Possible Worlds Semantics
for Classical Logic

december 1981

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1 Intuitionistic versus Classical Logic

The common conversational ploy 'that classical logic is more *elegant* than intuitionistic logic' is losing its force. From various points of view (notably that of natural deduction), it is rather intuitionistic logic which has the advantage. From another angle, it was noted in van Benthem (1980) how a Henkin completeness proof for intuitionistic logic involves the natural totality of all consistent theories, whereas the classical proof has to pass on to an indeterminate maximally consistent theory. This same problem of the 'irrelevant extension' also occurs in classical applications of ultraproducts. Thus, e.g., non-standard analysis does not produce one structure of extended reals: one thinks of 'any' ultrapower of $\mathbb{R}$, often without using more than the filter structure of the set of tails on the index set $\mathbb{N}$.

Again, this situation may be improved by passing on to 'intuitionistic' models. Suppose we have a family of classical models $\{M_i \mid i \in I\}$, with a filter F on I. For the usual Łoś Equivalence, we need an ultrafilter $F^+ \supseteq F$, in order to show

$$\prod_{F^+} M_i \models \varphi\ [(\vec{f})_{F^+}] \quad \text{iff} \quad \{i \in I \mid M_i \models \varphi\ [\vec{f}(i)]\} \in F^+$$

And we need such an equivalence to show that the product still behaves like (many of) its components. Now, by changing over to an 'intuitionistic product', we can have this same benefit for a model which is a unique mathematical object fixed by F. To see this, consider the family of reduced products $\prod_G M_i$, where each G is a filter on I containing F. There exist natural morphisms between these, as follows

$$\pi_{GG'}\ ((d_i)_i^G) := (d_i)_i^{G'} \qquad , \text{ for } G \subseteq G'.$$

Just by way of example, consider a first-order language with $\neg$ (*not*), $\wedge$ (*and*), $\exists$ (*there exists*). In the above multiple structure of reduced products, one now gives negation its intuitionist reading:

$\neg\varphi$ is true for $\overrightarrow{(d_i)_i}^G$ in $\Pi_F M_i$ if

φ is false for each $\overrightarrow{(d_i)_i}^{G'}$ in $\Pi_F M_i$ with $G' \supseteq G$

Then, a simple induction on the construction of formulas establishes

$$\Pi_F M_i \models \varphi\, [\overrightarrow{(d_i)_i}^G] \quad \text{iff} \quad \{i \in I \mid M_i \models \varphi\, [\vec{d_i}]\} \in G.$$

Thus, an intuitionistic point of view in classical model theory need not consist in mere imitation, or generalization of classical techniques: it may be an improvement upon these.

We shall investigate this point of view, although – perhaps surprisingly – retaining a classical logic in the studied first-order language. Why this is so will become clear in the next section.

2 Possible Worlds Semantics for Intuitionistic and Classical Logic

Kripke models for intuitionistic logic are partially ordered sets of worlds, each with a domain growing in the direction of the ordering relation. Thus, we have tuples

$$M = \langle W, \sqsubseteq, D, I \rangle,$$

where, for each $w \in W$, $\langle D_w, I_w \rangle$ is an ordinary structure, such that

- $w \sqsubseteq v$ only if $D_w \subseteq D_v$
- $w \sqsubseteq v$ only if, if $I_w(P)(\vec{d})$, then $I_v(P)(\vec{d})$, for all predicate letters P, and $\vec{d} \in D_w$

The truth definition then defines the notion

$M \models \varphi\, [w, \vec{d}\,]$ ('φ is true in M at w for $\vec{d}$ ($\in D_w$)')

through the clauses

$M \models P\vec{x}\, [w, \vec{d}\,]$	iff	$I_w(P)(\vec{d})$
$M \not\models \bot\, [w, \vec{d}\,]$		('the Falsum')
$M \models \varphi \to \psi\, [w, \vec{d}\,]$	iff	for all $v \sqsupseteq w$, if $M \models \varphi\, [v, \vec{d}\,]$, then $M \models \psi\, [v, \vec{d}\,]$
$M \models \varphi \wedge \psi\, [w, \vec{d}\,]$	iff	$M \models \varphi\, [w, \vec{d}\,]$ and $M \models \psi\, [w, \vec{d}\,]$
$M \models \varphi \vee \psi\, [w, \vec{d}\,]$	iff	$M \models \varphi\, [w, \vec{d}\,]$ or $M \models \psi\, [w, \vec{d}\,]$
$M \models \forall x\, \varphi(x)\, [w, \vec{d}\,]$	iff	for all $v \sqsupseteq w$, for all $d \in D_v$,

$$M \models \varphi(x)\,[v,\vec{d},d]$$

$M \models \exists x\varphi(x)\,[w,\vec{d}]$ iff there exists $d \in D_w$ such that $M \models \varphi(x)\,[w,\vec{d},d]$.

