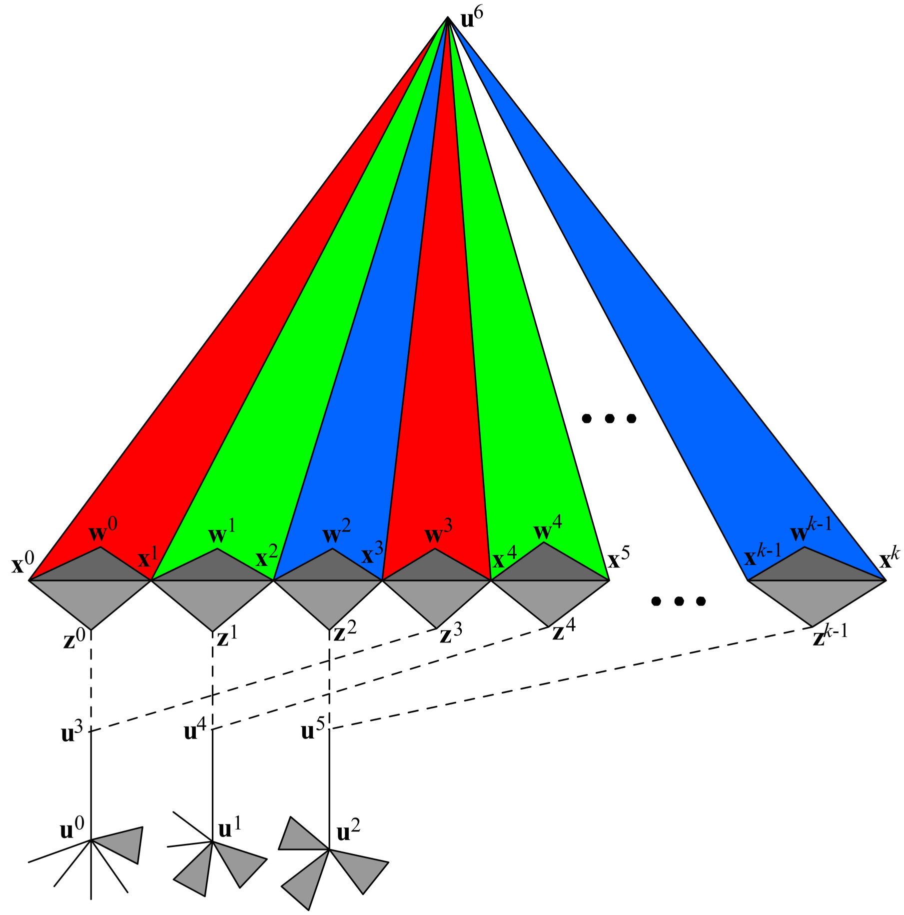


Figure 7.6: the complex Σ_k

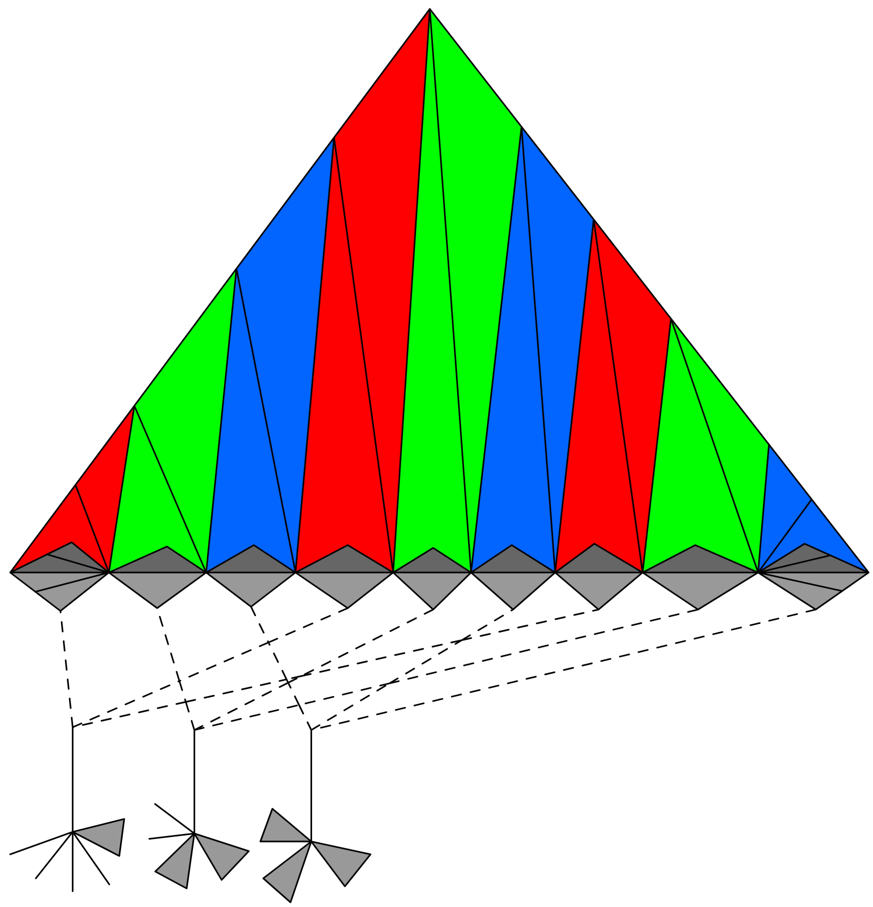


Figure 7.7: suggestion for a p-morphism from a subdivision of Σ_9 to Δ

$f(\mathbf{z}^i) \in \mathrm{star}(\Delta, \mathbf{y}^{j-3})$. Since f is monotone and maps all cells "above" $\mathbf{x}^0 \,\underline{\vee}\, \mathbf{x}^k$ to coloured triangles in Δ, it follows that

$$f\Big[\mathrm{sub}(\Gamma, \Sigma_k)^{-1}\big[\{\mathbf{x}^i \,\underline{\vee}\, \mathbf{x}^{i+1}\}\big]\Big] = \begin{cases} \{\mathbf{y}^5 \,\underline{\vee}\, \mathbf{y}^3\} & (i \cong 0 \,(\mathrm{mod}\ 3)) \\ \{\mathbf{y}^3 \,\underline{\vee}\, \mathbf{y}^4\} & (i \cong 1 \,(\mathrm{mod}\ 3)) \\ \{\mathbf{y}^4 \,\underline{\vee}\, \mathbf{y}^5\} & (i \cong 2 \,(\mathrm{mod}\ 3)). \end{cases}$$