Negation $\neg\varphi$ is defined in the usual way as $\varphi \to \bot$.

It will be clear that some aspects of this definition can be varied: notably, a suitable map from D_w into D_v (when $w \sqsubseteq v$) will serve just as well as actual inclusion. (After all, the intuitive picture behind the above semantics, which is that of increasing 'stages of knowledge' may allow for *identification* of old objects, besides construction of new ones.)

The effect of this truth definition is 'cumulation of knowledge'.

Heredity: if $M \models \varphi\,[w,\vec{d}]$ and $w \sqsubseteq v$, then $M \models \varphi\,[v,\vec{d}]$, for all formulas φ.

At a superficial level, all clauses in the above are 'classical', but for the cases $\to$, $\forall$. This is rather surprising: one would expect the typical 'constructive' aspect of intuitionism to reside in the treatment of $\vee$ and $\exists$. But, as was pointed out by Kit Fine, such comparisons need a proper setting for classical logic first. And the possible worlds clauses for *classical* $\vee, \exists$ will be rather different: not immediate choice, but 'eventual choice':

$M \models \varphi \text{ OR } \psi\,[w,\vec{d}]$ iff for all $v \sqsupseteq w$ there exists $u \sqsupseteq v$ with $M \models \varphi[u,\vec{d}]$ or $M \models \psi\,[u,\vec{d}]$,

and likewise for the existential quantifier. (Classical logic is in less of a hurry than intuitionistic logic.)

Now, such clauses by themselves do not make the logic classical at once. For, e.g., implication remains unaffected — and it is a well-known fact that intuitionistic implication does not satisfy the classical 'Law of Peirce'

$$((\varphi \to \psi) \to \varphi) \to \varphi$$

But, the new clauses suggest a new feature of 'classical knowledge': 'eventual truth implies actual truth' (or equivalently, 'actual non-truth implies possible refutation'):

Cofinality: if for all $v \sqsupseteq w$ there exists $u \sqsupseteq v$ such that $M \models \varphi[u, \vec{d}]$, then $M \models \varphi[w, \vec{d}]$, for all formulas φ.

The effect of this condition (when valid) is to verify the Double Negation law

$\neg\neg\varphi \to \varphi$;

and the validity of Peirce's Law then follows from the observation that

$((\varphi \to \psi) \to \varphi) \to \neg\neg\varphi$

is intuitionistically valid.

Thus, classical possible worlds models are structures like above, with one additional requirement:

- $\forall v \sqsupseteq w \, \exists u \sqsupseteq v \; I_u(P)(\vec{d})$ only if $I_w(P)(\vec{d})$, for all P and $\vec{d} \in D_w$.

The above clauses for $\bot, \to, \wedge$ and $\forall$ are retained.

Disjunction will be defined as follows

$\varphi \vee \psi =_{def} \neg(\neg\varphi \wedge \neg\psi)$,

and existential quantification through

$\exists x \, \varphi(x) =_{def} \neg\forall x \neg\varphi(x)$.

2.1 __Lemma__: Heredity holds for all formulas.

2.2 __Lemma__: Cofinality holds for all formulas.

__Proof__: Induction.

Case $\wedge$: $\neg\neg(\varphi \wedge \psi)$ implies $\neg\neg\varphi \wedge \neg\neg\psi$.

Case $\to$: $\neg\neg(\varphi \to \psi)$ implies $\neg\neg\varphi \to \neg\neg\psi$, and hence $\varphi \to \neg\neg\psi$ (by heredity and reflexivity) and hence $\varphi \to \psi$ (by the inductive hypothesis for ψ)

Case $\forall$: Let $d \in D_v$, $v \sqsupseteq w$. We want to show that $\varphi(x)(\vec{d}, d)$ at v, assuming that $\neg\neg\forall x \, \varphi(x)(\vec{d})$ at w. By the inductive hypothesis, it suffices to prove that $\neg\neg\varphi(x)(\vec{d}, d)$. But this follows because $\neg\neg\forall x \, \varphi(x)(\vec{d})$ holds at v as well, by heredity. □

2.3 __Corollary__: The definitions of $\vee, \exists$ work out according to the cofinality clauses presented on p.3.

The usual methods of generic branches establish a connection between these possible worlds models and old-fashioned classical models. Let M be a classical possible worlds model.

2.4 Definition. A *generic branch* is a —— $\sqsubseteq$-chain of worlds such that, for each formula φ and each $\vec{d}$ occurring along the chain,

- either $\varphi(\vec{d})$ at some world on the branch, or else $\neg\varphi(\vec{d})$,
- if $\neg\forall x\,(\varphi(x)(\vec{d})$ at some world on the branch, then $\neg\varphi(x)(\vec{d},d)$ for some d at some world on the branch.