Define a relation A on the set of coloured triangles in Γ by $\tau_0 A \tau_1$ iff τ_0 and τ_1 share an edge and have the same color. Let $\sim$ be the equivalence relation generated by A. For a fixed $i < k$, all colored triangles that have an edge in $\mathbf{x}^i \,\underline{\vee}\, \mathbf{x}^{i+1}$ are related by $\sim$. Call E_i the respective equivalence class. However, assuming $0 < i < k-1$, triangles in E_{i-1} have a different color than triangles in E_{i+1}. Thinking of E_{i-1} as an area $\bigcup E_i$, this means that this area has neighbouring areas in different colours. Since no vertex of Γ can have three successors that are triangles in all the colors red, green and blue, it follows that $\bigcup E_i$ must intersect $T := (\mathbf{x}^0 \,\underline{\vee}\, \mathbf{u}^6) \cup (\mathbf{u}^6 \,\underline{\vee}\, \mathbf{x}^k)$. Hence Γ has at least $k+1$ vertices in T. By Lemma 2.51-2, the complex Σ_k^{+n} has $2^{n+1}+1$ vertices in T. If $n < \log_2(k) - 1$, then $2^{n+1}+1 < k+1$, so $\Gamma \neq \Sigma_k^{+n}$. This proves part 2.