Each (generic) branch β induces a single classical model M_β, viz. the *union* of this chain.

2.5 Theorem: $M_\beta \models \varphi[\vec{d}]$ iff $M \models \varphi[w,\vec{d}]$ for some $w \in \beta$.

Proof: The definition of genericity has prepared the way for an obvious induction on φ. $\square$

2.6 Corollary. $M \models \varphi[w,\vec{d}]$ iff $M_\beta \models \varphi[\vec{d}]$ for all generic $\beta \ni w$.

Proof: This inversion of theorem 2.5 is immediate:

'Only if': directly from 2.5,

'If': if $M \not\models \varphi[w,\vec{d}]$, then, by cofinality, $M \models \neg\varphi[v,\vec{d}]$ for some $v \sqsupseteq w$, and any generic β through w,v falsifies φ. $\square$

That there is an abundant supply of generic branches is guaranteed by the usual enumeration method, in combination with the truth clauses for negation and universal quantification.

For each model M, one may define its *generic extension* $GE(M)$, adding classical generic end points following generic branches.

2.7 Theorem: For all φ, $w \in W$,

$$GE(M) \models \varphi[w,\vec{d}] \quad \text{iff} \quad M \models \varphi[w,\vec{d}]$$

Thus, generic extensions preserve truth.

3 The Tree of Knowledge

The mother model of possible worlds semantics is the *Henkin model* of all consistent sets of formulas in the language, representing all expressible states of knowledge. Maybe the most natural example is the following: take all *finitely axiomatized consistent* sets (with respect to classical derivability), ordered by ordinary *inclusion*. The earlier truth clauses correspond precisely to the following decomposition properties:

$$\Sigma \vdash \varphi \to \psi \quad \text{iff} \quad \forall \Sigma' \supseteq \Sigma, \text{ if } \Sigma' \vdash \varphi \text{ then } \Sigma' \vdash \psi$$

$$\Sigma \vdash \varphi \wedge \psi \quad \text{iff} \quad \Sigma \vdash \varphi \text{ and } \Sigma \vdash \psi$$

$$\Sigma \vdash \forall x\, \varphi(x) \quad \text{iff} \quad \forall \Sigma' \supseteq \Sigma\ \forall \text{constants } c\ \ \Sigma' \vdash \varphi(c).$$

(We are presupposing a language with a countable infinity of individual constants here.) $\vee, \exists$ do not decompose in this way; but, e.g.,

$$\Sigma \vdash \varphi \vee \psi \quad \text{iff} \quad \forall \Sigma' \supseteq \Sigma\ \exists \Sigma'' \supseteq \Sigma'\ \ \Sigma'' \vdash \varphi \text{ or } \Sigma'' \vdash \psi.$$

The observations embody well-known facts about classical deduction: the 'deduction theorem' is behind that concerning $\to$, the 'constant lemma' behind that for $\forall$.

Notice that in the intuitionistic case, just consistent sets will not do: constructive disjunction requires instantaneous decomposition of $\vee$, whence intuitionistic Henkin completeness proofs require 'splitting' sets.

The interest of the Henkin model does not stop here. One feature is its *inclusion structure*, which is much richer than a mere partial ordering. E.g., there is a lattice structure for meet and 'almost' for join. (Not every deductive union yields a consistent set; but one might easily allow some 'exploding' inconsistent world to stream-line the structure.) For instance, one might add the existence of greatest lower bounds to the requirements on classical models; which would facilitate some of the developments to follow. In fact, an intriguing question is the following

Precisely what is the first-order inclusion theory of the Henkin model?

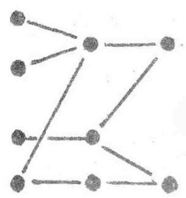
If it is recursively axiomatizable, then we have an effective means of mechanically producing logical meta-theorems.

Here is one partial answer.

3.1 <u>Theorem</u>: The universal first-order theory of the Henkin model with inclusion is axiomatized by the partial order axioms.

<u>Proof</u>: Each purely universal non-theorem of these axioms is refutable in some *finite* partial order (by the completeness theorem, and an obvious preservation result). And each finite partial order can be isomorphically embedded into the above Henkin model.

Instead of a full proof, here is an example.



goes to

$$\begin{array}{lll} \{p_1\} & & \\ & \{p_1, p_2, p_4\} & \{p_1, p_2, p_3, p_4, p_5\} \\ \{p_2\} & & \\ \{p_3\} & \{p_3, p_5\} & \\ \{p_4\} & \{p_4, p_6\} & \{p_3, p_4, p_5, p_6\} \end{array}$$

$\square$

How characteristic is the Henkin model for *all* classical models of some theory? In a global sense, the connection is this:

each branch (generic or not) induces a classical model,

each model determines a branch, through its first-order theory.