Next, to prove 1, let $\mathbf{P}_k = |\Sigma_k| \in \mathbf{plhdr}_2^3$, so that $\blacktriangleleft\mathbf{P}_k\blacktriangleright \in \mathbf{plhdr}_3^4$. We have $\chi(\Delta) \notin \mathrm{Log}_\varnothing(\mathbf{P}_k) \supseteq \mathrm{Log}(\blacktriangleleft\mathbf{P}_k\blacktriangleright)$ by Theorem 5.38. However, if $n < \log_2(k) - 1$, we claim that $\chi(\Delta)$ is true in $(\blacktriangleleft\Sigma_k\blacktriangleright)^{+n} \setminus \{\varnothing\}$. To check this, let $\mathbf{x} \in \mathrm{vtc}\big((\blacktriangleleft\Sigma_k\blacktriangleright)^{+n}\big)$. If $\mathbf{x} \in \{\mathbf{e}, -\mathbf{e}\}$, then $\uparrow(\mathbf{x}) \cong \Sigma_k^{+n}$ by Proposition 5.19, Lemma 5.24 and Remark 5.25. Hence what we showed in the previous paragraph implies that $\chi(\Delta)$ holds in $\uparrow(\mathbf{x})$. Next we suppose that $\mathbf{x} \notin \{\mathbf{e}, -\mathbf{e}\}$. Then by Lemma 7.22, $\chi(\Delta)$ is again true in $\uparrow(\mathbf{x})$. □

Let VALID$_{\chi(\Delta)}$ be the problem: given a polyhedron $\mathbf{P}$, do we have $\chi(\Delta) \in \mathrm{Log}_\varnothing(\mathbf{P})$? The proof of Theorem 7.23 shows that, in order to solve VALID$_{\chi(\Delta)}$, one has to do something cleverer than taking an arbitrary triangulation Σ of $\mathbf{P}$ and checking the truth of $\chi(\Delta)$ in Σ^{+n} for some constant n. In fact, proving (un)decidability of VALID$_{\chi(\Delta)}$ might be a first step towards solving Open problem 7.20. This is of particular importance because, if VALID$_\varnothing(\mathbf{plhdr})$ is decidable, then every problem defined in this chapter is decidable.

Chapter 8

Conclusions and future work

We defined two forms of polyhedral semantics for modal logic, one of which is novel (Definition 2.69), and investigated various of their properties.

We have seen that some phenomena in polyhedral semantics are much better behaved than others. This goes for the type of problem as well as the class of polyhedra under consideration. In chapter 3, we saw that the logic $\mathrm{Log}(\mathbf{P})$ of a manifold-with-boundary $\mathbf{P} \in \mathbf{plhdr}$ is readily determined by the dimension of **P** (Theorem 3.20). In contrast, in chapter 6 we constructed two-dimensional polyhedra $|\Lambda_1|\blacktriangleright, |\Lambda_2|\blacktriangleright, \ldots$ that formed an "antichain" with respect to p-morphisms between any triangulations. This gave rise to $2^{\aleph_0}$ polyhedrally-complete logics (Theorem 6.32).

The theoretical setting of polyhedral semantics opted for in this work, is all about fine subdivisions of simplicial complexes. We reinforced this view by proving that p-morphisms can be transferred along the process of subdividing complexes (Theorem 4.1). However, we did not quantitatively investigate how fine of a subdivision is required in order to recreate a p-morphism (cf. Discussion 4.3). This may be an interesting starting point for nontrivial future work.

Furthermore, in the last chapter we showed that fine subdivisions may indeed be required even if one restricts attention to a single formula of modal logic (Theorem 7.23). We indicated how this highlights some computational challenges for polyhedral semantics which future research may be able to tackle (cf. Open problems 7.18, 7.19 and 7.20). In particular, validity checking is decidable in three-dimensional space (Corollary 7.17), but the algorithm does not work in four-dimensional space.

In chapter 6 we showed how to do polyhedral semantics with graphs. Using pyramids (as in Theorem 5.38), we showed that this approach does not only apply to the new polyhedral semantics but also the traditional polyhedral semantics. It may be worthwhile to further investigate which graph-theoretical properties can be expressed by polyhedral semantics, and to see which results from graph theory can teach us something about polyhedral semantics (like we used Kruskal's theorem to prove Theorem 6.27).

More generally, one can take any class $\mathfrak{p}$ of polyhedra and wonder if there exists a nice axiomatization for $\mathrm{Log}(\mathfrak{p})$ or $\mathrm{Log}_{\varnothing}(\mathfrak{p})$. It is apparent from this work that properties of $\mathfrak{p}$ are often reflected by properties of these logics, and vice versa. Hence, it is imaginable that one day polyhedral semantics would in turn help us to prove results in polyhedral geometry.

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