But there exist more intimate connections.

3.2 <u>Theorem</u>: For each φ, and each model M,

$M \models \varphi$ iff M contains some countable elementary substructure M' which is a branch model for some generic $\beta \ni \{\varphi\}$.

Two words of explanation:

'elementary' with respect to the language of φ (leaving infinitely many further individual constants in the language of the Henkin model),

'branch model' in the sense that the elements in the domain are interpretations of the constants occurring on the branch.

<u>Proof</u>:

'If': φ holds in such branch models M' (by theorem 2.5), and hence in M.

'Only if': starting with $\{\varphi\}$, one constructs a generic branch β guided by what is true in M. Thus, along the enumeration $\varphi_1, \varphi_2, \ldots$

φ_γ is chosen when true, while - in case $\varphi_\gamma = \neg\forall x\, \psi(x)$ - a suitable new individual constant c is interpreted as some individual in M falsifying $\psi(x)$. The totality of all individuals in M chosen in this way forms the required countable submodel M'. That it is elementary follows from the fact that both M and M' verify the same formulas on the branch. $\square$

This tells us something about the connection between the Henkin model and ordinary classical models. But, the former may also have an important position among all possible worlds models for classical logic.
In its 'full' formulation with all theories, finitely axiomatized or not, the Henkin model has various saturation properties which make it universal among saturated possible worlds models. (This may be proven in a manner analogous to Fine (1975), which treats the case of modal logic.)

4 Fundamental Operations on Possible Worlds Models

The clauses of the truth definition only refer to other worlds which are $\sqsubseteq$-successors of those at which evaluation takes place. This remark inspires the following notion, well-known from modal logic.

4.1 Definition: M' is a *generated submodel* of M (notation: $M' \subseteq\!\!\!\rightarrow M$) if it is a submodel satisfying the additional conditions

- for $w \in W'$, $D'_w = D_w$, ('same domains')
- for $w \in W'$, $v \in W$, $w \sqsubseteq v$ only if $v \in W'$ ('$\sqsubseteq$-closure')

The pertinent semantic fact is the so-called 'generation theorem':

4.2 Lemma: For all formulas φ, $w \in W'$, $\vec{d} \in D'_w$,

$M' \models \varphi[w, \vec{d}]$ iff $M \models \varphi[w, \vec{d}]$

The term 'generation' derives from the fact that each w generates a smallest generated submodel (M, w), with domain $\{v \in W \mid w \sqsubseteq v\}$.

4.3 Corollary: A *disjoint union* of models $\{M_\gamma \mid \gamma \in I\}$ verifies a formula $\varphi(\vec{d})$ at $w \in W_\gamma$ iff M_γ verifies $\varphi(\vec{d})$ at w.

Proof: Each single M_γ lies as a generated submodel in such a disjoint union. $\square$

Another application that is easily visualized is 'rooting': take any family of models (M_i, w_i), made disjoint in some fashion, but having a non-empty intersection $\bigcap_{i \in I} D_{w_i}$. Add one new world w with precisely this domain, connecting it to all w_i. Finally, set $I_w(P)(\vec{d})$ iff $I_{w_i}(P)(\vec{d})$ for all $i \in I$. One easily sees that the models (M_i, w_i) remain generated, preserving the same formulas. Moreover, a routine induction shows that

4.4 <u>Lemma:</u> The root w verifies exactly those formulas $\varphi(\vec{d})$ which are true in all worlds w_i $(i \in I)$.

It follows straightaway that rooting preserves Heredity and Cofinality.

Another relation between models which has proven useful in modal logic is that of a 'p-morphism' or 'p-relation' (cf. Van Benthem (1982)). This is a kind of zigzagging connection between $\sqsubseteq$-successive worlds, which may be extended to the predicate-logical case by adding a Fraissé-type zigzagging idea for individuals:

4.5 <u>Definition:</u> M is *zigzag-connected* with M' (notation: $M \rightleftarrows M'$) through C if C is a binary relation between sequences of the form $\langle w, \vec{d} \rangle$ (w a world, $\vec{d}$ a finite sequence from D_w) such that

(onto) • the domain of C consists of all such sequences from M, its range of all such sequences from M',

(zig-zag) • if $\langle w, \vec{d} \rangle \, C \, \langle v, \vec{e} \rangle$ and $w \sqsubseteq w', d \in D_{w'}$, then there exist $v' \sqsupseteq v$, $e \in D_{v'}$ such that $\langle v, \vec{d}, d \rangle \, C \, \langle v', \vec{e}, e \rangle$, and conversely. (The special case *without* d, e is included.)

(partial isomorphism) • if $\langle w, \vec{d} \rangle \, C \, \langle v, \vec{e} \rangle$, then $I_w(P)(\vec{d})$ iff $I_v(P)(\vec{e})$ for all predicate letters P

Again, the latter equivalence may be lifted by induction on φ to prove

4.6 <u>Lemma:</u> If C is a zigzag connection between M and M', and $\langle w, \vec{d} \rangle \, C \, \langle v, \vec{e} \rangle$, then, for all formulas φ,
$M \models \varphi[w, \vec{d}]$ iff $M' \models \varphi[v, \vec{e}]$.

Again, one simple application concerns roots:

4.7 <u>Corollary</u>: Each model M is zigzag-connected to the disjoint union of all its rooted submodels (M,w) $(w \in W)$.

<u>Proof</u>: The obvious zigzag connection works. $\square$

Finally, we turn to a more complex operation, viz. the formation of *filter products*. Briefly, the idea of the following notion is that of a two-sorted ('worlds', 'individuals') family of reduced products.

4.8 <u>Definition</u>: Let $\{M_i \mid i \in I\}$ be a family of models, and F some filter on I. The *filter product* $\prod_F M_i$ is the model $\langle W, \sqsubseteq, D, I \rangle$ where ('w', 'v', 'd' now denote the appropriate *functions*)

- W consists of all indexed equivalence classes $w^G = \{ v \mid w \sim_G v \}$ where G is a filter extending F, and $w \sim_G v$ iff $\{i \in I \mid w_i = v_i\} \in G$,
- $w^G \sqsubseteq v^{G'}$ iff $G \subseteq G'$ and $\{i \in I \mid w_i \sqsubseteq_i v_i\} \in G'$,
- D assigns to w^G as its domain all equivalence classes d^G for which $\{i \in I \mid d_i \in D^i_{w_i}\} \in G$. (This is well-defined, because G is a filter.)
- $I_{w^G}(P)(\vec{d}^G)$ if and only if $\{i \in I \mid I^i_{w_i}(P)(\vec{d_i})\} \in G$.

The usual induction on φ then establishes the Łoś Equivalence.

4.9 <u>Theorem</u>:

$$\prod_F M_i \models \varphi[w^G, \vec{d}^G] \quad \text{iff} \quad \{i \in I \mid M_i \models \varphi[w_i, \vec{d_i}]\} \in G.$$

<u>Proof</u>:

φ is atomic: by definition.

φ is $\varphi_1 \wedge \varphi_2$: by the intersection property of filters.

φ is $\varphi_1 \rightarrow \varphi_2$:

'If': Suppose that φ_1 holds at $v^{G'} \sqsupseteq w^G$. Then G' contains $\{i \in I \mid w_i \sqsubseteq_i v_i\}$, $\{i \in I \mid M_i \models \varphi_1 \rightarrow \varphi_2 [w_i, \vec{d_i}]\}$ (by the assumption, plus $G \subseteq G'$) as well as $\{i \in I \mid M_i \models \varphi_1 [v_i, \vec{d_i}]\}$ (by the inductive hypothesis). So, G' contains their intersection, which is a subset of $\{i \in I \mid M_i \models \varphi_2 [v_i, \vec{d_i}]\}$.

'Only if': Suppose that $\{i \in I \mid M_i \models \varphi_1 \rightarrow \varphi_2 [w_i, \vec{d_i}]\} \notin G$.

Then $G \cup \{\{i \in I \mid M_i \not\models \varphi_1 \to \varphi_2 [w_i, \vec{d_i}]\}\}$ generates a new filter G'.
For each i in this new set, choose $v_i \sqsupseteq_i w_i$ such that φ_1 holds at v_i, while φ_2 is refuted. Choosing values at other coordinates arbitrarily, this yields a function v for which it follows, by the inductive hypothesis, that φ_1 is true at $v^{G'}$, while φ_2 is false. Moreover, obviously, $w^G \sqsubseteq v^{G'}$: and hence $\varphi_1 \to \varphi_2$ has been refuted at w^G.
φ is $\forall x\, \varphi(x)$: this case may be treated likewise, this time also choosing a suitable individual in the chosen world. □

Notice that there exists an obvious morphism from D_{w^G} into $D_{v^{G'}}$, when $w^G \sqsubseteq v^{G'}$. For d^G in D_{w^G}, one uses the availability in G' of $\{i \in I \mid d_i \in D^i_{w_i}\} \cap \{i \in I \mid w_i \sqsubseteq_i v_i\}$, in combination with the domain condition in the coordinates M_i. But notice also that this morphism may well identify individuals (cf. the remark on page 3). In a perfectly general treatment, this feature should be built in right from the start, of course.

Finally, the two conditions on classical models should be checked.

4.10 <u>Lemma:</u> Filter products satisfy Heredity and Cofinality.

<u>Proof:</u> Heredity is straightforward from heredity for the coordinates, using the Łoś Equivalence. As for Cofinality, if φ fails at w^G, say $\{i \in I \mid M_i \models \varphi[w_i, \vec{d_i}]\} \notin G$, then, by cofinality for the coordinates, $\{i \in I \mid M_i \models \neg\varphi[v_i, \vec{d_i}]$ for some $v_i \sqsupseteq_i w_i\}$ contains $\{i \in I \mid M_i \not\models \varphi[w_i, \vec{d_i}]\}$. The latter may be added consistently to G, to obtain a new filter G' verifying $\neg\varphi$ at $v, \vec{d}$, while $w^G \sqsubseteq v^{G'}$. □

As an example of such filter products, one may like to think of either some small finite example, say with $I = \{1,2\}$, $M_1 = M_2 =$ the single world structure $(\bullet \longrightarrow \bullet)$, and $F = \{I\}$; or of some mysterious whole such as $\prod_F \mathbb{R}$, where F is the Fréchet Filter of all tails over the index set $\mathbb{N}$.

Finally, the above proofs did not use the totality of *all* filters extending F. It suffices to have *some* set of such filters closed under 'finite additions'. We will relax the definition of 'filter product' accordingly.

Lest it be objected that this introduces a new form of indeterminacy, whereas this was precisely one of our objections in the introduction, it should be added that there exists a minimal set of filters containing F having the above closure property. Another uniquely defined mathematical object is the set of all *countably generated* filters over F, and this will suffice in the sequel.

5 Classical definability of classes of structures

Various model-theoretic questions arise in the perspective presented here. Indeed, what becomes of ordinary model theory in this new setting? The perspective also generates questions of its own. E.g., one would like to see a mathematical characterization of a new operation such as 'generic extension'. In this report, we confine ourselves to checking that the above three operations on models are indeed 'characteristic', by proving a Keisler-type definability result.

5.1 Theorem: A class of possible worlds models is definable by means of some set of classical first-order sentences if and only if it is closed under the formation of generated submodels, disjoint unions, zigzag images, filter products and filter bases.

Explanation: the set Σ defines the class $\mathcal{K}$ if $\mathcal{K}$ consists of all models at all of whose worlds every formula in Σ is true;

if $\prod_F M$ is a 'filter power' of M (i.e., each M_i in the family equals M), then M is a *filter base* of $\prod_F M$.

Proof: By the earlier propositions, definable classes possess all five closure properties. The converse of the theorem requires some familiar patterns from model theory.

Let $M \models Th(\mathcal{K}) =_{def} \{\varphi \mid \varphi$ is true throughout each model in $\mathcal{K}\}$. We want to show that M actually belongs to $\mathcal{K}$. For, then, $\mathcal{K} = MOD(Th(\mathcal{K}))$, and we are done.

For a start, here is a useful reduction following from corollary 4.7. Each rooted submodel (M,w) verifies $Th(\mathcal{K})$ as well, and M is a zigzag image of the disjoint union of all these. Hence, by the assumed closure properties, it suffices to show that $(M,w) \in \mathfrak{K}$.

A First, we find $(N,v) \in \mathfrak{K}$ verifying (at v) precisely the same formulas as (M,w) (at w). Let Δ^+ consist of all formulas true at w, Δ^- all others. Each finite Δ_0^+, Δ_0^- is realized in some model $(M_{\Delta_0}, v_{\Delta_0}) \in \mathfrak{K}$.

For, let $\varphi \in \Delta_0^-$. If $\Delta_0^+, \{\varphi\}^-$ is not realized anywhere in $\mathfrak{K}$, then $\Pi\Delta_0^+ \rightarrow \varphi$ belongs to $Th(\mathfrak{K})$, and hence it would be true in (M,w): quod non. But then, we can 'realize/omit' Δ_0^+, Δ_0^- simultaneously in $\mathfrak{K}$, by taking a finite filter product of such models for $\Delta_0^+, \{\varphi\}^-$, letting the filter consist of the entire index set only. At its root, Δ_0^+ will still be verified (being true at all coordinates), while no $\varphi \in \Delta_0^-$ is. (Notice how such filter products behave a little like *direct products*, but with more 'upward transmission of truth' from the factors to the whole.) That there are always rooted realizations follows from lemma 4.2.

Next, the desired model (N,v) is found as in a (non-ultrafilter) proof of the compactness theorem. For each finite Δ_0, we have $(M_{\Delta_0}, v_{\Delta_0})$. The filter product may now be taken over this index set, with respect to the *regular* filter F containing all sets of the form

$$\{\Delta_0' \text{ finite} \mid \Delta_0'^+ \supseteq \Delta_0^+,\ \Delta_0'^- \supseteq \Delta_0^-\}.$$

$\Pi_F(M_{\Delta_0}, v_{\Delta_0})$ is still rooted, and by the Łoś Equivalence, it verifies Δ_0^+, while omitting all of Δ_0^-.

B This starting point is now used to erect two filter powers $\Pi_F(M,w)$, $\Pi_F(N,v)$, where F is the Fréchet filter over $\mathbb{N}$, and all filters involved are countably generated over F (cf. p.12). In a diagram:

$$\begin{tikzcd}
(M,w) \arrow[rr, phantom, "\frac{\text{same}}{\text{theory}}"] \arrow[d, squiggly] & & (N,v) \arrow[d, squiggly] \\
\Pi_F(M,w) & & \Pi_F(N,v)
\end{tikzcd}$$

Using certain saturation properties of these filter powers, we shall show that there exists a zigzagging connection bridging the gap between them.

C The relevant definition of a relation C is simply

$\langle u,\vec{d}\rangle C \langle t,\vec{e}\rangle$ if these sequences verify the same theory.

Notice that C does hold between $\langle w,\emptyset\rangle, \langle v,\emptyset\rangle$

What we have to establish is the zigzag property. (All others follow from this by the definition of C, and the fact that these filter powers are generated by their roots.) Evidently, it suffices to consider one direction only.

So, let $\langle u,\vec{d}\rangle C \langle t,\vec{e}\rangle$, and suppose that $u \sqsubseteq s$, $d \in D_s$

We want to find r, e such that $t \sqsubseteq r$, $e \in D_t$ and $\langle s,\vec{d},d\rangle C \langle t,\vec{e},e\rangle$.

In other words, exactly the type of $\vec{d},d$ in s is to be realized in some successor of t. Again, we distinguish a 'finite' and a 'total' case.

For each *finite* Δ_0^+, Δ_0^- realized at $\langle s,\vec{d},d\rangle$, it is easily seen, as before, that $\Delta_0^+, \{\varphi\}^-$ is realizable at some $\sqsubseteq$-successor of t (for each $\varphi \in \Delta_0^-$). (Consider the formula $\forall x\,(\prod\Delta_0^+ \to \varphi)$.) This time, the 'pasting together' to one single successor verifying Δ_0^+ while omitting all of Δ_0^- is more involved.

D We are going to use a *mixing construction*, after various combinatorial preliminaries. The situation is as follows:

$t = t^G$ has $\sqsubseteq$-successors $t^{1,G_1},\ldots,t^{k,G_k}$, each verifying Δ_0^+, while not verifying φ_j ($1 \leq j \leq k$, respectively) for $\vec{e}, e^j$ ($1 \leq j \leq k$)

(Here Δ_0^- is taken to be $\{\varphi_1,\ldots,\varphi_k\}$.) Now define

$$I^j =_{def} \{i \in N \mid t_i \sqsubseteq_i t_i^j \text{ and } \Delta_0^+(t_i^j, \vec{e}(i), e_i^j) \text{ and not } \varphi^j(t_i^j, \vec{e}(i), e_i^j)\}$$

By earlier results, I^j is at least consistent with G_j.

The heuristic idea of the following construction is to mix $t^1,\ldots,t^k$ in proportions $I^1,\ldots,I^k$ (and likewise with $e^1,\ldots,e^k$) in such a way that the resulting sequence becomes a $\sqsubseteq$-successor of t with respect to the minimal filter generated by $G \cup \{\bigcup_{j=1}^{k} I^j\}$, which verifies Δ_0^+, while omitting Δ_0^-. But, as they stand, $I^1,\ldots,I^k$ may have too much overlap for a smooth definition, whence the following pruning procedure.

We define a sequence $I_*^1,\ldots,I_*^k$ of *disjoint* sets,

where $I^j_* \subseteq I^j$ $(1 \leq j \leq k)$, and each I^j_* is consistent with G.
If we succeed in doing this, then the above construction goes through.
For, then, choose t^*, e^* as indicated: copying t^{j,G_j}, e^{j,G_j} along I^j_* $(1 \leq j \leq k)$. Clearly, Δ^+_0 will hold with respect to the filter generated by $G \cup \{\bigcup_{j=1}^{k} I^j_*\}$ (thanks to 4.9), while all of Δ^-_0 will fail. For, if $\varphi_n \in \Delta^-_0$ were to be true, then $\{i \in N \mid i \in \bigcup_{j=1}^{k} I^j_*$ and φ is true at $t^*_i, \vec{e}_i, e^*_i\}$ will belong to the filter, whence $\bigcup_{j=1, j\neq n}^{k} I^j_*$ does. By the construction of the filter, this implies that some $X \in G$ has an intersection with the whole $\bigcup_{j=1}^{k} I^j_*$ avoiding I^n_*. Consequently, I^n_* is not consistent with G, contrary to the above assumption.

E It remains to really find the above sequence.
Start with I^1. Next, as for the iterative step,
Suppose that $I^1_*, \ldots, I^n_*$ have been found, satisfying the above conditions.
Consider I^{n+1}. There are two cases. First, if $I^{n+1} - \bigcup_{j=1}^{n} I^j_*$ is consistent with G, then I^{n+1}_* may be taken to be this remainder.
Otherwise, $I^{n+1} \cap \bigcup_{j=1}^{n} I^j_*$ must be consistent with G (since, at least, I^{n+1} is consistent with G: it belongs to a filter G_{n+1} extending G).
By a familiar argument, it follows that $I^{n+1} \cap I^j_*$ must be consistent with G for at least one j $(1 \leq j \leq n)$. We will split the infinite (!) set I^j_* into two infinite sets, both consistent with G, allotting one (suitable) half as I^{n+1}_*, leaving the remainder for the new I^j_*.

To do this, we notice that the following proposition holds generally for filters G in the present special type of filter product.
<u>Claim</u>: if X is consistent with G, then there exist disjoint X_1, X_2 both consistent with G such that $X = X_1 \cup X_2$.
This need not always be the case; but here we know that G is generated by the Fréchet filter of all tails together with some *countable* set of additions $Y_1, Y_2, \ldots$. Now, pick two disjoint sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ as follows.
First, take distinct a_1, b_1 in $X \cap Y_1$. (Because X is consistent with G, $X \cap Y_1$ is non-empty; indeed it is infinite, as G is a free filter.)

Next, suppose that distinct $a_1,\dots,a_n$; $b_1,\dots,b_n$ have been found such that $a_i, b_i \in X \cap Y_1 \cap \dots \cap Y_i$ $(1 \leq i \leq n)$. Consider $X \cap Y_1 \cap \dots \cap Y_n \cap Y_{n+1}$: this set is still infinite, and so one may continue, picking new a_{n+1}, b_{n+1}.
Thus, two sets $A = \{a_1, a_2, \dots\}$, $B = \{b_1, b_2, \dots\}$ are obtained, both intersecting all generators of G, and hence both consistent with G.
So, for the required X_1, X_2 one can take, say, $X_1 = A$, $X_2 = X - A$.

Finally, again the intersection of I^{n+1} with one of these sets must be consistent with G: that intersection becomes I^{n+1}_*, while the rest of the set becomes the new I^j_*.

Thus, the 'finite' case of part C (page 14) has been established.

F A saturation argument will now lift this to the 'total' case.
Fix enumerations $\psi_1, \psi_2, \dots$ of Δ^+, $\varphi_1, \varphi_2, \dots$ of Δ^- (recall $\langle s, \vec{d}, d \rangle$).
For each k, $\{\psi_1, \dots, \psi_k\}^+, \{\varphi_1, \dots, \varphi_k\}^-$ is realized at some $\sqsubseteq$-successor r of t, with respect to $\vec{e}, e^k$. It follows that $\{ i \in N \mid N_i$ realizes $\{\psi_1, \dots, \psi_k\}^+, \{\varphi_1, \dots, \varphi_k\}^-$ for $\vec{e}_i, e^k_i$ at some $r_i \sqsupseteq_i t_i \}$ is consistent with G, and hence (all filters being free) so is the intersection X_k of this set with $[k, \infty)$.
Thus, we have a descending sequence $X_1 \supseteq X_2 \supseteq \dots$ whose intersection is empty, all of whose members are consistent with G.
Together with G, these generate a filter G', whose contribution to $\Pi_F(N, v)$ is the following. For $i \in N$, let $y(i)$ be the greatest l such that $i \in X_l$, $y(i) = 0$ if no such l exists. (Notice that $y(i) \geq m$ for $i \in X_m$.)
For each i with $y(i) > 0$ choose some $r_i \sqsupseteq_i t_i$ with $\vec{e_i}, e_i$ realizing $\{\psi_1, \dots, \psi_{y(i)}\}^+, \{\varphi_1, \dots, \varphi_{y(i)}\}^-$. Now, we can see that $r \sqsupseteq t$, such that Δ^+ is verified at $(r, \vec{e}, e)^{G'}$, while all of Δ^- is omitted.
The zigzag property of C has been established.

G The final argument to show that $(M,w) \in \mathcal{K}$ merely follows the schema of B, clockwise:

$$\begin{array}{lcccl} \text{(filter base)} & (M,w)\in\mathcal{K} & & (N,v)\in\mathcal{K} & \text{(by part A)} \\ & \big\downarrow & \circlearrowright & \big\downarrow & \\ \text{(zigzag image)} & \Pi_F(M,w)\in\mathcal{K} & \leftrightarrows & \Pi_F(N,v)\in\mathcal{K} & \text{(filter product)} \end{array}$$

$\square$

This completes a first exercise in the new classical model theory arising from possible worlds semantics.

6 References

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